Numerical and Statistical Methods

for COMPUTER ENGINEERING

Gujarat Technological University 2017

Second Edition

About the Authors

Ravish R Singh is presently Academic Advisor at Thakur Educational Trust, Mumbai. He obtained a BE degree from University of Mumbai in 1991, an MTech degree from IIT Bombay in 2001, and a PhD degree from Faculty of Technology, University of Mumbai, in 2013. He has published several books with McGraw Hill Education (India) Private Limited on varied subjects like *Engineering Mathematics (I and II), Applied Mathematics, Electrical Engineering, Electrical and Electronics Engineering*, etc., for all-India curricula as well as regional curricula of some universities like Gujarat Technological University, Mumbai University, Pune University, Jawaharlal Nehru Technological University, Anna University, Uttarakhand Technical University, and Dr A P J Abdul Kalam Technical University (formerly known as UPTU). Dr Singh is a member of IEEE, ISTE, and IETE, and has published research papers in national and international journals. His fields of interest include Circuits, Signals and Systems, and Engineering Mathematics.

Mukul Bhatt is presently Assistant Professor, Department of Humanities and Sciences, at Thakur College of Engineering and Technology, Mumbai. She obtained her MSc (Mathematics) from H N B Garhwal University in 1992. She has published several books with McGraw Hill Education (India) Private Limited on *Engineering Mathematics (I and II)* and *Applied Mathematics* for all-India curricula as well as regional curricula of some universities like Gujarat Technological University, Mumbai University, Pune University, Jawaharlal Nehru Technological University, Anna University, Uttarakhand Technical University, and Dr A P J Abdul Kalam Technical University (formerly known as UPTU). She has seventeen years of teaching experience at various levels in engineering colleges in Mumbai and her fields of interest include Integral Calculus, Complex Analysis, and Operation Research. She is a member of ISTE.

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Ravish R Singh
Academic Advisor
Thakur Educational Trust
Mumbai, Maharashtra

Mukul Bhatt
Assistant Professor
Department of Humanities and Sciences
Thakur College of Engineering and Technology
Mumbai, Maharashtra

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Dedicated
To Our Parents

Late Shri Ramsagar Singh
and
Late Shrimati Premsheela Singh

Ravish R Singh

Late Shri Ved Prakash Sharma
and
Late Shrimati Vidyavati Hemdan

Mukul Bhatt

Contents

Preface

Mathematics is a key area of study in any engineering course. A sound knowledge of this subject will help engineering students develop analytical skills, and thus enable them to solve numerical problems encountered in real life, as well as apply mathematical principles to physical problems, particularly in the field of engineering.

Users

This book is designed for the 4^{th} semester GTU Computer Engineering students pursuing the course *Numerical and Statistical Methods (CODE 2140706)*. It covers the complete GTU syllabus for the course on Numerical and Statistical Methods for computer engineering branches.

Objective

The crisp and complete explanation of topics will help students easily understand the basic concepts. The tutorial approach (i.e., teach by example) followed in the text will enable students develop a logical perspective to solving problems.

Features

Each topic has been explained from the examination point-of-view, wherein the theory is presented in an easy-to-understand student-friendly style. Full coverage of concepts is supported by numerous solved examples with varied complexity levels, which is aligned to the latest GTU syllabus. Fundamental and sequential explanation of topics is well aided by examples and exercises. The solutions of examples are set following a 'tutorial' approach, which will make it easy for students from any background to easily grasp the concepts. Exercises with answers immediately follow the solved examples enforcing a practice-based approach. We hope that the students will gain logical understanding from solved problems and then reiterate it through solving similar exercise problems themselves. The unique blend of theory and application caters to the requirements of both the students and the faculty. Solutions of GTU examination questions are incorporated within the text appropriately.

Highlights

- Crisp content strictly as per the latest GTU syllabus of *Numerical and Statistical Methods* (Regulation 2014)
- Comprehensive coverage with lucid presentation style
- Each section concludes with an exercise to test understanding of topics
- Solutions of GTU examination papers from 2010 to 2015 present appropriately within the chapters
- Solution of 2016 GTU examination paper can be accessible through weblink.
- Rich exam-oriented pedagogy:
  - ➢ Solved Examples within chapters: 420
  - ➢ Solved GTU questions tagged within chapters: 112
  - ➢ Unsolved Exercises: 148

Online Learning Center

All the C Programs included in *Numerical and Statistical Methods (Computer Engineering)* are available on OLC link *http://www.mhhe.com/singh/nsm2e/cse/gtu2017*

Chapter Organization

The content spans the following ten chapters which wholly and sequentially cover each module of the syllabus.

❏ **Chapter 1** introduces Error Analysis.

❏ **Chapter 2** discusses Roots of Equations.

❏ **Chapter 3** presents Systems of Linear Algebraic Equations.

❏ **Chapter 4** covers Interpolation.

❏ **Chapter 5** deals with Curve Fitting.

❏ **Chapter 6** presents Numerical Integration.

❏ **Chapter 7** explains Ordinary Differential Equations.

❏ **Chapter 8** discusses Statistical Methods.

❏ **Chapter 9** deals with Correlation and Regression.

❏ **Chapter 10** introduces Trend Analysis.

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Ravish R Singh
Mukul Bhatt

Publisher's Note

Remember to write to us. We look forward to receiving your feedback, comments, and ideas to enhance the quality of this book. You can reach us at *info.india@mheducation.com*. Please mention the title and authors' name as the subject. In case you spot piracy of this book, please do let us know.

ROADMAP TO THE SYLLABUS

This text is useful for
Numerical and Statistical Methods (Code 2140706)
For Computer Engineering

Module 1: Mathematical Modelling and Engineering Problem-Solving

Approximations and errors; Significant figures; Accuracy and precision; Errors; Round-off and truncation errors; Error propagation

GO TO
CHAPTER 1: Error Analysis

Module 2: Roots of Equations

Mathematical background; Bisection; Regula falsi method; Newton–Raphson method; Secant method; Successive approximation method; Budan's theorem; Barristow's method; Case studies

GO TO
CHAPTER 2: Roots of Equations

Module 3: Systems of Linear Algebraic Equations

Mathematical background; Gauss elimination; Pitfalls and techniques for improvement; Matrix inversion and Gauss–Seidel methods; Ill-conditional equations; Predictor-corrector methods; Case studies

GO TO
CHAPTER 3: Systems of Linear Algebraic Equations

Module 4: Curve Fitting

Mathematical background; Least squares method; Linear and polynomial regression; Lagrange's interpolating polynomials; Spline interpolation; Case studies

GO TO
CHAPTER 4: Interpolation
CHAPTER 5: Curve Fitting

Module 5: Numerical Integration

Newton–Cotes integration formulae; Trapezoidal rule and Simpson's rules; Interpolation; Case studies

GO TO
CHAPTER 6: Numerical Integration

Module 6: Ordinary Differential Equations

Euler's method; Runge–Kutta methods; General methods for boundary-value problems; Case studies

GO TO
CHAPTER 7: Ordinary Differential Equations

Module 7: Statistical Methods

Frequency distributions; Data analysis; Expectations and moments; Corelation and regression; Trend analysis; Seasonal effects; Cyclical fluctuation; Moving average; MSE; Predictions; Non-parametric statistics; Computer-based resampling techniques; Confidence intervals and statistical significance

GO TO
CHAPTER 8: Statistical Methods
CHAPTER 9: Correlation and Regression
CHAPTER 10: Trend Analysis

CHAPTER 1

Error Analysis

Chapter Outline

1.1 INTRODUCTION

The main goal of numerical analysis is to develop efficient algorithms for computing precise numerical values of mathematical quantities, including functions, integrals, solutions of algebraic equations, solutions of differential equations, etc. Often the numerical data and the methods used are approximate ones. Hence, the error in a computed result may be caused by the errors in the data, or the errors in the method, or both. In any numerical computation, there are four key sources of errors:

(i) Inexactness of mathematical model for the underlying physical phenomenon
(ii) Errors in measurements of parameters entering the model
(iii) Round-off errors in computer arithmetic
(iv) Approximations used to solve the mathematical systems

1.2 ACCURACY AND PRECISION

Measurements and calculations can be characterized with regard to their accuracy and precision. *Accuracy* refers to how closely a computed or measured value agrees with the true value. *Precision* refers to how closely individually computed or measured values agree with each other. *Inaccuracy* is the systematic deviation from the truth. *Imprecision* refers to the magnitude of scatter. Figure 1.1 illustrates the concepts of accuracy and precision.

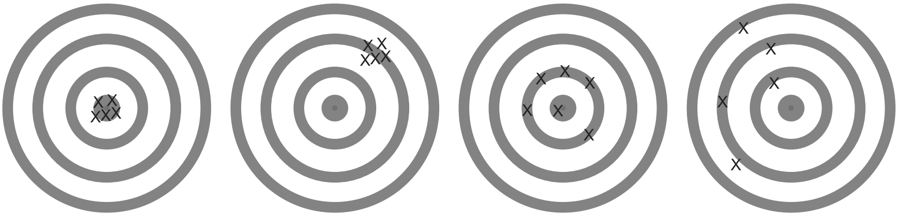

Fig. 1.1

The term *error* represents the imprecision and inaccuracy of a numerical computation.

1.3 TYPES OF ERRORS

There are various types of errors in measurements and calculations:

(i) Absolute error
(ii) Relative error
(iii) Percentage error

Absolute Error It is the difference between the measured or calculated value and true value. If x_{exact} is the true or exact value and x_{approx} is the measured or calculated or approximate value, the absolute error δx is given by

$$\in_a = \delta x = \left| x_{\text{exact}} - x_{\text{approx}} \right|$$

Relative Error It is the ratio of absolute error and true value of the quantity.

$$\in_r = \frac{\delta x}{x} = \left| \frac{x_{\text{exact}} - x_{\text{approx}}}{x_{\text{exact}}} \right|$$

Percentage Error It is relative error expressed in terms of per 100.

$$\in_p = \frac{\delta x}{x} \times 100 = \left| \frac{x_{\text{exact}} - x_{\text{approx}}}{x_{\text{exact}}} \right| \times 100$$

1.4 SOURCES OF ERRORS

There are three sources of errors, namely, inherent error, truncation error, and round-off error.

Inherent Error It is the error that pre-exist in the problem statement itself before its solution is obtained. Such errors arise in the values of data from the real world or by uncertainty in measurements or due to the given data being approximate or due to the limitations of mathematical tables, calculators, digital computer, etc. These errors cannot be completely eliminated but can be minimized if better data is selected or high-precision computer computations are employed, e.g., representation of irrational numbers such as π, e, $\sqrt{2}$ cannot be represented with a finite number of digits. Even a simple fraction in many cases has no exact representation, such as a rational number $\frac{1}{3}$.

Truncation Error It is the error that results from using an approximation in place of exact mathematical expressions. It is caused by truncating a finite number of terms. The most common example is the truncation of an infinite series to a finite number of terms, e.g., sin x is represented by the Maclaurin series as

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \infty = x_{\text{exact}}$$

But, if sin x is calculated by terminating the series up to x^7 or x^9,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} = x_{\text{approx}}$$

Hence, truncation error $= x_{\text{exact}} - x_{\text{approx}}$

Round-off Error It is the error that results due to chopping or rounding or arithmetic operations using normalized floating-point numbers. It is due to the inaccuracies that arise because of a finite number of digits of precision used to represent numbers. All computers represent numbers, except for integer and some fractions, with imprecision. Digital computers use floating-point numbers of fixed word length. This type of representation will not express the exact or true values correctly. Error introduced by the omission of significant figures due to computer imperfection is called *round-off error*.

1.5 SIGNIFICANT FIGURES

The significant figures of a number are digits that carry meaning contributing to its measurement resolution. This includes all digits except (i) all leading zeros, and (ii) all trailing zeros when they are merely placeholders to indicate the scale of the number.

Rules for Identifying Significant Figures

(i) All nonzero digits are considered significant, e.g., 93 has two significant figures, i.e., 9 and 3, while 135.76 has five significant figures, i.e., 1, 3, 5, 7, and 6.

(ii) All zeros between two nonzero digits are significant, e.g., 205.1308 has seven significant figures, i.e., 2, 0, 5, 1, 3, 0 and 8.

(iii) Leading zeros are not significant, e.g., 0.00075 has two significant figures, 7 and 5.

(iv) Trailing zeros in a number containing a decimal point are significant, e.g., 13.4000 has six significant figures, i.e., 1, 3, 4, 0, 0, and 0. The number 0.000134000 still has only six significant figures (the zeros before the 1 are not significant). The number 120.00 has five significant figures since it has three trailing zeros. The number of significant figures in 8200 is at least two, but it could be three or four because it is not clear if the zeros are significant or not. To avoid uncertainty, scientific notation is used to place zeros behind a decimal point, i.e., 8.200×10^3 has four significant figures, whereas 8.2×10^3 has two significant figures.

Example 1

Find the relative error and percentage error if 0.005998 *is truncated to three decimal digits.*

Solution

$$x_{\text{exact}} = 0.005998$$

$$x_{\text{approx}} = 0.005$$

$$\text{Relative error } = \left|\frac{x_{\text{exact}} - x_{\text{approx}}}{x_{\text{exact}}}\right|$$

$$= \left|\frac{0.005998 - 0.005}{0.005998}\right|$$

$$= 0.1664$$

$$\text{Percentage error } = \left|\frac{x_{\text{exact}} - x_{\text{approx}}}{x_{\text{exact}}}\right| \times 100$$

$$= 0.1664 \times 100$$

$$= 16.64\%$$

Example 2

For $a = 3.141592$ *and an approximation value of* a *as* 3.14, *evaluate absolute error, relative error, and percentage error.*

Solution

$$a_{\text{exact}} = 3.141592, \qquad a_{\text{approx}} = 3.14$$

$$\text{Absolute error} = \delta a = a_{\text{exact}} - a_{\text{approx}} = 3.141592 - 3.14 = 0.001592$$

$$\text{Relative error } = \frac{\delta a}{a} = \frac{0.001592}{3.141592} = 5.0675 \times 10^{-4}$$

$$\text{Percentage error } = \frac{\delta a}{a} \times 100 = 5.0675 \times 10^{-4} \times 100 = 0.05067\%$$

Example 3

If the approximate solution of a problem is $x_0 = 35.25$ with relative error of at the most 2%, find the range of values correct up to four decimal digits in which the exact value of the solution lies.

Solution

$$x_{0_{\text{approx}}} = 35.25$$

$$\epsilon_r = 2\% \text{ of } 35.25 = 0.705$$

$$\epsilon_r = \frac{x_{0_{\text{exact}}} - x_{0_{\text{approx}}}}{x_{0_{\text{exact}}}}$$

$$0.705 = \frac{x_{0_{\text{exact}}} - 35.25}{x_{0_{\text{exact}}}}$$

$$(1 - 0.705)\, x_{0_{\text{exact}}} = 35.25$$

$$x_{0_{\text{exact}}} = 119.4915$$

The range of values correct up to four decimal digits in which the exact value of the solution lies is [119.4915, 119.50].

Example 4

The approximate solution of a problem is 3.436. If the absolute error in the solution is less than 0.01 then find the interval within which the exact solution lies.

Solution

$$x_{\text{approx}} = 3.436$$

$$\delta x < 0.01$$

$$\delta x = \frac{x_{\text{exact}} - x_{\text{approx}}}{x_{\text{exact}}}$$

$$\frac{x_{\text{exact}} - 3.436}{x_{\text{exact}}} < 0.01$$

$$x_{\text{exact}}(1 - 0.01) < 3.436$$

$$x_{\text{exact}} < 3.4707$$

Hence, the exact solution lies in the interval [3.47, 3.48].

Example 5

If $u = 2v^6 - 5v$, find the percentage error in u at $v = 1$ if the error in v is 0.05.

Solution

$$u = 2v^6 - 5v$$

$$\delta u = 12v^5\ \delta v - 5\delta v$$

$$\frac{\delta u}{u}\times 100 = \frac{1}{u}\left[12\ v^5\ \delta v \times 100 - 5\ \delta v \times 100\right]$$

$$= \frac{1}{2v^6 - 5v}\left[12v^5\ \delta v \times 100 - 5\ \delta v \times 100\right]$$

Putting $\delta v = 0.05, \quad v = 1$

$$\frac{\delta u}{u}\times 100 = \frac{1}{2(1)^6 - 5(1)}\left[12(1)^5\ (0.05)\ (100) - 5(0.05)\ (100)\right]$$

$$= -11.67\%$$

Hence, the percentage error in $u = -11.67\%$.

Example 6

Given the trigonometric function $f(x) = \sin x$,

(i) expand $f(x)$ about $x = 0$ using the Taylor series

(ii) truncate the series to $n = 6$ terms

(iii) find the relative error at $x = \frac{\pi}{4}$ due to truncation.

Solution

(i) $f(x) = \sin x$

By the Taylor series,

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

(ii) Truncation of the Taylor series to $n = 6$ terms

$$f_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

(iii) Relative error at $x = \frac{\pi}{4}$ due to truncation

$$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$$

$$f_6\left(\frac{\pi}{4}\right)=\frac{\pi}{4}-\frac{\left(\frac{\pi}{4}\right)^3}{3!}+\frac{\left(\frac{\pi}{4}\right)^5}{5!}$$

$$\begin{aligned}\in_r &=\left|\frac{f\left(\frac{\pi}{4}\right)-f_6\left(\frac{\pi}{4}\right)}{f\left(\frac{\pi}{4}\right)}\right| \\ &=\left|\frac{\sin\left(\frac{\pi}{4}\right)-\left(\frac{\pi}{4}-\frac{\pi^3}{384}+\frac{\pi^5}{122880}\right)}{\sin\left(\frac{\pi}{4}\right)}\right| \\ &=5.1286\times10^{-5}\end{aligned}$$

Example 7

Given the function $f(x) = e^{-x}$,

(i) expand $f(x)$ about $x = 0$ using the Taylor series

(ii) truncate the series to $n = 5$ terms

(iii) find the relative error at $x = 1$ due to truncation.

Solution

(i) $f(x) = e^{-x}$

By the Taylor series,

$$e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\cdots$$

(ii) Truncation of the Taylor series to $n = 5$ terms

$$f_5(x)=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}$$

(iii) The relative error at $x = 1$ due to truncation

$$f_5(1)=1-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}=\frac{3}{8}$$

$$f(1)=e^{-1}$$

$$\begin{aligned}\in_r &=\left|\frac{f(1)-f_5(1)}{f(1)}\right| \\ &=\left|\frac{e^{-1}-\frac{3}{8}}{e^{-1}}\right| \\ &=0.0194 \\ &=1.94\%\end{aligned}$$

Example 8

If $R = x^3\, y^2\, z^2$ and 0.03, 0.01, 0.02 *are errors in x, y, z respectively at* $x = 1$, $y = 1$, $z = 2$. *Calculate the absolute error and percentage error in calculation of R.*

Solution

$$R = x^3\, y^2\, z^2$$

Taking logarithm on both the sides,

$$\begin{aligned}\log R &= \log x^3 + \log y^2 + \log z^2 \\ &= 3\log x + 2\log y + 2\log z\end{aligned}$$

$$\frac{1}{R}\delta_R = \frac{3}{x}\delta x + \frac{2}{y}\delta y + \frac{2}{z}\delta z$$

$$\frac{\delta_R}{R} = 3\frac{\delta x}{x} + 2\frac{\delta y}{y} + 2\frac{\delta z}{z}$$

Putting $\delta x = 0.03$, $\delta y = 0.01$, $\delta z = 0.02$, $x = 1$, $y = 1$, $z = 2$,

$$R = (1)^3(1)^2(2)^2 = 4$$

$$\frac{\delta R}{4} = 3\left(\frac{0.03}{1}\right) + 2\left(\frac{0.01}{1}\right) + 2\left(\frac{0.02}{1}\right)$$

$$\frac{\delta R}{4} = 0.15$$

$$\delta R = 0.6$$

Hence, absolute error = 0.6.

$$\begin{aligned}\text{Percentage error in } R &= \frac{\delta R}{R} \times 100 \\ &= 0.15 \times 100 \\ &= 15\%\end{aligned}$$

Example 9

Find the percentage error in calculating the area of a rectangle when an error of 3% *is made in measuring each of its sides.*

Solution

Let a and b be the sides of the rectangle and A be its area.

$$A = ab$$

Taking logarithm on both the sides,

$$\log A = \log a + \log b$$

$$\frac{1}{A}\delta A = \frac{1}{a}\delta a + \frac{1}{b}\delta b$$

$$\frac{\delta A}{A}\times 100 = \frac{\delta a}{a}\times 100 + \frac{\delta b}{b}\times 100$$

Putting $\quad \dfrac{\delta a}{a}\times 100 = 3, \quad \dfrac{\delta b}{b}\times 100 = 3,$

$$\frac{\delta A}{A}\times 100 = 3+3$$

$$= 6$$

Hence, percentage error in calculating area = 6%.

Example 10

Find the percentage error in the area of an ellipse when errors of 2% and 3% are made in measuring its major and minor axes respectively.

Solution

Let $2a$ and $2b$ be the major and minor axes of the ellipse and A be its area.

$$A = \pi ab$$

Taking logarithm on both the sides,

$$\log A = \log \pi + \log a + \log b$$

$$\frac{1}{A}\delta A = 0 + \frac{1}{a}\delta a + \frac{1}{b}\delta b$$

$$\frac{\delta A}{A}\times 100 = \frac{\delta a}{a}\times 100 + \frac{\delta b}{b}\times 100$$

Putting $\quad \dfrac{\delta a}{a}\times 100 = 2, \quad \dfrac{\delta b}{b}\times 100 = 3,$

$$\frac{\delta A}{A}\times 100 = 2+3$$

$$= 5$$

Hence, percentage error in area of ellipse = 5%.

Example 11

The focal length of a mirror is found from the formula $\dfrac{2}{f} = \dfrac{1}{v} - \dfrac{1}{u}$. *Find the percentage error in f if u and v are both in error by 2% each.*

Solution

$$\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$$

$$-\frac{2}{f^2}\delta f = -\frac{1}{v^2}\delta v + \frac{1}{u^2}\delta u$$

$$-\frac{2}{f}\frac{\delta f}{f}\times 100 = -\frac{1}{v}\frac{\delta v}{v}\times 100 + \frac{1}{u}\frac{\delta u}{u}\times 100$$

Putting $$\frac{\delta u}{u}\times 100 = 2, \quad \frac{\delta v}{v}\times 100 = 2,$$

$$-\frac{2}{f}\frac{\delta f}{f}\times 100 = -\frac{1}{v}(2) + \frac{1}{u}(2)$$

$$= -2\left(\frac{1}{v} - \frac{1}{u}\right) = -2\left(\frac{2}{f}\right)$$

$$\frac{\delta f}{f}\times 100 = 2$$

Hence, percentage error in $f = 2\%$.

Example 12

Find the possible percentage error in computing the parallel resistance R of two resistances R_1 and R_2 if R_1, R_2 are each in error by 2%.

Solution

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$-\frac{1}{R^2}\delta R = -\frac{1}{R_1^2}\delta R_1 - \frac{1}{R_2^2}\delta R_2$$

$$\frac{1}{R}\frac{\delta R}{R}\times 100 = \frac{1}{R_1}\frac{\delta R_1}{R_1}\times 100 + \frac{1}{R_2}\frac{\delta R_2}{R_2}\times 100$$

Putting $$\frac{\delta R_1}{R_1}\times 100 = 2, \frac{\delta R_2}{R_2}\times 100 = 2,$$

$$\frac{1}{R}\frac{\delta R}{R}\times 100 = \frac{1}{R_1}(2) + \frac{1}{R_2}(2)$$

$$= 2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$= 2\left(\frac{1}{R}\right)$$

$$\therefore \quad \frac{\delta R}{R} \times 100 = 2$$

Hence, percentage error in $R = 2\%$

Example 13

The resonant frequency in a series electrical circuit is given by $f = \dfrac{1}{2\pi\sqrt{LC}}$. *If the measurement of L and C are in error by* 2% *and* −1% *respectively, find the percentage error in f.*

Solution

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Taking logarithm on both the sides,

$$\log f = \log \frac{1}{2\pi} - \frac{1}{2}\log L - \frac{1}{2}\log C$$

$$\frac{1}{f}\delta f = 0 - \frac{1}{2}\frac{1}{L}\delta L - \frac{1}{2}\frac{1}{C}\delta C$$

$$\frac{\delta f}{f} \times 100 = -\frac{1}{2}\frac{\delta L}{L} \times 100 - \frac{1}{2}\frac{\delta C}{C} \times 100$$

Putting $$\frac{\delta L}{L} \times 100 = 2, \quad \frac{\delta C}{C} \times 100 = -1$$

$$\frac{\delta f}{f} \times 100 = -\frac{1}{2}(2) - \frac{1}{2}(-1)$$

$$= -0.5$$

Hence, percentage error in $f = -0.5\%$

Example 14

In calculating the volume of a right circular cone, errors of 2% *and* 1% *are made in the height and radius of base respectively. Find the percentage error in the calculating the volume.*

Solution

Let r and h be the radius of base and height of the right circular cone and V be its volume.

$$V = \frac{1}{3}\pi r^2 h$$

Taking logarithm on both the sides,

$$\log V = \log \frac{\pi}{3} + 2\log r + \log h$$

$$\frac{1}{V}\delta V = 0 + \frac{2}{r}\delta r + \frac{1}{h}\delta h$$

$$\frac{\delta V}{V}\times 100 = 2\frac{\delta r}{r}\times 100 + \frac{\delta h}{h}\times 100$$

Putting $\quad \frac{\delta r}{r}\times 100 = 1, \quad \frac{\delta h}{h}\times 100 = 2,$

$$\frac{\delta V}{V}\times 100 = 2(1) + 2$$

$$= 4$$

Hence, percentage error in volume = 4%

Example 15

In calculating the volume of a right circular cylinder, errors of 2% and 1% are found in measuring the height and base radius respectively. Find the percentage error in the calculated volume of the cylinder.

Solution

Let r and h be the base radius and height of the right circular cylinder and V be its volume.

$$V = \pi r^2 h$$

Taking logarithm on both the sides,

$$\log V = \log \pi + 2\log r + \log h$$

$$\frac{1}{V}\delta V = 0 + \frac{2}{r}\delta r + \frac{1}{h}\delta h$$

$$\frac{\delta V}{V}\times 100 = 2\frac{\delta r}{r}\times 100 + \frac{\delta h}{h}\times 100$$

Putting $\quad \frac{\delta r}{r}\times 100 = 1, \frac{\delta h}{h}\times 100 = 2,$

$$\frac{\delta V}{V}\times 100 = 2(1) + 2$$

$$= 4$$

Hence, percentage error in volume = 4%.

Example 16

Evaluate $[(3.82)^2 + 2(2.1)^3]^{\frac{1}{5}}$ *using the theory of approximation.*

Solution

Let $$z = (x^2 + 2y^3)^{\frac{1}{5}}$$

$$\delta z = \frac{1}{5}(x^2 + 2y^3)^{-\frac{4}{5}}(2x)\delta x + \frac{1}{5}(x^2 + 2y^3)^{-\frac{4}{5}}(6y^2)\delta y$$

$$= \frac{1}{5}(x^2 + 2y^3)^{-\frac{4}{5}}(2x\,\delta x + 6y^2\,\delta y)$$

Putting $x = 4$, $y = 2$,

$$\delta x = 3.82 - 4 = -0.18,$$

$$\delta y = 2.1 - 2 = 0.1$$

$$(x^2 + 2y^3) = 4^2 + 2(2)^3$$

$$= 32$$

and $$\delta z = \frac{1}{5}\cdot(32)^{-\frac{4}{5}}[2(4)(-0.18) + 6(2)^2(0.1)]$$

$$= 0.012$$

Approximate value $= z + \delta z$

$$= (32)^{1/5} + 0.012$$

$$= 2.012$$

Example 17

Find the approximate value of $\left[(0.98^2 + (2.01)^2 + (1.94)^2\right]^{\frac{1}{2}}$.

Solution

Let $$u = \sqrt{x^2 + y^2 + z^2}$$

$$u^2 = x^2 + y^2 + z^2$$

$$2u\delta u = 2x\delta x + 2y\delta y + 2z\delta z$$

$$u\delta u = x\delta x + y\delta y + z\delta z$$

Putting $x = 1$, $y = 2$, $z = 2$,

$$\delta x = 0.98 - 1 = -0.02,$$

$$\delta y = 2.01 - 2 = 0.01,$$

$$\delta z = 1.94 - 2 = -0.06$$

$$u = \sqrt{1^2 + 2^2 + 2^2}$$
$$= 3$$

and $$u\delta u = 1(-0.02) + 2(0.01) + 2(-0.06)$$
$$= -0.12$$
$$\delta u = -0.04$$

Approximate value $= u + \delta u$
$= 3 - 0.04$
$= 2.96$

Example 18

Evaluate $(1.99)^2(3.01)^3(0.98)^{\frac{1}{10}}$ *using approximation.*

Solution

Let $$u = x^2\ y^3\ z^{\frac{1}{10}}$$

$$\log u = 2\log x + 3\log y + \frac{1}{10}\log z$$

$$\frac{1}{u}\delta u = 2\frac{1}{x}\delta x + 3\frac{1}{y}\delta y + \frac{1}{10}\frac{1}{z}\delta z$$

Putting $$x = 2, \quad y = 3, \quad z = 1,$$
$$\delta x = 1.99 - 2 = -0.01,$$
$$\delta y = 3.01 - 3 = 0.01,$$
$$\delta z = 0.98 - 1 = -0.02$$
$$u = 2^2\ 3^3\ 1^{\frac{1}{10}} = 108$$

and $$\frac{1}{108}\delta u = 2\cdot\left(\frac{1}{2}\right)(-0.01) + 3\left(\frac{1}{3}\right)(0.01) + \frac{1}{10}\left(\frac{1}{1}\right)(-0.02)$$
$$\delta u = -0.216$$

Approximate value $= u + \delta u$
$= 108 - 0.216$
$= 107.784.$

EXERCISE 1.1

1. Round off the following numbers to 4-significant digits:

(i) 2.36345 (ii) 0.34176 (iii) 3.40087 (iv) 0.000143479

[**Ans.:** (i) 2.363 (ii) 0.3417 (iii) 3.401 (iv) 0.0001435]

2. Find the error and relative error in the following cases:

(i) $x_{\text{exact}} = 1000000$, $x_{\text{approx}} = 999996$

(ii) $x_{\text{exact}} = 0.000012$, $x_{\text{approx}} = 0.000009$

[**Ans.:** (i) 4, 0.000004 (ii) 0.000003, 0.25]

3. Find the relative error of $x - y$ for $x = 12.05$ and $y = 0.802$ having absolute error $\delta x = 0.005$ and $\delta y = 0.001$.

[**Ans.:** 0.00029]

4. Find absolute error, relative error, and percentage error if $\frac{2}{3}$ is approximated to 4-significant digits.

[**Ans.:** 0.000033, 0.0000495, 0.005%]

5. If the approximate value of $\frac{\pi}{4}$ is 0.7854, calculate (i) absolute error, (ii) relative error, and (iii) percentage error.

[**Ans.:** (i) 0.00031, (ii) 0.00039, (iii) 0.04%]

6. If $\delta x = 0.005$ and $\delta y = 0.001$ be the absolute errors in $x = 2.11$ and $y = 4.15$, find the relative error in computation of $x + y$.

[**Ans.:** 0.000958]

7. In calculating the volume of right circular cone, errors of 2.75% and 1.25% are made in height and radius of the base. Find the percentage error in volume.

[**Ans.:** 5.25%]

8. The height of a cone is $H = 30$ cm, the radius of base $R = 10$ cm. How will the volume of the cone change if H is increasing by 3 mm while R is decreasing by 1 mm?

[**Ans.:** decreased by 10π cm^3]

9. Find the percentage error in calculating the area of a rectangle when an error of 2% is made in measuring each of its sides.

[**Ans.:** 4%]

10. If R_1 and R_2 are two resistances in parallel, their resistance R is given by $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$. If there is an error of 2% in both R_1 and R_2, find percentage error in R.

[**Ans.:** 2%]

11. One side of a rectangle is $a = 10$ cm and the other side $b = 24$ cm. How will the diagonal l of the rectangle change if a is increased by 4 mm and b is decreased by 1 mm?

$\left[\textbf{Ans.: } \frac{4}{65}\text{ cm}\right]$

12. The resistance R of circuit was found by using the formula $I=\frac{E}{R}$. If there is an error of 0.1 ampere in reading I and 0.5 volts in reading E, find the corresponding percentage error in R when I = 15 amperes and E = 100 volts.

[**Ans.:** −0.167%]

13. The voltage V across a resistor is measured with error h, and the resistance R is measured with an error R. Show that the error in calculating the power $W=\frac{V^2}{R}$ is $\frac{V}{R^2}(2Rh-VR)$. If V can be measured to an accuracy of 0.5% and to an accuracy of 1%, what is the approximate possible percentage error in W?

[**Ans.:** 0%]

14. The radius and height of a cone are 4 cm and 6 cm respectively. What is the error in its volume if the scale used in taking the measurement is short by 0.01 cm per cm?

[**Ans.:** 0.96π cm^3]

15. Show that the error in calculating the time period of a pendulum at any place is zero if an error of $\mu\%$ is made in measuring its length and gravity at that place.

16. The diameter and the altitude of a right circular cylinder are measured as 24 cm and 30 cm respectively. There is an error of 0.1 cm in each measurement. Find the possible error in the volume of the cylinder.

[**Ans.:** 50.4π cm]

17. If the measurements of base radius and height of a right circular cone are changed by −1% and 2%, show that there will be no error in the volume.

18. If $f = x^2y^3z^{\frac{1}{10}}$, find the approximate value of f when $x = 1.99$, $y = 3.01$ and $z = 0.98$.

[Ans.: 107.784]

19. If $f = x^3\, y^2\, z^4$, find the approximate value of f when $x = 1.99$, $y = 3.01$, $z = 0.99$.

[Ans.: 68.5202]

20. If $f = (160 - x^3 - y^3)^{\frac{1}{3}}$, find the approximate value of $f\,(2.1, 2.9) - f\,(2, 3)$

[Ans.: 0.016]

21. If $f = e^{xyz}$, find the approximate value of f when $x = 0.01$, $y = 1.01$, $z = 2.01$.

[Ans.: 1.02]

22. Find $[(2.92)^3 + (5.87)^3]^{\frac{1}{5}}$ approximately by using the theory of approximation.

[Ans.: 2.96]

23. Find $[(11.99)^2 + (5.01)^2]^{\frac{1}{2}}$ approximately by using the theory of approximation.

[Ans.: 12.99]

24. Find $(1.04)^{3.01}$ by using theory of approximation.

[Ans.: 1.1253]

25. If $f(x, y) = (50 - x^2 - y^2)^{\frac{1}{2}}$ find the approximate value of $[f(3, 4) - f(3.1, 3.9)]$

[Ans.: –0.018]

26. Find $\log\left[\sqrt[3]{1.04} + \sqrt[4]{0.97} - 1\right]$ approximately by using the theory of approximation.

[Ans.: 0.0058]

Points to Remember

Accuracy and Precision

Accuracy refers to how closely a computed or measured value agrees with the true value. *Precision* refers to how closely individually computed or measured values agree with each other.

Types of Errors

Absolute Error It is the difference between the measured or calculated value and true value.

$$\in_a = \delta x = \left|x_{\text{exact}} - x_{\text{approx}}\right|$$

Relative Error It is the ratio of absolute error and true value of the quantity.

$$\in_r = \frac{\delta x}{x} = \left|\frac{x_{\text{exact}} - x_{\text{approx}}}{x_{\text{exact}}}\right|$$

Percentage Error It is relative error expressed in terms of per 100.

$$\in_p = \frac{\delta x}{x} \times 100 = \left|\frac{x_{\text{exact}} - x_{\text{approx}}}{x_{\text{exact}}}\right| \times 100$$

Sources of Errors

Inherent Error It is the error that pre-exist in the problem statement itself before its solution is obtained.

Truncation Error It is the error that results from using an approximation in place of exact mathematical expressions.

Round-off Error It is the error that results due to chopping or rounding or arithmetic operations using normalized floating-point numbers.

Significant Figures

The significant figures of a number are digits that carry meaning contributing to its measurement resolution. This includes all digits except (i) all leading zeros, and (ii) all trailing zeros when they are merely placeholders to indicate the scale of the number.

CHAPTER 2

Roots of Equations

Chapter Outline

2.1 INTRODUCTION

An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$, where a_0, a_1, a_2, ..., a_n are constants and n is a positive integer, is called an *algebraic polynomial of degree n* if $a_0 \neq 0$. The equation $f(x) = 0$ is called an algebraic equation if $f(x)$ is an algebraic polynomial, e.g., $x^3 - 4x - 9 = 0$. If $f(x)$ contains functions such as trigonometric, logarithmic, exponential, etc., then $f(x) = 0$ is called a *transcendental equation*, e.g., $2x^3 - \log(x + 3)\tan x + e^x = 0$.

In general, an equation is solved by factorization. But in many cases, the method of factorization fails. In such cases, numerical methods are used. There are some methods to solve the equation $f(x) = 0$ such as

(i) Bisection method
(ii) Regula Falsi method
(iii) Newton–Raphson method
(iv) Secant method

2.2 BISECTION METHOD

Let $f(x) = 0$ be the given equation. Let x_0 and x_1 be two real values of x at P and Q respectively such that $f(x_1)$ is positive and $f(x_0)$ is negative or vice versa (Fig. 2.1). Then there is one root of the equation $f(x) = 0$ between x_0 and x_1. Now, this interval $[x_0, x_1]$ is divided into two sub-intervals $[x_0, x_2]$ and $[x_2, x_1]$, where $x_2 = \dfrac{x_0 + x_1}{2}$.

If $f(x_0)$ and $f(x_2)$ are of opposite signs then the interval $[x_0, x_2]$ is divided into $[x_0, x_3]$ and $[x_3, x_2]$, where $x_3 = \dfrac{x_0 + x_2}{2}$. However, if $f(x_0)$ and $f(x_2)$ are of the same sign then $f(x_1)$ and $f(x_2)$ will be opposite signs and the interval $[x_1, x_2]$ is divided into $[x_1, x_3]$ and $[x_3, x_2]$, where $x_3 = \dfrac{x_1 + x_2}{2}$. This process is continued till the desired accuracy is obtained.

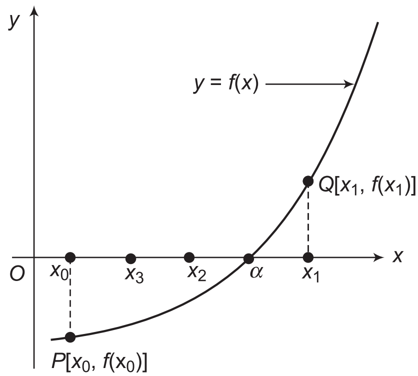

Fig. 2.1

Example 1

Find the positive root of $x^3 - 2x - 5 = 0$, *correct up to two decimal places.*

Solution

Let $$f(x) = x^3 - 2x - 5$$

$$f(1) = -6 \text{ and } f(2) = -1, f(3) = 16$$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

$$x_1 = \frac{2+3}{2} = 2.5$$
$$f(x_1) = f(2.5) = 5.625$$

Since $f(2.5) > 0$ and $f(2) < 0$, the root lies between 2.5 and 2.

$$x_2 = \frac{2.5+2}{2} = 2.25$$
$$f(x_2) = f(2.25) = 1.8906$$

Since $f(2.25) > 0$ and $f(2) < 0$, the root lies between 2.25 and 2.

$$x_3 = \frac{2.25+2}{2} = 2.125$$

$$f(x_3) = f(2.125) = 0.3457$$

Since $f(2.125) > 0$ and $f(2) < 0$, the root lies between 2.125 and 2.

$$x_4 = \frac{2.125+2}{2} = 2.0625$$

$$f(x_4) = f(2.0625) = -0.3513$$

Since $f(2.0625) < 0$ and $f(2.125) > 0$, the root lies between 2.0625 and 2.125.

$$x_5 = \frac{2.0625+2.125}{2} = 2.09375$$

$$f(x_5) = f(2.09375) = -0.0089$$

Since $f(2.09375) < 0$ and $f(2.125) > 0$, the root lies between 2.09375 and 2.125.

$$x_6 = \frac{2.09375+2.125}{2} = 2.109375$$

$$f(x_6) = f(2.109375) = 0.1668$$

Since $f(2.109375) > 0$ and $f(2.09375) < 0$, the root lies between 2.109375 and 2.09375.

$$x_7 = \frac{2.109375+2.09375}{2} = 2.10156$$

Since x_6 and x_7 are same up to two decimal places, the positive root is 2.10.

Example 2

Find a root of $x^3 - 5x + 3 = 0$ *by the bisection method correct up to four decimal places.* **[Summer 2015]**

Solution

Let $f(x) = x^3 - 5x + 3$

$$f(0) = 3 \text{ and } f(1) = -1$$

Since $f(0) > 0$ and $f(1) < 0$, the root lies between 0 and 1.

$$x_1 = \frac{0+1}{2} = 0.5$$

$$f(x_1) = f(0.5) = 0.625$$

Since $f(0.5) > 0$ and $f(1) < 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5+1}{2} = 0.75$$

$$f(x_2) = f(0.75) = -0.3281$$

Since $f(0.75) < 0$ and $f(0.5) > 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75+0.5}{2} = 0.625$$
$$f(x_3) = f(0.625) = 0.1191$$

Since $f(0.625) > 0$ and $f(0.75) < 0$, the root lies between 0.625 and 0.75.

$$x_4 = \frac{0.625+0.75}{2} = 0.6875$$
$$f(x_4) = f(0.6875) = -0.1125$$

Since $f(0.6875) < 0$ and $f(0.625) > 0$, the root lies between 0.6875 and 0.625.

$$x_5 = \frac{0.6875+0.625}{2} = 0.65625$$
$$f(x_5) = f(0.65625) = 0.00137$$

Since $f(0.65625) > 0$ and $f(0.6875) < 0$, the root lies between 0.65625 and 0.6875.

$$x_6 = \frac{0.65625+0.6875}{2} = 0.67188$$
$$f(x_6) = f(0.67188) = -0.0561$$

Since $f(0.67188) < 0$ and $f(0.65625) > 0$, the root lies between 0.67188 and 0.65625.

$$x_7 = \frac{0.67188+0.65625}{2} = 0.66407$$
$$f(x_7) = f(0.66407) = -0.02750$$

Since $f(0.66407) < 0$ and $f(0.65625) > 0$, the root lies between 0.66407 and 0.65625.

$$x_8 = \frac{0.66407+0.65625}{2} = 0.66016$$
$$f(x_8) = f(0.66016) = -0.01309$$

Since $f(0.66016) < 0$ and $f(0.65625) > 0$, the root lies between 0.66016 and 0.65625.

$$x_9 = \frac{0.66016+0.65625}{2} = 0.65821$$
$$f(x_9) = f(0.65821) = -0.00589$$

Since $f(0.65821) < 0$ and $f(0.65625) > 0$, the root lies between 0.65821 and 0.65625.

$$x_{10} = \frac{0.65821+0.65625}{2} = 0.65723$$
$$f(x_{10}) = f(0.65723) = -0.0023$$

Since $f(0.65723) < 0$ and $f(0.65625) > 0$, the root lies between 0.65723 and 0.65625.

$$x_{11} = \frac{0.65723 + 0.65625}{2} = 0.65674$$
$$f(x_{11}) = f(0.65674) = -0.00044$$

Since $f(0.65674) < 0$ and $f(0.65625) > 0$, the root lies between 0.65674 and 0.65625.

$$x_{12} = \frac{0.65674 + 0.65625}{2} = 0.6565$$
$$f(x_{12}) = f(0.6565) = 0.00044$$

Since $f(0.6565) > 0$ and $f(0.65674) < 0$, the root lies between 0.6565 and 0.65674.

$$x_{13} = \frac{0.6565 + 0.65674}{2} = 0.6566$$
$$f(x_{13}) = f(0.6566) = 0.00075$$

Since $f(0.6566) > 0$ and $f(0.65674) < 0$, the root lies between 0.6566 and 0.65674.

$$x_{14} = \frac{0.6566 + 0.65674}{2} = 0.65667$$

Since x_{13} and x_{14} are same up to four decimal places, the root is 0.6566.

Example 3

Perform the five iterations of the bisection method to obtain a root of the equation $f(x) = x^3 - x - 1 = 0$.

Solution

Let $$f(x) = x^3 - x - 1$$
$$f(1) = -1 \text{ and } f(2) = 5$$

Since $f(1) < 0$ and $f(2) > 0$, the root lies between 1 and 2.

$$x_1 = \frac{1+2}{2} = 1.5$$
$$f(x_1) = f(1.5) = 0.875$$

Since $f(1.5) > 0$ and $f(1) < 0$, the root lies between 1.5 and 1.

$$x_2 = \frac{1.5+1}{2} = 1.25$$
$$f(x_2) = f(1.25) = -0.2968$$

Since $f(1.25) < 0$ and $f(1.5) > 0$, the root lies between 1.25 and 1.5.

$$x_3 = \frac{1.25+1.5}{2} = 1.375$$
$$f(x_3) = f(1.375) = 0.2246$$

Since $f(1.375) > 0$ and $f(1.25) < 0$, the root lies between 1.375 and 1.25.

$$x_4 = \frac{1.375+1.25}{2} = 1.3125$$
$$f(x_4) = f(1.3125) = -0.0515$$

Since $f(1.3125) < 0$ and $f(1.375) > 0$, the root lies between 1.3125 and 1.375.

$$x_5 = \frac{1.3125+1.375}{2} = 1.3438$$

Hence, the root is 1.3438 up to five iterations.

Example 4

Find the approximate solution of $x^3 + x - 1 = 0$ *correct to three decimal places.* **[Winter 2013]**

Solution

Let $f(x) = x^3 + x - 1$

$$f(0) = -1 \text{ and } f(1) = 1$$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

$$x_1 = \frac{0+1}{2} = 0.5$$
$$f(x_1) = f(0.5) = -0.375$$

Since $f(0.5) < 0$ and $f(1) > 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5+1}{2} = 0.75$$
$$f(x_2) = f(0.75) = 0.1719$$

Since $f(0.75) > 0$ and $f(0.5) < 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75+0.5}{2} = 0.625$$
$$f(x_3) = f(0.625) = -0.1309$$

Since $f(0.625) < 0$ and $f(0.75) > 0$, the root lies between 0.625 and 0.75.

$$x_4 = \frac{0.625+0.75}{2} = 0.6875$$
$$f(x_4) = f(0.6875) = 0.01245$$

Since $f(0.6875) > 0$ and $f(0.625) < 0$, the root lies between 0.6875 and 0.625.

$$x_5 = \frac{0.6875+0.625}{2} = 0.6563$$
$$f(x_5) = f(0.6563) = -0.0644$$

Since $f(0.6563) < 0$ and $f(0.6875) > 0$, the root lies between 0.6563 and 0.6875.

$$x_6 = \frac{0.6563 + 0.6875}{2} = 0.6719$$
$$f(x_6) = f(0.6719) = -0.0248$$

Since $f(0.6719) < 0$ and $f(0.6875) > 0$, the root lies between 0.6719 and 0.6875.

$$x_7 = \frac{0.6719 + 0.6875}{2} = 0.6797$$
$$f(x_7) = f(0.6797) = -0.0141$$

Since $f(0.6797) < 0$ and $f(0.6875) > 0$, the root lies between 0.6797 and 0.6875.

$$x_8 = \frac{0.6797 + 0.6875}{2} = 0.6836$$
$$f(x_8) = f(0.6836) = 0.0031$$

Since $f(0.6836) > 0$ and $f(0.6797) < 0$, the root lies between 0.6836 and 0.6797.

$$x_9 = \frac{0.6836 + 0.6797}{2} = 0.6817$$
$$f(x_9) = f(0.6817) = -0.0015$$

Since $f(0.6817) < 0$ and $f(0.6836) > 0$, the root lies between 0.6817 and 0.6836.

$$x_{10} = \frac{0.6817 + 0.6836}{2} = 0.6827$$
$$f(x_{10}) = f(0.6827) = 0.00089$$

Since $f(0.6827) > 0$ and $f(0.6817) < 0$, the root lies between 0.6827 and 0.6817.

$$x_{11} = \frac{0.6827 + 0.6817}{2} = 0.6822$$

Since x_{10} and x_{11} are same up to three decimal points, the root is 0.682.

Example 5

Find a root of the equation $x^3 - 4x - 9 = 0$ *using the bisection method in four stages.*

Solution

Let $\quad f(x) = x^3 - 4x - 9$

$$f(2) = -9 \quad \text{and} \quad f(3) = 6$$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

$$x_1 = \frac{2+3}{2} = 2.5$$
$$f(x_1) = f(2.5) = -3.375$$

Since $f(2.5) < 0$ and $f(3) > 0$, the root lies between 2.5 and 3.

$$x_2 = \frac{2.5+3}{2} = 2.75$$
$$f(x_2) = f(2.75) = 0.7969$$

Since $f(2.75) > 0$ and $f(2.5) < 0$, the root lies between 2.75 and 2.5.

$$x_3 = \frac{2.75+2.5}{2} = 2.625$$
$$f(x_3) = f(2.625) = -1.4121$$

Since $f(2.625) < 0$ and $f(2.75) > 0$, the root lies between 2.625 and 2.75.

$$x_4 = \frac{2.625+2.75}{2} = 2.6875$$

Hence, the root is 2.6875 up to four stages.

Example 6

Find the negative root of $x^3 - 7x + 3$ by the bisection method up to three decimal places.

Solution

Let $$f(x) = x^3 - 7x + 3$$
$$f(-2) = 9 \quad \text{and} \quad f(-3) = -3$$

Since $f(-2) > 0$ and $f(-3) < 0$, the root lies between -2 and -3.

$$x_1 = \frac{-2-3}{2} = -2.5$$
$$f(x_1) = f(-2.5) = 4.875$$

Since $f(-2.5) > 0$ and $f(-3) < 0$, the root lies between -2.5 and -3.

$$x_2 = \frac{-2.5-3}{2} = -2.75$$
$$f(x_2) = f(-2.75) = 1.4531$$

Since $f(-2.75) > 0$ and $f(-3) < 0$, the root lies between -2.75 and -3.

$$x_3 = \frac{-2.75-3}{2} = -2.875$$
$$f(x_3) = f(-2.875) = -0.6387$$

Since $f(-2.875) < 0$ and $f(-2.75) > 0$, the root lies between –2.875 and –2.75.

$$x_4 = \frac{-2.875 - 2.75}{2} = -2.8125$$
$$f(x_4) = f(-2.8125) = 0.4402$$

Since $f(-2.8125) > 0$ and $f(-2.875) < 0$, the root lies between –2.8125 and –2.875.

$$x_5 = \frac{-2.8125 - 2.875}{2} = -2.8438$$
$$f(x_5) = f(-2.8438) = -0.0918$$

Since $f(-2.8438) < 0$ and $f(-2.8125) > 0$, the root lies between –2.8438 and –2.8125.

$$x_6 = \frac{-2.8438 - 2.8125}{2} = -2.8282$$
$$f(x_6) = f(-2.8282) = 0.1754$$

Since $f(-2.8282) > 0$ and $f(-2.8438) < 0$, the root lies between –2.8282 and –2.8438.

$$x_7 = \frac{-2.8282 - 2.8438}{2} = -2.836$$
$$f(x_7) = f(-2.836) = 0.0423$$

Since $f(-2.836) > 0$ and $f(-2.8438) < 0$, the root lies between –2.836 and –2.8438.

$$x_8 = \frac{-2.836 - 2.8438}{2} = -2.8399$$
$$f(x_8) = f(-2.8399) = -0.0246$$

Since $f(-2.8399) < 0$ and $f(-2.836) > 0$, the root lies between –2.8399 and –2.836.

$$x_9 = \frac{-2.8399 - 2.836}{2} = -2.838$$
$$f(x_9) = f(-2.838) = 0.0081$$

Since $f(-2.838) > 0$ and $f(-2.8399) < 0$, the root lies between –2.838 and –2.8399.

$$x_{10} = \frac{-2.838 - 2.8399}{2} = -2.8389$$

Since x_9 and x_{10} are same up to three decimal places, the negative root is –2.838.

Example 7

Perform three iterations of the bisection method to obtain the root of the equation $2 \sin x - x = 0$, *correct up to three decimal places.*

[Summer 2015]

Solution

Let $f(x) = 2 \sin x - x$

$f(1) = 0.6829$ and $f(2) = -0.1814$

Since $f(1) > 0$ and $f(2) < 0$, the root lies between 1 and 2.

$$x_1 = \frac{1+2}{2} = 1.5$$

$$f(x_1) = f(1.5) = 0.4949$$

Since $f(1.5) > 0$ and $f(2) < 0$, the root lies between 1.5 and 2.

$$x_2 = \frac{1.5+2}{2} = 1.75$$

$$f(x_2) = f(1.75) = 0.2179$$

Since $f(1.75) > 0$ and $f(2) < 0$, the root lies between 1.75 and 2.

$$x_3 = \frac{1.75+2}{2} = 1.875$$

Hence, the root is 1.875 up to three iterations.

Example 8

Solve $x = \cos x$ *by the bisection method correct to two decimal places.*

[Summer 2014]

Solution

Let $f(x) = x - \cos x$

$f(0) = -1$ and $f(1) = 0.4597$

Since $f(0) < 1$ and $f(1) > 0$, the root lies between 0 and 1.

$$x_1 = \frac{0+1}{2} = 0.5$$

$$f(x_1) = f(0.5) = -0.3776$$

Since $f(0.5) < 0$ and $f(1) > 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5+1}{2} = 0.75$$

$$f(x_2) = f(0.75) = 0.0183$$

Since $f(0.75) > 0$ and $f(0.5) < 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75+0.5}{2} = 0.625$$

$$f(x_3) = f(0.625) = -0.186$$

Since $f(0.625) < 0$ and $f(0.75) > 0$, the root lies between 0.625 and 0.75.

$$x_4 = \frac{0.625 + 0.75}{2} = 0.6875$$
$$f(x_4) = f(0.6875) = -0.0853$$

Since $f(0.6875) < 0$ and $f(0.75) > 0$, the root lies between 0.6875 and 0.75.

$$x_5 = \frac{0.6875 + 0.75}{2} = 0.71875$$
$$f(x_5) = f(0.71875) = -0.0338$$

Since $f(0.71875) < 0$ and $f(0.75) > 0$, the root lies between 0.71875 and 0.75.

$$x_6 = \frac{0.71875 + 0.75}{2} = 0.7344$$
$$f(x_6) = f(0.7344) = -0.0078$$

Since $f(0.7344) < 0$ and $f(0.75) > 0$, the root lies between 0.7344 and 0.75.

$$x_7 = \frac{0.7344 + 0.75}{2} = 0.7422$$
$$f(x_7) = f(0.7422) = 0.0052$$

Since $f(0.7422) > 0$ and $f(0.7344) < 0$, the root lies between 0.7422 and 0.7344.

$$x_8 = \frac{0.7422 + 0.7344}{2} = 0.7383$$
$$f(x_8) = f(0.7383) = -0.0013$$

Since $f(0.7383) < 0$ and $f(0.7422) > 0$, the root lies between 0.7383 and 0.7422.

$$x_9 = \frac{0.7383 + 0.7422}{2} = 0.74025$$
$$f(x_9) = f(0.74025) = 0.00195$$

Since $f(0.74025) > 0$ and $f(0.7383) < 0$, the root lies between 0.74025 and 0.7383.

$$x_{10} = \frac{0.74025 + 0.7383}{2} = 0.7393$$
$$f(x_{10}) = f(0.7393) = 0.0004$$

Since $f(0.7393) > 0$ and $f(0.7383) < 0$, the root lies between 0.7393 and 0.7383.

$$x_{11} = \frac{0.7393 + 0.7383}{2} = 0.7388$$

Since x_{10} and x_{11} are the same up to two decimal places, the root is 0.73.

Example 9

Find a real root between 0 *and* 1 *of the equation* $e^{-x} - x = 0$, *correct up to three decimal places.*

Solution

Let $\quad f(x) = e^{-x} - x$

$$f(0) = 1 \text{ and } f(1) = -0.63$$

Since $f(0) > 0$ and $f(1) < 0$, the root lies between 0 and 1.

$$x_1 = \frac{0+1}{2} = 0.5$$

$$f(x_1) = f(0.5) = 0.1065$$

Since $f(0.5) > 0$ and $f(1) < 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5+1}{2} = 0.75$$

$$f(x_2) = f(0.75) = -0.2776$$

Since $f(0.75) < 0$ and $f(0.5) > 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75+0.5}{2} = 0.625$$

$$f(x_3) = f(0.625) = -0.0897$$

Since $f(0.625) < 0$ and $f(0.5) > 0$, the root lies between 0.625 and 0.5.

$$x_4 = \frac{0.625+0.5}{2} = 0.5625$$

$$f(x_4) = f(0.5625) = 7.28 \times 10^{-3}$$

Since $f(0.5625) > 0$ and $f(0.625) < 0$, the root lies between 0.5625 and 0.625.

$$x_5 = \frac{0.5625+0.625}{2} = 0.5938$$

$$f(x_5) = f(0.5938) = -0.0416$$

Since $f(0.5938) < 0$ and $f(0.5625) > 0$, the root lies between 0.5938 and 0.5625.

$$x_6 = \frac{0.5938+0.5625}{2} = 0.5782$$

$$f(x_6) = f(0.5782) = -0.0173$$

Since $f(0.5782) < 0$ and $f(0.5625) > 0$, the root lies between 0.5782 and 0.5625.

$$x_7 = \frac{0.5782+0.5625}{2} = 0.5704$$

$$f(x_7) = f(0.5704) = -5.1007 \times 10^{-3}$$

Since $f(0.5704) < 0$ and $f(0.5625) > 0$, the root lies between 0.5704 and 0.5625.

$$x_8 = \frac{0.5704 + 0.5625}{2} = 0.5665$$

$$f(x_8) = f(0.5665) = 1.008 \times 10^{-3}$$

Since $f(0.5665) > 0$ and $f(0.5704) < 0$, the root lies between 0.5665 and 0.5704.

$$x_9 = \frac{0.5665 + 0.5704}{2} = 0.5685$$

$$f(x_9) = f(0.5685) = -2.1256 \times 10^{-3}$$

Since $f(0.5685) < 0$ and $f(0.5665) > 0$, the root lies between 0.5685 and 0.5665.

$$x_{10} = \frac{0.5685 + 0.5665}{2} = 0.5675$$

$$f(x_{10}) = f(0.5675) = -5.5898 \times 10^{-4}$$

Since $f(0.5675) < 0$ and $f(0.5665) > 0$, the root lies between 0.5675 and 0.5665.

$$x_{11} = \frac{0.5675 + 0.5665}{2} = 0.567$$

Since x_{10} and x_{11} are the same up to three decimal places, the root is 0.567.

Example 10

Find the root of $\cos x - xe^x = 0$ *in four steps.*

Solution

Let $\quad f(x) = \cos x - xe^x$

$$f(0) = 1 \quad \text{and} \quad f(1) = -2.18$$

Since $f(0) > 0$ and $f(1) < 0$, the root lies between 0 and 1.

$$x_1 = \frac{0 + 1}{2} = 0.5$$

$$f(x_1) = f(0.5) = 0.0532$$

Since $f(0.5) > 0$ and $f(1) < 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5 + 1}{2} = 0.75$$

$$f(x_2) = f(0.75) = -0.8561$$

Since $f(0.75) < 0$ and $f(0.5) > 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75 + 0.5}{2} = 0.625$$

$$f(x_3) = f(0.625) = -0.3567$$

Since $f(0.625) < 0$ and $f(0.5) > 0$, the root lies between 0.625 and 0.5.

$$x_4 = \frac{0.625 + 0.5}{2} = 0.5625$$

$$f(x_4) = f(0.5625) = -0.1413$$

Since $f(0.5625) < 0$ and $f(0.5) > 0$, the root lies between 0.5625 and 0.5.

$$x_5 = \frac{0.5625 + 0.5}{2} = 0.53125$$

Hence, the root is 0.53125 in four steps.

EXERCISE 2.1

Find a positive root of the following equations correct to four decimal places using the bisection method:

1. $x^3 - 4x - 9 = 0$

   [Ans.: 2.7065**]**

2. $x^3 + 3x - 1 = 0$

   [Ans.: 0.3222**]**

3. $x^3 + x^2 - 1 = 0$

   [Ans.: 0.7549**]**

4. $x^4 - x^3 - x^2 - 6x - 4 = 0$

   [Ans.: 2.5528**]**

5. $3x = \sqrt{1 + \sin x}$

   [Ans.: 0.3918**]**

6. $3x = \cos x + 1$

   [Ans.: 0.6071**]**

7. $x - \cos x = 0$

   [Ans.: 0.7391**]**

8. $xe^x = 1$

   [Ans.: 0.5671**]**

9. $x \log_{10} x = 1.2$ lying between 2 and 3

   [Ans.: 2.7406**]**

2.3 REGULA FALSI METHOD

This method resembles the bisection method. In this method, two points x_0 and x_1 are chosen such that $f(x_0)$ and $f(x_1)$ are of opposite signs, i.e., the graph of $y = f(x)$ crosses the x-axis between these points. Hence, a root lies between x_0 and x_1 and $f(x_0) f(x_1) < 0$ (Fig. 2.2).

The equation of the chord joining the points $P[x_0, f(x_0)]$ and $Q[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

In this method, the curve PQ is replaced by the chord PQ and the point of intersection of the chord with the x-axis is taken as an approximation to the root.

If x_2 is the point of intersection of the x-axis and the line joining $P[x_0, f(x_0)]$ and $Q[x_1, f(x_1)]$ then x_2 is closer to the root α than x_0 and x_1.

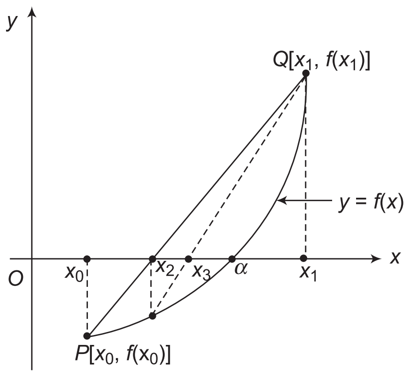

Fig. 2.2

Using the slope formula,

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_2) - f(x_0)}{x_2 - x_0} = \frac{0 - f(x_0)}{x_2 - x_0}$$

$$x_2 - x_0 = -\frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

which is an approximation to the root.

If $f(x_0)$ and $f(x_2)$ are of opposite signs, the root lies between x_0 and x_2, and the next approximation x_3 is obtained as

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

If the root lies between x_1 and x_2, the next approximation x_3 is obtained as

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

This process is repeated till the root is obtained to the desired accuracy. This iteration process is known as the method of false position or *regula falsi method*.

Example 1

Find a positive root of $x^3 - 4x + 1$ *correct up to three decimal places.*
[Summer 2015]

Solution

Let $$f(x) = x^3 - 4x + 1$$

$$f(0) = 1 \quad \text{and} \quad f(1) = -2$$

Since $f(0) > 0$ and $f(1) < 0$, the root lies between 0 and 1.

Let $x_0 = 0$, $\quad x_1 = 1$

$$\begin{aligned} x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 0 - \frac{1-0}{-2-1}(1) \\ &= 0.3333 \end{aligned}$$

$$f(x_2) = f(0.3333) = -0.2962$$

Since $f(0.3333) < 0$ and $f(0) > 0$, the root lies between 0.3333 and 0, i.e., x_2 and x_0.

$$\begin{aligned} x_3 &= x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0) \\ &= 0 - \frac{0.3333-0}{-0.2962-1}(1) \\ &= 0.2571 \end{aligned}$$

$$f(x_3) = f(0.2571) = -0.0114$$

Since $f(0.2571) < 0$ and $f(0) > 0$, the root lies between 0.2571 and 0, i.e., x_3 and x_0.

$$\begin{aligned} x_4 &= x_0 - \frac{x_3 - x_0}{f(x_3) - f(x_0)} f(x_0) \\ &= 0 - \frac{0.2571-0}{-0.0114-1}(1) \\ &= 0.2542 \end{aligned}$$

$$f(x_4) = f(0.2542) = -0.0004$$

Since $f(0.2542) < 0$ and $f(0) > 0$, the root lies between 0.2542 and 0, i.e., x_4 and x_0.

$$\begin{aligned} x_5 &= x_0 - \frac{x_4 - x_0}{f(x_4) - f(x_0)} f(x_0) \\ &= 0 - \frac{0.2542-0}{-0.0004-1}(1) \\ &= 0.2541 \end{aligned}$$

Since x_4 and x_5 are same up to three decimal places, a positive root is 0.254.

Example 2

Find the root of the equation $2x - \log_{10}x = 7$, *which lies between* 3.5 *and* 4, *correct up to five places of decimal.*

Solution

Let $\qquad f(x) = 2x - \log_{10}x - 7$

$$f(3.5) = -0.54407 \quad \text{and} \quad f(4) = 0.39794$$

Since $f(3.5) < 0$ and $f(4) > 0$, the root lies between 3.5 and 4.

Let $x_0 = 3.5, \qquad x_1 = 4$

$$\begin{aligned} x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 3.5 - \frac{4 - 3.5}{0.39794 + 0.54407}(-0.54407) \\ &= 3.78878 \\ f(x_2) &= f(3.78878) = -0.00094 \end{aligned}$$

Since $f(3.78878) < 0$ and $f(4) > 0$, the root lies between 3.78878 and 4, i.e., x_2 and x_1.

$$\begin{aligned} x_3 &= x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2) \\ &= 3.78878 - \frac{4 - 3.78878}{0.39794 + 0.00094}(-0.00094) \\ &= 3.78928 \\ f(x_3) &= f(3.78928) = 0.000003 \end{aligned}$$

Since $f(3.78928) > 0$ and $f(3.78878) < 0$, the root lies between 3.78928 and 3.78878, i.e., x_3 and x_2.

$$\begin{aligned} x_4 &= x_2 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_2) \\ &= 3.78878 - \frac{3.78928 - 3.78878}{0.000003 + 0.00094}(-0.00094) \\ &= 3.78928 \end{aligned}$$

Since x_3 and x_4 are same up to five decimal places, the root is 3.78928.

Example 3

Find a real root of the equation $x \log_{10} x = 1.2$ *by the regula falsi method.*

[Summer 2015]

Solution

Let $f(x) = x \log_{10} x - 1.2$

$$f(2) = -0.5979 \quad \text{and} \quad f(3) = 0.2314$$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

Let $x_0 = 2$, $x_1 = 3$

$$\begin{aligned} x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 2 - \frac{3-2}{0.2314 + 0.5979}(-0.5979) \\ &= 2.721 \\ f(x_2) &= f(2.721) = -0.0171 \end{aligned}$$

Since $f(2.721) < 0$ and $f(3) > 0$, the root lies between 2.721 and 3, i.e., x_2 and x_1.

$$\begin{aligned} x_3 &= x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2) \\ &= 2.721 - \frac{3 - 2.721}{0.2314 + 0.0171}(-0.0171) \\ &= 2.7402 \\ f(x_3) &= f(2.7402) = -0.0004 \end{aligned}$$

Since $f(2.7402) < 0$ and $f(3) > 0$, the root lies between 2.7402 and 3, i.e., x_3 and x_1.

$$\begin{aligned} x_4 &= x_1 - \frac{x_3 - x_1}{f(x_3) - f(x_1)} f(x_1) \\ &= 3 - \frac{2.7042 - 3}{-0.0004 - 0.2314}(0.2314) \\ &= 2.7406 \end{aligned}$$

Since x_3 and x_4 are same up to three decimal places, a real root is 2.740.

Example 4

Solve the equation $x \tan x = -1$, *starting with* $x_0 = 2.5$ and $x_1 = 3$, *correct up to three decimal places.*

Solution

Let $f(x) = x \tan x + 1$

$f(2.5) = -0.8676$ and $f(3) = 0.5724$

Since $f(2.5) < 0$ and $f(3) > 0$, the root lies between 2.5 and 3.

Let $x_0 = 2.5$, $x_1 = 3$

$$\begin{aligned} x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 2.5 - \frac{3 - 2.5}{0.5724 + 0.8676}(-0.8676) \\ &= 2.8013 \end{aligned}$$

$$f(x_2) = f(2.8013) = 0.0082$$

Since $f(2.8013) > 0$ and $f(2.5) < 0$, the root lies between 2.8013 and 2.5, i.e., x_2 and x_0.

$$\begin{aligned} x_3 &= x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0) \\ &= 2.5 - \frac{2.8013 - 2.5}{0.0082 + 0.8676}(-0.8676) \\ &= 2.7985 \end{aligned}$$

$$f(x_3) = f(2.7985) = 0.0003$$

Since $f(2.7985) > 0$ and $f(2.5) < 0$, the root lies between 2.7985 and 2.5, i.e., x_3 and x_0.

$$\begin{aligned} x_4 &= x_0 - \frac{x_3 - x_0}{f(x_3) - f(x_0)} f(x_0) \\ &= 2.5 - \frac{2.7985 - 2.5}{0.0003 + 0.8676}(-0.8676) \\ &= 2.7984 \end{aligned}$$

Since x_3 and x_4 are same up to three decimal places, the root is 2.798.

Example 5

Find the real root of the equation $\log_{10} x - \cos x = 0$, *correct to four decimal places.*

Solution

Let $f(x) = \log_{10} x - \cos x$

$f(1) = -0.5403$ and $f(1.5) = 0.10535$

Since $f(1) < 0$ and $f(1.5) > 0$, the root lies between 1 and 1.5.

Let $x_0 = 1$, $\quad x_1 = 1.5$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$
$$= 1 - \frac{1.5 - 1}{0.10535 + 0.5403}(-0.5403)$$
$$= 1.41842$$
$$f(x_2) = f(1.41842) = 0.00002$$

Since $f(1.41842) > 0$ and $f(1) < 0$, the root lies between 1.41842 and 1, i.e., x_2 and x_0.

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$
$$= 1 - \frac{1.41842 - 1}{0.00002 + 0.5403}(-0.5403)$$
$$= 1.41840$$

Since x_2 and x_3 are same up to four decimal places, the real root is 1.4184.

Example 6

Find the smallest root of an equation $x - e^{-x} = 0$ correct to three significant digits. **[Summer 2015]**

Solution

Let $\quad f(x) = x - e^{-x}$

$$f(0) = -1 \quad \text{and} \quad f(1) = 0.6321$$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 0$, $\quad x_1 = 1$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$
$$= 0 - \frac{1 - 0}{0.6321 + 1}(-1)$$
$$= 0.6127$$
$$f(x_2) = f(0.6127) = 0.0708$$

Since $f(0.6127) > 0$ and $f(0) < 0$, the root lies between 0.6127 and 0, i.e., x_2 and x_0.

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$
$$= 0 - \frac{0.6127 - 0}{0.0708 + 1}(-1)$$
$$= 0.5722$$
$$f(x_3) = f(0.5722) = 0.0079$$

Since $f(0.5722) > 0$ and $f(0) < 0$, the root lies between 0.5722 and 0, i.e., x_3 and x_0.

$$x_4 = x_0 - \frac{x_3 - x_0}{f(x_3) - f(x_0)} f(x_0)$$
$$= 0 - \frac{0.5722 - 0}{0.0079 + 1}(-1)$$
$$= 0.5677$$
$$f(x_4) = f(0.5677) = 0.0009$$

Since $f(0.5677) > 0$ and $f(0) < 0$, the root lies between 0.5677 and 0, i.e., x_4 and x_0.

$$x_5 = x_0 - \frac{x_4 - x_0}{f(x_4) - f(x_0)} f(x_0)$$
$$= 0 - \frac{0.5677 - 0}{0.0009 + 1}(-1)$$
$$= 0.5672$$

Since x_4 and x_5 are same up to three significant digits, a positive root is 0.567.

Example 7

Find the root of the equation $\cos x - xe^x = 0$ *correct up to three decimal places, lying between* 0.5 *and* 0.7.

Solution

Let $f(x) = \cos x - xe^x$

$$f(0.5) = 0.0532 \quad \text{and} \quad f(0.7) = -0.6448$$

Since $f(0.5) > 0$ and $f(0.7) < 0$, the root lies between 0.5 and 0.7.

Let $x_0 = 0.5$, $\quad x_1 = 0.7$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$
$$= 0.5 - \frac{0.7 - 0.5}{-0.6448 - 0.0532}(0.0532)$$
$$= 0.5152$$
$$f(x_2) = f(0.5152) = 0.0078$$

Since $f(0.5152) > 0$ and $f(0.7) < 0$, the root lies between 0.5152 and 0.7, i.e., x_2 and x_1.

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$
$$= 0.5152 - \frac{0.7 - 0.5152}{-0.6448 - 0.0078}(0.0078)$$
$$= 0.5174$$
$$f(x_3) = f(0.5174) = 0.0011$$

Since $f(0.5174) > 0$ and $f(0.7) < 0$, the root lies between 0.5174 and 0.7, i.e., x_3 and x_1.

$$x_4 = x_3 - \frac{x_1 - x_3}{f(x_1) - f(x_3)} f(x_3)$$

$$= 0.5174 - \frac{0.7 - 0.5174}{-0.6448 - 0.0011}(0.0011)$$

$$= 0.5177$$

Since x_3 and x_4 are same up to three decimal places, the root is 0.517.

EXERCISE 2.2

Find a real root of the following equations correct to three decimal places using the regula falsi method:

1. $x^3 + x - 1 = 0$

[Ans.: 0.682**]**

2. $x^3 - 4x - 9 = 0$

[Ans.: 2.707**]**

3. $x^3 - 5x - 7 = 0$

[Ans.: 2.747**]**

4. $xe^3 = 3$

[Ans.: 1.050**]**

5. $e^{-x} - \sin x = 0$

[Ans.: 0.5885**]**

6. $2x = \cos x + 3$

[Ans.: 1.524**]**

7. $x^2 - \log_e x = 12$

[Ans.: 3.646**]**

8. $e^x = 3x$

[Ans.: 1.512**]**

2.4 NEWTON—RAPHSON METHOD

Let $f(x) = 0$ be the given equation and x_0 be an approximate root of the equation. If $x_1 = x_0 + h$ be the exact root then $f(x_1) = 0$.

i.e., $$f(x_0 + h) = 0$$

$$f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\cdots=0 \qquad \text{[By Taylor's series]}$$

Since h is small, neglecting h^2 and higher powers of h,

$$f(x_0)+h\,f'(x_0)=0$$

$$h=-\frac{f(x_0)}{f'(x_0)}$$

$$\therefore \qquad x_1=x_0+h=x_0-\frac{f(x_0)}{f'(x_0)}$$

Similarly, starting with x_1, a still better approximation x_2 is obtained.

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

In general, $$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

This equation is known as the *Newton–Raphson formula* or *Newton's iteration formula*.

2.4.1 Geometrical Interpretation

Let x_0 be a point near the root α of the equation $f(x)=0$ (Fig. 2.3). The equation of the tangent at $P_0[x_0, f(x_0)]$ is

$$y-f(x_0)=f'(x_0)(x-x_0)$$

This line cuts the x-axis at x_1.

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$

which is a first approximation to the root α.

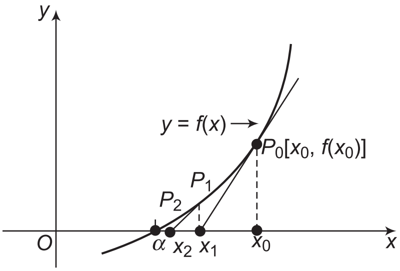

Fig. 2.3

If P_1 is the point corresponding to x_1 on the curve then the tangent at P_1 will cut the x-axis at x_2 which is nearer to α and is the second approximation to the root. Repeating this process, the root α is approached quite rapidly. Thus, this method consists of replacing the part of the curve between the point P_0 and the x-axis by means of the tangent to the curve at P_0.

2.4.2 Convergence of the Newton—Raphson Method

By the Newton–Raphson method,

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=\phi(x_n) \qquad \text{...(2.1)}$$

The Newton–Raphson method converges if $|\phi'(x)| < 1$.

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

$$\phi'(x) = 1 - \left[\frac{[f'(x)]^2 - f(x)\,f''(x)}{[f'(x)]^2}\right] = \frac{f(x)\,f''(x)}{[f'(x)]^2}$$

$$|\phi'(x)| = \left|\frac{f(x)\,f''(x)}{[f'(x)]^2}\right|$$

Hence, the Newton–Raphson method converges if

$$\left|\frac{f(x)\,f''(x)}{[f'(x)]^2}\right| < 1$$

$$|f(x)\,f''(x)| < [f'(x)]^2 \qquad \text{...(2.2)}$$

If α is the actual root of $f(x) = 0$, a small interval should be selected in which $f(x)$, $f'(x)$ and $f''(x)$ are all continuous and the condition given by Eq. (2.2) is satisfied.

Hence, the Newton–Raphson method always converges provided the initial approximation x_0 is taken very close to the actual root α.

2.4.3 Rate of Convergence of the Newton—Raphson Method

Let α be exact root of $f(x) = 0$ and let x_n, x_{n+1} be two successive approximations to the actual root. If $\in_n$ and $\in_{n+1}$ are the corresponding errors then

$$x_n = \alpha + \in_n$$

$$x_{n+1} = \alpha + \in_{n+1}$$

Substituting in Eq. (2.1),

$$\alpha + \in_{n+1} = \alpha + \in_n - \frac{f(\alpha + \in_n)}{f'(\alpha + \in_n)}$$

$$\in_{n+1} = \in_n - \frac{f(\alpha + \in_n)}{f'(\alpha + \in_n)}$$

$$= \in_n - \frac{f(\alpha) + \in_n f'(\alpha) + \frac{\in_n^2}{2!} f''(\alpha) + \cdots}{f'(\alpha) + \in_n f''(\alpha) + \cdots} \qquad \text{[By Taylor's series]}$$

$$= \in_n - \frac{\in_n f'(\alpha) + \frac{\in_n^2}{2} f''(\alpha) + \cdots}{f'(\alpha) + \in_n f''(\alpha)} \qquad [\because \; f(\alpha) = 0]$$

Neglecting the derivatives of order higher than two,

$$\in_{n+1} = \in_n - \frac{\in_n f'(\alpha) + \frac{\in_n^2}{2} f''(\alpha)}{f'(\alpha) + \in_n f''(\alpha)}$$

$$= \frac{1}{2}\left[\frac{\in_n^2 f''(\alpha)}{f'(\alpha) + \in_n f''(\alpha)}\right]$$

$$= \frac{\in_n^2}{2}\left[\frac{\frac{f''(\alpha)}{f'(\alpha)}}{1 + \in_n \frac{f''(\alpha)}{f'(\alpha)}}\right]$$

$$\approx \frac{\in_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \qquad ...(2.3)$$

Equation (2.3) shows that the error at each stage is proportional to the square of the error in the previous stage. Hence, the Newton–Raphson method has a quadratic convergence and the convergence is of the order 2.

Example 1

Find the root of the equation $x^3 + x - 1 = 0$, correct up to four decimal places.

Solution

Let $f(x) = x^3 + x - 1$

$f(0) = -1$ and $f(1) = 1$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 1$

$$f'(x) = 3x^2 + 1$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_0) = f(1) = 1$$

$$f'(x_0) = f'(1) = 4$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{1}{4}$$

$$= 0.75$$

$$f(x_1) = f(0.75) = 0.171875$$
$$f'(x_1) = f'(0.75) = 2.6875$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 0.75 - \frac{0.171875}{2.6875}$$
$$= 0.68605$$
$$f(x_2) = f(0.68605) = 0.00894$$
$$f'(x_2) = f'(0.68605) = 2.41198$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 0.68605 - \frac{0.00894}{2.41198}$$
$$= 0.68234$$
$$f(x_3) = f(0.68234) = 0.000028$$
$$f'(x_3) = f'(0.68234) = 2.39676$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$= 0.68234 - \frac{0.000028}{2.39676}$$
$$= 0.68233$$

Since x_3 and x_4 are same up to four decimal places, the root is 0.6823.

Example 2

Find a root of $x^4 - x^3 + 10x + 7 = 0$, *correct up to three decimal places between –2 and –1 by the Newton–Raphson method.*

Solution

Let $\quad f(x) = x^4 - x^3 + 10x + 7$

The root lies between –2 and –1.

Let $x_0 = -2$

$$f'(x) = 4x^3 - 3x^2 + 10$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$f(x_0) = f(-2) = 11$$

$$
\begin{aligned}
f'(x_0) &= f'(-2) = -34 \\
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
&= -2 - \frac{11}{(-34)} \\
&= -1.6765 \\
f(x_1) &= f(-1.6765) = 2.8468 \\
f'(x_1) &= f'(-1.6765) = -17.2802 \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\
&= -1.6765 - \frac{2.8468}{(-17.2802)} \\
&= -1.5118 \\
f(x_2) &= f(-1.5118) = 0.561 \\
f'(x_2) &= f'(-1.5118) = -10.6777 \\
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\
&= -1.5118 - \frac{0.561}{(-10.6777)} \\
&= -1.4593 \\
f(x_3) &= f(-1.4593) = 0.0497 \\
f'(x_3) &= f(-1.4593) = -8.8193 \\
x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} \\
&= -1.4593 - \frac{0.0497}{(-8.8193)} \\
&= -1.4537 \\
f(x_4) &= f(-1.4537) = 0.0008 \\
f'(x_4) &= f'(-1.4537) = -8.6278 \\
x_5 &= x_4 - \frac{f(x_4)}{f'(x_4)} \\
&= -1.4537 - \frac{0.0008}{(-8.6278)} \\
&= -1.4536
\end{aligned}
$$

Since x_4 and x_5 are same up to three decimal places, a root is –1.453.

Example 3

Find the root of $x^4 - x - 10 = 0$, *correct up to three decimal places.*

Solution

Let $f(x) = x^4 - x - 10$

$$f(1) = -10, \quad \text{and} \quad f(2) = 4$$

Since $f(1) < 0$ and $f(2) > 0$, the root lies between 1 and 2.

Let $x_0 = 2$

$$f'(x) = 4x^3 - 1$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_0) = f(2) = 4$$
$$f'(x_0) = f'(2) = 31$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 2 - \frac{4}{31}$$
$$= 1.871$$

$$f(x_1) = f(1.871) = 0.3835$$
$$f'(x_1) = f'(1.871) = 25.1988$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 1.871 - \frac{0.3835}{25.1988}$$
$$= 1.8558$$

$$f(x_2) = f(1.8558) = 5.2922 \times 10^{-3}$$
$$f'(x_2) = f'(1.8558) = 24.5655$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 1.8558 - \frac{5.2922 \times 10^{-3}}{24.5655}$$
$$= 1.8556$$

Since x_2 and x_3 are same up to three decimal places, the root is 1.855.

Example 4

Find the real root of $x \log_{10} x - 1.2 = 0$, *correct up to three decimal places.* **[Summer 2015]**

Solution

Let $$f(x) = x \log_{10} x - 1.2$$

$$f(1) = -1.2,\ f(2) = -0.5979 \quad \text{and} \quad f(3) = 0.2314$$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

Let $$x_0 = 3$$

$$f'(x) = \log_{10} x + x\frac{1}{x \log_e 10} = \log_{10} x + \log_{10} e = \log_{10} x + 0.4343$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_0) = f(3) = 0.2314$$

$$f'(x_0) = f'(3) = 0.9114$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{0.2314}{0.9114}$$

$$= 2.7461$$

$$f(x_1) = f(2.7461) = 4.759 \times 10^{-3}$$

$$f'(x_1) = f'(2.7461) = 0.8730$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.7461 - \frac{4.759 \times 10^{-3}}{0.8730}$$

$$= 2.7406$$

$$f(x_2) = f(2.7406) = -4.0202 \times 10^{-5}$$

$$f'(x_2) = f'(2.7406) = 0.8721$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.7406 - \frac{(-4.0202 \times 10^{-5})}{0.8721}$$

$$= 2.7406$$

Since x_2 and x_3 are the same up to three decimal places, the real root is 2.7406.

Example 5

Find a root between 0 *and* 1 *of the equation* $e^x \sin x = 1$, *correct up to four decimal places.*

Solution

Let $$f(x) = e^x \sin x - 1$$

$$f(0) = -1 \quad \text{and} \quad f(1) = 1.28$$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 0$

$$f'(x) = e^x (\cos x + \sin x)$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_0) = f(0) = -1$$

$$f'(x_0) = f'(0) = 1$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0 - \frac{(-1)}{1}$$

$$= 1$$

$$f(x_1) = f(1) = 1.2874$$

$$f'(x_1) = f'(1) = 3.7560$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1 - \frac{1.2874}{3.7560}$$

$$= 0.6572$$

$$f(x_2) = f(0.6572) = 0.1787$$

$$f'(x_2) = f'(0.6572) = 2.7062$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.6572 - \frac{0.1787}{2.7062}$$

$$= 0.5912$$

$$f(x_3) = f(0.5912) = 6.6742 \times 10^{-3}$$
$$f'(x_3) = f'(0.5912) = 2.5063$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$= 0.5912 - \frac{6.6742 \times 10^{-3}}{2.5063}$$
$$= 0.5885$$
$$f(x_4) = f(0.5885) = -8.1802 \times 10^{-5}$$
$$f'(x_4) = f'(0.5885) = 2.4982$$
$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$$
$$= 0.5885 - \frac{-8.1802 \times 10^{-5}}{2.4982}$$
$$= 0.5885$$

Since x_4 and x_5 are the same up to four decimal places, the root is 0.5885.

Example 6

Find the real root of the equation $3x = \cos x + 1$, *correct up to four decimal places.*

Solution

Let $f(x) = 3x - \cos x - 1$

$f(0) = -2$ and $f(1) = 1.4597$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 1$

$$f'(x) = 3 + \sin x$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$f(x_0) = f(1) = 1.4597$$
$$f'(x_0) = f'(1) = 3.8415$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 1 - \frac{1.4597}{3.8415}$$
$$= 0.62$$

$$
\begin{aligned}
f(x_1) &= f(0.62) = 0.0461 \\
f'(x_1) &= f'(0.62) = 3.5810 \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\
&= 0.62 - \frac{0.0461}{3.5810} \\
&= 0.6071 \\
f(x_2) &= f(0.6071) = -5.8845 \times 10^{-6} \\
f'(x_2) &= f'(0.6071) = 3.5705 \\
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\
&= 0.6071 - \frac{-5.8845 \times 10^{-6}}{3.5705} \\
&= 0.6071
\end{aligned}
$$

Since x_2 and x_3 are the same up to four decimal places, the real root is 0.6071.

Example 7

Find the real positive root of the equation $x \sin x + \cos x = 0$, *which is near* $x = \pi$ *correct up to four significant digits.* **[Summer 2015]**

Solution

Let $f(x) = x \sin x + \cos x$

Let $x_0 = \pi$

$$f'(x) = x \cos x + \sin x - \sin x = x \cos x$$

By the Newton–Raphson method,

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
f(x_0) &= f(\pi) = -1 \\
f'(x_0) &= f'(\pi) = -\pi \\
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
&= \pi - \frac{(-1)}{(-\pi)} \\
&= 2.82328
\end{aligned}
$$

$$f(x_1) = f(2.82328) = -0.06618$$
$$f'(x_1) = f'(2.823287) = -2.68145$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 2.82328 - \frac{(-0.06618)}{(-2.68145)}$$
$$= 2.7986$$
$$f(x_2) = f(2.7986) = -0.00056$$
$$f'(x_2) = f'(2.7986) = -2.63559$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 2.7986 - \frac{(-0.00056)}{(-2.63559)}$$
$$= 2.79839$$
$$f(x_3) = f(2.79839) = -0.0001$$
$$f'(x_3) = f'(2.79839) = -2.63519$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$= 2.79839 - \frac{(-0.0001)}{(-2.63519)}$$
$$= 2.79839$$

Since x_3 and x_4 are same up to four decimal point, the root is 2.7983.

Example 8

Find the positive root of $x = \cos x$ using Newton's method correct to three decimal places.

Solution

Let $f(x) = x - \cos x$

$f(0) = -1$ and $f(1) = 0.4597$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 1$

$f'(x) = 1 + \sin x$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_0) = f(1) = 0.4597$$

$$f'(x_0) = f'(1) = 1.8415$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{0.4597}{1.8415}$$

$$= 0.7504$$

$$f(x_1) = f(0.7504) = 0.019$$

$$f'(x_1) = f'(0.7504) = 1.6819$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.7504 - \frac{0.019}{1.6819}$$

$$= 0.7391$$

$$f(x_2) = f(0.7391) = 0.00002$$

$$f'(x_2) = f'(0.7391) = 1.6736$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.7391 - \frac{0.00002}{1.6736}$$

$$= 0.7391$$

Since x_2 and x_3 are same up to three decimal places, the root is 0.739.

Example 9

Derive the iteration formula for $\sqrt{N}$ *and, hence, find*

(i) $\sqrt{28}$ **[Summer 2015]**

(ii) $\sqrt{65}$ **[Winter 2014]**

(iii) $\sqrt{3}$ **[Winter 2014]**

correct up to three decimal places.

Solution

Let $$x = \sqrt{N}$$
$$x^2 - N = 0$$

Let $$f(x) = x^2 - N$$
$$f'(x) = 2x$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^2 - N}{2x_n}$$
$$= \frac{x_n^2 + N}{2x_n}$$

This is the iteration formula for $\sqrt{N}$.

(i) For $N = 28$, $f(x) = x^2 - 28$

$$f(5) = -3 \quad \text{and} \quad f(6) = 8$$

Since $f(5) < 0$ and $f(6) > 0$, the root lies between 5 and 6.
Let $x_0 = 5$

$$x_{n+1} = \frac{x_n^2 + 28}{2x_n}$$
$$x_1 = \frac{x_0^2 + 28}{2x_0} = 5.3$$
$$x_2 = \frac{x_1^2 + 28}{2x_1} = 5.2915$$
$$x_3 = \frac{x_2^2 + 28}{2x_2} = 5.2915$$

Since x_2 and x_3 are same up to three decimal places,

$$\sqrt{28} = 5.2915$$

(ii) For $N = 65$, $f(x) = x^2 - 65$

$$f(8) = -1 \quad \text{and} \quad f(9) = 16$$

Since $f(8) < 0$ and $f(9) > 0$, the root lies between 8 and 9.

Let $x_0 = 8$

$$x_{n+1} = \frac{x_n^2 + 65}{2x_n}$$

$$x_1 = \frac{x_0^2 + 65}{2x_0} = 8.0625$$

$$x_2 = \frac{x_1^2 + 65}{2x_1} = 8.0623$$

Since x_1 and x_2 are same up to three decimal places,

$$\sqrt{65} = 8.0623$$

(iii) For $\quad N = \sqrt{3}, f(x) = x^2 - 3$

$$f(1) = -2 \quad \text{and} \quad f(2) = 1$$

Since $f(1) < 0$ and $f(2) > 0$, the root lies between 1 and 2.

Let $x_0 = 2$

$$x_{n+1} = \frac{x_n^2 + 3}{2x_n}$$

$$x_1 = \frac{x_0^2 + 3}{2x_0} = 1.75$$

$$x_2 = \frac{x_1^2 + 3}{2x_1} = 1.7321$$

$$x_3 = \frac{x_2^2 + 3}{2x_2} = 1.7321$$

Since x_2 and x_3 are same up to three decimal places,

$$\sqrt{3} = 1.7321$$

Example 10

Find an iterative formula for $\sqrt[k]{N}$ *, where N is a positive number and hence, evaluate (i)* $\sqrt[3]{11}$*, and (ii)* $\sqrt[3]{58}$ **[Summer 2015]**

Solution

Let $\quad x = \sqrt[k]{N}$

$$x^k - N = 0$$

Let $$f(x) = x^k - N$$
$$f'(x) = kx^{k-1}$$

By the Newton–Raphson method,

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
&= x_n - \frac{x_n^k - N}{k\,x_n^{k-1}} \\
&= \frac{(k-1)x_n^k + N}{k\,x_n^{k-1}}
\end{aligned}$$

This is the iterative formula for $\sqrt[k]{N}$.

(i) When $N = 11$ and $k = 3$,

$$f(x) = x^3 - 11$$
$$f(2) = -3 \quad \text{and} \quad f(3) = 16$$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

Let $x_0 = 3$

$$\begin{aligned}
x_{n+1} &= \frac{2x_n^3 + 11}{3x_n^2} \\
x_1 &= \frac{2x_0^3 + 11}{3x_0^2} = 2.4074 \\
x_2 &= \frac{2x_1^3 + 11}{3x_1^2} = 2.2376 \\
x_3 &= \frac{2x_2^3 + 11}{3x_2^2} = 2.2240 \\
x_4 &= \frac{2x_3^3 + 11}{3x_3^2} = 2.2240
\end{aligned}$$

Since x_3 and x_4 are same up to four decimal places,

$$\sqrt[3]{11} = 2.2240$$

(ii) When $N = 58$ and $k = 3$,

$$f(x) = x^3 - 58$$
$$f(3) = -31 \quad \text{and} \quad f(4) = 6$$

Since $f(3) < 0$ and $f(4) > 0$, the root lies between 3 and 4.

Let $x_0 = 4$

$$x_{n+1} = \frac{2x_n^3 + 58}{3x_n^2}$$

$$x_1 = \frac{2x_0^3 + 58}{3x_0^2} = 3.875$$

$$x_2 = \frac{2x_1^3 + 58}{3x_1^2} = 3.8709$$

$$x_3 = \frac{2x_2^3 + 58}{3x_2^2} = 3.8709$$

Since x_2 and x_3 are same up to four decimal places,

$$\sqrt[3]{58} = 3.8709$$

EXERCISE 2.3

I. Find the roots of the following equations using the Newton–Raphson method:

1. $x^3 - x - 1 = 0$

[**Ans.:** 1.3247]

2. $x^3 + 2x^2 + 50x + 7 = 0$

[**Ans.:** -0.1407]

3. $x^3 - 5x + 3 = 0$

[**Ans.:** 0.6566]

4. $x^4 - x - 9 = 0$

[**Ans.:** 1.8134]

5. $\cos x - xe^x = 0$

[**Ans.:** 0.5177]

6. $x \log_{10} x = 4.772393$

[**Ans.:** 6.0851]

7. $x - 2\sin x = 0$

[**Ans.:** 1.8955]

8. $x \tan x = 1.28$

[**Ans.:** 6.4783]

9. $\cos x = x^2$

[**Ans.:** 0.8241]

II. Find the values of the following:

1. $\sqrt{35}$

[Ans.: 5.916]

2. $\sqrt[3]{24}$

[Ans.: 2.884]

3. $\dfrac{1}{\sqrt{14}}$

[Ans.: 0.2673]

2.5 SECANT METHOD

The Newton–Raphson method requires the evaluation of two functions (the function and its derivative) per iteration. For complicated expressions, the method takes a large amount of time. Hence, it is desirable to have a method that converges as fast as the Newton–Raphson method but involves only evaluation of the function.

Let $f(x) = 0$ be the given equation. Let x_0 and x_1 be the approximate roots of the equation $f(x) = 0$ and $f(x_0)$ and $f(x_1)$ are their function values respectively. If x_2 is the point of intersection of the x-axis and the line joining points $P[x_0, f(x_0)]$ and $Q[x_1, f(x_1)]$ then x_2 is closer to the root α than x_0 and x_1 (Fig. 2.4).

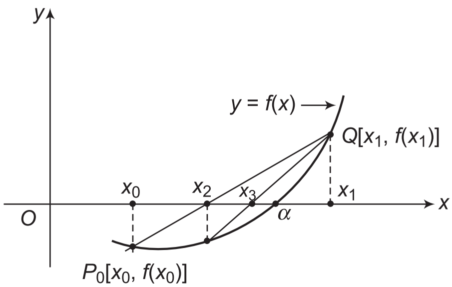

Fig. 2.4

Using the slope formula,

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0 - f(x_1)}{x_2 - x_1}$$

$$x_2 - x_1 = -\frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

Using x_1 and x_2, the process is repeated to obtain x_3.

In general,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n), \qquad n \geq 1$$

This method is similar to the regula falsi method. This method starts with two initial approximations x_0 and x_1 and calculates x_2 by the same formula as in the regula falsi method but proceeds to the next iteration without considering any root bracketing, i.e., the condition $f(x_0)\,f(x_1) < 0$.

Convergence of the Secant Method

By the Secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \qquad \ldots(2.4)$$

Let α be the exact root of $f(x) = 0$ and let x_n, x_{n+1} be two successive approximations to the actual root.

If $\in_n$, $\in_{n-1}$, are the corresponding error then

$$x_n = \alpha + \in_n$$

$$x_{n-1} = \alpha + \in_{n-1}$$

$$x_{n+1} = \alpha + \in_{n+1}$$

Substituting in Eq. (2.4),

$$\alpha + \in_{n+1} = \alpha + \in_n - \frac{\in_n - \in_{n-1}}{f(\alpha + \in_n) - f(\alpha + \in_{n-1})} f(\alpha + \in_n)$$

$$\in_{n+1} = \in_n - \frac{\in_n - \in_{n-1}}{f(\alpha + \in_n) - f(\alpha + \in_{n-1})} f(\alpha + \in_n)$$

$$= \in_n - \frac{(\in_n - \in_{n-1})\left[f(\alpha) + \in_n f'(\alpha) + \frac{\in_n^2}{2} f''(\alpha) + \cdots\right]}{(\in_n - \in_{n-1}) f'(\alpha) + \frac{1}{2}(\in_n^2 - \in_{n-1}^2) f''(\alpha) + \cdots} \qquad \text{[By Taylor's series]}$$

$$= \in_n - \frac{\left(\in_n - \in_{n-1}\right)\left[\in_n f'(\alpha) + \frac{\in_n^2}{2} f''(\alpha)\right]}{\left(\in_n - \in_{n-1}\right) f'(\alpha) + \frac{1}{2}\left(\in_n^2 - \in_{n-1}^2\right) f''(\alpha)} \qquad \left[\begin{array}{l}\because\ f(\alpha) = 0 \\ \text{and neglecting} \\ \text{higher order} \\ \text{derivatives of } f(\alpha)\end{array}\right]$$

$$= \in_n - \frac{\left(\in_n - \in_{n-1}\right) \in_n f'(\alpha)\left[1 + \frac{\in_n}{2} \frac{f''(\alpha)}{f(\alpha)}\right]}{\left(\in_n - \in_{n-1}\right) f'(\alpha) + \left[1 + \frac{\left(\in_n + \in_{n-1}\right)}{2} \frac{f''(\alpha)}{f(\alpha)}\right]}$$

$$= \in_n - \frac{\in_n \left(1 + c \in_n\right)}{1 + c\left(\in_n + \in_{n-1}\right)}$$

$$\text{where } c = \frac{1}{2} \frac{f''(\alpha)}{f(\alpha)} \qquad \ldots(2.5)$$

$$\begin{aligned}\epsilon_{n+1} &= \frac{\epsilon_n + c\,\epsilon_n^2 + c\,\epsilon_n\epsilon_{n-1} - \epsilon_n - c\,\epsilon_n^2}{1 + c\left(\epsilon_n + \epsilon_{n-1}\right)} \\ &= \frac{c\,\epsilon_n\epsilon_{n-1}}{1 + c\left(\epsilon_n + \epsilon_{n-1}\right)} \\ &\approx c\,\epsilon_n\epsilon_{n-1} \qquad \left[\because\ 1 + c\left(\epsilon_n + \epsilon_{n-1}\right) \approx 1\right] \qquad \text{...(2.6)}\end{aligned}$$

Equation (2.5) is a nonlinear difference equation which can be solved by letting $\epsilon_{n+1} = A\,\epsilon_n^p$ or $\epsilon_n = A\,\epsilon_{n-1}^p$.

$$\therefore \qquad \epsilon_{n-1} = \epsilon_n^{\frac{1}{p}}\, A^{-\frac{1}{p}}$$

Substituting in Eq. (2.6),

$$A\,\epsilon_n^p = c\,\epsilon_n\,\epsilon_n^{\frac{1}{p}}\,A^{-\frac{1}{p}}$$

$$\epsilon_n^p = c\,A^{-\left(1+\frac{1}{p}\right)}\,\epsilon_n^{1+\frac{1}{p}}$$

Equating the power of ϵ_n on both the sides,

$$p = 1 + \frac{1}{p}$$

$$p^2 - p - 1 = 0$$

$$p = \frac{1}{2}(1 \pm \sqrt{5})$$

Taking the positive sign only,

$$p = 1.618$$

$$\epsilon_{n+1} = A\,\epsilon_n^{1.618}$$

Hence, the rate of convergence of the secant method is 1.618 which is lesser than the Newton–Raphson method. The secant method evaluates the function only once in each iteration, whereas the Newton–Raphson method evaluates two functions $f(x)$ and $f'(x)$ in each iteration. Hence, the secant method is more efficient than the Newton–Raphson method.

Example 1

Find the approximate root of $x^3 - 2x - 1 = 0$, *starting from* $x_0 = 1.5$, $x_1 = 2$, *correct up to three decimal places.*

Solution

Let $\quad f(x) = x^3 - 2x - 1$

$$x_0 = 1.5,\ x_1 = 2$$

$$f(x_0) = f(1.5) = -0.625 \text{ and } f(x_1) = f(2) = 3$$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$= 2 - \frac{2 - 1.5}{3 + 0.625}(3)$$

$$= 1.5862$$

$$f(x_2) = f(1.5862) = -0.1815$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 1.5862 - \frac{1.5862 - 2}{-0.1815 - 3}(-0.1815)$$

$$= 1.6098$$

$$f(x_3) = f(1.6098) = -0.0479$$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$= 1.6098 - \frac{1.6098 - 1.5862}{-0.0479 + 0.1815}(-0.0479)$$

$$= 1.6183$$

$$f(x_4) = f(1.6183) = 0.0016$$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4)$$

$$= 1.6183 - \frac{1.6183 - 1.6098}{0.0016 + 0.0479}(0.0016)$$

$$= 1.6181$$

Since x_4 and x_5 are same up to three decimal places, the root is 1.618.

Example 2

Find the approximate root of the equation $x^3 + x^2 - 3x - 3 = 0$, *correct up to five decimal places.*

Solution

Let $f(x) = x^3 + x^2 - 3x - 3 = 0$

Let $x_0 = 1, x_1 = 2$

$f(x_0) = f(1) = -4$ and $f(x_1) = f(2) = 3$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$= 2 - \frac{2-1}{3+4}(3)$$

$$= 1.57143$$

$$f(x_2) = f(1.57143) = -1.36442$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 1.57143 - \frac{1.57143 - 2}{-1.36442 - 3}(-1.36442)$$

$$= 1.70541$$

$$f(x_3) = f(1.70541) = -0.24775$$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$= 1.70541 - \frac{1.70541 - 1.57143}{-0.24775 + 1.36442}(-0.24775)$$

$$= 1.73514$$

$$f(x_4) = f(1.73514) = 0.0293$$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4)$$

$$= 1.73514 - \frac{1.73514 - 1.70541}{0.0293 + 0.24775}(0.0293)$$

$$= 1.732$$

$$f(x_5) = f(1.732) = -0.00048$$

$$x_6 = x_5 - \frac{x_5 - x_4}{f(x_5) - f(x_4)} f(x_5)$$

$$= 1.732 - \frac{1.732 - 1.73514}{-0.00048 - 0.0293}(-0.00048)$$

$$= 1.73205$$

$$f(x_6) = f(1.73205) = -0.000008$$

$$\begin{aligned} x_7 &= x_6 - \frac{x_6 - x_5}{f(x_6) - f(x_5)} f(x_6) \\ &= 1.73205 - \frac{1.73205 - 1.732}{-0.000008 + 0.00048}(-0.000008) \\ &= 1.73205 \end{aligned}$$

Since x_6 and x_7 are same up to five decimal places, the root is 1.73205.

