Numerical and Statistical Methods

for CIVIL ENGINEERING

Gujarat Technological University 2017

Second Edition

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Dedicated
To Our Parents

Late Shri Ramsagar Singh
and
Late Shrimati Premsheela Singh

Ravish R Singh

Late Shri Ved Prakash Sharma
and
Late Shrimati Vidyavati Hemdan

Mukul Bhatt

Contents

Preface *xi*
Roadmap to the Syllabus *xv*

1. Probability **1.1–1.59**

1.1 Introduction *1.1*
1.2 Some Important Terms and Concepts *1.1*
1.3 Definitions of Probability *1.3*
1.4 Theorems on Probability *1.13*
1.5 Bayes' Theorem *1.47*
Points to Remember *1.57*

2. Random Variables and Probability Distributions **2.1–2.155**

2.1 Introduction *2.1*
2.2 Random Variables *2.2*
2.3 Discrete Probability Distribution *2.3*
2.4 Discrete Distribution Function *2.4*
2.5 Measures of Central Tendency for Discrete Probability Distribution *2.18*
2.6 Continuous Probability Distribution *2.33*
2.7 Continuous Distribution Function *2.33*
2.8 Measures of Central Tendency for Continuous Probability Distribution *2.56*
2.9 Binomial Distribution *2.76*
2.10 Poisson Distribution *2.102*
2.11 Normal Distribution *2.127*
Points to Remember *2.153*

3. Statistics **3.1–3.87**

3.1 Introduction *3.1*
3.2 Measures of Central Tendency *3.2*
3.3 Arithmetic Mean *3.2*
3.4 Median *3.24*
3.5 Mode *3.34*
3.6 Geometric Mean *3.45*
3.7 Harmonic Mean *3.47*

3.8 Standard Deviation *3.52*
3.9 Skewness *3.71*
Points to Remember 3.82

4. Correlation and Regression **4.1–4.58**

4.1 Introduction *4.1*
4.2 Correlation *4.2*
4.3 Types of Correlations *4.2*
4.4 Methods of Studying Correlation *4.3*
4.5 Scatter Diagram *4.4*
4.6 Simple Graph *4.5*
4.7 Karl Pearson's Coefficient of Correlation *4.5*
4.8 Properties of Coefficient of Correlation *4.6*
4.9 Rank Correlation *4.22*
4.10 Regression *4.29*
4.11 Types of Regression *4.30*
4.12 Methods of Studying Regression *4.30*
4.13 Lines of Regression *4.31*
4.14 Regression Coefficients *4.31*
4.15 Properties of Regression Coefficients *4.34*
4.16 Properties of Lines of Regression (Linear Regression) *4.35*
Points to Remember 4.56

5. Curve Fitting **5.1–5.27**

5.1 Introduction *5.1*
5.2 Least Square Method *5.2*
5.3 Fitting of Linear Curves *5.2*
5.4 Fitting of Quadratic Curves *5.10*
5.5 Fitting of Exponential and Logarithmic Curves *5.18*
Points to Remember 5.26

6. Finite Differences and Interpolation **6.1–6.88**

6.1 Introduction *6.1*
6.2 Finite Differences *6.2*
6.3 Different Operators and their Relations *6.6*
6.4 Interpolation *6.19*
6.5 Newton's Forward Interpolation Formula *6.19*
6.6 Newton's Backward Interpolation Formula *6.30*
6.7 Central Difference Interpolation *6.39*
6.8 Gauss's Forward Interpolation Formula *6.40*
6.9 Gauss's Backward Interpolation Formula *6.44*
6.10 Stirling's Formula *6.48*
6.11 Interpolation with Unequal Intervals *6.55*
6.12 Lagrange's Interpolation Formula *6.56*
6.13 Divided Differences *6.70*

6.14 Newton's Divided Difference Formula *6.71*
6.15 Inverse Interpolation *6.84*
Points to Remember *6.86*

7. Numerical Integration **7.1–7.41**

7.1 Introduction *7.1*
7.2 Newton–Cotes Quadrature Formula *7.1*
7.3 Trapezoidal Rule *7.2*
7.4 Simpson's 1/3 Rule *7.9*
7.5 Simpson's 3/8 Rule *7.19*
7.6 Gaussian Quadrature Formulae *7.31*
Points to Remember *7.40*

8. Solutions of a System of Linear Equations **8.1–8.62**

8.1 Introduction *8.1*
8.2 Solutions of a System of Linear Equations *8.2*
8.3 Elementary Transformations *8.2*
8.4 Numerical Methods for Solutions of a System of Linear Equations *8.3*
8.5 Gauss Elimination Method *8.4*
8.6 Gauss Elimination Method with Partial Pivoting *8.15*
8.7 Gauss–Jordan Method *8.20*
8.8 Gauss–Jacobi Method *8.31*
8.9 Gauss–Siedel Method *8.37*
Points to Remember *8.61*

9. Roots of Algebraic and Transcendental Equations **9.1–9.50**

9.1 Introduction *9.1*
9.2 Bisection Method *9.2*
9.3 Regula Falsi Method *9.15*
9.4 Newton–Raphson Method *9.22*
9.5 Secant Method *9.39*
Points to Remember *9.49*

10. Numerical Solutions of Ordinary Differential Equations **10.1–10.64**

10.1 Introduction *10.1*
10.2 Taylor's Series Method *10.2*
10.3 Euler's Method *10.9*
10.4 Modified Euler's Method *10.16*
10.5 Runge–Kutta Methods *10.28*
10.6 Milne's Predictor-Corrector Method *10.53*
Points to Remember *10.63*

Index ***I.1–I.3***

Preface

Mathematics is a key area of study in any engineering course. A sound knowledge of this subject will help engineering students develop analytical skills, and thus enable them to solve numerical problems encountered in real life, as well as apply mathematical principles to physical problems, particularly in the field of engineering.

Users

This book is designed for the 4th semester GTU Civil Engineering students pursuing the course *Numerical and Statistical Methods (CODE 2140606)*. It covers the complete GTU syllabus for the course on Numerical and Statistical Methods for the civil engineering branches.

Objective

The crisp and complete explanation of topics will help students easily understand the basic concepts. The tutorial approach (i.e., teach by example) followed in the text will enable students to develop a logical perspective to solving problems.

Features

Each topic has been explained from the examination point of view, wherein the theory is presented in an easy-to-understand student-friendly style. Full coverage of concepts is supported by numerous solved examples with varied complexity levels, which is aligned to the latest GTU syllabus. Fundamental and sequential explanation of topics is well aided by examples and exercises. The solutions of examples are set following a 'tutorial' approach, which will make it easy for students from any background to easily grasp the concepts. Exercises with answers immediately follow the solved examples enforcing a practice-based approach. We hope that the students will gain logical understanding from solved problems and then reiterate it through solving similar exercise problems themselves. The unique blend of theory and application caters to the requirements of both the students and the faculty. Solutions of GTU examination questions are incorporated within the text appropriately.

Highlights

- Crisp content strictly as per the latest GTU syllabus of *Numerical and Statistical Methods* (Regulation 2014)
- Comprehensive coverage with lucid presentation style
- Each section concludes with an exercise to test understanding of topics
- Solutions of GTU examination papers from 2010 to 2015 present appropriately within the chapters
- Solution of 2016 GTU examination paper can be accessible through weblink.
- Rich exam-oriented pedagogy:
  - ➢ Solved Examples within chapters: 443
  - ➢ Solved GTU questions tagged within chapters: 130
  - ➢ Unsolved Exercises: 468

Chapter Organization

The content spans the following ten chapters which wholly and sequentially cover each module of the syllabus.

- ❑ **Chapter 1** introduces Probability.
- ❑ **Chapter 2** discusses Random Variables and Probability Distributions.
- ❑ **Chapter 3** presents Statistics.
- ❑ **Chapter 4** covers Correlation and Regression.
- ❑ **Chapter 5** deals with Curve Fitting.
- ❑ **Chapter 6** presents Finite Differences and Interpolation.
- ❑ **Chapter 7** explains Numerical Integration.
- ❑ **Chapter 8** discusses Solutions of a System of Linear Equations.
- ❑ **Chapter 9** deals with Roots of Algebraic and Transcendental Equations.
- ❑ **Chapter 10** introduces Numerical Solutions of Ordinary Differential Equations.

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Ravish R Singh
Mukul Bhatt

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ROADMAP TO THE SYLLABUS

This text is useful for

Numerical and Statistical Methods (Code 2140606)

(For Civil Engineering)

Module 1: Reorientation

Definition of probability; Exhaustive events; Pairwise independent events; Multiplicative law of probability; Conditional probability; Bayes' theorem

GO TO

CHAPTER 1: Probability

Module 2: Probability Distributions

Random variable; Mathematical expectation; Standard deviation; Binomial, Poisson, and normal distributions; Mean; Median; Mode

GO TO

CHAPTER 2: Random Variables and Probability Distributions

Module 3: Descriptive Statistics

Mean; Median; Mode; Standard deviation; Skewness

GO TO

CHAPTER 3: Statistics

Module 4: Correlation and Regression

Bivariate distribution; Correlation coefficients; Regression lines; Formulas for regression coefficients; Rank correlation

GO TO

CHAPTER 4: Correlation and Regression

Module 5: Curve Fitting

Fitting of linear, quadratic, exponential, and logarithmic curves; Least squares method

GO TO

CHAPTER 5: Curve Fitting

Module 6: Finite Differences and Interpolation

Finite differences; Forward, backward, and central operators; Interpolation by polynomials: Newton's forward and backward interpolation formulae; Gauss and Stirling's central difference formulae; Newton's divided and Lagrange's formulae for unequal intervals

GO TO

CHAPTER 6: Finite Differences and Interpolation

Module 7: Numerical Integration

Newton–Cotes formula; Trapezoidal and Simpson's formulae; Error formulae; Gaussian quadrature formulae

GO TO

CHAPTER 7: Numerical Integration

Module 8: Solution of a System of Linear Equations

Gauss elimination; Partial pivoting; Gauss–Jacobi and Gauss–Seidel methods

GO TO

CHAPTER 8: Solutions of a System of Linear Equations

Module 9: Roots of Algebraic and Transcendental Equations

Bisection; False position; Secant and Newton–Raphson methods; Rate of convergence

GO TO

CHAPTER 9: Roots of Algebraic and Transcendental Equations

Module 10: Numerical Solution of Ordinary Differential Equations

Taylor series method; Euler method; Runge–Kutta method of order four; Milne's predictor-corrector method

GO TO

CHAPTER 10: Numerical Solutions of Ordinary Differential Equations

CHAPTER 1

Probability

Chapter Outline

1.1 Introduction
1.2 Some Important Terms and Concepts
1.3 Definitions of Probability
1.4 Theorems on Probability
1.5 Bayes' Theorem

1.1 INTRODUCTION

The concept of probability originated from the analysis of the games of chance. Even today, a large number of problems exist which are based on the games of chance, such as tossing of a coin, throwing of dice, and playing of cards. The utility of probability in business and economics is most emphatically revealed in the field of predictions for the future. Probability is a concept which measures the degree of uncertainty and that of certainty as a corollary.

The word *probability* or 'chance' is used commonly in day-to-day life. Daily, we come across the sentences like, 'it may rain today', 'India may win the forthcoming cricket match against Sri Lanka', 'the chances of making profits by investing in shares of Company A are very bright, etc. Each of the above sentences involves an element of uncertainty. A numerical measure of uncertainty is provided by a very important branch of mathematics called *theory of probability*. Before we study the probability theory in detail, it is appropriate to explain certain terms which are essential for the study of the theory of probability.

1.2 SOME IMPORTANT TERMS AND CONCEPTS

1. Random Experiment If an experiment is conducted, any number of times, under identical conditions, there is a set of all possible outcomes associated with it.

If the outcome is not unique but may be any one of the possible outcomes, the experiment is called a random experiment, e.g., tossing a coin, throwing a die.

2. Outcome The result of a random experiment is called an outcome. For example, consider the following:

(a) Suppose a random experiment is ‘a coin is tossed’. This experiment gives two possible outcomes—head or tail.

(b) Suppose a random experiment is ‘a die is thrown’. This experiment gives six possible outcomes—1, 2, 3, 4, 5 or 6—on the uppermost face of a die.

3. Trial and Event Any particular performance of a random experiment is called a trial and outcome. A combination of outcomes is called an event. For example, consider the following:

(a) Tossing of a coin is a trial, and getting a head or tail is an event.

(b) Throwing of a die is a trial and getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

4. Exhaustive Event The total number of possible outcomes of a random experiment is called an exhaustive event. For example, consider the following:

(a) In tossing of a coin, there are two exhaustive events, viz., head and tail.

(b) In throwing of a die, there are six exhaustive events, getting 1 or 2 or 3 or 4 or 5 or 6.

5. Mutually Exclusive Events Events are said to be mutually exclusive if the occurrence of one of them precludes the occurrence of all others in the same trial, i.e., they cannot occur simultaneously. For example, consider the following:

(a) In tossing a coin, the events head or tail are mutually exclusive since both head and tail cannot occur at the same time.

(b) In throwing a die, all the six events, i.e., getting 1 or 2 or 3 or 4 or 5 or 6 are mutually exclusive events.

6. Equally Likely Events The outcomes of a random experiment are said to be equally likely if the occurrence of none of them is expected in preference to others. For example, consider the following:

(a) In tossing a coin, head or tail are equally likely events.

(b) In throwing a die, all the six faces are equally likely events.

7. Independent Events Events are said to be independent if the occurrence of an event does not have any effect on the occurrence of other events. For example, consider the following:

(a) In tossing a coin, the event of getting a head in the first toss is independent of getting a head in the second, third, and subsequent tosses.

(b) In throwing a die, the result of the first throw does not affect the result of the second throw.

8. Favourable Events The favourable events in a random experiment are the number of outcomes which entail the occurrence of the event. For example, consider the following:

In throwing of two dice, the favourable events of getting the sum 5 is (1, 4), (4, 1), (2, 3), (3, 2), i.e., 4.

1.3 DEFINITIONS OF PROBABILITY

1.3.1 Classical Definition of Probability

Let n be the number of equally likely, mutually exclusive, and exhaustive outcomes of a random experiment. Let m be number of the outcomes which are favourable to the occurrence of an event A. The probability of event A occurring, denoted by $P(A)$, is given by

$$P(A) = \frac{\text{Number of outcomes favourable to } A}{\text{Number of exhaustive outcomes}} = \frac{m}{n}$$

1.3.2 Empirical or Statistical Definition of Probability

If an experiment is repeated a large number of times under identical conditions, the limiting value of the ratio of the number of times the event A occurs to the total number of trials of the experiment as the number of trials increase indefinitely is called the probability of occurrence of the event A.

Let $P(A)$ be the probability of occurrence of the event A. Let m be the number of times in which an event A occurs in a series of n trials.

$$P(A) = \lim_{n \to \infty} \frac{m}{n}, \text{ provided the limit is finite and unique.}$$

1.3.3 Axiomatic Definition of Probability

Before discussing the axiomatic definition of probability, it is necessary to explain certain concepts that are necessary to its understanding.

1. Sample Space A set of all possible outcomes of a random experiment is called a sample space. Each element of the set is called a *sample point* or a *simple event* or an *elementary event*.

The sample space of a random experiment is denoted by S. For example, consider the following:

(a) In a random experiment of tossing of a coin, the sample space consists of two elementary events.

$S = \{\text{H, T}\}$

(b) In a random experiment of throwing of a die, the sample space consists of six elementary events.

$S = \{1, 2, 3, 4, 5, 6\}$

The elements of S can either be single elements or ordered pairs. If two coins are tossed, each element of the sample space consists of the following ordered pairs:

$S = \{(H, H), (H, T), (T, H), (T, T)\}$

2. Event Any subset of a sample space is called an event. In the experiment of throwing of a die, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Let A be the event that an odd number appears on the die. Then $A = \{1, 3, 5\}$ is a subset of S. Similarly, let B be the event of getting a number greater than 3. Then $B = \{4, 5, 6\}$ is another subset of S.

Definition of Probability Let S be a sample space of an experiment and A be any event of this sample space. The probability $P(A)$ of the event A is defined as the real-value set function which associates a real value corresponding to a subset A of the sample space S. The probability $P(A)$ satisfies the following three axioms.

Axiom I: $P(A) \geq 0$, i.e., the probability of an event is a nonnegative number.

Axiom II: $P(S) = 1$, i.e., the probability of an event that is certain to occur must be equal to unity.

Axiom III: If $A_1, A_2, ..., A_n$ are finite mutually exclusive events then

$$P(A_1 \cup A_2 \cup ... \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$
$$= \sum_{i=1}^{n} P(A_i)$$

i.e., the probability of a union of mutually exclusive events is the sum of probabilities of the events themselves.

Example 1

What is the probability that a leap year selected at random will have 53 Sundays?

Solution

A leap year has 366 days, i.e., 52 weeks and 2 days. These 2 days can occur in the following possible ways:

(i) Monday and Tuesday
(ii) Tuesday and Wednesday
(iii) Wednesday and Thursday
(iv) Thursday and Friday
(v) Friday and Saturday
(vi) Saturday and Sunday
(vii) Sunday and Monday

Number of exhaustive cases $n = 7$

Number of favourable cases $m = 2$

Let A be the event of getting 53 Sundays in a leap year.

$$P(A) = \frac{m}{n} = \frac{2}{7}$$

Example 2

Three unbiased coins are tossed. Find the probability of getting (i) exactly two heads, (ii) at least one tail, (iii) at most two heads, (iv) a head on the second coin, and (v) exactly two heads in succession.

Solution

When three coins are tossed, the sample space S is given by

$$S = \{\text{HHH, HTH, THH, HHT, TTT, THT, TTH, HTT}\}$$

$$n(s) = 8$$

(i) Let A be the event of getting exactly two heads.

$$A = \{\text{HTH, THH, HHT}\}$$

$$n(A) = 3$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$$

(ii) Let B be the event of getting at least one tail.

$$B = \{\text{HTH, THH, HHT, TTT, THT, TTH, HTT}\}$$

$$n(B) = 7$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{7}{8}$$

(iii) Let C be the event of getting at most two heads.

$$C = \{\text{HTH, THH, HHT, TTT, THT, TTH, HTT}\}$$

$$n(C) = 7$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{7}{8}$$

(iv) Let D be the event of getting a head on the second coin.

$$D = \{\text{HHH, THH, HHT, THT}\}$$

$$n(D) = 4$$

$$P(D) = \frac{n(D)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

(v) Let E be the event of getting two heads in succession.

$$E = \{\text{HHH, THH, HHT}\}$$
$$n(E) = 3$$
$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$$

Example 3

A fair dice is thrown. Find the probability of getting (i) an even number, (ii) a perfect square, and (iii) an integer greater than or equal to 3.

Solution

When a dice is thrown, the sample space S is given by

$$S = \{1, 2, 3, 4, 5, 6\}$$
$$n(S) = 6$$

(i) Let A be the event of getting an even number.

$$A = \{2, 4, 6\}$$
$$n(A) = 3$$
$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

(ii) Let B be the event of getting a perfect square.

$$B = \{1, 4\}$$
$$n(B) = 2$$
$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

(iii) Let C be the event of getting an integer greater than or equal to 3.

$$C = \{3, 4, 5, 6\}$$
$$n(C) = 4$$
$$P(C) = \frac{n(C)}{n(S)} = \frac{4}{6} = \frac{2}{3}$$

Example 4

A card is drawn from a well-shuffled pack of 52 cards. Find the probability of (i) getting a king card, (ii) getting a face card, (iii) getting a red card, (iv) getting a card between 2 and 7, both inclusive, and (v) getting a card between 2 and 8, both exclusive.

Solution

Total number of cards = 52

One card out of 52 cards can be drawn in ways.

$$n(S) = {}^{52}C_1 = 52$$

(i) Let A be the event of getting a king card. There are 4 king cards and one of them can be drawn in ${}^{4}C_1$ ways.

$$n(A) = {}^{4}C_1 = 4$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

(ii) Let B be the event of getting a face card. There are 12 face cards and one of them can be drawn in ${}^{12}C_1$ ways.

$$n(B) = {}^{12}C_1 = 12$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{12}{52} = \frac{3}{13}$$

(iii) Let C be the event of getting a red card. There are 26 red cards and one of them can be drawn in ${}^{26}C_1$ ways.

$$n(C) = {}^{26}C_1 = 26$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{26}{52} = \frac{1}{2}$$

(iv) Let D be the event of getting a card between 2 and 7, both inclusive. There are 6 such cards in each suit giving a total of $6 \times 4 = 24$ cards. One of them can be drawn in ${}^{24}C_1$ ways.

$$n(D) = {}^{24}C_1 = 24$$

$$P(D) = \frac{n(D)}{n(S)} = \frac{24}{52} = \frac{6}{13}$$

(v) Let E be the event of getting a card between 2 and 8, both exclusive. There are 5 such cards in each suit giving a total of $5 \times 4 = 20$ cards. One of them can be drawn in ${}^{20}C_1$ ways.

$$n(E) = {}^{20}C_1 = 20$$

$$= \frac{n(E)}{n(S)} = \frac{20}{52} = \frac{5}{13}$$

Example 5

A bag contains 2 black, 3 red, and 5 blue balls. Three balls are drawn at random. Find the probability that the three balls drawn (i) are blue (ii) consist of 2 blue and 1 red ball, and (iii) consist of exactly one black ball.

Solution

Total number of balls = 10

3 balls out of 10 balls can be drawn in ${}^{10}C_3$ ways.

$$n(S) = {}^{10}C_3 = 120$$

(i) Let A be the event that the three balls drawn are blue. 3 blue balls out of 5 blue balls can be drawn in ${}^{5}C_3$ ways.

$$n(A) = {}^{5}C_3 = 10$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{10}{120} = \frac{1}{12}$$

(ii) Let B be the event that the three balls drawn consist of 2 blue and 1 red ball.
2 blue balls out of 5 blue balls can be drawn in ${}^{5}C_2$ ways. 1 red ball out of 3 red balls can be drawn in ${}^{3}C_1$ ways.

$$n(B) = {}^{5}C_2 \times {}^{3}C_1 = 30$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{30}{120} = \frac{1}{4}$$

(iii) Let C be the event that three balls drawn consist of exactly one black ball, i.e., remaining two balls can be drawn from 3 red and 5 blue balls. One black ball can be drawn from 2 black balls in ${}^{2}C_1$ ways and the remaining 2 balls can be drawn from 8 balls in ${}^{8}C_2$ ways.

$$n(C) = {}^{2}C_1 \times {}^{8}C_2 = 56$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{56}{120} = \frac{7}{15}$$

Example 6

A class consists of 6 girls and 10 boys. If a committee of three is chosen at random from the class, find the probability that (i) three boys are selected, and (ii) exactly two girls are selected.

Solution

Total number of students = 16

A committee of 3 students from 16 students can be selected in ${}^{16}C_3$ ways.

$$n(S) = {}^{16}C_3 = 560$$

(i) Let A be the event that 3 boys are selected.

$$n(A) = {}^{10}C_3 = 120$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{120}{560} = \frac{3}{14}$$

(ii) Let B be the event that exactly 2 girls are selected. 2 girls from 6 girls can be selected in ${}^6C_2$ ways and one boy from 10 boys can be selected in ${}^{10}C_1$ ways.

$$n(B) = {}^6C_2 \times {}^{10}C_1 = 150$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{150}{560} = \frac{15}{16}$$

Example 7

From a collection of 10 *bulbs, of which* 4 *are defective,* 3 *bulbs are selected at random and fitted into lamps. Find the probability that (i) all three bulbs glow, and (ii) the room is lit.*

Solution

Total number of bulbs = 10

3 bulbs can be selected from 10 bulbs in ${}^{10}C_3$ ways.

$$n(S) = {}^{10}C_3 = 120$$

(i) Let A be event that all three bulbs glow. This event will occur when 3 bulbs are selected from 6 nondefective bulbs in ${}^6C_3$ ways.

$$n(A) = {}^6C_3 = 20$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{20}{120} = \frac{1}{6}$$

(ii) Let B be the event that the room is lit. Let $\overline{B}$ be the event that the room is dark. The event $\overline{B}$ will occur when 3 bulbs are selected from 4 defective bulbs in ${}^4C_3$ ways.

$$n(\overline{B}) = {}^4C_3 = 4$$

$$P(\overline{B}) = \frac{n(\overline{B})}{n(S)} = \frac{4}{120} = \frac{1}{30}$$

$$\therefore \quad P(B) = 1 - P(\overline{B}) = 1 - \frac{1}{30} = \frac{29}{30}$$

Example 8

There are 20 *tickets numbered* 1, 2, ..., 20. *One ticket is drawn at random. Find the probability that the ticket bears a number which is (i) even, (ii) a perfect square, and (iii) multiple of* 3.

Solution

There are 20 tickets numbered from 1 to 20.

$n(S) = 20$

(i) Let A be the event that a ticket bears a number which is even.

$$A = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$$

$$n(A) = 10$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{10}{20} = \frac{1}{2}$$

(ii) Let B the event that a ticket bears a number which is a perfect square.

$$B = \{1, 4, 9, 16\}$$

$$n(B) = 4$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{20} = \frac{1}{5}$$

(iii) Let C be the event that a ticket bears a number which is a multiple of 3.

$$C = \{3, 6, 9, 12, 15, 18\}$$

$$n(C) = 6$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{6}{20} = \frac{3}{10}$$

Example 9

Four letters of the word 'THURSDAY' are arranged in all possible ways. Find the probability that the word formed is 'HURT'.

Solution

Total number of letters in the word 'THURSDAY' = 8

Four letters from 8 letters can be arranged in ${}^8P_4$ ways.

$$n(S) = {}^8P_4 = 1680$$

Let A be the event that the word formed is 'HURT'. The word 'HURT' can be formed in one way only.

$$n(A) = 1$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{1680}$$

Example 10

A bag contains 5 red, 4 blue, and m green balls. If the probability of getting two green balls when two balls are selected at random is $\frac{1}{7}$, *find m.*

Solution

Total number of balls = 5 + 4 + m = 9 + m

2 balls out of 9 + m balls can be drawn in ${}^{9+m}C_2$ ways.

$$n(S) = {}^{9+m}C_2$$

Let A be the event that both the balls drawn are green.

2 green balls out of m green balls can be drawn in ${}^{m}C_2$ ways.

$$n(A) = {}^{m}C_2$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{{}^{m}C_2}{{}^{9+m}C_2}$$

But $\quad P(A) = \frac{1}{7}$

$$\frac{{}^{m}C_2}{{}^{9+m}C_2} = \frac{1}{7}$$

$$\frac{m(m-1)}{(m+9)(m+8)} = \frac{1}{7}$$

$$(m+9)\,(m+8) = 7\,m(m-1)$$

$$m^2 + 17m + 72 = 7m^2 - 7m$$

$$6m^2 - 24m - 72 = 0$$

$$3m^2 - 12m - 36 = 0$$

$$3m^2 - 18m + 6m - 36 = 0$$

$$3m(m-6) + 6(m-6) = 0$$

$$(3m+6)(m-6) = 0$$

$$3m + 6 = 0 \quad \text{or} \quad m - 6 = 0$$

$$m = -2 \quad \text{or} \quad m = 6$$

But $\quad m \neq -2$

$\therefore \quad m = 6$

EXERCISE 1.1

1. A card is drawn at random from a pack of 52 cards. Find the probability that the card drawn is (i) an ace card, and (ii) a club card.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{1}{13} \quad \text{(ii) } \frac{1}{4}\right]$$

2. An unbiased coin is tossed twice. Find the probability of (i) exactly one head, (ii) at most one head, (iii) at least one head, and (iv) same face on both the coins.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{1}{2} \text{ (ii) } \frac{3}{4} \text{ (iii) } \frac{3}{4} \text{ (iv) } \frac{1}{2}\right]$$

3. A fair dice is thrown thrice. Find the probability that the sum of the numbers obtained is 10.

$$\left[\textbf{Ans.:} \frac{1}{8}\right]$$

4. A ball is drawn at random from a box containing 12 red, 18 white, 19 blue, and 15 orange balls. Find the probability that (i) it is red or blue, and (ii) it is white, blue, or orange.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{2}{5} \text{ (ii) } \frac{43}{55}\right]$$

5. Eight boys and three girls are to sit in a row for a photograph. Find the probability that no two girls are together.

$$\left[\textbf{Ans.:} \frac{28}{55}\right]$$

6. If four persons are chosen from a group of 3 men, 2 women, and 4 children, find the probability that exactly two of them will be children.

$$\left[\textbf{Ans.:} \frac{10}{21}\right]$$

7. A box contains 2 white, 3 red, and 5 black balls. Three balls are drawn at random. What is the probability that they will be of different colours?

$$\left[\textbf{Ans.:} \frac{1}{4}\right]$$

8. Two cards are drawn from a well-shuffled pack of 52 cards. Find the probability of getting (i) 2 king cards, (ii) 1 king card and 1 queen card, and (iii) 1 king card and 1 spade card.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{1}{221} \text{ (ii) } \frac{8}{663} \text{ (iii) } \frac{1}{26}\right]$$

9. A four-digit number is to be formed using the digits 0, 1, 2, 3, 4, 5. All the digits are to be different. Find the probability that the digit formed is (i) odd, (ii) greater than 4000, (iii) greater than 3400, and (iv) a multiple of 5.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{12}{25} \text{ (ii) } \frac{2}{5} \text{ (iii) } \frac{12}{25} \text{ (iv) } \frac{9}{25}\right]$$

10. 3 books of physics, 4 books of chemistry, and 5 books of mathematics are arranged in a shelf. Find the probability that (i) no physics books are together, (ii) chemistry books are always together, and (iii) books of the same subjects are together.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{6}{11} \quad \text{(ii) } \frac{1}{55} \quad \text{(iii) } \frac{1}{4620}\right]$$

11. 8 boys and 2 girls are to be seated at random in a row for a photograph. Find the probability that (i) the girls sit together, and (ii) the girls occupy 3^{rd} and 7^{th} seats.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{1}{5} \quad \text{(ii) } \frac{1}{45}\right]$$

12. A committee of 4 is to be formed from 15 boys and 3 girls. Find the probability that the committee contains (i) 2 boys and 2 girls, (ii) exactly one girl, (iii) one particular girl, and (iv) two particular girls.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{7}{68} \quad \text{(ii) } \frac{91}{204} \quad \text{(iii) } \frac{2}{9} \quad \text{(iv) } \frac{2}{51}\right]$$

13. If the letters of the word REGULATIONS are arranged at random, what is the probability that there will be exactly four letters between *R* and *E*?

$$\left[\textbf{Ans.:} \ \frac{6}{55}\right]$$

14. Find the probability that there will be 5 Sundays in the month of October.

$$\left[\textbf{Ans.:} \ \frac{3}{7}\right]$$

1.4 THEOREMS ON PROBABILITY

Theorem 1 The probability of an impossible event is zero, i.e., $P(\phi) = 0$, where ϕ is a null set.

Proof An event which has no sample points is called an impossible event and is denoted by ϕ.

For a sample space S of an experiment,

$$S \cup \phi = S$$

Taking probability of both the sides,

$$P(S \cup \phi) = P(S)$$

Since S and ϕ are mutually exclusive events,

$$P(S) + P(\phi) = P(S) \qquad \text{[Using Axiom III]}$$

$$\therefore \quad P(\phi) = 0$$

Theorem 2 The probability of the complementary event $\bar{A}$ of A is

$$P(\bar{A}) = 1 - P(A)$$

Proof Let A be an event in the sample space S.

$$A \cup \bar{A} = S$$

$$P(A \cup \bar{A}) = P(S)$$

Since A and $\bar{A}$ are mutually exclusive events,

$$P(A) + P(\bar{A}) = P(S)$$

$$P(A) + P(\bar{A}) = 1 \qquad [\because \quad P(S) = 1]$$

$$\therefore \quad P(\bar{A}) = 1 - P(A)$$

Note Since A and $\bar{A}$ are mutually exclusive events,

$A \cup \bar{A} = S$ and $A \cap \bar{A} = \phi$

Corollary Probability of an event is always less than or equal to one, i.e., $P(A) \le 1$

Proof $P(A) = 1 - P(\bar{A})$

$$P(A) \le 1 \qquad [\because \quad P(\bar{A}) \ge 0 \text{ by Axiom I}]$$

De Morgan's Laws Since an event is a subset of a sample space, De Morgan's laws are applicable to events.

$$P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B})$$

$$P(\overline{A \cap B}) = P(\bar{A} \cup \bar{B})$$

Theorem 3 For any two events A and B in a sample space S,

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

Proof From the Venn diagram (Fig. 1.1),

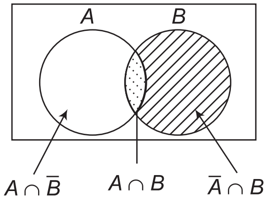


Fig. 1.1

$$B = (A \cap B) \cup (\bar{A} \cap B)$$

$$P(B) = P\left[(A \cap B) \cup (\bar{A} \cap B)\right]$$

Since $(A \cap B)$ and $(\bar{A} \cap B)$ are mutually exclusive events,

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

Similarly, it can be shown that

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

Theorem 4 Additive Law of Probability (Addition Theorem)

The probability that at least one of the events A and B will occur is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof From the Venn diagram (Fig. 1.1),

$$A \cup B = A \cup (\bar{A} \cap B)$$

$$P(A \cup B) = P\left[A \cup (\bar{A} \cap B\right]$$

Since A and $(\bar{A} \cap B)$ are mutually exclusive events,

$$\begin{aligned} P(A \cup B) &= P(A) + P(\bar{A} \cap B) && \text{[Using Axiom III]} \\ &= P(A) + P(B) - P(A \cap B) && \text{[Using Theorem 3]} \end{aligned}$$

Remarks

1. If A and B are mutually exclusive events, i.e., $A \cap B = \phi$ then $P(A \cap B) = 0$ according to Theorem 1.

   Hence, $P(A \cup B) = P(A) + P(B)$

2. The event $A \cup B$ (i.e., A or B) denotes the occurrence of either A or B or both. Alternately, it implies the occurrence of at least one of the two events.

   $A \cup B = A + B$

3. The event $A \cap B$ (i.e., A and B) is a compound or joint event that denotes the simultaneous occurrence of the two events.

   $A \cap B = AB$

Corollary 1 From the Venn diagram (Fig. 1.1),

$$P(A \cup B) = 1 - P(\bar{A} \cap \bar{B})$$

where $P(\bar{A} \cap \bar{B})$ is the probability that none of the events A and B occur simultaneously.

Corollary 2 P(Exactly one of A and B occurs) = $P\left[(A \cap \bar{B}) \cup (\bar{A} \cap B)\right]$

$$\begin{aligned} &= P(A \cap \bar{B}) + P(\bar{A} \cap B) && \left[\because \ (A \cap \bar{B}) \cap (\bar{A} \cap B) = \phi\right] \\ &= P(A) - P(A \cap B) + P(B) - P(A \cap B) && \text{[Using Theorem 3]} \\ &= P(A) + P(B) - 2P(A \cap B) \\ &= P(A \cup B) - P(A \cap B) && \text{[Using Theorem 4]} \end{aligned}$$

$= P$ (at least one of the two events occur)

$\quad - P$ (the two events occur simultaneously)

Corollary 3 The addition theorem can be applied for more than two events. If A, B, and C are three events of a sample space S then the probability of occurrence of at least one of them is given by

$$\begin{aligned}P(A\cup B\cup C) &= P\left[A\cup(B\cup C)\right]\\ &= P(A)+P(B\cup C)-P\left[A\cap(B\cup C)\right]\\ &= P(A)+P(B\cup C)-P\left[A\cap B)\cup(A\cap C)\right]\\ &= P(A)+P(B)+P(C)-P(B\cap C)-P(A\cap B)-P(A\cap C)+P(A\cap B\cap C)\\ &\qquad\left[\text{Applying Theorem 4 on second and third term}\right]\end{aligned}$$

Alternately, the probability of occurrence of at least one of the three events can also be written as

$$P(A\cup B\cup C)=1-P(\bar{A}\cap\bar{B}\cap\bar{C})$$

If A, B, and C are mutually exclusive events,

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)$$

Corollary 4 The probability of occurrence of at least two of the three events is given by

$$\begin{aligned}P\left[A\cap B)\cup(B\cap C)\cup(A\cap C)\right] &= P(A\cap B)+P(B\cap C)+P(A\cap C)-3P(A\cap B\cap C)\\ &\quad+P(A\cap B\cap C) \qquad \text{[Using Corollary 3]}\\ &= P(A\cap B)+P(B\cap C)+P(A\cap C)-2P(A\cap B\cap C)\end{aligned}$$

Corollary 5 The probability of occurrence of exactly two of the three events is given by

$$\begin{aligned}&P\left[A\cap B\cap\bar{C})\cup(A\cap\bar{B}\cap C)\cup(\bar{A}\cap B\cap C)\right]\\ &= P\left[(A\cap B)\cup(B\cap C)\cup(A\cap C)\right]-P(A\cap B\cap C) \qquad \text{[Using Corollary 2]}\\ &= P(A\cap B)+P(B\cap C)+P(A\cap C)-3P(A\cap B\cap C) \qquad \text{[Using Corollary 4]}\end{aligned}$$

Corollary 6 The probability of occurrence of exactly one of the three events is given by

$$P\left[(A\cap\bar{B}\cap\bar{C})\cup(\bar{A}\cap B\cap\bar{C})\cup(\bar{A}\cap\bar{B}\cap C)\right]$$

= P(at least one of the three event occur) – P (at least two of the three events occur)

$$= P(A)+P(B)+P(C)-2P(A\cap B)-2P(B\cap C)-2P(A\cap C)+3P(A\cap B\cap C)$$

Example 1

A card is drawn from a well-shuffled pack of cards. What is the probability that it is either a spade or an ace?

Solution

Let A and B be the events of getting a spade and an ace card respectively.

$$P(A) = \frac{{}^{13}C_1}{{}^{52}C_1} = \frac{13}{52}$$

$$P(B) = \frac{{}^{4}C_1}{{}^{52}C_1} = \frac{4}{52}$$

$$P(A \cap B) = \frac{{}^{1}C_1}{{}^{52}C_1} = \frac{1}{52}$$

Probability of getting either a spade or an ace card

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} \\ &= \frac{4}{13} \end{aligned}$$

Example 2

Two cards are drawn from a pack of cards. Find the probability that they will be both red or both pictures.

Solution

Let A and B be the events that both cards drawn are red and pictures respectively.

$$P(A) = \frac{{}^{26}C_2}{{}^{52}C_2} = \frac{325}{1326}$$

$$P(B) = \frac{{}^{12}C_2}{{}^{52}C_2} = \frac{66}{1326}$$

$$P(A \cap B) = \frac{{}^{6}C_2}{{}^{52}C_2} = \frac{15}{1326}$$

Probability that both cards drawn are red or pictures

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{325}{1326} + \frac{66}{1326} - \frac{15}{1326}$$
$$= \frac{188}{663}$$

Example 3

The probability that a contractor will get a plumbing contract is $\frac{2}{3}$ *and the probability that he will not get an electric contract is* $\frac{5}{9}$. *If the probability of getting any one contract is* $\frac{4}{5}$, *what is the probability that he will get both the contracts?*

Solution

Let A and B be the events that the contractor will get plumbing and electric contracts respectively.

$$P(A) = \frac{2}{3}, \ P(\overline{B}) = \frac{5}{9}, \ P(A \cup B) = \frac{4}{5}$$
$$P(B) = 1 - P(\overline{B}) = 1 - \frac{5}{9} = \frac{4}{9}$$

Probability that the contractor will get any one contract

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Probability that the contractor will get both the contracts

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$
$$= \frac{2}{3} + \frac{4}{9} - \frac{4}{5}$$
$$= \frac{14}{45}$$

Example 4

A person applies for a job in two firms A and B, the probability of his being selected in the firm A is 0.7 and being rejected in the firm B is 0.5. The probability of at least one of the applications being rejected is 0.6. What is the probability that he will be selected in one of the two firms?

Solution

Let A and B be the events that the person is selected in firms A and B respectively.

$$P(A) = 0.7, \qquad P(\overline{B}) = 0.5, \qquad P(\overline{A} \cup \overline{B}) = 0.6$$

$$P(\overline{A}) = 1 - P(A) = 1 - 0.7 = 0.3$$

$$P(B) = 1 - P(\overline{B}) = 1 - 0.5 = 0.5$$

$$P(\overline{A} \cup \overline{B}) = P(\overline{A}) + P(\overline{B}) - P(\overline{A} \cap \overline{B}) \qquad \ldots(1)$$

Probability that the person will be selected in one of the two firms

$$\begin{aligned} P(A \cup B) &= 1 - P(\overline{A} \cap \overline{B}) \\ &= 1 - \left[P(\overline{A}) + P(\overline{B}) - P(\overline{A} \cup \overline{B})\right] \qquad \text{[Using Eq. (1)]} \\ &= 1 - (0.3 + 0.5 - 0.6) \\ &= 0.8 \end{aligned}$$

Example 5

In a group of 1000 *persons, there are* 650 *who can speak Hindi,* 400 *can speak English, and* 150 *can speak both Hindi and English. If a person is selected at random, what is the probability that he speaks (i) Hindi only, (ii) English only, (iii) only of the two languages, and (iv) at least one of the two languages?*

Solution

Let A and B be the events that a person selected at random speaks Hindi and English respectively.

$$P(A) = \frac{650}{1000}, \quad P(B) = \frac{400}{1000}, P(A \cap B) = \frac{150}{1000}$$

(i) Probability that a person selected at random speaks Hindi only

$$\begin{aligned} P(A \cap \overline{B}) &= P(A) - P(A \cap B) \\ &= \frac{650}{1000} - \frac{150}{1000} \\ &= \frac{1}{2} \end{aligned}$$

(ii) Probability that a person selected at random speaks English only

$$\begin{aligned} P(\overline{A} \cap B) &= P(B) - P(A \cap B) \\ &= \frac{400}{1000} - \frac{150}{1000} \\ &= \frac{1}{4} \end{aligned}$$

(iii) Probability that a person selected at random speaks only one of the languages.

$$P\left[(A\cap\bar{B})\cup(\bar{A}\cap B)\right]=P(A)+P(B)-2P(A\cap B)$$
$$=\frac{650}{1000}+\frac{400}{1000}-2\left(\frac{150}{1000}\right)$$
$$=\frac{3}{4}$$

(iv) Probability that a person selected at random speaks at least one of the two languages

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$=\frac{650}{1000}+\frac{400}{1000}-\frac{150}{1000}$$
$$=\frac{9}{10}$$

Example 6

A box contains 4 white, 6 red, 5 black balls, and 5 balls of other colours. Two balls are drawn from the box at random. Find the probability that (i) both are white or both are red, and (ii) both are red or both are black.

Solution

Let A, B, and C be the events of drawing white, red and black balls from the box respectively.

$$P(A)=\frac{{}^{4}C_2}{{}^{20}C_2}=\frac{3}{95}$$

$$P(B)=\frac{{}^{6}C_2}{{}^{20}C_2}=\frac{3}{38}$$

$$P(C)=\frac{{}^{5}C_2}{{}^{20}C_2}=\frac{1}{19}$$

(i) Probability that the both balls are white or both are red

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$=\frac{3}{95}+\frac{3}{38}-0$$
$$=\frac{21}{190}$$

(ii) Probability that both balls are red or both are black

$$\begin{aligned} P(B \cup C) &= P(B) + P(C) - P(B \cap C) \\ &= \frac{3}{38} + \frac{1}{19} - 0 \\ &= \frac{5}{38} \end{aligned}$$

Example 7

Three students A, B, C are in a running race. A and B have the same probability of winning and each is twice as likely to win as C. Find the probability that B or C wins.

Solution

Let A, B, and C be the events that students A, B, and C win the race respectively.

$$P(A) = P(B) = 2P(C)$$

$$\begin{aligned} P(A) + P(B) + P(C) &= 1 \\ 2P(C) + 2P(C) + P(C) &= 1 \\ P(C) &= \frac{1}{5} \end{aligned}$$

$$\therefore \qquad P(A) = \frac{2}{5} \text{ and } P(B) = \frac{2}{5}$$

Probability that student B or C wins

$$\begin{aligned} P(B \cup C) &= P(B) + P(C) - P(B \cap C) \\ &= \frac{2}{5} + \frac{1}{5} - 0 \\ &= \frac{3}{5} \end{aligned}$$

Example 8

A card is drawn from a pack of 52 *cards. Find the probability of getting a king or a heart or a red card.*

Solution

Let A, B and C be the events that the card drawn is a king, a heart and a red card respectively.

$$P(A)=\frac{{}^{4}C_{1}}{{}^{52}C_{1}}=\frac{4}{52}$$

$$P(B)=\frac{{}^{13}C_{1}}{{}^{52}C_{1}}=\frac{13}{52}$$

$$P(C)=\frac{{}^{26}C_{1}}{{}^{52}C_{1}}=\frac{26}{52}$$

$$P(A\cap B)=\frac{{}^{1}C_{1}}{{}^{52}C_{1}}=\frac{1}{52}$$

$$P(B\cap C)=\frac{{}^{13}C_{1}}{{}^{52}C_{1}}=\frac{13}{52}$$

$$P(A\cap C)=\frac{{}^{2}C_{1}}{{}^{52}C_{1}}=\frac{2}{52}$$

$$P(A\cap B\cap C)=\frac{{}^{1}C_{1}}{{}^{52}C_{1}}=\frac{1}{52}$$

Probability that the card drawn is a king or a heart or a red card.

$$\begin{aligned}P(A\cup B\cup C)&=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)\\&\qquad+P(A\cap B\cap C)\\&=\frac{4}{52}+\frac{13}{52}+\frac{26}{52}-\frac{1}{52}-\frac{13}{52}-\frac{2}{52}+\frac{1}{52}\\&=\frac{7}{3}\end{aligned}$$

Example 9

From a city, 3 newspapers A, B, C are being published. A is read by 20%, B is read by 16%, C is read by 14%, both A and B are read by 8%, both A and C are read by 5%, both B and C are read by 4% and all three A, B, C are read by 2%. What is the probability that a randomly chosen person (i) reads at least one of these newspapers, and (ii) reads one of these newspapers?

Solution

Let A, B, and C be the events that the person reads newspapers A, B, and C respectively.

$$P(A)=0.2,\qquad P(B)=0.16\qquad P(C)=0.14$$

$$P(A\cap B)=0.08,\qquad P(A\cap B)=0.05,\qquad P(B\cap C)=0.04$$

$$P(A\cap B\cap C)=0.02$$

(i) Probability that the person reads at least one of these newspapers

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= 0.2 + 0.16 + 0.14 - 0.08 - 0.05 - 0.04 + 0.02$$

$$= 0.35$$

(ii) Probability that the person reads none of these newspapers

$$P(\overline{A} \cap \overline{B} \cap \overline{C}) = 1 - P(A \cup B \cup C)$$

$$= 1 - 0.35$$

$$= 0.65$$

Alternatively, the problem can be solved by a Venn diagram (Fig. 1.2).

(i) P(the person reads at least one paper) = $1 - \dfrac{65}{100} = 0.35$

(ii) P(the person reads none of these papers) = 0.65

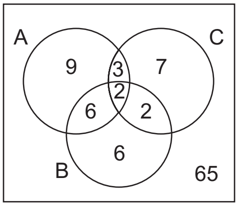


Fig. 1.2

EXERCISE 1.2

1. The probability that a student passes a Physics test is $\frac{2}{3}$ and the probability that he passes both Physics and English tests is $\frac{14}{45}$. The probability that he passes at least one test is $\frac{4}{5}$. What is the probability that the student passes the English test?

$\left[\textbf{Ans.:} \ \frac{4}{9}\right]$

2. What is the probability of drawing a black card or a king from a well-shuffled pack of playing cards?

$\left[\textbf{Ans.:} \ \frac{7}{13}\right]$

3. A pair of unbiased dice is thrown. Find the probability that (i) the sum of spots is either 5 or 10, and (ii) either there is a doublet or a sum less than 6.

$\left[\textbf{Ans.:} \text{ (i) } \frac{7}{36} \quad \text{(ii) } \frac{7}{18}\right]$

4. From a pack of well-shuffled cards, a card is drawn at random. What is the probability that the card drawn is a diamond card or a king card?

$$\left[\textbf{Ans.:}\ \frac{4}{13}\right]$$

5. A bag contains 6 red, 5 blue, 3 white, and 4 black balls. A ball is drawn at random. Find the probability that the ball is (i) red or black, and (ii) neither red or black.

$$\left[\textbf{Ans.:}\ \text{(i)}\ \frac{5}{9}\quad \text{(ii)}\ \frac{4}{9}\right]$$

6. There are 100 lottery tickets, numbered from 1 to 100. One of them is drawn at random. What is the probability that the number on it is a multiple of 5 or 7?

$$\left[\textbf{Ans.:}\ \frac{8}{25}\right]$$

7. From a group of 6 boys and 4 girls, a committee of 3 is to be formed. Find the probability that the committee will include (i) all three boys or all three girls, (ii) at most two girls, and (iii) at least one girl.

$$\left[\textbf{Ans.:}\ \text{(i)}\ \frac{1}{5}\quad \text{(ii)}\ \frac{29}{30}\quad \text{(iii)}\ \frac{5}{6}\right]$$

8. From a pack of 52 cards, three cards are drawn at random. Find the probability that (i) all three will be aces or all three kings, (ii) all three are pictures or all three are aces, (iii) none is a picture, (iv) at least one is a picture, (v) none is a spade, (vi) at most two are spades, and (vii) at least one is a spade.

$$\left[\begin{array}{l}\textbf{Ans.:}\ \text{(i)}\ \dfrac{2}{5225}\quad \text{(ii)}\ \dfrac{56}{5225}\quad \text{(iii)}\ \dfrac{38}{85}\quad \text{(iv)}\ \dfrac{47}{85}\\ \qquad\text{(v)}\ \dfrac{703}{1700}\quad \text{(vi)}\ \dfrac{839}{850}\quad \text{(vii)}\ \dfrac{997}{1700}\end{array}\right]$$

9. From a set of 16 cards numbered 1 to 16, one card is drawn at random. Find the probability that (i) the number obtained is divisible by 3 or 7, and (ii) not divisible by 3 and 7.

$$\left[\textbf{Ans.:}\ \text{(i)}\ \frac{7}{16}\quad \text{(ii)}\ \frac{9}{16}\right]$$

10. There are 12 bulbs in a basket of which 4 are working. A person tries to fit them in 3 sockets choosing 3 of the bulbs at random. What is the probability that there will be (i) some light, and (ii) no light in the room?

$$\left[\textbf{Ans.:} \text{ (i) } \frac{41}{55} \quad \text{(ii) } \frac{14}{55}\right]$$

Theorem 5 Multiplicative Law or Compound Law of Probability

A compound event is the result of the simultaneous occurrence of two or more event. The probability of a compound event depends upon whether the events are independent or not. Hence, there are two theorems:

(a) Conditional Probability Theorem

(b) Multiplicative Theorem for Independent Events

(a) Conditional Probability Theorem For any two events A and B in a sample space S, the probability of their simultaneous occurrence, i.e., both the events occurrings simultaneously is given by

$$P(A \cap B) = P(A)\, P(B/A)$$

or $$P(A \cap B) = P(B)\, P(A/B)$$

where $P(B/A)$ is the conditional probability of B given that A has already occurred. $P(A/B)$ is the conditional probability of A given that B has already occurred.

(b) Multiplicative Theorem for Independent Events If A and B are two independent events, the probability of their simultaneous occurrence is given by

$$P(A \cap B) = P(A)\, P(B)$$
$$P(A \cap B) = P(B)\, P(A/B) \quad \text{...(1.1)}$$

Proof $A = (A \cap B) \cup (A \cap \overline{B})$

Since $(A \cap B)$ and $(A \cap \overline{B})$ are mutually exclusive events,

$$P(A) = P(A \cap B) + P(A \cap \overline{B}) \qquad \text{[Using Axiom III]}$$
$$= P(B)\, P(A/B) + P(\overline{B})\, P(A/\overline{B})$$

If A and B are independent events, the proportion of A's in B is equal to proportion of A's in $\overline{B}$, i.e., $P(A/B) = P(A/\overline{B})$.

$$P(A) = P(A/B)\left[P(B) + P(\overline{B})\right]$$
$$= P(A/B)$$

Substituting in Eq. (1.1),

$\therefore \quad P(A \cap B) = P(A)\, P(B)$

Remark The additive law is used to find the probability of A or B, i.e., $P(A \cup B)$. The multiplicative law is used to find the probability of A and B, i.e., $P(A \cap B)$.

Corollary 1 If A, B and C are three events then

$$P(A \cap B \cap C) = P(A)\, P(B/A)\, P\left[C/(A \cap B)\right]$$

If A, B and C are independent events,

$$P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$$

Corollary 2 If A and B are independent events then A and $\bar{B}$, $\bar{A}$ and B, $\bar{A}$ and $\bar{B}$ are also independent.

Corollary 3 The probability of occurrence of at least one of the events A, B, C is given by

$$P(A \cup B \cup C) = 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$$

If A, B, and C are independent events, their complements will also be independent.

$$P(A \cup B \cup C) = 1 - P(\bar{A})\, P(\bar{B})\, P(\bar{C})$$

Pairwise Independence and Mutual Independence The events A, B and C are mutually independent if the following conditions are satisfied simultaneously:

$$P(A \cap B) = P(A)\, P(B)$$
$$P(B \cap C) = P(B)\, P(C)$$
$$P(A \cap C) = P(A)\, P(C)$$

and $$P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$$

If the last condition is not satisfied, the events are said to be pairwise independent. Hence, mutually independent events are always pairwise independent but not vices versa.

Example 1

If A and B are two events such that $P(A) = \dfrac{2}{3}, P(\bar{A} \cap B) = \dfrac{1}{6}$ *and* $P(A \cap B) = \dfrac{1}{3}$, *find* $P(B), P(A \cup B), P(A/B), P(B/A), P(\bar{A} \cup B)$ *and* $P(\bar{B})$. *Also, examine whether the events A and B are (i) equally likely, (ii) exhaustive, (iii) mutually exclusive, and (iv) independent.*

Solution

$$\begin{aligned} P(B) &= P(\bar{A} \cap B) + P(A \cap B) \\ &= \frac{1}{6} + \frac{1}{3} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
&= \frac{2}{3} + \frac{1}{2} - \frac{1}{3} \\
&= \frac{5}{6} \\
P(A/B) &= \frac{P(A \cap B)}{P(B)} \\
&= \frac{\left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)} \\
&= \frac{2}{3} \\
P(B/A) &= \frac{P(A \cap B)}{P(A)} \\
&= \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)} \\
&= \frac{1}{2} \\
P(\overline{A} \cup B) &= P(\overline{A}) + P(B) - P(\overline{A} \cap B) \\
&= \frac{1}{3} + \frac{1}{2} - \frac{1}{6} \\
&= \frac{2}{3} \\
P(\overline{A} \cap \overline{B}) &= 1 - P(A \cup B) \\
&= 1 - \frac{5}{6} \\
&= \frac{1}{6} \\
P(\overline{B}) &= 1 - P(B) \\
&= 1 - \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

(i) Since $P(A) \neq P(B)$, A and B are not equally like events.

(ii) Since $P(A \cup B) \neq 1$, A and B are not exhaustive events.

(iii) Since $P(A \cap B) \neq 0$, A and B are not mutually exclusive events.

(iv) Since $P(A \cap B) = P(A)\,P(B)$, A and B are independent events.

Example 2

If A and B are two events such that $P(A) = 0.3$, $P(B) = 0.4$, $P(A \cap B) = 0.2$, *find (i)* $P(A \cup B)$, *(ii)* $P(\overline{A}/B)$, *and (iii)* $P(A/\overline{B})$.

Solution

(i) $$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.3 + 0.4 - 0.2 \\ &= 0.5 \end{aligned}$$

(ii) $$\begin{aligned} P(\overline{A}/B) &= \frac{P(\overline{A} \cap B)}{P(B)} \\ &= \frac{P(B) - P(A \cap B)}{P(B)} \\ &= \frac{0.4 - 0.2}{0.4} \\ &= 0.5 \end{aligned}$$

(iii) $$\begin{aligned} P(A/\overline{B}) &= \frac{P(A \cap \overline{B})}{P(\overline{B})} \\ &= \frac{P(A) - P(A \cap B)}{1 - P(B)} \\ &= \frac{0.3 - 0.2}{1 - 0.4} \\ &= \frac{1}{6} \end{aligned}$$

Example 3

If A and B are two events with $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$, $P(A \cap B) = \frac{1}{12}$.

Find (i) $P(A/B)$, *(ii)* $P(B/A)$, *(iii)* $P(B/\overline{A})$, *and (iv)* $P(A \cap \overline{B})$.

Solution

(i) $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$$

(ii) $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{\frac{1}{12}}{\frac{1}{3}} = \dfrac{1}{4}$

(iii) $$P(B/\bar{A}) = \frac{P(B \cap \bar{A})}{P(\bar{A})}$$
$$= \frac{P(B) - P(B \cap A)}{1 - P(A)}$$
$$= \frac{\frac{1}{4} - \frac{1}{12}}{1 - \frac{1}{3}}$$
$$= \frac{1}{4}$$

(iv) $$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$
$$= \frac{1}{3} - \frac{1}{12}$$
$$= \frac{1}{4}$$

Example 4

Find the probability of drawing a queen and a king from a pack of cards in two consecutive draws, the cards drawn not being replaced.

Solution

Let A be the event that the card drawn is a queen.

$$P(A) = \frac{{}^4C_1}{{}^{52}C_1} = \frac{4}{52} = \frac{1}{13}$$

Let B be the event that the cards drawn are a king in the second draw given that the first card drawn is a queen.

$$P(B/A) = \frac{{}^4C_1}{{}^{51}C_1} = \frac{4}{51}$$

Probability that the cards drawn are a queen and a king

$$P(A \cap B) = P(A)\, P(B/A)$$
$$= \frac{4}{52} \times \frac{4}{51}$$
$$= \frac{4}{663}$$

Example 5

A bag contains 3 red and 4 white balls. Two draws are made without replacement. What is the probability that both the balls are red?

Solution

Let A be the event that the ball drawn is red in the first draw.

$$P(A) = \frac{3}{7}$$

Let B be the event that the ball drawn is red in the second draw given that the first ball drawn is red.

$$P(B/A) = \frac{2}{6}$$

Probability that both the balls are red

$$\begin{aligned} P(A \cap B) &= P(A)\, P(B/A) \\ &= \frac{3}{7} \times \frac{2}{6} \\ &= \frac{1}{7} \end{aligned}$$

Example 6

A bag contains 8 red and 5 white balls. Two successive draws of 3 balls each are made such that (i) the balls are replaced before the second trial, and (ii) the balls are not replaced before the second trial. Find the probability that the first draw will give 3 white and the second, 3 red balls.

Solution

Let A be the event that all 3 balls obtained at the first draw are white, and B be the event that all the 3 balls obtained at the second draw are red.

(i) When balls are replaced before the second trial,

$$P(A) = \frac{{}^{5}C_3}{{}^{13}C_3} = \frac{5}{143}$$

$$P(B) = \frac{{}^{8}C_3}{{}^{13}C_3} = \frac{28}{143}$$

Probability that the first draw will give 3 white and the second, 3 red balls

$$P(A \cap B) = P(A)\,P(B)$$
$$= \frac{5}{143} \times \frac{28}{143}$$
$$= \frac{140}{20449}$$

(ii) When the balls are not replaced before the second trial

$$P(B/A) = \frac{8_{C_3}}{10_{C_3}} = \frac{7}{15}$$

Probability that the first draw will give 3 white and the second, 3 red balls

$$P(A \cap B) = P(A)\,P(B/A)$$
$$= \frac{5}{143} \times \frac{7}{15}$$
$$= \frac{7}{429}$$

Example 7

From a bag containing 4 white and 6 black balls, two balls are drawn at random. If the balls are drawn one after the other without replacements, find the probability that the first ball is white and the second ball is black.

Solution

Let A be the event that the first ball drawn is white and B be the event that the second ball drawn is black given that the first ball drawn is white.

$$P(A) = \frac{4}{10}$$
$$P(B/A) = \frac{6}{9}$$

Probability that the first ball is white and the second ball is black.

$$P(A \cap B) = P(A)\,P(B/A)$$
$$= \frac{4}{10} \times \frac{6}{9}$$
$$= \frac{4}{15}$$

Example 8

Data on readership of a certain magazine show that the proportion of male readers under 35 is 0.40 and that over 35 is 0.20. If the proportion of readers under 35 is 0.70, find the probability of subscribers that are females over 35 years. Also, calculate the probability that a randomly selected male subscriber is under 35 years of age.

Solution

Let A be the event that the reader of the magazine is a male. Let B be the event that reader of the magazine is over 35 years of age.

$$P(A \cap \overline{B}) = 0.40, \qquad P(A \cap B) = 0.20, \qquad P(\overline{B}) = 0.7$$

$$\begin{aligned} P(B) &= 1 - P(\overline{B}) \\ &= 1 - 0.7 \\ &= 0.3 \end{aligned}$$

(i) Probability of subscribers that are females over 35 years

$$\begin{aligned} P(\overline{A} \cap B) &= P(B) - P(A \cap B) \\ &= 0.3 - 0.2 \\ &= 0.1 \end{aligned}$$

(ii) Probability that a randomly selected male subscriber is under 35 years of age

$$\begin{aligned} P(\overline{B}/A) &= \frac{P(A \cap \overline{B})}{P(A)} \\ &= \frac{P(A \cap \overline{B})}{P(A \cap B) + P(A \cap \overline{B})} \\ &= \frac{0.4}{0.2 + 0.4} \\ &= \frac{0.4}{0.6} \\ &= \frac{2}{3} \end{aligned}$$

Example 9

From a city population, the probability of selecting (a) a male or a smoker is $\frac{7}{10}$*, (b) a male smoker is* $\frac{2}{5}$*, and (c) a male, if a smoker is* *already selected, is* $\frac{2}{3}$. *Find the probability of selecting (i) a nonsmoker, (ii) a male, and (iii) a smoker, if a male is first selected.*

Solution

Let A be the event that a male is selected. Let B be the event that a smoker is selected.

$$P(A \cup B) = \frac{7}{10},\ P(A \cap B) = \frac{2}{5},\ P(A/B) = \frac{2}{3}$$

(i) Probability of selecting a nonsmoker

$$\begin{aligned} P(\overline{B}) &= 1 - P(B) \\ &= 1 - \frac{P(A \cap B)}{P(A/B)} \\ &= 1 - \frac{\left(\frac{2}{5}\right)}{\left(\frac{2}{3}\right)} \\ &= \frac{2}{5} \end{aligned}$$

(ii)
$$\begin{aligned} P(B) &= 1 - P(\overline{B}) \\ &= 1 - \frac{2}{5} \\ &= \frac{3}{5} \end{aligned}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad \text{...(1)}$$

Probability of selecting a male

$$\begin{aligned} P(A) &= P(A \cup B) + P(A \cap B) - P(B) \qquad \text{[Using Eq. (1)]} \\ &= \frac{7}{10} + \frac{2}{5} - \frac{3}{5} \\ &= \frac{1}{2} \end{aligned}$$

(iii) Probability of selecting a smoker if a male is first selected

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{\left(\frac{2}{5}\right)}{\left(\frac{1}{2}\right)}$$

$$= \frac{4}{5}$$

Example 10

Sixty per cent of the employees of the XYZ corporation are college graduates. Of these, ten percent are in sales. Of the employee who did not graduate from college, eighty percent are in sales. What is the probability that

(i) an employee selected at random is in sales?

(ii) an employee selected at random is neither in sales nor a college graduate?

Solution

Let A be the event that an employee is a college graduate. Let B be the event that an employee is in sales.

$$P(A) = 0.6,\ \ P(B/A) = 0.10,\ \ P(B/\overline{A}) = 0.8$$

$$P(\overline{A}) = 1 - P(A) = 1 - 0.60 = 0.40$$

(i) Probability that an employee is in sales

$$\begin{aligned} P(B) &= P(A \cap B) + P(\overline{A} \cap B) \\ &= P(A)\ P(B/A) + P(\overline{A})\ P(B/\overline{A}) \\ &= (0.6 \times 0.1) + (0.40 \times 0.80) \\ &= 0.38 \end{aligned}$$

(ii) Probability that an employee is neither in sales nor a college graduate

$$\begin{aligned} P(\overline{A} \cap \overline{B}) &= 1 - P(A \cup B) \\ &= 1 - \left[P(A) + P(B) - P(A \cap B)\right] \\ &= 1 - \left[P(A) + P(B) - P(A)\ P(B/A)\right] \\ &= 1 - \left[0.60 + 0.38 - (0.60 \times 0.10)\right] \\ &= 0.08 \end{aligned}$$

Example 11

If A and B are two events such that $P(A)=\frac{3}{8}, P(B)=\frac{5}{8}$ *and* $P(A\cup B)=\frac{3}{4}$*, find P(A/B) and P(B/A). Show whether A and B are independent.*

Solution

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$\frac{3}{4}=\frac{3}{8}+\frac{5}{8}-P(A\cap B)$$

$$P(A\cap B)=\frac{1}{4}$$

$$P(A/B)=\frac{P(A\cap B)}{P(B)}$$

$$=\frac{\left(\frac{1}{4}\right)}{\left(\frac{5}{8}\right)}$$

$$=\frac{2}{5}$$

$$P(B/A)=\frac{P(A\cap B)}{P(A)}$$

$$=\frac{\left(\frac{1}{4}\right)}{\left(\frac{3}{8}\right)}$$

$$=\frac{2}{3}$$

$$P(A)\,P(B)=\frac{3}{8}\times\frac{5}{8}=\frac{15}{64}$$

$$P(A\cap B)\neq P(A)\,P(B)$$

Hence, the events *A* and *B* are not independent.