Example 3

Find the root of $x \log_{10} x - 1.9 = 0$, *correct up to three decimal places with* $x_0 = 3$ *and* $x_1 = 4$.

Solution

Let $\quad f(x) = x \log_{10} x - 1.9$

$$x_0 = 3,\ x_1 = 4$$

$$f(x_0) = f(3) = -0.4686 \text{ and } f(x_1) = f(4) = 0.5082$$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$\begin{aligned} x_2 &= x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) \\ &= 4 - \frac{4-3}{0.5082 + 0.4686}(0.5082) \\ &= 3.4797 \end{aligned}$$

$$f(x_2) = f(3.4797) = -0.0156$$

$$\begin{aligned} x_3 &= x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) \\ &= 3.4797 - \frac{3.4797 - 4}{-0.0156 - 0.5082}(-0.0156) \\ &= 3.4952 \end{aligned}$$

$$f(x_3) = f(3.4952) = -0.0005$$

$$\begin{aligned} x_4 &= x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) \\ &= 3.4952 - \frac{3.4952 - 3.4797}{-0.0005 + 0.0156}(-0.0005) \\ &= 3.4957 \end{aligned}$$

Since x_3 and x_4 are same up to three decimal places, the root is 3.495.

Example 4

Find the positive solution of $x - 2\sin x = 0$, *correct up to three decimal places starting from* $x_0 = 2$ *and* $x_1 = 1.9$. **[Summer 2014]**

Solution

Let
$$f(x) = x - 2\sin x$$
$$x_0 = 2,\ x_1 = 1.9$$
$$f(x_0) = f(2) = 0.1814 \text{ and } f(x_1) = f(1.9) = 0.0074$$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$
$$= 1.9 - \frac{1.9 - 2}{0.0074 - 0.1814}(0.0074)$$
$$= 1.8957$$
$$f(x_2) = f(1.8957) = 0.00034$$
$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$
$$= 1.8957 - \frac{1.8957 - 1.9}{0.00034 - 0.0074}(0.00034)$$
$$= 1.8955$$

Since x_2 and x_3 are same up to three decimal places, the positive root is 1.895.

Example 5

Solve $xe^x - 1 = 0$, *correct up to three decimal places between* 0 *and* 1.

Solution

Let
$$f(x) = xe^x - 1$$

Let
$$x_0 = 0,\ x_1 = 1$$
$$f(x_0) = f(0) = -1 \text{ and } f(x_1) = f(1) = 1.7183$$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$\begin{aligned}
x_2 &= x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) \\
&= 1 - \frac{1-0}{1.7183+1}(1.7183) \\
&= 0.3679 \\
f(x_2) &= f(0.3679) = -0.4685 \\
x_3 &= x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) \\
&= 0.3679 - \frac{0.3679-1}{-0.4685-1.7183}(-0.4685) \\
&= 0.5033 \\
f(x_3) &= f(0.5033) = -0.1675 \\
x_4 &= x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) \\
&= 0.5033 - \frac{0.5033-0.3679}{-0.1675+0.4685}(-0.1675) \\
&= 0.5786 \\
f(x_4) &= f(0.5786) = 0.032 \\
x_5 &= x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4) \\
&= 0.5786 - \frac{0.5786-0.5033}{0.032+0.1675}(0.032) \\
&= 0.5665 \\
f(x_5) &= f(0.5665) = -0.0018 \\
x_6 &= x_5 - \frac{x_5 - x_4}{f(x_5) - f(x_4)} f(x_5) \\
&= 0.5665 - \frac{0.5665-0.5786}{-0.0018-0.032}(-0.0018) \\
&= 0.5671 \\
f(x_6) &= f(0.5671) = -0.0001 \\
x_7 &= x_6 - \frac{x_6 - x_5}{f(x_6) - f(x_5)} f(x_6) \\
&= 0.5671 - \frac{0.5671-0.5665}{-0.0001+0.0018}(-0.0001) \\
&= 0.5671
\end{aligned}$$

Since x_6 and x_7 are same up to three decimal places, the root is 0.567.

Example 6

Find the root of $\cos x - xe^x = 0$, *correct up to three decimal places.*

Solution

Let $f(x) = \cos x - xe^x$

Let $x_0 = 0, x_1 = 1$

$$f(x_0) = f(0) = 1 \text{ and } f(x_1) = f(1) = -2.178$$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$= 1 - \frac{1-0}{-2.178-1}(-2.178)$$

$$= 0.3147$$

$$f(x_2) = f(0.3147) = 0.5198$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 0.3147 - \frac{0.3147-1}{0.5198+2.178}(0.5198)$$

$$= 0.4467$$

$$f(x_3) = f(0.4467) = 0.2036$$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$= 0.4467 - \frac{0.4467-0.3147}{0.2036-0.5198}(0.2036)$$

$$= 0.5317$$

$$f(x_4) = f(0.5317) = -0.0429$$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4)$$

$$= 0.5317 - \frac{0.5317-0.4467}{-0.0429-0.2036}(-0.0429)$$

$$= 0.5169$$

$$f(x_5) = f(0.5169) = 0.0026$$

$$\begin{aligned}
x_6 &= x_5 - \frac{x_5 - x_4}{f(x_5) - f(x_4)} f(x_5) \\
&= 0.5169 - \frac{0.5169 - 0.5317}{0.0026 + 0.0429}(0.0026) \\
&= 0.5177 \\
f(x_6) &= f(0.5177) = 0.0002 \\
x_7 &= x_6 - \frac{x_6 - x_5}{f(x_6) - f(x_5)} f(x_6) \\
&= 0.5177 - \frac{0.5177 - 0.5169}{0.0002 - 0.0026}(0.0002) \\
&= 0.5178
\end{aligned}$$

Since x_6 and x_7 are same up to three decimal places, the root is 0.517.

EXERCISE 2.4

Find a real root of the following equations correct up to three decimal places using the secant method:

1. $x^3 - 2x^2 + 3x - 4 = 0$

[Ans.: 1.650**]**

2. $x^3 + 3x^2 - 3 = 0$

[Ans.: 0.879**]**

3. $e^x - 4x = 0$

[Ans.: 0.357**]**

4. $\sin x = e^x - 3x$

[Ans.: 0.360**]**

5. $2x - 7 - \log_{10} x = 0$

[Ans.: 3.789**]**

6. $e^x \tan x = 1$

[Ans.: 3.183**]**

7. $3x - 6 = \log_{10} x$

[Ans.: 2.108**]**

2.6 SUCCESSIVE APPROXIMATION METHOD (ITERATION METHOD)

Consider an equation $f(x) = 0$.

Rewriting the equation,

$$x = \phi(x) \qquad \text{...(2.7)}$$

Assuming x_0 to be the starting approximate value to the actual root α of $x = \phi(x)$, the first approximation is

$$x_1 = \phi(x_0)$$

The second approximation is

$$x_2 = \phi(x_1)$$

The third approximation is

$$x_3 = \phi(x_2)$$

In general, the n^{th} approximation is

$$x_{n+1} = \phi(x_n), \qquad n = 0, 1, 2, \ldots$$

The sequence of approximate roots $x_1, x_2, \ldots, x_n$ if it converges to α is taken as the root of the equation $f(x) = 0$.

Condition for the Convergence

Let I be the interval containing the root $x = \alpha$ of the equation $x = \phi(x)$. If $|\phi'(x)| < 1$ for all x in I then the sequence of approximations $x_0, x_1, x_2, \ldots, x_n$ will converge to α, if the initial starting value x_0 is chosen in I.

Example 1

Find the positive root of an equation $x^3 + x^2 - 1 = 0$ by the iteration method correct up to four decimal places. **[Summer 2015]**

Solution

Let $\quad f(x) = x^3 + x^2 - 1$

$$f(0) = -1 \text{ and } f(1) = 1$$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Rewriting the equation,

$$x^3 + x^2 - 1 = 0$$

$$x^2(x+1) - 1 = 0$$

$$x^2(x+1) = 1$$

$$x^2 = \frac{1}{x+1}$$

$$x = \frac{1}{\sqrt{x+1}} = \phi(x)$$

$$\phi'(x) = \frac{1}{2(x+1)^{\frac{3}{2}}}$$

$$\phi'(x) < 1 \quad \text{for } 0 < x < 1$$

Hence, the iteration method can be applied.

By the iteration method,

$$x_{n+1} = \frac{1}{\sqrt{x_n + 1}}, \qquad n = 0, 1, 2, \ldots$$

Let $\quad x_0 = 0.5$

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{0.5+1}} = 0.81649$$

$$x_2 = \phi(x_1) = \frac{1}{\sqrt{0.81649+1}} = 0.74197$$

$$x_3 = \phi(x_2) = \frac{1}{\sqrt{0.74197+1}} = 0.75767$$

$$x_4 = \phi(x_3) = \frac{1}{\sqrt{0.75767+1}} = 0.75428$$

$$x_5 = \phi(x_4) = \frac{1}{\sqrt{0.75428+1}} = 0.75501$$

$$x_6 = \phi(x_5) = \frac{1}{\sqrt{0.75501+1}} = 0.75485$$

$$x_7 = \phi(x_6) = \frac{1}{\sqrt{0.75485+1}} = 0.75489$$

Since x_6 and x_7 are same up to four decimal places, the positive root is 0.7548.

Example 2

Find a real root of $x^3 - x - 1 = 0$ *correct to three decimal places by the iteration method.*

Solution

Let $\quad f(x) = x^3 - x - 1$

$$f(1) = -1 \qquad \text{and} \qquad f(2) = 5$$

Since $f(1) < 0$ and $f(2) > 0$, the root lies between 1 and 2.

Rewriting the equation,

$$x^3 - x - 1 = 0$$
$$x^3 = x + 1$$
$$x = (x+1)^{\frac{1}{3}} = \phi(x)$$
$$\phi'(x) = \frac{1}{3}(x+1)^{-\frac{2}{3}}$$
$$\phi'(x) < 1 \text{ for } 1 < x < 2$$

Hence, the iteration method can be applied.

By the iteration method,

$$x_{n+1} = (x_n + 1)^{\frac{1}{3}}, \qquad n = 0, 1, 2, ...$$

Let $x_0 = 1.2$

$$x_1 = \phi(x_0) = (1.2+1)^{\frac{1}{3}} = 1.3006$$
$$x_2 = \phi(x_1) = (1.3006+1)^{\frac{1}{3}} = 1.3201$$
$$x_3 = \phi(x_2) = (1.3201+1)^{\frac{1}{3}} = 1.3238$$
$$x_4 = \phi(x_3) = (1.3238+1)^{\frac{1}{3}} = 1.3245$$
$$x_5 = \phi(x_4) = (1.3245+1)^{\frac{1}{3}} = 1.3247$$

Since x_4 and x_5 are same up to three decimal places, the real root is 1.324.

Example 3

Find the root of the equation $2x - \log_{10}x - 7 = 0$ *correct up to four decimal places using the iteration method.* **[Winter 2012]**

Solution

Let $f(x) = 2x - \log_{10}x - 7$

$f(3) = -1.4471$ and $f(4) = 0.3979$

Since $f(3) < 0$ and $f(4) > 0$, the root lies between 3 and 4.

Rewriting the equation,

$$2x - \log_{10} x - 7 = 0$$
$$x = \frac{1}{2}(\log_{10} x + 7) = \phi(x)$$

$$\phi'(x) = \frac{1}{2}\left(\frac{1}{x}\log_{10} e\right)$$

$$\phi'(x) < 1 \text{ for } 3 < x < 4$$

Hence, the iteration method can be applied.
By the iteration method,

$$x_{n+1} = \frac{1}{2}(\log_{10} x_n + 7), \qquad n = 0, 1, 2, ...$$

Let $x_0 = 3.6$

$$x_1 = \phi(x_0) = \frac{1}{2}(\log_{10} 3.6 + 7) = 3.77815$$

$$x_2 = \phi(x_1) = \frac{1}{2}(\log_{10} 3.77815 + 7) = 3.78864$$

$$x_3 = \phi(x_2) = \frac{1}{2}(\log_{10} 3.78864 + 7) = 3.78924$$

$$x_4 = \phi(x_3) = \frac{1}{2}(\log_{10} 3.78924 + 7) = 3.78928$$

Since x_3 and x_4 are same up to four decimal places, the root is 3.7892.

Example 4

Find a real root of the equation $\cos x + 1 = 3x$ *correct up to three decimal places by the iteration method.*

Solution

Let $f(x) = \cos x - 3x + 1$

$f(0) = 2$ and $f\left(\frac{\pi}{2}\right) = -3.7124$

Since $f(0) > 0$ and $f\left(\frac{\pi}{2}\right) < 0$, the root lies between 0 and $\frac{\pi}{2}$.

Rewriting the equation,

$$\cos x - 3x + 1 = 0$$

$$x = \frac{1}{3}(\cos x + 1) = \phi(x)$$

$$\phi'(x) = -\frac{\sin x}{3}$$

$$|\phi'(x)| = \left|-\frac{\sin x}{3}\right| < 1 \text{ for } 0 < x < \frac{\pi}{2}$$

Hence, the iteration method can be applied.

By the iteration method,

$$x_{n+1} = \frac{1}{3}(\cos x_n + 1), \ \ n = 0, 1, 2, ...$$

Let $x_0 = 0.5$

$$x_1 = \phi(x_0) = \frac{1}{3}(\cos 0.5 + 1) \qquad = 0.6258$$

$$x_2 = \phi(x_1) = \frac{1}{3}(\cos 0.6258 + 1) = 0.6035$$

$$x_3 = \phi(x_2) = \frac{1}{3}(\cos 0.6035 + 1) = 0.6078$$

$$x_4 = \phi(x_3) = \frac{1}{3}(\cos 0.6078 + 1) = 0.607$$

Since x_3 and x_4 are same up to three decimal places, the real root is 0.607.

Example 5

Find a real root of $e^{-x} = 10x$ correct to four decimal places by the iteration method.

Solution

Let $f(x) = e^{-x} - 10x$

$f(0) = 1$ and $f(1) = -9.6321$

Since $f(0) > 0$ and $f(1) < 0$, the root lies between 0 and 1.

Rewriting the equation,

$$e^{-x} - 10x = 0$$

$$x = \frac{e^{-x}}{10} = \phi(x)$$

$$\phi'(x) = -\frac{e^{-x}}{10}$$

$$|\phi'(x)| = \left|-\frac{e^{-x}}{10}\right| < 1 \ \text{ for } \ 0 < x < 1$$

Hence, the iteration method can be applied.

By the iteration method,

$$x_{n+1} = \frac{e^{-x_n}}{10}, \quad n = 0, 1, 2, ...$$

Let $x_0 = 0.1$

$$x_1 = \phi(x_0) = \frac{e^{-0.1}}{10} = 0.09048$$

$$x_2 = \phi(x_1) = \frac{e^{-0.09048}}{10} = 0.09135$$

$$x_3 = \phi(x_2) = \frac{e^{-0.09135}}{10} = 0.09127$$

$$x_4 = \phi(x_3) = \frac{e^{-0.09127}}{10} = 0.09128$$

Since x_3 and x_4 are same up to four decimal places, the real root is 0.0912.

Example 6

Find a positive root of $3x - \sqrt{1+\sin x} = 0$ *by the iteration method.*

Solution

Let $\quad f(x) = 3x - \sqrt{1+\sin x}$

$$f(0) = -1 \qquad \text{and} \qquad f(1) = 1.643$$

Since $f(0) < 1$ and $f(1) > 0$, the root lies between 0 and 1.

Rewriting the equation,

$$3x - \sqrt{1+\sin x} = 0$$

$$x = \frac{1}{3}\sqrt{1+\sin x} = \phi(x)$$

$$\phi'(x) = \frac{\cos x}{6\sqrt{1+\sin x}}$$

$$|\phi'(x)| = \left|\frac{\cos x}{6\sqrt{1+\sin x}}\right| < 1 \quad \text{for} \quad 0 < x < 1$$

Hence, the iteration method can be applied.

By the iteration method,

$$x_{n+1} = \frac{1}{3}\sqrt{1+\sin x_n}, \quad n = 0, 1, 2, \ldots$$

Let $\quad x_0 = 0.4$

$$x_1 = \phi(x_0) = \frac{1}{3}\sqrt{1+\sin(0.4)} = 0.39291$$

$$x_2 = \phi(x_1) = \frac{1}{3}\sqrt{1+\sin(0.39291)} = 0.39199$$

$$x_3 = \phi(x_2) = \frac{1}{3}\sqrt{1+\sin(0.39199)} = 0.39187$$

$$x_4 = \phi(x_3) = \frac{1}{3}\sqrt{1+\sin(0.39187)} = 0.39185$$

Since x_3 and x_4 are same up to four decimal places, the positive root is 0.3918.

EXERCISE 2.5

Solve the following equations by the iteration method:

1. $x^3 + x + 1 = 0$

[Ans.: –0.682]

2. $x^3 + x^2 - 100 = 0$

[Ans.: 4.3311]

3. $x^3 + 2x^2 + 10x - 20 = 0$

[Ans.: 1.3688]

4. $\sin x = \dfrac{x+1}{x-1}$

[Ans.: –0.4204]

5. $2 \sin x = x$

[Ans.: 1.8955]

6. $3x - \cos x - 2 = 0$

[Ans.: 0.879]

7. $3x + \sin x = e^x$

[Ans.: 0.3604]

8. $3x = 6 + \log_{10}x$

[Ans.: 2.108]

2.7 DESCARTES' RULE OF SIGNS

Descartes's rule of signs is another theorem that is often used to obtain information about the roots of a polynomial function. In Descartes' rule of signs, the number of variations in the sign of the coefficients of a function $f(x)$ or $f(-x)$ refers to the sign changes in the coefficients from positive to negative or negative to positive in successive terms of the function. The terms are assumed to appear in order of descending powers of x.

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ be a polynomial function with real coefficients $a_0, a_1, a_2, ..., a_n$ and with the terms arranged in order of decreasing powers of x. The number of positive roots of $f(x) = 0$ is equal to the number of variations in the sign of $f(x)$ or to that number decreased by an even integer. The number of negative roots of $f(x) = 0$ is equal to the number of variations in the sign of $f(-x)$ or to that number decreased by an even integer.

Example 1

Use Descartes' rule of signs to determine both the number of possible positive and negative roots of each polynomial function:

(i) $f(x) = x^4 - 5x^3 + 5x^2 + 5x - 6$

(ii) $f(x) = 2x^5 + 3x^3 + 5x^2 + 8x + 7$

Solution

(i) $$f(x) = x^4 \underset{1}{-} 5x^3 \underset{2}{+} 5x^2 + 5x \underset{3}{-} 6$$

There are three variations in the sign of $f(x)$. By Descartes' rule of signs, there are either three or 3 – 2, i.e., one positive root of $f(x) = 0$.

$$\begin{aligned} f(-x) &= (-x)^4 - 5(-x)^3 + 5(-x)^2 + 5(-x) - 6 \\ &= x^4 + 5x^3 + 5x^2 \underset{1}{-} 5x - 6 \end{aligned}$$

There is one variation in the sign of $f(-x)$. By Descartes' rule of signs, there is one negative root of $f(x) = 0$.

(ii) $$f(x) = 2x^5 + 3x^3 + 5x^2 + 8x + 7$$

There is no variation in the sign of $f(x)$. Hence, there are no positive roots of $f(x) = 0$.

$$\begin{aligned} f(-x) &= 2(-x)^5 + 3(-x)^3 + 5(-x)^2 + 8(-x) + 7 \\ &= -2x^5 - 3x^3 \underset{1}{+} 5x^2 \underset{2}{-} 8x \underset{3}{+} 7 \end{aligned}$$

There are three variations in the sign of $f(-x)$. Hence, there are either three or 3 – 2, i.e., one negative root of $f(x) = 0$.

Example 2

Apply Descartes' rule of signs to

$$P_5(x) = 8x^5 + 12x^4 - 10x^3 + 17x + 5 = 0$$

Solution

$$P_5(x) = 8x^5 + 12x^4 \underset{1}{-} 10x^3 \underset{2}{+} 17x + 5$$

There are two variations in the sign of $P_5(x)$. By Descartes' rule of signs, there are either two or 2–2, i.e., zero positive roots of $P_5(x) = 0$.

$$P_5(-x) = 8(-x^5) + 12(-x)^4 - 10(-x)^3 + 17(-x) + 5$$
$$= \underbrace{-8x^5 + 12x^4}_{1} \underbrace{+10x^3 - 17x}_{2} \underbrace{\;\;\,}_{3}\!\!+ 5$$

There are three variations in the sign of $P_5(x)$. By Descartes' rule of signs, there are either three or 3–2, i.e., one negative root of $P_5(x) = 0$.

Example 3

Apply Descartes' rule of signs to the polynomial

$$f(x) = x^5 - x^4 + 3x^3 + 9x^2 - x + 5$$

Solution

$$f(x) = x^5 - x^4 + 3x^3 + 9x^2 - x + 5$$

There are four variations in the sign of $f(x)$. By Descartes' rule of signs, there are either 4, 2, or no positive roots of $f(x) = 0$.

$$f(-x) = (-x)^5 - (-x)^4 + 3(-x)^3 + 9(-x)^2 - (-x) + 5$$
$$= -x^5 - x^4 \underbrace{- 3x^3 + 9x^2}_{1} + x + 5$$

There is one variation in the sign of $f(-x)$. By Descartes' rule of signs, there is one negative root of $f(x) = 0$.

Example 4

Apply Descartes' rule of signs to the polynomial

$$f(x) = x^7 + x^6 - x^4 - x^3 - x^2 + x - 1$$

Solution

$$f(x) = x^7 + x^6 - x^4 - x^3 - x^2 + x - 1$$

There are three variations in the sign of $f(x)$. By Descartes' rule of signs, there are either 3 or 3–2, i.e., one positive root of $f(x) = 0$.

$$f(-x) = (-x)^7 + (-x)^6 - (-x)^4 - (-x)^3 - (-x)^2 + (-x) - 1$$
$$= -x^7 + x^6 - x^4 + x^3 - x^2 - x - 1$$

There are four variations in the sign of $f(-x)$. By Descartes' rule of signs, there are 4 or 2 or 0 negative roots of $f(x) = 0$.

2.8 BUDAN'S THEOREM

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ be a polynomial function with real coefficients $a_0, a_1, a_2, \ldots, a_n$. Let $v(c)$ be the number of variations of signs in the sequence $f(x), f'(x), f''(x), \ldots, f^{(n)}(x)$ when $x = c$, where c is any real number. The number of roots of $f(x)$ in the interval $[a, b]$, counted with their order of multiplicity is equal to

$$v(a) - v(b) - 2m, \text{ for some } m \in \mathbb{N}$$

i.e., the number of roots of $f(x)$ is equal to $v(a) - v(b)$ or $v(a) - v(b)$ decreased by an even integer.

Example 1

Apply Budan's theorem to find the number of roots of the equation $f(x) = x^4 - 4x^3 + 3x^2 - 10x + 8 = 0$ *in the interval* $[-1, 0]$ *and* $[0, 1]$.

[Winter 2013, Summer 2013]

Solution

$$f(x) = x^4 - 4x^3 + 3x^2 - 10x + 8$$
$$f'(x) = 4x^3 - 12x^2 + 6x$$
$$f''(x) = 12x^2 - 24x + 6$$
$$f'''(x) = 24x - 24$$
$$f^{\text{iv}}(x) = 24$$

The signs of these functions for $x = -1$, 0, and 1 are shown in the following table:

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{\text{iv}}(x)$	No. of variations of sign $v(x)$
–1	+	–	+	–	+	4
0	+	+	+	–	+	2
1	–	–	–	+	+	1

No. of variations of sign in the interval $[-1, 0] = v(-1) - v(0) = 4 - 2 = 2$.

Hence, the number of roots of $f(x)$ in the interval $[-1, 0]$ is either two or zero.

No. of variations of sign in the interval $[0, 1] = v(0) - v(1) = 2 - 1 = 1$.

Hence, the number of roots of $f(x)$ in the interval $[0, 1]$ is one.

Example 2

Apply Budan's theorem to find the number of roots of the equation $x^3 - 3x^2 - 4x + 13$ *in the interval* [–3, –2], [–2, –1], [–1, 0], [0, 1], [1, 2], *and* [2, 3].

Solution

Let $f(x) = x^3 - 3x^2 - 4x + 13$

$$f'(x) = 3x^2 - 6x - 4$$

$$f''(x) = 6x - 6$$

$$f'''(x) = 6$$

The signs of these functions for $x = -3, -2, -1, 0, 1, 2$, and 3 are shown in the following table:

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	No. of variations of sign $v(x)$
–3	–	+	–	+	3
–2	+	+	–	+	2
–1	+	+	–	+	2
0	+	–	–	+	2
1	+	–	+	+	2
2	+	–	+	+	2
3	+	+	+	+	0

No. of variations of sign in the interval [–3, –2] = $v(-3) - v(-2) = 3 - 2 = 1$.

Hence, the number of roots of $f(x)$ in the interval [–3, –2] is one.

No. of variations of sign in the interval [–2, –1] = $v(-2) - v(-1) = 2 - 2 = 0$.

Hence, the number of roots of $f(x)$ in the interval [–2, –1] is zero.

No. of variations of sign in the interval [–1, 0] = $v(-1) - v(0) = 2 - 2 = 0$.

Hence, the number of roots of $f(x)$ in the interval [–1, 0] is zero.

No. of variations of sign in the interval [0, 1] = $v(0) - v(1) = 2 - 2 = 0$.

Hence, the number of roots of $f(x)$ in the interval [0, 1] is zero.

No. of variations of sign in the interval [1, 2] = $v(1) - v(2) = 2 - 2 = 0$.

Hence, the number of roots of $f(x)$ in the interval [1, 2] is zero.

No. of variations of sign in the interval [2, 3] = $v(2) - v(3) = 2 - 0 = 2$.

Hence, the number of roots of $f(x)$ in the interval [2, 3] is either two or zero.

Example 3

Apply Budan's theorem to find the number of roots of the equation $x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$ *in the interval* [–2, –1], [0, 1], and [1, 2].

Solution

Let

$$f(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$$
$$f'(x) = 5x^4 + 4x^3 - 12x^2 - 6x + 3$$
$$f''(x) = 20x^3 + 12x^2 - 24x - 6$$
$$f'''(x) = 60x^2 + 24x - 24$$
$$f^{\text{iv}}(x) = 120x + 24$$
$$f^{\text{v}}(x) = 120$$

The signs of these functions for x = –2, –1, 0, 1, and 2 are shown in the following table:

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{iv}(x)$	$f^{v}(x)$	No. of variations of sign $v(x)$
–2	–	+	–	+	–	+	5
–1	–	–	+	+	–	+	3
0	+	+	–	–	+	+	2
1	–	–	+	+	+	+	1
2	+	+	+	+	+	+	0

No. of variations of sign in the interval [–2, –1] = $v(-2) - v(-1) = 5 - 3 = 2$.

Hence, the number of roots of $f(x)$ in the interval [–2, –1] is either two or zero.

No. of variations of sign in the interval [–1, 0] = $v(-1) - v(0) = 3 - 2 = 1$.

Hence, the number of roots of $f(x)$ in the interval [–1, 0] is one.

No. of variations of sign in the interval [0, 1] = $v(0) - v(1) = 2 - 1 = 1$.

Hence, the number of roots of $f(x)$ in the interval [0, 1] is one.

No. of variations of sign in the interval [1, 2] = $v(1) - v(2) = 1 - 0 = 1$.

Hence, the number of roots of $f(x)$ in the interval [1, 2] is one.

Example 4

Apply Budan's theorem to find the roots of the equation

$$f(x) = x^5 - x^4 + 3x^3 + 9x^2 - x + 5$$

in the interval [–3, –2], [–2, –1], [–1, 0], *and* [0, 1].

Solution

$$f(x) = x^5 - x^4 + 3x^3 + 9x^2 - x + 5$$
$$f'(x) = 5x^4 - 4x^3 + 9x^2 + 18x - 1$$
$$f''(x) = 20x^3 - 12x^2 + 18x + 18$$
$$f'''(x) = 60x^2 - 24x + 18$$
$$f^{\text{iv}}(x) = 120x - 24$$
$$f^{\text{v}}(x) = 120$$

The signs of these functions for $x = -3, -2, -1, 0$, and 1 are shown in the following table:

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{\text{iv}}(x)$	$f^{\text{v}}(x)$	No. of variations of sign $v(x)$
–3	–	+	–	+	–	+	5
–2	–	+	–	+	–	+	5
–1	+	–	–	+	–	+	4
0	+	–	+	+	–	+	4
1	+	+	+	+	+	+	0

No. of variations of sign in the interval [–3, –2] = $v(-3) - v(-2) = 5 - 5 = 0$.

Hence, the number of roots of $f(x)$ in the interval [–3, –2] is zero.

No. of variations of sign in the interval [–2, –1] = $v(-2) - v(-1) = 5 - 4 = 1$.

Hence, the number of roots of $f(x)$ in the interval [–2, –1] is one.

No. of variations of sign in the interval [–1, 0] = $v(-1) - v(0) = 4 - 4 = 0$.

Hence, the number of roots of $f(x)$ in the interval [–1, 0] is zero.

No. of variations of sign in the interval [0, 1] = $v(0) - v(1) = 4 - 0 = 4$.

Hence, the number of roots of $f(x)$ in the interval [0, 1] is either four or two or zero.

EXERCISE 2.6

1. Apply Descartes' rule of signs to determine the number of positive and negative roots of the equations:

(i) $6x^5 - 3x^4 - 5x^3 - 6x^2 + 9x + 5 = 0$

(ii) $x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1 = 0$

(iii) $4x^7 + 3x^6 + x^5 + 2x^4 - x^3 + 9x^2 + x + 1 = 0$

(iv) $x^5 + x^4 + 4x^3 + 3x^2 + x + 1 = 0$

(v) $2x^4 - x^3 + 4x^2 - 5x + 3 = 0$

[**Ans.:** (i) 2 or 0 positive roots, 3 or 1 negative roots
(ii) 2 or 0 positive roots, 3 or 1 negative roots
(iii) 2 or 0 positive roots, 5 or 3 or 1 negative roots
(iv) 0 positive roots, 5 or 3 or 1 negative roots
(v) 4 or 2 or 0 positive roots, 0 negative root]

2. Apply Budan's theorem to find the number of roots of the equation $f(x) = x^4 - 4x^3 - 5x^2 + 3x + 2$ in the intervals [–1, 0] and [0, 1].

[**Ans.:** One root each in the interval [–1, 0] and [0, 1]]

3. Apply Budan's theorem to find the number of roots of the equation $x^5 + x^4 + 4x^3 - 3x^2 + 3x + 1 = 0$ in the intervals [–2, –1], [–1, 0], and [0, 1].

[**Ans.:** One root in the interval [–2, – 1]
3 or 1 roots in the interval [–1, 0]
2 or 0 roots in the interval [0, 1]]

2.9 BAIRSTOW'S METHOD

Bairstow's method is useful for finding the quadratic factors of a polynomial of degree n. Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ be a polynomial of degree n where $a_0 \neq 0$. When $f(x)$ is divided by a quadratic factor $x^2 - px - q$, the quotient will be a polynomial of degree $n - 2$, i.e., $b_0 x^{n-2} + b_1 x^{n-3} + \cdots + b_{n-2}$ and the remainder will be a first-degree polynomial of the form $b_{n-1}(x - p) + b_n$

$$\begin{aligned} f(x) &= a_0x^n + a_1 x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n \\ &= (x^2 - px - q)(b_0x^{n-2} + b_1x^{n-3} + \cdots + b_{n-2}) + b_{n-1}(x - p) + b_n \qquad \text{...(2.8)} \end{aligned}$$

Equating the coefficients of different powers of x,

$$
\begin{aligned}
a_0 &= b_0 && \text{i.e.,} \quad b_0 = a_0 \\
a_1 &= b_1 - pb_0 && \text{i.e.,} \quad b_1 = a_1 + pb_0 \\
a_2 &= b_2 - pb_1 - qb_0 && \text{i.e.,} \quad b_2 = a_2 + pb_1 + qb_0 \\
&\vdots && \qquad\quad \vdots \\
a_{n-1} &= b_{n-1} - pb_{n-2} - qb_{n-3}, && \text{i.e.,} \quad b_{n-1} = a_{n-1} + pb_{n-2} + qb_{n-3} \\
a_n &= b_n - pb_{n-1} - qb_{n-2}, && \text{i.e.,} \quad b_n = a_n + pb_{n-1} + qb_{n-2}
\end{aligned}
\qquad ...(2.9)
$$

Thus, b's are functions of p and q. If $x^2 - px - q$ is an exact factor of $f(x)$ then

$$b_n(p, q) = 0 \qquad ...(2.10)$$

and $$b_{n-1}(p, q) = 0 \qquad ...(2.11)$$

If p_0 and q_0 are initial approximations of p and q,

$$p = p_0 + \Delta p$$

and $$q = q_0 + \Delta q$$

where Δp and Δq are small.

Substituting in Eqs (2.10) and (2.11),

$$b_n(p_0 + \Delta p, q_0 + \Delta q) = 0 \qquad ...(2.12)$$

and $$b_{n-1}(p_0 + \Delta p, q_0 + \Delta q) = 0 \qquad ...(2.13)$$

By Taylor's series expansion of Eqs (2.12) and (2.13), and neglecting terms of higher powers,

$$b_n + \frac{\partial b_n}{\partial p}\Delta p + \frac{\partial b_n}{\partial q}\Delta q = 0 \qquad (2.14)$$

and $$b_{n-1} + \frac{\partial b_{n-1}}{\partial p}\Delta p + \frac{\partial b_{n-1}}{\partial q}\Delta q = 0 \qquad ...(2.15)$$

where b_n, b_{n-1} and the partial derivatives are evaluated at $p = p_0$, $q = q_0$.

Let $$\frac{\partial b_n}{\partial p} = c_{n-1} \text{ and } \frac{\partial b_n}{\partial q} = c_{n-2} \qquad ...(2.16)$$

Substituting Eq (2.16) in Eqs (2.14) and (2.15),

$$b_n + c_{n-1}\,\Delta p + c_{n-2}\,\Delta q = 0 \qquad ...(2.17)$$

$$b_{n-1} + c_{n-2}\,\Delta p + c_{n-3}\,\Delta q = 0 \qquad ...(2.18)$$

Solving Eqs (2.17) and (2.18), Δp and Δq can be calculated.

Hence, the first approximations for p and q are

$$p_1 = p_0 + \Delta p, \qquad q_1 = q_0 + \Delta q$$

This procedure is repeated till the desired degree of accuracy is achieved.

The first, second, third, etc., approximations can be computed by the following procedure which is similar to synthetic division:

	a_0	a_1	a_2	$\cdots$	a_{n-1}	a_n
p	–	pb_0	pb_1	$\cdots$	pb_{n-2}	pb_{n-1}
q	–	–	qb_0	$\cdots$	qb_{n-3}	qb_{n-2}
	$b_0(=a_0)$	b_1	b_2	$\cdots$	b_{n-1}	b_n
p	–	pc_0	pc_1	$\cdots$	pc_{n-2}	–
q	–	–	qc_0	$\cdots$	qc_{n-3}	–
	$c_0(=a_0)$	c_1	c_2	$\cdots$	c_{n-1}	

(i) The first-row elements are the coefficients of the given equation. p, q are the current values of the actual p and q.

(ii) The second-row elements (from second column onwards) are obtained by multiplying $b_0, b_1, b_2, ..., b_{n-1}$ by p respectively.

(iii) The third-row elements (from third column onwards) are obtained by multiplying $b_0, b_1, b_2, ..., b_{n-2}$ by q.

$b_0, b_1, b_2, b_3, ..., b_n$ are the column totals.

(iv) Again, proceed similarly for the fifth and sixth row up to b_{n-1} since the last c value is c_{n-1}.

Example 1

Find the roots of the equation $x^3 - 2x^2 + x - 2 = 0$ using Lin-Bairstow's method up to second iteration with $p_0 = q_0 = 0$.

Solution

Let $$f(x) = x^3 - 2x^2 + x - 2$$

$$p_0 = 0, \qquad q_0 = 0$$

First iteration

	1	–2	1	–2
0	–	0	0	0
0	–	–	0	0
	$1 = b_0$	$-2 = b_1$	$1 = b_2$	$-2 = b_3$
0	–	0	0	
0	–	–	0	
	$1 = c_0$	$-2 = c_1$	$1 = c_2$	

Hence, the equations are

$$1\,\Delta p - 2\Delta q = 2 \qquad ...(1)$$

$$-2\Delta p + 1\Delta q = -1 \qquad ...(2)$$

Solving Eqs (1) and (2),

$$\Delta p = 0$$
$$\Delta q = 1$$

Hence, the first approximation is

$$p_1 = p_0 + \Delta p = 0 + 0 = 0$$
$$q_1 = q_0 + \Delta q = 0 + 1 = 1$$

Second iteration

$$\begin{array}{c|cccc}
 & 1 & -2 & 1 & -2 \\
0 & - & 0 & 0 & 0 \\
1 & - & - & 1 & -2 \\
\hline
 & 1 = b_0 & -2 = b_1 & 2 = b_2 & -4 = b_3 \\
0 & - & 0 & 0 & \\
1 & - & - & 1 & \\
\hline
 & 1 = c_0 & -2 = c_1 & 3 = c_2 &
\end{array}$$

Hence, the equations are

$$3\,\Delta p - 2\Delta q = 4 \qquad \text{...(3)}$$
$$-2\Delta p + 1\Delta q = -2 \qquad \text{...(4)}$$

Solving Eqs (3) and (4),

$$\Delta p = 0$$
$$\Delta q = -2$$

Hence, the second approximation is

$$p_2 = p_1 + \Delta p = 0 + 0 = 0$$
$$q_2 = q_1 + \Delta q = 1 - 2 = -1$$

Hence, the quadratic factor is $x^2 + 0x + 1$, i.e., $x^2 + 1$.

The other factor is $x - 2$.

Hence, the roots are 2, i, and $-i$.

Example 2

Find all the roots of the equation $x^3 + x^2 - x + 2$ using Lin–Bairstow method. Start with the initial factor $x^2 - 0.9x + 0.9$. **[Winter 2014]**

Solution

Let $f(x) = x^3 + x^2 - x + 2$

Comparing $x^2 - 0.9x + 0.9$ with $x^2 - px - q$,

$$p = 0.9, \qquad q = -0.9$$

Let $p_0 = 0.9$ and $q_0 = -0.9$ be the initial approximations of p and q.

First iteration

	1	1	−1	2
0.9	–	0.9	1.71	−0.171
−0.9	–	–	−0.9	−1.71
	$1 = b_0$	$1.9 = b_1$	$-0.19 = b_2$	$0.119 = b_3$
0.9	–	0.9	2.52	
−0.9	–	–	−0.9	
	$1 = c_0$	$2.8 = c_1$	$1.43 = c_2$	

Hence, the equations are

$$1.43\ \Delta p + 2.8\ \Delta q = -0.119 \quad \text{...(1)}$$

$$2.8\ \Delta p + 1\ \Delta q = 0.19 \quad \text{...(2)}$$

Solving Eqs (1) and (2),

$$\Delta p = 0.1016$$

$$\Delta q = -0.0944$$

Hence, the first approximation is

$$p_1 = p_0 + \Delta p = 0.9 + 0.1016 = 1.0016$$

$$q_1 = q_0 + \Delta p = -0.9 - 0.0944 = -0.9944$$

Second iteration

	1	1	−1	2
1.0016	–	1.0016	2.0048	0.0104
−0.9944	–	–	−0.9944	−1.9904
	$1 = b_0$	$2.0016 = b_1$	$0.0104 = b_2$	$0.02 = b_3$
1.0016	–	1.0016	3.008	
−0.9944	–	–	−0.9944	
	$1 = c_0$	$3.0032 = c_1$	$2.024 = c_2$	

Hence, the equations are

$$2.024\ \Delta p + 3.0032 = -0.02 \quad \text{...(3)}$$

$$3.0032\ \Delta p + 1\ \Delta q = -0.0104 \quad \text{...(4)}$$

Solving Eqs (3) and (4),

$$\Delta p = -0.0016$$

$$\Delta q = -0.0056$$

Hence, the second approximation is

$$p_2 = p_1 + \Delta p = 1.0016 - 0.0016 = 1$$

$$q_2 = q_1 + \Delta q = -0.9944 - 0.0056 = -1$$

Hence, the quadratic factor is $x^2 - x + 1$.

The other factor is $x + 2$.

Hence, all the roots are $-2, -0.5 \pm 0.866\, i$.

Example 3

Find the roots of $x^3 - x - 1 = 0$. *Start with the initial factor* $x^2 + x + 1$.

Solution

Let $\qquad f(x) = x^3 - x - 1$

Comparing $x^2 + x + 1$ with $x^2 - px - q$,

$$p = -1, \qquad q = -1$$

Let $p_0 = -1$ and $q_0 = -1$ be the initial approximations of p and q.

First iteration

	1	0	–1	–1
–1	–	–1	1	1
–1	–	–	–1	1
	$1 = b_0$	$-1 = b_1$	$-1 = b_2$	$1 = b_3$
–1	1	–1	2	
–1	–	–	–1	
	$1 = c_0$	$-2 = c_1$	$0 = c_2$	

Hence, the equations are

$$-2\,\Delta q = -1 \qquad \text{...(1)}$$

$$-2\Delta p + 1\,\Delta q = 1 \qquad \text{...(2)}$$

Solving Eqs (1) and (2),

$$\Delta p = -0.25$$

$$\Delta q = 0.5$$

Hence, the first approximation is

$$p_1 = p_0 + \Delta p = -1 - 0.25 = -1.25$$

$$q_1 = q_0 + \Delta q = -1 + 0.5 = -0.5$$

Second iteration

	1	0	–1	–1
–1.25	–	–1.25	1.5625	–0.0781
–0.5	–		–0.5	0.625
	$1 = b_0$	$-1.25 = b_1$	$0.0625 = b_2$	$-0.4531 = b_3$
–1.25	–	–1.25	3.125	
–0.5	–	–	–0.5	
	$1 = c_0$	$-2.5 = c_1$	$2.6875 = c_2$	

Hence, the equations are

$$2.6875\ \Delta p - 2.5\ \Delta q = 0.4531 \qquad ...(3)$$

$$-2.5\Delta p + 1\ \Delta q = -0.0625 \qquad ...(4)$$

Solving Eqs (3) and (4),

$$\Delta p = -0.0833$$

$$\Delta q = -0.2708$$

Hence, the second approximation is

$$p_2 = p_1 + \Delta p = -1.25 - 0.0833 = -1.3333$$

$$q_2 = q_1 + \Delta q = -0.5 - 0.2708 = -0.7708$$

Hence, the quadratic factor is $x^2 + 1.3333\,x + 0.7708$.

The other factor is $x - 1.3247$.

Hence, the roots are 1.3247, $-0.6624 \pm 0.5623\,i$.

Example 4

Find all the roots of the equation $x^4 + x^3 + 2x^2 + x + 1 = 0$ using the Lin–Bairstow method. Start with the initial factor $x^2 + 0.9x + 0.9$.

[Summer 2014]

Solution

Let $f(x) = x^4 + x^3 + 2x^2 + x + 1$

Comparing $x^2 + 0.9x + 0.9$ with $x^2 - px - q$,

$$p = -0.9, \qquad q = -0.9$$

Let $p_0 = -0.9$ and $q_0 = -0.9$ be the initial approximations of p and q.

First iteration

	1	1	2	1	1
−0.9	–	−0.9	−0.09	−0.909	−0.0009
−0.9	–	–	−0.9	−0.09	−0.909
	$1 = b_0$	$0.1 = b_1$	$1.01 = b_2$	$0.001 = b_3$	$0.0901 = b_4$
−0.9	–	−0.9	0.72	−0.747	
−0.9	–	–	−0.9	0.72	
	$1 = c_0$	$-0.8 = c_1$	$0.83 = c_2$	$-0.026 = c_3$	

Hence, the equations are

$$-0.026\ \Delta p + 0.83\ \Delta q = -0.0901 \qquad ...(1)$$

$$0.83\ \Delta p - 0.8\ \Delta q = -0.001 \qquad ...(2)$$

Solving Eqs (1) and (2),

$$\Delta p = -\,0.1091$$

$$\Delta q = -\,0.112$$

Hence, the first approximation is

$$p_1 = p_0 + \Delta p = -\,0.9 - 0.1091 = -\,1.0091$$

$$q_1 = q_0 + \Delta q = -\,0.9 - 0.112 \quad = -1.012$$

Second iteration

	1	1	2	1	1
−1.0091	–	−1.0091	−0.0092	−0.9877	−0.0217
−1.012	–	–	−1.012	0.0092	−0.9905
	$1 = b_0$	$-0.0091 = b_1$	$0.9788 = b_2$	$0.0215 = b_3$	$-0.0122 = b_4$
−1.0091	–	−1.0091	1.0275	−1.0033	
−1.012	–	–	−1.012	1.0304	
	$1 = c_0$	$-1.0182 = c_1$	$0.9943 = c_2$	$0.0486 = c_3$	

Hence, the equations are

$$0.0486\,\Delta p + 0.9943\,\Delta q = 0.0122 \qquad \text{...(3)}$$

$$0.9943\,\Delta p - 1.0182\,\Delta q = -0.0215 \qquad \text{...(4)}$$

Solving Eqs (3) and (4),

$$\Delta p = -0.0086$$

$$\Delta q = 0.0127$$

Hence, the second approximation is

$$p_2 = p_1 + \Delta p = -1.0091 - 0.0086 = -1.0177$$

$$q_2 = q_1 + \Delta q = -1.012 + 0.0127 = -0.9993$$

Hence, the quadratic factor is $x^2 + 1.0177x + 0.9993$, i.e., $x^2 + x + 1$.

The other factor is $x^2 + 1$.

Hence, all the roots are $\pm i$ and $-0.5 \pm 0.866\,i$.

Example 5

$x^4 - 8x^3 + 39x^2 - 62x + 50 = 0$ *by using Lin–Bairstow method up to third iteration with* $p_0 = q_0 = 0$. **[Summer 2013]**

Solution

Let $\quad f(x) = x^4 - 8x^3 + 39x^2 - 62x + 50$

$$p_0 = 0, \qquad q_0 = 0$$

First iteration

	1	−8	39	−62	50
0	–	0	0	0	0
0	–	–	0	0	0
	$1 = b_0$	$-8 = b_1$	$39 = b_2$	$-62 = b_3$	$50 = b_4$
0	–	0	0	0	
0	–	–	0	0	
	$1 = c_0$	$-8 = c_1$	$39 = c_2$	$-62 = c_3$	

Hence, the equations are

$$-62\ \Delta p + 39\ \Delta q = -50 \quad ...(1)$$

$$39\ \Delta p + -8\ \Delta q = 62 \quad ...(2)$$

Solving Eqs (1) and (2),

$$\Delta p = 1.9688$$

$$\Delta q = 1.8478$$

Hence, the first approximation is

$$p_1 = p_0 + \Delta p = 0 + 1.9688 = 1.9688$$

$$q_1 = q_0 + \Delta q = 0 + 1.8478 = 1.8478$$

Second iteration

	1	−8	39	−62	50
1.9688	–	1.9688	−11.8742	57.0432	−31.703
1.8478	–	–	1.8478	−11.1459	53.5374
	$1 = b_0$	$-6.0312 = b_1$	$28.9736 = b_2$	$-16.1027 = b_3$	$71.8344 = b_4$
1.9688	–	1.9688	−7.9981	44.9345	
1.8478	–	–	1.8478	−7.5065	
	$1 = c_0$	$-4.0624 = c_1$	$22.8233 = c_2$	$21.3253 = c_3$	

Hence, the equations are

$$21.3253\ \Delta p + 22.8233\ \Delta q = -71.8344 \quad ...(3)$$

$$22.8233\ \Delta p - 4.0624\ \Delta q = 16.1027 \quad ...(4)$$

Solving Eqs (3) and (4),

$$\Delta p = 0.1246$$

$$\Delta q = -3.2638$$

Hence, the second approximation is

$$p_2 = p_1 + \Delta p = 1.9688 + 0.1246 = 2.0934$$

$$q_2 = q_1 + \Delta q = 1.8478 - 3.2638 = -1.416$$

Third iteration

	1	–8	39	–62	50
2.0934	–	2.0934	–12.3649	52.7937	–1.3877
–1.416	–	–	–1.416	8.3634	–35.7102
	$1 = b_0$	$-5.9066 = b_1$	$25.2191 = b_2$	$-0.6629 = b_3$	$12.9021 = b_4$
2.0934	–	2.0934	–7.9826	33.1186	
–1.416	–	–	–1.416	5.3995	
	$1 = c_0$	$-3.8132 = c_1$	$15.8205 = c_2$	$37.8552 = c_3$	

Hence, the equations are

$$37.8552\ \Delta p + 15.8205\ \Delta q = -12.9021 \quad \text{...(5)}$$

$$15.8205\ \Delta p - 3.8132\ \Delta q = 0.6629 \quad \text{...(6)}$$

Solving Eqs (5) and (6),

$$\Delta p = -0.0981$$

$$\Delta q = -0.5808$$

Hence, the third approximation is

$$p_3 = p_2 + \Delta p = 2.0934 - 0.0981 = 1.9953$$

$$q_3 = q_2 + \Delta q = -1.416 - 0.5808 = -1.9968$$

Hence, the quadratic factor is $x^2 - 1.9953\,x + 1.9968$, i.e., $x^2 - 2x + 2$.

The other quadratic factor is $x^2 - 6x + 25$.

Hence, the complex roots are $1 \pm i$, $3 \pm 4i$.

EXERCISE 2.7

1. Find the complex roots of $x^3 + x^2 - 2 = 0$ starting with $p_0 = q_0 = 0$.

   [Ans.: $-1 \pm i$**]**

2. Using the approximate factor $x^2 + 2x + 2$ of $x^4 - 3x^3 + 20x^2 + 44x + 54$, find the quadratic factor performing two iterations.

   [Ans.: $x^2 + 1.94x + 1.95$**]**

3. Find the quadratic factor of $x^4 - 1.1x^3 + 2.3x^2 + 0.5x + 3.3 = 0$ starting with the approximation $x^2 + x + 1 = 0$.

   [Ans.: $(x^2 + 0.9x + 1.1)(x^2 - 2x + 3)$**]**

4. Find the roots of the equation $x^4 + 9x^3 + 36x^2 + 51x + 27 = 0$ to three decimal places.

   [Ans.: $-0.759, -1.42, -3.411 \pm 2.903\,i$**]**

5. Find a quadratic factor of the equation $x^4 + 5x^3 + 3x^2 - 5x - 9$ starting with $x^2 + 3x - 5$.

   [Ans.: $x^2 + 2.9026x - 4.9176$**]**

Points to Remember

Bisection Method

In this method, two points x_0 and x_1 are chosen such that $f(x_0)$ and $f(x_1)$ are of opposite signs. The first approximation to the root is

$$x_2 = \frac{x_0 + x_1}{2}$$

If $f(x_0)$and $f(x_2)$ are of opposite signs, the root lies between x_0 and x_2 and the next approximation x_3 is obtained as

$$x_3 = \frac{x_0 + x_2}{2}$$

This process is repeated till the root is obtained to the desired accuracy.

Regula Falsi Method

In this method, two points x_0 and x_1 are chosen such that $f(x_0)$ and $f(x_1)$ are of opposite signs.

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

which is an approximation to the root.

If $f(x_0)$ and $f(x_2)$ are of opposite signs, the root lies between x_0 and x_2, and the next approximation x_3 is obtained as

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

If the root lies between x_1 and x_2, the next approximation x_3 is obtained as

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

This process is repeated till the root is obtained to the desired accuracy.

Newton–Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The Newton–Raphson method has a quadratic convergence and the convergence is of the order 2.

Secant Method

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

The rate of convergence of the secant method is 1.618.

Successive Approximation Method

$$x_{n+1} = \phi(x_n), \qquad n = 0, 1, 2, \ldots$$

Descartes' Rule of Signs

The number of variations in the sign of the coefficients of a function $f(x)$ or $f(-x)$ refers to the sign changes in the coefficients from positive to negative or negative to positive in successive terms of the function.

The number of positive roots of $f(x) = 0$ is equal to the number of variations in the sign of $f(x)$ or to that number decreased by an even integer. The number of negative roots of $f(x) = 0$ is equal to the number of variations in the sign of $f(-x)$ or to that number decreased by an even integer.

Budan's Theorem

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ be a polynomial function with real coefficients $a_0, a_1, a_2, \ldots, a_n$. Let $v(c)$ be the number of variations of signs in the sequence $f(x), f'(x), f''(x), \ldots, f^{(n)}(x)$ when $x = c$, where c is any real number. The number of roots of $f(x)$ in the interval $[a, b]$, counted with their order of multiplicity is equal to

$$v(a) - v(b) - 2m, \text{ for some } m \in \mathbb{N}$$

i.e., the number of roots of $f(x)$ is equal to $v(a) - v(b)$ or $v(a) - v(b)$ decreased by an even integer.

Bairstow's Method

$$b_n + c_{n-1}\,\Delta p + c_{n-2}\,\Delta q = 0$$

$$b_{n-1} + c_{n-2}\,\Delta p + c_{n-3}\,\Delta q = 0$$

CHAPTER 3

Systems of Linear Algebraic Equations

Chapter Outline

3.1 INTRODUCTION

A system of m nonhomogenous linear equations in n variables $x_1, x_2, ..., x_n$ or simply a linear system, is a set of m linear equations, each in n variables. A linear system is represented by

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Writing these equations in matrix form,

$$A\mathbf{x} = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is called the *coefficient matrix* of order $m \times n$,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ is any vector of order } n \times 1$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \text{ is any vector of order } m \times 1$$

3.2 SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS

For a system of m linear equations in n variables, there are three possibilities of the solutions to the system:

(i) The system has a unique solution.
(ii) The system has infinite solutions.
(iii) The system has no solution.

When the system of linear equations has one or more solutions, the system is said to be consistent, otherwise it is inconsistent.

The matrix
$$[A : B] = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the *augmented matrix* of the given system of linear equations.

To solve a system of linear equations, elementary transformations are used to reduce the augmented matrix to echelon form.

3.3 ELEMENTARY TRANSFORMATIONS

Elementary transformation is any one of the following operations on a matrix.

(i) The interchange of any two rows (or columns)
(ii) The multiplication of the elements of any row (or column) by any nonzero number
(iii) The addition or subtraction of k times the elements of a row (or column) to the corresponding elements of another row (or column), where $k \neq 0$

Symbols to be used for elementary transformation:

(i) R_{ij}: Interchange of i^{th} and j^{th} row
(ii) kR_i: Multiplication of i^{th} row by a nonzero number k
(iii) $R_i + kR_j$: Addition of k times the j^{th} row to the i^{th} row

The corresponding column transformations are denoted by C_{ij}, kC_i, and $C_i + kC_j$ respectively.

3.3.1 Elementary Matrices

A matrix obtained from a unit matrix by subjecting it to any row or column transformation is called an elementary matrix.

3.3.2 Equivalence of Matrices

If B be an $m \times n$ matrix obtained from an $m \times n$ matrix by elementary transformation of A then A is equivalent to B. Symbolically, we can write $A \sim B$.

3.3.3 Echelon Form of a Matrix

A matrix A is said to be in echelon form if it satisfies the following properties:

(i) Every zero row of the matrix A occurs below a nonzero row.

(ii) In a nonzero row the first nonzero number from the left is 1. This is called a leading 1.

(iii) For each nonzero row, the leading 1 appears to the right of any leading 1 in preceding rows.

The following matrices are in echelon form:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

3.4 NUMERICAL METHODS FOR SOLUTION OF A SYSTEM OF LINEAR EQUATIONS

There are two methods to solve linear algebraic equations:

(i) Direct methods

(ii) Iterative methods

3.4.1 Direct Methods

Direct methods transform the original equations into equivalent equations that can be solved easily. The transformation of the original equations is carried out by applying elementary row transformations to the augmented matrix of the system of equations.

We will discuss two direct methods:

(i) Gauss elimination method

(ii) Gauss–Jordan method

3.4.2 Iterative Methods

The direct methods lead to exact solutions in many cases but are subject to errors due to roundoff and other factors. In the iterative method, an approximation to the true solution is assumed initially to start the method. By applying the method repeatedly, better and better approximations are obtained. For large systems, iterative methods are faster than direct methods and round-off errors are also smaller. Any error made at any stage of computation gets automatically corrected in the subsequent steps.

We will discuss two iterative methods.

(i) Gauss–Jacobi method

(ii) Gauss–Seidel method

3.5 GAUSS ELIMINATION METHOD

This method solves a given system of equations by transforming the augmented matrix to an echelon form. The corresponding linear system of equations is then solved for the unknowns by back substitution.

Consider the system of equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The matrix form of the system is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Reducing the augmented matrix to echelon form by using elementary row transformations,

$$[A : B] \xrightarrow[\text{row transformations}]{\text{elementary}} \left[\begin{array}{ccc:c} c_{11} & c_{12} & c_{13} & d_1 \\ 0 & c_{22} & c_{23} & d_2 \\ 0 & 0 & c_{33} & d_3 \end{array}\right]$$

The corresponding system of equations is

$$c_{11}x + c_{12}y + c_{13}z = d_1$$

$$c_{22}y + c_{23}z = d_2$$
$$c_{33}z = d_3$$

The solution of the system is obtained by solving these equations by back substitution.

Working Rule

(i) Write the matrix form of the system of equations.

(ii) Write the augmented matrix.

(iii) Obtain the echelon form of the augmented matrix by using elementary row transformations.

(iv) Write the corresponding linear system of equations from the echelon form.

(v) Solve the corresponding linear system of equations by back substitution.

Example 1

Solve the following system of equations:

$$\begin{aligned} x + 3y + 2z &= 5 \\ 2x + 4y - 6z &= -4 \\ x + 5y + 3z &= 10 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 10 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 1 & 3 & 2 & 5 \\ 2 & 4 & -6 & -4 \\ 1 & 5 & 3 & 10 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$R_2 - 2R_1, R_3 - R_1$$

$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & 3 & 2 & 5 \\ 0 & -2 & -10 & -14 \\ 0 & 2 & 1 & 5 \end{array}\right]$$

$$\left(-\frac{1}{2}\right)R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 3 & 2 & 5 \\ 0 & 1 & 5 & 7 \\ 0 & 2 & 1 & 5 \end{array}\right]$$

$$R_3 - 2R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 3 & 2 & 5 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & -9 & -9 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x + 3y + 2z &= 5 \\ y + 5z &= 7 \\ -9z &= -9 \end{aligned}$$

Solving these equations by back substitution,

$$\begin{aligned} &z = 1 \\ &y = 7 - 5z = 7 - 5(1) = 2 \\ &x = 5 - 3y - 2z = 5 - 3(2) - 2(1) = -3 \end{aligned}$$

Hence, the solution is

$$x = -3,\ y = 2,\ z = 1$$

Example 2

Solve the following system of equations:

$$\begin{aligned} 2x + y + z &= 10 \\ 3x + 2y + 3z &= 18 \\ x + 4y + 9z &= 16 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B]=\left[\begin{array}{ccc:c}2&1&1&10\\3&2&3&18\\1&4&9&16\end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$R_{13}$$

$$[A:B]\sim\left[\begin{array}{ccc:c}1&4&9&16\\3&2&3&18\\2&1&1&10\end{array}\right]$$

$$R_2-3R_1, R_3-2R_1$$

$$\sim\left[\begin{array}{ccc:c}1&4&9&16\\0&-10&-24&-30\\0&-7&-17&-22\end{array}\right]$$

$$\left(-\frac{1}{10}\right)R_2$$

$$\sim\left[\begin{array}{ccc:c}1&4&9&16\\0&1&\frac{24}{10}&3\\0&-7&-17&-22\end{array}\right]$$

$$R_3+7R_2$$

$$\sim\left[\begin{array}{ccc:c}1&4&9&16\\0&1&\frac{24}{10}&3\\0&0&-\frac{1}{5}&-1\end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned}x+4y+\quad 9z&=16\\ y+\frac{24}{10}z&=3\\ -\frac{1}{5}z&=-1\end{aligned}$$

Solving these equations by back substitution,

$$z = 5$$

$$y = 3 - \frac{24}{10}z = 3 - \frac{24}{10}(5) = -9$$

$$x = 16 - 4y - 9z = 16 - 4(-9) - 9(5) = 7$$

Hence, the solution is

$$x = 7,\ y = -9,\ z = 5$$

Example 3

Solve the following system of equations:

$$\begin{aligned} 6x - y - z &= 19 \\ 3x + 4y + z &= 26 \\ x + 2y + 6z &= 22 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 6 & -1 & -1 \\ 3 & 4 & 1 \\ 1 & 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 19 \\ 26 \\ 22 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[A : B\right] = \left[\begin{array}{ccc:c} 6 & -1 & -1 & 19 \\ 3 & 4 & 1 & 26 \\ 1 & 2 & 6 & 22 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$R_{13}$$

$$\left[A : B\right] \sim \left[\begin{array}{ccc:c} 1 & 2 & 6 & 22 \\ 3 & 4 & 1 & 26 \\ 6 & -1 & -1 & 19 \end{array}\right]$$

$$R_2 - 3R_1,\ R_3 - 6R_1$$

$$\sim\left[\begin{array}{ccc:c} 1 & 2 & 6 & 22 \\ 0 & -2 & -17 & -40 \\ 0 & -13 & -37 & -113 \end{array}\right]$$

$$\left(-\frac{1}{2}\right)R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 2 & 6 & 22 \\ 0 & 1 & \frac{17}{2} & 20 \\ 0 & -13 & -37 & -113 \end{array}\right]$$

$$R_3 + 13R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 2 & 6 & 22 \\ 0 & 1 & \frac{17}{2} & 20 \\ 0 & 0 & \frac{147}{2} & 147 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x + 2y + 6z &= 22 \\ y + \frac{17}{2}z &= 20 \\ \frac{147}{2}z &= 147 \end{aligned}$$

Solving these equations by back substitution,

$$z = 2$$

$$y = 20 - \frac{17}{2}z = 20 - \frac{17}{2}(2) = 3$$

$$x = 22 - 2y - 6z = 22 - 2(3) - 6(2) = 4$$

Hence, the solution is

$$x = 4,\ y = 3,\ z = 2$$

Example 4

Solve the following system of equations:

$$\begin{aligned} 5x + 5y + 2z &= 12 \\ 2x + 4y + 5z &= 2 \\ 39x + 43y + 45z &= 74 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 5 & 5 & 2 \\ 2 & 4 & 5 \\ 39 & 43 & 45 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \\ 74 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 5 & 5 & 2 & 12 \\ 2 & 4 & 5 & 2 \\ 39 & 43 & 45 & 74 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$\left(\frac{1}{5}\right)R_1$$

$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & 1 & \frac{2}{5} & \frac{12}{5} \\ 2 & 4 & 5 & 2 \\ 39 & 43 & 45 & 74 \end{array}\right]$$

$$R_2 - 2R_1, R_3 - 39R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & \frac{2}{5} & \frac{12}{5} \\ 0 & 2 & \frac{21}{5} & -\frac{14}{5} \\ 0 & 4 & \frac{147}{5} & -\frac{98}{5} \end{array}\right]$$

$$\left(\frac{1}{2}\right)R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 1 & \frac{2}{5} & \frac{12}{5} \\ 0 & 1 & \frac{21}{10} & -\frac{14}{10} \\ 0 & 4 & \frac{147}{5} & -\frac{98}{5} \end{array}\right]$$

$R_3 - 4R_2$

$$\sim\left[\begin{array}{ccc:c} 1 & 1 & \frac{2}{5} & \frac{12}{5} \\ 0 & 1 & \frac{21}{10} & -\frac{14}{10} \\ 0 & 0 & 21 & -14 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x + y + \frac{2}{5}z &= \frac{12}{5} \\ y + \frac{21}{10}z &= -\frac{14}{10} \\ 21z &= -14 \end{aligned}$$

Solving these equations by back substitution,

$$\begin{aligned} z &= -\frac{14}{21} = -\frac{2}{3} \\ y &= -\frac{14}{10} - \frac{21}{10}z = -\frac{14}{10} - \frac{21}{10}\left(-\frac{2}{3}\right) = 0 \\ x &= \frac{12}{5} - y - \frac{2}{5}z = \frac{12}{5} - \frac{2}{5}\left(-\frac{2}{3}\right) = \frac{8}{3} \end{aligned}$$

Hence, the solution is

$$x = \frac{8}{3},\ y = 0,\ z = -\frac{2}{3}$$

Example 5

Use the Gauss elimination method to solve the following equations:

$$\begin{aligned} x + 4y - z &= -5 \\ x + y - 6z &= -12 \\ 3x - y - z &= 4 \end{aligned}$$

[Summer 2015]

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 1 & 1 & -6 & -12 \\ 3 & -1 & -1 & 4 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$R_2 - R_1,\ R_3 - 3R_1$$

$$[A:B] \sim \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & -7 \\ 0 & -13 & 2 & 19 \end{array}\right]$$

$$\left(-\frac{1}{3}\right)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 0 & 1 & \frac{5}{3} & \frac{7}{3} \\ 0 & -13 & 2 & 19 \end{array}\right]$$

$$R_3 + 13R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 0 & 1 & \frac{5}{3} & \frac{7}{3} \\ 0 & 0 & \frac{71}{3} & \frac{148}{3} \end{array}\right]$$

The corresponding system of equations is

$$x + 4y - z = -5$$

$$y + \frac{5}{3}z = \frac{7}{3}$$

$$\frac{71}{3}z = \frac{148}{3}$$

Solving these equations by back substitution,

$$z=\frac{148}{71}$$

$$y=\frac{7}{3}-\frac{5}{3}z=\frac{7}{3}-\frac{5}{3}\left(\frac{148}{71}\right)=-\frac{81}{71}$$

$$x=-5-4y+z=-5-4\left(-\frac{81}{71}\right)+\frac{148}{71}=\frac{117}{71}$$

Hence, the solution is

$$x=\frac{117}{71},\quad y=-\frac{81}{71},\quad z=\frac{148}{71}$$

Example 6

Solve the following system of linear equations:

$$\begin{aligned}8y+2z&=-7\\3x+5y+2z&=8\\6x+2y+8z&=26\end{aligned}$$

[Summer 2014]

Solution

The matrix form of the system is

$$A\mathbf{x}=B$$

$$\begin{bmatrix}0&8&2\\3&5&2\\6&2&8\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-7\\8\\26\end{bmatrix}$$

The augmented matrix of the system is

$$[A:B]=\left[\begin{array}{ccc|c}0&8&2&-7\\3&5&2&8\\6&2&8&26\end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$\underset{\sim}{R_{12}}$$

$$[A:B]\sim\left[\begin{array}{ccc|c}3&5&2&8\\0&8&2&-7\\6&2&8&26\end{array}\right]$$

$$\left(\frac{1}{3}\right)R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{5}{3} & \frac{2}{3} & \frac{8}{3} \\ 0 & 8 & 2 & -7 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

$$R_3-6R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{5}{3} & \frac{2}{3} & \frac{8}{3} \\ 0 & 8 & 2 & -7 \\ 0 & -8 & 4 & 10 \end{array}\right]$$

$$\left(\frac{1}{8}\right)R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{5}{3} & \frac{2}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{4} & -\frac{7}{8} \\ 0 & -8 & 4 & 10 \end{array}\right]$$

$$R_3+8R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{5}{3} & \frac{2}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{4} & -\frac{7}{8} \\ 0 & 0 & 6 & 3 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x+\frac{5}{3}y+\frac{2}{3}z&=\frac{8}{3} \\ y+\frac{1}{4}z&=-\frac{7}{8} \\ 6z&=3 \end{aligned}$$

Solving these equations by back substitution,

$$z=\frac{1}{2}$$

$$y=-\frac{7}{8}-\frac{1}{4}z=-\frac{7}{8}-\frac{1}{4}\left(\frac{1}{2}\right)=-1$$

$$x=\frac{8}{3}-\frac{5}{3}y-\frac{2}{3}z=\frac{8}{3}-\frac{5}{3}(-1)-\frac{2}{3}\left(\frac{1}{2}\right)=4$$

Hence, the solution is

$$x = 4, \quad y = -1, \quad z = \frac{1}{2}$$

3.6 GAUSS ELIMINATION METHOD WITH PARTIAL PIVOTING

For a large system of linear equations, the Gaussian elimination method can involve a large number of arithmetic computations, each of which can produce rounding errors. This is due to the fact that every computation is dependent on previous results.

Consequently, an error in the early step will tend to propagate, i.e., it will cause errors in subsequent steps, and the final solution will become inaccurate. The rounding error can be reduced by the Gaussian elimination method with partial pivoting.

Consider the system of equations:

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

The matrix form of the system is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

For the partial pivoting process, the left column is searched for the largest absolute-value entry. This entry is called the *pivot*. The row interchange is performed, if necessary, to bring the pivot in the first row. The first row is divided by the pivot and elementary row operations are used to reduce the remaining entries in the first column to zero. The completion of these steps is called a *pass*. After performing the first pass, the first row and first column are ignored and the process is repeated on the remaining submatrix. This process is continued until the matrix is in the row echelon form.

The term *partial* in partial pivoting refers to the fact that in each pivot search, only entries in the left column of the matrix or submatrix are considered. This search can be extended to include every entry in the coefficient matrix or submatrix. The resulting method is called the *Gaussian elimination method with complete pivoting*. Generally, partial pivoting is preferred because complete pivoting becomes very complicated.

Example 1

Solve the following system of equations using partial pivoting by the Gauss elimination method:

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &= 6 \\ 4x_1 + 2x_2 + 3x_3 &= 4 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

[Summer 2015]

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 2 & 2 & 1 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 6 \\ 4 & 2 & 3 & 4 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

In the left column, 4 is the pivot because it is the entry that has the largest absolute value.

$$R_{12}$$

$$[A : B] \sim \left[\begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 2 & 2 & 1 & 6 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\left(\frac{1}{4}\right)R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{4} & 1 \\ 2 & 2 & 1 & 6 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 - 2R_1,\ R_3 - R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & \frac{1}{2} & \frac{3}{4} & 1 \\ 0 & 1 & -\frac{1}{2} & 4 \\ 0 & \frac{1}{2} & \frac{1}{4} & -1 \end{array}\right]$$

This completes the first pass. For the second pass, the pivot is 1 in the submatrix formed by deleting the first row and first column.

$$R_3 - \frac{1}{2}R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & \frac{1}{2} & \frac{3}{4} & 1 \\ 0 & 1 & -\frac{1}{2} & 4 \\ 0 & 0 & \frac{1}{2} & -3 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{3}{4}x_3 &= 1 \\ x_2 - \frac{1}{2}x_3 &= 4 \\ \frac{1}{2}x_3 &= -3 \end{aligned}$$

Solving these equations by back substitution,

$$\begin{aligned} x_3 &= -6 \\ x_2 &= 4 + \frac{1}{2}x_3 = 4 + \frac{1}{2}(-6) = 1 \\ x_1 &= 1 - \frac{1}{2}x_2 - \frac{3}{4}x_3 = 1 - \frac{1}{2}(1) - \frac{3}{4}(-6) = 5 \end{aligned}$$

Hence, the solution is

$$x_1 = 5, \qquad x_2 = 1, \qquad x_3 = -6$$

Example 2

Solve the following system of equations using the Gauss elimination method with partial pivoting.

$$\begin{aligned} x+\ y+\ z&=7 \\ 3x+3y+4z&=24 \\ 2x+\ y+3z&=16 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x}=B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 7 \\ 24 \\ 16 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B]=\left[\begin{array}{ccc:c} 1 & 1 & 1 & 7 \\ 3 & 3 & 4 & 24 \\ 2 & 1 & 3 & 16 \end{array}\right]$$

In the left column, 3 is the pivot because it is the entry that has largest absolute value.

$$R_{12}$$

$$[A:B]\sim\left[\begin{array}{ccc:c} 3 & 3 & 4 & 24 \\ 1 & 1 & 1 & 7 \\ 2 & 1 & 3 & 16 \end{array}\right]$$

$$\left(\frac{1}{3}\right)R_1$$

$$\sim\left[\begin{array}{ccc:c} 1 & 1 & \frac{4}{3} & 8 \\ 1 & 1 & 1 & 7 \\ 2 & 1 & 3 & 16 \end{array}\right]$$

$$R_2-R_1,\ R_3-2R_1$$

$$\sim\left[\begin{array}{ccc:c} 1 & 1 & \frac{4}{3} & 8 \\ 0 & 0 & -\frac{1}{3} & -1 \\ 0 & -1 & \frac{1}{3} & 0 \end{array}\right]$$

This completes the first pass. For the second pass, the pivot is –1 in the submatrix formed by deleting the first row and first column.

$$\overset{R_{23}}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & \frac{4}{3} & 8 \\ 0 & -1 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} & -1 \end{array}\right]$$

$$\overset{(-1)\,R_2}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & \frac{4}{3} & 8 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} & -1 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x+y+\frac{4}{3}z &= 8 \\ y-\frac{1}{3}z &= 0 \\ -\frac{1}{3}z &= -1 \end{aligned}$$

Solving these equations by back substitution,

$$\begin{aligned} z &= 3 \\ y &= \frac{1}{3}z = \frac{1}{3}(3) = 1 \\ x &= 8-y-\frac{4}{3}z = 8-1\frac{4}{3}(3) = 3 \end{aligned}$$

Hence, the solution is

$$x = 3, \qquad y = 1, \qquad z = 3$$

EXERCISE 3.1

Solve the following systems of equations by the Gauss elimination method:

1. $\begin{aligned} x - y + z &= 1 \\ -3x + 2y - 3z &= -6 \\ 2x - 5y + 4z &= 5 \end{aligned}$

$\left[\textbf{Ans.: } x = -2, y = 3, z = 6\right]$

2. $$\begin{aligned} x + 3y - 2z &= 5 \\ 2x + y - 3z &= 1 \\ 3x + 2y - z &= 6 \end{aligned}$$

$\left[\textbf{Ans.:}\ x = 1, y = 2, z = 1\right]$

3. $$\begin{aligned} 6x + 3y + 6z &= 30 \\ 2x + 3y + 3z &= 17 \\ x + 2y + 2z &= 11 \end{aligned}$$

$\left[\textbf{Ans.:}\ x = 1, y = 2, z = 3\right]$

4. $$\begin{aligned} 2x + y + z &= 4 \\ 3y - 3z &= 0 \\ -y + 2z &= 1 \end{aligned}$$

$\left[\textbf{Ans.:}\ x = 1, y = 1, z = 1\right]$

5. $$\begin{aligned} 2x + 2y + z &= 12 \\ 3x + 2y + 2z &= 8 \\ 5x + 10y - 8z &= 10 \end{aligned}$$

$\left[\textbf{Ans.:}\ x = -12.75, y = 14.375, z = 8.75\right]$

6. $$\begin{aligned} 3x + 4y + 5z &= 18 \\ 2x - y + 8z &= 13 \\ 5x - 2y + 7z &= 20 \end{aligned}$$

$\left[\textbf{Ans.:}\ x = 3, y = 1, z = 1\right]$

7. $$\begin{aligned} 2x + 6y - z &= -12 \\ 5x - y + z &= 11 \\ 4x - y + 3z &= 10 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x = \frac{113}{69}, y = -\frac{172}{69}, z = \frac{22}{69}\right]$$

3.7 GAUSS–JORDAN METHOD

This method is a modification of the Gauss elimination method. This method solves a given system of equations by transforming the coefficient matrix into a unit matrix.

Consider the system of equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The matrix form of the system is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Applying elementary row transformations to augmented matrix to reduce coefficient matrix to unit matrix,

$$[A : B] \xrightarrow[\text{row transformations}]{\text{elementary}} \left[\begin{array}{ccc:c} 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & d_3 \end{array}\right]$$

The corresponding system of equations is

$$x = d_1$$
$$y = d_2$$
$$z = d_3$$

Hence, the solution is

$$x = d_1,\ y = d_2,\ z = d_3$$

Working Rule

(i) Write the matrix form of the system of equations.
(ii) Write the augmented matrix.
(iii) Reduce the coefficient matrix to unit matrix by applying elementary row transformations to the augmented matrix.
(iv) Write the corresponding linear system of equations to obtain the solution.

Example 1

Solve the following system of equations:

$$\begin{aligned} x + 3y + 2z &= 17 \\ x + 2y + 3z &= 16 \\ 2x - y + 4z &= 13 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ 16 \\ 13 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 3 & 2 & 17 \\ 1 & 2 & 3 & 16 \\ 2 & -1 & 4 & 13 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$R_2 - R_1, R_3 - 2R_1$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 3 & 2 & 17 \\ 0 & -1 & 1 & -1 \\ 0 & -7 & 0 & -21 \end{array}\right]$$

$$(-1)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 2 & 17 \\ 0 & 1 & -1 & 1 \\ 0 & -7 & 0 & -21 \end{array}\right]$$

$$R_1 - 3R_2, R_3 + 7R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 5 & 14 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -7 & -14 \end{array}\right]$$

$$\left(-\frac{1}{7}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 5 & 14 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_1 - 5R_3, R_2 + R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The corresponding system of equations is

$$x = 4$$
$$y = 3$$
$$z = 2$$

Hence, the solution is

$$x = 4, y = 3, z = 2$$

Example 2

Solve the following system of equations:

$$\begin{aligned} 3x - 2y + 5z &= 2 \\ 4x + \ \ y + 2z &= 4 \\ 2x - \ \ y + 4z &= 7 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 3 & -2 & 5 \\ 4 & 1 & 2 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 3 & -2 & 5 & 2 \\ 4 & 1 & 2 & 4 \\ 2 & -1 & 4 & 7 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$R_1 - R_3$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & -5 \\ 4 & 1 & 2 & 4 \\ 2 & -1 & 4 & 7 \end{array}\right]$$

$$R_2 - 4R_1, R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & -5 \\ 0 & 5 & -2 & 24 \\ 0 & 1 & 2 & 17 \end{array}\right]$$

$$\underset{R_{23}}{\sim}\left[\begin{array}{ccc:c} 1 & -1 & 1 & -5 \\ 0 & 1 & 2 & 17 \\ 0 & 5 & -2 & 24 \end{array}\right]$$

$$\underset{R_1+R_2,\,R_3-\,5R_2}{\sim}\left[\begin{array}{ccc:c} 1 & 0 & 3 & 12 \\ 0 & 1 & 2 & 17 \\ 0 & 0 & -12 & -61 \end{array}\right]$$

$$\underset{\left(-\frac{1}{12}\right)R_3}{\sim}\left[\begin{array}{ccc:c} 1 & 0 & 3 & 12 \\ 0 & 1 & 2 & 17 \\ 0 & 0 & 1 & \frac{61}{12} \end{array}\right]$$

$$\underset{R_1-\,3R_3,\,R_2-\,2R_3}{\sim}\left[\begin{array}{ccc:c} 1 & 0 & 0 & -\frac{13}{4} \\ 0 & 1 & 0 & \frac{41}{6} \\ 0 & 0 & 1 & \frac{61}{12} \end{array}\right]$$

The corresponding system of equations is

$$x = -\frac{13}{4}$$

$$y = \frac{41}{6}$$

$$z = \frac{61}{12}$$

Hence, the solution is

$$x = -\frac{13}{4},\ y = \frac{41}{6},\ z = \frac{61}{12}$$

Example 3

Solve the following system of equations:

$$\begin{aligned} x - 2y \quad\quad &= -\ 4 \\ -\ 5y + \ z &= -\ 9 \\ 4x \quad\quad - 3z &= -\ 10 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -5 & 1 \\ 4 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -9 \\ -10 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 1 & -2 & 0 & -4 \\ 0 & -5 & 1 & -9 \\ 4 & 0 & -3 & -10 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$R_3 - 4R_1$$

$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & -2 & 0 & -4 \\ 0 & -5 & 1 & -9 \\ 0 & 8 & -3 & 6 \end{array}\right]$$

$$\left(-\frac{1}{5}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & -2 & 0 & -4 \\ 0 & 1 & -\frac{1}{5} & \frac{9}{5} \\ 0 & 8 & -3 & 6 \end{array}\right]$$

$$R_1 + 2R_2, R_3 - 8R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -\frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{9}{5} \\ 0 & 0 & -\frac{7}{5} & -\frac{42}{5} \end{array}\right]$$

$$\left(-\frac{5}{7}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -\frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{9}{5} \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$R_1 + \left(\frac{2}{5}\right)R_3,\ R_2 + \left(\frac{1}{5}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

The corresponding system of equations is

$$x = 2$$
$$y = 3$$
$$z = 6$$

Hence, the solution is

$$x = 2,\ y = 3,\ z = 6$$

Example 4

Solve the following system of equations:

$$\begin{aligned} 2x - 6y + 8z &= 24 \\ 5x + 4y - 3z &= 2 \\ 3x + y + 2z &= 16 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 2 & -6 & 8 & 24 \\ 5 & 4 & -3 & 2 \\ 3 & 1 & 2 & 16 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$\left(\frac{1}{2}\right)R_1$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & -3 & 4 & 12 \\ 5 & 4 & -3 & 2 \\ 3 & 1 & 2 & 16 \end{array}\right]$$

$$R_2 - 5R_1, R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & -3 & 4 & 12 \\ 0 & 19 & -23 & -58 \\ 0 & 10 & -10 & -20 \end{array}\right]$$

$$\left(\frac{1}{19}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & -3 & 4 & 12 \\ 0 & 1 & -\frac{23}{19} & -\frac{58}{19} \\ 0 & 10 & -10 & -20 \end{array}\right]$$

$$R_1 + 3R_2, R_3 - 10R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & \frac{7}{19} & \frac{54}{19} \\ 0 & 1 & -\frac{23}{19} & -\frac{58}{19} \\ 0 & 0 & \frac{40}{19} & \frac{200}{19} \end{array}\right]$$

$$\left(\frac{19}{40}\right)R_3$$

$$\sim\left[\begin{array}{ccc:c} 1 & 0 & \frac{7}{19} & \frac{54}{19} \\ 0 & 1 & -\frac{23}{19} & -\frac{58}{19} \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$R_1-\left(\frac{7}{19}\right)R_3, R_2+\left(\frac{23}{19}\right)R_3$$

$$\sim\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

The corresponding system of equations is

$$x = 1$$
$$y = 3$$
$$z = 5$$

Hence, the solution is

$$x = 1, y = 3, z = 5$$

Example 5

Solve the following system of linear equations:

$$\begin{aligned} 2x + 5y - 3z &= 1 \\ 5x + y + 4z &= 2 \\ 7x + 3y + z &= 4 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 2 & 5 & -3 \\ 5 & 1 & 4 \\ 7 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B]=\left[\begin{array}{ccc:c} 2 & 5 & -3 & 1 \\ 5 & 1 & 4 & 2 \\ 7 & 3 & 1 & 4 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$[A:B] \underset{\left(\frac{1}{2}\right)R_1}{\sim} \left[\begin{array}{ccc:c} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 5 & 1 & 4 & 2 \\ 7 & 3 & 1 & 4 \end{array}\right]$$

$$R_2 - 5R_1,\ \ R_3 - 7R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{23}{2} & \frac{23}{2} & -\frac{1}{2} \\ 0 & -\frac{29}{2} & \frac{23}{2} & \frac{1}{2} \end{array}\right]$$

$$\left(-\frac{2}{23}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & -1 & \frac{1}{23} \\ 0 & -\frac{29}{2} & \frac{23}{2} & \frac{1}{2} \end{array}\right]$$

$$R_1 - \frac{5}{2}R_2,\ R_3 + \frac{29}{2}R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & \frac{9}{23} \\ 0 & 1 & -1 & \frac{1}{23} \\ 0 & 0 & -3 & \frac{26}{23} \end{array}\right]$$

$$\left(-\frac{1}{3}\right)R_3$$

$$\sim\left[\begin{array}{ccc:c} 1 & 0 & 1 & \frac{9}{23} \\ 0 & 1 & -1 & \frac{1}{23} \\ 0 & 0 & 1 & -\frac{26}{69} \end{array}\right]$$

$$R_1 - R_3,\ R_2 + R_3$$

$$\sim\left[\begin{array}{ccc:c} 1 & 0 & 0 & \frac{53}{69} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{26}{69} \end{array}\right]$$

The corresponding system of equations is

$$x = \frac{53}{69}$$

$$y = -\frac{1}{3}$$

$$z = -\frac{26}{69}$$

Hence, the solution is

$$x = \frac{53}{69},\ y = -\frac{1}{3},\ z = -\frac{26}{69}$$

EXERCISE 3.2

Solve the following systems of equations by the Gauss–Jordan method:

1. $x + 2y + z = 3$
$2x + 3y + 3z = 10$
$3x - y + 2z = 13$

$$\left[\textbf{Ans.: } x = 2, y = -1, z = 3\right]$$

2. $$\begin{aligned} 2x + 3y - z &= 5 \\ 4x + 4y - 3z &= 3 \\ 2x - 3y + 2z &= 2 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x = 1, y = 2, z = 3\right]$$

3. $$\begin{aligned} 10x + y + z &= 12 \\ 2x + 10y + z &= 13 \\ x + y + 5z &= 7 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x = 1, y = 1, z = 1\right]$$

4. $$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 11 \\ 4x_1 - 2x_2 + 3x_3 &= 8 \\ -2x_1 + 2x_2 - x_3 &= -6 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x_1 = 3, x_2 = -1, x_3 = -2\right]$$

5. $$\begin{aligned} 2x_1 + 6x_2 + x_3 &= 7 \\ x_1 + 2x_2 - x_3 &= -1 \\ 5x_1 + 7x_2 - 4x_3 &= 9 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x_1 = 10, x_2 = -3, x_3 = 5\right]$$

6. $$\begin{aligned} 2x + y + 4z &= 12 \\ 8x - 3y + 2z &= 20 \\ 4x + 11y - z &= 33 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x = 3, y = 2, z = 1\right]$$

7. $$\begin{aligned} x + y + z &= 1 \\ 4x + 3y - z &= 6 \\ 3x + 5y + 3z &= 4 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x = 1, y = \frac{1}{2}, z = -\frac{1}{2}\right]$$

3.8 GAUSS—JACOBI METHOD

This method is applicable to the system of equations in which leading diagonal elements of the coefficient matrix are dominant (large in magnitude) in their respective rows.

Consider the system of equations

$$\left.\begin{aligned} a_{11}\ x + a_{12}\ y + a_{13}\ z &= b_1 \\ a_{21}\ x + a_{22}\ y + a_{23}\ z &= b_2 \\ a_{31}\ x + a_{32}\ y + a_{33}\ z &= b_3 \end{aligned}\right\} \quad \ldots (3.1)$$

where $|a_{11}|$, $|a_{22}|$, $|a_{33}|$ are large as compared to the other coefficients in the corresponding row and satisfy the condition of convergence as follows:

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

Rewriting the equations for x, y, and z respectively,

$$\left.\begin{aligned} x &= \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z) \\ y &= \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z) \\ z &= \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y) \end{aligned}\right\} \qquad \dots (3.2)$$

Iteration 1

Assuming $x = x_0$, $y = y_0$, $z = z_0$ as initial approximation and substituting in Eq. (3.2),

$$x_1 = \frac{1}{a_{11}}\left(b_1 - a_{12}y_0 - a_{13}z_0\right)$$

$$y_1 = \frac{1}{a_{22}}\left(b_2 - a_{21}x_0 - a_{23}z_0\right)$$

$$z_1 = \frac{1}{a_{33}}\left(b_3 - a_{31}x_0 - a_{32}y_0\right)$$

Again substituting these values of x, y, z in Eq. (3.2), the next approximation is obtained.

The above iteration process is continued until two successive approximations are nearly equal.

Working Rule

(i) Arrange the equations in such a manner that the leading diagonal elements are large in magnitude in their respective rows satisfying the conditions

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.

(iii) Start the iteration 1 by assuming the initial values of (x, y, z) as (x_0, y_0, z_0) and obtain (x_1, y_1, z_1).

(iv) Start the iteration 2 by putting $x = x_1$, $y = y_1$, $z = z_1$ in equations of x, y, z and obtain (x_2, y_2, z_2).

(v) The above process is repeated for the next iterations and it continues until two successive approximations are nearly equal.

Example 1

Solve the following system of equations:

$$\begin{aligned} 6x + 2y - z &= 4 \\ x + 5y + z &= 3 \\ 2x + y + 4z &= 27 \end{aligned}$$

Solution

Rewriting the equations,

$$\left.\begin{aligned} x &= \frac{1}{6}(4 - 2y + z) \\ y &= \frac{1}{5}(3 - x - z) \\ z &= \frac{1}{4}(27 - 2x - y) \end{aligned}\right\} \qquad \ldots (1)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and putting in Eq. (1),

$$x_1 = \frac{2}{3} = 0.67$$

$$y_1 = \frac{3}{5} = 0.6$$

$$z_1 = \frac{27}{4} = 6.75$$

Iteration 2: Putting x_1, y_1, z_1 in Eq. (1),

$$x_2 = \frac{1}{6}\left[4 - 2(0.6) + 6.75\right] = 1.59$$

$$y_2 = \frac{1}{5}\left[3 - 0.67 - 6.75\right] = -0.884$$

$$z_2 = \frac{1}{4}\left[27 - 2(0.67) - 0.6\right] = 6.265$$

Iteration 3: Putting x_2, y_2, z_2 in Eq. (1),

$$x_3 = \frac{1}{6}\left[4 - 2(-0.884) + 6.265\right] = 2.005$$

$$y_3 = \frac{1}{5}\left[3 - 1.59 - 6.265\right] = -0.971$$

$$z_3 = \frac{1}{4}\left[27 - 2(1.59) - (-0.884)\right] = 6.176$$

Iteration 4: Putting x_3, y_3, z_3 in Eq. (1),

$$x_4 = \frac{1}{6}\left[4 - 2(-0.971) + 6.176\right] = 2.01$$

$$y_4 = \frac{1}{5}\left[3 - 2.005 - 6.176\right] = -1.03$$

$$z_4 = \frac{1}{4}\left[27 - 2(2.005) - (-0.971)\right] = 5.99$$

Iteration 5: Putting x_4, y_4, z_4 in Eq. (1),

$$x_5 = \frac{1}{6}\left[4 - 2(-1.03) + 5.99\right] = 2.00$$

$$y_5 = \frac{1}{5}\left[3 - 2.01 - 5.99\right] = -1.00$$

$$z_5 = \frac{1}{4}\left[27 - 2(2.01) - (-1.03)\right] = 6.00$$

Since the fourth and fifth iteration values are nearly equal, the approximate solution is

$$x = 2,\ y = -1,\ z = 6$$

Example 2

Solve the following system of equations:

$$\begin{aligned} 8x - y + 2z &= 13 \\ x - 10y + 3z &= 17 \\ 3x + 2y + 12z &= 25 \end{aligned}$$

Solution

Since absolute values of all diagonal elements are large as compared to absolute values of other coefficients, rewriting the equations,

$$\left.\begin{aligned} x &= \frac{1}{8}(13 + y - 2z) \\ y &= -\frac{1}{10}(17 - x - 3z) \\ z &= \frac{1}{12}(25 - 3x - 2y) \end{aligned}\right\} \quad \ldots(1)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as first approximation and putting in Eq. (1),

$$x_1 = \frac{13}{8} = 1.625$$

$$y_1 = -\frac{17}{10} = -1.7$$

$$z_1 = \frac{25}{12} = 2.08$$

Iteration 2: Putting x_1, y_1, z_1 in Eq. (1),

$$x_2 = \frac{1}{8}\left[13 - 1.7 - 2(2.08)\right] = 0.8925$$

$$y_2 = -\frac{1}{10}\left[17 - 1.625 - 3(2.08)\right] = -0.9135$$

$$z_2 = \frac{1}{12}\left[25 - 3(1.625) - 2(-1.7)\right] = 1.9604$$

Iteration 3: Putting x_2, y_2, z_2 in Eq. (1),

$$x_3 = \frac{1}{8}\left[13 - 0.9135 - 2(1.9604)\right] = 1.0207$$

$$y_3 = -\frac{1}{10}\left[17 - 0.8925 - 3(1.9604)\right] = -1.0226$$

$$z_3 = \frac{1}{12}\left[25 - 3(0.8925) - 2(-0.9135)\right] = 2.0124$$

Iteration 4: Putting x_3, y_3, z_3 in Eq. (1),

$$x_4 = \frac{1}{8}\left[13 - 1.0226 - 2(2.0124)\right] = 0.9941$$

$$y_4 = -\frac{1}{10}\left[17 - 1.0207 - 3(2.0124)\right] = -0.9942$$

$$z_4 = \frac{1}{12}\left[25 - 3(1.0207) - 2(-1.0226)\right] = 1.9985$$

Since the third and fourth iteration values are nearly equal, the approximate solution is

$$x = 1, y = -1, z = 2$$

The above method can also be represented in tabular form as follows:

Iteration number	$x = \frac{1}{8}(13 + y - 2z)$	$y = -\frac{1}{10}(17 - x - 3z)$	$z = \frac{1}{12}(25 - 3x - 2y)$
1	$x_0 = 0$ $x_1 = 1.625$	$y_0 = 0$ $y_1 = -1.7$	$z_0 = 0$ $z_1 = 2.08$
2	$x_2 = 0.8925$	$y_2 = -0.9135$	$z_2 = 1.9604$
3	$x_3 = 1.0207$	$y_3 = -1.0226$	$z_3 = 2.0124$
4	$x_4 = 0.9941$	$y_4 = -0.9942$	$z_4 = 1.9985$

EXERCISE 3.3

Solve the following system of equations by using the Gauss–Jacobi method:

1. $4x + y + 3z = 17$
$x + 5y + z = 14$
$2x - y + 8z = 12$

[**Ans.:** $x = 3, y = 2, z = 1$]

2. $10x + y + 2z = 13$
$2x + 10y + 3z = 15$
$x + 3y + 10z = 14$

[**Ans.:** $x = 1, y = 1, z = 1$]

3. $10x - 2y - 3z = 205$
$2x - 10y + 2z = -154$
$2x + y - 10z = -120$

[**Ans.:** $x = 32, y = 26, z = 21$]

4. $12x + 2y + z = 27$
$2x + 15y - 3z = 16$
$2x - 3y + 25z = 23$

[**Ans.:** $x = 2.0148, y = 0.9731, z = 0.8756$]

5. $$\begin{aligned} 20x + y - 2z &= 17 \\ 3x + 20y - z &= -18 \\ 2x - 3y + 20z &= 25 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x = 1, y = -1, z = 1\right]$$

6. $$\begin{aligned} 10x - 5y - 2z &= 3 \\ 4x - 10y + 3z &= -3 \\ x + 6y + 10z &= -3 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x = 0.342, y = 0.285, z = -0.505\right]$$

7. $$\begin{aligned} 8x - 3y + 2z &= 20 \\ 4x + 11y - z &= 33 \\ 6x + 3y + 12z &= 35 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x = 3.0168, y = 1.9859, z = 0.9118\right]$$

8. $$\begin{aligned} x + y + 54z &= 110 \\ 27x + 6y - z &= 85 \\ 6x + 15y + 2z &= 72 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x = 2.425, y = 3.573, z = 1.926\right]$$

3.9 GAUSS—SIEDEL METHOD

This method is applicable to the system of equations in which leading diagonal elements of the coefficient matrix are dominant (large in magnitude) in their respective rows.

Consider the system of equations

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \qquad \ldots(3.3)$$

where $|a_{11}|, |a_{22}|, |a_{33}|$ are large as compared to the other coefficients in the corresponding row and satisfy the condition of convergence as follows:

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

Rewriting the equations for x, y, and z respectively,

$$\left.\begin{aligned} x &= \frac{1}{a_{11}}\left(b_1 - a_{12}y - a_{13}z\right) \\ y &= \frac{1}{a_{22}}\left(b_2 - a_{21}x - a_{23}z\right) \\ z &= \frac{1}{a_{33}}\left(b_3 - a_{31}x - a_{32}y\right) \end{aligned}\right\} \quad \ldots(3.4)$$

Iteration 1

Assuming $x = x_0$, $y = y_0$, $z = z_0$ as initial approximations and substituting in the equation of x,

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}y_0 - a_{13}z_0)$$

Now, substituting $x = x_1$, $z = z_0$ in the equation of y,

$$y_1 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}z_0)$$

Substituting $x = x_1$, $y = y_1$ in the equation of z,

$$z_1 = \frac{1}{a_{33}}(b_3 - a_{31}x_1 - a_{32}y)$$

Iteration 2

Substituting $y = y_1$, $z = z_1$ in the equation of x,

$$x_2 = \frac{1}{a_{11}}(b_1 - a_{12}y_1 - a_{13}z_1)$$

Substituting $x = x_2$, $z = z_1$ in the equation of y,

$$y_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_2 - a_{23}z_1)$$

Substituting $x = x_2$, $y = y_2$ in the equation of z,

$$z_2 = \frac{1}{a_{33}}(b_3 - a_{31}x_2 - a_{32}y_2)$$

The above iteration process is continued until two successive approximations are nearly equal.

Working Rule

(i) Arrange the equations in such a manner that the leading diagonal elements are large in magnitude in their respective rows such that

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.

(iii) Start the iteration 1 by assuming the initial values of (x, y, z) as (x_0, y_0, z_0).

(iv) In the iteration 1, put $y = y_0$, $z = z_0$ in the equation of x to obtain x_1, put $x = x_1$, $z = z_0$ in the equation of y to obtain y_1, put $x = x_1$, $y = y_1$ in the equation of z to obtain z_1.

(v) The above process is repeated for the next iterations and it continues until two successive approximations are nearly equal.

Example 1

Solve the following system of equations:

$$\begin{aligned} 3x - 0.1y - 0.2z &= 7.85 \\ 0.1x + 7y - 0.3z &= -19.3 \\ 0.3x - 0.2y + 10z &= 71.4 \end{aligned}$$

Solution

Since diagonal elements are largest, the Gauss–Siedel method can be applied. Rewriting the equations.

$$\left.\begin{aligned} x &= \frac{1}{3}(7.85 + 0.1y + 0.2z) \\ y &= \frac{1}{7}(-19.3 - 0.1x + 0.3z) \\ z &= \frac{1}{10}(71.4 - 0.3x + 0.2y) \end{aligned}\right\} \qquad \ldots(1)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{3}(7.85) = 2.6167$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{7}(-19.3 - 0.1x_1 + 0.3z_0) \\ &= \frac{1}{7}\left[-19.3 - 0.1(2.6167) + 0.3(0)\right] \\ &= -2.7945 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$z_1 = \frac{1}{10}(71.4 - 0.3x_1 + 0.2y_1)$$

$$= \frac{1}{10}[71.4 - 0.3(2.6167) + 0.2(-2.7945)]$$

$$= 7.0056$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$x_2 = \frac{1}{3}(7.85 + 0.1y_1 + 0.2z_1)$$

$$= \frac{1}{3}[7.85 + 0.1(-2.7945) + 0.2(7.0056)]$$

$$= 2.9906$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$y_2 = \frac{1}{7}(-19.3 - 0.1x_2 + 0.3z_1)$$

$$= \frac{1}{7}[-19.3 - 0.1(2.9906) + 0.3(7.0056)]$$

$$= -2.4996$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$z_2 = \frac{1}{10}(71.4 - 0.3x_2 + 0.2y_2)$$

$$= \frac{1}{10}[71.4 - 0.3(2.9906) + 0.2(-2.4996)]$$

$$= 7.0003$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$x_3 = \frac{1}{3}(7.85 + 0.1y_2 + 0.2z_2)$$

$$= \frac{1}{3}[7.85 + 0.1(-2.4996) + 0.2(7.0003)]$$

$$= 3.000$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$y_3 = \frac{1}{7}(-19.3 - 0.1x_3 + 0.3z_2)$$

$$= \frac{1}{7}[-19.3 - 0.1(3) + 0.3(7.0003)]$$

$$= -2.4999$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$z_3 = \frac{1}{10}(71.4 - 0.3x_3 + 0.2y_3)$$

$$= \frac{1}{10}[71.4 - 0.3(3) + 0.2(-2.4999)]$$

$$= 7.0000$$

Since the second and third iteration values are nearly equal, the approximate solution is

$$x = 3,\ y = -2.5,\ z = 7$$

Example 2

Solve the following system of equations:

$$\begin{aligned} 5x + y - z &= 10 \\ 2x + 4y + z &= 14 \\ x + y + 8z &= 20 \end{aligned}$$

Solution

Since diagonal elements are largest, the Gauss–Siedel method can be applied.
Rewriting the equations,

$$x = \frac{1}{5}(10 - y + z)$$

$$y = \frac{1}{4}(14 - 2x - z)$$

$$z = \frac{1}{8}(20 - x - y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{5}(10) = 2$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$y_1 = \frac{1}{4}(14 - 2x_1 - z_0)$$

$$= \frac{1}{4}[14 - 2(2) - 0]$$

$$= 2.5$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= \frac{1}{8}(20 - x_1 - y_1) \\ &= \frac{1}{8}(20 - 2 - 2.5) \\ &= 1.9375 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{5}(10 - y_1 + z_1) \\ &= \frac{1}{5}(10 - 2.5 + 1.9375) \\ &= 1.8875 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$\begin{aligned} y_2 &= \frac{1}{4}(14 - 2x_2 - z_1) \\ &= \frac{1}{4}[14 - 2(1.8875) - 1.9375] \\ &= 2.0719 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned} z_2 &= \frac{1}{8}(20 - x_2 - y_2) \\ &= \frac{1}{8}(20 - 1.8875 - 2.0719) \\ &= 2.0050 \end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$\begin{aligned} x_3 &= \frac{1}{5}(10 - y_2 + z_2) \\ &= \frac{1}{5}(10 - 2.0719 + 2.0050) \\ &= 1.9866 \end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned} y_3 &= \frac{1}{4}(14 - 2x_3 - z_2) \\ &= \frac{1}{4}[14 - 2(1.9866) - 2.005] \\ &= 2.0055 \end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$\begin{aligned} z_3 &= \frac{1}{8}(20 - x_3 - y_3) \\ &= \frac{1}{8}(20 - 1.9866 - 2.0055) \\ &= 2.0009 \end{aligned}$$

Iteration 4: Putting $y = y_3$, $z = z_3$ in the equation of x,

$$\begin{aligned} x_4 &= \frac{1}{5}(10 - y_3 + z_3) \\ &= \frac{1}{5}(10 - 2.0055 + 2.0009) \\ &= 1.9991 \end{aligned}$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$\begin{aligned} y_4 &= \frac{1}{4}(14 - 2x_4 - z_3) \\ &= \frac{1}{4}\left[14 - 2(1.9991) - 2.0009\right] \\ &= 2.0002 \end{aligned}$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$\begin{aligned} z_4 &= \frac{1}{8}(20 - x_4 - y_4) \\ &= \frac{1}{8}(20 - 1.9991 - 2.0002) \\ &= 2.0001 \end{aligned}$$

Since the third and fourth iteration values are nearly equal, the approximate solution is

$$x = 2,\ y = 2,\ z = 2$$

Example 3

Solve the following system of linear equations:

$$\begin{aligned} 8x + y + z &= 5 \\ x + 8y + z &= 5 \\ x + y + 8z &= 5 \end{aligned}$$

[Summer 2015, Winter 2013]

Solution

Since diagonal elements are largest, the Gauss–Seidel method can be applied.
Rewriting the equations,

$$x = \frac{1}{8}(5 - y - z)$$

$$y = \frac{1}{8}(5 - x - z)$$

$$z = \frac{1}{8}(5 - x - y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x.

$$x_1 = \frac{1}{8}(5) = 0.625$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{8}(5 - x_1 - z_0) \\ &= \frac{1}{8}(5 - 0.625 - 0) \\ &= 0.5469 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= \frac{1}{8}(5 - x_1 - y_1) \\ &= \frac{1}{8}(5 - 0.625 - 0.5469) \\ &= 0.4785 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{8}(5 - y_1 - z_1) \\ &= \frac{1}{8}(5 - 0.5469 - 0.4785) \\ &= 0.4968 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$\begin{aligned} y_2 &= \frac{1}{8}(5 - x_2 - z_1) \\ &= \frac{1}{8}(5 - 0.4968 - 0.4785) \\ &= 0.5031 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned} z_2 &= \frac{1}{8}(5 - x_2 - y_2) \\ &= \frac{1}{8}(5 - 0.4968 - 0.5031) \\ &= 0.5 \end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation x,

$$\begin{aligned} x_3 &= \frac{1}{8}(5 - y_2 - z_2) \\ &= \frac{1}{8}(5 - 0.5031 - 0.5) \\ &= 0.4996 \end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned} y_3 &= \frac{1}{8}(5 - x_3 - z_2) \\ &= \frac{1}{8}(5 - 0.4996 - 0.5) \\ &= 0.5001 \end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$\begin{aligned} z_3 &= \frac{1}{8}(5 - x_3 - y_3) \\ &= \frac{1}{8}(5 - 0.4996 - 0.5001) \\ &= 0.5 \end{aligned}$$

Since the second and third iteration values are nearly equal, the approximate solution is

$$x = 0.5,\ y = 0.5,\ z = 0.5$$

Example 4

Use the Gauss–Siedel method to solve

$$\begin{aligned} 6x + y + z &= 105 \\ 4x + 8y + 3z &= 155 \\ 5x + 4y - 10z &= 65 \end{aligned}$$

[Summer 2015]

Solution

Since diagonal elements are largest, the Gauss–Seidel method can be applied.
Rewriting the equations,

$$x = \frac{1}{6}(105 - y - z)$$

$$y = \frac{1}{8}(155 - 4x - 3z)$$

$$z = -\frac{1}{10}(65 - 5x - 4y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{6}(105) = 17.5$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{8}(155 - 4x_1 - 3z_0) \\ &= \frac{1}{8}[155 - 4(17.5) - 3(0)] \\ &= 10.625 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= -\frac{1}{10}(65 - 5x_1 - 4y_1) \\ &= -\frac{1}{10}[65 - 5(17.5) - 4(10.625)] \\ &= 6.5 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{6}(105 - y_1 - z_1) \\ &= \frac{1}{6}(105 - 10.625 - 6.5) \\ &= 14.6458 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$\begin{aligned} y_2 &= \frac{1}{8}(155 - 4x_2 - 3z_1) \\ &= \frac{1}{8}[155 - 4(14.6458) - 3(6.5)] \\ &= 9.6146 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned}
z_2 &= -\frac{1}{10}(65 - 5x_2 - 4y_2) \\
&= -\frac{1}{10}[65 - 5(14.6458) - 4(9.6146)] \\
&= 4.6687
\end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$\begin{aligned}
x_3 &= \frac{1}{6}(105 - y_2 - z_2) \\
&= \frac{1}{6}(105 - 9.6146 - 4.6687) \\
&= 15.1195
\end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned}
y_3 &= \frac{1}{8}(155 - 4x_3 - 3z_2) \\
&= \frac{1}{8}[155 - 4(15.1195) - 3(4.6687)] \\
&= 10.0645
\end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$\begin{aligned}
z_3 &= -\frac{1}{10}(65 - 5x_3 - 4y_3) \\
&= -\frac{1}{10}[65 - 5(15.1195) - 4(10.0645)] \\
&= 5.0856
\end{aligned}$$

Iteration 4: Putting $y = y_3$, $z = z_3$ in the equation of y,

$$\begin{aligned}
x_4 &= \frac{1}{6}(105 - y_3 - z_3) \\
&= \frac{1}{6}(105 - 10.0645 - 5.0856) \\
&= 14.975
\end{aligned}$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$\begin{aligned}
y_4 &= \frac{1}{8}(155 - 4x_4 - 3z_3) \\
&= \frac{1}{8}[155 - 4(14.975) - 3(5.0856)] \\
&= 9.9804
\end{aligned}$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$\begin{aligned} z_4 &= -\frac{1}{10}(65 - 5x_4 - 4y_4) \\ &= -\frac{1}{10}[65 - 5(14.975) - 4(9.9804)] \\ &= 4.9797 \end{aligned}$$

Iteration 5: Putting $y = y_4$, $z = z_4$ in the equation of x,

$$\begin{aligned} x_5 &= \frac{1}{6}(105 - y_4 - z_4) \\ &= \frac{1}{6}(105 - 9.9804 - 4.9797) \\ &= 15.0067 \end{aligned}$$

Putting $x = x_5$, $z = z_4$ in the equation of y,

$$\begin{aligned} y_5 &= \frac{1}{8}(155 - 4x_5 - 3z_4) \\ &= \frac{1}{8}[155 - 4(15.0067) - 3(4.9797)] \\ &= 10.0043 \end{aligned}$$

Putting $x = x_5$, $y = y_5$ in the equation of z,

$$\begin{aligned} z_5 &= -\frac{1}{10}(65 - 5x_5 - 4y_5) \\ &= -\frac{1}{10}[65 - 5(15.0067) - 4(10.0043)] \\ &= 5.0051 \end{aligned}$$

Since the fourth and fifth iteration values are nearly equal, the approximate solution is

$$x = 15,\ y = 10,\ z = 5$$

Example 5

Solve the following system of equations:

$$\begin{aligned} 25x + 2y - 3z &= 48 \\ 3x + 27y - 2z &= 56 \\ x + 2y + 23z &= 52 \end{aligned}$$

starting with (1, 1, 0).