Example 12

The probability that a student A solves a mathematics problem is $\frac{2}{5}$ *and the probability that a student B solves it is* $\frac{2}{3}$*. What is the probability* *that (i) the problem is not solved, (ii) the problem is solved, and (iii) both A and B, working independently of each other, solve the problem?*

Solution

Let A and B be events that students A and B solve the problem respectively.

$$P(A)=\frac{2}{5},\ \ P(B)=\frac{2}{3}$$

Events A and B are independent.

Probability that the student A does not solve the problem

$$\begin{aligned} P(\overline{A}) &= 1-P(A) \\ &= 1-\frac{2}{5} \\ &= \frac{3}{5} \end{aligned}$$

Probability that the student B does not solve the problem

$$\begin{aligned} P(\overline{B}) &= 1-P(B) \\ &= 1-\frac{2}{3} \\ &= \frac{1}{3} \end{aligned}$$

(i) Probability that the problem is not solved

$$\begin{aligned} P(\overline{A}\cap\overline{B}) &= P(\overline{A})\,P(\overline{B}) \\ &= \frac{3}{5}\times\frac{1}{3} \\ &= \frac{1}{5} \end{aligned}$$

(ii) Probability that the problem is solved

$$\begin{aligned} P(A\cup B) &= 1-P(\overline{A}\cap\overline{B}) \\ &= 1-\frac{1}{5} \\ &= \frac{4}{5} \end{aligned}$$

(iii) Probability that both A and B solve the problem

$$\begin{aligned} P(A\cap B) &= P(A)\,P(B) \\ &= \frac{2}{5}\times\frac{2}{3} \\ &= \frac{4}{15} \end{aligned}$$

Example 13

The probability that the machine A will perform a usual function in 5 years' time is $\frac{1}{4}$, *while the probability that the machine B will perform the function in 5 years' time is* $\frac{1}{3}$. *Find the probability that both machines will perform the usual function.*

Solution

Let A and B be the events that machines A and B will perform the usual function respectively.

$$P(A) = \frac{1}{4}$$

$$P(B) = \frac{1}{3}$$

Events A and B are independent.

Probability that both machines will perform the usual function

$$\begin{aligned} P(A \cap B) &= P(A)\, P(B) \\ &= \frac{1}{4} \times \frac{1}{3} \\ &= \frac{1}{12} \end{aligned}$$

Example 14

A person A is known to hit a target in 3 *out of* 4 *shots, whereas another person B is known to hit the same target in* 2 *out of* 3 *shots. Find the probability of the target being hit at all when they both try.*

[Summer 2015]

Solution

Let A and B be the events that the persons A and B hit the target respectively.

$$P(A) = \frac{3}{4}$$

$$P(B) = \frac{2}{3}$$

Events A and B are independent.

Probability that the person A will not hit the target $= P(\overline{A}) = 1 - P(A) = 1 - \frac{3}{4} = \frac{1}{4}$

Probability that the person B will not hit the target $= P(\overline{B}) = 1 - P(B) = 1 - \frac{2}{3} = \frac{1}{3}$

Probability that the target is not hit at all

$$P(\overline{A} \cap \overline{B}) = P(\overline{A})\, P(\overline{B})$$
$$= \frac{1}{4} \times \frac{1}{3}$$
$$= \frac{1}{12}$$

Probability that the target is hit at all when they both try

$$P(A \cup B) = 1 - P(\overline{A} \cap \overline{B})$$
$$= 1 - \frac{1}{12}$$
$$= \frac{11}{12}$$

Aliter

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= P(A) + P(B) - P(A)\, P(B) \quad [\because \quad A \text{ and } B \text{ independent}]$$
$$= \frac{3}{4} + \frac{2}{3} - \frac{3}{4} \times \frac{2}{3}$$
$$= \frac{11}{12}$$

Example 15

The odds against A speaking the truth are 4 : 6 *while the odds in favour of B speaking the truth are* 7 : 3. *What is the probability that A and B contradict each other in stating the same fact?*

Solution

Let A and B be events that A and B speak the truth respectively.

$$P(A) = \frac{6}{10}$$
$$P(B) = \frac{7}{10}$$

Events A and B are independent.

Probability that A speaks a lie $= P(\overline{A}) = 1 - P(A) = 1 - \frac{6}{10} = \frac{4}{10}$

Probability that B speaks a lie $= P(\overline{B}) = 1 - P(B) = 1 - \frac{7}{10} = \frac{3}{10}$

Probability that A and B contradict each other

$$P\left[(A \cap \bar{B}) \cup (\bar{A} \cap B)\right] = P(A \cap \bar{B}) + P(\bar{A} \cap B) \quad \left[\begin{array}{l}\because \ (A \cap \bar{B}) \text{ and } (\bar{A} \cap B) \text{ are} \\ \text{mutually exclusive events}\end{array}\right]$$

$$= P(A)\, P(\bar{B}) + P(\bar{A})\, P(B)$$

$$= \frac{6}{10} \times \frac{3}{10} + \frac{4}{10} \times \frac{7}{10}$$

$$= \frac{23}{50}$$

Example 16

An urn contains 10 red, 5 white and 5 blue balls. Two balls are drawn at random. Find the probability that they are not of the same colour.

Solution

Let A, B, and C be the events that two balls drawn at random be of the same colour, i.e., red, white, and blue respectively.

$$P(A) = \frac{10_{C_2}}{20_{C_2}} = \frac{9}{38}$$

$$P(B) = \frac{5_{C_2}}{20_{C_2}} = \frac{1}{19}$$

$$P(C) = \frac{5_{C_2}}{20_{C_2}} = \frac{1}{19}$$

Events A, B, and C are independent.

Probability that both balls drawn are of same colour

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

$$= \frac{9}{38} + \frac{1}{19} + \frac{1}{19}$$

$$= \frac{13}{38}$$

Probability that both balls drawn are not of the same colour

$$P(\bar{A} \cap \bar{B} \cap \bar{C}) = 1 - P(A \cup B \cup C)$$

$$= 1 - \frac{13}{38}$$

$$= \frac{25}{38}$$

Example 17

A problem in statistics is given to three students A, B and C, whose chances of solving it independently are $\frac{1}{2}, \frac{1}{3}$, and $\frac{1}{4}$ respectively. Find the probability that

(i) the problem is solved

(ii) at least two of them are able to solve the problem

(iii) exactly two of them are able to solve the problem

(iv) exactly one of them is able to solve the problem

Solution

Let A, B, and C be the events that students A, B, and C solve the problem respectively.

$$P(A) = \frac{1}{2},\ \ P(B) = \frac{1}{3},\ \ P(C) = \frac{1}{4}$$

Events A, B, and C are independent.

(i) Probability that the problem is solved or at least one of them is able to solve the problem is same.

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &\qquad + P(A \cap B \cap C) \\ &= P(A) + P(B) + P(C) - P(A)\ P(B) - P(A)\ P(C) - P(B)\ P(C) \\ &\qquad + P(A)\ P(B)\ P(C) \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \left(\frac{1}{2} \times \frac{1}{3}\right) - \left(\frac{1}{2} \times \frac{1}{4}\right) - \left(\frac{1}{3} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}\right) \\ &= \frac{3}{4} \end{aligned}$$

(ii) Probability that at least two of them are able to solve the problem

$$\begin{aligned} P\left[(A \cap B) \cup (B \cap C) \cup (A \cap C)\right] &= P(A \cap B) + P(B \cap C) + P(A \cap C) - 2P(A \cap B \cap C) \\ &= P(A)\ P(B) + P(B)\ P(C) + P(A)\ P(C) \\ &\qquad - 2P(A)\ P(B)\ P(C) \\ &= \left(\frac{1}{2} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) - 2\left(\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}\right) \\ &= \frac{7}{24} \end{aligned}$$

(iii) Probability that exactly two of them are able to solve the problem

$$
\begin{aligned}
P\left[(A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C \cup (\bar{A} \cap B \cap C\right] \\
&= P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C) \\
&= P(A)\, P(B) + P(B)\, P(C) + P(A)\, P(C) - 3P(A)\, P(B)\, P(C) \\
&= \left(\frac{1}{2} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) - 3\left(\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}\right) \\
&= \frac{1}{4}
\end{aligned}
$$

(iv) Probability that exactly one of them is able to solve the problem

$$
\begin{aligned}
P\left[A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)\right] \\
&= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(A \cap C) + 3P(A \cap B \cap C) \\
&= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - 2\left(\frac{1}{2} \times \frac{1}{3}\right) - 2\left(\frac{1}{3} \times \frac{1}{4}\right) - 2\left(\frac{1}{2} \times \frac{1}{4}\right) + 3\left(\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}\right) \\
&= \frac{11}{24}
\end{aligned}
$$

Example 18

A husband and wife appeared in an interview for two vacancies in an office. The probability of the husband's selection is $\frac{1}{7}$ *and that of the wife's selection is* $\frac{1}{5}$. *Find the probability that (i) both of them are selected, (ii) only one of them is selected, (iii) none of them is selected, and (iv) at least one of them is selected.*

Solution

Let A and B be the events that the husband and wife are selected respectively.

$$
P(A) = \frac{1}{7}, \qquad P(B) = \frac{1}{5}
$$

Events A and B are independent.

(i) Probability that both of them are selected

$$
\begin{aligned}
P(A \cap B) &= P(A)\, P(B) \\
&= \frac{1}{7} \times \frac{1}{5} \\
&= \frac{1}{35}
\end{aligned}
$$

(ii) Probability that at least one of them is selected

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{7} + \frac{1}{5} - \frac{1}{35}$$

$$= \frac{11}{35}$$

(iii) Probability that none of them is selected

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$$

$$= 1 - \frac{11}{35}$$

$$= \frac{24}{35}$$

(iv) Probability that only one of them is selected

$$P\left[A \cap \bar{B}) \cup (\bar{A} \cap B)\right] = P(A \cup B) - P(A \cap B)$$

$$= \frac{11}{35} - \frac{1}{35}$$

$$= \frac{10}{35}$$

$$= \frac{2}{7}$$

Example 19

There are two bags. The first contains 2 red and 1 white ball, whereas the second bag has only 1 red and 2 white balls. One ball is taken out at random from the first bag and put in the second. Then a ball is chosen at random from the second bag. What is the probability that this last ball is red?

Solution

There are two mutually exclusive cases.

Case I: A red ball is transferred from the first bag to the second bag and a red ball is drawn from it.

Case II: A white ball is transferred from the first bag to the second bag and then a red ball is drawn from it.

Let A be the event of transferring a red ball from the first bag, and B be the event of transferring a white ball from the first bag.

$$P(A) = \frac{2}{3}$$

$$P(B) = \frac{1}{3}$$

Let E be the event of drawing a red ball from the second bag.

$$P(E/A) = \frac{2}{4}$$

$$P(E/B) = \frac{1}{4}$$

$$\begin{aligned} P(\text{Case I}) &= P(A \cap E) \\ &= P(A)\, P(E/A) \\ &= \frac{2}{3} \times \frac{2}{4} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} P(\text{Case II}) &= P(B \cap E) \\ &= P(B)\, P(E/B) \\ &= \frac{1}{3} \times \frac{1}{4} \\ &= \frac{1}{12} \end{aligned}$$

$$\begin{aligned} P\left[(A \cap E) \cup (B \cap E)\right] &= P(A \cap E) + P(B \cap E) \\ &= \frac{1}{3} + \frac{1}{12} \\ &= \frac{5}{12} \end{aligned}$$

Example 20

An urn contains four tickets marked with numbers 112, 121, 211, *and* 222, *and one ticket is drawn. Let* A_i $(i = 1, 2, 3)$ *be the event that the* i^{th} *digit of the ticket drawn is* 1. *Show that the events* A_1, A_2, A_3 *are pairwise independent but not mutually independent.*

Solution

$$A_1 = \{112, 121\},\ \ A_2 = \{112, 211\},\ \ A_3 = \{121, 211\}$$

$$A_1 \cap A_2 = \{112\},\ \ A_1 \cap A_3 = \{121\},\ \ A_2 \cap A_3 = \{211\}$$

$$P(A_1) = \frac{2}{4} = \frac{1}{2} = P(A_2) = P(A_3)$$

$$P(A_1 \cap A_2) = \frac{1}{4} = P(A_1 \cap A_3) = P(A_2 \cap A_3)$$

$$P(A_1 \cap A_2) = P(A_1)\, P(A_2) = \frac{1}{4}$$

$$P(A_2 \cap A_3) = P(A_2)\, P(A_3) = \frac{1}{4}$$

$$P(A_1 \cap A_3) = P(A_1)\, P(A_3) = \frac{1}{4}$$

Hence, events A_1, A_2, and A_3 are pairwise independent.

$$P(A_1 \cap A_2 \cap A_3) = P(\phi) = 0$$

$$P(A_1 \cap A_2 \cap A_3) \neq P(A_1)\, P(A_2)\, P(A_3)$$

Hence, events A_1, A_2, and A_3 are not mutually independent.

EXERCISE 1.3

1. Find the probability of drawing 2 red balls in succession from a bag containing 4 red and 5 black balls when the ball that is drawn first is (i) not replaced, and (ii) replaced.

$\left[\textbf{Ans.:} \text{ (i) } \frac{1}{6} \quad \text{(ii) } \frac{16}{81}\right]$

2. Two aeroplanes bomb a target in succession. The probability of each correctly scoring a hit is 0.3 and 0.2 respectively. The second will bomb only if the first misses the target. Find the probability that (i) the target is hit, and (ii) both fail to score hits.

[**Ans.:** (i) 0.44 (ii) 0.56]

3. Box *A* contains 5 red and 3 white marbles and Box *B* contains 2 red and 6 white marbles. If a marble is drawn from each box, what is the probability that they are both of the same colour?

[**Ans.:** 0.109]

4. Two marbles are drawn in succession from a box containing 10 red, 30 white, 20 blue, and 15 orange marbles, with replacement being made after each draw. Find the probability that (i) both are white, and (ii) the first is red and the second is white.

$\left[\textbf{Ans.:} \text{ (i) } \frac{4}{25} \quad \text{(ii) } \frac{4}{75}\right]$

5. *A*, *B*, *C* are aiming to shoot a balloon. *A* will succeed 4 times out of 5 attempts. The chance of *B* to shoot the balloon is 3 out of 4, and that of C is 2 out of 3. If the three aim the balloon simultaneously, find the probability that at least two of them hit the balloon.

$$\left[\textbf{Ans.:} \frac{5}{6}\right]$$

6. There are 12 cards numbered 1 to 12 in a box. If two cards are selected, what is the probability that the sum is odd (i) with replacement, and (ii) without replacement?

$$\left[\textbf{Ans.:} \text{(i) } \frac{1}{2} \quad \text{(ii) } \frac{6}{11}\right]$$

7. Two cards are drawn from a well-shuffled pack of 52 cards. Find the probability that they are both aces if the first card is (i) replaced, and (ii) not replaced.

$$\left[\textbf{Ans.:} \text{(i) } \frac{1}{169} \quad \text{(ii) } \frac{1}{221}\right]$$

8. A can hit a target 2 times in 5 shots; B, 3 times in 4 shots; and C, 2 times in 3 shots. They fire a volley. What is the probability that at least 2 shots hit the target?

$$\left[\textbf{Ans.:} \frac{2}{3}\right]$$

9. There are two bags. The first bag contains 5 red and 7 white balls and the second bag contains 3 red and 12 white balls. One ball is taken out at random from the first bag and is put in the second bag. Now, a ball is drawn from the second bag. What is the probability that this last ball is red?

$$\left[\textbf{Ans.:} \frac{41}{192}\right]$$

10. In a shooting competition, the probability of A hitting the target is $\frac{1}{2}$; of B, is $\frac{2}{3}$; and of C, is $\frac{3}{4}$. If all of them fire at the target, find the probability that (i) none of them hits the target, and (ii) at least one of them hits the target.

$$\left[\textbf{Ans.:} \text{(i) } \frac{1}{24} \quad \text{(ii) } \frac{23}{24}\right]$$

11. The odds against a student X solving a statistics problem are 12 to 10 and the odds in favour of a student Y solving the problem are 6 to 9.

What is the probability that the problem will be solved when both try independently of each other?

$$\left[\textbf{Ans.: } \frac{37}{55}\right]$$

12. A bag contains 6 white and 9 black balls. Four balls are drawn at random twice. Find the probability that the first draw will give 4 white balls and the second draw will give 4 black balls if (i) the balls are replaced, and (ii) the balls are not replaced before the second draw.

$$\left[\textbf{Ans.: } \text{(i) } \frac{6}{5915} \quad \text{(ii) } \frac{3}{715}\right]$$

13. An urn contains 10 white and 3 black balls. Another urn contains 3 white and 5 black balls. Two balls are transferred from the first urn to the second urn and then one ball is drawn from the latter. What is the probability that the ball drawn is white?

$$\left[\textbf{Ans.: } \frac{5}{26}\right]$$

14. A man wants to marry a girl having the following qualities: fair complexion—the probability of getting such a girl is $\frac{1}{20}$, handsome dowry—the probability is $\frac{1}{50}$, westernized manners and etiquettes—the probability of this is $\frac{1}{100}$. Find the probability of his getting married to such a girl when the possessions of these three attributes are independent.

$$\left[\textbf{Ans.: } \frac{1}{100000}\right]$$

15. A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98 and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the fire engine and ambulance will be available.

$$\left[\textbf{Ans.: } 0.9016\right]$$

16. In a certain community, 36% of the families own a dog and 22% of the families that own a dog also own a cat. In addition, 30% of the families own a cat. What is the probability that (i) a randomly selected family owns both a dog and a cat, and (ii) a randomly selected family owns a dog given that it owns a cat?

[**Ans.:** (i) 0.0792 (ii) 0.264]

1.5 BAYES' THEOREM

Let $A_1, A_2, ..., A_n$ be n mutually exclusive and exhaustive events with $P(A_i) \neq 0$ for $i = 1, 2, ..., n$ in a sample space S. Let B be an event that can occur in combination with any one of the events $A_1, A_2, ..., A_n$ with $P(B) \neq 0$. The probability of the event A_i when the event B has actually occurred is given by

$$P(A_i/B) = \frac{P(A_i)\, P(B/A_i)}{\sum_{i=1}^{n} P(A_i)\, P(B/A_i)}$$

Proof Since $A_1, A_2, ..., A_n$ are n mutually exclusive and exhaustive events of the sample space S,

$$S = A_1 \cup A_2 \cup ... \cup A_n$$

Since B is another event that can occur in combination with any of the mutually exclusive and exhaustive events $A_1, A_2, ..., A_n$,

$$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$$

Taking probability of both the sides,

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$$

The events $(A_1 \cap B), (A_2 \cap B),$ etc., are mutually exclusive.

$$P(B) = \sum_{i=1}^{n} P(A_i \cap B) = \sum_{i=1}^{n} P(A_i)\, P(B/A_i)$$

The conditional probability of an event A given that B has already occurred is given by

$$P(A_i/B) = \frac{P(A_i \cap B)}{P(B)}$$

$$= \frac{P(A_i)\, P(B/A_i)}{P(B)}$$

$$= \frac{P(A_i)\, P(B/A_i)}{\sum_{i=1}^{n} P(A_i)\, P(B/A_i)}$$

Example 1

A company has two plants to manufacture hydraulic machines. Plant I manufactures 70% of the hydraulic machines, and Plant II manufactures 30%. At Plant I, 80% of hydraulic machines are rated standard quality; and at Plant II, 90% of hydraulic machines are rated standard quality. A machine is picked up at random and is found to be of standard quality. What is the chance that it has come from Plant I? **[Summer 2015]**

Solution

Let A_1 and A_2 be the events that the hydraulic machines are manufactured in Plant I and Plant II respectively. Let B be the event that the machine picked up is found to be of standard quality.

$$P(A_1) = \frac{70}{100} = 0.7$$

$$P(A_2) = \frac{30}{100} = 0.3$$

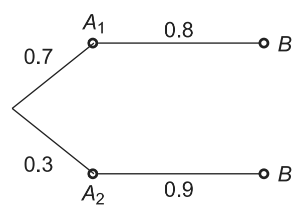


Fig. 1.3

Probability that the machine is of standard quality given that it is manufactured in Plant I

$$P(B/A_1) = \frac{80}{100} = 0.8$$

Probability that the machine is of standard quality given that it is manufactured in Plant II

$$P(B/A_2) = \frac{90}{100} = 0.9$$

Probability that a machine is manufactured in Plant I given that it is of standard quality

$$P(A_1/B) = \frac{P(A_1)\,P(B/A_1)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)}$$

$$= \frac{0.7 \times 0.8}{0.7 \times 0.8 + 0.3 \times 0.9}$$

$$= 0.6747$$

Example 2

A bag A contains 2 white and 3 red balls, and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that the red ball is drawn from the bag B.

Solution

Let A_1 and A_2 be the events that the ball is drawn from bags A and B respectively. Let B be the event that the ball drawn is red.

$$P(A_1) = \frac{1}{2}$$

$$P(A_2) = \frac{1}{2}$$

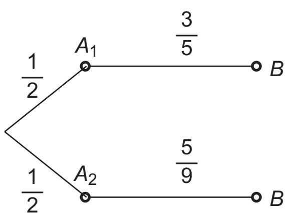


Fig. 1.4

Probability that the ball drawn is red given that it is drawn from the bag A

$$P(B/A_1) = \frac{3}{5}$$

Probability that the ball drawn is red given that it is drawn from the bag B

$$P(B/A_2) = \frac{5}{9}$$

Probability that the ball is drawn from the bag B given that it is red

$$P(A_2/B) = \frac{P(A_2)\,P(B/A_2)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)}$$

$$= \frac{\frac{1}{2} \times \frac{5}{9}}{\left(\frac{1}{2} \times \frac{3}{5}\right) + \left(\frac{1}{2} \times \frac{5}{9}\right)}$$

$$= \frac{25}{52}$$

Example 3

The chances that Doctor A will diagnose a disease X correctly is 60%. The chances that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of Doctor A, who had the disease X, died. What is the chance that his disease was diagnosed correctly?

Solution

Let A_1 be the event that the disease X is diagnosed correctly by Doctor A. Let A_2 be the event that the disease X is not diagnosed correctly by Doctor A. Let B be the event that a patient of Doctor A who has the disease X, dies.

$$P(A_1) = \frac{60}{100} = 0.6$$
$$P(A_2) = P(\overline{A}_1) = 1 - P(A_1) = 0.4$$

Probability that the patient of Doctor A who has the disease X dies given that the disease X is diagnosed correctly

$$P(B/A_1) = \frac{40}{100} = 0.4$$

Probability that the patient of Doctor A who has the disease X dies given that the disease X is not diagnosed correctly

$$P(B/A_2) = \frac{70}{100} = 0.7$$

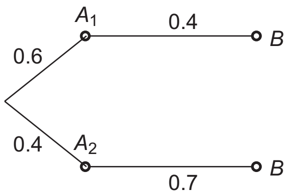


Fig. 1.5

Probability that the disease X is diagnosed correctly given that a patient of Doctor A who has the disease X dies

$$P(A_1/B) = \frac{P(A_1)\,P(B/A_1)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)}$$
$$= \frac{0.6 \times 0.4}{(0.6 \times 0.4) + (0.4 \times 0.7)}$$
$$= \frac{6}{13}$$

Example 4

In a bolt factory, machines A, B, C manufacture 25%, 35%, and 40% of the total output and out of the total manufacturing, 5%, 4%, and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. Find the probabilities that it is manufactured from (i) Machine A, (ii) Machine B, and (iii) Machine C.

Solution

Let A_1, A_2 and A_3 be the events that bolts are manufactured by machines A, B, and C respectively. Let B be the event that the bolt drawn is defective.

$$P(A_1) = \frac{25}{100} = 0.25$$
$$P(A_2) = \frac{35}{100} = 0.35$$
$$P(A_3) = \frac{40}{100} = 0.4$$

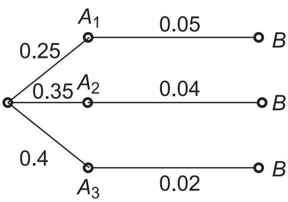


Fig. 1.6

Probability that the bolt drawn is defective given that it is manufactured from Machine A

$$P(B/A_1) = \frac{5}{100} = 0.05$$

Probability that the bolt drawn is defective given that it is manufactured from Machine B

$$P(B/A_2) = \frac{4}{100} = 0.04$$

Probability that the bolt drawn is defective given that it is manufactured from Machine C

$$P(B/A_3) = \frac{2}{100} = 0.02$$

(i) Probability that a bolt is manufactured from Machine A given that it is defective

$$\begin{aligned} P(A_1/B) &= \frac{P(A_1)\,P(B/A_1)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + P(A_3)\,P(B/A_3)} \\ &= \frac{0.25 \times 0.05}{(0.25 \times 0.05) + (0.35 \times 0.04) + (0.4 \times 0.02)} \\ &= 0.3623 \end{aligned}$$

(ii) Probability that a bolt is manufactured from Machine B given that it is defective

$$\begin{aligned} P(A_2/B) &= \frac{P(A_2)\,P(B/A_2)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + P(A_3)\,P(B/A_3)} \\ &= \frac{0.35 \times 0.04}{(0.25 \times 0.05) + (0.35 \times 0.04) + (0.4 \times 0.02)} \\ &= 0.4058 \end{aligned}$$

(iii) Probability that a bolt is manufactured from Machine C given that it is defective

$$\begin{aligned} P(A_3/B) &= \frac{P(A_3)\,P(B/A_3)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + P(A_3)\,P(B/A_3)} \\ &= \frac{0.4 \times 0.02}{(0.25 \times 0.05) + (0.35 \times 0.04) + (0.4 \times 0.02)} \\ &= 0.2319 \end{aligned}$$

Example 5

A businessman goes to hotels X, Y, Z for 20%, 50%, 30% *of the time respectively. It is known that* 5%, 4%, 8% *of the rooms in X, Y, Z hotels have faulty plumbings. What is the probability that the businessman's room having faulty plumbing is assigned to Hotel Z?*

Solution

Let A_1, A_2 and A_3 be the events that the businessman goes to hotels X, Y, Z respectively. Let B be the event that the rooms have faulty plumbings.

$$P(A_1) = \frac{20}{100} = 0.2$$

$$P(A_2) = \frac{50}{100} = 0.5$$

$$P(A_3) = \frac{30}{100} = 0.3$$

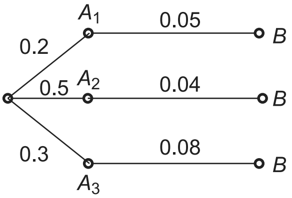


Fig. 1.7

Probability that rooms have faulty plumbings given that rooms belong to Hotel X

$$P(B/A_1) = \frac{5}{100} = 0.05$$

Probability that rooms have faulty plumbing given that rooms belong to Hotel Y

$$P(B/A_2) = \frac{4}{100} = 0.04$$

Probability that rooms have faulty plumbings given that rooms belong to Hotel Z

$$P(B/A_3) = \frac{8}{100} = 0.08$$

Probability that the businessman's room belongs to Hotel Z given that the room has faulty plumbing

$$P(A_3/B) = \frac{P(A_3)\,P(B/A_3)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + P(A_3)\,P(B/A_3)}$$

$$= \frac{0.3 \times 0.08}{(0.2 \times 0.05) + (0.5 \times 0.04) + (0.3 \times 0.08)}$$

$$= \frac{4}{9}$$

Example 6

Of three persons the chances that a politician, a businessman, or an academician would be appointed the Vice Chancellor (VC) of a university are 0.5, 0.3, 0.2 *respectively. Probabilities that research is promoted by these persons if they are appointed as VC are* 0.3, 0.7, 0.8 *respectively.*

(i) Determine the probability that research is promoted.

(ii) If research is promoted, what is the probability that the VC is an academician?

Solution

Let A_1, A_2 and A_3 be the events that a politician, a businessman or an academician will be appointed as the VC respectively. Let B be the event that research is promoted by these persons if they are appointed as VC.

$$P(A_1) = 0.5$$
$$P(A_2) = 0.3$$
$$P(A_3) = 0.2$$

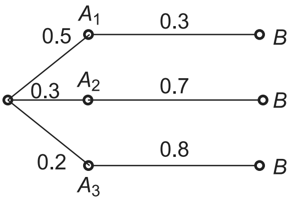


Fig. 1.8

Probability that research is promoted given that a politician is appointed as VC

$$P(B/A_1) = 0.3$$

Probability that research is promoted given that a businessman is promoted as VC

$$P(B/A_2) = 0.7$$

Probability that research is promoted given that an academician is appointed as VC

$$P(B/A_3) = 0.8$$

(i) Probability that research is promoted

$$\begin{aligned} P(B) &= P(A_1)\, P(B/A_1) + P(A_2)\, P(B/A_2) + P(A_3)\, P(B/A_3) \\ &= (0.5 \times 0.3) + (0.3 \times 0.7) + (0.2 \times 0.8) \\ &= 0.52 \end{aligned}$$

(ii) Probability that the VC is an academician given that research is promoted by him

$$\begin{aligned} P(A_3/B) &= \frac{P(A_3)\, P(B/A_3)}{P(A_1)\, P(B/A_1) + P(A_2)\, P(B/A_2) + P(A_3)\, P(B/A_3)} \\ &= \frac{0.2 \times 0.8}{0.52} \\ &= \frac{4}{13} \end{aligned}$$

Example 7

The contents of urns I, II, and III are as follows:

1 *white,* 2 *red, and* 3 *black balls,*

2 *white,* 3 *red, and* 1 *black ball, and*

3 *white,* 1 *red, and* 2 *black balls.*

One urn is chosen at random and two balls are drawn. They happen to be white and red. Find the probability that they came from (i) Urn I, (ii) Urn II, and (iii) Urn III.

Solution

Let A_1, A_2, and A_3 be the events that urns I, II and III are chosen respectively. Let B be the event that 2 balls drawn are white and red.

$$P(A_1) = \frac{1}{3}$$

$$P(A_2) = \frac{1}{3}$$

$$P(A_3) = \frac{1}{3}$$

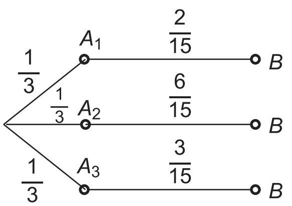


Fig. 1.9

Probability that 2 balls drawn are white and red given that they are chosen from the urn I

$$P(B/A_1) = \frac{{}^1C_1 \times {}^2C_1}{{}^6C_2} = \frac{1 \times 2}{15} = \frac{2}{15}$$

Probability that 2 balls drawn are white and red given that they are chosen from the urn II

$$P(B/A_2) = \frac{{}^2C_1 \times {}^3C_1}{{}^6C_2} = \frac{2 \times 3}{15} = \frac{6}{15}$$

Probability that 2 balls drawn are white and red given that they are chosen from the urn III

$$P(B/A_3) = \frac{{}^3C_1 \times {}^1C_1}{{}^6C_2} = \frac{3 \times 1}{15} = \frac{3}{15}$$

(i) Probability that 2 balls came from the urn I given that they are white and red

$$P(A_1/B) = \frac{P(A_1)\, P(B/A_1)}{P(A_1)\, P(B/A_1) + P(A_2)\, P(B/A_2) + P(A_3)\, P(B/A_3)}$$

$$= \frac{\frac{1}{3} \times \frac{2}{15}}{\left(\frac{1}{3} \times \frac{2}{15}\right) + \left(\frac{1}{3} \times \frac{6}{15}\right) + \left(\frac{1}{3} \times \frac{3}{15}\right)}$$

$$= \frac{2}{11}$$

(ii) Probability that 2 balls came from the urn II given that they are white and red

$$P(A_2/B) = \frac{P(A_2)\, P(B/A_2)}{P(A_1)\, P(B/A_1) + P(A_2)\, P(B/A_2) + P(A_3)\, P(B/A_3)}$$

$$= \frac{\frac{1}{3} \times \frac{6}{15}}{\left(\frac{1}{3} \times \frac{2}{15}\right) + \left(\frac{1}{3} \times \frac{6}{15}\right) + \left(\frac{1}{3} \times \frac{3}{15}\right)}$$

$$= \frac{6}{11}$$

(iii) Probability that 2 balls came from the urn III given that they are white and red

$$P(A_3/B) = \frac{P(A_3)\,P(B/A_3)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + P(A_3)\,P(B/A_3)}$$

$$= \frac{\frac{1}{3} \times \frac{3}{15}}{\left(\frac{1}{3} \times \frac{2}{15}\right) + \left(\frac{1}{3} \times \frac{6}{15}\right) + \left(\frac{1}{3} \times \frac{3}{15}\right)}$$

$$= \frac{3}{11}$$

EXERCISE 1.4

1. There are 4 boys and 2 girls in Room *A* and 5 boys and 3 girls in Room *B*. A girl from one of the two rooms laughed loudly. What is the probability the girl who laughed was from Room *B*?

$\left[\textbf{Ans.:}\ \frac{9}{17}\right]$

2. The probability of *X*, *Y*, and *Z* becoming managers are $\frac{4}{9}, \frac{2}{9}$, and $\frac{1}{3}$ respectively. The probabilities that the bonus scheme will be introduced if *X*, *Y*, and *Z* become managers are $\frac{3}{10}, \frac{1}{2}$, and $\frac{4}{5}$ respectively. (i) What is the probability that the bonus scheme will be introduced? (ii) If the bonus scheme has been introduced, what is the probability that the manager appointed was *X*?

$\left[\textbf{Ans.:}\ \text{(i)}\ \frac{23}{45}\quad \text{(ii)}\ \frac{6}{23}\right]$

3. A factory has two machines, *A* and *B*. Past records show that the machine *A* produces 30% of the total output and the machine *B*, the remaining 70%. Machine *A* produces 5% defective articles and Machine *B* produces 1% defective items. An item is drawn at random and found to be defective. What is the probability that it was produced (i) by the machine *A*, and (ii) by the Machine *B*?

[**Ans.:** (i) 0.682 (ii) 0.318]

4. A company has two plants to manufacture scooters. Plant I manufactures 80% of the scooters, and Plant II manufactures 20%. At Plant I, 85 out of 100 scooters are rated standard quality or better. At Plant II, only 65 out of 100 scooters are rated standard quality or better. What is the probability that a scooter selected at random came from (i) Plant I, and (ii) Plant II if it is known that the scooter is of standard quality?

[**Ans.:** (i) 0.84 (ii) 0.16]

5. A new pregnancy test was given to 100 pregnant women and 100 non-pregnant women. The test indicated pregnancy in 92 of the 100 pregnant women and in 12 of the 100 non-pregnant women. If a randomly selected woman takes this test and the test indicates she is pregnant, what is the probability she was not pregnant?

$$\left[\textbf{Ans.:}\ \frac{3}{26}\right]$$

6. An insurance company insured 2000 scooter drivers, 4000 car drivers, and 6000 truck drivers. The probability of an accident is 0.01, 0.03, and 0.15 in the respective category. One of the insured drivers meets with an accident. What is the probability that he is a scooter driver?

$$\left[\textbf{Ans.:}\ \frac{1}{52}\right]$$

7. Consider a population of consumers consisting of two types. The upper-income class of consumers comprise 35% of the population and each member has a probability of 0.8 of purchasing Brand *A* of a product. Each member of the rest of the population has a probability of 0.3 of purchasing Brand *A* of the product. A consumer, chosen at random, is found to be the buyer of Brand *A*. What is the probability that the buyer belongs to the middle-income and lower-income classes of consumers?

$$\left[\textbf{Ans.:}\ \frac{39}{95}\right]$$

8. There are two boxes of identical appearance, each containing 4 spark plugs. It is known that the box I contains only one defective spark plug, while all the four spark plugs of the box II are non-defective. A spark plug drawn at random from a box, selected at random, is found to be non-defective. What is the probability that it came from the box I?

$$\left[\textbf{Ans.:}\ \frac{3}{7}\right]$$

9. Vijay has 5 one-rupee coins and one of them is known to have two heads. He takes out a coin at random and tosses it 5 times—it always falls head upward. What is the probability that it is a coin with two heads?

$$\left[\textbf{Ans.: } \frac{8}{9}\right]$$

10. Stores A, B, and C have 50, 75, and 100 employees and, respectively 50, 60, 70 per cent of these are women. Resignations are equally likely among all employees, regardless of sex. One employee resigns and this is a woman. What is the probability that she works in Store C?

$$\left[\textbf{Ans.: } 0.5\right]$$

Points to Remember

Theorems on Probability

Theorem 1

The probability of an impossible event is zero, i.e., $P(\phi) = 0$, where ϕ is a null set.

Theorem 2

The probability of the complementary event $\bar{A}$ of A is

$$P(\bar{A}) = 1 - P(A)$$

De Morgan's Laws

$$P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B})$$

$$P(\overline{A \cap B}) = P(\bar{A} \cup \bar{B})$$

Theorem 3

For any two events A and B in a sample space S,

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

Theorem 4 Additive Law of Probability (Addition Theorem)

The probability that at least one of the events A and B will occur is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Corollary 1 From the Venn diagram,

$$P(A \cup B) = 1 - P(\bar{A} \cap \bar{B})$$

Corollary 2 $P(\text{Exactly one of } A \text{ and } B \text{ occurs}) = P(A \cup B) - P(A \cap B)$

Corollary 3 If A, B, and C are three events of a sample space S then the probability of occurrence of at least one of them is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

Alternately,

$$P(A \cup B \cup C) = 1 - P(\overline{A} \cap \overline{B} \cap \overline{C})$$

If A, B, and C are mutually exclusive events,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

Corollary 4 The probability of occurrence of at least two of the three events is given by

$$P\left[A \cap B) \cup (B \cap C) \cup (A \cap C)\right] = P(A \cap B) + P(B \cap C) + P(A \cap C) - 2P(A \cap B \cap C)$$

Corollary 5 The probability of occurrence of exactly two of the three events is given by

$$P\left[A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C)\right]$$

$$= P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C)$$

Corollary 6 The probability of occurrence of exactly one of the three events is given by

$$P\left[(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)\right]$$

$$= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(A \cap C) + 3P(A \cap B \cap C)$$

Theorem 5 Multiplicative Law or Compound Law of Probability

(a) Conditional Probability Theorem

$$P(A \cap B) = P(A)\, P(B/A)$$

$$P(A \cap B) = P(B)\, P(A/B)$$

(b) Multiplicative Theorem for Independent Events

$$P(A \cap B) = P(A)\, P(B)$$

Corollary 1 If A, B and C are three events then

$$P(A \cap B \cap C) = P(A)\, P(B/A)\, P\left[C/(A \cap B)\right]$$

If A, B and C are independent events,

$$P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$$

Corollary 2 If A and B are independent events then A and $\overline{B}$, $\overline{A}$ and B, $\overline{A}$ and $\overline{B}$ are also independent.

Corollary 3 The probability of occurrence of at least one of the events A, B, C is given by

$$P(A \cup B \cup C) = 1 - P(\overline{A} \cap \overline{B} \cap \overline{C})$$

If A, B, and C are independent events, their complements will also be independent.

$$P(A \cup B \cup C) = 1 - P(\overline{A})\, P(\overline{B})\, P(\overline{C})$$

Pairwise Independence and Mutual Independence The events A, B and C are mutually independent if the following conditions are satisfied simultaneously:

$$P(A \cap B) = P(A)\, P(B)$$
$$P(B \cap C) = P(B)\, P(C)$$
$$P(A \cap C) = P(A)\, P(C)$$

and $$P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$$

If the last condition is not satisfied, the events are said to be pairwise independent.

Bayes' Theorem

The probability of the event A_i when the event B has actually occurred is given by

$$P(A_i/B) = \frac{P(A_i)\, P(B/A_i)}{\sum_{i=1}^{n} P(A_i)\, P(B/A_i)}$$

CHAPTER 2

Random Variables and Probability Distributions

Chapter Outline

2.1 Introduction
2.2 Random Variables
2.3 Discrete Probability Distribution
2.4 Discrete Distribution Function
2.5 Measures of Central Tendency for Discrete Probability Distribution
2.6 Continuous Probability Distribution
2.7 Continuous Distribution Function
2.8 Measures Of Central Tendency For Continuous Probability Distribution
2.9 Binomial Distribution
2.10 Poisson Distribution
2.11 Normal Distribution

2.1 INTRODUCTION

The outcomes of random experiments are, in general, abstract quantities or, in other words, most of the time they are not in any numerical form. However, the outcomes of a random experiment can be expressed in quantitative terms, in particular, by means of real numbers. Hence, a function can be defined that takes a definite real value corresponding to each outcome of an experiment. This gives a rationale for the concept of random variables about which probability statements can be made.

In probability and statistics, a probability distribution assigns a probability to each measurable subset of the possible outcomes of a random experiment. Important and commonly encountered probability distributions include binomial distribution, Poisson distribution, and normal distribution.

2.2 RANDOM VARIABLES

A random variable X is a real-valued function of the elements of the sample space of a random experiment. In other words, a variable which takes the real values, depending on the outcome of a random experiment is called a *random variable*, e.g.,

(i) When a fair coin is tossed, $S = \{H, T\}$. If X is the random variable denoting the number of heads,

$X(H) = 1$ and $X(T) = 0$

Hence, the random variable X can take values 0 and 1.

(ii) When two fair coins are tossed, $S = \{HH, HT, TH, TT\}$. If X is the random variable denoting the number of heads,

$X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0.$

Hence, the random variable X can take values 0, 1, and 2.

(iii) When a fair die is tossed, $S = \{1, 2, 3, 4, 5, 6\}$.

If X is the random variable denoting the square of the number obtained,

$X(1) = 1, X(2) = 4, X(3) = 9, X(4) = 16, X(5) = 25, X(6) = 36$

Hence, the random variable X can take values 1, 4, 9, 16, 25, and 36.

Types of Random Variables

There are two types of random variables:

(i) Discrete random variables
(ii) Continuous random variables

Discrete Random Variables A random variable X is said to be discrete if it takes either finite or countably infinite values. Thus, a discrete random variable takes only isolated values, e.g.,

(i) Number of children in a family
(ii) Number of cars sold by different companies in a year
(iii) Number of days of rainfall in a city
(iv) Number of stars in the sky
(v) Profit made by an investor in a day

Continuous Random Variables A random variable X is said to be continuous if it takes any values in a given interval. Thus, a continuous random variable takes uncountably infinite values, e.g.,

(i) Height of a person in cm
(ii) Weight of a bag in kg
(iii) Temperature of a city in degree Celsius

(iv) Life of an electric bulb in hours
(v) Volume of a gas in cc.

Example 1

Identify the random variables as either discrete or continuous in each of the following cases:

(i) A page in a book can have at most 300 *words*
X = Number of misprints on a page
(ii) Number of students present in a class of 50 *students*
(iii) A player goes to the gymnasium regularly
X = Reduction in his weight in a month
(iv) Number of attempts required by a candidate to clear the IAS examination
(v) Height of a skyscraper

Solution

(i) X = Number of misprints on a page
The page may have no misprint or 1 misprint or 2 misprint … or 300 misprints. Thus, X takes values 0, 1, 2, …, 300. Hence, X is a discrete random variable.
(ii) Let X be the random variable denoting the number of students present in a class. X takes values 0, 1, 2, …, 50. Hence, X is a discrete random variable.
(iii) Reduction in weight cannot take isolated values 0, 1, 2, etc., but it takes any continuous value.
Hence, X is a continuous random variable.
(iv) Let X be a random variable denoting the number of attempts required by a candidate. Thus, X takes values 1, 2, 3, …. Hence, X is a discrete random variable.
(v) Since height can have any fractional value, it is a continuous random variable.

2.3 DISCRETE PROBABILITY DISTRIBUTION

Probability distribution of a random variable is the set of its possible values together with their respective probabilities. Let X be a discrete random variable which takes the values $x_1, x_2, \dots x_n$. The probability of each possible outcome x_i is $p_i = p(x_i) = P(X = x_i)$ for $i = 1, 2, \dots, n$. The number $p(x_i)$, $i = 1,2, \dots$. must satisfy the following conditions:

(i) $p(x_i) \geq 0$ for all values of i

(ii) $$\sum_{i=1}^{\infty} p(x_i) = 1$$

The function $p(x_i)$ is called the probability function or probability mass function or probability density function of the random variable X. The set of pairs $\{x, p(x_i)\}$, $i = 1, 2, \ldots, n$ is called the probability distribution of the random variable which can be displayed in the form of a table as shown below:

$X = x_i$	x_1	x_2	x_3	$\ldots x_i$	$\ldots x_n$
$p(x_i) = P(X = x_i)$	$p(x_1)$	$p(x_2)$	$p(x_3)$	$\ldots p(x_i)$	$\ldots p(x_n)$

2.4 DISCRETE DISTRIBUTION FUNCTION

Let X be a discrete random variable which takes the values x_1, x_2, … such that $x_1 < x_2 < \ldots$ with probabilities $p(x_1)$, $p(x_2)$ … such that $p(x_i) \geq 0$ for all values of i and $\sum_{i=1}^{x} p(x_i) = 1$.

The distribution function $F(x)$ of the discrete random variable X is defined by

$$F(x) = P(X \leq x) = \sum_{i=1}^{x} p(x_i)$$

where x is any integer. The function $F(x)$ is also called the cumulative distribution function. The set of pairs $\{x_i, F(x)\}$, $i = 1, 2, \ldots$ is called the cumulative probability distribution.

X	x_1	x_2	...
$F(x)$	$p(x_1)$	$p(x_1) + p(x_2)$	...

Example 1

A fair die is tossed once. If the random variable is getting an even number, find the probability distribution of X.

Solution

When a fair die is tossed,

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let X be the random variable of getting an even number. Hence, X can take the values 0 and 1.

$$P(X = 0) = P(1, 3, 5) = \frac{3}{6} = \frac{1}{2}$$

$$P(X = 1) = P(2, 4, 6) = \frac{3}{6} = \frac{1}{2}$$

Hence, the probability distribution of X is

$X = x$	0	1
$P(X = 1)$	$\frac{1}{2}$	$\frac{1}{2}$

Also, $\sum P(X = x) = \frac{1}{2} + \frac{1}{2} = 1$

Example 2

Find the probability distribution of the number of heads when three coins are tossed.

Solution

When three coins are tossed,

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Let X be the random variable of getting heads in tossing of three coins. Hence X can take the values 0, 1, 2, 3.

$$P(X = 0) = P(\text{no head}) = P(\text{TTT}) = \frac{1}{8}$$

$$P(X = 1) = P(\text{one head}) = P(\text{HTT, THT, TTH}) = \frac{3}{8}$$

$$P(X = 2) = P(\text{two heads}) = P(\text{HHT, THH, HTH}) = \frac{3}{8}$$

$$P(X = 3) = P(\text{three heads}) = P(\text{HHH}) = \frac{1}{8}$$

Hence, the probability distribution of X is

$X = x$	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Also, $\sum P(X = x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

Example 3

State with reasons whether the following represent the probability mass function of a random variable:

(i)

$X = x$	0	1	2	3
$P(X = x)$	0.4	0.3	0.2	0.1

(ii)

$X = x$	0	1	2	3
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{4}$

(iii)

$X = x$	0	1	2	3
$P(X = x)$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$

Solution

(i) Here, $0 \le P(X = x) \le 1$ is satisfied for all values of X.

$$\begin{aligned}\sum P(X = x) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)\\ &= 0.4 + 0.3 + 0.2 + 0.1\\ &= 1\end{aligned}$$

Since $\sum P(X = x) = 1$, it represents probability mass function.

(ii) Here, $0 \le P(X = x) \le 1$ is satisfied for all values of X.

$$\begin{aligned}\sum P(X = x) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)\\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{4}\\ &= \frac{5}{4} > 1\end{aligned}$$

Since $\sum (P(X = x) > 1$, it does not represent a probability mass function.

(iii) Here, $0 \le P(X = x) \le 1$ is not satisfied for all the values of X as $P(X = 0) = -\frac{1}{2}$.

Hence, $P(X = x)$ does not represent a probability mass function.

Example 4

Verify whether the following functions can be regarded as the probability mass function for the given values of X:

(i) $P(X = x) = \dfrac{1}{5} \quad \text{for} \quad x = 0, 1, 2, 3, 4$

$\qquad\qquad = 0 \quad \text{for} \quad \text{otherwise}$

(ii) $P(X = x) = \dfrac{x-2}{5} \quad \text{for} \quad x = 1, 2, 3, 4, 5$

$\qquad\qquad = 0 \quad \text{for} \quad \text{otherwise}$

(iii) $P(X = x) = \dfrac{x^2}{30} \quad \text{for} \quad x = 0, 1, 2, 3, 4$

$\qquad\qquad = 0 \quad \text{for} \quad \text{otherwise}$

Solution

(i) $P(X = 0) = P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = \dfrac{1}{5}$

$P(X = x) \geq 0$ for all values of x

$$\begin{aligned}\sum P(X = x) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)\\ &= \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}\\ &= 1\end{aligned}$$

Hence, $P(X = x)$ is a probability mass function.

(ii) $P(X = 1) = \dfrac{1-2}{5} = -\dfrac{1}{5} < 0$

Hence, $P(X = x)$ is not a probability mass function.

(iii) $P(X = 0) = 0$

$P(X = 1) = \dfrac{1}{30}$

$P(X = 2) = \dfrac{4}{30}$

$P(X = 3) = \dfrac{9}{30}$

$P(X = 4) = \dfrac{16}{30}$

$P(X = x) \geq 0$ for all values of x

$$\sum P(X=x) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$$
$$= 0+\frac{1}{30}+\frac{4}{30}+\frac{9}{30}+\frac{16}{30}$$
$$=1$$

Hence, $P(X=x)$ is a probability mass function.

Example 5

A random variable X has the probability mass function given by

X	1	2	3	4
$P(X=x)$	0.1	0.2	0.5	0.2

Find (i) $P(2 \le x < 4)$, *(ii)* $P(X > 2)$, *(iii)* $P(X$ *is odd*$)$, *and (iv)* $P(X$ *is even*$)$.

Solution

(i) $P(2 \le X < 4) = P(X=2)+P(X=3)$
$= 0.2 + 0.5$
$= 0.7$

(ii) $P(X > 2) = P(X=3)+P(X=4)$
$= 0.5 + 0.2$
$= 0.7$

(iii) $P(X \text{ is odd}) = P(X=1)+P(X=3)$
$= 0.1 + 0.5$
$= 0.6$

(iv) $P(X \text{ is even}) = P(X=2)+P(X=4)$
$= 0.2 + 0.2$
$= 0.4$

Example 6

If the random variable X takes the value 1, 2, 3, *and* 4 *such that* $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. *Find the probability distribution.*

Solution

Let $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = k$

$$P(X=1) = \frac{k}{2}$$

$$P(X=2) = \frac{k}{3}$$

$$P(X = 3) = k$$

$$P(X = 4) = \frac{k}{5}$$

Since $\sum(P(X = x) = 1$,

$$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$k = \frac{30}{61}$$

Hence, the probability distribution is

X	1	2	3	4
$P(X = x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

Example 7

A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6	7
$P(X = x)$	a	$4a$	$3a$	$7a$	$8a$	$10a$	$6a$	$9a$

(i) Find the value of a.

(ii) Find $P(X < 3)$.

(iii) Find the smallest value of m for which $P(X \leq m) \geq 0.6$.

Solution

(i) Since $P(X = x)$ is a probability distribution function,

$$\sum(P(X = x) = 1$$

$$P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7) = 1$$

$$a + 4a + 3a + 7a + 8a + 10a + 6a + 9a = 1$$

$$a = \frac{1}{48}$$

(ii) $$P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= a + 4a + 3a$$

$$= 8a$$

$$= 8\left(\frac{1}{48}\right)$$

$$= \frac{1}{6}$$

(iii) $P(X \le 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$$= a + 4a + 3a + 7a + 8a$$
$$= 23a$$
$$= 23\left(\frac{1}{48}\right)$$
$$= 0.575$$

$$P(X \le 5) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$$
$$= a + 4a + 3a + 7a + 8a + 10a$$
$$= 33a$$
$$= 33\left(\frac{1}{48}\right)$$
$$= 0.69$$

Hence, the smallest value of m for which $P(X \le m) \ge 0.6$ is 5.

Example 8

The probability mass function of a random variable X is zero except at the points X = 0, 1, 2. At these points, it has the values $P(X = 0) = 3c^3$, $P(X = 1) = 4c - 10c^2$, $P(X = 2) = 5c - 1$. *Find (i) c, (ii)* $P(X < 1)$, *(iii)* $P(1 < X \le 2)$, *and (iv)* $P(0 < X \le 2)$.

Solution

(i) Since $P(X = x)$ is a probability mass function,

$$\sum(P(X = x) = 1$$
$$P(X = 0) + P(X = 1) + P(X = 2) = 1$$
$$3c^3 + 4c - 10c^2 + 5c - 1 = 1$$
$$3c^3 - 10c^2 + 9c - 2 = 0$$
$$(3c - 1)(c - 2)(c - 1) = 0$$
$$c = \frac{1}{3}, \quad 2, \quad 1$$

But $c < 1$, otherwise given probabilities will be greater than one or less than zero.

$$\therefore \qquad c = \frac{1}{3}$$

Hence, the probability distribution is

X	0	1	2
$P(X = x)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{3}$

(ii) $P(X < 1) = P(X = 0) = \dfrac{1}{9}$

(iii) $P(1 < X \le 2) = P(X = 2) = \dfrac{2}{3}$

(iv) $P(0 < X \le 2) = P(X = 1) + P(X = 2)$

$$= \frac{2}{9} + \frac{2}{3}$$

$$= \frac{8}{9}$$

Example 9

From a lot of 10 items containing 3 defectives, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. Find the probability distribution of X.

Solution

The random variable X can take the value 0, 1, 2, or 3.

Total number of items = 10

Number of good items = 7

Number of defective items = 3

$$P(X = 0) = P(\text{no defective}) = \frac{{}^{7}C_4}{{}^{10}C_4} = \frac{1}{6}$$

$$P(X = 1) = P(\text{one defective and three good items}) = \frac{{}^{3}C_1\,{}^{7}C_3}{{}^{10}C_4} = \frac{1}{2}$$

$$P(X = 2) = P(\text{two defectives and two good items}) = \frac{{}^{3}C_2\,{}^{7}C_2}{{}^{10}C_4} = \frac{3}{10}$$

$$P(X = 3) = P(\text{three defectives and one good item}) = \frac{{}^{3}C_3\,{}^{7}C_1}{{}^{10}C_4} = \frac{1}{30}$$

Hence, the probability distribution of the random variable is

X	0	1	2	3
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

Example 10

Construct the distribution function of the discrete random variable X whose probability distribution is as given below:

X	1	2	3	4	5	6	7
$P(X=x)$	0.1	0.15	0.25	0.2	0.15	0.1	0.05

Solution

Distribution function of X

X	$P(X=x)$	$F(x)$
1	0.1	0.1
2	0.15	0.25
3	0.25	0.5
4	0.2	0.7
5	0.15	0.85
6	0.1	0.95
7	0.05	1

Example 11

A random variable X has the probability function given below:

X	0	1	2
$P(X=x)$	k	$2k$	$3k$

Find (i) k, (ii) $P(X < 2)$, $P(X \leq 2)$, $P(0 < X < 2)$, *and (iii) the distribution function.*

Solution:

(i) Since $P(X = x)$ is a probability density function,

$$\sum(P(X=x) = 1$$
$$k + 2k + 3k = 1$$
$$6k = 1$$
$$k = \frac{1}{6}$$

Hence, the probability distribution is

X	0	1	2
$P(X=x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

(ii) $P(X < 2) = P(X = 0) + P(X = 1) = \frac{1}{6} + \frac{2}{6} = \frac{1}{2}$

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} = 1$$

$$P(0 < X < 2) = P(X = 1) = \frac{1}{3}$$

(iii) Distribution function

X	$P(X = x)$	$F(x)$
0	$\frac{1}{6}$	$\frac{1}{6}$
1	$\frac{2}{6}$	$\frac{1}{2}$
2	$\frac{3}{6}$	1

Example 12

A random variable X takes the values –3, –2, –1, 0, 1, 2, 3, such that
$P(X = 0) = P(X > 0) = P(X < 0)$,
$P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$.
Obtain the probability distribution and the distribution function of X.

Solution

Let $P(X = 0) = P(X > 0) = P(X < 0) = k_1$

Since $\sum P(X = x) = 1$

$$k_1 + k_1 + k_1 = 1$$

$$\therefore \quad k_1 = \frac{1}{3}$$

$$P(X = 0) = P(X > 0) = P(X < 0) = \frac{1}{3}$$

Let $\quad P(X = 1) = P(X = 2) = P(X = 3) = k_2$

$$P(X > 0) = P(X = 1) + P(X = 2) + P(X = 3)$$

$$\frac{1}{3} = k_2 + k_2 + k_2$$

$$\therefore \quad k_2 = \frac{1}{9}$$

$$P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{9}$$

Similarly, $P(X = -3) = P(X = -2) = P(X = -1) = \dfrac{1}{9}$

Probability distribution and distribution function

X	$P(X = x)$	$F(x)$
–3	$\frac{1}{9}$	$\frac{1}{9}$
–2	$\frac{1}{9}$	$\frac{2}{9}$
–1	$\frac{1}{9}$	$\frac{3}{9}$
0	$\frac{1}{3}$	$\frac{6}{9}$
1	$\frac{1}{9}$	$\frac{7}{9}$
2	$\frac{1}{9}$	$\frac{8}{9}$
3	$\frac{1}{9}$	1

Example 13

A discrete random variable X has the following distribution function:

$$F(x) = \begin{cases} 0 & x < 1 \\ \dfrac{1}{3} & 1 \le x < 4 \\ \dfrac{1}{2} & 4 \le x < 6 \\ \dfrac{5}{6} & 6 \le x < 10 \\ 1 & x \ge 10 \end{cases}$$

Find (i) $P(2 < X \le 6)$, *(ii)* $P(X = 5)$, *(iii)* $P(X = 4)$, *(iv)* $P(X \le 6)$, *and (v)* $P(X = 6)$.

Solution

(i) $P(2 < X \le 6) = F(6) - F(2) = \frac{5}{6} - \frac{1}{3} = \frac{3}{6} = \frac{1}{2}$

(ii) $P(X = 5) = P(X \le 5) - P(X < 5) = F(5) - P(X < 5) = \frac{1}{2} - \frac{1}{2} = 0$

(iii) $P(X = 4) = P(X \le 4) - P(X < 4) = F(4) - P(X < 4) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

(iv) $P(X \le 6) = F(6) = \frac{5}{6}$

(v) $P(X = 6) = P(X \le 6) - P(X < 6) = F(6) - P(X < 6) = \frac{5}{6} - \frac{1}{2} = \frac{1}{3}$

EXERCISE 2.1

1. Verify whether the following functions can be considered as probability mass functions:

(i) $P(X = x) = \frac{x^2 + 1}{18}, x = 0, 1, 2, 3$ **[Ans.:** Yes)

(ii) $P(X = x) = \frac{x^2 - 2}{8}, x = 1, 2, 3$ **[Ans.:** No]

(iii) $P(X = x) = \frac{2x + 1}{18}, x = 0, 1, 2, 3$ **[Ans.:** No]

2. The probability density function of a random variable X is

X	0	1	2	3	4	5	6
$P(X = x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find $P(X < 4)$ and $P(3 < X \le 6)$.

$\left[\textbf{Ans.:} \frac{16}{49}, \frac{33}{49}\right]$

3. A random variable X has the following probability distribution:

X	1	2	3	4	5	6	7
$P(X = x)$	k	$2k$	$3k$	k^2	$k^2 + k$	$2k^2$	$4k^2$

Find (i) k, (ii) $P(X < 5)$, (iii) $P(X > 5)$, and (iv) $P(0 \le X \le 5)$

$\left[\textbf{Ans.:} \frac{1}{8} \text{ (ii) } \frac{49}{64} \text{ (iii) } \frac{3}{32} \text{ (iv) } \frac{29}{32}\right]$

4. A discrete random variable X has the following probability distribution:

X	-2	-1	0	1	2	3
$P(X = x)$	0.1	k	0.2	$2k$	0.3	$3k$

Find (i) k, (ii) $P(X \geq 2)$, and (iii) $P(-2 < X < 2)$.

$$\left[\textbf{Ans.: } \frac{1}{15} \text{ (ii) } \frac{1}{2} \text{ (iii) } \frac{2}{5}\right]$$

5. Given the following probability function of a discrete random variable X:

X	0	1	2	3	4	5	6	7
$P(X = x)$	0	c	$2c$	$2c$	$3c$	c^2	$2c^2$	$7c^2 + c$

Find (i) c, (ii) $P(X \geq 6)$, (iii) $P(X < 6)$, and (iv) Find k if $P(X \leq k) > \frac{1}{2}$, where k is a positive integer.

[**Ans.:** (i) 0.1 (ii) 0.19 (iii) 0.81 (iv) 4]

6. A random variable X assumes four values with probabilities $\frac{1+3x}{4}, \frac{1-x}{4}, \frac{1+2x}{4}$ and $\frac{1-4x}{4}$. For what value of x do these values represent the probability distribution of X?

$$\left[\textbf{Ans.: } -\frac{1}{3} \leq X \leq \frac{1}{4}\right]$$

7. Let X denote the number of heads in a single toss of 4 fair coins.

Determine (i) $P(X < 2)$, and (ii) $P(1 < X \leq 3)$.

$$\left[\textbf{Ans.: (i) } \frac{5}{16} \text{ (ii) } \frac{5}{8}\right]$$

8. If 3 cars are selected from a lot of 6 cars containing 2 defective cars, find the probability distribution of the number of defective cars.

Ans.:

X	0	1	2
$P(X = x)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{2}{5}$

9. Five defective bolts are accidentally mixed with 20 good ones. Find the probability distribution of the number of defective bolts, if four bolts are drawn at random from this lot.

Ans.:

X	0	1	2	3	4
$P(X = x)$	$\frac{969}{2530}$	$\frac{1140}{2530}$	$\frac{380}{2530}$	$\frac{40}{2530}$	$\frac{1}{2530}$

10. Two dice are rolled at once. Find the probability distribution of the sum of the numbers on them.

Ans.:

X	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

11. A random variable X takes three values 0, 1, and 2 with probabilities $\frac{1}{3}, \frac{1}{6}$, and $\frac{1}{2}$ respectively. Obtain the distribution function of X.

$$\left[\textbf{Ans.: } F(0) = \frac{1}{3}, F(1) = \frac{1}{2}, F(2) = 1\right]$$

12. A random variable X has the following probability function:

x	0	1	2	3	4
$P(X = x)$	k	$3k$	$5k$	$7k$	$9k$

Find (i) the value of k, (ii) $P(X < 3)$, $P(X \geq 3)$, $P(0 < X < 4)$, and (iii) distribution function of X.

$$\left[\begin{array}{l}\textbf{Ans.: } \text{(i) } \frac{1}{25}, \text{(ii) } \frac{9}{25}, \frac{16}{25}, \frac{3}{5} \\ \qquad \text{(iii) } F(0) = \frac{1}{25}, F(1) = \frac{4}{25}. F(2) = \frac{9}{25}, F(3) = \frac{16}{25}, F(4) = 1\end{array}\right]$$

13. A random variable X has the probability function

X	–2	–1	0	1	2	3
$P(X = x)$	0.1	k	0.2	$2k$	0.3	k

Find (i) k, (ii) $P(X \leq 1)$, (iii) $P(-2 < X < 1)$, and (iv) obtain the distribution function of X.

[**Ans.:** (i) 0.1 (ii) 0.6 (iii) 0.3]

14. The following is the distribution function $F(x)$ of a discrete random variable X:

X	–3	–2	–1	0	1	2	3
$P(X = x)$	0.08	0.2	0.4	0.65	0.8	0.9	1

Find (i) the probability distribution of X, (ii) $P(-2 \le X \le 1)$, and (iii) $P(X \ge 1)$.

Ans.: (i)

X	–3	–2	–1	0	1	2	3
$P(X = x)$	0.08	0.12	0.2	0.25	0.15	0.1	0.1

(ii) 0.72 (ii) 0.35

2.5 MEASURES OF CENTRAL TENDENCY FOR DISCRETE PROBABILITY DISTRIBUTION

The behaviour of a random variable is completely characterized by the distribution function $F(x)$ or density function $p(x)$. Instead of a function, a more compact description can be made by a single numbers such as mean, median, mode, variance, and standard deviation known as measures of central tendency of the random variable X.