Solution

Since diagonal elements are largest, the Gauss–Siedel method can be applied.
Rewriting the equations,

$$x = \frac{1}{25}(48 - 2y + 3z)$$

$$y = \frac{1}{27}(56 - 3x + 2z)$$

$$z = \frac{1}{23}(52 - x - 2y)$$

Iteration 1: It is given that $x_0 = 1$, $y_0 = 1$, $z_0 = 0$. Putting $y = y_0$, $z = z_0$ in the equation of x,

$$\begin{aligned} x_1 &= \frac{1}{25}(48 - 2y_0 + 3z_0) \\ &= \frac{1}{25}[48 - 2(1) + 3(0)] \\ &= 1.84 \end{aligned}$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{27}(56 - 3x_1 + 2z_0) \\ &= \frac{1}{27}[56 - 3(1.84) + 2(0)] \\ &= 1.8696 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= \frac{1}{23}(52 - x_1 - 2y_1) \\ &= \frac{1}{23}[52 - 1.84 - 2(1.8696)] \\ &= 2.0183 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$, in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{25}(48 - 2y_1 + 3z_1) \\ &= \frac{1}{25}[48 - 2(1.8696) + 3(2.0183)] \\ &= 2.0126 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$y_2 = \frac{1}{27}(56 - 3x_2 + 2z_1)$$

$$= \frac{1}{27}\left[56 - 3(2.0126) + 2(2.0183)\right]$$
$$= 1.9999$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$z_2 = \frac{1}{23}(52 - x_2 - 2y_2)$$
$$= \frac{1}{23}\left[52 - 2.0126 - 2(1.9999)\right]$$
$$= 1.9994$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$x_3 = \frac{1}{25}(48 - 2y_2 + 3z_2)$$
$$= \frac{1}{25}\left[48 - 2(1.9999) + 3(1.9994)\right]$$
$$= 1.9999$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$y_3 = \frac{1}{27}(56 - 3x_3 + 2z_2)$$
$$= \frac{1}{27}\left[56 - 3(1.9999) + 2(1.9994)\right]$$
$$= 1.9999$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$z_3 = \frac{1}{23}(52 - x_3 - 2y_3)$$
$$= \frac{1}{23}\left[52 - 1.9999 - 2(1.9999)\right]$$
$$= 2.0000$$

Since the second and third iteration values are nearly equal, the approximate solution is

$$x = 2,\ y = 2,\ z = 2$$

Example 6

Solve the following system of equations, by the Gauss–Seidel method:

$$\begin{aligned} 2x + y + 6z &= 9 \\ 8x + 3y + 2z &= 13 \\ x + 5y + z &= 7 \end{aligned}$$

[Summer 2015]

Solution

Since diagonal elements are not largest in their respective rows, rearranging the equations, we have

$$\begin{aligned} 8x + 3y + 2z &= 13 \\ x + 5y + z &= 7 \\ 2x + y + 6z &= 9 \end{aligned}$$

Now, diagonal elements are largest. Rewriting the equations,

$$x = \frac{1}{8}(13 - 3y - 2z)$$

$$y = \frac{1}{5}(7 - x - z)$$

$$z = \frac{1}{6}(9 - 2x - y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{8}(13) = 1.625$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{5}(7 - x_1 - z_0) \\ &= \frac{1}{5}(7 - 1.625 - 0) \\ &= 1.075 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= \frac{1}{6}(9 - 2x_1 - y_1) \\ &= \frac{1}{6}[9 - 2(1.625) - 1.075] \\ &= 0.7792 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{8}(13 - 3y_1 - 2z_1) \\ &= \frac{1}{8}[13 - 3(1.075) - 2(0.7792)] \\ &= 1.0271 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$\begin{aligned} y_2 &= \frac{1}{5}(7 - x_2 - z_1) \\ &= \frac{1}{5}(7 - 1.0271 - 0.7792) \\ &= 1.0387 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned} z_2 &= \frac{1}{6}(9 - 2x_2 - y_2) \\ &= \frac{1}{6}[9 - 2(1.0271) - 1.0387] \\ &= 0.9845 \end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$\begin{aligned} x_3 &= \frac{1}{8}(13 - 3y_2 - 2z_2) \\ &= \frac{1}{8}[13 - 3(1.0387) - 2(0.9845)] \\ &= 0.9894 \end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned} y_3 &= \frac{1}{5}(7 - x_3 - z_2) \\ &= \frac{1}{5}(7 - 0.9894 - 0.9845) \\ &= 1.0052 \end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$\begin{aligned} z_3 &= \frac{1}{6}(9 - 2x_3 - y_3) \\ &= \frac{1}{6}[9 - 2(0.9894) - 1.0052] \\ &= 1.0027 \end{aligned}$$

Iteration 4: Putting $y = y_3$, $z = z_3$ in the equation of x,

$$\begin{aligned} x_4 &= \frac{1}{8}(13 - 3y_3 - 2z_3) \\ &= \frac{1}{8}[13 - 3(1.0052) - 2(1.0027)] \\ &= 0.9974 \end{aligned}$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$\begin{aligned} y_4 &= \frac{1}{5}(7 - x_4 - z_3) \\ &= \frac{1}{5}(7 - 0.9974 - 1.0027) \\ &= 1 \end{aligned}$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$\begin{aligned} z_4 &= \frac{1}{6}(9 - 2x_4 - y_4) \\ &= \frac{1}{6}[9 - 2(0.9974) - 1] \\ &= 1.0009 \end{aligned}$$

Since the third and fourth iteration values are nearly equal, the approximate solution is

$$x = 1,\ y = 1,\ z = 1$$

Example 7

Solve the following system of equations:

$$\begin{aligned} x + 2y + z &= 0 \\ 3x + y - z &= 0 \\ x - y + 4z &= 3 \end{aligned}$$

starting with (1, 1, 1).

Solution

Since diagonal elements are not largest in their respective rows, rearranging the equations,

$$\begin{aligned} 3x + y - z &= 0 \\ x + 2y + z &= 0 \\ x - y + 4z &= 3 \end{aligned}$$

Now, diagonal elements are largest. Rewriting the equations,

$$x = \frac{1}{3}(-y + z)$$

$$y = \frac{1}{2}(-x - z)$$

$$z = \frac{1}{4}(3 - x + y)$$

Iteration 1: Assuming $x_0 = 1$, $y_0 = 1$, $z_0 = 1$ as initial approximation and substituting in the equation of x,

$$\begin{aligned} x_1 &= \frac{1}{3}(-y_0 + z_0) \\ &= \frac{1}{3}(-1+1) \\ &= 0 \end{aligned}$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{2}(-x_1 - z_0) \\ &= \frac{1}{2}(-0-1) \\ &= -0.5 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= \frac{1}{4}(3 - x_1 + y_1) \\ &= \frac{1}{4}(3-0-0.5) \\ &= 0.625 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{3}(-y_1 + z_1) \\ &= \frac{1}{3}\left[-(-0.5)+0.625\right] \\ &= 0.375 \end{aligned}$$

Putting $x = x_2$, $z = z_1$, in the equation of y,

$$\begin{aligned} y_2 &= \frac{1}{2}(-x_2 - z_1) \\ &= \frac{1}{2}(-0.375-0.625) \\ &= -0.5 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned} z_2 &= \frac{1}{4}(3 - x_2 + y_2) \\ &= \frac{1}{4}(3-0.375-0.5) \\ &= 0.5313 \end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$\begin{aligned} x_3 &= \frac{1}{3}(-y_2 + z_2) \\ &= \frac{1}{3}[-(-0.5) + 0.5313] \\ &= 0.3438 \end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned} y_3 &= \frac{1}{2}(-x_3 - z_2) \\ &= \frac{1}{2}(-0.3438 - 0.5313) \\ &= -0.4376 \end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$\begin{aligned} z_3 &= \frac{1}{4}(3 - x_3 + y_3) \\ &= \frac{1}{4}(3 - 0.3438 - 0.4376) \\ &= 0.5547 \end{aligned}$$

Iteration 4: Putting $y = y_3$, $z = z_3$ in the equation of x,

$$\begin{aligned} x_4 &= \frac{1}{3}(-y_3 + z_3) \\ &= \frac{1}{3}[-(-0.4376) + 0.5547] \\ &= 0.3307 \end{aligned}$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$\begin{aligned} y_4 &= \frac{1}{2}(-x_4 - z_3) \\ &= \frac{1}{2}(-0.3307 - 0.5547) \\ &= -0.4427 \end{aligned}$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$\begin{aligned} z_4 &= \frac{1}{4}(3 - x_4 + y_4) \\ &= \frac{1}{4}(3 - 0.3307 - 0.4427) \\ &= 0.5566 \end{aligned}$$

Iteration 5: Putting $y = y_4$, $z = z_4$ in the equation of z,

$$\begin{aligned} x_5 &= \frac{1}{3}(-y_4 + z_4) \\ &= \frac{1}{3}[-(-0.4427) + 0.5566] \\ &= 0.3331 \end{aligned}$$

Putting $x = x_5$, $z = z_4$ in the equation of y,

$$\begin{aligned} y_5 &= \frac{1}{2}(-x_5 - z_4) \\ &= \frac{1}{2}(-0.3331 - 0.5566) \\ &= -0.4449 \end{aligned}$$

Putting $x = x_5$, $y = y_5$ in the equation of z,

$$\begin{aligned} z_5 &= \frac{1}{4}(3 - x_5 + y_5) \\ &= \frac{1}{4}(3 - 0.3331 - 0.4449) \\ &= 0.5555 \end{aligned}$$

Iteration 6: Putting $y = y_5$, $z = z_5$ in the equation of z,

$$\begin{aligned} x_6 &= \frac{1}{3}(-y_5 + z_5) \\ &= \frac{1}{3}[-(-0.4449) + 0.5555] \\ &= 0.3335 \end{aligned}$$

Putting $x = x_6$, $z = z_5$ in the equation of y,

$$\begin{aligned} y_6 &= \frac{1}{2}(-x_6 - z_5) \\ &= \frac{1}{2}(-0.3335 - 0.5555) \\ &= -0.4445 \end{aligned}$$

Putting $x = x_6$, $y = y_6$ in the equation of z,

$$\begin{aligned} z_6 &= \frac{1}{4}(3 - x_6 + y_6) \\ &= \frac{1}{4}(3 - 0.3335 - 0.4445) \\ &= 0.5555 \end{aligned}$$

Since the fifth and sixth iteration values are nearly equal, the approximate solution is

$$x = 0.333,\ y = -0.444,\ z = 0.555$$

Example 8

Solve the following system of equations:

$$\begin{aligned} 2x - 15y + 6z &= 72 \\ -x + 6y - 27z &= 85 \\ 54x + y + z &= 110 \end{aligned}$$

Solution

Since diagonal elements are not largest in their respective rows, rearranging the equations, we have

$$\begin{aligned} 54x + y + z &= 110 \\ 2x - 15y + 6z &= 72 \\ -x + 6y - 27z &= 85 \end{aligned}$$

Now, diagonal elements are largest. Rewriting the equations,

$$x = \frac{1}{54}(110 - y - z)$$

$$y = -\frac{1}{15}(72 - 2x - 6z)$$

$$z = -\frac{1}{27}(85 + x - 6y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{54}(110) = 2.037$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= -\frac{1}{15}(72 - 2x_1 - 6z_0) \\ &= -\frac{1}{15}\left[72 - 2(2.037) - 6(0)\right] \\ &= -4.5284 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$z_1 = -\frac{1}{27}(85 + x_1 - 6y_1)$$

$$= -\frac{1}{27}[85 + 2.037 - 6(-4.5284)]$$

$$= -4.2299$$

Iteration 2: Putting $y = y_1$, $z = z_1$, in the equation of x,

$$x_2 = \frac{1}{54}(110 - y_1 - z_1)$$

$$= \frac{1}{54}[110 - (-4.5284) - (-4.2299)]$$

$$= 2.1992$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$y_2 = -\frac{1}{15}(72 - 2x_2 - 6z_1)$$

$$= -\frac{1}{15}[72 - 2(2.1992) - 6(-4.2299)]$$

$$= -6.1987$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$z_2 = -\frac{1}{27}(85 + x_2 - 6y_2)$$

$$= -\frac{1}{27}[85 + 2.1992 - 6(-6.1987)]$$

$$= -4.6071$$

Iteration 3: Putting $y = y_2$, $z = z_2$, in the equation of x,

$$x_3 = \frac{1}{54}(110 - y_2 - z_2)$$

$$= \frac{1}{54}[110 - (-6.1987) - (-4.6071)]$$

$$= 2.2371$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$y_3 = -\frac{1}{15}(72 - 2x_3 - 6z_2)$$

$$= -\frac{1}{15}[72 - 2(2.2371) - 6(-4.6071)]$$

$$= -6.3446$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$z_3 = -\frac{1}{27}(85 + x_3 - 6y_3)$$

$$= -\frac{1}{27}\left[85 + 2.2371 - 6(-6.3446)\right]$$

$$= -4.6409$$

Iteration 4: Putting $y = y_3$, $z = z_3$, in the equation of x,

$$x_4 = \frac{1}{54}(110 - y_3 - z_3)$$

$$= \frac{1}{54}\left[110 - (-6.3446) - (-4.6409)\right]$$

$$= 2.2405$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$y_4 = -\frac{1}{15}(72 - 2x_4 - 6z_3)$$

$$= -\frac{1}{15}\left[72 - 2(2.2405) - 6(-4.6409)\right]$$

$$= -6.3576$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$z_4 = -\frac{1}{27}(85 + x_4 - 6y_4)$$

$$= -\frac{1}{27}\left[85 + 2.2405 - 6(-6.3576)\right]$$

$$= -4.6439$$

Iteration 5: Putting $y = y_4$, $z = z_4$, in the equation of x,

$$x_5 = \frac{1}{54}(110 - y_4 - z_4)$$

$$= \frac{1}{54}\left[110 - (-6.3576) - (-4.6439)\right]$$

$$= 2.2408$$

Putting $x = x_5$, $z = z_4$ in the equation of y,

$$y_5 = -\frac{1}{15}(72 - 2x_5 - 6z_4)$$

$$= -\frac{1}{15}\left[72 - 2(2.2408) - 6(-4.6439)\right]$$

$$= -6.3588$$

Putting $x = x_5$, $y = y_5$ in the equation of z,

$$\begin{aligned} z_5 &= -\frac{1}{27}(85 + x_5 - 6y_5) \\ &= -\frac{1}{27}[85 + 2.2408 - 6(-6.3588)] \\ &= -4.6442 \end{aligned}$$

Iteration 6: Putting $y = y_5$, $z = z_5$, in the equation of x,

$$\begin{aligned} x_6 &= \frac{1}{54}(110 - y_5 - z_5) \\ &= \frac{1}{54}[110 - (-6.3588) - (-4.6442)] \\ &= 2.2408 \end{aligned}$$

Putting $x = x_6$, $z = z_5$ in the equation of y,

$$\begin{aligned} y_6 &= -\frac{1}{15}(72 - 2x_6 - 6z_5) \\ &= -\frac{1}{15}[72 - 2(2.2408) - 6(-4.6442)] \\ &= -6.3589 \end{aligned}$$

Putting $x = x_6$, $y = y_6$ in the equation of z,

$$\begin{aligned} z_6 &= -\frac{1}{27}(85 + x_6 - 6y_6) \\ &= -\frac{1}{27}[85 + 2.2408 - 6(-6.3589)] \\ &= -4.6442 \end{aligned}$$

Since the fifth and sixth iteration values are nearly equal, the approximate solution is

$$x = 2.2408,\ y = -6.3589,\ z = -4.6442$$

The above method can also be represented in tabular form as follows:

Iteration number	$x = \frac{1}{54}(110 - y - z)$	$y = -\frac{1}{15}(72 - 2x - 6z)$	$z = -\frac{1}{27}(85 + x - 6y)$
1	$x_0 = 0$ $x_1 = 2.037$	$y_0 = 0$ $y_1 = -4.5284$	$z_0 = 0$ $z_1 = -4.2299$
2	$x_2 = 2.1992$	$y_2 = -6.1987$	$z_2 = -4.6071$
3	$x_3 = 2.2371$	$y_3 = -6.3446$	$z_3 = -4.6409$
4	$x_4 = 2.2405$	$y_4 = -6.3576$	$z_4 = -4.6439$
5	$x_5 = 2.2408$	$y_5 = -6.3588$	$z_5 = -4.6442$
6	$x_6 = 2.2408$	$y_6 = -6.3589$	$z_6 = -4.6442$

EXERCISE 3.4

Solve the following system of equations by using the Gauss–Seidel method:

1. $$\begin{aligned} 54x + y + z &= 110 \\ 2x + 15y + 6z &= 72 \\ -x + 6y + 27z &= 85 \end{aligned}$$

$$\left[\textbf{Ans.: } x = 1.92,\ y = 3.57,\ z = 2.42\right]$$

2. $$\begin{aligned} 20x + y - 2z &= 17 \\ 3x + 20y - z &= -18 \\ 2x - 3y + 20z &= 25 \end{aligned}$$

$$[\textbf{Ans.: } x = 1,\ y = -1,\ z = 1]$$

3. $$\begin{aligned} 10x + y + z &= 12 \\ 2x + 10y + z &= 13 \\ 2x + 2y + 10z &= 14 \end{aligned}$$

$$\left[\textbf{Ans.: } x = 1,\ y = 1,\ z = 1\right]$$

4. $$\begin{aligned} 27x + 6y - z &= 85 \\ 6x + 15y + 2z &= 72 \\ x + y + 54z &= 110 \end{aligned}$$

$$\left[\textbf{Ans.: } x = 2.43,\ y = 3.57,\ z = 1.92\right]$$

5. $$\begin{aligned} 28x + 4y - z &= 32 \\ 2x + 17y + 4z &= 35 \\ x + 3y + 10z &= 24 \end{aligned}$$

$$[\textbf{Ans.: } x = 0.99,\ y = 1.51,\ z = 1.85]$$

3.10 ILL-CONDITIONED SYSTEMS

An ill-conditioned system is one in which a small change in any of the elements of the system causes a large change in the solution of the system. Since ill-conditioned systems are extremely sensitive to small changes in the elements of the system, they are also extremely sensitive to round-off errors.

A well-conditioned system is one in which a small change in any of the elements of the system causes only a small change in the solution on the system.

Consider the following system of two linear equations in two unknowns:

$$\begin{bmatrix} 400 & -201 \\ -800 & 401 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 200 \\ -200 \end{bmatrix}$$

This system can be solved by any previously discussed methods and the solution is

$$x_1 = -100, \quad x_2 = -200$$

If one of the elements of the coefficient matrix, say a_{11} is changed from 400 to 401, i.e.,

$$\begin{bmatrix} 400 & -201 \\ -800 & 401 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 200 \\ -200 \end{bmatrix}$$

then the solution is

$$x_1 = 40000, \quad x_2 = 79800$$

With a small change in one of the coefficient, there is significant change in solution of the system. The solution is very sensitive to the values of the coefficient matrix. Such a system is called ill-conditioned system.

Points to Remember

Gauss Elimination Method

(i) Write the matrix form of the system of equations.
(ii) Write the augmented matrix.
(iii) Obtain the echelon form of the augmented matrix by using elementary row transformations.
(iv) Write the corresponding linear system of equations from the echelon form.
(v) Solve the corresponding linear system of equations by back substitution.

Gauss–Jordan Method

(i) Write the matrix form of the system of equations.
(ii) Write the augmented matrix.
(iii) Reduce the coefficient matrix to unit matrix by applying elementary row transformations to the augmented matrix.
(iv) Write the corresponding linear system of equations to obtain the solution.

Gauss–Jacobi Method

(i) Arrange the equations in such a manner that the leading diagonal elements are large in magnitude in their respective rows satisfying the conditions

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.

(iii) Start the iteration 1 by assuming the initial values of (x, y, z) as (x_0, y_0, z_0) and obtain (x_1, y_1, z_1).

(iv) Start the iteration 2 by putting $x = x_1$, $y = y_1$, $z = z_1$ in equations of x, y, z and obtain (x_2, y_2, z_2).

(v) The above process is repeated for the next iterations and it continues until two successive approximations are nearly equal.

Gauss–Siedel Method

(i) Arrange the equations in such a manner that the leading diagonal elements are large in magnitude in their respective rows such that

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.

(iii) Start the iteration 1 by assuming the initial values of (x, y, z) as (x_0, y_0, z_0).

(iv) In the iteration 1, put $y = y_0$, $z = z_0$ in the equation of x to obtain x_1, put $x = x_1$, $z = z_0$ in the equation of y to obtain y_1, put $x = x_1$, $y = y_1$ in the equation of z to obtain z_1.

(v) The above process is repeated for the next iterations and it continues until two successive approximations are nearly equal.

CHAPTER 4

Interpolation

Chapter Outline

4.1 INTRODUCTION

Interpolation is the process of reading between the lines of a table. It is the process of computing intermediate values of a function from a given set of tabular values of the function. Extrapolation is used to denote the process of finding the values outside the given interval.

In the interpolation process, the given set of tabular values are used to find an expression for $f(x)$ and then using it to find its required value for a given value of x. But it is difficult to find an exact form of $f(x)$ using the limited values in the table. Hence, $f(x)$ is replaced by another function $\phi(x)$, which matches with $f(x)$ at the discrete values in the table. This function $\phi(x)$ is known as the *interpolating function*.

When the interpolating function is a polynomial function, the process is known as *polynomial interpolation*. Polynomial interpolations are preferred because of the following reasons:

(i) They are simple forms of functions which can be easily manipulated.

(ii) Polynomials are free from singularities whereas rational functions or other types have singularities.

4.2 FINITE DIFFERENCES

Let the function $y = f(x)$ be tabulated for the equally spaced values $y_0 = f(x_0)$, $y_1 = f(x_0 + h)$, $y_2 = f(x_0 + 2h)$, …, $y_n = f(x_0 + nh)$, as

x	x_0	$x_0 + h$	$x_0 + 2h$	....	$x_0 + nh$	....
$y = f(x)$	y_0	y_1	y_2	....	y_n	....

To determine the values of $f(x)$ for some intermediate values of x, the following three types of differences can be used.

4.2.1 Forward Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the first forward differences of y and are denoted by $\Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1}$ respectively.

$$\begin{aligned}
\Delta y_0 &= y_1 - y_0 \\
\Delta y_1 &= y_2 - y_1 \\
\vdots \quad &\quad \vdots \\
\Delta y_{n-1} &= y_n - y_{n-1}
\end{aligned}$$

where Δ is called the *forward difference operator*. The differences of the first forward differences are called *second forward differences* and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots, \Delta^2 y_{n-1}$. Similarly, third forward differences, fourth forward differences, etc., can be defined.

$$\begin{aligned}
\Delta^2 y_0 &= \Delta\,(\Delta y_0) \\
&= \Delta\,(y_1 - y_0) \\
&= \Delta y_1 - \Delta y_0 \\
&= y_2 - y_1 - (y_1 - y_0) \\
&= y_2 - 2y_1 + y_0 \\
\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 \\
&= (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) \\
&= y_3 - 3y_2 + 3y_1 - y_0
\end{aligned}$$

$$\begin{aligned}\Delta^4 y_0 &= \Delta^3 y_1 - \Delta^3 y_0\\ &= (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0)\\ &= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0\end{aligned}$$

Since the coefficients occurring on the right-hand side are the binomial coefficients, the general form is

$$\Delta^n y_0 = y_n - {}^n c_1\, y_{n-1} + {}^n c_2\, y_{n-2} - \cdots + (-1)^n\, y_0$$

Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		$\Delta y_0 = y_1 - y_0$			
$x_0 + h = x_1$	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
$x_0 + 2h = x_2$	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
$x_0 + 3h = x_3$	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
		$\Delta y_3 = y_4 - y_3$			
$x_0 + 4h = x_4$	y_4				

In a difference table, x is called the *argument*, and y is called the *function* or *entry*.

Note When $(n + 1)$ values are given, the highest-order difference is n, e.g., when 5 values are given, the highest-order difference is 4.

4.2.2 Backward Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the *first backward differences* of y and are denoted by $\nabla y_1, \nabla y_2, \ldots., \nabla y_n$, respectively.

$$\begin{aligned}\nabla y_1 &= y_1 - y_0\\ \nabla y_2 &= y_2 - y_1\\ \vdots\quad &\quad \vdots \quad \vdots\\ \nabla y_n &= y_n - y_{n-1}\end{aligned}$$

where ∇ is called the *backward difference operator.* Similarly, backward differences of higher order can be defined.

$$\nabla^2 y_2 = \nabla\, (\nabla y_2)$$

$$\begin{aligned} &= \nabla (y_2 - y_1) \\ &= \nabla y_2 - \nabla y_1 \\ &= y_2 - y_1 - (y_1 - y_0) \\ &= y_2 - 2y_1 + y_0 \\ \nabla^3 y_3 &= \nabla^2 y_3 - \nabla^2 y_2 \\ &= y_3 - 3y_2 + 3y_1 - y_0 \text{ etc.} \end{aligned}$$

Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		$\nabla y_1 = y_1 - y_0$			
x_1	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$		
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$	
x_2	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$		$\nabla^4 y_4 = \nabla^3 y_4 - \nabla^3 y_3$
		$\nabla y_3 = y_3 - y_2$		$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$	
x_3	y_3		$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$		
		$\nabla y_4 = y_4 - y_3$			
x_4	y_4				

4.2.3 Central Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the *central differences* of y and are denoted by $\delta y_{\frac{1}{2}}, \delta y_{\frac{3}{2}}, \ldots, \delta y_{n-\frac{1}{2}}$ respectively.

$$\begin{aligned} \delta y_{\frac{1}{2}} &= y_1 - y_0 \\ \delta y_{\frac{3}{2}} &= y_2 - y_1 \\ \vdots \quad & \quad \vdots \quad \quad \vdots \\ \delta y_{n-\frac{1}{2}} &= y_n - y_{n-1} \end{aligned}$$

where δ is called the *central difference operator*. Similarly, higher-order central differences can be defined.

$$\begin{aligned}\delta^2 y_1 &= \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}}\\ &= (y_2 - y_1) - (y_1 - y_0)\\ &= y_2 - 2y_1 + y_0\\ \delta^3 y_{\frac{3}{2}} &= \delta^2 y_2 - \delta^2 y_1 \text{ etc.}\end{aligned}$$

Central Difference Table

x	y	δ	δ^2	δ^3	δ^4
x_0	y_0				
		$\delta y_{\frac{1}{2}} = y_1 - y_0$			
x_1	y_1		$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}}$		
		$\delta y_{\frac{3}{2}} = y_2 - y_1$		$\delta^3 y_{\frac{3}{2}} = \delta^2 y_2 - \delta^2 y_1$	
x_2	y_2		$\delta^2 y_2 = \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}}$		$\delta^4 y_2 = \delta^3 y_{\frac{5}{2}} - \delta^3 y_{\frac{3}{2}}$
		$\delta y_{\frac{5}{2}} = y_3 - y_2$		$\delta^3 y_{\frac{5}{2}} = \delta^2 y_3 - \delta^2 y_2$	
x_3	y_3		$\delta^2 y_3 = \delta y_{\frac{7}{2}} - \delta y_{\frac{5}{2}}$		
		$\delta y_{\frac{7}{2}} = y_4 - y_3$			
x_4	y_4				

From the central difference table, it is clear that the central differences on the same horizontal line have the same suffix. Also, the differences of odd orders are known only for half values of the suffix and those of even orders, for only integral values of the suffix.

Note It is clear from the three difference tables that it is only the notations which change and not the differences.

$$y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{\frac{1}{2}}$$

4.3 DIFFERENT OPERATORS AND THEIR RELATIONS

1. **Forward Difference Operator** The forward difference operator is denoted by Δ and is defined as

$$\Delta f(x) = f(x+h) - f(x)$$

or
$$\Delta y_r = y_{r+1} - y_r$$

where h is known as the *interval of differencing*.

2. **Backward Difference Operator** The backward difference operator is denoted by ∇ and is defined as

$$\nabla f(x) = f(x) - f(x-h)$$

or
$$\nabla y_r = y_r - y_{r-1}$$

3. **Central Difference Operator** The central difference operator is denoted by δ and is defined as

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

or
$$\delta y_r = y_{r+\frac{1}{2}} - y_{r-\frac{1}{2}}$$

4. **Shift Operator** The shift operator is denoted by E and is defined as

$$E\,[f(x)] = f(x+h)$$

or
$$E\,y_r = y_{r+1}$$

Similarly,
$$E^{-1}\,[f(x)] = f(x-h)$$

or
$$E^{-1}\,y_r = y_{r-1}$$

5. **Averaging Operator** The averaging operator is denoted by μ and is defined as

$$\mu f(x) = \frac{f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)}{2}$$

6. **Differential Operator** The differential operator is denoted by D and is defined as

$$D\,f(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x)$$

4.3.1 Relations between Operators

1. Relation between Δ and *E*

[Summer 2015, Winter 2014, Summer 2014, Summer 2013]

$$\begin{aligned}\Delta f(x) &= f(x+h)-f(x)\\ &= Ef(x)-f(x) \qquad [\because Ef(x)=f(x+h)]\\ &= (E-1)f(x)\end{aligned}$$

$$\therefore \quad \Delta \equiv E-1 \quad \text{or} \quad E \equiv 1+\Delta$$

2. Relation between ∇ and *E* **[Winter 2014, Winter 2013]**

$$\begin{aligned}\nabla f(x) &= f(x)-f(x-h)\\ &= f(x)-E^{-1}f(x) \qquad [\because E^{-1}f(x)=f(x-h)]\\ &= (1-E^{-1})f(x)\end{aligned}$$

$$\therefore \quad \nabla \equiv 1-E^{-1}$$

3. Relation between δ and *E*

$$\begin{aligned}\delta f(x) &= f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)\\ &= E^{\frac{1}{2}} f(x)-E^{-\frac{1}{2}} f(x)\\ &= \left(E^{\frac{1}{2}}-E^{-\frac{1}{2}}\right)f(x)\end{aligned}$$

$$\therefore \quad \delta \equiv E^{\frac{1}{2}}-E^{-\frac{1}{2}}$$

4. Relation between μ and *E*

$$\begin{aligned}\mu f(x) &= \frac{f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)}{2}\\ &= \frac{E^{\frac{1}{2}} f(x)+E^{-\frac{1}{2}} f(x)}{2}\end{aligned}$$

$$\therefore \quad \mu \equiv \frac{E^{\frac{1}{2}}+E^{-\frac{1}{2}}}{2}$$

5. Relation between *D* and *E*

[Winter 2014, Summer 2014]

$$
\begin{aligned}
E\,f(x) &= f(x+h)\\
&= f(x) + h\,f'(x) + \frac{h^2}{2!}f''(x) + \cdots \qquad \text{[By Taylor's series]}\\
&= f(x) + hD\,f(x) + \frac{h^2}{2!}D^2 f(x) + \cdots\\
&= \left(1 + hD + \frac{h^2}{2!}D^2 + \cdots\right) f(x)\\
&= e^{hD} f(x)
\end{aligned}
$$

$$\therefore \quad E \equiv e^{hD}$$

Also, $$hD \equiv \log E \equiv \log(1+\Delta)$$

Corollary $E^{-\frac{1}{2}} \equiv e^{-\frac{hD}{2}}$

Proof

$$
\begin{aligned}
E^{-\frac{1}{2}} f(x) &= f\left(x - \frac{h}{2}\right)\\
&= f(x) - \frac{h}{2}f'(x) + \frac{\left(\frac{h}{2}\right)^2}{2!}f''(x) - \cdots\\
&= \left[1 - \frac{h}{2}D + \frac{\left(\frac{h}{2}\right)^2}{2!}D^2 - \cdots\right] f(x)\\
&= e^{-\frac{hD}{2}} f(x)
\end{aligned}
$$

$$\therefore \quad E^{-\frac{1}{2}} \equiv e^{-\frac{hD}{2}}$$

6. Relation between μ and δ

$$\mu \equiv \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}$$

$$
\begin{aligned}
\mu^2 &\equiv \left(\frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}\right)^2\\
&\equiv \frac{E + 2 + E^{-1}}{4}
\end{aligned}
$$

$$\equiv \frac{4+(E-E^{-1})^2}{4}$$

$$\equiv 1+\frac{\delta^2}{4}$$

$$\mu \equiv \sqrt{1+\frac{\delta^2}{4}}$$

Example 1

Prove that $(1 + \Delta)\,(1 - \nabla) = 1$

Solution

$$E \equiv 1+\Delta$$

$$\nabla \equiv 1 - E^{-1}$$

$$E^{-1} \equiv 1-\nabla$$

$$(1+\Delta)(1-\nabla) \equiv EE^{-1} = 1$$

Example 2

Prove that $\delta \equiv 2\sinh\dfrac{hD}{2}$

Solution

$$\delta \equiv f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$\equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

$$\equiv e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}$$

$$\equiv 2\sinh\left(\frac{hD}{2}\right)$$

Example 3

Prove that $hD \equiv \sinh^{-1}(\mu\delta)$

Solution

$$\mu\delta \equiv \left(\frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}\right)\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)$$

$$\equiv \frac{1}{2}(E - E^{-1})$$

$$2\,\mu\delta \equiv E - E^{-1}$$

$$\equiv e^{hD} - e^{-hD}$$

$$\equiv 2\sinh(hD)$$

$$\therefore \qquad hD \equiv \sinh^{-1}(\mu\delta)$$

Example 4

Prove that $\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$

Solution

$$\Delta \log f(x) = \log f(x+h) - \log f(x)$$

$$= \log \frac{f(x+h)}{f(x)}$$

$$= \log \frac{E\, f(x)}{f(x)}$$

$$= \log \frac{(1+\Delta)\, f(x)}{f(x)}$$

$$= \log\left[\frac{f(x) + \Delta f(x)}{f(x)}\right]$$

$$= \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

Example 5

Evaluate (*a*) $\Delta(x^2 + \sin x)$, *and* (*b*) $\Delta^2 \cos 3x$, *the interval of differencing being h.*

Solution

(i)
$$\begin{aligned}\Delta(x^2+\sin x) &= [(x+h)^2+\sin(x+h)]-(x^2+\sin x)\\ &= h^2+2hx+\sin(x+h)-\sin x\\ &= h^2+2hx+2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}\end{aligned}$$

(ii)
$$\begin{aligned}\Delta^2\cos 3x &= \Delta(\Delta\cos 3x)\\ &= \Delta[\cos 3(x+h)-\cos 3x]\\ &= \Delta\cos 3(x+h)-\Delta\cos 3x\\ &= \cos 3[(x+h)+h]-\cos 3(x+h)-\cos 3(x+h)+\cos 3x\\ &= \cos 3(x+2h)-2\cos 3(x+h)+\cos 3x\\ &= \cos 3(x+2h)+\cos 3x-2\cos 3(x+h)\end{aligned}$$

Example 6

Prove that $\left(\dfrac{\Delta^2}{E}\right)e^x \cdot \dfrac{E\,e^x}{\Delta^2\,e^x} = e^x$

Solution

$$\begin{aligned}\left(\frac{\Delta^2}{E}\right)e^x\cdot\frac{E\,e^x}{\Delta^2\,e^x} &= \frac{(E-1)^2}{E}e^x\cdot\frac{e^{x+h}}{\Delta(e^{x+h}-e^x)}\\ &= \left(\frac{E^2-2E+1}{E}\right)e^x\,\frac{e^{x+h}}{\left(e^{x+2h}-e^{x+h}-e^{x+h}+e^x\right)}\\ &= (E-2+E^{-1})e^x\cdot\frac{e^{x+h}}{\left(e^{x+2h}-2e^{x+h}+e^x\right)}\\ &= \frac{\left(e^{x+h}-2e^x+e^{x-h}\right)e^{x+h}}{\left(e^{x+2h}-2e^{x+h}+e^x\right)}\end{aligned}$$

$$= \frac{e^x\left(e^{x+2h} - 2e^{x+h} + e^x\right)}{e^{x+2h} - 2e^{x+h} + e^x}$$

$$= e^x$$

Example 7

Prove that $\Delta\nabla \equiv (\Delta - \nabla)$

Solution:

$$\begin{aligned}
\Delta\nabla f(x) &= \Delta[f(x) - f(x-h)] \\
&= \Delta f(x) - \Delta f(x-h) \\
&= \Delta f(x) - [f(x) - f(x-h)] \\
&= \Delta f(x) - \nabla f(x) \\
&= (\Delta - \nabla) f(x)
\end{aligned}$$

$\therefore$ $$\Delta\nabla \equiv (\Delta - \nabla)$$

Example 8

Prove that $\Delta \equiv E\nabla \equiv \nabla E$

Solution

$$\Delta f(x) = f(x+h) - f(x) \qquad ...(1)$$

$$\begin{aligned}
E\nabla f(x) &= E\left\{f(x) - f(x-h)\right\} \\
&= E f(x) - E f(x-h) \\
&= f(x+h) - f(x)
\end{aligned} \qquad ...(2)$$

$$\begin{aligned}
\nabla E f(x) &= \nabla f(x+h) \\
&= f(x+h) - f(x)
\end{aligned} \qquad ...(3)$$

From Eqs (1), (2), and (3),

$$\Delta \equiv E\nabla \equiv \nabla E$$

Example 9

Find the missing term in the following table:

x	1	2	3	4	5
y	7	–	13	21	37

Solution

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	7				
		$y_1 - 7$			
2	y_1		$20 - 2y_1$		
		$13 - y_1$		$3y_1 - 25$	
3	13		$y_1 - 5$		$38 - 4y_1$
		8		$13 - y_1$	
4	21		8		
		16			
5	37				

Since only four entries are given, the fourth-order difference will be zero.

$$\Delta^4 y_0 = 0$$
$$38 - 4y_1 = 0$$
$$y_1 = 9.5$$

Example 10

Obtain the estimate of missing terms in the following table:

x	1	2	3	4	5	6	7	8
y	2	4	8	–	32	–	128	256

Solution

Let $$f(4) = a, \quad f(6) = b$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1	2						
		2					
2	4		2				
		4		$a - 14$			
3	8		$a - 12$		$66 - 4a$		
		$a - 8$		$52 - 3a$		$10a + b - 222$	
4	a		$40 - 2a$		$b + 6a - 156$		$706 - 20a - 6b$
		$32 - a$		$b + 3a - 104$		$484 - 10a - 5b$	
5	32		$b + a - 64$		$328 - 4a - 4b$		$15b + 15a - 1196$
		$b - 32$		$224 - 3b - a$		$10b + 5a - 712$	
6	b		$160 - 2b$		$6b + a - 384$		
		$128 - b$		$3b - 160$			
7	128		b				
		128					
8	256						

Since only six values are given,

$$\Delta^6 y_0 = 0$$

$$20a + 6b = 706 \qquad \ldots(1)$$

$$15a + 15b = 1196 \qquad \ldots(2)$$

Solving Eqs (1) and (2),

$$a = 16.26$$

$$b = 63.48$$

Example 11

The following table gives the value of y which is a polynomial of degree five. It is known that y = f(3) is in error. Correct the error.

x	0	1	2	3	4	5	6
y	1	2	33	254	1025	3126	7777

Solution

Let $$f(3) = a$$

Since y is a polynomial of degree 5,

$$\Delta^6 y_0 = 0$$

$$(E - 1)^6 y_0 = 0$$

$$(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)\, y_0 = 0$$

$$y^6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0 = 0$$

$$7777 - 6(3126) + 15(1025) - 20a + 15(33) - 6(2) + 1 = 0$$

$$-20a = -4880$$

$$a = 244$$

$$\text{Error} = 254 - 244 = 10$$

Example 12

If u_x is a function for which the fifth difference is constant and $u_1 + u_7 = -784$, $u_2 + u_6 = 686$, $u_3 + u_5 = 1088$, find u_4.

Solution

Since the fifth difference is constant,

$$\Delta^6 u_1 = 0$$

$$(E - 1)^6 u_1 = 0$$

$$(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)\, u_1 = 0$$

$$u_7 - 6u_6 + 15u_5 - 20u_4 + 15u_3 - 6u_2 + u_1 = 0$$

$$(u_7 + u_1) - 6(u_6 + u_2) + 15(u_5 + u_3) - 20u_4 = 0$$

$$-784 - 6(686) + 15(1088) - 20u_4 = 0$$

$$20u_4 = 11420$$

$$u_4 = 571$$

4.3.2 Factorial Notation

A product of the form $x(x - 1)\,(x - 2)\ldots(x - n + 1)$ is called a factorial polynomial or function and is denoted by $[x]^n$.

$$[x]^n = x\,(x - 1)\,(x - 2)\,\ldots(x - n + 1)$$

If the interval of differencing is h then

$$[x]^n = x\,(x - h)\,(x - 2h)\,\ldots\,\{x - (n - 1)\,h\}$$

The factorial notation is of special utility in the theory of finite differences. It is useful in finding the successive differences of a polynomial directly by simple rule of differentiation.

Example 1

Write $f(x) = x^4 - 2x^3 + x^2 - 2x + 1$ *in factorial notation and find* $\Delta^4 f(x)$.

Solution

$$f(x) = x^4 - 2x^3 + x^2 - 2x + 1$$

Let $$f(x) = A[x]^4 + B[x]^3 + C[x]^2 + D[x]^1 + E$$

Using synthetic division,

1	1	–2	1	–2	1 = E
	0	1	–1	0	
2	1	–1	0	–2 = D	
	0	2	2		
3	1	1	2 = C		
	0	3			
	1 = A	4 = B			

$\therefore$ $$f(x) = [x]^4 + 4[x]^3 + 2[x]^2 - 2[x]^1 + 1$$

$$\Delta f(x) = 4[x]^3 + 12[x]^2 + 4[x] - 2$$

$$\Delta^2 f(x) = 12[x]^2 + 24[x] + 4$$

$$\Delta^3 f(x) = 24[x] + 24$$

$$\Delta^4 f(x) = 24$$

Example 2

Express $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$ *and its successive differences in terms of factorial polynomials. Also, find the function whose first difference is* $f(x)$.

Solution

$$f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$$

Let $$f(x) = A[x]^4 + B[x]^3 + C[x]^2 + D[x]^1 + E$$

Using synthetic division,

$$\begin{array}{c|cccc|l}
1 & 1 & -12 & 42 & -30 & 9 = E \\
 & 0 & 1 & -11 & 31 & \\
\hline
2 & 1 & -11 & 31 & 1 = D & \\
 & 0 & 2 & -18 & & \\
\hline
3 & 1 & -9 & 13 = C & & \\
 & 0 & 3 & & & \\
\hline
 & 1 = A & -6 = B & & &
\end{array}$$

$\therefore$
$$f(x) = [x]^4 - 6[x]^3 + 13[x]^2 + [x]^1 + 9$$
$$\Delta f(x) = 4[x]^3 - 18[x]^2 + 26[x]^1 + 1$$
$$\Delta^2 f(x) = 12[x]^2 - 36[x]^1 + 26$$
$$\Delta^3 f(x) = 24[x]^1 - 36$$
$$\Delta^4 f(x) = 24$$
$$\Delta^5 f(x) = 0$$

By integrating $f(x)$, the function $\phi(x)$ whose first difference is $f(x)$, is obtained.

$$\phi(x) = \frac{[x]^5}{5} - \frac{6[x]^4}{4} + \frac{13[x]^3}{3} + \frac{[x]^2}{2} + 9[x]^1 + c$$

Example 3

Express $f(x) = 2x^3 - 3x^2 + 3x - 10$ in factorial polynomial and, hence, show that $\Delta^3 f(x) = 12$.

Solution

$$f(x) = 2x^3 - 3x^2 + 3x - 10$$

Let
$$f(x) = A[x]^3 + B[x]^2 + C[x]^1 + D$$

Using synthetic division,

$$\begin{array}{c|ccc|l}
1 & 2 & -3 & 3 & -10 = D \\
 & 0 & 2 & -1 & \\
\hline
2 & 2 & -1 & 2 = C & \\
 & 0 & 4 & & \\
\hline
 & 2 = A & 3 = B & &
\end{array}$$

$$\therefore \qquad f(x) = 2[x]^3 + 3[x]^2 + 2[x]^1 - 10$$

$$\Delta f(x) = 6[x]^2 + 6[x]^1 + 2$$

$$\Delta^2 f(x) = 12[x]^1 + 6$$

$$\Delta^3 f(x) = 12$$

EXERCISE 4.1

1. Prove the following identities:

(i) $\Delta\nabla \equiv \nabla\Delta$

(ii) $\nabla \equiv E^1 \Delta$

(iii) $E\nabla \equiv \nabla E$

(iv) $hD \equiv -\log(1-\nabla)$

(v) $\Delta + \nabla \equiv \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta}$

(vi) $\left(E^{\frac{1}{2}} - E^{\frac{-1}{2}}\right)(1-\nabla)^{\frac{1}{2}} \equiv \nabla$

2. Find

(i) $\Delta\left[\dfrac{2^x}{(x+1)!}\right]$

(ii) $\Delta \tan^{-1} x$

(iii) $\Delta^n e^{ax}$

(iv) $\Delta(x + \cos x)$

(v) $\Delta^4 (ax^4 + bx^2 + cx + d)$

$\left[\textbf{Ans.:} \text{ (i) } -\dfrac{x \cdot 2^x}{(x+2)!} \quad \text{(ii) } \tan^{-1}\left(\dfrac{h}{1+hx+x^2}\right) \quad \text{(iii) } (e^a - 1)^n \cdot e^x \right.$

$\left. \text{(iv) } 1 - 2\sin\left(x + \dfrac{1}{2}\right)\sin x \quad \text{(v) } a^4 \cdot 4! \right]$

3. Evaluate $\left(\dfrac{\Delta^2}{E}\right)\sin x$, where the interval of difference is h.

[**Ans.:** $\sin(x+h) - 2\sin x + \sin(x-h)$]

4. Prove that

(i) $\Delta[x\,f(x)] = (x+h)\,\Delta f(x) + hf(x)$

(ii) $(\Delta + \nabla)^2 (x^2 + x) = 8h^2$

5. Prove that $f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$.

6. Find $\dfrac{\Delta^2(x^2)}{E(x^2)}$ and $\left(\dfrac{\Delta^2}{E}\right)x^2$.

$$\left[\textbf{Ans.: } \frac{2}{(1+x)^2}, 2\right]$$

7. If $y = a3^x + b(-2)^x$ and $h = 1$, prove that $(\Delta^2 + \Delta - 6)y = 0$.

8. Find the missing term from the following data:

x	0	1	2	3	4
y	1	3	9	–	81

[Ans.: 31**]**

9. From the following table, estimate y at $x = 2$.

x	4	6	8	10	12
y	6	7	13	32	77

[Ans.: 7**]**

10. If $u_0 = -10$, $u_1 = -6$, $u_2 = 2$, $u_3 = 12$, $u_4 = 26$, $u_5 = 42$, find u_6.

[Ans.: 46**]**

11. If $u_3 = 4$, $u_4 = 12$, $u_5 = 22$, $u_6 = 37$, $u_7 = 55$, find u_8.

[Ans.: 69**]**

12. From the following table, find $(15)^3$.

x	3	5	7	9	11
y	27	125	343	721	1331

[Ans.: 3375**]**

4.4 INTERPOLATION

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to the values $x_0, x_1, x_2, \ldots, x_n$ of x. The process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called *interpolation*. Thus, interpolation is a technique of finding the value of a function for any intermediate value of the independent variable. The process of computing the value of the function outside the range of given values of the variable is called *extrapolation*. The study of interpolation is based on the concept of finite differences which were discussed in the preceding section.

4.5 NEWTON'S FORWARD INTERPOLATION FORMULA

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots.$ corresponding to the values $x_0, x_1, x_2, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + rh$, where r is any real number.

$$\begin{aligned} y_r &= f(x_0 + rh) \\ &= E^r f(x_0) \\ &= (1+\Delta)^r f(x_0) \\ &= (1+\Delta)^r y_0 \\ &= \left[1 + r\Delta + \frac{r(r-1)}{2!}\Delta^2 + \frac{r(r-1)(r-2)}{3!}\Delta^3 + ...\right] y_0 \end{aligned}$$

[Using Binomial theorem]

$$= y_0 + r\,\Delta\, y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots \quad ...(4.1)$$

Equation (4.1) is known as *Newton's forward interpolation formula*.

Note This formula is used for evaluating the value of y near the initial tabulated value of x, i.e., near x_0.

Example 1

Compute cosh (0.56) *using Newton's forward difference formula from the following table:*

x	0.5	0.6	0.7	0.8
$f(x)$	1.127626	1.185465	1.255169	1.337435

Solution

Let $x = 0.56$, $x_0 = 0.5$, $h = 0.1$

$$r = \frac{x - x_0}{h} = \frac{0.56 - 0.5}{0.1} = 0.6$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0.5	1.127626			
		0.057839		
0.6	1.185465		0.011865	
		0.069704		0.000697
0.7	1.255169		0.012562	
		0.082266		
0.8	1.337435			

By Newton's forward difference formula,

$$f(x_0+rh) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \frac{r(r-1)(r-2)}{3!}\Delta^3 f(x_0) + \cdots$$

$$\begin{aligned}\cosh(0.56) &= 1.127626 + 0.6(0.057839) + \frac{0.6(0.6-1)}{2!}(0.011865) \\ &\quad + \frac{0.6(0.6-1)(0.6-2)}{3!}(0.000697) \\ &= 1.127626 + 0.034703 - 0.001424 + 0.000039 \\ &= 1.160944\end{aligned}$$

Example 2

Find the value of sin 52° *using Newton's forward interpolation formula from the following table:*

$\theta°$	45°	50°	55°	60°
sin $\theta°$	0.7071	0.7660	0.8192	0.8660

Solution

Let $x = 52°, x_0 = 45°, h = 5°$

$$r = \frac{x - x_0}{h} = \frac{52° - 45°}{5°} = 1.4°$$

Difference Table

$x = \theta°$	$y = \sin\theta°$	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0.7071			
		0.0589		
50°	0.7660		−0.0057	
		0.0532		−0.0007
55°	0.8192		−0.0064	
		0.0468		
60°	0.8660			

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

$$\sin 52° = 0.7071 + 1.4(0.0589) + \frac{1.4(1.4-1)}{2!}(-0.0057) + \frac{1.4(1.4-1)(1.4-2)}{3!}(-0.0007)$$

$$= 0.7071 + 0.0825 - 0.0016 + 0.00004$$

$$= 0.7880$$

Example 3

Using Newton's forward interpolation formula, find the value of f(1.6).

x	1	1.4	1.8	2.2
$f(x)$	3.49	4.82	5.96	6.5

Solution

Let $x = 1.6,\ x_0 = 1,\ h = 0.4$

$$r = \frac{x - x_0}{h} = \frac{1.6-1}{0.4} = 1.5$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	3.49			
		1.33		
1.4	4.82		−0.19	
		1.14		−0.41
1.8	5.96		−0.6	
		0.54		
2.2	6.5			

By Newton's forward interpolation formula,

$$f(x_0 + rh) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \frac{r(r-1)(r-2)}{3!}\Delta^3 f(x)_0 + \cdots$$

$$f(1.6) = 3.49 + 1.5(1.33) + \frac{1.5(1.5-1)}{2!}(-0.19) + \frac{1.5(1.5-1)(1.5-2)}{3!}(-0.41)$$

$$= 3.49 + 1.995 - 0.0713 + 0.0256$$

$$= 5.4393$$

Example 4

Use Newton's forward difference method to find the approximate value of f(1.3) from the following data:

x	1	2	3	4
$f(x)$	1.1	4.2	9.3	16.4

Solution

Let $x = 1.3,\ x_0 = 1,\ h = 1$

$$r = \frac{x - x_0}{h} = \frac{1.3 - 1}{1} = 0.3$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1.1			
		3.1		
2	4.2		2	
		5.1		0
3	9.3		2	
		7.1		
4	16.4			

By Newton's forward interpolation formula,

$$f(x) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \frac{r(r-1)(r-2)}{3!}\Delta^3 f(x)_0 + \cdots$$

$$f(1.3) = 1.1 + 0.3(3.1) + \frac{0.3(0.3-1)}{2!}(2) + 0$$

$$= 1.1 + 0.93 - 0.21$$

$$= 1.82$$

Example 5

Use Newton's forward difference method to find the approximate value of f(2.3) from the following data:

x	2	4	6	8
$f(x)$	4.2	8.2	12.2	16.2

Solution

Let $x = 2.3,\ x_0 = 2,\ h = 2$

$$r = \frac{x - x_0}{h} = \frac{2.3 - 2}{2} = 0.15$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
2	4.2		
		4	
4	8.2		0
		4	
6	12.2		0
		4	
8	16.2		

By Newton's forward interpolation formula,

$$f(x) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \cdots$$

$$\begin{aligned} f(2.3) &= 4.2 + 0.15(4) + 0 \\ &= 4.2 + 0.6 \\ &= 4.8 \end{aligned}$$

Example 6

Using Newton's forward interpolation formula, find the value of $f(218)$.

x	100	150	200	250	300	350	400
$f(x)$	10.63	13.03	15.04	16.81	18.42	19.90	21.27

[Summer 2014]

Solution

Let $x = 218,\quad x_0 = 100,\quad h = 50$

$$r = \frac{x - x_0}{h} = \frac{218 - 100}{50} = 2.36$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
100	10.63						
		2.4					
150	13.03		−0.39				
		2.01		0.15			
200	15.04		−0.24		−0.07		
		1.77		0.08		0.02	
250	16.81		−0.16		−0.05		0.02
		1.61		0.03		0.04	
300	18.42		−0.13		−0.01		
		1.48		0.02			
350	19.90		−0.11				
		1.37					
400	21.27						

By Newton's forward interpolation formula,

$$
\begin{aligned}
f(x) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \frac{r(r-1)(r-2)}{3!}\Delta^3 f(x_0) \\
+ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 f(x_0) + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 f(x_0) \\
+ \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 f(x_0) + \cdots
\end{aligned}
$$

$$
\begin{aligned}
f(218) &= 10.63 + 2.36(2.4) + \frac{2.36(2.36-1)}{2!}(-0.39) + \frac{2.36(2.36-1)(2.36-2)}{3!}(0.15) \\
&\quad + \frac{2.36(2.36-1)(2.36-2)(2.36-3)}{4!}(-0.07) \\
&\quad + \frac{2.36(2.36-1)(2.36-2)(2.36-3)(2.36-4)}{5!}(0.02) \\
&\quad + \frac{2.36(2.36-1)(2.36-2)(2.36-3)(2.36-4)(2.36-5)}{6!}(0.02) \\
&= 10.63 + 5.664 - 0.6259 + 0.0289 + 0.0022 + 0.0002 - 0.00009 \\
&= 15.6993
\end{aligned}
$$

Example 7

From the following table, estimate the number of students who obtained marks between 40 *and* 45:

Marks	30–40	40–50	50–60	60–70	70–80
Number of students	31	42	51	35	31

[Summer 2015]

Solution

Cumulative Frequency Table

Marks less than (x)	40	50	60	70	80
Number of students (y)	31	73	124	159	190

Since $x = 45$ is nearer to the beginning of the table, Newton's forward interpolation formula is used.

Let $x = 45, x_0 = 40, h = 10$

$$r = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		–25	
60	124		–16		37
		35		12	
70	159		–4		
		31			
80	190				

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots$$

$$y(45) = 31 + 0.5(42) + \frac{0.5(0.5-1)}{2!}(9) + \frac{0.5(0.5-1)(0.5-2)}{3!}(-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}(37)$$

$$= 31 + 21 - 1.1250 - 1.5625 - 1.4453$$

$$= 47.8672$$

$$\approx 48$$

The number of students with marks less than 45 = 48
The number of students with marks less than 40 = 31
Hence, the number of students obtaining marks between 40 and 45 = 48 – 31 = 17

Example 8

Determine the polynomial by Newton's forward difference formula from the following table:

x	0	1	2	3	4	5
y	–10	–8	–8	–4	10	40

Solution

Let $x_0 = 0, h = 1$

$$r = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	–10				
		2			
1	–8		–2		
		0		6	
2	–8		4		0
		4		6	
3	–4		10		0
		14		6	
4	10		16		
		30			
5	40				

By Newton's forward difference formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0$$

$$= -10 + x(2) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(6) + 0$$

$$= -10 + 2x - x(x-1) + x(x-1)(x-2)$$

$$= -10 + 2x - x^2 + x + x(x^2 - 3x + 2)$$

$$= -10 + 2x - x^2 + x + x^3 - 3x^2 + 2x$$

$$= x^3 - 4x^2 + 5x - 10$$

Example 9

Find a polynomial of degree 2 which takes the following values:

x	0	1	2	3	4	5	6	7
y	1	2	4	7	11	16	22	29

Solution

Let $x_0 = 0,\ h = 1$

$$r = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		1	
		2		0
2	4		1	
		3		0
3	7		1	
		4		0
4	11		1	
		5		0
5	16		1	
		6		
6	22		1	
		7		
7	29			

By Newton's forward interpolation formula,

$$\begin{aligned}
y(x) &= y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots \\
&= 1 + x(1) + \frac{x(x-1)}{2!}(1) + 0 \\
&= 1 + x + \frac{x^2 - x}{2} \\
&= 1 + x + \frac{x^2}{2} - \frac{x}{2} \\
&= \frac{1}{2}(x^2 + x + 2)
\end{aligned}$$

Example 10

Construct Newton's forward interpolation polynomial for the following data:

x	4	6	8	10
y	1	3	8	16

[Summer 2015]

Solution

Let $x_0 = 4,\ h = 2$

$$r = \frac{x - x_0}{h} = \frac{x-4}{2}$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1 + \left(\frac{x-4}{2}\right)(2) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2}-1\right)}{2!}(3) + 0$$

$$= 1 + (x-4) + \frac{(x-4)(x-6)}{8}(3)$$

$$= x - 3 + \frac{(x^2 - 10x + 24)}{8}(3)$$

$$= x - 3 + \frac{3x^2}{8} - \frac{15x}{4} + 9$$

$$= \frac{3x^2}{8} - \frac{11x}{4} + 6$$

$$y(5) = \frac{3(25)}{8} - \frac{11(5)}{4} + 6 = \frac{13}{8}$$

4.6 NEWTON'S BACKWARD INTERPOLATION FORMULA

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots.$ corresponding to the values $x_0, x_1, x_2, \ldots.$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + rh$, where r is any real number.

$$\begin{aligned} y_r &= f(x_n + rh) \\ &= E^r f(x_n) \\ &= (E^{-1})^{-r} y_n \\ &= (1-\nabla)^{-r} y_n \\ &= \left[1 + r\nabla + \frac{r(r+1)}{2!}\nabla^2 + \frac{r(r+1)(r+2)}{3!}\nabla^3 + \cdots\right] y_n \end{aligned}$$

[Using Binomial theorem]

$$= y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots \qquad \text{...(4.2)}$$

Equation (4.2) is known as *Newton's backward interpolation formula*.

Note This formula is used for evaluating the value of y near to the end of tabulated value of x, i.e., near x_n.

Example 1

Consider the following tabular values:

x	50	100	150	200	250
y	618	724	805	906	1032

Determine $y(300)$ *using Newton's backward interpolation formula.*

Solution

Let $x = 300, x_n = 250, h = 50$

$$r = \frac{x - x_n}{h} = \frac{300 - 250}{50} = 1$$

Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
50	618				
		106			
100	724		–25		
		81		45	
150	805		20		–40
		101		5	
200	906		25		
		126			
250	1032				

By Newton's backward interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

$$y(300) = 1032 + 1(126) + \frac{1(2)}{2!}(25) + \frac{1(2)(3)}{3!}(5) + \frac{1(2)(3)(4)}{4!}(-40)$$

$$= 1032 + 126 + 25 + 5 - 40$$

$$= 1148$$

Example 2

The area A of a circle of diameter d is given for the following values:

d	80	85	90	95	100
A	5026	5674	6362	7088	7854

Calculate the area of a circle of a diameter of 105 *units using Newton's interpolation formula.* **[Summer 2015]**

Solution

Since $x = d = 105$ is near to the end of the table, Newton's backward interpolation formula is used.

Let $\quad x = 105,\ x_n = 100,\quad h = 5$

$$r = \frac{x - x_n}{h} = \frac{105 - 100}{5} = 1$$

Difference Table

$x = d$	$y = A$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		648			
85	5674		40		
		688		–2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

By Newton's backward interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

$$\begin{aligned} y(105) &= 7854 + 1(766) + \frac{1(2)}{2!}(40) + \frac{1(2)(3)}{3!}(2) + \frac{1(2)(3)(4)}{4!}(4) \\ &= 7854 + 766 + 40 + 2 + 4 \\ &= 8666 \end{aligned}$$

Example 3

From the following table, find P when t = 142°C and 175°C using appropriate Newton's interpolation formula.

Temperature t°C	140	150	160	170	180
Pressure P	3685	4845	6302	8076	10225

[Winter 2014]

Solution

Since $x = 142$ is nearer to the beginning of the table, Newton's forward interpolation formula is used.

Let $x = 142,\quad x_0 = 140,\quad h = 10$

$$r = \frac{x - x_0}{h} = \frac{142 - 140}{10} = 0.2$$

Difference Table

$x = t°$	$y = P$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	3685				
		1169			
150	4854		279		
		1448		47	
160	6302		326		2
		1774		49	
170	8076		375		
		2149			
180	10225				

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots$$

$$P(142) = y(142) = 3685 + 0.2(1169) + \frac{0.2(0.2-1)}{2!}(279) + \frac{0.2(0.2-1)(0.2-2)}{3!}(47) + \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{4!}(2)$$

$$= 3685 + 233.8 - 22.32 + 2.256 - 0.0672$$

$$= 3898.6688$$

Since $x = 175$ is near to the end of the table, Newton's backward interpolation formula is used.

$$x = 175, \quad x_n = 180, \quad h = 10$$

$$r = \frac{x - x_n}{h} = \frac{175 - 180}{10} = -0.5$$

By Newton's backward interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

$$\begin{aligned} P(175) = y(175) &= 10225 + (-0.5)(2149) + \frac{(-0.5)(-0.5+1)}{2!}(375) \\ &\quad + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(49) \\ &\quad + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}(2) \\ &= 10225 - 1074.5 - 46.875 - 3.0625 - 0.0781 \\ &= 9100.4844 \end{aligned}$$

Example 4

The population of a town is given below. Estimate the population for the year 1895 and 1930 using suitable interpolation.

Year	x	1891	1901	1911	1921	1931
Population (in thousand)	y	46	66	81	93	101

[Summer 2015]

Solution

Since $x = 1895$ is near to the beginning of the table, Newton's forward interpolation formula is used.

Let $x = 1895, \quad x_0 = 1891, \quad h = 10$

$$r = \frac{x - x_0}{h} = \frac{1895 - 1891}{10} = 0.4$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		20			
1901	66		−5		
		15		2	
1911	81		−3		−3
		12		−1	
1921	93		−4		
		8			
1931	101				

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots$$

$$y(1895) = 46 + 0.4(20) + \frac{0.4(0.4-1)}{2!}(-5) + \frac{0.4(0.4-1)(0.4-2)}{3!}(2) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!}(-3)$$

$$= 46 + 8 + 0.6 + 0.128 + 0.1248$$

$$= 54.8528 \text{ thousands}$$

Since $x = 1930$ is near to the end of the table, Newton's backward interpolation formula is used.

Let $\quad x = 1930,\ x_n = 1931,\quad h = 10$

$$r = \frac{x - x_n}{h} = \frac{1930 - 1931}{10} = -0.1$$

By Newton's forward interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

$$y(1930) = 101 + (-0.1)(8) + \frac{(-0.1(-0.1+1)}{2!}(-4) + \frac{(-0.1)(-0.1+1)(-0.1+2)}{3!}(-1)$$
$$+ \frac{(-0.1)(-0.1+1)(-0.1+2)(-0.1+3)}{4!}(-3)$$
$$= 101 - 0.8 + 0.18 + 0.0285 + 0.062$$
$$= 100.4705 \text{ thousands}$$

Example 5

In the table below, the values of y are consecutive terms of a series of which 23.6 *is the sixth term. Find the first and tenth terms of the series:*

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Solution

To find the first term, Newton's forward interpolation formula is used.

Let $x = 1, x_0 = 3, h = 1$

$$r = \frac{x - x_0}{h} = \frac{1-3}{1} = -2$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots$$

$$y(1) = 4.8 + (-2)(3.6) + \frac{(-2)(-3)}{2!}(2.5) + \frac{(-2)(-3)(-4)}{3!}(0.5)$$

$$= 4.8 + 7.2 + 7.5 - 2$$

$$= 3.1$$

To find the tenth term, Newton's backward interpolation formula is used.

Let $x = 10,\ x_n = 9,\ h = 1$

$$r = \frac{x - x_n}{h} = \frac{10-9}{1} = 1$$

By Newton's backward interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

$$y(10) = 73.9 + 1(21.1) + \frac{1(2)}{2!}(4.5) + \frac{1(2)(3)}{3!}(0.5)$$

$$= 73.9 + 21.1 + 4.5 + 0.5$$

$$= 100$$

EXERCISE 4.2

1. Find tan 67° 20′ from the table:

θ	65°	66°	67°	68°	69°
tan θ	2.1445	2.2460	2.3559	2.4751	2.6051

[Ans.: 2.3932]

2. Find $(5.5)^3$ from the following table:

x	3	5	7	9	11
$y = x^3$	27	125	343	729	1331

[Ans.: 166.375]

3. Calculate $e^{1.85}$ from the following table:

x	1.7	1.8	1.9	2	2.1	2.2	2.3
e^x	5.474	6.050	6.686	7.389	8.166	9.025	9.974

[Ans.: 6.36]

4. Find $\sqrt{x}$ at $x = 2.52$ and $x = 2.62$ from the table:

x	2.5	2.55	2.6	2.65	2.7	2.75
$\sqrt{x}$	1.58114	1.59687	1.61245	1.62788	1.64317	1.65831

[Ans.: 1.58745, 1.6186]

5. The values of specific heat (C_p) at constant pressure of a gas are tabulated below for various temperatures. Find the specific heat at 900°C.

Temperature °C	500	1000	1500	2000
C_p	31.23	35.01	39.18	43.75

[Ans.: 34.223]

6. P and V are connected by the following data:

V	10	20	30	40
P	1.1	2	4.4	7.9

Determine P when $V = 25$ and $V = 45$. **[Ans.:** 3.0375, 9.9375]

7. Find the number of persons getting wages less than ₹15 from the following data:

Wages in ₹	0–10	10–20	20–30	30–40
Number of persons	9	30	35	42

[Ans.: 24]

8. Find the number of students getting marks less than 70 from the following data:

Marks	0–20	20–40	40–60	60–80	80–100
Number of students	41	62	65	50	17

[Ans.: 196]

9. From the following table, estimate the profit in the year 1925.

Year	1891	1901	1911	1921	1931
Profit in lakhs	46	66	81	93	101

[Ans.: ₹ 96.8365 lakhs]

10. Find the polynomial of degree three which takes the same values as $y = 2^x + 2x + 1$ at $x = -1, 0, 1, 2$.

$$\left[\textbf{Ans.: } \frac{1}{12}(x^3 + 3x^2 + 32x + 24)\right]$$

11. Obtain the cubic polynomial which takes the values

x	0	1	2	3
y	1	2	1	0

and, hence, find $f(4)$. **[Ans.:** $2x^3 - 7x^2 + 6x + 1$, 41]

12. Find a polynomial of degree 4 which takes the values

x	2	4	6	8	10
y	0	0	1	0	0

$$\left[\textbf{Ans.: } \frac{1}{64}(x^4 - 24x^3 + 196x^2 - 624x + 640)\right]$$

13. Given $u_1 = 40$, $u_3 = 45$, $u_5 = 54$, find u_2 and u_4. **[Ans.:** 42, 49]

14. Given $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 200$, $y_4 = 100$, $y_5 = 8$. Without forming the difference table, find $\Delta^5 y_0$. **[Ans.:** 755]

15. Find the polynomial of least degree passing through the points (0,–1), (1,1), (2,1), and (3,–2).

$$\left[\textbf{Ans.:}\ -\frac{1}{6}(x^3 + 3x^2 - 16x + 6)\right]$$

4.7 CENTRAL DIFFERENCE INTERPOLATION

Central difference interpolation formulae are used for interpolation near the middle of the tabulated values. If x takes the values $x_0 - 2h$, $x_0 - h$, x_0, $x_0 + h$, $x_0 + 2h$ and the corresponding values of $y = f(x)$ are y_{-2}, y_{-1}, y_0, y_1, y_2, the difference tables in the two notations are given as follows:

x	y	First Difference	Second Difference	Third Difference	Fourth Difference
$x_0 - 2h$	y_{-2}				
		$\Delta y_{-2} = \delta y_{-\frac{3}{2}}$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} = \delta^2 y_{-1}$		
		$\Delta y_{-1} = \delta y_{-\frac{1}{2}}$		$\Delta^3 y_{-2} = \delta^3 y_{-\frac{1}{2}}$	
x_0	y_0		$\Delta^2 y_{-1} = \delta^2 y_0$		$\Delta^4 y_{-2} = \delta^4 y_0$
		$\Delta y_0 = \delta y_{\frac{1}{2}}$		$\Delta^3 y_{-1} = \delta^3 y_{\frac{1}{2}}$	
$x_0 + h$	y_1		$\Delta^2 y_0 = \delta^2 y_1$		
		$\Delta y_1 = \delta y_{\frac{3}{2}}$			
$x_0 + 2h$	y_2				

4.8 GAUSS'S FORWARD INTERPOLATION FORMULA

By Newton's forward interpolation formula,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots \qquad ...(4.3)$$

where $r = \dfrac{x - x_0}{h}$

$$\Delta^2 y_0 = \Delta^2 Ey_{-1} = \Delta^2 (1+\Delta) y_{-1} = \Delta^2 y_{-1} + \Delta^3 y_{-1} \qquad ...(4.4)$$

$$\Delta^3 y_0 = \Delta^3 Ey_{-1} = \Delta^3 (1+\Delta) y_{-1} = \Delta^3 y_{-1} + \Delta^4 y_{-1} \qquad ...(4.5)$$

$$\Delta^4 y_0 = \Delta^4 Ey_{-1} = \Delta^4 (1+\Delta) y_{-1} = \Delta^4 y_{-1} + \Delta^5 y_{-1} \qquad ...(4.6)$$

Also,

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}, \text{ etc.} \qquad ...(4.7)$$

Substituting the values of $\Delta^2 y_0$ $\Delta^3 y_0$, ... in Eq. (4.3),

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{r(r-1)(r-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{r(r-1)(r-2)(r-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

[Using Eqs (4.4), (4.5), and (4.6)]

$$= y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \left[\frac{r(r-1)}{2!}\Delta^3 y_{-1} + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_{-1}\right] + \left[\frac{r(r-1)(r-2)}{3!}\Delta^4 y_{-1} + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_{-1} + \cdots\right]$$

$$= y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-1} + \cdots$$

$$= y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\left(\Delta^4 y_{-2} + \Delta^5 y_{-2}\right) + \cdots$$

[Using Eq. (4.7)]

$$= y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-2} + \cdots \qquad ...(4.8)$$

Equation (4.8) is known as *Gauss's forward interpolation formula*.

Corollary In the central difference notation,

$$y_r = y_0 + r\,\delta y_{\frac{1}{2}} + \frac{r(r-1)}{2!}\delta^2 y_0 + \frac{(r+1)r(r-1)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{(r+1)r(r-1)(r-2)}{4!}\delta^4 y_0 + \cdots$$

Notes

(i) This formula involves odd differences below the central line and even differences on the central line.

$$\begin{array}{ccccccccccccccc} y_0 & & \cdots & & \Delta^2 y_{-1} & & \cdots & & \Delta^4 y_{-2} & & \cdots & & \Delta^6 y_{-3} & & \cdots \quad \text{Central line} \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\ & & \Delta y_0 & & & & \Delta^3 y_{-1} & & & & \Delta^5 y_{-2} & & & & \Delta^7 y_{-3} \end{array}$$

(ii) This formula is used to evaluate the values of y for r between 0 and 1.

Example 1

Find $y(32)$ from the following table:

x	25	30	35	40
$y = f(x)$	0.2707	0.3027	0.3386	0.3794

Solution

Let $x = 32,\ x_0 = 30,\ h = 5$

$$r = \frac{x - x_0}{h} = \frac{32 - 30}{5} = 0.4$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$
25	–1	0.2707			
			0.0320		
30	0	0.3027		0.0039	
			0.0359		0.0010
35	1	0.3386		0.0049	
			0.0408		
40	2	0.3794			

By Gauss's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \cdots$$

$$y(32) = 0.3027 + 0.4(0.0359) + \frac{0.4(0.4-1)}{2!}(0.0039) + \frac{(0.4+1)(0.4)(0.4-1)}{3!}(0.0010)$$

$$= 0.3027 + 0.0144 - 0.0005 - 0.0001$$

$$= 0.3165$$

Example 2

Use Gauss's forward interpolation formula to find y(3.3) from the following table:

x	1	2	3	4	5
y	15.3	15.1	15	14.5	14

Solution

Let $x = 3.3,\ x_0 = 3,\ h = 1$

$$r = \frac{x - x_0}{h} = \frac{3.3 - 3}{1} = 0.3$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	–2	15.3				
			–0.2			
2	–1	15.1		0.1		
			–0.1		–0.5	
3	0	15		–0.4		0.9
			–0.5		0.4	
4	1	14.5		0		
			–0.5			
5	2	14				

By Gauss's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-2} + \cdots$$

$$y(3.3) = 15 + 0.3(-0.5) + \frac{0.3(0.3-1)}{2!}(-0.4) + \frac{(0.3+1)(0.3)(0.3-1)}{3!}(0.4) + \frac{(0.3+1)(0.3)(0.3-1)(0.3-2)}{4!}(0.9)$$

$$= 15 - 0.15 + 0.042 - 0.0182 + 0.0174$$

$$= 14.8912$$

Example 3

Find the polynomial which fits the data in the following table using Gauss's forward interpolation formula.

x	3	5	7	9	11
y	6	24	58	108	174

Solution

Let $x_0 = 7,\ h = 2$

$$r = \frac{x - x_0}{h} = \frac{x-7}{2}$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	–2	6				
			18			
5	–1	24		16		
			34		0	
7	0	58		16		0
			50		0	
9	1	108		16		
			66			
11	2	174				

By Gauss's forward interpolation formula,

$$\begin{aligned} y(x) &= y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \cdots \\ &= 58 + \left(\frac{x-7}{2}\right)(50) + \frac{1}{2}\left(\frac{x-7}{2}\right)\left(\frac{x-7}{2} - 1\right)(16) \\ &= 58 + 25(x-7) + 2(x-7)(x-9) \\ &= 58 + 25x - 175 + 2x^2 - 32x + 126 \\ &= 2x^2 - 7x + 9 \end{aligned}$$

4.9 GAUSS'S BACKWARD INTERPOLATION FORMULA

By Newton's forward interpolation formula,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots \quad \text{...(4.9)}$$

where $r = \dfrac{x - x_0}{h}$

$$\Delta y_0 = \Delta E y_{-1} = \Delta(1+\Delta)y_{-1} = \Delta y_{-1} + \Delta^2 y_{-1} \quad \text{...(4.10)}$$

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1} \quad \text{...(4.11)}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \quad \text{...(4.12)}$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \quad \text{...(4.13)}$$

Also, $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}, \text{ etc.} \quad \text{...(4.14)}$$

Substituting the values of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, …. in Eq. (4.9),

$$y_r = y_0 + r(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{r(r-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{r(r-1)(r-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{r(r-1)(r-2)(r-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

[Using Eqs (4.10), (4.11), and (4.12)]

$$= y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-1} + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^5 y_{-1} + \cdots$$

$$= y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}(\Delta^3 y_{-2} + \Delta^4 y_{-2})$$
$$+ \frac{(r+1)r(r-1)(r-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \dots$$

[Using Eq. (4.14)]

$$= y_0 + r\,\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\,\Delta^3 y_{-2}$$
$$+ \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + \cdots \qquad \text{...(4.15)}$$

Equation (4.15) is known as *Gauss's backward interpolation formula*.

Corollary In the central difference notation,

$$y_r = y_0 + r\,\delta y_{-\frac{1}{2}} + \frac{(r+1)r}{2!}\delta^2 y_0 + \frac{(r+1)r(r-1)}{3!}\delta^3 y_{-\frac{1}{2}} + \frac{(r+2)(r+1)r(r-1)}{4!}\delta^4 y_0 + \dots$$

Notes

(i) This formula involves odd differences above the central line and even differences on the central line.

$$\begin{tikzcd}[column sep=small]
 & \Delta y_{-1} \arrow[dr] & & \Delta^3 y_{-2} \arrow[dr] & & \Delta^5 y_{-3} \arrow[dr] & & \\
y_0 \arrow[ur] & \dots & \Delta^2 y_{-1} \arrow[ur] & \dots & \Delta^4 y_{-2} \arrow[ur] & \dots & \Delta^6 y_{-3} & \text{Central line}
\end{tikzcd}$$

(ii) This formula is used to evaluate the values of y for r between –1 and 0.

Example 1

Using Gauss's backward interpolation formula, find the population for the year 1936 *given that*

Year (x)	1901	1911	1921	1931	1941	1951
Population in thousands (y)	12	15	20	27	39	52

Solution

Let $\qquad x = 1936,\ x_0 = 1941,\ h = 10$

$$r = \frac{x - x_0}{h} = \frac{1936 - 1941}{10} = -0.5$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1901	–4	12					
			3				
1911	–3	15		2			
			5		0		
1921	–2	20		2		3	
			7		3		–10
1931	–1	27		5		–7	
			12		–4		
1941	0	39		1			
			13				
1951	1	52					

By Gauss's backward interpolation formula,

$$y(x) = y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \cdots$$

$$\begin{aligned} y(1936) &= 39 + (-0.5)(12) + \frac{(-0.5+1)(-0.5)}{2!}(1) + \frac{(-0.5+1)(-0.5)(-0.5-1)}{3!}(-4) \\ &= 39 - 6 - 0.1250 - 0.25 \\ &= 32.625 \text{ thousands} \end{aligned}$$

Example 2

Find $y(2.36)$ from the following table:

x	1.6	1.8	2	2.2	2.4	2.6
y	4.95	6.05	7.39	9.03	11.02	13.46

Solution

Let $\quad x = 2.36,\ x_0 = 2.4,\ h = 0.2$

$$r = \frac{x - x_0}{h} = \frac{2.36 - 2.4}{0.2} = -0.2$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.6	−4	4.95					
			1.1				
1.8	−3	6.05		0.24			
			1.34		0.06		
2	−2	7.39		0.3		−0.01	
			1.64		0.05		0.06
2.2	−1	9.03		0.35		0.05	
			1.99		0.1		
2.4	0	11.02		0.45			
			2.44				
2.6	1	13.46					

By Gauss's backward interpolation formula,

$$y(x) = y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \cdots$$

$$y(2.36) = 11.02 + (-0.2)(1.99) + \frac{(-0.2+1)(-0.2)}{2!}(0.45) + \frac{(-0.2+1)(-0.2)(-0.2-1)}{3!}(0.1)$$

$$= 11.02 - 0.398 - 0.036 + 0.0032$$

$$= 10.5892$$

Example 3

From the following table, find y when x = 38.

x	30	35	40	45	50
y	15.9	14.9	14.1	13.3	12.5

Solution

Let $x = 38, \quad x_0 = 40, \quad h = 5$

$$r = \frac{x - x_0}{h} = \frac{38 - 40}{5} = -0.4$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
30	–2	15.9				
			-1			
35	–1	14.9		0.2		
			-0.8		-0.2	
40	0	14.1		0		0.2
			-0.8		0	
45	1	13.3		0		
			-0.8			
50	2	12.5				

By Gauss's backward interpolation formula,

$$y(x) = y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + \cdots$$

$$\begin{aligned} y(38) &= 14.1 + (-0.4)(-0.8) + \frac{(-0.4+1)(-0.4)}{2!}(0) + \frac{(-0.4+1)(-0.4)(-0.4-1)}{3!}(-0.2) \\ &\quad + \frac{(-0.4+2)(-0.4+1)(-0.4)(-0.4-1)}{4!}(0.2) \\ &= 14.1 + 0.32 + 0 - 0.0112 + 0.0045 \\ &= 14.4133 \end{aligned}$$

4.10 STIRLING'S FORMULA

By Gauss's forward interpolation formula,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-2} + \cdots \quad ...(4.16)$$

By Gauss's backward interpolation formula,

$$y_r = y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + \cdots \quad (4.17)$$

Adding Eqs (4.16) and (4.17) and then dividing by 2,

$$y_r = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \frac{r^2(r^2-1)}{4!}\Delta^4 y_{-2} + \cdots \quad \text{...(4.18)}$$

Equation (4.18) is known as *Stirling's formula*.

Corollary In the central difference notation,

$$y_r = y_0 + r\,\mu\,\delta y_0 + \frac{r^2}{2!}\delta^2 y_0 + \frac{r(r^2+1^2)}{3!}\mu\,\delta^3 y_0 + \frac{r^2(r^2-1^2)}{4!}\delta^4 y_0 + \cdots$$

$$\frac{1}{2}(\Delta y_{-1} + \Delta y_0) = \frac{1}{2}\left(\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}}\right) = \mu\,\delta y_0$$

$$\frac{1}{2}(\Delta^3 y_{-2} + \Delta^3 y_{-1}) = \frac{1}{2}\left(\delta^3 y_{\frac{1}{2}} + \delta^3 y_{-\frac{1}{2}}\right) = \mu\,\delta^3 y_0, \quad \text{etc.}$$

Notes

(i) This formula involves means of the odd differences just above and below the central line and even differences on the central line.

$$y_0 \begin{pmatrix} \Delta y_{-1} \\ \Delta y_0 \end{pmatrix} \cdots \Delta^2 y_{-1} \cdots \begin{pmatrix} \Delta^3 y_{-2} \\ \Delta^3 y_{-1} \end{pmatrix} \cdots \Delta^4 y_{-2} \cdots \text{Central line}$$

(ii) This formula gives fairly accurate values of y for r between –0.25 and 0.2 but can be used for r between –0.5 to 0.5.

Example 1

Using Stirling's formula, estimate the value of tan 16°.

x	0°	5°	10°	15°	20°	25°	30°
$y = \tan x$	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Solution

Let $x = 16°, x_0 = 15°, h = 5°$

$$r = \frac{x - x_0}{h} = \frac{16° - 15°}{5°} = 0.2$$

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0°	–3	0						
			0.0875					
5°	–2	0.0875		0.0013				
			0.0888		0.0015			
10°	–1	0.1763		0.0028		0.0002		
			0.0916		0.0017		–0.0002	
15°	0	0.2679		0.0045		0		0.0011
			0.0961		0.0017		0.0009	
20°	1	0.3640		0.0062		0.0009		
			0.1023		0.0026			
25°	2	0.4663		0.0088				
			0.1111					
30°	3	0.5774						

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right)$$
$$+ \frac{r^2(r^2-1)}{4!}\Delta^4 y_{-2} + \frac{r(r^2-1)(r^2-4)}{5!}\left(\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2}\right)$$
$$+ \frac{r^2(r^2-1)(r^2-4)}{6!}\Delta^6 y_{-3} + \cdots$$

$$y(16) = 0.2679 + 0.2\left(\frac{0.0916 + 0.0961}{2}\right) + \frac{(0.2)^2}{2!}(0.0045)$$
$$+ \frac{(0.2)(0.2^2-1)}{3!}\left(\frac{0.0017 + 0.0017}{2}\right) + 0$$
$$+ \frac{(0.2)(0.2^2-1)(0.2^2-4)}{5!}\left(\frac{-0.0002 + 0.0009}{2}\right)$$
$$+ \frac{(0.2)^2(0.2^2-1)(0.2^2-4)}{6!}(0.0011)$$
$$= 0.2679 + 0.0188 + (9\times 10^{-5}) - (5.44\times 10^{-5})$$
$$+ 0 + (2.2176\times 10^{-6}) + (2.3232\times 10^{-7})$$
$$= 0.2867$$

Example 2

Employ Stirling's formula to compute y(35) from the following table:

x	20	30	40	50
y	512	439	346	243

Solution

Let $x = 35,\ x_0 = 30,\ h = 10$

$$r = \frac{x - x_0}{h} = \frac{35 - 30}{10} = 0.5$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	–1	512			
			–73		
30	0	439		–20	
			–93		10
40	1	346		–10	
			–103		
50	2	243			

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2 - 1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \cdots$$

$$\begin{aligned} y(35) &= 439 + 0.5\left(\frac{-73 - 93}{2}\right) + \frac{(0.5)^2}{2!}(-20) + \frac{0.5(0.5^2 - 1)}{3!}\left(\frac{10}{2}\right) \\ &= 439 - 41.5 - 2.5 - 0.3125 \\ &= 394.6875 \end{aligned}$$

Example 3

Let $f(40) = 836$, $f(50) = 682$, $f(60) = 436$, $f(70) = 272$. *Use Stirling's formula to find* $f(55)$.

Solution

Let $x = 55, \quad x_0 = 50, \quad h = 10$

$$r = \frac{x - x_0}{h} = \frac{55 - 50}{10} = 0.5$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$
40	–1	836			
			–154		
50	0	682		–92	
			–246		174
60	1	436		–82	
			–164		
70	2	272			

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \cdots$$

$$\begin{aligned} y(55) &= 682 + 0.5\left(\frac{-154-246}{2}\right) + \frac{(0.5)^2}{2!}(-92) + \frac{0.5(0.5^2-1)}{3!}\left(\frac{174}{2}\right) \\ &= 682 - 100 - 11.5 - 5.4375 \\ &= 565.0625 \end{aligned}$$

Example 4

Using Stirling's formula, find y(25) from the following table:

x	20	24	28	32
y	0.01427	0.01581	0.01772	0.01996

Solution

Let $\quad x = 25, \quad x_0 = 24, \quad h = 4$

$$r = \frac{x - x_0}{h} = \frac{25 - 24}{4} = 0.25$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	–1	0.01427			
			0.00154		
24	0	0.01581		0.00037	
			0.00191		–0.00004
28	1	0.01772		0.00033	
			0.00224		
32	2	0.01996			

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \cdots$$

$$y(25) = 0.01581 + 0.25\left(\frac{0.00154 + 0.00191}{2}\right) + \frac{(0.25)^2}{2!}(0.00037) + \frac{0.25(0.25^2-1)}{3!}\left(\frac{-0.00004}{2}\right)$$

$$= 0.01581 + 4.3125\times 10^{-4} + 1.15625\times 10^{-5} + 7.8125\times 10^{-7}$$

$$= 0.01625$$

Example 5

Find the value of $y(1.63)$ *from the following table using Stirling's formula:*

x	1.5	1.6	1.7	1.8	1.9
$y = f(x)$	17.609	20.412	23.045	25.527	27.875

Solution

Let $x = 1.63, \quad x_0 = 1.6, \quad h = 0.1$

$$r = \frac{x - x_0}{h} = \frac{1.63 - 1.6}{0.1} = 0.3$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5	–1	17.609				
			2.803			
1.6	0	20.412		−0.17		
			2.633		0.019	
1.7	1	23.045		−0.151		–0.002
			2.482		0.017	
1.8	2	25.527		−0.134		
			2.348			
1.9	3	27.875				

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \cdots$$

$$y(1.63) = 20.412 + 0.3\left(\frac{2.803 + 2.633}{2}\right) + \frac{(0.3)^2}{2!}(-0.17)$$

$$+ \frac{0.3(0.3^2-1)}{3!}\left(\frac{0.019}{2}\right)$$

$$= 20.412 + 0.8154 - 7.65\times 10^{-3} - 4.3225\times 10^{-4}$$

$$= 21.2193$$

EXERCISE 4.3

1. Use Gauss's interpolation formula to find y_{16}.

x	5	10	15	20	25
y	26.782	19.951	14.001	8.762	4.163

[Ans.: 12.901]

2. Find $e^{-1.7425}$ by Gauss's forward formula.

x	1.72	1.73	1.74	1.75	1.76
e^{-x}	0.17907	0.17728	0.17552	0.17377	0.17204

[Ans.: 0.17508]

3. Find $f(25)$ given $f(20) = 14$, $f(24) = 32$, $f(28) = 35$, and $f(32) = 40$ using Gauss's formula.

[Ans.: 33.41]

4. Apply Gauss's backward formula to find the population in 1926.

Year	x	1911	1921	1931	1941	1951
Population in lacs	y	15	20	27	39	52

[Ans.: 22.898 lacs]

5. Apply Gauss's backward interpolation formula to find sin 45°.

$x°$	20	30	40	50	60	70
sin $x°$	0.34202	0.50200	0.64279	0.76604	0.86603	0.93969

[Ans.: 0.705990]

6. Use Gauss's backward formula, find $f(5.8)$ given that $f(x)$ is a polynomial of degree four and $f(4) = 270$, $f(5) = 648$, $\Delta f(5) = 682$, $\Delta^2 f(4) = 132$

[Ans.: 1163]

7. Using Stirling's formula, find $y(5)$ from the following table:

x	0	4	8	12
y	14.27	15.81	17.72	19.96

[Ans.: 16.25]

8. Find $\sqrt{1.12}$ using Stirling's formula from the following table:

x	1.0	1.05	1.10	1.15	1.20	1.25	1.30
$f(x)$	1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.14017

[Ans.: 1.05830]

9. Use Stirling's formula to find tan 89° 26′ from the table:

x	89° 21′	89° 23′	89° 25′	89° 27′	89° 29′
tan x	88.14	92.91	98.22	104.17	110.90

[Ans.: 101.107]

4.11 INTERPOLATION WITH UNEQUAL INTERVALS

If the values of x are unequally spaced then interpolation formulae for equally spaced points cannot be used. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. There are two such formulae for unequally spaced values of x.

(i) Lagrange's interpolation formula

(ii) Newton's interpolation formula with divided difference

4.12 LAGRANGE'S INTERPOLATION FORMULA

Let $y = f(x)$ be a function which take the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x = x_0, x_1, x_2, \ldots, x_n$. Since there are $(n + 1)$ values of x and y, $f(x)$ can be represented by a polynomial in x of degree n.

$$\begin{aligned} y = f(x) = a_0 (x - x_1)(x - x_2) \ldots (x - x_n) + a_1 (x - x_0)(x - x_2) \ldots (x - x_n) + \ldots \\ + a_n (x - x_0)(x - x_1) \ldots (x - x_{n-1}) \end{aligned} \qquad \ldots(4.19)$$

where $a_0, a_1, a_2, ..., a_n$ are constants.

Putting $x = x_0$, $y = y_0$ in Eq. (4.19),

$$y_0 = a_0 (x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)$$

$$a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}$$

Similarly, putting $x = x_1$, $y = y_1$ in Eq. (4.19),

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)}$$

Proceeding in the same way,

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}$$

Substituting the values of $a_0, a_1, a_2 \ldots, a_n$ in Eq. (4.19),

$$\begin{aligned} f(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} y_1 + \ldots \\ + \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} y_n \end{aligned} \qquad \ldots(4.20)$$

Equation (4.20) is known as *Lagrange's interpolation formula*.

Note This formula can also be used to split the given function into partial fractions. Dividing both sides of Eq. (4.20) by $(x - x_0)(x - x_1) \ldots (x - x_n)$,

$$\begin{aligned} \frac{f(x)}{(x - x_0)(x - x_1)\ldots(x - x_n)} &= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} \left(\frac{1}{x - x_0} \right) \\ &+ \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} \left(\frac{1}{x - x_1} \right) + \ldots \\ &+ \frac{y_n}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} \left(\frac{1}{x - x_n} \right) \end{aligned}$$

Example 1

Compute f(9.2) by using Lagrange's interpolation method from the following data:

x	9	9.5	11
$f(x)$	2.1972	2.2513	2.3979

[Summer 2013]

Solution

By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$f(9.2) = \frac{(9.2-9.5)(9.2-11)}{(9-9.5)(9-11)}(2.1972) + \frac{(9.2-9)(9.2-11)}{(9.5-9)(9.5\text{-}11)}(2.2513) + \frac{(9.2-9)(9.2-9.5)}{(11-9)(11-9.5)}(2.3979)$$

$$= 1.1865 + 1.0806 - 0.048$$

$$= 2.2191$$

Example 2

Find the value of y when x = 10 *from the following table:*

x	5	6	9	11
y	12	13	14	16

Solution

By Lagrange's interpolation formula,

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13) + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= 2 - 4.3333 + 11.6666 + 5.3333$$

$$= 14.6666$$

Example 3

Compute f(4) from the tabular values given:

x	2	3	5	7
$f(x)$	0.1506	0.3001	0.4517	0.6259

using Lagrange's interpolation formula. **[Winter 2012]**

Solution

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$f(4)=\frac{(4-3)(4-5)(4-7)}{(2-3)(2-5)(2-7)}(0.1506)+\frac{(4-2)(4-5)(4-7)}{(3-2)(3-5)(3-7)}(0.3001)$$
$$+\frac{(4-2)(4-3)(4-7)}{(5-2)(5-3)(5-7)}(0.4517)+\frac{(4-2)(4-3)(4-5)}{(7-2)(7-3)(7-5)}(0.6259)$$
$$=-0.0301+0.2251+0.2259-0.0313$$
$$=0.3896$$

Example 4

Compute f(2) by using Lagrange's interpolation method from the following data:

x	–1	0	1	3
$f(x)$	2	1	0	–1

[Winter 2013, Summer 2015]

Solution

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$
\begin{aligned}
f(2) &= \frac{(2-0)(2-1)(2-3)}{(-1-0)(-1-1)(-1-3)}(2)+\frac{(2+1)(2-1)(2-3)}{(0+1)(0-1)(0-3)}(1)\\
&\quad+\frac{(2+1)(2-0)(2-3)}{(1+1)(1-0)(1-3)}(0)+\frac{(2+1)(2-0)(2-1)}{(3+1)(3-0)(3-1)}(-1)\\
&= 0.5-1+0-0.25\\
&= -0.75
\end{aligned}
$$

Example 5

By using Lagrange's formula, find y when x = 10.

x	5	6	9	11
y	12	13	14	16

[Summer 2015]

Solution

By Lagrange's interpolation formula,

$$
\begin{aligned}
y(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1\\
&\quad+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3\\
y(10) &= \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12)+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)\\
&\quad+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14)+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)\\
&= 2-4.3333+11.6667+5.3333\\
&= 14.6667
\end{aligned}
$$

Example 6

Evaluate f(9) by using Lagrange's interpolation method from the following data:

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

[Summer 2014]

Solution

By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} f(x_0)$$
$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} f(x_1)$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} f(x_2)$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} f(x_3)$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} f(x_4)$$

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}(150)$$
$$+ \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}(392)$$
$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}(1452)$$
$$+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}(2366)$$
$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}(5202)$$
$$= -16.6667 + 209.0667 + 1290.6667 - 788.6667 + 115.6$$
$$= 810$$

Example 7

Determine $y(12)$ by using Lagrange's interpolation method from the following data:

x	11	13	14	18	20	23
y	25	47	68	82	102	124

[Winter 2014]

Solution

By Lagrange's interpolation formula,

$$\begin{aligned}
y(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}y_0 \\
&+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}y_1 \\
&+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}y_2 \\
&+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}y_3 \\
&+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}y_4 \\
&+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}y_5 \\
y(12) &= \frac{(12-13)(12-14)(12-18)(12-20)(12-23)}{(11-13)(11-14)(11-18)(11-20)(11-23)}(25) \\
&= \frac{(12-11)(12-14)(12-18)(12-20)(12-23)}{(13-11)(13-14)(13-18)(13-20)(13-23)}(47) \\
&= \frac{(12-11)(12-13)(12-18)(12-20)(12-23)}{(14-11)(14-13)(14-18)(14-20)(14-23)}(68) \\
&= \frac{(12-11)(12-13)(12-14)(12-20)(12-23)}{(18-11)(18-13)(18-14)(18-20)(18-23)}(82) \\
&= \frac{(12-11)(12-13)(12-14)(12-18)(12-23)}{(20-11)(20-13)(20-14)(20-18)(20-23)}(102) \\
&= \frac{(12-11)(12-13)(12-14)(12-18)(12-20)}{(23-11)(23-13)(23-14)(23-18)(23-20)}(124) \\
&= 5.8201 + 70.9029 - 55.4074 + 10.3086 - 5.9365 + 0.7348 \\
&= 26.4225
\end{aligned}$$

Example 8

Find a second-degree polynomial passing through the points (0, 0), (1, 1) *and* (2, 20) *using Larange's interpolation.* **[Summer 2015]**

Solution

Let $x_0 = 0, \quad x_1 = 1, \quad x_2 = 2$

$f(x_0) = 0, \quad f(x_1) = 1, \quad f(x_2) = 20$

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$
$$=\frac{(x-1)(x-2)}{(0-1)(0-2)}(0)+\frac{(x-0)(x-2)}{(1-0)(1-2)}(1)+\frac{(x-0)(x-1)}{(2-0)(2-1)}(20)$$
$$=0-x(x-2)+10x(x-1)$$
$$=9x^2-8x$$

Example 9

Using Lagrange's interpolation formula, find the interpolating polynomial for the following table:

x	0	1	2	5
$f(x)$	2	3	12	147

Solution

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$
$$=\frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2)+\frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$
$$+\frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12)+\frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$
$$=\frac{(x-1)(x^2-7x+10)}{-5}+\frac{3x(x^2-7x+10)}{4}+\frac{2x(x^2-6x+5)}{-1}+\frac{49x(x^2-3x+2)}{20}$$
$$=\frac{\left\{\begin{array}{l}-4(x^3-8x^2+17x-10)+(15x^3-105x^2+150x)\\ \qquad\qquad -20(2x^3-12x^2+10x)+(49x^3-147x^2+98x)\end{array}\right\}}{20}$$
$$=\frac{20x^3+20x^2-20x+40}{20}$$
$$=x^3+x^2-x+2$$

Example 10

Find the Lagrange interpolating polynomial from the following data:

x	0	1	4	5
$f(x)$	1	3	24	39

Solution

By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$= \frac{(x-1)(x-4)(x-5)}{(0-1)(0-4)(0-5)}(1) + \frac{(x-0)(x-4)(x-5)}{(1-0)(1-4)(1-5)}(3)$$

$$+ \frac{(x-0)(x-1)(x-5)}{(4-0)(4-1)(4-5)}(24) + \frac{(x-0)(x-1)(x-4)}{(5-0)(5-1)(5-4)}(39)$$

$$= -\frac{(x-1)(x-4)(x-5)}{20} + \frac{x(x-4)(x-5)}{4}$$

$$-2x(x-1)(x-5) + \frac{39x(x-1)(x-4)}{20}$$

$$= -\frac{x^3-10x^2+29x-20}{20} + \frac{x^3-9x^2+20x}{4}$$

$$-(2x^3-12x^2+10x) + \frac{39(x^3-5x^2+4x)}{20}$$

$$= \frac{1}{20}(3x^3+10x^2+27x+20)$$

Example 11

Use Lagrange's formula to fit a polynomial to the data:

x	–1	0	2	3
y	8	3	1	12

and hence, find $y(2)$.