1. Mean The mean or average value (μ) of the probability distribution of a discrete random variable X is called as expectation and is denoted by $E(X)$.

$$\mu = E(X) = \sum_{i=1}^{\infty} x_i\ p(x_i) = \sum x\ p(x)$$

where $p(x)$ is the probability density function of the discrete random variable X. Expectation of any function $\phi(x)$ of a random variable X is given by

$$E\left[\phi(x)\right] = \sum_{i=1}^{\infty} \phi(x_i)\, p(x_i) = \sum \phi(x)\, p(x)$$

Some important results on expectation:

(i) $E(X + k) = E(X) + k$

(ii) $E(aX \pm b) = aE(X) \pm b$

(iii) $E(X + Y) = E(X) + E(Y)$ provided $E(X)$ and $E(Y)$ exists.

(iv) $E(XY) = E(X)\, E(Y)$ if X and Y are two independent random variables.

2. Variance Variance characterizes the variability in the distributions since two distributions with same mean can still have different dispersion of data about their means. Variance of the probability distribution of a discrete random variable X is given by

$$\begin{aligned}
\text{Var}(X) = \sigma^2 &= E(X - \mu)^2 \\
&= E(X^2 - 2X\mu + \mu^2) \\
&= E(X^2) - E(2X\mu) + E(\mu^2) \\
&= E(X^2) - 2\mu\, E(X) + \mu^2 \qquad [\because E(\text{constant}) = (\text{constant})] \\
&= E(X^2) - 2\mu\mu + \mu^2
\end{aligned}$$

$$= E(X^2) - \mu^2$$
$$= E(X^2) - [E(X)]^2$$

Some important results on variance:

(i) Var $(k) = 0$

(ii) Var $(kX) = k^2$ Var (X)

(iii) Var $(X + k) =$ Var (X)

(iv) Var $(aX + b) = a^2$ Var(X)

3. Standard Deviation Standard deviation is the positive square root of the variance.

$$\text{SD} = \sigma = \sqrt{\sum_{i=1}^{\infty} x_i^2 \; p(x_i) - \mu^2}$$
$$= \sqrt{E\,(X^2) - \mu^2}$$
$$= \sqrt{E\,(X^2) - [E(X)]^2}$$

Example 1

A random variable X has the following distribution:

X	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Find (i) mean, (ii) variance, and (iii) $P(1 < X < 6)$.

Solution

(i) Mean $= \mu = \sum xp(x)$

$$= 1\left(\frac{1}{36}\right) + 2\left(\frac{3}{36}\right) + 3\left(\frac{5}{36}\right) + 4\left(\frac{7}{36}\right) + 5\left(\frac{9}{36}\right) + 6\left(\frac{11}{36}\right)$$
$$= \frac{161}{36}$$
$$= 4.47$$

(ii) Variance $= \sigma^2 = \sum x^2 p(x) - \mu^2$

$$= 1\left(\frac{1}{36}\right) + 4\left(\frac{3}{36}\right) + 9\left(\frac{5}{36}\right) + 16\left(\frac{7}{36}\right) + 25\left(\frac{9}{36}\right)$$
$$+ 36\left(\frac{11}{36}\right) - (4.47)^2$$

$$= \frac{791}{36} - 19.98$$
$$= 1.99$$

(iii) $P(1 < X < 6) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

$$= \frac{3}{36} + \frac{5}{36} + \frac{7}{36} + \frac{9}{36}$$
$$= \frac{24}{36}$$
$$= 0.67$$

Example 2

The probability distribution of a random variable X is given below. Find (i) E(X), (ii) Var(X), *(iii) E(2X – 3), and (iv)* Var $(2X - 3)$

X	–2	–1	0	1	2
$P(X = x)$	0.2	0.1	0.3	0.3	0.1

Solution

(i) $E(X) = \Sigma x\, p(x)$

$= -2(0.2) - 1(0.1) + 0 + (0.3) + 2(0.1)$

$= 0$

(ii) $\text{Var}(X) = \Sigma x^2\, p(x) - [E(X)]^2$

$= 4(0.2) + 1(0.1) + 0 + 1(0.3) + 4(0.1) - 0$

$= 1.6$

(iii) $E(2X - 3) = 2E(X) - 3$

$= 2(0) - 3$

$= -3$

(iv) $\text{Var}\,(2X - 3) = (2)^2\, \text{Var}\,(X)$

$= 4(1.6)$

$= 6.4$

Example 3

Mean and standard deviation of a random variable X are 5 *and* 4 *respectively. Find E(X^2) and standard deviation of* $(5 - 3X)$.

Solution

$E(X) = \mu = 5$

$\text{SD} = \sigma = 4$

$\therefore \quad \text{Var}(X) = \sigma^2 = 16$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$
$$16 = E(X^2) - (5)^2$$
$$\therefore \quad E(X^2) = 41$$
$$\text{Var}\,(5 - 3X) = \text{Var}\,(5) - (-3)^2\ \text{Var}\,(X)$$
$$= 0 + 9(16)$$
$$= 144$$
$$\text{SD}\,(5 - 3X) = \sqrt{\text{Var}\,(5 - 3X)}$$
$$= \sqrt{144}$$
$$= 12$$

Example 4

A machine produces an average of 500 items during the first week of the month and on average of 400 items during the last week of the month, the probability for these being 0.68 and 0.32 respectively. Determine the expected value of the production.

[Summer 2015]

Solution

Let X be the random variable which denotes the items produced by the machine. The probability distribution is

X	500	400
$P(X = x)$	0.68	0.32

Expected value of the production $E(X) = \sum x\, p(x)$

$$= 500(0.68) + 400(0.32)$$
$$= 468$$

Example 5

The monthly demand for Allwyn watches is known to have the following probability distribution:

Demand (x)	1	2	3	4	5	6	7	8
Probability $p(x)$	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Find the expected demand for watches. Also, compute the variance.

Solution

$$\begin{aligned}E(X) &= \Sigma\, x\, p(x)\\ &= 1(0.08)+2(0.12)+3(0.19)+4(0.24)+5(0.16)\\ &\qquad +6(0.10)+7(0.07)+8(0.04)\\ &= 4.06\end{aligned}$$

$$\begin{aligned}\text{Var}(X) &= E(X^2)-[E(X)]^2\\ &= \Sigma\, x^2\; p(x)-[E(X)]^2\\ &= 1(0.08)+4(0.12)+9(0.19)+16(0.24)+25(0.16)\\ &\qquad +36(0.10)+49(0.07)+64(0.04)-(4.06)^2\\ &= 19.7-16.48\\ &= 3.21\end{aligned}$$

Example 6

A discrete random variable has the probability mass function given below:

X	–2	–1	0	1	2	3
$P(X=x)$	0.2	k	0.1	$2k$	0.1	$2k$

Find k, mean, and variance.

Solution

Since $P(X=x)$ is a probability mass function,

$$\sum P(X=x)=1$$

$$\begin{aligned}0.2+k+0.1+2k+0.1+2k &= 1\\ 5k+0.4 &= 1\\ 5k &= 0.6\\ k &= \frac{0.6}{5}=\frac{3}{25}\end{aligned}$$

Hence, the probability distribution is

X	–2	–1	0	1	2	3
$P(X=x)$	$\frac{2}{10}$	$\frac{3}{25}$	$\frac{1}{10}$	$\frac{6}{25}$	$\frac{1}{10}$	$\frac{6}{25}$

Mean = $E(X)=\sum x\, p(x)$

$$=(-2)\left(\frac{2}{10}\right)+(-1)\left(\frac{3}{25}\right)+0+1\left(\frac{6}{25}\right)+2\left(\frac{1}{10}\right)+3\left(\frac{6}{25}\right)$$

$$=\frac{6}{25}$$

$$\text{Variance} = \text{Var}(X) = E(X^2) - [E(X)]^2$$
$$= \sum x^2\, p(x) - [E(X)]^2$$
$$= 4\left(\frac{2}{10}\right) + 1\left(\frac{3}{25}\right) + 0 + 1\left(\frac{6}{25}\right) + 4\left(\frac{1}{10}\right) + 9\left(\frac{6}{25}\right) - \left(\frac{6}{25}\right)^2$$
$$= \frac{73}{250} - \frac{36}{625}$$
$$= \frac{293}{625}$$

Example 7

A random variable X has the following probability function:

x	0	1	2	3	4	5	6	7
$p(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) *Determine k.* (ii) *Evaluate* $P(X < 6)$, $P(X \geq 6)$, $P(0 < X < 5)$ and $P(0 \leq X \leq 4)$. (iii) *Determine the distribution function of X.* (iv) *Find the mean.* (v) *Find the variance.*

Solution

(i) Since $p(x)$ is a probability mass function,

$$\sum p(x) = 1$$
$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$
$$10k^2 + 9k - 1 = 1$$
$$(10k - 1)(k + 1) = 0$$
$$k = \frac{1}{10} \text{ or } k = -1$$
$$k = \frac{1}{10} = 0.1 \; [\because \; p(x) \geq 0, k \neq -1]$$

Hence, the probability function is

X	0	1	2	3	4	5	6	7
$P(X = x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

(ii) $P(X<6)=P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$$=0+0.1+0.2+0.2+0.3+0.01$$
$$=0.81$$

$$P(X\geq 6)=1-P(X<6)$$
$$=1-0.81$$
$$=0.19$$

$$P(0<X<5)=P(X=1)+P(X=2)+P(X=3)+P(X=4)$$
$$=0.1+0.2+0.2+0.3$$
$$=0.8$$

$$P(0\leq X\leq 4)=P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$$
$$=0+0.1+0.2+0.2+0.3$$
$$=0.8$$

(iii) Distribution function of X

x	$p(x)$	$F(x)$
0	0	0
1	0.1	0.1
2	0.2	0.3
3	0.2	0.5
4	0.3	0.8
5	0.01	0.81
6	0.02	0.83
7	0.17	1

(iv) $\mu=\sum xp(x)$

$$=0+1(0.1)+2(0.2)+3(0.2)+4(0.3)+5(0.01)+6(0.02)+7(0.17)$$
$$=3.66$$

(v) $\text{Var}(X)=\sigma^2=\sum x^2\,p(x)-\mu^2$

$$=0+1(0.1)+4(0.2)+9(0.2)+16(0.3)+25(0.01)+36(0.02)$$
$$+49(0.17)-(3.66)^2$$
$$=3.4044$$

Example 8

A fair die is tossed. Let the random variable X denote the twice the number appearing on the die. Write the probability distribution of X. Calculate mean and variance.

Solution

Let X be the random variable which denotes twice the number appearing on the die.

(i) Probability distribution of X

x	2	4	6	8	10	12
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

(ii) Mean $= \mu = \sum xp(x)$

$$= 2\left(\frac{1}{6}\right)+4\left(\frac{1}{6}\right)+6\left(\frac{1}{6}\right)+8\left(\frac{1}{6}\right)+10\left(\frac{1}{6}\right)+12\left(\frac{1}{6}\right)$$

$$= 7$$

(iii) Variance $= \sigma^2 = \sum x^2 p(x) - \mu^2$

$$= 4\left(\frac{1}{6}\right)+16\left(\frac{1}{6}\right)+36\left(\frac{1}{6}\right)+64\left(\frac{1}{6}\right)+100\left(\frac{1}{6}\right)+144\left(\frac{1}{6}\right)-(7)^2$$

$$= 11.67$$

Example 9

Two unbiased dice are thrown at random. Find the probability distribution of the sum of the numbers on them. Also, find mean and variance.

Solution

Let X be the random variable which denotes the sum of the numbers on two unbiased dice. The random variable X can take values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. The probability distribution is

X	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mean $= \mu = \sum x\, p(x)$

$$= 2\left(\frac{1}{36}\right)+3\left(\frac{2}{36}\right)+4\left(\frac{3}{36}\right)+5\left(\frac{4}{36}\right)+6\left(\frac{5}{36}\right)+7\left(\frac{6}{36}\right)+8\left(\frac{5}{36}\right)$$
$$+9\left(\frac{4}{36}\right)+10\left(\frac{3}{36}\right)+11\left(\frac{2}{36}\right)+12\left(\frac{1}{36}\right)$$

$$= \frac{252}{36}$$

$$= 7$$

$$\begin{aligned}\text{Variance} = \sigma^2 &= \sum x^2\, p(x) - \mu^2 \\ &= 4\left(\frac{1}{36}\right) + 9\left(\frac{2}{36}\right) + 16\left(\frac{3}{36}\right) + 25\left(\frac{4}{36}\right) + 36\left(\frac{5}{36}\right) \\ &\qquad + 49\left(\frac{6}{36}\right) + 64\left(\frac{5}{36}\right) + 81\left(\frac{4}{36}\right) + 100\left(\frac{3}{36}\right) \\ &\qquad + 121\left(\frac{2}{36}\right) + 144\left(\frac{1}{36}\right) - (7)^2 \\ &= \frac{1974}{36} - 49 \\ &= 5.83\end{aligned}$$

Example 10

A sample of 3 items is selected at random from a box containing 10 items of which 4 are defective. Find the expected number of defective items.

Solution

Let X be the random variable which denotes the defective items.

Total number of items = 10

Number of good items = 6

Number of defective items = 4

$$P(X=0) = P(\text{no defective item}) = \frac{{}^6C_3}{{}^{10}C_3} = \frac{1}{6}$$

$$P(X=1) = P(\text{one defective item}) = \frac{{}^6C_2\ {}^4C_1}{{}^{10}C_3} = \frac{1}{2}$$

$$P(X=2) = P(\text{two defective items}) = \frac{{}^6C_1\ {}^4C_2}{{}^{10}C_3} = \frac{3}{10}$$

$$P(X=3) = P(\text{three defective items}) = \frac{{}^4C_3}{{}^{10}C_3} = \frac{1}{30}$$

Hence, the probability distribution is

X	0	1	2	3
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

Expected number of defective items $= E(X) = \sum x\, p(x)$

$$= 0 + 1\left(\frac{1}{2}\right) + 2\left(\frac{3}{10}\right) + 3\left(\frac{1}{30}\right)$$

$$= 1.2$$

Example 11

A player tosses two fair coins. He wins ₹ 100 if a head appears and ₹ 200 if two heads appear. On the other hand, he loses ₹ 500 if no head appears. Determine the expected value of the game. Is the game favourable to the players?

Solution

Let X be the random variable which denotes the number of heads appearing in tosses of two fair coins.

$$S = \{\text{HH, HT, TH, TT}\}$$

$$p(x_1) = P(X = 0) = P(\text{no heads}) = \frac{1}{4}$$

$$p(x_2) = P(X = 1) = P(\text{one head}) = \frac{2}{4} = \frac{1}{2}$$

$$p(x_3) = P(X = 2) = P(\text{two heads}) = \frac{1}{4}$$

Amount to be lost if no head appears $= x_1 = -$ ₹ 500
Amount to be won if one head appears $= x_2 =$ ₹ 100
Amount to be won if two heads appear $= x_3 =$ ₹ 200
Expected value of the game $= \mu = \sum x\, p(x)$

$$= x_1\, p(x_1) + x_2\, p(x_2) + x_3\, p(x_3)$$

$$= -500\left(\frac{1}{4}\right) + 100\left(\frac{1}{2}\right) + 200\left(\frac{1}{4}\right)$$

$$= ₹\ -25$$

Hence, the game is not favourable to the player.

Example 12

Amit plays a game of tossing a die. If a number less than 3 appears, he gets ₹ a, otherwise he has to pay ₹ 10. If the game is fair, find a.

Solution

Let X be the random variable which denotes tossing of a die.

Probability of getting a number less than 3, i.e., 1 or 2 = $p(x_1) = \frac{2}{6} = \frac{1}{3}$

Probability of getting number more than or equal to 3, i.e., 3, 4, 5, or 6 $= p(x_2) = \frac{4}{6} = \frac{2}{3}$

Amount to be received for number less than 3 = x_1 = ₹ a
Amount to be paid for numbers more than or equal to 3 = x_2 = ₹ –10

$$\begin{aligned}
E(X) &= \sum x\,p(x) \\
&= x_1\,p(x_1) + x_2\,p(x_2) \\
&= a\left(\frac{1}{3}\right) + (-10)\left(\frac{2}{3}\right) \\
&= \frac{a}{3} - \frac{20}{3}
\end{aligned}$$

For a pair game, $E(x) = 0$.

$$\frac{a}{3} - \frac{20}{3} = 0$$
$$a = 20$$

Example 13

A man draws 2 balls from a bag containing 3 white and 5 black balls. If he is to receive ₹ 14 for every white ball which he draws and ₹ 7 for every black ball, what is his expectation?

Solution

Let X be the random variable which denotes the balls drawn from a bag. 2 balls drawn may be either (i) both white, or (ii) both black, or (iii) one white and one black.

Probability of drawing 2 white balls = $p(x_1) = \frac{{}^3C_2}{{}^8C_2} = \frac{3}{28}$

Probability of drawing 2 black balls = $p(x_2) = \frac{{}^5C_2}{{}^8C_2} = \frac{10}{28}$

Probability of drawing 1 white and 1 black ball = $p(x_3) = \frac{{}^3C_1\,{}^5C_1}{{}^8C_2} = \frac{15}{28}$

Amount to be received for 2 white balls = x_1 = ₹ 14 × 2 = ₹ 28
Amount to be received for 2 black balls = x_2 = ₹ 7 × 2 = ₹ 14
Amount to be received for 1 white and 1 black ball = x_3 = ₹ 14 + ₹ 7 = ₹ 21

$$\text{Expectation} = E(X) = \sum x\, p(x)$$
$$= x_1\, p(x_1) + x_2\, p(x_2) + x_3\, p(x_3)$$
$$= 28\left(\frac{3}{28}\right) + 14\left(\frac{10}{28}\right) + 21\left(\frac{15}{28}\right)$$
$$= ₹\ 19.25$$

Example 14

The probability that there is at least one error in an account statement prepared by A is 0.2 *and for B and C, they are* 0.25 *and* 0.4 *respectively. A, B, and C prepared* 10, 16, *and* 20 *statements respectively. Find the expected number of correct statements in all.*

Solution

Let $p(x_1)$, $p(x_2)$ and $p(x_3)$ be the probabilities of the events that there is no error in the account statements prepared by A, B, and C respectively.

$$p(x_1) = 1 - (\text{Probability of at least one error in the account statement prepared by } A)$$
$$= 1 - 0.2$$
$$= 0.8$$

Similarly, $p(x_2) = 1 - 0.25 = 0.75$

$p(x_3) = 1 - 0.4 = 0.6$

Also, $x_1 = 10$, $x_2 = 16$, $x_3 = 20$

$$\text{Expected number of correct statements} = E(X) = \sum x\, p(x)$$
$$= x_1\, p(x_1) + x_2\, p(x_2) + x_3\, p(x_3)$$
$$= 10(0.8) + 16(0.75) + 20(0.6)$$
$$= 32$$

Example 15

A man has the choice of running either a hot-snack stall or an ice-cream stall at a seaside resort during the summer season. If it is a fairly cool summer, he should make ₹ 5000 *by running the hot-snack stall, but if the summer is quite hot, he can only expect to make* ₹ 1000. *On the other hand, if he operates the ice-cream stall, his profit is estimated at* ₹ 6500, *if the summer is hot, but only* ₹ 1000 *if it is cool. There is a* 40 *percent chance of the summer being hot. Should he opt for running the hot-snack stall or the ice-cream stall?*

Solution

Let X and Y be the random variables which denote the income from the hot-snack and ice-cream stalls respectively.

Probability of hot summer $= p_1 = 40\% = 0.4$

Probability of cool summer $= p_2 = 1 - p_1 = 1 - 0.4 = 0.6$

$$x_1 = 1000, \qquad x_2 = 5000, \qquad y_1 = 6500, \qquad y_2 = 1000$$

$$\begin{aligned}\text{Expected income from hot-snack stall } &= E(X)\\ &= x_1\, p_1 + x_2\, p_2\\ &= 1000\,(0.4) + 5000(0.6)\\ &= ₹\ 3400\end{aligned}$$

$$\begin{aligned}\text{Expected income from ice-cream stall } &= E(Y)\\ &= y_1\, p_1 + y_2\, p_2\\ &= 6500\,(0.4) + 1000(0.6)\\ &= ₹\ 3200\end{aligned}$$

Hence, he should opt for running the hot-snack stall.

EXERCISE 2.2

1. The probability distribution of a random variable X is given by

X	–2	–1	0	1	2	3
$P(X = x)$	0.1	k	0.2	$2k$	0.3	k

Find k, the mean, and variance.

[**Ans.:** 0.1, 0.8, 2.16]

2. Find the mean and variance of the following distribution:

X	4	5	6	8
$P(X = x)$	0.1	0.3	0.4	0.2

[**Ans.:** 5.9, 1.49]

3. Find the value of k from the following data:

X	0	10	15
$P(X = x)$	$\frac{k-6}{5}$	$\frac{2}{k}$	$\frac{14}{5k}$

Also, find the distribution function and expectation of X.

Ans.: 8,

X	0	10	15
$F(X)$	$\frac{2}{5}$	$\frac{13}{20}$	1

, $\frac{31}{4}$

4. For the following distribution,

X	–3	–2	–1	0	1	2
$P(X = x)$	0.01	0.1	0.2	0.3	0.2	0.15

Find (i) $P(X \geq 1)$, (ii) $P(X < 0)$, (iii) $E(X)$, and (iv) Var(X)

[**Ans.:** (i) 0.35 (ii) 0.35 (iii) 0.05 (iv) 1.8475]

5. A random variable X has the following probability function:

X	0	1	2	3	4	5	6	7	8
$P(X = x)$	$\frac{k}{45}$	$\frac{k}{15}$	$\frac{k}{9}$	$\frac{k}{5}$	$\frac{2k}{45}$	$\frac{6k}{45}$	$\frac{7k}{45}$	$\frac{8k}{45}$	$\frac{4k}{45}$

Determine (i) k, (ii) mean, (iii) variance, and (iv) SD.

[**Ans.:** (i) 1 (ii) 0.4622 (iii) 4.9971 (iv) 2.24]

6. A fair coin is tossed until a head or five tails appear. Find (i) discrete probability distribution, and (ii) mean of the distribution.

Ans.: (i)

X	1	2	3	4	5
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

(ii) 1.9

7. Let X denotes the minimum of two numbers that appear when a pair of fair dice is thrown once. Determine (i) probability distribution, (ii) expectation, and (iii) variance.

Ans.: (i)

X	1	2	3	4	5	6
$P(X = x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

(ii) 2.5278 (iii) 1.9713

8. For the following probability distribution,

X	–3	–2	–1	0	1	2	3
$P(X = x)$	0.001	0.01	0.1	?	0.1	0.01	0.001

Find (i) missing probability, (ii) mean, and (iii) variance.

[**Ans.:** (i) 0.778 (ii) 0.2 (iii) 0.258]

9. A discrete random variable can take all integer values from 1 to k each with the probability of $\frac{1}{k}$. Show that its mean and variance are $\frac{k+1}{2}$ and $\frac{k^2+1}{2}$ respectively.

10. An urn contains 6 white and 4 black balls; 3 balls are drawn without replacement. What is the expected number of black balls that will be obtained?

[**Ans.:** $\frac{6}{5}$]

11. A six-faced die is tossed. If a prime number occurs, Anil wins that number of rupees but if a nonprime number occurs, he loses that number of rupees. Determine whether the game is favourable to the player.

[**Ans.:** The game is favourable to Anil]

12. A man runs an ice-cream parlour at a holiday resort. If the summer is mild, he can sell 2500 cups of ice cream; if it is hot, he can sell 4000 cups; if it is very hot, he can sell 5000 cups. It is known that for any year, the probability of summer to be mild is $\frac{1}{7}$ and to be hot is $\frac{4}{7}$. A cup of ice cream costs ₹ 2 and is sold for ₹ 3.50. What is his expected profit?

[**Ans.:** ₹ 6107.14]

13. A player tosses two fair coins. He wins ₹ 1 or ₹ 2 as 1 tail or 1 head appears. On the other hand, he loses ₹ 5 if no head appears. Find the expected gain or loss of the player.

[**Ans.:** Loss of ₹ 0.25]

14. A bag contains 2 white balls and 3 black balls. Four persons *A*, *B*, *C*, *D* in the order named each draws one ball and does not replace it. The first to draw a white ball receives ₹ 20. Determine their expectations.

[**Ans.:** ₹ 8, ₹ 6, ₹ 4, ₹ 2]

2.6 CONTINUOUS PROBABILITY DISTRIBUTION

Let X be a continuous random variable such that the probability of the variable X falling in the small interval $x-\frac{1}{2}\mathrm{d}x$ to $x+\frac{1}{2}\mathrm{d}x$ is $f(x)\,\mathrm{d}x$, i.e.,

$$P\left(x-\frac{1}{2}\mathrm{d}x \le X \le x+\frac{1}{2}\mathrm{d}x\right) = f(x)\,\mathrm{d}x$$

The function $f(x)$ is called the probability density function of the random variable X and the continuous curve $y = f(x)$ is called the probability curve.

Properties of Probability Density Function

(i) $f(x) \ge 0, \qquad -\infty < x < \infty$

(ii) $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$

(iii) $P(a < x < b) = \int_a^b f(x)\,\mathrm{d}x$

2.7 CONTINUOUS DISTRIBUTION FUNCTION

If X is a continuous random variable having the probability density function $f(x)$ then the function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,\mathrm{d}x, \quad -\infty < x < \infty$$

is called the distribution function or cumulative distribution function of the random variable X.

Properties of Distribution Function

(i) $F(-\infty) = 0$

(ii) $F(\infty) = 1$

(iii) $0 \le F(x) \le 1, \quad -\infty < x < \infty$

(iv) $P(a < X < b) = F(b) - F(a)$

(v) $F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x), \quad f(x) \ge 0$

Example 1

Show that the function f(x) defined by

$$f(x) = \frac{1}{7} \qquad 1 < x < 8$$
$$= 0 \qquad \textit{otherwise}$$

is a probability density function for a random variable. Hence, find $P(3 < X < 10)$.

Solution

$$f(x) \geq 0 \qquad \text{in} \qquad 1 < x < 8$$

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{1} f(x)\,\mathrm{d}x + \int_{1}^{8} f(x)\,\mathrm{d}x + \int_{8}^{\infty} f(x)\,\mathrm{d}x$$
$$= 0 + \int_{1}^{8} \frac{1}{7}\,\mathrm{d}x + 0$$
$$= \frac{1}{7} |x|_1^8$$
$$= \frac{1}{7}(8-1)$$
$$= 1$$

Hence, $f(x)$ is a probability density function.

$$P(3 < X < 10) = \int_{3}^{10} f(x)\,\mathrm{d}x$$
$$= \int_{3}^{8} f(x)\,\mathrm{d}x + \int_{8}^{10} f(x)\,\mathrm{d}x$$
$$= \int_{3}^{8} \frac{1}{7}\,\mathrm{d}x + 0$$
$$= \frac{1}{7} |x|_3^8$$
$$= \frac{1}{7}(8-3)$$
$$= \frac{5}{7}$$

Example 2

Is the function $f(x)$ defined by

$$f(x) = e^{-x} \qquad x \geq 0$$
$$= 0 \qquad x < 0$$

is a probability density function. If so, find the probability that the variate having this density falls in the interval $(1, 2)$.

Solution

$$f(x) \geq 0 \quad \text{in} \quad (0, \infty)$$

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{\infty} f(x)\,\mathrm{d}x$$
$$= 0 + \int_{0}^{\infty} e^{-x}\,\mathrm{d}x$$
$$= \left|-e^{-x}\right|_{0}^{\infty}$$
$$= -e^{-\infty} + 1$$
$$= 1$$

Hence, $f(x)$ is a probability density function.

$$P(1 \leq X \leq 2) = \int_{1}^{2} f(x)\,\mathrm{d}x$$
$$= \int_{1}^{2} e^{-x}\,\mathrm{d}x$$
$$= \left|-e^{-x}\right|_{1}^{2}$$
$$= -e^{-2} + e^{-1}$$
$$= 0.233$$

Example 3

If a random variable has the probability density function $f(x)$ as

$$f(x) = 2e^{-2x} \qquad x > 0$$
$$= 0 \qquad x \leq 0$$

Find the probabilities that it will take on a value (i) between 1 *and* 3, *and (ii) greater than 0.5.*

Solution

(i) Probability that the variable will take a value between 1 and 3

$$P(1 < X < 3) = \int_1^3 f(x)\,dx$$

$$= \int_1^3 2e^{-2x}\,dx$$

$$= 2\left|\frac{e^{-2x}}{-2}\right|_1^3$$

$$= -(e^{-6} - e^{-2})$$

$$= e^{-2} - e^{-6}$$

(ii) Probability that the variable will take a value greater than 0.5

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx$$

$$= \int_{0.5}^{\infty} 2e^{-2x}\,dx$$

$$= 2\left|\frac{e^{-2x}}{-2}\right|_{0.5}^{\infty}$$

$$= -(e^{-\infty} - e^{-1})$$

$$= e^{-1}$$

Example 4

Find the constant k such that the function

$$\begin{aligned} f(x) &= kx^2 && 0 < x < 3 \\ &= 0 && \textit{otherwise} \end{aligned}$$

is a probability density function and compute (i) $P(1 < x < 2)$, (ii) $P(X < 2)$, and (iii) $P(X \geq 2)$.

Solution

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_0^3 f(x)\,dx + \int_3^{\infty} f(x)\,dx = 1$$

$$0+\int_0^3 kx^2\,\mathrm{d}x+0=1$$

$$k\left|\frac{x^3}{3}\right|_0^3=1$$

$$\frac{k}{3}(27-0)=1$$

$$9k=1$$

$$k=\frac{1}{9}$$

Hence, $f(x)=\frac{1}{9}x^2 \qquad 0<x<3$

$\qquad\qquad = 0 \qquad\qquad$ otherwise

(i) $P(1<X<2)=\int_1^2 f(x)\,\mathrm{d}x$

$$=\int_1^2 \frac{1}{9}x^2\,\mathrm{d}x$$

$$=\frac{1}{9}\left|\frac{x^3}{3}\right|_1^2$$

$$=\frac{1}{27}(8-1)$$

$$=\frac{7}{27}$$

(ii) $P(X<2)=\int_{-\infty}^2 f(x)\,\mathrm{d}x$

$$=\int_{-\infty}^0 f(x)\,\mathrm{d}x+\int_0^2 f(x)\,\mathrm{d}x$$

$$=0+\int_0^2 \frac{1}{9}x^2\,\mathrm{d}x$$

$$=\frac{1}{9}\int_0^2 x^2\,\mathrm{d}x$$

$$=\frac{1}{9}\left|\frac{x^3}{3}\right|_0^2$$

$$=\frac{1}{27}(8-0)$$

$$=\frac{8}{27}$$

(iii) $$P(X \geq 2) = 1 - P(X < 2)$$
$$= 1 - \frac{8}{27}$$
$$= \frac{19}{27}$$

Example 5

If the probability density function of a random variable is given by

$$f(x) = k(1 - x^2) \quad 0 < x < 1$$
$$= 0 \quad \textit{otherwise}$$

Find the value of k and the probabilities that a random variable having this probability density will take on a value (i) between 0.1 *and* 0.2, *and (ii) greater than* 0.5.

Solution

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$\int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{1} f(x)\,\mathrm{d}x + \int_{1}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$0 + \int_{0}^{1} k(1 - x^2)\,\mathrm{d}x + 0 = 1$$

$$k\left|x - \frac{x^3}{3}\right|_0^1 = 1$$

$$k\left(1 - \frac{1}{3}\right) = 1$$

$$k = \frac{3}{2}$$

Hence, $$f(x) = \frac{3}{2}(1 - x^2) \quad 0 < x < 1$$
$$= 0 \quad \text{otherwise}$$

(i) Probability that the variable will take on a value between 0.1 and 0.2

$$P(0.1 < X < 0.2) = \int_{0.1}^{0.2} f(x)\,\mathrm{d}x$$

$$= \int_{0.1}^{0.2} \frac{3}{2}(1 - x^2)\,\mathrm{d}x$$

$$= \frac{3}{2}\left| x - \frac{x^3}{3} \right|_{0.1}^{0.2}$$

$$= \frac{3}{2}\left[\left(0.2 - \frac{0.008}{3}\right) - \left(0.1 - \frac{0.001}{3}\right)\right]$$

$$= 0.1465$$

(ii) Probability that the variable will take on a value greater than 0.5

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,\mathrm{d}x$$

$$= \int_{0.5}^{1} f(x)\,\mathrm{d}x + \int_{1}^{\infty} f(x)\,\mathrm{d}x$$

$$= \int_{0.5}^{1} \frac{3}{2}(1 - x^2)\,\mathrm{d}x + 0$$

$$= \frac{3}{2}\left| x - \frac{x^3}{3} \right|_{0.5}^{1}$$

$$= \frac{3}{2}\left[\left(1 - \frac{1}{3}\right) - \left(0.5 - \frac{0.125}{3}\right)\right]$$

$$= 0.3125$$

Example 6

If X is a continuous random variable with pdf

$$f(x) = x^2 \quad 0 \le x \le 1$$
$$= 0 \quad \textit{otherwise}$$

If $P(a \le X \le 1) = \dfrac{19}{81}$, *find the value of a.*

Solution

$$P(a \le X \le 1) = \frac{19}{81}$$

$$\int_a^1 f(x)\,\mathrm{d}x = \frac{19}{81}$$

$$\int_a^1 x^2\,\mathrm{d}x = \frac{19}{81}$$

$$\left|\frac{x^3}{3}\right|_a^1 = \frac{19}{81}$$

$$\frac{1}{3}(1-a) = \frac{19}{81}$$

$$1-a = \frac{19}{27}$$

$$a = \frac{46}{27}$$

Example 7

Let X be a continuous random variable with pdf

$$f(x) = kx\,(1 - x),\ 0 \le x \le 1$$

Find k and determine a number b such that $P(X \le b) = P(X \ge b)$.

Solution

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x) = 1$$

$$\int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{1} f(x)\,\mathrm{d}x + \int_{1}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$0 + \int_{0}^{1} kx(1-x)\,\mathrm{d}x + 0 = 1$$

$$k\int_{0}^{1} (x - x^2)\,\mathrm{d}x = 1$$

$$k\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = 1$$

$$k\left[\left(\frac{1}{2} - \frac{1}{3}\right) - (0-0)\right] = 1$$

$$k\left(\frac{1}{6}\right) = 1$$

$$k = 6$$

Hence, $f(x) = 6(x - x^2) \qquad 0 \le x \le 1$

Since total probability is 1 and $P(X \le b) = P(X \ge b)$,

$$P(X \le b) = \frac{1}{2}$$

$$\int_0^b f(x)\,\mathrm{d}x = \frac{1}{2}$$

$$6\int_0^b (x - x^2)\,\mathrm{d}x = \frac{1}{2}$$

$$6\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^b = \frac{1}{2}$$

$$\frac{b^2}{2} - \frac{b^3}{3} = \frac{1}{12}$$

$$6b^2 - 4b^3 = 1$$

$$4b^3 - 6b^2 + 1 = 0$$

$$(2b-1)(2b^2 - 2b - 1) = 0$$

$$b = \frac{1}{2} \text{ or } b = \frac{1 \pm \sqrt{3}}{2}$$

b lies in (0, 1).

$\therefore \quad b = \dfrac{1}{2}$

Example 8

The length of time (in minutes) that a certain lady speaks on the telephone is found to be a random phenomenon, with a probability function specified by the function

$$f(x) = A\,e^{-\frac{x}{5}} \quad x \ge 0$$
$$= 0 \quad \textit{otherwise}$$

(i) Find the value of A that makes f(x) a probability density function.
(ii) What is the probability that the number of minutes that she will take over the phone is more than 10 *minutes?*

Solution

(i) For $f(x)$ to be a probability density function,

$$\int_{-\infty}^{\infty} f(x)\, x = 1$$

$$\int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$0+\int_0^\infty A e^{-\frac{x}{5}}\,dx = 1$$

$$A\left|\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right|_0^\infty = 1$$

$$-5A(e^{-\infty}-e^{-0}) = 1$$

$$-5A(0-1) = 1$$

$$5A = 1$$

$$A = \frac{1}{5}$$

Hence, $f(x) = \frac{1}{5}e^{-\frac{x}{5}} \qquad x \ge 0$

$= 0 \qquad$ otherwise

(ii) $P(X > 10) = \int_{10}^{\infty} f(x)\,dx$

$$= \int_{10}^{\infty} \frac{1}{5}e^{-\frac{x}{5}}\,dx$$

$$= \frac{1}{5}\left|\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right|_{10}^\infty$$

$$= -\left(e^{-\infty}-e^{-2}\right)$$

$$= -\left(0-e^{-2}\right)$$

$$= e^{\frac{1}{2}}$$

Example 9

A continuous random variable X has a pdf $f(x)^2 = 3x^2$, $0 \le x \le 1$. *Find a and b such that*

(i) $P(X \le a) = P(X > a)$ and

(ii) $P(X > b) = 0.05$

Solution

Since total probability is 1 and $P(X \le a) = P(X > a)$,

$$P(X \le a) = \frac{1}{2}$$

$$\int_0^a f(x)\,dx = \frac{1}{2}$$

$$\int_0^a 3x^2 dx = \frac{1}{2}$$

$$3\left|\frac{x^3}{3}\right|_0^a = \frac{1}{2}$$

$$a^3 = \frac{1}{2}$$

$$a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

$$P(X > b) = 0.05$$

$$\int_b^1 f(x)\,dx = 0.05$$

$$\int_b^1 3x^2\,dx = 0.05$$

$$3\left|\frac{x^3}{3}\right|_b^1 = 0.05$$

$$1 - b^3 = 0.05$$

$$b^3 = \frac{19}{20}$$

$$b = \left(\frac{19}{20}\right)^{\frac{1}{3}}$$

Example 10

Let the continuous random variable X have the probability density function

$$f(x) = \frac{2}{x^3} \qquad 1 < x < \infty$$

$$= 0 \qquad \textit{otherwise}$$

Find F(x).

Solution

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{1} f(x)\,\mathrm{d}x + \int_{1}^{x} f(x)\,\mathrm{d}x \\
&= 0 + \int_{1}^{x} \frac{2}{x^3}\,\mathrm{d}x \\
&= 2\left|\frac{x^{-2}}{-2}\right|_1^x \\
&= -\left|\frac{1}{x^2}\right|_1^x \\
&= -\left(\frac{1}{x^2} - 1\right) \\
&= 1 - \frac{1}{x^2}
\end{aligned}
$$

Hence, $F(x) = 1 - \dfrac{1}{x^2} \qquad 1 < x < \infty$

$\qquad = 0 \qquad$ otherwise

Example 11

Verify that the function $F(x)$ *is a distribution function.*

$$
\begin{aligned}
F(x) &= 0 & x < 0 \\
&= 1 - e^{-\frac{x}{4}} & x \ge 0
\end{aligned}
$$

Also, find the probabilities $P(X \le 4), P(X \ge 8), P(4 \le X \le 8)$.

Solution

For the function $F(x)$,

(i) $F(-\infty) = 0$

(ii) $F(\infty) = 1 - e^{-\infty} = 1 - 0 = 1$

(iii) $0 \le F(x) \le 1 \qquad -\infty < x < \infty$

If $f(x)$ is the corresponding probability density function,

$$
\begin{aligned}
f(x) = F'(x) &= 0 & x < 0 \\
&= \frac{1}{4} e^{-\frac{x}{4}} & x \ge 0
\end{aligned}
$$

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{\infty} f(x)\,\mathrm{d}x$$

$$= 0 + \int_{0}^{\infty} \frac{1}{4} e^{-\frac{x}{4}}\,\mathrm{d}x$$

$$= \frac{1}{4}\left|\frac{e^{-\frac{x}{4}}}{-\frac{1}{4}}\right|_0^{\infty}$$

$$= -\left|e^{-\frac{x}{4}}\right|_0^{\infty}$$

$$= -(0-1)$$

$$= 1$$

Hence, $F(x)$ is a distribution function.

$$P(X \le 4) = F(4)$$

$$= 1 - e^{-1}$$

$$= 1 - \frac{1}{e}$$

$$= \frac{e-1}{e}$$

$$P(X \ge 8) = 1 - P(X \le 8)$$

$$= 1 - F(8)$$

$$= 1 - (1 - e^{-2})$$

$$= e^{-2}$$

$$= \frac{1}{e^2}$$

$$P(4 \le X \le 8) = F(8) - F(4)$$

$$= (1 - e^{-2}) - (1 - e^{-1})$$

$$= e^{-1} - e^{-2}$$

$$= \frac{1}{e} - \frac{1}{e^2}$$

$$= \frac{e-1}{e^2}$$

Example 12

The troubleshooting capacity of an IC chip in a circuit is a random variable X whose distribution function is given by

$$F(x) = 0 \qquad x \le 3$$
$$= 1 - \frac{9}{x^2} \qquad x > 3$$

where x denotes the number of years. Find the probability that the IC chip will work properly (i) less than 8 years, (i) beyond 8 years, (iii) between 5 to 7 years, and (iv) anywhere from 2 to 5 years.

Solution

(i) $P(X \le 8) = F(8)$
$$= 1 - \frac{9}{8^2}$$
$$= 0.8594$$

(ii) $P(X > 8) = 1 - P(X \le 8)$
$$= 1 - F(8)$$
$$= 1 - 0.8594$$
$$= 0.1406$$

(iii) $P(5 \le X \le 7) = F(7) - F(5)$
$$= \left(1 - \frac{9}{7^2}\right) - \left(1 - \frac{9}{5^2}\right)$$
$$= 0.1763$$

(iv) $P(2 \le X \le 5) = F(5) - F(2)$
$$= \left(1 - \frac{9}{5^2}\right) - 0$$
$$= 0.64$$

Example 13

The probability density function of a continuous random variable X is given by

$$f(x) = \begin{cases} ax & 0 \le x \le 1 \\ a & 1 \le x \le 2 \\ 3a - ax & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find the value of a, and (ii) find the cdf of X.

Solution

(i) Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$\int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{1} f(x)\,\mathrm{d}x + \int_{1}^{2} f(x)\,\mathrm{d}x + \int_{2}^{3} f(x)\,\mathrm{d}x = 1$$

$$0 + \int_{0}^{1} ax\,\mathrm{d}x + \int_{1}^{2} a\,\mathrm{d}x + \int_{2}^{3} (3a - ax)\,\mathrm{d}x = 1$$

$$a\left|\frac{x^2}{2}\right|_0^1 + a\left|\,x\,\right|_1^2 + \left|3ax - \frac{ax^2}{2}\right|_2^3 = 1$$

$$a\left(\frac{1}{2} - 0\right) + a(2-1) + \left[\left(9a - \frac{9a}{2}\right) - (6a - 2a)\right] = 1$$

$$\frac{1}{2}a + a + \frac{9a}{2} - 4a = 1$$

$$2a = 1$$

$$a = \frac{1}{2}$$

(ii) $F(x) = \displaystyle\int_{-\infty}^{x} f(x)\,\mathrm{d}x$

For $0 \le x \le 1$,

$$F(x) = \int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{x} f(x)\,\mathrm{d}x$$

$$= 0 + \int_{0}^{x} ax\,\mathrm{d}x$$

$$= a\left|\frac{x^2}{2}\right|_0^x$$

$$= \frac{ax^2}{2}$$

For $1 \le x \le 2$,

$$F(x) = \int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{1} f(x)\,\mathrm{d}x + \int_{1}^{x} f(x)\,\mathrm{d}x$$

$$= 0 + \int_{0}^{1} ax\,\mathrm{d}x + \int_{1}^{x} a\,\mathrm{d}x$$

$$= a\left|\frac{x^2}{2}\right|_0^1 + a|x|_1^x$$

$$= a\left(\frac{1}{2} - 0\right) + a(x-1)$$

$$= \frac{a}{2} + ax - a$$

$$= ax - \frac{a}{2}$$

For $2 \le x \le 3$,

$$F(x) = \int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_0^1 f(x)\,\mathrm{d}x + \int_1^2 f(x)\,\mathrm{d}x + \int_2^x f(x)\,\mathrm{d}x$$

$$= 0 + \int_0^1 ax\,\mathrm{d}x + \int_1^2 a\,\mathrm{d}x + \int_2^x (3a - ax)\,\mathrm{d}x$$

$$= a\left|\frac{x^2}{2}\right|_0^1 + a|x|_1^2 + \left|3ax - \frac{ax^2}{2}\right|_2^x$$

$$= a\left(\frac{1}{2} - 0\right) + a(2-1) + \left[\left(3ax - \frac{ax^2}{2}\right) - (6a - 2a)\right]$$

$$= \frac{a}{2} + a + 3ax - \frac{ax^2}{2} - 4a$$

$$= 3ax - \frac{ax^2}{2} - \frac{5a}{2}$$

Hence,
$$\begin{aligned} F(x) &= \frac{ax^2}{2} && 0 \le x \le 1 \\ &= ax - \frac{a}{2} && 1 \le x \le 2 \\ &= 3ax - \frac{ax^2}{2} - \frac{5a}{2} && 2 \le x \le 3 \end{aligned}$$

Example 14

The pdf of a continuous random variable X is

$$f(x) = \frac{1}{2}e^{-|x|}$$

Find cdf F(x).

Solution

$$f(x) = \frac{1}{2}e^{x} \qquad -\infty < x < 0$$
$$= \frac{1}{2}e^{-x} \qquad 0 < x < \infty$$

$$F(x) = \int_{-\infty}^{x} f(x)\,\mathrm{d}x$$

For $x \le 0$,

$$F(x) = \int_{-\infty}^{x} \frac{1}{2}e^{x}\mathrm{d}x$$
$$= \frac{1}{2}\left|e^{x}\right|_{-\infty}^{x}$$
$$= \frac{1}{2}(e^{x} - e^{-\infty})$$
$$= \frac{1}{2}e^{x}$$

For $x > 0$,

$$F(x) = \int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{x} f(x)\,\mathrm{d}x$$
$$= \int_{-\infty}^{0} \frac{1}{2}e^{x}\mathrm{d}x + \int_{0}^{x} \frac{1}{2}e^{-x}\ \mathrm{d}x$$
$$= \frac{1}{2}\left|e^{x}\right|_{-\infty}^{0} + \frac{1}{2}\left|-e^{-x}\right|_{0}^{x}$$
$$= \frac{1}{2}(1 - e^{-\infty}) + \frac{1}{2}(-e^{-x} + e^{0})$$
$$= \frac{1}{2} - \frac{1}{2}e^{-x} + \frac{1}{2}$$
$$= 1 - \frac{1}{2}e^{-x}$$

Hence, $F(x) = \frac{1}{2}e^{x} \qquad x \le 0$

$$= 1 - \frac{1}{2}e^{-x} \qquad x > 0$$

Example 15

Find the value of k and the distribution function F(x) given the probability density function of a random variable X as

$$f(x) = \frac{k}{x^2+1} \qquad -\infty < x < \infty$$

Solution

Since $f(x)$ is the probability density function,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$\int_{-\infty}^{\infty} \frac{k}{x^2+1}\,\mathrm{d}x = 1$$

$$k\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = 1$$

$$k\left|\tan^{-1} x\right|_{-\infty}^{\infty} = 1$$

$$k\left[\tan^{-1}\infty - \tan^{-1}(-\infty)\right] = 1$$

$$k\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$$

$$k\pi = 1$$

$$k = \frac{1}{\pi}$$

Hence, $f(x) = \dfrac{1}{\pi}\dfrac{1}{x^2+1} \qquad -\infty < x < \infty$

$$F(x) = \int_{-\infty}^{x} f(x)\,\mathrm{d}x$$

$$= \frac{1}{\pi}\int_{-\infty}^{x} \frac{1}{x^2+1}\,\mathrm{d}x$$

$$= \frac{1}{\pi}\left|\tan^{-1} x\right|_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1} x - \tan^{-1}(-\infty)\right]$$

$$= \frac{1}{\pi}\left(\tan^{-1} x + \frac{\pi}{2}\right)$$

Example 16

Find the constant k such that

$$f(x) = kx^2 \quad 0 < x < 3$$
$$= 0 \quad \textit{otherwise}$$

is a probability function. Also, find the distribution function F(x) and $P(1 < X \le 2)$.

Solution

Since $f(x)$ is probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{3} kx^2\,dx + 0 = 1$$

$$k\left|\frac{x^3}{3}\right|_0^3 = 1$$

$$k(9-0) = 1$$

$$k = \frac{1}{9}$$

Hence, $f(x) = \frac{1}{9}x^2 \quad 0 < x < 3$

$$= 0 \quad \text{otherwise}$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_0^x \frac{1}{9}x^2\,dx$$

$$= \frac{1}{9}\left|\frac{x^3}{3}\right|_0^x$$

$$= \frac{1}{27}x^3$$

Hence, $F(x) = \frac{1}{27}x^3 \qquad 0 < x < 3$

$\qquad\qquad = 0 \qquad\qquad \text{otherwise}$

$$P(1 < x \le 2) = \int_1^2 f(x)\,\mathrm{d}x$$

$$= \int_1^2 \frac{1}{9}x^2\,\mathrm{d}x$$

$$= \frac{1}{9}\left|\frac{x^3}{3}\right|_1^2$$

$$= \frac{1}{27}(8-1)$$

$$= \frac{7}{27}$$

EXERCISE 2.3

1. Verify whether the following functions are probability density functions:

(i) $f(x) = k\,e^{-kx} \qquad x \ge 0, k > 0$

(ii) $f(x) = \frac{1}{2}\,e^{-|x|} \qquad -\infty < x < \infty$

(iii) $f(x) = \frac{2}{9}x\left(2 - \frac{x}{2}\right) \qquad 0 \le x \le 3$

[**Ans.:** (i) Yes (ii) Yes (iii) Yes]

2. Find the value of k if the following are probability density functions:

(i) $f(x) = k(1+x) \qquad 2 \le x \le 5$

(ii) $f(x) = k(x - x^2) \qquad 0 \le x \le 1$

(iii) $f(x) = kx\,e^{-4x^2} \qquad 0 \le x \le \infty$

(iv) $f(x) = kx\,e^{-\frac{x^2}{4}} \qquad 0 \le x \le \infty$

$\left[\textbf{Ans.:} \text{ (i) } \frac{2}{27} \text{ (ii) } 6 \text{ (iii) } 8 \text{ (iv) } \frac{1}{2}\right]$

3. A function is defined as

$$f(x) = \begin{cases} 0 & x < 2 \\ \dfrac{2x+3}{18} & 2 \le x \le 4 \\ 0 & x > 4 \end{cases}$$

Show that $f(x)$ is a probability density function and find $P(2 < X < 3)$.

$$\left[\textbf{Ans.: } \frac{4}{9}\right]$$

4. Let X be a continuous random variable with probability distribution

$$f(x) = \begin{cases} \dfrac{x}{6} + k & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Find k, and $P(1 \le X \le 2)$.

$$\left[\textbf{Ans.: } 1, \frac{1}{3}\right]$$

5. Find the value of k such that $f(x)$ is a probability density function. Find also, $P(X \le 1.5)$.

$$f(x) = \begin{cases} kx & 0 \le x \le 1 \\ k & 1 \le x \le 2 \\ k(3-x) & 2 \le x \le 3 \end{cases}$$

$$\left[\textbf{Ans.: } \frac{1}{2}, \frac{1}{2}\right]$$

6. If X is a continuous random variable whose probability density function is given by

$$\begin{aligned} f(x) &= k(4x - 2x^2) \qquad && 0 < x < 2 \\ &= 0 && \text{otherwise} \end{aligned}$$

Find (i) the value of k, and (ii) $P(X > 1)$.

$$\left[\textbf{Ans.: } \text{(i) } \frac{3}{8} \text{ (ii) } \frac{1}{2}\right]$$

7. If a random variable has the probability density function

$$\begin{aligned} f(x) &= k(x^2 - 1) \qquad && -1 \le x \le 3 \\ &= 0 && \text{otherwise} \end{aligned}$$

Find (i) the value of k, and (ii) $P\left(\frac{1}{2} \le X \le \frac{5}{2}\right)$.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{3}{28} \text{ (ii) } \frac{19}{56}\right]$$

8. The probability density function is

$$\begin{aligned} f(x) &= k(3x^2 - 1) & -1 \le x \le 2 \\ &= 0 & \text{otherwise} \end{aligned}$$

Find (i) the value of k, and (ii) $P(-1 \le X \le 0)$.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{1}{6} \text{ (ii) } 0\right]$$

9. Is the function defined by

$$\begin{aligned} f(x) &= 0 & x < 2 \\ &= \frac{1}{18}(2x+3) & 2 \le x \le 4 \\ &= 0 & x > 4 \end{aligned}$$

a probability density function? Find the probability that a variate having $f(x)$ as density function will fall in the interval $2 \le X \le 3$.

$$\left[\textbf{Ans.:} \text{ Yes, } \frac{4}{9}\right]$$

10. A random variable X gives measurements x between 0 and 1 with a probability function

$$\begin{aligned} f(x) &= 12x^3 - 21x^2 + 10x & 0 \le x \le 1 \\ &= 0 & \text{otherwise} \end{aligned}$$

(i) Find $P\left(X \le \frac{1}{2}\right)$ and $P\left(X > \frac{1}{2}\right)$.

(ii) Find a number k such that $P(X \le k) = \frac{1}{2}$.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{7}{16} \text{ (ii) } 0.452\right]$$

11. The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 1 - e^{-x^2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function.

$$\left[\begin{aligned} \textbf{Ans.: } f(x) &= 2xe^{-x^2} & x > 0 \\ &= 0 & \text{otherwise} \end{aligned}\right]$$

12. The cdf of a continuous random variable X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

Find the pdf and $P\left(\frac{1}{2} \le X \le \frac{4}{5}\right)$.

[**Ans.:** 0.195]

13. Find the distribution function corresponding to the following probability density functions:

(i) $f(x) = \begin{cases} \frac{1}{2}x^2 e^{-x} & 0 \le x < \infty \\ 0 & \text{otherwise} \end{cases}$

(ii) $f(x) = x \qquad 0 \le x \le 1$

$\quad = 2 - x \qquad 1 \le x \le 2$

$\quad = 0 \qquad \text{otherwise}$

(iii) $f(x) = \lambda(x-1)^4 \qquad 1 \le x \le 3, \lambda > 0$

$\quad = 0 \qquad \text{otherwise}$

Ans.: (i) $F(x) = \begin{cases} 1 - e^{-x}\left(1 + x + \frac{x^2}{2}\right) & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$

(ii) $F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \le x \le 1 \\ 2x - 0.5x^2 - 1 & 1 \le x \le 2 \\ 1 & x > 2 \end{cases}$

(iii) $\lambda = \frac{5}{32}, \quad F(x) = \begin{cases} 0 & x \le 1 \\ \frac{5}{32}(x-1)^4 & 1 \le x \le 3 \\ 1 & x \ge 3 \end{cases}$

14. A continuous random variable X has the following probability density function

$$f(x) = \frac{a}{x^5} \qquad 2 \le x \le 10$$

Determine the constant a, distribution function of X, and find the probability of the event $4 \le x \le 7$.

$$\left[\textbf{Ans.:}\ \frac{2500}{39},\ F(x) = \frac{625}{39}\left(\frac{1}{16} - \frac{1}{x^4}\right),\ 0.056\right]$$

2.8 MEASURES OF CENTRAL TENDENCY FOR CONTINUOUS PROBABILITY DISTRIBUTION

1. Mean The mean or average value (μ) of the probability distribution of a continuous random variable X is called the expectation and is denoted by $E(X)$.

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, \mathrm{d}x$$

where $f(x)$ is the probability density function of the continuous random variable.
Expectation of any function $\phi(x)$ of a continuous random variable X is given by

$$E\left[\phi(x)\right] = \int_{-\infty}^{\infty} \phi(x)\ f(x)\ \mathrm{d}x$$

2. Median The median is the point which divides the entire distribution into two equal parts. In case of a continuous distribution, the median is the point which divides the total area into two equal parts. Thus, if a continuous random variable X is defined from a to b and M is the median,

$$\int_{a}^{M} f(x)\ \mathrm{d}x = \int_{M}^{b} f(x)\ \mathrm{d}x = \frac{1}{2}$$

By solving any one of this equation, the median is obtained.

3. Mode The mode is value of x for which $f(x)$ is maximum. Mode is given by

$$f'(x) = 0 \text{ and } f''(x) < 0 \quad \text{for} \quad a < x < b$$

4. Variance The variance of the probability distribution of a continuous random variable X is given by

$$\begin{aligned}\mathrm{Var}(X) = \sigma^2 &= \int_{-\infty}^{\infty} (x-\mu)^2\ f(x)\ \mathrm{d}x \\ &= \int_{-\infty}^{\infty} x^2\ f(x)\ \mathrm{d}x - \mu^2\end{aligned}$$

5. Standard Deviation The standard deviation of the probability distribution of a continuous random variable X is given by

$$\mathrm{SD} = \sqrt{\mathrm{Var}(X)} = \sigma$$

Example 1

For the continuous random variable having pdf

$$f(x) = 4x^3 \quad 0 \le x \le 1$$
$$= 0 \quad \textit{otherwise}$$

Find the mean and variance of X.

Solution

$$\text{Mean} = \mu = \int_{-\infty}^{\infty} x\, f(x)\, \mathrm{d}x$$

$$= \int_{-\infty}^{0} x\, f(x)\, \mathrm{d}x + \int_{0}^{1} x\, f(x)\, \mathrm{d}x + \int_{1}^{\infty} x\, f(x)\, \mathrm{d}x$$

$$= 0 + \int_{0}^{1} x(4x^3)\, \mathrm{d}x + 0$$

$$= 4\int_{0}^{1} x^4\, \mathrm{d}x$$

$$= 4\left|\frac{x^5}{5}\right|_0^1$$

$$= 4\left(\frac{1}{5} - 0\right)$$

$$= \frac{4}{5}$$

$$\text{Var}(X) = \int_{-\infty}^{\infty} x^2\, f(x)\, \mathrm{d}x - \mu^2$$

$$= \int_{-\infty}^{0} x^2 f(x)\, \mathrm{d}x + \int_{0}^{1} x^2 f(x)\, \mathrm{d}x + \int_{1}^{\infty} x^2 f(x)\, \mathrm{d}x - \mu^2$$

$$= 0 + \int_{0}^{1} x^2\, (4x^3)\, \mathrm{d}x + 0 - \left(\frac{4}{5}\right)^2$$

$$= 4\int_{0}^{1} x^5\, \mathrm{d}x - \frac{16}{25}$$

$$= 4\left|\frac{x^6}{6}\right|_0^1 - \frac{16}{25}$$

$$= \frac{4}{6} - \frac{16}{25}$$
$$= \frac{2}{75}$$

Example 2

For the triangular distribution

$$\begin{aligned} f(x) &= x && 0 < x \le 1 \\ &= 2 - x && 1 \le x \le 2 \\ &= 0 && \textit{otherwise} \end{aligned}$$

Find the mean and variance.

Solution

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, \mathrm{d}x$$
$$= \int_{-\infty}^{0} x\ f(x)\ \mathrm{d}x + \int_{0}^{1} x\, f(x)\ \mathrm{d}x + \int_{1}^{2} x\, f(x) + \int_{2}^{\infty} x\, f(x)\ \mathrm{d}x$$
$$= 0 + \int_{0}^{1} x \cdot x\ \mathrm{d}x + \int_{1}^{2} x(2 - x)\ \mathrm{d}x + 0$$
$$= \int_{0}^{1} x^2\ \mathrm{d}x + \int_{1}^{2} (2x - x^2)\ \mathrm{d}x$$
$$= \left|\frac{x^3}{3}\right|_0^1 + \left|2\frac{x^2}{2} - \frac{x^3}{3}\right|_1^2$$
$$= \left(\frac{1}{3} - 0\right) + \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right]$$
$$= \frac{1}{3} + \frac{4}{3} - \frac{2}{3}$$
$$= 1$$

$$\text{Var }(X) = \int_{-\infty}^{\infty} x^2\ f(x)\, \mathrm{d}x - \mu^2$$
$$= \int_{-\infty}^{0} x^2 f(x)\ \mathrm{d}x + \int_{0}^{1} x^2\ f(x)\ \mathrm{d}x + \int_{1}^{2} x^2\ f(x)\ \mathrm{d}x + \int_{2}^{\infty} x^2 f(x)\ \mathrm{d}x - \mu^2$$

$$= 0 + \int_0^1 x^2 \cdot x \, dx + \int_1^2 x^2 (2 - x) \, dx + 0 - 1$$

$$= \int_0^1 x^3 \, dx + \int_1^2 (2x^2 - x^3) \, dx - 1$$

$$= \left| \frac{x^4}{4} \right|_0^1 + \left| \frac{2x^3}{3} - \frac{x^4}{4} \right|_1^2 - 1$$

$$= \left(\frac{1}{4} - 0 \right) + \left[\left(\frac{16}{3} - \frac{16}{4} \right) - \left(\frac{2}{3} - \frac{1}{4} \right) \right] - 1$$

$$= \frac{7}{6} - 1$$

$$= \frac{1}{6}$$

Example 3

If the probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{x}{2} & 0 < x \le 1 \\ \dfrac{1}{2} & 1 < x \le 2 \\ \dfrac{3-x}{2} & 2 < x < 3 \\ 0 & \textit{otherwise} \end{cases}$$

Find the expected value of $f(x) = x^2 - 5x + 3$.

Solution

$$E\left[E\phi(x)\right] = \int_{-\infty}^{\infty} \phi(x) \, f(x) \, dx$$

$$E(x^2 - 5x + 3) = \int_{-\infty}^{\infty} (x^2 - 5x + 3) \, f(x) \, dx$$

$$= \int_0^1 (x^2 - 5x + 3)\frac{x}{2}\,\mathrm{d}x + \int_1^2 (x^2 - 5x + 3)\frac{1}{2}\,\mathrm{d}x + \int_2^3 (x^2 - 5x + 3)\left(\frac{3-x}{2}\right)\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1 (x^3 - 5x^2 + 3x)\,\mathrm{d}x + \frac{1}{2}\int_1^2 (x^2 - 5x + 3)\,\mathrm{d}x + \frac{1}{2}\int_2^3 (-x^3 + 8x^2 - 18x + 9)\,\mathrm{d}x$$

$$= \frac{1}{2}\left|\frac{x^4}{4} - \frac{5x^3}{3} + \frac{3x^2}{2}\right|_0^1 + \frac{1}{2}\left|\frac{x^3}{3} - \frac{5x^2}{2} + 3x\right|_1^2 + \frac{1}{2}\left|-\frac{x^4}{4} + \frac{8x^3}{3} - \frac{18x^2}{2} + 9x\right|_2^3$$

$$= \frac{1}{2}\left(\frac{1}{4} - \frac{5}{3} + \frac{3}{2}\right) + \frac{1}{2}\left(\frac{8}{3} - 10 + 6 - \frac{1}{3} + \frac{5}{2} - 3\right) + \frac{1}{2}\left(-\frac{81}{4} + \frac{216}{3} - \frac{162}{2} + 27 + \frac{16}{4} - \frac{64}{3} + \frac{72}{2} - 18\right)$$

$$= \frac{1}{24} - \frac{13}{12} - \frac{19}{24}$$

$$= -\frac{11}{6}$$

Example 4

A continuous random variable has the probability density function

$$f(x) = kxe^{-\lambda x} \qquad x \ge 0,\ \lambda > 0$$
$$= 0 \qquad \textit{otherwise}$$

Determine (i) k, (ii) mean, and (iii) variance.