Solution

By Lagrange's interpolation formula,

$$
\begin{aligned}
y(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 \\
&\quad + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3 \\
&= \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(8) + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3) \\
&\quad + \frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1) + \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}(12) \\
&= -\frac{2x(x-2)(x-3)}{3} + \frac{(x+1)(x-2)(x-3)}{2} \\
&\quad - \frac{(x+1)(x)(x-3)}{6} + (x+1)(x)(x-2) \\
&= \frac{1}{3}(2x^3 + 2x^2 - 15x + 9) \\
y(2) &= \frac{1}{3}\left[2(8) + 2(4) - 15(2) + 9\right] = 1
\end{aligned}
$$

Example 12

Express the given rational function as a sum of partial fractions:

$$
y = \frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)}
$$

Solution

Let $f(x) = 3x^2 + x + 1$.
For $x = 1$, $x = 2$ and $x = 3$, the table is

x	1	2	3
$f(x)$	5	15	31

By Lagrange's interpolation formula,

$$
\begin{aligned}
f(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2) \\
&= \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31) \\
&= \frac{5}{2}(x-2)(x-3) - 15(x-1)(x-3) + \frac{31}{2}(x-1)(x-2)
\end{aligned}
$$

$$\therefore \quad y = \frac{f(x)}{(x-1)(x-2)(x-3)}$$

$$= \frac{5}{2(x-1)} - \frac{15}{x-2} + \frac{31}{2(x-3)}$$

Example 13

Express the function $\dfrac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)}$ *as a sum of partial fractions using Lagrange's formula.*

Solution

Let $f(x) = 3x^2 - 12x + 11$.
For $x = 1$, $x = 2$ and $x = 3$, the table is

x	1	2	3
$f(x)$	2	–1	2

By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= \frac{(x-2)(x-3)}{(1-2)(1-3)}(2) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(-1) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(2)$$

$$= (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)$$

$$\therefore \quad y = \frac{f(x)}{(x-1)(x-2)(x-3)}$$

$$= \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}$$

Example 14

The following values of the function $f(x)$ *are given as* $f(1) = 4$, $f(2) = 5$, $f(7) = 5$, $f(8) = 4$. *Find the value of* $f(6)$ *and also the value of* x *for which* $f(x)$ *is maximum or minimum.*

Solution

Tabular form of the data is

x	1	2	7	8
$f(x)$	4	5	5	4

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$=\frac{(x-2)(x-7)(x-8)}{(1-2)(1-7)(1-8)}(4)+\frac{(x-1)(x-7)(x-8)}{(2-1)(2-7)(2-8)}(5)+\frac{(x-1)(x-2)(x-8)}{(7-1)(7-2)(7-8)}(5)$$

$$+\frac{(x-1)(x-2)(x-7)}{(8-1)(8-2)(8-7)}(4)$$

$$=-\frac{2}{21}(x^3-17x^2+86x-112)+\frac{1}{6}(x^3-16x^2+71x-56)$$

$$-\frac{1}{6}(x^3-11x^2+26x-16)+\frac{2}{21}(x^3-10x^2+23x-14)$$

$$=-\frac{1}{6}x^2+\frac{3}{2}x+\frac{8}{3}$$

$$f(6)=-\frac{1}{6}(6)^2+\frac{3}{2}(6)+\frac{8}{3}=\frac{17}{3}$$

For extreme values,
$$f'(x)=0$$
$$-\frac{1}{3}x+\frac{3}{2}=0$$
$$x=4.5$$

Also,
$$f''(x)=-\frac{1}{3}<0$$

Since $f''(x)$ is negative, $f(x)$ is maximum at $x = 4.5$.

Example 15

A body moving with velocity v at any time t satisfies the data

t	0	1	3	4
v	21	15	12	10

Obtain the distance travelled in 4 seconds and acceleration at the end of 4 *seconds.*

Solutions

By Lagrange's interpolation formula,

$$
\begin{aligned}
v &= \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)}v_0 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_1 \\
&\quad + \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)}v_2 + \frac{(t-t_0)(t-t_1)(t-t_2)}{(t_3-t_0)(t_3-t_1)(t_3-t_2)}v_3 \\
&= \frac{(t-1)(t-3)(t-4)}{(0-1)(0-3)(0-4)}(21) + \frac{(t-0)(t-3)(t-4)}{(1-0)(1-3)(1-4)}(15) \\
&\quad + \frac{(t-0)(t-1)(t-4)}{(3-0)(3-1)(3-4)}(12) + \frac{(t-0)(t-1)(t-3)}{(4-0)(4-1)(4-3)}(10) \\
&= \frac{1}{12}(-5t^3+38t^2-105t+252)
\end{aligned}
$$

If s is the distance travelled in time t,

$$
\begin{aligned}
v &= \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{12}(-5t^3+38t^2-105t+252) \\
s &= \int_0^4 v\,\mathrm{d}t \\
&= \frac{1}{12}\int_0^4 (-5t^3+38t^2-105t+252) \\
&= \frac{1}{12}\left| -\frac{5t^4}{4} + \frac{38t^3}{3} - \frac{105t^2}{2} + 252t \right|_0^4 \\
&= \frac{1}{12}\left[-\frac{5}{4}\times 256 + \frac{38}{3}\times 64 - \frac{105}{2}\times 16 + 1008 \right] \\
&= 54.88
\end{aligned}
$$

Hence, the distance travelled in 4 seconds = 54.88

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{12}(-15t^2+76t-105)$$

At $t = 4$,

$$a = \frac{1}{12}(-15\times 16 + 76\times 4 - 105) = 3.416$$

EXERCISE 7.4

1. From the table given below, find $y(x = 2)$.

x	0	1	3	4
y	5	6	50	105

[Ans.: 19]

2. Use Lagrange's formula to find the velocity of the particle $v = f(t)$ at $t = 3.5$ from the following table:

t	0	1	2	3
v	21	15	12	10

[Ans.: 8.75]

3. Find $f(27)$ from the following table:

x	14	17	31	35
$f(x)$	68.7	64.0	44.0	39.1

[Ans.: 49.3]

4. Find $f(6)$ from the following table:

x	2	5	7	10	12
$f(x)$	18	180	448	1210	2028

[Ans.: 294]

5. Find $f(9)$ from the following table:

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

[Ans.: 809.997]

6. If $y_0 = 4.3315$, $y_1 = 7.4046$, $y_3 = 5.6713$, $y_5 = 7.1154$, find the curve passing through these points. Hence, find y_2 and y_4.

[Ans.: 5.1420, 6.3199]

7. If $f(1) = 3$, $f(2) = -5$, $f(-4) = 4$, find the three-point Lagrange's interpolation polynomial that takes the same values.

$$\left[\textbf{Ans.: } \frac{1}{20}(-39x^2 - 123x + 252)\right]$$

8. A third-degree polynomial passes through the points (0,–1), (1,1), (2,1), (3,–2). Find the polynomial.

$$\left[\textbf{Ans.: } \frac{1}{6}(-x^3 - 3x^2 + 16x - 6)\right]$$

9. If $y = a_0 + a_1x + a_2x^2 + a_3x^3$ passes through the points

x	1	3	5	7
y	0	50	236	654

find a_0, a_1, a_2 and a_3.

[**Ans.:** –4, 3, –1, 2]

10. Find the polynomial of degree 3 which takes the same values as $y = 2^x + 2x + 1$ at $x = -1, 0, 1, 2$.

$$\left[\textbf{Ans.: } \frac{1}{12}\left(x^3 + 3x^2 + 32x + 24\right)\right]$$

11. Find the polynomial which takes the values $f(1) = 1$, $f(2) = 9$, $f(3) = 25$, $f(4) = 55$, $f(5) = 105$.

[**Ans.:** $x^3 - 2x^2 + 7x - 5$]

12. Find $f(x)$ from the following table:

x	0	2	3	6
$f(x)$	659	705	729	804

$$\left[\textbf{Ans.: } \frac{1}{72}(-x^3 + 29x^2 + 1604x + 47448)\right]$$

13. Observe the following table:

x	1	3	4	6
$f(x)$	–3	9	30	132

Express $f(x)$ as a third-degree polynomial in x. Also, find $f'(x)$, $f''(x)$ at $x = 1$.

[**Ans.:** $x^3 - 3x^2 + 5x - 6$, 2, 0]

14. Using Lagrange's formula for unequal intervals, express the function $\dfrac{x^2 + 6x - 1}{(x^2 - 1)(x - 4)(x - 6)}$ as a sum of partial fractions.

$$\left[\textbf{Ans.: } \frac{1}{5(x-1)} + \frac{3}{35(x+1)} - \frac{13}{10(x-4)} + \frac{71}{70(x-6)}\right]$$

4.13 DIVIDED DIFFERENCES

In Lagrange's interpolation formula, if another interpolation value is added then the interpolation coefficients are required to be recalculated. To avoid this recalculation, Newton's general interpolation formula is used.

If $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ …. be given points then the first divided difference for x_0, x_1 is defined by the relation,

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly, $[x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1}$, etc.

The second divided difference for x_0, x_1, x_2 is defined as

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

The third divided difference for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$

Notes

(i) The divided differences are symmetrical in their arguments, i.e., independent of the order of arguments:

$$\begin{aligned}[x_0, x_1] &= \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0} \\ &= [x_1, x_0]\end{aligned}$$

$$\begin{aligned}[x_0, x_1, x_2] &= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)} \\ &= [x_1, x_2, x_0] \text{ or } [x_2, x_0, x_1]\end{aligned}$$

(ii) The n^{th} divided differences of a polynomial of the n^{th} degree are constant.

Let the arguments be equally spaced so that $x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$

$$\begin{aligned}[x_0, x_1] &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= \frac{\Delta y_0}{h}\end{aligned}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

$$= \frac{1}{2h}\left(\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right)$$

$$= \frac{1}{2!h^2}\Delta^2 y_0$$

In general,

$$[x_0, x_1, x_2 ..., x_n] = \frac{1}{n!h^n}\Delta^n y_0$$

If the tabulated function is an n^{th} degree polynomial, $\Delta^n y_0$ will be constant. Hence, the n^{th} divided differences will also be constant.

4.14 NEWTON'S DIVIDED DIFFERENCE FORMULA

Let the function $y = f(x)$ take values $y_0, y_1, y_2, \dots, y_n$ corresponding to $x_0, x_1, x_2, \dots, x_n$ respectively. According to the definition of divided differences,

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

$$y = y_0 + (x - x_0)[x, x_0] \qquad ...(4.21)$$

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

$$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$$

Substituting the value of $[x, x_0]$ in Eq. (4.21),

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x, x_0, x_1] \qquad ...(4.22)$$

Also, $\quad [x, x_0, x_1, x_2] = \dfrac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$

$$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)\,[x, x_0, x_1, x_2]$$

Substituting the value of $[x, x_0, x_1]$ in Eq. (4.22),

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2]$$
$$+ (x - x_0)\,(x - x_1)\,(x - x_2)\,[x, x_0, x_1, x_2]$$

Proceeding in the same manner,

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2]$$
$$+ (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3] + \dots$$
$$+ (x - x_0)\,(x - x_1)\,\dots\,(x - x_{n-1})\,[x, x_0, x_1, \dots, x_n] \qquad \dots(4.23)$$

Equation (4.23) is known as *Newton's general interpolation formula* with divided differences.

Example 1

If $f(x) = \dfrac{1}{x}$, *find the divided difference* $[a, b]$ *and* $[a, b, c]$.

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference
a	$\dfrac{1}{a}$		
		$\dfrac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\dfrac{1}{ab}$	
b	$\dfrac{1}{b}$		$\dfrac{-\frac{1}{bc}+\frac{1}{ab}}{c-a} = \dfrac{1}{abc}$
		$\dfrac{\frac{1}{c}-\frac{1}{b}}{c-b} = -\dfrac{1}{bc}$	
c	$\dfrac{1}{c}$		

$$[a,b] = -\frac{1}{ab}$$

$$[a,b,c] = \frac{1}{abc}$$

Example 2

Find the second divided difference for the argument $x = 1, 2, 5,$ *and* 7 *for the function* $f(x) = x^2$. **[Summer 2015]**

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	1			
		$\frac{4-1}{2-1} = 3$		
2	4		$\frac{7-3}{5-1} = 1$	
		$\frac{25-4}{5-2} = 7$		0
5	25		$\frac{12-7}{7-2} = 1$	
		$\frac{49-25}{7-5} = 12$		
7	49			

Example 3

Find the third divided difference with arguments 2, 4, 9, 10 *of the function* $f(x) = x^3 - 2x$.

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
2	4			
		$\frac{56-4}{4-2} = 26$		
4	56		$\frac{131-26}{9-2} = 15$	
		$\frac{711-56}{9-4} = 131$		$\frac{23-15}{10-2} = 1$
9	711		$\frac{269-131}{10-4} = 23$	
		$\frac{980-711}{10-9} = 269$		
10	980			

Example 4

Construct the divided difference for the data given below:

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335

[Summer 2015]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
-4	1245				
		$\frac{33-1245}{-1+4}=-404$			
-1	33		$\frac{-28+404}{0+4}=94$		
		$\frac{5-33}{0+1}=-28$		$\frac{10-94}{2+4}=-14$	
0	5		$\frac{2+28}{2+1}=10$		$\frac{13+14}{5+1}=13$
		$\frac{9-5}{2-0}=2$		$\frac{88-10}{5+1}=13$	
2	9		$\frac{442-2}{5-0}=88$		
		$\frac{1335-9}{5-2}=442$			
5	1335				

Example 5

Complete $f(9.2)$ from the following data by using Newton's divided difference interpolation formula.

x	8	9	9.5	11
$f(x)$	2.079442	2.197225	2.251292	2.397895

[Winter 2013]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
8	2.079442			
		0.117783		
9	2.197225		−0.006433	
		0.108134		0.000411
9.5	2.251292		−0.005200	
		0.097735		
11	2.397895			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$\begin{aligned} f(9.2) &= 2.079442 + (9.2 - 8)(0.117783) + (9.2 - 8)(9.2 - 9)(-0.006433) \\ &\quad + (9.2 - 8)(9.2 - 9)(9.2 - 9.5)(0.00041) \\ &= 2.079442 + 0.141340 - 0.001544 - 0.000030 \\ &= 2.219208 \end{aligned}$$

Example 6

Using Newton's divided difference formula, compute $f(10.5)$ from the following data:

x	10	11	13	17
$f(x)$	2.3026	2.3979	2.5649	2.8332

[Summer 2013]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
10	2.3026			
		0.0953		
11	2.3979		−0.0039	
		0.0835		0.0002
13	2.5649		−0.0027	
		0.0671		
17	2.8332			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$\begin{aligned} f(10.5) &= 2.3026 + (10.5 - 10)\,(0.0953) + (10.5 - 10)\,(10.5 - 11)\,(-0.0039) \\ &\quad + (10.5 - 10)\,(10.5 - 11)\,(10.5 - 13)\,(0.0002) \\ &= 2.3026 + 0.0477 + 0.00098 + 0.00013 \\ &= 2.3514 \end{aligned}$$

Example 7

Using Newton's divided difference interpolation, compute the value of $f(6)$ from the table given below:

x	1	2	7	8
$f(x)$	1	5	5	4

[Summer 2015]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	1			
		4		
2	5		$-\frac{2}{3}$	
		0		$\frac{1}{14}$
7	5		$-\frac{1}{6}$	
		−1		
8	4			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3]$$

$$f(6) = 1 + (6 - 1)(4) + (6 - 1)(6 - 2)\left(-\frac{2}{3}\right) + (6 - 1)(6 - 2)(6 - 7)\left(\frac{1}{14}\right)$$

$$= 1 + 20 - 13.3333 - 1.4286$$

$$= 6.2381$$

Example 8

Evaluate $f(9)$ using the following table:

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

[Summer 2014]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		42		
		709			
17	5202				

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)(x - x_2)(x - x_3)[x_0, x_1, x_2, x_3, x_4]$$

$$f(9) = 150 + (9 - 5)(121) + (9 - 5)(9 - 7)(24) + (9 - 5)(9 - 7)(9 - 11)(1) + 0$$

$$= 150 + 484 + 192 - 16$$

$$= 810$$

Example 9

Compute $f(8)$ from the following values using Newton's divided difference formula:

x	4	5	7	10	11	13
$f(x)$	48	100	244	900	1210	2028

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
4	48				
		52			
5	100		15		
		97		1	
7	244		21		0
		202		1	
10	900		27		0
		310		1	
11	1210				
		409	33		
13	2028				

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$f(8) = 48 + (8 - 4)(52) + (8 - 4)(8 - 5)(15) + (8 - 4)(8 - 5)(8 - 7)(1) + 0$$
$$= 48 + 208 + 180 + 12$$
$$= 448$$

Example 10

From the following table, find f(x) using Newton's divided difference formula:

x	1	2	7	8
$f(x)$	1	5	5	4

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	1			
		4		
2	5		$-\frac{2}{3}$	
		0		$\frac{1}{14}$
7	5		$-\frac{1}{6}$	
		–1		
8	4			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$= 1 + (x - 1)\,4 + (x - 1)(x - 2)\left(-\frac{2}{3}\right) + (x - 1)(x - 2)(x - 7)\left(\frac{1}{14}\right)$$

$$= 1 + 4x - 4 - \frac{2}{3}(x^2 - 3x + 2) + \frac{1}{14}(x^3 - 10x^2 + 23x - 14)$$

$$= \frac{1}{14}x^3 - \frac{29}{21}x^2 + \frac{107}{14}x - \frac{16}{3}$$

Example 11

Using Newton's divided difference formula, find a polynomial and also, find f(–1) and f(6).

x	1	2	4	7
$f(x)$	10	15	67	430

[Summer 2015]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	10			
		5		
2	15		7	
		26		2
4	67		19	
		121		
7	430			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$= 10 + (x-1)(5) + (x-1)(x-2)(7) + (x-1)(x-2)(x-4)2$$

$$= 10 + 5x - 5 + 7x^2 - 21x + 14 + 2x^3 - 14x^2 + 28x - 16$$

$$= 2x^3 - 7x^2 + 12x + 3$$

$$f(-1) = 2(-1)^3 - 7(-1)^2 + 12(-1) + 3 = -18$$

$$f(6) = 2(6)^3 - 7(6)^2 + 12(6) + 3 = 255$$

Example 12

Establish a cubic polynomial of the curve $y = f(x)$ passing through the points (1, –3), (3, 9), (4, 30), (6, 132). *Hence, find* $f(2)$.

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	–3			
		6		
3	9		5	
		21		1
4	30		10	
		51		
6	132			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$= -3 + (x - 1)\,(6) + (x - 1)\,(x - 3)\,(5) + (x - 1)\,(x - 3)\,(x - 4)\,(1)$$

$$= -3 + 6x - 6 + 5x^2 - 20x + 15 + x^3 - 8x^2 + 19x - 12$$

$$= x^3 - 3x^2 + 5x - 6$$

$$f(2) = (2)^3 - 3(2)^2 + 5(2) - 6 = 0$$

Example 13

The shear stress in kilopound per square foot (ksf) for 5 *specimens in a clay stratum are as follows:*

Depth (*m*)	1.9	3.1	4.2	5.1	5.8
Stress (*ksf*)	0.3	0.6	0.4	0.9	0.7

Use Newton's dividend difference interpolating polynomial to compute the stress at 4.5 *m depth.* **[Winter 2012]**

Solution

Divided Difference Table

Depth x	Stress y	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
1.9	0.3				
		0.25			
3.1	0.6		–0.1877		
		–0.1818		0.1739	
4.2	0.4		0.3687		–0.1295
		0.5556		–0.3313	
5.1	0.9		–0.5258		
		–0.2857			
5.8	0.7				

By Newton's divided difference formula,

$$y(x) = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3] + (x - x_0)\,(x - x_1)\,(x - x_2)\,(x - x_3)\,[x_0, x_1, x_2, x_3, x_4]$$

$$y(4.5) = 0.3 + (4.5 - 1.9)(0.25) + (4.5 - 1.9)(4.5 - 3.1)(-0.1877) + (4.5 - 1.9)(4.5 - 3.1)(4.5 - 4.2)(0.1739) + (4.5 - 1.9)(4.5 - 3.1)(4.5 - 4.2)(4.5 - 5.1)(-0.1295)$$

$$= 0.3 + 0.65 - 0.6832 + 0.1899 + 0.0848$$

$$= 0.5415 \text{ ksf}$$

EXERCISE 4.5

1. If $f(x) = \dfrac{1}{x^2}$, find the divided differences $f(a, b)$, $f(a, b, c)$, and $f(a, b, c, d)$.

$$\left[\textbf{Ans.: } -\frac{(a+b)}{a^2b^2}, \frac{ab+bc+ca}{a^2b^2c^2}, -\frac{(abc+bcd+acd+abd)}{a^2b^2c^2d^2}\right]$$

2. Find the third divided difference of $f(x)$ with arguments 2, 4, 9, 10 where $f(x) = x^3 - 2x$.

[**Ans.:** 1]

3. Obtain the value of $\log_{10} 656$ given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$ and $\log_{10} 666 = 2.8202$.

[**Ans.:** 2.8169]

4. Find $f(5)$ from the following table:

x	0	1	3	6
$f(x)$	1	4	88	1309

[**Ans.:** 636]

5. Find $y(x = 20)$ from the following table:

x	12	18	22	24	32
$y(x)$	146	836	19481	2796	9236

[**Ans.:** 1305.36]

6. Find a polynomial $f(x)$ of lowest degree which takes the values 3, 7, 9, and 19 when $x = 2, 4, 5, 10$.

[**Ans.:** $2x - 1$]

7. Using the divided difference table, find $f(x)$ which takes the values 1, 4, 40, 85 as $x = 0, 1, 3, 4$.

[**Ans.:** $x^3 + x^2 + x + 1$]

8. Find $f(x)$ as a polynomial by using Newton's formula:

x	–1	0	3	6	7
$f(x)$	3	–6	39	822	1611

[**Ans.:** $x^4 - 3x^3 + 5x^2 - 6$]

9. Find the polynomial $y = f(x)$ passing through (5, 1355), (2, 9), (0, 5), (–1, 33), and (–4, 1245).

[Ans.: $3x^4 - 5x^3 + 6x^2 + 14x + 5$]

10. Find the polynomial equation of degree 4 passing through the points (8, 1515), (7, 778), (5, 138), (4, 43), and (2, 3).

[Ans.: $x^4 - 10x^3 + 36x^2 - 36x - 5$]

11. Find the function $y(x)$ in powers of $(x - 1)$ given $y(0) = 8$, $y(1) = 11$, $y(4) = 68$, $y(5) = 123$.

[Ans.: $11 + 4(x - 1) + 2(x - 1)^2 + (x - 1)^3$]

12. Using the following table, find $f(x)$ as a polynomial in powers of $(x - 6)$.

x	–1	0	2	3	7	10
$f(x)$	–11	1	1	1	141	561

[Ans.: $73 + 54(x - 6) + 13(x - 6)^2 + (x - 6)^3$]

4.15 INVERSE INTERPOLATION

The process of evaluating the value of x for a value of y (which is not in the table) is called *inverse interpolation*. Lagrange's formula is a relation between two variables, either of which may be taken as the independent variable. On interchanging x and y in the Lagrange's interpolation formula,

$$x = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_1)}x_0 + \frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)}x_1 + \ldots$$
$$+ \frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots(y_n-y_{n-1})}x_n \quad \ldots(4.24)$$

Equation (4.24) is used for inverse interpolation.

Example 1

From the data given, find the value of x when y = 13.5.

x	93	96.2	100	104.2	108.7
y	11.38	12.80	14.70	17.07	19.91

Solution

By Lagrange's formula for inverse interpolation,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y-y_4)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_4)}x_1$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_3)(y-y_4)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_4)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_4)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)(y_3-y_4)}x_3$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_3)}x_4$$

$$
\begin{aligned}
&= \frac{(13.5-12.80)(13.5-14.70)(13.5-17.07)(13.5-19.91)}{(11.38-12.80)(11.38-14.70)(11.38-17.07)(11.38-19.91)}(93) \\
&\quad + \frac{(13.5-11.38)(13.5-14.70)(13.5-17.07)(13.5-19.91)}{(12.80-11.38)(12.80-14.70)(12.80-17.07)(12.80-19.91)}(96.2) \\
&\quad + \frac{(13.5-11.38)(13.5-12.80)(13.5-17.07)(13.5-19.91)}{(14.70-11.38)(14.70-12.80)(14.70-17.07)(14.70-19.91)}(100) \\
&\quad + \frac{(13.5-11.38)(13.5-12.80)(13.5-14.70)(13.5-19.91)}{(17.07-11.38)(17.07-12.80)(17.07-14.70)(17.07-19.91)}(104.2) \\
&\quad + \frac{(13.5-11.38)(13.5-12.80)(13.5-14.70)(13.5-17.07)}{(19.91-11.38)(19.91-12.80)(19.91-14.70)(19.91-17.07)}(108.7)
\end{aligned}
$$

$$
\begin{aligned}
x &= -7.8137 + 68.4669 + 43.6076 - 7.2758 + 0.7711 \\
&= 97.7561
\end{aligned}
$$

Example 2

Find the root of the equation $f(x) = 0$, *given that* $f(30) = -30$, $f(34) = -13$, $f(38) = 3$, *and* $f(42) = 18$.

Solution

Let $x_0 = 30,\quad x_1 = 34, x_2 = 38, x_3 = 42$

$y_0 = -30, y_1 = -13, y_2 = 3, y_3 = 18$

It is required to find x for $y = f(x) = 0$.

By Lagrange's formula for inverse interpolation,

$$
\begin{aligned}
x &= \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1 \\
&\quad + \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3 \\
&= \frac{(0+13)(0-3)(0-18)}{(-30+13)(-30-3)(-30-18)}(30) + \frac{(0+30)(0-3)(0-18)}{(-13+30)(-13-3)(-13-18)}(34) \\
&\quad + \frac{(0+30)(0+13)(0-18)}{(3+30)(3+13)(3-18)}(38) + \frac{(0+30)(0+13)(0-3)}{(18+30)(18+13)(18-3)}(42) \\
&= -0.782 + 6.5323 + 33.6818 - 2.2016 \\
&= 37.2305
\end{aligned}
$$

Hence, the root of $f(x) = 0$ is 37.2305.

EXERCISE 4.6

1. Find x given $y = 0.3887$ from the following data:

x	21	23	25
y	0.3706	0.4068	0.4433

[Ans.: 22]

2. Find x corresponding to $y = 85$ from the following table:

x	2	5	8	14
y	94.8	87.9	81.3	68.7

[Ans.: 6.5928]

3. Find x corresponding to $y = 100$ from the following table:

x	3	5	7	9	11
y	6	24	58	108	174

[Ans.: 8.656]

4. Find the value of θ given $f(\theta) = 0.3887$ where $f(\theta) = \int_0^\theta \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}}$ using the table:

θ	21°	23°	25°
$f(\theta)$	0.3706	0.4068	0.4433

[Ans.: 22.0020°]

5. Find the age corresponding to the annuity value 13.6 from the given table:

Age (x)	30	35	40	45	50
Annuity value (y)	15.9	14.9	14.1	13.3	12.5

[Ans.: 43]

4.16 CUBIC SPLINE INTERPOLATION

In the polynomial interpolation method discussed till now, the complete set of tabulated values were approximated by a single higher degree polynomial. But, for many functions, the corresponding interpolation polynomial may tend to oscillate more and more between nodes (end points of sub-intervals) as the degree of the polynomial increases. Such oscillations are avoided using the method of splines in which piecewise polynomial approximations are used.

In spline interpolation, the complete interval is divided into sub-intervals and the function is approximated by lower degree polynomials called *spline functions*.

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, ..., y_n$ corresponding to $x = x_0, x_1, x_2, ..., x_n$. This function is approximated by cubic splines in each interval $x_i - x_{i-1} = h$, $i = 1, 2, ..., n$. Since there are n equal intervals, n cubic polynomials are found. A cubic polynomial $S(x)$ satisfies the following conditions:

(i) $S(x)$ is a polynomial of degree one for $x < x_0$ and $x > x_n$.

(ii) $S(x)$ is at most a cubic polynomial in each interval (x_{i-1}, x_i), $i = 1, 2, ..., n$.

(iii) $S(x)$, $S'(x)$ and $S''(x)$ are continuous at each point (x_i, y_i), $i = 0, 1, 2, ..., n$.

(iv) $S(x_i) = y_i$, $i = 0, 1, 2, ..., n$.

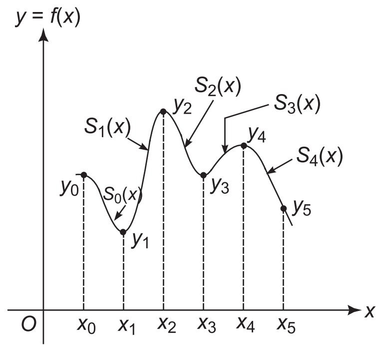

Fig. 4.1

Since $S(x)$ is a cubic polynomial, $S''(x)$ is linear in each interval (x_{i-1}, x_i), $i = 1, 2, ..., n$.

Let $$S''(x) = \frac{1}{h}\left[(x_i - x)S''(x_{i-1}) + (x - x_{i-1})S''(x_i)\right] \qquad ...(4.25)$$

Integrating Eq. (4.25) twice w.r.t. x,

$$S(x) = \frac{1}{h}\left[\frac{(x_i - x)^3}{3!}S''(x_{i-1}) + \frac{(x - x_{i-1})^3}{3!}S''(x_i)\right] + a_i(x_i - x) + b_i(x - x_{i-1}) \qquad ...(4.26)$$

where a_i and b_i are constants to be found out by the condition

$$S(x_i) = y_i, \qquad i = 0, 1, 2, ..., n \qquad (4.27)$$

Putting $x = x_{i-1}$ in Eq. (4.26),

$$y_{i-1} = \frac{1}{h}\left[\frac{h^3}{3!}S''(x_{i-1})\right] + ha_i$$

$$a_i = \frac{1}{h}\left[y_{i-1} - \frac{h^2}{3!}S''(x_{i-1})\right] \qquad ...(4.28)$$

Similarly, putting $x = x_i$ in Eq. (4.26),

$$b_i = \frac{1}{h}\left[y_i - \frac{h^2}{3!}S''(x_i)\right] \qquad ...(4.29)$$

Substituting the values of a_i and b_i in Eq. (4.26),

$$S(x) = \frac{1}{h}\left[\frac{(x_i - x)^3}{3!}S''(x_{i-1}) + \frac{(x - x_{i-1})^3}{3!}S''(x_i)\right] + \frac{1}{h}(x_i - x)\left[y_{i-1} - \frac{h^2}{3!}S''(x_{i-1})\right] + \frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{3!}S''(x_i)\right]$$

Writing $S''(x_i) = M_i$,

$$S(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right]$$

$$+\frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6}M_i\right] \qquad ...(4.30)$$

Differentiating Eq. (4.30) w.r.t. x,

$$S'(x) = \frac{1}{6h}\left[3(x_i - x)^2(-M_{i-1}) + 3(x - x_{i-1})^2 M_i\right]$$

$$+\frac{1}{h}\left[-y_{i-1} + \frac{h^2}{6}M_{i-1}\right] + \frac{1}{h}\left[y_i - \frac{h^2}{6}M_i\right]$$

Since $S'(x)$ is continuous,

$$S'(x_i^-) = \frac{h}{3}M_i + \frac{h}{6}M_{i-1} + \frac{1}{h}(y_i - y_{i-1}) \qquad ...(4.31)$$

Similarly,

$$S'(x_i^+) = -\frac{h}{3}M_i - \frac{h}{6}M_{i+1} + \frac{1}{h}(y_{i+1} - y_i) \qquad ...(4.32)$$

Equating Eqs (4.31) and (4.32),

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), \qquad i = 1, 2, ..., n \qquad ...(4.33)$$

Since $S(x)$ is linear for $x < x_0$ and $x > x_n$, $S''(x) = 0$ at $x = x_0$ and $x = x_n$.

Hence, $M_0 = 0$, $M_n = 0$ ...(4.34)

Equations (4.33) and (4.34) give $(n + 1)$ equations in $(n + 1)$ unknowns $M_0, M_1, M_2, ..., M_n$. Substituting the values of $M_0, M_1, M_2, ..., M_n$ in Eq. (4.30), the cubic spline in each interval is obtained.

Example 1

Test whether the following functions are cubic spline or not.

(*i*) $S_1(x) = x^2 - x + 1 \qquad 1 \le x \le 2$

$S_2(x) = 3x - 3 \qquad 2 \le x \le 3$

(*ii*) $S_1(x) = -2x^2 + x^3 \qquad -1 \le x \le 0$

$S_2(x) = x^2 - 2x^3 \qquad 0 \le x \le 1$

Solution

Each polynomial is at most a cubic polynomial in each sub-interval.

(i) $S_1(2) = 3 = S_2(2)$

$S_1'(2) = 3 = S_2'(2)$

$S_1''(2) = 2, \; S_2''(2) = 0$

$S''(x)$ is not continuous at $x = 2$.
Hence, the functions are not cubic splines.

(ii) $S_1(0) = 0 = S_2(0)$

$S_1'(0) = 0 = S_2'(0)$

$S_1''(0) = -4, S_2''(0) = 2$

$S''(x)$ is not continuous at $x = 0$.
Hence, the functions are not cubic splines.

Example 2

From the following data,

x	1	2	3
y	–8	–1	18

Compute $y(1.5)$ *and* $y'(1)$ *jusing cubic splines.*

Solution

$$h = 1, \qquad n = 2$$

Also, $M_0 = 0$ and $M_2 = 0$ (assumption)

For cubic spline interpolation,

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), \;\; i = 1, 2, ..., n-1$$

For $i = 1$,

$$\begin{aligned} M_0 + 4M_1 + M_2 &= 6(y_0 - 2y_1 + y_2) \\ &= 6[-8 - 2(-1) + 18] \\ &= 72 \\ 4M_1 &= 72 \qquad [\because \;\; M_0 = M_2 = 0] \\ M_1 &= 18 \end{aligned}$$

The cubic spline functions are given by

$$S(x)=\frac{1}{6h}\left[(x_i-x)^3 M_{i-1}+(x-x_{i-1})^3 M_i\right]+\frac{1}{h}(x_i-x)\left(y_{i-1}-\frac{h^2}{6}M_{i-1}\right)$$
$$+\frac{1}{h}(x-x_{i-1})\left(y_i-\frac{h^2}{6}M_i\right),\quad i=1,2,\ldots,n$$

For $i = 1$ in the interval $1 \le x \le 2$,

$$S(x)=\frac{1}{6}\left[(x_1-x)^3 M_0+(x-x_0)^3 M_1+(x_1-x)\left(y_0-\frac{1}{6}M_0\right)+(x-x_0)\left(y_1-\frac{1}{6}M_1\right)\right]$$
$$=\frac{1}{6}\left[(2-x)^3(0)+(x-1)^3(18)\right]+(2-x)\left[-8-\frac{1}{6}(0)\right]+(x-1)\left[-1-\frac{1}{6}(18)\right]$$
$$=\frac{1}{6}\left[18(x-1)^3\right]+(2-x)(-8)+(x-1)(-4)$$
$$=3x^3-9x^2+9x-3-16+8x-4x+4$$
$$=3x^3-9x^2+13x-15$$

$$y(1.5)=S(1.5)=3(1.5)^3-9(1.5)^2+13(1.5)-15=-5.625$$
$$y'=S'(x)=9x^2-18x+13$$
$$y'(1)=S'(1)=9(1)^2-18(1)+13=4$$

Example 3

Using cubic splines, find $y(0.5)$ and $y'(1)$, given $M_0 = M_2 = 0$

x	0	1	2
y	–5	–4	3

Solution

$$h=1,\qquad n=2,\qquad M_0=0,\qquad M_2=0$$

For cubic spline interpolation,

$$M_{i-1}+4M_i+M_{i+1}=\frac{6}{h^2}(y_{i-1}-2y_i+y_{i+1}),\ \ n=1,2,\ldots,n-1$$

For $i = 1$,

$$M_0+4M_1+M_2=6(y_0-2y_1+y_2)$$
$$=6[-5-2(-4)+3]$$
$$=36$$

$$4M_1 = 36 \qquad [\because \quad M_0 = M_2 = 0]$$
$$M_1 = 9$$

The cubic splines function are given by

$$S(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)\left(y_{i-1} - \frac{h^2}{6}M_{i-1}\right) + \frac{1}{h}(x - x_{i-1})\left(y_i - \frac{h^2}{6}M_i\right)$$

For $i = 1$ in the interval $0 \le x \le 1$,

$$S(x) = \frac{1}{6}\left[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1\right] + (x_1 - x)\left(y_0 - \frac{1}{6}M_0\right) + (x - x_0)\left(y_1 - \frac{1}{6}M_1\right)$$
$$= \frac{1}{6}\left[(1 - x)^3(0) + (x - 0)^3(9)\right] + (1 - x)\left[-5 - \frac{1}{6}(0)\right] + (x - 0)\left[-4 - \frac{1}{6}(9)\right]$$
$$= \frac{3}{2}x^3 - 5(1 - x) - \frac{33}{6}x$$
$$= \frac{3}{2}x^3 - \frac{x}{2} - 5$$

$$y(0.5) = S(0.5) = \frac{3}{2}(0.5)^3 - \frac{(0.5)}{2} - 5 = -5.0625$$
$$y' = S'(x) = \frac{9}{2}x^2 - \frac{1}{2}$$
$$y'(1) = S'(1) = \frac{9}{2}(1)^2 - \frac{1}{2} = 4$$

Example 4

Obtain the cubic splines for every sub-interval from the following data:

x	0	1	2	3
$y = f(x)$	1	2	33	244

Hence, find estimate of $f(2.5)$. Assume $M(0) = 0$, $M(3) = 0$.

[Summer 2013, Winter 2013, Summer 2014]

Solution

$$h = 1, \qquad n = 3, \qquad M_0 = 0, \qquad M_3 = 0$$

For cubic spline interpolation,

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), \quad i = 1, 2, ..., n-1$$

For $i = 1$,

$$\begin{aligned} M_0 + 4M_1 + M_2 &= 6(y_0 - 2y_1 + y_2) \\ &= 6[1 - 2(2) + 33] \\ &= 180 \end{aligned}$$

$$4M_1 + M_2 = 180 \qquad [\because \quad M_0 = 0] \qquad ...(1)$$

For $i = 2$,

$$\begin{aligned} M_1 + 4M_2 + M_3 &= 6(y_1 - 2y_2 + y_3) \\ &= 6[2 - 2(33) + 244] \\ &= 1080 \end{aligned}$$

$$M_1 + 4M_2 = 1080 \qquad [\because \quad M_3 = 0] \qquad ...(2)$$

Solving Eqs (1) and (2),

$$M_1 = -24, \qquad M_2 = 276$$

The cubic spline functions are given by

$$S(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)\left(y_{i-1} - \frac{h^2}{6}M_{i-1}\right)$$
$$+ \frac{1}{h}(x - x_{i-1})\left(y_i - \frac{h^2}{6}M_i\right), \qquad i = 1, 2, ..., n$$

For $i = 1$ in the interval $0 \le x \le 1$,

$$\begin{aligned} S(x) &= \frac{1}{6}\left[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1\right] + (x_1 - x)\left(y_0 - \frac{1}{6}M_0\right) + (x - x_0)\left(y_1 - \frac{1}{6}M_1\right) \\ &= \frac{1}{6}\left[(1 - x)^3(0) + (x - 0)^3(-24)\right] + (1 - x)\left[1 - \frac{1}{6}(0)\right] + (x - 0)\left[2 - \frac{1}{6}(-24)\right] \\ &= -4x^3 + (1 - x) + 6x \\ &= -4x^3 + 5x + 1 \end{aligned}$$

For $i = 2$ in the interval $1 \le x \le 2$,

$$\begin{aligned} S(x) &= \frac{1}{6}\left[(x_2 - x)^3 M_1 + (x - x_1)^3 M_2\right] + (x_2 - x)\left(y_1 - \frac{1}{6}M_1\right) + (x - x_1)\left(y_2 - \frac{1}{6}M_2\right) \\ &= \frac{1}{6}\left[(2 - x)^3(-24) + (x - 1)^3(276)\right] + (2 - x)\left[2 - \frac{1}{6}(-24)\right] + (x - 1)\left[33 - \frac{1}{6}(276)\right] \\ &= -4(2 - x)^3 + 46(x - 1)^3 + 6(2 - x) - 13(x - 1) \end{aligned}$$

$$= 4x^3 - 24x^2 + 48x - 32 + 46x^3 - 138x^2 + 138x - 46 + 12 - 6x - 13x + 13$$
$$= 50x^3 - 162x^2 + 167x - 53$$

For $i = 3$ in the interval $2 \le x \le 3$,

$$S(x) = \frac{1}{6}\left[(x_3 - x)^3 M_2 + (x - x_2)^3 M_3\right] + (x_3 - x)\left(y_2 - \frac{1}{6}M_2\right) + (x - x_2)\left(y_3 - \frac{1}{6}M_3\right)$$
$$= \frac{1}{6}\left[(3 - x)^3(276) + (x - 2)^3(0)\right] + (3 - x)\left[33 - \frac{1}{6}(276)\right] + (x - 2)\left[244 - \frac{1}{6}(0)\right]$$
$$= \frac{1}{6}\left[(3 - x)^3(276)\right] + (3 - x)(-13) + (x - 2)(244)$$
$$= 46(27 - 27x + 9x^2 - x^3) - 39 + 13x + 244x - 488$$
$$= -46x^3 + 414x^2 - 985x + 715$$

$$f(2.5) = S(2.5) = -46(2.5)^3 + 414(2.5)^2 - 985(2.5) + 715 = 121.25$$

Example 5

Obtain the cubic splines and evaluate $y(1.5)$ and $y'(3)$ for the following data:

x	1	2	3	4
y	1	2	5	11

[Summer 2015, Winter 2012, 2014]

Solution

$h = 1, \quad n = 3,$

Also, $M_0 = 0$ and $M_3 = 0$ (assumption)

For cubic spline interpolation,

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}) \qquad i = 1, 2, ..., n-1$$

For $i = 1$,

$$M_0 + 4M_1 + M_2 = 6(y_0 - 2y_1 + y_2)$$
$$= 6[1 - 2(2) + 5]$$
$$= 12$$

$$4M_1 + M_2 = 12 \qquad [\because \quad M_0 = 0] \qquad ...(1)$$

For $i = 2$,

$$M_1 + 4M_2 + M_3 = 6(y_1 - 2y_2 + y_3)$$
$$= 6[2 - 2(5) + 11]$$
$$= 18$$
$$M_1 + 4M_2 = 18 \qquad [\because \quad M_3 = 0] \qquad \text{...(2)}$$

Solving Eqs (1) and (2),

$$M_1 = 2, \qquad M_2 = 4$$

The cubic spline functions are given by

$$S(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)\left(y_{i-1} - \frac{h^2}{6}M_{i-1}\right)$$
$$+ \frac{1}{h}(x - x_{i-1})\left(y_i - \frac{h^2}{6}M_i\right), \qquad i = 1, 2, ..., n$$

For $i = 1$, in the interval $1 \le x \le 2$,

$$S(x) = \frac{1}{6}\left[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1\right] + (x_1 - x)\left(y_0 - \frac{1}{6}M_0\right) + (x - x_0)\left(y_1 - \frac{1}{6}M_1\right)$$
$$= \frac{1}{6}\left[(2-x)^3(0) + (x-1)^3(2)\right] + (2-x)\left[1 - \frac{1}{6}(0)\right] + (x-1)\left[2 - \frac{1}{6}(2)\right]$$
$$= \frac{1}{3}(x-1)^3 + (2-x) + \frac{5}{3}(x-1)$$
$$= \frac{1}{3}[x^3 - 3x^2 + 3x - 1 + 6 - 3x + 5x - 5]$$
$$= \frac{1}{3}(x^3 - 3x^2 + 5x)$$

For $i = 2$ in the interval $2 \le x \le 3$,

$$S(x) = \frac{1}{6}\left[(x_2 - x)^3 M_1 + (x - x_1)^3 M_2\right] + (x_2 - x)\left(y_1 - \frac{1}{6}M_1\right) + (x - x_1)\left(y_2 - \frac{1}{6}M_2\right)$$
$$= \frac{1}{6}\left[(3-x)^3(2) + (x-2)^3(4)\right] + (3-x)\left[2 - \frac{1}{6}(2)\right] + (x-2)\left[5 - \frac{1}{6}(4)\right]$$
$$= \frac{1}{3}(3-x)^3 + \frac{2}{3}(x-2)^3 + \frac{5}{3}(3-x) + \frac{13}{3}(x-2)$$
$$= \frac{1}{3}\left[(3-x)^3 + 2(x-2)^3 + 5(3-x) + 13(x-2)\right]$$
$$= \frac{1}{3}\left[(27 - 27x + 9x^2 - x^3) + (2x^3 - 12x^2 + 24x + 15) + (15 - 5x) + (13x - 26)\right]$$
$$= \frac{1}{3}\left[x^3 - 3x^2 + 5x\right]$$

For $i = 3$ in the interval $3 \le x \le 4$,

$$S(x) = \frac{1}{6}\left[(x_3 - x)^3 M_2 + (x - x_2)^3 M_3\right] + (x_3 - x)\left(y_2 - \frac{1}{6}M_2\right) + (x - x_2)\left(y_3 - \frac{1}{6}M_3\right)$$

$$= \frac{1}{6}\left[(4 - x)^3(4) + (x - 3)^3(0)\right] + (4 - x)\left[5 - \frac{1}{6}(4)\right] + (x - 3)\left[11 - \frac{1}{6}(0)\right]$$

$$= \frac{2}{3}(4 - x)^3 + \frac{13}{3}(4 - x) + 11(x - 3)$$

$$= \frac{1}{3}\left[-2x^3 + 24x^2 - 96x + 128 + 52 - 13x + 33x - 99\right]$$

$$= \frac{1}{3}(-2x^3 + 24x^2 - 76x + 81)$$

$$y(1.5) = S(1.5) = \frac{1}{3}\left[(1.5)^3 - 3(1.5)^2 + 5(1.5)\right] = \frac{11}{8}$$

In the interval $2 \le x \le 3$,

$$y' = S'(x) = \frac{1}{3}(3x^2 - 6x + 5)$$

$$y'(3) = S'(3) = \frac{1}{3}\left[3(3)^2 - 6(3) + 5\right] = \frac{14}{3}$$

In the interval $3 \le x \le 4$,

$$y' = S'(x) = \frac{1}{3}(-6x^2 + 48x - 76)$$

$$y'(3) = S'(3) = \frac{1}{3}\left[-6(3)^2 + 48(3) - 76\right] = \frac{14}{3}$$

Example 6

Find the cubic spline in the interval [0, 2] *for the following data:*

x	0	2	4	6
y	1	9	41	41

Given $M_0 = 0$ *and* $M_3 = -12$.

Solution

$$h = 2, \qquad n = 3, \qquad M_0 = 0, \qquad M_3 = -12$$

For cubic spline interpolation,

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}), \quad i = 1, 2, ..., n-1$$

For $i = 1$,

$$M_0 + 4M_1 + M_2 = \frac{6}{4}(y_0 - 2y_1 + y_2)$$

$$= \frac{3}{2}[1 - 2(9) + 41]$$

$$4M_1 + M_2 = 36 \qquad [\because \quad M_0 = 0] \qquad \text{...(1)}$$

For $i = 2$,

$$M_1 + 4M_2 + M_3 = \frac{6}{4}(y_1 - 2y_2 + y_3)$$

$$= \frac{3}{2}[9 - 2(41) + 41]$$

$$= -48$$

$$M_1 + 4M_2 = -36 \quad [\because \quad M_3 = -12] \qquad \text{...(2)}$$

Solving Eqs (1) and (2),

$$M_1 = 12, \qquad M_2 = -12$$

The cubic spline function are given by

$$S(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)\left(y_{i-1} - \frac{h^2}{6}M_{i-1}\right) + \frac{1}{h}(x - x_{i-1})\left(y_i - \frac{h^2}{6}M_i\right)$$

For $i = 1$ in the interval $0 \le x \le 2$,

$$S(x) = \frac{1}{12}\left[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1\right] + \frac{1}{2}(x_1 - x)\left(y_0 - \frac{4}{6}M_0\right) + \frac{1}{2}(x - x_0)\left(y_1 - \frac{4}{6}M_1\right)$$

$$= \frac{1}{12}\left[(2 - x)^3(0) + (x - 0)^3(12)\right] + \frac{1}{2}(2 - x)\left[1 - \frac{4}{6}(0)\right] + \frac{1}{2}(x - 0)\left[9 - \frac{4}{6}(12)\right]$$

$$= \frac{1}{12}(12x^3) + 1 - \frac{x}{2} + \frac{x}{2}$$

$$= 1 + x^3$$

EXERCISE 4.7

1. Find the natural cubic spline for the following data:

x	0	1	2
y	0	1	0

$$\left[\begin{array}{l}\textbf{Ans.: } M_1 = -3,\ S_1(x) = \frac{1}{2}(3x - x^3) \text{ for } 0 < x < 1 \\ \quad y(0.5) = \frac{11}{6},\ y'(1) = 0\end{array}\right]$$

2. Find the natural cubic spline in [0, 1] for the following data:

x	0	1	2
y	0	2	6

Also, find $y(0.5)$ and $y'(1)$.

$$\left[\begin{array}{l}\textbf{Ans.: } M_1 = 3,\ S_1(x) = \frac{1}{2}(3x + x^3) \text{ for } 0 < x < 1 \\ \quad y(0.5) = \frac{13}{16},\ y'(1) = 3\end{array}\right]$$

3. Find the cubic spline in the interval [3, 4] for the function given by the following data under the conditions $M(1) = 0$, $M(4) = 0$:

x	1	2	3	4
y	3	10	29	65

$$\left[\begin{array}{l}\textbf{Ans.: } M_1 = \frac{62}{5},\ M_2 = \frac{112}{5}, \\ S_3(x) = \frac{1}{15}(-56x^3 + 672x^2 - 2092x + 2175) \text{ for } 3 < x < 4\end{array}\right]$$

4. Find the cubic spline for the following data under the conditions $M(0) = 0$, $M(3) = 0$ in the interval [1, 2]. Hence, find $y(1.5)$.

x	0	1	2	3
y	1	4	10	8

$$\left[\begin{array}{l}\textbf{Ans.: } M_1 = 8,\ M_2 = -14, \\ \quad S_2(x) = \frac{1}{3}(-11x^3 + 45x^2 - 40x + 18) \text{ for } 1 < x < 2\end{array}\right]$$

5. Find the natural cubic spline for the data in the interval [1, 2] and, hence, find $y(1.5)$.

x	1	2	3	4
y	1	5	11	8

$$\left[\begin{aligned}&\textbf{Ans.: } M_1 = \frac{34}{5},\ M_2 = -\frac{76}{5},\\ &\quad S_1(x) = \frac{1}{15}(17x^3 - 51x^2 + 94x - 45) \text{ for } 1 < x < 2\\ &\quad y(1.5) = 2.575\end{aligned}\right]$$

6. Test whether the following functions are cubic splines or not.

$$S_1(x) = -x^2 + 20x^3, \qquad -1 \le x < 0$$
$$S_2(x) = -x^2 + 6x^3, \qquad 0 \le x \le 1$$

[**Ans.:** Yes]

7. Obtain the cubic splines, given $f(-1) = 0$, $f(0) = 4$, $f(1) = 0$ and $M_0 = 24$, $M_2 = 24$.

$$\left[\begin{aligned}&\textbf{Ans.: } S_1(x) = 4 - 12x^2 - 8x^3, \qquad -1 \le x < 0\\ &\qquad\quad S_2(x) = 4 - 12x^2 + 8x^3, \qquad 0 \le x \le 1\end{aligned}\right]$$

8. Find the cubic spline corresponding to the interval [2, 3] from the following data:

x	1	2	3	4	5
$f(x)$	30	15	32	18	25

Hence, find $f(2.5)$ and $f'(3)$.

$$\left[\begin{aligned}&\textbf{Ans.: } S(x) = \frac{1}{6}\left[-142.9x^3 + 1058.4x^2 - 2475.2x + 1950\right],\\ &\quad f(2.5) = -24.03,\ f'(3) = 2.817\end{aligned}\right]$$

9. Find the cubic spline $S(x)$, given $f(0) = 3$, $f(2) = 5$, $f(4) = 31$, $M_0 = 1$ and $M_2 = 21$. Also, find $f(1)$ and $f(3)$.

$$\left[\begin{aligned}&\textbf{Ans.: } S_0(x) = 3 + x - 2x^2 + x^3, \qquad 0 \le x \le 2\\ &\qquad\quad S_1(x) = 5 + 5(x-2) + 4(x-2)^2, \quad 2 \le x < 4\\ &\qquad\quad f(1) = 3,\ f(3) = 14\end{aligned}\right]$$

10. Find the cubic splines for the following data:

x	0	1	2	3
y	1	0	–1	0

Given $S''(0) = M_0 = 0$, $S''(3) = M_3 = -6$. Also, find $y(0.5)$.

$$\left[\begin{array}{ll} \textbf{Ans.: } S_0(x) = 1 - x^2, & 0 \le x \le 1 \\ S_1(x) = -2(x-1) - (x-1)^2 + 2(x-1)^3, & 1 \le x \le 2 \\ S_2(x) = -1 + 2(x-2) + 5(x-2)^2 - 6(x-2)^3, & 2 \le x \le 3 \\ y(0.5) = \dfrac{3}{4} & \end{array}\right]$$

Points to Remember

Forward Differences

$$\Delta y_{n-1} = y_n - y_{n-1}$$

Backward Differences

$$\nabla y_n = y_n - y_{n-1}$$

Central Differences

$$\delta y_{n-\frac{1}{2}} = y_n - y_{n-1}$$

Newton's Forward Interpolation Formula

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

Newton's Backward Interpolation Formula

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

Gauss's Forward Interpolation Formula

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-2} + \cdots$$

Gauss's Backward Interpolation Formula

$$y_r = y_0 + r\,\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Stirling's Formula

$$y_r = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \frac{r^2(r^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Lagrange's Interpolation Formula

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}y_1 + \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}y_n$$

Newton's Divided Difference Formula

$$y = y_0 + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3] + \ldots + (x - x_0)\,(x - x_1)\,\ldots\,(x - x_{n-1})\,[x, x_0, x_1, \ldots, x_n]$$

Inverse Interpolation

$$x = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_1)}x_0 + \frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)}x_1 + \ldots + \frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots(y_n-y_{n-1})}x_n$$

Cubic Spline Interpolation

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}),\ \ i = 1, 2, \ldots, n-1$$

$$S(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)\left(y_{i-1} - \frac{h^2}{6}M_{i-1}\right) + \frac{1}{h}(x - x_{i-1})\left(y_i - \frac{h^2}{6}M_i\right), \quad i = 1, 2, \ldots, n$$

CHAPTER 5

Curve Fitting

Chapter Outline

5.1 INTRODUCTION

Curve fitting is the process of finding the 'best-fit' curve for a given set of data. It is the representation of the relationship between two variables by means of an algebraic equation. On the basis of this mathematical equation, predictions can be made in many statistical problems.

Suppose a set of n points of values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ of the two variables x and y are given. These values are plotted on a rectangular coordinate system, i.e., the xy-plane. The resulting set of points is known as a *scatter diagram* (Fig. 5.1). The scatter diagram exhibits the trend and it is possible to visualize a smooth curve approximating the data. Such a curve is known as an *approximating curve*.

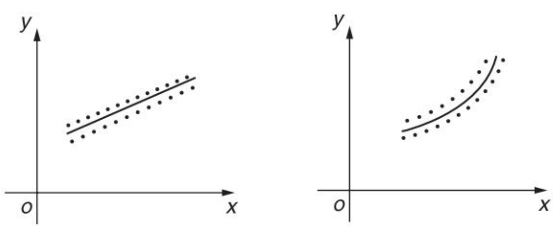

Fig. 5.1

5.2 LEAST SQUARE METHOD

From a scatter diagram, generally, more than one curve may be seen to be appropriate to the given set of data. The method of least squares is used to find a curve which passes through the maximum number of points.

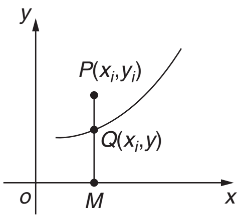

Fig. 5.2

Let P (x_i, y_i) be a point on the scatter diagram (Fig. 5.2). Let the ordinate at P meet the curve $y = f(x)$ at Q and the x-axis at M.

Distance
$$\begin{aligned} QP &= MP - MQ \\ &= y_i - y \\ &= y_i - f(x_i) \end{aligned}$$

The distance QP is known as *deviation*, *error*, or *residual* and is denoted by d_i. It may be positive, negative, or zero depending upon whether P lies above, below, or on the curve. Similar residuals or errors corresponding to the remaining $(n-1)$ points may be obtained. The sum of squares of residuals, denoted by E, is given as

$$E = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} [y_i - f(x_i)]^2$$

If $E = 0$ then all the n points will lie on $y = f(x)$. If $E \neq 0, f(x)$ is chosen such that E is minimum, i.e., the best fitting curve to the set of points is that for which E is minimum. This method is known as the least square method. This method does not attempt to determine the form of the curve $y = f(x)$ but it determines the values of the parameters of the equation of the curve.

5.3 FITTING OF LINEAR CURVES

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be the set of n values and let the relation between x and y be $y = a + bx$. The constants a and b are selected such that the straight line is the best fit to the data.

The residual at $x = x_i$ is

$$\begin{aligned} d_i &= y_i - f(x_i) \\ &= y_i - (a + bx_i) \qquad i = 1, 2, ..., n \\ E &= \sum_{i=1}^{n} d_i^2 \\ &= \sum_{i=1}^{n} \left[y_i - (a + bx_i) \right]^2 \\ &= \sum_{i=1}^{n} (y_i - a - bx_i)^2 \end{aligned}$$

For E to be minimum,

(i) $\frac{\partial E}{\partial a} = 0$

$$\sum_{i=1}^{n} 2(y_i - a - bx_i)(-1) = 0$$

$$\sum_{i=1}^{n} (y_i - a - bx_i) = 0$$

$$\sum_{i=1}^{n} y_i = a\sum_{i=1}^{n} 1 + b\sum_{i=1}^{n} x_i$$

$$\sum y = na + b\sum x$$

(ii) $\frac{\partial E}{\partial b} = 0$

$$\sum_{i=1}^{n} 2(y_i - a - bx_i)(-x_i) = 0$$

$$\sum_{i=1}^{n} (x_i y_i - ax_i - bx_i^2) = 0$$

$$\sum_{i=1}^{n} x_i\, y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2$$

$$\sum xy = a\sum x + b\sum x^2$$

These two equations are known as *normal equations*. These equations can be solved simultaneously to give the best values of a and b. The best fitting straight line is obtained by substituting the values of a and b in the equation $y = a + bx$.

Example 1

Fit a straight line to the following data:

x	1	2	3	4	6	8
y	2.4	3	3.6	4	5	6

Solution

Let the straight line to be fitted to the data be

$$y = a + bx$$

The normal equations are

$$\sum y = na + b\sum x \quad \text{...(1)}$$

$$\sum xy = a\sum x + b\sum x^2 \quad \text{...(2)}$$

Here, $n = 6$

x	y	x^2	xy
1	2.4	1	2.4
2	3	4	6
3	3.6	9	10.8
4	4	16	16
6	5	36	30
8	6	64	48
$\sum x = 24$	$\sum y = 24$	$\sum x^2 = 130$	$\sum xy = 113.2$

Substituting these values in Eqs (1) and (2),

$$24 = 6a + 24b \qquad \ldots(3)$$

$$113.2 = 24a + 130b \qquad \ldots(4)$$

Solving Eqs (3) and (4),

$$a = 1.9764$$

$$b = 0.5059$$

Hence, the required equation of the straight line is

$$y = 1.9764 + 0.5059x$$

Example 2

Fit a straight line to the following data. Also, estimate the value of y at $x = 2.5$.

x	0	1	2	3	4
y	1	1.8	3.3	4.5	6.3

Solution

Let the straight line to be fitted to the data be

$$y = a + bx$$

The normal equations are

$$\sum y = na + b\sum x \qquad \ldots(1)$$

$$\sum xy = a\sum x + b\sum x^2 \qquad \ldots(2)$$

Here, $n = 5$

x	y	x^2	xy
0	1	0	0
1	1.8	1	1.8
2	3.3	4	6.6
3	4.5	9	13.5
4	6.3	16	25.2
$\sum x = 10$	$\sum y = 16.9$	$\sum x^2 = 30$	$\sum xy = 47.1$

Substituting these values in Eqs (1) and (2),

$$16.9 = 5a + 10b \qquad \dots(3)$$

$$47.1 = 10a + 30b \qquad \dots(4)$$

Solving Eqs (3) and (4),

$$a = 0.72$$

$$b = 1.33$$

Hence, the required equation of the straight line is

$$y = 0.72 + 1.33x$$

At $x = 2.5$,

$$y\,(2.5) = 0.72 + 1.33\,(2.5) = 4.045$$

Example 3

A simply supported beam carries a concentrated load P(lb) at its midpoint. Corresponding to various values of P, the maximum deflection Y(in) is measured. The data is given below:

P	100	120	140	160	180	200
Y	0.45	0.55	0.60	0.70	0.80	0.85

Find a law of the form $Y = a + bP$ using the least square method.

[Summer 2015]

Solution

Let the straight line to be fitted to the data be

$$Y = a + bP$$

The normal equations are

$$\sum Y = na + b\sum P \qquad \dots(1)$$

$$\sum PY = a\sum P + b\sum P^2 \quad ...(2)$$

Here, $n = 6$

P	Y	P^2	PY
100	0.45	10000	45
120	0.55	14400	66
140	0.60	19600	84
160	0.70	25600	112
180	0.80	32400	144
200	0.85	40000	170
$\sum P = 900$	$\sum Y = 3.95$	$\sum P^2 = 142000$	$\sum PY = 621$

Substituting these values in Eqs (1) and (2),

$$3.95 = 6a + 900\,b \quad ...(3)$$

$$621 = 900\,a + 142000\,b \quad ...(4)$$

Solving Eqs (3) and (4),

$$a = 0.0476$$

$$b = 0.0041$$

Hence, the required equation of the straight line is

$$Y = 0.0476 + 0.0041\,P$$

Example 4

Fit a straight line to the following data. Also, estimate the value of y at $x = 70$.

x	71	68	73	69	67	65	66	67
y	69	72	70	70	68	67	68	64

Solution

Since the values of x and y are larger, we choose the origin for x and y at 69 and 67 respectively,

Let $X = x - 69$ and $Y = y - 67$

Let the straight line to be fitted to the data be

$$Y = a + bX$$

The normal equations are

$$\sum Y = na + b\sum X \quad \ldots(1)$$

$$\sum XY = a\sum X + b\sum X^2 \quad \ldots(2)$$

Here, $n = 8$

x	y	X	Y	X^2	XY
71	69	2	2	4	4
68	72	–1	5	1	–5
73	70	4	3	16	12
69	70	0	3	0	0
67	68	–2	1	4	–2
65	67	–4	0	16	0
66	68	–3	1	9	–3
67	64	–2	–3	4	6
		$\sum X = -6$	$\sum Y = 12$	$\sum X^2 = 54$	$\sum XY = 12$

Substituting these values in Eqs (1) and (2),

$$12 = 8a - 6b \quad \ldots(3)$$

$$12 = -6a + 54b \quad \ldots(4)$$

Solving Eqs (3) and (4),

$$a = 1.8182$$
$$b = 0.4242$$

Hence, the required equation of the straight line is

$$Y = 1.8182 + 0.4242X$$

$$y - 67 = 1.8182 + 0.4242(x - 69)$$

$$y = 0.4242x + 39.5484$$

$$y(x = 70) = 0.4242(70) + 39.5484 = 69.2424$$

Example 5

Fit a straight line to the following data taking x as the dependent variable.

x	1	3	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9

Solution

If x is considered the dependent variable and y the independent variable, the equation of the straight line to be fitted to the data is

$$x = a + by$$

The normal equations are

$$\sum x = na + b\sum y \quad \ldots(1)$$

$$\sum xy = a\sum y + b\sum y^2 \quad \ldots(2)$$

Here, $n = 8$

x	y	y^2	xy
1	1	1	1
3	2	4	6
4	4	16	16
6	4	16	24
8	5	25	40
9	7	49	63
11	8	64	88
14	9	81	126
$\sum x = 56$	$\sum y = 40$	$\sum y^2 = 256$	$\sum xy = 364$

Substituting these values in Eqs (1) and (2),

$$56 = 8a + 40b \quad \ldots(3)$$

$$364 = 40a + 256b \quad \ldots(4)$$

Solving Eqs (3) and (4),

$$a = -0.5$$

$$b = 1.5$$

Hence, the required equation of the straight line is

$$x = -0.5 + 1.5y$$

Example 6

If P is the pull required to lift a load W by means of a pulley block, find a linear law of the form P = mW + c connecting P and W using the following data:

P	12	15	21	25
W	50	70	100	120

where P and W are taken in kg-wt. Compute P when W = 150 kg.

Solution

Let the linear curve (straight line) fitted to the data be

$$P = mW + c = c + mW$$

The normal equations are

$$\sum P = nc + mW \qquad \text{...(1)}$$

$$\sum PW = c\sum W + m\sum W^2 \qquad \text{...(2)}$$

Here, $n = 4$

P	W	W^2	PW
12	50	2500	600
15	70	4900	1050
21	100	10000	2100
25	120	14400	3000
$\sum P = 73$	$\sum W = 340$	$\sum W^2 = 31800$	$\sum PW = 6750$

Substituting these values in Eqs (1) and (2),

$$73 = 4c + 340\,m \qquad \text{...(3)}$$

$$6750 = 340\,c + 31800\,m \qquad \text{...(4)}$$

Solving Eqs (3) and (4),

$$c = 2.2759$$

$$m = 0.1879$$

Hence, the required equation of the straight line is

$$P = 0.1879\,W + 2.2759$$

When $W = 150$ kg,

$$P = 0.1879(150) + 2.2759 = 30.4609$$

EXERCISE 5.1

1. Fit the line of best fit to the following data:

x	0	5	10	15	20	25
y	12	15	17	22	24	30

$\left[\textbf{Ans.: } y = 0.7x + 11.28\right]$

2. The results of a measurement of electric resistance R of a copper bar at various temperatures $t°$C are listed below:

$t°C$	19	25	30	36	40	45	50
R	76	77	79	80	82	83	85

Find a relation $R = a + bt$ where a and b are constants to be determined.

$\left[\textbf{Ans.: } R = 70.0534 + 0.2924t\right]$

3. Fit a straight line to the following data:

x	1.53	1.78	2.60	2.95	3.42
y	33.50	36.30	40.00	45.85	53.40

$$\left[\textbf{Ans.: } y = 19 + 9.7x\right]$$

4. Fit a straight line to the following data:

x	100	120	140	160	180	200
y	0.45	0.55	0.60	0.70	0.80	0.85

$$\left[\textbf{Ans.: } y = 0.0475 + 0.00407x\right]$$

5. Find the relation of the type $R = aV + b$, when some values of R and V obtained from an experiment are

V	60	65	70	75	80	85	90
R	109	114	118	123	127	130	133

$$\left[\textbf{Ans.: } R = 0.8071\text{V} + 61.4675\right]$$

5.4 FITTING OF QUADRATIC CURVES

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be the set of n values and let the relation between x and y be $y = a + bx + cx^2$. The constants a, b, and c are selected such that the parabola is the best fit to the data. The residual at $x = x_i$ is

$$\begin{aligned} d_i &= y_i - f(x_i) \\ &= y_i - \left(a + bx_i + cx_i^2\right) \end{aligned}$$

$$\begin{aligned} E &= \sum_{i=1}^{n} d_i^2 \\ &= \sum_{i=1}^{n} \left[y_i - \left(a + bx_i + cx_i^2\right)\right]^2 \\ &= \sum_{i=1}^{n} \left(y_i - a - bx_i - cx_i^2\right)^2 \end{aligned}$$

For E to be minimum,

(i) $\dfrac{\partial E}{\partial a} = 0$

$$\sum_{i=1}^{n} 2\left(y_i - a - bx_i - cx_i\right)(-1) = 0$$

$$\sum_{i=1}^{n}\left(y_i - a - bx_i - cx_i\right) = 0$$

$$\sum_{i=1}^{n} y_i = a\sum_{i=1}^{n} 1 + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2$$

$$\sum y_i = na + b\sum x + c\sum x^2$$

(ii) $\dfrac{\partial E}{\partial b} = 0$

$$\sum_{i=1}^{n} 2(y_i - a - bx_i - cx_i)(-x_i) = 0$$

$$\sum_{i=1}^{n}\left(x_i y_i - ax_i - bx_i^2 - cx_i^3\right) = 0$$

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i^3$$

$$\sum xy = na + b\sum x^2 + c\sum x^3$$

(iii) $\dfrac{\partial E}{\partial c} = 0$

$$\sum_{i=1}^{n} 2(y_i - a - bx_i - cx_i^2)(x_i^2) = 0$$

$$\sum_{i=1}^{n} x_i^2 y_i - ax_i^2 - bx_i^3 - cx_i^4 = 0$$

$$\sum_{i=1}^{n} x_i^2 y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^4$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4$$

These equations are known as *normal equations*. These equations can be solved simultaneously to give the best values of a, b, and c. The best fitting parabola is obtained by substituting the values of a, b, and c in the equation $y = a + bx + cx^2$.

Example 1

Fit a least squares quadratic curve to the following data:

x	1	2	3	4
y	1.7	1.8	2.3	3.2

Estimate $y(2.4)$.

Solution

Let the equation of the least squares quadratic curve (parabola) be $y = a + bx + cx^2$. The normal equations are

$$\sum y = na + b\sum x + c\sum x^2 \qquad \ldots(1)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \qquad \ldots(2)$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4 \qquad \ldots(3)$$

Here, $n = 4$

x	y	x^2	x^3	x^4	xy	x^2y
1	1.7	1	1	1	1.7	1.7
2	1.8	4	8	16	3.6	7.2
3	2.3	9	27	81	6.9	20.7
4	3.2	16	64	256	12.8	51.2
$\Sigma x = 10$	$\Sigma y = 9$	$\Sigma x^2 = 30$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$	$\Sigma xy = 25$	$\Sigma x^2 y = 80.8$

Substituting these values in Eqs (1), (2), and (3),

$$9 = 4a + 10b + 30c \qquad \ldots(4)$$

$$25 = 10a + 30b + 100c \qquad \ldots(5)$$

$$80.8 = 30a + 100b + 354c \qquad \ldots(6)$$

Solving Eqs (4), (5), and (6),

$$a = 2$$

$$b = -0.5$$

$$c = 0.2$$

Hence, the required equation of least squares quadratic curve is

$$y = 2 - 0 \cdot 5x + 0 \cdot 2x^2$$

$$y(2 \cdot 4) = 2 - 0 \cdot 5(2 \cdot 4) + 0 \cdot 2(2.4)^2 = 1 \cdot 952$$

Example 2

Fit a second-degree polynomial using least square method to the following data:

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

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Solution

Let the equation of the least squares quadratic curve be $y = a + bx + cx^2$. The normal equations are

$$\sum y = na + b\sum x + c\sum x^2 \quad ...(1)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \quad ...(2)$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4 \quad ...(3)$$

Here, $n = 5$

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	1.8	1	1	1	1.8	1.8
2	1.3	4	8	16	2.6	5.2
3	2.5	9	27	81	7.5	22.5
4	6.3	16	64	256	25.2	100.8
$\sum x = 10$	$\sum y = 12.9$	$\sum x^2 = 30$	$\sum x^3 = 100$	$\sum x^4 = 354$	$\sum xy = 37.1$	$\sum x^2y = 130.3$

Substituting these values in Eqs (1), (2), and (3),

$$12.9 = 5a + 10b + 30\,c \quad ...(4)$$

$$37.1 = 10a + 30b + 100c \quad ...(5)$$

$$130.3 = 30a + 100b + 354c \quad ...(6)$$

Solving Eqs (4), (5), and (6),

$$a = 1.42$$

$$b = -1.07$$

$$c = 0.55$$

Hence, the required equation of the least squares quadratic curve is

$$y = 1.42 - 1.07\,x + 0.55\,x^2$$

Example 3

By the method of least squares, fit a parabola to the following data:

x	1	2	3	4	5
y	5	12	26	60	97

Also, estimate y at x = 6.

Solution

Let the equation of the parabola be $y = a + bx + cx^2$. The normal equations are

$$\sum y = na + b\sum x + c\sum x^2 \quad ...(1)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \qquad \text{...(2)}$$

$$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4 \qquad \text{...(3)}$$

Here, $n = 5$

x	y	x^2	x^3	x^4	xy	x^2y
1	5	1	1	1	5	5
2	12	4	8	16	24	48
3	26	9	27	81	78	234
4	60	16	64	256	240	960
5	97	25	125	625	485	2425
$\sum x = 15$	$\sum y = 200$	$\sum x^2 = 55$	$\sum x^3 = 225$	$\sum x^4 = 979$	$\sum xy = 832$	$\sum x^2y = 3672$

Substituting these values in Eqs (1), (2), and (3),

$$200 = 5a + 15b + 55\,c \qquad \text{...(4)}$$

$$832 = 15a + 55b + 225c \qquad \text{...(5)}$$

$$3672 = 55a + 225b + 979c \qquad \text{...(6)}$$

Solving Eqs (4), (5), and (6),

$$a = 10.4$$

$$b = -11.0857$$

$$c = 5.7143$$

Hence, the required equation of the parabola is

$$y = 10.4 - 11.0857\,x + 5.7143\,x^2$$

$$y(6) = 10.4 - 11.0857(6) + 5.7143(6)^2 = 149.6006$$

Example 4

Fit a second-degree parabolic curve to the following data.

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

Solution

Let

$$X = x - 5$$

$$Y = y - 10$$

Let the equation of the parabola be $Y = a + bX + cX^2$.

The normal equations are

$$\sum Y = na + b\sum X + c\sum X^2 \qquad \text{...(1)}$$

$$\sum XY = a\sum X + b\sum X^2 + c\sum X^3 \quad \ldots(2)$$

$$\sum X^2Y = a\sum X^2 + b\sum X^3 + c\sum X^4 \quad \ldots(3)$$

Here, $n = 9$

x	y	X	Y	X^2	X^3	X^4	XY	X^2Y
1	2	–4	–8	16	–64	256	32	–128
2	6	–3	–4	9	–27	81	12	–36
3	7	–2	–3	4	–8	16	6	–12
4	8	–1	–2	1	–1	1	2	–2
5	10	0	0	0	0	0	0	0
6	11	1	1	1	1	1	1	1
7	11	2	1	4	8	16	2	4
8	10	3	0	9	27	81	0	0
9	9	4	–1	16	64	256	–4	–16
		$\Sigma X = 0$	$\Sigma Y = -16$	$\Sigma X^2 = 60$	$\Sigma X^3 = 0$	$\Sigma X^4 = 708$	$\Sigma XY = 51$	$\Sigma X^2Y = -189$

Substituting these values in Eqs (1), (2), and (3),

$$-16 = 9a + 60c \quad \ldots(4)$$

$$51 = 60b \quad \ldots(5)$$

$$-189 = 60a + 708c \quad \ldots(6)$$

Solving Eqs (4), (5), and (6),

$$a = 0.0043$$

$$b = 0.85$$

$$c = -0.2673$$

Hence, the required equation of the parabola is

$$Y = 0.0043 + 0.85X - 0.2673X^2$$

$$y - 10 = 0.0043 + 0.85(x-5) - 0.2673(x-5)^2$$

$$\begin{aligned} y &= 10 + 0.0043 + 0.85(x-5) - 0.2673(x^2 - 10x + 25) \\ &= 10 + 0.0043 + 0.85x - 4.25 - 0.2673x^2 + 2.673x - 6.6825 \\ &= -0.9282 + 3.523x - 0.2673x^2 \end{aligned}$$

Example 5

Fit a second-degree parabola $y = a + bx^2$ to the following data:

x	1	2	3	4	5
y	1.8	5.1	8.9	14.1	19.8

Solution

Let the curve to be fitted to the data be

$$y = a + bx^2$$

The normal equations are

$$\sum y = na + b\sum x^2 \qquad \text{...(1)}$$

$$\sum x^2 y = a\sum x^2 + b\sum x^4 \qquad \text{...(2)}$$

Here, $n = 5$

x	y	x^2	x^4	x^2y
1	1.8	1	1	1.8
2	5.1	4	16	20.4
3	8.9	9	81	80.1
4	14.1	16	256	225.6
5	19.8	25	625	495
	$\sum y = 49.7$	$\sum x^2 = 55$	$\sum x^4 = 979$	$\sum x^2 y = 822.9$

Substituting these values in Eqs (1) and (2),

$$49.7 = 5a + 55b \qquad \text{...(3)}$$

$$822.9 = 55a + 979\,b \qquad \text{...(4)}$$

Solving Eqs (3) and (4),

$$a = 1.8165$$

$$b = 0.7385$$

Hence, the required equation of the curve is

$$y = 1.8165 + 0.7385\,x^2$$

Example 6

Fit a curve $y = ax + bx^2$ *for the following data:*

x	1	2	3	4	5	6
y	2.51	5.82	9.93	14.84	20.55	27.06

Solution

Let the curve to be fitted to the data be

$$y = ax + bx^2$$

The normal equations are

$$\sum xy = a\sum x^2 + b\sum x^3 \qquad \text{...(1)}$$

$$\sum x^2 y = a\sum x^3 + b\sum x^4 \qquad \ldots(2)$$

x	y	x^2	x^3	x^4	xy	x^2y
1	2.51	1	1	1	2.51	2.51
2	5.82	4	8	16	11.64	23.28
3	9.93	9	27	81	29.79	89.37
4	14.84	16	64	256	59.36	237.44
5	20.55	25	125	625	102.75	513.75
6	27.06	36	216	1296	162.36	974.16
		$\Sigma x^2 = 91$	$\Sigma x^3 = 441$	$\Sigma x^4 = 2275$	$\Sigma xy = 368.41$	$\Sigma x^2 y = 1840.51$

Substituting these values in Eqs (1) and (2),

$$368 \cdot 41 = 91a + 441b \qquad \ldots(3)$$

$$1840 \cdot 51 = 441\,a + 2275\,b \qquad \ldots(4)$$

Solving Eqs (3) and (4),

$$a = 2.11$$

$$b = 0.4$$

Hence, the required equation of the curve is

$$y = 2 \cdot 11x + 0 \cdot 4x^2$$

EXERCISE 5.2

1. Fit a parabola to the following data:

x	–2	–1	0	1	2
y	1.0	1.8	1.3	2.5	6.3

[**Ans.:** $y = 1.48 + 1.13x + 0.55x^2$]

2. Fit a curve $y = ax + bx^2$ to the following data:

x	–2	–1	0	1	2
y	–72	–46	–12	35	93

[**Ans.:** $y = 41.1x + 2.147x^2$]

3. Fit a parabola $y = a + bx + cx^2$ to the following data:

x	0	2	5	10
y	4	7	6.4	–6

[**Ans.:** $y = 4.1 + 1.979x - 0.299x^2$]

4. Fit a curve $y = a_0 + a_1x + a_2x^2$ for the given data:

x	3	5	7	9	11	13
y	2	3	4	6	5	8

[**Ans.:** $y = 0.7897 + 0.4004x + 0.0089x^2$]

5.5 FITTING OF EXPONENTIAL AND LOGARITHMIC CURVES

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be the set of n values and let the relation between x and y be $y = ab^x$.

Taking logarithm on both the sides of the equation $y = ab^x$,

$$\log_e y = \log_e a + x \log_e b$$

Putting $\log_e y = Y$, $\log_e a = A$, $x = X$, and $\log_e b = B$,

$$Y = A + BX$$

This is a linear equation in X and Y. The normal equations are

$$\sum Y = nA + B\sum X$$

$$\sum XY = A\sum X + B\sum X^2$$

Solving these equations, A and B, and, hence, a and b can be found. The best fitting exponential curve is obtained by substituting the values of a and b in the equation $y = ab^x$.

Similarly, the best fitting exponential curves for the relation $y = ax^b$ and $y = ae^{bx}$ can be obtained.

Example 1

Find the law of the form $y = ab^x$ to the following data:

x	1	2	3	4	5	6	7	8
y	1	1.2	1.8	2.5	3.6	4.7	6.6	9.1

Solution

$$y = ab^x$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + x \log_e b$$

Putting $\log_e y = Y$, $\log_e a = A$, $x = X$ and $\log_e b = B$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + B\sum X \quad \ldots(1)$$

$$\sum XY = A\sum X + B\sum X^2 \quad \ldots(2)$$

Here, $n = 8$

x	y	X	Y	X^2	XY
1	1	1	0.0000	1	0.0000
2	1.2	2	0.1823	4	0.3646
3	1.8	3	0.5878	9	1.7634
4	2.5	4	0.9163	16	3.6652
5	3.6	5	1.2809	25	6.4045
6	4.7	6	1.5476	36	9.2856
7	6.6	7	1.8871	49	13.2097
8	9.1	8	2.2083	64	17.6664
		$\sum X = 36$	$\sum Y = 8.6103$	$\sum X^2 = 204$	$\sum XY = 52.3594$

Substituting these values in Eqs (1) and (2),

$$8.6103 = 8A + 36B \quad \ldots(3)$$

$$52.3594 = 36A + 204B \quad \ldots(4)$$

Solving Eqs (3) and (4),

$$A = -0.3823$$

$$B = 0.3241$$

$$\log_e a = A$$

$$\log_e a = -0.3823$$

$$a = 0.6823$$

$$\log_e b = B$$

$$\log_e b = 0.3241$$

$$b = 1.3828$$

Hence, the required law is

$$y = 0.6823\ (1.3828)^x$$

Example 2

Fit a curve of the form $y = ab^x$ to the following data by the method of least squares:

x	1	2	3	4	5	6	7
y	87	97	113	129	202	195	193

Solution

$$y = ab^x$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + x \log_e b$$

Putting $\log_e y = Y$, $\log_e a = A$, $x = X$ and $\log_e b = B$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + B\sum X \qquad ...(1)$$

$$\sum XY = A\sum X + B\sum X^2 \qquad ...(2)$$

Here, $n = 7$

x	y	X	Y	X^2	XY
1	87	1	4.4659	1	4.4659
2	97	2	4.5747	4	9.1494
3	113	3	4.7274	9	14.1822
4	129	4	4.8598	16	19.4392
5	202	5	5.3083	25	26.5415
6	195	6	5.2730	36	31.6380
7	193	7	5.2627	49	36.8389
		$\sum X = 28$	$\sum Y = 34.4718$	$\sum X^2 = 140$	$\sum XY = 142.2551$

Substituting these values in Eqs (1) and (2),

$$34.4718 = 7A + 28\,B \qquad ...(3)$$

$$142.2551 = 28\,A + 140\,B \qquad ...(4)$$

Solving Eqs (3) and (4),

$$A = 4.3006$$

$$B = 0.156$$

$$\log_e a = A$$
$$\log_e a = 4.3006$$
$$a = 73.744$$
$$\log_e b = B$$
$$\log_e b = 0.156$$
$$b = 1.1688$$

Hence, the required curve is

$$y = 73.744\ (1.1688)^x$$

Example 3

Fit a curve of the form $y = ax^b$ to the following data:

x	20	16	10	11	14
y	22	41	120	89	56

Solution

$$y = ax^b$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + b \log_e x$$

Putting $\log_e y = Y$, $\log_e a = A$, $b = B$ and $\log_e x = X$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + B\sum X \qquad \ldots(1)$$

$$\sum XY = A\sum X + B\sum X^2 \qquad \ldots(2)$$

Here, $n = 5$

x	y	X	Y	X^2	XY
20	22	2.9957	3.0910	8.9742	9.2597
16	41	2.7726	3.7136	7.6873	10.2963
10	120	2.3026	4.7875	5.3019	11.0237
11	89	2.3979	4.4886	5.7499	10.7632
14	56	2.6391	4.0254	6.9648	10.6234
		$\sum X = 13.1079$	$\sum Y = 20.1061$	$\sum X^2 = 34.6781$	$\sum XY = 51.9663$

Substituting these values in Eqs (1) and (2),

$$20.1061 = 5A + 13.1079\ B \qquad \ldots(3)$$

$$51.9663 = 13.1079\ A + 34.6781\ B \qquad \ldots(4)$$

Solving Eqs (3) and (4),

$$A = 10.2146$$
$$B = -2.3624$$
$$\log_e a = A$$
$$\log_e a = 10.2146$$
$$a = 27298 \cdot 8539$$
$$\text{and } b = B = -2.3624$$

Hence, the required equation of the curve is

$$y = 27298.8539\, x^{-2.3624}$$

Example 4

Fit a curve of the form $y = ae^{bx}$ to the following data:

x	1	3	5	7	9
y	115	105	95	85	80

Solution

$$y = ae^{bx}$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + bx \log_e e$$
$$= \log_e a + bx$$

Putting $\log_e y = Y,\ \log_e a = A,\ b = B$ and $x = X$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + B\sum X \quad \ldots(1)$$
$$\sum XY = A\sum X + B\sum X^2 \quad \ldots(2)$$

Here, $n = 5$

x	y	X	Y	X^2	XY
1	115	1	4.7449	1	4.7449
3	105	3	4.6539	9	13.9617
5	95	5	4.5539	25	22.7695
7	85	7	4.4427	49	31.0989
9	80	9	4.3820	81	39.438
		$\sum X = 25$	$\sum Y = 22.7774$	$\sum X^2 = 165$	$\sum XY = 112.013$

Substituting these values in Eqs (1) and (2),

$$22.7774 = 5A + 25B \qquad \text{...(3)}$$

$$112.013 = 25A + 165B \qquad \text{...(4)}$$

Solving Eqs (3) and (4),

$$A = 4.7897$$

$$B = -0.0469$$

$$\log_e a = A$$

$$\log_e a = 4.7897$$

$$a = 120.2653$$

and

$$b = B = -0.0469$$

Hence, the required equation of the curve is

$$y = 120.2653\, e^{-0.0469x}$$

Example 5

Fit the exponential curve $y = ae^{bx}$ to the following data:

x	0	2	4	6	8
y	150	63	28	12	5.6

[Summer 2015]

Solution

$$y = ae^{bx}$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + bx \log_e e$$

$$= \log_e a + bx$$

Putting $\log_e y = Y$, $\log_e a = A$, $b = B$ and $x = X$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + b\sum X \qquad \text{...(1)}$$

$$\sum XY = A\sum X + B\sum X^2 \qquad \text{...(2)}$$

Here, $n = 5$

x	y	X	Y	X^2	XY
0	150	0	5.0106	0	0
2	63	2	4.1431	4	8.2862
4	28	4	3.3322	16	13.3288
6	12	6	2.4849	36	14.9094
8	5.6	8	1.7228	64	13.7824
		$\sum X = 20$	$\sum Y = 16.6936$	$\sum X^2 = 120$	$\sum XY = 50.3068$

Substituting these values in Eqs (1) and (2),

$$16.6936 = 5A + 20B \qquad ...(3)$$

$$50.3068 = 20A + 120B \qquad ...(4)$$

Solving Eqs (3) and (4),

$$A = 4.9855$$

$$B = -0.4117$$

$$\log_e a = A$$

$$\log_e a = 4.9855$$

$$a = 146.28$$

and $\quad b = B = -0.4117$

Hence, the required equation of the curve is

$$y = 146.28\, e^{-0.4117x}$$

Example 6

The pressure and volume of a gas are related by the equation $PV^{\gamma} = c$. Fit this curve to the following data:

P	0.5	1.0	1.5	2.0	2.5	3.0
V	1.62	1.00	0.75	0.62	0.52	0.46

Solution

$$PV^{\gamma} = c$$

Taking logarithm on both the sides,

$$\log_e P + \gamma \log_e V = \log_e c$$

$$\log_e V = \frac{1}{\gamma}\log_e c - \frac{1}{\gamma}\log_e P$$

Putting $\log_e V = y, \frac{1}{\gamma}\log_e c = a, \log_e P = x, -\frac{1}{\gamma} = b,$

$$y = a + bx$$

The normal equations are

$$\sum y = na + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

Here, $n = 6$

P	V	x	y	x^2	xy
0.5	1.62	–0.6931	0.4824	0.4804	–0.3343
1.0	1.00	0	0	0	0
1.5	0.75	0.4055	–0.2877	0.1644	–0.1166
2.0	0.62	0.6931	–0.4780	0.4804	–0.3313
2.5	0.52	0.9163	–0.6539	0.8396	–0.5992
3.0	0.46	1.0986	–0.7765	1.2069	–0.8531
		$\sum x = 2.4204$	$\sum y = -1.7137$	$\sum x^2 = 3.1717$	$\sum xy = -2.2345$

Substituting these values in Eqs (1) and (2),

$$-1.7137 = 6a + 2.4204\, b \qquad \text{...(3)}$$
$$-2.2345 = 2.4204a + 3.1717\, b \qquad \text{...(4)}$$

Solving Eqs (3) and (4),

$$a = -0.002$$
$$b = -0.7029$$
$$-\frac{1}{\gamma} = b$$
$$\gamma = 1.4227$$
$$\frac{1}{\gamma}\log_e c = a$$
$$\frac{1}{1.4227}\log_e c = -0.002$$
$$c = 0.9972$$

Hence, the required equation of the curve is

$$PV^{(1.4227)} = 0.9972$$

EXERCISE 5.3

1. Fit the curve $y = ab^x$ to the following data:

x	2	3	4	5	6
y	144	172.3	207.4	248.8	298.5

[Ans.: $y = 100\,(1.2)^x$**]**

2. Fit the curve $y = ae^{bx}$ to the following data:

x	0	2	4
y	5.012	10	31.62

[Ans.: $y = 4.642e^{0.46x}$**]**

3. Fit the curve $y = ax^b$ to the following data:

x	1	2	3	4
y	2.50	8.00	19.00	50.00

[Ans.: $y = 2.227x^{2.09}$**]**

4. Estimate γ by fitting the ideal gas law $PV^{\gamma} = c$ to the following data:

P	16.6	39.7	78.5	115.5	195.3	546.1
V	50	30	20	15	10	5

[Ans.: $\gamma = 1.504$**]**

Points to Remember

Fitting of Linear Curves

(i) The normal equations for the straight line $y = a + bx$ are

$$\sum y = na + b\sum x$$

$$\sum xy = a\sum x + b\sum x^2$$

(ii) The normal equations for the straight line $x = a + by$ are

$$\sum x = na + b\sum y$$

$$\sum xy = a\sum y + b\sum y^2$$

Fitting of Quadratic Curves

(i) The normal equations for the least squares quadratic curve (parabola) $y = a + bx + cx^2$ are

$$\sum y = na + b\sum x + c\sum x^2$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4$$

(ii) The normal equations for the curve $y = a + bx^2$ are

$$\sum y = na + b\sum x^2$$

$$\sum x^2 y = a\sum x^2 + b\sum x^4$$

(iii) The normal equations for the curve $y = ax + bx^2$ are

$$\sum xy = a\sum x^2 + b\sum x^3$$

$$\sum x^2 y = a\sum x^3 + b\sum x^4$$

Fitting of Exponential and Logarithmic Curves

For the curve $y = ab^x$,

Taking logarithm on both the sides of the equation $y = ab^x$,

$$\log_e y = \log_e a + x\log_e b$$

Putting $\log_e y = Y$, $\log_e a = A$, $x = X$, and $\log_e b = B$,

$$Y = A + BX$$

This is a linear equation in X and Y. The normal equations are

$$\sum Y = nA + B\sum X$$

$$\sum XY = A\sum X + B\sum X^2$$

Similarly, the best fitting exponential curves for the relation $y = ax^b$ and $y = ae^{bx}$ can be obtained.

CHAPTER 6
Numerical Integration

Chapter Outline

6.1 INTRODUCTION

The process of evaluating a definite integral from a set of tabulated values of $f(x)$ is called *numerical integration*. This process when applied to a function of a single variable is known as *quadrature*. In numerical integration, $f(x)$ is represented by an interpolation formula and then it is integrated between the given limits. In this way, the quadrature formula is derived for approximate integration of a function defined by a set of numerical values only.

6.2 NEWTON–COTES QUADRATURE FORMULA

Let the function $y = f(x)$ takes values $y_0, y_1, y_2 \ldots, y_n$ for $x_0, x_1, x_2 \ldots, x_n$ respectively (Fig. 6.1).

Let $I = \int_a^b f(x)\,dx$. Dividing the interval (a, b) into n sub-intervals of width h such that

$x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots, x_n = x_0 + nh = b$

$$\therefore \qquad \int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$$

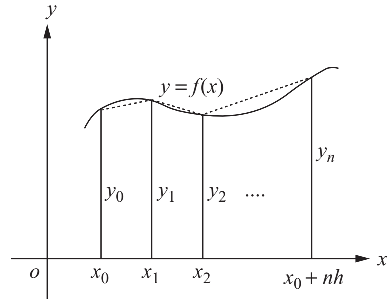

Fig. 6.1

Putting $x = x_0 + rh$, $dx = h dr$
When $x = x_0$, $r = 0$
When $x = x_0 + nh$, $r = n$

$$\int_a^b f(x)\,dx = h\int_0^n f(x_0 + rh)\,dr$$

$$= h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots \right] dr$$

[By Newton's forward interpolation formula]

$$= h\left| r y_0 + \frac{r^2}{2}\Delta y_0 + \left(\frac{\frac{r^3}{3} - \frac{r^2}{2}}{2} \right)\Delta^2 y_0 + \left(\frac{\frac{r^4}{4} - r^3 + r^2}{6} \right)\Delta^3 y_0 + \cdots \right|_0^n$$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right]$$

This equation is known as the *Newton–Cotes quadrature formula*.

6.3 TRAPEZOIDAL RULE

By the Newton–Cotes quadrature formula,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right] \quad \ldots(6.1)$$

Putting $n = 1$ in Eq. (6.1) and ignoring the differences of order higher than one,

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0 \right)$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0) \right]$$

$$= \frac{h}{2}(y_0 + y_1)$$

Similarly,

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}(y_1 + y_2)$$

$$\vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding all these integrals,

$$\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{h}{2}\Big[\big(y_0 + y_n\big) + 2\big(y_1 + y_2 + \cdots + y_{n-1}\big)\Big]$$

$$= \frac{h}{2}[X + 2R]$$

where X = extreme terms, R = remaining terms
This is known as the *trapezoidal rule*.

Errors in the Trapezoidal Rule

Expanding $y = f(x)$ in the neighbourhood of $x = x_0$ by Taylor's series,

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \cdots \qquad \ldots(6.2)$$

where $y_0' = [y'(x)]_{x=x_0}$, and so on.

$$\int_{x_0}^{x_1} y\,\mathrm{d}x = \int_{x_0}^{x_1}\left[y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \cdots\right]\mathrm{d}x$$

$$= \left| y_0 x + \frac{(x-x_0)^2}{2!}y_0' + \frac{(x-x_0)^3}{3!}y_0'' + \cdots \right|_{x_0}^{x_1}$$

$$= y_0(x_1 - x_0) + \frac{(x_1-x_0)^2}{2!}y_0' + \frac{(x_1-x_0)^3}{3!}y_0'' + \cdots$$

$$= h\,y_0 + \frac{h^2}{2!}y_0' + \frac{h^3}{3!}y_0'' + \cdots \qquad \ldots(6.3)$$

where $\qquad x_1 - x_0 = h$

Also, $\displaystyle\int_{x_0}^{x_1} y\,\mathrm{d}x \approx \frac{h}{2}(y_0 + y_1)$ = Area of the first trapezium = A_1 ...(6.4)

Putting $x = x_1$ in Eq. (6.2),

$$y(x_1) = y_1 = y_0 + (x_1 - x_0)\,y_0' + \frac{(x_1-x_0)^2}{2!}y_0'' + \cdots$$

$$= y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots \qquad \ldots(6.5)$$

Substituting Eq. (6.5) in Eq. (6.4),

$$A_1 \approx \frac{h}{2}\left[y_0 + y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots\right]$$

$$\approx h\,y_0 + \frac{h^2}{2}y_0' + \frac{h^3}{2(2!)}y_0'' + \cdots \qquad \ldots(6.6)$$

Subtracting Eq. (6.6) from Eq. (6.3),

$$\int_{x_0}^{x_1} y\,dx - A_1 = h^3 y_0''\left[\frac{1}{3!} - \frac{1}{2(2!)}\right] + \cdots$$

$$= -\frac{1}{12}h^3 y_0'' + \cdots$$

Hence, the error in the first interval (x_0, x_1), neglecting other terms, is $-\frac{1}{12}h^3\ y_0''$.

Similarly, the error in the interval (x_1, x_2) is $-\frac{1}{12}h^3\ y_1''$ and the error in the interval (x_{n-1}, x_n) is $-\frac{1}{12}h^3 y_{n-1}''$.

Hence, the total error is

$$E = -\frac{1}{12}h^3(y_0'' + y_1'' + \cdots + y_{n-1}'')$$

Let $y''(\xi)$ be the largest value of $y_0'', y_1'', \ldots, y_{n-1}''$ where $x_0 < \xi < x_n$.

$$E < -\frac{1}{12}nh^3\ y''(\xi)$$

$$< -\frac{(x_n - x_0)}{12}h^2 y''(\xi) \qquad [\because\ nh = x_n - x_0]$$

Example 1

Find the area bounded by the curve and the x-axis from x = 7.47 to x = 7.52 from the following table, by using the trapezoidal rule.

x	7.47	7.48	7.49	7.50	7.51	7.52
$f(x)$	1.93	1.95	1.98	2.01	2.03	2.06

Solution

$$a = 7.47,\ b = 7.52,\ h = 0.01$$

$$\text{Area} = \int_{7.47}^{7.52} f(x)\,\mathrm{d}x$$

By the trapezoidal rule,

$$\int_{7.47}^{7.52} f(x)\,\mathrm{d}x = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\right]$$

$$= \frac{0.01}{2}\left[(1.93 + 2.06) + 2(1.95 + 1.98 + 2.01 + 2.03)\right]$$

$$= 0.0996$$

Example 2

Consider the following tabular values:

x	25.0	25.1	25.2	25.3	25.4	25.5	25.6
$f(x)$	3.205	3.217	3.232	3.245	3.256	3.268	3.280

Determine the area bounded by the given curve and the x-axis between $x = 25$ *and* $x = 25.6$ *by the trapezoidal rule.*

Solution

$$a = 25, \quad b = 25.6, \quad h = 0.1$$

By the trapezoidal rule,

$$\int_{25}^{25.6} y\,\mathrm{d}x = \frac{h}{2}\left[\left(y_0 + y_6\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5\right)\right]$$

$$= \frac{0.1}{2}\left[(3.205 + 3.280) + 2(3.217 + 3.232 + 3.245 + 3.256 + 3.268)\right]$$

$$= 1.9461$$

Example 3

Given the data below, find the isothermal work done on the gas if it is compressed from $v_1 = 22$ L to $v_2 = 2$ L.

Use $W = -\int_{v_1}^{v_2} p\,\mathrm{d}v$

v, L	2	7	12	17	22
P, atm	12.20	3.49	2.049	1.44	1.11

[Winter 2012]

Solution

$$v_1 = 22, v_2 = 2, h = 5$$

By the trapezoidal rule,

$$W = -\int_{v_1}^{v_2} p\,\mathrm{d}v$$

$$= -\int_{22}^{2} p\,\mathrm{d}v$$

$$= \int_{2}^{22} p\,\mathrm{d}v$$
$$= \frac{h}{2}\left[\left(y_0 + y_4\right) + 2\left(y_1 + y_2 + y_3\right)\right]$$
$$= \frac{5}{2}\left[(12.20 + 1.11) + 2(3.49 + 2.049 + 1.44)\right]$$
$$= 68.17$$

Example 4

Use trapezoidal rule to evaluate $\int_{0}^{2} \frac{x}{\sqrt{2+x^2}}\,\mathrm{d}x$, *dividing the interval into four equal parts.*

Solution

$$a = 0,\ b = 2,\ n = 4$$
$$h = \frac{x_n - x_0}{n} = \frac{2-0}{4} = 0.5$$
$$y = f(x) = \frac{x}{\sqrt{2+x^2}}$$

x	0	0.5	1	1.5	2
$y = f(x)$	0	0.3333	0.5774	0.7276	0.8165
	y_0	y_1	y_2	y_3	y_4

By the trapezoidal rule,

$$\int_{0}^{2} \frac{x}{\sqrt{2+x^2}}\,\mathrm{d}x = \frac{h}{2}\left[\left(y_0 + y_4\right) + 2\left(y_1 + y_2 + y_3\right)\right]$$
$$= \frac{0.5}{2}\left[(0 + 0.8165) + 2(0.3333 + 0.5774 + 0.7276)\right]$$
$$= 1.0233$$

Example 5

Evaluate $\int_{0}^{1} e^x\,\mathrm{d}x$, *with* $n = 10$ *using the trapezoidal rule.*

Solution

$$a = 0, b = 1, n = 10$$

$$h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$$

$$y = f(x) = e^x$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.10
$f(x)$	1	1.1052	1.2214	1.3499	1.4918	1.6487	1.8221	2.0138	2.2255	2.4596	2.7183
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By the trapezoidal rule,

$$\int_0^1 e^x \, dx = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)\right]$$

$$= \frac{0.1}{2}\left[(1+2.7183) + 2(1.1052 + 1.2214 + 1.3499 + 1.4918 + 1.6487 + 1.8211 + 2.0138 + 2.2255 + 2.4596)\right]$$

$$= 1.7196$$

Example 6

Calculate $\int_0^1 2\, e^x \, dx$ *with* $n = 10$ *using the trapezoidal rule.*

[Winter 2014]

Solution

$$a = 0, b = 1, n = 10$$

$$h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$$

$$y = f(x) = 2e^x$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$f(x)$	2	2.2103	2.4428	2.6997	2.9836	3.2974	3.6442	4.0275	4.4511	4.9192	5.4365
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By the trapezoidal rule,

$$\int_0^1 2e^x \mathrm{d}x = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)\right]$$

$$= \frac{0.1}{2}\left[(2 + 5.4365) + 2(2.2103 + 2.4428 + 2.6997 + 2.9836 + 3.2974 + 3.6442 + 4.0275 + 4.4511 + 4.9192)\right]$$

$$= 3.4394$$

Example 7

Compute the integral $\int_{-1}^{1} e^x \,\mathrm{d}x$ *using the trapezoidal rule for* $n = 4$.