Solution

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$\int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_0^{\infty} f(x)\,\mathrm{d}x = 1$$

$$0 + \int_0^{\infty} k\,x\,e^{-\lambda x}\,\mathrm{d}x = 1$$

$$k\int_0^\infty x e^{-\lambda x}\,\mathrm{d}x = 1$$

$$k\left| x\frac{e^{-\lambda x}}{-\lambda} - 1\frac{e^{-\lambda x}}{\lambda^2}\right|_0^\infty = 1$$

$$k\left[(0-0)-\left(0-\frac{1}{\lambda^2}\right)\right] = 1$$

$$k = \lambda^2$$

Hence, $f(x) = \lambda^2 x e^{-\lambda x} \qquad x \ge 0, \lambda = 0$

$= 0 \qquad$ otherwise

(ii) Mean $= \mu = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$

$$= \int_{-\infty}^{0} x\, f(x)\,\mathrm{d}x + \int_0^\infty x\, f(x)\,\mathrm{d}x$$

$$= 0 + \int_0^\infty x\,\lambda^2\, x e^{-\lambda x}\,\mathrm{d}x$$

$$= \lambda^2\int_0^\infty x^2\, e^{-\lambda x}\,\mathrm{d}x$$

$$= \lambda^2\left| x^2\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 2x\left(\frac{e^{-\lambda x}}{\lambda^2}\right) + 2\left(\frac{e^{-\lambda x}}{-\lambda^3}\right)\right|_0^\infty$$

$$= \lambda^2\left[(0-0+0)-\left(0-0-\frac{2}{\lambda^3}\right)\right]$$

$$= \frac{2}{\lambda}$$

(iii) Variance $= \sigma^2 = \int_{-\infty}^{\infty} x^2\, f(x)\,\mathrm{d}x - \mu^2$

$$= \int_{-\infty}^{0} x^2\, f(x)\,\mathrm{d}x + \int_0^\infty x^2\, f(x)\,\mathrm{d}x - \mu^2$$

$$= 0 + \int_0^\infty x^2\,\lambda^2\, x e^{-\lambda x}\,\mathrm{d}x - \left(\frac{2}{\lambda}\right)^2$$

$$= \lambda^2\int_0^\infty x^3\, e^{-\lambda x}\,\mathrm{d}x - \frac{4}{\lambda^2}$$

$$= \lambda^2 \left| x^3 \left(\frac{e^{-\lambda x}}{-\lambda x}\right) - 3x^2\left(\frac{e^{-\lambda x}}{\lambda^2}\right) + 6x\left(\frac{e^{-\lambda x}}{-\lambda^3}\right) - 6\left(\frac{e^{-\lambda x}}{\lambda^4}\right)\right|_0^\infty - \frac{4}{\lambda^2}$$

$$= \lambda^2\left[(0-0+0-0) - \left(0-0+0-\frac{6}{\lambda^4}\right)\right] - \frac{4}{\lambda^2}$$

$$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2}$$

$$= \frac{2}{\lambda^2}$$

Example 5

The probability density $f(x)$ of a continuous random variable is given by $f(x) = k\, e^{-|x|}$, $-\infty < x < \infty$ (i) show that $k = \frac{1}{2}$, and (ii) find the mean and variance of the distribution. (iii) Also, find the probability that the variate lies between 0 and 4.

Solution

(i) Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$\int_{-\infty}^{\infty} k\,e^{-|x|}\,\mathrm{d}x = 1$$

$$k\int_{-\infty}^{\infty} e^{-|x|}\,\mathrm{d}x = 1$$

$$2k\int_{0}^{\infty} e^{-|x|}\,\mathrm{d}x = 1 \qquad \left[\because\ e^{-|x|} \text{ is an even function}\right]$$

$$2k\int_{0}^{\infty} e^{-x}\,\mathrm{d}x = 1 \qquad \left[\because\ |x| = x \quad 0 \le x \le \infty\right]$$

$$2k\left|-e^{-x}\right|_0^\infty = 1$$

$$-2k\,(0-1) = 1$$

$$k = \frac{1}{2}$$

Hence, $f(x) = \frac{1}{2}e^{-|x|} \qquad -\infty < x < \infty$

(ii) $\mu = \int_{-\infty}^{\infty} x\, f(x)\, \mathrm{d}x$

$= \dfrac{1}{2} \int_{-\infty}^{\infty} x e^{-|x|}\, \mathrm{d}x$

$= 0 \qquad [\because \text{ the integrand is an odd function}]$

(iii) $\mathrm{Var}(X) = \sigma^2 = \int_{-\infty}^{\infty} x^2\, f(x)\, \mathrm{d}x - \mu^2$

$= \dfrac{1}{2} \int_{-\infty}^{\infty} x^2\, e^{-|x|}\, \mathrm{d}x - 0$

$= 2\left(\dfrac{1}{2}\right) \int_{0}^{\infty} x^2\, e^{-|x|}\, \mathrm{d}x \quad [\because \text{ the integrand is an even function}]$

$= \int_{0}^{\infty} x^2\, e^{-|x|}\, \mathrm{d}x$

$= \left| x^2\, \dfrac{e^{-x}}{-1} - 2x \dfrac{e^{-x}}{1} + 2 \dfrac{e^{-x}}{-1} \right|_0^{\infty}$

$= 0 - (-2)$

$= 2$

(iii) Probability that the variate lies between 0 and 4

$P(0 < X < 4) = \int_0^4 f(x)\, \mathrm{d}x$

$= \dfrac{1}{2} \int_0^4 e^{-|x|}\, \mathrm{d}x$

$= \dfrac{1}{2} \int_0^4 e^{-x}\, \mathrm{d}x \qquad [\because \ |x| = x \quad 0 < x < 4]$

$= -\dfrac{1}{2} \left| e^{-x} \right|_0^4$

$= -\dfrac{1}{2}(e^{-4} - 1)$

$= 0.4908$

Example 6

The daily consumption of electric power is a random variable X with probability density function

$$
\begin{aligned}
f(x) &= k\,x\,e^{-\frac{x}{3}} && x > 0 \\
&= 0 && x \le 0
\end{aligned}
$$

Find the value of k, the expectation of X, and the probability that on a given day, the electric consumption is more than the expected value.

Solution

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$\int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$0 + \int_{0}^{\infty} k\,x\,e^{-\frac{x}{3}}\,\mathrm{d}x = 1$$

$$k\left| x\left(\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}}\right) - (1)\left(\frac{e^{-\frac{x}{3}}}{\frac{1}{9}}\right)\right|_0^{\infty} = 1$$

$$k\left[(0-0)-(0-9)\right] = 1$$

$$9k = 1$$

$$k = \frac{1}{9}$$

Hence,
$$
\begin{aligned}
f(x) &= \frac{1}{9}xe^{-\frac{x}{3}} && x > 0 \\
&= 0 && x \le 0
\end{aligned}
$$

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x\,f(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{0} x\,f(x)\,\mathrm{d}x + \int_{0}^{\infty} x\,f(x)\,\mathrm{d}x \\
&= 0 + \int_{0}^{\infty} x\cdot\frac{1}{9}\,x\,e^{-\frac{x}{3}}\,\mathrm{d}x \\
&= \frac{1}{9}\int_{0}^{\infty} x^2\,e^{-\frac{x}{3}}\,\mathrm{d}x
\end{aligned}
$$

$$= \frac{1}{9}\left| x^2\left(\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}}\right) - 2x\left(\frac{e^{-\frac{x}{3}}}{\frac{1}{9}}\right) + 2\left(\frac{e^{-\frac{x}{3}}}{-\frac{1}{27}}\right)\right|_0^\infty$$

$$= \frac{1}{9}(0 - 0 + 0 + 54)$$

$$= 6$$

$$P(X > 6) = \int_0^6 f(x)\,\mathrm{d}x$$

$$= \int_0^6 \frac{1}{9} x e^{-\frac{x}{3}}\,\mathrm{d}x$$

$$= \frac{1}{9}\int_0^6 x e^{-\frac{x}{3}}\,\mathrm{d}x$$

$$= \frac{1}{9}\left| x\left(\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}}\right) - 1\left(\frac{e^{-\frac{x}{3}}}{\frac{1}{9}}\right)\right|_0^\infty$$

$$= \frac{1}{9}\left[(0-0) - \left(-18e^{-2} - 9e^{-2}\right)\right]$$

$$= 3\,e^{-2}$$

$$= 0.406$$

Example 7

Let X be a random variable with E(X) = 10 *and Var(X) =* 25*. Find the positive values of a and b such that Y = aX – b has an expectation of* 0 *and a variance of* 1.

Solution

$$E(Y) = E(aX - b)$$

$$0 = aE(X) - b$$

$$= a(10) - b$$

$$10a - b = 0$$

$$\mathrm{Var}(Y) = \mathrm{Var}(aX - b)$$

$$1 = a^2\ \mathrm{Var}(X)$$

$$= a^2(25)$$

$$25a^2 = 1$$
$$a = \frac{1}{5}$$
$$b = 2$$

Example 8

A continuous random variable X is distributed over the interval [0, 1] *with pdf* $f(x) = ax^2 + bx$*, where a, b are constants. If the mean of X is* 0.5*, find the values of a and b.*

Solution

Since $f(x)$ is probability density function,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,\mathrm{d}x + \int_{1}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$0 + \int_{0}^{1} (ax^2 + bx)\,\mathrm{d}x + 0 = 1$$

$$\left| \frac{ax^3}{3} + \frac{bx^2}{2} \right|_0^1 = 1$$

$$\frac{a}{3} + \frac{b}{2} = 1$$

$$2a + 3b = 6 \qquad \text{...(1)}$$

Also, $\mu = 0.5$

$$\int_{0}^{1} x\, f(x)\,\mathrm{d}x = 0.5$$

$$\int_{0}^{1} x\,(ax^2 + bx)\,\mathrm{d}x = 0.5$$

$$\int_{0}^{1} (ax^3 + bx^2)\,\mathrm{d}x = 0.5$$

$$\left| \frac{ax^4}{4} + \frac{bx^3}{3} \right|_0^1 = 0.5$$

$$\frac{a}{4} + \frac{b}{3} = 0.5$$

$$3a + 4b = 6 \qquad \text{...(2)}$$

Solving Eqs (1) and (2),

$$a = -6, \qquad b = 6$$

Example 9

A continuous random variable X has the pdf defined by f (x) = A + Bx, $0 \le x \le 1$. *If the mean of the distribution is* $\frac{1}{3}$, *find A and B.*

Solution

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$\int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{1} f(x)\,\mathrm{d}x + \int_{1}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$0 + \int_{0}^{1} (A + Bx)\,\mathrm{d}x + 0 = 1$$

$$\left| Ax + \frac{Bx^2}{2} \right|_0^1 = 1$$

$$A + \frac{B}{2} = 1 \qquad \text{...(1)}$$

Also,

$$\mu = \frac{1}{3}$$

$$\int^{\infty} x\, f(x)\,\mathrm{d}x = \frac{1}{3}$$

$$\int_{-\infty}^{0} x\, f(x)\,\mathrm{d}x + \int_{0}^{1} x\, f(x)\,\mathrm{d}x = \frac{1}{3}$$

$$0 + \int_{0}^{1} x\,(A + Bx)\,\mathrm{d}x = \frac{1}{3}$$

$$\int_{0}^{1} (Ax + Bx^2)\,\mathrm{d}x = \frac{1}{3}$$

$$\left| \frac{Ax^2}{2} + \frac{Bx^3}{3} \right|_0^1 = \frac{1}{3}$$

$$\frac{A}{2} + \frac{B}{3} = \frac{1}{3}$$

$$3A + 2B = 2 \qquad \text{...(2)}$$

Solving Eqs (1) and (2),

$$A = 2, \qquad B = -2$$

Example 10

A continuous random variable has probability density function $f(x) = 6(x - x^2) \qquad 0 \le x \le 1$.

Find the (i) mean, (ii) variance, (iii) median, and (iv) mode.

Solution

(i) $$\mu = \int_{-\infty}^{\infty} x\, f(x)\, \mathrm{d}x$$

$$= \int_{-\infty}^{0} x\, f(x)\, \mathrm{d}x + \int_{0}^{1} x\, f(x)\, \mathrm{d}x + \int_{1}^{\infty} x\, f(x)\, \mathrm{d}x$$

$$= 0 + \int_{0}^{1} x\, 6(x - x^2)\, \mathrm{d}x + 0$$

$$= 6\int_{0}^{1} (x^2 - x^3)\, \mathrm{d}x$$

$$= 6\left|\frac{x^3}{3} - \frac{x^4}{4}\right|_0^1$$

$$= 6\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{1}{2}$$

(ii) $$\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)\, \mathrm{d}x - \mu^2$$

$$= \int_{-\infty}^{0} x^2 f(x)\, \mathrm{d}x + \int_{0}^{1} x^2\, f(x)\, \mathrm{d}x + \int_{1}^{\infty} x^2 f(x)\, \mathrm{d}x - \mu^2$$

$$= 0 + \int_0^1 x^2\, 6(x - x^2)\, \mathrm{d}x + 0 - \frac{1}{4}$$

$$= 6\int_0^1 (x^3 - x^4)\, \mathrm{d}x - \frac{1}{4}$$

$$= 6\left|\frac{x^4}{4} - \frac{x^5}{5}\right|_0^1 - \frac{1}{4}$$

$$= 6\left(\frac{1}{4} - \frac{1}{5}\right) - \frac{1}{4}$$

$$= \frac{6}{20} - \frac{1}{4}$$

$$= \frac{1}{20}$$

(iii)
$$\int_a^M f(x)\,\mathrm{d}x = \int_M^b f(x)\,\mathrm{d}x = \frac{1}{2}$$

$$\int_0^M 6(x - x^2)\,\mathrm{d}x = \frac{1}{2}$$

$$6\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^M = \frac{1}{2}$$

$$6\left(\frac{M^2}{2} - \frac{M^3}{3}\right) = \frac{1}{2}$$

$$3M^2 - 2M^3 = \frac{1}{2}$$

$$4M^3 - 6M^2 + 1 = 0$$

$$(2M - 1)(2M^2 - 2M - 1) = 0$$

$$M = \frac{1}{2} \quad \text{or} \quad M = \frac{1 \pm \sqrt{3}}{2}$$

$$M = \frac{1}{2} \text{ lies in } (0, 1)$$

Hence, median $M = \frac{1}{2}$

(iv) Mode is the value of x for which $f(x)$ is maximum. For $f(x)$ to be maximum, $f'(x) = 0$ and $f''(x) < 0$.

$$f'(x) = 0$$

$$6(1 - 2x) = 0$$

$$x = \frac{1}{2}$$

$$f''(x) = -12\,x$$

At $x = \frac{1}{2}$, $f''(x) = -12 < 0$

Hence, $f(x)$ is maximum at $x = \frac{1}{2}$.

Mode $= \frac{1}{2}$

Example 11

The probability density function of a random variable X is

$$f(x) = \frac{1}{2}\sin x \quad 0 \le x \le \pi$$
$$= 0 \quad \textit{otherwise}$$

Find the mean, mode, and median of the distribution and also, find the probability between 0 and $\frac{\pi}{2}$.

Solution

(i) $\mu = \int_{-\infty}^{\infty} f(x)\,dx$

$$= \int_{-\infty}^{0} x\,f(x)\,dx + \int_{0}^{\pi} x\,f(x)\,dx + \int_{\pi}^{\infty} x\,f(x)\,dx$$

$$= 0 + \int_{0}^{\pi} x\left(\frac{1}{2}\sin x\right) dx + 0$$

$$= \frac{1}{2}\int_{0}^{\pi} x\sin x\,dx$$

$$= \frac{1}{2}\left|-x\cos x + \sin x\right|_{0}^{\pi}$$

$$= \frac{\pi}{2}$$

(ii) Mode is the value of x for which $f(x)$ is maximum. For $f(x)$ to be maximum, $f'(x) = 0$ and $f''(x) < 0$.

$$f'(x) = 0$$
$$\cos x = 0$$
$$x = \frac{\pi}{2}$$
$$f''(x) = -\frac{1}{2}\sin x$$

At $x = \frac{\pi}{2}$, $f''(x) = -\frac{1}{2} < 0$

Hence, $f(x)$ is maximum of $x = \frac{\pi}{2}$.

Mode = $\frac{\pi}{2}$

(iii) $$\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = \frac{1}{2}$$

$$\int_0^M \frac{1}{2}\sin x\,dx = \int_M^\pi \frac{1}{2}\sin x\,dx = \frac{1}{2}$$

$$\int_0^M \frac{1}{2}\sin x\,dx = \frac{1}{2}$$

$$-\frac{1}{2}|\cos x|_0^M = \frac{1}{2}$$

$$-\frac{1}{2}(\cos M - 1) = \frac{1}{2}$$

$$1 - \cos M = 0$$

$$\cos M = 0$$

$$M = \frac{\pi}{2}$$

Hence, median $M = \dfrac{\pi}{2}$

(iv) $$P\left(0 < X < \frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} f(x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin x\,dx$$

$$= -\frac{1}{2}|\cos x|_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{2}(0-1)$$

$$= \frac{1}{2}$$

Example 12

The cumulative distribution function of a continuous random variable X is $F(x) = 1 - e^{-2x} \quad x \ge 0$

$= 0 \quad x < 0$

Find the (i) the probability density function, (ii) mean, and (iii) variance.

Solution

(i) $f(x) = \dfrac{\mathrm{d}}{\mathrm{d}x} F(x)$

$$f(x) = \frac{1}{2} e^{-2x} \qquad x \ge 0$$
$$= 0 \qquad x < 0$$

(ii) $\mu = \int_{-\infty}^{\infty} x\, f(x)\, \mathrm{d}x$

$$= \int_{-\infty}^{0} x\, f(x)\, \mathrm{d}x + \int_{0}^{\infty} x\, f(x)\, \mathrm{d}x$$
$$= 0 + \int_{0}^{\infty} x \cdot \frac{1}{2} e^{-2x}\, \mathrm{d}x$$
$$= \frac{1}{2} \int_{0}^{\infty} x\, e^{-2x}\, \mathrm{d}x$$
$$= \frac{1}{2} \left| x\left(\frac{e^{-2x}}{-2}\right) - 1\left(\frac{e^{-2x}}{4}\right) \right|_{0}^{\infty}$$
$$= \frac{1}{2}\left[(0-0) - \left(0 - \frac{1}{4}\right)\right]$$
$$= \frac{1}{8}$$

(iii) $\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2\, f(x)\, \mathrm{d}x - \mu^2$

$$= \int_{-\infty}^{0} x^2\, f(x)\, \mathrm{d}x + \int_{0}^{\infty} x^2\, f(x)\, \mathrm{d}x - \mu^2$$
$$= 0 + \int_{0}^{\infty} x^2 \cdot \frac{1}{2} e^{-2x}\, \mathrm{d}x - \left(\frac{1}{8}\right)^2$$
$$= \frac{1}{2} \int_{0}^{\infty} x^2\, e^{-2x} \mathrm{d}x - \frac{1}{64}$$
$$= \frac{1}{2} \left| x^2\left(\frac{e^{-2x}}{-2}\right) - 2x\left(\frac{e^{-2x}}{4}\right) + 2\left(\frac{e^{-2x}}{-8}\right) \right|_{0}^{\infty} - \frac{1}{64}$$
$$= \frac{1}{2}\left[(0-0-0) - \left(0 - 0 - \frac{1}{4}\right)\right] - \frac{1}{64}$$
$$= \frac{1}{8} - \frac{1}{64}$$
$$= \frac{7}{64}$$

Example 13

A continuous random variable X has the distribution function

$$\begin{aligned} F(x) &= 0 & x \le 1 \\ &= k(x-1)^4 & 1 < x \le 3 \\ &= 1 & x > 3 \end{aligned}$$

Determine (i) f(x), (ii) k, and (iii) mean.

Solution

(i) $f(x) = \dfrac{\mathrm{d}}{\mathrm{d}x} F(x)$

$$\begin{aligned} f(x) &= 0 & x \le 1 \\ &= 4k(x-1)^3 & 1 < x \le 3 \\ &= 0 & x > 3 \end{aligned}$$

(ii) Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$\int_{-\infty}^{1} f(x)\,\mathrm{d}x + \int_{1}^{3} f(x)\,\mathrm{d}x + \int_{3}^{\infty} f(x)\,\mathrm{d}x = 1$$

$$0 + \int_{1}^{3} 4k(x-1)^3\,\mathrm{d}x + 0 = 1$$

$$4k\left|\frac{(x-1)^4}{4}\right|_1^3 = 1$$

$$k(16-0) = 1$$

$$k = \frac{1}{16}$$

Hence,
$$\begin{aligned} f(x) &= 0 & x \le 1 \\ &= \frac{1}{4}(x-1)^3 & 1 < x \le 3 \\ &= 0 & x > 3 \end{aligned}$$

(iii) $\mu = \displaystyle\int_{-\infty}^{\infty} x\, f(x)\,\mathrm{d}x$

$$= \int_{-\infty}^{1} x\, f(x)\,\mathrm{d}x + \int_{1}^{3} x\, f(x)\,\mathrm{d}x + \int_{3}^{\infty} x\, f(x)\,\mathrm{d}x$$

$$= 0 + \int_{1}^{3} x \cdot \frac{1}{4}(x-1)^3\,\mathrm{d}x + 0$$

$$= \frac{1}{4}\int_{1}^{3} x(x-1)^3\,\mathrm{d}x$$

$$= \frac{1}{4}\int_0^2 (t+1)\, t^3 \ \mathrm{d}t \qquad \begin{bmatrix} \text{Putting} \quad x-1=t \\ \text{When } x=1, t=0 \\ \text{When } x=3, t=2 \end{bmatrix}$$

$$= \frac{1}{4}\int_0^2 (t^4+t^3)\ \mathrm{d}t$$

$$= \frac{1}{4}\left|\frac{t^5}{5}+\frac{t^4}{4}\right|_0^2$$

$$= \frac{1}{4}\left[\left(\frac{2^5}{5}+\frac{2^4}{4}\right)-(0)\right]$$

$$= 2.6$$

EXERCISE 2.4

1. If the probability density function is given by

$$\begin{aligned} f(x) &= kx^2(1-x^3) && 0 \le x \le 1 \\ &= 0 && \text{otherwise} \end{aligned}$$

Find (i) k, (ii) $P\left(0 < X < \frac{1}{2}\right)$, (iii) $\overline{X}$, and (iv) σ^2.

$$\left[\textbf{Ans.:} \text{ (i) } 6 \text{ (ii) } \frac{15}{64} \text{ (iii) } \frac{9}{14} \text{ (iv) } \frac{9}{245}\right]$$

2. If the probability density function of a random variable is given by

$$\begin{aligned} f(x) &= kx && 0 \le x \le 2 \\ &= 2k && 2 \le x \le 4 \\ &= 6k - kx && 4 \le x \le 6 \end{aligned}$$

Find (i) k, (ii) $P(1 \le X \le 3)$, and (iii) $\overline{X}$.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{1}{2} \text{ (ii) } \frac{1}{3} \text{ (iii) } \frac{383}{36}\right]$$

3. If the probability density of a random variable is given by

$$\begin{aligned} f(x) &= k\,x\,e^{-\frac{x}{3}} && x > 0 \\ &= 0 && x \le 0 \end{aligned}$$

Find (i) k, (ii) $\overline{X}$, and (iii) σ^2.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{1}{9} \text{ (ii) } 6 \text{ (iii) } 18\right]$$

4. A continuous random variable has the probability density function

$$\begin{aligned} f(x) &= 2e^{-2x} & x &> 0 \\ &= 0 & x &\le 0 \end{aligned}$$

Find (i) $E(X)$, (ii) $E(\overline{X})$, (iii) Var (X), and (iv) SD of X.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{1}{2} \text{ (ii) } \frac{1}{2} \text{ (iii) } \frac{1}{4} \text{ (iv) } \frac{1}{2}\right]$$

5. A random variable X has the pdf

$$f(x) = \frac{k}{1+x^2}, -\infty < x < \infty$$

Determine (i) k, (ii) $P(X \ge 0)$, (iii) mean, and (iv) variance.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{1}{\pi} \text{ (ii) } \frac{1}{2} \text{ (iii) 0 (iv) does not exist}\right]$$

6. The distribution function of a continuous random variable X is given by $F(x) = 1-(1+x)e^{-x}$, $x \ge 0$. Find (i) pdf, (ii) mean, and (iii) variance.

$$\left[\textbf{Ans.:} \text{ (i) } f(x) = xe^{-x}, x \ge 0 \text{ (ii) 2 (iii) 2}\right]$$

7. If $f(x)$ is the probability density function of a continuous random variable, find k, mean, and variance.

$$\begin{aligned} f(x) &= kx^2 & 0 &\le x \le 1 \\ &= (2-x)^2 & 1 &\le x \le 2 \end{aligned}$$

$$\left[\textbf{Ans.:} \text{ 2, } \frac{11}{12}, 0.626\right]$$

8. A continuous random variable X has the probability density function given by

$$\begin{aligned} f(x) &= 2ax + b & 0 &\le x \le 2 \\ &= 0 & &\text{otherwise} \end{aligned}$$

If the mean of the distribution is 3, find the constants a and b.

$$\left[\textbf{Ans.:} \frac{3}{2}, -\frac{5}{2}\right]$$

9. If X is a continuous random variable with probability density function given by

$$\begin{aligned} f(x) &= k(x - x^3) & 0 &\le x \le 1 \\ &= 0 & &\text{otherwise} \end{aligned}$$

Find (i) k, (ii) mean, (iii) variance, and (iv) median.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{1}{2} \text{ (ii) } 0.06 \text{ (iii) } 0.04 \text{ (iv) } 2\right]$$

10. The probability density function of a random variable is given by

$$\begin{aligned} f(x) &= 0 && x < 2 \\ &= \frac{2x+3}{18} && 2 \le x \le 4 \\ &= 0 && x > 4 \end{aligned}$$

Find the mean and variance.

$$\left[\textbf{Ans.:} \text{ (i) } \frac{83}{27}, 0.33\right]$$

11. A continuous random variable X has the probability density function

$$\begin{aligned} f(x) &= x^3 && 0 \le x \le 1 \\ &= (2-x)^3 && 1 \le x \le 2 \\ &= 0 && \text{otherwise} \end{aligned}$$

Find $P(0.5 \le X \le 1.5)$ and mean of the distribution.

$$\left[\textbf{Ans.:} \frac{15}{32}, \frac{1}{2}\right]$$

12. The probability density function of a continuous random variable X is given by

$$f(x) = kx\,(2-x) \qquad 0 \le x \le 2$$

Find k, mean, and variance.

$$\left[\textbf{Ans.:} \frac{3}{4}, 1, \frac{1}{5}\right]$$

2.9 BINOMIAL DISTRIBUTION

Consider n independent trials of a random experiments which results in either success or failure. Let p be the probability of success remaining constant every time and $q = 1 - p$ be the probability of failure. The probability of x successes and $n - x$ failures is given by $p^x q^{n-x}$ (multiplication theorem of probability). But these x successes and $n - x$ failures can occur in any of the ${}^nC_x$ ways in each of which the probability is same. Hence, the probability of x successes is ${}^nC_x p^x q^{n-x}$.

$$P(X = x) = {}^nC_x\, p^x\, q^{n-x}, \quad x = 0, 1, 2, \ldots, n, \text{ where } p + q = 1$$

A random variable X is said to follow the binomial distribution if the probability of x is given by

$$P(X = x) = p(x) = {}^nC_x\ p^x\ q^{n-x}, \quad x = 0, 1, 2, ..., n \text{ and } q = 1 - p$$

The two constants n and p are called the parameters of the distribution.

2.9.1 Examples of Binomial Distribution

(i) Number of defective bolts in a box containing n bolts.
(ii) Number of post-graduates in a group of n people.
(iii) Number of oil wells yielding natural gas in a group of n wells test drilled.
(iv) Number of machines lying idle in a factory having n machines.

2.9.2 Conditions for Binomial Distribution

The binomial distribution holds under the following conditions:
(i) The number of trials n is finite.
(ii) There are only two possible outcomes, success or failure.
(iii) The trials are independent of each other.
(iv) The probability of success p is constant for each trial.

2.9.3 Constants of the Binomial Distribution

1. Mean of the Binomial Distribution

$$\begin{aligned} E(X) &= \sum x\ p(x) \\ &= \sum_{x=0}^{n} x\ {}^nC_x\ p^x\ q^{n-x} \\ &= 0 \cdot {}^nC_0\ p^0\ q^n + 1 \cdot {}^nC_1\ p q^{n-1} + 2 \cdot {}^nC_2\ p^2\ q^{n-2} + \cdots + n\ p^n \\ &= np\,[q^{n-1} + {}^{(n-1)}C_1\ q^{n-2}\ p + {}^{(n-1)}C_2\ q^{n-3}\ p^2 + \cdots + p^{n-1}] \\ &= np\,(q+p)^{\,n-1} \\ &= np \qquad\qquad [\because \quad p + q = 1] \end{aligned}$$

2. Variance of the Binomial Distribution

$$\begin{aligned} \text{Var}(X) &= E(X^2) - \mu^2 \\ &= \sum_{x=0}^{n} x^2\ p(x) - \mu^2 \\ &= \sum_{x=0}^{n} x^2\ {}^nC_x\ p^x\ q^{n-x} - \mu^2 \\ &= \sum_{x=0}^{n} \left[x + x(x-1)\right] {}^nC_x\ p^x\ q^{n-x} - \mu^2 \\ &= \sum_{x=0}^{n} x\ {}^nC_x\ p^x\ q^{n-x} + \sum_{x=0}^{n} x\,(x-1)\ {}^nC_x\ p^x\ q^{n-x} - \mu^2 \end{aligned}$$

$$= np + \sum_{x=0}^{n} x(x-1)\frac{n(n-1)}{x(x-1)}\cdot {}^{(n-2)}C_{x-2} p^x \, q^{n-x} - \mu^2$$

$$= np + \sum_{x=0}^{n} n(n-1)\cdot {}^{(n-2)}C_{x-2} p^2 p^{x-2} \, q^{n-x} - \mu^2$$

$$= np + n(n-1)\, p^2 \sum_{x=0}^{n} {}^{(n-2)}C_{x-2} \, p^{x-2} \, q^{n-x} - \mu^2$$

$$= np + n(n-1)p^2 \cdot (q+p)^{n-2} - \mu^2$$

$$= np + n(n-1)\, p^2 - \mu^2 \qquad [\because \quad p+q=1]$$

$$= np\,[1+(n-1)p] - \mu^2$$

$$= np\,[1-p+np] - \mu^2$$

$$= np\,[q+np] - \mu^2 \qquad [\because \quad 1-p=q]$$

$$= np\,(q+np) - (np)^2$$

$$= npq$$

3. Standard Deviation of the Binomial Distribution

$$\text{SD} = \sqrt{\text{Variance}} = \sqrt{npq}$$

4. Mode of the Binomial Distribution

Mode of the binomial distribution is the value of x at which $p(x)$ has maximum value.

Mode = integral part of $(n + 1)p$, if $(n + 1)p$ is not an integer

= $(n + 1)\, p$ and $(n + 1)\, p - 1$, if $(n + 1)\, p$ is an integer.

2.9.4 Recurrence Relation for the Binomial Distribution

For the binomial distribution,

$$P(X=x) = {}^nC_x \, p^x \, q^{n-x}$$

$$P(X=x+1) = {}^nC_{x+1} \, p^{x+1} \, q^{n-x-1}$$

$$\frac{P(X=x+1)}{P(X=x)} = \frac{{}^nC_{x+1} \, p^{x+1} \, q^{n-x-1}}{{}^nC_x \, p^x \, q^{n-x}}$$

$$= \frac{n!}{(x+1)!\,(n-x-1)!} \times \frac{x!\,(n-x)!}{n!} \cdot \frac{p}{q}$$

$$= \frac{(n-x)\,(n-x-1)!\,x!}{(x+1)\,x!\,(n-x-1)!} \cdot \frac{p}{q}$$

$$= \frac{n-x}{x+1} \cdot \frac{p}{q}$$

$$P(X = x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \cdot P(X = x)$$

2.9.5 Binomial Frequency Distribution

If n independent trials constitute one experiment and this experiment is repeated N times, the frequency of x successes is $N\,P(X = x)$, i.e., $N\ {}^nC_x\ p^x\ q^{n-x}$. This is called expected or theoretical frequency $f(x)$ of a success.

$$\sum_{x=0}^{n} f(x) = N \sum_{x=0}^{n} P(X = x) = N \qquad \left[\because \quad \sum_{x=0}^{n} P(X = x) = 1\right]$$

The expected or theoretical frequencies $f(0), f(1), f(2), ..., f(n)$ of 0, 1, 2, ..., n, successes are respectively the first, second, third, ..., $(n + 1)^{\text{th}}$ term in the expansion of $N(q + p)^n$. The possible number of successes and their frequencies is called a binomial frequency distribution. In practice, the expected frequencies differ from observed frequencies due to chance factor.

Example 1

The mean and standard deviation of a binomial distribution are 5 and 2. Determine the distribution.

Solution

$$\mu = np = 5$$

$$\text{SD} = \sqrt{npq} = 2$$

$$npq = 4$$

$$\frac{npq}{np} = \frac{4}{5}$$

$$\therefore \quad q = \frac{4}{5}$$

$$p = 1 - q = 1 - \frac{4}{5} = \frac{1}{5}$$

$$np = 5$$

$$n\left(\frac{1}{5}\right) = 5$$

$$\therefore \quad n = 25$$

Hence, the binomial distribution is

$$P(X = x) = {}^nC_x\ p^x\ q^{n-x}$$

$$= {}^{25}C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{25-x}, \qquad x = 0, 1, 2, ..., 25$$

Example 2

The mean and variance of a binomial variate are 8 *and* 6. *Find* $P(X \geq 2)$.

Solution

$$\mu = np = 8$$

$$\sigma^2 = npq = 6$$

$$\frac{npq}{np} = \frac{6}{8} = \frac{3}{4}$$

$$\therefore \quad q = \frac{3}{4}$$

$$p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$np = 8$$

$$n\left(\frac{1}{4}\right) = 8$$

$$\therefore \quad n = 32$$

$$P(X = x) = {}^nC_x \; p^x \; q^{n-x}$$

$$= {}^{32}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{32-x}, \quad x = 0, 1, 2, ..., 32$$

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - \left[P(X = 0) + P(X = 1)\right]$$

$$= 1 - \sum_{x=0}^{1} P(X = x)$$

$$= 1 - \sum_{x=0}^{1} {}^{32}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{32-x}$$

$$= 0.9988$$

Example 3

Suppose $P(X = 0) = 1 - P(X = 1)$. *If* $E(X) = 3\,Var\,(X)$, *find* $P(X = 0)$.

Solution

$$E(X) = 3\,\text{Var}\,(X)$$

$$np = 3\,npq$$

$$1 = 3q$$

$$\therefore \quad q = \frac{1}{3}$$

$$p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$$

Let $\quad P(\text{X} = 1) = p$

$$P(X = 0) = 1 - P(X = 1)$$
$$= 1 - p$$
$$= 1 - \frac{2}{3}$$
$$= \frac{1}{3}$$

Example 4

The mean and variance of a binomial distribution are 4 *and* $\dfrac{4}{3}$ *respectively. Find* $P(X \geq 1)$.

Solution

$$\mu = np = 4$$

$$\sigma^2 = npq = \frac{4}{3}$$

$$\frac{npq}{np} = \frac{\frac{4}{3}}{4} = \frac{1}{3}$$

$$\therefore \quad q = \frac{1}{3}$$

$$p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$$

$$np = 4$$

$$n\left(\frac{2}{3}\right) = 4$$

$$\therefore \quad n = 6$$

$$P(X = x) = {}^nC_x \; p^x \; q^{n-x}$$
$$= {}^6C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{6-x}, \quad x = 0, 1, 2, \ldots, 6$$

$$P(X \geq 1) = 1 - P(X < 1)$$
$$= 1 - P(X = 0)$$

$$= 1 - {}^6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^6$$
$$= 0.9986$$

Example 5

A discrete random variable X has mean 6 *and variance* 2*. If it is assumed that the distribution is binomial, find the probability that* $5 \le X \le 7$.

Solution

$$\mu = np = 6$$
$$\sigma^2 = npq = 2$$
$$\frac{npq}{np} = \frac{2}{6} = \frac{1}{3}$$
$$\therefore \quad q = \frac{1}{3}$$
$$p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$$
$$np = 6$$
$$n\left(\frac{2}{3}\right) = 6$$
$$\therefore \quad n = 9$$
$$P(X = x) = {}^nC_x \; p^x \; q^{n-x}$$
$$= {}^9C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{9-x}, \quad x = 0, 1, 2, ..., 9$$
$$P(5 \le X \le 7) = P(X = 5) + P(X = 6) + P(X = 7)$$
$$= \sum_{x=5}^{7} P(X = x)$$
$$= \sum_{x=5}^{7} {}^9C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{9-x}$$
$$= \frac{4672}{6561}$$
$$= 0.7121$$

Example 6

With the usual notation, find p for a binomial distribution if $n = 6$ *and* $9P(X = 4) = P(X = 2)$.

Solution

For the binomial distribution,

$$P(X=x) = {}^nC_x\ p^x\ q^{n-x},\quad x = 0, 1, 2, \ldots, n$$

$$n = 6$$

$$9P(X=4) = P(X=2)$$

$$9\ {}^6C_4\ p^4\ q^2 = {}^6C_2\ p^2 q^4$$

$$9p^2 = q^2 = (1-p)^2$$

$$9p^2 = 1 - 2p + p^2$$

$$8p^2 + 2p - 1 = 0$$

$$p = \frac{-2 \pm \sqrt{4+32}}{2 \times 8} = \frac{-2 \pm 6}{16} = -\frac{1}{2}, \frac{1}{4}$$

Since probability connot be negative, $p = \frac{1}{4}$.

Example 7

In a binomial distribution consisting of 5 *independent trials, the probability of* 1 *and* 2 *successes are* 0.4096 *and* 0.2048 *respectively. Find the parameter p of the distribution.*

Solution

$$n = 5,\quad P(X=1) = 0.4096,\quad P(X=2) = 0.2048$$

Probability of getting x successes out of 5 trials

$$P(X=x) = {}^nC_x\ p^x\ q^{n-x} = {}^5C_x\ p^x\ q^{5-x},\ x = 0, 1, 2, \ldots, 5$$

$$P(X=1) = {}^5C_1\ pq^4 = 0.4096 \qquad \ldots(1)$$

$$P(X=2) = {}^5C_2\ p^2\ q^3 = 0.2048 \qquad \ldots(2)$$

Dividing Eq. (2) by Eq. (1),

$$\frac{{}^5C_2\ p^2\ q^3}{{}^5C_1\ p\ q^4} = \frac{0.2048}{0.4096}$$

$$\frac{10p}{5q} = \frac{1}{2}$$

$$\frac{p}{q} = \frac{1}{4}$$

$$4p = q = 1 - p$$

$$5p = 1$$

$$p = \frac{1}{5}$$

Example 8

In a binomial distribution, the sum and product of the mean and variance are $\frac{25}{3}$ and $\frac{50}{3}$ respectively. Determine the distribution.

Solution

For the binomial distribution,

$$np + npq = \frac{25}{3}$$

$$np(1+q) = \frac{25}{3} \qquad \text{...(1)}$$

and

$$np(npq) = \frac{50}{3}$$

$$n^2 p^2 q = \frac{50}{3} \qquad \text{...(2)}$$

Squaring Eq. (1) and then dividing by Eq. (2),

$$\frac{n^2 p^2 (1+q)^2}{n^2 p^2 q} = \frac{\frac{625}{9}}{\frac{50}{3}}$$

$$\frac{1+2q+q^2}{q} = \frac{25}{6}$$

$$6(q^2+2q+1) = 25q$$

$$6q^2 - 13q + 6 = 0$$

$$(2q-3)(3q-2) = 0$$

$$q = \frac{3}{2} \text{ or } q = \frac{2}{3}$$

Since q can not be greater than 1,

$$q = \frac{2}{3}$$

$$p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$$

From Eq. (1),

$$n\left(\frac{1}{3}\right)\left(1+\frac{2}{3}\right) = \frac{25}{3}$$

$$\therefore \qquad n = 15$$

Hence, the binomial distribution is

$$P(X = x) = {}^{15}C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{15-x}, \qquad x = 0, 1, 2, \ldots 15$$

Example 9

If the probability of a defective bolt is $\frac{1}{8}$*, find the (i) mean, and (ii) variance for the distribution of* 640 *defective bolts.*

Solution

$$p = \frac{1}{8}, \quad n = 640$$

$$\mu = np = \frac{640}{8} = 80$$

$$q = 1 - p = 1 - \frac{1}{8} = \frac{7}{8}$$

Variance of the distribution = $npq = 640\left(\frac{1}{8}\right)\left(\frac{7}{8}\right) = 70$

Example 10

In eight throws of a die, 5 *or* 6 *is considered as a success. Find the mean number of success and the standard deviation.*

Solution

Let p be the probability of success.

$$p = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

$$n = 8$$

$$\mu = np = 8\left(\frac{1}{3}\right) = \frac{8}{3}$$

$$\text{SD} = \sqrt{npq} = \sqrt{8\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)} = \frac{4}{3}$$

Example 11

4 coins are tossed simultaneously. What is the probability of getting (i) 2 heads? (ii) at least 2 heads? (iii) at most 2 heads?

Solution

Let p be the probability of getting a head in the toss of a coin.

$$p = \frac{1}{2}, \quad q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}, \quad n = 4$$

The probability of getting x heads when 4 coins are tossed

$$P(X = x) = {}^nC_x \ p^x q^{n-x} = {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}, \ \ x = 0, 1, 2, 3, 4$$

(i) Probability of getting 2 heads when 4 coins are tossed

$$P(X = 2) = {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

(ii) Probability of getting at least two heads when 4 coins are tossed

$$\begin{aligned} P(X \geq 2) &= P(X = 2) + P(X = 3) + P(X = 4) \\ &= \sum_{x=2}^{4} P(X = x) \\ &= \sum_{x=2}^{4} {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x} \\ &= \frac{11}{16} \end{aligned}$$

(iii) Probability getting at most 2 heads when 4 coins are tossed

$$\begin{aligned} P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= \sum_{x=0}^{2} P(X = x) \\ &= \sum_{x=0}^{2} {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x} \\ &= \frac{11}{16} \end{aligned}$$

Example 12

Two dice are thrown five times. Find the probability of getting the sum as 7 (i) at least once, (ii) two times, and (iii) $P(1 < X < 15)$.

Solution

In a single throw of two dice, a sum of 7 can occur in 6 ways out of 6 × 6 = 36 ways.

(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)

Let p be the probability of getting the sum as 7 in a single throw of a pair of dice.

$$p = \frac{6}{36} = \frac{1}{6}, \quad q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}, \quad n = 5$$

Probability of getting the sum x times in 5 throws of a pair of dice

$$P(X = x) = {}^nC_x\, p^x\, q^{n-x} = {}^5C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{5-x}, \quad x = 0, 1, 2, ..., 5$$

(i) Probability of getting the sum as 7 at least once in 5 throws of two dice

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - {}^5C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^5 \\ &= 1 - \frac{3125}{7776} \\ &= \frac{4651}{7776} \end{aligned}$$

(ii) Probability of getting the sum as 7 two times in 5 throws of two dice

$$P(X = 2) = {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 = \frac{625}{3888}$$

(iii) Probability of getting the sum as 7 for $P(1 < X < 5)$ in 5 throws of two dice

$$\begin{aligned} P(1 < X < 5) &= P(X = 2) + P(X = 3) + P(X = 4) \\ &= \sum_{x=2}^{4} P(X = x) \\ &= \sum_{x=2}^{4} {}^5C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{5-x} \\ &= \frac{1525}{7776} \end{aligned}$$

Example 13

If 10% of the screws produced by a machine are defective, find the probability that out of 5 screws chosen at random, (i) none is defective, (ii) one is defective, and (iii) at most two are defective.

Solution

Let p be the probability of defective screws.

$$p = 0.1, \quad q = 1 - p = 1 - 0.1 = 0.9, \quad n = 5$$

Probability that x screws out of 5 screws are defective

$$P(X = x) = {}^nC_x\ p^x\ q^{n-x} = {}^5C_x\ (0.1)^x\ (0.9)^{5-x},\ x = 0, 1, 2, ..., 5$$

(i) Probability that none of the screws out of 5 screws is defective
$P(X = 0) = {}^5C_0\ (0.1)^0\ (0.9)^5 = 0.5905$

(ii) Probability that one screw out of 5 screws is defective
$P(X = 1) = {}^5C_1\ (0.1)^1\ (0.9)^4 = 0.3281$

(iii) Probability that at most 2 screws out of 5 screws are defective

$$\begin{aligned} P(X \le 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= \sum_{x=0}^{2} P(X = x) \\ &= \sum_{x=0}^{2} {}^5C_x (0.1)^x (0.9)^{5-x} \\ &= 0.9914 \end{aligned}$$

Example 14

A multiple-choice test consists of 8 *questions with* 3 *answers to each question (of which only one is correct). A student answers each question by rolling a balanced die and checking the first answer if he gets 1 or 2, the second answer if he gets* 3 *or* 4, *and the third answer if he gets* 5 *or* 6. *To get a distinction, the student must secure at least* 75% *correct answers. If there is no negative making, what is the probability that the student secures a distinction?* **[Summer 2015]**

Solution

Let p be the probability of getting an answer to a question correctly. There are three answers to each question, out of which only one is correct.

$$p = \frac{1}{3}, \quad q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}, \quad n = 8$$

Probability of getting x correct answers in an 8 questions test

$$P(X = x) = {}^nC_x\ p^x\ q^{n-x} = {}^8C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{8-x},\ x = 0, 1, 2, ..., 8$$

Probability of securing a distinction, i.e., getting at least 6 correct answers out of the 8 questions

$$\begin{aligned} P(X \le 6) &= P(X = 6) + P(X = 7) + P(X = 8) \\ &= \sum_{x=6}^{8} P(X = x) \end{aligned}$$

$$= \sum_{x=6}^{8} {}^8C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{8-x}$$
$$= \frac{43}{2187}$$
$$= 0.0197$$

Example 15

A and B play a game in which their chances of winning are in the ratio 3:2. Find A's chance of winning at least three games out of the five games played.

Solution

Let p be the probability that A wins the game.

$$p = \frac{3}{3+2} = \frac{3}{5}, \quad q = 1 - p = 1 - \frac{3}{5} = \frac{2}{5}, \quad n = 5$$

Probability that A wins x games out of 5 games

$$P(X = x) = {}^nC_x \, p^x \, q^{n-x} = {}^5C_x \left(\frac{3}{5}\right)^x \left(\frac{2}{5}\right)^{5-x}, \ x = 0, 1, 2, ..., 5$$

Probability that A wins at least 3 games

$$P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5)$$
$$= \sum_{x=3}^{5} P(X = x)$$
$$= \sum_{x=3}^{5} {}^5C_x \left(\frac{3}{5}\right)^x \left(\frac{2}{5}\right)^{5-x}$$
$$= \frac{2133}{3125}$$
$$= 0.6826$$

Example 16

It has been claimed that in 60% of all solar heat installations the utility bill is reduced by at least one-third. Accordingly, what are the probabilities that the utility bill will be reduced by at least one third in (i) four of five installations? (ii) at least four of five installations?

Solution

Let p be the probability that the utility bill is reduced by one-third in the solar heat installations.

$$p = 60\% = 0.6, \qquad q = 1 - p = 1 - 0.6 = 0.4, \qquad n = 5$$

Probability that the utility bill is reduced by one-third in x installations out of 5 installations

$$P(X = x) = {}^nC_x\ p^x\ q^{n-x} = {}^5C_x (0.6)^x\ (0.4)^{5-x},\ x = 0, 1, 2, ..., 5$$

Probability that the utility bill is reduced by one-third in 4 of 5 installations

$$P(X = 5) = {}^5C_4\ (0.6)^4\ (0.4)^1 = \frac{162}{625}$$

Probability that the utility bill is reduced by one-third in at least 4 of 5 installations

$$\begin{aligned} P(X \ge 4) &= P(X = 4) + P(X = 5) \\ &= \sum_{x=4}^{5} P(X = x) \\ &= \sum_{x=4}^{5}\ {}^5C_x\ (0.6)^x (0.4)^{5-x} \\ &= \frac{1053}{3125} \\ &= 0.337 \end{aligned}$$

Example 17

The incidence of an occupational disease in an industry is such that the workers have a 20% chance of suffering from it. What is the probability that out of 6 workers chosen at random, four or more will suffer from the disease?

Solution

Let p be the probability of a worker suffering from the disease.

$$p = 0.2, \qquad q = 1 - p = 1 - 0.2 = 0.8, \qquad n = 6$$

Probability that x workers will suffer from the disease

$$P(X = x) = {}^nC_x\ p^x\ q^{n-x} = {}^6C_x\ (0.2)^x\ (0.8)^{6-x},\ \ x = 0, 1, 2, ..., 6$$

Probability that 4 or more workers will suffer from the disease

$$\begin{aligned} P(X \ge 4) &= P(X = 4) + P(X = 5) + P(X = 6) \\ &= \sum_{x=4}^{6} P(X = x) \\ &= \sum_{x=4}^{6}\ {}^6C_x\ (0.2)^x (0.8)^{6-x} \\ &= \frac{53}{3125} \\ &= 0.017 \end{aligned}$$

Example 18

The probability that a man aged 60 will live up to 70 is 0.65. What is the probability that out of 10 such men now at 60 at least 7 will live up to 70?

Solution

Let p be the probability that a man will live up to 70.

$$p = 0.65, \qquad q = 1 - p = 1 - 0.65 = 0.35, \qquad n = 10$$

Probability that x men out of 10 will live up to 70

$$P(X = x) = {}^nC_x\ p^x\ q^{n-x} = {}^{10}C_x (0.65)^x\ (0.35)^{10-x},\ x = 0, 1, 2, ..., 10$$

Probability that at least 7 men out of 10 will live up to 70

$$\begin{aligned} P(X \geq 7) &= P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10) \\ &= \sum_{x=7}^{10} P(X = x) \\ &= \sum_{x=7}^{10}\ {}^{10}C_x\ (0.65)^x\ (0.35)^{10-x} \\ &= 0.5138 \end{aligned}$$

Example 19

In a multiple-choice examination, there are 20 questions. Each question has 4 alternative answers following it and the student must select one correct answer. 4 marks are given for a correct answer and 1 mark is deducted for a wrong answer. A student must secure at least 50% of the maximum possible marks to pass the examination. Suppose a student has not studied at all, so that he answers the questions by guessing only. What is the probability that he will pass the examination?

Solution

Since there are 20 questions and each carries with 4 marks, the maximum marks are 80. If the student solves 12 questions correctly and 8 questions wrongly, he gets 48 – 8 = 40 marks required for passing. If he gets more than 12 correct answers, he gets more than 40 marks. Let p be the probability of getting a correct answer.

$$p = \frac{1}{4},\ \ q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4},\ \ n = 20$$

Probability of getting x correct answers out of 20 answers

$$P(X = x) = {}^nC_x\ p^x\ q^{n-x} = {}^{20}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{20-x},\ \ x = 0, 1, 2, ..., 20$$

Probability of passing the examination, i.e., probability of getting at least 12 correct answers out of 20 answers

$$
\begin{aligned}
P(X \geq 12) &= \sum_{x=12}^{20} P(X = x) \\
&= \sum_{x=12}^{20} {}^{20}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{20-x} \\
&= 9.3539 \times 10^{-4}
\end{aligned}
$$

Example 20

The probability of a man hitting a target is $\frac{1}{3}$. (i) If he fires 5 times, what is the probability of his hitting the target at least twice? (ii) How many times must he fire so that the probability of his hitting the target at least once is more than 90%?

Solution

Let p be probability of hitting a target.

$$p = \frac{1}{3}, \quad q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}, \quad n = 5$$

Probability of hitting the target x times out of 5 times

$$P(X = x) = {}^nC_x\, p^x\, q^{n-x} = {}^5C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{5-x}, \quad x = 0, 1, 2, \ldots, 5$$

(i) Probability of hitting the target at least twice out of 5 times

$$
\begin{aligned}
P(X \geq 2) &= P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) \\
&= \sum_{x=2}^{5} P(X = x) \\
&= \sum_{x=2}^{5} {}^5C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{5-x} \\
&= \frac{131}{243} \\
&= 0.5391
\end{aligned}
$$

(ii) Probability of hitting the target at least once out of 5 times

$$
\begin{aligned}
P(X \geq 1) &> 0.9 \\
1 - P(X = 0) &> 0.9 \\
1 - {}^nC_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^n &> 0.9 \\
1 - \left(\frac{2}{3}\right)^n &> 0.9
\end{aligned}
$$

For $n = 6,\ 1-\left(\frac{2}{3}\right)^6 = 0.9122$

Hence, the man must fire 6 times so that the probability of hitting the target at lest once is more than 90%.

Example 21

In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 *is* 2. *Out of* 1000 *such samples, how many would be expected to contain exactly two defective parts?* **[Summer 2015]**

Solution

Let p be the probability of parts being defective.

$$\mu = np = 2, \qquad n = 20, \qquad N = 1000$$

$$np = 2$$

$$20(p) = 2$$

$$\therefore \quad p = 0.1$$

$$q = 1 - p = 1 - 0.1 = 0.9$$

Probability that the samples contain x defective parts out of 20 parts

$$P(X = x) = {}^nC_x p^x\ q^{n-x} = {}^{20}C_x (0.1)^x (0.9)^{20-x},\ x = 0, 1, 2, ..., 20$$

Probability that the samples contain exactly 2 defective parts

$$P(X = 2) = {}^{20}C_2 (0.1)^2\ (0.9)^{18} = 0.2852$$

Expected number of samples to contain exactly 2 defective parts $= N\,P(X = 2)$

$$= 1000\ (0.2852)$$

$$= 285.2$$

$$\approx 285$$

Example 22

An irregular 6*-faced die is thrown such that the probability that it gives* 3 *even numbers in* 5 *throws is twice the probability that it gives* 2 *even numbers in* 5 *throws. How many sets of exactly* 5 *trials can be expected to give no even number out of* 2500 *sets?*

Solution

Let p be the probability of getting an even number in a throw of a die.

$$n = 5, \qquad N = 2500$$

Probability of getting x even numbers in 5 throws of a die

$$P(X = x) = {}^nC_x \; p^x \; q^{n-x} = {}^5C_x \; p^x q^{5-x}, \; x = 0, 1, 2, ..., 5$$

$$P(X = 3) = 2\; P(X = 2)$$

$${}^5C_3 \; p^3 \; q^2 = 2\left({}^5C_2 \; p^2 \; q^3\right)$$

$$10\; p^3 \; q^2 = 20\; p^2 \; q^3$$

$$p = 2q$$

$$p = 2(1 - p) = 2 - 2p$$

$$\therefore \quad p = \frac{2}{3}$$

$$q = 1 - p = 1 - \frac{2}{3} = \frac{1}{3}$$

Probability of getting no even number in 5 throws of a die

$$P(X = 0) = {}^5C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^5 = \frac{1}{243}$$

Expected number of sets $= NP\;(X = 0)$

$$= \frac{2500}{243}$$

Example 23

Out of 800 families with 5 children each, how many would you expect to have (i) 3 boys? (ii) 5 girls? (iii) either 2 or 3 boys? (iv) at least one boy? Assume equal probabilities for boys and girls.

Solution

Let p be the probability of having a boy in each family.

$$p = \frac{1}{2}, \quad q = 1 - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}, \quad n = 5, \quad N = 800$$

Probability of having x boys out of 5 children in each family

$$P(X = x) = {}^nC_x \; p^x \; q^{n-x} = {}^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}, \; x = 0, 1, 2, ..., 5$$

(i) Probability of having 3 boys out of 5 children in each family

$$P(X = 3) = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 = \frac{5}{16}$$

Expected number of families having 3 boys out of 5 children $= N\,P(X = 3)$

$$= 800\left(\frac{5}{16}\right)$$

$$= 250$$

(ii) Probability of having 5 girls, i.e., no boys out of 5 children in each family

$$P(X=0) = {}^5C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

Expected number of families 5 girls out of 5 children $= NP(X=0)$

$$= 800\left(\frac{1}{32}\right)$$

$$= 25$$

(iii) Probability of having either 2 or 3 boys out of 5 children in each family

$$P(X=2)+P(X=3) = \sum_{x=2}^{3} P(X=x)$$

$$= \sum_{x=2}^{3} {}^5C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{5-x}$$

$$= \frac{5}{8}$$

Expected number of families having either 2 of 3 boys out of 5 children

$$= N[P(X=2)+P(X=3)]$$

$$= 800\left(\frac{5}{8}\right)$$

$$= 500$$

(iv) Probability of having at least one boy out of 5 children in each family

$$P(X\ge 1) = P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$$

$$= \sum_{x=1}^{5} P(X=x)$$

$$= \sum_{x=1}^{5} {}^5C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{5-x}$$

$$= \frac{31}{32}$$

Expected number of families having at least-one boy out of 5 children

$$= NP(X\ge 1)$$

$$= 800\left(\frac{31}{32}\right)$$

$$= 775$$

Example 24

If hens of a certain breed lay eggs on 5 *days a week on an average, find how many days during a season of* 100 *days a will poultry keeper with* 5 *hens of this breed expect to receive at least* 4 *eggs.*

Solution

Let p be the probability of hen laying an egg on any day of a week.

$$p = \frac{5}{7}, \quad q = 1 - p = 1 - \frac{5}{7} = \frac{2}{7}, \quad n = 5, \quad N = 100$$

Probability of x hens laying eggs on any day of a week

$$P(X = x) = {}^nC_x \; p^x \; q^{n-x} = {}^5C_x \left(\frac{5}{7}\right)^x \left(\frac{2}{7}\right)^{5-x}, \; x = 0, 1, 2, ..., 5$$

Probability of receiving at least 4 eggs on any day of a week

$$\begin{aligned} P(X \geq 4) &= P(X = 4) + P(X = 5) \\ &= \sum_{x=4}^{5} P(X = x) \\ &= \sum_{x=4}^{5} {}^5C_x \left(\frac{5}{7}\right)^x \left(\frac{2}{7}\right)^{5-x} \\ &= 0.5578 \end{aligned}$$

Expected number of days during a season of 100 days, a poultry keeper with 5 hens of this breed will receive at least 4 eggs

$$\begin{aligned} &= N\,P(X \geq 4) \\ &= 100\,(0.5578) \\ &= 55.78 \\ &\approx 56 \end{aligned}$$

Example 25

Seven unbiased coins are tossed 128 times and the number of heads obtained is noted as given below:

No. of heads	0	1	2	3	4	5	6	7
Frequency	7	6	19	35	30	23	7	1

Fit a binomial distribution to the data.

Solution

Since the coin is unbiased,

$$p = \frac{1}{2}, \quad q = \frac{1}{2}, n = 7, N = 128$$

For binomial distribution,

$$P(X = x) = {}^nC_x \; p^x \; q^{n-x} = {}^7C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{7-x}, \; x = 0, 1, 2, ..., 7$$

Theoretical or expected frequency $f(x) = N\,P(X = x)$

$$f(x) = 128\ {}^7C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{7-x} = 128\ {}^7C_x\left(\frac{1}{2}\right)^7$$

$$f(0) = 128\ {}^7C_0\left(\frac{1}{2}\right)^7 = 1$$

$$f(1) = 128\ {}^7C_1\left(\frac{1}{2}\right)^7 = 7$$

$$f(2) = 128\ {}^7C_2\left(\frac{1}{2}\right)^7 = 21$$

$$f(3) = 128\ {}^7C_3\left(\frac{1}{2}\right)^7 = 35$$

$$f(4) = 128\ {}^7C_4\left(\frac{1}{2}\right)^7 = 35$$

$$f(5) = 128\ {}^7C_5\left(\frac{1}{2}\right)^7 = 21$$

$$f(6) = 128\ {}^7C_6\left(\frac{1}{2}\right)^7 = 7$$

$$f(7) = 128\ {}^7C_7\left(\frac{1}{2}\right)^7 = 1$$

Binomial distribution

No. of heads x	0	1	2	3	4	5	6	7
Expected binomial frequency $f(x)$	1	7	21	35	35	21	7	1

Example 26

Fit a binomial distribution to the following data:

x	0	1	2	3	4	5
f	2	14	20	34	22	8

Solution

$$\text{Mean} = \frac{\sum fx}{\sum f}$$

$$= \frac{2(0)+14(1)+20(2)+34(3)+22(4)+8(5)}{2+14+20+34+22+8}$$

$$= \frac{284}{100}$$
$$= 2.84$$

For binomial distribution,

$$n = 5$$
$$\mu = np = 2.84$$
$$5p = 2.84$$
$$\therefore \quad p = 0.568$$
$$q = 1 - p = 1 - 0.568 = 0.432$$

$$P(X = x) = {}^nC_x\ p^x\ q^{n-x} = {}^5C_x (0.568)^x (0.432)^{5-x}, \quad x = 0, 1, 2, ..., 5$$
$$N = \sum f = 100$$

Theoretical or expected frequency $f(x) = N\ P(X = x)$

$$f(x) = 100\ {}^5C_x\ (0.568)^x (0.432)^{5-x}$$

$$f(0) = 100\ {}^5C_0 (0.568)^0\ (0.432)^5 = 1.505 \approx 1.5$$
$$f(1) = 100\ {}^5C_1 (0.568)^1\ (0.432)^4 = 9.89 \approx 10$$
$$f(2) = 100\ {}^5C_2 (0.568)^2\ (0.432)^3 = 26.01 \approx 26$$
$$f(3) = 100\ {}^5C_2 (0.568)^3\ (0.432)^2 = 34.2 \approx 34$$
$$f(4) = 100\ {}^5C_2 (0.568)^4\ (0.432)^1 = 22.48 \approx 22$$
$$f(5) = 100\ {}^5C_2 (0.568)^5\ (0.432)^0 = 5.91 \approx 6$$

Binomial Distribution

x	0	1	2	3	4	5
Expected binomial frquency	1.5	10	26	34	22	6

EXERCISE 2.5

1. Find the fallacy if any in the following statements:

(a) The mean of a binomial distribution is 6 and SD is 4.

(b) The mean of a binomial distribution is 9 and its SD is 4.

$$\left[\begin{array}{r} \textbf{Ans.:}\ \text{(a) False, } q = \dfrac{8}{3} \text{ is impossible} \\ \text{(b) False, } q = \dfrac{19}{9} \text{ is impossible} \end{array}\right]$$

2. The mean and variance of a binomial distribution are 3 and 1.2 respectively. Find n, p, and $P(X < 4)$.

$$\left[\textbf{Ans.: } 5, 0.6, \frac{2068}{3125}\right]$$

3. Find the binomial distribution if the mean is 5 and the variance is $\frac{10}{3}$. Find $P(X = 2)$.

$$\left[\textbf{Ans.: } P(X = x) = {}^{25}C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{25-x}, 0.003\right]$$

4. In a binomial distribution, the mean and variance are 4 and 3 respectively. Find $P(X \geq 1)$.

$$\left[\textbf{Ans.: } 0.9899\right]$$

5. The odds in favour of X winning a game against Y are 4:3. Find the probability of Y winning 3 games out of 7 played.

$$\left[\textbf{Ans.: } 0.0929\right]$$

6. On an average, 3 out of 10 students fail in an examination. What is the probability that out of 10 students that appear for the examination none will fail?

$$\left[\textbf{Ans.: } 0.0282\right]$$

7. If on the average rain falls on 10 days in every thirty, find the probability (i) that the first three days of a week will be fine and remaining wet, and (ii) that rain will fall on just three days of a week.

$$\left[\textbf{Ans.: } \text{(i) } \frac{8}{2187} \text{ (ii) } \frac{280}{2187}\right]$$

8. Two unbiased dice are thrown three times. Find the probability that the sum nine would be obtained (i) once, and (ii) twice.

$$\left[\textbf{Ans.: } \text{(i) } 0.26 \text{ (ii) } 0.03\right]$$

9. For special security in a certain protected area, it was decided to put three lightbulbs on each pole. If each bulb has probability p of burning out in the first 100 hours of service, calculate the probability that at least one of them is still good after 100 hours. If $p = 0.3$, how many bulbs would be needed on each pole to ensure with 99% safety that at least one is good after 100 hours?

$$\left[\textbf{Ans.: } \text{(i) } 1 - p^3 \text{ (ii) } 4\right]$$

10. It is known from past records that 80% of the students in a school do their homework. Find the probability that during a random check of 10 students, (i) all have done their homework, (ii) at the most two have not done their homework, and (iii) at least one has not done the homework.

[**Ans.:** (i) 0.1074 (ii) 0.6778 (iii) 0.8926]

11. An insurance salesman sells policies to 5 men, all of identical age and good health. According to the actuarial tables, the probability that a man of this particular age will be alive 30 years hence is $\frac{2}{3}$. Find the probability that 30 years hence (i) at least 1 man will be alive, (ii) at least 3 men will be alive, and (iii) all 5 men will be alive.

[**Ans.:** (i) $\frac{242}{243}$ (ii) $\frac{64}{81}$ (iii) $\frac{32}{243}$]

12. A company has appointed 10 new secretaries out of which 7 are trained. If a particular executive is to get three secretaries selected at random, what is the chance that at least one of them will be untrained?

[**Ans.:** 0.7083]

13. The overall pass rate in a university examination is 70%. Four candidates take up such an examination. What is the probability that (i) at least one of them will pass? (ii) all of them will pass the examination?

[**Ans.:** (i) 0.9919 (ii) 0.7599]

14. The normal rate of infection of a certain disease in animals is known to be 25%. In an experiment with a new vaccine, it was observed that none of the animals caught the infection. Calculate the probability of the observed result.

[**Ans.:** $\frac{729}{4096}$]

15. Suppose that weather records show that on the average, 5 out of 31 days in October are rainy days. Assuming a binomial distribution with each day of October as an independent trial, find the probability that the next October will have at most three rainy days.

[**Ans.:** 0.2403]

16. Assuming that half the population of a village is female and assuming that 100 samples each of 10 individuals are taken, how many samples would you expect to have 3 or less females?

[**Ans.:** 17]

17. Assuming that half the population of a town is vegetarian so that the chance of an individual being vegetarian is $\frac{1}{2}$, and assuming that 100 investigators can take a sample of 10 individuals to see whether they are vegetarians, how many investigators would you expect to report that three people or less in the sample were vegetarians?