Solution

$$a = -1,\ b = 1,\ n = 4$$

$$h = \frac{x_n - x_0}{n} = \frac{1-(-1)}{4} = 0.5$$

$$y = f(x) = e^x$$

x	–1	–0.5	0	0.5	1
$f(x)$	0.3679	0.6065	1	1.6487	2.7183
	y_0	y_1	y_2	y_3	y_4

By the trapezoidal rule,

$$\int_{-1}^{1} e^x \,\mathrm{d}x = \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{0.5}{2}\ [(0.3679 + 2.7183) + 2(0.6065 + 1 + 1.6487)]$$

$$= 2.39916$$

Example 8

Evaluate $\int_0^1 e^{-x^2} \,\mathrm{d}x$ *with* $n = 10$ *using the trapezoidal rule.*

Solution

$$a = 0,\ b = 1,\ n = 10$$

$$h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$$

$$y = f(x) = e^{-x^2}$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.10
$f(x)$	1	0.99	0.9608	0.9139	0.8521	0.7788	0.6977	0.6126	0.5273	0.4449	0.3679
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By the trapezoidal rule,

$$\int_0^1 e^{-x^2}\,dx = \frac{h}{2}\left[\left(y_0 + y_{10}\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9\right)\right]$$

$$= \frac{0.1}{2}\left[(1 + 0.3679) + 2(0.99 + 0.9608 + 0.9139 + 0.8521 + 0.7788 + 0.6977 + 0.6126 + 0.5273 + 0.4449)\right]$$

$$= 0.7462$$

6.4 SIMPSON'S 1/3 RULE

By the Newton–Cotes quadrature formula,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right] \quad \ldots(6.7)$$

Putting $n = 2$ in Eq. (6.7) and ignoring the differences of order higher than 2,

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + \left(y_1 - y_0\right) + \left(\frac{y_2 - 2y_1 + y_0}{6}\right)\right]$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\left(y_2 + 4y_3 + y_4\right)$$

$$\vdots$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right)$$

Adding all these integrals,

$$\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{h}{3}\left[\left(y_0 + y_n\right) + 4\left(y_1 + y_3 + \cdots + y_{n-1}\right) + 2\left(y_2 + y_4 + \cdots + y_{n-2}\right)\right]$$

$$= \frac{h}{3}\left[X + 4O + 2E\right]$$

where X = extreme terms, O = odd terms, E = even terms
This is known as Simpson's 1/3 rule.

Note To apply this rule, the number of sub-intervals must be a multiple of 2.

Errors in Simpson's 1/3 Rule

Expanding $y = f(x)$ in the neighbourhood of $x = x_0$ by Taylor's series,

$$y(x) = y_0 + (x - x_0)\, y_0' + \frac{(x - x_0)^2}{2!}\, y_0'' + \frac{(x - x_0)^3}{3!}\, y_0''' + \frac{(x - x_0)^4}{4!}\, y_0^{\text{iv}} + \cdots \qquad \text{...(6.8)}$$

where $y_0' = [y'(x)]_{x=x_0}$, and so on.

$$\int_{x_0}^{x_2} y\,\mathrm{d}x = \int_{x_0}^{x_2} \left[y_0 + (x - x_0)\, y_0' + \frac{(x - x_0)^2}{2!}\, y_0'' + \frac{(x - x_0)^3}{3!}\, y_0''' + \frac{(x - x_0)^4}{4!}\, y_0^{\text{iv}} + \cdots \right] \mathrm{d}x$$

$$= \left| y_0 x + \frac{(x - x_0)^2}{2!}\, y_0' + \frac{(x - x_0)^3}{3!}\, y_0'' + \frac{(x - x_0)^4}{4!}\, y_0''' + \frac{(x - x_0)^5}{5!}\, y_0^{\text{iv}} + \cdots \right|_{x_0}^{x_2}$$

$$= y_0 (x_2 - x_0) + \frac{(x_2 - x_0)^2}{2!}\, y_0' + \frac{(x_2 - x_0)^3}{3!}\, y_0''$$

$$+ \frac{(x_2 - x_0)^4}{4!}\, y_0''' + \frac{(x_2 - x_0)^5}{5!}\, y_0^{\text{iv}} + \cdots$$

$$= 2h\, y_0 + \frac{4h^2}{2!}\, y_0' + \frac{8h^3}{3!}\, y_0'' + \frac{16\, h^4}{4!}\, y_0''' + \frac{32\, h^5}{5!}\, y_0^{\text{iv}} + \cdots$$

$$= 2h\, y_0 + 2h^2 y_0' + \frac{4h^3}{3}\, y_0'' + \frac{2h^4}{3}\, y_0''' + \frac{4\, h^5}{15!}\, y_0^{\text{iv}} + \cdots \qquad \text{...(6.9)}$$

where $x_2 - x_0 = 2h$

Also, $$\int_{x_0}^{x_2} y\,\mathrm{d}x = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) = \text{Area in the interval } \left(x_0, x_2\right) = A_1 \qquad \text{...(6.10)}$$

Putting $x = x_1$ in Eq. (6.8),

$$y(x_1) = y_1 = y_0 + (x_1 - x_0)\, y_0' + \frac{(x_1 - x_0)^2}{2!}\, y_0'' + \frac{(x_1 - x_0)^3}{3!}\, y_0''' + \frac{(x_1 - x_0)^4}{4!}\, y_0^{\text{iv}} + \cdots$$

$$= y_0 + h y_0' + \frac{h^2}{2!}\, y_0'' + \frac{h^3}{3!}\, y_0''' + \frac{h^4}{4!}\, y_0^{\text{iv}} + \cdots \qquad \text{...(6.11)}$$

Putting $x = x_2$ in Eq. (6.8),

$$y(x_2) = y_2 = y_0 + 2h y_0' + \frac{4h^2}{2!}\, y_0'' + \frac{8h^3}{3!}\, y_0''' + \frac{16h^4}{4!}\, y_0^{\text{iv}} + \cdots \qquad \text{...(6.12)}$$

Substituting Eq. (6.11) and (6.12) in Eq. (6.10),

$$A_1 = 2h\, y_0 + 2h^2 y_0' + \frac{4h^3}{3}\, y_0'' + \frac{2h^4}{3}\, y_0''' + \frac{5h^5}{18}\, y_0^{\text{iv}} + \cdots \qquad \text{...(6.13)}$$

Subtracting Eq. (6.13) from Eq. (6.9),

$$\int_{x_0}^{x_2} y\,\mathrm{d}x - A_1 = \left(\frac{4}{15} - \frac{5}{18}\right) h^5\, y_0^{\text{iv}} + \cdots$$

$$= -\frac{1}{90}\, h^5 y_0^{\text{iv}} + \cdots$$

Hence, the error in the interval $(x_0,\ x_2)$, neglecting higher powers of h, is $-\dfrac{1}{90}\, h^5\, y_0^{\text{iv}} + \cdots$

Similarly, the error in the interval (x_2, x_4) is $-\dfrac{1}{90}\, h^5 y_2^{\text{iv}}$.

Hence, the total error is

$$E = -\frac{1}{90} h^5 \left(y_0^{\text{iv}} + y_2^{\text{iv}} + \cdots\right)$$

Let $y^{\text{iv}}(\xi)$ be the largest value of $y_0^{\text{iv}}, y_2^{\text{iv}}, \ldots, y_{2n-2}^{\text{iv}}$ where $x_0 < \xi < x_{2n}$.

$$E < -\frac{1}{90} n h^5\; y^{\text{iv}}(\xi)$$

$$< -\frac{(x_{2n} - x_0)}{180}\, h^4\; y^{\text{iv}}\,(\xi) \qquad [\because\ \ 2nh = x_{2n} - x_0]$$

Example 1

Consider the following values:

x	10	11	12	13	14	15	16
y	1.02	0.94	0.89	0.79	0.71	0.62	0.55

Find $\int_{10}^{16} y\,dx$ *by Simpson's* 1/3 *rule.*

Solution

$$a = 10, \quad b = 16, \quad h = 1$$

By Simpson's 1/3 rule,

$$\int_{10}^{16} y\,dx = \frac{h}{3}\left[\left(y_0 + y_6\right) + 4\left(y_1 + y_3 + y_5\right) + 2\left(y_2 + y_4\right)\right]$$

$$= \frac{1}{3}\left[(1.02 + 0.55) + 4(0.94 + 0.79 + 0.62) + 2(0.89 + 0.71)\right]$$

$$= 4.7233$$

Example 2

A rocket is launched from the ground. Its acceleration is registered during the first 80 *seconds and is given as follows:*

t (s)	0	10	20	30	40	50	60	70	80
a (m/s^2)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

By Simpson's 1/3 *rule, find the velocity at t = 80 s.*

Solution

$$a = 0, \quad b = 80, \quad h = 10$$

By Simpson's 1/3 rule,

$$\text{Velocity} = \int_0^{80} a\,dt$$

$$= \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{10}{3}\left[(30 + 50.67) + 4(31.63 + 35.47 + 40.33 + 46.69) + 2(33.34 + 37.75 + 43.25)\right]$$

$$= 3086.1 \text{ m/s}$$

Example 3

A river is 80 metres wide. The depth 'd' in metres at a distance x metres from one bank is given by the following table. Calculate the area of cross section of the river using Simpson's 1/3 rule. **[Summer 2015]**

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	7

Solution

$$a = 0, \quad b = 80, \quad h = 10$$

$$A = \int_0^{80} y \, \mathrm{d}x$$

By Simpson's 1/3 rule,

$$\int_0^{80} y \, \mathrm{d}x = \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{10}{3}\left[(0+7) + 4(4+9+15+8) + 2(7+12+14)\right]$$

$$= 723.33 \text{ m}^2$$

Example 4

Evaluate $\int_0^6 \frac{1}{1+x} \mathrm{d}x$ *taking* $h = 1$ *using Simpson's* 1/3 *rule. Hence, obtain an approximate value of* log 7. **[Winter 2013]**

Solution

$$a = 0, \quad b = 6, \quad h = 1$$

$$n = \frac{b-a}{h} = \frac{6-0}{1} = 6$$

$$y = f(x) = \frac{1}{1+x}$$

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.3333	0.25	0.2	0.1667	0.1429
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_0^6 \frac{1}{1+x}\,\mathrm{d}x = \frac{h}{3}\left[\left(y_0 + y_6\right) + 4\left(y_1 + y_3 + y_5\right) + 2\left(y_2 + y_4\right)\right]$$

$$= \frac{1}{3}\left[(1 + 0.1429) + 4(0.5 + 0.25 + 0.1667) + 2(0.3333 + 0.2)\right]$$

$$= 1.9588 \qquad \text{...(1)}$$

By direct integration,

$$\int_0^6 \frac{1}{1+x}\,\mathrm{d}x = \left|\log(1+x)\right|_0^6 = \log 7 \qquad \text{...(2)}$$

From Eqs (1) and (2),

$$\log 7 = 1.9588$$

Example 5

Evaluate $\int_0^5 \frac{\mathrm{d}x}{4x+5}$ *by using Simpson's* 1/3 *rule, taking* 10 *equal parts. Hence, find the approximate value of* $\log_e 5$.

Solution

$$a = 0, \quad b = 5, \quad n = 10$$

$$h = \frac{b-a}{n} = \frac{5-0}{10} = 0.5$$

$$y = f(x) = \frac{1}{4x+5}$$

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	0.2	0.1428	0.1111	0.0910	0.0769	0.0667	0.0588	0.0526	0.0476	0.0435	0.04
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By Simpson's 1/3 rule,

$$\int_0^5 \frac{\mathrm{d}x}{4x+5} = \frac{h}{3}\left[\left(y_0 + y_{10}\right) + 4\left(y_1 + y_3 + y_5 + y_7 + y_9\right) + 2\left(y_2 + y_4 + y_6 + y_8\right)\right]$$

$$= \frac{0.5}{3}\left[(0.2 + 0.04) + 4(0.1428 + 0.0910 + 0.0667 + 0.0526 + 0.0435) + 2(0.1111 + 0.0769 + 0.0588 + 0.0476)\right]$$

$$= 0.4026 \qquad \text{...(1)}$$

By the direct method,

$$\int_0^5 \frac{dx}{4x+5} = \left|\frac{\log_e(4x+5)}{4}\right|_0^5$$

$$= \frac{1}{4}(\log_e 25 - \log_e 5)$$

$$= \frac{1}{4}\log_e \frac{25}{5}$$

$$= \frac{1}{4}\log_e 5 \qquad \ldots(2)$$

Equating Eqs (1) and (2),

$$\frac{1}{4}\log_e 5 = 0.4026$$

$$\log_e 5 = 1.6104$$

Example 6

Evaluate the integral $\int_{-2}^{6} (1+x^2)^{\frac{3}{2}}\, dx$ *by Simpson's* 1/3 *rule with taking* 6 *sub-intervals. Use four digits after the decimal point for calculations.*

[Winter 2012]

Solution

$$a = -2, \quad b = 6, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{6-(-2)}{6} = \frac{4}{3}$$

$$y = f(x) = (1+x^2)^{\frac{3}{2}}$$

x	-2	$-\frac{2}{3}$	$\frac{2}{3}$	2	$\frac{10}{3}$	$\frac{14}{3}$	6
$f(x)$	11.1803	1.7360	1.7360	11.1803	42.1479	108.7094	225.0622
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_{-2}^{6}(1+x^2)^{\frac{3}{2}}\,dx=\frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$
$$=\frac{4}{9}\left[(11.1803+225.0622)+4(1.7360+11.1803+108.7094)+2(1.7360+42.1479)\right]$$
$$=360.2280$$

Example 7

Using Simpson's 1/3 *rule, find* $\int_{0}^{0.6} e^{-x^2}\,dx$ *by taking* $n = 6$.

[Summer 2015]

Solution

$$a=0,\quad b=0.6,\quad n=6$$
$$h=\frac{b-a}{n}=\frac{0.6-0}{6}=0.1$$
$$y=f(x)=e^{-x^2}$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	1	0.99	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_{0}^{0.6} e^{-x^2}\,dx=\frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$
$$=\frac{0.1}{3}\left[(1+0.6977)+4(0.99+0.9139+0.7788)+2(0.9608+0.8521)\right]$$
$$=0.5351$$

Example 8

Estimate $\int_{0}^{3}\cos^2 x\,dx$ *by using Simpson's* 1/3 *rule with* 6 *intervals.*

Solution

$$a = 0, \quad b = 3, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$$

$$y = f(x) = \cos^2 x$$

x	0	0.5	1.0	1.5	2.0	2.5	3.0
$f(x)$	1	0.9999	0.9996	0.9993	0.9988	0.9981	0.9973
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_0^3 \cos^2 x \, dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{0.5}{3}\left[(1 + 0.9973) + 4(0.9999 + 0.9993 + 0.9981) + 2(0.9996 + 0.9988)\right]$$

$$= 2.9978$$

Example 9

Compute the integral $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx$ *for* $n = 6$ *with an accuracy to five decimal places using Simpson's* 1/3 *rule.*

Solution

$$a = 0, \quad b = \frac{\pi}{2}, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

$$y = f(x) = \sqrt{\sin x}$$

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$f(x)$	0	0.5087	0.7071	0.8409	0.9306	0.9828	1.0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{\pi}{36}\left[(0+1) + 4(0.5087 + 0.8409 + 0.9828) + 2(0.7071 + 0.9306)\right]$$

$$= 1.1873$$

Example 10

The speed v metres per second, of a car, t seconds after it starts, is shown in the following table:

t	0	12	24	36	48	60	72	84	96	108	120
v	0	3.60	10.08	18.90	21.60	18.54	10.26	4.50	4.5	5.4	9.0

Using Simpson's 1/3 *rule, find the distance travelled by the car in* 2 *minutes.*

Solution

Let s (metres) distance be travelled in t (seconds).

$$\frac{ds}{dt} = v$$

$$\int ds = \int v\, dt$$

$$s = \int v\, dt$$

The distance travelled in 2 minutes, i.e., 120 seconds is

$$s = \int_0^{120} v\, dt$$

Also, $h = 12$ seconds

By Simpson's 1/3 rule,

$$\int_0^{120} v\, dt = \frac{h}{3}\left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$= \frac{12}{3}\left[(0 + 9.0) + 4(3.60 + 18.90 + 18.54 + 4.50 + 5.4) + 2(10.08 + 21.60 + 10.26 + 4.5)\right]$$

$$= 1222.56 \text{ metres}$$

6.5 SIMPSON'S 3/8 RULE

By the Newton–Cotes quadrature formula,

$$\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right] \quad \ldots(6.14)$$

Putting $n = 3$ in Eq. (6.14) and ignoring the differences of order higher than 3,

$$\int_{x_0}^{x_0+3h} f(x)\,\mathrm{d}x = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right)$$

Similarly, $$\int_{x_0+3h}^{x_0+6h} f(x)\,\mathrm{d}x = \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$

$$\vdots$$

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{3h}{8}\left(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)$$

Adding all these integrals,

$$\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{3h}{8}\left[\left(y_0 + y_n\right) + 2\left(y_3 + y_6 + \cdots + y_{n-3}\right) + 3\left(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}\right)\right]$$

$$= \frac{3h}{8}\left[X + 2T + 3R\right]$$

where X = extreme terms, T = multiple of three terms, R = remaining terms
This is known as Simpson's 3/8 rule.

Note To apply this rule, the number of sub-intervals must be a multiple of 3.

Errors in Simpson's 3/8 Rule

Expanding $y = f(x)$ in the neighbourhood of $x = x_0$ by Taylor's series,

$$y(x) = y_0 + (x - x_0)\,y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0^{\mathrm{iv}} + \cdots \quad \ldots(6.15)$$

where $y_0' = [y'(x)]_{x=x_0}$, and so on.

$$\int_{x_0}^{x_3} y\,dx = \int_{x_0}^{x_3}\left[y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0^{iv} + \cdots\right]$$

$$= \left| y_0 x + \frac{(x-x_0)^2}{2!}y_0' + \frac{(x-x_0)^3}{3!}y_0'' + \frac{(x-x_0)^4}{4!}y_0''' + \frac{(x-x_0)^5}{5!}y_0^{iv} + \cdots \right|_{x_0}^{x_3}$$

$$= y_0(x_3-x_0) + \frac{(x_3-x_0)^2}{2!}y_0' + \frac{(x_3-x_0)^3}{3!}y_0'' + \frac{(x_3-x_0)^4}{4!}y_0''' + \frac{(x_3-x_0)^5}{5!}y_0^{iv} + \cdots$$

$$= 3h\,y_0 + \frac{9h^2}{2!}y_0' + \frac{27h^3}{3!}y_0'' + \frac{81h^4}{4!}y_0''' + \frac{243h^5}{5!}y_0^{iv} + \cdots \qquad ...(6.16)$$

where $x_3 - x_0 = 3h$

Also, $\displaystyle\int_{x_0}^{x_3} y\,dx = \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3) = \text{Area in the interval } (x_0, x_3) = A_1$ ...(6.17)

Putting $x = x_1$ in Eq. (6.15),

$$y(x_1) = y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots \qquad ...(6.18)$$

Putting $x = x_2$ in Eq. (6.16),

$$y(x_2) = y_2 = y_0 + 2hy_0' + \frac{4h^2}{2!}y_0'' + \frac{8h^3}{3!}y_0''' + \frac{16h^4}{4!}y_0^{iv} + \cdots \qquad ...(6.19)$$

Putting $x = x_3$ in Eq. (6.17),

$$y(x_3) = y_3 = y_0 + 3h\,y_0' + \frac{9h^2}{2!}y_0'' + \frac{27h^3}{3!}y_0''' + \frac{81h^4}{4!}y_0^{iv} + \cdots \qquad ...(6.20)$$

Substituting Eqs (6.18), (6.19) and (6.20) in Eq. (6.17),

$$A_1 = 3h\,y_0 + \frac{9h^2}{2!}y_0' + \frac{27h^3}{3!}y_0'' + \frac{81h^4}{4!}y_0''' + \frac{33h^5}{16}y^{iv} + \cdots \qquad ...(6.21)$$

Subtracting Eq. (6.21) from Eq. (6.16),

$$\int_{x_0}^{x_3} y\,dx - A_1 = \left(\frac{81}{40} - \frac{33}{16}\right)h^5 y_0^{iv} + \cdots$$

$$= -\frac{3}{80}h^5 y_0^{iv} + \cdots$$

Hence, the error in the interval (x_0, x_3), neglecting higher powers of h, is $-\dfrac{3}{80}h^5 y_0^{iv}$.

Similarly, the error in the interval (x_3, x_6) is $-\dfrac{3}{80}h^5 y_3^{iv}$.

Hence, the total error is

$$E = -\frac{3}{80}h^5\left(y_0^{\text{iv}} + y_3^{\text{iv}} + \cdots + y_{3n-3}^{\text{iv}}\right)$$

Let $y^{iv}(\xi)$ be the largest value of $y_0^{\text{iv}}, y_3^{\text{iv}}, \ldots, y_{n-3}^{\text{iv}}$ where $x_0 < \xi < x_{3n}$.

$$E < -\frac{3}{80}nh^5\, y^{\text{iv}}(\xi)$$

$$< -\frac{(x_{3n} - x_0)}{80}h^4 y^{\text{iv}}(\xi) \qquad \left[\because\ 3nh = x_{3n} - x_0\right]$$

Example 1

Evaluate $\int_0^3 \frac{1}{1+x}\mathrm{d}x$ *with* $n = 6$ *by using Simpson's* 3/8 *rule and, hence, calculate* log 2. **[Summer 2014]**

Solution

$$a = 0, \quad b = 3, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$$

$$y = f(x) = \frac{1}{1+x}$$

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	1	0.6667	0.5	0.4	0.3333	0.2857	0.25
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^3 \frac{1}{1+x}\mathrm{d}x = \frac{3h}{8}\left[\left(y_0 + y_6\right) + 2\left(y_3\right) + 3\left(y_1 + y_2 + y_4 + y_5\right)\right]$$

$$= \frac{3(0.5)}{8}\left[(1+0.25) + 2(0.4) + 3(0.6667 + 0.5 + 0.3333 + 0.2857)\right]$$

$$= 1.3888 \qquad \text{...(1)}$$

By direct integration,

$$\int_0^3 \frac{1}{1+x}\mathrm{d}x = \left|\log(1+x)\right|_0^3$$

$$= \log 4$$

$$= \log(2)^2$$

$$= 2\log 2 \qquad \text{...(2)}$$

From Eqs (1) and (2),

$$2 \log 2 = 1.3888$$

$$\log 2 = 0.6944$$

Example 2

Evaluate $\int_0^{\pi} \frac{\sin^2 x}{5+4\cos x}\,\mathrm{d}x$ *by using Simpson's 3/8 rule.*

Solution

$$a = 0, \quad b = \pi$$

Dividing the interval into six equal parts, i.e., $n = 6$,

$$h = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

$$y = f(x) = \frac{\sin^2 x}{5 + 4\cos x}$$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$f(x)$	0	0.02954	0.10714	0.2	0.25	0.16277	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^{\pi} \frac{\sin^2 x}{5+4\cos x}\,\mathrm{d}x = \frac{3h}{8}\left[\left(y_0 + y_6\right) + 2\left(y_3\right) + 3\left(y_1 + y_2 + y_4 + y_5\right)\right]$$

$$= \frac{3}{8}\left(\frac{\pi}{6}\right)\left[(0+0) + 2(0.2) + 3(0.02954 + 0.10714 + 0.25 + 0.16277)\right]$$

$$= \frac{\pi}{16}(2.04835)$$

$$= 0.40219$$

Example 3

Find $\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{1}{2}\sin^2 t}\,\mathrm{d}t$ *using one of the methods of numerical integration.*

Solution

Dividing the interval $\left[0, \frac{\pi}{2}\right]$ into six equal parts and applying Simpson's 3/8 rule,

$$a = 0, \quad b = \frac{\pi}{2}, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

$$y = f(t) = \sqrt{1 - \frac{1}{2}\sin^2 t}$$

t	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$f(x)$	1	0.9831	0.9354	0.8660	0.7906	0.7304	0.7071
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{1}{2}\sin^2 t}\, dt = \frac{3h}{8}\left[\left(y_0 + y_6\right) + 2\left(y_3\right) + 3\left(y_1 + y_2 + y_4 + y_5\right)\right]$$

$$= \frac{3}{8}\left(\frac{\pi}{12}\right)[(1 + 0.7071) + 2(0.8660)$$
$$+ 3(0.9831 + 0.9354 + 0.7906 + 0.7304)]$$
$$= 1.3496$$

Example 4

Find $\int_0^{\frac{\pi}{2}} e^{\sin\theta} d\theta$ *by Simpson's 3/8 rule, dividing the interval* $\left[0, \frac{\pi}{12}\right]$ *into six equal parts.*

Solution

$$a = 0, \quad b = \frac{\pi}{2}, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

$$y = f(\theta) = e^{\sin\theta}$$

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$f(\theta)$	1	1.2953	1.6487	2.0281	2.3773	2.6247	2.7182
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^{\frac{\pi}{2}} e^{\sin\theta} d\theta = \frac{3h}{8}\left[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3}{8}\left(\frac{\pi}{12}\right)[(1+2.7182)+2(2.0281)$$

$$+3(1.2953+1.6487+2.3773+2.6247)]$$

$$= 3.1012$$

Example 5

By Simpson's 3/8 rule, evaluate $\int_0^1 \frac{\sin x}{x}$ *taking* $h = \frac{1}{6}$.

Solution

$$a = 0, \quad b = 1, \quad h = \frac{1}{6}$$

$$n = \frac{b-a}{h} = \frac{1-0}{\frac{1}{6}} = 6$$

$$y = f(x) = \frac{\sin x}{x}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$f(x)$	1	0.9954	0.9816	0.9589	0.9276	0.8882	0.8415
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^1 \frac{\sin x}{x} dx = \frac{3h}{8}\left[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3}{8}\left(\frac{1}{6}\right)\big[(1 + 0.8415) + 2(0.9589)$$

$$+ 3(0.9954 + 0.9816 + 0.9276 + 0.8882)\big]$$

$$= 0.9461$$

Example 6

The velocity of a train which starts from rest is given by the following table, the time being reckoned in minutes from the start and speed in km/h.

Time	3	6	9	12	15	18
Velocity	22	29	31	20	4	0

Estimate approximately the distance covered in 18 minutes by Simpson's 3/8 *rule.*

Solution

Let s km distance be covered in t minutes.

$$\frac{ds}{dt} = v$$

$$\int ds = \int v\,dt$$

$$s = \int v\,dt$$

The distance covered in 18 minutes is

$$s = \int_0^{18} v\,dt$$

Since the train starts from rest, at $t = 0$, $v = 0$ $\quad \therefore \; y_0 = 0$

Time (t)	0	3	6	9	12	15	18
Velocity (v)	0	22	29	31	20	4	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Also, $h = 3$ minutes $= \frac{3}{60} = \frac{1}{20}$ hours

By Simpson's 3/8 rule,

$$\int_0^{18} v\,dt = \frac{3h}{8}\left[\left(y_0 + y_6\right) + 2\left(y_3\right) + 3\left(y_1 + y_2 + y_4 + y_5\right)\right]$$

$$= \frac{3}{8}\left(\frac{1}{20}\right)\left[(0+0)+2(31)+3(22+29+20+4)\right]$$
$$= 5.38125$$

Example 7

Find the volume of a solid of revolution formed by rotating about the x-axis the area bounded by the lines $x = 0$, $x = 1.5$, $y = 0$, and the curve passing through the following points:

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50
y	1.00	0.9826	0.9589	0.9089	0.8415	0.7624	0.7589

Solution

Volume is given by

$$V = \int \pi y^2 \, dx$$

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50
y^2	1.00	0.9655	0.9195	0.8261	0.7081	0.5812	0.5759
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$h = 0.25$$

By Simpson's 3/8 rule,

$$\int y^2 \, dx = \frac{3h}{8}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\right]$$
$$= \frac{3(0.25)}{8}\left[(1.00 + 0.5759) + 2(0.8261)\right.$$
$$\left. + 3(0.9655 + 0.9195 + 0.7081 + 0.5812)\right]$$
$$= 1.1954$$
$$\text{Volume} = \pi \int y^2 \, dx$$
$$= \pi(1.1954)$$
$$= 3.7555$$

Example 8

Evaluate $\int_4^{5.2} \log x$ *using the trapezoidal rule and Simpson's* 3/8 *rule, take* $h = 0.2$.

Solution

$$a = 4, \quad b = 5.2, \quad h = 0.2$$

$$n = \frac{b-a}{h} = \frac{5.2-4}{0.2} = 6$$

$$y = f(x) = \log x$$

x	4	4.2	4.4	4.6	4.8	5.0	5.2
$f(x)$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By the trapezoidal rule,

$$\int_4^{5.2} \log x\,dx = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{0.2}{2}\left[(1.3863 + 1.6487) + 2(1.4351 + 1.4816 + 1.5261 + 1.5686 + 1.6094)\right]$$

$$= 1.8277$$

By Simpson's 3/8 rule,

$$\int_4^{5.2} \log dx = \frac{3h}{8}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\right]$$

$$= \frac{3(0.2)}{8}\left[(1.3863 + 1.6487) + 2(1.5261) + 3(1.4351 + 1.4816 + 1.5686 + 1.6094)\right]$$

$$= 1.8278$$

Example 9

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ *taking* $h = \frac{1}{6}$ *using Simpson's 3/8 rule and the trapezoidal rule.*

Solution

$$a = 0, \quad b = 1, \quad h = \frac{1}{6}$$

$$n = \frac{b-a}{h} = \frac{1-0}{\frac{1}{6}} = 6$$

$$y = f(x) = \frac{1}{1+x^2}$$

x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$f(x)$	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\begin{aligned}
\int_0^1 \frac{dx}{1+x^2} &= \frac{3h}{8}\left[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)\right] \\
&= \frac{3}{8}\left(\frac{1}{6}\right)\left[\left(1+\frac{1}{2}\right)+2\left(\frac{4}{5}\right)+3\left(\frac{36}{37}+\frac{9}{10}+\frac{9}{13}+\frac{36}{61}\right)\right] \\
&= 0.7854
\end{aligned}$$

By the trapezoidal rule,

$$\begin{aligned}
\int_0^1 \frac{dx}{1+x^2} &= \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right] \\
&= \frac{1}{12}\left[\left(1+\frac{1}{2}\right)+2\left(\frac{36}{37}+\frac{9}{10}+\frac{4}{5}+\frac{9}{13}+\frac{36}{61}\right)\right] \\
&= 0.7842
\end{aligned}$$

Example 10

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ *by using (i) trapezoidal rule, (ii) Simpson's* 1/3 *rule, (iii) Simpson's* 3/8 *rule.* **[Summer 2014]**

Solution

$$a = 0,\ b = 6$$

Dividing the interval into six equal parts, i.e., $n = 6$,

$$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$$

$$y = f(x) = \frac{1}{1+x^2}$$

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By the trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[\left(y_0+y_6\right)+2\left(y_1+y_2+y_3+y_4+y_5\right)\right]$$

$$= \frac{1}{2}\left[\left(1+0.027\right)+2\left(0.5+0.2+0.1+0.0588+0.0385\right)\right]$$

$$= 1.4108$$

(ii) By Simpson's 1/3 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[\left(y_0+y_6\right)+4\left(y_1+y_3+y_5\right)+2\left(y_2+y_4\right)\right]$$

$$= \frac{1}{3}\left[(1+0.027)+4(0.5+0.1+0.0385)+2(0.2+0.0588)\right]$$

$$= 1.3662$$

(iii) By Simpson's 3/8 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[\left(y_0+y_6\right)+2\left(y_3\right)+3\left(y_1+y_2+y_4+y_5\right)\right]$$

$$= \frac{3}{8}\left[(1+0.027)+2(0.1)+3(0.5+0.2+0.0588+0.0385)\right]$$

$$= 1.3571$$

EXERCISE 6.1

1. Evaluate $\int_1^2 \frac{dx}{1+x^2}$ taking h = 0.2, using trapezoidal rule.

[Ans.: 0.3228**]**

2. Evaluate the value of $\int_0^{0.3} \sqrt{1-8x^3}\,dx$ using Simpson's 3/8 rule.

[Ans.: 0.2916**]**

3. Evaluate $\int_0^{\frac{\pi}{2}} e^{\sin x} dx$ by Simpson's 3/8 rule.

[Ans.: 3.1044]

4. Evaluate $\int_0^1 \frac{dx}{1+x}$ by using (i) trapezoidal rule, (ii) Simpson's 1/3 rule, and (iii) Simpson's 3/8 rule. Take h = 0.25.

[Ans.: (i) 0.6970 (ii) 0.6932 (iii) 0.6932]

5. Calculate $\int_0^{\frac{\pi}{2}} \sin x \, dx$ by dividing the interval into ten equal parts, using the trapezoidal rule and Simpson's 1/3 rule. **[Ans.:** 0.9981, 1.0006]

6. Find the value of $\log 2^{\frac{1}{3}}$ from $\int_0^1 \frac{x^2}{1+x^3}$ using Simpson's 1/3 rule with h = 0.25. **[Ans.:** 0.2311]

7. Compute the value of $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\, dx$ taking h = 0.2 and using the trapezoidal rule, and Simpson's rule. **[Ans.:** 4.0715, 4.0521]

8. Evaluate $\int_{0.5}^{0.7} \sqrt{x}\, e^{-x} dx$ using Simpson's 3/8 rule.

[Ans.: 0.0841]

9. Evaluate $\int_0^1 \frac{dx}{x^3+x+1}$ using Simpson's 1/3 rule, taking h = 0.25.

[Ans.: 0.6305]

10. A curve is drawn to pass through the points given by the following table:

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Obtain the area bounded by the curve, the x-axis, and the lines x = 1 and x = 4 by any method.

[Ans.: 7.7833]

6.6 GAUSSIAN QUADRATURE FORMULAE

An n-point Gaussian quadrature formula is a quadrature formula constructed to give an exact result for polynomials of degree $2n - 1$ or less by a suitable choice of the points x_i and weights w_i for $i = 1, 2, ..., n$. Gauss quadrature formula can be expressed as

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \sum_{i=1}^{n} w_i f(x_i) \qquad ...(6.22)$$

6.6.1 One-point Gaussian Quadrature Formula

Consider a function $f(x)$ over the interval $[-1, 1]$ with sampling point x_1 and weight w_1. The one-point Gaussian quadrature formula is

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = w_1 f(x_1) \qquad ...(6.23)$$

This formula will be exact for polynomials of degrees up to $2n - 1 = 2(1) - 1 = 1$, i.e., it is exact for $f(x) = 1$ and x.

Substituting $f(x)$ in Eq. (6.23) successively,

$$\int_{-1}^{1} 1\,\mathrm{d}x = w_1$$

$$\left| x \right|_{-1}^{1} = w_1$$

$$2 = w_1 \qquad ...(6.24)$$

$$\int_{-1}^{1} x\,\mathrm{d}x = w_1 x_1$$

$$\left| \frac{x^2}{2} \right|_{-1}^{1} = w_1 x_1$$

$$0 = w_1 x_1 \qquad ...(6.25)$$

Solving Eqs (6.24) and (6.25),

$$w_1 = 2$$

$$x_1 = 0$$

Hence, $$\int_{-1}^{1} f(x)\,\mathrm{d}x = 2f(0) \qquad ...(6.26)$$

Equation (6.26) is known as *one-point Gaussian quadrature formula*. This formula is exact for polynomials up to degree one.

6.6.2 Two-Point Gaussian Quadrature Formula

Consider a function $f(x)$ over the interval [–1, 1] with sampling points x_1, x_2 and weights w_1, w_2 respectively. The two-point Gaussian quadrature formula is

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = w_1 f(x_1) + w_2 f(x_2) \qquad \text{...(6.27)}$$

This formula will be exact for polynomials of degrees up to $2n - 1 = 2(2) - 1 = 3$, i.e., it is exact for $f(x) = 1, x, x^2$ and x^3.

Substituting $f(x)$ in Eq. (6.27), successively,

$$\int_{-1}^{1} 1\,\mathrm{d}x = w_1 + w_2$$

$$\left| x \right|_{-1}^{1} = w_1 + w_2$$

$$2 = w_1 + w_2 \qquad \text{...(6.28)}$$

$$\int_{-1}^{1} x\,\mathrm{d}x = w_1 x_1 + w_2 x_2$$

$$\left| \frac{x^2}{2} \right|_{-1}^{1} = w_1 x_1 + w_2 x_2$$

$$0 = w_1 x_1 + w_2 x_2 \qquad \text{...(6.29)}$$

$$\int_{-1}^{1} x^2\,\mathrm{d}x = w_1 x_1^2 + w_2 x_2^2$$

$$\left| \frac{x^3}{3} \right|_{-1}^{1} = w_1 x_1^2 + w_2 x_2^2$$

$$\frac{2}{3} = w_1 x_1^2 + w_2 x_2^2 \qquad \text{...(6.30)}$$

$$\int_{-1}^{1} x^3\,\mathrm{d}x = w_1 x_1^3 + w_2 x_2^3$$

$$\left| \frac{x^4}{4} \right|_{-1}^{1} = w_1 x_1^3 + w_2 x_2^3$$

$$0 = w_1 x_1^3 + w_2 x_2^3 \qquad \text{...(6.31)}$$

Solving Eqs (6.28), (6.29), (6.30), and (6.31),

$$w_1 = w_2 = 1$$

$$x_1 = -\frac{1}{\sqrt{3}},\ x_2 = \frac{1}{\sqrt{3}}$$

Hence, $$\int_{-1}^{1} f(x)\,\mathrm{d}x = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \quad ...(6.32)$$

Equation (6.32) is known as the two-point Gaussian quadrature formula. This formula is exact for polynomials up to degree three.

6.6.3 Three-Point Gaussian Quadrature Formula

Consider a function $f(x)$ over the interval [–1, 1] with sampling points x_1, x_2, x_3 and weights w_1, w_2, w_3 respectively. The three-point Gaussian Quadrature formula is

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) \quad ...(6.33)$$

This formula will be exact for polynomials of degrees up to $2n - 1 = 2(3) - 1 = 5$, i.e., it is exact for $f(x) = 1, x, x^2, x^3, x^4$ and x^5.

Substituting $f(x)$ in Eq. (6.33) successively,

$$\int_{-1}^{1} 1\,\mathrm{d}x = w_1 + w_2 + w_3$$

$$\left|x\right|_{-1}^{1} = w_1 + w_2 + w_3$$

$$0 = w_1 + w_2 + w_3 \quad ...(6.34)$$

$$\int_{-1}^{1} x\,\mathrm{d}x = w_1x_1 + w_2x_2 + w_3x_3$$

$$\left|\frac{x^2}{2}\right|_{-1}^{1} = w_1x_1 + w_2x_2 + w_3x_3$$

$$0 = w_1x_1 + w_2x_2 + w_3x_3 \quad ...(6.35)$$

$$\int_{-1}^{1} x^2\,\mathrm{d}x = w_1x_1^2 + w_2x_2^2 + w_3x_3^2$$

$$\left|\frac{x^3}{3}\right|_{-1}^{1} = w_1x_1^2 + w_2x_2^2 + w_3x_3^2$$

$$\frac{2}{3} = w_1x_1^2 + w_2x_2^2 + w_3x_3^2 \quad ...(6.36)$$

$$\int_{-1}^{1} x^3 \, dx = w_1 x_1^3 + w_2 x_2^3 + w_3 x_3^3$$

$$\left| \frac{x^4}{4} \right|_{-1}^{1} = w_1 x_1^3 + w_2 x_2^3 + w_3 x_3^3$$

$$0 = w_1 x_1^3 + w_2 x_2^3 + w_3 x_3^3 \qquad \text{...(6.37)}$$

$$\int_{-1}^{1} x^4 \, dx = w_1 x_1^4 + w_2 x_2^4 + w_3 x_3^4$$

$$\left| \frac{x^5}{5} \right|_{-1}^{1} = w_1 x_1^4 + w_2 x_2^4 + w_3 x_3^4$$

$$\frac{2}{5} = w_1 x_1^4 + w_2 x_2^4 + w_3 x_3^4 \qquad \text{...(6.38)}$$

$$\int_{-1}^{1} x^5 \, dx = w_1 x_1^5 + w_2 x_2^5 + w_3 x_3^5$$

$$\left| \frac{x^6}{6} \right|_{-1}^{1} = w_1 x_1^5 + w_2 x_2^5 + w_3 x_3^5$$

$$0 = w_1 x_1^5 + w_2 x_2^5 + w_3 x_3^5 \qquad \text{...(6.39)}$$

Solving Eqs (6.34), (6.35), (6.36), (6.37), (6.38), and (6.39),

$$w_1 = \frac{5}{9}, \quad w_2 = \frac{8}{9}, \quad w_3 = \frac{5}{9}$$

$$x_1 = -\sqrt{\frac{3}{5}}, \; x_2 = 0, \; x_3 = \sqrt{\frac{3}{5}}$$

Hence, $$\int_{-1}^{1} f(x) \, dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) \qquad \text{...(6.40)}$$

Equation (6.40) is known as the three-point Gaussian quadrature formula. This formula is exact for polynomials up to degree 5.

Example 1

Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ *by one-point, two-point, and three-point Gaussian formulae.*

Solution

$$f(x) = \frac{1}{1+x^2}$$

By the one-point Gaussian formula,

$$\int_{-1}^{1} \frac{dx}{1+x^2} = 2f(0)$$
$$= 2\left(\frac{1}{1+0}\right)$$
$$= 2$$

By the two-point Gaussian formula,

$$\int_{-1}^{1} \frac{dx}{1+x^2} = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}}$$
$$= 1.5$$

By the three-point Gaussian formula,

$$\int_{-1}^{1} \frac{dx}{1+x^2} = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$
$$= \frac{5}{9}\left(\frac{1}{1+\frac{3}{5}}\right) + \frac{8}{9}\left(\frac{1}{1+0}\right) + \frac{5}{9}\left(\frac{1}{1+\frac{3}{5}}\right)$$
$$= 1.5833$$

Example 2

Evaluate $\int_{0}^{1} \frac{dt}{1+t}$ *by one-point, two-point, and three-point Gaussian formula.*

Solution

Let $$t = \frac{b-a}{2}x + \frac{b+a}{2}$$

Here, $a = 0$, $b = 1$

$$t = \frac{1}{2}x + \frac{1}{2} = \frac{1}{2}(x+1)$$

$$\mathrm{d}t = \frac{1}{2}\,\mathrm{d}x$$

When $t = 0$, $x = -1$

When $t = 1$, $x = 1$

$$\int_0^1 \frac{\mathrm{d}t}{1+t} = \frac{1}{2}\int_{-1}^{1} \frac{\mathrm{d}x}{1+\frac{1}{2}(x+1)}$$

$$= \int_{-1}^{1} \frac{\mathrm{d}x}{x+3}$$

$$f(x) = \frac{1}{x+3}$$

By the one-point Gaussian formula,

$$\int_0^1 \frac{\mathrm{d}t}{1+t} = \int_{-1}^{1} \frac{\mathrm{d}x}{x+3}$$

$$= 2f(0)$$

$$= 2\left(\frac{1}{0+3}\right)$$

$$= 0.6667$$

By the two-point Gaussian formula,

$$\int_0^1 \frac{\mathrm{d}t}{1+t} = \int_{-1}^{1} \frac{\mathrm{d}x}{x+3}$$

$$= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{-\sqrt{\frac{1}{3}}+3} + \frac{1}{\sqrt{\frac{1}{3}}+3}$$

$$= 0.6923$$

By the three-point Gaussian formula,

$$\int_0^1 \frac{\mathrm{d}t}{1+t} = \int_{-1}^{1} \frac{\mathrm{d}x}{x+3}$$

$$= \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)$$

$$= \frac{5}{9}\left(\frac{1}{-\sqrt{\frac{3}{5}}+3}\right) + \frac{8}{9}\left(\frac{1}{0+3}\right) + \frac{5}{9}\left(\frac{1}{\sqrt{\frac{3}{5}}+3}\right)$$
$$= 0.6931$$

Example 3

Evaluate the integral $\int_{-2}^{6} (1+x^2)^{\frac{3}{2}}\,\mathrm{d}x$ *by the Gaussian formula for* $n = 3$.

[Winter 2012]

Solution

Let $\quad x = \frac{b-a}{2}t + \frac{b+a}{2}$

Here, $\quad a = -2, \quad b = 6$

$$x = 4t + 2$$
$$\mathrm{d}x = 4\mathrm{d}t$$

When $\quad x = -2, \quad t = -1$

When $\quad x = 6, \quad t = 1$

$$\int_{-2}^{6} (1+x^2)^{\frac{3}{2}}\,\mathrm{d}x = \int_{-1}^{1} \left[1+(4t+2)^2\right]^{\frac{3}{2}} 4\,\mathrm{d}t$$
$$= 4\int_{-1}^{1} (16t^2+16t+5)^{\frac{3}{2}}\,\mathrm{d}t$$
$$f(t) = (16\,t^2 + 16t + 5)^{\frac{3}{2}}$$

By the three-point Gaussian formula,

$$\int_{-2}^{6} (1+x^2)^{\frac{3}{2}}\,\mathrm{d}x = 4\int_{-1}^{1} (16t^2+16t+5)^{\frac{3}{2}}\,\mathrm{d}t$$
$$= 4\left[\frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)\right]$$
$$= 4\left[\frac{5}{9}\left\{16\left(\frac{3}{5}\right)+16\left(-\sqrt{\frac{3}{5}}\right)+5\right\}^{\frac{3}{2}} + \frac{8}{9}(5)^{\frac{3}{2}} + \frac{5}{9}\left\{16\left(\frac{3}{5}\right)+16\left(\sqrt{\frac{3}{5}}\right)+5\right\}^{\frac{3}{2}}\right]$$
$$= 358.6928$$

Example 4

Evaluate $\int_0^1 e^{-x^2}\,\mathrm{d}x$ *by using the Gaussian quadrature formula with* $n = 3$. **[Winter 2014, Summer 2015]**

Solution

Let
$$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

Here, $a = 0$, $b = 1$

$$x = \frac{1}{2}t + \frac{1}{2} = \frac{1}{2}(t+1)$$

$$\mathrm{d}x = \frac{1}{2}\,\mathrm{d}t$$

When $x = 0$, $t = -1$

When $x = 1$, $t = 1$

$$\int_0^1 e^{-x^2}\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} e^{-\frac{1}{4}(t+1)^2}\,\mathrm{d}t$$

$$f(x) = e^{-\frac{1}{4}(t+1)^2}$$

By the three-point Gaussian quadrature formula,

$$\begin{aligned}\int_0^1 e^{-x^2}\,\mathrm{d}x &= \frac{1}{2}\int_{-1}^{1} e^{-\frac{1}{4}(t+1)^2}\,\mathrm{d}t\\
&= \frac{1}{2}\left[\frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)\right]\\
&= \frac{1}{2}\left[\frac{5}{9}e^{-\frac{1}{4}\left(-\sqrt{\frac{3}{5}}+1\right)^2} + \frac{8}{9}e^{-\frac{1}{4}(0+1)^2} + \frac{5}{9}e^{-\frac{1}{4}\left(\sqrt{\frac{3}{5}}+1\right)^2}\right]\\
&= 0.746815\end{aligned}$$

Example 5

Evaluate $\int_0^{\frac{\pi}{2}} \sin t\,\mathrm{d}t$ *by the two-point Gaussian formula.*

Solution

Let $$t = \frac{b-a}{2}x + \frac{b+a}{2}$$

Here, $a = 0$, $b = \frac{\pi}{2}$

$$t = \frac{\pi}{4}x + \frac{\pi}{4} = \frac{\pi}{4}(x+1)$$

$$\mathrm{d}t = \frac{\pi}{4}\,\mathrm{d}x$$

When $t = 0$, $x = -1$

When $t = \frac{\pi}{2}$, $x = 1$

$$\int_0^{\frac{\pi}{2}} \sin t \,\mathrm{d}t = \frac{\pi}{4}\int_{-1}^{1} \sin\frac{\pi}{4}(x+1)\,\mathrm{d}x$$

$$f(x) = \sin\frac{\pi}{4}(x+1)$$

By the two-point Gaussian formula,

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin t \,\mathrm{d}t &= \frac{\pi}{4}\int_{-1}^{1} \sin\frac{\pi}{4}(x+1)\,\mathrm{d}x \\
&= \frac{\pi}{4}\left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right] \\
&= \frac{\pi}{4}\left[\sin\frac{\pi}{4}\left(-\frac{1}{\sqrt{3}}+1\right) + \sin\frac{\pi}{4}\left(\frac{1}{\sqrt{3}}+1\right)\right] \\
&= 0.99847
\end{aligned}$$

EXERCISE 6.2

Evaluate the following integrals by using Gaussian quadrature formulae:

1. $\int_0^1 e^x \,\mathrm{d}x$ (2 points)

[Ans.: 2.342696]

2. $\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^4}}$ (2 points)

[Ans.: 1.311028]

3. $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\sin^2\theta}}$ (2 points)

[**Ans.:** 1.226]

4. $\int_0^{\frac{\pi}{2}} \log(1+x)\,dx$ (2 points)

[**Ans.:** 0.858]

5. $\int_0^{3} x^2 \cos x\,dx$ (3 points)

[**Ans.:** –4.936]

6. $\int_1^{2} e^x\,dx$ (3 points)

[**Ans.:** 4.67077]

Points to Remember

Newton–Cotes Quadrature Formula

$$\int_{x_0}^{x_0+nh} f(x)\,dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

Trapezoidal Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \cdots + y_{n-1}\right)\right]$$

Simpson's 1/3 Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[\left(y_0 + y_n\right) + 4\left(y_1 + y_3 + \cdots + y_{n-1}\right) + 2\left(y_2 + y_4 + \cdots + y_{n-2}\right)\right]$$

Simpson's 3/8 Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\Big[\left(y_0 + y_n\right) + 2\left(y_3 + y_6 + \cdots + y_{n-3}\right)$$
$$+3\left(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}\right)\Big]$$

Gaussian Quadrature Formulae

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \sum_{i=1}^{n} w_i f(x_i)$$

1. One-point Gaussian Quadrature Formula

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = 2f(0)$$

2. Two-Point Gaussian Quadrature Formula

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

3. Three-Point Gaussian Quadrature Formula

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

CHAPTER 7

Ordinary Differential Equations

Chapter Outline

7.1 INTRODUCTION

Many problems in science and engineering can be reduced to the problem of solving differential equations satisfying certain given conditions. The analytical method of solutions of differential equations can be applied to solve only a selected class of differential equations. In many physical and engineering problems, these methods cannot be used and, hence, numerical methods are used to solve such differential equations.

Consider the first-order differential equations

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition $y(x_0) = y_0$

A number of numerical methods yield solutions either as a power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y. Picard's and Taylor's series methods belong to the former class of solutions, whereas those of Euler, Runge–Kutta, Milne, etc., belong to the latter class. In these later methods, the values of y are calculated in short steps for equal intervals of x and are, therefore, termed step-by-step methods. In the Euler and Runge–Kutta methods, the interval length h should be kept small and, hence, these methods can be applied for tabulating y over a limited range only. If, however, the function values are desired over a wide range, the Milne method may be used. These later methods require starting values which are found by Picard's or Taylor series or Runge–Kutta methods.

7.2 TAYLOR'S SERIES METHOD

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{...(7.1)}$$

with the initial condition $y(x_0) = y_0$.
If $y(x)$ is the exact solution of Eq. (7.1) then the Taylor's series for $y(x)$ around $x = x_0$ is given by

$$y_1 = y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots \qquad \text{...(7.2)}$$

Putting $x - x_0 = h$ in Eq. (7.2),

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \qquad \text{...(7.3)}$$

Similarly, Taylor series for $y(x)$ around $x = x_1$ is given by

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \qquad \text{...(7.4)}$$

Proceeding in the same way,

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \cdots$$

Example 1

Solve $\frac{dy}{dx} = x + y$ *by the Taylor's series method. Start from* $x = 1$, $y = 0$, and carry to $x = 1.2$ with $h = 0.1$. **[Summer 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = x + y$$

(i) Given: $x_0 = 1$, $y_0 = 0$, $h = 0.1$, $x_1 = x_0 + h = 1 + 0.1 = 1.1$

$y' = x + y \qquad y_0' = 1 + 0 = 1$

$y'' = 1 + y' \qquad y_0'' = 1 + 1 = 2$

$$y''' = y'' \qquad y_0''' = 2$$

$$y^{\text{iv}} = y''' \qquad y_0^{\text{iv}} = 2$$

By Taylor's series,

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{\text{iv}} + \cdots$$

$$y_1 = y(1.1) = 0 + 0.1(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2) + \cdots$$

$$= 0.1103$$

(ii) Now, $x_1 = 1.1$, $y_1 = 0.1103$, $h = 0.1$, $x_2 = x_1 + h = 1.1 + 0.1 = 1.2$

$$y_1' = 1.1 + 0.1103 = 1.2103$$

$$y_1'' = 1 + 1.2103 = 2.2103$$

$$y_1''' = 2.2103$$

$$y_1^{\text{iv}} = 2.2103$$

By Taylor's series,

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{\text{iv}} + \cdots$$

$$y_2 = y(1.2) = 0.1103 + 0.1(1.2103) + \frac{(0.1)^2}{2!}(2.2103) + \frac{(0.1)^3}{3!}(2.2103) + \frac{(0.1)^4}{4!}(2.2103) + \cdots$$

$$= 0.2428$$

Example 2

Solve $\dfrac{dy}{dx} = 2y + 3e^x$ *with initial conditions* $x_0 = 0$, $y_0 = 1$ *by the Taylor's series method. Find the approximate value of y for* $x = 0.1$ and $x = 0.2$.

Solution

$$\frac{dy}{dx} = f(x, y) = 2y + 3e^x$$

(i) Given: $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $h = x_1 - x_0 = 0.1 - 0 = 0.1$

$$y' = 2y + 3e^x \qquad y_0' = 2(1) + 3e^0 = 5$$

$$y'' = 2y' + 3e^x \qquad y_0'' = 2(5) + 3e^0 = 13$$

$$y''' = 2y'' + 3e^x \qquad y_0''' = 2(13) + 3e^0 = 29$$

$$y^{\text{iv}} = 2y''' + 3e^x \qquad y_0^{\text{iv}} = 2(29) + 3e^0 = 61$$

By Taylor's series,

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{\text{iv}} + \cdots$$

$$y_1 = y(0.1) = 1 + 0.1(5) + \frac{(0.1)^2}{2!}(13) + \frac{(0.1)^3}{3!}(29) + \frac{(0.1)^4}{4!}(61) + \cdots$$

$$= 1.5700$$

(ii) Now, $x_1 = 0.1, y_1 = 1.5700, x_2 = 0.2, h = x_2 - x_1 = 0.2 - 0.1 = 0.1$

$$y_1' = 2(1.5700) + 3e^{0.1} = 6.4555$$

$$y_1'' = 2(6.4555) + 3e^{0.1} = 16.2265$$

$$y_1''' = 2(16.2265) + 3e^{0.1} = 35.7685$$

$$y_1^{\text{iv}} = 2(35.7685) + 3e^{0.1} = 74.8525$$

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{\text{iv}} + \cdots$$

$$y_2 = y(0.2) = 1.5700 + 0.1(6.4555) + \frac{(0.1)^2}{2!}(16.2265) + \frac{(0.1)^3}{3!}(35.7685) + \frac{(0.1)^4}{4!}(74.8525) + \cdots$$

$$= 2.303$$

Example 3

Solve $\dfrac{dy}{dx} = 1 + y^2$ *with initial conditions* $x_0 = 0$, $y_0 = 0$ *by the Taylor's series method. Find the approximate value of y for* $x = 0.2$ *and* $x = 0.4$.

Solution

$$\frac{dy}{dx} = f(x, y) = 1 + y^2$$

(i) Given: $x_0 = 0, \quad y_0 = 0, \quad x_1 = 0.2, \quad h = x_1 - x_0 = 0.2 - 0 = 0.2$

$$y' = 1 + y^2 \qquad y_0' = 1 + 0 = 1$$

$$y'' = 2yy' \qquad y_0'' = 0$$

$$y''' = 2yy'' + 2(y')^2 \qquad y_0''' = 0 + 2(1)^2 = 2$$

$$y^{\text{iv}} = 2y'y'' + 2yy''' + 4y'y'' = 2yy''' + 6y'y'' \qquad y_0^{\text{iv}} = 0$$

By Taylor's series,

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{\text{iv}} + \cdots$$

$$y_1 = y(0.2) = 0 + 0.2(1) + 0 + \frac{(0.2)^3}{3!}(2) + 0 + \cdots$$

$$= 0.2027$$

(ii) Now, $x_1 = 0.2$, $y_1 = 0.2027$, $x_2 = 0.4$, $h = x_2 - x_1 = 0.4 - 0.2 = 0.2$

$$y_1' = 1 + (0.2027)^2 = 1.0411$$

$$y_1'' = 2(0.2027)(1.0411) = 0.4221$$

$$y_1''' = 2(0.2027)(0.4221) + 2(1.0411)^2 = 2.3389$$

$$y_1^{\text{iv}} = 2(0.2027)(2.3389) + 6(1.0411)(0.4221) = 3.5849$$

By Taylor's series,

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{\text{iv}} + \cdots$$

$$y_2 = y(0.4) = 0.2027 + 0.2(1.0411) + \frac{(0.2)^2}{2!}(0.4221) + \frac{(0.2)^3}{3!}(2.3389) + \frac{(0.2)^4}{4!}(3.5849) + \cdots$$

$$= 0.4227$$

Example 4

Use the Taylor's series method to solve $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$. *Also find* $y(0.03)$.

Solution

$$\frac{dy}{dx} = f(x, y) = x^2y - 1$$

Given: $x_0 = 0$, $y_0 = 1$, $x = 0.03$, $h = x - x_0 = 0.03 - 0 = 0.03$

$$y' = x^2y - 1 \qquad y_0' = 0 - 1 = -1$$

$$y'' = 2xy + x^2y' \qquad y_0'' = 0$$

$$y''' = 2y + 4xy' + x^2y'' \qquad y_0''' = 2(1) + 0 + 0 = 2$$

$$y^{\text{iv}} = 6y' + 6xy'' + x^2y''' \qquad y_0^{\text{iv}} = 6(-1) + 0 + 0 = -6$$

By Taylor's series.

$$y(x) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{\text{iv}} + \cdots$$

$$y(0.03) = 1 + 0.03(-1) + 0 + \frac{(0.03)^3}{3!}(2) + \frac{(0.03)^4}{4!}(-6) + \cdots$$

$$= 0.970009$$

Example 5

Using the Taylor's series method, find correct to four decimal places, the value of $y(0.1)$, *given* $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2$ *and* $y(0) = 1$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = x^2 + y^2$$

Given: $x_0 = 0$, $y_0 = 1$, $x = 0.1$, $h = x - x_0 = 0.1 - 0 = 0.1$

$$y' = x^2 + y^2 \qquad y_0' = 0 + 1 = 1$$

$$y'' = 2x + 2yy' \qquad y_0'' = 2(0) + 2(1)(1) = 2$$

$$y''' = 2 + 2yy'' + 2(y')^2 \qquad y_0''' = 2 + 2(1)(2) + 2(1)^2 = 8$$

$$y^{\text{iv}} = 6y'y'' + 2yy''' \qquad y_0^{\text{iv}} = 6(1)(2) + 2(1)(8) = 28$$

By Taylor's series,

$$y(x) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{\text{iv}} + \cdots$$

$$y(0.1) = 1 + 0.1(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(8) + \frac{(0.1)^4}{4!}(28) + \cdots$$

$$= 1.1115$$

Example 6

Using the Taylor's series method, find $y(1.1)$ *correct to four decimal places given that* $\frac{\mathrm{d}y}{\mathrm{d}x} = xy^{\frac{1}{3}}$, $y(1) = 1$, $h = 0.1$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = xy^{\frac{1}{3}}$$

Given: $x_0 = 1,\quad y_0 = 1,\quad h = 0.1,\quad x = x_0 + h = 1 + 0.1 = 1.1$

$$y' = xy^{\frac{1}{3}} \qquad y_0' = 1(1)^{\frac{1}{3}} = 1$$

$$y'' = \frac{1}{3}xy^{-\frac{2}{3}}y' + y^{\frac{1}{3}} = \frac{1}{3}x^2y^{-\frac{1}{3}} + y^{\frac{1}{3}} \qquad y_0'' = \frac{1}{3}(1)(1)^{-\frac{2}{3}} + (1)^{\frac{1}{3}} = \frac{4}{3}$$

$$y''' = \frac{1}{3}x^2\left(-\frac{1}{3}\right)y^{-\frac{4}{3}}y' + \frac{2}{3}xy^{-\frac{1}{3}} + \frac{1}{3}y^{-\frac{2}{3}}y' \qquad y_0''' = \frac{1}{3}(1)^2\left(-\frac{1}{3}\right)(1)^{-\frac{4}{3}}(1) + \frac{2}{3}(1)(1)^{-\frac{1}{3}} + \frac{1}{3}(1)^{-\frac{2}{3}}(1)$$

$$= -\frac{1}{9} + \frac{2}{3} + \frac{1}{3} = \frac{8}{9}$$

By Taylor's series,

$$y(x) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$y(1.1) = 1 + 0.1(1) + \frac{(0.1)^2}{2!}\left(\frac{4}{3}\right) + \frac{(0.1)^3}{3!}\left(\frac{8}{9}\right) + \cdots$$

$$= 1.1068$$

Example 7

Evaluate y(0.1) correct to four decimal places using the Taylor's series method if $\frac{dy}{dx} = y^2 + x,\ y(0) = 1.$

[Summer 2015]

Solution

$$\frac{dy}{dx} = f(x, y) = y^2 + x$$

Given: $x_0 = 0,\quad y_0 = 1,\quad x = 0.1,\quad h = x - x_0 = 0.1 - 0 = 0.1$

$$y' = y^2 + x \qquad y_0' = (1)^2 + 0 = 1$$

$$y'' = 2yy' + 1 \qquad y_0'' = 2(1)(1) + 1 = 3$$

$$y''' = 2yy'' + 2(y')^2 \qquad y_0''' = 2(1)(3) + 2(1)^2 = 8$$

$$y^{iv} = 2yy''' + 2y'y'' + 4y'y'' = 2yy''' + 6y'y'' \qquad y_0^{iv} = 2(1)(8) + 6(1)(3) = 34$$

By Taylor's series,

$$y(x) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y^{\text{iv}} + \cdots$$

$$\begin{aligned} y(0.1) &= 1 + 0.1(1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(8) + \frac{(0.1)^4}{4!}(34) + \cdots \\ &= 1.1165 \end{aligned}$$

EXERCISE 7.1

Solve the following differential equations:

1. $\frac{dy}{dx} = x^2 + y^2$ with $x_0 = 0, y_0 = 0$ at $x = 0.4$

   [Ans.: 0.0215]

2. $\frac{dy}{dx} = y - xy$ with $x_0 = 0, y_0 = 2$

   $\left[\textbf{Ans.:}\ 2 + 2x - \frac{2x^3}{3} - \frac{x^4}{6} + \cdots\right]$

3. $\frac{dy}{dx} = x - y^2$ with $x_0 = 0, y_0 = 1$ at $x = 0.1$

   [Ans.: 0.9138]

4. $\frac{dy}{dx} = y\sin x + \cos x$ with $x_0 = 0, y_0 = 0$

   $\left[\textbf{Ans.:}\ x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\right]$

5. $\frac{dy}{dx} = xy - 1$ with $x_0 = 1, y_0 = 2$ at $x = 1.02$

   [Ans.: 2.0206]

6. $\frac{dy}{dx} = \frac{1}{x^2 + y^2}$ with $x_0 = 4, y_0 = 4$ at $x = 4.1$

   [Ans.: 4.0031]

7. $\frac{dy}{dx} = 3x + \frac{1}{2}y$ with $x_0 = 0, y_0 = 1$ at $x = 0.1$

   [Ans.: 1.0065]

8. $\frac{dy}{dx} = 3x + y^2$ with $y(0) = 1$ at $x = 0.1$

   [Ans.: 1.1272]

9. $\frac{dy}{dx} = e^x - y^2$ with $y(0) = 1$ at $x = 0.1$

   [Ans.: 1.005]

10. $\dfrac{dy}{dx} = -xy$ with $x_0 = 0, y_0 = 1$

$$\left[\textbf{Ans.:}\ 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \cdots\right]$$

7.3 EULER'S METHOD

Consider the differential equation

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition $y(x_0) = y_0$.

The solution of the differential equation is represented by the curve as shown in Fig. 7.1. The point $P_0(x_0, y_0)$ lies on the curve.

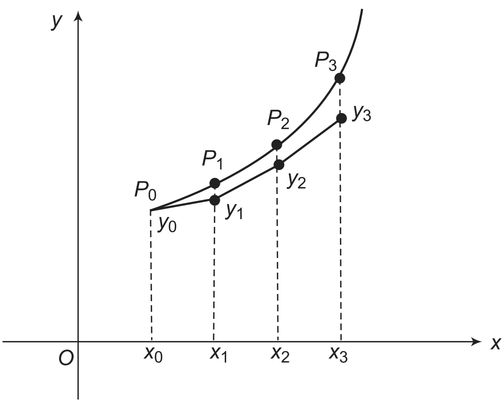

Fig. 7.1

At $x = x_0$, $\left.\dfrac{dy}{dx}\right|_{x=x_0} = f(x_0, y_0)$

The equation of the tangent to the curve at the point (x_0, y_0) is given by

$$y - y_0 = \left(\left.\frac{dy}{dx}\right|_{x=x_0}\right)(x - x_0)$$

$$= f(x_0, y_0)(x - x_0)$$

$$y = y_0 + f(x_0, y_0)(x - x_0)$$

If the point x_1 is very close to x_0, the curve is approximated by the tangent line in the interval (x_0, x_1). Hence, the value of y on the curve is approximately equal to the value of y on the tangent at the point (x_0, y_0) corresponding to $x = x_1$.

$$\therefore \quad y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + h\, f(x_0, y_0) \qquad \text{where } h = x_1 - x_0$$

At $x = x_1$, $\left.\dfrac{dy}{dx}\right|_{x=x_1} = f(x_1, y_1)$

Again the curve is approximated by the tangent line through the point (x_1, y_1).

$$y_2 = y_1 + h\, f(x_1, y_1)$$

Hence, $\quad y_{n+1} = y_n + h\, f(x_n, y_n)$

This formula is known as *Euler's formula*. In this method, the actual curve is approximated by a sequence of short straight lines. As the step size h increases, the straight line deviates much from the actual curve.

Hence, accuracy cannot be obtained.

Example 1

Using Euler's method, find $y(0.2)$ *given* $\dfrac{dy}{dx} = y - \dfrac{2x}{y}$, $y(0) = 1$ *with* $h = 0.1$.

Solution

$$\frac{dy}{dx} = f(x, y) = y - \frac{2x}{y}$$

Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad x = 0.2$

$$n = \frac{x - x_0}{h} = \frac{0.2 - 0}{0.1} = 2$$

$$x_1 = 0.1$$

$$\begin{aligned} y_1 &= y_0 + h\,f(x_0, y_0) \\ &= 1 + 0.1 f(0, 1) \\ &= 1 + 0.1\left[1 - \frac{2(0)}{1}\right] \\ &= 1.1 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + h\,f(x_1, y_1) \\ &= 1.1 + 0.1 f(0.1, 1.1) \\ &= 1.1 + 0.1\left[1.1 - \frac{2(0.1)}{1.1}\right] \\ &= 1.1918 \end{aligned}$$

Hence, $y_2 = y(0.2) = 1.1918$

Example 2

Find the value of y for $\dfrac{dy}{dx} = x + y$, $y(0) = 1$ *when* x = 0.1, 0.2 *with step size* h = 0.05. **[Summer 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = x + y$$

Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.05, \quad x = 0.2$

$$n = \frac{x - x_0}{h} = \frac{0.2 - 0}{0.05} = 4$$

$$x_1 = 0.05, \quad x_2 = 0.1, \quad x_3 = 0.15$$

$$\begin{aligned} y_1 &= y_0 + h\,f(x_0, y_0) \\ &= 1 + 0.05 f(0, 1) \\ &= 1 + 0.05(0 + 1) \\ &= 1.05 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + h\,f(x_1, y_1) \\ &= 1.05 + 0.05 f(0.05, 1.05) \\ &= 1.05 + 0.05(0.05 + 1.05) \\ &= 1.105 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + h\,f(x_2, y_2) \\ &= 1.105 + 0.05 f(0.1, 1.105) \\ &= 1.105 + 0.05(0.1 + 1.105) \\ &= 1.16525 \end{aligned}$$

$$\begin{aligned} y_4 &= y_3 + h\,f(x_3, y_3) \\ &= 1.16525 + 0.05 f(0.15, 1.16525) \\ &= 1.16525 + 0.05(0.15 + 1.16525) \\ &= 1.231 \end{aligned}$$

Hence,
$$y_2 = y(0.1) = 1.105$$
$$y_4 = y(0.2) = 1.231$$

Example 3

Solve the initial-value problem $\frac{dy}{dx} = x\sqrt{y},\ y(1) = 1$ *and, hence, find* $y(1.5)$ *by taking* $h = 0.1$ *using Euler's method.* **[Summer 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = x\sqrt{y}$$

Given:
$$x_0 = 1, \quad y_0 = 1, \quad h = 0.1, \quad x = 1.5$$

$$n = \frac{x - x_0}{h} = \frac{1.5 - 1}{0.1} = 5$$

$$x_1 = 1.1, \quad x_2 = 1.2, \quad x_3 = 1.3, \quad x_4 = 1.4$$

$$\begin{aligned}
y_1 &= y_0 + h\,f(x_0, y_0)\\
&= 1 + 0.1 f(1,1)\\
&= 1 + 0.1(1\sqrt{1})\\
&= 1.1\\
y_2 &= y_1 + h\,f(x_1, y_1)\\
&= 1.1 + 0.1 f(1.1, 1.1)\\
&= 1.1 + 0.1(1.1\sqrt{1.1})\\
&= 1.2154\\
y_3 &= y_2 + h\,f(x_2, y_2)\\
&= 1.2154 + 0.1 f(1.2, 1.2154)\\
&= 1.2154 + 0.1(1.2\sqrt{1.2154})\\
&= 1.3477\\
y_4 &= y_3 + h\,f(x_3, y_3)\\
&= 1.3477 + 0.1 f(1.3, 1.3477)\\
&= 1.3477 + 0.1(1.3\sqrt{1.3477})\\
&= 1.4986\\
y_5 &= y_4 + h\,f(x_4, y_4)\\
&= 1.4986 + 0.1 f(1.4, 1.4986)\\
&= 1.4986 + 0.1(1.4\sqrt{1.4986})\\
&= 1.67
\end{aligned}$$

Hence, $y_5 = y(0.5) = 1.67$

Example 4

Using Euler's method, find the approximate value of y at x = 1.5 taking $h = 0.1$. Given $\dfrac{dy}{dx} = \dfrac{y-x}{\sqrt{xy}}$ and $y(1) = 2$.

Solution

$$\frac{dy}{dx} = f(x,y) = \frac{y-x}{\sqrt{xy}}$$

Given: $x_0 = 1, \quad y_0 = 2, \quad h = 0.1, \quad x = 1.5$

$$n = \frac{x - x_0}{h} = \frac{1.5-1}{0.1} = 5$$

$$x_1 = 1.1, \quad x_2 = 1.2, \quad x_3 = 1.3, \quad x_4 = 1.4$$

$$
\begin{aligned}
y_1 &= y_0 + h\,f(x_0, y_0) \\
&= 2 + 0.1f(1, 2) \\
&= 2 + 0.1\left[\frac{2-1}{\sqrt{1(2)}}\right] \\
&= 2.0707 \\
y_2 &= y_1 + h\,f(x_1, y_1) \\
&= 2.0707 + 0.1(f(1.1, 2.0707) \\
&= 2.0707 + 0.1\left[\frac{2.0707 - 1.1}{\sqrt{1.1(2.0707)}}\right] \\
&= 2.1350 \\
y_3 &= y_2 + h\,f(x_2, y_2) \\
&= 2.1350 + 0.1f(1.2, 2.1350) \\
&= 2.1350 + 0.1\left[\frac{2.1350 - 1.2}{\sqrt{1.2(2.1350)}}\right] \\
&= 2.1934 \\
y_4 &= y_3 + h\,f(x_3, y_3) \\
&= 2.1934 + 0.1f(1.3, 2.1934) \\
&= 2.1934 + 0.1\left[\frac{2.1934 - 1.3}{\sqrt{1.3(2.1934)}}\right] \\
&= 2.2463 \\
y_5 &= y_4 + h\,f(x_{4,}\ y_4) \\
&= 2.2463 + 0.1f\,(1.4, 2.2463) \\
&= 2.2463 + 0.1\left[\frac{2.2463 - 1.4}{\sqrt{1.4(2.2463)}}\right] \\
&= 2.2940
\end{aligned}
$$

Hence, $y_5 = y(1.5) = 2.2940$

Example 5

Using Euler's method, find the approximate value of y at $x = 1$ taking $h = 0.2$. Given $\frac{dy}{dx} = x^2 + y^2$ and $y\,(0) = 1$.

Solution

$$\frac{dy}{dx} = f(x, y) = x^2 + y^2$$

Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.2, \quad x = 1$

$$n = \frac{x - x_0}{h} = \frac{1-0}{0.2} = 5$$

$$x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8$$

$$\begin{aligned}
y_1 &= y_0 + h\,f(x_0, y_0) \\
&= 1 + 0.2 f(0, 1) \\
&= 1 + 0.2[(0)^2 + (1)^2] \\
&= 1.2 \\
y_2 &= y_1 + h\,f(x_1, y_1) \\
&= 1.2 + 0.2 f(0.2, 1.2) \\
&= 1.2 + 0.2\left[(0.2)^2 + (1.2)^2\right] \\
&= 1.496 \\
y_3 &= y_2 + h\,f(x_2, y_2) \\
&= 1.496 + 0.2 f(0.4, 1.496) \\
&= 1.496 + 0.2\left[(0.4)^2 + (1.496)^2\right] \\
&= 1.9756 \\
y_4 &= y_3 + h\,f(x_3, y_3) \\
&= 1.9756 + 0.2 f(0.6, 1.9756) \\
&= 1.9756 + 0.2\left[(0.6)^2 + (1.9756)^2\right] \\
&= 2.8282 \\
y_5 &= y_4 + h\,f(x_4, y_4) \\
&= 2.8282 + 0.2 f(0.8, 2.8282) \\
&= 2.8282 + 0.2\left[(0.8)^2 + (2.8282)^2\right] \\
&= 4.5559
\end{aligned}$$

Hence, $y_5 = y(1) = 4.5559$

Example 6

Given $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$ *with the initial condition* $y = 1$ *at* $x = 0$. *Find* y *at* $x = 0.1$ *in five steps.*

Solution

$$\frac{dy}{dx} = f(x, y) = \frac{y-x}{y+x}$$

Given: $x_0 = 0, \quad y_0 = 1, \quad n = 5, \quad x = 0.1$

$$h = \frac{x - x_0}{n} = \frac{0.1 - 0}{5} = 0.02$$

$$x_1 = 0.02, \quad x_2 = 0.04, \quad x_3 = 0.06, \quad x_4 = 0.08$$

$$\begin{aligned}
y_1 &= y_0 + h\, f(x_0, y_0)\\
&= 1 + 0.02 f(0, 1)\\
&= 1 + 0.02\left(\frac{1-0}{1+0}\right)\\
&= 1.02\\
y_2 &= y_1 + h\, f(x_1, y_1)\\
&= 1.02 + 0.02 f(0.02, 1.02)\\
&= 1.02 + 0.02\left(\frac{1.02 - 0.02}{1.02 + 0.02}\right)\\
&= 1.0392\\
y_3 &= y_2 + h\, f(x_2, y_2)\\
&= 1.0392 + 0.02 f(0.04, 1.0392)\\
&= 1.0392 + 0.02\left(\frac{1.0392 - 0.04}{1.0392 + 0.04}\right)\\
&= 1.0577\\
y_4 &= y_3 + h\, f(x_3, y_3)\\
&= 1.0577 + 0.02 f(0.06, 1.0577)\\
&= 1.0577 + 0.02\left(\frac{1.0577 - 0.06}{1.0577 + 0.06}\right)\\
&= 1.0756\\
y_5 &= y_4 + h\, f(x_4, y_4)\\
&= 1.0756 + 0.02 f(0.08, 1.0756)\\
&= 1.0756 + 0.02\left(\frac{1.0756 - 0.08}{1.0756 + 0.08}\right)\\
&= 1.0928
\end{aligned}$$

Hence, $y_5 = y(0.1) = 1.0928$

EXERCISE 7.2

Solve the following differential equations using Euler's method:

1. $\frac{dy}{dx} = xy$ with $y(0) = 2$, $h = 0.2$ at $x = 1$

[Ans.: 2.9186**]**

2. $\frac{dy}{dx} = \frac{y-x}{x}$ with $y(1) = 2$ at $x = 2$ taking $h = 0.2$

[Ans.: 2.6137**]**

3. $\frac{dy}{dx} = y^2 - \frac{y}{x}$ with $y(1) = 1$ taking $h = 0.1$ at $x = 1.3$ and $x = 1.5$

[Ans.: 1.0268, 1.0889**]**

4. $\frac{dy}{dx} = x + y^2$ with $y(0) = 1$ taking $h = 0.1$ at $x = 0.2$

[Ans.: 1.231**]**

5. $\frac{dy}{dx} = 1 - 2xy$ with $y(0) = 0$ taking $h = 0.2$ at $x = 0.6$

[Ans.: 0.5226**]**

6. $\frac{dy}{dx} = x + \sqrt{y}$ with $y(2) = 4$ taking $h = 0.2$ at $x = 3$

[Ans.: 8.7839**]**

7. $\frac{dy}{dx} = x + y + xy$ with $y(0) = 1$ taking $h = 0.025$ at $x = 0.1$

[Ans.: 1.1117**]**

8. $\frac{dy}{dx} = 1 - y^2$ with $y(0) = 0$ taking $h = 0.2$ at $x = 1$

[Ans.: 0.8007**]**

7.4 MODIFIED EULER'S METHOD

The Euler's method is very easy to implement but it cannot give accurate solutions. A very small step size is required to get any meaningful result. Since the starting point of each sub-interval is used to find the slope of the solution curve, the solution would be correct only if the function is linear. In the modified Euler's method, the arithmetic average of the slopes is used to approximate the solution curve.

In the modified Euler's method, $y_1^{(0)}$ is first calculated from the Euler's method.

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0)$$

This value is improved by making use of average slopes at (x_0, y_0) and $(x_1, y_1^{(0)})$. The first approximation to y_1 is written as

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

This value of $y_1^{(1)}$ is further improved by the equation

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

which is the second approximation to y_1.

In general,

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right], n = 0, 1, 2, \ldots$$

where $y_1^{(n)}$ is the n^{th} approximation to y_1.

The procedure will be terminated depending on the accuracy required. If two consecutive values of $y_1^{(k)}$ and $y_1^{(k+1)}$ are equal, $y_1 = y_1^{(k)}$.

Now, $y_2^{(0)}$ is calculated from the Euler's method.

$$y_2^{(0)} = y_1 + h\,f(x_1, y_1)$$

Better approximation to y_2 is obtained as

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

This procedure is repeated till two approximation to y_2 are equal. Proceeding in the same manner, other values, i.e., y_3, y_4, etc., can be calculated.

Example 1

Determine the value of y when $x = 0.1$ *correct up to four decimal places by taking* $h = 0.05$. *Given that* $y(0) = 1$ *and* $\frac{dy}{dx} = x^2 + y$.

Solution

$$\frac{dy}{dx} = f(x, y) = x^2 + y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.05, \quad x_1 = 0.05$

$$f(x_0, y_0) = 0 + 1 = 1$$

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0) = 1 + 0.05(1) = 1.05$$

First approximation to y_1

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \\ &= 1 + \frac{0.05}{2}\left[1 + f(0.05, 1.05)\right] \\ &= 1 + \frac{0.05}{2}\left[1 + \{(0.05)^2 + 1.05\}\right] \\ &= 1.0513 \end{aligned}$$

Second approximation to y_1

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 1 + \frac{0.05}{2}[1 + f(0.05, 1.0513)]$$
$$= 1 + \frac{0.05}{2}\left[1 + \{(0.05)^2 + 1.0513\}\right]$$
$$= 1.0513$$

Since the values of $y_1^{(1)}$ and $y_1^{(2)}$ are equal,

$$y_1 = y(0.05) = 1.0513$$

(ii) Now, $x_1 = 0.05, \quad y_1 = 1.0513, \quad h = 0.05, \quad x_2 = 0.1$

$$f(x_1, y_1) = (0.05)^2 + 1.0513 = 1.0538$$
$$y_2^{(0)} = y_1 + h\,f(x_1, y_1) = 1.0513 + 0.05(1.0538) = 1.1040$$

First approximation to y_2

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$
$$= 1.0513 + \frac{0.05}{2}[1.0538 + f(0.1, 1.1040)]$$
$$= 1.0513 + \frac{0.05}{2}\left[1.0538 + \{(0.1)^2 + 1.1040\}\right]$$
$$= 1.1055$$

Second approximation to y_2

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$
$$= 1.0513 + \frac{0.05}{2}[1.0538 + f(0.1, 1.1055)]$$
$$= 1.0513 + \frac{0.05}{2}\left[1.0538 + \{(0.1)^2 + 1.1055\}\right]$$
$$= 1.1055$$

Since the values of $y_2^{(1)}$ and $y_2^{(2)}$ are equal,

$$y_2 = y(0.1) = 1.1055$$

Example 2

Using the modified Euler's method, solve $\frac{dy}{dx} = 1 - y$ *with the initial condition* $y(0) = 0$ *at* $x = 0.1, 0.2$.

Solution

$$\frac{dy}{dx} = f(x, y) = 1 - y$$

(i) Given: $x_0 = 0, \quad y_0 = 0, \quad h = x_1 - x_0 = 0.1, \quad x_1 = 0.1$

$$f(x_0, y_0) = 1 - 0 = 1$$

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0) = 0 + 0.1(1) = 0.1$$

First approximation to y_1

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \\ &= 0 + \frac{0.1}{2}\left[1 + f(0.1, 0.1)\right] \\ &= 0 + \frac{0.1}{2}\left[1 + (1 - 0.1)\right] \\ &= 0.095 \end{aligned}$$

Second approximation to y_1

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \\ &= 0 + \frac{0.1}{2}\left[1 + f(0.1, 0.095)\right] \\ &= 0 + \frac{0.1}{2}\left[1 + (1 - 0.095)\right] \\ &= 0.0953 \end{aligned}$$

Third approximation to y_1

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \\ &= 0 + \frac{0.1}{2}\left[1 + f(0.1, 0.0953)\right] \\ &= 0 + \frac{0.1}{2}\left[1 + (1 - 0.0953)\right] \\ &= 0.0952 \end{aligned}$$

Fourth approximation to y_1

$$\begin{aligned} y_1^{(4)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right] \\ &= 0 + \frac{0.1}{2}\left[1 + f(0.1, 0.0952)\right] \\ &= 0 + \frac{0.1}{2}\left[1 + (1 - 0.0952)\right] \\ &= 0.0952 \end{aligned}$$

Since the values of $y_1^{(3)}$ and $y_1^{(4)}$ are equal,

$$y_1 = y(0.1) = 0.0952$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 0.0952, \quad h = 0.1, \quad x_2 = 0.2$

$$f(x_1, \quad y_1) = 1 - 0.0952 = 0.9048$$

$$y_2^{(0)} = y_1 + h\,f(x_1, y_1) = 0.0952 + 0.1(0.9048) = 0.1857$$

First approximation to y_2

$$\begin{aligned} y_2^{(1)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right] \\ &= 0.0952 + \frac{0.1}{2}\left[0.9048 + f(0.2, 0.1857)\right] \\ &= 0.0952 + \frac{0.1}{2}\left[0.9048 + (1 - 0.1857)\right] \\ &= 0.1812 \end{aligned}$$

Second approximation to y_2

$$\begin{aligned} y_2^{(2)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] \\ &= 0.0952 + \frac{0.1}{2}\left[0.9048 + f(0.2, 0.1812)\right] \\ &= 0.0952 + \frac{0.1}{2}\left[0.9048 + (1 - 0.1812)\right] \\ &= 0.1814 \end{aligned}$$

Third approximation to y_2

$$\begin{aligned} y_2^{(3)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right] \\ &= 0.0952 + \frac{0.1}{2}\left[0.9048 + f(0.2, 0.1814)\right] \\ &= 0.952 + \frac{0.1}{2}\left[0.9048 + (1 - 0.1814)\right] \\ &= 0.1814 \end{aligned}$$

Since the values of $y_2^{(2)}$ and $y_2^{(3)}$ are equal,

$$y_2 = y(0.2) = 0.1814$$

Example 3

Apply the modified Euler's method to solve the initial-value problem $y' = x + y$ *with* $y(0) = 0$ *choosing* $h = 0.2$ *and compute* y *for* $x = 0.2$, $x = 0.4$. **[Winter 2014]**

Solution

$$\frac{dy}{dx} = f(x, y) = x + y$$

(i) Given: $x_0 = 0, \quad y_0 = 0, \quad h = 0.2, \quad x_1 = 0.2$

$$f(x_0, y_0) = 0 + 0 = 0$$

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0) = 0 + 0.2(0) = 0$$

First approximation to y_1

$$\begin{aligned}
y_1^{(1)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \\
&= 0 + \frac{0.2}{2}\left[0 + f(0.2, 0)\right] \\
&= 0 + \frac{0.2}{2}\left[0 + (0.2 + 0)\right] \\
&= 0.02
\end{aligned}$$

Second approximation to y_1

$$\begin{aligned}
y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \\
&= 0 + \frac{0.2}{2}\left[0 + f(0.2, 0.02)\right] \\
&= 0 + \frac{0.2}{2}\left[0 + (0.2 + 0.02)\right] \\
&= 0.022
\end{aligned}$$

Third approximation to y_1

$$\begin{aligned}
y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \\
&= 0 + \frac{0.2}{2}\left[0 + f(0.2, 0.022)\right] \\
&= 0 + \frac{0.2}{2}\left[0 + (0.2 + 0.022)\right] \\
&= 0.0222
\end{aligned}$$

Fourth approximation to y_1

$$\begin{aligned}
y_1^{(4)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right] \\
&= 0 + \frac{0.2}{2}\left[0 + f(0.2, 0.0222)\right] \\
&= 0 + \frac{0.2}{2}\left[0 + (0.2 + 0.0222)\right] \\
&= 0.0222
\end{aligned}$$

Since the values of $y_1^{(3)}$ and $y_1^{(4)}$ are equal,

$$y_1 = y(0.2) = 0.0222$$

(ii) Now, $x_1 = 0.2, \quad y_1 = 0.0222, \quad h = 0.2, \quad x_2 = 0.4$

$$f(x_1, y_1) = 0.2 + 0.0222 = 0.2222$$

$$y_2^{(0)} = y_1 + h\, f(x_1, y_1) = 0.0222 + \frac{0.2}{2}(0.2222) = 0.0444$$

First approximation to y_2

$$\begin{aligned} y_2^{(1)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + f(0.4, 0.0444)\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + (0.4 + 0.0444)\right] \\ &= 0.0889 \end{aligned}$$

Second approximation to y_2

$$\begin{aligned} y_2^{(2)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + f(0.4, 0.0889)\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + (0.4 + 0.0889)\right] \\ &= 0.0933 \end{aligned}$$

Third approximation to y_2

$$\begin{aligned} y_2^{(3)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + f(0.4, 0.0933)\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + (0.4 + 0.0933)\right] \\ &= 0.0938 \end{aligned}$$

Fourth approximation to y_2

$$\begin{aligned} y_2^{(4)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(3)})\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + f(0.4, 0.0938)\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + (0.4 + 0.0938)\right] \\ &= 0.0938 \end{aligned}$$

Since the value of $y_2^{(3)}$ and $y_2^{(4)}$ are equal,

$$y_2 = y(0.4) = 0.0938$$

Example 4

Use modified Euler's method to find the value of y satisfying the equation $\dfrac{dy}{dx} = \log(x + y)$ *for* $x = 1.2$ *and* $x = 1.4$*, correct up to four decimal places by taking* $h = 0.2$*. Given that* $y(1) = 2$*.*

Solution

$$\frac{dy}{dx} = f(x, y) = \log(x + y)$$

(i) Given: $x_0 = 1, \quad y_0 = 2, \quad h = 0.2, \quad x_1 = 1.2$

$$f(x_0, y_0) = \log(1 + 2) = 1.0986$$

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0) = 2 + 0.2(1.0986) = 2.2197$$

First approximation to y_1

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \\ &= 2 + \frac{0.2}{2}[1.0986 + f(1.2, 2.2197)] \\ &= 2 + \frac{0.2}{2}[1.0986 + \log(1.2 + 2.2197)] \\ &= 2.2328 \end{aligned}$$

Second approximation to y_1

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \\ &= 2 + \frac{0.2}{2}[1.0986 + f(1.2, 2.2328)] \\ &= 2 + \frac{0.2}{2}[1.0986 + \log(1.2 + 2.2328)] \\ &= 2.2332 \end{aligned}$$

Third approximation to y_1

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \\ &= 2 + \frac{0.2}{2}[1.0986 + f(1.2, 2.2332)] \end{aligned}$$

$$= 2 + \frac{0.2}{2}\left[1.0986 + \log(1.2 + 2.2332)\right]$$

$$= 2.2332$$

Since the values of $y_1^{(2)}$ and $y_1^{(3)}$ are equal,

$$y_1 = y(1.2) = 2.2332$$

(ii) Now, $x_1 = 1.2, \quad y_1 = 2.2332, \quad h = 0.2, \quad x_2 = 1.4$

$$f(x_1, y_1) = \log(1.2 + 2.2332) = 1.2335$$

$$y_2^{(0)} = y_1 + h\,f(x_1, y_1) = 2.2332 + 0.2(1.2335) = 2.4799$$

First approximation to y_2

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + f(1.4, 2.4799)\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + \log(1.4 + 2.4799)\right]$$

$$= 2.4291$$

Second approximation to y_2

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + f(1.4, 2.4921)\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + \log(1.4 + 2.4921)\right]$$

$$= 2.4924$$

Third approximation to y_2

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + f(1.4, 2.4924)\right]$$

$$= 2.2332 + \frac{0.2}{2}\left[1.2335 + \log(1.4 + 2.4924)\right]$$

$$= 2.4924$$

Since the values of $y_2^{(2)}$ and $y_2^{(3)}$ are equal,

$$y_2 = y(1.4) = 2.4924$$

Example 5

Solve $\frac{dy}{dx} = 2 + \sqrt{xy}$ *with* $x_0 = 1.2$, $y_0 = 1.6403$ *by Euler's modified method for* $x = 1.6$, *correct up to four decimal places by taking* $h = 0.2$.

Solution

$$\frac{dy}{dx} = f(x, y) = 2 + \sqrt{xy}$$

(i) Given: $x_0 = 1.2, \quad y_0 = 1.6403, \quad h = 0.2, \quad x_1 = 1.4$

$$f(x_0, y_0) = 2 + \sqrt{(1.2)(1.6403)} = 3.4030$$

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0) = 1.6403 + 0.2(3.4030) = 2.3209$$

First approximation to y_1

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \\ &= 1.6403 + \frac{0.2}{2}[3.4030 + f(1.4, 2.3209)] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + \left\{2 + \sqrt{(1.4)(2.3209)}\right\}\right] \\ &= 2.3609 \end{aligned}$$

Second approximation to y_1

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \\ &= 1.6403 + \frac{0.2}{2}[3.4030 + f(1.4, 2.3609)] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + \left\{2 + \sqrt{(1.4)(2.3609)}\right\}\right] \\ &= 2.3624 \end{aligned}$$

Third approximation to y_1

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \\ &= 1.6403 + \frac{0.2}{2}[3.4030 + f(1.4, 2.3624)] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + \left\{2 + \sqrt{(1.4)(2.3624)}\right\}\right] \\ &= 2.3625 \end{aligned}$$

Fourth approximation to y_1

$$\begin{aligned}
y_1^{(4)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right] \\
&= 1.6403 + \frac{0.2}{2}[3.4030 + f(1.4, 2.3625)] \\
&= 1.6403 + \frac{0.2}{2}\left[3.4030 + \left\{2 + \sqrt{(1.4)(2.3625)}\right\}\right] \\
&= 2.3625
\end{aligned}$$

Since the values of $y_1^{(3)}$ and $y_1^{(4)}$ are equal,

$$y_1 = y(1.4) = 2.3625$$

(ii) Now, $x_1 = 1.4, \quad y_1 = 2.3625, \quad h = 0.2, \quad x_2 = 1.6$

$$f(x_1, y_1) = 2 + \sqrt{(1.4)\,(2.3625)} = 3.8187$$

$$y_2^{(0)} = y_1 + h\,f(x_1, y_1) = 2.3625 + 0.2\,(3.8187) = 3.1262$$

First approximation to y_2

$$\begin{aligned}
y_2^{(1)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right] \\
&= 2.3625 + \frac{0.2}{2}[3.8187 + f\,(1.6, 3.1262)] \\
&= 2.3625 + \frac{0.2}{2}\left[3.8187 + \left\{2 + \sqrt{(1.6)(3.1262)}\right\}\right] \\
&= 3.1680
\end{aligned}$$

Second approximation to y_2

$$\begin{aligned}
y_2^{(2)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] \\
&= 2.3625 + \frac{0.2}{2}[3.8187 + f(1.6, 3.1680)] \\
&= 2.3625 + \frac{0.2}{2}\left[3.8187 + \left\{2 + \sqrt{(1.6) + 3.1680)}\right\}\right] \\
&= 3.1695
\end{aligned}$$

Third approximation to y_2

$$\begin{aligned}
y_2^{(3)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right] \\
&= 2.3625 + \frac{0.2}{2}[3.8187 + f(1.6, 3.1695)] \\
&= 2.3625 + \frac{0.2}{2}\left[3.8187 + \left\{2 + \sqrt{(1.6)(3.1695)}\right\}\right] \\
&= 3.1696
\end{aligned}$$

Fourth approximation to y_2

$$\begin{aligned} y_2^{(4)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(3)})\right] \\ &= 2.3625 + \frac{0.2}{2}[3.8187 + f(1.6, 3.1696)] \\ &= 2.3625 + \frac{0.2}{2}\left[3.8187 + \left\{2 + \sqrt{(1.6)(3.1696)}\right\}\right] \\ &= 3.1696 \end{aligned}$$

Since the values of $y_2^{(3)}$ and $y_2^{(4)}$ are equal,

$$y_2 = y(1.6) = 3.1696$$

EXERCISE 7.3

Solve the following differential equations by the modified Euler's method:

1. $\frac{dy}{dx} = x + 3y$ with $x_0 = 0, y_0 = 1$ taking h = 0.05 at x = 0.1

[Ans.: 1.3548**]**

2. $\frac{dy}{dx} = x - y^2$ with $x_0 = 0, y_0 = 1$ taking h = 0.05 at x = 0.1

[Ans.: 0.9137**]**

3. $\frac{dy}{dx} = x + y$ with $x_0 = 0, y_0 = 1$ taking h = 0.05 at x = 0.1

[Ans.: 1.1104**]**

4. $\frac{dy}{dx} = -xy^2$ with $y(0) = 2$ for $x = 0.2$ by taking h = 0.1

[Ans.: 1.9238**]**

5. $\frac{dy}{dx} = 1 + \frac{y}{x}$ with $y(1) = 2$ for $x = 1.2$

[Ans.: 2.6182**]**

6. $\frac{dy}{dx} = x + \sqrt{y}$ with $y(0) = 1$ for $x = 0.2$

[Ans.: 1.2309**]**

7. $\frac{dy}{dx} = y^2 - \frac{y}{x}$ with $y(1) = 1$ for $x = 1.1$ taking h = 0.05

[Ans.: 1.0073**]**

8. $\frac{dy}{dx} = y - x$ with $y(0) = 2$ for $x = 0.2$

[Ans.: 2.4222**]**

7.5 RUNGE—KUTTA METHODS

Runge–Kutta methods do not require the determination of higher order derivatives. These methods require only the function values at different points on the sub-interval. The main advantage of Runge–Kutta methods is the self-starting feature and, consequently, the ease of programming. One disadvantage of Runge–Kutta methods is the requirement that the function must be evaluated at different values of x and y in every step of the function. This repeated determination of the function may result in a less efficient method with respect to computing time than other methods of comparable accuracy in which previously determined values of the dependent variable are used in the subsequent steps.

7.5.1 First-Order Runge—Kutta Method

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

with the initial condition $y(x_0) = y_0$

By Euler's method,

$$y_{n+1} = y_n + h\,f(x_n, y_n)$$

Expanding LHS by Taylor's series,

$$y_{n+1} = y_n + h\,y_n' + \frac{h^2}{2!}\,y_n'' + \cdots$$

Euler's method is known as the first-order Runge–Kutta method.

7.5.2 Second-Order Runge—Kutta Method (Heun Method)

The second order Rungta-Kutta method is given by the equations

$$k_1 = h\,f(x_n, y_n)$$

$$k_2 = h\,f\,(x_n + h, y_n + k_1)$$

$$k = \frac{1}{2}(k_1 + k_2)$$

$$y_{n+1} = y_n + k$$

7.5.3 Third-Order Runge—Kutta Method

The third-order Runge–Kutta method is given by the equations

$$k_1 = h\,f(x_n, y_n)$$

$$k_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h\,f(x_n + h,\, y_n + 2k_2 - k_1)$$
$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$
$$y_{n+1} = y_n + k$$

7.5.4 Fourth-Order Runge—Kutta Method

This method is mostly used and is often referred to as the Runge–Kutta method only without reference of the order. The fourth-order Runge–Kutta method is given by the equations

$$k_1 = h\,f(x_n, y_n)$$
$$k_2 = h\,f\left(x_n + \frac{h}{2},\, y_n + \frac{k_1}{2}\right)$$
$$k_3 = h\,f\left(x_n + \frac{h}{2},\, y_n + \frac{k_2}{2}\right)$$
$$k_4 = h\,f(x_n + h,\, y_n + k_3)$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$y_{n+1} = y_n + k$$

Example 1

Given that $y = 1.3$ *when* $x = 1$ *and* $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x + y$. *Use the second-order Runge–Kutta method (i.e., Heun method) to approximate y when* $x = 1.2$. *Use a step size of* 0.1. **[Winter 2012]**

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = 3x + y$$

(i) Given: $x_0 = 1, \quad y_0 = 1.3, \quad h = 0.1, \quad n = 0$

$$\begin{aligned} k_1 &= h\,f(x_0, y_0) \\ &= 0.1\,f(1, 1.3) \\ &= 0.1\big[3(1) + 1.3\big] \\ &= 0.43 \\ k_2 &= h\,f(x_0 + h,\, y_0 + k_1) \\ &= 0.1\,f(1 + 0.1,\, 1.3 + 0.43) \end{aligned}$$

$$= 0.1\, f(1.1, 1.73)$$
$$= 0.1\,[3(1.1) + 1.73]$$
$$= 0.503$$
$$k = \frac{1}{2}(k_1 + k_2)$$
$$= \frac{1}{2}(0.43 + 0.503)$$
$$= 0.4665$$
$$y_1 = y_0 + k$$
$$= 1.3 + 0.4665$$
$$= 1.7665$$

(ii) Now, $x_1 = 1.1, \quad y_1 = 1.7665, \quad h = 0.1, \quad n = 1$

$$k_1 = h\,f(x_1, y_1)$$
$$= 0.1\, f(1.1,\ 1.7665)$$
$$= 0.1\ [3(1.1) + 1.7665]$$
$$= 0.5067$$
$$k_2 = h\,f(x_1 + h,\ y_1 + k_1)$$
$$= 0.1\ f(1.1 + 0.1, 1.7665 + 05067)$$
$$= 0.1\ f(1.2, 2.2732)$$
$$= 0.1[3(1.2) + 2.2732)]$$
$$= 0.5873$$
$$k = \frac{1}{2}(k_1 + k_2)$$
$$= \frac{1}{2}(0.5067 + 0.5873)$$
$$= 0.5470$$
$$y_2 = y_1 + k$$
$$= 1.7665 + 0.5470$$
$$= 2.3135$$

Hence, $y_2 = y(1.2) = 2.3135$

Example 2

Use the second-order Runge–Kutta method to find an approximate value of y given that $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y^2$ *and* $y(0) = 1$ *at* $x = 0.2$ *taking* $h = 0.1$.

Solution

$$\frac{dy}{dx} = f(x, y) = x - y^2$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$\begin{aligned} k_1 &= h\,f(x_0, y_0) \\ &= 0.1\,f(0, 1) \\ &= 0.1\left[0 - (1)^2\right] \\ &= -0.1 \end{aligned}$$

$$\begin{aligned} k_2 &= h\,f(x_0 + h, y_0 + k_1) \\ &= 0.1\,f[0 + 0.1, 1 + (-0.1)] \\ &= 0.1\,f(0.1, 0.9) \\ &= 0.1\left[0.1 - (0.9)^2\right] \\ &= -0.071 \end{aligned}$$

$$\begin{aligned} k &= \frac{1}{2}(k_1 + k_2) \\ &= \frac{1}{2}(-0.1 - 0.071) \\ &= -0.0855 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + k \\ &= 1 - 0.0855 \\ &= 0.9145 \end{aligned}$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 0.9145, \quad h = 0.1, \quad n = 1$

$$\begin{aligned} k_1 &= h\,f(x_1, y_1) \\ &= 0.1\,f(0.1, 0.9145) \\ &= 0.1\left[0.1 - (0.9145)^2\right] \\ &= -0.0736 \end{aligned}$$

$$\begin{aligned} k_2 &= h\,f(x_1 + h, y_1 + k_1) \\ &= 0.1\,f\left[0.1 + 0.1, 0.9145 - 0.0736\right] \\ &= 0.1\,f\,(0.2, 0.8408) \\ &= 0.1\left[0.2 - (0.8408)^2\right] \\ &= -0.0507 \end{aligned}$$

$$
\begin{aligned}
k &= \frac{1}{2}(k_1 + k_2) \\
&= \frac{1}{2}(-0.0736 - 0.0507) \\
&= -0.0622 \\
y_2 &= y_1 + k \\
&= 0.9145 - 0.0622 \\
&= 0.8523
\end{aligned}
$$

Hence, $y_2 = y(0.2) = 0.8523$

Example 3

Obtain the values of y at $x = 0.1, 0.2$ *using the Runge–Kutta method of third order for the differential equation* $\frac{dy}{dx} + y = 0,\ y(0) = 1$.

Solution

$$\frac{dy}{dx} = f(x, y) = -y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$
\begin{aligned}
k_1 &= h\, f(x_0, y_0) \\
&= 0.1\, f(0, 1) \\
&= 0.1\,(-1) \\
&= -0.1 \\
k_2 &= h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\
&= 0.1\, f\left(0 + \frac{0.1}{2}, 1 - \frac{0.1}{2}\right) \\
&= 0.1\, f(0.05, 0.95) \\
&= 0.1(-0.95) \\
&= -0.095 \\
k_3 &= h\, f(x_0 + h, y_0 + 2k_2 - k_1) \\
&= 0.1\, f\left[0 + 0.1, 1 + 2(-0.095) + 0.1\right] \\
&= 0.1\, f(0.1, 0.91) \\
&= 0.1(-0.91) \\
&= -0.091
\end{aligned}
$$

$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$
$$= \frac{1}{6}[-0.1 + 4(-0.095) - 0.091]$$
$$= -0.0952$$
$$y_1 = y_0 + k$$
$$= 1 - 0.0952$$
$$= 0.9048$$

Hence, $y_1 = y(0.1) = 0.9048$

(ii) Now, $x_1 = 0.1, \quad y_1 = 0.9048, \quad h = 0.1, \quad n = 1$

$$k_1 = h\,f(x_1, y_1)$$
$$= 0.1\,f(0.1, 0.9048)$$
$$= 0.1(-0.9048)$$
$$= -0.0905$$
$$k_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.1\,f\left(0.1 + \frac{0.1}{2}, 0.9048 - \frac{0.0905}{2}\right)$$
$$= 0.1\,f(0.15, 0.8596)$$
$$= 0.1(-0.8596)$$
$$= -0.086$$
$$k_3 = h\,f(x_1 + h, y_1 + 2k_2 - k_1)$$
$$= 0.1\,f[0.1 + 0.1, 0.9048 + 2(-0.086) + 0.0905]$$
$$= 0.1\,f(0.2, 0.8233)$$
$$= 0.1\,(-0.8233)$$
$$= -0.0823$$
$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$
$$= \frac{1}{6}[-0.0905 + 4(-0.086) - 0.0823]$$
$$= -0.0861$$
$$y_2 = y_1 + k$$
$$= 0.9048 - 0.0861$$
$$= 0.8187$$

Hence, $y_2 = y(0.2) = 0.8187$

Example 4

Apply the third-order Runge–Kutta method to the initial-value problem $\frac{dy}{dx} = x^2 - y,\ y(0) = 1$ *over the interval* $(0, 0.2)$ *taking* $h = 0.1$.