$\left[\textbf{Ans.: } 17\right]$

18. The probability of failure in a physics practical examination is 20%. If 25 batches of 6 students each take the examination, in how many batches of 4 or more students would pass?

$\left[\textbf{Ans.: } 23\right]$

19. A lot contains 1% defective items. What should be the number of items in a lot so that the probability of finding at least one defective item in it is at least 0.95?

$\left[\textbf{Ans.: } 299\right]$

20. The probability that a bomb will hit the target is 0.2. Two bombs are required to destroy the target. If six bombs are used, find the probability that the target will be destroyed.

$\left[\textbf{Ans.: } 0.3447\right]$

21. Out of 1000 families with 4 children each, how many would you expect to have (i) 2 boys and 2 girls? (ii) at least one boy? (iii) no girl? (iv) at most 2 girls?

$\left[\textbf{Ans.: } \text{(i) 375 (ii) 938 (iii) 63 (iv) 69}\right]$

22. In a sampling of a large number of parts produced by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many samples would you expect to contain at least 3 defectives?

$\left[\textbf{Ans.: } 323\right]$

23. Five pair coins are tossed 3200 times, find the frequency distribution of the number of heads obtained. Also, find the mean and SD.

$\left[\textbf{Ans.: } \text{(i) 100, 500, 1000, 1000, 500, 100 (ii) 1600 (iii) 28.28}\right]$

24. Fit a binomial distribution to the following data:

x	0	1	2	3	4
f	12	66	109	59	10

$\left[\textbf{Ans.: } 17, 67, 96, 61, 15\right]$

2.10 POISSON DISTRIBUTION

Poisson distribution is a limiting case of binomial distribution under the following conditions:

(i) The number of trials should be infinitely large, i.e., $n \to \infty$.
(ii) The probability of successes p for each trial should be very small, i.e., $p \to 0$.
(iii) $np = \lambda$ should be finite where λ is a constant.

The binomial distribution is

$$P(X = x) = {}^nC_x\, p^x\, q^{n-x}$$

$$= {}^nC_x \left(\frac{p}{q}\right)^x q^n$$

$$= {}^nC_x \left(\frac{p}{1-p}\right)^x (1-p)^n$$

Putting $p = \dfrac{\lambda}{n}$,

$$P(X = x) = \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!}\left(\frac{\frac{\lambda}{n}}{1-\frac{\lambda}{n}}\right)^x \left(1-\frac{\lambda}{x}\right)^n$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!}\frac{\lambda^x}{n^x}\frac{1}{\left(1-\frac{\lambda}{n}\right)^x}\left(1-\frac{\lambda}{x}\right)^n$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!}\frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left[1-\left(\frac{x-1}{n}\right)\right]}{x!}\lambda^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

Since $\lim\limits_{n\to\infty}\left(1-\dfrac{\lambda}{n}\right)^{n-x} = e^{-\lambda}$

and $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right) = \lim\limits_{n\to\infty}\left(1-\dfrac{2}{n}\right) = 1$

Taking the limits of both the sides as $n \to \infty$,

$$P(X = x) = \frac{\lambda^x\, e^{-\lambda}}{x!}, \quad x = 0, 1, 2, ..., \infty$$

A random variable X is said to follow poisson distribution if the probability of x is given by

$$P(X = x) = p(x) = \frac{e^{-\lambda}\,\lambda^{x}}{x!}, \quad x = 0, 1, 2, \ldots$$

where λ is called the *parameter of the distribution*.

2.10.1 Examples of Poisson Distribution

(i) Number of defective bulbs produced by a reputed company
(ii) Number of telephone calls per minute at a switchboard
(iii) Number of cars passing a certain point in one minute
(iv) Number of printing mistakes per page in a large text
(v) Number of persons born blind per year in a large city

2.10.2 Conditions of Poisson Distribution

The Poisson distribution holds under the following conditions:
(i) The random variable X should be discrete.
(ii) The numbers of trials n is very large.
(iii) The probability of success p is very small (very close to zero).
(iv) $\lambda = np$ is finite.
(v) The occurrences are rare.

2.10.3 Constants of the Poisson Distribution

1. Mean of the Poisson Distribution

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{\infty} x\, p(x) \\
&= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\,\lambda^{x}}{x!} \\
&= \sum_{x=0}^{\infty} \frac{x e^{-\lambda}\,\lambda\,\lambda^{x-1}}{x!} \\
&= e^{-\lambda}\cdot\lambda \sum_{x=1}^{\infty} \frac{x\,\lambda^{x-1}}{x!} \\
&= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \qquad \left[\because \quad \frac{x}{x!} = \frac{1}{(x-1)!}\right] \\
&= \lambda e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \cdots\right) \\
&= \lambda e^{-\lambda}\, e^{\lambda} \\
&= \lambda
\end{aligned}
$$

2. Variance of the Poisson Distribution

$$
\begin{aligned}
\operatorname{Var}(X) &= E(X^2) - \mu^2 \\
&= \sum_{x=0}^{\infty} x^2 \; p(x) - \mu^2 \\
&= \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!} - \lambda^2 \\
&= \sum_{x=0}^{\infty} x\big[(x-1)+x\big]\frac{e^{-\lambda}\lambda^x}{x!} - \lambda^2 \\
&= \sum_{x=0}^{\infty} \frac{x(x-1)e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{x e^{-\lambda}\lambda^x}{x!} - \lambda^2 \\
&= \sum_{x=0}^{\infty} \frac{x(x-1)e^{-\lambda}\,\lambda^{x-2}\,\lambda^2}{x(x-1)(x-2)\cdots 1} + \lambda - \lambda^2 \\
&= e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda - \lambda^2 \\
&= e^{-\lambda}\,\lambda^2 \left(1 + \lambda + \frac{\lambda^2}{2!} + \cdots\right) + \lambda - \lambda^2 \\
&= -e^{\lambda}\, e^{-\lambda}\, \lambda^2 + \lambda - \lambda^2 \\
&= \lambda^2 + \lambda - \lambda^2 \\
&= \lambda
\end{aligned}
$$

3. Standard Deviation of the Poisson Distribution

$$\text{SD} = \sqrt{\text{Variance}} = \sqrt{\lambda}$$

4. Mode of the Poisson Distribution

Mode is the value of x for which the probability $p(x)$ is maximum.

$$p(x) \ge p(x+1) \text{ and } p(x) \ge p(x-1)$$

When $p(x) \ge p(x+1)$,

$$
\begin{aligned}
\frac{e^{-\lambda}\,\lambda^x}{x!} &\ge \frac{e^{-\lambda}\,\lambda^{x+1}}{(x+1)!} \\
1 &\ge \frac{\lambda}{x+1} \\
(x+1) &\ge \lambda \\
x &\ge \lambda - 1 \qquad \text{...(2.1)}
\end{aligned}
$$

Similarly, for $p(x) \ge p(x-1)$,

$$x \le \lambda \qquad \text{...(2.2)}$$

Combining Eqs (2.1) and (2.2),

$$\lambda - 1 \le x \le \lambda$$

Hence, the mode of the Poisson distribution lies between $\lambda - 1$ and λ.

Case I If λ is an integer then $\lambda - 1$ is also an integer. The distribution is bimodal and the two modes are $\lambda - 1$ and λ.

Case II If λ is not an integer, the distribution is unimodal and the mode of the Poisson distribution is an integral part of λ. The mode is the integer between $\lambda - 1$ and λ.

2.10.4 Recurrence Relation for the Poisson Distribution

For the Poisson distribution,

$$p(x) = \frac{e^{-\lambda}\,\lambda^x}{x!}$$

$$p(x+1) = \frac{e^{-\lambda}\,\lambda^{x+1}}{(x+1)!}$$

$$\frac{p(x+1)}{p(x)} = \frac{e^{-\lambda}\,\lambda^{x+1}}{(x+1)!} \cdot \frac{x!}{e^{-\lambda}\,\lambda^x}$$

$$= \frac{\lambda}{x+1}$$

$$p(x+1) = \frac{\lambda}{x+1}\,p(x)$$

Example 1

Find out the fallacy if any in the statement. "The mean of a Poisson distribution is 2 and the variance is 3."

Solution

In a Poisson distribution, the mean and variance are same. Hence, the above statement is false.

Example 2

If the mean of the Poisson distribution is 4, find

$$P(\lambda - 2\sigma < X < \lambda + 2\sigma).$$

Solution

For a Poisson distribution,

$$\text{Variance} = \lambda$$

$$\text{Mean} = \lambda = 4, \qquad \sigma = 2$$

$$P(X = x) = \frac{e^{-\lambda}\ \lambda^x}{x!} = \frac{e^{-4}4^x}{x!}, \qquad x = 0, 1, 2, \ldots$$

$$\begin{aligned} P(\lambda - 2\sigma < X < \lambda + 2\sigma) &= P(0 < X < 8) \\ &= \sum_{x=1}^{7} P(X = x) \\ &= \sum_{x=1}^{7} \frac{e^{-4}4^x}{x!} \\ &= 0.9306 \end{aligned}$$

Example 3

If the mean of a Poisson variable is 1.8*, find (i) P(X >* 1)*, (ii) P(X =* 5)*,* and *(iii)* $P(0 < X < 5)$.

Solution

For a Poisson distribution,

$$\lambda = 1.8$$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-1.8}1.8^x}{x!}, \quad x = 0, 1, 2, \ldots$$

(i) $$\begin{aligned} P(X > 1) &= 1 - P(X \le 1) \\ &= 1 - \left[P(X = 0) + P(X = 1)\right] \\ &= 1 - \sum_{x=0}^{1} P(X = x) \\ &= 1 - \sum_{x=0}^{1} \frac{e^{-1.8}1.8^x}{x!} \\ &= 0.5372 \end{aligned}$$

(ii) $$P(X = 5) = \frac{e^{-1.8}1.8^5}{5!} = 0.026$$

(iii) $$\begin{aligned} P(0 < X < 5) &= P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) \\ &= \sum_{x=1}^{4} P(X = x) \\ &= \sum_{x=1}^{4} \frac{e^{-1.8}1.8^x}{x!} \\ &= 0.7983 \end{aligned}$$

Example 4

If a random variable has a Poisson distribution such that $P(X = 1) = P(X = 2)$, find (i) the mean of the distribution, (ii) $P(X = 4)$, (iii) $P(X \geq 1)$, and (iv) $P(1 < X < 4)$.

Solution

For a Poisson distribution,

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

(i) $P(X = 1) = P(X = 2)$

$$\frac{e^{-\lambda}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!}$$

$$\lambda^2 = 2\lambda$$

$$\lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0$$

$$\lambda = 0 \text{ or } \lambda = 2$$

Since $\lambda \neq 0$, $\quad \lambda = 2$

Hence, $P(X = x) = \dfrac{e^{-\lambda}\lambda^x}{x!} = \dfrac{e^{-2}\,2^x}{x!}, \qquad x = 0, 1, 2, \ldots$

(ii) $$P(X = 4) = \frac{e^{-2}2^4}{4!} = 0.9022$$

(iii) $$\begin{aligned} P(X \geq 1) &= 1 - P(X < 1) \\ &= 1 - P(X = 0) \\ &= 1 - \frac{e^{-2}2^0}{0!} \\ &= 0.8647 \end{aligned}$$

(iv) $$\begin{aligned} P(1 < X < 4) &= P(X = 2) + P(X = 3) \\ &= \sum_{x=2}^{3} P(X = x) \\ &= \sum_{x=2}^{3} \frac{e^{-2}2^x}{x!} \\ &= 0.4511 \end{aligned}$$

Example 5

If X is a Poisson variate such that P(X = 0) = P(X = 1), find P(X = 0) and using recurrence relation formula, find the probabilities at x = 1, 2, 3, 4, *and* 5.

Solution

For a Poisson distribution,

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

$$P(X = 0) = P(X = 1)$$

$$\frac{e^{-\lambda}\lambda^0}{0!} = \frac{e^{-\lambda}\lambda^1}{1!}$$

$$\lambda = 1$$

Hence, $$P(X = x) = \frac{e^{-\lambda}1^x}{x!}, \qquad x = 0, 1, 2, \ldots$$

(i) $$P(X = 0) = \frac{e^{-\lambda}\lambda^0}{0!} = 0.3678$$

(ii) By recurrence relation,

$$p(x+1) = \frac{\lambda}{x+1}p(x)$$

$$p(x+1) = \frac{1}{x+1}p(x) \qquad [\because \quad \lambda = 1]$$

$$p(1) = p(0) = 0.3678$$

$$p(2) = \frac{1}{2}p(1) = \frac{1}{2}(0.3678) = 0.1839$$

$$p(3) = \frac{1}{3}p(2) = \frac{1}{3}(0.1839) = 0.0613$$

$$p(4) = \frac{1}{4}p(3) = \frac{1}{4}(0.0613) = 0.015325$$

$$p(5) = \frac{1}{5}p(4) = \frac{1}{5}(0.015325) = 0.003065$$

Example 6

If the variance of a Poisson variate is 3, *find the probability that (i)* $X = 0$, *(ii)* $0 < X \le 3$, *and (iii)* $1 \le X < 4$.

Solution

For a Poisson distribution,

Variance = Mean = $\lambda = 3$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-3}3^x}{x!}, \quad x = 0, 1, 2, \ldots$$

(i) $P(X = 0) = \frac{e^{-3}3^0}{0!} = 0.0498$

(ii)
$$\begin{aligned} P(0 < X \le 3) &= P(X = 1) + P(X = 2) + P(X = 3) \\ &= \sum_{x=1}^{3} P(X = x) \\ &= \sum_{x=1}^{3} \frac{e^{-3}3^x}{x!} \\ &= 0.5974 \end{aligned}$$

(iii)
$$\begin{aligned} P(1 \le X < 4) &= P(X = 1) + P(X = 2) + P(X = 3) \\ &= \sum_{x=1}^{3} P(X = x) \\ &= \sum_{x=1}^{3} \frac{e^{-3}3^x}{x!} \\ &= 0.5974 \end{aligned}$$

Example 7

If a Poisson distribution is such that $\frac{3}{2}P(X=1) = P(X=3)$, *find (i)* $P(X \ge 1)$, *(ii)* $P(X \le 3)$, *and (iii)* $P(2 \le X \le 5)$.

Solution

For a Poisson distribution,

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

$$\frac{3}{2}P(X = 1) = P(X = 3)$$

$$\frac{3}{2}\frac{e^{-\lambda}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^3}{3!}$$

$$\frac{3}{2}\lambda = \frac{\lambda^3}{6}$$

$$\lambda^3 - 9\lambda = 0$$

$$\lambda(\lambda^2 - 9) = 0$$
$$\lambda = 0, 3, -3$$

Since $\lambda > 0$, $\lambda = 3$

Hence, $P(x = x) = \dfrac{e^{-3} 3^x}{x!}, \qquad x = 0, 1, 2, ...$

(i) $$\begin{aligned} P(X \geq 1) &= 1 - P(X < 1) \\ &= 1 - P(X = 0) \\ &= 1 - \frac{e^{-3} 3^0}{0!} \\ &= 0.9502 \end{aligned}$$

(ii) $$\begin{aligned} P(X \leq 3) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\ &= \sum_{x=0}^{3} P(X = x) \\ &= \sum_{x=0}^{3} \frac{e^{-3} 3^x}{x!} \\ &= 0.6472 \end{aligned}$$

(iii) $$\begin{aligned} P(2 \leq X \leq 5) &= P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) \\ &= \sum_{x=2}^{5} P(X = x) \\ &= \sum_{x=2}^{5} \frac{e^{-3} 3^x}{x!} \\ &= 0.7169 \end{aligned}$$

Example 8

If X is a Poisson variate such that

$$P(X = 2) = 9\, P(X = 4) + 90\, P(X = 6)$$

Find (i) the mean of X, (ii) the variance of X, (iii) P(X $<$ *2), (iv) P(X* $>$ *4), and (v) P(X* $\geq$ *1).*

Solution

For a Poisson distribution,

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, \qquad x = 0, 1, 2, ...$$

$$P(X = 2) = 9P(X = 4) + 90P(X = 6)$$

$$\frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$$

$$= e^{-\lambda}\lambda^2\left(\frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!}\right)$$

$$\frac{1}{2} = \frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!}$$

$$\frac{1}{2} = \frac{3\lambda^2}{8} + \frac{\lambda^4}{8}$$

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

$$\lambda^2 = -\frac{3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2} = 1, -4$$

Since $\lambda > 0$, $\lambda^2 = 1$

(i) Mean $= \lambda = 1$

(ii) Variance $= \lambda = 1$

$$P(X = x) = \frac{e^{-1}1^x}{x!}, \qquad x = 0, 1, 2, \ldots$$

(iii) $P(X < 2) = P(X = 0) + P(X = 1)$

$$= \sum_{x=0}^{1} \frac{e^{-1}1^x}{x!}$$

$$= 0.7358$$

(iv) $P(X > 4) = 1 - P(X \le 4)$

$$= 1 - [P(X = 0) + (X = 1) + P(X = 2) + P(X = 3) + P(X = 4)]$$

$$= 1 - \sum_{x=0}^{4} \frac{e^{-1}1^x}{x!}$$

$$= 0.00366$$

(v) $P(X \ge 1) = 1 - P(X = 0)$

$$= 1 - \frac{e^{-1}1^0}{1!}$$

$$= 0.6321$$

Example 9

If a Poisson distribution is such that $\frac{3}{2}P(X = 1) = P(X = 3)$, *find (i)* $P(X \ge 1)$, *(ii)* $P(X \le 3)$, *and (iii)* $P(2 \le X \le 5)$.

Solution

$$\frac{3}{2}P(X=1)=P(X=3)$$

$$\frac{3}{2}\frac{e^{-\lambda}\lambda^1}{1!}=\frac{e^{-\lambda}\lambda^3}{3!}$$

$$\frac{3}{2}=\frac{\lambda^2}{6}$$

$$\lambda^2=9$$

$$\lambda=\pm 3$$

Since $\lambda>0$, $\lambda=3$

$$P(X=x)=\frac{e^{-3}3^x}{x!}, \qquad x=0,1,2,\ldots$$

(i) $$P(X\geq 1)=1-P(X<1)$$
$$=1-P(X=0)$$
$$=1-\frac{e^{-3}3^0}{0!}$$
$$=0.9502$$

(ii) $$P(X\leq 3)=P(X=0)+P(X=1)+P(X=2)+P(X=3)$$
$$=\sum_{x=0}^{3}P(X=x)$$
$$=\sum_{x=0}^{3}\frac{e^{-3}3^x}{x!}$$
$$=0.6472$$

(iii) $$P(2\leq X\leq 5)=P(X=2)+P(X=3)+P(X=4)+P(X=5)$$
$$=\sum_{x=2}^{5}P(X=x)$$
$$=\sum_{x=2}^{5}\frac{e^{-3}3^x}{x!}$$
$$=0.7169$$

Example 10

If X is a Poisson variate such that

$$3P(X=4)=\frac{1}{2}P(X=2)+P(X=0)$$

Find (i) the mean of X, and (ii) $P(X\leq 2)$.

Solution

(i) For a Poisson distribution,

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

$$3P(X = 4) = \frac{1}{2}P(X = 2) + P(X = 0)$$

$$3\frac{e^{-\lambda}\lambda^4}{4!} = \frac{1}{2}\frac{e^{-\lambda}\lambda^2}{2!} + \frac{e^{-\lambda}\lambda^0}{0!}$$

$$\lambda^4 - 2\lambda^2 - 8 = 0$$

$$(\lambda^2 - 4)(\lambda^2 + 2) = 0$$

$$\lambda = \pm 2 \qquad (\because \quad \lambda \text{ is real})$$

$$\lambda = 2 \qquad (\because \quad \lambda > 0)$$

Mean $= \lambda = 2$

Hence, $\quad P(X = x) = \dfrac{e^{-2}2^x}{x!}, \qquad x = 0, 1, 2, \ldots$

(ii) $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= \sum_{x=0}^{2} P(X = x)$$

$$= \sum_{x=0}^{2} \frac{e^{-2}2^x}{x!}$$

$$= 0.6766$$

Example 11

A manufacturer of cotterpins knows that 5% *of his products are defective. If he sells cotterpins in boxes of* 100 *and guarantees that not more than* 10 *pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality?*

Solution

Let p be the probability of a pin being defective.

$$p = 5\% = 0.05, \qquad n = 100$$

Since p is very small and n is large, Poisson distribution is used.

$$\lambda = np = 100 \times 0.05 = 5$$

Let X be the random variable which denotes the number of defective pins in a box of 100.

Probability of x defective pins in a box of 100

$$P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}=\frac{e^{-5}5^x}{x!}, \quad x=0,1,2,...$$

Probability that a box will fail to meet the guaranteed quality

$$\begin{aligned}P(X>10)&=1-P(X\le 10)\\&=1-\sum_{x=0}^{10}P(X=x)\\&=1-\sum_{x=0}^{10}\frac{e^{-5}5^x}{x!}\\&=0.0137\end{aligned}$$

Example 12

A car-hire firm has two cars, which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with a mean of 1.5. *Calculate the proportion of days on which (i) neither car is used, and (ii) the proportion of days on which some demand is refused.*

Solution

$$\lambda = 1.5$$

Let X be the random variable which denotes the number of demands for a car on each day.

Probability of days on which there are x demands for a car

$$P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}=\frac{e^{-1.5}1.5^x}{x!}, \quad x=0,1,2,...$$

(i) Proportion or probability of days on which neither car is used

$$P(X=0)=\frac{e^{-1.5}1.5^0}{0!}=0.2231$$

(ii) Proportion or probability of days on which some demand is refused

$$\begin{aligned}P(X>2)&=1-P(X\le 2)\\&=1-\sum_{x=0}^{2}P(X=x)\\&=1-\sum_{x=0}^{2}\frac{e^{-1.5}1.5^x}{x!}\\&=0.1912\end{aligned}$$

Example 13

Six coins are tossed 6400 times. Using the Poisson distribution, what is the approximate probability of getting six heads 10 times?

Solution

Let p be the probability of getting one head with one coin.

$$p = \frac{1}{2}$$

Probability of getting 6 heads with 6 coins = $\left(\frac{1}{2}\right)^6 = \frac{1}{64}$

$$n = 6400$$

$$\lambda = np = 6400\left(\frac{1}{64}\right) = 100$$

Probability of getting x heads

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-100}100^x}{x!}, \qquad x = 0, 1, 2, \ldots$$

Probability of getting 6 heads 10 times

$$P(X = 10) = \frac{e^{-100}100^{10}}{10!} = 1.025 \times 10^{-30}$$

Example 14

If 2% *of lightbulbs are defective, find the probability that (i) at least one is defective, and (ii) exactly* 7 *are defective. Also, find* $P(1 < X < 8)$ *in a sample of* 100.

Solution

Let p be the probability of defective bulb.

$$p = 2\% = 0.02$$

$$n = 100$$

Since p is very small and n is large, Poisson distribution is used.

$$\lambda = np = 100(0.02) = 2$$

Let X be the random variable which denotes the number of defective bulbs in a sample of 100.

Probability of x defective bulb in a sample of 100

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-2}2^x}{x!}, \quad x = 0, 1, 2, \ldots$$

(i) Probability that at least one bulb is defective

$$P(X \geq 1) = 1 - P(X = 0)$$
$$= 1 - \frac{e^{-2} 2^0}{0!}$$
$$= 0.8647$$

(ii) Probability that exactly 7 bulbs are defective

$$P(X = 7) = \frac{e^{-2} 2^7}{7!} = 0.0034$$

(iii) $P(1 < X < 8) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7)$

$$= \sum_{x=2}^{7} P(X = x)$$
$$= \sum_{x=2}^{7} \frac{e^{-2} 2^x}{x!}$$
$$= 0.5929$$

Example 15

An insurance company insured 4000 *people against loss of both eyes in a car accident. Based on previous data, the rates were computed on the assumption that on the average,* 10 *persons in* 100000 *will have car accidents each year that result in this type of injury. What is the probability that more than* 3 *of the insured will collect on their policy in a given year?*

Solution

Let p be the probability of loss of both eyes in a car accident.

$$p = \frac{10}{100000} = 0.0001$$
$$n = 4000$$

Since p is very small and n is large, Poisson distribution is used.

$$\lambda = np = 4000(0.0001) = 0.4$$

Let X be the random variable which denotes the number of car accidents in a group of 4000 people.

Probability of x car accidents in a group of 4000 people

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-0.4} 0.4^x}{x!}, \quad x = 0, 1, 2, \ldots$$

Probability that more than 3 of the insured will collect on their policy, i.e., probability of more than 3 car accidents in a group of 4000 people

$$
\begin{aligned}
P(X>3) &= 1-P(X\le 3)\\
&= 1-\left[P(X=0)+(X=1)+P(X=2)+P(X=3)\right]\\
&= 1-\sum_{x=0}^{3} P(X=x)\\
&= 1-\sum_{x=0}^{3}\frac{e^{-0.4}0.4^{x}}{x!}\\
&= 0.00077
\end{aligned}
$$

Example 16

If two cards are drawn from a pack of 52 cards which are diamonds, using Poisson distribution, find the probability of getting two diamonds at least 3 times in 51 consecutive trials of two cards drawing each time.

Solution

Let p be the probability of getting two diamonds from a pack of 52 cards.

$$p=\frac{{}^{13}C_2}{{}^{52}C_2}=\frac{3}{51},\qquad n=51$$

Since p is very small and n is large, Poisson distribution is used.

$$\lambda = np = 51\left(\frac{3}{51}\right)=3$$

Let X be the random variable which denotes the drawing of two diamond cards.

Probability of x trials of drawing two diamond cards in 51 trials

$$P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!}=\frac{e^{-3}3^{x}}{x!},\qquad x=0,1,2,...$$

Probability of getting two diamond cards at least 3 times in 51 trials

$$
\begin{aligned}
P(X\ge 3) &= 1-P(X<3)\\
&= 1-[P(X=0)+P(X=1)+P(X=2)]\\
&= 1-\sum_{x=0}^{2}\frac{e^{-3}3^{x}}{x!}\\
&= 0.5768
\end{aligned}
$$

Example 17

Suppose a book of 585 pages contains 43 typographical errors. If these errors are randomly distributed throughout the book, what is the probability that 10 pages, selected at random, will be free from errors?

Solution

Let p be the probability of errors in a page.

$$p = \frac{43}{585} = 0.0735, \quad n = 10$$

Since p is very small and n is large, Poisson distribution is used.

$$\lambda = np = 10(0.0735) = 0.735$$

Let X be the random variable which denotes the errors in the pages.

Probability of x errors in a page in a book of 585 pages

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.735}0.735^x}{x!}, \quad x = 0, 1, 2, \ldots$$

Probability that a random sample of 10 pages will contain no error.

$$P(X = 0) = \frac{e^{-0.735}0.735^0}{0!} = 0.4795$$

Example 18

A hospital switchboard receives an average of 4 *emergency calls in a* 10-*minute interval. What is the probability that (i) there are at most* 2 *emergency calls? (ii) there are exactly* 3 *emergency calls in an interval of* 10 *minutes?*

Solution

Let p be the probability of receiving emergency calls per minute.

$$p = \frac{4}{10} = 0.4, \quad n = 10$$

$$\lambda = np = 10(0.4) = 4$$

Let X be the random variable which denotes the number of emergency calls per minute.

Probability of x emergency calls per minute

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-4}4^x}{x!}, \quad x = 0, 1, 2, \ldots$$

Probability that there are at most 2 emergency calls

$$\begin{aligned} P(X \le 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= \sum_{x=0}^{2} P(X = x) \\ &= \sum_{x=0}^{2} \frac{e^{-4}4^x}{x!} \\ &= 0.238 \end{aligned}$$

Probability that there are exactly 3 emergency calls

$$P(X=3)=\frac{e^{-4}4^3}{3!}=0.1954$$

Example 19

A manufacturer, who produces medicine bottles, finds that 0.1% of the bottles are defective. The bottles are packed in boxes containing 500 bottles. A drug manufacturer buys 100 boxes from the producer of bottles. Using Poisson distribution, find how many boxes will contain (i) no defective bottles and (ii) at least 2 defective bottles.

Solution

Let p be the probability of deflective bottles.

$$p=0.1\%=0.001$$
$$n=500$$
$$\lambda=np=500(0.001)=0.5$$

Let X be the random variable which denotes the number of defective bottles in a box of 500.

Probability of x defective bottles in a box of 500

$$P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}=\frac{e^{-0.5}\,0.5^x}{x!},\quad x=0,1,2,\ldots$$

(i) Probability of no defective bottles in a box

$$P(X=0)=\frac{e^{-0.5}\,0.5^0}{0!}=0.6065$$

Number of boxes containing no defective bottles

$f(x)=N\,P(x=0)=100(0.6065)\approx 61$

(ii) Probability of at least 2 defective bottles

$$\begin{aligned}P(X\ge 2)&=1-P(X<2)\\&=1-\left[P(X=0)+P(X=1)\right]\\&=1\sum_{x=0}^{1}P(X=x)\\&=1-\sum_{x=0}^{1}\frac{e^{-0.5}\,0.5^x}{x!}\\&=0.0902\end{aligned}$$

Number of boxes containing at least 2 defective bottles

$f(x)=N\,P(X\ge 2)=100\,(0.0902)\approx 9$

Example 20

In a certain factory turning out blades, there is a small chance of $\dfrac{1}{500}$ *for any blade to be defective. The blades are supplied in packets of* 10. *Use the Poisson distribution to calculate the approximate number of packets containing no defective, one defective, and two defective blades in a consignment of* 10000 *packets.*

Solution

Let p be the probability of defective blades in a packet.

$$p = \frac{1}{500}, \qquad n = 10, \qquad N = 10000$$

$$\lambda = np = 10\left(\frac{1}{500}\right) = 0.02$$

Let X be the random variable which denotes the number of defective blades in a packet.

Probability of x defective blades in a packet

$$P(X = x) = \frac{e^{-\lambda}\ \lambda^x}{x!} = \frac{e^{-0.02}\ 0.02^x}{x!}, \quad x = 0, 1, 2, \ldots$$

(i) Probability of no defective blades in a packet

$$P(X = 0) = \frac{e^{-0.02}\ 0.02^0}{0!} = 0.9802$$

Number of packets with no defective blades

$f(x) = N\,P(X = 0) = 10000(0.9802) = 9802$

(ii) Probability of one defective blade in a packet

$$P(X = 1) = \frac{e^{-0.02}\ 0.02^1}{1!} = 0.0196$$

Number of packets with one defective blade

$f(x) = N\,P(X = 1) = 10000\ (0.0196) = 196$

(iii) Probability of two defective blades in a packet

$$P(X = 2) = \frac{e^{-0.02}\ 0.02^2}{2!} = 1.96 \times 10^{-4}$$

Number of packets with 2 defective blades

$f(x) = N\,P(X = 2) = 10000\ (1.96 \times 10^{-4}) = 1.96 \approx 2$

Example 21

The number of accidents in a year attributed to taxi drivers in a city follows Poisson distribution with a mean of 3. *Out of* 1000 *taxi drivers,* *find approximately the number of drivers with (i) no accidents in a year, and (ii) more than 3 accidents in a year.*

Solution

For a Poisson distribution,

$$\lambda = 3, N = 1000$$

Probably of x accidents in year

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-3} 3^x}{x!}, \quad x = 0, 1, 2, \ldots$$

(i) Probability of no accidents in a year

$$P(X = 0) = \frac{e^{-3} 3^0}{0!} = 0.0498$$

Number of drivers with no accidents

$$f(x) = N\,P(X = 0) = 1000(0.0498) = 49.8 \approx 50$$

(ii) Probability of more than 3 accidents in a year

$$\begin{aligned} P(X > 3) &= 1 - P(X \le 3) \\ &= 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)] \\ &= 1 - \sum_{x=0}^{3} P(X = x) \\ &= 1 - \sum_{x=0}^{3} \frac{e^{-3} 3^x}{x!} \\ &= 0.3528 \end{aligned}$$

Number of drivers with more than 3 accidents

$$f(x) = N\,P(X > 3) = 1000\,(0.3528) = 3528 \approx 353$$

Example 22

Fit a Poisson distribution to the following data:

Number of deaths (x)	0	1	2	3	4
Frequency (f)	122	60	15	2	1

Solution

$$\begin{aligned} \text{Mean} &= \frac{\sum fx}{\sum f} \\ &= \frac{122(0) + 60(1) + 15(2) + 2(3) + 1(4)}{122 + 60 + 15 + 2 + 1} \\ &= \frac{100}{200} \\ &= 0.5 \end{aligned}$$

For a Poisson distribution,

$$\lambda = 0.5$$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.5}\,0.5^x}{x!}, \qquad x = 0, 1, 2, 3, 4$$

$$N = \sum f = 100$$

Theoretical or expected frequency $f(x) = N\,P(X = x)$

$$f(x) = \frac{200\,e^{-0.5}\,0.5^x}{x!}$$

$$f(0) = \frac{200\,e^{-0.5}\,0.5^0}{0!} = 121.31 \approx 121$$

$$f(1) = \frac{200\,e^{-0.5}\,0.5^1}{1!} = 60.65 \approx 61$$

$$f(2) = \frac{200\,e^{-0.5}\,0.5^2}{2!} = 15.16 \approx 15$$

$$f(3) = \frac{200\,e^{-0.5}\,0.5^3}{3!} = 2.53 \approx 3$$

$$f(4) = \frac{200\,e^{-0.5}\,0.5^4}{4!} = 0.32 \approx 0$$

Poisson Distribution

Number of deaths (x)	0	1	2	3	4
Expected Poisson frequency $f(x)$	121	61	15	3	0

Example 23

Assuming that the typing mistakes per page committed by a typist follows a Poisson distribution, find the expected frequencies for the following distribution of typing mistakes:

Number of mistakes per page	0	1	2	3	4	5
Number of pages	40	30	20	15	10	5

Solution

$$\text{Mean} = \frac{\sum fx}{\sum f}$$

$$= \frac{40(0) + 30(1) + 20(2) + 15(3) + 10(4) + 5(5)}{40 + 30 + 20 + 15 + 10 + 5}$$

$$= \frac{180}{120}$$
$$= 1.5$$

For a Poisson distribution,

$$\lambda = 1.5$$

$$P(X = x) = \frac{e^{-\lambda}\,\lambda^x}{x!} = \frac{e^{-1.5}\,1.5^x}{x!}, \quad x = 0, 1, 2, 3, 4, 5$$

$$N = \sum f = 120$$

Expected frequency $f(x) = N\,P(X = x)$

$$f(x) = \frac{120\,e^{-1.5}\,1.5^x}{x!}$$

$$f(0) = \frac{120\,e^{-1.5}\,1.5^0}{0!} = 26.78 \approx 27$$

$$f(1) = \frac{120\,e^{-1.5}\,1.5^1}{1!} = 40.16 \approx 40$$

$$f(2) = \frac{120\,e^{-1.5}\,1.5^2}{2!} = 30.12 \approx 30$$

$$f(3) = \frac{120\,e^{-1.5}\,1.5^3}{3!} = 15.06 \approx 15$$

$$f(4) = \frac{120\,e^{-1.5}\,1.5^4}{4!} = 5.65 \approx 6$$

$$f(5) = \frac{120\,e^{-1.5}\,1.5^5}{5!} = 1.69 \approx 2$$

EXERCISE 2.6

1. The mean and variance of a probability distribution is 2. Write down the distribution.

$\left[\textbf{Ans.: } P(X = x) = \frac{e^{-2}\,2^x}{x!}, \quad x = 0, 1, 2, \ldots\right]$

2. In a Poisson distribution, the probability $P(X = 0)$ is 20 per cent. Find the mean of the distribution.

[**Ans.:** 2.9957]

3. If X is a Poisson variate and $P(X = 0) = 6\,P(X = 3)$, find $P(X = 2)$.

[**Ans.:** 0.1839]

4. The standard deviation of a Poisson distribution is 3. Find the probability of getting 3 successes.

[**Ans.:** 0.0149]

5. The probability that a Poisson variable X takes a positive value is $1 - e^{-1.5}$. Find the variance and the probability that X lies between -1.5 and 1.5.

[**Ans.:** 1.5, 0.5578]

6. If 2 per cent bulbs are known to be defective bulbs, find the probability that in a lot of 300 bulbs, there will be 2 or 3 defective bulbs using Poisson distribution.

[**Ans.:** 0.1338]

7. In a certain manufacturing process, 5% of the tools produced turn out to be defective. Find the probability that in a sample of 40 tools, at most 2 will be defective.

[**Ans.:** 0.675]

8. If the probability that an individual suffers a bad reaction from a particular injection is 0.001, determine the probability that out of 2000 individuals (i) exactly three, and (ii) more than two individuals suffer a bad reaction.

[**Ans.:** (i) 0.1804 (ii) 0.3233]

9. It is known from past experience that in a certain plant, there are on the average 4 industrial accidents per year. Find the probability that in a given year, there will be less than 4 accidents. Assume Poisson distribution.

[**Ans.:** 0.43]

10. Find the probability that at most 5 defective fuses will be found in a box of 200 fuses, if experience shows that 2% of such fuses are defective.

[**Ans.:** 0.7851]

11. Assume that the probability of an individual coal minor being killed in a mine accident during a year is $\frac{1}{2400}$. Use appropriate statistical distribution to calculate the probability that in a mine employing 200 miners, there will be at least one fatal accident every year.

[**Ans.:** 0.07]

12. Between the hours of 2 and 4 p.m., the average number of phone calls per minute coming into the switchboard of a company is 2.5. Find the probability that during a particular minute, there will be (i) no phone call at all, (ii) 4 or less calls, and (iii) more than 6 calls.

$\left[\textbf{Ans.:} \text{ (i) } 0.0821 \text{ (ii) } 0.8909 \text{ (iii) } 0.0145\right]$

13. Suppose that a local appliances shop has found from experience that the demand for tubelights is roughly distributed as Poisson with a mean of 4 tubelights per week. If the shop keeps 6 tubelights during a particular week, what is the probability that the demand will exceed the supply during that week?

$\left[\textbf{Ans.:} \ 0.1106\right]$

14. The distribution of the number of road accidents per day in a city is Poisson with a mean of 4. Find the number of days out of 100 days when there will be (i) no accident, (i) at least 2 accidents, and (iii) at most 3 accidents.

$\left[\textbf{Ans.:} \text{ (i) } 2 \text{ (ii) } 91 \text{ (iii) } 44\right]$

15. A manufacturer of electric bulbs sends out 500 lots each consisting of 100 bulbs. If 5% bulbs are defective, in how many lot can we expect (i) 97 or more good bulbs? (ii) less than 96 good bulbs?

$\left[\textbf{Ans.:} \text{ (i) } 62 \text{ (ii) } 132\right]$

16. A firm produces articles, 0.1 per cent of which are defective. It packs them in cases containing 500 articles. If a wholesaler purchases 100 such cases, how many cases can be expectecd (i) to be free from defects? (ii) to have one defective article?

$\left[\textbf{Ans.:} \text{ (i) } 16 \text{ (ii) } 30\right]$

17. In a certain factory producing certain articles, the probability that an article is defective is $\frac{1}{500}$. The articles are supplied in packets of 20. Find approximately the number of packets containing no defective, one defective, two defectives in a consignment of 20000 packets.

$\left[\textbf{Ans.:} \ 19200, 768, 15\right]$

18. In a certain factory manufacturing razor blades, there is a small chance, $\frac{1}{50}$ for any blade to be defective. The blades are placed in packets, each containing 10 blades. Using the Poisson distribution, calculate the approximate number of packets containing not more than 2 defective blades in a consignment of 10000 packets.

$\left[\textbf{Ans.:} \ 9988\right]$

19. It is known that 0.5% of ballpen refills produced by a factory are defective. These refills are dispatched in packaging of equal numbers. Using a Poisson distribution, determine the number of refills in a packing to be sure that at least 95% of them contain no defective refills.

[**Ans.:** 10]

20. A manufacturer finds that the average demand per day for the mechanics to repair his new product is 1.5 over a period of one year and the demand per day is distributed as a Poisson variate. He employs two mechanics. On how many days in one year (i) would both mechanics would be free? (ii) some demand is refused?

[**Ans.:** (i) 81.4 days (ii) 69.8 days]

21. Fit a Poisson distribution to the following data:

X	0	1	2	3	4
f	211	90	19	5	0

[**Ans.:** $\lambda = 0.44$, Frequencies : 209, 92, 20, 3, 1]

22. Fit a Poisson distribution to the following data:

No. of defects per piece	0	1	2	3	4
No. of pieces	43	40	25	10	2

[**Ans.:** Frequencies: 42, 44, 24, 8, 2]

23. Fit a Poisson distribution to the following data:

X	0	1	2	3	4	5
f	142	156	69	27	5	1

[**Ans.:** Frequencies: 147, 147, 74, 24, 6, 2]

24. Fit a Poisson distribution to the following data:

X	0	1	2	3	4	5	6	7	8
f	56	156	132	92	37	22	4	0	1

[**Ans.:** Frequency : 70, 137, 135, 89, 44, 17, 6, 2, 0]

2.11 NORMAL DISTRIBUTION

A continuous random variable X is said to follow normal distribution with mean μ and variance σ^2, if its probability function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$-\infty < X < \infty, -\infty < \mu < \infty, \sigma > 0$$

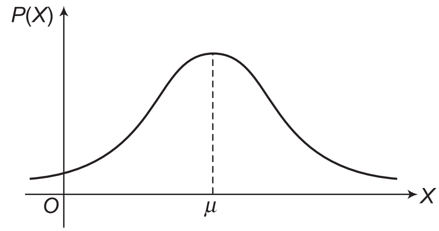


Fig. 2.1

where μ and σ are called parameters of the normal distribution. The curve representing the normal distribution is called the normal curve (Fig. 2.1).

2.11.1 Properties of the Normal Distribution

A normal probability curve, or normal curve, has the following properties:

(i) It is a bell-shaped symmetrical curve about the ordinate $X = \mu$. The ordinate is maximum at $X = \mu$.

(ii) It is a unimodal curve and its tails extend infinitely in both the directions, i.e., the curve is asymptotic to X-axis in both the directions.

(iii) All the three measures of central tendency coincide, i.e.,

mean = median = mode

(iv) The total area under the curve gives the total probability of the random variable X taking values between $-\infty$ to ∞. Mathematically,

$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \mathrm{d}x = 1$$

(v) The ordinate at $X = \mu$ divides the area under the normal curve into two equal parts, i.e.,

$$\int_{-\infty}^{\mu} f(x)\,\mathrm{d}x = \int_{\mu}^{\infty} f(x)\,\mathrm{d}x = \frac{1}{2}$$

(vi) The value of $f(x)$ is always nonnegative for all values of X, i.e., the whole curve lies above the X-axis.

(vii) The points of inflexion (the point at which curvature changes) of the curve are at $X = \mu + \sigma$ and the curve changes from concave to convex at $X = \mu + \sigma$ to $X = \mu - \sigma$.

(viii) The area under the normal curve (Fig. 2.2) is distributed as follows:

(a) The area between the ordinates at $\mu - \sigma$ and $\mu + \sigma$ is 68.27%

(b) The area between the ordinates at $\mu - 2\sigma$ and $\mu + 2\sigma$ is 95.45%

(c) The area between the ordinates at $\mu - 3\sigma$ and $\mu + 3\sigma$ is 99.74%

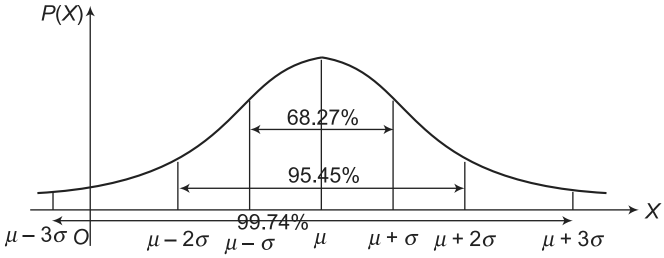


Fig. 2.2

2.11.2 Constants of the Normal Distribution

1. Mean of the Normal Distribution

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\,\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \mathrm{d}x$$

Putting $\dfrac{x-\mu}{\sigma} = t,\ \mathrm{d}x = \sigma\,\mathrm{d}t$

$$E(X) = \int_{-\infty}^{\infty} (\mu + \sigma \mathrm{t}) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\, \mathrm{d}t$$

$$= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\, \mathrm{d}t + \int_{-\infty}^{\infty} \sigma \frac{t}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\, \mathrm{d}t$$

Putting $t^2 = u$ in the second integral,

$2t\ \mathrm{d}t = \mathrm{d}u$

When $t \to \infty, \quad u \to \infty$

When $t \to -\infty, \quad u \to \infty$

$$E(X) = \mu \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} + \int_{\infty}^{\infty} \sigma \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u} \frac{\mathrm{d}u}{2} \qquad \left[\because \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} \mathrm{d}t = \sqrt{2\pi}\right]$$

$$= \mu + 0 \qquad [\because \text{ the limits of integration are same}]$$

$$= \mu$$

2. Variance of the Normal Distribution

$$\mathrm{Var}(X) = E(X - \mu)^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \; f(x)\,\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \,\mathrm{d}x$$

Putting $\dfrac{x-\mu}{\sigma} = t, \mathrm{d}x = \sigma\,\mathrm{d}t$

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} \sigma^2 t^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\,\mathrm{d}t$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^2 e^{-\frac{1}{2}t^2}\,\mathrm{d}t$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} t^2 e^{-\frac{1}{2}t^2}\,\mathrm{d}t \qquad [\because \text{ integral is an even function}]$$

Putting $\dfrac{t^2}{2} = u,$

$$t = \sqrt{2u}$$

$$\mathrm{d}t = \sqrt{2}\frac{1}{2\sqrt{u}}\,\mathrm{d}u = \frac{1}{\sqrt{2u}}\,\mathrm{d}u$$

When $t = 0,\quad u = 0$

When $t = \infty,\quad u = \infty$

$$\mathrm{Var}(X) = \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} 2u\,e^{-u} \frac{1}{\sqrt{2u}}\,\mathrm{d}u$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-u}\, u^{\frac{1}{2}}\,\mathrm{d}u$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \overline{\left|\frac{3}{2}\right.} \qquad \left[\because \int_{0}^{\infty} e^{-x} x^{n-1}\,\mathrm{d}x = \overline{|n}\right]$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \overline{\left|\frac{1}{2}\right.}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2}\sqrt{\pi}$$

$$= \sigma^2$$

3. Standard Deviation of the Normal Distribution

$$\mathrm{SD} = \sigma$$

4. Mode of the Normal Distribution

Mode is the value of x for which $f(x)$ is maximum. Mode is given by

$$f'(x) = 0 \text{ and } f''(x) < 0$$

For normal distribution,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Differentiating w.r.t. x,

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \left[-\left(\frac{x-\mu}{\sigma^2}\right)\right]$$

$$= -\frac{x-\mu}{\sigma^2} f(x)$$

When $\quad f'(x) = 0, \quad x - \mu = 0$

$$x = \mu$$

$$f''(x) = -\frac{1}{\sigma^2}\left[(x-\mu)f'(x) + f(x)\right]$$

$$= -\frac{1}{\sigma^2}\left[(x-\mu)\left\{-\frac{(x-\mu)}{\sigma^2} f(x)\right\} + f(x)\right]$$

$$= -\frac{1}{\sigma^2} f(x)\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right]$$

At $x = \mu$,

$$f''(x) = \frac{f(x)}{\sigma^2} = -\frac{1}{\sigma^3\sqrt{2\pi}} < 0$$

Hence, $x = \mu$ is the mode of the normal distribution.

5. Median of the Normal Distribution

If M is median of the normal distribution,

$$\int_{-\infty}^{M} f(x)\, dx = \frac{1}{2}$$

$$\int_{-\infty}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2} \qquad \text{...(2.3)}$$

Putting $\dfrac{x-\mu}{\sigma} = t$ in the first integral,

$$dx = \sigma\, dt$$

When $x = -\infty$, $t = -\infty$

When $x = \mu$, $t = 0$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}t^2} \sigma\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}t^2} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}t^2} dt \qquad \text{[By symmetry]}$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}}$$

$$= \frac{1}{2} \qquad \text{...(2.4)}$$

From Eqs (2.3) and (2.4),

$$\frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0$$

$$\int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0$$

$$\mu = M \left[\because \quad \text{if} \int_{a}^{b} f(x)\, dx = 0 \text{ then } a = b \text{ where } f(x) > 0\right]$$

Hence, mean = median for the normal distribution.

Note For normal distribution,

mean = median = mode = μ

Hence, the normal distribution is symmetrical.

2.11.3 Probability of a Normal Random Variable in an Interval

Let X be a normal random variable with mean μ and standard deviation σ. The probability of X lying in the interval (x_1, x_2) (Fig. 2.3) is given by

$$P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \mathrm{d}x$$

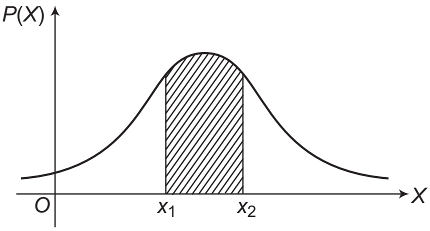


Fig. 2.3

Hence, the probability is equal to the area under the normal curve between the ordinates $X = x_1$ and $X = x_2$ respectively. $P(x_1 < X < x_2)$ can be evaluated easily by converting a normal random variable into another random variable.

Let $\quad Z = \dfrac{X-\mu}{\sigma}$ be a new random variable.

$$E(Z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}\left[E(X)-\mu\right] = 0$$

$$\text{Var}(Z) = \text{Var}\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}\text{Var}(X-\mu) = \frac{1}{\sigma^2}\text{Var}(X) = 1$$

The distribution of Z is also normal. Thus, if X is a normal random variable with mean μ and standard deviation σ then $Z = \dfrac{X-\mu}{\sigma}$ is a normal random variable with mean 0 and standard deviation 1. Since the parameters of the distribution of Z are fixed, it is a known distribution and is termed *standard normal distribution*. Further, Z is termed as a *standard normal variate*. Thus, the distribution of any normal variate X can always be transformed into the distribution of the standard normal variate Z.

$$P(x_1 \le X \le x_2) = P\left[\left(\frac{x_1-\mu}{\sigma}\right) \le \left(\frac{X-\mu}{\sigma}\right) \le \left(\frac{x_2-\mu}{\sigma}\right)\right]$$

$$= P(z_1 \le Z \le z_2)$$

where $\quad z_1 = \dfrac{x_1-\mu}{\sigma}$ and $z_2 = \dfrac{x_2-\mu}{\sigma}$

This probability is equal to the area under the standard normal curve between the ordinates at $Z = z_1$ and $Z = z_2$.

Case I If both z_1 and z_2 are positive (or both negative) (Fig. 2.4),

$$P(x_1 \le X \le x_2)$$
$$= P(z_1 \le Z \le z_2)$$
$$= P(0 \le Z \le z_2) - P(0 \le Z \le z_1)$$
$$= (\text{Area under the normal curve from } 0 \text{ to } z_2)$$
$$- (\text{Area under the normal curve from } 0 \text{ to } z_1)$$

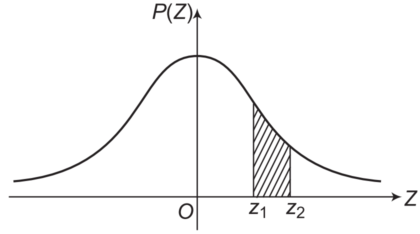


Fig. 2.4

Case II If $z_1 < 0$ and $z_2 > 0$ (Fig. 2.5),

$$P(x_1 \le X \le x_2)$$
$$= P(-z_1 \le Z \le z_2)$$
$$= P(-z_1 \le Z \le 0) + P(0 \le Z \le z_2)$$
$$= P(0 \le Z \le z_1) + P(0 \le Z \le z_2) \quad \text{[By symmetry]}$$
$$= (\text{Area under the normal curve from } 0 \text{ to } z_1)$$
$$+ (\text{Area under the normal curve from } 0 \text{ to } z_2)$$

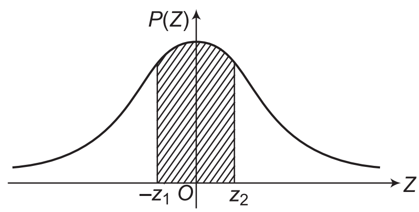


Fig. 2.5

When $X > x_1$, $Z > z_1$, the probability $P(Z > z_1)$ can be found for two cases as follows:

Case I If $z_1 > 0$ (Fig. 2.6),

$$P(X > x_1) = P(Z > z_1)$$
$$= 0.5 - P(0 \le Z \le z_1)$$
$$= 0.5 - (\text{Area under the curve from } 0 \text{ to } z_1)$$

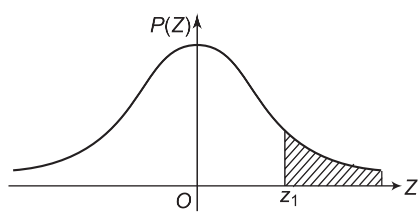


Fig. 2.6

Case II If $z_1 < 0$ (Fig. 2.7),

$$P(X > x_1) = P(Z > -z_1)$$
$$= 0.5 + P(-z_1 < Z < 0)$$
$$= 0.5 + P(0 < Z < z_1) \quad \text{[By symmetry)}$$
$$= 0.5 + (\text{Area under the curve from } 0 \text{ to } z_1)$$

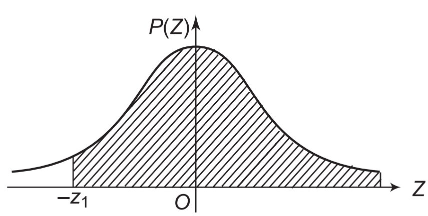


Fig. 2.7

When $X < x_1$, $Z < z_1$, the probability $P(Z < z_1)$ can be found for two cases as follows:

Case I If $z_1 > 0$ (Fig. 2.8),

$$\begin{aligned} P(X < x_1) &= P(Z < z_1) \\ &= 1 - P(Z \geq z_1) \\ &= 1 - \left[0.5 - P(0 < Z < z_1)\right] \\ &= 0.5 + P(0 < Z < z_1) \\ &= 0.5 + (\text{Area under the curve from 0 to } z_1) \end{aligned}$$

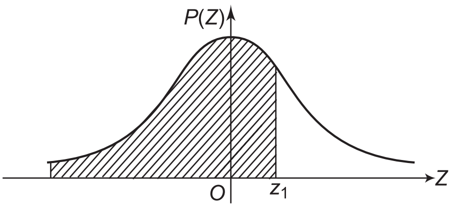


Fig. 2.8

Case II If $z_1 < 0$ (Fig. 2.9),

$$\begin{aligned} P(X < x_1) &= P(Z < -z_1) \\ &= 1 - P(Z \geq -z_1) \\ &= 1 - \left[0.5 + P(-z_1 \leq Z \leq 0)\right] \\ &= 1 - \left[0.5 + P(0 \leq Z \leq z_1)\right] \quad \text{[By symmetry]} \\ &= 0.5 - P(0 \leq Z \leq z_1) \\ &= 0.5 - (\text{Area under the curve from 0 to } z_1) \end{aligned}$$

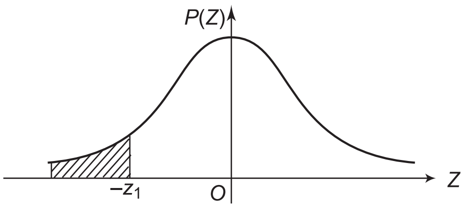


Fig. 2.9

Note

(i) $P(X < x_1) = F(x_1) = \int\limits_{-\infty}^{x_1} f(x)\,\mathrm{d}x$

Hence, $P(X < x_1)$ represents the area under the curve from $X = -\infty$ to $X = x_1$.

(ii) If $P(X < x_1) < 0.5$, the point x_1 lies to the left of $X = \mu$ and the corresponding value of standard normal variate will be negative (Fig. 2.10).

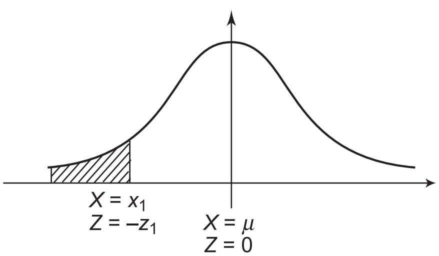


Fig. 2.10

(iii) If $P(X < x_1) > 0.5$, the point x_1 lies to the right of $x = \mu$ and the corresponding value of standard normal variate will be positive (Fig. 2.11).

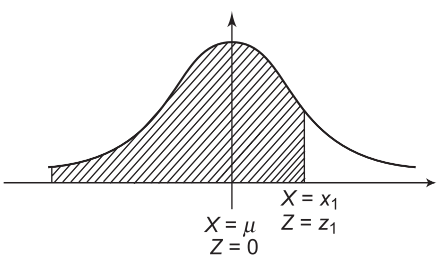
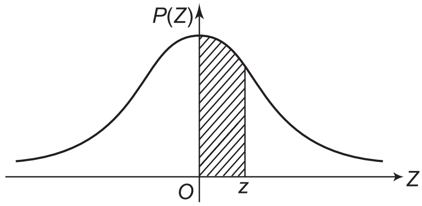


Fig. 2.11

Standard Normal (*Z*) Table, Area between 0 and *z*



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3990	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4115	0.4131	0.4147	0.4162
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

2.11.4 Uses of Normal Distribution

(i) The normal distribution can be used to approximate binomial and Poisson distributions.

(ii) It is used extensively in sampling theory. It helps to estimate parameters from statistics and to find confidence limits of the parameter.

(iii) It is widely used in testing statistical hypothesis and tests of significance in which it is always assumed that the population from which the samples have been drawn should have normal distribution.

(iv) It serves as a guiding instrument in the analysis and interpretation of statistical data.

(v) It can be used for smoothing and graduating a distribution which is not normal simply by contracting a normal curve.

Example 1

What is the probability that a standard normal variate Z will be (i) greater than 1.09*? (ii) less than* –1.65*? (iii) lying between* –1 *and* 1.96*? (iv) lying between* 1.25 *and* 2.75*?*

Solution

(i) $Z > 1.09$ (Fig. 2.12)

$$P(Z > 1.09) = 0.5 - P(0 \le Z \le 1.09)$$
$$= 0.5 - 0.3621$$
$$= 0.1379$$

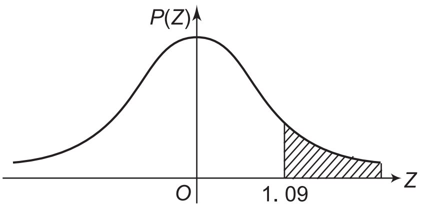


Fig. 2.12

(ii) $Z \le -1.65$ (Fig. 2.13)

$$P(Z \le -1.65) = 1 - P(Z > -1.65)$$
$$= 1 - [0.5 + P(-1.65 < Z < 0)]$$
$$= 1 - [0.5 + P(0 < Z < 1.65)]$$
[By symmetry]
$$= 0.5 - P(0 < Z < 1.65)$$
$$= 0.5 - 0.4505$$
$$= 0.0495$$

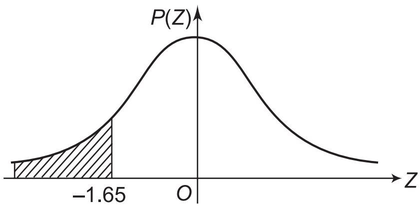


Fig. 2.13

(iii) $-1 < Z < 1.96$ (Fig. 2.14)

$$P(-1 < Z < 1.96)$$
$$= P(-1 < Z < 0) + P(0 < Z < 1.96)$$
$$= P(0 < Z < 1) + P(0 < Z < 1.96)$$
[By symmetry]
$$= 0.3413 + 0.4750$$
$$= 0.8163$$

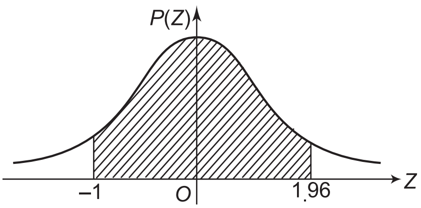


Fig. 2.14

(iv) $1.25 < Z < 2.75$ (Fig. 2.15)

$$P(1.25 < Z < 275)$$
$$= P(0 < Z < 2.75) - P(0 < Z < 1.25)$$
$$= 0.4970 - 0.3944$$
$$= 0.1026$$

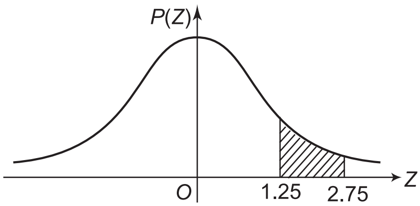


Fig. 2.15

Example 2

If X is a normal variate with a mean of 30 *and an SD of* 5, *find the probabilities that (i)* $26 \le X \le 40$, *and (ii)* $X \ge 45$.

Solution

$$\mu = 30, \qquad \sigma = 5$$

$$Z = \frac{X - \mu}{\sigma}$$

(i) When $X = 26, Z = \dfrac{26 - 30}{5} = -0.8$

When $X = 40, Z = \dfrac{40 - 30}{5} = 2$

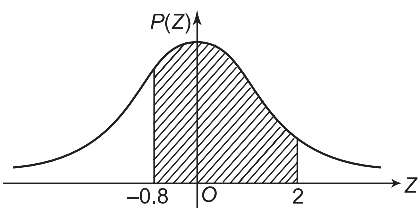


Fig. 2.16

$$P(26 \le X \le 40) = P(-0.8 \le Z \le 2) \text{ (Fig. 2.16)}$$
$$= P(-0.8 \le Z \le 0) + P(0 \le Z \le 2)$$
$$= P(0 \le Z \le 0.8) + P(0 \le Z \le 2) \qquad \text{[By symmetry]}$$
$$= 0.2881 + 0.4772$$
$$= 0.7653$$

(ii) When $X = 45, Z = \dfrac{45 - 30}{5} = 3$

$$P(X \ge 45) = P(Z \ge 3) \text{ (Fig. 2.17)}$$
$$= 0.5 - P(0 < Z < 3)$$
$$= 0.5 - 0.4987$$
$$= 0.0013$$

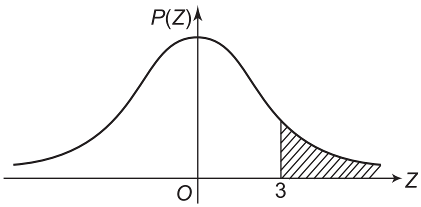


Fig. 2.17

Example 3

X is normally distributed and the mean of X is 12 *and the SD is* 4. *Find out the probability of the following:*

(i) $X \ge 20$ *(ii)* $X \le 20$ *(iii)* $0 \le X \le 12$.

Solution

$$\mu = 12, \qquad \sigma = 4$$

$$Z = \frac{X-\mu}{\sigma}$$

(i) When $X = 20, Z = \frac{20-12}{4} = 2$

$$P(X \geq 20) = P(Z \geq 2) \text{ (Fig. 2.18)}$$
$$= 0.5 - P(0 < Z < 2)$$
$$= 0.5 - 0.4772$$
$$= 0.0228$$

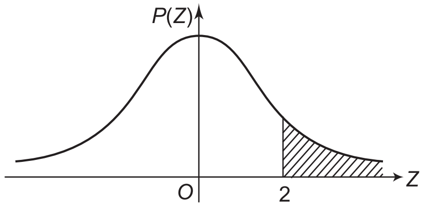


Fig. 2.18

(ii) $P(X \leq 20) = 1 - P(X > 20)$

$$= 1 - 0.0228$$
$$= 0.9772$$

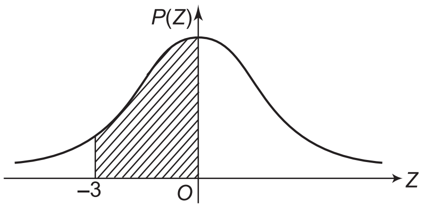


Fig. 2.19

(iii) When $X = 0, Z = \frac{0-12}{4} = -3$

When $X = 12, Z = \frac{12-12}{4} = 0$

$$P(0 \leq X \leq 12) = P(-3 \leq Z \leq 0) \text{ (Fig. 2.19)}$$
$$= P(0 \leq Z \leq 3) \qquad \text{[By symmetry]}$$
$$= 0.4987$$

Example 4

If X is normally distributed with a mean of 2 *and an SD of* 0.1, *find* $P(|X-2|) \geq 0.01)$?