Solution

$$\frac{dy}{dx} = f(x, y) = x^2 - y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$\begin{aligned}
k_1 &= h\, f(x_0, y_0) \\
&= 0.1\, f(0, 1) \\
&= 0.1(0 - 1) \\
&= -0.1 \\
k_2 &= h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\
&= 0.1\, f\left(0 + \frac{0.1}{2}, 1 - \frac{0.1}{2}\right) \\
&= 0.1\, f(0.05, 0.95) \\
&= 0.1\left[(0.05)^2 - 0.95\right] \\
&= -0.0948 \\
k_3 &= h\, f(x_0 + h, y_0 + 2k_2 - k_1) \\
&= 0.1\, f\left[0 + 0.1, 1 + 2(-0.0948) + 0.1\right] \\
&= 0.1\, f(0.1, 0.9104) \\
&= 0.1\left[(0.1)^2 - 0.9104\right] \\
&= -0.09 \\
k &= \frac{1}{6}(k_1 + 4k_2 + k_3) \\
&= \frac{1}{6}\left[-0.1 + 4(-0.0948) - 0.09\right] \\
&= -0.0949 \\
y_1 &= y_0 + k \\
&= 1 - 0.0949 \\
&= 0.9051
\end{aligned}$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 0.9051, \quad h = 0.1, \quad n = 1$

$$
\begin{aligned}
k_1 &= h f(x_1, y_1) \\
&= 0.1\, f(0.1, 0.9051) \\
&= 0.1\left[(0.1)^2 - 0.9051\right] \\
&= -0.0895 \\
k_2 &= h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) \\
&= 0.1\, f\left(0.1 + \frac{0.1}{2}, 0.9051 - \frac{0.0895}{2}\right) \\
&= 0.1\, f(0.15, 0.8604) \\
&= 0.1\left[(0.15)^2 - 0.8604\right] \\
&= -0.0838 \\
k_3 &= h f(x_1 + h, y_1 + 2k_2 - k_1) \\
&= 0.1\, f\left[0.1 + 0.1, 0.9051 + 2(-0.0838) + 0.0895\right] \\
&= 0.1\, f(0.2, 0.827) \\
&= 0.1\left[(0.2)^2 - 0.827\right] \\
&= -0.0787
\end{aligned}
$$

$$
\begin{aligned}
k &= \frac{1}{6}(k_1 + 4k_2 + k_3) \\
&= \frac{1}{6}\left[-0.0895 + 4(-0.0838) - 0.0787\right] \\
&= -0.0839 \\
y_2 &= y_1 + k \\
&= 0.9051 - 0.0839 \\
&= 0.8212
\end{aligned}
$$

Example 5

Solve the differential equation $\dfrac{dy}{dx} = x + y$, *with the fourth-order Runge–Kutta method, where* $y(0) = 1$, *with* $x = 0$ *to* $x = 0.2$ *with* $h = 0.1$.

[Winter 2012]

Solution

$$\frac{dy}{dx} = f(x, y) = x + y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$
\begin{aligned}
k_1 &= h\,f(x_0, y_0)\\
&= 0.1\, f(0, 1)\\
&= 0.1\,(0+1)\\
&= 0.1\\
k_2 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)\\
&= 0.1\, f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)\\
&= 0.1\, f(0.05, 1.05)\\
&= 0.1\,(0.05 + 1.05)\\
&= 0.11\\
k_3 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)\\
&= 0.1\, f\left(0 + \frac{0.1}{2}, 1 + \frac{0.11}{2}\right)\\
&= 0.1\, f(0.05, 1.055)\\
&= 0.1(0.05 + 1.055)\\
&= 0.1105\\
k_4 &= h\,f(x_0 + h, y_0 + k_3)\\
&= 0.1\, f(0 + 0.1, 1 + 0.1105)\\
&= 0.1\, f(0.1, 1.1105)\\
&= 0.1(0.1 + 1.1105)\\
&= 0.1211\\
k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\\
&= \frac{1}{6}\left[0.1 + 2(0.11) + 2(0.1105) + 0.1211\right]\\
&= 0.1103\\
y_1 &= y_0 + k\\
&= 1 + 0.1103\\
&= 1.1103
\end{aligned}
$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 1.1103, \quad h = 0.1, \quad n = 1$

$$
\begin{aligned}
k_1 &= h\,f(x_1, y_1)\\
&= 0.1\, f(0.1, 1.1103)\\
&= 0.1(0.1 + 1.1103)\\
&= 0.1210
\end{aligned}
$$

$$k_2 = h\,f\left(x_1 + \frac{h}{2},\, y_1 + \frac{k_1}{2}\right)$$
$$= 0.1\,f\left(0.1 + \frac{0.1}{2},\, 1.1103 + \frac{0.1210}{2}\right)$$
$$= 0.1\,f(0.15, 1.1708)$$
$$= 0.1\,(0.15 + 1.1708)$$
$$= 0.1321$$
$$k_3 = h\,f\left(x_1 + \frac{h}{2},\, y_1 + \frac{k_2}{2}\right)$$
$$= 0.1\,f\left(0.1 + \frac{0.1}{2},\, 1.1103 + \frac{0.1321}{2}\right)$$
$$= 0.1\,f(0.15, 1.1764)$$
$$= 0.1\,(0.15 + 1.1764)$$
$$= 0.1326$$
$$k_4 = h\,f(x_1 + h,\, y_1 + k_3)$$
$$= 0.1\,f(0.1 + 0.1, 1.1103 + 0.1326)$$
$$= 0.1\,f(0.2, 1.2429)$$
$$= 0.1(0.2 + 1.2429)$$
$$= 0.1443$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}\left[0.1210 + 2(0.1321) + 2(0.1326) + 0.1443\right]$$
$$= 0.1325$$
$$y_2 = y_1 + k$$
$$= 1.1103 + 0.1325$$
$$= 1.2428$$

Example 6

Using the Runge–Kutta method of fourth-order, solve $10\dfrac{dy}{dx} = x^2 + y^2$, $y(0) = 1$ *at* $x = 0.1$ *and* $x = 0.2$ *taking* $h = 0.1$. **[Summer 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = \frac{x^2 + y^2}{10} = 0.1(x^2 + y^2)$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$
\begin{aligned}
k_1 &= h\,f(x_0, y_0) \\
&= 0.1\,f(0,1) \\
&= 0.1(0.1)(0+1) \\
&= 0.01 \\
k_2 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\
&= 0.1\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0.01}{2}\right) \\
&= 0.1\,f(0.05, 1.005) \\
&= 0.1(0.1)\left[(0.05)^2 + (1.005)^2\right] \\
&= 0.0101 \\
k_3 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\
&= 0.1\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0.0101}{2}\right) \\
&= 0.1\,f(0.05, 1.0051) \\
&= 0.1(0.1)\left[(0.05)^2 + (1.0051)^2\right] \\
&= 0.0101 \\
k_4 &= h\,f(x_0 + h, y_0 + k_3) \\
&= 0.1\,f(0 + 0.1, 1 + 0.0101) \\
&= 0.1\,f(0.1, 1.0101) \\
&= 0.1(0.1)\left[(0.1)^2 + (1.0101)^2\right] \\
&= 0.0103 \\
k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&= \frac{1}{6}[0.01 + 2(0.0101) + 2(0.0101) + 0.0103] \\
&= 0.0101 \\
y_1 &= y_0 + k \\
&= 1 + 0.0101 \\
&= 1.0101
\end{aligned}
$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 1.0101, \quad h = 0.1, \quad n = 1$

$$\begin{aligned}
k_1 &= h\,f(x_1, y_1) \\
&= h\,f(0.1, 1.0101) \\
&= 0.1(0.1)\left[(0.1)^2 + (1.0101)^2\right] \\
&= 0.0103 \\
k_2 &= h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) \\
&= 0.1\,f\left(0.1 + \frac{0.1}{2}, 1.0101 + \frac{0.0103}{2}\right) \\
&= 0.1\,f(0.15, 1.0153) \\
&= 0.1(0.1)\left[(0.15)^2 + (1.0153)^2\right] \\
&= 0.0105 \\
k_3 &= h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) \\
&= 0.1\,f\left(0.1 + \frac{0.1}{2}, 1.0101 + \frac{0.0105}{2}\right) \\
&= 0.1\,f(0.15, 1.0154) \\
&= 0.1(0.1)\left[(0.15)^2 + (1.0154)^2\right] \\
&= 0.0105 \\
k_4 &= h\,f(x_1 + h, y_1 + k_3) \\
&= 0.1\,f(0.1 + 0.1, 1.0101 + 0.0105) \\
&= 0.1\,f(0.2, 1.0206) \\
&= 0.1\,(0.1)\left[(0.2)^2 + (1.0206)^2\right] \\
&= 0.0108 \\
k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&= \frac{1}{6}\left[0.0103 + 2(0.0105) + 2(0.0105) + 0.0108\right] \\
&= 0.0105 \\
y_2 &= y_1 + k \\
&= 1.0101 + 0.0105 \\
&= 1.0206
\end{aligned}$$

Example 7

Use the fourth order Runge–Kutta method to find the value of y at $x = 1$*, given that* $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$, $y(0)=1$ *with* $h = 0.5$. **[Summer 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = \frac{y-x}{y+x}$$

(i) Given: $x_0 = 0,\quad y_0 = 1,\quad h = 0.5,\quad n = 0$

$$\begin{aligned}
k_1 &= h\,f(x_0, y_0)\\
&= 0.5\ f(0, 1)\\
&= 0.5\left(\frac{1-0}{1+0}\right)\\
&= 0.5\\
k_2 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)\\
&= 0.5\ f\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2}\right)\\
&= 0.5\ f(0.25, 1.25)\\
&= 0.5\left(\frac{1.25-0.25}{1.25+0.25}\right)\\
&= 0.3333\\
k_3 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)\\
&= 0.5\ f\left(0 + \frac{0.5}{2}, 1 + \frac{0.3333}{2}\right)\\
&= 0.5\ f(0.25, 1.1667)\\
&= 0.5\left(\frac{1.1667-0.25}{1.1667+0.25}\right)\\
&= 0.3235\\
k_4 &= h\,f(x_0 + h, y_0 + k_3)\\
&= 0.5\ f(0+0.5, 1+0.3235)\\
&= 0.5(0.5, 1.3235)\\
&= 0.5\left(\frac{1.3235-0.5}{1.3235+0.5}\right)\\
&= 0.2258
\end{aligned}$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.5 + 2(0.3333) + 2(0.3235) + 0.2258]$$
$$= 0.3399$$
$$y_1 = y_0 + k$$
$$= 1 + 0.3399$$
$$= 1.3399$$

(ii) Now, $x_1 = 0.5, \quad y_1 = 1.3399, \quad h = 0.5, \quad n = 1$

$$k_1 = h\,f(x_1, y_1)$$
$$= 0.5\ f(0.5, 1.3399)$$
$$= 0.5\left(\frac{1.3399 - 0.5}{1.3399 + 0.5}\right)$$
$$= 0.2282$$
$$k_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.5 f\left(0.5 + \frac{0.5}{2}, 1.3399 + \frac{0.2282}{2}\right)$$
$$= 0.5\ f(0.75, 1.454)$$
$$= 0.5\left(\frac{1.454 - 0.75}{1.454 + 0.75}\right)$$
$$= 0.1597$$
$$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.5\ f\left(0.5 + \frac{0.5}{2}, 1.3399 + \frac{0.1597}{2}\right)$$
$$= 0.5\ f(0.75, 1.4198)$$
$$= 0.5\left(\frac{1.4198 - 0.75}{1.4198 + 0.75}\right)$$
$$= 0.1543$$
$$k_4 = h\,f(x_1 + h, y_1 + k_3)$$
$$= 0.5\ f(0.5 + 0.5, 1.3399 + 0.1543)$$
$$= 0.5\ f(1, 1.4942)$$
$$= 0.5\left(\frac{1.4942 - 1}{1.4942 + 1}\right)$$
$$= 0.0991$$

$$
\begin{aligned}
k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&= \frac{1}{6}\left[0.2282 + 2(0.1597) + 2(0.1543) + 0.0991\right] \\
&= 0.1592 \\
y_2 &= y_1 + k \\
&= 1.3399 + 0.1592 \\
&= 1.4991
\end{aligned}
$$

Hence, $y_2 = y(1) = 1.4991$

Example 8

Using the fourth order Runge–Kutta method, find y at x = 0.1 for differential equation $\frac{dy}{dx} = 3e^x + 2y,\ y(0) = 0$ *by taking* $h = 0.1$.

[Summer 2015]

Solution

$$\frac{dy}{dx} = f(x, y) = 3e^x + 2y$$

(i) Given: $x_0 = 0, \quad y_0 = 0, \quad h = 0.1, \quad n = 0$

$$
\begin{aligned}
k_1 &= h\,f(x_0, y_0) \\
&= 0.1\,f(0, 0) \\
&= 0.1(3e^0 + 0) \\
&= 0.3 \\
k_2 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\
&= 0.1\,f\left(0 + \frac{0.1}{2}, 0 + \frac{0.3}{2}\right) \\
&= 0.1\,f(0.05, 0.15) \\
&= 0.1\left[3e^{0.05} + 2(0.15)\right] \\
&= 0.3454 \\
k_3 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\
&= 0.1\,f\left(0 + \frac{0.1}{2}, 0 + \frac{0.3454}{2}\right) \\
&= 0.1\,f(0.05, 0.1727)
\end{aligned}
$$

$$= 0.1\left[3e^{0.05} + 2(0.1727)\right]$$
$$= 0.3499$$
$$k_4 = h\,f(x_0 + h,\, y_0 + k_3)$$
$$= 0.1\,f(0 + 0.1,\, 0 + 0.3499)$$
$$= 0.1\,f(0.1,\, 0.3499)$$
$$= 0.1\left[3e^{0.1} + 2(0.3499)\right]$$
$$= 0.4015$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}\left[0.3 + 2(0.3454) + 2(0.3499) + 0.4015\right]$$
$$= 0.3487$$
$$y_1 = y_0 + k$$
$$= 0 + 0.3487$$
$$= 0.3487$$

Hence, $y_2 = y(0.1) = 0.3487$

Example 9

Determine $y(0.1)$ *and* $y(0.2)$ *correct to four decimal places from* $\frac{dy}{dx} = 2x + y,\ y(0) = 1$ *with* $h = 0.1$.

Solution

$$\frac{dy}{dx} = f(x, y) = 2x + y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$k_1 = h\,f(x_0, y_0)$$
$$= 0.1\,f(0, 1)$$
$$= 0.1\left[2(0) + 1\right]$$
$$= 0.1$$
$$k_2 = h\,f\left(x_0 + \frac{h}{2},\, y_0 + \frac{k_1}{2}\right)$$
$$= 0.1\,f\left(0 + \frac{0.1}{2},\, 1 + \frac{0.1}{2}\right)$$
$$= 0.1\,f(0.05,\, 1.05)$$

$$= 0.1[2(0.05) + 1.05]$$
$$= 0.115$$
$$k_3 = h\,f\left(x_0 + \frac{h}{2},\, y_0 + \frac{k_2}{2}\right)$$
$$= 0.1\,f\left(0 + \frac{0.1}{2},\, 1 + \frac{0.115}{2}\right)$$
$$= 0.1\,f(0.05, 1.0575)$$
$$= 0.1[2(0.05) + 1.0575]$$
$$= 0.11575$$
$$k_4 = h\,f(x_0 + h,\, y_0 + k_3)$$
$$= 0.1\,f(0 + 0.1, 1 + 0.11575)$$
$$= 0.1\,f(0.1, 1.11575)$$
$$= 0.1[2(0.1) + 1.11575]$$
$$= 0.13158$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.1 + 2(0.115) + 2(0.11575) + 0.13158]$$
$$= 0.1155$$
$$y_1 = y_0 + k$$
$$= 1 + 0.1155$$
$$= 1.1155$$

Hence, $y_1 = y(0.1) = 1.1155$

(ii) Given: $x_1 = 0.1, \quad y_1 = 1.1155, \quad h = 0.1, \quad n = 1$

$$k_1 = h\,f(x_1, y_1)$$
$$= 0.1\,f(0.1, 1.1155)$$
$$= 0.1[2(0.1) + 1.1155]$$
$$= 0.13165$$
$$k_2 = h\,f\left(x_1 + \frac{h}{2},\, y_1 + \frac{k_1}{2}\right)$$
$$= 0.1\,f\left(0.1 + \frac{0.1}{2},\, 1.1155 + \frac{0.13165}{2}\right)$$
$$= 0.1\,f(0.15, 1.1813)$$
$$= 0.1[2(0.15) + 1.1813]$$
$$= 0.14813$$

$$
\begin{aligned}
k_3 &= h\,f\left(x_1+\frac{h}{2},\, y_1+\frac{k_2}{2}\right)\\
&= 0.1\,f\left(0.1+\frac{0.1}{2},\, 1.1155+\frac{0.14813}{2}\right)\\
&= 0.1\,f(0.15, 1.18965)\\
&= 0.1\left[2(0.15)+1.18965\right]\\
&= 0.149\\
k_4 &= h(x_1+h,\, y_1+k_3)\\
&= 0.1\,f(0.1+0.1, 1.1155+0.149)\\
&= 0.1\,f(0.2, 1.2645)\\
&= 0.1\left[2(0.2)+1.2645\right]\\
&= 0.16645\\
k &= \frac{1}{6}[k_1+2k_2+2k_3+k_4)\\
&= \frac{1}{6}[0.13165+2(0.14813)+2(0.149)+0.16645]\\
&= 0.1487\\
y_2 &= y_1+k\\
&= 1.1155+0.1487\\
&= 1.2642
\end{aligned}
$$

Hence, $y_2 = y(0.2) = 1.2642$

Example 10

Apply the Runge–Kutta method of fourth order to find an approximate value of y at x $= 0.6$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x+y},\ y(0.4) = 0.41$ *in two steps.*

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = \sqrt{x+y}$$

(i) Given: $x_0 = 0.4,\quad y_0 = 0.41,\quad h = 0.1,\quad n = 0$

$$
\begin{aligned}
k_1 &= hf(x_0, y_0)\\
&= 0.1\,f(0.4, 0.41)\\
&= 0.1\sqrt{0.4+0.41}\\
&= 0.09
\end{aligned}
$$

$$\begin{aligned}
k_2 &= hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)\\
&= 0.1f\left(0.4+\frac{0.1}{2}, 0.41+\frac{0.09}{2}\right)\\
&= 0.1\, f(0.45, 0.455)\\
&= 0.1\sqrt{0.45+0.455}\\
&= 0.0951\\
k_3 &= hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)\\
&= 0.1\, f\left(0.4+\frac{0.1}{2}, 0.41+\frac{0.0951}{2}\right)\\
&= 0.1\, f(0.45, 0.4576)\\
&= 0.1\sqrt{0.45+0.4576}\\
&= 0.0953\\
k_4 &= hf(x_0+h, y_0+k_3)\\
&= 0.1\, f(0.4+0.1, 0.41+0.0953)\\
&= 0.1\, f(0.5, 0.5053)\\
&= 0.1\sqrt{0.5+0.5053}\\
&= 0.1003\\
k &= \frac{1}{6}(k_1+2k_2+2k_3+k_4)\\
&= \frac{1}{6}\left[0.09+2(0.0951)+2(0.0953)+0.1003\right]\\
&= 0.0952\\
y_1 &= y_0+k\\
&= 0.41+0.0952\\
&= 0.5052
\end{aligned}$$

(ii) Now, $x_1 = 0.5, \quad y_1 = 0.5052, \quad h = 0.1, \quad n = 1$

$$\begin{aligned}
k_1 &= h\, f(x_1, y_1)\\
&= 0.1\, f(0.5, 0.5052)\\
&= 0.1\sqrt{0.5+0.5052}\\
&= 0.1003\\
k_2 &= h\, f\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right)\\
&= 0.1\, f\left(0.5+\frac{0.1}{2}, 0.5052+\frac{0.1003}{2}\right)
\end{aligned}$$

$$= 0.1\, f(0.55, 0.5554)$$
$$= 0.1\sqrt{0.55 + 0.5554}$$
$$= 0.1051$$
$$k_3 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.1\, f\left(0.5 + \frac{0.1}{2}, 0.5052 + \frac{0.1051}{2}\right)$$
$$= 0.1\, f(0.55, 0.5578)$$
$$= 0.1\sqrt{0.55 + 0.5578}$$
$$= 0.1053$$
$$k_4 = h\, f(x_1 + h, y_1 + k_3)$$
$$= 0.1\, f(0.5 + 0.1, 0.5052 + 0.1053)$$
$$= 0.1\, f(0.6, 0.6105)$$
$$= 0.1\sqrt{0.6 + 0.6105}$$
$$= 0.1100$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}\left[0.1003 + 2(0.1051) + 2(0.1053) + 0.1100\right]$$
$$= 0.1052$$
$$y_2 = y_1 + k$$
$$= 0.5052 + 0.1052$$
$$= 0.6104$$

Hence, $y_2 = y(0.6) = 0.6104$

Example 11

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x+y}$, $x_1 = 0$, $y_1 = 1$ *for the interval* (0, 1) *choosing* $h = 0.5$ *by the Runge–Kutta method of fourth order.*

Solution

$$\frac{dy}{dx} = f(x, y) = \frac{1}{x+y}$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.5, \quad n = 0$

$$\begin{aligned}
k_1 &= h\,f(x_0, y_0)\\
&= 0.5 f(0, 1)\\
&= 0.5\left(\frac{1}{0+1}\right)\\
&= 0.5\\
k_2 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)\\
&= 0.5\ f\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2}\right)\\
&= 0.5\ f(0.25, 1.25)\\
&= 0.5\left(\frac{1}{0.25 + 1.25}\right)\\
&= 0.3333\\
k_3 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)\\
&= 0.5\ f\left(0 + \frac{0.5}{2}, 1 + \frac{0.3333}{2}\right)\\
&= 0.5\ f(0.25, 1.1666)\\
&= 0.5\left(\frac{1}{0.25 + 1.1666}\right)\\
&= 0.3529\\
k_4 &= h\,f(x_0 + h, y_0 + k_2)\\
&= 0.5\ f(0 + 0.5, 1 + 0.3529)\\
&= 0.5\ f(0.5, 1.3529)\\
&= 0.5\left(\frac{1}{0.5 + 1.3529}\right)\\
&= 0.2698\\
k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\\
&= \frac{1}{6}\big[0.5 + 2(0.3333) + 2(0.3529) + 0.2698\big]\\
&= 0.3570\\
y_1 &= y_0 + k\\
&= 1 + 0.3570\\
&= 1.3570
\end{aligned}$$

(ii) Now, $x_1 = 0.5, \quad y_1 = 1.3570, \quad h = 0.5, \quad n = 1$

$$\begin{aligned}
k_1 &= h\,f(x_1, y_1)\\
&= 0.5\ f(0.5, 1.3570)\\
&= 0.5\left(\frac{1}{0.5+1.3570}\right)\\
&= 0.2692\\
k_2 &= h\,f\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right)\\
&= 0.5\ f\left(0.5+\frac{0.5}{2}, 1.3570+\frac{0.2692}{2}\right)\\
&= 0.5\ f(0.75, 1.4916)\\
&= 0.5\left(\frac{1}{0.75+1.4916}\right)\\
&= 0.2230\\
k_3 &= h\,f\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right)\\
&= 0.5\ f\left(0.5+\frac{0.5}{2}, 1.3570+\frac{0.2230}{2}\right)\\
&= 0.5\ f(0.75, 1.4685)\\
&= 0.5\left(\frac{1}{0.75+1.4685}\right)\\
&= 0.2253\\
k_4 &= h\,f(x_1+h, y_1+k_3)\\
&= 0.5\ f(0.5+0.5, 1.3570+0.2253)\\
&= 0.5\ f(1, 1.5823)\\
&= 0.5\left(\frac{1}{1+1.5823}\right)\\
&= 0.1936\\
k &= \frac{1}{6}(k_1+2k_2+2k_3+k_4)\\
&= \frac{1}{6}\big[0.2692+2(0.2230)+2(0.2253)+0.1936\big]\\
&= 0.2265\\
y_2 &= y_1+k\\
&= 1.3570+0.2265\\
&= 1.5835
\end{aligned}$$

Example 12

Apply the Runge–Kutta method of fourth order to find an approximate value of y at $x = 0.2$ *if* $\frac{dy}{dx} = x + y^2$, *given that* $y = 1$ *when* $x = 0$ *in steps of* $h = 0.1$. **[Summer 2014]**

Solution

$$\frac{dy}{dx} = f(x, y) = x + y^2$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$\begin{aligned}
k_1 &= h\,f(x_0, y_0) \\
&= 0.1\,f(0, 1) \\
&= 0.1(0 + 1^2) \\
&= 0.1 \\
k_2 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\
&= 0.1\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right) \\
&= 0.1\,f(0.05, 1.05) \\
&= 0.1\left[0.05 + (1.05)^2\right] \\
&= 0.1152 \\
k_3 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\
&= 0.1\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1152}{2}\right) \\
&= 0.1\,f(0.05, 1.0576) \\
&= 0.1\left[0.05 + (1.0576)^2\right] \\
&= 0.1168 \\
k_4 &= h\,f(x_0 + h, y_0 + k_3) \\
&= 0.1\,f(0 + 0.1, 1 + 0.1168) \\
&= 0.1\,f(0.1, 1.1168) \\
&= 0.1\left[0.1 + (1.1168)^2\right] \\
&= 0.1347
\end{aligned}$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.1 + 2(0.1152) + 2(0.1168) + 0.1347]$$
$$= 0.1164$$
$$y_1 = y_0 + k$$
$$= 1 + 0.1164$$
$$= 1.1164$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 1.1164, \quad h = 0.1, \quad n = 1$

$$k_1 = h\,f(x_1, y_1)$$
$$= 0.1\,f(0.1, 1.1164)$$
$$= 0.1\left[0.1 + (1.1164)^2\right]$$
$$= 0.1346$$
$$k_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.1\,f\left(0.1 + \frac{0.1}{2}, 1.1164 + \frac{0.1346}{2}\right)$$
$$= 0.1\,f(0.15, 1.1837)$$
$$= 0.1\left[0.15 + (1.1837)^2\right]$$
$$= 0.1551$$
$$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.1\,f\left(0.1 + \frac{0.1}{2}, 1.1164 + \frac{0.1551}{2}\right)$$
$$= 0.1\,f(0.15, 1.1939)$$
$$= 0.1\left[0.15 + (1.1939)^2\right]$$
$$= 0.1575$$
$$k_4 = h\,f(x_1 + h, y_1 + k_3)$$
$$= 0.1\,f(0.1 + 0.1, 1.1164 + 0.1575)$$
$$= 0.1\,f(0.2, 1.2739)$$
$$= 0.1\left[0.2 + (1.2739)^2\right]$$
$$= 0.1822$$

$$\begin{aligned} k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ &= \frac{1}{6}\left[0.1346 + 2(0.1551) + 2(0.1575) + 0.1822\right] \\ &= 0.157 \\ y_2 &= y_1 + k \\ &= 1.1164 + 0.157 \\ &= 1.2734 \end{aligned}$$

Hence, $y_2 = y(0.2) = 1.2734$

EXERCISE 7.4

Solve the following differential equations by the Runge–Kutta method:

1. $\frac{dy}{dx} = x + y$ with $x_0 = 0, y_0 = 1$ at $x = 0.2$

[Ans.: 1.2424]

2. $\frac{dy}{dx} = xy$ with $y(1) = 2$ at $x = 1.2, x = 1.4$

[Ans.: 2.4921, 3.2311]

3. $\frac{dy}{dx} = x^2 + y^2$ with $x_0 = 1,\ y_0 = 1.5,\ h = 0.1$ at $x = 1.2$

[Ans.: 2.5043]

4. $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$, at $x = 0.2$ and $x = 0.4$

[Ans.: 1.8310, 2.0214]

5. $\frac{dy}{dx} = \frac{y - x}{y + x}$ with $x_0 = 0, y_0 = 1$ at $x = 0.2$

[Ans.: 1.1678]

6. $\frac{dy}{dx} = 1 + y^2$ with $x_0 = 0, y_0 = 0$ at $x = 0.2, 0.4$ and 0.6

[Ans.: 0.2027, 0.4228, 0.6891]

7. $\frac{dy}{dx} = xy^2$ with $x_0 = 2, y_0 = 1$ for $x = 2.2$ taking $h = 0.2$

[Ans.: −1.7241]

8. $\frac{dy}{dx} = x - y^2$ with $x_0 = 0, y_0 = 1$ at $x = 0.2$ taking $h = 0.1$

[Ans.: 0.8512]

9. $\dfrac{dy}{dx} = \dfrac{x-y}{xy}$ with $x_0 = 1, y_0 = 1$ at $x = 1.1$

[Ans.: 1.0045]

10. $\dfrac{dy}{dx} = \dfrac{y^2-2x}{y^2+x}$ with $x_0 = 0, y_0 = 1$ at $x = 0.1, 0.2, 0.3,$ and 0.4

[Ans.: 1.0911, 1.1677, 1.2352, 1.2902]

7.6 MILNE'S PREDICTOR-CORRECTOR METHOD

Consider the differential equation

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition $y(x_0) = y_0$.

By Taylor's series method,

$$y_1 = y(x_0 + h)$$
$$y_2 = y(x_0 + 2h)$$
$$y_3 = y(x_0 + 3h)$$

Also,

$$f_0 = f(x_0, y_0)$$
$$f_1 = f(x_0 + h, y_1)$$
$$f_2 = f(x_0 + 2h, y_2)$$
$$f_3 = f(x_0 + 3h, y_3)$$

By Newton's forward interpolation formula,

$$f(x, y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 f_0 + \cdots$$

Now,

$$y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x, y)\, dx$$

$$= y_0 + \int_{x_0}^{x_0+4h} \left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 f_0 + \cdots \right) dx$$

Putting $x = x_0 + nh$, $\quad dx = h\,dn$

When $x = x_0$, $\quad n = 0$

When $x = x_0 + 4h$ $\quad n = 4$

$$y_4 = y_0 + h\int_0^4 \left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 f_0 + \cdots \right) dn$$

$$= y_0 + h\left| f_0 n + \frac{n^2}{2}\Delta f_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 f_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 f_0 + \cdots \right|_0^4$$

$$= y_0 + h\left[4f_0 + 8\Delta f_0 + \frac{1}{2}\left(\frac{64}{3} - 8\right)\Delta^2 f_0 + \frac{1}{6}(64 - 64 + 16)\Delta^3 f_0 + \cdots\right]$$

$$= y_0 + h\left[4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + \cdots\right]$$

Neglecting fourth and higher order differences and expressing Δf_0, $\Delta^2 f_0$ and $\Delta^3 f_0$ in terms of the function values,

$$y_{4p} = y_0 + h\left[4f_0 + 8(f_1 - f_0) + \frac{20}{3}(f_2 - 2f_1 + f_0) + \frac{8}{3}(f_3 - 3f_2 + 3f_1 - f_0)\right]$$

$$= y_0 + h\left[\left(4 - 8 + \frac{20}{3} - \frac{8}{3}\right)f_0 + \left(8 - \frac{40}{3} + 8\right)f_1 + \left(\frac{20}{3} - 8\right)f_2 + \frac{8}{3}f_3\right]$$

$$= y_0 + h\left(\frac{8}{3}f_1 - \frac{4}{3}f_2 + \frac{8}{3}f_3\right)$$

$$= y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

This equation is known as *predictor.*

In general,

$$y_{(n+1)p} = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n)$$

From y_4, a first approximation to $f_4 = f(x_0 + 4h, y_4)$ is obtained.

A better value of y_4 is obtained by Simpson's rule.

$$y_{4c} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

This equation is known as *corrector.*

In general,

$$y_{(n+1)c} = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1})$$

Then an improved value of f_4 is calculated using y_{4c} and again the corrector is applied to find a still better value of y_{4c}. This step is repeated till two consecutive values of y_{4c} are same.

Once y_4 and f_4 are obtained to the desired degree of accuracy, the next value of y is obtained from predictor-corrector equations.

This method is known as *Milne's predictor-corrector* method.

Example 1

Given $\dfrac{dy}{dx}=\dfrac{1}{2}(1+x^2)y^2$ *and* $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$, *evaluate* $y(0.4)$.

Solution

$$x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3,\ x_4 = 0.4$$
$$y_0 = 1,\ y_1 = 1.06,\ y_2 = 1.12,\ y_3 = 1.21,\ h = 0.1$$
$$\frac{dy}{dx} = f(x, y) = \frac{1}{2}(1+x^2)\,y^2$$
$$f_1 = \frac{1}{2}(1+x_1^2)\,y_1^2 = \frac{1}{2}\left[1+(0.1)^2\right](1.06)^2 = 0.5674$$
$$f_2 = \frac{1}{2}(1+x_2^2)\,y_2^2 = \frac{1}{2}\left[1+(0.2)^2\right](1.12)^2 = 0.6523$$
$$f_3 = \frac{1}{2}(1+x_3^2)\,y_3^2 = \frac{1}{2}\left[1+(0.3)^2\right](1.21)^2 = 0.7979$$

By Milne's predictor method,

$$\begin{aligned} y_{4p} &= y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \\ &= 1 + \frac{4(0.1)}{3}\left[2(0.5674) - 0.6523 + 2(0.7979)\right] \\ &= 1.2771 \end{aligned}$$

$$\begin{aligned} f_4 &= \frac{1}{2}(1+x_4^2)\,y_{4p}^2 \\ &= \frac{1}{2}\left[1+(0.4)^2\right](1.2771)^2 \\ &= 0.9460 \end{aligned}$$

By Milne's corrector method,

$$\begin{aligned} y_{4c} &= y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \\ &= 1.12 + \frac{0.1}{3}\left[0.6523 + 4(0.7979 + 0.9460\right] = 1.2797 \end{aligned}$$

Again,
$$\begin{aligned} f_4 &= \frac{1}{2}(1+x_4^2)\,y_{4c}^2 \\ &= \frac{1}{2}\left[1+(0.4)^2\right](0.2797)^2 \\ &= 0.9498 \end{aligned}$$

By Milne's corrector method,

$$y_{4c} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 1.12 + \frac{0.1}{3}\left[0.6523 + 4(0.7979) + 0.9498\right]$$

$$= 1.2798$$

$\therefore \quad y(0.4) = 1.2798$

Example 2

Find $y(4.4)$ *given* $5xy' + y^2 - 2 = 0$ with $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$.

Solution

$$x_0 = 4,\ x_1 = 4.1,\ x_2 = 4.2,\ x_3 = 4.3,\ x_4 = 4.4$$

$$y_0 = 1,\ y_1 = 1.0049,\ y_2 = 1.0097,\ y_3 = 1.0143,\ h = 0.1$$

$$\frac{dy}{dx} = f(x, y) = \frac{2 - y^2}{5x}$$

$$f_1 = \frac{2 - y_1^2}{5x_1} = \frac{2 - (1.0049)^2}{5(4.1)} = 0.0483$$

$$f_2 = \frac{2 - y_2^2}{5x_2} = \frac{2 - (1.0097)^2}{5(4.2)} = 0.0467$$

$$f_3 = \frac{2 - y_3^2}{5x_3} = \frac{2 - (1.0143)^2}{5(4.3)} = 0.0452$$

By Milne's predictor method,

$$y_{4p} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.0483) - 0.0467 + 2(0.0452)\right]$$

$$= 1.0187$$

$$f_4 = \frac{2 - y_{4p}^2}{5x_4} = \frac{2 - (1.0187)^2}{5(4.4)} = 0.0437$$

By Milne's corrector method,

$$y_{4c} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 1.0097 + \frac{0.1}{3}\left[0.0467 + 4(0.0452) + 0.0437)\right]$$

$$= 1.0187$$

$\therefore \quad y(4.4) = 1.0187$

Example 3

Given $y' = x(x^2 + y^2)e^{-x}$, $y(0) = 1$, find y at 0.1, 0.2, and 0.3 by Taylor's series method and compute $y(0.4)$ by Milne's method.

Solution

$$\frac{dy}{dx} = f(x, y) = x(x^2 + y^2)e^{-x}$$

(i) Given: $x_0 = 0$, $y_0 = 1$, $h = 0.1$, $x_1 = x_0 + h = 0 + 0.1 = 0.1$

$$y' = x(x^2 + y^2)e^{-x} \qquad y_0' = 0$$

$$y'' = \left[(x^3 + xy^2)(-e^{-x}) + 3x^2 + y^2 + x(2y)\right]e^{-x}$$

$$= e^{-v}(-x^3 - xy^2 + 3x^2 + y^2 + 2xyy') \qquad y_0'' = 1$$

$$y''' = -e^{-x}\begin{bmatrix} -x^3 - xy^2 + 3x^2 + y^2 + 2xyy' + 3x^2 + y^2 \\ + 2xyy' - 6x - 2yy' - 2x(y')^2 - 2xyy' \end{bmatrix} \qquad y_0''' = -2$$

By Taylor's series,

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$y_1 = y(0.1) = 1 + 0.1(0) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(-2) + \cdots$$

$$= 1.0047$$

(ii) Given: $x_1 = 0.1$, $y_1 = 1.0047$, $h = 0.1$, $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$$y_1' = 0.0922$$

$$y_1'' = 0.849$$

$$y_1''' = -1.247$$

By Taylor's series,

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots$$

$$y_2 = y(0.2) = 1.0047 + 0.1(0.0922) + \frac{(0.1)^2}{2!}(0.849) + \frac{(0.1)^3}{3!}(-1.247) + \cdots$$

$$= 1.018$$

(iii) Given: $x_2 = 0.2$, $y_2 = 1.018$, $h = 0.1$, $x_3 = x_2 + h = 0.2 + 0.1 = 0.3$

$$y_2' = 0.176$$

$$y_2'' = 0.77$$

$$y_2''' = 0.819$$

By Taylor's series,

$$y_3 = y(x_3) = y_2 + hy_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \cdots$$

$$y_3 = y(0.3) = 1.018 + 0.1(0.176) + \frac{(0.1)^2}{2!}(0.77) + \frac{(0.1)^3}{3!}(0.819) + \cdots$$

$$= 1.04$$

For Milne's method,

$$\begin{array}{lll} x_0 = 0 & y_0 = 1 & \\ x_1 = 0.1 & y_1 = 1.0047 & f_1 = 0.092 \\ x_2 = 0.2 & y_2 = 1.018 & f_2 = 0.176 \\ x_3 = 0.3 & y_3 = 1.04 & f_3 = 0.26 \end{array}$$

By Milne's predictor method,

$$y_{4p} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.092) - 0.176 + 2(0.26)\right]$$

$$= 1.09$$

$$x_4 = 0.4,\ y_{4p} = 1.09$$

$$f_4 = x_4\left(x_4^2 + y_{4p}^2\right)e^{-x_4}$$

$$= 0.4\left[(0.4)^2 + (1.09)^2\right]e^{-0.4}$$

$$= 0.3615$$

By Milne's corrector method,

$$y_{4c} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 1.018 + \frac{0.1}{3}\left[0.176 + 4(0.26) + 0.3615\right]$$

$$= 1.071$$

$\therefore \qquad y(0.4) = 1.071$

Example 4

Determine the value of $y(0.4)$ *using the predictor-corrector method, given* $\frac{dy}{dx} = xy + y^2,\ y(0) = 1.$ *Use Taylor series to get the values of* $y(0.1)$, $y(0.2)$, $y(0.3)$. *Take* $h = 0.1$. **[Summer 2013, 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = xy + y^2$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad x_1 = x_0 + h = 0 + 0.1 = 0.1$

$$y' = xy + y^2 \qquad y'_0 = 0 + (1)^2 = 1$$

$$y'' = xy' + y + 2yy' \qquad y''_0 = 0 + 1 + 2(1)(1) = 3$$

$$y''' = xy'' + 2y' + 2yy'' + 2(y')^2 \qquad y'''_0 = 0 + 2(1) + 2(1)(3) + 2(1)^2 = 10$$

By Taylor's series,

$$y_1 = y(x_1) = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \cdots$$

$$y_1 = y(0.1) = 1 + 0.1(1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(10) + \cdots$$

$$= 1.1167$$

(ii) Given: $x_1 = 0.1, \quad y_1 = 1.1167, \quad h = 0.1, \quad x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$$y'_1 = 0.1(1.1167) + (1.1167)^2 = 1.3587$$

$$y''_1 = 0.1(1.3587) + 1.1167 + 2(1.1167)(1.3587) = 4.2871$$

$$y'''_1 = 0.1(4.2871) + 2(1.3587) + 2(1.1167)(4.2871) + 2(1.3587)^2$$

$$= 16.4131$$

By Taylor's series,

$$y_2 = y(x_2) = y_1 + hy'_1 + \frac{h^2}{2!}y''_1 + \frac{h^3}{3!}y'''_1 + \cdots$$

$$y_2 = y(0.2) = 1.1167 + 0.1(1.3587) + \frac{(0.1)^2}{2!}(4.2871) + \frac{(0.1)^3}{3!}(16.4131)$$

$$= 1.2767$$

(iii) Given: $x_2 = 0.2, \quad y_2 = 1.2767, \quad h = 0.1, \quad x_3 = x_2 + h = 0.2 + 0.1 = 0.3$

$$y'_2 = 0.2(1.2767) + (1.2767)^2 = 1.8853$$

$$y''_2 = 0.2(1.8853) + 1.2767 + 2(1.2767)(1.8853) = 6.4677$$

$$y'''_2 = 0.2(6.4677) + 2(1.8853) + 2(1.2767)(6.4677) + 2(1.8853)^2$$

$$= 28.6875$$

By Taylor's series,

$$y_3 = y(x_3) = y_2 + hy'_2 + \frac{h^2}{2!}y''_2 + \frac{h^3}{3!}y'''_2 + \cdots$$

$$y_3 = y(0.3) = 1.2767 + 0.1(1.8853) + \frac{(0.1)^2}{2!}(6.4677) + \frac{(0.1)^3}{3!}(28.6875) + \cdots$$

$$= 1.5023$$

For Milne's method,

$$
\begin{array}{lll}
x_0 = 0 & y_0 = 1 & \\
x_1 = 0.1 & y_1 = 1.1167 & f_1 = (0.1)(1.1167) + (1.1167)^2 = 1.3587 \\
x_2 = 0.2 & y_2 = 1.2767 & f_2 = (0.2)(1.2767) + (1.2767)^2 = 1.8853 \\
x_3 = 0.3 & y_3 = 1.5023 & f_3 = (0.3)(1.5023) + (1.5023)^2 = 2.7076
\end{array}
$$

By Milne's predictor method,

$$
\begin{aligned}
y_{4p} &= y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \\
&= 1 + \frac{4(0.1)}{3}\left[2(1.3587) - 1.8853 + 2(2.7076)\right] \\
&= 1.833
\end{aligned}
$$

$$x_4 = 0.4, \qquad y_{4p} = 1.833$$

$$f_4 = (0.4)(1.833) + (1.833)^2 = 4.093$$

By Milne's corrector method,

$$
\begin{aligned}
y_{4c} &= y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \\
&= 1.2767 + \frac{0.1}{3}\left[1.8853 + 4(2.7076) + 4.093\right] \\
&= 1.83699
\end{aligned}
$$

$\therefore \qquad y(0.4) = 1.83699$

Example 5

Using Taylor's series method, compute the approximate values of y at x = 0.2, 0.4, *and* 0.6 *for the differential equation* $\dfrac{dy}{dx} = x - y^2$ *with the initial condition* $y(0) = 0$. *Now, apply Milne's predictor-corrector method to find y at x =* 0.8. **[Winter 2012]**

Solution

$$\frac{dy}{dx} = f(x, y) = x - y^2$$

(i) Given: $x_0 = 0, \quad y_0 = 0, \quad h = 0.2, \quad x_1 = x_0 + h = 0 + 0.2 = 0.2$

$$
\begin{array}{ll}
y' = x - y^2 & y_0' = 0 \\
y'' = 1 - 2yy' & y_0'' = 1 - 2(0)(-1) = 1 \\
y''' = -2yy'' - 2(y')^2 & y_0''' = -2(0)(3) - 2(0)^2 = 0
\end{array}
$$

By Taylor's series,

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$y_1 = y(0.2) = 0 + 0.2(0) + \frac{(0.2)^2}{2!}(1) + \frac{(0.2)^3}{3!}(0) + \cdots$$

$$= 0.02$$

(ii) Given: $x_1 = 0.2,\quad y_1 = 0.02,\quad h = 0.2,\quad x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$$y_1' = 0.2 - (0.02)^2 = 0.1996$$

$$y_1'' = 1 - 2(0.02)(0.1996) = 0.9920$$

$$y_1''' = -2(0.02)(0.9920) - 2(0.1996)^2 = -0.1194$$

By Taylor's series,

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots$$

$$y_2 = y(0.4) = 0.02 + 0.2(0.1996) + \frac{(0.2)^2}{2!}(0.9920) + \frac{(0.2)^3}{3!}(-0.1194) + \cdots$$

$$= 0.0796$$

(iii) Given: $x_2 = 0.4,\quad y_2 = 0.0796,\quad h = 0.2,\quad x_3 = x_2 + h = 0.4 + 0.2 = 0.6$

$$y_2' = 0.4 - (0.0796)^2 = 0.3937$$

$$y_2'' = 1 - 2(0.0796)(0.3937) = 0.9373$$

$$y_2''' = -2(0.0796)(0.9373) - 2(0.3937)^2 = -0.4592$$

By Taylor's series,

$$y_3 = y(x_3) = y_2 + hy_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \cdots$$

$$= 0.0796 + 0.2(0.3937) + \frac{(0.2)^2}{2!}(0.9373) + \frac{(0.2)^3}{3!}(-0.4592) + \cdots$$

$$= 0.1765$$

For Milne's method,

$x_0 = 0$	$y_0 = 0$	
$x_1 = 0.2$	$y_1 = 0.02$	$f_1 = 0.2 - (0.02)^2 = 0.7996$
$x_2 = 0.4$	$y_2 = 0.0796$	$f_2 = 0.4 - (0.0796)^2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1765$	$f_3 = 0.6 - (0.1765)^2 = 0.5688$

By Milne's predictor method,

$$\begin{aligned}y_{4p} &= y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)\\ &= 0 + \frac{4(0.2)}{3}\left[2(0.1996) - 0.3937 + 2(0.5688)\right]\\ &= 0.3048\end{aligned}$$

$$x_4 = 0.8, \qquad y_{4p} = 0.3048$$

$$f_4 = 0.8 - (0.3048)^2 = 0.7071$$

By Milne's corrector method,

$$\begin{aligned}y_{4c} &= y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)\\ &= 0.0796 + \frac{0.2}{3}\left[0.3937 + 4(0.5688) + 0.7071\right]\\ &= 0.3047\end{aligned}$$

$$\therefore \quad y(0.8) = 0.3047$$

EXERCISE 7.4

1. Find $y(2)$ if $y(x)$ is the solution of $\frac{dy}{dx} = \frac{1}{2}(x + y)$ given $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$ and $y(1.5) = 4.968$.

[Ans.: 6.8732**]**

2. Find $y(0.8)$ given $y' = y - x^2$, $y(0) = 1$, $y(0.2) = 1.12186$, $y(0.4) = 1.46820$, $y(0.6) = 1.73790$.

[Ans.: 2.01105**]**

3. Given $y' = x^2 - y$, $y(0) = 1$, $y(0.1) = 0.9052$, $y(0.2) = 0.8213$, find $y(0.3)$ by Taylor series. Also, find $y(0.4)$ and $y(0.5)$.

[Ans.: 0.6897, 0.6435**]**

4. If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$, $y(0.3) = 2.090$, find $y(0.4)$ and $y(0.5)$.

[Ans.: 2.1621, 2.546**]**

5. Given $y' = \frac{1}{x + y}$, $y(0) = 2, y(0.2) = 2.0933, y(0.4) = 2.1755, y(0.6) = 2.2493$, find $y(0.8)$.

[Ans.: 2.3164**]**

Points to Remember

Taylor's Series Method

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''_n + \cdots$$

Euler's Method

$$y_{n+1} = y_n + h\,f(x_n, y_n)$$

Modified Euler's Method

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0)$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right], n = 0, 1, 2, \ldots$$

$$y_2^{(0)} = y_1 + h\,f(x_1, y_1)$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

Runge–Kutta Methods

1. First-Order Runge–Kutta Method

$$y_{n+1} = y_n + h\,y'_n + \frac{h^2}{2!} y''_n + \cdots$$

2. Second-Order Runge–Kutta Method (Heun Method)

$$k_1 = h\,f(x_n, y_n)$$

$$k_2 = h\,f\,(x_n + h, y_n + k_1)$$

$$k = \frac{1}{2}(k_1 + k_2)$$

$$y_{n+1} = y_n + k$$

3. Third-Order Runge–Kutta Method

$$k_1 = h\,f(x_n, y_n)$$

$$k_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h\,f(x_n + h, y_n + 2k_2 - k_1)$$

$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$y_{n+1} = y_n + k$$

4. Fourth-Order Runge–Kutta Method

$$k_1 = h\,f(x_n, y_n)$$

$$k_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h\,f(x_n + h, y_n + k_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{n+1} = y_n + k$$

Milne's Predictor-Corrector Method

$$y_{(n+1)p} = y_{n-3} + \frac{4h}{3}\left(2f_{n-2} - f_{n-1} + 2f_n\right)$$

$$y_{(n+1)c} = y_{n-1} + \frac{h}{3}\left(f_{n-1} + 4f_n + f_{n+1}\right)$$

CHAPTER 8
Statistical Methods

Chapter Outline

8.1 INTRODUCTION

Statistics is the science which deals with the collection, presentation, analysis, and interpretation of numerical data. Statistics should possess the following characteristics:

(i) Statistics are aggregates of facts.
(ii) Statistics are affected by a large number of causes.
(iii) Statistics are always numerically expressed.
(iv) Statistics should be enumerated or estimated.
(v) Statistics should be collected in a systematic manner.
(vi) Statistics should be collected for a pre-determined purpose.
(vii) Statistics should be placed in relation to each other.

The use of statistical methods help in presenting a complex mass of data in a simplified form so as to facilitate the process of comparison of characteristics in two or more situations. Statistics also provide important techniques for the study of relationship between two or more characteristics (or variables) in forecasting, testing of hypothesis, quality control, decision making, etc.

8.2 DATA ANALYSIS

The collection and analysis of data constitute the main stages of execution of any statistical investigation. The procedure for collection of data depends upon various considerations such as objective, scope, nature of investigation, etc. Data may be collected for each and every unit of the whole lot (population), which will ensure greater accuracy. Data may also be collected for a sample of population and conclusions that can be drawn on the basis of this sample are taken to hold for the population.

8.3 CLASSIFICATION OF DATA

The collected data are a complex and unorganized mass of figures which is very difficult to analyze and interpret. Therefore, it becomes necessary to organize the data so that it is easier to grasp its broad features. In order to analyze the data, it is essential that the data are arranged in a definite form. This task is accomplished by the process of classification. The main objectives of any classification are

(i) To present the data in a condensed form.
(ii) To bring out the relationship between variables.
(iii) To prepare data for tabulation and analysis.
(iv) To highlight the effect of one variable by eliminating the effect of others.

Consider the raw data relating to marks obtained in mathematics by a group of 60 students:

38, 11, 40, 0, 26, 15, 5, 40, 31, 12, 35, 0, 7, 20, 5, 28, 8, 21, 7, 28, 48, 45, 42, 17, 2, 38, 41, 18, 16, 16, 0, 19, 10, 7, 5, 1, 17, 22, 35, 44, 28, 46, 9, 16, 29, 34, 31, 27, 4, 12, 35, 39, 41, 8, 6, 13, 14, 17, 19, 20.

This data can be grouped and shown in tabular form as follows:

Class interval	Frequency	Cumulative frequency
0–6	10	10
7–13	11	21
14–20	13	34
21–27	4	38
28–34	7	45
35–41	10	55
42–48	5	60

Thus, the 60 values have been put into only 7 groups, called the classes. The width of the class is called the *class interval* and the number in that interval is called the *frequency*. The mid-point or the mid-value of the class is called the *class mark*.

8.4 FREQUENCY DISTRIBUTION

A table in which the frequencies and the associated values of a variable are written side by side, is known as a *frequency distribution*. A frequency distribution can be discrete or continuous depending upon whether the variable is discrete or continuous. A frequency distribution has the following parameters:

(i) Number of class intervals
(ii) Width of a class interval
(iii) Mid-value of a class
(iv) Cumulative frequency

8.4.1 Class Intervals

The class intervals can be exclusive or inclusive. In the exclusive class interval, the upper limit of a class is taken to be equal to the lower limit of the next class. To keep various class intervals as mutually exclusive, the observations with magnitude greater than or equal to lower limit but less than the upper limit of a class are included in it. For example, if the lower limit of a class is 20 and its upper limit is 30 then this class, written as 20–30, includes all the observations which are greater than or equal to 20 but less than 30. The observations with magnitude 30 will be included in the next class.

Class intervals	Frequency
0–10	5
10–20	17
20–30	25
30–40	12
40–50	8

In the inclusive class interval, all the observations with magnitude greater than or equal to lower limit and less than or equal to upper limit of a class are included in it.

Class intervals	Frequency
0–9	12
10–19	9
20–29	18
30–39	35
40–49	20

Inclusive class intervals can be converted into exclusive class intervals by the following procedures:

(i) Find the difference between the lower limit of the second class and the upper limit of the first class.

(ii) Divide the difference by 2.

(iii) Subtract the value so obtained from all the lower limits and add the value to all the uppper limits.

In the above example, the lower limit of the second class is 10 and the upper limit of the first class is 9. Hence, $\frac{10-9}{2} = 0.5$ is subtracted from all the lower limits and added to all the upper limits as follows:

Class intervals	Frequency
−0.5–9.5	12
9.5–19.5	9
19.5–29.5	18
29.5–39.5	35
39.5–49.5	20

8.4.2 Mid-value of a Class

In exclusive types of class intervals, the mid-value of a class is defined as the arithmetic mean of its lower and upper limits.

8.4.3 Cumulative Frequency

There are two types of cumulative frequency distributions:

(i) *Less than cumulative frequency*: Less than cumulative frequency for any value of the variable/class is obtained by adding successively the frequencies of all the previous classes, including the frequency of the class, against which the total are written provided the values are written in ascending order of magnitude.

(ii) *More (or greater) than cumulative frequency*: More than cumulative frequency for any value of the variable/class is obtained by adding successively the frequencies of all the succeeding classes, including the frequency of the class, against which the total are written provided values are written in ascending order of magnitude.

8.5 GRAPHICAL REPRESENTATION

A frequency distribution is conveniently represented by means of a graph. Graphs are good visual aids. It makes the raw data readily intelligible and leaves a more lasting impression on the mind of the observer. But it does not give accurate measurements of the variable as are given by the table. Some important types of graphs are given below:

1. Histogram A histogram is drawn by erecting rectangles over the class intervals, such that the areas of the rectangles are proportional to the class frequencies. If the class intervals are of equal size, the height of the rectangles will be proportional to the class frequencies. For drawing a histogram, all the class intervals are marked off along the *x*-axis on a suitable scale and frequencies are marked off along the *y*-axis on a suitable scale. If, however, the classes are of unequal width then the height of the rectangle will be proportional to the ratio of the frequencies to the width of the classes. The diagram of continuous rectangles so obtained is called a histogram. If the grouped frequency distribution is not continuous, first it is to be converted into a continuous distribution and then the histogram is drawn. The frequency distribution and corresponding histogram are shown below:

Class intervals	Frequency
30–42	7
42–54	4
54–66	8
66–78	9
78–90	5
90–102	5
102–114	2

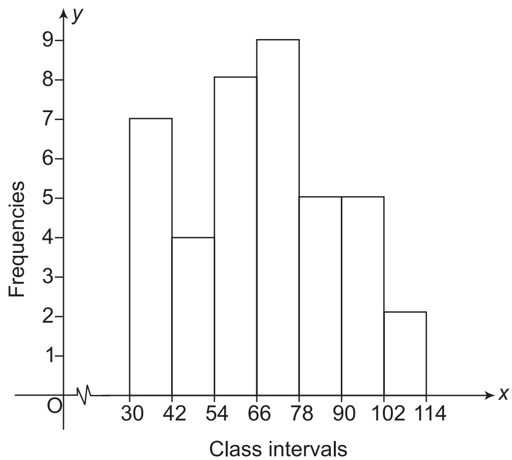

Fig. 8.1

2. Frequency Polygon A frequency polygon for an ungrouped frequency distribution is obtained by joining points plotted with the variable values as abscissae and the frequencies as the ordinates. For a grouped frequency distribution, the abscissae of the points are mid-values of the class intervals. For equal class intervals, the frequency polygon can be obtained by joining the middle points of the upper sides of the adjacent rectangles of the histogram by straight lines. If the class intervals are of small width, the polygon can be obtained by drawing a smooth curve through the vertices of the frequency polygon and is called the frequency curve.

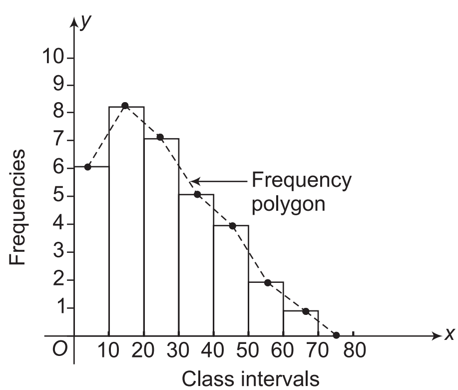

Fig. 8.2

3. Cumulative Frequency Curve or Ogive A cumulative frequency curve or ogive is obtained by plotting cumulative frequencies above or below a given value. Since a cumulative frequency distribution can be of 'less than' or 'more than' type and accordingly there are two types of ogives—'less than' ogive or 'more than' ogive.

A 'less than' ogive is obtained by plotting the points with the upper limits of the classes as abscissae and the corresponding less than cumulative frequency as ordinates and joining these points by a freehand smooth curve. A 'more than' ogive is obtained by plotting the points with the lower limits of the classes as abscissae and the corresponding more than cumulative frequency as ordinates and joining these points by a freehand smooth curve.

An ogive is used to determine certain positional averages like median, quartiles, deciles, percentiles, etc. Various frequency distributions can be compared on the basis of their ogives.

Example 1

Draw a histogram and frequency curve for the following data:

Profit (₹ in thousands)	0–15	15–30	30–45	45–60	60–75	75–90	90–105	105–120	120–135
No. of companies	3	7	18	25	20	12	6	5	2

Solution

Histogram and Frequency curve

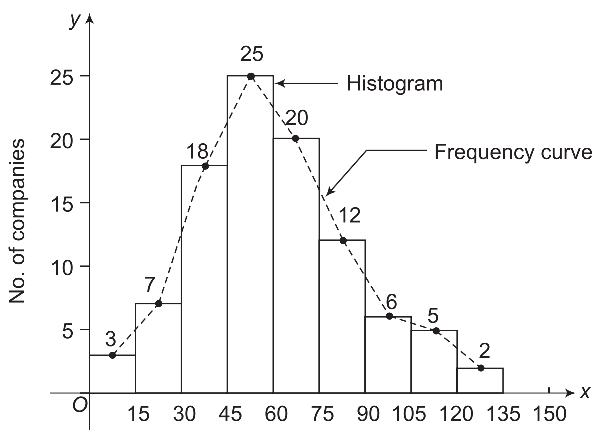

Fig. 8.3

Example 2

Draw a histogram and a frequency polygon for the following data:

Output (units per worker)	500–509	510–519	520–529	530–539	540–549	550–559	560–569
No. of workers	8	18	23	37	47	26	16

Solution

The data is presented in the form of inclusive class intervals. It can be converted into exclusive class intervals. The difference between the lower limit of the second class interval and the upper limit of the first class interval is 510 – 509 = 1. The new classes will be formed by subtracting $\frac{1}{2}$ from the lower limit and adding $\frac{1}{2}$ to the upper limit.

Class intervals	No. of workers (frequency)
499.5–509.5	8
509.5–519.5	18
519.5–529.5	23
529.5–539.5	37
539.5–549.5	47
549.5–559.5	26
559.5–569.5	16

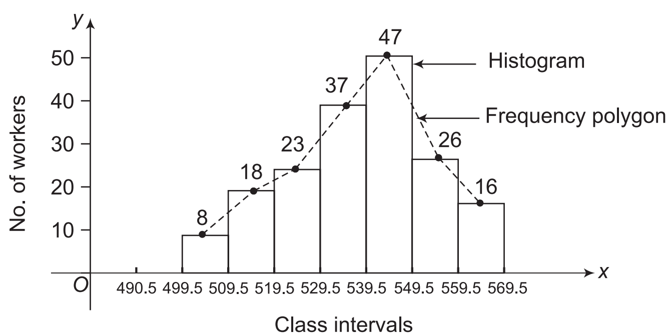

Fig. 8.4

Example 3

Construct a histogram and a frequency polygon for the following frequency distribution:

Marks (mid-value)	100	120	140	160	180	200
No. of students	5	6	4	6	4	5

Solution

The given data of mid-points is first converted into class interval form. The difference between two mid-values is 20. Hence, $\frac{20}{2}$ is subtracted from each mid-value to get the lower limit and $\frac{20}{2}$ is added to each mid-value to get the upper limit of a class interval.

Class intervals	No. of students
90–110	5
110–130	6
130–150	4
150–170	6
170–190	4
190–210	5

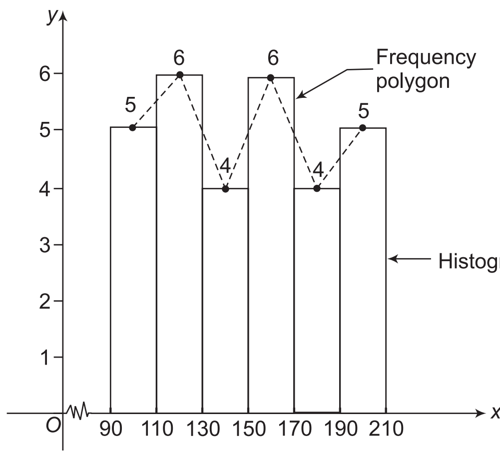

Fig. 8.5

Example 4

The following are the scores of two groups of a class in a test of reading ability:

Scores	Group A	Group B
50–52	4	2
47–49	10	3
44–46	15	4
41–43	18	8
38–40	20	12
35–37	12	17
32–34	13	22

Construct a frequency polygon for each group on the same axes.

Solution

For both the groups, i.e., group *A* and group *B*, the two hypothetical intervals with zero frequencies, one at the beginning and the other at the end with frequencies zero (53–55) and (29–31) are created.

Table for Group A

Scores	Class marks	Frequency	Points
53–55	54	0	(54, 0)
50–52	51	4	(51, 4)
47–49	48	10	(48, 10)
44–46	45	15	(45, 15)
41–43	42	18	(42, 18)
38–40	39	20	(39, 20)
35–37	36	12	(36, 12)
32–34	33	13	(33, 13)
29–31	30	0	(30, 0)

Table for Group B

Scores	Class marks	Frequency	Points
53–55	54	0	(54, 0)
50–52	51	2	(51, 2)
47–49	48	3	(48, 3)
44–46	45	4	(45, 4)
41–43	42	8	(42, 8)
38–40	39	12	(39, 12)
35–37	36	17	(36, 17)
32–34	33	22	(33, 22)
29–31	30	0	(30, 0)

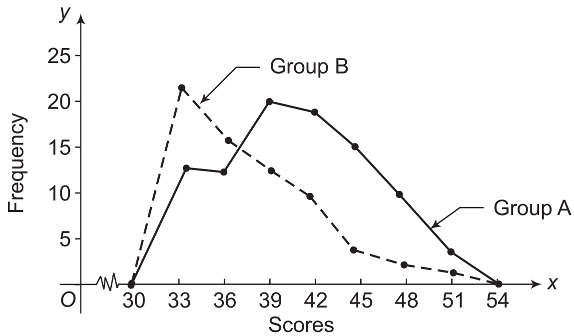

Fig. 8.6

Example 5

Draw 'less than' and 'more than' ogive distributions of monthly salary of 250 *families.*

Income intervals	0–500	500–1000	1000–1500	1500–2000	2000–2500	2500–3000	3000–3500	3500–4000
No. of families	50	80	40	25	25	15	10	5

Solution

'Less than' and 'More than' Frequency Distributions

Income intervals	No. of families	Less than cumulative frequency	More than cumulative frequency
0–500	50	50	250
500–1000	80	130	200
1000–1500	40	170	120
1500–2000	25	195	80
2000–2500	25	220	55
2500–3000	15	235	30
3000–3500	10	245	5
3500–4000	5	250	5

A 'less than' ogive is obtained by plotting the points (500, 50), (1000, 130), (1500, 170), (2000, 195), (2500, 220), (3000, 235), (3500, 245), (4000, 250) and joining them by a freehand curve.

A 'more than' ogive is obtained by plotting the points (0, 250), (500, 200), (1000, 120), (1500, 80), (2000, 55), (2500, 30), (3000, 15), (3500, 5) and joining them by a freehand curve.

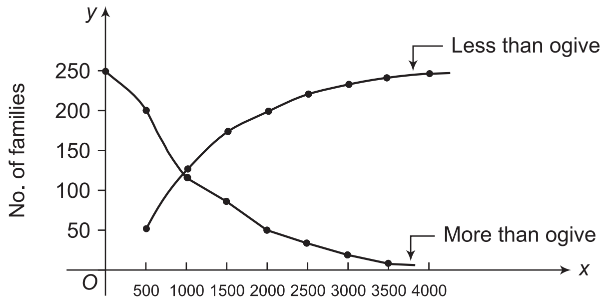

Fig. 8.7

Example 6

Draw the 'less than' ogive for the following distribution:

Age (in years)	0–9	10–19	20–29	30–39	40–49	50–59	60–69
No. of persons	5	15	20	25	15	12	8

Solution

The given frequency distribution is not continuous. It is first converted into continuous or exclusive class intervals.

Age (in years)	Class intervals	No. of persons	Cumulative frequency
0–9	–0.5–9.5	5	5
10–19	9.5–19.5	15	20
20–29	19.5–29.5	20	40
30–39	29.5–39.5	25	65
40–49	39.5–49.5	15	80
50–59	49.5–59.5	12	92
60–69	59.5–69.5	8	100

A 'less than' ogive is obtained by plotting points (9.5, 5), (19.5, 20), (29.5, 40), (39.5, 65), (49.5, 80), (59.5, 92), (69.5, 100) and joining them by a freehand smooth curve.

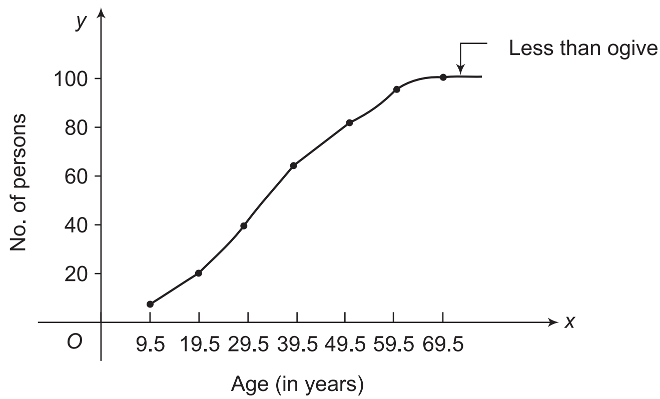

Fig. 8.8

Example 7

Convert the following distribution into a more than frequency distribution:

Weekly wages less than (₹)	20	40	60	80	100
No. of workers	41	92	156	194	201

For the data given, draw 'less than' and 'more than' ogives.

Solution

'Less than' and 'more than' frequency distribution.

Weekly wages (₹)	No. of workers f	Less than cumulative frequency	More than cumulative frequency
0–20	41	41	201
20–40	92–41 = 51	92	160
40–60	156–92 = 64	156	109
60–80	194–156 = 38	194	45
80–100	201–194 = 7	201	7

A 'less than' ogive is obtained by plotting the points (20, 41), (40, 92), (60, 156), (80, 194), (100, 201) and joining them by a freehand curve.

A 'more than' ogive is obtained by plotting the points (0, 201), (20, 160), (40, 109), (60, 45), (80, 7) and joining them by a freehand curve.

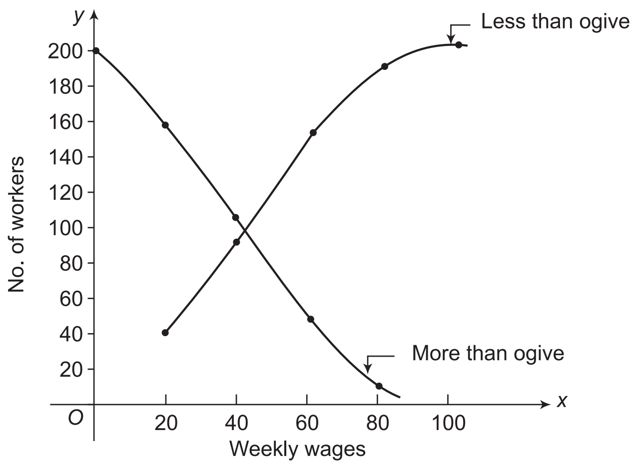

Fig. 8.9

EXERCISE 8.1

1. The following are the monthly rents in rupees of 40 shops. Tabulate the data by grouping in intervals of ₹ 8.

38, 42, 49, 35, 82, 35, 77, 60, 50, 75, 84, 75, 63, 40, 70, 42, 36, 65, 51, 48, 74, 47, 50, 55, 64, 67, 72, 77, 82, 51, 31, 38, 43, 75, 67, 70, 43, 64, 84, 71.

2. The following table shows the distribution of the number of students per teacher in 750 colleges:

Students	1	4	7	10	13	16	19	22	25	28
Frequency	7	46	165	195	189	89	28	19	9	3

3. Draw a histogram for the following data:

Age (in years)	2–5	5–11	11–12	12–14	14–15	15–16
No. of boys	6	6	2	5	1	3

4. Draw the histogram and frequency polygon for the following data:

Monthly wages (₹ in thousands)	11–13	13–15	15–17	17–19	19–21	21–23	23–25
No. of workers	6	53	85	56	21	16	8

5. Draw the histogram and frequency polygon for the following distribution:

Class interval	0-99	100-199	200-299	300-399	400-499	500-599	600-699	700-799
Frequency	10	54	184	264	246	40	1	1

6. Represent the following distribution by (i) histogram and (ii) frequency polygon:

Scores	Frequency
30–39	1
40–49	3
50–59	14
60–69	20
70–79	22
80–89	12
90–99	2

7. Represent the following distribution by an ogive:

Marks	No. of students	Marks	No. of students
0–10	5	50–60	4
10–20	13	60–70	1
20–30	12	70–80	3
30–40	11	80–90	1
40–50	8	90–100	2

8. The following table gives the distribution of monthly income of 600 middle-class families in a certain city:

Monthly income in ₹	Frequency	Monthly income in ₹	Frequency
Below 76	69	300–375	58
76–150	167	375–450	25
150–225	207	450 and over	10
225–300	65		

Draw 'less than' and 'more than' ogive for the above data.

9. Draw an ogive by less than method for the following data:

No. of rooms	1	2	3	4	5	6	7	8	9	10
No. of houses	4	9	22	28	24	12	8	6	5	2

10. Draw histogram, frequency polygon and ogive for the following data:

Marks	Frequency	Marks	Frequency
0-10	4	40-50	20
10-20	10	50-60	18
20-30	16	60-70	8
30-40	22	70-80	2

8.6 MEASURES OF CENTRAL TENDENCY

Summarization of data is a necessary function of any statistical analysis. The data is summarized in the form of tables and frequency distributions. In order to bring the characteristics of the data, these tables and frequency distributions need to be summarized further. A measure of central tendency or an average is very essential and an important summary measure in any statistical analysis.

An *average* is a single value which can be taken as a representative of the whole distribution. There are five types of measures of central tendency or averages which are commonly used.

(i) Arithmetic mean
(ii) Median
(iii) Mode
(iv) Geometric mean
(v) Harmonic mean

A good measure of average must have the following characteristics:

(i) It should be rigidly defined so that different persons obtain the same value for a given set of data.
(ii) It should be easy to understand and easy to calculate.
(iii) It should be based on all the observations of the data.
(iv) It should be easily subjected to further mathematical calculations.
(v) It should not be much affected by the fluctuations of sampling.
(vi) It should not be unduly affected by extreme observations.
(vii) It should be easy to interpret.

8.7 ARITHMETIC MEAN

The *arithmetic mean* of a set of observations is their sum divided by the number of observations. Let $x_1, x_2, ..., x_n$ be n observations. Then their average or arithmetic mean is given by

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum x}{n}$$

For example, the marks obtained by 10 students in Class XII in a physics examination are 25, 30, 21, 55, 40, 45, 17, 48, 35, 42. The arithmetic mean of the marks is given by

$$\bar{x} = \frac{\sum x}{n} = \frac{25+30+21+55+40+45+17+48+35+42}{10} = \frac{358}{10} = 35.8$$

If n observations consist of n distinct values denoted by $x_1, x_2, ..., x_n$ of the observed variable x occurring with frequencies $f_1, f_2, ..., f_n$ respectively then the arithmetic mean is given by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_ix_i}{N} = \frac{\sum fx}{N}$$

where $\quad N = \sum_{i=1}^{n} f_i = f_1 + f_2 + \cdots + f_n$

8.7.1 Arithmetic Mean of Grouped Data

In case of a grouped or continuous frequency distribution, the arithmetic mean is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i} = \frac{\sum fx}{N}, \text{ where } N = \sum_{i=1}^{n} f_i$$

and x is taken as the mid-value of the corresponding class.

Example 1

Find the arithmetic mean from the following frequency distribution:

x	5	6	7	8	9	10	11	12	13	14
f	25	45	90	165	112	96	81	26	18	12

Solution

x	f	fx
5	25	125
6	45	270
7	90	630
8	165	1320
9	112	1008
10	96	960
11	81	891
12	26	312
13	18	234
14	12	168
	$\sum f = 670$	$\sum fx = 5918$

$$N = \sum f = 670$$

$$\bar{x} = \frac{\sum fx}{N} = \frac{5918}{670} = 8.83$$

Example 2

Find the arithmetic mean of the marks from the following data:

Marks	0–10	10–20	20–30	30–40	40–50	50–60
Number of students	12	18	27	20	15	8

Solution

Marks	Number of students (f)	Mid-value (x)	fx
0–10	12	5	60
10–20	18	15	270
20–30	27	25	675
30–40	20	35	700
40–50	15	45	675
50–60	8	55	440
	$\sum f = 100$		$\sum fx = 2820$

$$N = \sum f = 100$$

$$\bar{x} = \frac{\sum fx}{N} = \frac{2820}{100} = 28.20$$

8.7.2 Arithmetic Mean from Assumed Mean

If the values of x and (or) f are large, the calculation of mean becomes quite time-consuming and tedious. In such cases, the provisional mean 'a' is taken as that value of x (mid-value of the class interval) which corresponds to the highest frequency or which comes near the middle value of the frequency distribution. This number is called the *assumed mean*.

Let

$$d = x - a$$

$$fd = f(x-a) = fx - af$$

$$\sum fd = \sum fx - a\sum f$$

$$= \sum fx - aN$$

Dividing both the sides by n,

$$\frac{\sum fd}{N} = \frac{\sum fx}{N} - a$$

$$= \bar{x} - a$$

$$\therefore \qquad \bar{x} = a + \frac{\sum fd}{N}$$

Example 1

Ten coins were tossed together and the number of tails resulting from them were observed. The operation was performed 1050 *times and the frequencies thus obtained for different numbers of tail* (x) *are shown in the following table. Calculate the arithmetic mean.*

x	0	1	2	3	4	5	6	7	8	9	10
f	2	8	43	133	207	260	213	120	54	9	1

Solution

Let $a = 5$ be the assumed mean.

$d = x - a = x - 5$

x	f	$d = x - 5$	fd
0	2	–5	–10
1	8	–4	–32
2	43	–3	–129
3	133	–2	–266
4	207	–1	–207
5	260	0	0
6	213	1	213
7	120	2	240
8	54	3	162
9	9	4	36
10	1	5	5
	$\sum f = 1050$		$\sum fd = 12$

$$N = \sum f = 1050$$

$$\bar{x} = a + \frac{\sum fd}{N}$$

$$= 5 + \frac{12}{1050}$$
$$= 5.0114$$

Example 2

Calculate the mean for the following frequency distribution

Class	0–8	8–16	16–24	24–32	32–40	40–48
Frequency	8	7	16	24	15	7

Solution

Let $a = 28$ be the assumed mean.

$d = x - a = x - 28$

Class	Frequency	Mid-value (x)	$d = x - 28$	fd
0–8	8	4	–24	–192
8–16	7	12	–16	–112
16–24	16	20	–8	–128
24–32	24	28	0	0
32–40	15	36	8	120
40–48	7	44	16	112
	$\sum f = 77$			$\sum fd = -200$

$$N = \sum f = 77$$
$$\bar{x} = a + \frac{\sum fd}{N}$$
$$= 28 + \frac{(-200)}{77}$$
$$= 25 \cdot 403$$

8.7.3 Arithmetic Mean by the Step-Deviation Method

When the class intervals in a grouped data are equal, calculation can be simplified by the step-deviation method. In such cases, deviation of the variate x from the assumed mean a (i.e., $d = x - a$) are divided by the common factor h which is equal to the width of the class interval.

Let $d = \frac{x - a}{h}$

$$\bar{x} = a + h\frac{\sum fd}{\sum f} = a + h\frac{\sum fd}{N}$$

where a is the assumed mean

$d = \dfrac{x-a}{h}$ is the deviation of any variate x from a

h is the width of the class interval

N is the number of observations

Example 1

Calculate the arithmetic mean of the following marks obtained by students in mathematics:

Marks (x)	5	10	15	20	25	30	35	40	45	50
Number of students (f)	20	43	75	67	72	45	39	9	8	6

Solution

Let $a = 30$ be the assumed mean and $h = 5$ be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-30}{5}$$

x	f	$d = \frac{x-30}{5}$	fd
5	20	–5	–100
10	43	–4	–172
15	75	–3	–225
20	67	–2	–134
25	72	–1	–72
30	45	0	0
35	39	1	39
40	9	2	18
45	8	3	24
50	6	4	24
	$\sum f = 384$		$\sum fd = -598$

$$N = \sum f = 384$$

$$\begin{aligned}\bar{x} &= a + h\frac{\sum fd}{N} \\ &= 30 + 5\left(\frac{-598}{384}\right) \\ &= 22.214\end{aligned}$$

Example 2

The following table gives the distribution of companies according to size of capital. Find the mean size of the capital of a company.

Capital (₹ in lacs)	<5	<10	<15	<20	<25	<30
No. of companies	20	27	29	38	48	53

Solution

This is a 'less than' type of frequency distribution. This will be first converted into class intervals. Let $a = 12.5$ be the assumed mean and $h = 5$ be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-12.5}{5}$$

Class intervals	Frequency f	Mid-value x	$d = \dfrac{x-12.5}{5}$	fd
0–5	20	2.5	–2	–40
5–10	7	7.5	–1	–7
10–15	2	12.5	0	0
15–20	9	17.5	1	9
20–25	10	22.5	2	20
25–30	5	27.5	3	15
	$\sum f = 53$			$\sum fd = -3$

$$N = \sum f = 53$$

$$\begin{aligned}\bar{x} &= a + h\frac{\sum fd}{N} \\ &= 12.5 + 5\left(\frac{-3}{53}\right) \\ &= 12.22 \text{ lacs}\end{aligned}$$

Example 3

Following is the distribution of marks obtained by 60 students in a mathematics test:

Marks	Number of students
More than 0	60
More than 10	56
More than 20	40
More than 30	20
More than 40	10
More than 50	3

Calculate the arithmetic mean.

Solution

This is a 'more than' type of frequency distribution. This will be first converted into class intervals. Let $a = 35$ be the assumed mean and $h = 10$ be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-35}{10}$$

Marks	No. of students f	Mid-value x	$d = \frac{x-35}{10}$	fd
0–10	4	5	–3	–12
10–20	16	15	–2	–32
20–30	20	25	–1	–20
30–40	10	35	0	0
40–50	7	45	1	7
50–60	3	55	2	6
	$\sum f = 60$			$\sum fd = -51$

$$N = \sum f = 60$$

$$\bar{x} = a + h\frac{\sum fd}{N}$$

$$= 35 + 10\left(\frac{-51}{60}\right)$$

$$= 26.5$$

EXERCISE 8.2

1. Find the mean of the following marks obtained by students of a class:

Marks	15	20	25	30	35	40
No. of students	9	7	12	14	15	6

[**Ans.:** 25.58]

2. The following table gives the distribution of total household expenditure (in rupees) of manual workers in a city:

Expenditure (in ₹)	100–150	150–200	200–250	250–300	300–350	350–400	400–450	450–500
Frequency	24	40	33	28	30	22	16	7

Find the average expenditure (in ₹) per household.

[**Ans.:** ₹ 266.25]

3. Calculate the mean for the following data:

Heights (in cm)	135–140	140–145	145–150	150–155	155–160	160–165	165–170	170–175
No. of boys	4	9	18	28	24	10	5	2

[**Ans.:** 153.45 cm]

4. The weights in kilograms of 60 workers in a factory are given below. Find the mean weight of a worker.

Weight (in kg)	60	61	62	63	64	65
No. of workers	5	8	14	16	10	7

[**Ans.:** 62.65 kg]

5. Calculate the mean from the following data:

Marks less than/up to	10	20	30	40	50	60
No. of students	10	30	60	110	150	180

[**Ans.:** 35]

6. Calculate the mean from the following data:

Marks more than	0	10	20	30	40	50	60
No. of students	180	170	150	120	70	30	0

[**Ans.:** 35]

7. Calculate the mean from the following data:

Marks	1–5	6–10	11–15	16–20	21–25	26–30	31–35	36–40	41–45
No. of students	7	10	16	30	24	17	10	5	1

[**Ans.:** 20.33]

8.8 MEDIAN

Median is the central value of the variable when the values are arranged in ascending or descending order of magnitude. It divides the distribution into two equal parts. When the observations are arranged in the order of their size, median is the value of that item which has equal number of observations on either side.

In case of ungrouped data, if the number of observations is odd then the median is the middle value after the values have been arranged in ascending or descending order of magnitude. If the number of observations is even, there are two middle terms and the median is obtained by taking the arithmetic mean of the middle terms.

Examples

(i) The median of the values 20, 15, 25, 28, 18, 16, 30, i.e., 15, 16, 18, 20, 25, 28, 30 is 20 because $n = 7$, i.e., odd and the median is the middle value, i.e., 20.

(ii) The median of the values 8, 20, 50, 25, 15, 30, i.e., 8, 15, 20, 25, 30, 50 is the arithmetic mean of the middle terms, i.e., $\frac{20+25}{2} = 22.5$ because $n = 6$, i.e., even.

In case of discrete frequency distribution, the median is obtained by considering the cumulative frequencies. The steps for calculating the median are given below:

(i) Arrange the values of the variables in ascending or descending order of magnitudes.

(ii) Find $\frac{N}{2}$, where $N = \sum f$

(iii) Find the cumulative frequency just greater than $\frac{N}{2}$ and determine the corresponding value of the variable.

(iv) The corresponding value of x is the median.

Example 1

The following table represents the marks obtained by a batch of 12 *students in certain class tests in physics and chemistry.*

Marks (Physics)	53	54	32	30	60	46	28	25	48	72	33	65
Marks (Chemistry)	55	41	48	49	27	25	23	20	28	60	43	67

Indicate the subject in which the level of achievement is higher.

Solution

The level of achievement is higher in that subject for which the median marks are more.

Arranging the marks in two subjects in ascending order,

Marks (Physics)	25	28	30	32	33	46	48	53	54	60	65	72
Marks (Chemistry)	20	23	25	27	28	41	43	48	49	55	60	67

Since the number of students is 12, the median is the arithmetic mean of the middle terms.

Median marks in physics = $\dfrac{46+48}{2} = 47$

Median marks in chemistry = $\dfrac{41+43}{2} = 42$

Since the median marks in physics are greater than the median marks in chemistry, the level of achievement is higher in physics.

Example 2

Obtain the median for the following frequency distribution.

x	0	1	2	3	4	5	6	7
f	7	14	18	36	51	54	52	18

Solution

x	f	Cumulative frequency
0	7	7
1	14	21
2	18	39
3	36	75
4	51	126
5	54	180
6	52	232
7	18	250

$$N = 250$$

$$\frac{N}{2} = \frac{250}{2} = 125$$

The cumulative frequency just greater than $\dfrac{N}{2}$ = 125 is 126 and the value of x corresponding to 126 is 4. Hence, the median is 4.

Median for Continuous Frequency Distribution

In case of continuous frequency distribution (less than frequency distribution), the class corresponding to the cumulative frequency just greater than $\frac{N}{2}$, is called the *median class*, and the value of the median is given by

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

where l is the lower limit of the median class
f is the frequency of the median class
h is the width of the median class
c is the cumulative frequency of the class preceding the median class
N is sum of frequencies, i.e., $N = \sum f$

In case of 'more than' or 'greater than' type of frequency distributions, the value of the median is given by

$$\text{Median} = u - \frac{h}{f}\left(\frac{N}{2} - c\right)$$

where u is the upper limit of the median class
f is the frequency of the median class
h is the width of the median class
c is the cumulative frequency of the class succeeding the median class

Example 1

The following table gives the weekly expenditures of 100 *workers. Find the median weekly expenditure.*

Weekly expenditure (in ₹)	0–10	10–20	20–30	30–40	40–50
Number of workers	14	23	27	21	15

Solution

Weekly expenditure (in ₹)	Number of workers (f)	Cumulative frequency
0–10	14	14
10–20	23	37
20–30	27	64
30–40	21	85
40–50	15	100

$N = 100$

$$\frac{N}{2}=\frac{100}{2}=50$$

The cumulative frequency just greater than $\frac{N}{2}$ = 50 is 64 and the corresponding class 20–30 is the median class.

Here, $\frac{N}{2}=50, \quad l=20, \quad h=10, \quad f=27, \quad c=37$

$$\begin{aligned}\text{Median } &= l+\frac{h}{f}\left(\frac{N}{2}-c\right)\\ &= 20+\frac{10}{27}(50-37)\\ &= 24.815\end{aligned}$$

Example 2

From the following data, calculate the median:

Marks (Less than)	5	10	15	20	25	30	35	40	45
No. of students	29	224	465	582	634	644	650	653	655

[Summer 2015]

Solution

This is a 'less than' type of frequency distribution. This will be first converted into class intervals.

Class intervals	Frequency	Less than CF
0–5	29	29
5–10	195	224
10–15	241	465
15–20	117	582
20–25	52	634
25–30	10	644
30–35	6	650
35–40	3	653
40–45	2	655

$$N=655$$

Since $\frac{N}{2}=\frac{655}{2}=327.5$, the median class is 10–15.

Here, $l = 10, \quad h = 5, \quad f = 241, \quad c = 224$

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

$$= 10 + \frac{5}{241}(327.5 - 224)$$

$$= 12.147$$

Example 3

Find the mean of the following data:

Age greater than (in years)	0	10	20	30	40	50	60	70
No. of persons	230	218	200	165	123	73	28	8

Solution

This is a 'greater than' type of frequency distribution. This will be first converted into class intervals.

Class intervals	Frequency	Greater than CF
0–10	12	230
10–20	18	218
20–30	35	200
30–40	42	165
40–50	50	123
50–60	45	73
60–70	20	28
70 and above	8	8

$N = 230$

Since $\frac{N}{2} = \frac{230}{2} = 115,$ the median class is 40–50.

Here, $u = 50, \quad h = 10, \quad f = 50, \quad c = 73$

$$\text{Median} = u - \frac{h}{f}\left(\frac{N}{2} - c\right)$$

$$= 50 - \frac{10}{50}(115 - 73)$$

$$= 41.6 \text{ years}$$

Example 4

The following table gives the marks obtained by 50 students in mathematics. Find the median.

Marks	10–14	15–19	20–24	25–29	30–34	35–39	40–44	45–49
No. of students	4	6	10	5	7	3	9	6

Solution

Since the class intervals are inclusive, it is necessary to convert them into exclusive series.

Marks	No. of students	Cumulative frequency
9.5–14.5	4	4
14.5–19.5	6	10
19.5–24.5	10	20
24.5–29.5	5	25
29.5–34.5	7	32
34.5–39.5	3	35
39.5–44.5	9	44
44.5–49.5	6	50

$N = 50$

Since $\frac{N}{2} = \frac{50}{2} = 25,$ the median class is 24.5–29.5.

Here, $l = 24.5, \quad h = 5, \quad f = 5, \quad c = 20$

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

$$= 24.5 + \frac{5}{5}(25 - 20)$$

$$= 29.5$$

Example 5

Find the median of the following distribution:

Mid-values	1500	2500	3500	4500	5500	6500	7500
Frequency	27	32	65	78	58	32	8

Solution

The difference between two mid-values is 1000. On subtracting and adding half of this, i.e., 500 to each of the mid-values, the lower and upper limits of the respective class intervals are obtained.

Class intervals	Frequency	Cumulative frequency
1000–2000	27	27
2000–3000	32	59
3000–4000	65	124
4000–5000	78	202
5000–6000	58	260
6000–7000	32	292
7000–8000	8	300

$N = 300$

Since $\frac{N}{2} = 150,$ the median class is 4000–5000.

Here, $l = 4000$, $h = 1000$, $f = 78$, $c = 124$

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

$$= 4000 + \frac{1000}{78}(150 - 124)$$

$$= 4333.33$$

EXERCISE 8.3

1. The heights (in cm) of 15 students of Class XII are 152, 147, 156, 149, 151, 159, 148, 160, 153, 154, 150, 143, 155, 157, 161. Find the median.

[**Ans.:** 153 cm]

2. The median of the following observations are arranged in the ascending order: 11, 12, 14, 18, *x* + 2, *x* + 4, 30, 32, 35, 41 is 24. Find *x*.

[**Ans.:** 21]

3. Find the median of the following frequency distribution:

x	10	11	12	13	14	15	16
f	8	15	25	20	12	10	5

[**Ans.:** 12]

4. Find the median of the following frequency distribution:

Wages (in ₹)	20–30	30–40	40–50	50–60	60–70
No. of workers	3	5	20	10	5

[Ans.: 46.75]

5. Calculate the median of the following data:

x	3–4	4–5	5–6	6–7	7–8	8–9	9–10	10–11
f	3	7	12	16	22	20	13	7

[Ans.: 7.55]

6. The weekly wages of 1000 workers of a factory are shown in the following table:

Weekly wages (less than)	425	475	525	575	625	675	725	775	825	875
No. of workers	2	10	43	123	293	506	719	864	955	1000

[Ans.: 673.59]

7. Calculate the mean of the following distribution of marks obtained by 50 students in advanced engineering mathematics.

Marks more than	0	10	20	30	40	50
No. of students	50	46	40	20	10	3

[Ans.: 27.5]

8. Calculate the median from the following data:

Mid-values	115	125	135	145	155	165	175	185	195
Frequency	6	25	48	72	116	60	38	22	3

[Ans.: 153.79]

8.9 MODE

Mode is the value which occurs most frequently in a set of observations and around which the other items of the set are heavily distributed. In other words, mode is the value of the variable which is most frequent or predominant in the series. In case of a discrete frequency distribution, mode is the value of x corresponding to the maximum frequency.

Examples

(i) In the series 6, 5, 3, 4, 3, 7, 8, 5, 9, 5, 4, the value 5 occurs most frequently. Hence, the mode is 5.

(ii) Consider the following frequency distribution:

x	1	2	3	4	5	6	7	8
f	4	9	16	25	22	15	7	3

The value of x corresponding to the maximum frequency, viz., 25, is 4. Hence, the mode is 4.

For an asymmetrical frequency distribution, the difference between the mean and the mode is approximately three times the difference between the mean and the median.

$$\text{Mean} - \text{Mode} = 3\,(\text{Mean} - \text{Median})$$

$$\text{Mode} = 3\text{ Median} - 2\text{ Mean}$$

This is known as the *empirical formula for calculation of the mode*.

Mode for a Continuous Frequency Distribution

In case of a continuous frequency distribution, the class in which the mode lies is called the *modal class* and the value of the mode is given by

$$\text{Mode} = l + h\left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right)$$

where l is the lower limit of the modal class

h is the width of the modal class

f_m is the frequency of the modal class

f_1 is the frequency of the class preceding the modal class

f_2 is the frequency of the class succeeding the modal class

This method of finding the mode is called the *method of interpolation*. This formula is applicable only to a unimodal frequency distribution.

Example 1

Find the mode for the following data:

Profit per shop	0–100	100–200	200–300	300–400	400–500	500–600
No. of shops	12	18	27	20	17	6

Solution

Since the maximum frequency is 27, which lies in the class 200–300, the modal class is 200–300.

Here, $l = 200$, $h = 100$, $f_m = 27$, $f_1 = 18$, $f_2 = 20$

$$\text{Mode} = l + h\left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right)$$

$$= 200 + 100\left[\frac{27 - 18}{2(27) - 18 - 20}\right]$$

$$= 256.25$$

Example 2

The frequency distribution of marks obtained by 60 *students of a class in a college is given by*

Marks	30–34	35–39	40–44	45–49	50–54	55–59	60–64
Frequency	3	5	12	18	14	6	2

Find the mode of the distribution.

Solution

The class intervals are first converted into a continuous exclusive series as shown in the following table:

Marks	Frequency
29.5–34.5	3
34.5–39.5	5
39.5–44.5	12
44.5–49.5	18
49.5–54.5	14
54.5–59.5	6
59.5–64.5	2

Since the maximum frequency is 18 which lies in the interval 44.5–49.5, the modal class is 44.5–49.5.

Here, $l = 44.5$, $h = 5$, $f_m = 18$, $f_1 = 12$, $f_2 = 14$

$$\text{Mode} = l + h\left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right)$$

$$= 44.5 + 5\left[\frac{18 - 12}{2(18) - 12 - 14}\right]$$

$$= 47.5$$

Example 3

Find the mode for the following distribution:

Class intervals	0–10	10–20	20–30	30–40	40–50
Frequency	45	20	14	7	3

Solution

Since the highest frequency occurs in the first class interval, the interpolation formula is not applicable. Thus, empirical formula is used for calculation of mode.

Class intervals	Frequency	CF	Mid-value	$d = \dfrac{x-25}{10}$	fd
0–10	45	45	5	–2	–90
10–20	20	65	15	–1	–20
20–30	14	79	25	0	0
30–40	7	86	35	1	7
40–50	3	89	45	2	6
	$\sum f = 89$				$\sum fd = -97$

$$N = \sum f = 89$$

Since $\dfrac{N}{2} = \dfrac{89}{2} = 44.5$, the median class is 0–10.

Here, $l = 0$, $h = 10$, $f = 45$, $c = 0$

$$\begin{aligned}
\text{Median} &= l + \frac{h}{f}\left(\frac{N}{2} - c\right) \\
&= 0 + \frac{10}{45}(44.5 - 0) \\
&= 9.89
\end{aligned}$$

$$\begin{aligned}
\text{Mean} &= a + h\frac{\sum fd}{N} \\
&= 25 + 10\left(\frac{-97}{89}\right) \\
&= 14.1
\end{aligned}$$

$$\begin{aligned}
\text{Hence, mode} &= 3\,\text{Median} - 2\,\text{Mean} \\
&= 3(9.89) - 2(14.1) \\
&= 1.47
\end{aligned}$$

EXERCISE 8.4

1. Calculate the mode for the following distribution:

x	6	12	18	24	30	36
f	12	24	36	38	37	6

[Ans.: 24]

2. Calculate the mode for the following distribution:

x	10	20	30	40	50	60	70
f	17	22	31	39	27	15	13

[Ans.: 40]

3. Calculate the mode for the following distribution:

Class interval	0–4	4–8	8–12	12–16
Frequency	4	8	5	6

[Ans.: 6.28]

4. Calculate the mode of the following distribution:

x	0–5	5–10	10–15	15–20	20–25	25–30	30–35	35–40	40–45
f	20	24	32	28	20	16	37	10	18

[Ans.: 13.33]

5. Calculate the mode for the following data:

Class	10–20	20–30	30–40	40–50	50–60	60–70	70–80
f	24	42	56	66	108	130	154

[Ans.: 71.348]

6. Find the mode of the following distribution:

Class	55–64	65–74	75–84	85–94	95–104	105–114	115–124	125–134	135–144
f	1	2	9	22	33	22	8	2	1

[Ans.: 99.5]

7. Calculate the modal marks from the following distribution of marks of 100 students of a class:

Marks (more than)	90	80	70	60	50	40	30	20	10
No. of students	0	4	15	33	53	76	92	98	100

[Ans.: 47]

8.10 STANDARD DEVIATION

Standard deviation is the positive square root of the arithmetic mean of the squares of the deviations of the given values from their arithmetic mean. It is denoted by the Greek letter σ. Let X be a random variable which takes on values, viz., $x_1, x_2, ..., x_n$. The standard deviation of these n observations is given by

$$\sigma = \sqrt{\frac{\sum (x - \overline{x})^2}{n}}$$

where $\overline{x} = \frac{\sum x}{n}$ is the arithmetic mean of these observations.

This equation can be modified further.

$$\sigma = \sqrt{\frac{\sum (x^2 - 2x\overline{x} + \overline{x})^2}{n}}$$

$$= \sqrt{\frac{\sum x^2 - 2\overline{x}\sum x + \overline{x}^2 \sum 1}{n}}$$

$$= \sqrt{\frac{\sum x^2}{n} - 2\frac{\sum x}{n}\frac{\sum x}{n} + \left(\frac{\sum x}{n}\right)^2 \cdot \frac{n}{n}} \qquad \left[\because \sum 1 = n\right]$$

$$= \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

$$= \sqrt{\text{Mean of squares} - \text{Square of mean}}$$

In case of a frequency distribution consisting of n observations $x_1, x_2, ..., x_n$ with respective frequencies $f_1, f_2, ..., f_n$, the standard deviation is given by

$$\sigma = \sqrt{\frac{\sum f(x - \overline{x})^2}{N}}$$

This equation can also be modified.

$$\sigma = \sqrt{\frac{\sum f\,(x^2 - 2x\bar{x} + \bar{x}^2)}{N}}$$

$$= \sqrt{\frac{\sum fx^2}{N} - \frac{2\bar{x}\sum fx}{N} + \bar{x}^2\frac{\sum f}{N}}$$

$$= \sqrt{\frac{\sum fx^2}{N} - 2\frac{\sum fx}{N}\frac{\sum fx}{N} + \left(\frac{\sum fx}{N}\right)^2} \qquad \left[\because \quad \sum f = N \quad \text{and} \quad \bar{x} = \frac{\sum fx}{N}\right]$$

$$= \sqrt{\frac{\sum fx^2}{N} - \left(\frac{\sum fx}{N}\right)^2}$$

8.10.1 Variance

The *variance* is the square of the standard deviation and is denoted by σ^2. The method for calculating variance is same as that given for the standard deviation.

Example 1

Calculate the standard deviation of the weights of ten persons.

Weight (in kg)	45	49	55	50	41	44	60	58	53	55

Solution

$$n = 10$$

$$\sum x = 45 + 49 + 55 + 50 + 41 + 44 + 60 + 58 + 53 + 55 = 510$$

$$\sum x^2 = 45^2 + 49^2 + 55^2 + 50^2 + 41^2 + 44^2 + 60^2 + 58^2 + 53^2 + 55^2 = 26366$$

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

$$= \sqrt{\frac{26366}{10} - \left(\frac{510}{10}\right)^2}$$

$$= 5.967$$

Aliter:

$$\bar{x} = \frac{\sum x}{n} = \frac{510}{10} = 51$$

x	$x-\overline{x}$	$(x-\overline{x})^2$
45	–6	36
49	–2	4
55	4	16
50	–1	1
41	–10	100
44	–7	49
60	9	81
58	7	49
53	2	4
55	4	16
		$\sum(x-\overline{x})^2 = 356$

$$\sigma = \sqrt{\frac{\sum(x-\overline{x})^2}{n}}$$

$$= \sqrt{\frac{356}{10}}$$

$$= 5.967$$

Example 2

Calculate the standard deviation of the following data:

x	10	11	12	13	14	15	16	17	18
f	2	7	10	12	15	11	10	6	3

Solution

x	f	fx	x^2	fx^2
10	2	20	100	200
11	7	77	121	847
12	10	120	144	1440
13	12	156	169	2028
14	15	210	196	2940
15	11	165	225	2475
16	10	160	256	2560
17	6	102	289	1734
18	3	54	324	972
	$\sum f = 76$	$\sum fx = 1064$		$\sum fx^2 = 15196$

$$N = \sum f = 76$$

$$\sigma = \sqrt{\frac{\sum fx^2}{N} - \left(\frac{\sum fx}{N}\right)^2}$$

$$= \sqrt{\frac{15196}{76} - \left(\frac{1064}{76}\right)^2}$$

$$= 1.987$$

Aliter:

$$N = \sum f = 76$$

$$\overline{x} = \frac{\sum fx}{N} = \frac{1064}{76} = 14$$

x	f	$x - \overline{x}$	$(x - \overline{x})^2$	$f(x - \overline{x})^2$
10	2	–4	16	32
11	7	–3	9	63
12	10	–2	4	40
13	12	–1	1	12
14	15	0	0	0
15	11	1	1	11
16	10	2	4	40
17	6	3	9	54
18	3	4	16	48
				$\sum f(x - \overline{x})^2 = 300$

$$\sigma = \sqrt{\frac{\sum f(x - \overline{x})^2}{N}}$$

$$= \sqrt{\frac{300}{76}}$$

$$= 1.987$$

8.10.2 Standard Deviation from the Assumed Mean

If the values of x and f are large, the calculation of fx, fx^2 becomes tedious. In such a case, the assumed mean a is taken to simplify the calculation.

Let a be the assumed mean.

$$d = x - a$$

$$x = a + d$$

$$\sum fx = \sum f(a+d) = Na + \sum fd$$

Dividing both the sides by N,

$$\frac{\sum fx}{N} = a + \frac{\sum fd}{N}$$

$$\bar{x} = a + \bar{d}$$

$$x - \bar{x} = d - \bar{d}$$

$$\sigma_x = \sqrt{\frac{\sum f(x-\bar{x})^2}{N}}$$

$$= \sqrt{\frac{\sum f(d-\bar{d})^2}{N}}$$

$$= \sigma_d$$

Hence, the standard deviation is independent of change of origin.

$$\therefore \qquad \sigma_x = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

Example 1

Find the standard deviation from the following data:

Size of the item	10	11	12	13	14	15	16
Frequency	2	7	11	15	10	4	1

Solution

Let a = 13 be the assumed mean.

$$d = x - a = x - 13$$

Size of item (x)	Frequency (f)	$d = x - a$	d^2	fd	fd^2
10	2	–3	9	–6	18
11	7	–2	4	–14	28
12	11	–1	1	–11	11
13	15	0	0	0	0
14	10	1	1	10	10
15	4	2	4	8	16
16	1	3	9	3	9
	$\sum f = 50$			$\sum fd = -10$	$\sum fd^2 = 92$

$$N = \sum f = 50$$

$$\sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

$$= \sqrt{\frac{92}{50} - \left(\frac{-10}{50}\right)^2}$$

$$= 1.342$$

8.10.3 Standard Deviation by Step-Deviation Method

Let a be the assumed mean and h be the width of the class interval.

$$d = \frac{x-a}{N}$$

$$x = a + hd$$

$$\sum fx = \sum f\,(a+hd) = Na + h\sum fd$$

Dividing both the sides by N,

$$\frac{\sum fx}{N} = a + h\frac{\sum fd}{N}$$

$$\overline{x} = a + h\overline{d}$$

$$x - \overline{x} = h(d - \overline{d})$$

$$\sigma_x = \sqrt{\frac{\sum f\,(x-\overline{x})^2}{N}}$$

$$= \sqrt{\frac{\sum f\,h^2\,(d-\overline{d})^2}{N}}$$

$$= h\sqrt{\frac{\sum f\,(d-\overline{d})^2}{N}}$$

$$= h\,\sigma_d$$

Hence, the standard deviation is independent of change of origin but not of scale.

$$\therefore \qquad \sigma_x = h\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

Example 1

Find the standard deviation for the following distribution:

Marks	10–20	20–30	30–40	40–50	50–60	60–70	70–80
Number of students	5	12	15	20	10	4	2

Solution

Let $a = 45$ be the assumed mean and $h = 10$ be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-45}{10}$$

Marks	Number of students f	Mid-value x	$d = \frac{x-45}{10}$	d^2	fd	fd^2
10–20	5	15	–3	9	–15	45
20–30	12	25	–2	4	–24	48
30–40	15	35	–1	1	–15	15
40–50	20	45	0	0	0	0
50–60	10	55	1	1	10	10
60–70	4	65	2	4	8	16
70–80	2	75	3	9	6	18
	$\sum f = 68$				$\sum fd = -30$	$\sum fd^2 = 152$

$$N = \sum f = 68$$

$$\sigma = h\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

$$= 10\sqrt{\frac{152}{68} - \left(\frac{-30}{68}\right)^2}$$

$$= 14.285$$

Example 2

Find the mean and standard deviation of the following distribution:

Age (in years)	No. of persons
less than 20	0
less than 25	170
less than 30	280
less than 35	360
less than 40	405
less than 45	445
less than 50	480

Solution

This is a 'less than' type of frequency distribution. This is first converted into an exclusive series. Let $a = 32.5$ be the assumed mean and $h = 5$ be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-32.5}{5}$$

Class intervals	No. of persons f	Mid-value x	$d = \frac{x-32.5}{5}$	fd	fd^2
20–25	170	22.5	–2	–340	680
25–30	110	27.5	–1	–110	110
30–35	80	32.5	0	0	0
35–40	45	37.5	1	45	45
40–45	40	42.5	2	80	160
45–50	35	47.5	3	105	315
	$\sum f = 480$			$\sum fd = -220$	$\sum fd^2 = 1310$

$$N = \sum f = 480$$

$$\bar{x} = a + h\frac{\sum fd}{N}$$

$$= 32.5 + 5\left(\frac{-220}{480}\right)$$

$$= 30.21 \text{ years}$$

$$\sigma = h\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

$$= 5\sqrt{\frac{1310}{480} - \left(\frac{-220}{480}\right)^2}$$

$$= 7.94 \text{ years}$$

8.10.4 Coefficient of Variation

The *standard deviation* is an absolute measure of dispersion. The coefficient of variation is a relative measure of dispersion and is denoted by CV.

$$\text{CV} = \frac{\sigma}{\bar{x}} \times 100$$

where σ is the standard deviation and $\bar{x}$ is the mean of the given series. The coefficient of variation has great practical significance and is the best measure of comparing the variability of two series. The series or groups for which the coefficient of variation is greater is said to be more variable or less consistent. On the other hand, the series for which the variation is lesser is said to be less variable or more consistent.

Example 1

The arithmetic mean of the runs scored by three batsmen Amit, Sumeet, and Nayan in the series are 50, 48, *and* 12 *respectively. The standard deviations of their runs are* 15, 12, *and* 2 *respectively. Who is the more consistent of the three?*

Solution

Let $\bar{x}_1, \bar{x}_2, \bar{x}_3$ be the arithmetic means and $\sigma_1, \sigma_2, \sigma_3$ be the standard deviations of the runs scored by Amit, Sumeet, and Nayan.

$$\bar{x}_1 = 50, \bar{x}_2 = 48, \bar{x}_3 = 12, \sigma_1 = 15, \sigma_2 = 12, \sigma_3 = 2$$

$$\text{CV}_1 = \frac{\sigma_1}{\bar{x}_1} \times 100$$

$$= \frac{15}{50} \times 100$$

$$= 30\%$$

$$\text{CV}_2 = \frac{\sigma_2}{\bar{x}_2} \times 100$$

$$= \frac{12}{48} \times 100$$

$$= 25\%$$

$$\begin{aligned}\text{CV}_3 &= \frac{\sigma_3}{\bar{x}_3}\times 100\\ &= \frac{2}{12}\times 100\\ &= 16.67\%\end{aligned}$$

Since the coefficient of variation of Nayan is least, he is the most consistent.

Example 2

The runs scored by two batsmen A and B in 9 consecutive matches are given below:

A	85	20	62	28	74	5	69	4	13
B	72	4	15	30	59	15	49	27	26

Which of the batsmen is more consistent?

Solution

$$n = 9$$

For the batsman A,

$$\sum x_A = 85+20+62+28+74+5+69+4+13 = 360$$

$$\sum x_A^2 = 85^2+20^2+62^2+28^2+74^2+5^2+69^2+4^2+13^2 = 22700$$

$$\begin{aligned}\sigma_A &= \sqrt{\frac{\sum x_A^2}{n} - \left(\frac{\sum x_A}{n}\right)^2}\\ &= \sqrt{\frac{22700}{9} - \left(\frac{360}{9}\right)^2}\\ &= 30.37\end{aligned}$$

$$\bar{x}_A = \frac{\sum x_A}{n} = \frac{360}{9} = 40$$

$$\begin{aligned}\text{CV}_A &= \frac{\sigma_A}{\bar{x}_A}\times 100\\ &= \frac{30.37}{40}\times 100\\ &= 75.925\%\end{aligned}$$

For the batsman B,

$$\sum x_B = 72+4+15+30+59+15+49+27+26 = 297$$

$$\sum x_B^2 = 72^2+4^2+15^2+30^2+59^2+15^2+49^2+27^2+26^2 = 13837$$

$$\begin{aligned}\sigma_B &= \sqrt{\frac{\sum x_B^2}{n} - \left(\frac{\sum x_B}{n}\right)^2} \\ &= \sqrt{\frac{13837}{9} - \left(\frac{297}{9}\right)^2} \\ &= 21.18\end{aligned}$$

$$\bar{x}_B = \frac{\sum x_B}{n} = \frac{297}{9} = 33$$

$$\begin{aligned}\text{CV}_B &= \frac{\sigma_B}{\bar{x}_B} \times 100 \\ &= \frac{21.18}{33} \times 100 \\ &= 64.18\%\end{aligned}$$

Since $\text{CV}_B < \text{CV}_A$, the batsman B is more consistent.

Example 3

Two automatic filling machines A and B are used to fill a mixture of cement concrete in a beam. A random sample of beams on each machine showed the following information:

Machine A	32	28	47	63	71	39	10	60	96	14
Machine B	19	31	48	53	67	90	10	62	40	80

Find the standard deviation of each machine and also comment on the performances of the two machines.

[Summer 2015]

Solution

$$n = 10$$

$$\bar{x} = \frac{\sum x}{n} = \frac{460}{10} = 46$$

$$\bar{y} = \frac{\sum y}{n} = \frac{500}{10} = 50$$

Machine A			Machine B		
x	$x-\bar{x}$	$(x-\bar{x})^2$	y	$y-\bar{y}$	$(y-\bar{y})^2$
32	–14	196	19	–31	961
28	–18	324	31	–19	361
47	1	1	48	–2	4
63	17	289	53	3	9
71	25	625	67	17	289
39	–7	49	90	40	1600
10	–36	1296	10	–40	1600
60	14	196	62	12	144
96	50	2500	40	–10	100
14	–32	1024	80	30	900
$\sum x = 460$		$\Sigma(x-\bar{x})^2 = 6500$	$\sum y = 500$		$\Sigma(y-\bar{y})^2 = 5968$

$$\sigma_A = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{6500}{10}}$$

$$= 25.495$$

$$\sigma_B = \sqrt{\frac{\sum(y-\bar{y})^2}{n}}$$

$$= \sqrt{\frac{5968}{10}}$$

$$= 24.429$$

$$\text{CV}_A = \frac{\sigma_A}{\bar{x}} \times 100$$

$$= \frac{25.495}{46} \times 100$$

$$= 55.423\%$$

$$\text{CV}_B = \frac{\sigma_B}{\bar{y}} \times 100$$

$$= \frac{24.429}{50} \times 100$$

$$= 48.858\%$$

Since $\text{CV}_B < \text{CV}_A$, there is less variability in the performance of the machine B.

EXERCISE 8.5

1. Find the standard deviation of 10 persons whose income in rupees is given below:

   312, 292, 227, 235, 269, 255, 333, 348, 321, 299

   [**Ans.:** 39.24]

2. Calculate the standard deviation from the following data:

Heights in cm	150	155	160	165	170	175	180
No. of students	15	24	32	33	24	16	6

   [**Ans.:** 8.038 cm]

3. Find the standard deviation of the following data:

Size of items	10	11	12	13	14	15	16
Frequency	2	7	11	15	10	4	1

   [**Ans.:** 1.342]

4. Calculate the standard deviation for the following frequency distribution:

Class interval	0–4	4–8	8–12	12–16
Frequency	4	8	2	1

   [**Ans.:** 3.27]

5. Calculate the standard deviation of the following series:

Marks	0–10	10–20	20–30	30–40	40–50
Frequency	10	8	15	8	4

   [**Ans.:** 12.37]

6. Calculate the SD for the following distributions of 300 telephone calls according to their durations in seconds:

Duration (in seconds)	0–30	30–60	60–90	90–120	120–150	150–180	180–210
No. of calls	9	17	43	82	81	44	24

   [**Ans.:** 42.51]

7. Calculate the standard deviation from the following data:

Age less than (in years)	10	20	30	40	50	60	70	80
No. of persons	15	30	53	75	100	110	115	125

[**Ans.:** 19.75]

8. Find the standard deviation from the following data:

Mid-value	30	35	40	45	50	55	60	65	70	75	80
Frequency	1	2	4	7	9	13	17	12	7	6	3

[**Ans.:** 11.04]

9. Two cricketers scored the following runs in ten innings. Find who is a better run-getter and who is a more consistent player.

A	42	17	83	59	72	76	64	45	40	32
B	28	70	31	0	59	108	82	14	3	95

[**Ans.:** *A* is a better run-getter and *B* is more consistent.]

10. Two workers on the same job show the following results over a long period of time:

	Worker *A*	Worker *B*
Mean time (in minutes)	30	25
Standard deviation (in minutes)	6	4

[**Ans.:** *B* is more consistent]

8.11 MOMENTS

Moment is the arithmetic mean of the various powers of the deviations of items from their assumed mean or actual mean. If the deviations of the items are taken from the arithmetic mean of the distribution, it is known as *central moment*. If the mean of the first power of deviations are taken, the first moment about the mean is obtained and is denoted by μ_1. The mean of the second power of the deviations gives the second moment about the mean and is denoted by μ_2. Similarly, the mean of the cubes of deviations gives third moment about the mean and is denoted by μ_3. The mean of the fourth power of the deviations from the mean gives the fourth moment about the mean and is denoted by μ_4. Thus, the mean of the r^{th} power of deviations gives the r^{th} moment about mean or r^{th} central moment and is denoted by μ_r.

8.11.1 Central Moments or Moments about Actual Mean

Let $x_1, x_2,..., x_n$ be n observations with arithmetic mean $\bar{x}$. The various moments about actual mean are given by by the following:

First moment about the mean $\mu_1 = \dfrac{\sum (x-\bar{x})}{n}$

Second moment about the mean $\mu_2 = \dfrac{\sum (x-\bar{x})^2}{n}$

Third moment about the mean $\mu_3 = \dfrac{\sum (x-\bar{x})^3}{n}$

Fourth moment about the mean $\mu_4 = \dfrac{\sum (x-\bar{x})^4}{n}$

In general,

r^{th} moment about the mean $\mu_r = \dfrac{\sum (x-\bar{x})^r}{n}$

In case of a frequency distribution consisting of n observations $x_1, x_2,..., x_n$ with respective frequencies $f_1, f_2,..., f_n$ having arithmetic mean $\bar{x}$,

$$N = \sum f$$

$$\bar{x} = \frac{\sum fx}{N}$$

The various moments about the actual mean are given by the following:

First moment about the mean $\mu_1 = \dfrac{\sum f(x-\bar{x})}{N}$

Second moment about the mean $\mu_2 = \dfrac{\sum f(x-\bar{x})^2}{N}$

Third moment about the mean $\mu_3 = \dfrac{\sum f(x-\bar{x})^3}{N}$

Fourth moment about the mean $\mu_4 = \dfrac{\sum f(x-\bar{x})^4}{N}$

In general,

r^{th} moment about the mean $\mu_r = \dfrac{\sum f(x-\bar{x})^r}{N}$

8.11.2 Properties of Central Moments

(i) The first moment about the mean is always zero, i.e., $\mu_1 = 0$.

(ii) The second moment about the mean measures variance, i.e.,

$$\mu_2 = \sigma^2 \quad \text{or} \quad \text{SD} = \sigma = \pm\sqrt{\mu_2}$$

(iii) The third moment about the mean measures skewness.
If $\mu_3 > 0$, the distribution is positively skewed.
If $\mu_3 < 0$, the distribution is negatively skewed.
If $\mu_3 = 0$, the distribution is symmetrical.

$$\text{Skewness} \ \beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

(iv) The fourth moment about the mean measures kurtosis. It gives information on the peakedness or height of the peak of a frequency distribution, i.e., whether it is more peaked or more flat topped than a normal curve.

$$\text{Kurtosis} \ \beta_2 = \frac{\mu_4}{\mu_2^2}$$

(v) In a symmetric distribution, all odd moments are zero, i.e., $\mu_1 = \mu_3 = \mu_5 = \cdots = \mu_{2r+1} = 0$.

8.11.3 Raw Moments or Moments about Arbitrary Origin

When the actual mean of a distribution is a fraction, it is tedious to calculate central moments. In such cases, moments about an arbitrary origin '*a*' is calculated and then these moments are converted into the moments about actual mean. The moments about the arbitrary origin are known as raw moments and are denoted by μ'_r. Thus, μ'_1 denotes the first moment about an arbitrary origin, μ'_2 denotes the second moment about an arbitrary origin and so on.

The various raw moments are given by the following:

First moment about the arbitrary origin $\mu'_1 = \dfrac{\sum(x-a)}{n}$

Second moment about the arbitrary origin $\mu'_2 = \dfrac{\sum(x-a)^2}{n}$

Third moment about the arbitrary origin $\mu'_3 = \dfrac{\sum(x-a)^3}{n}$

Fourth moment about the arbitrary origin $\mu'_4 = \dfrac{\sum(x-a)^4}{n}$

In general,

r^{th} moment about the arbitrary origin $\mu'_r = \dfrac{\sum(x-a)^r}{n}$

In case of frequency distribution consisting of n observations x_1, x_2, ..., x_n with respective frequencies $f_1, f_2, ..., f_n$ having the arbitrary origin a,

$$N = \sum f$$

$$d = x - a$$

The various moments about the arbitrary origin are given by the following:

First moment about the arbitrary origin $\mu_1' = \dfrac{\sum fd}{N}$

Second moment about the arbitrary origin $\mu_2' = \dfrac{\sum fd^2}{N}$

Third moment about the arbitrary origin $\mu_3' = \dfrac{\sum fd^3}{N}$

Fourth moment about the arbitrary origin $\mu_4' = \dfrac{\sum fd^4}{N}$

In general,

r^{th} moment about the arbitrary origin $\mu_r' = \dfrac{\sum fd^r}{N}$

In case of frequency distribution with 'a' as arbitrary origin and h as width of the class interval,

$$N = \sum f$$

$$d = \frac{x - a}{h}$$

The various moments about the arbitrary origin are given by the following:

First moment about the arbitrary origin $\mu_1' = h\dfrac{\sum fd}{N}$

Second moment about the arbitrary origin $\mu_2' = h\dfrac{\sum fd^2}{N}$

Third moment about the arbitrary origin $\mu_3' = h^3\dfrac{\sum fd^3}{N}$

Fourth moment about the arbitrary origin $\mu_4' = h^4\dfrac{\sum fd^4}{N}$

In general,

r^{th} moment about the arbitrary origin $\mu_r' = h^r\dfrac{\sum fd^r}{N}$

8.11.4 Relation between Central Moments and Raw Moments

The moments about the actual mean, i.e., central moments and moments about the arbitrary origin, i.e., raw moments are related with each other by the following equations:

First central moment $\mu_1 = \mu_1' - \mu_1' = 0$

Second central moment $\mu_2 = \mu_2' - (\mu_1')^2$

Third central moment $\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2(\mu_1')^3$

Fourth central moment $\mu_4 = \mu_4' - 4\mu_3' \mu_1' + 6\mu_2' (\mu_1')^2 - 3(\mu_1')^4$

Similarly, the raw moments can be expressed in terms of central moments.

First raw moment $\mu_1' = \bar{x} - a$

Second raw moment $\mu_2' = \mu_2 + (\mu_1')^2$

Third raw moment $\mu_3' = \mu_3 + 3\mu_2 \mu_1' + (\mu_1')^3$

Fourth raw moment $\mu_4' = \mu_4 + 4\mu_3 \mu_1' + 6\mu_2(\mu_1')^2 + (\mu_1')^4$

8.11.5 Moments about Zero

The moments about zero are denoted by v_1, v_2, v_3, v_4, etc. The various moments about zero are given by the following:

First moment about zero $v_1 = \dfrac{\sum fx}{N}$

Second moment about zero $v_2 = \dfrac{\sum fx^2}{N}$

Third moment about zero $v_3 = \dfrac{\sum fx^3}{N}$

Fourth moment about zero $v_4 = \dfrac{\sum fx^4}{N}$

In general,

r^{th} moment about zero $v_r = \dfrac{\sum fx^r}{N}$

8.11.6 Relation between Moments about Zero and Central Moments

The moments about zero and central moments are related by the following equations:

First moment about zero $v_1 = a + \mu_1' = \bar{x}$

Second moment about zero $v_2 = \mu_2 + (v_1)^2$

Third moment about zero $v_3 = \mu_3 + 3v_1v_2 - 2(v_1)^3$

Fourth moment about zero $v_4 = \mu_4 + 4v_1v_3 - 6(v_1)^2v_2 + 3(v_1)^4$

Example 1

Find the first four moments for the set of numbers 2, 4, 6, 8.

Solution

$$n = 4$$

$$\bar{x} = \frac{2+4+6+8}{4} = \frac{20}{4} = 5$$

x	$x-\bar{x}$	$(x-\bar{x})^2$	$(x-\bar{x})^3$	$(x-\bar{x})^4$
2	–3	9	–27	81
4	–1	1	–1	1
6	1	1	1	1
8	3	9	27	81
$\sum x = 20$	$\sum(x-\bar{x}) = 0$	$\sum(x-\bar{x})^2 = 20$	$\sum(x-\bar{x})^3 = 0$	$\sum(x-\bar{x})^4 = 164$

Moments about the actual mean:

$$\mu_1 = \frac{\sum(x-\bar{x})}{n} = \frac{0}{4} = 0$$

$$\mu_2 = \frac{\sum(x-\bar{x})^2}{n} = \frac{20}{4} = 5$$

$$\mu_3 = \frac{\sum(x-\bar{x})^3}{n} = \frac{0}{4} = 0$$

$$\mu_4 = \frac{\sum(x-\bar{x})^4}{n} = \frac{164}{4} = 41$$

Example 2

Calculate the first four moments from the following data:

x	0	1	2	3	4	5	6	7	8
f	5	10	15	20	25	20	15	10	5

Also, calculate the values of β_1 *and* β_2.

Solution

$$N = \sum f = 125$$
$$\overline{x} = \frac{\sum fx}{N} = \frac{500}{125} = 4$$

x	f	fx	$x-\overline{x}$	$f(x-\overline{x})$	$f(x-\overline{x})^2$	$f(x-\overline{x})^3$	$f(x-\overline{x})^4$
0	5	0	–4	20	80	–320	1280
1	10	10	–3	–30	90	–270	810
2	15	30	–2	–30	60	–120	240
3	20	60	–1	–20	20	–20	20
4	25	100	0	0	0	0	0
5	20	100	1	20	20	20	20
6	15	90	2	30	60	120	240
7	10	70	3	30	90	270	810
8	5	40	4	20	80	320	1280
	$\sum f = 125$	$\sum fx = 500$		$\sum f(x-\overline{x}) = 0$	$\sum f(x-\overline{x})^2 = 500$	$\sum f(x-\overline{x})^3 = 0$	$\sum f(x-\overline{x})^4 = 4700$

Moments about the actual mean:

$$\mu_1 = \frac{\sum f(x-\overline{x})}{N} = \frac{0}{125} = 0$$
$$\mu_2 = \frac{\sum f(x-\overline{x})^2}{N} = \frac{500}{125} = 4$$
$$\mu_3 = \frac{\sum f(x-\overline{x})^3}{N} = \frac{0}{125} = 0$$
$$\mu_4 = \frac{\sum f(x-\overline{x})^4}{N} = \frac{4700}{125} = 37.6$$
$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{0}{64} = 0$$
$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{37.6}{16} = 2.35$$

Example 3

Calculate the first four moments of the following distribution about the mean:

x	0	1	2	3	4	5	6	7	8
f	1	8	28	56	70	56	28	8	1

Also, evaluate β_1 *and* β_2.

Solution

Let $a = 4$ be the arbitrary origin.

x	f	$x - a$	$f(x - a)$	$f(x - a)^2$	$f(x - a)^3$	$f(x - a)^4$
0	1	–4	–4	16	–64	256
1	8	–3	–24	72	–216	648
2	28	–2	–56	112	–224	448
3	56	–1	–56	56	–56	56
4	70	0	0	0	0	0
5	56	1	56	56	56	56
6	28	2	56	112	224	448
7	8	3	24	72	216	648
8	1	4	4	16	64	256
	$\sum f = 256$		$\sum f(x-a) = 0$	$\sum f(x-a)^2 = 512$	$\sum f(x-a)^3 = 0$	$\sum f(x-a)^4 = 2816$

$N \sum f = 256$

Moments about the arbitrary origin:

$$\mu_1' = \frac{\sum f(x-a)}{N} = \frac{0}{256} = 0$$

$$\mu_2' = \frac{\sum f(x-a)^2}{N} = \frac{512}{256} = 2$$

$$\mu_3' = \frac{\sum f(x-a)^3}{N} = \frac{0}{256} = 0$$

$$\mu_4' = \frac{\sum f(x-a)^4}{N} = \frac{2816}{256} = 11$$

Moments about the actual mean:

$$\mu_1 = 0$$

$$\begin{aligned}\mu_2 &= \mu_2' - (\mu_1')^2 \\ &= 2 - 0 \\ &= 2\end{aligned}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$
$$= 0 - 3(2)(0) + 2(0)^3$$
$$= 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$
$$= 11 - 4(0)(0) + 6(2)(0)^2 - 3(0)^4$$
$$= 11$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{11}{(2)^2} = 2.75$$

Example 4

Find the first four moments of the following data about the assumed mean 25 and actual mean:

Class limit	0–10	10–20	20–30	30–40
Frequency	1	3	4	2

Solution

Let $a = 25$ be the assumed mean and $h = 10$ be the width of the class limits.

$$d = \frac{x-a}{h} = \frac{x-25}{10}$$

Class limit	Frequency f	Mid-value x	$d = \frac{x-25}{10}$	fd	fd^2	fd^3	fd^4
0–10	1	5	–2	–2	4	–8	16
10–20	3	15	–1	–3	3	–3	3
20–30	4	25	0	0	0	0	0
30–40	2	35	1	2	2	2	2
	$\sum f = 10$			$\sum fd = -3$	$\sum fd^2 = 9$	$\sum fd^3 = -9$	$\sum fd^4 = 21$

$$N = \sum f = 10$$

Moments about the assumed mean:

$$\mu_1' = h\frac{\sum fd}{N} = 10\left(\frac{-3}{10}\right) = -3$$

$$\mu_2' = h^2\frac{\sum fd^2}{N} = 100\left(\frac{9}{10}\right) = 90$$

$$\mu_3' = h^3\frac{\sum fd^3}{N} = 1000\left(\frac{-9}{10}\right) = -900$$

$$\mu_4' = h^4\frac{\sum fd^4}{N} = 10000\left(\frac{21}{10}\right) = 21000$$

Moments about the actual mean:

$$\mu_1 = 0$$

$$\begin{aligned}\mu_2 &= \mu_2' - (\mu_1')^2\\ &= 90 - (-3)^2\\ &= 81\end{aligned}$$

$$\begin{aligned}\mu_3 &= \mu_3' - 3\mu_2'\,\mu_1' + 2(\mu_1')^3\\ &= -900 - 3(90)(-3) + 2(-3)^3\\ &= -144\end{aligned}$$

$$\begin{aligned}\mu_4 &= \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2'\,(\mu_1')^2 - 3(\mu_1')^4\\ &= 21000 - 4(-900)(-3) + 6(90)(-3)^2 - 3(-3)^4\\ &= 14817\end{aligned}$$

Example 5

Find the first four central moments of the following distribution:

Class-limits	100–104.9	105–109.9	110–114.9	115–119.9	120–124.9
Frequency	7	13	25	25	30

Solution

Let a = 112.45 be the arbitrary origin and h = 5 be the width of class limits.

$$d = \frac{x-a}{N} = \frac{x-112.45}{5}$$

Class limits	Frequency f	Mid-value x	$d = \dfrac{x - 112.45}{5}$	fd	fd^2	fd^3	fd^4
100–104.9	7	102.45	–2	–14	28	–56	112
105–109.9	13	107.45	–1	–13	13	–13	13
110–114.9	25	112.45	0	0	0	0	0
115–119.9	25	117.45	1	25	25	25	25
120–124.9	30	122.45	2	60	120	240	480
	$\sum f = 100$			$\Delta fd = 58$	$\sum fd^2 = 186$	$\sum fd^3 = 196$	$\sum fd^4 = 630$

$$N = \sum f = 100$$

Moments about the arbitrary origin:

$$\mu_1' = h\frac{\sum fd}{N} = 5\left(\frac{58}{100}\right) = 2.9$$

$$\mu_2' = h^2\frac{\sum fd^2}{N} = 25\left(\frac{186}{100}\right) = 46.5$$

$$\mu_3' = h^3\frac{\sum fd^3}{N} = 125\left(\frac{196}{100}\right) = 245$$

$$\mu_4' = h^4\frac{\sum fd^4}{N} = 625\left(\frac{630}{100}\right) = 3937.5$$

Moments about the actual mean:

$$\mu_1 = 0$$

$$\begin{aligned}\mu_2 &= \mu_2' - (\mu_1')^2 \\ &= 46.5 - (2.9)^2 \\ &= 38.09\end{aligned}$$

$$\begin{aligned}\mu_3 &= \mu_3' - 3\mu_2'\,\mu_1' + 2(\mu_1')^3 \\ &= 245 - 3(46.5)(2.9) + 2(2.9)^3 \\ &= -110.772\end{aligned}$$

$$\begin{aligned}\mu_4 &= \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2'\,(\mu_1')^2 - 3(\mu_1')^4 \\ &= 3937.5 - 4(245)(2.9) + 6(46.5)(2.9)^2 - 3(2.9)^4 \\ &= 3229.7057\end{aligned}$$

Example 6

The first four moments of distribution about $x = 2$ are 1, 2.5, 5.5, and 16. Calculate the four moments about $\overline{x}$ and about zero.

Solution

$$\mu_1' = 1, \quad \mu_2' = 2.5, \; \mu_3' = 5.5, \quad \mu_4' = 16$$

Moments about the mean:

$$\mu_1 = 0$$

$$\begin{aligned}
\mu_2 &= \mu_2' - (\mu_1')^2 \\
&= 2.5 - (1)^2 \\
&= 1.5
\end{aligned}$$

$$\begin{aligned}
\mu_3 &= \mu_3' - 3\mu_2\mu_1' + 2(\mu_1')^3 \\
&= 5.5 - 3(2.5)(1) + 2(1)^3 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mu_4 &= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 \\
&= 16 - 4(5.5)(1) + 6(2.5)(1)^2 - 3(1)^4 \\
&= 6
\end{aligned}$$

Moments about zero:

$$\begin{aligned}
\mu_1' &= 1 \\
\bar{x} - a &= 1 \\
\bar{x} - 2 &= 1 \\
\bar{x} &= 3
\end{aligned}$$

$$v_1 = \bar{x} = 3$$

$$\begin{aligned}
v_2 &= \mu_2 + (v_1)^2 \\
&= 1.5 + (3)^2 \\
&= 10.5
\end{aligned}$$

$$\begin{aligned}
v_3 &= \mu_3 + 3v_1v_2 - 2(v_1)^3 \\
&= 0 + 3(3)(10.5) - 2(3)^3 \\
&= 40.5
\end{aligned}$$

$$\begin{aligned}
v_4 &= \mu_4 + 4v_1v_3 - 6v_1^2\, v_2 + 3(v_1)^4 \\
&= 6 + 4(3)(40.5) - 6(3)^2(10.5) + 3(3)^4 \\
&= 168
\end{aligned}$$

Example 7

The first three moments of a distribution about the value 2 *of the variables are* 1, 16*, and* –40. *Show that the mean* = 3, *variance* = 15 *and* $\mu_3 = -86$.

Solution

$$a = 2, \quad \mu_1' = 1, \quad \mu_2' = 16, \quad \mu_2' = 16, \quad \mu_3' = -40$$

$$\mu_1' = \bar{x} - a$$
$$1 = \bar{x} - 2$$
$$\therefore \quad \bar{x} = 3$$

$$\text{Mean} = 3$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$
$$= 16 - (1)^2$$
$$= 15$$

$$\text{Variance} = \mu_2 = 15$$

$$\mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2(\mu_1')^3$$
$$= -40 - 3(16)(1) + 2(1)^3$$
$$= -86$$

EXERCISE 8.6

1. Calculate the first four moments about the mean from the following data:

x	1	2	3	4	5
f	2	3	5	4	1

[Ans.: 0, 1.262, 0.722, 3.795]

2. Calculate the first four moments about the mean and also the value of β_2 from the following table:

x	0	1	2	3	4	5	6	7	8
f	1	8	28	156	170	56	28	8	1

[Ans.: 0, 1.294, 0.642, 0.582, 3.93]

3. Calculate the first four moments about mean from the following data:

Class interval	0–10	10–20	20–30	30–40	40–50
Frequency	2	2	3	2	1

[Ans.: 0, 156, 144, 49392]

4. The first four moments of a distribution about the value 4 of the variables are 1, 4, 10, and 45. Show that the mean = 5, variance = 3, and μ_3 = 0.

5. The first four central moments of a distribution are 0, 2.5, 0.7, and 18.75. Calculate β_1 and β_2.

[Ans.: 0.031, 3]

6. The values of μ_1, μ_2, μ_3 and μ_4 are 0, 9.2, 3.6, and 1.22 respectively. Find skewness and kurtosis of the distribution.

[Ans.: 0.129, 1.4]

7. The first four moments about the working mean 28.5 of a distribution are 0.294, 7.144, 14.409, and 454.98. Calculate the moments about the mean. Also, evaluate β_1 and β_2.

[Ans.: 28.794, 7.058, 36.151, 408.738, 3.717, 8.205]

8.12 RANDOM VARIABLES

A random variable X is a real-valued function of the elements of the sample space of a random experiment. In other words, a variable which takes the real values, depending on the outcome of a random experiment, is called a *random variable*, e.g.,

(i) When a fair coin is tossed, $S = \{H, T\}$. If X is the random variable denoting the number of heads,

$X(H) = 1$ and $X(T) = 0$

Hence, the random variable X can take values 0 and 1.

(ii) When two fair coins are tossed, $S = \{HH, HT, TH, TT\}$. If X is the random variable denoting the number of heads,

$X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0.$

Hence, the random variable X can take values 0, 1, and 2.

(iii) When a fair die is tossed, $S = \{1, 2, 3, 4, 5, 6\}$.

If X is the random variable denoting the square of the number obtained,

$X(1) = 1, X(2) = 4, X(3) = 9, X(4) = 16, X(5) = 25, X(6) = 36$

Hence, the random variable X can take values 1, 4, 9, 16, 25, and 36.

Types of Random Variables

There are two types of random variables:

(i) Discrete random variables
(ii) Continuous random variables

1. Discrete Random Variables A random variable X is said to be discrete if it takes either finite or countably infinite values. Thus, a discrete random variable takes only isolated values, e.g.,

(i) Number of children in a family
(ii) Number of cars sold by different companies in a year
(iii) Number of days of rainfall in a city
(iv) Number of stars in the sky
(v) Profit made by an investor in a day

2. Continuous Random Variables A random variable X is said to be continuous if it takes any values in a given interval. Thus, a continuous random variable takes uncountably infinite values, e.g.,

(i) Height of a person in cm
(ii) Weight of a bag in kg
(iii) Temperature of a city in degree Celsius
(iv) Life of an electric bulb in hours
(v) Volume of a gas in cc

Example 1

Identify the random variables as either discrete or continuous in each of the following cases:

(i) A page in a book can have at most 300 *words*
X = Number of misprints on a page
(ii) Number of students present in a class of 50 *students*
(iii) A player goes to the gymnasium regularly
X = Reduction in his weight in a month
(iv) Number of attempts required by a candidate to clear the IAS examination
(v) Height of a skyscraper

Solution

(i) X = Number of misprints on a page
The page may have no misprint or 1 misprint or 2 misprint … or 300 misprints. Thus, X takes values 0, 1, 2, …, 300. Hence, X is a discrete random variable.
(ii) Let X be the random variable denoting the number of students present in a class. X takes values 0, 1, 2, …, 50. Hence, X is a discrete random variable.
(iii) Reduction in weight cannot take isolated values 0, 1, 2, etc., but it takes any continuous value.
Hence, X is a continuous random variable.
(iv) Let X be a random variable denoting the number of attempts required by a candidate. Thus, X takes values 1, 2, 3, …. Hence, X is a discrete random variable.
(v) Since height can have any fractional value, it is a continuous random variable.

8.13 DISCRETE PROBABILITY DISTRIBUTION

Probability distribution of a random variable is the set of its possible values together with their respective probabilities. Let X be a discrete random variable which takes the values $x_1, x_2, \ldots x_n$. The probability of each possible outcome x_i is

$p_i = p(x_i) = P(X = x_i)$ for $i = 1, 2, \ldots, n$. The number $p(x_i)$, $i = 1,2, \ldots$. must satisfy the following conditions:

(i) $p(x_i) \geq 0$ for all values of i

(ii) $\sum_{i=1}^{\infty} p(x_i) = 1$

The function $p(x_i)$ is called the probability function or probability mass function or probability density function of the random variable X. The set of pairs $\{x, p(x_i)\}$, $i = 1, 2, \ldots, n$ is called the probability distribution of the random variable which can be displayed in the form of a table as shown below:

$X = x_i$	x_1	x_2	x_3	$\ldots x_i$	$\ldots x_n$
$p(x_i) = P(X = x_i)$	$p(x_1)$	$p(x_2)$	$p(x_3)$	$\ldots p(x_i)$	$\ldots p(x_n)$

8.14 DISCRETE DISTRIBUTION FUNCTION

Let X be a discrete random variable which takes the values $x_1, x_2, \ldots$ such that $x_1 < x_2 < \ldots$ with probabilities $p(x_1), p(x_2) \ldots$ such that $p(x_i) \geq 0$ for all values of i and $\sum_{i=1}^{x} p(x_i) = 1.$

The distribution function $F(x)$ of the discrete random variable X is defined by

$$F(x) = P(X \leq x) = \sum_{i=1}^{x} p(x_i)$$

where x is any integer. The function $F(x)$ is also called the cumulative distribution function. The set of pairs $\{x_i, F(x)\}$, $i = 1, 2, \ldots$ is called the cumulative probability distribution.

X	x_1	x_2	...
$F(x)$	$p(x_1)$	$p(x_1) + p(x_2)$	...

Example 1

A fair die is tossed once. If the random variable is getting an even number, find the probability distribution of X.

Solution

When a fair die is tossed,

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let X be the random variable of getting an even number. Hence, X can take the values 0 and 1.

$$P(X = 0) = P(1, 3, 5) = \frac{3}{6} = \frac{1}{2}$$

$$P(X = 1) = P(2, 4, 6) = \frac{3}{6} = \frac{1}{2}$$

Hence, the probability distribution of X is

$X = x$	0	1
$P(X = 1)$	$\frac{1}{2}$	$\frac{1}{2}$

Also, $\sum P(X = x) = \frac{1}{2} + \frac{1}{2} = 1$

Example 2

Find the probability distribution of the number of heads when three coins are tossed.

Solution

When three coins are tossed,

$$S = \{\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}\}$$

Let X be the random variable of getting heads in tossing of three coins. Hence, X can take the values 0, 1, 2, 3.

$$P(X = 0) = P(\text{no head}) = P(\text{TTT}) = \frac{1}{8}$$

$$P(X = 1) = P(\text{one head}) = P(\text{HTT, THT, TTH}) = \frac{3}{8}$$

$$P(X = 2) = P(\text{two heads}) = P(\text{HHT, THH, HTH}) = \frac{3}{8}$$

$$P(X = 3) = P(\text{three heads}) = P(\text{HHH}) = \frac{1}{8}$$

Hence, the probability distribution of X is

$X = x$	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Also, $\sum P(X = x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

Example 3

State with reasons whether the following represent the probability mass function of a random variable:

(i)

$X = x$	0	1	2	3
$P(X = x)$	0.4	0.3	0.2	0.1

(ii)

$X = x$	0	1	2	3
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{4}$

(iii)

$X = x$	0	1	2	3
$P(X = x)$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$

Solution

(i) Here, $0 \leq P(X = x) \leq 1$ is satisfied for all values of X.

$$\begin{aligned}\sum P(X = x) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\ &= 0.4 + 0.3 + 0.2 + 0.1 \\ &= 1\end{aligned}$$

Since $\sum P(X = x) = 1$, it represents the probability mass function.

(ii) Here, $0 \leq P(X = x) \leq 1$ is satisfied for all values of X.

$$\begin{aligned}\sum P(X = x) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{4} \\ &= \frac{5}{4} > 1\end{aligned}$$

Since $\sum (P(X = x) > 1$, it does not represent a probability mass function.

(iii) Here, $0 \leq P(X = x) \leq 1$ is not satisfied for all the values of X as $P(X = 0) = -\frac{1}{2}$.

Hence, $P(X = x)$ does not represent a probability mass function.

Example 4

Verify whether the following functions can be regarded as probability mass functions for the given values of X:

(i) $P(X = x) = \dfrac{1}{5}$ for $x = 0, 1, 2, 3, 4$

$= 0$, otherwise

(ii) $P(X = x) = \dfrac{x - 2}{5}$, $x = 1, 2, 3, 4, 5$

$= 0$, otherwise

(iii) $P(X = x) = \dfrac{x^2}{30}$, $x = 0, 1, 2, 3, 4$

$= 0$, otherwise

Solution

(i) $P(X = 0) = P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = \dfrac{1}{5}$

$P(X = x) \geq 0$ for all values of x

$$\sum P(X = x) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

$$= \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$$

$$= 1$$

Hence, $P(X = x)$ is a probability mass function.

(ii) $P(X = 1) = \dfrac{1 - 2}{5} = -\dfrac{1}{5} < 0$

Hence, $P(X = x)$ is not a probability mass function.

(iii) $P(X = 0) = 0$

$$P(X = 1) = \frac{1}{30}$$

$$P(X = 2) = \frac{4}{30}$$

$$P(X = 3) = \frac{9}{30}$$

$$P(X = 4) = \frac{16}{30}$$

$P(X = x) \geq 0$ for all values of x

$$\sum P(X = x) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$
$$= 0 + \frac{1}{30} + \frac{4}{30} + \frac{9}{30} + \frac{16}{30}$$
$$= 1$$

Hence, $P(X = x)$ is a probability mass function.

Example 5

A random variable X has the probability mass function given by

X	1	2	3	4
$P(X = x)$	0.1	0.2	0.5	0.2

Find (i) $P(2 \le x < 4)$, *(ii)* $P(X > 2)$, *(iii)* $P(X$ *is odd*$)$, *and (iv)* $P(X$ *is even*$)$.

Solution

(i) $P(2 \le X < 4) = P(X = 2) + P(X = 3)$
$= 0.2 + 0.5$
$= 0.7$

(ii) $P(X > 2) = P(X = 3) + P(X = 4)$
$= 0.5 + 0.2$
$= 0.7$

(iii) $P(X \text{ is odd}) = P(X = 1) + P(X = 3)$
$= 0.1 + 0.5$
$= 0.6$

(iv) $P(X \text{ is even}) = P(X = 2) + P(X = 4)$
$= 0.2 + 0.2$
$= 0.4$

Example 6

If the random variable X takes the value 1, 2, 3, *and* 4 *such that* $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4)$. *Find the probability distribution.*

Solution

Let $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4) = k$

$$P(X = 1) = \frac{k}{2}$$

$$P(X = 2) = \frac{k}{3}$$
$$P(X = 3) = k$$
$$P(X = 4) = \frac{k}{5}$$

Since $\sum(P(X = x) = 1$,

$$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$k = \frac{30}{61}$$

Hence, the probability distribution is

X	1	2	3	4
$P(X = x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

Example 7

A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6	7
$P(X = x)$	a	$4a$	$3a$	$7a$	$8a$	$10a$	$6a$	$9a$

(i) Find the value of a.

(ii) Find $P(X < 3)$.

(iii) Find the smallest value of m for which $P(X \leq m) \geq 0.6$.

Solution

(i) Since $P(X = x)$ is a probability distribution function,

$$\sum(P(X = x) = 1$$

$$P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7) = 1$$

$$a + 4a + 3a + 7a + 8a + 10a + 6a + 9a = 1$$

$$a = \frac{1}{48}$$

(ii) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$
$= a + 4a + 3a$
$= 8a$

$$= 8\left(\frac{1}{48}\right)$$

$$= \frac{1}{6}$$

(iii) $P(X \le 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$$= a + 4a + 3a + 7a + 8a$$

$$= 23a$$

$$= 23\left(\frac{1}{48}\right)$$

$$= 0.575$$

$$P(X \le 5) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$$

$$= a + 4a + 3a + 7a + 8a + 10a$$

$$= 33a$$

$$= 33\left(\frac{1}{48}\right)$$

$$= 0.69$$

Hence, the smallest value of m for which $P(X \le m) \ge 0.6$ is 5.

Example 8

The probability mass function of a random variable X is zero except at the points X = 0, 1, 2. At these points, it has the values $P(X = 0) = 3c^3$, $P(X = 1) = 4c - 10c^2$, $P(X = 2) = 5c - 1$. Find (i) c, (ii) $P(X < 1)$, (iii) $P(1 < X \le 2)$, and (iv) $P(0 < X \le 2)$.

Solution

(i) Since $P(X = x)$ is a probability mass function,

$$\sum(P(X = x) = 1$$

$$P(X = 0) + P(X = 1) + P(X = 2) = 1$$

$$3c^3 + 4c - 10c^2 + 5c - 1 = 1$$

$$3c^3 - 10c^2 + 9c - 2 = 0$$

$$(3c - 1)(c - 2)(c - 1) = 0$$

$$c = \frac{1}{3}, \quad 2, \quad 1$$

But $c < 1$, otherwise given probabilities will be greater than one or less than zero.

$$\therefore \qquad c = \frac{1}{3}$$

Hence, the probability distribution is

X	0	1	2
$P(X = x)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{3}$

(ii) $P(X < 1) = P(X = 0) = \dfrac{1}{9}$

(iii) $P(1 < X \le 2) = P(X = 2) = \dfrac{2}{3}$

(iv) $P(0 < X \le 2) = P(X = 1) + P(X = 2)$

$$= \frac{2}{9} + \frac{2}{3}$$

$$= \frac{8}{9}$$

Example 9

From a lot of 10 *items containing* 3 *defectives, a sample of* 4 *items is drawn at random. Let the random variable X denote the number of defective items in the sample. Find the probability distribution of X.*

Solution

The random variable X can take the value 0, 1, 2, or 3.

Total number of items = 10

Number of good items = 7

Number of defective items = 3

$$P(X = 0) = P(\text{no defective}) = \frac{{}^7C_4}{{}^{10}C_4} = \frac{1}{6}$$

$$P(X = 1) = P(\text{one defective and three good items}) = \frac{{}^3C_1\ {}^7C_3}{{}^{10}C_4} = \frac{1}{2}$$

$$P(X = 2) = P(\text{two defectives and two good items}) = \frac{{}^3C_2\ {}^7C_2}{{}^{10}C_4} = \frac{3}{10}$$

$$P(X = 3) = P(\text{three defectives and one good item}) = \frac{{}^3C_3\ {}^7C_1}{{}^{10}C_4} = \frac{1}{30}$$

Hence, the probability distribution of the random variable is

X	0	1	2	3
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

Example 10

Construct the distribution function of the discrete random variable X whose probability distribution is as given below:

X	1	2	3	4	5	6	7
$P(X = x)$	0.1	0.15	0.25	0.2	0.15	0.1	0.05

Solution

Distribution function of X

X	$P(X = x)$	$F(x)$
1	0.1	0.1
2	0.15	0.25
3	0.25	0.5
4	0.2	0.7
5	0.15	0.85
6	0.1	0.95
7	0.05	1

Example 11

A random variable X has the probability function given below:

X	0	1	2
$P(X = x)$	k	$2k$	$3k$

Find (i) k, (ii) $P(X < 2)$, $P(X \le 2)$, $P(0 < X < 2)$, *and (iii) the distribution function.*

Solution

(i) Since $P(X = x)$ is a probability density function,

$$\sum (P(X = x) = 1$$
$$k + 2k + 3k = 1$$
$$6k = 1$$

$$k = \frac{1}{6}$$

Hence, the probability distribution is

X	0	1	2
$P(X = x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

(ii) $P(X < 2) = P(X = 0) + P(X = 1) = \frac{1}{6} + \frac{2}{6} = \frac{1}{2}$

$$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} = 1$$

$$P(0 < X < 2) = P(X = 1) = \frac{1}{3}$$

(iii) Distribution function

X	$P(X = x)$	$F(x)$
0	$\frac{1}{6}$	$\frac{1}{6}$
1	$\frac{2}{6}$	$\frac{1}{2}$
2	$\frac{3}{6}$	1

Example 12

A random variable X takes the values –3, –2, –1, 0, 1, 2, 3, such that
$P(X = 0) = P(X > 0) = P(X < 0)$,
$P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$.
Obtain the probability distribution and the distribution function of X.

Solution

Let $P(X = 0) = P(X > 0) = P(X < 0) = k_1$

Since $\sum P(X = x) = 1$

$$k_1 + k_1 + k_1 = 1$$

$$\therefore \qquad k_1 = \frac{1}{3}$$

$$P(X = 0) = P(X > 0) = P(X < 0) = \frac{1}{3}$$

Let $\quad P(X = 1) = P(X = 2) = P(X = 3) = k_2$

$$P(X > 0) = P(X = 1) + P(X = 2) + P(X = 3)$$

$$\frac{1}{3} = k_2 + k_2 + k_2$$

$\therefore \quad k_2 = \frac{1}{9}$

$$P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{9}$$

Similarly, $P(X = -3) = P(X = -2) = P(X = -1) = \frac{1}{9}$

Probability distribution and distribution function

X	$P(X = x)$	$F(x)$
–3	$\frac{1}{9}$	$\frac{1}{9}$
–2	$\frac{1}{9}$	$\frac{2}{9}$
–1	$\frac{1}{9}$	$\frac{3}{9}$
0	$\frac{1}{3}$	$\frac{6}{9}$
1	$\frac{1}{9}$	$\frac{7}{9}$
2	$\frac{1}{9}$	$\frac{8}{9}$
3	$\frac{1}{9}$	1

Example 13

A discrete random variable X has the following distribution function:

$$F(x)=\begin{cases} 0 & x<1 \\ \dfrac{1}{3} & 1\le x<4 \\ \dfrac{1}{2} & 4\le x<6 \\ \dfrac{5}{6} & 6\le x<10 \\ 1 & x\ge 10 \end{cases}$$

Find (i) $P(2 < X \le 6)$, *(ii)* $P(X = 5)$, *(iii)* $P(X = 4)$, *(iv)* $P(X \le 6)$, *and (v)* $P(X = 6)$.

Solution

(i) $P(2<X\le 6)=F(6)-F(2)=\dfrac{5}{6}-\dfrac{1}{3}=\dfrac{3}{6}=\dfrac{1}{2}$

(ii) $P(X=5)=P(X\le 5)-P(X<5)=F(5)-P(X<5)=\dfrac{1}{2}-\dfrac{1}{2}=0$

(iii) $P(X=4)=P(X\le 4)-P(X<4)=F(4)-P(X<4)=\dfrac{1}{2}-\dfrac{1}{3}=\dfrac{1}{6}$

(iv) $P(X\le 6)=F(6)=\dfrac{5}{6}$

(v) $P(X=6)=P(X\le 6)-P(X<6)=F(6)-P(X<6)=\dfrac{5}{6}-\dfrac{1}{2}=\dfrac{1}{3}$

EXERCISE 8.7

1. Verify whether the following functions can be considered probability mass functions:

(i) $P(X = x) = \dfrac{x^2+1}{18}$, $x = 0, 1, 2, 3$ **[Ans.:** Yes)

(ii) $P(X = x) = \dfrac{x^2-2}{8}$, $x = 1, 2, 3$ **[Ans.:** No]

(iii) $P(X = x) = \dfrac{2x+1}{18}$, $x = 0, 1, 2, 3$ **[Ans.:** No]

2. The probability density function of a random variable X is

X	0	1	2	3	4	5	6
$P(X = x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find $P(X < 4)$ and $P(3 < X \leq 6)$.

$$\left[\textbf{Ans.:} \frac{16}{49}, \frac{33}{49}\right]$$

3. A random variable X has the following probability distribution:

X	1	2	3	4	5	6	7
$P(X = x)$	k	$2k$	$3k$	k^2	$k^2 + k$	$2k^2$	$4k^2$

Find (i) k, (ii) $P(X < 5)$, (iii) $P(X > 5)$, and (iv) $P(0 \leq X \leq 5)$

$$\left[\textbf{Ans.:} \frac{1}{8} \text{ (ii) } \frac{49}{64} \text{ (iii) } \frac{3}{32} \text{ (iv) } \frac{29}{32}\right]$$

4. A discrete random variable X has the following probability distribution:

X	–2	–1	0	1	2	3
$P(X = x)$	0.1	k	0.2	$2k$	0.3	$3k$

Find (i) k, (ii) $P(X \geq 2)$, and (iii) $P(-2 < X < 2)$.

$$\left[\textbf{Ans.:} \frac{1}{15} \text{ (ii) } \frac{1}{2} \text{ (iii) } \frac{2}{5}\right]$$

5. Given the following probability function of a discrete random variable X:

X	0	1	2	3	4	5	6	7
$P(X = x)$	0	c	$2c$	$2c$	$3c$	c^2	$2c^2$	$7c^2 + c$

Find (i) c, (ii) $P(X \geq 6)$, (iii) $P(X < 6)$, and (iv) find k if $P(X \leq k) > \frac{1}{2}$, where k is a positive integer.

[**Ans.:** (i) 0.1 (ii) 0.19 (iii) 0.81 (iv) 4]

6. A random variable X assumes four values with probabilities $\frac{1+3x}{4}, \frac{1-x}{4}, \frac{1+2x}{4}$ and $\frac{1-4x}{4}$. For what value of x do these values represent the probability distribution of X?

$$\left[\textbf{Ans.:} -\frac{1}{3} \leq X \leq \frac{1}{4}\right]$$

7. Let X denote the number of heads in a single toss of 4 fair coins. Determine (i) $P(X < 2)$, and (ii) $P(1 < X \le 3)$.

$$\left[\textbf{Ans.: (i) } \frac{5}{16} \text{ (ii) } \frac{5}{8}\right]$$

8. If 3 cars are selected from a lot of 6 cars containing 2 defective cars, find the probability distribution of the number of defective cars.

Ans.:

X	0	1	2
$P(X = x)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{2}{5}$

9. Five defective bolts are accidentally mixed with 20 good ones. Find the probability distribution of the number of defective bolts, if four bolts are drawn at random from this lot.

Ans.:

X	0	1	2	3	4
$P(X = x)$	$\frac{969}{2530}$	$\frac{1140}{2530}$	$\frac{380}{2530}$	$\frac{40}{2530}$	$\frac{1}{2530}$

10. Two dice are rolled at once. Find the probability distribution of the sum of the numbers on them.

Ans.:

X	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

11. A random variable X takes three values 0, 1, and 2 with probabilities $\frac{1}{3}, \frac{1}{6}$, and $\frac{1}{2}$ respectively. Obtain the distribution function of X.

$$\left[\textbf{Ans.: } F(0) = \frac{1}{3}, F(1) = \frac{1}{2}, F(2) = 1\right]$$

12. A random variable X has the following probability function:

x	0	1	2	3	4
$P(X = x)$	k	$3k$	$5k$	$7k$	$9k$

Find (i) the value of k, (ii) $P(X < 3)$, $P(X \geq 3)$, $P(0 < X < 4)$, and (iii) distribution function of X.

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) } \frac{1}{25}, \text{ (ii) } \frac{9}{25}, \frac{16}{25}, \frac{3}{5} \\ \qquad \text{(iii) } F(0) = \frac{1}{25}, F(1) = \frac{4}{25}. F(2) = \frac{9}{25}, F(3) = \frac{16}{25}, F(4) = 1\end{array}\right]$$

13. A random variable X has the probability function

X	–2	–1	0	1	2	3
$P(X = x)$	0.1	k	0.2	$2k$	0.3	k

Find (i) k, (ii) $P(X \leq 1)$, (iii) $P(-2 < X < 1)$, and (iv) obtain the distribution function of X.

[**Ans.:** (i) 0.1 (ii) 0.6 (iii) 0.3]

14. The following is the distribution function $F(x)$ of a discrete random variable X:

X	–3	–2	–1	0	1	2	3
$P(X = x)$	0.08	0.2	0.4	0.65	0.8	0.9	1

Find (i) the probability distribution of X, (ii) $P(-2 \leq X \leq 1)$, and (iii) $P(X \geq 1)$.

Ans.: (i)

X	–3	–2	–1	0	1	2	3
$P(X = x)$	0.08	0.12	0.2	0.25	0.15	0.1	0.1

(ii) 0.72 (ii) 0.35

8.15 MEASURES OF CENTRAL TENDENCY FOR A DISCRETE PROBABILITY DISTRIBUTION

The behaviour of a random variable is completely characterized by the distribution function $F(x)$ or density function $p(x)$. Instead of a function, a more compact description can be made by single numbers such as mean, median, mode, variance, and standard deviation known as measures of central tendency of the random variable X.

1. Mean The mean or average value (μ) of the probability distribution of a discrete random variable X is called *expectation* and is denoted by $E(X)$.

$$\mu = E(X) = \sum_{i=1}^{\infty} x_i \ p(x_i) = \sum x \ p(x)$$

where $p(x)$ is the probability density function of the discrete random variable X. Expectation of any function $\phi(x)$ of a random variable X is given by

$$E[\phi(x)] = \sum_{i=1}^{\infty} \phi(x_i)\, p(x_i) = \sum \phi(x)\, p(x)$$

Some important results on expectation:

(i) $E(X + k) = E(X) + k$
(ii) $E(aX \pm b) = aE(X) \pm b$
(iii) $E(X + Y) = E(X) + E(Y)$ provided $E(X)$ and $E(Y)$ exists
(iv) $E(XY) = E(X)\, E(Y)$ if X and Y are two independent random variables

2. Variance Variance characterizes the variability in the distributions since two distributions with same mean can still have different dispersion of data about their means. Variance of the probability distribution of a discrete random variable X is given by

$$\begin{aligned}
\text{Var}(X) = \sigma^2 &= E(X - \mu)^2 \\
&= E(X^2 - 2X\mu + \mu^2) \\
&= E(X^2) - E(2X\mu) + E(\mu^2) \\
&= E(X^2) - 2\mu\, E(X) + \mu^2 \qquad [\because E(\text{constant}) = (\text{constant})] \\
&= E(X^2) - 2\mu\mu + \mu^2 \\
&= E(X^2) - \mu^2 \\
&= E(X^2) - [E(X)]^2
\end{aligned}$$

Some important results on variance:

(i) Var $(k) = 0$
(ii) Var $(kX) = k^2$ Var (X)
(iii) Var $(X + k) =$ Var (X)
(iv) Var $(aX + b) = a^2$ Var(X)

3. Standard Deviation Standard deviation is the positive square root of the variance.

$$\begin{aligned}
\text{SD} = \sigma &= \sqrt{\sum_{i=1}^{\infty} x_i^2\; p(x_i) - \mu^2} \\
&= \sqrt{E\,(X^2) - \mu^2} \\
&= \sqrt{E\,(X^2) - [E(X)]^2}
\end{aligned}$$

Example 1

A random variable X has the following distribution:

X	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Find (i) mean, (ii) variance, and (iii) $P(1 < X < 6)$.

Solution

(i)
$$\text{Mean} = \mu = \sum xp(x)$$
$$= 1\left(\frac{1}{36}\right) + 2\left(\frac{3}{36}\right) + 3\left(\frac{5}{36}\right) + 4\left(\frac{7}{36}\right) + 5\left(\frac{9}{36}\right) + 6\left(\frac{11}{36}\right)$$
$$= \frac{161}{36}$$
$$= 4.47$$

(ii)
$$\text{Variance} = \sigma^2 = \sum x^2 p(x) - \mu^2$$
$$= 1\left(\frac{1}{36}\right) + 4\left(\frac{3}{36}\right) + 9\left(\frac{5}{36}\right) + 16\left(\frac{7}{36}\right) + 25\left(\frac{9}{36}\right) + 36\left(\frac{11}{36}\right) - (4.47)^2$$
$$= \frac{791}{36} - 19.98$$
$$= 1.99$$

(iii)
$$P(1 < X < 6) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$$
$$= \frac{3}{36} + \frac{5}{36} + \frac{7}{36} + \frac{9}{36}$$
$$= \frac{24}{36}$$
$$= 0.67$$

Example 2

The probability distribution of a random variable X is given below. Find (i) $E(X)$, *(ii)* $\text{Var}(X)$, *(iii)* $E(2X-3)$, *and (iv)* $\text{Var}(2X-3)$

X	–2	–1	0	1	2
$P(X=x)$	0.2	0.1	0.3	0.3	0.1

Solution

(i)
$$\begin{aligned} E(X) &= \Sigma x\, p(x) \\ &= -2(0.2) - 1(0.1) + 0 + (0.3) + 2(0.1) \\ &= 0 \end{aligned}$$

(ii)
$$\begin{aligned} \text{Var}(X) &= \Sigma x^2\ p(x) - [E(X)]^2 \\ &= 4(0.2) + 1(0.1) + 0 + 1(0.3) + 4(0.1) - 0 \\ &= 1.6 \end{aligned}$$

(iii)
$$\begin{aligned} E(2X-3) &= 2E(X) - 3 \\ &= 2(0) - 3 \\ &= -3 \end{aligned}$$

(iv)
$$\begin{aligned} \text{Var}\ (2X-3) &= (2)^2\ \text{Var}\ (X) \\ &= 4(1.6) \\ &= 6.4 \end{aligned}$$

Example 3

The mean and standard deviation of a random variable X are 5 *and* 4 *respectively. Find* $E(X^2)$ *and standard deviation of* $(5-3X)$.

Solution

$$\begin{aligned} E(X) &= \mu = 5 \\ \text{SD} &= \sigma = 4 \\ \therefore \quad \text{Var}(X) &= \sigma^2 = 16 \\ \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ 16 &= E(X^2) - (5)^2 \\ \therefore \quad E(X^2) &= 41 \\ \text{Var}\ (5-3X) &= \text{Var}\ (5) - (-3)^2\ \text{Var}\ (X) \\ &= 0 + 9(16) \\ &= 144 \\ \text{SD}\ (5-3X) &= \sqrt{\text{Var}\ (5-3X)} \\ &= \sqrt{144} \\ &= 12 \end{aligned}$$

Example 4

A machine produces an average of 500 *items during the first week of the month and on average of* 400 *items during the last week of the month, the probability for these being* 0.68 *and* 0.32 *respectively. Determine the expected value of the production.*

[Summer 2015]

Solution

Let X be the random variable which denotes the items produced by the machine. The probability distribution is

X	500	400
$P(X = x)$	0.68	0.32

Expected value of the production $E(X) = \sum x\, p(x)$

$$= 500(0.68) + 400(0.32)$$
$$= 468$$

Example 5

The monthly demand for Allwyn watches is known to have the following probability distribution:

Demand (x)	1	2	3	4	5	6	7	8
Probability $p(x)$	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Find the expected demand for watches. Also, compute the variance.

Solution

$$\begin{aligned}
E(X) &= \Sigma\, x\, p(x)\\
&= 1(0.08) + 2(0.12) + 3(0.19) + 4(0.24) + 5(0.16)\\
&\qquad + 6(0.10) + 7(0.07) + 8(0.04)\\
&= 4.06\\
\text{Var}(X) &= E(X^2) - [E(X)]^2\\
&= \Sigma\, x^2\ p(x) - [E(X)]^2\\
&= 1(0.08) + 4(0.12) + 9(0.19) + 16(0.24) + 25(0.16)\\
&\qquad + 36(0.10) + 49(0.07) + 64(0.04) - (4.06)^2\\
&= 19.7 - 16.48\\
&= 3.21
\end{aligned}$$

Example 6

A discrete random variable has the probability mass function given below:

X	–2	–1	0	1	2	3
$P(X = x)$	0.2	k	0.1	$2k$	0.1	$2k$

Find k, mean, and variance.

Solution

Since $P(X = x)$ is a probability mass function,

$$\sum P(X = x) = 1$$

$$0.2 + k + 0.1 + 2k + 0.1 + 2k = 1$$

$$5k + 0.4 = 1$$

$$5k = 0.6$$

$$k = \frac{0.6}{5} = \frac{3}{25}$$

Hence, the probability distribution is

X	-2	-1	0	1	2	3
$P(X = x)$	$\frac{2}{10}$	$\frac{3}{25}$	$\frac{1}{10}$	$\frac{6}{25}$	$\frac{1}{10}$	$\frac{6}{25}$

$$\text{Mean} = E(X) = \sum x\, p(x)$$

$$= (-2)\left(\frac{2}{10}\right) + (-1)\left(\frac{3}{25}\right) + 0 + 1\left(\frac{6}{25}\right) + 2\left(\frac{1}{10}\right) + 3\left(\frac{6}{25}\right)$$

$$= \frac{6}{25}$$

$$\text{Variance} = \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \sum x^2\, p(x) - [E(X)]^2$$

$$= 4\left(\frac{2}{10}\right) + 1\left(\frac{3}{25}\right) + 0 + 1\left(\frac{6}{25}\right) + 4\left(\frac{1}{10}\right) + 9\left(\frac{6}{25}\right) - \left(\frac{6}{25}\right)^2$$

$$= \frac{73}{250} - \frac{36}{625}$$

$$= \frac{293}{625}$$

Example 7

A random variable X has the following probability function:

x	0	1	2	3	4	5	6	7
$p(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Determine k. (ii) Evaluate $P(X < 6)$, $P(X \geq 6)$, $P(0 < X < 5)$, and $P(0 \leq X \leq 4)$. (iii) Determine the distribution function of X. (iv) Find the mean. (v) Find the variance.

Solution

(i) Since $p(x)$ is a probability mass function,

$$\sum p(x) = 1$$

$$0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$$

$$10k^2+9k-1=1$$

$$(10k-1)(k+1)=0$$

$$k=\frac{1}{10} \text{ or } k=-1$$

$$k=\frac{1}{10}=0.1 \left[\because \quad p(x)\geq 0, k\neq -1\right]$$

Hence, the probability function is

X	0	1	2	3	4	5	6	7
$P(X=x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

(ii)
$$\begin{aligned}
P(X<6) &= P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)\\
&= 0+0.1+0.2+0.2+0.3+0.01\\
&= 0.81\\
P(X\geq 6) &= 1-P(X<6)\\
&= 1-0.81\\
&= 0.19\\
P(0<X<5) &= P(X=1)+P(X=2)+P(X=3)+P(X=4)\\
&= 0.1+0.2+0.2+0.3\\
&= 0.8\\
P(0\leq X\leq 4) &= P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)\\
&= 0+0.1+0.2+0.2+0.3\\
&= 0.8
\end{aligned}$$

(iii) Distribution function of X

x	$p(x)$	$F(x)$
0	0	0
1	0.1	0.1
2	0.2	0.3
3	0.2	0.5
4	0.3	0.8
5	0.01	0.81
6	0.02	0.83
7	0.17	1

(iv) $\mu = \sum xp(x)$

$$= 0 + 1(0.1) + 2(0.2) + 3(0.2) + 4(0.3) + 5(0.01) + 6(0.02) + 7(0.17)$$
$$= 3.66$$

(v) $\text{Var}(X) = \sigma^2 = \sum x^2\, p(x) - \mu^2$

$$= 0 + 1(0.1) + 4(0.2) + 9(0.2) + 16(0.3) + 25(0.01) + 36(0.02) + 49(0.17) - (3.66)^2$$
$$= 3.4044$$

Example 8

A fair die is tossed. Let the random variable X denote twice the number appearing on the die. Write the probability distribution of X. Calculate mean and variance.

Solution

Let X be the random variable which denotes twice the number appearing on the die.

(i) Probability distribution of X

x	2	4	6	8	10	12
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

(ii) Mean $= \mu = \sum x\, p(x)$

$$= 2\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) + 8\left(\frac{1}{6}\right) + 10\left(\frac{1}{6}\right) + 12\left(\frac{1}{6}\right)$$
$$= 7$$

(iii) Variance $= \sigma^2 = \sum x^2\, p(x) - \mu^2$

$$= 4\left(\frac{1}{6}\right) + 16\left(\frac{1}{6}\right) + 36\left(\frac{1}{6}\right) + 64\left(\frac{1}{6}\right) + 100\left(\frac{1}{6}\right) + 144\left(\frac{1}{6}\right) - (7)^2$$
$$= 11.67$$

Example 9

Two unbiased dice are thrown at random. Find the probability distribution of the sum of the numbers on them. Also, find the mean and variance.

Solution

Let X be the random variable which denotes the sum of the numbers on two unbiased dice. The random variable X can take values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

The probability distribution is

X	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\begin{aligned}
\text{Mean} = \mu = \sum x\, p(x) \\
&= 2\left(\frac{1}{36}\right)+3\left(\frac{2}{36}\right)+4\left(\frac{3}{36}\right)+5\left(\frac{4}{36}\right)+6\left(\frac{5}{36}\right)+7\left(\frac{6}{36}\right)+8\left(\frac{5}{36}\right) \\
&\quad +9\left(\frac{4}{36}\right)+10\left(\frac{3}{36}\right)+11\left(\frac{2}{36}\right)+12\left(\frac{1}{36}\right) \\
&= \frac{252}{36} \\
&= 7
\end{aligned}$$

$$\begin{aligned}
\text{Variance} = \sigma^2 &= \sum x^2\, p(x) - \mu^2 \\
&= 4\left(\frac{1}{36}\right)+9\left(\frac{2}{36}\right)+16\left(\frac{3}{36}\right)+25\left(\frac{4}{36}\right)+36\left(\frac{5}{36}\right) \\
&\quad +49\left(\frac{6}{36}\right)+64\left(\frac{5}{36}\right)+81\left(\frac{4}{36}\right)+100\left(\frac{3}{36}\right) \\
&\quad +121\left(\frac{2}{36}\right)+144\left(\frac{1}{36}\right)-(7)^2 \\
&= \frac{1974}{36}-49 \\
&= 5.83
\end{aligned}$$

Example 10

A sample of 3 items is selected at random from a box containing 10 items of which 4 are defective. Find the expected number of defective items.

Solution

Let X be the random variable which denotes the defective items.

Total number of items = 10

Number of good items = 6

Number of defective items = 4

$$P(X=0) = P(\text{no defective item}) = \frac{{}^6C_3}{{}^{10}C_3} = \frac{1}{6}$$

$$P(X=1) = P(\text{one defective item}) = \frac{{}^6C_2\ {}^4C_1}{{}^{10}C_3} = \frac{1}{2}$$

$$P(X=2) = P(\text{two defective items}) = \frac{{}^6C_1\ {}^4C_2}{{}^{10}C_3} = \frac{3}{10}$$

$$P(X=3) = P(\text{three defective items}) = \frac{{}^4C_3}{{}^{10}C_3} = \frac{1}{30}$$

Hence, the probability distribution is

X	0	1	2	3
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

Expected number of defective items $= E(X) = \sum x\, p(x)$

$$= 0 + 1\left(\frac{1}{2}\right) + 2\left(\frac{3}{10}\right) + 3\left(\frac{1}{30}\right)$$

$$= 1.2$$

Example 11

A player tosses two fair coins. He wins ₹ 100 if a head appears and ₹ 200 if two heads appear. On the other hand, he loses ₹ 500 if no head appears. Determine the expected value of the game. Is the game favourable to the players?

Solution

Let X be the random variable which denotes the number of heads appearing in tosses of two fair coins.

$$S = \{\text{HH, HT, TH, TT}\}$$

$$p(x_1) = P(X=0) = P(\text{no heads}) = \frac{1}{4}$$

$$p(x_2) = P(X=1) = P(\text{one head}) = \frac{2}{4} = \frac{1}{2}$$

$$p(x_3) = P(X=2) = P(\text{two heads}) = \frac{1}{4}$$

Amount to be lost if no head appears $= x_1 = -$ ₹ 500

Amount to be won if one head appears = x_2 = ₹ 100
Amount to be won if two heads appear = x_3 = ₹ 200

$$\text{Expected value of the game} = \mu = \sum x\,p(x)$$
$$= x_1\,p(x_1) + x_2\,p(x_2) + x_3\,p(x_3)$$
$$= -500\left(\frac{1}{4}\right) + 100\left(\frac{1}{2}\right) + 200\left(\frac{1}{4}\right)$$
$$= ₹\ -25$$

Hence, the game is not favourable to the player.

Example 12

Amit plays a game of tossing a die. If a number less than 3 appears, he gets ₹ a, otherwise he has to pay ₹ 10. If the game is fair, find a.

Solution

Let X be the random variable which denotes tossing of a die.

Probability of getting a number less than 3, i.e., 1 or 2 = $p(x_1) = \frac{2}{6} = \frac{1}{3}$

Probability of getting number more than or equal to 3, i.e., 3, 4, 5, or 6 $= p(x_2) = \frac{4}{6} = \frac{2}{3}$

Amount to be received for number less than 3 = x_1 = ₹ a
Amount to be paid for numbers more than or equal to 3 = x_2 = ₹ –10

$$E(X) = \sum x\,p(x)$$
$$= x_1\,p(x_1) + x_2\,p(x_2)$$
$$= a\left(\frac{1}{3}\right) + (-10)\left(\frac{2}{3}\right)$$
$$= \frac{a}{3} - \frac{20}{3}$$

For a pair game, $E(x) = 0$.

$$\frac{a}{3} - \frac{20}{3} = 0$$
$$a = 20$$

Example 13

A man draws 2 balls from a bag containing 3 white and 5 black balls. If he is to receive ₹ 14 for every white ball which he draws and ₹ 7 for every black ball, what is his expectation?

Solution

Let X be the random variable which denotes the balls drawn from a bag. 2 balls drawn may be either (i) both white, or (ii) both black, or (iii) one white and one black.

Probability of drawing 2 white balls = $p(x_1) = \dfrac{{}^3C_2}{{}^8C_2} = \dfrac{3}{28}$

Probability of drawing 2 black balls = $p(x_2) = \dfrac{{}^5C_2}{{}^8C_2} = \dfrac{10}{28}$

Probability of drawing 1 white and 1 black ball = $p(x_3) = \dfrac{{}^3C_1\ {}^5C_1}{{}^8C_2} = \dfrac{15}{28}$

Amount to be received for 2 white balls = x_1 = ₹ 14 × 2 = ₹ 28

Amount to be received for 2 black balls = x_2 = ₹ 7 × 2 = ₹ 14

Amount to be received for 1 white and 1 black ball = x_3 = ₹ 14 + ₹ 7 = ₹ 21

$$
\begin{aligned}
\text{Expectation} = E(X) &= \sum x\, p(x) \\
&= x_1\, p(x_1) + x_2\, p(x_2) + x_3\, p(x_3) \\
&= 28\left(\frac{3}{28}\right) + 14\left(\frac{10}{28}\right) + 21\left(\frac{15}{28}\right) \\
&= ₹\ 19.25
\end{aligned}
$$

Example 14

The probability that there is at least one error in an account statement prepared by A is 0.2 and for B and C, they are 0.25 and 0.4 respectively. A, B, and C prepared 10, 16, and 20 statements respectively. Find the expected number of correct statements in all.

Solution

Let $p(x_1)$, $p(x_2)$ and $p(x_3)$ be the probabilities of the events that there is no error in the account statements prepared by A, B, and C respectively.

$$
\begin{aligned}
p(x_1) &= 1 - (\text{Probability of at least one error in the account} \\
&\qquad \text{statement prepared by } A) \\
&= 1 - 0.2 \\
&= 0.8
\end{aligned}
$$

Similarly, $p(x_2) = 1 - 0.25 = 0.75$

$p(x_3) = 1 - 0.4 = 0.6$

Also, $x_1 = 10, \quad x_2 = 16, \quad x_3 = 20$

Expected number of correct statements $= E(X) = \sum x\, p(x)$

$$= x_1\, p(x_1) + x_2\, p(x_2) + x_3\, p(x_3)$$
$$= 10(0.8) + 16(0.75) + 20(0.6)$$
$$= 32$$

Example 15

A man has the choice of running either a hot-snack stall or an ice-cream stall at a seaside resort during the summer season. If it is a fairly cool summer, he should make ₹ 5000 *by running the hot-snack stall, but if the summer is quite hot, he can only expect to make* ₹ 1000. *On the other hand, if he operates the ice-cream stall, his profit is estimated at* ₹ 6500, *if the summer is hot, but only* ₹ 1000 *if it is cool. There is a* 40 *percent chance of the summer being hot. Should he opt for running the hot-snack stall or the ice-cream stall?*

Solution

Let X and Y be the random variables which denote the income from the hot-snack and ice-cream stalls respectively.

Probability of hot summer $= p_1 = 40\% = 0.4$

Probability of cool summer $= p_2 = 1 - p_1 = 1 - 0.4 = 0.6$

$$x_1 = 1000, \quad x_2 = 5000, \quad y_1 = 6500, \quad y_2 = 1000$$

Expected income from hot-snack stall $= E(X)$

$$= x_1\, p_1 + x_2\, p_2$$
$$= 1000(0.4) + 5000(0.6)$$
$$= ₹\ 3400$$

Expected income from ice-cream stall $= E(Y)$

$$= y_1\, p_1 + y_2\, p_2$$
$$= 6500\,(0.4) + 1000(0.6)$$
$$= ₹\ 3200$$

Hence, he should opt for running the hot-snack stall.

EXERCISE 8.8

1. The probability distribution of a random variable X is given by

X	-2	-1	0	1	2	3
$P(X = x)$	0.1	k	0.2	$2k$	0.3	k

Find k, the mean, and variance.

[**Ans.:** 0.1, 0.8, 2.16]

2. Find the mean and variance of the following distribution:

X	4	5	6	8
$P(X = x)$	0.1	0.3	0.4	0.2

[**Ans.:** 5.9, 1.49]

3. Find the value of k from the following data:

X	0	10	15
$P(X = x)$	$\frac{k-6}{5}$	$\frac{2}{k}$	$\frac{14}{5k}$

Also, find the distribution function and expectation of X.

[**Ans.:** 8,

X	0	10	15
$F(X)$	$\frac{2}{5}$	$\frac{13}{20}$	1

, $\frac{31}{4}$]

4. For the following distribution,

X	-3	-2	-1	0	1	2
$P(X = x)$	0.01	0.1	0.2	0.3	0.2	0.15

find (i) $P(X \geq 1)$, (ii) $P(X < 0)$, (iii) $E(X)$, and (iv) Var(X)

[**Ans.:** (i) 0.35 (ii) 0.35 (iii) 0.05 (iv) 1.8475]

5. A random variable X has the following probability function:

X	0	1	2	3	4	5	6	7	8
$P(X = x)$	$\frac{k}{45}$	$\frac{k}{15}$	$\frac{k}{9}$	$\frac{k}{5}$	$\frac{2k}{45}$	$\frac{6k}{45}$	$\frac{7k}{45}$	$\frac{8k}{45}$	$\frac{4k}{45}$

Determine (i) k, (ii) mean, (iii) variance, and (iv) SD.

[**Ans.:** (i) 1 (ii) 0.4622 (iii) 4.9971 (iv) 2.24]

6. A fair coin is tossed until a head or five tails appear. Find (i) discrete probability distribution, and (ii) mean of the distribution.

Ans.: (i)

X	1	2	3	4	5
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

(ii) 1.9

7. Let X denote the minimum of two numbers that appear when a pair of fair dice is thrown once. Determine (i) probability distribution, (ii) expectation, and (iii) variance.

Ans.: (i)

X	1	2	3	4	5	6
$P(X = x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

(ii) 2.5278 (iii) 1.9713

8. For the following probability distribution,

X	–3	–2	–1	0	1	2	3
$P(X = x)$	0.001	0.01	0.1	?	0.1	0.01	0.001

find (i) missing probability, (ii) mean, and (iii) variance.

[**Ans.:** (i) 0.778 (ii) 0.2 (iii) 0.258]

9. A discrete random variable can take all integer values from 1 to k each with the probability of $\frac{1}{k}$. Show that its mean and variance are $\frac{k+1}{2}$ and $\frac{k^2+1}{2}$ respectively.

10. An urn contains 6 white and 4 black balls; 3 balls are drawn without replacement. What is the expected number of black balls that will be obtained?

[**Ans.:** $\frac{6}{5}$]

11. A six-faced die is tossed. If a prime number occurs, Anil wins that number of rupees but if a non-prime number occurs, he loses that number of rupees. Determine whether the game is favourable to the player.

[**Ans.:** The game is favourable to Anil.]

12. A man runs an ice-cream parlour at a holiday resort. If the summer is mild, he can sell 2500 cups of ice cream; if it is hot, he can sell 4000 cups; if it is very hot, he can sell 5000 cups. It is known that for any year, the probability of summer to be mild is $\frac{1}{7}$ and to be hot is $\frac{4}{7}$. A cup of ice cream costs ₹ 2 and is sold for ₹ 3.50. What is his expected profit?

[**Ans.:** ₹ 6107.14]

13. A player tosses two fair coins. He wins ₹ 1 or ₹ 2 as 1 tail or 1 head appears. On the other hand, he loses ₹ 5 if no head appears. Find the expected gain or loss of the player.

[**Ans.:** Loss of ₹ 0.25]

14. A bag contains 2 white balls and 3 black balls. Four persons *A*, *B*, *C*, *D* in the order named each draws one ball and does not replace it. The first to draw a white ball receives ₹ 20. Determine their expectations.

[**Ans.:** ₹ 8, ₹ 6, ₹ 4, ₹ 2]

Points to Remember

Arithmetic Mean

The *arithmetic mean* of a set of observations is their sum divided by the number of observations. If $x_1, x_2, \ldots, x_n$ be n observations then their average or arithmetic mean is given by

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum x}{n}$$

If n observations consist of n distinct values denoted by $x_1, x_2, \ldots, x_n$ of the observed variable x occurring with frequencies $f_1, f_2, \ldots, f_n$ respectively then the arithmetic mean is given by

$$\bar{x} = \frac{f_1 x_1 + f_2 x_2 + \cdots + f_n x_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_i x_i}{N} = \frac{\sum fx}{N}$$

1. Arithmetic Mean of Grouped Data

In case of a grouped or continuous frequency distribution, the arithmetic mean is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{\sum fx}{N}, \text{ where } N = \sum_{i=1}^{n} f_i$$

and x is taken as the mid-value of the corresponding class.

2. Arithmetic Mean from Assumed Mean

$$\bar{x} = a + \frac{\sum fd}{N}$$

3. Arithmetic Mean by the Step-Deviation Method

$$\bar{x} = a + h\frac{\sum fd}{N}$$

4. Weighted Arithmetic Mean

$$\text{Weighted arithmetic mean} = \frac{w_1 x_1 + w_2 x_2 + \cdots + w_n x_n}{w_1 + w_2 + \cdots + w_n}$$

$$\bar{x}_w = \frac{\sum wx}{\sum w}$$

When the assumed mean is used for calculation,

$$\bar{x}_w = a + \frac{\sum wd}{\sum w}$$

When the step-deviation method is used for calculation,

$$\bar{x}_w = a + h\frac{\sum wd}{\sum w}$$

Combined Arithmetic Mean

If $\bar{x}_1, \bar{x}_2, ..., \bar{x}_k$ are the means of k series of sizes $n_1, n_2, ..., n_k$ respectively then the mean $\bar{x}$ of the composite series is given by

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\,\bar{x}_2 + \cdots + n_k\,\bar{x}_k}{n_1 + n_2 + \cdots + n_k}$$

$$= \frac{\sum_{i=1}^{k} n_i x_i}{\sum_{i=1}^{k} n_i}$$

Median

Median is the central value of the variable when the values are arranged in ascending or descending order of magnitude.

In case of ungrouped data, if the number of observations is odd then the median is the middle value after the values have been arranged in ascending or descending order of magnitude. If the number of observations is even, there are two middle terms and the median is obtained by taking the arithmetic mean of the middle terms.

In case of discrete frequency distribution, the median is obtained by considering the cumulative frequencies. The steps for calculating the median are given below:

(i) Arrange the values of the variables in ascending or descending order of magnitudes.

(ii) Find $\frac{N}{2}$, where $N = \sum f$

(iii) Find the cumulative frequency just greater than $\frac{N}{2}$ and determine the corresponding value of the variable.

(iv) The corresponding value of x is the median.

Median for Continuous Frequency Distribution

In case of a continuous frequency distribution (less than frequency distribution), the class corresponding to the cumulative frequency just greater than $\frac{N}{2}$, is called the *median class*, and the value of the median is given by

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

In case of 'more than' or 'greater than' type of frequency distributions, the value of the median is given by

$$\text{Median} = u - \frac{h}{f}\left(\frac{N}{2} - c\right)$$

where u is the upper limit of the median class
f is the frequency of the median class
h is the width of the median class
c is the cumulative frequency of the class succeeding the median class

Mode

Mode is the value which occurs most frequently in a set of observations and around which the other items of the set are heavily distributed.

Mode for a Continuous Frequency Distribution

In case of a continuous frequency distribution, the class in which the mode lies is called the *modal class* and the value of the mode is given by

$$\text{Mode} = l + h\left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right)$$

where l is the lower limit of the modal class
h is the width of the modal class
f_m is the frequency of the modal class
f_1 is the frequency of the class preceding the modal class
f_2 is the frequency of the class succeeding the modal class

Harmonic Mean

The *harmonic mean* of a number of observations, none of which is zero, is the reciprocal of the arithmetic mean of the reciprocals of the given values.

$$\text{HM} = \frac{1}{\frac{1}{n}\sum\left(\frac{1}{x}\right)}$$

$$= \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}$$

In case of a frequency distribution consisting of n observations x_1, x_2, ..., x_n with respective frequencies $f_1, f_2, ..., f_n$, the harmonic mean is given by

$$\text{HM} = \frac{f_1+f_2+\cdots+f_n}{\frac{f_1}{x_1}+\frac{f_2}{x_2}+\cdots+\frac{f_n}{x_n}}$$

$$= \frac{\sum f}{\sum\left(\frac{f}{x}\right)}$$

If x_1, x_2, ..., x_n are n observations with weights w_1, w_2, ..., w_n respectively, their weighted harmonic mean is given by

$$\text{HM} = \frac{\sum w}{\sum\left(\frac{w}{x}\right)}$$

Relation between Arithmetic Mean, Geometric Mean, and Harmonic Mean

$$\text{AM} \ge \text{GM} \ge \text{HM}$$

For two observations x_1 and x_2 of a series,

$$\text{GM} = \sqrt{\text{AM}\cdot\text{HM}}$$

Standard Deviation

Standard deviation is the positive square root of the arithmetic mean of the squares of the deviations of the given values from their arithmetic mean.

$$\sigma = \sqrt{\frac{\sum (x - \overline{x})^2}{n}}$$

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

In case of a frequency distribution consisting of n observations $x_1, x_2, ..., x_n$ with respective frequencies $f_1, f_2, ..., f_n$, the standard deviation is given by

$$\sigma = \sqrt{\frac{\sum fx^2}{N} - \left(\frac{\sum fx}{N}\right)^2}$$

1. Standard Deviation from the Assumed Mean

$$\sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

2. Standard Deviation by Step-Deviation Method

$$\sigma = h\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

3. Variance

The *variance* is the square of the standard deviation and is denoted by σ^2. The method for calculating variance is same as that given for the standard deviation.

4. Coefficient of Variation

The *standard deviation* is an absolute measure of dispersion. The coefficient of variation is a relative measure of dispersion and is denoted by CV.

$$\text{CV} = \frac{\sigma}{\overline{x}} \times 100$$

Random Variables

A random variable X is a real-valued function of the elements of the sample space of a random experiment. In other words, a variable which takes the real values, depending on the outcome of a random experiment is called a *random variable*,

1. Discrete Random Variables A random variable X is said to be discrete if it takes either finite or countably infinite values.

2. Continuous Random Variables A random variable X is said to be continuous if it takes any values in a given interval.

Discrete Probability Distribution

Probability distribution of a random variable is the set of its possible values together with their respective probabilities.

Discrete Distribution Function

$$F(x) = P(X \le x) = \sum_{i=1}^{x} p(x_i)$$

Measures of Central Tendency for Discrete Probability Distribution

1. Mean

$$\mu = E(X) = \sum_{i=1}^{\infty} x_i \; p(x_i) = \sum x \; p(x)$$

2. Variance

$$\begin{aligned} \text{Var}(X) = \sigma^2 &= E(X-\mu)^2 \\ &= E(X^2) - [E(X)]^2 \end{aligned}$$

3. Standard Deviation

$$\begin{aligned} \text{SD} = \sigma &= \sqrt{\sum_{i=1}^{\infty} x_i^2 \; p(x_i) - \mu^2} \\ &= \sqrt{E\,(X^2) - \mu^2} \\ &= \sqrt{E\,(X^2) - [E(X)]^2} \end{aligned}$$

CHAPTER 9

Correlation and Regression

Chapter Outline

9.1 INTRODUCTION

Correlation and regression are the most commonly used techniques for investigating the relationship between two quantitative variables. *Correlation* refers to the relationship of two or more variables. It measures the closeness of the relationship between the variables. *Regression* establishes a functional relationship between the variables. In correlation, both the variables x and y are random variables, whereas in regression, x is a random variable and y is a fixed variable. The coefficient of correlation is a relative measure whereas the regression coefficient is an absolute figure.

9.2 CORRELATION

Correlation is the relationship that exists between two or more variables. Two variables are said to be correlated if a change in one variable affects a change in the other variable. Such a data connecting two variables is called *bivariate data*. Thus, correlation is a statistical analysis which measures and analyses the degree or extent to which two variables fluctuate with reference to each other. Some examples of such a relationship are as follows:

1. Relationship between heights and weights.
2. Relationship between price and demand of commodity.
3. Relationship between rainfall and yield of crops.
4. Relationship between age of husband and age of wife.

9.3 TYPES OF CORRELATIONS

Correlation is classified into four types:

1. Positive and negative correlations
2. Simple and multiple correlations
3. Partial and total correlations
4. Linear and nonlinear correlations

9.3.1 Positive and Negative Correlations

Depending on the variation in the variables, correlation may be positive or negative.

1. Positive Correlation If both the variables vary in the same direction, the correlation is said to be positive. In other words, if the value of one variable increases, the value of the other variable also increases, or, if value of one variable decreases, the value of the other variable decreases, e.g., the correlation between heights and weights of group of persons is a positive correlation.

Height (cm)	150	152	155	160	162	165
Weight (kg)	60	62	64	65	67	69

2. Negative Correlation If both the variables vary in the opposite direction, correlation is said to be negative. In other words, if the value of one variable increases, the value of the other variable decreases, or, if the value of one variable decreases, the value of the other variable increases, e.g., the correlation between the price and demand of a commodity is a negative correlation.

Price (₹ per unit)	10	8	6	5	4	1
Demand (units)	100	200	300	400	500	600

9.3.2 Simple and Multiple Correlations

Depending upon the study of the number of variables, correlation may be simple or multiple.

1. Simple Correlation When only two variables are studied, the relationship is described as simple correlation, e.g., the quantity of money and price level, demand and price, etc.

2. Multiple Correlation When more than two variables are studied, the relationship is described as multiple correlation, e.g., relationship of price, demand, and supply of a commodity.

9.3.3 Partial and Total Correlations

Multiple correlation may be either partial or total.

1. Partial Correlation When more than two variables are studied excluding some other variables, the relationship is termed as partial correlation.

2. Total Correlation When more than two variables are studied without excluding any variables, the relationship is termed total correlation.

9.3.4 Linear and Nonlinear Correlations

Depending upon the ratio of change between two variables, the correlation may be linear or nonlinear.

1. Linear Correlation If the ratio of change between two variables is constant, the correlation is said to be linear. If such variables are plotted on a graph paper, a straight line is obtained, e.g.,

Milk (*l*)	5	10	15	20	25	30
Curg (kg)	2	4	6	8	10	12

2. Nonlinear Correlation If the ratio of change between two variables is not constant, the correlation is said to nonlinear. The graph of a nonlinear or curvilinear relationship will be a curve, e.g.,

Advertising expenses (₹ in lacs)	3	6	9	12	15
Sales (₹ in lacs)	10	12	15	15	16

9.4 METHODS OF STUDYING CORRELATION

There are two different methods of studying correlation, (1) Graphic methods (2) Mathematical methods.

Graphic methods are (a) scatter diagram, and (b) simple graph.

Mathematical methods are (a) Karl Pearson's coefficient of correlation, and (b) Spearman's rank coefficient of correlation.

9.5 SCATTER DIAGRAM

The scatter diagram is a diagrammatic representation of bivariate data to find the correlation between two variables. There are various correlationships between two variables represented by the following scatter diagrams.

1. Perfect Positive Correlation If all the plotted points lie on a straight line rising from the lower left-hand corner to the upper right-hand corner, the correlation is said to be perfectly positive (Fig. 9.1).

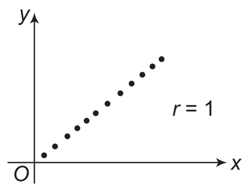

Fig. 9.1

2. Perfect Negative Correlation If all the plotted points lie on a straight line falling from the upper-left hand corner to the lower right-hand corner, the correlation is said to be perfectly negative (Fig. 9.2).

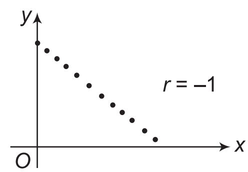

Fig. 9.2

3. High Degree of Positive Correlation If all the plotted points lie in the narrow strip, rising from the lower left-hand corner to the upper right-hand corner, it indicates a high degree of positive correlation (Fig. 9.3).

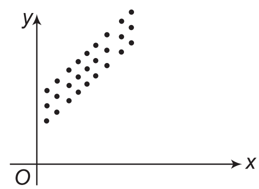

Fig. 9.3

4. High Degree of Negative Correlation If all the plotted points lie in a narrow strip, falling from the upper left-hand corner to the lower right-hand corner, it indicates the existence of a high degree of negative correlation (Fig. 9.4).

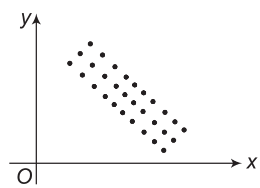

Fig. 9.4

5. No Correlation If all the plotted points lie on a straight line parallel to the *x*-axis or *y*-axis or in a haphazard manner, it indicates the absence of any relationship between the variables (Fig. 9.5).

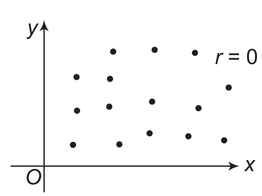

Fig. 9.5

Merits of a Scatter Diagram

1. It is simple and nonmathematical method to find out the correlation between the variables.

2. It gives an indication of the degree of linear correlation between the variables.
3. It is easy to understand.
4. It is not influenced by the size of extreme items.

9.6 SIMPLE GRAPH

A *simple graph* is a diagrammatic representation of bivariate data to find the correlation between two variables. The values of the two variables are plotted on a graph paper. Two curves are obtained, one for the variable x and the other for the variable y. If both the curves move in the same direction, the correlation is said to be positive. If both the curves move in the opposite direction, the correlation is said to be negative. This method is used in the case of a time series. It does not reveal the extent to which the variables are related.

9.7 KARL PEARSON'S COEFFICIENT OF CORRELATION

The coefficient of correlation is the measure of correlation between two random variables X and Y, and is denoted by r.

$$r = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}$$

where cov (X, Y) is covariance of variables X and Y,

σ_X is the standard deviation of variable X,

and σ_Y is the standard deviation of variable Y.

This expression is known as Karl Pearson's coefficient of correlation or Karl Pearson's product-moment coefficient of correlation.

$$\text{cov}(X,Y) = \frac{1}{n}\sum (x-\overline{x})\,(y-\overline{y})$$

$$\sigma_X = \sqrt{\frac{\sum (x-\overline{x})^2}{n}}$$

$$\sigma_Y = \sqrt{\frac{\sum (y-\overline{y})^2}{n}}$$

$$\therefore \qquad r = \frac{\sum (x-\overline{x})\,(y-\overline{y})}{\sqrt{\sum (x-\overline{x})^2}\sqrt{\sum (y-\overline{y})^2}}$$

The above expression can be further modified.

Expanding the terms,

$$r = \frac{\sum (xy - x\bar{y} - \bar{x}y + \bar{x}\,\bar{y})}{\sqrt{\sum \left(x^2 - 2x\bar{x} + \bar{x}^2\right)}\sqrt{\sum \left(y^2 - 2y\bar{y} + \bar{y}^2\right)}}$$

$$= \frac{\sum xy - \bar{y}\sum x - \bar{x}\sum y + \bar{x}\,\bar{y}\sum 1}{\sqrt{\sum x^2 - 2\bar{x}\sum x + \bar{x}^2 \sum 1}\ \sqrt{\sum y^2 - 2\bar{y}\sum y + \bar{y}^2 \sum 1}}$$

$$= \frac{\sum xy - \frac{\sum y}{n}\sum x - \frac{\sum x}{n}\sum y + \frac{\sum x}{n}\,\frac{\sum y}{n}\cdot n}{\sqrt{\sum x^2 - 2\frac{\sum x}{n}\sum x + \left(\frac{\sum x}{n}\right)^2 n}\ \sqrt{\sum y^2 - 2\frac{\sum y}{n}\sum y + \left(\frac{\sum y}{n}\right)^2 n}}$$

$$= \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}\ \sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}}$$

9.8 PROPERTIES OF COEFFICIENT OF CORRELATION

1. The coefficient of correlation lies between −1 and 1, i.e., $-1 \le r \le 1$.

Proof Let $\bar{x}$ and $\bar{y}$ be the mean of x and y series and σ_x and σ_y be their respective standard deviations.

Let $\sum\left(\frac{x-\bar{x}}{\sigma_x} \pm \frac{y-\bar{y}}{\sigma_y}\right)^2 \ge 0$ $\left[\because \begin{array}{l}\text{sum of squares of real quantities}\\ \text{cannot be negative}\end{array}\right]$

$$\frac{\sum (x-\bar{x})^2}{\sigma_x^2} + \frac{\sum (y-\bar{y})^2}{\sigma_y^2} \pm \frac{2\sum (x-\bar{x})(y-\bar{y})}{\sigma_x \sigma_y} \ge 0$$

$$n + n \pm 2nr \ge 0$$

$$2n \pm 2nr \ge 0$$

$$2n(1 \pm r) \ge 0$$

$$1 \pm r \ge 0$$

i.e., $1 + r \ge 0$ or $1 - r \ge 0$

$r \ge -1$ or $r \le 1$

Hence, the coefficient of correlation lies between –1 and 1, i.e., $-1 \le r \le 1$.

2. Correlation coefficient is independent of change of origin and change of scale.

Proof Let $d_x = \dfrac{x-a}{h}, \quad d_y = \dfrac{y-b}{k}$

$$x = a + hd_x, \quad y = b + kd_y$$

where a, b, h (>0) and k(>0) are constants.

$$x = a + hd_x \Rightarrow \bar{x} = a + h\bar{d}_x \Rightarrow x - \bar{x} = h(d_x - \bar{d}_x)$$
$$y = b + kd_y \Rightarrow \bar{y} = b + h\,\bar{d}_y \Rightarrow y - \bar{y} = k(d_y - \bar{d}_y)$$

$$\begin{aligned}
r_{xy} &= \frac{\sum (x-\bar{x})\,(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2}\,\sqrt{\sum (y-\bar{y})^2}} \\
&= \frac{\sum h(d_x - \bar{d}_x)\,k(d_y - \bar{d}_y)}{\sqrt{\sum h^2 (d_x - \bar{d}_x)^2}\,\sqrt{\sum k^2 (d_y - \bar{d}_y)^2}} \\
&= \frac{\sum (d_x - \bar{d}_x)\,(d_y - \bar{d}_y)}{\sqrt{\sum (d_x - \bar{d}_x)^2}\,\sqrt{(d_y - \bar{d}_y)^2}} \\
&= r_{d_x d_y}
\end{aligned}$$

Hence, the correlation coefficient is independent of change of origin and change of scale.

Note Since correlation coefficient is independent of change of origin and change of scale,

$$r = \frac{\sum d_x d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \dfrac{\left(\sum d_x\right)^2}{n}}\,\sqrt{\sum d_y^2 - \dfrac{\left(\sum d_y\right)^2}{n}}}$$

3. Two independent variables are uncorrelated.

Proof If random variables X and Y are independent,

$$\sum (x-\bar{x})\,(y-\bar{y}) = 0 \quad \text{or} \quad \operatorname{cov}(X, Y) = 0$$

$$\therefore \qquad r = 0$$

Thus, if X and Y are independent variables, they are uncorrelated.

Note The converse of the above property is not true, i.e., two uncorrelated variables may not be independent.

Example 1

Calculate the correlation coefficient between x and y using the following data:

x	2	4	5	6	8	11
y	18	12	10	8	7	5

Solution

$n = 6$

x	y	x^2	y^2	xy
2	18	4	324	36
4	12	16	144	48
5	10	25	100	50
6	8	36	64	48
8	7	64	49	56
11	5	121	25	55
$\sum x = 36$	$\sum y = 60$	$\sum x^2 = 266$	$\sum y^2 = 706$	$\sum xy = 293$

$$r = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}\sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}}$$

$$= \frac{293 - \frac{(36)(60)}{6}}{\sqrt{266 - \frac{(36)^2}{6}}\sqrt{706 - \frac{(60)^2}{6}}}$$

$$= -0.9203$$

Example 2

Calculate the coefficient of correlation from the following data:

x	12	9	8	10	11	13	7
y	14	8	6	9	11	12	3

Solution

$n = 7$

x	y	x^2	y^2	xy
12	14	144	196	168
9	8	81	64	72
8	6	64	36	48
10	9	100	81	90
11	11	121	121	121
13	12	169	144	156
7	3	49	9	21
$\sum x = 70$	$\sum y = 63$	$\sum x^2 = 728$	$\sum y^2 = 651$	$\sum xy = 676$

$$r = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}\sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}}$$

$$= \frac{676 - \frac{(70)\,(63)}{7}}{\sqrt{728 - \frac{(70)^2}{7}}\sqrt{651 - \frac{(63)^2}{7}}}$$

$$= 0.949$$

Example 3

Calculate the coefficient of correlation for the following data:

x	9	8	7	6	5	4	3	2	1
y	15	16	14	13	11	12	10	8	9

Solution

$n = 9$

x	y	x^2	y^2	xy
9	15	81	225	135
8	16	64	256	128
7	14	49	196	98
6	13	36	169	78
5	11	25	121	55
4	12	16	144	48
3	10	9	100	30
2	8	4	64	16
1	9	1	81	9
$\sum x = 45$	$\sum y = 108$	$\sum x^2 = 285$	$\sum y^2 = 1356$	$\sum xy = 597$

$$r = \frac{\sum x\,y - \frac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}\sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}}$$

$$= \frac{597 - \frac{(45)(108)}{9}}{\sqrt{285 - \frac{(45)^2}{9}}\sqrt{1356 - \frac{(108)^2}{9}}}$$

$$= 0.95$$

Example 4

Calculate the correlation coefficient between the following data:

x	5	9	13	17	21
y	12	20	25	33	35

Solution

$$n = 5$$

$$\bar{x} = \frac{\sum x}{n} = \frac{65}{5} = 13$$

$$\bar{y} = \frac{\sum y}{n} = \frac{125}{5} = 25$$

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
5	12	–8	–13	64	169	104
9	20	–4	–5	16	25	20
13	25	0	0	0	0	0
17	33	4	8	16	64	32
21	35	8	10	64	100	80
$\sum x = 65$	$\sum y = 125$	$\sum(x-\bar{x}) = 0$	$\sum(y-\bar{y}) = 0$	$\sum(x-\bar{x})^2 = 160$	$\sum(y-\bar{y})^2 = 358$	$\sum(x-\bar{x})(y-\bar{y}) = 236$

$$r = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2}\sqrt{\sum(y-\bar{y})^2}}$$

$$= \frac{236}{\sqrt{160}\sqrt{358}}$$

$$= 0.986$$

Example 5

Calculate the correlation coefficient between for the following values of demand and the corresponding price of a commodity:

Demand in quintals	65	66	67	67	68	69	70	72
Price in rupees per kg	67	68	65	68	72	72	69	71

Solution

Let the demand in quintal be denoted by x and the price in rupees per kg be denoted by y.

$$n = 8$$

$$\overline{x} = \frac{\sum x}{n} = \frac{544}{8} = 68$$

$$\overline{y} = \frac{\sum y}{n} = \frac{552}{8} = 69$$

x	y	$x-\overline{x}$	$y-\overline{y}$	$(x-\overline{x})^2$	$(y-\overline{y})^2$	$(x-\overline{x})(y-\overline{y})$
65	67	–3	–2	9	4	6
66	68	–2	–1	4	1	2
67	65	–1	–4	1	16	4
67	68	–1	–1	1	1	1
68	72	0	3	0	9	0
69	72	1	3	1	9	3
70	69	2	0	4	0	0
72	71	4	2	16	4	8
$\sum x = 544$	$\sum y = 552$	$\sum(x-\overline{x}) = 0$	$\sum(y-\overline{y}) = 0$	$\sum(x-\overline{x})^2 = 36$	$\sum(y-\overline{y})^2 = 44$	$\sum(x-\overline{x})(y-\overline{y}) = 24$

$$r = \frac{\sum(x-\overline{x})(y-\overline{y})}{\sqrt{\sum(x-\overline{x})^2}\sqrt{\sum(y-\overline{y})^2}}$$

$$= \frac{24}{\sqrt{36}\sqrt{44}}$$

$$= 0.603$$

Example 6

Calculate the coefficient of correlation for the following pairs of x and y:

x	17	19	21	26	20	28	26	27
y	23	27	25	26	27	25	30	33

Solution

Let $a = 23$ and $b = 27$ be the assumed means of x and y series respectively.

$$d_x = x - a = x - 23$$
$$d_y = y - b = y - 27$$
$$n = 8$$

x	y	d_x	d_y	d_x^2	d_y^2	$d_x d_y$
17	23	–6	–4	36	16	24
19	27	–4	0	16	0	0
21	25	–2	–2	4	4	4
26	26	3	–1	9	1	–3
20	27	–3	0	9	0	0
28	25	5	–2	25	4	–10
26	30	3	3	9	9	9
27	33	4	6	16	36	24
		$\Sigma d_x = 0$	$\Sigma d_y = 0$	$\Sigma d_x^2 = 124$	$\Sigma d_y^2 = 70$	$\Sigma d_x d_y = 48$

$$r = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \frac{\left(\sum dx\right)^2}{n}}\sqrt{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}}$$

$$= \frac{48 - 0}{\sqrt{124 - 0}\sqrt{70 - 0}}$$

$$= 0.515$$

Example 7

Calculate the correlation coefficient from the following data:

x	23	27	28	29	30	31	33	35	36	39
y	18	22	23	24	25	26	28	29	30	32

Solution

Let $a = 30$ and $b = 25$ be the assumed means of x and y series respectively.

$$d_x = x - a = x - 30$$
$$d_y = y - b = x - 25$$
$$n = 10$$

x	y	d_x	d_y	d_x^2	d_y^2	$d_x d_y$
23	18	–7	–7	49	49	49
27	22	–3	–3	9	9	9
28	23	–2	–2	4	4	4
29	24	–1	–1	1	1	1
30	25	0	0	0	0	0
31	26	1	1	1	1	1
33	28	3	3	9	9	9
35	29	5	4	25	16	20
36	30	6	5	36	25	30
39	32	9	7	81	49	63
		$\Sigma d_x = 11$	$\Sigma d_y = 7$	$\Sigma d_x^2 = 215$	$\Sigma d_y^2 = 163$	$\Sigma d_x d_y = 186$

$$r = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \frac{\left(\sum d_x\right)^2}{n}}\sqrt{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}}$$

$$= \frac{186 - \frac{(11)(7)}{10}}{\sqrt{215 - \frac{(11)^2}{10}}\sqrt{163 - \frac{(7)^2}{10}}}$$

$$= 0.996$$

Example 8

Calculate the coefficient of correlation between the ages of cars and annual maintenance costs.