Solution:

$$\mu = 2, \qquad \sigma = 0.1$$

$$Z = \frac{X-\mu}{\sigma}$$

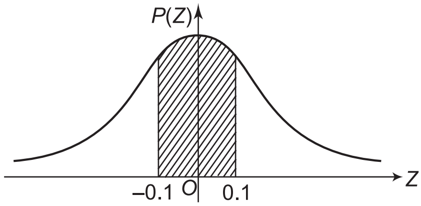


Fig. 2.20

When $X = 1.99, Z = \frac{1.99-2}{0.1} = -0.1$

When $X = 2.01, Z = \frac{2.01-2}{0.1} = 0.1$

$$
\begin{aligned}
P(|X-2| \le 0.01) &= P(1.99 \le X \le 2.01) \text{ (Fig. 2.20)} \\
&= P(-0.1 \le Z \le 0.1) \\
&= P(-0.1 \le Z \le 0) + P(0 \le Z \le 0.1) \\
&= P(0 \le Z \le 0.1) + P(0 \le Z \le 0.1) \qquad \text{[By symmetry]} \\
&= 2P(0 < Z \le 0.1) \\
&= 2(0.0398) \\
&= 0.0796 \\
P(|X-2| \ge 0.01) &= 1 - P(|X-2| < 0.01) \\
&= 1 - 0.0796 \\
&= 0.9204
\end{aligned}
$$

Example 5

If X is a normal variate with a mean of 120 *and a standard deviation of* 10*, find c such that (i)* $P(X > c) = 0.02$*, and (ii)* $P(X < c) = 0.05$.

Solution

For normal variate X,

$$\mu = 120, \quad \sigma = 10$$

$$Z = \frac{X-\mu}{\sigma}$$

(i) $P(X > c) = 0.02$

$$
\begin{aligned}
P(X < c) &= 1 - P(X \ge c) \\
&= 1 - 0.02 \\
&= 0.98
\end{aligned}
$$

Since $P(X < c) > 0.5$, the corresponding value of Z will be positive.

$$
\begin{aligned}
P(X > c) &= P(Z > z_1) \text{ (Fig. 2.21)} \\
0.02 &= 0.5 - P(0 \le Z \le z_1) \\
P(0 \le Z \le z_1) &= 0.48
\end{aligned}
$$

$\therefore$ $z_1 = 2.05$ [From normal table]

$$Z = \frac{c-120}{10} = z_1 = 2.05$$

$$c = 2.05(10) + 120 = 140.05$$

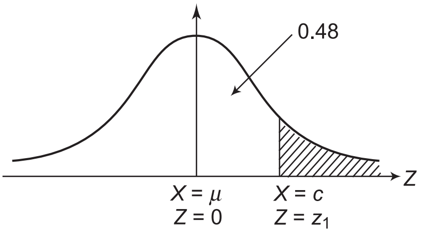


Fig. 2.21

(ii) Since $P(X < c) < 0.5$, the corresponding value of Z will be negative.

$$
\begin{aligned}
P(X < c) &= P(Z < -z_1) \text{ (Fig. 2.22)} \\
0.05 &= 1 - P(Z \ge -z_1) \\
0.05 &= 1 - [0.5 + P(-z_1 \le Z \le 0)]
\end{aligned}
$$

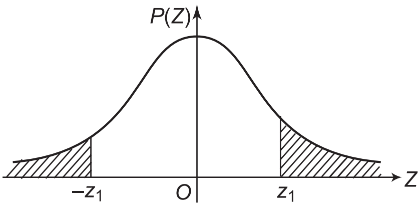


Fig. 2.22

$$0.05 = 1 - \left[0.5 + P(0 \le Z \le z_1)\right] \qquad \text{[By symmetry]}$$
$$0.05 = 0.5 - P(0 \le Z \le z_1)$$
$$P(0 \le Z \le z_1) = 0.5 - 0.05 = 0.45$$
$$\therefore \qquad z_1 = -1.64 \quad \text{[From normal table]}$$
$$Z = \frac{c-120}{10} = z_1 = -1.64$$
$$c = 10(-1.64) + 120 = 103.6$$

Example 6

A manufacturer knows from his experience that the resistances of resistors he produces is normal with $\mu = 100$ *ohms and SD* $= \sigma = 2$ *ohms. What percentage of resistors will have resistances between* 98 *ohms and* 102 *ohms?*

Solution

Let X be the random variable which denotes the resistances of the resistors.

$$\mu = 100, \qquad \sigma = 2$$
$$Z = \frac{X-\mu}{\sigma}$$

When $X = 98$, $\quad Z = \dfrac{98-100}{2} = -1$

When $X = 102$, $\quad Z = \dfrac{102-100}{2} = 1$

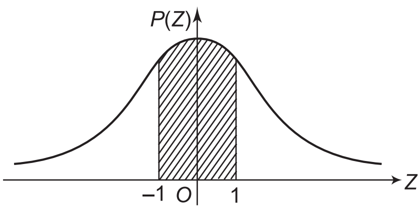


Fig. 2.23

$$\begin{aligned} P(98 \le X \le 102) &= P(-1 \le Z \le 1) \quad \text{(Fig. 2.23)} \\ &= P(-1 \le Z \le 0) + P(0 \le Z \le 1) \\ &= P(0 \le Z \le 1) + P(0 \le Z \le 1) \qquad \text{[By symmetry]} \\ &= 2P(0 \le Z \le 1) \\ &= 2(0.3413) \\ &= 0.6826 \end{aligned}$$

Hence, the percentage of resistors have resistances between 98 ohms and 102 ohms = 68.26%.

Example 7

The average seasonal rainfall in a place is 16 *inches with an SD of* 4 *inches. What is the probability that the rainfall in that place will be between* 20 *and* 24 *inches in a year?*

Solution

Let X be the random variable which denotes the seasonal rainfall in a year.

$$\mu = 16, \qquad \sigma = 4$$

$$Z = \frac{X-\mu}{\sigma}$$

When $X = 20$, $\quad Z = \dfrac{20-16}{4} = 1$

When $X = 24$, $\quad Z = \dfrac{24-16}{4} = 2$

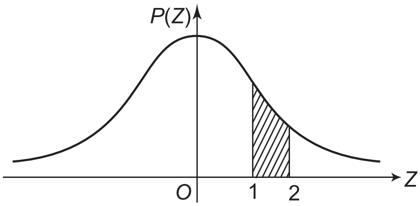


Fig. 2.24

$$\begin{aligned} P(20 < X < 24) &= P(1 < Z < 2) \quad \text{(Fig. 2.24)} \\ &= P(0 < Z < 2) - P(0 < Z < 1) \\ &= 0.4772 - 0.3413 \\ &= 0.1359 \end{aligned}$$

Example 8

The lifetime of a certain kind of batteries has a mean life of 400 *hours and the standard deviation as* 45 *hours. Assuming the distribution of lifetime to be normal, find (i) the percentage of batteries with a lifetime of at least* 470 *hours, (ii) the proportion of batteries with a lifetime between* 385 *and* 415 *hours, and (iii) the minimum life of the best* 5% *of batteries.*

Solution

Let X be the random variable which denotes the lifetime of a certain kind of batteries.

$$\mu = 400, \qquad \sigma = 45$$

$$Z = \frac{X-\mu}{\sigma}$$

(i) When $X = 470$,

$$Z = \frac{470-400}{45} = 1.56$$

$$\begin{aligned} P(X \ge 470) &= P(Z \ge 1.56) \quad \text{(Fig. 2.25)} \\ &= 0.5 - P(0 < Z < 1.56) \\ &= 0.5 - 0.4406 \\ &= 0.0594 \end{aligned}$$

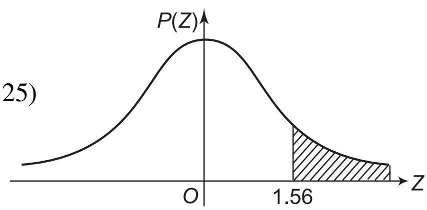


Fig. 2.25

Hence, the percentage of batteries with a lifetime of at least 470 hours = 5.94%.

(ii) When $X = 385$,

$$Z = \frac{385-400}{45} = -0.33$$

When $X = 415$,

$$Z = \frac{415-400}{45} = 0.33$$

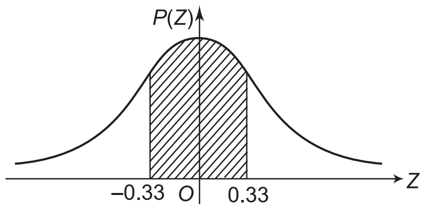


Fig. 2.26

$$\begin{aligned} P(385 < X < 415) &= P(-0.33 < Z < 0.33) \quad \text{(Fig. 2.26)} \\ &= P(-0.33 < Z < 0) + P(0 < Z < 0.33) \\ &= P(0 < Z < 0.33) + P(0 < Z < 0.33) \qquad \text{[By symmetry]} \\ &= 2P(0 < Z < 0.33) \\ &= 2(0.1293) \\ &= 0.2586 \end{aligned}$$

Hence, the proportion of batteries with a lifetime between 385 and 415 hours = 25.86%.

(iii) $P(X > x_1) = 0.05$ (Fig. 2.27)

$P(X > x_1) = P(Z > z_1)$

$0.05 = 0.5 - P(0 \le Z \le z_1)$

$P(0 \le Z \le z_1) = 0.5 - 0.05 = 0.45$

$\therefore$ $z_1 = 1.65$ [From normal table]

$$Z = \frac{x_1 - 400}{45} = z_1 = 1.65$$

$\therefore$ $x_1 = 1.65(45) + 400 = 474.25$ hours

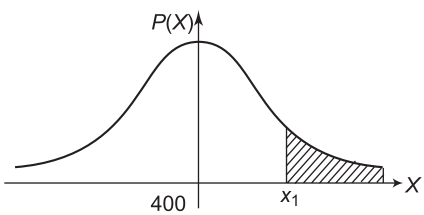


Fig. 2.27

Example 9

If the weights of 300 *students are normally distributed with a mean of* 68 *kg and a standard deviation of* 3 *kg, how many students have weights (i) greater than* 72 *kg? (ii) less than or equal to* 64 *kg? (iii) between* 65 *kg and* 71 *kg inclusive?*

Solution

Let X be the random variable which denotes the weight of a student.

$$\mu = 68, \quad \sigma = 3, \quad N = 300$$

$$Z = \frac{X - \mu}{\sigma}$$

(i) When $X = 72$, $Z = \dfrac{72-68}{3} = 1.33$

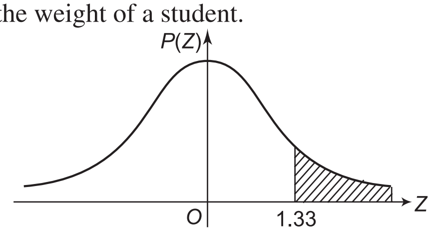


Fig. 2.28

$$P(X > 72) = P(Z > 1.33) \quad \text{(Fig. 2.28)}$$
$$= 0.5 - P(0 \le Z \le 1.33)$$
$$= 0.5 - 0.4082$$
$$= 0.0918$$

Number of students with weights more than 72 kg $= N\ P(X > 72)$

$$= 300(0.0918)$$
$$= 27.54$$
$$\approx 28$$

(ii) When $X = 64, \quad Z = \dfrac{64-68}{3} = -1.33$

$$P(X \le 64) = P(Z \le -1.33) \quad \text{(Fig. 2.29)}$$
$$= P(Z \ge 1.33) \qquad \text{[By symmetry]}$$
$$= 0.5 - P(0 < Z < 1.33)$$
$$= 0.5 - 0.4082$$
$$= 0.0918$$

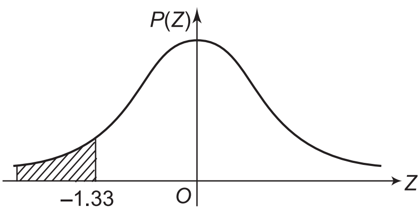


Fig. 2.29

Number of students with weights less than or equal to 64 kg

$$= N\ P(X \le 64)$$
$$= 300(0.0918)$$
$$= 27.54$$
$$\approx 28$$

(iii) When $X = 65, \quad Z = \dfrac{65-68}{3} = -1$

When $X = 71, \quad Z = \dfrac{71-68}{3} = 1$

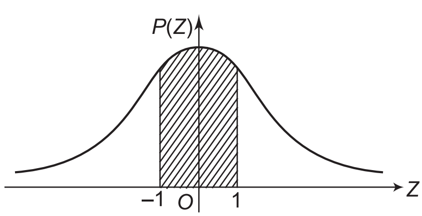


Fig. 2.30

$$P(65 \le X \le 71) = P(-1 \le Z \le 1) \quad \text{(Fig. 2.30)}$$
$$= P(-1 \le Z \le 0) + P(0 \le Z \le 1)$$
$$= P(0 \le Z \le 1) + P(0 \le Z \le 1) \qquad \text{[By symmetry]}$$
$$= 2P(0 \le Z \le 1)$$
$$= 2(0.3413)$$
$$= 0.6826$$

Number of students with weights between 65 and 71 kg $= N\ P(65 \le X \le 71)$

$$= 300(0.6826)$$
$$= 204.78$$
$$\approx 205$$

Example 10

The mean yield for a one-acre plot is 662 *kg with an SD of* 32 *kg. Assuming normal distribution, how many one-acre plots in a batch of* 1000 *plots would you expect to have yields (i) over* 700 *kg? (ii) below* 650 *kg? (iii) What is the lowest yield of the best* 100 *plots?*

Solution

Let X be the random variable which denotes the yield for the one-acre plot.

$$\mu = 662, \quad \sigma = 32, \quad N = 1000$$

$$Z = \frac{X-\mu}{\sigma}$$

(i) When $X = 700, \quad Z = \dfrac{700-662}{32} = 1.19$

$$\begin{aligned} P(X > 700) &= P(Z > 1.19) \quad \text{(Fig. 2.31)} \\ &= 0.5 - P(0 \le Z \le 1.19) \\ &= 0.5 - 0.3830 \\ &= 0.1170 \end{aligned}$$

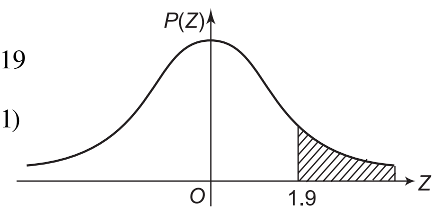


Fig. 2.31

Expected number of plots with yields over 700 kg $= N\,P(X > 700)$

$$= 1000(0.1170)$$

$$= 117$$

(ii) When $X = 650$,

$$Z = \frac{650-662}{32} = -0.38$$

$$\begin{aligned} P(X < 650) &= P(Z < -0.38) \quad \text{(Fig. 2.32)} \\ &= P(Z > 0.38) \\ &\qquad \text{[By symmetry]} \\ &= 0.5 - P(0 \le Z \le 0.38) \\ &= 0.5 - 0.1480 \\ &= 0.352 \end{aligned}$$

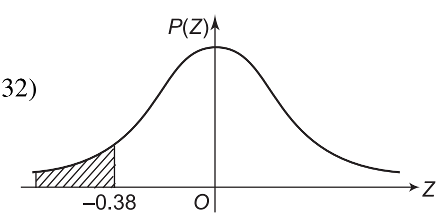


Fig. 2.32

Expected number of plots with yields below 650 kg $= N\,P(X < 650)$

$$= 1000(0.352)$$

$$= 352$$

(iii) The lowest yield, say, x_1 of the best 100 plots is given by

$$P(X > x_1) = \frac{100}{1000} = 0.1$$

When $X = x_1, \quad Z = \dfrac{x_1 - 662}{32} = z_1$

$$P(X > x_1) = P(Z > z_1)$$
$$0.1 = 0.5 - P(0 \le Z \le z_1)$$
$$P(0 \le Z \le z_1) = 0.4$$
$$\therefore \quad z_1 = 1.2 \text{ (approx.) [From normal table]}$$
$$\frac{x_1 - 662}{32} = 1.28$$
$$x_1 = 702.96$$

Hence, the best 100 plots have yields over 702.96 kg.

Example 11

Assume that the mean height of Indian soldiers is 68.22 *inches with a variance of* 10.8 *inches. How many soldiers in a regiment of* 1000 *would you expect to be over* 6 *feet tall?*

Solution

Let X be the continuous random variable which denotes the heights of Indian soldiers.

$$\mu = 68.22, \quad \sigma^2 = 10.8, \quad N = 1000$$
$$\sigma = 3.29$$
$$Z = \frac{X - \mu}{\sigma}$$

When X = 6 feet = 72 inches,

$$Z = \frac{72 - 68.22}{3.29} = 1.15$$

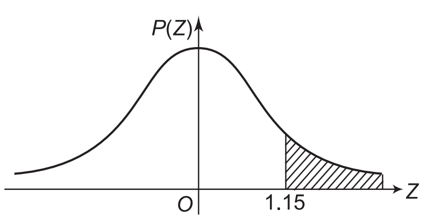


Fig. 2.33

$$\begin{aligned} P(X > 72) &= P(Z > 1.15) \quad \text{(Fig. 2.33)} \\ &= 0.5 - P(0 \le Z \le 1.15) \\ &= 0.5 - 0.3749 \\ &= 0.1251 \end{aligned}$$

Expected number of Indian soldiers having heights over 6 feet (72 inches)

$$\begin{aligned} &= N\ P(X > 72) \\ &= 1000(0.1251) \\ &= 125.1 \\ &\approx 125 \end{aligned}$$

Example 12

The marks obtained by students in a college are normally distributed with a mean of 65 and a variance of 25. If 3 students are selected at random from this college, what is the probability that at least one of them would have scored more than 75 marks?

Solution

Let X be the continuous random variable which denotes the marks of a student.

$$\mu = 65, \quad \sigma^2 = 25$$

$$\sigma = 5$$

$$Z = \frac{X - \mu}{\sigma}$$

When $X = 75, \quad Z = \dfrac{75 - 65}{5} = 2$

$$\begin{aligned} P(X > 75) &= P(Z > 2) \ \text{(Fig. 2.34)} \\ &= 0.5 - P(0 \le Z \le 2) \\ &= 0.5 - 0.4772 \\ &= 0.0228 \end{aligned}$$

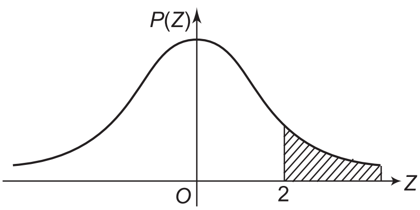


Fig. 2.34

If p is the probability of scoring more than 75 marks,

$p = 0.0228, \ q = 1 - p = 1 - 0.0228 = 0.9772$

P(at least one student would have scored more than 75 marks)

$$\begin{aligned} &= \sum_{x=1}^{3} {}^3C_x \ p^x \ q^{n-x} \\ &= \sum_{x=1}^{3} {}^3C_x (0.0228)^x (0.9772)^{3-x} \\ &= 0.0668 \end{aligned}$$

Example 13

Find the mean and standard deviation in which 7% *of items are under* 35 *and* 89% *are under* 63.

Solution

Let μ be the mean and σ be standard deviation of the normal curve.

$$P(X < 35) = 0.07$$

$$P(X < 63) = 0.89$$

$$P(X > 63) = 1 - P(X < 63) = 1 - 0.89 = 0.11$$

$$Z = \frac{X - \mu}{\sigma}$$

Since $P(X < 35) < 0.5$, the corresponding value of Z will be negative.

When $X = 35$, $Z = \dfrac{35 - \mu}{\sigma} = -z_1$ (say)

Since $P(X < 63) > 0.5$, the corresponding value of Z will be positive.

When $X = 63$, $Z = \dfrac{63 - \mu}{\sigma} = z_2$ (say)

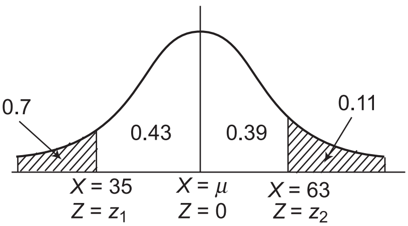


Fig. 2.35

From Fig. 2.35,

$$P(Z < -z_1) = 0.07$$

$$P(Z > z_2) = 0.11$$

$$\begin{aligned} P(0 < Z < z_1) &= P(-z_1 < Z < 0) \\ &= 0.5 - P(Z \le -z_1) \\ &= 0.5 - 0.07 \\ &= 0.43 \\ z_1 &= 1.48 \qquad \text{[From normal table]} \\ P(0 < Z < z_2) &= 0.5 - P(Z \ge z_2) \\ &= 0.5 - 0.11 \\ &= 0.39 \\ z_2 &= 1.23 \qquad \text{[From normal table]} \end{aligned}$$

Hence, $\dfrac{35 - \mu}{\sigma} = -1.48$

$$-1.48\,\sigma + \mu = 35 \qquad \text{...(1)}$$

and $\dfrac{63 - \mu}{\sigma} = 1.23$

$$1.23\,\sigma + \mu = 63 \qquad \text{...(2)}$$

Solving Eqs (1) and (2),

$$\mu = 50.29, \qquad \sigma = 10.33$$

Example 14

In an examination, it is laid down that a student passes if he secures 40 % *or more. He is placed in the first, second, and third division according to whether he secures* 60% *or more marks, between* 50% *and* 60% *marks and between* 40% *and* 50% *marks respectively. He gets a distinction in case he secures* 75% *or more. It is noticed from the result that* 10% *of* *the students failed in the examination, whereas 5% of them obtained distinction. Calculate the percentage of students placed in the second division. (Assume normal distribution of marks.)*

Solution

Let X be the random variable which denotes the marks of students in the examination. Let μ be the mean and σ be the standard deviation of the normal distribution of marks.

$$P(X < 40) = 0.10$$
$$P(X \geq 75) = 0.05$$
$$P(X < 75) = 1 - P(X \geq 75) = 1 - 0.05 = 0.95$$
$$Z = \frac{X - \mu}{\sigma}$$

Since $P(X < 40) < 0.5$, the corresponding value of Z will be negative.

When $X = 40$, $\quad Z = \dfrac{40 - \mu}{\sigma} = -z_1$ (say)

Since $P(X < 75) < 0.5$, the corresponding value of Z will be positive.

When $X = 75$, $\quad Z = \dfrac{75 - \mu}{\sigma} = z_2$ (say)

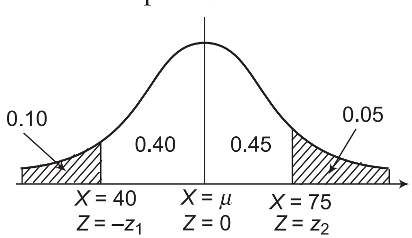


Fig. 2.36

From Fig. 2.36,

$$P(Z < -z_1) = 0.10$$
$$P(Z > z_2) = 0.05$$
$$\begin{aligned} P(0 < Z < z_1) &= P(-z_1 < Z < 0) \\ &= 0.5 - P(Z \leq -z_1) \\ &= 0.5 - 0.10 \\ &= 0.40 \end{aligned}$$
$$z_1 = 1.28 \quad \text{[From normal table]}$$
$$\begin{aligned} P(0 < Z < z_2) &= 0.5 - P(Z \geq z_2) \\ &= 0.5 - 0.05 \\ &= 0.45 \end{aligned}$$
$$z_2 = 1.64 \quad \text{[From normal table]}$$

Hence, $\quad \dfrac{40 - \mu}{\sigma} = -1.28$

$$\mu - 1.28\,\sigma = 40 \qquad \text{...(1)}$$

and $\quad \dfrac{75 - \mu}{\sigma} = 1.64$

$$\mu + 1.64\,\sigma = 75 \qquad \text{...(2)}$$

Solving Eqs (1) and (2),

$$\mu = 55.34 \approx 55$$
$$\sigma = 11.98 \approx 12$$

Probability that a student is placed in the second division is equal to the probability that his score lies between 50 and 60

When $X = 50$, $Z = \dfrac{50-55}{12} = -0.42$

When $X = 60$, $Z = \dfrac{60-55}{12} = 0.42$

$$\begin{aligned} P(50 < X < 60) &= P(-0.42 < Z < 0.42) \\ &= P(-0.42 < Z < 0) + P(0 < Z < 0.42) \\ &= P(0 < Z < 0.42) + P(0 < Z < 0.42) \qquad \text{[By symmetry]} \\ &= 2\,P(0 < Z < 0.42) \\ &= 2\,(0.1628) \\ &= 0.3256 \\ &\approx 0.32 \end{aligned}$$

Hence, the percentage of students placed in the second division = 32%.

2.11.5 Fitting a Normal Distribution

Fitting a normal distribution or a normal curve to the data means to find the equation of the curve in the form $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ which will be as close as possible to the points given. There are two purposes of fitting a normal curve:

(i) To judge the whether the normal curve is the best fit to the sample data.

(ii) To use the normal curve to estimate the characteristics of a population.

The area method for fitting a normal curve is given by the following steps:

(i) Find the mean μ and standard deviation σ for the given data if not given.

(ii) Write the class intervals and lower limits X of class intervals in two columns.

(iii) Find $Z = \dfrac{X-\mu}{\sigma}$ for each class interval.

(iv) Find the area corresponding to each Z from the normal table.

(v) Find the area under the normal curve between the successive values of Z. These are obtained by subtracting the successive areas when the corresponding Z's have the same sign and adding them when the corresponding Z's have the same sign and adding them when the corresponding Z's opposite sign.

(vi) Find the expected frequencies by multiplying the relative frequencies by the number of observations.

Example 1

Fit a normal curve from the following distribution. It is given that the mean of the distribution is 43.7 *and its standard distribution is* 14.8.

Class interval	11–20	21–30	31–40	41–50	51–60	61–70	71–80
Frequency	20	28	40	60	32	20	8

Solution

$\mu = 43.7, \qquad \sigma = 14.8 \qquad N = \Sigma f = 200$

The series is converted into an inclusive series.

Class Interval	Lower class	$Z = \dfrac{X-\mu}{\sigma}$	Area from 0 to Z	Area in class Interval	Expected Frequencies
10.5–20.5	10.5	–2.24	0.4875	0.0457	9.14 ≈ 9
20.5–30.5	20.5	–1.57	0.4418	0.1285	25.7 ≈ 26
30.5–40.5	30.5	–0.89	0.3133	0.2262	45.24 ≈ 45
40.5–50.5	40.5	–0.22	0.0871	0.2643	52.86 ≈ 53
50.5–60.5	50.5	0.46	0.1772	0.1957	39.14 ≈ 39
60.5–70.5	60.5	1.14	0.3729	0.092	18.4 ≈ 18
70.5–80.5	70.5	1.81	0.4649	0.0287	5.74 ≈ 65
	80.5	2.49	0.4936		

Example 2

Fit a normal distribution to the following data:

X	125	135	145	155	165	175	185	195	205
Y	1	1	14	22	25	19	13	3	2

It is given that $\mu = 165.5$ *and* $\sigma = 15.26$.

Solution

$\mu = 165.5, \qquad \sigma = 15.26 \qquad N = \Sigma f = 100$

The data is first converted into class intervals with inclusive series.

Class Interval	Lower class	$Z = \dfrac{X - \mu}{\sigma}$	Area from 0 to Z	Area in class Interval	Expected Frequencies
120–130	120	–2.98	0.4986	0.0085	$0.85 \approx 1$
130–140	130	–2.33	0.4901	0.0376	$3.74 \approx 4$
140–150	140	–1.67	0.4525	0.1064	$10.64 \approx 11$
150–160	150	–1.02	0.3461	0.2055	$20.55 \approx 21$
160–170	160	–0.36	0.1406	0.2547	$25.47 \approx 25$
170–180	170	0.29	0.1141	0.2148	$21.48 \approx 21$
180–190	180	0.95	0.3289	0.1174	$11.74 \approx 12$
190–200	190	1.61	0.4463	0.0418	$4.18 \approx 4$
200–210	200	2.26	0.4881	0.0101	$1.01 \approx 1$
210–220	210	2.92	0.4982		

EXERCISE 2.7

1. If X is normally distributed with a mean and standard deviation of 4, find (i) $P(5 \le X \le 10)$, (ii) $P(X \ge 15)$, (iii) $P(10 \le X \le 15)$, and (iv) $P(X \le 5)$.

[**Ans.:** (i) 0.3345 (ii) 0.003 (iii) 0.0638 (iv) 0.4013]

2. A normal distribution has a mean of 5 and a standard deviation of 3. What is the probability that the deviation from the mean of an item taken at random will be negative?

[**Ans.:** 0.0575]

3. If X is a normal variate with a mean of 30 and an SD of 6, find the value of $X = x_1$ such that $P(X \ge x_1) = 0.05$.

[**Ans.:** 39.84]

4. If X is a normal variate with a mean of 25 and SD of 5, find the value of $X = x_1$ such that $P(X \le x_1) = 0.01$.

[**Ans.:** 11.02]

5. The weights of 4000 students are found to be normally distributed with a mean of 50 kg and an SD of 5 kg. Find the probability that a student selected at random will have weight (i) less than 45 kg, and (ii) between 45 and 60 kg.

[**Ans.:** (i) 0.1587 (ii) 0.8185]

6. The daily sales of a firm are normally distributed with a mean of ₹ 8000 and a variance of ₹ 10000. (i) What is the probability that on a certain day the sales will be less than ₹ 8210? (ii) What is the percentage of days on which the sales will be between ₹ 8100 and ₹ 8200?

[**Ans.:** (i) 0.482 (ii) 14%]

7. The mean height of Indian soldiers is 68.22″ with a variance of 10.8″. Find the expected number of soldiers in a regiment of 1000 whose height will be more than 6 feet.

[**Ans.:** 125]

8. The life of army shoes is normally distributed with a mean of 8 months and a standard deviation of 2 months. If 5000 pairs are issued, how many pairs would be expected to need replacement after 12 months?

[**Ans.:** 2386]

9. In an intelligence test administered to 1000 students, the average was 42 and the standard deviation was 24. Find the number of students (i) exceeding 50, (ii) between 30 and 54, and (iii) the least score of top 1000 students.

[**Ans.:** (i) 129 (ii) 383 (iii) 72.72]

10. In a test of 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average of life of 2040 hours and a standard deviation of 60 hours. Estimate the number of bulbs likely to burn for (i) more than 2150 hours, and (ii) less than 1950 hours.

[**Ans.:** (i) 67 (ii) 184]

11. The marks of 1000 students of a university are found to be normally distributed with a mean of 70 and a standard of deviation 5. Estimate the number of students whose marks will be (i) between 60 and 75, (ii) more than 75, and (iii) less than 68.

[**Ans.:** (i) 910 (ii) 23 (iii) 37]

12. In a normal distribution, 31% items are under 45 and 8% are over 64. Find the mean and standard deviation. Find also, the percentage of items lying between 30 and 75.

[**Ans.:** 50, 10, 0.957]

13. Of a large group of men, 5% are under 60 inches in height and 40% are between 60 and 65 inches. Assuming a normal distribution, find the mean and standard deviation of distribution.

[**Ans.:** 65.42, 3.27]

14. The marks obtained by students in an examination follow a normal distribution. If 30% of the students got marks below 35 and 10% got marks above 60, find the mean and percentage of students who got marks between 40 and 50.

[**Ans.:** 42.23, 13.88, 28%]

15. Fit a normal distribution to the following data:

Class	60–65	65–70	70–75	75–80	80–85	85–90	90–95	95–100
Frequency	3	21	150	335	326	135	26	4

[**Ans.:** Expected frequency: 3, 31, 148, 322, 319, 144, 30, 3]

Points to Remember

Random Variables

A random variable X is a real-valued function of the elements of the sample space of a random experiment. In other words, a variable which takes the real values, depending on the outcome of a random experiment is called a *random variable*,

Discrete Random Variables: A random variable X is said to be discrete if it takes either finite or countably infinite values.

Continuous Random Variables: A random variable X is said to be continuous if it takes any values in a given interval.

Discrete Probability Distribution

Probability distribution of a random variable is the set of its possible values together with their respective probabilities.

Discrete Distribution Function

$$F(x) = P(X \le x) = \sum_{i=1}^{x} p(x_i)$$

Measures of Central Tendency for Discrete Probability Distribution

1. Mean

$$\mu = E(X) = \sum_{i=1}^{\infty} x_i\ p(x_i) = \sum x\ p(x)$$

2. Variance

$$\begin{aligned}\text{Var}(X) &= \sigma^2 = E(X-\mu)^2 \\ &= E(X^2) - [E(X)]^2\end{aligned}$$

3. Standard deviation

$$\text{SD} = \sigma = \sqrt{\sum_{i=1}^{\infty} x_i^2 \ p(x_i) - \mu^2}$$

$$= \sqrt{E\,(X^2) - \mu^2}$$

$$= \sqrt{E\,(X^2) - [E(X)]^2}$$

Continuous Distribution Function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,\mathrm{d}x, \quad -\infty < x < \infty$$

Measures of Central Tendency for Continuous Probability Distribution

1. Mean

$$\mu = E(X) = \int_{-\infty}^{\infty} x\,f(x)\,\mathrm{d}x$$

2. Median

$$\int_{a}^{M} f(x)\,\mathrm{d}x = \int_{M}^{b} f(x)\,\mathrm{d}x = \frac{1}{2}$$

3. Variance

$$\text{Var}\,(X) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \ f(x)\,\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} x^2 \ f(x)\,\mathrm{d}x - \mu^2$$

3. Standard Deviation

$$\text{SD} = \sqrt{\text{Var}(X)} = \sigma$$

Binomial Distribution

$$P(X = x) = p(x) = {}^{n}C_x \ p^x \ q^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

1. Mean of the Binomial Distribution

$$E(X) = np$$

2. Variance of the Binomial Distribution

$$\text{Var}(X) = npq$$

3. Standard Deviation of the Binomial Distribution

$$\text{SD} = \sqrt{\text{Variance}} = \sqrt{npq}$$

Recurrence Relation for the Binomial Distribution

$$P(X = x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \cdot P(X = x)$$

Binomial Frequency Distribution

$$\sum_{x=0}^{n} f(x) = N \sum_{x=0}^{n} P(X = x) = N$$

Poisson Distribution

$$P(X = x) = p(x) = \frac{e^{-\lambda}\,\lambda^{x}}{x!}, \quad x = 0, 1, 2, \ldots$$

1. Mean of the Poisson Distribution

$$E(X) = \lambda$$

2. Variance of the Poisson Distribution

$$\text{Var}\,(X) = \lambda$$

3. Standard Deviation of the Poisson Distribution

$$\text{SD} = \sqrt{\text{Variance}} = \sqrt{\lambda}$$

Recurrence Relation for the Poisson Distribution

$$p(x+1) = \frac{\lambda}{x+1}\, p(x)$$

Normal Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < X < \infty, -\infty < \mu < \infty, \sigma > 0$$

1. Mean of the Normal Distribution

$$E(X) = \mu$$

2. Variance of the Normal Distribution

$$\text{Var}(X) = \sigma^2$$

3. Standard Deviation of the Normal Distribution

$$\text{SD} = \sigma$$

5. Median of the Normal Distribution

$$\mu = M$$

CHAPTER 3

Statistics

Chapter Outline

3.1 Introduction
3.2 Measures of Central Tendency
3.3 Arithmetic Mean
3.4 Median
3.5 Mode
3.6 Geometric Mean
3.7 Harmonic Mean
3.8 Standard Deviation
3.9 Skewness

3.1 INTRODUCTION

Statistics is the science which deals with the collection, presentation, analysis, and interpretation of numerical data. Statistics should possess the following characteristics:

(i) Statistics are aggregates of facts.
(ii) Statistics are affected by a large number of causes.
(iii) Statistics are always numerically expressed.
(iv) Statistics should be enumerated or estimated.
(v) Statistics should be collected in a systematic manner.
(vi) Statistics should be collected for a pre-determined purpose.
(vii) Statistics should be placed in relation to each other.

The use of statistical methods help in presenting a complex mass of data in a simplified form so as to facilitate the process of comparison of characteristics in two or more situations. Statistics also provide important techniques for the study of relationship between two or more characteristics (or variables) in forecasting, testing of hypothesis, quality control, decision making, etc.

3.2 MEASURES OF CENTRAL TENDENCY

Summarization of data is a necessary function of any statistical analysis. The data is summarized in the form of tables and frequency distributions. In order to bring the characteristics of the data, these tables and frequency distributions need to be summarized further. A measure of central tendency or an average is very essential and an important summary measure in any statistical analysis.

An *average* is a single value which can be taken as a representative of the whole distribution. There are five types of measures of central tendency or averages which are commonly used.

(i) Arithmetic mean
(ii) Median
(iii) Mode
(iv) Geometric mean
(v) Harmonic mean

A good measure of average must have the following characteristics:

(i) It should be rigidly defined so that different persons obtain the same value for a given set of data.
(ii) It should be easy to understand and easy to calculate.
(iii) It should be based on all the observations of the data.
(iv) It should be easily subjected to further mathematical calculations.
(v) It should not be much affected by the fluctuations of sampling.
(vi) It should not be unduly affected by extreme observations.
(vii) It should be easy to interpret.

3.3 ARITHMETIC MEAN

The *arithmetic mean* of a set of observations is their sum divided by the number of observations. Let $x_1, x_2, ..., x_n$ be n observations. Then their average or arithmetic mean is given by

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum x}{n}$$

For example, the marks obtained by 10 students in Class XII in a physics examination are 25, 30, 21, 55, 40, 45, 17, 48, 35, 42. The arithmetic mean of the marks is given by

$$\bar{x} = \frac{\sum x}{n} = \frac{25+30+21+55+40+45+17+48+35+42}{10} = \frac{358}{10} = 35.8$$

If n observations consist of n distinct values denoted by $x_1, x_2, ..., x_n$ of the observed variable x occurring with frequencies $f_1, f_2, ..., f_n$ respectively then the arithmetic mean is given by

$$\overline{x} = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_ix_i}{N} = \frac{\sum fx}{N}$$

where $\quad N = \sum_{i=1}^{n} f_i = f_1 + f_2 + \cdots + f_n$

3.3.1 Arithmetic Mean of Grouped Data

In case of grouped or continuous frequency distribution the arithmetic mean is given by

$$\overline{x} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i} = \frac{\sum fx}{N}, \text{ where } N = \sum_{i=1}^{n} f_i$$

and x is taken as the midvalue of the corresponding class.

Example 1

Find the arithmetic mean from the following frequency distribution:

x	5	6	7	8	9	10	11	12	13	14
f	25	45	90	165	112	96	81	26	18	12

Solution

x	f	fx
5	25	125
6	45	270
7	90	630
8	165	1320
9	112	1008
10	96	960
11	81	891
12	26	312
13	18	234
14	12	168
	$\sum f = 670$	$\sum fx = 5918$

$$N = \sum f = 670$$

$$\bar{x} = \frac{\sum fx}{N} = \frac{5918}{670} = 8.83$$

Example 2

Find the arithmetic mean of the marks from the following data:

Marks	0–10	10–20	20–30	30–40	40–50	50–60
Number of students	12	18	27	20	15	8

Solution

Marks	Number of students (f)	Midvalue (x)	fx
0–10	12	5	60
10–20	18	15	270
20–30	27	25	675
30–40	20	35	700
40–50	15	45	675
50–60	8	55	440
	$\sum f = 100$		$\sum fx = 2820$

$$N = \sum f = 100$$

$$\bar{x} = \frac{\sum fx}{N} = \frac{2820}{100} = 28.20$$

Example 3

A company is planning to improve plant safety. For this, accident data for the last 50 *weeks was compiled. These data are grouped into the frequency distribution as shown below. Calculate the arithmetic mean of the number of accidents per week.*

Number of accidents (x)	0–4	5–9	10–14	15–19	20–24
Number of weeks (f)	5	22	13	8	2

Solution

The given class intervals are inclusive. However, they need not be converted into exclusive class intervals for the calculation of mean.

Number of Accidents	Number of Weeks	Midvalue (x)	fx
0–4	5	2	10
5–9	22	7	154
10–14	13	12	156
15–19	8	17	136
20–24	2	22	44
	$\sum f = 50$		$\sum fx = 500$

$$N = \sum f = 50$$

$$\bar{x} = \frac{\sum fx}{N} = \frac{500}{50} = 10$$

3.3.2 Arithmetic Mean from Assumed Mean

If the values of x and (or) f are large, the calculation of mean becomes quite time-consuming and tedious. In such cases, the provisional man 'a' is taken as that value of x (midvalue of the class interval) which corresponds to the highest frequency or which comes near the middle value of the frequency distribution. This number is called the *assumed mean*.

Let
$$d = x - a$$
$$fd = f(x-a) = fx - af$$
$$\sum fd = \sum fx - a\sum f$$
$$= \sum fx - aN$$

Dividing both the sides by n,

$$\frac{\sum fd}{N} = \frac{\sum fx}{N} - a$$
$$= \bar{x} - a$$

$$\therefore \quad \bar{x} = a + \frac{\sum fd}{N}$$

Example 1

Ten coins were tossed together and the number of tails resulting from them were observed. The operation was performed 1050 *times and the frequencies thus obtained for different number of tail* (x) *are shown in the following table. Calculate the arithmetic mean.*

x	0	1	2	3	4	5	6	7	8	9	10
y	2	8	43	133	207	260	213	120	54	9	1

Solution

Let $a = 5$ be the assumed mean.

$d = x - a = x - 5$

x	f	$d = x - 5$	fd
0	2	–5	–10
1	8	–4	–32
2	43	–3	–129
3	133	–2	–266
4	207	–1	–207
5	260	0	0
6	213	1	213
7	120	2	240
8	54	3	162
9	9	4	36
10	1	5	5
	$\sum f = 1050$		$\sum fd = 12$

$$N = \sum f = 1050$$

$$\bar{x} = a + \frac{\sum fd}{N}$$

$$= 5 + \frac{12}{1050}$$

$$= 5.0114$$

Example 2

The daily earnings (in rupees) of employees working on a daily basis in a firm are

Daily earnings (₹)	100	120	140	160	180	200	220
Number of employees	3	6	10	15	24	42	75

Calculate the mean of daily earnings.

Solution

Let $a = 160$ be the assumed mean.

$d = x - a = x - 160$

Daily Earnings x	Number of Employees f	$d = x - 160$	fd
100	3	–60	–180
120	6	–40	–240
140	10	–20	–200
160	15	0	0
180	24	20	480
200	42	40	1680
220	75	60	4500
	$\sum f = 175$		$\sum fd = 6040$

$$N = \sum f = 175$$

$$\bar{x} = a + \frac{\sum fd}{N}$$

$$= 160 + \frac{6040}{175}$$

$$= 194.51$$

Example 3

Calculate the mean for the following frequency distribution

Class	0–8	8–16	16–24	24–32	32–40	40–48
Frequency	8	7	16	24	15	7

Solution

Let $a = 28$ be the assumed mean.

$d = x - a = x - 28$

Class	Frequency	Midvalue (x)	$d = x - 28$	fd
0–8	8	4	–24	–192
8–16	7	12	–16	–112
16–24	16	20	–8	–128
24–32	24	28	0	0
32–40	15	36	8	120
40–48	7	44	16	112
	$\sum f = 77$			$\sum fd = -200$

$$N = \sum f = 77$$

$$\bar{x} = a + \frac{\sum fd}{N}$$

$$= 28 + \frac{(-200)}{77}$$

$$= 25 \cdot 403$$

Example 4

Calculate the arithmetic mean of the following distribution:

Class Interval	0–10	10–20	20–30	30–40	40–50	50–60	60–70	70–80
Frequency	3	8	12	15	18	16	11	5

Solution

Let $a = 35$ be the assumed mean.
$d = x - a = x - 35$

Class Interval	Frequency f	Midvalue x	$d = x - 35$	fd
0–10	3	5	–30	–90
10–20	8	15	–20	–160
20–30	12	25	–10	–120
30–40	15	35	0	0
40–50	18	45	10	180
50–60	16	55	20	320
60–70	11	65	30	330
70–80	5	75	40	200
	$\sum f = 88$			$\sum fd = 660$

$$N = \sum f = 88$$

$$\bar{x} = a + \frac{\sum fd}{N}$$

$$= 35 + \frac{660}{88}$$

$$= 42.5$$

3.3.3 Arithmetic Mean by the Step-Deviation Method

When the class intervals in a grouped data are equal, calculation can be simplified by the step-deviation method. In such cases, deviation of variate x from the assumed mean

a (i.e., $d = x - a$) are divided by the common factor h which is equal to the width of the class interval.

Let $$d = \frac{x-a}{h}$$

$$\overline{x} = a + h\frac{\sum fd}{\sum f} = a + h\frac{\sum fd}{N}$$

where a is the assumed mean

$d = \frac{x-a}{h}$ is the deviation of any variate x from a

h is the width of the class interval

N is the number of observations

Example 1

Calculate the arithmetic mean of the following marks obtained by students in mathematics:

Marks (x)	5	10	15	20	25	30	35	40	45	50
Number of students (f)	20	43	75	67	72	45	39	9	8	6

Solution

Let $a = 30$ be the assumed mean and $h = 5$ be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-30}{5}$$

x	f	$d = \frac{x-30}{5}$	fd
5	20	–5	–100
10	43	–4	–172
15	75	–3	–225
20	67	–2	–134
25	72	–1	–72
30	45	0	0
35	39	1	39
40	9	2	18
45	8	3	24
50	6	4	24
	$\sum f = 384$		$\sum fd = -598$

$$N = \sum f = 384$$

$$\overline{x} = a + h\frac{\sum fd}{N}$$

$$= 30 + 5\left(\frac{-598}{384}\right)$$

$$= 22.214$$

Example 2

Calculate the average overtime work done per employee for the following distribution which gives the pattern of overtime work done by 100 employees of a company.

Overtime hours	10–15	15–20	20–25	25–30	30–35	35–40
Number of employees	11	20	35	20	8	6

Solution

Let $a = 22.5$ be the assumed mean and $h = 5$ be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-22.5}{5}$$

Overtime hours	Number of employees f	Midvalue x	$d = \frac{x-22.5}{5}$	fd
10–15	11	12.5	–2	–22
15–20	20	17.5	–1	–20
20–25	35	22.5	0	0
25–30	20	27.5	1	20
30–35	8	32.5	2	16
35–40	6	37.5	3	18
	$\Sigma f = 100$			$\Sigma fd = 12$

$$N = \sum f = 100$$

$$\overline{x} = a + h\frac{\sum fd}{N}$$

$$= 22.5 + 5\left(\frac{12}{100}\right)$$

$$= 23.1 \text{ hours}$$

Example 3

The following table gives the distribution of companies according to size of capital. Find the mean size of the capital of a company.

Capital (₹ in lacs)	<5	<10	<15	<20	<25	<30
No. of companies	20	27	29	38	48	53

Solution

This is a 'less than' type of frequency distribution. This will be first converted into class intervals. Let $a = 12.5$ be the assumed mean and $h = 5$ be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-12.5}{5}$$

Class intervals	Frequency f	Midvalue x	$d = \frac{x-12.5}{5}$	fd
0–5	20	2.5	–2	–40
5–10	7	7.5	–1	–7
10–15	2	12.5	0	0
15–20	9	17.5	1	9
20–25	10	22.5	2	20
25–30	5	27.5	3	15
	$\sum f = 53$			$\sum fd = -3$

$$N = \sum f = 53$$

$$\begin{aligned}\bar{x} &= a + h\frac{\sum fd}{N}\\ &= 12.5 + 5\left(\frac{-3}{53}\right)\\ &= 12.22 \text{ lacs}\end{aligned}$$

Example 4

Find the arithmetic mean from the following data:

Marks less than	10	20	30	40	50	60
No. of students	10	30	60	110	150	180

Solution

This is a 'less than' type of frequency distribution. This will be first converted into class intervals. Let a = 45 be the assumed mean and h = 10 be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-45}{10}$$

Marks	No. of students f	Midvalue x	$d = \frac{x-45}{10}$	fd
0–10	10	5	–4	–40
10–20	20	15	–3	–60
20–30	30	25	–2	–60
30–40	50	35	–1	–50
40–50	40	45	0	0
50–60	30	55	1	30
	$\sum f = 180$			$\sum fd = -180$

$$N = \sum f = 180$$

$$\begin{aligned}\overline{x} &= a + h\frac{\sum fd}{N}\\ &= 45 + 10\left(\frac{-180}{180}\right)\\ &= 35\end{aligned}$$

Example 5

Following is the distribution of marks obtained by 60 students in a mathematics test:

Marks	Number of students
More than 0	60
More than 10	56
More than 20	40
More than 30	20
More than 40	10
More than 50	3

Calculate the arithmetic mean.

Solution

This is a 'more than' type of frequency distribution. This will be first converted into class intervals. Let $a = 35$ be the assumed mean and $h = 10$ be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-35}{10}$$

Marks	No. of students f	Midvalue x	$d = \dfrac{x-35}{10}$	fd
0–10	4	5	–3	–12
10–20	16	15	–2	–32
20–30	20	25	–1	–20
30–40	10	35	0	0
40–50	7	45	1	7
50–60	3	55	2	6
	$\sum f = 60$			$\sum fd = -51$

$$N = \sum f = 60$$

$$\bar{x} = a + h\frac{\sum fd}{N}$$

$$= 35 + 10\left(\frac{-51}{60}\right)$$

$$= 26.5$$

Example 6

Find the average marks of students from the following table:

Marks	No. of students	Marks	No. of students
Above 0	80	Above 60	23
Above 10	77	Above 70	16
Above 20	72	Above 80	10
Above 30	65	Above 90	8
Above 40	55	Above 100	0
Above 50	43		

Solution

This is a 'more than' type of frequency distribution. This will be first converted into class intervals. Let $a = 45$ be the assumed mean and $h = 10$ be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-45}{10}$$

Marks	No. of students f	Midvalue x	$d = \frac{x-45}{10}$	fd
0–10	3	5	–4	–12
10–20	5	15	–3	–15
20–30	7	25	–2	–14
30–40	10	35	–1	–10
40–50	12	45	0	0
50–60	20	55	1	20
60–70	7	65	2	14
70–80	6	75	3	18
80–90	2	85	4	8
90–100	8	95	5	40
	$\sum f = 80$			$\sum fd = 49$

$$N = \sum f = 80$$

$$\begin{aligned} \bar{x} &= a + h\frac{\sum fd}{N} \\ &= 45 + 10\left(\frac{49}{80}\right) \\ &= 51.125 \end{aligned}$$

Example 7

Find the arithmetic mean of the following data obtained during study on patients:

Age (in years)	10–19	20–29	30–39	40–49	50–59	60–69	70–79	80–89
No. of cases	1	0	1	10	17	38	9	3

Solution

This is an inclusive series. The inclusive series can be converted into exclusive series by subtracting half the difference between the upper limit of a class and the lower limit of the next class from the lower limit of the class and adding the same to the upper limit of the class. Let a = 44.5 be the assumed mean and h = 10 be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-44.5}{10}$$

Age (in years)	No. of cases f	Midvalue x	$d = \frac{x - 44.5}{10}$	fd
9.5–19.5	1	14.5	–3	–3
19.5–29.5	0	24.5	–2	0
29.5–39.5	1	34.5	–1	–1
39.5–49.5	10	44.5	0	0
49.5–59.5	17	54.5	1	17
59.5–69.5	38	64.5	2	26
69.5–79.5	9	74.5	3	27
79.5–89.5	3	84.5	4	12
	$\sum f = 79$			$\sum fd = 128$

$$N = \sum f = 79$$

$$\bar{x} = a + h\frac{\sum fd}{N}$$

$$= 44.5 + 10\left(\frac{128}{79}\right)$$

$$= 60.7$$

3.3.4 Weighted Arithmetic Mean

In the calculation of arithmetic mean, equal importance is given to all the items. If all the items are not of equal importance, a simple arithmetic mean will not be a good representative of the given data. In such a case, proper weightage is to be given to various items. The weights are assigned to different items depending upon their importance, i.e., more important items are assigned more weight.

If $w_1, w_2, ..., w_n$ are the weights assigned to the values $x_1, x_2, ..., x_n$ respectively then the weighted arithmetic mean is given by

$$\text{Weighted arithmetic mean} = \frac{w_1x_1 + w_2x_2 + \cdots + w_nx_n}{w_1 + w_2 + \cdots + w_n}$$

$$\bar{x}_w = \frac{\sum wx}{\sum w}$$

When the assumed mean is used for calculation,

$$\bar{x}_w = a + \frac{\sum wd}{\sum w}$$

When the step-deviation method is used for calculation,

$$\bar{x}_w = a + h\frac{\sum wd}{\sum w}$$

Example 1

A candidate obtains the following percentages in an examination: English—46%, Mathematics—67%, Physics—72%, Chemistry—58%. It is agreed to give double weights to marks in English and Mathematics compared to other subjects. What is the weighted mean?

Solution

Subjects	Marks x	Weights w	wx
English	46	2	92
Mathematics	67	2	134
Physics	72	1	72
Chemistry	58	1	58
		$\sum w = 6$	$\sum wx = 356$

Weighted mean $\bar{x}_w = \frac{\sum wx}{\sum w} = \frac{356}{6} = 59.33$

Example 2

Comment on the performance of the students of two universities given below:

	GTU		Mumbai University	
Course	Pass %	No. of Students (in hundreds)	Pass %	No. of Students (in hundreds)
BE (Computer)	65	200	60	190
BE (Civil)	75	150	80	120
BE (IT)	55	180	60	130
BE (Mechanical)	60	130	65	150

Solution

Course	GTU			Mumbai University		
	Pass % x	No. of Students (in hundreds) w	wx	Pass % x	No. of Students (in hundreds) w	wx
BE (Computer)	65	200	13000	60	190	9000
BE (Civil)	75	150	11250	80	120	9600
BE (IT)	55	180	9900	60	130	7800
BE (Mechanical)	60	130	7800	65	150	9750
		$\sum w = 660$	$\sum wx =$ 41950		$\sum w = 590$	$\sum wx =$ 36150

$$\text{GTU} : \bar{x}_w = \frac{\sum wx}{\sum w} = \frac{41950}{660} = 63.56$$

$$\text{Mumbai University} : \bar{x}_w = \frac{\sum wx}{\sum w} = \frac{36150}{590} = 61.27$$

Hence, the performance of students of GTU is better than that of Mumbai University.

3.3.5 Properties of Arithmetic Mean

1. The algebraic sum of deviations from the mean is zero. If the mean of n observations $x_1, x_2, ..., x_n$ is $\bar{x}$ then $(x_1 - \bar{x}) + (x_2 - \bar{x}) + \cdots + (x_n - \bar{x}) = 0$, i.e., $\sum (x - \bar{x}) = 0$.

2. The sum of squares of the deviations is minimum when taken about the mean.

3. If $\bar{x}_1, \bar{x}_2, ..., \bar{x}_k$ are the means of k series of sizes $n_1, n_2, ..., n_k$ respectively then the mean $\bar{x}$ of the composite series is given by

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\,\bar{x}_2 + \cdots + n_k\,\bar{x}_k}{n_1 + n_2 + \cdots + n_k}$$

$$= \frac{\sum_{i=1}^{k} n_i x_i}{\sum_{i=1}^{k} n_i}$$

Example 1

Find the value of p for the following distribution whose mean is 11.37.

x	5	7	p	11	13	16	20
f	2	4	29	54	11	8	4

Solution

$$\bar{x} = \frac{\sum fx}{\sum f}$$

$$11.37 = \frac{(5\times 2)+(7\times 4)+29p+(11\times 54)+(13\times 11)+(16\times 8)+(20\times 4)}{2+4+29+54+11+8+4}$$

$$= \frac{10+28+29p+594+143+128+80}{112}$$

$$= \frac{983+29p}{112}$$

$$\therefore \quad p = 10.015 \approx 10$$

Example 2

The following is the distribution of weights (in lbs) of 60 *students of a class:*

Weight	No. of students	Weight	No. of students
93–97	2	113–117	14
98–102	5	118–122	?
103–107	12	123–127	3
108–112	?	128–132	1

If the mean weight of the students is 110.917, *find the missing frequencies.*

Solution

Let f_1 be the frequency of the class 108–112.

The frequency of the class 118–122 = 60 – (2 + 5 +12 + 14 + 3 + 1 + f_1)

= 23 – f_1

Let a = 110 be the assumed mean and h = 5 be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-110}{5}$$

Weights (lbs)	No. of students	Midvalue x	$d = \frac{x-110}{5}$	fd
93–97	2	95	–3	–6
98–102	5	100	–2	–10
103–107	12	105	–1	–12
108–112	f_1	110	0	0
113–117	14	115	1	14
118–122	$23 - f_1$	120	2	$46 - 2f_1$
123–127	3	125	3	9
128–132	1	130	4	4
	$\sum f = 60$			$\sum fd = 45 - 2f_1$

$$N = \sum f = 60$$

$$\bar{x} = a + h\frac{\sum fd}{N}$$

$$110.917 = 110 + 5\left(\frac{45 - 2f_1}{60}\right)$$

$$f_1 = 17$$

Hence, the frequency of the class 108–112 is 17 and the frequency of the class 118–122 is 23–17 = 6.

Example 3

The average salary of male employees in a company is ₹ 5200 *and that of females is* ₹ 4200. *The mean salary of all the employees is* ₹ 5000. *Find the percentage of male and female employees.*

Solution

Let $\bar{x}_1$ and $\bar{x}_2$ be the average salary of male and female employees respectively. Let n_1 and n_2 be the number of male and female employees in the company respectively. Let $\bar{x}$ be the average salary of all the employees in the company.

$$\bar{x}_1 = 5200, \bar{x}_2 = 4200, \bar{x} = 5000$$

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

$$5000 = \frac{5200\, n_1 + 4200\, n_2}{n_1 + n_2}$$

$$5000(n_1+n_2)=5200n_1+4200n_2$$
$$200n_1=800n_2$$
$$\frac{n_1}{n_2}=\frac{4}{1}$$

Hence, the percentage of male employees = $\frac{4}{4+1}\times 100=80$

and the percentage of female employees = $\frac{1}{4+1}\times 100=20$

Example 4

There are 50 *students in a class of which* 40 *are boys and the rest girls. The average weight of the class is* 44 *kg and the average weight of the girls is* 40 *kg. Find the average weight of the boys.*

Solution

Let $\bar{x}_1$ and $\bar{x}_2$ be the average weight of boys and girls respectively. Let n_1 and n_2 be the number of boys and girls in the class respectively. Let $\bar{x}$ be the average weight of all the boys and girls in the class.

$$n_1=40,\quad n_2=10,\quad \bar{x}=44,\quad \bar{x}_2=40$$
$$\bar{x}=\frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}$$
$$44=\frac{40\bar{x}_1+(10\times 40)}{40+10}$$
$$\therefore\quad \bar{x}_1=45$$

Example 5

The mean marks scored by 100 *students was found to be* 40. *Later on, it was discovered that a score of* 53 *was misread as* 83. *Find the correct mean.*

Solution

$$\bar{x}=40,\quad n=100$$
$$\bar{x}=\frac{\sum x}{n}$$
$$40=\frac{\sum x}{100}$$

$$\sum x = 4000$$

$$\text{Incorrect} \sum x = 4000$$

$$\text{Correct} \sum x = \text{Incorrect} \sum x - \text{Incorrect Item} + \text{Correct Item}$$

$$= 4000 - 83 + 53 = 3970$$

$$\therefore \quad \text{Correct mean} = \frac{\text{Correct} \sum x}{n} = \frac{3970}{100} = 397$$

EXERCISE 3.1

1. Find the mean of the following marks obtained by students of a class:

Marks	15	20	25	30	35	40
No. of Students	9	7	12	14	15	6

[Ans.: 25.58**]**

2. The following table gives the distribution of total household expenditure (in rupees) of manual workers in a city:

Expenditure (in ₹)	100–150	150–200	200–250	250–300	300–350	350–400	400–450	450–500
Frequency	24	40	33	28	30	22	16	7

Find the average expenditure (in ₹) per household.

[Ans.: ₹ 266.25**]**

3. Calculate the mean for the following data:

Heights (in cm)	135–140	140–145	145–150	150–155	155–160	160–165	165–170	170–175
No. of boys	4	9	18	28	24	10	5	2

[Ans.: 153.45 cm**]**

4. The weights in kilograms of 60 workers in a factory are given below. Find the mean weight of a worker.

Weight (in kg)	60	61	62	63	64	65
No. of workers	5	8	14	16	10	7

[Ans.: 62.65 kg**]**

5. Calculate the mean from the following data:

Marks less than/up to	10	20	30	40	50	60
No. of students	10	30	60	110	150	180

[**Ans.:** 35]

6. Calculate the mean from the following data:

Marks more than	0	10	20	30	40	50	60
No. of students	180	170	150	120	70	30	0

[**Ans.:** 35]

7. Calculate the mean from the following data:

Marks	1–5	6–10	11–15	16–20	21–25	26–30	31–35	36–40	41–45
No. of students	7	10	16	30	24	17	10	5	1

[**Ans.:** 20.33]

8. Find the missing frequency p for the following distribution whose mean is 50.

x	10	30	50	70	90
f	17	p	32	24	19

[**Ans.:** 28]

9. Find the missing value of p for the following distribution whose mean is 12.58.

x	5	8	10	12	p	20	25
f	2	5	8	22	7	4	2

[**Ans.:** 15]

10. The mean of the following frequency table is 50. But the frequencies f_1 and f_2 in the classes 20–40 and 60–80 are missing. Find the missing frequencies.

Class	0–20	20–40	40–60	60–80	80–100	Total
Frequency	17	f_1	32	f_2	19	120

[**Ans.:** $f_1 = 28, f_2 = 24$]

11. 100 students appeared in an examination. The result of the examination is as under:

Marks	4	5	6	7	8	9
No. of students	16	20	18	12	8	6

If the combined mean of the marks obtained by 100 students is 5.16, calculate the combined mean of the marks obtained by all the unsuccessful students.

[**Ans.:** 2.1]

12. Ram purchased equity shares of a company in 4 successive months. Find the average price per share.

Month	Price per share (in ₹)	No. of shares
Dec. 14	100	200
Jan. 15	150	250
Feb. 15	200	280
March 15	125	300

[**Ans.:** ₹ 146.60]

13. From the following results of two colleges *A* and *B*, find out which of the two is better.

Examination	College A		College B	
	Appeared	Passed	Appeard	Passed
M.Sc.	60	40	200	160
M.A.	100	60	240	200
B.Sc.	200	150	200	140
B.A.	120	75	160	100

[**Ans.:** College *B* is better]

14. The average rainfall for a week, excluding Sunday, was 10 cm. Due to heavy rainfall on Sunday, the average for the week rose to 15 cm. How much rainfall was recorded on Sunday?

[**Ans.:** 45 cm]

15. The arithmetic mean of 50 items of a series was calculated by a student as 20. However, it was later discovered that an item of 25 was misread as 35. Find the correct value of mean.

[**Ans.:** 19.8]

16. The sales of a balloon seller on seven days of a week are as given below:

Days	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Sales (in ₹)	100	150	125	140	160	200	250

If the profit is 20% of sales, find the average profit per day.

[**Ans.:** ₹ 32.14]

3.4 MEDIAN

Median is the central value of the variable when the values are arranged in ascending or descending order of magnitude. It divides the distribution into two equal parts. When the observations are arranged in the order of their size, median is the value of that item which has equal number of observations on either side.

In case of ungrouped data, if the number of observations is odd then the median is the middle value after the values have been arranged in ascending or descending order of magnitude. If the number of observations is even, there are two middle terms and the median is obtained by taking the arithmetic mean of the middle terms.

Examples

(i) The median of the values 20, 15, 25, 28, 18, 16, 30, i.e., 15, 16, 18, 20, 25, 28, 30 is 20 because $n = 7$, i.e., odd and the median is the middle value, i.e., 20.

(ii) The median of the values 8, 20, 50, 25, 15, 30, i.e., 8, 15, 20, 25, 30, 50 is the arithmetic mean of the middle terms, i.e., $\frac{20+25}{2} = 22.5$ because $n = 6$, i.e., even.

In case of discrete frequency distribution, the median is obtained by considering the cumulative frequencies. The steps for calculating the median are given below:

(i) Arrange the values of the variables in ascending or descending order of magnitudes.

(ii) Find $\frac{N}{2}$ where $N = \sum f$

(iii) Find the cumulative frequency just greater than $\frac{N}{2}$ and determine the corresponding value of the variable.

(iv) The corresponding value of x is the median.

Example 1

The following table represents the marks obtained by a batch of 12 *students in certain class tests in physics and chemistry.*

Marks (Physics)	53	54	32	30	60	46	28	25	48	72	33	65
Marks (Chemistry)	55	41	48	49	27	25	23	20	28	60	43	67

Indicate the subject in which the level of achievement is higher.

Solution

The level of achievement is higher in that subject for which the median marks are more. Arranging the marks in two subjects in ascending order,

Marks (Physics)	25	28	30	32	33	46	48	53	54	60	65	72
Marks (Chemistry)	20	23	25	27	28	41	43	48	49	55	60	67

Since the number of students is 12, the median is the arithmetic mean of the middle terms.

Median marks in Physics $= \dfrac{46+48}{2} = 47$

Median marks in Chemistry $= \dfrac{41+43}{2} = 42$

Since the median marks in physics are greater than the median marks in chemistry, the level of achievement is higher in physics.

Example 2

Obtain the median for the following frequency distribution.

x	0	1	2	3	4	5	6	7
f	7	14	18	36	51	54	52	18

Solution

x	f	Cumulative Frequency
0	7	7
1	14	21
2	18	39
3	36	75
4	51	126
5	54	180
6	52	232
7	18	250

$N = 250$

$\dfrac{N}{2} = \dfrac{250}{2} = 125$

The cumulative frequency just greater than $\frac{N}{2}$ = 125 is 126 and the value of x corresponding to 126 is 4. Hence, the median is 4.

Example 3

Find the median of the following distribution:

x	5	7	9	12	14	17	19	21
f	6	5	3	6	5	3	2	3

Solution

x	f	Cumulative Frequency
5	6	6
7	5	11
9	3	14
12	6	20
14	5	25
17	3	28
19	2	30
21	3	33

$$N = 33$$

$$\frac{N}{2} = \frac{33}{2} = 16.5$$

The cumulative frequency just greater than $\frac{N}{2}$ = 16.5 is 20 and the value of x corresponding to 20 is 12. Hence, the median is 12.

Median for Continuous Frequency Distribution

In case of continuous frequency distribution (less than frequency distribution), the class corresponding to the cumulative frequency just greater than $\frac{N}{2}$, is called the *median class*, and the value of the median is given by

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

where l is the lower limit of the median class

f is the frequency of the median class

h is the width of the median class

c is the cumulative frequency of the class preceding the median class

N is sum of frequencies, i.e., $N = \sum f$

In case of 'more than' or 'greater than' type of frequency distributions, the value of the median is given by

$$\text{Median} = u - \frac{h}{f}\left(\frac{N}{2} - c\right)$$

where u is the upper limit of the median class
f is the frequency of the median class
h is the width of the median class
c is the cumulative frequency of the class succeeding the median class

Example 1

The following table gives the weekly expenditures of 100 workers. Find the median weekly expenditure.

Weekly Expenditure (in ₹)	0–10	10–20	20–30	30–40	40–50
Number of workers	14	23	27	21	15

Solution

Weekly Expenditure (in ₹)	Number of Workers i.e., frequency (f)	Cumulative Frequency
0–10	14	14
10–20	23	37
20–30	27	64
30–40	21	85
40–50	15	100

$$N = 100$$

$$\frac{N}{2} = \frac{100}{2} = 50$$

The cumulative frequency just greater than $\frac{N}{2} = 50$ is 64 and the corresponding class 20–30 is the median class.

Here, $\frac{N}{2} = 50, \quad l = 20, \quad h = 10, \quad f = 27, \quad c = 37$

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

$$= 20 + \frac{10}{27}(50 - 37)$$

$$= 24.815$$

Example 2

From the following data, calculate the median:

Marks (Less than)	5	10	15	20	25	30	35	40	45
No. of Students	29	224	465	582	634	644	650	653	655

[Summer 2015]

Solution

This is a 'less than' type of frequency distribution. This will be first converted into class intervals.

Class Intervals	Frequency	Less than CF
0–5	29	29
5–10	195	224
10–15	241	465
15–20	117	582
20–25	52	634
25–30	10	644
30–35	6	650
35–40	3	653
40–45	2	655

$$N = 655$$

Since $\frac{N}{2} = \frac{655}{2} = 327.5$, the median class is 10–15.

Here, $l = 10$, $h = 5$, $f = 241$, $c = 224$

$$\begin{aligned}\text{Median} &= l + \frac{h}{f}\left(\frac{N}{2} - c\right)\\ &= 10 + \frac{5}{241}(327.5 - 224)\\ &= 12.147\end{aligned}$$

Example 3

Find the mean of the following data:

Age greater than (in years)	0	10	20	30	40	50	60	70
No. of Persons	230	218	200	165	123	73	28	8

Solution

This is a 'greater than' type of frequency distribution. This will be first converted into class intervals.

Class Intervals	Frequency	Greater than CF
0–10	12	230
10–20	18	218
20–30	35	200
30–40	42	165
40–50	50	123
50–60	45	73
60–70	20	28
70 and above	8	8

$N = 230$

Since $\frac{N}{2} = \frac{230}{2} = 115$, the median class is 40–50.

Here, $u = 50$, $h = 10$, $f = 50$, $c = 73$

$$\text{Median} = u - \frac{h}{f}\left(\frac{N}{2} - c\right)$$

$$= 50 - \frac{10}{50}(115 - 73)$$

$$= 41.6 \text{ years}$$

Example 4

The following table gives the marks obtained by 50 students in mathematics. Find the median.

Marks	10–14	15–19	20–24	25–29	30–34	35–39	40–44	45–49
No. of Students	4	6	10	5	7	3	9	6

Solution

Since the class intervals are inclusive, it is necessary to convert them into exclusive series.

Marks	No. of Students	Cumulative Frequency
9.5–14.5	4	4
14.5–19.5	6	10
19.5–24.5	10	20
24.5–29.5	5	25
29.5–34.5	7	32
34.5–39.5	3	35
39.5–44.5	9	44
44.5–49.5	6	50

$N = 50$

Since $\frac{N}{2} = \frac{50}{2} = 25$, the median class is 24.5–29.5.

Here, $l = 24.5$, $h = 5$, $f = 5$, $c = 20$

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

$$= 24.5 + \frac{5}{5}(25 - 20)$$

$$= 29.5$$

Example 5

Find the median of the following distribution:

Midvalues	1500	2500	3500	4500	5500	6500	7500
Frequency	27	32	65	78	58	32	8

Solution

The difference between two midvalues is 1000. On subtracting and adding half of this, i.e., 500 to each of the midvalues, the lower and upper limits of the respective class intervals are obtained.

Class Intervals	Frequency	Cumulative Frequency
1000–2000	27	27
2000–3000	32	59
3000–4000	65	124
4000–5000	78	202
5000–6000	58	260
6000–7000	32	292
7000–8000	8	300

$N = 300$

Since $\frac{N}{2} = 150$, the median class is 4000–5000.

Here, $l = 4000$, $h = 1000$, $f = 78$, $c = 124$

$$\text{Median } = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

$$= 4000 + \frac{1000}{78}(150 - 124)$$

$$= 4333.33$$

Example 6

The following table gives the distribution of daily wages of 900 workers. However, the frequencies of classes 40–50 and 60–70 are missing. If the median of the distribution is ₹ 59.25, find the missing frequencies.

Wages (in ₹)	30–40	40–50	50–60	60–70	70–80
No. of workers	120	?	200	?	185

Solution

Let f_1 and f_2 be the frequencies of the classes 40–50 and 60–70 respectively.

$f_1 + f_2 = 900 - (120 + 200 + 185) = 395$

Class Intervals	Frequency	Less than CF
30–40	120	120
40–50	f_1	$120 + f_1$
50–60	200	$320 + f_1$
60–70	f_2	$320 + f_1 + f_2$
70–80	185	900

$$N = 900$$

Since the median is 59.25, the median class is 50–60.

Here, $\frac{N}{2} = 450$, $l = 50$, $h = 10$, $f = 200$, $c = 120 + f_1$

$$\text{Median } = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

$$59.25 = 50 + \frac{10}{200}\left[450 - (120 + f_1)\right]$$

$$\therefore \quad f_1 = 145$$

$$f_2 = 395 - 145 = 250$$

EXERCISE 3.2

1. The heights (in cm) of 15 students of Class XII are 152, 147, 156, 149, 151, 159, 148, 160, 153, 154, 150, 143, 155, 157, 161. Find the median.

[**Ans.:** 153 cm]

2. The median of the following observations are arranged in the ascending order: 11, 12, 14, 18, x + 2, x + 4, 30, 32, 35, 41 is 24. Find x.

[**Ans.:** 21]

3. Find the median of the following frequency distribution:

x	10	11	12	13	14	15	16
f	8	15	25	20	12	10	5

[**Ans.:** 12]

4. Find the median of the following frequency distribution:

Wages (in ₹)	20–30	30–40	40–50	50–60	60–70
No. of workers	3	5	20	10	5

[**Ans.:** 46.75]

5. Calculate the median of the following data:

x	3–4	4–5	5–6	6–7	7–8	8–9	9–10	10–11
f	3	7	12	16	22	20	13	7

[Ans.: 7.55**]**

6. The weekly wages of 1000 workers of a factory are shown in the following table:

Weekly wages (less than)	425	475	525	575	625	675	725	775	825	875
No. of Workers	2	10	43	123	293	506	719	864	955	1000

[Ans.: 673.59**]**

7. Calculate the mean of the following distribution of marks obtained by 50 students in advanced engineering mathematics.

Marks more than	0	10	20	30	40	50
No. of Students	50	46	40	20	10	3

[Ans.: 27.5**]**

8. Calculate the median from the following data:

Midvalues	115	125	135	145	155	165	175	185	195
Frequency	6	25	48	72	116	60	38	22	3

[Ans.: 153.79**]**

9. The following incomplete table gives the number of students in different age groups of a town. If the median of the distribution is 11 years, find out the missing frequencies.

Age group	0–5	5–10	10–15	15–20	20–25	25–30	Total
No. of Students	15	125	?	66	?	4	300

[Ans.: 50, 40**]**

10. An incomplete frequency distribution is given as follows:

Variable	10–20	20–30	30–40	40–50	50–60	60–70	70–80	Total
Frequency	12	30	?	65	?	25	18	229

Given that the median value is 46, calculate the missing frequencies.

[Ans.: 34, 45**]**

3.5 MODE

Mode is the value which occurs most frequently in a set of observations and around which the other items of the set are heavily distributed. In other words, mode is the value of the variable which is most frequent or predominant in the series. In case of a discrete frequency distribution, mode is the value of x corresponding to the maximum frequency.

Examples

(i) In the series 6, 5, 3, 4, 3, 7, 8, 5, 9, 5, 4, the value 5 occurs most frequently. Hence, the mode is 5.

(ii) Consider the following frequency distribution:

x	1	2	3	4	5	6	7	8
f	4	9	16	25	22	15	7	3

The value of x corresponding to the maximum frequency, viz., 25, is 4. Hence, the mode is 4.

For an asymmetrical frequency distribution, the difference between the mean and the mode is approximately three times the difference between the mean and the median.

$$\text{Mean} - \text{Mode} = 3\,(\text{Mean} - \text{Median})$$

$$\text{Mode} = 3\text{ Median} - 2\text{ Mean}$$

This is known as the *empirical formula for calculation of the mode*.

3.5.1 Mode by Method of Grouping

The grouping method is used when the frequency distribution is not regular. In such cases, the difference between the maximum frequency and the frequency preceding or succeeding it is very small, and the items are heavily concentrated on either side. In such cases, the value of the mode is determined by preparing a grouping table and an analysis table.

Grouping Table

A grouping table has the following six columns:

Column I: This column has original frequencies.

Column II: In this column, the frequencies of Column I are combined ‘two by two’.

Column III: In this column, the first frequency of Column I is left out and the remaining frequencies of Column I are combined ‘two by two’.

Column IV: In this column, the frequencies of Column I are combined ‘three by three’.

Column V: In this column, the first frequency of Column I is left out and the remaining frequencies of Column I are combined ‘three by three’.

Column VI: In this column, the first two frequencies of Column I are left out and the remaining frequencies of Column I are combined ‘three by three’.

The maximum frequency in each column is identified and circled.

Analysis Table

After preparing the grouping table, an analysis table is prepared. The column numbers are put on the left-hand side and various probable values of the mode, i.e., x are put on the right-hand side. The values against which frequencies are marked maximum in the grouping table are entered in the analysis table corresponding to the values they represent.

Example 1

Calculate the mode from the following frequency distribution:

x	50	51	52	53	54	55	56	57	58	59	60
f	2	4	5	6	8	5	4	7	11	5	3

Solution

Since the frequency distribution is not regular, the method of grouping is used for calculation of the mode.

Grouping Table

x	(I) f	(II) Column of two	(III) Column of two leaving out the first	(IV) Column of three	(V) Column of three leaving out the first	(VI) Column of three leaving out the first two
50	2	6		11		
51	4		9		15	
52	5	11				19
53	6		14	19		
54	8	13			17	
55	5		9			16
56	4	11		22		
57	7		18		23	
58	11	16				19
59	5		8			
60	3					

Analysis Table

Columns ↓ \ $x \rightarrow$	50	51	52	53	54	55	56	57	58	59	60
I									1		
II									1	1	
III								1	1		
IV							1	1	1		
V								1	1	1	
VI			1	1	1				1	1	1
Σ	0	0	1	1	1	0	1	3	6	3	1

Since the value 58 has occurred the maximum number of times, the mode of the frequency distribution is 58.

Example 2

Find the mode of the following frequency distribution:

x	10	11	12	13	14	15	16	17	18	19
f	8	15	20	100	98	95	90	75	50	30

Solution

Since the frequency distribution is not regular, the method of grouping is used for calculation of the mode.

Grouping Table

x	(I) f	(II) Column of two	(III) Column of two leaving out the first	(IV) Column of three	(V) Column of three leaving out the first	(VI) Column of three leaving out the first two
10	8	23		43		
11	15		35		135	
12	20	120				218
13	100		198	293		
14	98	193			283	
15	95		185			260
16	90	165		215		
17	75		125		155	
18	50	80				
19	30					

Analysis Table

Columns ↓ / $x \rightarrow$	10	11	12	13	14	15	16	17	18	19
I				1						
II					1	1				
III				1	1					
IV				1	1	1				
V					1	1	1			
VI						1	1	1		
$\sum$	0	0	0	3	4	4	2	1	0	0

Since the values 14 and 15 have occurred the maximum number of times, the mode is ill-defined. In such a case, mode = 3 median – 2 mean.

Calculation of Mean and Median

x	f	fx	CF
10	8	80	8
11	15	165	23
12	20	240	43
13	100	1300	143
14	98	1372	241
15	95	1425	336
16	90	1440	426
17	75	1275	501
18	50	900	551
19	30	570	581
	$\sum f = 581$	$\sum fx = 8767$	

$$N = \sum f = 581$$

$$\bar{x} = \frac{\sum fx}{N} = \frac{8767}{581} = 15.09$$

$$\frac{N}{2} = \frac{581}{2} = 290.5$$

The cumulative frequency just greater than $\frac{N}{2} = 290.5$ is 95 and the value of x corresponding to 95 is 15. Hence, the median is 15.

Mode = 3 × 15 – 2 × 15.09 = 14.82

3.5.2 Mode for a Continuous Frequency Distribution

In case of a continuous frequency distribution, the class in which the mode lies is called the *modal class* and the value of the mode is given by

$$\text{Mode} = l + h\left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right)$$

where l is the lower limit of the modal class

h is the width of the modal class

f_m is the frequency of the modal class

f_1 is the frequency of the class preceding the modal class

f_2 is the frequency of the class succeeding the modal class

This method of finding mode is called the *method of interpolation*. This formula is applicable only to a unimodal frequency distribution.

Example 1

Find the mode for the following data:

Profit per shop	0–100	100–200	200–300	300–400	400–500	500–600
No. of Shops	12	18	27	20	17	6

Solution

Since the maximum frequency is 27 which lies in the class 200–300, the modal class is 200–300.

Here, $l = 200$, $h = 100$, $f_m = 27$, $f_1 = 18$, $f_2 = 20$

$$\text{Mode} = l + h\left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right)$$

$$= 200 + 100\left[\frac{27 - 18}{2(27) - 18 - 20}\right]$$

$$= 256.25$$

Example 2

The frequency distribution of marks obtained by 60 students of a class in a college is given by

Marks	30–34	35–39	40–44	45–49	50–54	55–59	60–64
Frequency	3	5	12	18	14	6	2

Find the mode of the distribution.

Solution

The class intervals are first converted into a continuous exclusive series as shown in the following table:

Marks	Frequency
29.5–34.5	3
34.5–39.5	5
39.5–44.5	12
44.5–49.5	18
49.5–54.5	14
54.5–59.5	6
59.5–64.5	2

Since the maximum frequency is 18 which lies in the interval 44.5–49.5, the modal class is 44.5–49.5.

Here, $l = 44.5$, $h = 5$, $f_m = 18$, $f_1 = 12$, $f_2 = 14$

$$\text{Mode} = l + h\left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right)$$

$$= 44.5 + 5\left[\frac{18-12}{2(18)-12-14}\right]$$

$$= 47.5$$

Example 3

Calculate the mode of the following distribution:

Midvalues	5	15	25	35	45	55	65	75
Frequency	7	15	18	30	31	4	3	1

Solution

Since the frequency distribution is not regular, the method of grouping is used for calculation of the mode.

The difference between the two midvalues is 10. On subtracting and adding half of this, i.e., 5, to each of the midvalues, the lower and upper limits of the respective class intervals are obtained.

Grouping Table

	I	II	III	IV	V	VI
Class Intervals	f	Column of two	Column of two leaving out the first	Column of three	Column of three leaving out the first	Column of three leaving out the first two
0–10	7	22		40		
10–20	15		33		63	
20–30	18	48				79
30–40	30		61	65		
40–50	31	35			38	
50–60	4		7			8
60–70	3	4				
70–80	1					

Analysis Table

Columns ↓	$x \rightarrow$ 10–20	20–30	30–40	40–50	50–60
I				1	
II		1	1		
III			1	1	
IV			1	1	1
V	1	1	1		
VI		1	1	1	
	1	3	5	4	1

From the analysis table, the modal class is 30–40.

Here, $l = 30$, $h = 10$, $f_m = 30$, $f_1 = 18$, $f_2 = 31$

$$\text{Mode} = l + h\left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right)$$

$$= 30 + 10\left[\frac{30 - 18}{2(30) - 18 - 31}\right]$$

$$= 40.91$$

Example 4

Find the mode for the following distribution:

Class intervals	0–10	10–20	20–30	30–40	40–50
Frequency	45	20	14	7	3

Solution

Since the highest frequency occurs in the first class interval, the interpolation formula is not applicable. Thus, empirical formula is used for calculation of mode.

Class intervals	Frequency	CF	Midvalue	$d = \frac{x - 25}{10}$	fd
0–10	45	45	5	–2	–90
10–20	20	65	15	–1	–20
20–30	14	79	25	0	0
30–40	7	86	35	1	7
40–50	3	89	45	2	6
	$\sum f = 89$				$\sum fd = -97$

$$N = \sum f = 89$$

Since $\frac{N}{2} = \frac{89}{2} = 44.5,$ the median class is 0–10.

Here, $l = 0, \quad h = 10, \quad f = 45, \quad c = 0$

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

$$= 0 + \frac{10}{45}(44.5 - 0)$$

$$= 9.89$$

$$\text{Mean} = a + h\frac{\sum fd}{N}$$

$$= 25 + 10\left(\frac{-97}{89}\right)$$

$$= 14.1$$

$$\text{Hence, mode} = 3\,\text{Median} - 2\,\text{Mean}$$

$$= 3(9.89) - 2(14.1)$$

$$= 1.47$$

Example 5

The following table gives the incomplete income distribution of 300 *workers of a company, where the frequencies of the classes* 3000–4000 *and* 5000–6000 *are missing. If the mode of the distribution is* ₹ 4428.57, *find the missing frequencies.*

Monthly Income (in ₹)	No. of Workers
1000–2000	30
2000–3000	35
3000–4000	?
4000–5000	75
5000–6000	?
6000–7000	30
7000–8000	15

Solution

Let f_1 and f_2 be the frequencies of the classes 3000–4000 and 5000–6000 respectively.

$$f_1 + f_2 = 300 - (30 + 35 + 75 + 30 + 15) = 115$$

Since the mode is 4428.57, the modal class is 4000–5000.

Here, $l = 4000$, $h = 1000$, $f_m = 75$

$$\text{Mode} = l + h\left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right)$$

$$= l + h\left[\frac{f_m - f_1}{2f_m - (f_1 + f_2)}\right]$$

$$4428.57 = 4000 + 1000\left[\frac{75 - f_1}{2(75) - 115}\right]$$

$$\therefore \quad f_1 = 60$$

$$f_2 = 115 - 60 = 55$$

EXERCISE 3.3

1. Calculate the mode for the following distribution:

x	6	12	18	24	30	36
f	12	24	36	38	37	6

[**Ans.:** 24]

2. Calculate the mode for the following distribution:

x	10	20	30	40	50	60	70
f	17	22	31	39	27	15	13

[**Ans.:** 40]

3. Calculate the mode for the following distribution:

Class interval	0–4	4–8	8–12	12–16
Frequency	4	8	5	6

[**Ans.:** 6.28]

4. Calculate the mode of the following distribution:

x	0–5	5–10	10–15	15–20	20–25	25–30	30–35	35–40	40–45
f	20	24	32	28	20	16	37	10	18

[**Ans.:** 13.33]

5. Calculate the mode for the following data:

Class	10–20	20–30	30–40	40–50	50–60	60–70	70–80
f	24	42	56	66	108	130	154

[**Ans.:** 71.348]

6. Find the mode of the following distribution:

Class	55–64	65–74	75–84	85–94	95–104	105–114	115–124	125–134	135–144
f	1	2	9	22	33	22	8	2	1

[**Ans.:** 99.5]

7. Calculate the modal marks from the following distribution of marks of 100 students of a class:

Marks (more than)	90	80	70	60	50	40	30	20	10
No. of Students	0	4	15	33	53	76	92	98	100

[**Ans.:** 47]

8. If the mode and mean of a moderately asymmetrical series are 80 and 68, what will be the most probable median?

[**Ans.:** 72]

9. Calculate the mode from the following data:

Midpoint	1	2	3	4	5	6	7	8
Frequency	5	50	45	30	20	10	15	5

[**Ans.:** 2.875]

10. Calculate the mode from the following series:

Class intervals	10–19	20–29	30–39	40–49	50–59	60–69
Frequency	4	6	8	5	4	2

[**Ans.:** 33.5]

11. Calculate the mode from the following distribution:

Marks (less than)	7	14	21	28	35	42	49
No. of Students	20	25	33	41	45	50	52

[**Ans.:** 11.26]

12. Calculate the mode from the following distribution:

Class intervals	6–10	11–15	16–20	21–25	26–30
Frequency	20	30	50	40	10

[**Ans.:** 18.83]

3.6 GEOMETRIC MEAN

The *geometric mean* of a set of n observations is the n^{th} root of their product. If there are n observations, $x_1, x_2, .., x_n$ such that $x_i > 0$ for each i, their geometric mean GM is given by

$$\text{GM} = (x_1 \cdot x_2 \cdot \cdots \cdot x_n)^{\frac{1}{n}}$$

The n^{th} root is calculated with the help of logarithms. Taking logarithms of both the sides,

$$\begin{aligned}\log \text{GM} &= \log(x_1 \cdot x_2 \cdots x_n)^{\frac{1}{n}}\\ &= \frac{1}{n}\log(x_1 \cdot x_2 \cdots x_n)\\ &= \frac{1}{n}(\log x_1 + \log x_2 + \cdots + \log x_n)\\ &= \frac{\sum \log x}{n}\end{aligned}$$

$$\text{GM} = \text{antilog}\left(\frac{\sum \log x}{n}\right)$$

In case of a frequency distribution consisting of n observations $x_1, x_2, .., x_n$ with respective frequencies $f_1, f_2, .., f_n$, the geometric mean is given by

$$\text{GM} = (x_1^{f_1} \cdot x_2^{f_2} \cdots x_n^{f_n})^{\frac{1}{N}}, \text{ where } N = \sum f$$

Taking logarithms of both the sides,

$$\begin{aligned}\log \text{GM} &= \frac{1}{N}(f_1 \log x_1 + f_2 \log x_2 + \cdots + f_n \log x_n)\\ &= \frac{\sum f \log x}{N}\end{aligned}$$

$$\text{GM} = \text{antilog}\left(\frac{\sum f \log x}{N}\right)$$

Thus, the geometric mean is the antilog of the weighted mean of the different values of log x_i whose weights are their frequencies f_i.

In case of a continuous or grouped frequency distribution, x is taken to be the value corresponding to the midpoints of the class intervals.

Example 1

Calculate the geometric mean of the following data:

10, 110, 120, 50, 52, 80

Solution

$$\text{GM} = \text{antilog}\left(\frac{\sum \log x}{n}\right)$$

$$= \text{antilog}\left(\frac{\log 10 + \log 110 + \log 120 + \log 50 + \log 52 + \log 80}{6}\right)$$

$$= \text{antilog}\left(\frac{2.3026 + 4.7005 + 4.7875 + 3.9120 + 3.9512 + 4.3820}{6}\right)$$

$$= \text{antilog}\ (4.006)$$

$$= 54.9267$$

Example 2

Find the geometric mean of the following data:

x	5	10	15	20	25	30
f	13	18	50	40	10	6

Solution

x	f	$\log x$	$f \log x$
5	13	1.6094	20.9227
10	18	2.3026	41.4465
15	50	2.7081	135.4025
20	40	2.9957	119.8293
25	10	3.2189	32.1888
30	6	3.4012	20.4072
	$\sum f = 137$		$\sum f \log x = 370.197$

$$N = \sum f = 137$$

$$\text{GM} = \text{antilog}\left(\frac{\sum f \log x}{N}\right)$$
$$= \text{antilog}\left(\frac{370.197}{137}\right)$$
$$= 14.912$$

Example 3

Find the geometric mean of the following data:

Marks	0–10	10–20	20–30	30–40
No. of Students	5	8	3	4

Solution

Marks	No. of Students f	Midvalue x	$\log x$	$f \log x$
0–10	5	5	1.6094	8.047
10–20	8	15	2.7081	21.6648
20–30	3	25	3.2189	9.6567
30–40	4	35	3.5553	14.2212
	$\sum f = 20$			$\sum f \log x = 53.5897$

$$N = \sum f = 20$$
$$\text{GM} = \text{antilog}\left(\frac{\sum f \log x}{N}\right)$$
$$= \text{antilog}\left(\frac{53.5897}{20}\right)$$
$$= 14.5776$$

3.7 HARMONIC MEAN

The *harmonic mean* of a number of observations, none of which is zero, is the reciprocal of the arithmetic mean of the reciprocals of the given values.

The harmonic mean of n observations $x_1, x_2, ..., x_n$ is given by

$$\text{HM} = \frac{1}{\frac{1}{n}\sum\left(\frac{1}{x}\right)}$$

$$= \frac{1}{\frac{1}{n}\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)}$$

$$= \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}$$

For example, the harmonic mean of 2, 4 and 5 is

$$\text{HM} = \frac{3}{\frac{1}{2}+\frac{1}{4}+\frac{1}{5}} = 3.16$$

In case of a frequency distribution consisting of n observations $x_1, x_2, ..., x_n$ with respective frequencies $f_1, f_2, ..., f_n$, the harmonic mean is given by

$$\text{HM} = \frac{f_1+f_2+\cdots+f_n}{\frac{f_1}{x_1}+\frac{f_2}{x_2}+\cdots+\frac{f_n}{x_n}}$$

$$= \frac{\sum f}{\sum\left(\frac{f}{x}\right)}$$

If $x_1, x_2, ..., x_n$ are n observations with weights $w_1, w_2, ..., w_n$ respectively, their weighted harmonic mean is given by

$$\text{HM} = \frac{\sum w}{\sum\left(\frac{w}{x}\right)}$$

Example 1

Calculate the harmonic mean of the following data:

x	20	21	22	23	24	25
f	4	2	7	1	3	1

Solution

x	f	$\frac{f}{x}$
20	4	0.2
21	2	0.095
22	7	0.318
23	1	0.043
24	3	0.125
25	1	0.04
	$\sum f = 18$	$\sum\left(\frac{f}{x}\right) = 0.821$

$$\text{HM} = \frac{\sum f}{\sum\left(\frac{f}{x}\right)} = \frac{18}{0.821} = 21.924$$

Example 2

Find the harmonic mean of the following distribution:

Class interval	0–10	10–20	20–30	30–40	40–50	50–60	60–70	70–80
Frequency	5	8	11	21	35	30	22	18

Solution

Class Interval	Frequency f	Midvalue x	$\frac{f}{x}$
0–10	5	5	1
10–20	8	15	0.533
20–30	11	25	0.44
30–40	21	35	0.6
40–50	35	45	0.778
50–60	30	55	0.545
60–70	22	65	0.338
70–80	18	75	0.24
	$\sum f = 150$		$\sum\left(\frac{f}{x}\right) = 4.474$

$$\text{HM} = \frac{\sum f}{\sum\left(\frac{f}{x}\right)} = \frac{150}{4.474} = 33.527$$

Relation between Arithmetic Mean, Geometric Mean, and Harmonic Mean

The arithmetic mean (AM), geometric mean (GM), and harmonic mean (HM) for a given set of observations of a series are related as

$$\text{AM} \geq \text{GM} \geq \text{HM}$$

For two observations x_1 and x_2 of a series,

$$\text{AM} = \frac{x_1 + x_2}{2}$$

$$\text{GM} = \sqrt{x_1\ x_2}$$

$$\text{HM} = \frac{2}{\frac{1}{x_1} + \frac{1}{x_2}} = \frac{2x_1\ x_2}{x_1 + x_2}$$

$$\text{AM} \cdot \text{HM} = \left(\frac{x_1 + x_2}{2}\right)\left(\frac{2x_1\ x_2}{x_1 + x_2}\right) = x_1\ x_2 = (\text{GM})^2$$

$$\therefore \qquad \text{GM} = \sqrt{\text{AM} \cdot \text{HM}}$$

Example 1

If the AM of two observations is 15 *and their GM is* 9, *find their HM and the two observations.*

Solution

$$\text{GM} = \sqrt{\text{AM} \cdot \text{HM}}$$

$$9 = \sqrt{15 \times \text{HM}}$$

$$\therefore \quad \text{HM} = 5.4$$

Let the two observations be x_1 and x_2.

$$\text{AM} = \frac{x_1 + x_2}{2} = 15$$

$$x_1 + x_2 = 30 \qquad \text{...(1)}$$

$$\text{GM} = \sqrt{x_1\ x_2} = 9$$

$$x_1\ x_2 = 81 \qquad \text{...(2)}$$

Solving Eqs (1) and (2),

$$x_1 = 27, x_2 = 3$$

EXERCISE 3.4

1. Calculate the geometric and harmonic means of the following series of monthly expenditure of a batch of students:

₹	125	130	75	10	45	0.5	0.4	500	1505

[**Ans.:** ₹ 22.98, ₹ 2.06]

2. Calculate the geometric mean of the following distribution:

Class intervals	5–15	15–25	25–35	35–45	45–55
Frequency	10	22	25	20	8

[**Ans.:** 26.65]

3. Calculate the harmonic mean of the following data:

x	10	11	12	13	14
f	5	8	10	9	6

[**Ans.:** 11.94]

4. An investor buys ₹ 1200 worth of shares in a company each month. During the first 5 months, he bought the shares at a price of ₹ 10, ₹ 12, ₹ 15, ₹ 20, ₹ 24 per share. After 5 months, what is the average price paid for the shares?

[**Ans.:** ₹ 14.63]

5. Calculate the geometric mean of the following distribution:

Marks (less than)	10	20	30	40	50
No. of Students	12	27	72	93	100

[**Ans.:** 21.35]

6. Calculate the GM and HM for the following data:

Class intervals	5–15	15–25	25–35	35–45	45–55
Frequency	6	9	15	8	4

[**Ans.:** 26.07, 22.92]

3.8 STANDARD DEVIATION

Standard deviation is the positive square root of the arithmetic mean of the squares of the deviations of the given values from their arithmetic mean. It is denoted by the Greek letter σ. Let X be a random variable which takes on values, viz., $x_1, x_2, ..., x_n$. The standard deviation of these n observations is given by

$$\sigma = \sqrt{\frac{\sum (x-\overline{x})^2}{n}}$$

where $\overline{x} = \dfrac{\sum x}{n}$ is the arithmetic mean of these observations.

This equation can be modified further.

$$\sigma = \sqrt{\frac{\sum (x^2 - 2x\overline{x} + \overline{x})^2}{n}}$$

$$= \sqrt{\frac{\sum x^2 - 2\overline{x}\sum x + \overline{x}^2 \sum 1}{n}}$$

$$= \sqrt{\frac{\sum x^2}{n} - 2\frac{\sum x}{n}\frac{\sum x}{n} + \left(\frac{\sum x}{n}\right)^2 \cdot \frac{n}{n}} \qquad \left[\because \sum 1 = n\right]$$

$$= \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

$$= \sqrt{\text{Mean of squares} - \text{Square of mean}}$$

In case of a frequency distribution consisting of n observations $x_1, x_2, ..., x_n$ with respective frequencies $f_1, f_2, ..., f_n$, the standard deviation is given by

$$\sigma = \sqrt{\frac{\sum f(x-\overline{x})^2}{N}}$$

This equation can also be modified.

$$\sigma = \sqrt{\frac{\sum f\,(x^2 - 2x\overline{x} + \overline{x}^2)}{N}}$$

$$= \sqrt{\frac{\sum fx^2}{N} - \frac{2\overline{x}\sum fx}{N} + \overline{x}^2\frac{\sum f}{N}}$$

$$= \sqrt{\frac{\sum fx^2}{N} - 2\frac{\sum fx}{N}\frac{\sum fx}{N} + \left(\frac{\sum fx}{N}\right)^2} \qquad \left[\because \sum f = N \text{ and } \overline{x} = \frac{\sum fx}{N}\right]$$

$$= \sqrt{\frac{\sum fx^2}{N} - \left(\frac{\sum fx}{N}\right)^2}$$

3.8.1 Variance

The *variance* is the square of the standard deviation and is denoted by σ^2. The method for calculating variance is same as that given for the standard deviation.

Example 1

Calculate the standard deviation of the weights of ten persons.

Weight (in kg)	45	49	55	50	41	44	60	58	53	55

Solution

$$n = 10$$

$$\sum x = 45 + 49 + 55 + 50 + 41 + 44 + 60 + 58 + 53 + 55 = 510$$

$$\sum x^2 = 45^2 + 49^2 + 55^2 + 50^2 + 41^2 + 44^2 + 60^2 + 58^2 + 53^2 + 55^2 = 26366$$

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

$$= \sqrt{\frac{26366}{10} - \left(\frac{510}{10}\right)^2}$$

$$= 5.967$$

Aliter:

$$\bar{x} = \frac{\sum x}{n} = \frac{510}{10} = 51$$

x	$x - \bar{x}$	$(x - \bar{x})^2$
45	–6	36
49	–2	4
55	4	16
50	–1	1
41	–10	100
44	–7	49
60	9	81
58	7	49
53	2	4
55	4	16
		$\sum(x - \bar{x})^2 = 356$

$$\sigma = \sqrt{\frac{\sum (x-\bar{x})^2}{n}}$$
$$= \sqrt{\frac{356}{10}}$$
$$= 5.967$$

Example 2

Calculate the standard deviation of the following data:

x	10	11	12	13	14	15	16	17	18
f	2	7	10	12	15	11	10	6	3

Solution

x	f	fx	x^2	fx^2
10	2	20	100	200
11	7	77	121	847
12	10	120	144	1440
13	12	156	169	2028
14	15	210	196	2940
15	11	165	225	2475
16	10	160	256	2560
17	6	102	289	1734
18	3	54	324	972
	$\sum f = 76$	$\sum fx = 1064$		$\sum fx^2 = 15196$

$$N = \sum f = 76$$
$$\sigma = \sqrt{\frac{\sum fx^2}{N} - \left(\frac{\sum fx}{N}\right)^2}$$
$$= \sqrt{\frac{15196}{76} - \left(\frac{1064}{76}\right)^2}$$
$$= 1.987$$

Aliter:

$$N = \sum f = 76$$
$$\bar{x} = \frac{\sum fx}{N} = \frac{1064}{76} = 14$$

x	f	$x-\overline{x}$	$(x-\overline{x})^2$	$f(x-\overline{x})^2$
10	2	–4	16	32
11	7	–3	9	63
12	10	–2	4	40
13	12	–1	1	12
14	15	0	0	0
15	11	1	1	11
16	10	2	4	40
17	6	3	9	54
18	3	4	16	48
				$\sum f(x-\overline{x})^2 = 300$

$$\sigma = \sqrt{\frac{\sum f\,(x-\overline{x})^2}{N}}$$

$$= \sqrt{\frac{300}{76}}$$

$$= 1.987$$

3.8.2 Standard Deviation from the Assumed Mean

If the values of x and f are large, the calculation of fx, fx^2 becomes tedious. In such a case, the assumed mean a is taken to simplify the calculation.

Let a be the assumed mean.

$$d = x - a$$

$$x = a + d$$

$$\sum fx = \sum f\,(a+d) = Na + \sum fd$$

Dividing both the sides by N,

$$\frac{\sum fx}{N} = a + \frac{\sum fd}{N}$$

$$\overline{x} = a + \overline{d}$$

$$x - \overline{x} = d - \overline{d}$$

$$\sigma_x = \sqrt{\frac{\sum f\,(x-\overline{x})^2}{N}}$$

$$= \sqrt{\frac{\sum f\,(d-\overline{d})^2}{N}}$$

$$= \sigma_d$$

Hence, the standard deviation is independent of change of origin.

$$\therefore \qquad \sigma_x = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

Example 1

Find the standard deviation from the following data:

Size of the item	10	11	12	13	14	15	16
Frequency	2	7	11	15	10	4	1

Solution

Let $a = 13$ be the assumed mean.

$$d = x - a = x - 13$$

Size of item (x)	Frequency (f)	$d = x - a$	d^2	fd	fd^2
10	2	–3	9	–6	18
11	7	–2	4	–14	28
12	11	–1	1	–11	11
13	15	0	0	0	0
14	10	1	1	10	10
15	4	2	4	8	16
16	1	3	9	3	9
	$\sum f = 50$			$\sum fd = -10$	$\sum fd^2 = 92$

$$N = \sum f = 50$$

$$\sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

$$= \sqrt{\frac{92}{50} - \left(\frac{-10}{50}\right)^2}$$

$$= 1.342$$

3.8.3 Standard Deviation by Step-Deviation Method

Let a be the assumed mean and h be the width of the class interval.

$$d = \frac{x - a}{N}$$

$$x = a + hd$$

$$\sum fx = \sum f\,(a + hd) = Na + h\sum fd$$

Dividing both the sides by N,

$$\frac{\sum fx}{N} = a + h\frac{\sum fd}{N}$$

$$\overline{x} = a + h\overline{d}$$

$$x - \overline{x} = h(d - \overline{d})$$

$$\sigma_x = \sqrt{\frac{\sum f(x-\overline{x})^2}{N}}$$

$$= \sqrt{\frac{\sum f\, h^2\, (d-\overline{d})^2}{N}}$$

$$= h\sqrt{\frac{\sum f\, (d-\overline{d})^2}{N}}$$

$$= h\,\sigma_d$$

Hence, the standard deviation is independent of change of origin but not of scale.

$$\therefore \qquad \sigma_x = h\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

Example 1

Find the standard deviation for the following distribution:

Marks	10–20	20–30	30–40	40–50	50–60	60–70	70–80
Number of Students	5	12	15	20	10	4	2

Solution

Let $a = 45$ be the assumed mean and $h = 10$ be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-45}{10}$$

Marks	Number of students f	Midvalue x	$d = \frac{x-45}{10}$	d^2	fd	fd^2
10–20	5	15	–3	9	–15	45
20–30	12	25	–2	4	–24	48
30–40	15	35	–1	1	–15	15
40–50	20	45	0	0	0	0
50–60	10	55	1	1	10	10
60–70	4	65	2	4	8	16
70–80	2	75	3	9	6	18
	$\sum f = 68$				$\sum fd = -30$	$\sum fd^2 = 152$

$$N = \sum f = 68$$

$$\sigma = h\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

$$= 10\sqrt{\frac{152}{68} - \left(\frac{-30}{68}\right)^2}$$

$$= 14.285$$

Example 2

Find the standard deviation for the following data:

Class interval	0–10	10–20	20–30	30–40	40–50	50–60	60–70
Frequency	6	14	10	8	1	3	8

Solution

Let $a = 35$ be the assumed mean and $h = 10$ be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-35}{10}$$

Class interval	f	Midvalue x	$d = \frac{x-35}{10}$	d^2	fd	fd^2
0–10	6	5	–3	9	–18	54
10–20	14	15	–2	4	–28	56
20–30	10	25	–1	1	–10	10
30–40	8	35	0	0	0	0
40–50	1	45	1	1	1	1
50–60	3	55	2	4	6	12
60–70	8	65	3	9	24	72
	$\sum f = 50$				$\sum fd = -25$	$\sum fd^2 = 205$

$$N = \sum f = 50$$

$$\sigma = h\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

$$= 10\sqrt{\frac{205}{50} - \left(\frac{-25}{50}\right)^2}$$

$$= 19.62$$

Example 3

Find the standard deviation for the following data:

Age (in years)	10–19	20–29	30–39	40–49	50–59	60–69	70–79	80–89
Number of cases	1	0	1	10	17	38	9	3

Solution

Let $a = 44.5$ be the assumed mean and $h = 10$ be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-44.5}{10}$$

Age (in years)	No. of cases f	Midvalue x	$d = \frac{x-44.5}{10}$	fd	fd^2
10–19	1	14.5	–3	–3	9
20–29	0	24.5	–2	0	0
30–39	1	34.5	–1	–1	1
40–49	10	44.5	0	0	0
50–59	17	54.5	1	17	17
60–69	38	64.5	2	76	152
70–79	9	74.5	3	27	81
80–89	3	84.5	4	12	48
	$\sum f = 79$			$\sum fd = 128$	$\sum fd^2 = 308$

$$N = \sum f = 79$$

$$\sigma = h\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

$$= 10\sqrt{\frac{308}{79} - \left(\frac{128}{79}\right)^2}$$

$$= 11.285$$

Example 4

Find the mean and standard deviation of the following distribution:

Age (in years)	No. of Persons
less than 20	0
less than 25	170
less than 30	280
less than 35	360
less than 40	405
less than 45	445
less than 50	480

Solution

This is a 'less than' type of frequency distribution. This is first converted into an exclusive series. Let $a = 32.5$ be the assumed mean and $h = 5$ be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-32.5}{5}$$

Class intervals	No. of Persons f	Midvalue x	$d = \dfrac{x-32.5}{5}$	fd	fd^2
20–25	170	22.5	–2	–340	680
25–30	110	27.5	–1	–110	110
30–35	80	32.5	0	0	0
35–40	45	37.5	1	45	45
40–45	40	42.5	2	80	160
45–50	35	47.5	3	105	315
	$\sum f = 480$			$\sum fd = -220$	$\sum fd^2 = 1310$

$$N = \sum f = 480$$

$$\begin{aligned}\bar{x} &= a + h\frac{\sum fd}{N}\\ &= 32.5 + 5\left(\frac{-220}{480}\right)\\ &= 30.21 \text{ years}\end{aligned}$$

$$\sigma = h\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

$$= 5\sqrt{\frac{1310}{480} - \left(\frac{-220}{480}\right)^2}$$

$$= 7.94 \text{ years}$$

Example 5

A student obtained the mean and standard deviation of 100 *observations as* 40 *and* 5.1 *respectively. It was later discovered that he had wrongly copied down an observation as* 50 *instead of* 40. *Calculate the correct mean and standard deviation.*

Solution

$$n = 100, \quad \bar{x} = 40, \quad \sigma = 5.1$$

$$\bar{x} = \frac{\sum x}{n}$$

$$40 = \frac{\sum x}{100}$$

$$\therefore \quad \sum x = 4000$$

$$\text{Correct} \sum x = \text{Uncorrect} \sum x - \text{Wrong observation} + \text{Correct observation}$$

$$= 4000 - 50 + 40$$

$$= 3990$$

$$\text{Correct} \quad \bar{x} = \frac{\text{Correct} \sum x}{n}$$

$$= \frac{3990}{100}$$

$$= 39.9$$

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

$$5.1 = \sqrt{\frac{\sum x^2}{100} - \left(\frac{4000}{100}\right)^2}$$

$$\therefore \quad \sum x^2 = 162601$$

Correct $\sum x^2$ = Uncorrect $\sum x^2$ – Wrong observation + Correct observation

$= 162601 - (50)^2 + (40)^2$

$= 161701$

Correct $\quad \sigma = \sqrt{\dfrac{\text{Correct} \sum x^2}{n} - \left(\dfrac{\text{Correct} \sum x}{n}\right)^2}$

$$= \sqrt{\frac{161701}{100} - \left(\frac{3990}{100}\right)^2}$$

$$= 5$$

3.8.4 Coefficient of Variation

The *standard deviation* is an absolute measure of dispersion. The coefficient of variation is a relative measure of dispersion and is denoted by CV.

$$\text{CV} = \frac{\sigma}{\overline{x}} \times 100$$

where σ is the standard deviation and $\overline{x}$ is the mean of the given series. The coefficient of variation has great practical significance and is the best measure of comparing the variability of two series. The series or groups for which the coefficient of variation is greater is said to be more variable or less consistent. On the other hand, the series for which the variation is lesser is said to be less variable or more consistent.

Example 1

The arithmetic mean of the runs scored by three batsmen Amit, Sumeet, and Nayan in the series are 50, 48, *and* 12 *respectively. The standard deviations of their runs are* 15, 12, *and* 2 *respectively. Who is the more consistent of the three?*

Solution

Let $\overline{x}_1, \overline{x}_2, \overline{x}_3$ be the arithmetic means and σ_1, σ_2, σ_3 be the standard deviations of the runs scored by Amit, Sumeet, and Nayan.

$$\overline{x}_1 = 50, \overline{x}_2 = 48, \overline{x}_3 = 12, \sigma_1 = 15, \sigma_2 = 12, \sigma_3 = 2$$

$$\text{CV}_1 = \frac{\sigma_1}{\overline{x}_1} \times 100$$

$$= \frac{15}{50} \times 100$$

$$= 30\%$$

$$\begin{aligned}
CV_2 &= \frac{\sigma_2}{\bar{x}_2} \times 100 \\
&= \frac{12}{48} \times 100 \\
&= 25\%
\end{aligned}$$

$$\begin{aligned}
CV_3 &= \frac{\sigma_3}{\bar{x}_3} \times 100 \\
&= \frac{2}{12} \times 100 \\
&= 16.67\%
\end{aligned}$$

Since the coefficient of variation of Nayan is least, he is the most consistent.

Example 2

The runs scored by two batsmen A and B in 9 consecutive matches are given below:

A	85	20	62	28	74	5	69	4	13
B	72	4	15	30	59	15	49	27	26

Which of the batsmen is more consistent?

Solution

$$n = 9$$

For the batsman A,

$$\sum x_A = 85 + 20 + 62 + 28 + 74 + 5 + 69 + 4 + 13 = 360$$

$$\sum x_A^2 = 85^2 + 20^2 + 62^2 + 28^2 + 74^2 + 5^2 + 69^2 + 4^2 + 13^2 = 22700$$

$$\begin{aligned}
\sigma_A &= \sqrt{\frac{\sum x_A^2}{n} - \left(\frac{\sum x_A}{n}\right)^2} \\
&= \sqrt{\frac{22700}{9} - \left(\frac{360}{9}\right)^2} \\
&= 30.37
\end{aligned}$$

$$\bar{x}_A = \frac{\sum x_A}{n} = \frac{360}{9} = 40$$

$$\begin{aligned}\text{CV}_A &= \frac{\sigma_A}{\bar{x}_A} \times 100\\ &= \frac{30.37}{40} \times 100\\ &= 75.925\%\end{aligned}$$

For the batsman B,

$$\sum x_B = 72 + 4 + 15 + 30 + 59 + 15 + 49 + 27 + 26 = 297$$

$$\sum x_B^2 = 72^2 + 4^2 + 15^2 + 30^2 + 59^2 + 15^2 + 49^2 + 27^2 + 26^2 = 13837$$

$$\begin{aligned}\sigma_B &= \sqrt{\frac{\sum x_B^2}{n} - \left(\frac{\sum x_B}{n}\right)^2}\\ &= \sqrt{\frac{13837}{9} - \left(\frac{297}{9}\right)^2}\\ &= 21.18\end{aligned}$$

$$\bar{x}_B = \frac{\sum x_B}{n} = \frac{297}{9} = 33$$

$$\begin{aligned}\text{CV}_B &= \frac{\sigma_B}{\bar{x}_B} \times 100\\ &= \frac{21.18}{33} \times 100\\ &= 64.18\%\end{aligned}$$

Since $\text{CV}_B < \text{CV}_A$, the batsman B is more consistent.

Example 3

The following is the record goals scored by Team A in a football season:

No. of goals scored by Team A in the match	0	1	2	3	4
No. of matches played in the month	1	9	7	5	3

For Team B, the average number of goals scored per match was 2.5 *with a SD of* 1.25 *goals. Find which team may be considered more consistent.*

Solution

$$N = 1 + 9 + 7 + 5 + 3 = 25$$

For Team A,

$$\sum fx_A = (1\times 0)+(9\times 1)+(7\times 2)+(5\times 3)+(3\times 4) = 50$$

$$\sum fx_A^2 = (1\times 0^2)+(9\times 1^2)+(7\times 2^2)+(5\times 3^2)+(3\times 4^2) = 130$$

$$\sigma_A = \sqrt{\frac{\sum fx_A^2}{N} - \left(\frac{\sum fx_A}{N}\right)^2}$$

$$= \sqrt{\frac{130}{25} - \left(\frac{50}{25}\right)^2}$$

$$= 1.095$$

$$\bar{x}_A = \frac{\sum fx_A}{n} = \frac{50}{25} = 2$$

$$\text{CV}_A = \frac{\sigma_A}{\bar{x}_A}\times 100$$

$$= \frac{1.095}{2}\times 100$$

$$= 54.75\%$$

$$\sigma_B = 1.25, \quad \bar{x}_B = 2.5$$

$$\text{CV}_B = \frac{\sigma_B}{\bar{x}_B}\times 100$$

$$= \frac{1.25}{2.5}\times 100$$

$$= 50\%$$

Since $\text{CV}_B < \text{CV}_A$, Team B is more consistent.

Example 4

The number of matches played and goals scored by two teams A and B in World Cup Football 2002 *were as follows:*

Matches played by Team A	27	9	8	5	4
Matches played by Team B	17	9	6	5	3
No. of goals scored in a match	0	1	2	3	4

Find which team may be considered more consistent.

Solution

For Team A,

$$N_A = 27+9+8+5+4 = 53$$

$$\sum fx_A = (27\times 0)+(9\times 1)+(8\times 2)+(5\times 3)+(4\times 4) = 56$$

$$\sum fx_A^2 = (27\times 0^2)+(9\times 1^2)+(8\times 2^2)+(5\times 3^2)+(4\times 4^2) = 150$$

$$\sigma_A = \sqrt{\frac{\sum fx_A^2}{N_A} - \left(\frac{\sum fx_A}{N_A}\right)^2}$$

$$= \sqrt{\frac{150}{53} - \left(\frac{56}{53}\right)^2}$$

$$= 1.31$$

$$\bar{x}_A = \frac{\sum fx_A}{N_A} = \frac{56}{53} = 1.06$$

$$\text{CV}_A = \frac{\sigma_A}{\bar{x}_A}\times 100$$

$$= \frac{1.31}{1.06}\times 100$$

$$= 123.58\%$$

For Team B,

$$N_B = 17+9+6+5+3 = 40$$

$$\sum fx_B = (17\times 0)+(9\times 1)+(6\times 2)+(5\times 3)+(3\times 4) = 48$$

$$\sum fx_B^2 = (17\times 0^2)+(9\times 1^2)+(6\times 2^2)+(5\times 3^2)+(3\times 4^2) = 126$$

$$\sigma_B = \sqrt{\frac{\sum fx_B^2}{N_B} - \left(\frac{\sum fx_B}{N_B}\right)^2}$$

$$= \sqrt{\frac{126}{40} - \left(\frac{48}{40}\right)^2}$$

$$= 1.31$$

$$\bar{x}_B = \frac{\sum fx_B}{N_B} = \frac{48}{40} = 1.2$$

$$\text{CV}_B = \frac{\sigma_B}{\bar{x}_B}\times 100$$

$$= \frac{1.31}{1.2}\times 100$$

$$= 109.17\%$$

Since $CV_B < CV_A$, Team B is more consistent in performance.

Example 5

Two automatic filling machines A and B are used to fill a mixture of cement concrete in a beam. A random sample of beams on each machine showed the following information:

Machine A	32	28	47	63	71	39	10	60	96	14
Machine B	19	31	48	53	67	90	10	62	40	80

Find the standard deviation of each machine and also comment on the performances of the two machines.

[Summer 2015]

Solution

$$n = 10$$

$$\bar{x} = \frac{\sum x}{n} = \frac{460}{10} = 46$$

$$\bar{y} = \frac{\sum y}{n} = \frac{500}{10} = 50$$

Machine A			Machine B		
x	$x - \bar{x}$	$(x - \bar{x})^2$	y	$y - \bar{y}$	$(y - \bar{y})^2$
32	–14	196	19	–31	961
28	–18	324	31	–19	361
47	1	1	48	–2	4
63	17	289	53	3	9
71	25	625	67	17	289
39	–7	49	90	40	1600
10	–36	1296	10	–40	1600
60	14	196	62	12	144
96	50	2500	40	–10	100
14	–32	1024	80	30	900
$\sum x = 460$		$\Sigma(x - \bar{x})^2 = 6500$	$\sum y = 500$		$\Sigma(y - \bar{y})^2 = 5968$

$$\sigma_A = \sqrt{\frac{\sum (x - \bar{x})^2}{n}}$$
$$= \sqrt{\frac{6500}{10}}$$
$$= 25.495$$
$$\sigma_B = \sqrt{\frac{\sum (y - \bar{y})^2}{n}}$$
$$= \sqrt{\frac{5968}{10}}$$
$$= 24.429$$
$$\text{CV}_A = \frac{\sigma_A}{\bar{x}} \times 100$$
$$= \frac{25.495}{46} \times 100$$
$$= 55.423\%$$
$$\text{CV}_B = \frac{\sigma_B}{\bar{y}} \times 100$$
$$= \frac{24.429}{50} \times 100$$
$$= 48.858\%$$

Since $\text{CV}_B < \text{CV}_A$, there is less variability in the performance of the machine B.

3.8.5 Combined Standard Deviation

If $\bar{x}_1, \bar{x}_2, ..., \bar{x}_k$ be the arithmetic means, $\sigma_1, \sigma_2, ..., \sigma_k$ be the standard deviations, and $n_1, n_2, ..., n_k$ be the number of observations of k groups then the combined standard deviation is given by

$$\sigma = \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2) + \cdots + n_k(\sigma_k^2 + d_k^2)}{n_1 + n_2 + \cdots + n_k}}$$

where $\quad d_1 = \bar{x}_1 - \bar{x},\ d_2 = \bar{x}_2 - \bar{x}, ..., d_k = \bar{x}_k - x$

Example 1

A sample of 90 *values has a means of* 55 *and a standard deviation of* 3. *A second sample of* 110 *values has a mean of* 60 *and a standard deviation of* 2. *Find the mean and standard deviation of the combined sample of* 200 *values.*

Solution

$$n_1 = 90, \quad \bar{x}_1 = 55, \quad \sigma_1 = 3$$
$$n_2 = 110, \quad \bar{x}_2 = 60, \quad \sigma_2 = 2$$

Combined mean $\bar{x} = \dfrac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$

$$= \frac{(90 \times 55) + (110 \times 60)}{90 + 110}$$
$$= 57.75$$

$$d_1 = \bar{x}_1 - \bar{x} = 55 - 57.75 = -2.75$$
$$d_2 = x_2 - \bar{x} = 60 - 57.75 = 2.25$$

Combined standard deviation $\sigma = \sqrt{\dfrac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}}$

$$= \sqrt{\frac{90\left[3^2 + (-2.75)^2\right] + 110(2^2 + 2.25)^2}{90 + 110}}$$
$$= 3.53$$

EXERCISE 3.5

1. Find the standard deviation of 10 persons whose income in rupees is given below:

312, 292, 227, 235, 269, 255, 333, 348, 321, 299

[Ans.: 39.24]

2. Calculate the standard deviation from the following data:

Heights in cm	150	155	160	165	170	175	180
No. of students	15	24	32	33	24	16	6

[Ans.: 8.038 cm]

3. Find the standard deviation of the following data:

Size of items	10	11	12	13	14	15	16
Frequency	2	7	11	15	10	4	1

[Ans.: 1.342]

4. Calculate the standard deviation for the following frequency distribution:

Class interval	0–4	4–8	8–12	12–16
Frequency	4	8	2	1

[Ans.: 3.27**]**

5. Calculate the standard deviation of the following series:

Marks	0–10	10–20	20–30	30–40	40–50
Frequency	10	8	15	8	4

[Ans.: 12.37**]**

6. Calculate the SD for the following distributions of 300 telephone calls according to their durations in seconds:

Duration (in seconds)	0–30	30–60	60–90	90–120	120–150	150–180	180–210
No. of calls	9	17	43	82	81	44	24

[Ans.: 42.51**]**

7. Calculate the standard deviation from the following data:

Age less than (in years)	10	20	30	40	50	60	70	80
No. of Persons	15	30	53	75	100	110	115	125

[Ans.: 19.75**]**

8. Find the standard deviation from the following data:

Midvalue	30	35	40	45	50	55	60	65	70	75	80
Frequency u	1	2	4	7	9	13	17	12	7	6	3

[Ans.: 11.04**]**

9. Two cricketers scored the following runs in ten innings. Find who is a better run-getter and who is a more consistent player.

A	42	17	83	59	72	76	64	45	40	32
B	28	70	31	0	59	108	82	14	3	95

[Ans.: *A* is a better run-getter and is more consistent**]**

10. Two workers on the same job show the following results over a long period of time:

	Worker *A*	Worker *B*
Mean time (in minutes)	30	25
Standard deviation (in minutes)	6	4

[**Ans.:** *B* is more consistent]

11. The mean and standard deviation of 100 items are found to be 40 and 10. At the time of calculations, two items are wrongly taken as 30 and 72 instead of 3 and 27. Find the correct mean and correct standard deviation.

[**Ans.:** 39.28, 10.18]

12. The mean and standard deviation of distributions of 100 and 150 items are 50, 5 and 40, 6 respectively. Find the mean and standard deviation of all the 250 items taken together.

[**Ans.:** 44, 7.46]

3.9 SKEWNESS

Skewness is a measure that refers to the extent of symmetry or asymmetry in a distribution. A distribution is said to be *symmetrical* when its mean, median, and mode are equal, and the frequencies are symmetrically distributed about the mean. A symmetrical distribution when plotted on a graph will give a perfectly bell-shaped curve which is known as a *normal curve* (Fig. 3.1).

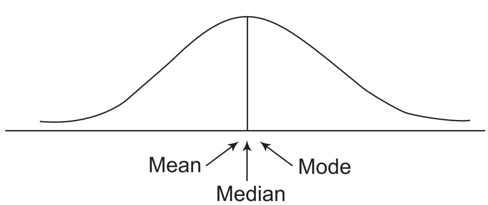


Fig. 3.1

A distribution is said to be asymmetrical or skewed when the mean, median, and mode are not equal, i.e., the mean median, and mode do not coincide. If the curve has a longer tail towards the left, it is said to be a negatively skewed distribution (Fig. 3.2*a*). If the curve has a longer tail towards the right, it is said to be positively skewed (Fig. 3.2*b*).

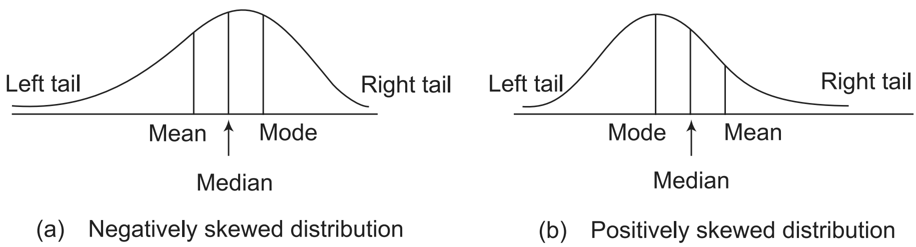


Fig. 3.2

Skewness gives an idea of the nature and degree of concentration of observations about the mean.

3.9.1 Measures of Skewness

A measure of skewness gives the extent and direction of skewness of a distribution. These measures can be absolute or relative. The absolute measures are also known as measures of skewness.

$$\text{Absolute skewness} = \text{Mean} - \text{Mode}$$

If the value of the mean is greater than the mode, the skewness will be positive and if the value of the mean is less than the mode, the skewness will be negative.

The relative measures of skewness is called the *coefficient of skewness*.

3.9.2 Karl Pearson's Coefficient of Skewness

Karl Pearson's coefficient of skewness denoted by S_k, is given by

$$S_k = \frac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}}$$

$$= \frac{\text{Mean} - \text{Mode}}{\sigma}$$

When the mode is ill-defined and the distribution is moderately skewed, the averages have the following relationship:

$$\text{Mode} = 3\ \text{Median} - 2\ \text{Mean}$$

$$S_k = \frac{\text{Mean} - (3\ \text{Median} - 2\ \text{Mean})}{\text{Standard Deviation}}$$

$$= \frac{3(\text{Mean} - \text{Median})}{\text{Standard Deviation}}$$

$$= \frac{3(\text{Mean} - \text{Median})}{\sigma}$$

The coefficient of skewness usually lies between –1 and 1.

For a positively skewed distribution, $S_k > 0$.

For a negatively skewed distribution, $S_k < 0$.
For a symmetrical distribution, $S_k = 0$.

Example 1

Calculate Karl Pearson's coefficient of skewness for the following data:

x	0	1	2	3	4	5	6	7
y	12	17	29	19	8	4	1	0

Solution

Let $a = 4$ be the assumed mean.
$d = x - a = x - 4$

x	f	d	d^2	fd	fd^2
0	12	–4	16	–48	192
1	17	–3	9	–51	153
2	29	–2	4	–58	116
3	19	–1	1	–19	19
4	8	0	0	0	0
5	4	1	1	4	4
6	1	2	4	2	4
7	0	3	9	0	0
	$\sum f = 90$			$\sum fd = -170$	$\sum fd^2 = 488$

$$N = \sum f = 90$$

$$\bar{x} = a + \frac{\sum fd}{N}$$

$$= 4 + \left(\frac{-170}{90}\right)$$

$$= 2.11$$

$$\sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

$$= \sqrt{\frac{488}{90} - \left(\frac{-170}{90}\right)^2}$$

$$= 1.36$$

Since the maximum frequency is 29, the mode is 2.

$$S_k = \frac{\text{Mean} - \text{Mode}}{\sigma}$$
$$= \frac{2.11 - 2}{1.36}$$
$$= 0.08$$

Example 2

Calculate Karl Pearson's coefficient of skewness from the following data:

Wages (₹)	10–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50
No. of Workers	8	16	30	45	62	32	15	6

Solution

Let $a = 32.5$ be the assumed mean and $h = 5$ be the width of the class interval.

$$d = \frac{x - a}{h} = \frac{x - 32.5}{5}$$

Wages (₹)	No. of workers f	Midvalue x	$d = \frac{x-32.5}{5}$	d^2	fd	fd^2
10–15	8	12.5	–4	16	–32	128
15–20	16	17.5	–3	9	–48	144
20–25	30	22.5	–2	4	–60	120
25–30	45	27.5	–1	1	–45	45
30–35	62	32.5	0	0	0	0
35–40	32	37.5	1	1	32	32
40–45	15	42.5	2	4	30	60
45–50	6	47.5	3	9	18	54
	$\sum f = 214$				$\sum fd = -105$	$\sum fd^2 = 583$

$$N = \sum f = 214$$
$$\bar{x} = a + h\frac{\sum fd}{N}$$
$$= 32.5 + 5\left(\frac{-105}{214}\right)$$
$$= 30.05$$

$$\sigma = h\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

$$= 5\sqrt{\frac{583}{214} - \left(\frac{-105}{214}\right)^2}$$

$$= 7.88$$

Since the maximum frequency is 32, the mode lies in the interval 30–35.

Here, $l = 30$, $h = 5$, $f_m = 62$, $f_1 = 45$, $f_2 = 32$

$$\text{Mode} = l + h\left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right)$$

$$= 30 + 5\left[\frac{62 - 45}{2(62) - 45 - 32}\right]$$

$$= 31.81$$

$$S_k = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

$$= \frac{30.05 - 31.81}{7.88}$$

$$= -0.223$$

Example 3

The scores at an aptitude test by 100 *candidates are given below. Calculate Karl Pearson's coefficient of skewness.*

Marks	0–10	10–20	20–30	30–40	40–50	50–60	60–70
No. of candidates	10	15	24	25	10	10	6

Solution

Let $a = 35$ be the assumed mean and $h = 10$ be the width of the class interval.

$$d = \frac{x - a}{h} = \frac{x - 35}{10}$$

Marks	No. of candidates f	Midvalue x	$d = \dfrac{x-35}{10}$	d^2	fd	fd^2
0–10	10	5	–3	9	–30	90
10–20	15	15	–2	4	–30	60
20–30	24	25	–1	1	–24	24
30–40	25	35	0	0	0	0
40–50	10	45	1	1	10	10
50–60	10	55	2	4	20	40
60–70	6	65	3	9	18	54
	$\sum f = 100$				$\sum fd = -36$	$\sum fd^2 = 278$

$$N = \sum f = 100$$

$$\bar{x} = a + h\left(\frac{\sum fd}{N}\right)$$

$$= 35 + 10\left(\frac{-36}{100}\right)$$

$$= 31.4$$

$$\sigma = h\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

$$= 10\sqrt{\frac{278}{100} - \left(\frac{-36}{100}\right)^2}$$

$$= 16.28$$

Since the maximum frequency is 25, the mode lies in the interval 30–40.