Age of cars (year)	2	4	6	7	8	10	12
Annual maintenance cost (₹)	1600	1500	1800	1900	1700	2100	2000

Solution

Let the ages of cars in years be denoted by x and annual maintenance costs in rupees be denoted by y.

Let $a = 7$ and $b = 1800$ be the assumed means of x and y series respectively.

Let $h = 1, \qquad k = 100$

$$d_x = \frac{x-a}{h} = \frac{x-7}{1} = x-7$$

$$d_y = \frac{y-b}{k} = \frac{y-1800}{100}$$

$$n = 7$$

x	y	d_x	d_y	d_x^2	d_y^2	d_xd_y
2	1600	–5	–2	25	4	10
4	1500	–3	3	9	9	9
6	1800	–1	0	1	0	0
7	1900	0	1	0	1	0
8	1700	1	–1	1	1	–1
10	2100	3	3	9	9	9
12	2000	5	2	25	4	10
		$\sum d_x = 0$	$\sum d_y = 0$	$\sum d_x^2 = 70$	$\sum d_y^2 = 28$	$\sum d_xd_y = 37$

$$r = \frac{\sum d_x\, d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \dfrac{\left(\sum d_x\right)^2}{n}}\sqrt{\sum d_y^2 - \dfrac{\left(\sum d_y\right)^2}{n}}}$$

$$= \frac{37-0}{\sqrt{70-0}\,\sqrt{28-0}}$$

$$= 0.836$$

Example 9

Calculate Karl Pearson's coefficient of correlation for the data given below:

x	10	14	18	22	26	30
y	18	12	24	6	30	36

Solution

Let $a = 22$ and $b = 24$ be the assumed means of x and y series respectively.

Let $h = 4$, $k = 6$

$$d_x = \frac{x-a}{h} = \frac{x-22}{4}$$

$$d_y = \frac{y-b}{k} = \frac{y-24}{6}$$

$$n = 6$$

x	y	d_x	d_y	d_x^2	d_y^2	$d_x d_y$
10	18	–3	–1	9	1	3
14	12	–2	–2	4	4	4
18	24	–1	0	1	0	0
22	6	0	–3	0	9	0
26	30	1	1	1	1	1
30	36	2	2	4	4	4
		$\sum d_x = -3$	$\sum d_y = -3$	$\sum d_x^2 = 19$	$\sum d_y^2 = 19$	$\sum d_x d_y = 12$

$$r = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \frac{\left(\sum d_x\right)^2}{n}}\sqrt{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}}$$

$$= \frac{12 - \frac{(-3)(-3)}{6}}{\sqrt{19 - \frac{(-3)^2}{6}}\sqrt{19 - \frac{(-3)^2}{6}}}$$

$$= 0.6$$

Example 10

The coefficient of correlation between two variables X and Y is 0.48. *The covariance is* 36. *The variance of X is* 16. *Find the standard deviation of Y.*

Solution

$$r = 0.48, \qquad \text{cov}(X, Y) = 36, \qquad \sigma_X^2 = 16$$

$$\therefore \qquad \sigma_X = 4$$

$$r = \frac{\text{cov}(X, Y)}{\sigma_X \, \sigma_Y}$$

$$0.48 = \frac{36}{4\sigma_Y}$$

$$\therefore \quad \sigma_Y = 18.75$$

Example 11

Given $n = 10$, $\sigma_X = 5.4$, $\sigma_Y = 6.2$, *and sum of the product of deviations from the mean of x and y is* 66. *Find the correlation coefficient.*

Solution

$$n = 10, \sigma_X = 5.4, \sigma_Y = 6.2$$

$$\sum (x - \overline{x})(y - \overline{y}) = 66$$

$$\sigma_X = \sqrt{\frac{\sum (x - \overline{x})^2}{n}}$$

$$5.4 = \sqrt{\frac{\sum (x - \overline{x})^2}{10}}$$

$$\therefore \quad \sum (x - \overline{x})^2 = 291.6$$

$$\sigma_Y = \sqrt{\frac{\sum (y - \overline{y})^2}{n}}$$

$$6.2 = \sqrt{\frac{\sum (y - \overline{y})^2}{10}}$$

$$\therefore \quad \sum (y - \overline{y})^2 = 384.4$$

$$r = \frac{\sqrt{\sum (x - \overline{x})(y - \overline{y})}}{\sqrt{\sum (x - \overline{x})^2} \sqrt{\sum (y - \overline{y})^2}}$$

$$= \frac{66}{\sqrt{291.6} \sqrt{384.4}}$$

$$= 0.197$$

Example 12

From the following information, calculate the value of n.

$$\sum x = 4, \sum y = 4, \sum x^2 = 44, \sum y^2 = 44, \sum xy = -40, r = -1$$

Solution

$$r = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}\sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}}$$

$$-1 = \frac{-40 - \frac{(4)(4)}{n}}{\sqrt{44 - \frac{(4)^2}{n}}\sqrt{44 - \frac{(4)^2}{n}}}$$

$$\therefore \quad n = 8$$

Example 13

From the following data, find the number of items n.

$$r = 0.5, \sum (x-\overline{x})(y-\overline{y}) = 120, \sigma_Y = 8, \sum (x-\overline{x})^2 = 90$$

Solution

$$\sigma_Y = \sqrt{\frac{\sum (y-\overline{y})^2}{n}}$$

$$8 = \sqrt{\frac{\sum (y-\overline{y})^2}{n}}$$

$$\sum (y-\overline{y})^2 = 64\,n$$

$$r = \frac{\sum (x-\overline{x})(y-\overline{y})}{\sqrt{\sum (x-\overline{x})^2}\sqrt{\sum (y-\overline{y})^2}}$$

$$0.5 = \frac{120}{\sqrt{90}\sqrt{64\,n}}$$

$$\therefore \quad n = 10$$

Example 14

Calculate the correlation coefficient between x and y from the following data:

$$n = 10, \sum x = 140, \sum y = 150, \sum (x-10)^2 = 180$$

$$\sum (y-15)^2 = 215, \sum (x-10)\,(y-15) = 60$$

Solution

$$\sum d_x^2 = \sum (x-10)^2 = 180$$

$$\sum d_y^2 = \sum (y-15)^2 = 215$$

$$\sum d_x\, d_y = \sum (x-10)\,(y-15) = 60$$

$$a = 10$$

$$b = 15$$

$$n = 10$$

$$\bar{x} = \frac{\sum x}{n} = \frac{140}{10} = 14$$

$$\bar{y} = \frac{\sum y}{n} = \frac{150}{10} = 15$$

$$\bar{x} = a + \frac{\sum d_x}{n}$$

$$14 = 10 + \frac{\sum d_x}{10}$$

$$\therefore \quad \sum d_x = 40$$

$$\bar{y} = b + \frac{\sum d_y}{n}$$

$$15 = 15 + \frac{\sum d_y}{10}$$

$$\therefore \quad \sum d_y = 0$$

$$r = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \frac{\left(\sum d_x\right)^2}{n}}\sqrt{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}}$$

$$= \frac{60 - \frac{(40)(0)}{10}}{\sqrt{180 - \frac{(40)^2}{10}}\sqrt{215 - \frac{0}{10}}}$$

$$= 0.915$$

Example 15

A computer operator while calculating the coefficient between two variates x and y for 25 pairs of observations obtained the following constants:

$$n = 25, \sum x = 125, \sum x^2 = 650, \sum y = 100,$$
$$\sum y^2 = 460, \sum xy = 508$$

It was later discovered at the time of checking that he had copied down two pairs as (6, 14) *and* (8, 6) *while the correct pairs were* (8,12) *and* (6, 8). *Obtain the correct value of the correlation coefficient.*

Solution

$$n = 25$$

$$\text{Corrected} \sum x = \text{Incorrect} \sum x - (\text{Sum of incorrect } x) + (\text{Sum of correct } x)$$
$$= 125 - (6+8) + (8+6)$$
$$= 125$$

Similarly,

$$\text{Corrected} \sum y = 100 - (14+6) + (12+8) = 100$$
$$\text{Corrected} \sum x^2 = 650 - (6^2 + 8^2) + (8^2 + 6^2) = 650$$
$$\text{Corrected} \sum y^2 = 460 - (14^2 + 6^2) + (12^2 + 8^2) = 436$$
$$\text{Corrected} \sum xy = 508 - (84+48) + (96+48) = 520$$

Correct value of correlation coefficient

$$r = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}} \sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}}$$
$$= \frac{520 - \frac{(125)(100)}{25}}{\sqrt{650 - \frac{(125)^2}{25}} \sqrt{436 - \frac{(100)^2}{25}}}$$
$$= 0.67$$

EXERCISE 9.1

1. Draw a scatter diagram to represent the following data:

x	2	4	5	6	8	11
y	18	12	10	8	7	5

Calculate the coefficient of correlation between *x* and *y*.

[Ans.: –0.92]

2. Find the coefficient of correlation between x and y for the following data:

x	10	12	18	24	23	27
y	13	18	12	25	30	10

[Ans.: 0.223**]**

3. From the following information relating to the stock exchange quotations for two shares *A* and *B*, ascertain by using Pearson's coefficient of correlation how shares *A* and *B* are correlated in their prices?

Price share (*A*) ₹	160	164	172	182	166	170	178
Price share (*B*) ₹	292	280	260	234	266	254	230

[Ans.: –0.96**]**

4. Find the correlation coefficient between the income and expenditure of a wage earner.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul
Income	46	54	56	56	58	60	62
Expenditure	36	40	44	54	42	58	54

[Ans.: 0.769**]**

5. From the following data, examine whether the input of oil and output of electricity can be said to be correlated.

Input of oil	6.9	8.2	7.8	4.8	9.6	8.0	7.7
Output of electricity	1.9	3.5	6.5	1.3	5.5	3.5	2.2

[Ans.: 0.696**]**

6. For the following data, show that cov $(x, x^2) = 0$.

x	–3	–2	–1	0	1	2	3
x^2	9	4	1	0	1	4	9

7. Find the coefficient of correlation between x and y for the following data:

x	62	64	65	69	70	71	72	74
y	126	125	139	145	165	152	180	208

[Ans.: 0.9032**]**

8. The following data gave the growth of employment in lacs in the organized sector in India between 1988 and 1995:

Year	1988	1989	1990	1991	1992	1993	1994	1995
Public sector	98	101	104	107	113	120	125	128
Private sector	65	65	67	68	68	69	68	68

Find the correlation coefficient between the employment in public and private sectors.

[Ans.: 0.77**]**

9. Calculate Karl Pearson's coefficient of correlation from the following data, using 20 as the working mean for price and 70 as working mean for demand.

Price	14	16	17	18	19	20	21	22	23
Demand	84	78	70	75	66	67	62	58	60

[Ans.: −0.954**]**

10. A sample of 25 pairs of values x and y lead to the following results:

$$\sum x = 127, \sum y = 100, \sum x^2 = 760, \sum y^2 = 449, \sum xy = 500$$

Later on, it was found that two pairs of values were taken as (8, 14) and (8, 6) instead of the correct values (8, 12) and (6, 8). Find the corrected coefficient between x and y.

[Ans.: −0.31**]**

9.9 RANK CORRELATION

Let a group of n individuals be arranged in order of merit with respect to some characteristics. The same group would give a different order (rank) for different characteristics. Considering the orders corresponding to two characteristics A and B, the correlation between these n pairs of ranks is called the *rank correlation* in the characteristics A and B for that group of individuals.

9.9.1 Spearman's Rank Correlation Coefficient

Let x, y be the ranks of the i^{th} individuals in two characteristics A and B respectively where $i = 1, 2, ..., n$. Assuming that no two individuals have the same rank either for x or y, each of the variables x and y take the values 1, 2, ..., n.

$$\bar{x} = \bar{y} = \frac{1+2+3+\cdots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

$$\begin{aligned}
\sum (x-\bar{x})^2 &= \sum (x^2 - 2x\,\bar{x} + \bar{x}^2) \\
&= \sum x^2 - 2\bar{x} \sum x + \bar{x}^2 \sum 1 \\
&= \sum x^2 - 2n\bar{x}^2 + n\bar{x}^2 \qquad \left[\because \quad \sum x = n\bar{x} \text{ and } \sum 1 = n\right] \\
&= \sum x^2 - n\bar{x}^2 \\
&= (1^2 + 2^2 + \cdots + n^2) - n\left(\frac{n+1}{2}\right)^2
\end{aligned}$$

$$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)^2}{4}$$

$$= \frac{1}{12}(n^3 - n)$$

Similarly, $\sum (y - \bar{y})^2 = \frac{1}{12}(n^3 - n)$

If d denotes the difference between the ranks of the i^{th} individuals in the two variables,

$$d = x - y = (x - \bar{x}) - (y - \bar{y}) \qquad [\because \quad \bar{x} = \bar{y}]$$

Squaring and summing over i from 1 to n,

$$\begin{aligned} \sum d^2 &= \sum [(x - \bar{x}) - (y - \bar{y})]^2 \\ &= \sum (x - \bar{x})^2 + \sum (y - \bar{y})^2 - 2\sum (x - \bar{x})(y - \bar{y}) \end{aligned}$$

$$\begin{aligned} \sum (x - \bar{x})(y - \bar{y}) &= \frac{1}{2}\left[\sum (x - \bar{x})^2 + \sum (y - \bar{y})^2 - \sum d^2\right] \\ &= \frac{1}{12}(n^3 - n) - \frac{1}{2}\sum d^2 \end{aligned}$$

Hence, the coefficient of correlation between these variables is

$$\begin{aligned} r &= \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 \sum (y - \bar{y})^2}} \\ &= \frac{\frac{1}{12}(n^3 - n) - \frac{1}{2}\sum d^2}{\frac{1}{12}(n^3 - n)} \\ &= 1 - \frac{6\sum d^2}{n^3 - n} \\ &= 1 - \frac{6\sum d^2}{n(n^2 - 1)} \end{aligned}$$

This is called Spearman's rank correlation coefficient and is denoted by ρ.

Note $\sum d = \sum (x - y) = \sum x - \sum y = n(\bar{x} - \bar{y}) = 0$

Example 1

Ten participants in a contest are ranked by two judges as follows:

x	1	3	7	5	4	6	2	10	9	8
y	3	1	4	5	6	9	7	8	10	2

Calculate the rank correlation coefficient.

Solution

$n = 10$

Rank by first Judge x	Rank by second Judge y	$d = x - y$	d^2
1	3	–2	4
3	1	2	4
7	4	3	9
5	5	0	0
4	6	–2	4
6	9	–3	9
2	7	–5	25
10	8	2	4
9	10	–1	1
8	2	6	36
		$\sum d = 0$	$\sum d^2 = 96$

$$r = 1 - \frac{6\sum d^2}{n(n^2 - 1)}$$

$$= 1 - \frac{6(96)}{10\left[(10)^2 - 1\right]}$$

$$= 0.418$$

Example 2

Ten competitors in a musical test were ranked by the three judges A, B, and C in the following order:

Rank by A	1	6	5	10	3	2	4	9	7	8
Rank by B	3	5	8	4	7	10	2	1	6	9
Rank by C	6	4	9	8	1	2	3	10	5	7

Using the rank correlation method, find which pair of judges has the nearest approach to common liking in music. **[Summer 2015]**

Solution

$n = 10$

Rank by A x	Rank by B y	Rank by C z	$d_1 = x - y$	$d_2 = y - z$	$d_3 = z - x$	d_1^2	d_2^2	d_3^2
1	3	6	–2	–3	5	4	9	25
6	5	4	1	1	–2	1	1	4
5	8	9	–3	–1	4	9	1	16
10	4	8	6	–4	–2	36	16	4
3	7	1	–4	6	–2	16	36	4
2	10	2	–8	8	0	64	64	0
4	2	3	2	–1	–1	4	1	1
9	1	10	8	–9	1	64	81	1
7	6	5	1	1	–2	1	1	4
8	9	7	–1	2	–1	1	4	1
			$\sum d_1 = 0$	$\sum d_2 = 0$	$\sum d_3 = 0$	$\sum d_1^2 = 200$	$\sum d_2^2 = 214$	$\sum d_3^2 = 60$

$$
\begin{aligned}
r(x, y) &= 1 - \frac{6\sum d_1^2}{n(n^2-1)} \\
&= 1 - \frac{6(200)}{10\left[(10)^2 - 1\right]} \\
&= -0.21 \\
r(y, z) &= 1 - \frac{6\sum d_2^2}{n(n^2-1)} \\
&= 1 - \frac{6(214)}{10\left[(10)^2 - 1\right]} \\
&= -0.296 \\
r(z, x) &= 1 - \frac{6\sum d_3^2}{n(n^2-1)} \\
&= 1 - \frac{6(60)}{10\left[(10)^2 - 1\right]} \\
&= 0.64
\end{aligned}
$$

Since $r(z, x)$ is maximum, the pair of judges A and C has the nearest common approach.

Example 3

Ten students got the following percentage of marks in mathematics and physics:

Mathematics (x)	8	36	98	25	75	82	92	62	65	35
Physics (y)	84	51	91	60	68	62	86	58	35	49

Find the rank correlation coefficient.

Solution

$n = 10$

x	y	Rank in mathematics x	Rank in physics y	$d = x - y$	d^2
8	84	10	3	7	49
36	51	7	8	–1	1
98	91	1	1	0	0
25	60	9	6	3	9
75	68	4	4	0	0
82	62	3	5	–2	4
92	86	2	2	0	0
62	58	6	7	–1	1
65	35	5	10	–5	25
35	49	8	9	–1	1
				$\sum d = 0$	$\sum d^2 = 90$

$$r = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

$$= 1 - \frac{6(90)}{10\left[(10)^2 - 1\right]}$$

$$= 0.455$$

Example 4

The coefficient of rank correlation of the marks obtained by 10 students in physics and chemistry was found to be 0.5. It was later discovered that the difference in ranks in the two subjects obtained by one of the students was wrongly taken as 3 instead of 7. Find the rank coefficient of the rank correlation.

Solution

$n = 10$

$$r = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

$$0.5 = 1 - \frac{6\sum d^2}{10(100-1)}$$

$$\therefore \qquad \sum d^2 = 82.5$$

$$\text{Correct } \sum d^2 = \text{Incorrect } \sum d^2 - (\text{Incorrect rank difference})^2 + (\text{Correct rank difference})^2$$

$$= 82.5 - (3)^2 + (7)^2$$

$$= 122.5$$

Correct coefficient of rank correlation $r = 1 - \dfrac{6(122.5)}{10(100-1)}$

$$= 0.26$$

9.9.2 Tied Ranks

If there is a tie between two or more individuals ranks, the rank is divided among equal individuals, e.g., if two items have fourth rank, the 4th and 5th rank is divided between them equally and is given as $\dfrac{4+5}{2} = 4.5^{\text{th}}$ rank to each of them. If three items have the same 4th rank, each of them is given $\dfrac{4+5+6}{3} = 5^{\text{th}}$ rank. As a result of this, the following adjustment or correction is made in the rank correlation formula. If m is the number of item having equal ranks then the factor $\dfrac{1}{12}(m^3 - m)$ is added to $\sum d^2$. If there are more than one cases of this type, this factor is added corresponding to each case.

$$r = 1 - \frac{6\left[\sum d^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \cdots\right]}{n(n^2-1)}$$

Example 1

Obtain the rank correlation coefficient from the following data:

x	10	12	18	18	15	40
y	12	18	25	25	50	25

Solution

Here, $n = 6$

x	y	Rank x	Rank y	$d = x - y$	d^2
10	12	1	1	0	0
12	18	2	2	0	0
18	25	4.5	4	0.5	0.25
18	25	4.5	4	0.5	0.25
15	50	3	6	–3	9
40	25	6	4	2	4
					$\sum d^2 = 13.5$

There are two items in the x series having equal values at the rank 4. Each is given the rank 4.5. Similarly, there are three items in the y series at the rank 3. Each of them is given the rank 4.

$$m_1 = 2, m_2 = 3$$

$$r = 1 - \frac{6\left[\sum d^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2)\right]}{n(n^2 - 1)}$$

$$= 1 - \frac{6\left[13.50 + \frac{1}{12}(8-2) + \frac{1}{12}(27-3)\right]}{6\left[(6)^2 - 1\right]}$$

$$= 0.5429$$

EXERCISE 9.2

1. Compute Spearman's rank correlation coefficient from the following data:

x	18	20	34	52	12
y	39	23	35	18	46

[Ans.: –0.9]

2. Two judges gave the following ranks to a series of eight one-act plays in a drama competition. Examine the relationship between their judgements.

Judge *A*	8	7	6	3	2	1	5	4
Judge *B*	7	5	4	1	3	2	6	8

[Ans.: 0.62]

3. From the following data, calculate Spearman's rank correlation between *x* and *y*.

x	36	56	20	42	33	44	50	15	60
y	50	35	70	58	75	60	45	80	38

[Ans.: 0.92**]**

4. Ten competitors in a voice test are ranked by three judges in the following order:

Rank by First Judge	6	10	2	9	8	1	5	3	4	7
Rank by Second Judge	5	4	10	1	9	3	8	7	2	6
Rank by Third Judge	4	8	2	10	7	6	9	1	3	6

Use the method of rank correlation to gauge which pairs of judges has the nearest approach to common liking in voice.

[Ans.: The first and third judge**]**

5. The following table gives the scores obtained by 11 students in English and Tamil translation. Find the rank correlation coefficient.

Scores in English	40	46	54	60	70	80	82	85	85	90	95
Scores in Tamil	45	45	50	43	40	75	55	72	65	42	70

[Ans.: 0.36**]**

6. Calculate Spearman's coefficient of rank correlation for the following data:

x	53	98	95	81	75	71	59	55
y	47	25	32	37	30	40	39	45

[Ans.: –0.905**]**

7. Following are the scores of ten students in a class and their IQ:

Score	35	40	25	55	85	90	65	55	45	50
IQ	100	100	110	140	150	130	100	120	140	110

Calculate the rank correlation coefficient between the score IQ.

[Ans.: 0.47**]**

9.10 REGRESSION

Regression is defined as a method of estimating the value of one variable when that of the other is known and the variables are correlated. *Regression analysis* is used to predict or estimate one variable in terms of the other variable. It is a highly valuable tool for prediction purpose in economics and business. It is useful in statistical estimation of demand curves, supply curves, production function, cost function, consumption function, etc.

9.11 TYPES OF REGRESSION

Regression is classified into two types:

1. Simple and multiple regressions
2. Linear and nonlinear regressions

9.11.1 Simple and Multiple Regressions

Depending upon the study of the number of variables, regression may be simple or multiple.

1. Simple Regression The regression analysis for studying only two variables at a time is known as simple regression.

2. Multiple Regression The regression analysis for studying more than two variables at a time is known as multiple regression.

9.11.2 Linear and Nonlinear Regressions

Depending upon the regression curve, regression may be linear or nonlinear.

1. Linear Regression If the regression curve is a straight line, the regression is said to be linear.

2. Nonlinear Regression If the regression curve is not a straight line i.e., not a first-degree equation in the variables x and y, the regression is said to be nonlinear or curvilinear. In this case, the regression equation will have a functional relation between the variables x and y involving terms in x and y of the degree higher than one, i.e., involving terms of the type x^2, y^2, x^3, y^3, xy, etc.

9.12 METHODS OF STUDYING REGRESSION

There are two methods of studying correlation:

(i) Method of scatter diagram
(ii) Method of least squares

9.12.1 Method of Scatter Diagram

It is the simplest method of obtaining the lines of regression. The data are plotted on a graph paper by taking the independent variable on the x-axis and the dependent variable on the y-axis. Each of these points are generally scattered in a narrow strip. If the correlation is perfect, i.e., if r is equal to one, positive, or negative, the points will lie on a line which is the line of regression.

9.12.2 Method of Least Squares

This is a mathematical method which gives an objective treatment to find a line of regression. It is used for obtaining the equation of a curve which fits best to a given set of observations. It is based on the assumption that the sum of squares of differences between the estimated values and the actual observed values of the observations is minimum.

9.13 LINES OF REGRESSION

If the variables, which are highly correlated, are plotted on a graph then the points lie in a narrow strip. If all the points in the scatter diagram cluster around a straight line, the line is called the *line of regression*. The line of regression is the line of best fit and is obtained by the principle of least squares.

Line of Regression of *y* on *x*

It is the line which gives the best estimate for the values of y for any given values of x. The regression equation of y on x is given by

$$y - \overline{y} = r\frac{\sigma_y}{\sigma_x}(x - \overline{x})$$

It is also written as

$$y = a + bx$$

Line of Regression of *x* on *y*

It is the line which gives the best estimate for the values of x for any given values of y. The regression equation for x on y is given by

$$x - \overline{x} = r\frac{\sigma_x}{\sigma_y}(y - \overline{y})$$

It is also written as

$$x = a + by$$

where $\overline{x}$ and $\overline{y}$ are means of x series and y series respectively, σ_x and σ_y are standard deviations of x series and y series respectively, r is the correlation coefficient between x and y.

9.14 REGRESSION COEFFICIENTS

The slope b of the line of regression of y on x is also called the *coefficient of regression* of y on x. It represents the increment in the value of y corresponding to a unit change in the value of x.

$$b_{yx} = \text{Regression coefficient of } y \text{ on } x$$

$$= r\frac{\sigma_y}{\sigma_x}$$

Similarly, the slope b of the line of regression of x on y is called the coefficient of regression of x on y. It represents the increment in the value of x corresponding to a unit change in the value of y.

$$b_{xy} = \text{Regression coefficient of } x \text{ on } y$$

$$= r\,\frac{\sigma_x}{\sigma_y}$$

Expressions for Regression Coefficients

(i) We know that

$$r = \frac{\sum (x-\bar{x})\,(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2}\ \sqrt{\sum (y-\bar{y})^2}}$$

$$\sigma_x = \sqrt{\frac{\sum (x-\bar{x})^2}{n}}$$

$$\sigma_y = \sqrt{\frac{\sum (y-\bar{y})^2}{n}}$$

$$b_{yx} = r\,\frac{\sigma_y}{\sigma_x}$$

$$= \frac{\sum (x-\bar{x})\,(y-\bar{y})}{\sum (x-\bar{x})^2}$$

and $$b_{xy} = r\,\frac{\sigma_x}{\sigma_y}$$

$$= \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (y-\bar{y})^2}$$

(ii) We know that

$$r = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}\ \sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}}$$

$$\sigma_x = \sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}$$

$$\sigma_y = \sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}$$

$$b_{yx} = r\,\frac{\sigma_y}{\sigma_x}$$

$$= \frac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}}$$

and $$b_{xy} = r\,\frac{\sigma_x}{\sigma_y}$$

$$= \frac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sum y^2 - \dfrac{\left(\sum y\right)^2}{n}}$$

(iii) We know that

$$r = \frac{\sum d_x d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \dfrac{\left(\sum d_x\right)^2}{n}}\sqrt{\sum d_y^2 - \dfrac{\left(\sum d_y\right)^2}{n}}}$$

$$\sigma_x = \sqrt{\sum d_x^2 - \frac{\left(\sum d_x\right)^2}{n}}$$

$$\sigma_y = \sqrt{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}$$

$$b_{yx} = r\,\frac{\sigma_y}{\sigma_x}$$

$$= \frac{\sum d_x d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \dfrac{\left(\sum d_x\right)^2}{n}}}$$

and $$b_{xy} = r\,\frac{\sigma_x}{\sigma_y}$$

$$= \frac{\sum d_x d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_y^2 - \dfrac{\left(\sum d_y\right)^2}{n}}}$$

9.15 PROPERTIES OF REGRESSION COEFFICIENTS

1. The coefficient of correlation is the geometric mean of the coefficients of regression, i.e., $r = \sqrt{b_{yx} b_{xy}}$.

Proof We know that

$$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

$$b_{xy} = r\frac{\sigma_x}{\sigma_y}$$

$$b_{yx}\, b_{xy} = r\frac{\sigma_y}{\sigma_x} \cdot r\frac{\sigma_x}{\sigma_y}$$

$$= r^2$$

$$r = \sqrt{b_{yx}\, b_{xy}}$$

2. If one of the regression coefficients is greater than one, the other must be less than one.

Proof Let $b_{yx} > 1$

We know that

$$r^2 \le 1 \text{ and } r^2 = b_{yx}\, b_{xy}$$

$$b_{yx}\, b_{xy} \le 1$$

$$b_{yx} \le \frac{1}{b_{xy}}$$

Hence, if $b_{yx} < 1$, $b_{xy} > 1$

3. The arithmetic mean of regression coefficients is greater than or equal to the coefficient of correlation.

Proof We have to prove that

$$\frac{1}{2}(b_{yx} + b_{xy}) \ge r$$

i.e., $$\frac{1}{2}\left(r\frac{\sigma_y}{\sigma_x} + r\frac{\sigma_x}{\sigma_y}\right) \ge r$$

i.e., $$\frac{\sigma_y}{\sigma_x} + \frac{\sigma_x}{\sigma_y} \ge 2$$

i.e., $$\sigma_y^2 + \sigma_x^2 - 2\sigma_x\sigma_y \ge 0$$

i.e., $(\sigma_y - \sigma_x)^2 \geq 0$

which is always true, since the square of a real quantity is $1 \geq 0$.

4. Regression Coefficients are independent of the change of origin but not of scale.

Proof Let $d_x = \dfrac{x-a}{h},\ d_y = \dfrac{y-b}{k}$

$$x = a + hd_x,\ \ y = b + kd_y$$

where a, b, h (> 0) and k(> 0) are constants.

$$r_{d_x d_y} = r_{xy},\ \sigma_{d_x}^2 = \frac{1}{h^2}\sigma_x^2,\ \sigma_{d_y}^2 = \frac{1}{k^2}\sigma_y^2$$

$$\begin{aligned} b_{d_x d_y} &= r_{d_x d_y}\frac{\sigma_{d_x}}{\sigma_{d_y}} \\ &= r_{xy}\frac{\sigma_x}{h}\frac{k}{\sigma_y} \\ &= \frac{k}{h} r_{xy}\frac{\sigma_x}{\sigma_y} \\ &= \frac{k}{h} b_{xy} \end{aligned}$$

Similarly, $b_{d_y d_x} = \dfrac{h}{k} b_{yx}$

5. Both regression coefficients will have the same sign i.e., either both are positive or both are negative.

6. The sign of correlation is same as that of the regression coefficients, i.e., $r > 0$ if $b_{xy} > 0$ and $b_{yx} > 0$; and $r < 0$ if $b_{xy} < 0$ and $b_{yx} < 0$.

9.16 PROPERTIES OF LINES OF REGRESSION (LINEAR REGRESSION)

1. The two regression lines x on y and y on x always intersect at their means $(\bar{x}, \bar{y})$.
2. Since $r^2 = b_{yx}\, b_{xy}$, i.e., $r = \sqrt{b_{yx}\, b_{xy}}$, therefore, r, b_{yx}, b_{xy} all have the same sign.
3. If $r = 0$, the regression coefficients are zero.
4. The regression lines become identical if $r = \pm 1$. It follows from the regression equations that $x = \bar{x}$ and $y = \bar{y}$. If $r = 0$, these lines are perpendicular to each other.

Example 1

The regression lines of a sample are $x + 6y = 6$ *and* $3x + 2y = 10$. *Find*
(i) sample means $\bar{x}$ and $\bar{y}$, *and*
(ii) the coefficient of correlation between x and y.
(iii) Also estimate y when $x = 12$.

Solution

(i) The regression lines pass through the point $(\bar{x}, \bar{y})$.

$$\bar{x} + 6\bar{y} = 6 \qquad ...(1)$$

$$3\bar{x} + 2\bar{y} = 10 \qquad ...(2)$$

Solving Eqs (1) and (2),

$$\bar{x} = 3, \quad \bar{y} = \frac{1}{2}$$

(ii) Let the line $x + 6y = 6$ be the line of regression of y on x.

$$6y = -x + 6$$

$$y = -\frac{1}{6}x + 1$$

$$\therefore \qquad b_{yx} = -\frac{1}{6}$$

Let the line $3x + 2y = 10$ be the line of regression of x on y.

$$3x = -2y + 10$$

$$x = -\frac{2}{3}y + \frac{10}{3}$$

$$\therefore \qquad b_{xy} = -\frac{2}{3}$$

$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{\left(-\frac{1}{6}\right)\left(-\frac{2}{3}\right)} = \frac{1}{3}$$

Since b_{yx} and b_{xy} are negative, r is negative.

$$r = -\frac{1}{3}$$

Estimated value of y when $x = 12$ is

$$y = -\frac{1}{6}(12) + 1 = -1$$

Example 2

If the two lines of regression are $4x - 5y + 30 = 0$ and $20x - 9y - 107 = 0$, which of these are lines of regression of x on y and y on x? Find r_{xy} and σ_y when $\sigma_x = 3$.

Solution

For the line $4x - 5y + 30 = 0,$

$$-5y = -4x - 30$$

$$y = 0.8x + 6$$

$$\therefore \quad b_{yx} = 0.8$$

For the line $20x - 9y - 107 = 0$

$$20x = 9y + 107$$

$$x = 0.45y + 5.35$$

$$\therefore \quad b_{xy} = 0.45$$

Both b_{yx} and b_{xy} are positive.

Hence, line $4x - 5y + 30 = 0$ is the line of regression of y one x and line $20x - 9y - 107 = 0$ is the line of regression of x on y.

$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{(0.8)(0.45)} = 0.6$$

$$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

$$0.8 = 0.6\left(\frac{\sigma_y}{3}\right)$$

$$\therefore \quad \sigma_y = 4$$

Example 3

The following data regarding the heights (y) and weights (x) of 100 college students are given:

$$\sum x = 15000, \quad \sum x^2 = 2272500, \quad \sum y = 6800$$

$$\sum y^2 = 463025, \quad \sum xy = 1022250$$

Find the coefficient of correlation between height and weight and also the equation of regression of height and weight.

Solution

$$n = 100$$

$$b_{yx} = \frac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}}$$

$$= \frac{1022250 - \dfrac{(15000)(6800)}{100}}{2272500 - \dfrac{(15000)^2}{100}}$$

$$= 0.1$$

$$b_{xy} = \frac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sum y^2 - \dfrac{\left(\sum y\right)^2}{n}}$$

$$= \frac{1022250 - \dfrac{(15000)(6800)}{100}}{463025 - \dfrac{(6800)^2}{100}}$$

$$= 3.6$$

$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{(0.1)(3.6)} = 0.6$$

$$\bar{x} = \frac{\sum x}{n} = \frac{15000}{100} = 150$$

$$\bar{y} = \frac{\sum y}{n} = \frac{6800}{100} = 68$$

The equation of the line of regression of y on x is

$$y - \bar{y} = b_{yx}\,(x - \bar{x})$$

$$y - 68 = 0.1(x - 150)$$

$$y = 0.1x + 53$$

The equation of the line of regression of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 150 = 3.6(y - 68)$$

$$x = 3.6y - 94.8$$

Example 4

For a bivariate data, the mean value of x is 20 and the mean value of y is 45. The regression coefficient of y on x is 4 and that of x on y is $\frac{1}{9}$*. Find*

(i) the coefficient of correlation, and

(ii) the standard deviation of x if the standard deviation of y is 12.

(iii) Also write down the equations of regression lines.

Solution

$$\bar{x} = 20, \quad \bar{y} = 45, \quad b_{yx} = 4, \quad b_{xy} = \frac{1}{9}$$

(i) $$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{(4)\left(\frac{1}{9}\right)} = \frac{2}{3} = 0.667$$

(ii) $$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

$$4 = \frac{2}{3}\left(\frac{12}{\sigma_x}\right)$$

$$\therefore \quad \sigma_x = 2$$

(iii) The equation of the regression line of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 45 = 4(x - 20)$$

$$y = 4x - 35$$

The equation of the regression line of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 20 = \frac{1}{9}(y - 45)$$

$$x = \frac{1}{9}y + 15$$

Example 5

From the following results, obtain the two regression equations and estimate the yield when the rainfall is 29 cm and the rainfall, when the yield is 600 kg:

	Yield in kg	Rainfall in cm
Mean	508.4	26.7
SD	36.8	4.6

The coefficient of correlation between yield and rainfall is 0.52.

Solution

Let rainfall in cm be denoted by x and yield in kg be denoted by y.

$$\bar{x} = 26.7, \quad \bar{y} = 508.4, \quad \sigma_x = 4.6, \quad \sigma_y = 36.8, \quad r = 0.52$$

$$\begin{aligned} b_{yx} &= r\frac{\sigma_y}{\sigma_x} \\ &= 0.52\left(\frac{36.8}{4.6}\right) \\ &= 4.16 \\ b_{xy} &= r\frac{\sigma_x}{\sigma_y} \\ &= 0.52\left(\frac{4.6}{36.8}\right) \\ &= 0.065 \end{aligned}$$

The equation of the line of regression of y on x is

$$\begin{aligned} y - \bar{y} &= b_{yx}(x - \bar{x}) \\ y - 508.4 &= 4.16(x - 26.7) \\ y &= 4.16x + 397.328 \end{aligned}$$

The equation of the line of regression of x on y is

$$\begin{aligned} x - \bar{x} &= b_{xy}(y - \bar{y}) \\ x - 26.7 &= 0.065(y - 508.4) \\ x &= 0.065y - 6.346 \end{aligned}$$

Estimated yield when the rainfall is 29 cm is

$$y = 4.16\,(29) + 397.328 = 517.968 \text{ kg}$$

Estimated rainfall when the yield is 600 kg is

$$x = 0.065\,(600) - 6.346 = 32.654 \text{ cm}$$

Example 6

Find the regression coefficients b_{yx} *and* b_{xy} *and hence, find the correlation coefficient between x and y for the following data:*

x	4	2	3	4	2
y	2	3	2	4	4

Solution

$n = 5$

x	y	x^2	y^2	xy
4	2	16	4	8
2	3	4	9	6
3	2	9	4	6
4	4	16	16	16
2	4	4	16	8
$\Sigma x = 15$	$\Sigma y = 15$	$\Sigma x^2 = 49$	$\Sigma y^2 = 49$	$\Sigma xy = 44$

$$b_{yx} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}$$

$$= \frac{44 - \frac{(15)(15)}{5}}{49 - \frac{(15)^2}{5}}$$

$$= -0.25$$

$$b_{xy} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}$$

$$= \frac{44 - \frac{(15)(15)}{5}}{49 - \frac{(15)^2}{5}}$$

$$= -0.25$$

$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{(-0.25)(-0.25)} = 0.25$$

Since b_{yx} and b_{xy} are negative, r is negative.

$$r = -0.25$$

Example 7

The following data give the experience of machine operators and their performance rating as given by the number of good parts turned out per 100 pieces.

Operator	1	2	3	4	5	6
Performance rating (x)	23	43	53	63	73	83
Experience (y)	5	6	7	8	9	10

Calculate the regression line of performance rating on experience and also estimate the probable performance if an operator has 11 *years of experience.* **[Summer 2015]**

Solution

$n = 6$

x	y	y^2	xy
23	5	25	115
43	6	36	258
53	7	49	371
63	8	64	504
73	9	81	657
83	10	100	830
$\sum x = 338$	$\sum y = 45$	$\sum y^2 = 355$	$\sum xy = 2735$

$$b_{xy} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}$$

$$= \frac{2735 - \frac{(338)(45)}{6}}{355 - \frac{(45)^2}{6}}$$

$$= 11.429$$

$$\bar{x} = \frac{\sum x}{n} = \frac{338}{6} = 56.33$$

$$\bar{y} = \frac{\sum y}{n} = \frac{45}{6} = 7.5$$

The equation of regression line of x on y is

$$x - \bar{x} = b_{xy}\,(y - \bar{y})$$

$$x - 56.33 = 11.429(y - 7.5)$$

$$x = 11.429\,y - 29.3875$$

Estimated performance if $y = 11$ is

$$x = 11.429(11) - 29.3875 = 96.3315$$

Example 8

The number of bacterial cells (y) per unit volume in a culture at different hours (x) is given below:

x	0	1	2	3	4	5	6	7	8	9
y	43	46	82	98	123	167	199	213	245	272

Fit lines of regression of y on x and x on y. Also, estimate the number of bacterial cells after 15 *hours.*

Solution

$n = 10$

x	y	x^2	xy	y^2
0	43	0	0	1849
1	46	1	46	2116
2	82	4	164	6724
3	98	9	294	9604
4	123	16	492	15129
5	167	25	835	27889
6	199	36	1194	39601
7	213	49	1491	45369
8	245	64	1960	60025
9	272	81	2448	73984
$\sum x = 45$	$\sum y = 1488$	$\sum x^2 = 285$	$\sum xy = 8924$	$\sum y^2 = 282290$

$$b_{yx} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}$$

$$= \frac{8924 - \frac{(45)(1488)}{10}}{285 - \frac{(45)^2}{10}}$$

$$= 27.0061$$

$$b_{xy} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}$$

$$= \frac{8924 - \frac{(45)(1488)}{10}}{282290 - \frac{(1488)^2}{10}}$$

$$= 0.0366$$

$$\overline{x} = \frac{\sum x}{n} = \frac{45}{10} = 4.5$$

$$\overline{y} = \frac{\sum y}{n} = \frac{1488}{10} = 148.8$$

The equation of the line of regression of y on x is

$$y - \overline{y} = b_{yx}(x - \overline{x})$$

$$y - 148.8 = 27.0061\,(x - 4.5)$$

$$y = 27.0061x + 27.2726$$

The equation of the line of regression of x on y is

$$x - \overline{x} = b_{xy}(y - \overline{y})$$

$$x - 4.5 = 0.0366(y - 148.8)$$

$$x = 0.366\,y - 0.9461$$

At $\quad x = 15$ hours,

$y = 27.0061\ (15) + 27.2726 = 432.3641$

Example 9

Find the regression coefficient of y on x for the following data:

x	1	2	3	4	5
y	160	180	140	180	200

Solution

$$n = 5$$

$$\overline{x} = \frac{\sum x}{n} = \frac{15}{5} = 3$$

$$\overline{y} = \frac{\sum y}{n} = \frac{860}{5} = 172$$

x	y	$x-\overline{x}$	$y-\overline{y}$	$(x-\overline{x})^2$	$(x-\overline{x})(y-\overline{y})$
1	160	–2	–12	4	24
2	180	–1	8	1	–8
3	140	0	–32	0	0
4	180	1	8	1	8
5	200	2	28	4	56
$\sum x = 15$	$\sum y = 860$	$\sum(x-\overline{x}) = 0$	$\sum(y-\overline{y}) = 0$	$\sum(x-\overline{x})^2 = 10$	$\sum(x-\overline{x})(y-\overline{y}) = 80$

$$b_{yx} = \frac{\sum(x-\overline{x})(y-\overline{y})}{\sum(x-\overline{x})^2}$$

$$= \frac{80}{10}$$

$$= 8$$

Example 10

Calculate the two regression coefficients from the data and find correlation coefficient.

x	7	4	8	6	5
y	6	5	9	8	2

Solution

$$n = 5$$

$$\bar{x} = \frac{\sum x}{n} = \frac{30}{5} = 6$$

$$\bar{y} = \frac{\sum y}{n} = \frac{30}{5} = 6$$

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
7	6	1	0	1	0	0
4	5	–2	–1	4	1	2
8	9	2	3	4	9	6
6	8	0	2	0	4	0
5	2	–1	–4	1	16	4
$\sum x = 30$	$\sum y = 30$	$\sum(x-\bar{x}) = 0$	$\sum(y-\bar{y}) = 0$	$\sum(x-\bar{x})^2 = 10$	$\sum(y-\bar{y})^2 = 30$	$\sum(x-\bar{x})(y-\bar{y}) = 12$

$$b_{yx} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$$

$$= \frac{12}{10}$$

$$= 1.2$$

$$b_{xy} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(y-\bar{y})^2}$$

$$= \frac{12}{30}$$

$$= 0.4$$

$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{(1.2)(0.4)} = 0.693$$

Example 11

Obtain the two regression lines from the following data and hence, find the correlation coefficient.

x	6	2	10	4	8
y	9	11	5	8	7

[Summer 2015]

Solution

$$n = 5$$

$$\overline{x} = \frac{\sum x}{n} = \frac{30}{5} = 6$$

$$\overline{y} = \frac{\sum y}{n} = \frac{40}{5} = 8$$

x	y	$x-\overline{x}$	$y-\overline{y}$	$(x-\overline{x})^2$	$(y-\overline{y})^2$	$(x-\overline{x})(y-\overline{y})$
6	9	0	1	0	1	0
2	11	–4	3	16	9	–12
10	5	4	–3	16	9	–12
4	8	–2	0	4	0	0
8	7	2	–1	4	1	–2
$\sum x = 30$	$\sum y = 40$	$\sum(x-\overline{x}) = 0$	$\sum(y-\overline{y}) = 0$	$\sum(x-\overline{x})^2 = 40$	$\sum(y-\overline{y})^2 = 20$	$\sum(x-\overline{x})(y-\overline{y})^2 = -26$

$$b_{yx} = \frac{\sum (x-\overline{x})(y-\overline{y})}{\sum (x-\overline{x})^2} = \frac{-26}{40} = -0.65$$

$$b_{xy} = \frac{\sum (x-\overline{x})(y-\overline{y})}{\sum (y-\overline{y})^2} = \frac{-26}{20} = -1.3$$

The equation of regression line of y on x is

$$y - \overline{y} = b_{yx}(x - \overline{x})$$

$$y - 8 = -0.65(x - 6)$$

$$y = -0.65\,x + 11.9$$

The equation of regression line of x on y is

$$x - \overline{x} = b_{xy}(y - \overline{y})$$

$$x - 6 = -1.3(y - 8)$$

$$x = -1.3\,y + 16.4$$

$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{(-0.65)(-1.3)} = 0.9192$$

Since b_{yx} and b_{xy} are negative, r is negative.

$$r = -0.9192.$$

Example 12

Calculate the regression coefficients and find the two lines of regression from the following data:

x	57	58	59	59	60	61	62	64
y	67	68	65	68	72	72	69	71

Find the value of y when x = 66.

Solution

$$n = 8$$

$$\bar{x} = \frac{\sum x}{n} = \frac{480}{8} = 60$$

$$\bar{y} = \frac{\sum y}{n} = \frac{552}{8} = 69$$

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
57	67	–3	–2	9	4	6
58	68	–2	–1	4	1	2
59	65	–1	–4	1	16	4
59	68	–1	–1	1	1	1
60	72	0	3	0	9	0
61	72	1	3	1	9	3
62	69	2	0	4	0	0
64	71	4	2	16	4	8
$\sum x = 480$	$\sum y = 552$	$\sum(x-\bar{x}) = 0$	$\sum(y-\bar{y}) = 0$	$\sum(x-\bar{x})^2 = 36$	$\sum(y-\bar{y})^2 = 44$	$\sum(x-\bar{x})(y-\bar{y}) = 24$

$$b_{yx} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$$

$$= \frac{24}{36}$$

$$= 0.667$$

$$b_{xy} = \frac{\sum (x-\overline{x})(y-\overline{y})}{\sum (y-\overline{y})^2}$$
$$= \frac{24}{44}$$
$$= 0.545$$

The equation of regression line of y on x is

$$y - \overline{y} = b_{yx}(x - \overline{x})$$
$$y - 69 = 0.667(x - 60)$$
$$y = 0.667x + 28.98$$

The equation of regression line of x on y is

$$x - \overline{x} = b_{xy}(y - \overline{y})$$
$$x - 60 = 0.545(y - 69)$$
$$x = 0.545y + 22.395$$

Value of y when $x = 66$ is

$$y = 0.667\ (66) + 28.98 = 73.002$$

Example 13

The following data represents rainfall (x) and yield of paddy per hectare (y) in a particular area. Find the linear regression of x on y.

x	113	102	95	120	140	130	125
y	1.8	1.5	1.3	1.9	1.1	2.0	1.7

Solution

Let $a = 120$ and $b = 1.8$ be the assumed means of x and y series respectively.

$$d_x = x - a = x - 120$$
$$d_y = y - b = y - 1.8$$
$$n = 7$$

x	y	d_x	d_y	d_y^2	$d_x d_y$
113	1.8	–7	0	0	0
102	1.5	–18	–0.3	0.09	5.4
95	1.3	–25	–0.5	0.25	12.5
120	1.9	0	0.1	0.01	0
140	1.1	20	–0.7	0.49	–14
130	2.0	10	0.2	0.04	2.0
125	1.7	5	–0.1	0.01	–0.5
$\sum x = 825$	$\sum y = 11.3$	$\sum d_x = -15$	$\sum d_y = -1.3$	$\sum d_y^2 = 0.89$	$\sum d_x d_y = 5.4$

$$b_{xy} = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}$$

$$= \frac{5.4 - \frac{(-15)(-1.3)}{7}}{0.89 - \frac{(-1.3)^2}{7}}$$

$$= 4.03$$

$$\overline{x} = \frac{\sum x}{n} = \frac{825}{7} = 117.86$$

$$\overline{y} = \frac{\sum y}{n} = \frac{11.3}{7} = 1.614$$

The equation of the regression line of x on y is

$$x - \overline{x} = b_{xy}(y - \overline{y})$$

$$x - 117.86 = 4.03\,(y - 1.614)$$

$$x = 4.03\,y + 111.36$$

Example 14

Find the two lines of regression from the following data:

Age of husband (x)	25	22	28	26	35	20	22	40	20	18
Age of wife (y)	18	15	20	17	22	14	16	21	15	14

Hence, estimate (i) the age of the husband when the age of the wife is 19, *and (ii) the age of the wife when the age of the husband is* 30.

Solution

Let $a = 26$ and $b = 17$ be the assumed means of x and y series respectively.

$$d_x = x - a = x - 26$$
$$d_y = y - b = y - 17$$
$$n = 10$$

x	y	d_x	d_y	d_x^2	d_y^2	$d_x d_y$
25	18	–1	1	1	1	–1
22	15	–4	–2	16	4	8
28	20	2	3	4	9	6
26	17	0	0	0	0	0
35	22	9	5	81	25	45
20	14	–6	–3	36	9	18
22	16	–4	–1	16	1	4
40	21	14	4	196	16	56
20	15	–6	–2	36	4	12
18	14	–8	–3	64	9	24
$\sum x = 256$	$\sum y = 172$	$\sum d_x = -4$	$\sum d_y = 2$	$\sum d_x^2 = 450$	$\sum d_y^2 = 78$	$\sum d_x d_y = 172$

$$b_{yx} = \frac{\sum d_x d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sum d_x^2 - \dfrac{\left(\sum d_x\right)^2}{n}}$$

$$= \frac{172 - \dfrac{(-4)(2)}{10}}{450 - \dfrac{(-4)^2}{10}}$$

$$= 0.385$$

$$b_{xy} = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}$$

$$= \frac{172 - \frac{(-4)(2)}{10}}{78 - \frac{(2)^2}{10}}$$

$$= 2.227$$

$$\overline{x} = \frac{\sum x}{n} = \frac{256}{10} = 25.6$$

$$\overline{y} = \frac{\sum y}{n} = \frac{172}{10} = 17.2$$

The equation of the regression line of y on x is

$$y - \overline{y} = b_{yx}(x - \overline{x})$$

$$y - 17.2 = 0.385(x - 25.6)$$

$$y = 0.385x + 7.344$$

The equation of the regression line of x on y is

$$x - \overline{x} = b_{xy}(y - \overline{y})$$

$$x - 25.6 = 2.227(y - 17.2)$$

$$x = 2.227y - 12.704$$

Estimated age of the husband when the age of the wife is 19 is

$$x = 2.227\,(19) - 12.704 = 29.601 \text{ or } 30 \text{ nearly}$$

Age of the husband = 30 years

Estimated age of the wife when the age of the husband is 30 is

$$y = 0.385\,(30) + 7.344 = 18.894 \text{ or } 19 \text{ nearly}$$

Age of the wife = 19 years

Example 15

From the following data, obtain the two regression lines and correlation coefficient.

Sales (x)	100	98	78	85	110	93	80
Purchase (y)	85	90	70	72	95	81	74

Solution

Let $a = 93$ and $b = 81$ be the assumed means of x and y series respectively.

$$d_x = x - a = x - 93$$

$$d_y = y - b = y - 91$$

$$n = 7$$

x	y	d_x	d_y	d_x^2	d_y^2	$d_x d_y$
100	85	7	4	49	16	28
98	90	5	9	25	81	45
78	70	–15	–11	225	121	165
85	72	–8	–9	64	81	72
110	95	17	14	289	196	238
93	81	0	0	0	0	0
80	74	–13	–7	169	49	91
$\sum x = 644$	$\sum y = 567$	$\sum d_x = -7$	$\sum d_y = 0$	$\sum d_x^2 = 821$	$\sum d_y^2 = 544$	$\sum d_x d_y = 639$

$$b_{yx} = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sum d_x^2 - \frac{\left(\sum d_x\right)^2}{n}}$$

$$= \frac{639 - \frac{(-7)(0)}{7}}{821 - \frac{(-7)^2}{7}}$$

$$= 0.785$$

$$b_{xy} = \frac{\sum d_x dy - \frac{\sum d_x \sum d_y}{n}}{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}$$

$$= \frac{639 - \frac{(-7)(0)}{7}}{544 - \frac{(0)^2}{7}}$$

$$= 1.1746$$

$$\bar{x} = \frac{\sum x}{n} = \frac{644}{7} = 92$$

$$\bar{y} = \frac{\sum y}{n} = \frac{567}{7} = 81$$

The equation of regression line of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 81 = 0.785(x - 92)$$

$$y = 0.785x + 8.78$$

The equation of regression line of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 92 = 1.1746(y - 81)$$

$$x = 1.1746 - 3.1426$$

$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{(0.785)(1.1746)} = 0.9602$$

EXERCISE 9.3

1. The following are the lines of regression $4y = x + 38$ and $9y = x + 288$. Estimate y when $x = 99$ and x when $y = 30$. Also, find the means of x and y.

[**Ans.:** $y = 43, x = 82, \bar{x} = 162, \bar{y} = 50$]

2. The equations of the two lines of regression are $x = 19.13 - 0.87\,y$ and $y = 11.64 - 0.50\,x$. Find (i) the means of x and y, and (ii) the coefficient of correlation between x and y.

[**Ans.:** $\bar{x} = 15.79, \bar{y} = 3.74$, (ii) $r = -0.66, b_{yx} = -0.5, b_{xy} = 0.87$]

3. Given var(x) = 25. The equations of the two lines of regression are $5x - y = 22$ and $64\,x - 45\,y = 24$. Find (i) $\bar{x}$ and $\bar{y}$, (ii) r, and (iii) σ_y.

[**Ans.:** $\bar{x} = 6, \bar{y} = 8$, (ii) $r = 1.87$ (iii) $\sigma_y = 0.2$]

4. In a partially destroyed laboratory record of analysis of correlation data the following results are legible. Variance = 9, the equations of the lines of regression $4x - 5y + 33 = 0$, $20\,x - 9\,y - 107 = 0$. Find (i) the mean values of x and y, (ii) the standard deviation of y, and (iii) the coefficient of correlation between x and y

[**Ans.:** (i) $\bar{x} = 13, \bar{y} = 17$, (ii) $\sigma_y = 4$, (iii) $r = 0.6$]

5. From a sample of 200 pairs of observation, the following quantities were calculated:

$$\sum x = 11.34, \sum y = 20.78, \sum x^2 = 12.16, \sum y^2 = 84.96, \sum xy = 22.13$$

From the above data, show how to compute the coefficients of the equation $y = a + bx$.

[Ans.: $a = 0.0005, b = 1.82$ **]**

6. In the estimation of regression equations of two variables x and y, the following results were obtained:

$$\bar{x} = 90, \bar{y} = 70, n = 10, \Sigma(x - \bar{x})^2 = 6360, \Sigma(y - \bar{y})^2 = 2860$$
$$\Sigma(x - \bar{x})(y - \bar{y}) = 3900$$

Obtain the two lines of regression.

[Ans.: $x = 1.361\, y - 5.27$, $y = 0.613\, x + 14.812$**]**

7. Find the likely production corresponding to a rainfall of 40 cm from the following data:

	Rainfall (in cm)	Output (in quintals)
mean	30	50
SD	5	10
$r = 0.8$		

[Ans.: 66 quintals**]**

8. The following table gives the age of a car of a certain make and annual maintenance cost. Obtain the equation of the line of regression of cost on age.

Age of a car	2	4	6	8
Maintenance	1	2	2.5	3

[Ans.: $x = 0.325\, y + 0.5$**]**

9. Obtain the equation of the line of regression of y on x from the following data and estimate y for $x = 73$.

x	70	72	74	76	78	80
y	163	170	179	188	196	220

[Ans.: $y = 5.31\, x - 212.57$, $y = 175.37$**]**

10. The heights in cm of fathers (x) and of the eldest sons (y) are given below:

x	165	160	170	163	173	158	178	168	173	170	175	180
y	173	168	173	165	175	168	173	165	180	170	173	178

Estimate the height of the eldest son if the height of the father is 172 cm and the height of the father if the height of the eldest son is 173 cm. Also, find the coefficient of correlation between the heights of fathers and sons.

[Ans.: (i) $y = 1.016\,x - 5.123$ (ii) $x = 0.476\,y + 98.98$

(iii) 169.97, 173.45 (iv) $r = 0.696$**]**

11. Find (i) the lines of regression, and (ii) coefficient of correlation for the following data:

x	65	66	67	67	68	69	70	72
y	67	68	65	66	72	72	69	71

[Ans.: (i) $y = 19.64 + 0.72\,x$, $x = 33.29 + 0.5\,y$, (ii) $r = 0.604$**]**

12. Find the line of regression for the following data and estimate y corresponding to $x = 15.5$.

x	10	12	13	16	17	20	25
y	19	22	24	27	29	33	37

[Ans.: $y = 1.21x + 7.71$, $y = 26.465$**]**

13. The following data give the heights in inches (x) and weights in lbs (y) of a random sample of 10 students:

x	61	68	68	64	65	70	63	62	64	67
y	112	123	130	115	110	125	100	113	116	126

Estimate the weight of a student of height 59 inches.

[Ans.: 126.4 lbs**]**

14. Find the regression equations of y on x from the data given below taking deviations from actual mean of x and y.

Price in rupees (x)	10	12	13	12	16	15
Demand (y)	40	38	43	45	37	43

Estimate the demand when the price is ₹20.

[Ans.: $y = -0.25\,x + 44.25$, $y = 39.25$**]**

Points to Remember

Karl Pearson's Coefficient of Correlation

(i) $$r = \frac{\operatorname{cov}(X, Y)}{\sigma_X \sigma_Y}$$

(ii) $$r = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2}\sqrt{\sum (y-\bar{y})^2}}$$

(iii) $$r = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}\sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}}$$

(iv) $$r = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \frac{\left(\sum d_x\right)^2}{n}}\sqrt{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}}$$

Spearman's Rank Correlation Coefficient

$$r = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

Spearman's Rank Correlation Coefficient for Tied Ranks

$$r = 1 - \frac{6\left[\sum d^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \cdots\right]}{n(n^2-1)}$$

Lines of Regression

Line of Regression of *y* on *x*

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

It is also written as

$$y = a + bx$$

Line of Regression of *x* on *y*

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

It is also written as

$$x = a + by$$

Regression Coefficients

$$b_{yx} = r\,\frac{\sigma_y}{\sigma_x}$$

$$b_{xy} = r\,\frac{\sigma_x}{\sigma_y}$$

Expressions for Regression Coefficients

(i) $$b_{yx} = \frac{\sum (x-\overline{x})\,(y-\overline{y})}{\sum (x-\overline{x})^2}$$

and $$b_{xy} = \frac{\sum (x-\overline{x})(y-\overline{y})}{\sum (y-\overline{y})^2}$$

(ii) $$b_{yx} = \frac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}}$$

and $$b_{xy} = \frac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sum y^2 - \dfrac{\left(\sum y\right)^2}{n}}$$

(iii) $$b_{yx} = \frac{\sum d_x d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \dfrac{\left(\sum d_x\right)^2}{n}}}$$

and $$b_{xy} = \frac{\sum d_x d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_y^2 - \dfrac{\left(\sum d_y\right)^2}{n}}}$$

CHAPTER 10

Trend Analysis

Chapter Outline

10.1 INTRODUCTION

A series of observations on a variable, recorded after successive intervals of time, is called a *time-series*. A time-series is an arrangement of statistical data in accordance to the time of occurrence in a chronological order. The data on the population of India is a time-series data where time interval between two successive figures is 10 years. Similarly, figures of national income, agricultural and industrial production, etc., are available on yearly basis. Time-series analysis is done primarily for the purpose of making forcasts for the future and also for the purpose of evaluating past performances. The analysis of time-series plays an important role in the study of all economic, business, and natural and social sciences. Thus, time-series analysis is helpful in studying any phenomenon whose values are arranged chronologicallly over successive intervals of time.

10.2 OBJECTIVES OF TIME-SERIES

1. Analysis It helps in the analysis of past behaviour of data. Analysis of past data gives information about various factors which affects the data.

2. Forcasting It helps to make forcasts for the future. The analysis of past data is the basis of forecasting the future behaviour of the variable under study.

3. Evaluation It helps in the evaluation of current achievements. The review and evaluation of progress made on the basis of a plan are done on the basis of time-series data.

4. Comparison It helps in making comparative studies. Once the data is arranged chronologically, the comparison can be done. It provides a scientific basis for making comparisons by studying the effects of various components of a time-series.

10.3 COMPONENTS OF A TIME-SERIES

There are four components of a time-series:

1. Secular trend, or trend
2. Seasonal variations
3. Cyclical variations
4. Irregular variations

10.3.1 Secular Trend

Secular trend, or simply trend, is the general tendency of the data to increase or decrease or stegnate over a long period of time. Most of the business and economic time-series would reveal a tendency to increase or to decrease over a number of years. An *upward tendency* is usually upward in time-series relating to population, production and sales, prices, incomes, and money in circulation, while a *downward tendency* is noticed in the data of deaths and epidemics as a result of advancement in medical sciences, illiteracy, etc. Thus, a trend is either upward or downward. Hence, secular trend is that irreversible movement which continues, in general, in the same direction for a considerable period of time. Further, it is not necessary that increase or decrease should be in the same direction throughout the given period.

10.3.2 Seasonal Variations

Seasonal variations refer to such movements in a time-series which repeat themselves periodically in every season. These variations repeat themselves in less than one year. Seasonal variations are usually measured in an interval. The main causes of seasonal variations are climatic conditions, customs, and traditions. The changes in climatic conditions affect the value of a time-series variable. For example, the sale of woolen garments is generally at its peak in the months of November–December. The customs and traditions of people also give rise to the seasonal variations in time-series. For example, the sale of garments and ornaments may be highest during the marriage season and Diwali. Both the causes are often repeated after a gap of less than or equal to one year.

10.3.3 Cyclical Variations

Cyclical variations are the oscillatory movements in a time-series with the period of oscillation greater than one year. These variations in a time-series are due to ups and downs recurring after a period greater than one year. Normally, the period of cyclical variations lies between 3 to 10 years.

The main objectives of measuring cyclical variations are:

(i) To analyse the behaviour of cyclical variations in the past
(ii) To predict the effect of cyclical variations for future business policies

10.3.4 Irregular Variations

Irregular variations do not exhibit any regular pattern of movements and there is no regular period or time of their occurrence. These variations are caused by random factors such as strikes, floods, fire, war, famines, etc. These are accidental changes which are purely random, unforseen, and unpredictable. Normally, they are short-term variations but sometimes their effect may be so intense that the value of trend may get permanently affected.

10.4 MEASUREMENT OF TREND

The following methods are used to measure the trend:

1. Freehand or graphic method
2. Method of semi-averages
3. Method of moving averages
4. Method of least squares

10.5 FREEHAND OR GRAPHIC METHOD

This is the simplest method of studying the trends. The given series data are plotted on a graph paper by taking time on the x-axis and the other variable on the y-axis. A smooth line or curve, drawn through the plotted points, gives the trend of the given data. It is a very simple method of estimating the trend which requires no mathematical calculations. It is a flexible method as compared to rigid mathematical trends and can be used to describe all types of trends. The strongest objection to this method is that it is highly subjective in nature. The values of trends, obtained by different persons would be different and, hence, not reliable. Predictions made on the basis of this method are of little value.

Example 1

Determine the trend of the following time-series by the graphical method:

Years	1978	1979	1980	1981	1982	1983	1984	1985	1986	1987	1988
Sales (in thousand)	60	80	70	100	80	120	110	140	130	160	150

Solution

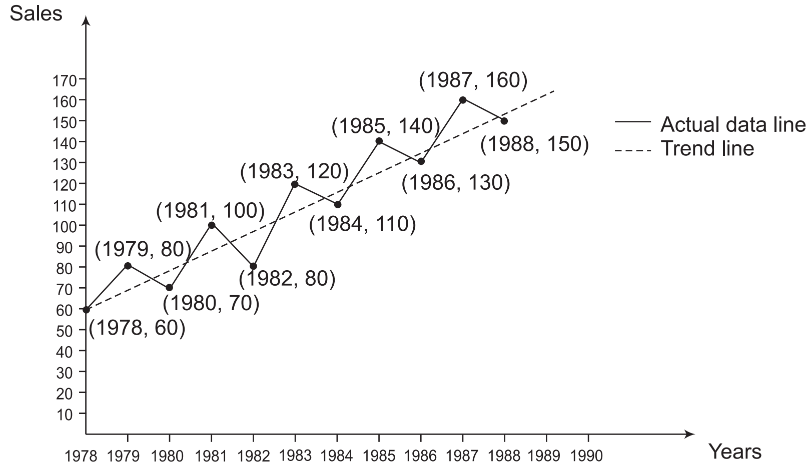

Fig. 10.1

Example 2

Determine the trend of the following times-series data by the freehand curve method:

Years	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991
Values	128	164	194	142	156	224	230	262	176	260

Solution

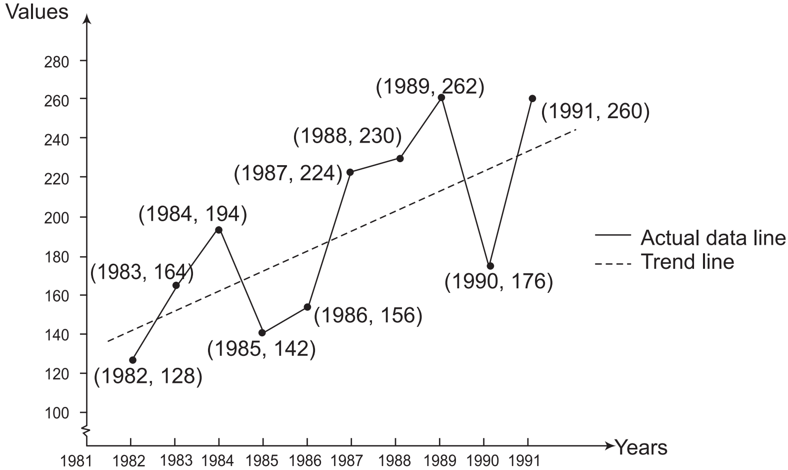

Fig. 10.2

10.6 METHOD OF SEMI-AVERAGES

In this method, semi-averages are calculated to find out the trend values. Semi-averages are the averages of the two halves of a series, i.e., the whole times series is classified into two equal parts with respect to time. Each semi-average is paired with the centre of time period of its part. The two pairs are then plotted on a graph paper and the points are joined by a straight line to find the trend. In case of odd number of observations the two equal parts are obtained by excluding the middle-most observation.

It is a simple method of measuring trends. It is an objective method because any two persons will get the same trend line from a set of figures. This method assumes the presence of *linear trend* which would not be true in many cases. The trend values obtained by this method and the predicted values for the future are not precise and reliable.

Example 1

Fit a trend line to the following data by the method of semi-averages:

Years	1982	1983	1984	1985	1986	1987	1988	1989	1990
Output (in tons)	30	40	35	55	45	50	64	50	60

Solution

Since the data are given for nine years, the middle year is excluded for the calculation.

Average output of the first four years = $\dfrac{30+40+35+55}{4} = 50$

Average output of the last four years = $\dfrac{50+64+50+60}{4} = 56$

The semi-average of first four years, i.e., 40 is plotted against the mid-point of the first four years, i.e., 1983.5 and the semi-average of the last four years, i.e., 56, is plotted against the mid-point of the last four years, i.e., 1988.5. By joining these two points, a trend line is obtained (Fig. 10.3).

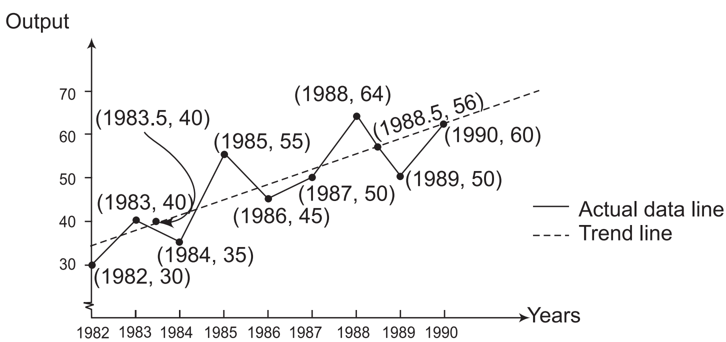

Fig. 10.3

Example 2

Draw a trend line by the method semi-averages.

Years	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991
Sales (in thousands)	210	200	215	205	220	235	210	235	225	245

Solution

Since the data are given for ten years, there are two halves of 5 years each.

Average sales of the first five years = $\dfrac{210+200+215+205+220}{5} = 210$

Average sales of the second five years = $\dfrac{235+210+235+245}{5} = 230$

The semi-average of first five years, i.e., 210 is plotted against the mid-point of the first five years, i.e., 1984. The semi-average of the second five years, i.e., 230 is plotted against the mid-point of the second five years, i.e., 1989. By joining these two points, a trend line is obtained (Fig. 10.4).

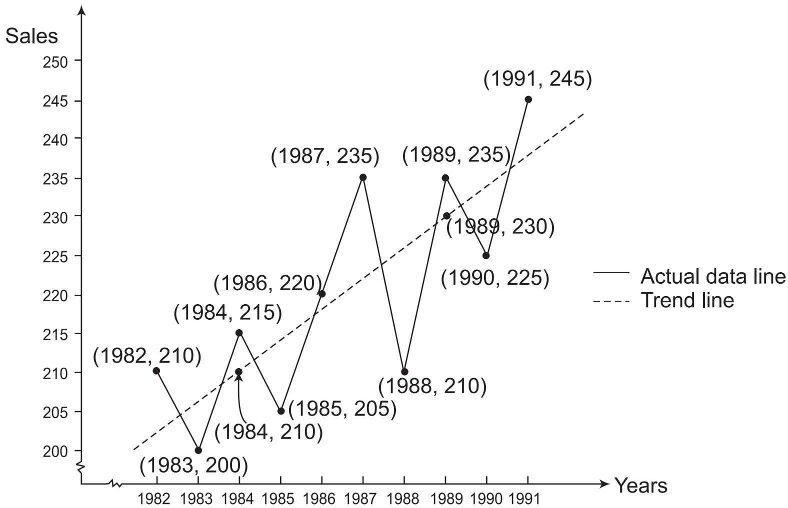

Fig. 10.4

10.7 METHOD OF MOVING AVERAGES

The method of moving averages is a simple method for reducing fluctuations and obtaining trend values with a fair degree of accuracy. In this method, the arithmetic mean of the values for a certain span of time is taken and then it is placed at the centre of the time span. The average value of a number of years is taken as the trend value for the middle point of the period of moving averages. The process of averaging smoothens the curve and reduces the fluctuations. The period of moving averages can be 3-yearly moving averages, 4-yearly moving averages, 5-yearly moving averages, 7-yearly moving averages, etc.

Let Y_1, Y_2, ..., Y_n be the n values of a time-series for successive time periods 1, 2, ..., n respectively. The calculation of 3-yearly moving averages and 4-yearly moving averages are shown in the following tables:

Time period	Values of Y	3-yearly moving average	Time period	Values of Y	4-yearly moving averages	Centred values
1	Y_1	–	1	Y_1	–	–
2	Y_2	$\frac{Y_1+Y_2+Y_3}{3}$	2	Y_2	–	–
					$\frac{Y_1+Y_2+Y_3+Y_4}{4}=A_1$	
3	Y_3	$\frac{Y_2+Y_3+Y_4}{3}$	3	Y_3		$\frac{A_1+A_2}{2}$
					$\frac{Y_2+Y_3+Y_4+Y_5}{4}=A_2$	
4	Y_4	$\frac{Y_3+Y_4+Y_5}{3}$	4	Y_4		$\frac{A_2+A_3}{2}$
					$\frac{Y_3+Y_4+Y_5+Y_6}{4}=A_3$	
5	Y_5	$\frac{Y_4+Y_5+Y_6}{3}$	5	Y_5		
⋮	⋮	⋮	⋮	⋮	⋮	⋮
n	Y_n	–	6	Y_n	–	–

In case of 3-yearly moving averages, it is not possible to get the moving averages for the first and the last period. Similarly, larger the period of the moving average, more will be the information loss at the end of a time-series.

When the period of moving average is even, the computed average will correspond to the middle of the two middle-most periods. These values should be centred by taking the arithmetic mean of the two successive averages. Hence, in case of even period of moving averages, the trend values are obtained after centering the averages.

Example 1

Calculate the 3-yearly moving averages of the following data:

Years	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989
Sales (millions of rupees)	3	4	8	6	7	11	9	10	14	12

Solution

3-yearly Moving Average

Year	Sales (millions of rupees)	3-yearly moving total	3-yearly moving average (trend value)
1980	3	–	–
1981	4	15	5
1982	8	18	6
1983	6	21	7
1984	7	24	8
1985	11	27	9
1986	9	30	10
1987	10	33	11
1988	14	36	12
1989	12	–	–

Example 2

Calculate the 5-yearly moving averages of the number of students passing from a college from the following data:

Year	Number of students	Year	Number of students
2003	332	2008	405
2004	317	2009	410
2005	357	2010	427
2006	392	2011	405
2007	402	2012	438

[Winter 2012]

Solution

5-yearly Moving Averages

Year	Number of students	Five-early moving total	Five-yearly moving average (Trend value)
2003	332	–	–
2004	317	–	–
2005	357	1800	360
2006	392	1873	374.6
2007	402	1966	393.2
2008	405	2036	407.2
2009	410	2049	409.8
2010	427	2085	417
2011	405	–	–
2012	438	–	–

Example 3

Calculate the 7-yearly moving averages for the following data showing the number of students of an engineering college clearing GATE:

Year	Number of students	Year	Number of students
1999	23	2007	9
2000	26	2008	13
2001	28	2009	11
2002	32	2010	14
2003	20	2011	12
2004	12	2012	9
2005	12	2013	3
2006	10	2014	1

[Summer 2014, Winter 2014]

Solution

7-yearly Moving Average

Year	Number of students	Seven-yearly moving total	Seven-yearly moving average (Trend value)
1999	23	–	–
2000	26	–	–
2001	28	–	–
2002	32	153	21.86
2003	20	140	20
2004	12	123	17.57
2005	12	108	15.43
2006	10	87	12.43
2007	9	81	11.57
2008	13	81	11.57
2009	11	78	11.14
2010	14	71	10.14
2011	12	63	9
2012	9	–	–
2013	3	–	–
2014	1	–	–

Example 4

Assume a four-yearly cycle and calculate the trend by the method of moving averages from the following data relating to the production in pen drives in India:

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Production (million kgs)	464	515	518	467	502	540	557	571	586	612

[Summer 2015]

Solution

Four-yearly Moving Average

Year	Production (million kgs)	Four Yearly Moving total	Four Yearly Moving Average	Centred Value (Trend value)
2000	465	–	–	–
2001	515	–	–	–
		1965	491.25	
2002	518			495.875
		2002	500.5	
2003	467			503.625
		2027	506.75	
2004	502			511.625
		2066	516.5	
2005	540			529.5
		2170	542.5	
2006	557			553
		2254	563.5	
2007	571			572.5
		2326	581.5	
2008	586	–	–	–
2009	612	–	–	–

Example 5

Compute the 4-yearly moving averages from the following data:

Year	1991	1992	1993	1994	1995	1996	1997	1998
Annual sales (₹ in crores)	36	43	43	34	44	54	34	24

Solution

4-yearly Moving Average

Year	Annual sales (₹ in crores)	4-yearly moving total	4-yearly moving average	4-yearly centred moving averages (Trend value)
1991	36	–	–	–
1992	43	–	–	–
		156	39	
1993	43			40
		164	41	
1994	34			42.375
		175	43.75	
1995	44			42.625
		166	41.50	
1996	54			40.25
		156	39	
1997	34	–	–	–
1998	24	–	–	–

Weighted Moving Average

The weighted moving average is obtained on dividing the weighted moving totals by the sum of weights. Let $x_1, x_2, ..., x_n$ occur with weights $w_1, w_2, ..., w_n$ respectively.

$$\text{Weighted moving average} = \frac{\sum W_i x_i}{\sum W_i} = \frac{\sum Wx}{W}$$

A weighted moving average with appropriate weights is used when the moving averages are strongly affected by extreme values.

Example 1

Find the trend for the following series using 3*-year weighted moving averages with weights* 1, 2, 1:

Year	1991	1992	1993	1994	1995	1996	1997
Values	2	4	5	7	8	10	13

Solution

Total weights = 1 + 2 + 1 = 4

Year	Values	3-yearly weighted moving total	3-yearly weighted moving average
1991	2	–	–
1992	4	(2 × 1) + (4 × 2) + (5 × 1) = 15	15 ÷ 4 = 3.75
1993	5	(4 × 1) + (5 × 2) + (7 × 1) = 21	21 ÷ 4 = 5.25
1994	7	(5 × 1) + (7 × 2) + (8 × 1) = 27	27 ÷ 4 = 6.75
1995	8	(7 × 1) + (8 × 2) + (10 × 1) = 33	33 ÷ 4 = 8.25
1996	10	(8 × 1) + (10 × 2) + (13 × 1) = 41	41 ÷ 4 = 10.25
1997	13	–	–

Example 2

Calculate 5*-year weighted moving averages for the following data using weights* 1, 1, 3, 2, 1 *respectively.*

Year	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Coded sales	40	33	72	81	76	68	91	87	98	97

Solution

Total weights = 1 + 1 + 3 + 2 + 1 = 8

Year	Sales	5-yearly weighted moving total	5-yearly weighted moving average
1	40	–	–
2	33	–	–
3	72	(40 × 1) + (33 × 1) + (72 × 3) + (81 × 2) + (76 × 1) = 527	527 ÷ 8 = 65.875
4	81	(33 × 1) + (72 × 1) + (81 × 3) + (76 × 2) + (68 × 1) = 568	568 ÷ 8 = 71
5	76	(72 ×1) + (81 ×1) + (76 × 3) + (68 × 2) + (91 × 1) = 630	630 ÷ 8 = 78.75
6	68	(81 × 1) + (76 × 1) + (68 × 3) + (91 × 2) + (87 × 1) = 630	630 ÷ 8 = 78.75
7	91	(76 × 1) + (68 × 1) + (91 × 3) + (87 × 2) + (98 × 1) = 689	689 ÷ 8 = 86.125
8	87	(68 × 1) + (91 × 1) + (87 × 3) + (98 × 2) + (97 × 1) = 713	713 ÷ 8 = 89.125
9	98	–	–
10	97	–	–

Example 3

Consider the following data and show that a 4-year centred moving average is equivalent to a 5-year weighted moving average with weights 1, 2, 2, 2, 1 *respectively.*

Year	1983	1984	1985	1986	1987	1988	1989	1990	1991	1992	1993
Annual sales	7	6	1	5	3	7	2	6	4	8	3

Solution

Total weights = 1 + 2 + 2 + 2 + 1 = 8

5-year Weighted Moving Averages

Year	Annual sales	5-year weighted moving total	5-year weighted moving average
1983	7	–	–
1984	6	–	–
1985	1	$(7 \times 1) + (6 \times 2) + (1 \times 2) + (5 \times 2) + (3 \times 1) = 34$	$34 \div 8 = 4.25$
1986	5	$(6 \times 1) + (1 \times 2) + (5 \times 2) + (3 \times 2) + (7 \times 1) = 31$	$31 \div 8 = 3.875$
1987	3	$(1 \times 1) + (5 \times 2) + (3 \times 2) + (7 \times 2) + (2 \times 1) = 33$	$33 \div 8 = 4.125$
1988	7	$(5 \times 1) + (3 \times 2) + (7 \times 2) + (2 \times 2) + (6 \times 1) = 35$	$35 \div 8 = 4.375$
1989	2	$(3 \times 1) + (7 \times 2) + (2 \times 2) + (6 \times 2) + (4 \times 1) = 37$	$37 \div 8 = 4.625$
1990	6	$(7 \times 1) + (2 \times 2) + (6 \times 2) + (4 \times 2) + (8 \times 1) = 39$	$39 \div 8 = 4.875$
1991	4	$(2 \times 1) + (6 \times 2) + (4 \times 2) + (8 \times 2) + (3 \times 1) = 41$	$41 \div 8 = 5.125$
1992	8	–	–
1993	3	–	–

4-year Centred Moving Averages

Year	Annual sales	4-years moving total	4-years moving average	4-years centered moving average
1983	7	–	–	–
1984	6	–	–	–
		19	4.75	
1985	1			4.25
		15	3.75	
1986	5			3.875
		16	4	
1987	3			4.125
		17	4.25	
1988	7			4.375
		18	4.5	
1989	2			4.625
		19	4.75	
1990	6			4.875
		20	5	
1991	4			5.125
		21	5.25	
1992	8	–	–	–
1993	3	–	–	–

From the two tables, it is clear that a 4-year centred moving average is equivalent to a 5-year moving average with weights 1, 2, 2, 2, 1, respectively.

10.8 METHOD OF LEAST SQUARES

This is the best method for obtaining the trend values. This method provides a line of best fit in a series. The line of best fit is a line from which the sum of the deviations of various points on its either side is zero and the sum of the squares of these deviations are minimum as compared to the sum of the squares of the deviations obtained by using other lines.

10.8.1 Fitting of Linear Trend

When the data is given for finding the trend, the straight-line trend equation fitted to the data is

$$Y = a + bX$$

where Y represents the estimated values of the trend, X represents the deviations in time period; and a and b are constants. The values of two constants a and b are estimated by solving two normal equations.

$$\sum Y = na + b\sum X$$
$$\sum XY = a\sum X + b\sum X^2$$

where n represents the number of years for which data is given. The variable X can be measured from any point of time as the origin. It is better to take the mid-point of time as the origin which gives $\sum X = 0$.

When $\sum X = 0$, the two normal equations are

$$\sum Y = na$$
$$\therefore \quad a = \frac{\sum Y}{n}$$
$$\text{and} \quad \sum XY = b\sum X^2$$
$$\therefore \quad b = \frac{\sum XY}{\sum X^2}$$

The constant a gives Y-intercept and the constant b gives the slope of the line which indicates the change in Y for each unit change in X.

10.8.2 Fitting a Straight-line Trend for Even Number of Years

If the number of years is even, there is no middle years. In this case, the mid-point which is taken as the origin, lies midway between the two middle years. For example, if the two middle years are 1997 and 1998, the midpoint lies midway between 1 January, 1997 and 1 January, 1998, which is 1 July, 1997. To avoid fractions, the unit of X is taken as $\frac{1}{2}$ year.

Example 1

Find the equation of a straight line which best fits the following data:

Year	2000	2001	2002	2003	2004
Sales (in ₹ thousand)	35	56	79	80	40

Compute the trend values for all the years from 2000 *to* 2004.

Solution

Let the origin = 2002

unit = 1 year

Let the straight-line trend equation fitted to the data be

$$Y = a + bX$$

The normal equations are

$$\sum Y = na + b\sum X \quad \text{...(1)}$$

$$\sum XY = a\sum X + b\sum X^2 \quad \text{...(2)}$$

Here, $n = 5$

Year x	Sales Y	$X = x - 2002$	X^2	XY
2000	35	–2	4	–70
2001	56	–1	1	–56
2002	79	0	0	0
2003	80	1	1	80
2004	40	2	4	80
	$\sum Y = 290$	$\sum X = 0$	$\sum X^2 = 10$	$\sum XY = 34$

Substituting these values in Eqs (1) and (2),

$$290 = 5a$$

$\therefore$ $a = 58$

and $34 = 10\,b$

$\therefore$ $b = 3.4$

Hence, the required equation of the straight-line trend is

$$Y = 58 + 3.4\,X$$

Trend values for the years from 2000 to 2004

Year x	X	Trend values Y
2000	–2	58 + 3.4(–2) = 51.2
2001	–1	58 + 3.4(–1) = 54.6
2002	0	58 + 3.4(0) = 58
2003	1	58 + 3.4(1) = 61.4
2004	2	58 + 3.4(2) = 64.8

Example 2

Using 1991 as the origin, obtain a linear trend equation by the method of least squares:

Year	1987	1989	1990	1991	1992	1993	1996
Value	140	144	160	152	168	176	180

Find the trend value for the missing year 1988.

Solution

Let the origin = 1991

unit = 1 year

Let the straight line trend equation fitted to the data be

$$Y = a + bX$$

The normal equations are

$$\sum Y = na + b\sum X \qquad ...(1)$$

$$\sum XY = a\sum X + b\sum X^2 \qquad ...(2)$$

Here, $n = 7$

Year x	Value Y	$X = x - 1991$	X^2	XY
1987	140	–4	16	– 560
1989	144	– 2	4	– 28
1990	160	–1	1	–160
1991	152	0	0	0
1992	168	1	1	168
1993	176	2	4	352
1996	180	5	25	90
	$\sum Y = 1120$	$\sum X = 1$	$\sum X^2 = 51$	$\sum XY = 412$

Substituting these values in Eqs (1) and (2),

$$1120 = 7a + b \qquad ...(3)$$

$$412 = a + 51b \qquad ...(4)$$

Solving Eqs (3) and (4),

$$a = 159.29$$

$$b = 4.96$$

Hence, the required equation of the straight-line trend is

$$Y = 159.29 + 4.96\ X$$

Estimated value for 1988

$$X = 1988 - 1991 = -3$$

$$Y = 159.29 + 4.96(-3) = 144.41$$

Example 3

The sales of a company in millions of rupees for the years 1994–2001 *are given below:*

Year	1994	1995	1996	1997	1998	1999	2000	2001
Sales	550	560	555	585	540	525	545	585

Find the linear trend equation. Estimate the sales for the year 1993. *Find the slope of the straight-line trend.*

[Summer 2015]

Solution

Let the origin = 1 July 1997

$$\text{unit} = \frac{1}{2}\text{ year}$$

Let the straight-line trend equation fitted to the data be

$$Y = a + bX$$

The normal equations are

$$\sum Y = na + b\sum X \quad \text{...(1)}$$

$$\sum XY = a\sum X + b\sum X^2 \quad \text{...(2)}$$

Here, $n = 8$

Year x	Sales Y	$d = x - 1997.5$	$X = 2d$	X^2	XY
1994	550	–3.5	–7	49	–3850
1995	560	–2.5	–5	25	–2800
1996	555	–1.5	–3	9	–1665
1997	585	–0.5	–1	1	–585
1998	540	0.5	1	1	540
1999	525	1.5	3	9	1575
2000	545	2.5	5	25	2725
2001	585	3.5	7	49	4095
	$\sum Y = 4445$		$\sum X = 0$	$\sum X^2 = 168$	$\sum XY = 35$

Substituting these values in Eqs (1) and (2),

$$4445 = 8a$$

$\therefore$ $a = 555.625$

and $35 = 168\,b$

$\therefore$ $b = 0.2083$

Hence, the required equation of the straight-line trend is

$$Y = 555.625 + 0.2083\,X$$

Estimate of Y when $x = 1993$

$$d = 1993 - 1997.5 = -4.5,\ X = -9$$

$$Y_{1993} = 555.625 + 0.2083\,(-9) = 553.7503$$

Slope of the straight-line trend = b = 0.2083.

Example 4

Fit a straight-line trend equation by the method of least squares and estimate the trend values.

Year	1981	1982	1983	1984	1985	1986	1987	1988
Value	80	90	92	83	94	99	92	104

Solution

Let the origin = 1 July 1984

$$\text{unit} = \frac{1}{2}\text{ year}$$

Let the straight-line trend equation fitted to the data be

$$Y = a + bX$$

The normal equations are

$$\sum Y = na + b\sum X \qquad \text{...(1)}$$

$$\sum XY = a\sum X + b\sum X^2 \qquad \text{...(2)}$$

Here, $n = 8$

Year x	Value Y	$d = x - 1984.5$	$X = 2d$	X^2	XY
1981	80	–3.5	–7	49	–560
1982	90	–2.5	–5	25	–450
1983	92	–1.5	–3	9	–276
1984	83	–0.5	–1	1	–89
1985	94	0.5	1	1	94
1986	99	1.5	3	9	297
1987	92	2.5	5	25	460
1988	104	3.5	7	49	728
	$\sum Y = 734$		$\sum X = 0$	$\sum X^2 = 168$	$\sum XY = 210$

Substituting these values in Eqs (1) and (2),

$$734 = 8a$$

$$\therefore \quad a = 91.75$$

and

$$210 = 168\, b$$

$$\therefore \quad b = 1.25$$

Hence, the required equation of the straight-line trend is

$$Y = 91.75 + 1.25\, X$$

Trend Values

Year	X	Trend values
1981	–7	91.75 + 1.25(–7) = 83
1982	–5	91.75 + 1.5(–5) = 85.5
1983	–3	91.75 + 1.5(–3) = 88
1984	–1	91.75 + 1.25(–1) = 90.5
1985	1	91.75 + 1.25(1) = 93
1986	3	91.75 + 1.25(3) = 95.5
1987	5	91.75 + 1.25(5) = 98
1988	7	91.75 + 1.25(7) = 100.5

10.9 MEASUREMENT OF SEASONAL VARIATIONS

The following methods are used to measure seasonal variations:

1. Method of simple averages
2. Method of ratio to trend
3. Method of ratio to moving average
4. Method of link relative

Out of the above methods, we will study the method of ratio to moving average.

10.10 METHOD OF RATIO TO MOVING AVERAGE

This method is also known as *percentage of moving averages method*. This method is most widely used for measuring seasonal variations/fluctuations. The steps involved in the computation of seasonal indices are as follows:

(i) Obtain the centred 12 months (4 quarters) moving average values for the given series.

(ii) Express each original value of the time-series as a percentage of the trend value.

(iii) Arrange these percentages seasonwise for all the years and then calculate the average of these percentages. The resultant percentages would be seasonal indices.

(iv) Obtain the adjusted seasonal index using the following formula:

$$\text{Adjusted Seasonal Index} = \frac{\text{Average Seasonal Index} \times 400}{\text{Total of Average of Seasonal Index}} \quad \text{[For quarterly data]}$$

$$\text{Adjusted Seasonal Index} = \frac{\text{Average Seasonal Index} \times 100}{\text{Total of Average of Seasonal Index}} \quad \text{[For monthly data]}$$

Example 1

Calculate seasonal indices by the ratio-to-moving average method from the following data:

Year	I Quarter	II Quarter	III Quarter	IV Quarter
1991	68	62	61	63
1992	65	58	66	61
1993	68	63	63	67

Solution

Calculation of Ratio to Moving Averages

Year	Quarter	Values	4-Quarters moving total	4-Quarters moving total centered	4-quarters moving average centered (Col 5 ÷ 8)	Given values as percentage of centered moving averages
1991	I	68	–	–	–	–
	II	62		–	_	–
			254			
	III	61		505	63.125	96.63
			251			
	IV	63		498	62.25	101.20
			247			
1992	I	65		499	62.375	104.21
			252			
	II	58		502	62.75	92.43
			250			
	III	66		503	62.875	104.97
			253			
	IV	61		511	63.875	95.5
			258			
1993	I	68		513	64.125	106.04
			255			
	II	63		516	64.5	97.67
			261			
	III	63	_	_	–	–
	IV	67	_	_	–	–

Calculation of Seasonal Indices

Year	I	II	III	IV
1991	–	–	96.63	101.20
1992	104.21	92.43	104.97	95.5
1993	106.04	97.67	–	–
Total	210.25	190.1	201.6	196.7
Average Seasonal Index	105.125	95.05	100.8	98.35
Adjusted Seasonal Index	105.3	95.21	100.97	98.52

The adjustment of seasonal index is required because the total of averages of seasonal index, i.e., 105.125 + 95.05 + 100.8 + 98.35 = 399.325 which is less than 400.

$$\text{Correction factor} = \frac{400}{399.325}$$

$$\text{Adjusted seasonal index for Quarter I} = \frac{105.125 \times 400}{399.325} = 105.3$$

$$\text{Adjusted seasonal index for Quarter II} = \frac{95.05 \times 400}{399.325} = 95.21$$

$$\text{Adjusted seasonal index for Quarter III} = \frac{100.8 \times 400}{399.325} = 100.97$$

$$\text{Adjusted seasonal index for Quarter IV} = \frac{98.35 \times 400}{399.325} = 98.52$$

Example 2

Obtain seasonal fluctuations from the following data using the moving-average method:

	Quarterly output of commodity (tons)			
Year	I	II	III	IV
1984	65	58	56	61
1985	68	63	63	67
1986	70	59	56	52
1987	60	55	51	58

Solution

Calculation of Ratio to Moving Averages

Year	Quarter	Output	4-Quarter moving total	4-Quarter moving total centered	4-Quarter moving average centered (Col. 5 ÷ 8)	Given output as percentage of centered of moving average
1984	I	65	–	–	–	–
	II	58		–	–	–
			240			
	III	56		483	60.375	92.75
			243			
	IV	61		491	61.375	99.39
			248			
1985	I	68		503	62.875	108.15
			255			
	II	63		516	64.5	97.67
			261			
	III	63		524	65.5	96.18
			263			
	IV	67		522	65.25	102.68
			259			
1986	I	70		511	63.875	109.59
			252			
	II	59		489	61.125	96.52
			237			
	III	56		464	58	96.55
			227			
	IV	52		450	56.25	92.44
			223			
1987	I	60		441	55.125	108.84
			218			
	II	55		442	55.25	99.55
			224			
	III	51	–	–	–	–
	IV	58	–	–	–	–

Calculation of Seasonal Indices

Year	I	II	III	IV
1984	–	–	92.75	99.39
1985	108.15	97.67	96.18	102.68
1986	109.59	96.52	96.55	92.44
1987	108.84	99.55	–	–
Total	326.58	293.74	285.48	294.51
Average Seasonal Index	108.86	97.91	95.16	98.17
Adjusted Seasonal Index	108.83	97.89	95.14	98.15

The adjustment of seasonal index is required because the total of averages of seasonal index, i.e., 108.86 + 97.91 + 95.16 + 98.17 = 400.1 which is more than 400.

$$\text{Correction factor} = \frac{400}{400.1}$$

$$\text{Adjusted seasonal index for Quarter I} = \frac{108.86 \times 400}{400.1} = 108.83$$

$$\text{Adjusted seasonal index for Quarter II} = \frac{97.91 \times 400}{400.1} = 97.89$$

$$\text{Adjusted seasonal index for Quarter III} = \frac{95.16 \times 400}{400.1} = 95.14$$

$$\text{Adjusted seasonal index for Quarter IV} = \frac{98.17 \times 400}{400.1} = 98.15$$

EXERCISE 10.1

1. Calculate the 3-yearly moving averages of the following data:

(i)

Year	1998	1999	2000	2001	2002	2003	2004	2005
Values	3	5	7	10	12	14	15	16

[**Ans.:** 5, 7.33, 9.67, 12, 13.67, 15]

(ii)

Year	2006	2007	2008	2009	2010	2011	2012
Values	2	4	5	7	8	10	13

[**Ans.:** 3.67, 5.33, 6.67, 8.33, 10.33]

2. Calculate 5-yearly moving averages of the following data:

(i)

Year	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998
Sales	57	62	54	68	63	74	60	71	68	66

[Ans.: 60.8, 64.2, 63.8, 67.2, 67.2, 67.8]

(ii)

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
No. of students	332	317	357	392	402	405	410	427	405	438

[Ans.: 360, 374.6, 393.2, 407.2, 409.8, 417.0]

(iii)

Year	1987	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002
Output	4	5	6	7	9	6	5	7	8	7	6	8	9	10	7	9

[Ans.: 6.2, 6.6, 6.6, 6.8, 7, 6.6, 6.6, 7.2, 7.6, 8, 8, 8.6]

3. The following table shows the average monthly production of coal in millions of tonnes for the year 2005:

Year	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
Production (in million tonnes)	50	36.5	43	44.5	38.9	38.1	32.6	41.7	41.1	33.8

Determine the trend values using the 4-yearly moving-average method.

[Ans.: 42.1, 40.9, 39.8, 38.15, 38.14, 37.85]

4. Calculate trend values from the following data relating to the production of tea in India by the moving-average method, on the assumption of a 4-yearly cycle:

Year	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006
Production	464	515	518	467	502	540	557	571	586	612

[Ans.: 495.8, 503.6, 511.6, 529.5, 553, 572.5]

5. From the following data, calculate the trend values using the four-yearly moving-average method:

Year	1989	1990	1991	1992	1993	1994	1995	1996	1997
Values	506	620	1036	673	588	696	1116	738	663

[Ans.: 708.75, 729.25, 748.25, 768.25, 784.5]

6. Find the 3-yearly weighted moving average with weights 1, 4, 1 for the following series:

Year	2001	2002	2003	2004	2005	2006	2007
Values	2	6	1	5	3	7	2

[Ans.: 4.5, 3.5, 4, 4, 5.5]

7. For the following data, verify that the 5-year weighted moving average with weights 1, 2, 3, 3, 1 respectively is equivalent to the 4-year centred moving average.

Year	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Sales (in lakhs)	5	3	7	6	4	8	9	10	8	9

8. For the following series, verify that the 6-yearly centred moving average is equivalent to a 7-yearly weighted moving average with weights 1, 2, 2, 2, 2, 2, 1.

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
Sales (in thousands)	2	4	3	6	7	9	4	6	7	8	10

9. Fit a linear trend equation to the following data. Hence, estimate the value of sales for year 2007.

Year	2001	2002	2003	2004	2005
Sales (in lakhs)	100	120	140	160	180

[Ans.: $Y = 140 + 20X$, 220 lakhs]

10. The following table shows the figures of production of a commodity during the years 1989–1996 in the state of Punjab:

Year	1989	1990	1991	1992	1993	1994	1995	1996
Production (in thousand tonnes)	38	40	65	72	69	60	87	95

Use the method of least squares to fit a straight line to the data.

[Ans.: $Y = 65.75 + 3.667\ X$]

11. Fit a straight-line trend equation by the method of least squares to the following data:

Year	1980	1981	1982	1983	1984	1985	1986	1987
Value	380	400	650	720	690	600	870	930

[Ans.: Y = 655 + 35.838]

12. Fit a straight-line trend to the data and estimate the profit for the year 1997.

Year	1990	1991	1992	1993	1994	1995	1996
Profit (in lakhs)	60	72	75	65	80	85	95

[Ans.: Y = 76 + 4.857 X, ₹ 95.428 lakhs]

13. Obtain the seasonal trend by the ratio-to-moving-average method from the following data:

	Quarters			
Year	I	II	III	IV
2002	40	35	38	40
2003	42	37	39	38
2004	41	35	38	42

[Ans.: 38.5, 39, 39.375, 39.25, 38.775, 38.5, 38.125, 38.5]

14. Compute seasonal fluctuations from the following time-series using the moving-average method:

Year Quarter	2002	2003	2004	2005
I	75	86	90	100
II	60	65	72	78
III	54	63	66	72
IV	59	80	85	93

[Ans.: 122.36, 92.43, 84.70, 100.51]

Points to Remember

Time-series

A series of observations on a variable, recorded after successive intervals of time, is called a *time-series*. A time-series is an arrangement of statistical data in accordance to the time of occurrence in a chronological order.

Components of a Time-Series

1. Secular trend, or trend
2. Seasonal variations
3. Cyclical variations
4. Irregular variations

1. Secular Trend Secular trend, or simply trend, is the general tendency of the data to increase or decrease or stegnate over a long period of time.

2. Seasonal Variations Seasonal variations refer to such movements in a time-series which repeat themselves periodically in every season. These variations repeat themselves in less than one year.

3. Cyclical Variations Cyclical variations are the oscillatory movements in a time-series with the period of oscillation greater than one year.

4. Irregular Variations Irregular variations do not exhibit any regular pattern of movements and there is no regular period or time of their occurrence. These variations are caused by random factors such as strikes, floods, fire, war, famines, etc.

Measurement of Trend

1. Freehand or graphic method
2. Method of semi-averages
3. Method of moving averages
4. Method of least squares

Method of Moving Averages

In this method, the arithmetic mean of the values for a certain span of time is taken and then it is placed at the centre of the time span. The average value of a number of years is taken as the trend value for the middle point of the period of moving averages.

Measurement of Seasonal Variations

1. Method of simple averages
2. Method of ratio to trend
3. Method of ratio to moving average
4. Method of link relative

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