Here, $l = 30$, $h = 10$, $f_m = 25$, $f_1 = 24$, $f_2 = 10$

$$\text{Mode } = l + h\left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right)$$

$$= 30 + 10\left[\frac{25 - 24}{2(25) - 24 - 10}\right]$$

$$= 30.625$$

$$S_k = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

$$= \frac{31.4 - 30.625}{16.28}$$

$$= 0.0476$$

Example 4

Calculate Karl Pearson's coefficient of skewness from the following data:

Weekly wages	40–50	50–60	60–70	70–80	80–90	90–100	100–110	110–120	120–130	130–140
No. of workers	5	6	8	10	25	30	36	50	60	70

Solution

The maximum frequency 70 occurs at the end of the frequency distribution. Hence, the mode is ill-defined and Karl Pearson's coefficient of skewness is obtained using the median. Let a = 85 be the assumed mean and h = 10 be the width of the class interval.

$$d = \frac{x-a}{h} = \frac{x-85}{10}$$

Weekly wages	No. of workers f	Midvalue x	$d = \frac{x-85}{10}$	d^2	fd	fd^2	CF
40–50	5	45	–4	16	–20	80	5
50–60	6	55	–3	9	–18	54	11
60–70	8	65	–2	4	–16	32	19
70–80	10	75	–1	1	–10	10	29
80–90	25	85	0	0	0	0	54
90–100	30	95	1	1	30	30	84
100–110	36	105	2	4	72	144	120
110–120	50	115	3	9	150	450	170
120–130	60	125	4	16	240	960	230
130–140	70	135	5	25	350	1750	300
	$\sum f = 300$				$\sum fd = 778$	$\sum fd^2 = 3510$	

$$N = \sum f = 300$$

$$\bar{x} = a + h\left(\frac{\sum fd}{N}\right)$$

$$= 85 + 10\left(\frac{778}{300}\right)$$

$$= 110.93$$

$$\sigma = h\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

$$= 10\sqrt{\frac{3510}{300} - \left(\frac{778}{300}\right)^2}$$

$$= 22.304$$

$$\frac{N}{2} = \frac{300}{2} = 150$$

The cumulative frequency just greater than 150 is 170 and the corresponding class 110–120 is the median class.

Here, $l = 110$, $h = 10$, $f = 50$, $c = 120$

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

$$= 110 + \frac{10}{50}(150 - 120)$$

$$= 116$$

$$S_k = \frac{3(\text{Mean} - \text{median})}{\sigma}$$

$$= \frac{3(110.93 - 116)}{22.304}$$

$$= -0.682$$

Example 5

From the marks scored by 100 *students in Section A and* 100 *students in Section B of a class, the following measures were obtained:*

Section *A*	$\bar{x}_A = 55$	$\sigma_A = 15.4$	Mode = 58.72
Section *B*	$\bar{x}_A = 53$	$\sigma_B = 15.4$	Mode = 48.83

Determine which distribution of marks is more skewed.

Solution

$$S_{k_A} = \frac{\text{Mean} - \text{Mode}}{\sigma_A} = \frac{55 - 58.72}{15.4} = -0.24$$

$$S_{k_B} = \frac{\text{Mean} - \text{Mode}}{\sigma_B} = \frac{53 - 48.83}{15.4} = 0.27$$

$$|0.27| > |-0.24|$$

Hence, the distribution of marks of Section *B* is more skewed.

Example 6

For a group of 10 *items,* $\sum x = 452$, $\sum x^2 = 24270$, *and mode* = 43.7. *Find Karl Pearson's coefficient of skewness.*

Solution

$n = 10, \quad \sum x = 452, \quad \sum x^2 = 24270, \quad \text{mode} = 43.7$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{452}{10} = 45.2$$

$$\sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

$$= \sqrt{\frac{24270}{10} - \left(\frac{452}{10}\right)^2}$$

$$= 19.59$$

$$S_k = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

$$= \frac{45.2 - 43.7}{19.59}$$

$$= 0.077$$

Example 7

In a distribution, the mean = 65, *median* = 70, *coefficient of skewness* = –0.6. *Find the mode and coefficient of variation.*

Solution

$$\bar{x} = 65, \quad \text{Median} = 70, \quad S_k = -0.6$$

$$\text{Mode} = 3\text{ Median} - 2\text{ Mean} = 3(70) - 2(65) = 80$$

$$S_k = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

$$-0.6 = \frac{65 - 80}{\sigma}$$

$$\therefore \quad \sigma = 25$$

$$\text{CV} = \frac{\sigma}{\bar{x}} \times 100 = \frac{25}{65} \times 100 = 38.64\%$$

Example 8

The following information was obtained from the records of a factory relating to wages:

Arithmetic mean = ₹ 56.8, Median = ₹ 59.5, *Standard deviation* = ₹ 12.4

Give the information about the distribution of wages.

Solution

$$\bar{x} = 56.8, \quad \text{Median} = 59.5, \quad \sigma = 12.4$$

$$S_k = \frac{3(\text{Mean} - \text{Median})}{\sigma} = \frac{3(56.8 - 59.5)}{12.4} = -0.65$$

$$\text{Mode} = 3\text{ Median} - 2\text{ Mean} = 3(59.5) - 2(56.8) = 64.9$$

Hence, the maximum wages is ₹ 64.9.

There is a negative skewness in wages.

Example 9

For a moderately skewed distribution of retail price for men's shoes, it is found that the mean price is ₹ 20 *and the median price is* ₹ 17. *If the coefficient of variation is* 20%, *find the Pearson's coefficient of skewness.*

Solution

$$\bar{x} = 20, \quad \text{Median} = 17, \quad \text{CV} = 20\%$$

$$\text{CV} = \frac{\sigma}{\bar{x}} \times 100$$

$$20 = \frac{\sigma}{20} \times 100$$

$$\therefore \quad \sigma = 4$$

$$S_k = \frac{3(\text{Mean} - \text{Median})}{\sigma} = \frac{3(20 - 17)}{4} = 2.25$$

EXERCISE 3.6

1. Calculate Karl Pearson's coefficient of skewness for the following data:

   25, 15, 23, 40, 27, 25, 23, 25, 30

   [Ans.: – 0.03**]**

2. Calculate Karl Pearson's coefficient of skewness for the following data:

Size	1	2	3	4	5	6	7
Frequency	10	18	30	25	12	3	2

[**Ans.:** 0.2075]

3. Find the coefficient of skewness for the following data:

Weekly wages (in ₹)	15	20	25	30	35	40	45
No. of earners	3	25	19	16	4	5	6

[**Ans.:** 0.88]

4. Find Karl Pearson's coefficient of skewness for the following data:

Marks	0–10	10–20	20–30	30–40	40–50	50–60	60–70
No. of students	10	12	18	25	16	14	8

[**Ans.:** 0.013]

5. Calculate the coefficient of skewness from the following data:

Marks less than	20	30	40	50	60	70	80
No. of students	10	25	40	65	80	95	100

[**Ans.:** – 0.089]

6. For the data given below, calculate Karl Pearson's coefficient of skewness:

x	70–80	80–90	90–100	100–110	110–120	120–130	130–140	140–150
f	12	18	35	42	50	45	20	8

[**Ans.:** 0.243]

7. Karl Pearson's measure of skewness of a distribution is 0.5. Its median and mode are respectively 42 and 36. Find the coefficient of variation.

[**Ans.:** 40]

8. From the marks scored by 120 students in Section *A* and 120 students in Section *B* of a class, the following measures are obtained:

Section *A*	$\overline{x}$ = 46.83	SD = 14.8	mode = 51.67
Section *B*	$\overline{x}$ = 47.83	SD = 14.8	mode = 47.07

Determine which distribution of marks is more skewed.

[**Ans.:** Section *A*]

9. For a moderately skewed data, the arithmetic mean is 200, the coefficient of variation is 8, and Karl Pearson's coefficient of skewness is 0.3. Find the mode and median.

[**Ans.:** 195.2, 198.4]

10. Karl Pearson's coefficient of skewness of a distribution is 0.32. Its standard deviation is 6.5 and the mean is 29.6. Find the mode and median for the distribution.

[**Ans.:** 27.52, 28.9]

11. The median, mode and coefficient of skewness for a certain distribution are respectively 17.4, 15.3, and 0.35. Find the coefficient of variation.

[**Ans.:** 48.78%]

12. In a distribution, mean = 65, median = 70, coefficient of skewness = –6. Find the mode and coefficient of variation.

[**Ans.:** 80, 39.78%]

Points to Remember

Arithmetic Mean

The *arithmetic mean* of a set of observations is their sum divided by the number of observations. If $x_1, x_2, \ldots, x_n$ be n observations then their average or arithmetic mean is given by

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum x}{n}$$

If n observations consist of n distinct values denoted by $x_1, x_2, \ldots, x_n$ of the observed variable x occurring with frequencies $f_1, f_2, \ldots, f_n$ respectively then the arithmetic mean is given by

$$\bar{x} = \frac{f_1 x_1 + f_2 x_2 + \cdots + f_n x_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_i x_i}{N} = \frac{\sum fx}{N}$$

1. Arithmetic Mean of Grouped Data

In case of grouped or continuous frequency distribution the arithmetic mean is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{\sum fx}{N}, \text{ where } N = \sum_{i=1}^{n} f_i$$

and x is taken as the midvalue of the corresponding class.

2. Arithmetic Mean from Assumed Mean

$$\bar{x} = a + \frac{\sum fd}{N}$$

3. Arithmetic Mean by the Step-Deviation Method

$$\bar{x} = a + h\frac{\sum fd}{N}$$

4. Weighted Arithmetic Mean

Weighted arithmetic mean = $\dfrac{w_1 x_1 + w_2 x_2 + \cdots + w_n x_n}{w_1 + w_2 + \cdots + w_n}$

$$\bar{x}_w = \frac{\sum wx}{\sum w}$$

When the assumed mean is used for calculation,

$$\bar{x}_w = a + \frac{\sum wd}{\sum w}$$

When the step-deviation method is used for calculation,

$$\bar{x}_w = a + h\frac{\sum wd}{\sum w}$$

Combined Arithmetic Mean

If $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ are the means of k series of sizes $n_1, n_2, \ldots, n_k$ respectively then the mean $\bar{x}$ of the composite series is given by

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\,\bar{x}_2 + \cdots + n_k\,\bar{x}_k}{n_1 + n_2 + \cdots + n_k}$$

$$= \frac{\sum_{i=1}^{k} n_i x_i}{\sum_{i=1}^{k} n_i}$$

Median

Median is the central value of the variable when the values are arranged in ascending or descending order of magnitude.

In case of ungrouped data, if the number of observations is odd then the median is the middle value after the values have been arranged in ascending or descending order of magnitude. If the number of observations is even, there are two middle terms and the median is obtained by taking the arithmetic mean of the middle terms.

In case of discrete frequency distribution, the median is obtained by considering the cumulative frequencies. The steps for calculating the median are given below:

(i) Arrange the values of the variables in ascending or descending order of magnitudes.

(ii) Find $\frac{N}{2}$ where $N = \sum f$

(iii) Find the cumulative frequency just greater than $\frac{N}{2}$ and determine the corresponding value of the variable.

(iv) The corresponding value of x is the median.

Median for Continuous Frequency Distribution

In case of continuous frequency distribution (less than frequency distribution), the class corresponding to the cumulative frequency just greater than $\frac{N}{2}$, is called the *median class*, and the value of the median is given by

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - c\right)$$

In case of 'more than' or 'greater than' type of frequency distributions, the value of the median is given by

$$\text{Median} = u - \frac{h}{f}\left(\frac{N}{2} - c\right)$$

where u is the upper limit of the median class

f is the frequency of the median class

h is the width of the median class

c is the cumulative frequency of the class succeeding the median class

Mode

Mode is the value which occurs most frequently in a set of observations and around which the other items of the set are heavily distributed.

Mode for a Continuous Frequency Distribution

In case of a continuous frequency distribution, the class in which the mode lies is called the *modal class* and the value of the mode is given by

$$\text{Mode} = l + h\left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right)$$

where l is the lower limit of the modal class

h is the width of the modal class

f_m is the frequency of the modal class

f_1 is the frequency of the class preceding the modal class

f_2 is the frequency of the class succeeding the modal class

Geometric Mean

The *geometric mean* of a set of n observations is the n^{th} root of their product.

$$\text{GM} = (x_1 \cdot x_2 \cdot \cdots \cdot x_n)^{\frac{1}{n}}$$

$$\text{GM} = \text{antilog}\left(\frac{\sum \log x}{n}\right)$$

In case of a frequency distribution consisting of n observations x_1, x_2, .., x_n with respective frequencies f_1, f_2, .., f_n, the geometric mean is given by

$$\text{GM} = (x_1^{f_1} \cdot x_2^{f_2} \cdots\cdot x_n^{f_n})^{\frac{1}{N}}, \text{ where } N = \sum f$$

$$\text{GM} = \text{antilog}\left(\frac{\sum f \log x}{N}\right)$$

Harmonic Mean

The *harmonic mean* of a number of observations, none of which is zero, is the reciprocal of the arithmetic mean of the reciprocals of the given values.

$$\text{HM} = \frac{1}{\frac{1}{n}\sum\left(\frac{1}{x}\right)}$$

$$= \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

In case of a frequency distribution consisting of n observations $x_1, x_2, ..., x_n$ with respective frequencies $f_1, f_2, ..., f_n$, the harmonic mean is given by

$$\text{HM} = \frac{f_1 + f_2 + \cdots + f_n}{\frac{f_1}{x_1} + \frac{f_2}{x_2} + \cdots + \frac{f_n}{x_n}}$$

$$= \frac{\sum f}{\sum\left(\frac{f}{x}\right)}$$

If $x_1, x_2, ..., x_n$ are n observations with weights $w_1, w_2, ..., w_n$ respectively, their weighted harmonic mean is given by

$$\text{HM} = \frac{\sum w}{\sum\left(\frac{w}{x}\right)}$$

Relation between Arithmetic Mean, Geometric Mean and Harmonic Mean

$$\text{AM} \geq \text{GM} \geq \text{HM}$$

For two observations x_1 and x_2 of a series,

$$\text{GM} = \sqrt{\text{AM} \cdot \text{HM}}$$

Standard Deviation

Standard deviation is the positive square root of the arithmetic mean of the squares of the deviations of the given values from their arithmetic mean.

$$\sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n}}$$

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

In case of a frequency distribution consisting of n observations $x_1, x_2, ..., x_n$ with respective frequencies $f_1, f_2, ..., f_n$, the standard deviation is given by

$$\sigma = \sqrt{\frac{\sum fx^2}{N} - \left(\frac{\sum fx}{N}\right)^2}$$

1. Standard Deviation from the Assumed Mean

$$\sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

2. Standard Deviation by Step-Deviation Method

$$\sigma = h\sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2}$$

3. Variance

The *variance* is the square of the standard deviation and is denoted by σ^2. The method for calculating variance is same as that given for the standard deviation.

4. Coefficient of Variation

The *standard deviation* is an absolute measure of dispersion. The coefficient of variation is a relative measure of dispersion and is denoted by CV.

$$\text{CV} = \frac{\sigma}{\bar{x}} \times 100$$

Skewness

Skewness is a measure that refers to the extent of symmetry or asymmetry in a distribution.

Karl Pearson's Coefficient of Skewness

Karl Pearson's coefficient of skewness denoted by S_k, is given by

$$S_k = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

When the mode is ill-defined and the distribution is moderately skewed, the averages have the following relationship:

$$S_k = \frac{\text{Mean} - (3\text{ Median} - 2\text{ Mean})}{\text{Standard Deviation}}$$

$$= \frac{3(\text{Mean} - \text{Median})}{\sigma}$$

CHAPTER 4

Correlation and Regression

Chapter Outline

4.1 Introduction
4.2 Correlation
4.3 Types of Correlations
4.4 Methods of Studying Correlation
4.5 Scatter Diagram
4.6 Simple Graph
4.7 Karl Pearson's Coefficient of Correlation
4.8 Properties of Coefficient of Correlation
4.9 Rank Correlation
4.10 Regression
4.11 Types of Regression
4.12 Methods of Studying Regression
4.13 Lines of Regression
4.14 Regression Coefficients
4.15 Properties of Regression Coefficients
4.16 Properties of Lines of Regression (Linear Regression)

4.1 INTRODUCTION

Correlation and regression are the most commonly used techniques for investigating the relationship between two quantitative variables. *Correlation* refers to the relationship of two or more variables. It measures the closeness of the relationship between the variables. *Regression* establishes a functional relationship between the variables. In correlation, both the variables x and y are random variables, whereas in regression, x is a random variable and y is a fixed variable. The coefficient of correlation is a relative measure whereas the regression coefficient is an absolute figure.

4.2 CORRELATION

Correlation is the relationship that exists between two or more variables. Two variables are said to be correlated if a change in one variable affects a change in the other variable. Such a data connecting two variables is called *bivariate data*. Thus, correlation is a statistical analysis which measures and analyses the degree or extent to which two variables fluctuate with reference to each other. Some examples of such a relationship are as follows:

1. Relationship between heights and weights.
2. Relationship between price and demand of commodity.
3. Relationship between rainfall and yield of crops.
4. Relationship between age of husband and age of wife.

4.3 TYPES OF CORRELATIONS

Correlation is classified into four types:

1. Positive and negative correlations
2. Simple and multiple correlations
3. Partial and total correlations
4. Linear and nonlinear correlations

4.3.1 Positive and Negative Correlations

Depending on the variation in the variables, correlation may be positive or negative.

1. Positive Correlation If both the variables vary in the same direction, the correlation is said to be positive. In other words, if the value of one variable increases, the value of the other variable also increases, or, if value of one variable decreases, the value of the other variable decreases, e.g., the correlation between heights and weights of group of persons is a positive correlation.

Height (cm)	150	152	155	160	162	165
Weight (kg)	60	62	64	65	67	69

2. Negative Correlation If both the variables vary in the opposite direction, correlation is said to be negative. In other words, if the value of one variable increases, the value of the other variable decreases, or, if the value of one variable decreases, the value of the other variable increases, e.g., the correlation between the price and demand of a commodity is a negative correlation.

Price (₹ per unit)	10	8	6	5	4	1
Demand (units)	100	200	300	400	500	600

4.3.2 Simple and Multiple Correlations

Depending upon the study of the number of variables, correlation may be simple or multiple.

1. Simple Correlation When only two variables are studied, the relationship is described as simple correlation, e.g., the quantity of money and price level, demand and price, etc.

2. Multiple Correlation When more than two variables are studied, the relationship is described as multiple correlation, e.g., relationship of price, demand, and supply of a commodity.

4.3.3 Partial and Total Correlations

Multiple correlation may be either partial or total.

1. Partial Correlation When more than two variables are studied excluding some other variables, the relationship is termed as partial correlation.

2. Total Correlation When more than two variables are studied without excluding any variables, the relationship is termed total correlation.

4.3.4 Linear and Nonlinear Correlations

Depending upon the ratio of change between two variables, the correlation may be linear or nonlinear.

1. Linear Correlation If the ratio of change between two variables is constant, the correlation is said to be linear. If such variables are plotted on a graph paper, a straight line is obtained, e.g.,

Milk (*l*)	5	10	15	20	25	30
Curg (kg)	2	4	6	8	10	12

2. Nonlinear Correlation If the ratio of change between two variables is not constant, the correlation is said to nonlinear. The graph of a nonlinear or curvilinear relationship will be a curve, e.g.,

Advertising expenses (₹ in lacs)	3	6	9	12	15
Sales (₹ in lacs)	10	12	15	15	16

4.4 METHODS OF STUDYING CORRELATION

There are two different methods of studying correlation, (1) Graphic methods (2) Mathematical methods.

Graphic methods are (a) scatter diagram, and (b) simple graph.

Mathematical methods are (a) Karl Pearson's coefficient of correlation, and (b) Spearman's rank coefficient of correlation.

4.5 SCATTER DIAGRAM

The scatter diagram is a diagrammatic representation of bivariate data to find the correlation between two variables. There are various correlationships between two variables represented by the following scatter diagrams.

1. Perfect Positive Correlation If all the plotted points lie on a straight line rising from the lower left-hand corner to the upper right-hand corner, the correlation is said to be perfectly positive (Fig. 4.1).

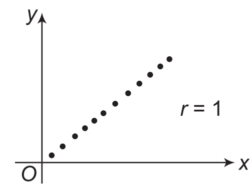


Fig. 4.1

2. Perfect Negative Correlation If all the plotted points lie on a straight line falling from the upper-left hand corner to the lower right-hand corner, the correlation is said to be perfectly negative (Fig. 4.2).

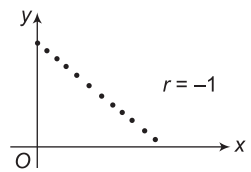


Fig. 4.2

3. High Degree of Positive Correlation If all the plotted points lie in the narrow strip, rising from the lower left-hand corner to the upper right-hand corner, it indicates a high degree of positive correlation (Fig. 4.3).

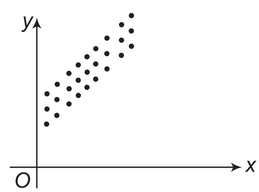


Fig. 4.3

4. High Degree of Negative Correlation If all the plotted points lie in a narrow strip, falling from the upper left-hand corner to the lower right-hand corner, it indicates the existence of a high degree of negative correlation (Fig. 4.4).

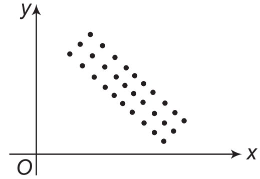


Fig. 4.4

5. No Correlation If all the plotted points lie on a straight line parallel to the x-axis or y-axis or in a haphazard manner, it indicates the absence of any relationship between the variables (Fig. 4.5).

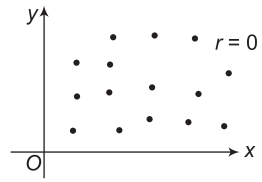


Fig. 4.5

Merits of a Scatter Diagram

1. It is simple and nonmathematical method to find out the correlation between the variables.

2. It gives an indication of the degree of linear correlation between the variables.
3. It is easy to understand.
4. It is not influenced by the size of extreme items.

4.6 SIMPLE GRAPH

A *simple graph* is a diagrammatic representation of bivariate data to find the correlation between two variables. The values of the two variables are plotted on a graph paper. Two curves are obtained, one for the variable x and the other for the variable y. If both the curves move in the same direction, the correlation is said to be positive. If both the curves move in the opposite direction, the correlation is said to be negative. This method is used in the case of a time series. It does not reveal the extent to which the variables are related.

4.7 KARL PEARSON'S COEFFICIENT OF CORRELATION

The coefficient of correlation is the measure of correlation between two random variables X and Y, and is denoted by r.

$$r = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}$$

where cov (X, Y) is covariance of variables X and Y,

σ_X is the standard deviation of variable X,

and σ_Y is the standard deviation of variable Y.

This expression is known as Karl Pearson's coefficient of correlation or Karl Pearson's product-moment coefficient of correlation.

$$\text{cov}(X,Y) = \frac{1}{n}\sum (x-\bar{x})\,(y-\bar{y})$$

$$\sigma_X = \sqrt{\frac{\sum (x-\bar{x})^2}{n}}$$

$$\sigma_Y = \sqrt{\frac{\sum (y-\bar{y})^2}{n}}$$

$$\therefore \qquad r = \frac{\sum (x-\bar{x})\,(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2}\sqrt{\sum (y-\bar{y})^2}}$$

The above expression can be further modified.

Expanding the terms,

$$r = \frac{\sum (xy - x\bar{y} - \bar{x}y + \bar{x}\,\bar{y})}{\sqrt{\sum \left(x^2 - 2x\bar{x} + \bar{x}^2\right)}\sqrt{\sum \left(y^2 - 2y\bar{y} + \bar{y}^2\right)}}$$

$$= \frac{\sum xy - \bar{y}\sum x - \bar{x}\sum y + \bar{x}\,\bar{y}\sum 1}{\sqrt{\sum x^2 - 2\bar{x}\sum x + \bar{x}^2 \sum 1}\sqrt{\sum y^2 - 2\bar{y}\sum y + \bar{y}^2 \sum 1}}$$

$$= \frac{\sum xy - \frac{\sum y}{n}\sum x - \frac{\sum x}{n}\sum y + \frac{\sum x}{n}\frac{\sum y}{n}\cdot n}{\sqrt{\sum x^2 - 2\frac{\sum x}{n}\sum x + \left(\frac{\sum x}{n}\right)^2 n}\sqrt{\sum y^2 - 2\frac{\sum y}{n}\sum y + \left(\frac{\sum y}{n}\right)^2 n}}$$

$$= \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}\sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}}$$

4.8 PROPERTIES OF COEFFICIENT OF CORRELATION

1. The coefficient of correlation lies between −1 and 1, i.e., $-1 \le r \le 1$.

Proof Let $\bar{x}$ and $\bar{y}$ be the mean of x and y series and σ_x and σ_y be their respective standard deviations.

Let $\sum \left(\frac{x-\bar{x}}{\sigma_x} \pm \frac{y-\bar{y}}{\sigma_y}\right)^2 \ge 0$ $\left[\because \begin{array}{l}\text{sum of squares of real quantities}\\ \text{cannot be negative}\end{array}\right]$

$$\frac{\sum (x-\bar{x})^2}{\sigma_x^2} + \frac{\sum (y-\bar{y})^2}{\sigma_y^2} \pm \frac{2\sum (x-\bar{x})(y-\bar{y})}{\sigma_x \sigma_y} \ge 0$$

$$n + n \pm 2nr \ge 0$$

$$2n \pm 2nr \ge 0$$

$$2n(1 \pm r) \ge 0$$

$$1 \pm r \ge 0$$

i.e., $1 + r \ge 0$ or $1 - r \ge 0$

$r \ge -1$ or $r \le 1$

Hence, the coefficient of correlation lies between –1 and 1, i.e., $-1 \le r \le 1$.

2. Correlation coefficient is independent of change of origin and change of scale.

Proof Let $d_x = \dfrac{x-a}{h}, \quad d_y = \dfrac{y-b}{k}$

$$x = a + hd_x, \quad y = b + kd_y$$

where a, b, h (>0) and k(>0) are constants.

$$x = a + hd_x \Rightarrow \bar{x} = a + h\bar{d}_x \Rightarrow x - \bar{x} = h(d_x - \bar{d}_x)$$
$$y = b + kd_y \Rightarrow \bar{y} = b + h\,\bar{d}_y \Rightarrow y - \bar{y} = k(d_y - \bar{d}_y)$$

$$\begin{aligned}
r_{xy} &= \frac{\sum (x-\bar{x})\,(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2}\ \sqrt{\sum (y-\bar{y})^2}} \\
&= \frac{\sum h(d_x - \bar{d}_x)\,k(d_y - \bar{d}_y)}{\sqrt{\sum h^2(d_x - \bar{d}_x)^2}\ \sqrt{\sum k^2(d_y - \bar{d}_y)^2}} \\
&= \frac{\sum (d_x - \bar{d}_x)\,(d_y - \bar{d}_y)}{\sqrt{\sum (d_x - \bar{d}_x)^2}\ \sqrt{(d_y - \bar{d}_y)^2}} \\
&= r_{d_x d_y}
\end{aligned}$$

Hence, the correlation coefficient is independent of change of origin and change of scale.

Note Since correlation coefficient is independent of change of origin and change of scale,

$$r = \frac{\sum d_x d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \dfrac{\left(\sum d_x\right)^2}{n}}\ \sqrt{\sum d_y^2 - \dfrac{\left(\sum d_y\right)^2}{n}}}$$

3. Two independent variables are uncorrelated.

Proof If random variables X and Y are independent,

$$\sum (x-\bar{x})\,(y-\bar{y}) = 0 \ \text{ or } \ \operatorname{cov}(X, Y) = 0$$

$$\therefore \qquad r = 0$$

Thus, if X and Y are independent variables, they are uncorrelated.

Note The converse of the above property is not true, i.e., two uncorrelated variables may not be independent.

Example 1

Calculate the correlation coefficient between x and y using the following data:

x	2	4	5	6	8	11
y	18	12	10	8	7	5

Solution

$n = 6$

x	y	x^2	y^2	xy
2	18	4	324	36
4	12	16	144	48
5	10	25	100	50
6	8	36	64	48
8	7	64	49	56
11	5	121	25	55
$\sum x = 36$	$\sum y = 60$	$\sum x^2 = 266$	$\sum y^2 = 706$	$\sum xy = 293$

$$r = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}\sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}}$$

$$= \frac{293 - \frac{(36)(60)}{6}}{\sqrt{266 - \frac{(36)^2}{6}}\sqrt{706 - \frac{(60)^2}{6}}}$$

$$= -0.9203$$

Example 2

Calculate the coefficient of correlation from the following data:

x	12	9	8	10	11	13	7
y	14	8	6	9	11	12	3

Solution

$n = 7$

x	y	x^2	y^2	xy
12	14	144	196	168
9	8	81	64	72
8	6	64	36	48
10	9	100	81	90
11	11	121	121	121
13	12	169	144	156
7	3	49	9	21
$\sum x = 70$	$\sum y = 63$	$\sum x^2 = 728$	$\sum y^2 = 651$	$\sum xy = 676$

$$r = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}\sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}}$$

$$= \frac{676 - \frac{(70)\,(63)}{7}}{\sqrt{728 - \frac{(70)^2}{7}}\sqrt{651 - \frac{(63)^2}{7}}}$$

$$= 0.949$$

Example 3

Calculate the coefficient of correlation for the following data:

x	9	8	7	6	5	4	3	2	1
y	15	16	14	13	11	12	10	8	9

Solution

$n = 9$

x	y	x^2	y^2	xy
9	15	81	225	135
8	16	64	256	128
7	14	49	196	98
6	13	36	169	78
5	11	25	121	55
4	12	16	144	48
3	10	9	100	30
2	8	4	64	16
1	9	1	81	9
$\sum x = 45$	$\sum y = 108$	$\sum x^2 = 285$	$\sum y^2 = 1356$	$\sum xy = 597$

$$r = \frac{\sum x\,y - \dfrac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}}\sqrt{\sum y^2 - \dfrac{\left(\sum y\right)^2}{n}}}$$

$$= \frac{597 - \dfrac{(45)(108)}{9}}{\sqrt{285 - \dfrac{(45)^2}{9}}\sqrt{1356 - \dfrac{(108)^2}{9}}}$$

$$= 0.95$$

Example 4

Calculate the correlation coefficient between the following data:

x	5	9	13	17	21
y	12	20	25	33	35

Solution

$$n = 5$$

$$\overline{x} = \frac{\sum x}{n} = \frac{65}{5} = 13$$

$$\overline{y} = \frac{\sum y}{n} = \frac{125}{5} = 25$$

x	y	$x-\overline{x}$	$y-\overline{y}$	$(x-\overline{x})^2$	$(y-\overline{y})^2$	$(x-\overline{x})(y-\overline{y})$
5	12	–8	–13	64	169	104
9	20	–4	–5	16	25	20
13	25	0	0	0	0	0
17	33	4	8	16	64	32
21	35	8	10	64	100	80
$\sum x = 65$	$\sum y = 125$	$\sum(x-\overline{x}) = 0$	$\sum(y-\overline{y}) = 0$	$\sum(x-\overline{x})^2 = 160$	$\sum(y-\overline{y})^2 = 358$	$\sum(x-\overline{x})(y-\overline{y}) = 236$

$$r = \frac{\sum(x-\overline{x})(y-\overline{y})}{\sqrt{\sum(x-\overline{x})^2}\sqrt{\sum(y-\overline{y})^2}}$$

$$= \frac{236}{\sqrt{160}\sqrt{358}}$$

$$= 0.986$$

Example 5

Calculate the correlation coefficient between for the following values of demand and the corresponding price of a commodity:

Demand in Quintals	65	66	67	67	68	69	70	72
Price in rupees per kg	67	68	65	68	72	72	69	71

Solution

Let the demand in quintal be denoted by x and the price in rupees per kg be denoted by y.

$$n = 8$$

$$\bar{x} = \frac{\sum x}{n} = \frac{544}{8} = 68$$

$$\bar{y} = \frac{\sum y}{n} = \frac{552}{8} = 69$$

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
65	67	–3	–2	9	4	6
66	68	–2	–1	4	1	2
67	65	–1	–4	1	16	4
67	68	–1	–1	1	1	1
68	72	0	3	0	9	0
69	72	1	3	1	9	3
70	69	2	0	4	0	0
72	71	4	2	16	4	8
$\sum x = 544$	$\sum y = 552$	$\sum(x-\bar{x}) = 0$	$\sum(y-\bar{y}) = 0$	$\sum(x-\bar{x})^2 = 36$	$\sum(y-\bar{y})^2 = 44$	$\sum(x-\bar{x})(y-\bar{y}) = 24$

$$r = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sqrt{\sum(x-\bar{x})^2}\,\sqrt{\sum(y-\bar{y})^2}}$$

$$= \frac{24}{\sqrt{36}\,\sqrt{44}}$$

$$= 0.603$$

Example 6

Calculate the coefficient of correlation for the following pairs of x and y:

x	17	19	21	26	20	28	26	27
y	23	27	25	26	27	25	30	33

Solution

Let $a = 23$ and $b = 27$ be the assumed means of x and y series respectively.

$$d_x = x - a = x - 23$$
$$d_y = y - b = y - 27$$
$$n = 8$$

x	y	d_x	d_y	d_x^2	d_y^2	$d_x d_y$
17	23	–6	–4	36	16	24
19	27	–4	0	16	0	0
21	25	–2	–2	4	4	4
26	26	3	–1	9	1	–3
20	27	–3	0	9	0	0
28	25	5	–2	25	4	–10
26	30	3	3	9	9	9
27	33	4	6	16	36	24
		$\Sigma d_x = 0$	$\Sigma d_y = 0$	$\Sigma d_x^2 = 124$	$\Sigma d_y^2 = 70$	$\Sigma d_x d_y = 48$

$$r = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \frac{\left(\sum dx\right)^2}{n}}\sqrt{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}}$$

$$= \frac{48 - 0}{\sqrt{124 - 0}\sqrt{70 - 0}}$$

$$= 0.515$$

Example 7

Calculate the correlation coefficient from the following data:

x	23	27	28	29	30	31	33	35	36	39
y	18	22	23	24	25	26	28	29	30	32

Solution

Let $a = 30$ and $b = 25$ be the assumed means of x and y series respectively.

$$d_x = x - a = x - 30$$
$$d_y = y - b = x - 25$$
$$n = 10$$

x	y	d_x	d_y	d_x^2	d_y^2	$d_x d_y$
23	18	–7	–7	49	49	49
27	22	–3	–3	9	9	9
28	23	–2	–2	4	4	4
29	24	–1	–1	1	1	1
30	25	0	0	0	0	0
31	26	1	1	1	1	1
33	28	3	3	9	9	9
35	29	5	4	25	16	20
36	30	6	5	36	25	30
39	32	9	7	81	49	63
		$\Sigma d_x = 11$	$\Sigma d_y = 7$	$\Sigma d_x^2 = 215$	$\Sigma d_y^2 = 163$	$\Sigma d_x d_y = 186$

$$r = \frac{\sum d_x d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \dfrac{\left(\sum d_x\right)^2}{n}}\sqrt{\sum d_y^2 - \dfrac{\left(\sum d_y\right)^2}{n}}}$$

$$= \frac{186 - \dfrac{(11)(7)}{10}}{\sqrt{215 - \dfrac{(11)^2}{10}}\sqrt{163 - \dfrac{(7)^2}{10}}}$$

$$= 0.996$$

Example 8

Calculate the coefficient of correlation between the ages of cars and annual maintenance costs.

Age of cars (year)	2	4	6	7	8	10	12
Annual maintenance cost (₹)	1600	1500	1800	1900	1700	2100	2000

Solution

Let the ages of cars in years be denoted by x and annual maintenance costs in rupees be denoted by y.

Let $a = 7$ and $b = 1800$ be the assumed means of x and y series respectively.

Let $h = 1$, $k = 100$

$$d_x = \frac{x-a}{h} = \frac{x-7}{1} = x-7$$

$$d_y = \frac{y-b}{k} = \frac{y-1800}{100}$$

$$n = 7$$

x	y	d_x	d_y	d_x^2	d_y^2	$d_x d_y$
2	1600	–5	–2	25	4	10
4	1500	–3	3	9	9	9
6	1800	–1	0	1	0	0
7	1900	0	1	0	1	0
8	1700	1	–1	1	1	–1
10	2100	3	3	9	9	9
12	2000	5	2	25	4	10
		$\sum d_x = 0$	$\sum d_y = 0$	$\sum d_x^2 = 70$	$\sum d_y^2 = 28$	$\sum d_x d_y = 37$

$$r = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \frac{\left(\sum d_x\right)^2}{n}}\sqrt{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}}$$

$$= \frac{37-0}{\sqrt{70-0}\sqrt{28-0}}$$

$$= 0.836$$

Example 9

Calculate Karl Pearson's coefficient of correlation for the data given below:

x	10	14	18	22	26	30
y	18	12	24	6	30	36

Solution

Let $a = 22$ and $b = 24$ be the assumed means of x and y series respectively.

Let $h = 4$, $k = 6$

$$d_x = \frac{x-a}{h} = \frac{x-22}{4}$$

$$d_y = \frac{y-b}{k} = \frac{y-24}{6}$$

$$n = 6$$

x	y	d_x	d_y	d_x^2	d_y^2	$d_x d_y$
10	18	–3	–1	9	1	3
14	12	–2	–2	4	4	4
18	24	–1	0	1	0	0
22	6	0	–3	0	9	0
26	30	1	1	1	1	1
30	36	2	2	4	4	4
		$\sum d_x = -3$	$\sum d_y = -3$	$\sum d_x^2 = 19$	$\sum d_y^2 = 19$	$\sum d_x d_y = 12$

$$r = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \frac{\left(\sum d_x\right)^2}{n}}\sqrt{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}}$$

$$= \frac{12 - \frac{(-3)(-3)}{6}}{\sqrt{19 - \frac{(-3)^2}{6}}\sqrt{19 - \frac{(-3)^2}{6}}}$$

$$= 0.6$$

Example 10

The coefficient of correlation between two variables X and Y is 0.48. The covariance is 36. The variance of X is 16. Find the standard deviation of Y.

Solution

$$r = 0.48, \qquad \text{cov}(X, Y) = 36, \qquad \sigma_X^2 = 16$$

$$\therefore \qquad \sigma_X = 4$$

$$r = \frac{\text{cov}\,(X, Y)}{\sigma_X\ \sigma_Y}$$

$$0.48 = \frac{36}{4\sigma_Y}$$

$$\therefore\quad \sigma_Y = 18.75$$

Example 11

Given n = 10*,* σ_X = 5.4*,* σ_Y = 6.2*, and sum of the product of deviations from the mean of x and y is* 66*. Find the correlation coefficient.*

Solution

$$n = 10,\ \sigma_X = 5.4,\ \sigma_Y = 6.2$$

$$\sum (x-\overline{x})(y-\overline{y}) = 66$$

$$\sigma_X = \sqrt{\frac{\sum (x-\overline{x})^2}{n}}$$

$$5.4 = \sqrt{\frac{\sum (x-\overline{x})^2}{10}}$$

$$\therefore\qquad \sum (x-\overline{x})^2 = 291.6$$

$$\sigma_Y = \sqrt{\frac{\sum (y-\overline{y})^2}{n}}$$

$$6.2 = \sqrt{\frac{\sum (y-\overline{y})^2}{10}}$$

$$\therefore\qquad \sum (y-\overline{y})^2 = 384.4$$

$$r = \frac{\sqrt{\sum (x-\overline{x})(y-\overline{y})}}{\sqrt{\sum (x-\overline{x})^2}\ \sqrt{\sum (y-\overline{y})^2}}$$

$$= \frac{66}{\sqrt{291.6}\ \sqrt{384.4}}$$

$$= 0.197$$

Example 12

From the following information, calculate the value of n.

$$\sum x = 4,\ \sum y = 4,\ \sum x^2 = 44,\ \sum y^2 = 44,\ \sum xy = -40,\ r = -1$$

Solution

$$r = \frac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}}\sqrt{\sum y^2 - \dfrac{\left(\sum y\right)^2}{n}}}$$

$$-1 = \frac{-40 - \dfrac{(4)(4)}{n}}{\sqrt{44 - \dfrac{(4)^2}{n}}\sqrt{44 - \dfrac{(4)^2}{n}}}$$

$$\therefore \qquad n = 8$$

Example 13

From the following data, find the number of items n.

$$r = 0.5, \sum (x - \overline{x})(y - \overline{y}) = 120, \sigma_Y = 8, \sum (x - \overline{x})^2 = 90$$

Solution

$$\sigma_Y = \sqrt{\frac{\sum (y - \overline{y})^2}{n}}$$

$$8 = \sqrt{\frac{\sum (y - \overline{y})^2}{n}}$$

$$\sum (y - \overline{y})^2 = 64\,n$$

$$r = \frac{\sum (x - \overline{x})(y - \overline{y})}{\sqrt{\sum (x - \overline{x})^2}\sqrt{\sum (y - \overline{y})^2}}$$

$$0.5 = \frac{120}{\sqrt{90}\sqrt{64\,n}}$$

$$\therefore \qquad n = 10$$

Example 14

Calculate the correlation coefficient between x and y from the following data:

$$n = 10, \sum x = 140, \sum y = 150, \sum (x - 10)^2 = 180$$

$$\sum (y - 15)^2 = 215, \sum (x - 10)\,(y - 15) = 60$$

Solution

$$\sum d_x^2 = \sum (x-10)^2 = 180$$
$$\sum d_y^2 = \sum (y-15)^2 = 215$$
$$\sum d_x\, d_y = \sum (x-10)\,(y-15) = 60$$
$$a = 10$$
$$b = 15$$
$$n = 10$$
$$\bar{x} = \frac{\sum x}{n} = \frac{140}{10} = 14$$
$$\bar{y} = \frac{\sum y}{n} = \frac{150}{10} = 15$$
$$\bar{x} = a + \frac{\sum d_x}{n}$$
$$14 = 10 + \frac{\sum d_x}{10}$$
$$\therefore \qquad \sum d_x = 40$$
$$\bar{y} = b + \frac{\sum d_y}{n}$$
$$15 = 15 + \frac{\sum d_y}{10}$$
$$\therefore \qquad \sum d_y = 0$$
$$r = \frac{\sum d_x d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \dfrac{\left(\sum d_x\right)^2}{n}}\ \sqrt{\sum d_y^2 - \dfrac{\left(\sum d_y\right)^2}{n}}}$$
$$= \frac{60 - \dfrac{(40)(0)}{10}}{\sqrt{180 - \dfrac{(40)^2}{10}}\ \sqrt{215 - \dfrac{0}{10}}}$$
$$= 0.915$$

Example 15

A computer operator while calculating the coefficient between two variates x and y for 25 pairs of observations obtained the following constants:

$$n = 25, \sum x = 125, \sum x^2 = 650, \sum y = 100,$$
$$\sum y^2 = 460, \sum xy = 508$$

It was later discovered at the time of checking that he had copied down two pairs as (6, 14) *and* (8, 6) *while the correct pairs were* (8,12) *and* (6, 8). *Obtain the correct value of the correlation coefficient.*

Solution

$$n = 25$$

$$\begin{aligned}\text{Corrected} \sum x &= \text{Incorrect} \sum x - (\text{Sum of incorrect } x) + (\text{Sum of correct } x)\\ &= 125 - (6+8) + (8+6)\\ &= 125\end{aligned}$$

Similarly,

$$\text{Corrected} \sum y = 100 - (14+6) + (12+8) = 100$$
$$\text{Corrected} \sum x^2 = 650 - (6^2 + 8^2) + (8^2 + 6^2) = 650$$
$$\text{Corrected} \sum y^2 = 460 - (14^2 + 6^2) + (12^2 + 8^2) = 436$$
$$\text{Corrected} \sum xy = 508 - (84+48) + (96+48) = 520$$

Correct value of correlation coefficient

$$r = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}\sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}}$$
$$= \frac{520 - \frac{(125)(100)}{25}}{\sqrt{650 - \frac{(125)^2}{25}}\sqrt{436 - \frac{(100)^2}{25}}}$$
$$= 0.67$$

EXERCISE 4.1

1. Draw a scatter diagram to represent the following data:

x	2	4	5	6	8	11
y	18	12	10	8	7	5

Calculate the coefficient of correlation between *x* and *y*.

[Ans.: −0.92]

2. Find the coefficient of correlation between x and y for the following data:

x	10	12	18	24	23	27
y	13	18	12	25	30	10

[Ans.: 0.223**]**

3. From the following information relating to the stock exchange quotations for two shares A and B, ascertain by using Pearson's coefficient of correlation how shares A and B are correlated in their prices?

Price share (A) ₹	160	164	172	182	166	170	178
Price share (B) ₹	292	280	260	234	266	254	230

[Ans.: –0.96**]**

4. Find the correlation coefficient between the income and expenditure of a wage earner.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul
Income	46	54	56	56	58	60	62
Expenditure	36	40	44	54	42	58	54

[Ans.: 0.769**]**

5. From the following data, examine whether the input of oil and output of electricity can be said to be correlated.

Input of oil	6.9	8.2	7.8	4.8	9.6	8.0	7.7
Output of Electricity	1.9	3.5	6.5	1.3	5.5	3.5	2.2

[Ans.: 0.696**]**

6. For the following data, show that cov $(x, x^2) = 0$.

x	–3	–2	–1	0	1	2	3
x^2	9	4	1	0	1	4	9

7. Find the coefficient of correlation between x and y for the following data:

x	62	64	65	69	70	71	72	74
y	126	125	139	145	165	152	180	208

[Ans.: 0.9032**]**

8. The following data gave the growth of employment in lacs in the organized sector in India between 1988 and 1995:

Year	1988	1989	1990	1991	1992	1993	1994	1995
Public sector	98	101	104	107	113	120	125	128
Private sector	65	65	67	68	68	69	68	68

Find the correlation coefficient between the employment in public and private sectors.

[Ans.: 0.77**]**

9. Calculate Karl Pearson's coefficient of correlation from the following data, using 20 as the working mean for price and 70 as working mean for demand.

Price	14	16	17	18	19	20	21	22	23
Demand	84	78	70	75	66	67	62	58	60

[Ans.: −0.954**]**

10. A sample of 25 pairs of values x and y lead to the following results:

$$\sum x = 127, \sum y = 100, \sum x^2 = 760, \sum y^2 = 449, \sum xy = 500$$

Later on, it was found that two pairs of values were taken as (8, 14) and (8, 6) instead of the correct values (8, 12) and (6, 8). Find the corrected coefficient between x and y.

[Ans.: −0.31**]**

4.9 RANK CORRELATION

Let a group of n individuals be arranged in order of merit with respect to some characteristics. The same group would give a different order (rank) for different characteristics. Considering the orders corresponding to two characteristics A and B, the correlation between these n pairs of ranks is called the *rank correlation* in the characteristics A and B for that group of individuals.

4.9.1 Spearman's Rank Correlation Coefficient

Let x, y be the ranks of the i^{th} individuals in two characteristics A and B respectively where $i = 1, 2, ..., n$. Assuming that no two individuals have the same rank either for x or y, each of the variables x and y take the values $1, 2, ..., n$.

$$\bar{x} = \bar{y} = \frac{1+2+3+\cdots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

$$\begin{aligned}
\sum (x-\bar{x})^2 &= \sum (x^2 - 2x\,\bar{x} + \bar{x}^2) \\
&= \sum x^2 - 2\bar{x}\sum x + \bar{x}^2 \sum 1 \\
&= \sum x^2 - 2n\bar{x}^2 + n\bar{x}^2 \qquad \left[\because \quad \sum x = n\bar{x} \text{ and } \sum 1 = n\right] \\
&= \sum x^2 - n\bar{x}^2 \\
&= (1^2 + 2^2 + \cdots + n^2) - n\left(\frac{n+1}{2}\right)^2
\end{aligned}$$

$$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)^2}{4}$$

$$= \frac{1}{12}(n^3 - n)$$

Similarly, $\sum (y - \bar{y})^2 = \frac{1}{12}(n^3 - n)$

If d denotes the difference between the ranks of the i^{th} individuals in the two variables,

$$d = x - y = (x - \bar{x}) - (y - \bar{y}) \qquad [\because \quad \bar{x} = \bar{y}]$$

Squaring and summing over i from 1 to n,

$$\sum d^2 = \sum \left[(x - \bar{x}) - (y - \bar{y})\right]^2$$

$$= \sum (x - \bar{x})^2 + \sum (y - \bar{y})^2 - 2\sum (x - \bar{x})(y - \bar{y})$$

$$\sum (x - \bar{x})(y - \bar{y}) = \frac{1}{2}\left[\sum (x - \bar{x})^2 + \sum (y - \bar{y})^2 - \sum d^2\right]$$

$$= \frac{1}{12}(n^3 - n) - \frac{1}{2}\sum d^2$$

Hence, the coefficient of correlation between these variables is

$$r = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sqrt{\sum (x - \bar{x})^2 \sum (y - \bar{y})^2}}$$

$$= \frac{\frac{1}{12}(n^3 - n) - \frac{1}{2}\sum d^2}{\frac{1}{12}(n^3 - n)}$$

$$= 1 - \frac{6\sum d^2}{n^3 - n}$$

$$= 1 - \frac{6\sum d^2}{n(n^2 - 1)}$$

This is called Spearman's rank correlation coefficient and is denoted by ρ.

Note $\sum d = \sum (x - y) = \sum x - \sum y = n(\bar{x} - \bar{y}) = 0$

Example 1

Ten participants in a contest are ranked by two judges as follows:

x	1	3	7	5	4	6	2	10	9	8
y	3	1	4	5	6	9	7	8	10	2

Calculate the rank correlation coefficient.

Solution

$n = 10$

Rank by first Judge x	Rank by second Judge y	$d = x - y$	d^2
1	3	–2	4
3	1	2	4
7	4	3	9
5	5	0	0
4	6	–2	4
6	9	–3	9
2	7	–5	25
10	8	2	4
9	10	–1	1
8	2	6	36
		$\sum d = 0$	$\sum d^2 = 96$

$$r = 1 - \frac{6\sum d^2}{n(n^2 - 1)}$$

$$= 1 - \frac{6(96)}{10\left[(10)^2 - 1\right]}$$

$$= 0.418$$

Example 2

Ten competitors in a musical test were ranked by the three judges A, B, and C in the following order:

Rank by A	1	6	5	10	3	2	4	9	7	8
Rank by B	3	5	8	4	7	10	2	1	6	9
Rank by C	6	4	9	8	1	2	3	10	5	7

Using the rank correlation method, find which pair of judges has the nearest approach to common liking in music. **[Summer 2015]**

Solution

$n = 10$

Rank by A x	Rank by B y	Rank by C z	$d_1 = x - y$	$d_2 = y - z$	$d_3 = z - x$	d_1^2	d_2^2	d_3^2
1	3	6	–2	–3	5	4	9	25
6	5	4	1	1	–2	1	1	4
5	8	9	–3	–1	4	9	1	16
10	4	8	6	–4	–2	36	16	4
3	7	1	–4	6	–2	16	36	4
2	10	2	–8	8	0	64	64	0
4	2	3	2	–1	–1	4	1	1
9	1	10	8	–9	1	64	81	1
7	6	5	1	1	–2	1	1	4
8	9	7	–1	2	–1	1	4	1
			$\sum d_1 = 0$	$\sum d_2 = 0$	$\sum d_3 = 0$	$\sum d_1^2 = 200$	$\sum d_2^2 = 214$	$\sum d_3^2 = 60$

$$
\begin{aligned}
r(x, y) &= 1 - \frac{6\sum d_1^2}{n(n^2 - 1)} \\
&= 1 - \frac{6(200)}{10\left[(10)^2 - 1\right]} \\
&= -0.21
\end{aligned}
$$

$$
\begin{aligned}
r(y, z) &= 1 - \frac{6\sum d_2^2}{n(n^2 - 1)} \\
&= 1 - \frac{6(214)}{10\left[(10)^2 - 1\right]} \\
&= -0.296
\end{aligned}
$$

$$
\begin{aligned}
r(z, x) &= 1 - \frac{6\sum d_3^2}{n(n^2 - 1)} \\
&= 1 - \frac{6(60)}{10\left[(10)^2 - 1\right]} \\
&= 0.64
\end{aligned}
$$

Since $r(z, x)$ is maximum, the pair of judges A and C has the nearest common approach.

Example 3

Ten students got the following percentage of marks in mathematics and physics:

Mathematics (x)	8	36	98	25	75	82	92	62	65	35
Physics (y)	84	51	91	60	68	62	86	58	35	49

Find the rank correlation coefficient.

Solution

$n = 10$

x	y	Rank in mathematics x	Rank in Physics y	$d = x - y$	d^2
8	84	10	3	7	49
36	51	7	8	–1	1
98	91	1	1	0	0
25	60	9	6	3	9
75	68	4	4	0	0
82	62	3	5	–2	4
92	86	2	2	0	0
62	58	6	7	–1	1
65	35	5	10	–5	25
35	49	8	9	–1	1
				$\sum d = 0$	$\sum d^2 = 90$

$$r = 1 - \frac{6 \sum d^2}{n(n^2 - 1)}$$

$$= 1 - \frac{6(90)}{10\left[(10)^2 - 1\right]}$$

$$= 0.455$$

Example 4

The coefficient of rank correlation of the marks obtained by 10 students in physics and chemistry was found to be 0.5. It was later discovered that the difference in ranks in the two subjects obtained by one of the students was wrongly taken as 3 instead of 7. Find the rank coefficient of the rank correlation.

Solution

$n = 10$

$$r = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

$$0.5 = 1 - \frac{6\sum d^2}{10(100-1)}$$

$$\therefore \quad \sum d^2 = 82.5$$

$$\text{Correct} \sum d^2 = \text{Incorrect} \sum d^2 - (\text{Incorrect rank difference})^2 + (\text{Correct rank difference})^2$$

$$= 82.5 - (3)^2 + (7)^2$$

$$= 122.5$$

Correct coefficient of rank correlation $r = 1 - \dfrac{6(122.5)}{10(100-1)}$

$$= 0.26$$

4.9.2 Tied Ranks

If there is a tie between two or more individuals ranks, the rank is divided among equal individuals, e.g., if two items have fourth rank, the 4th and 5th rank is divided between them equally and is given as $\frac{4+5}{2} = 4.5^{\text{th}}$ rank to each of them. If three items have the same 4th rank, each of them is given $\frac{4+5+6}{3} = 5^{\text{th}}$ rank. As a result of this, the following adjustment or correction is made in the rank correlation formula. If m is the number of item having equal ranks then the factor $\frac{1}{12}(m^3 - m)$ is added to $\sum d^2$. If there are more than one cases of this type, this factor is added corresponding to each case.

$$r = 1 - \frac{6\left[\sum d^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \cdots\right]}{n(n^2-1)}$$

Example 1

Obtain the rank correlation coefficient from the following data:

x	10	12	18	18	15	40
y	12	18	25	25	50	25

Solution

Here, $n = 6$

x	y	Rank x	Rank y	$d = x - y$	d^2
10	12	1	1	0	0
12	18	2	2	0	0
18	25	4.5	4	0.5	0.25
18	25	4.5	4	0.5	0.25
15	50	3	6	–3	9
40	25	6	4	2	4
					$\sum d^2 = 13.5$

There are two items in the x series having equal values at the rank 4. Each is given the rank 4.5. Similarly, there are three items in the y series at the rank 3. Each of them is given the rank 4.

$$m_1 = 2, m_2 = 3$$

$$r = 1 - \frac{6\left[\sum d^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2)\right]}{n(n^2 - 1)}$$

$$= 1 - \frac{6\left[13.50 + \frac{1}{12}(8-2) + \frac{1}{12}(27-3)\right]}{6\left[(6)^2 - 1\right]}$$

$$= 0.5429$$

EXERCISE 4.2

1. Compute Spearman's rank correlation coefficient from the following data:

x	18	20	34	52	12
y	39	23	35	18	46

[Ans.: –0.9]

2. Two judges gave the following ranks to a series of eight one-act plays in a drama competition. Examine the relationship between their judgements.

Judge A	8	7	6	3	2	1	5	4
Judge B	7	5	4	1	3	2	6	8

[Ans.: 0.62]

3. From the following data, calculate Spearman's rank correlation between x and y.

x	36	56	20	42	33	44	50	15	60
y	50	35	70	58	75	60	45	80	38

[Ans.: 0.92**]**

4. Ten competitors in a voice test are ranked by three judges in the following order:

Rank by First Judge	6	10	2	9	8	1	5	3	4	7
Rank by Second Judge	5	4	10	1	9	3	8	7	2	6
Rank by Third Judge	4	8	2	10	7	6	9	1	3	6

Use the method of rank correlation to gauge which pairs of judges has the nearest approach to common liking in voice.

[Ans.: The first and third judge**]**

5. The following table gives the scores obtained by 11 students in English and Tamil translation. Find the rank correlation coefficient.

Scores in English	40	46	54	60	70	80	82	85	85	90	95
Scores in Tamil	45	45	50	43	40	75	55	72	65	42	70

[Ans.: 0.36**]**

6. Calculate Spearman's coefficient of rank correlation for the following data:

x	53	98	95	81	75	71	59	55
y	47	25	32	37	30	40	39	45

[Ans.: –0.905**]**

7. Following are the scores of ten students in a class and their IQ:

Score	35	40	25	55	85	90	65	55	45	50
IQ	100	100	110	140	150	130	100	120	140	110

Calculate the rank correlation coefficient between the score IQ.

[Ans.: 0.47**]**

4.10 REGRESSION

Regression is defined as a method of estimating the value of one variable when that of the other is known and the variables are correlated. *Regression analysis* is used to predict or estimate one variable in terms of the other variable. It is a highly valuable tool for prediction purpose in economics and business. It is useful in statistical estimation of demand curves, supply curves, production function, cost function, consumption function, etc.

4.11 TYPES OF REGRESSION

Regression is classified into two types:

1. Simple and multiple regressions
2. Linear and nonlinear regressions

4.11.1 Simple and Multiple Regressions

Depending upon the study of the number of variables, regression may be simple or multiple.

1. Simple Regression The regression analysis for studying only two variables at a time is known as simple regression.

2. Multiple Regression The regression analysis for studying more than two variables at a time is known as multiple regression.

4.11.2 Linear and Nonlinear Regressions

Depending upon the regression curve, regression may be linear or nonlinear.

1. Linear Regression If the regression curve is a straight line, the regression is said to be linear.

2. Nonlinear Regression If the regression curve is not a straight line i.e., not a first-degree equation in the variables x and y, the regression is said to be nonlinear or curvilinear. In this case, the regression equation will have a functional relation between the variables x and y involving terms in x and y of the degree higher than one, i.e., involving terms of the type x^2, y^2, x^3, y^3, xy, etc.

4.12 METHODS OF STUDYING REGRESSION

There are two methods of studying correlation:

(i) Method of scatter diagram
(ii) Method of least squares

4.12.1 Method of Scatter Diagram

It is the simplest method of obtaining the lines of regression. The data are plotted on a graph paper by taking the independent variable on the x-axis and the dependent variable on the y-axis. Each of these points are generally scattered in a narrow strip. If the correlation is perfect, i.e., if r is equal to one, positive, or negative, the points will lie on a line which is the line of regression.

4.12.2 Method of Least Squares

This is a mathematical method which gives an objective treatment to find a line of regression. It is used for obtaining the equation of a curve which fits best to a given set of observations. It is based on the assumption that the sum of squares of differences between the estimated values and the actual observed values of the observations is minimum.

4.13 LINES OF REGRESSION

If the variables, which are highly correlated, are plotted on a graph then the points lie in a narrow strip. If all the points in the scatter diagram cluster around a straight line, the line is called the *line of regression*. The line of regression is the line of best fit and is obtained by the principle of least squares.

Line of Regression of *y* on *x*

It is the line which gives the best estimate for the values of y for any given values of x. The regression equation of y on x is given by

$$y - \overline{y} = r\frac{\sigma_y}{\sigma_x}(x - \overline{x})$$

It is also written as

$$y = a + bx$$

Line of Regression of *x* on *y*

It is the line which gives the best estimate for the values of x for any given values of y. The regression equation for x on y is given by

$$x - \overline{x} = r\frac{\sigma_x}{\sigma_y}(y - \overline{y})$$

It is also written as

$$x = a + by$$

where $\overline{x}$ and $\overline{y}$ are means of x series and y series respectively, σ_x and σ_y are standard deviations of x series and y series respectively, r is the correlation coefficient between x and y.

4.14 REGRESSION COEFFICIENTS

The slope b of the line of regression of y on x is also called the *coefficient of regression* of y on x. It represents the increment in the value of y corresponding to a unit change in the value of x.

$$b_{yx} = \text{Regression coefficient of } y \text{ on } x$$

$$= r\frac{\sigma_y}{\sigma_x}$$

Similarly, the slope b of the line of regression of x on y is called the coefficient of regression of x on y. It represents the increment in the value of x corresponding to a unit change in the value of y.

$$b_{xy} = \text{Regression coefficient of } x \text{ on } y$$
$$= r\,\frac{\sigma_x}{\sigma_y}$$

Expressions for Regression Coefficients

(i) We know that

$$r = \frac{\sum (x-\bar{x})\,(y-\bar{y})}{\sqrt{\sum (x-\bar{x})^2}\,\sqrt{\sum (y-\bar{y})^2}}$$

$$\sigma_x = \sqrt{\frac{\sum (x-\bar{x})^2}{n}}$$

$$\sigma_y = \sqrt{\frac{\sum (y-\bar{y})^2}{n}}$$

$$b_{yx} = r\,\frac{\sigma_y}{\sigma_x}$$
$$= \frac{\sum (x-\bar{x})\,(y-\bar{y})}{\sum (x-\bar{x})^2}$$

and $$b_{xy} = r\,\frac{\sigma_x}{\sigma_y}$$
$$= \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (y-\bar{y})^2}$$

(ii) We know that

$$r = \frac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}}\,\sqrt{\sum y^2 - \dfrac{\left(\sum y\right)^2}{n}}}$$

$$\sigma_x = \sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}$$

$$\sigma_y = \sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}$$

$$b_{yx} = r\,\frac{\sigma_y}{\sigma_x}$$

$$= \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}$$

and $$b_{xy} = r\,\frac{\sigma_x}{\sigma_y}$$

$$= \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}$$

(iii) We know that

$$r = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \frac{\left(\sum d_x\right)^2}{n}}\;\sqrt{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}}$$

$$\sigma_x = \sqrt{\sum d_x^2 - \frac{\left(\sum d_x\right)^2}{n}}$$

$$\sigma_y = \sqrt{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}$$

$$b_{yx} = r\,\frac{\sigma_y}{\sigma_x}$$

$$= \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \frac{\left(\sum d_x\right)^2}{n}}}$$

and $$b_{xy} = r\,\frac{\sigma_x}{\sigma_y}$$

$$= \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}}$$

4.15 PROPERTIES OF REGRESSION COEFFICIENTS

1. The coefficient of correlation is the geometric mean of the coefficients of regression, i.e., $r = \sqrt{b_{yx} b_{xy}}$.

Proof We know that

$$b_{yx} = r\,\frac{\sigma_y}{\sigma_x}$$

$$b_{xy} = r\,\frac{\sigma_x}{\sigma_y}$$

$$b_{yx}\, b_{xy} = r\,\frac{\sigma_y}{\sigma_x} \cdot r\,\frac{\sigma_x}{\sigma_y}$$

$$= r^2$$

$$r = \sqrt{b_{yx}\, b_{xy}}$$

2. If one of the regression coefficients is greater than one, the other must be less than one.

Proof Let $b_{yx} > 1$

We know that

$$r^2 \le 1 \text{ and } r^2 = b_{yx}\, b_{xy}$$

$$b_{yx}\, b_{xy} \le 1$$

$$b_{yx} \le \frac{1}{b_{xy}}$$

Hence, if $b_{yx} < 1$, $b_{xy} > 1$

3. The arithmetic mean of regression coefficients is greater than or equal to the coefficient of correlation.

Proof We have to prove that

$$\frac{1}{2}(b_{yx} + b_{xy}) \ge r$$

i.e., $$\frac{1}{2}\left(r\,\frac{\sigma_y}{\sigma_x} + r\,\frac{\sigma_x}{\sigma_y}\right) \ge r$$

i.e., $$\frac{\sigma_y}{\sigma_x} + \frac{\sigma_x}{\sigma_y} \ge 2$$

i.e., $$\sigma_y^2 + \sigma_x^2 - 2\sigma_x\sigma_y \ge 0$$

i.e., $(\sigma_y - \sigma_x)^2 \geq 0$

which is always true, since the square of a real quantity is 1 ≥ 0.

4. Regression Coefficients are independent of the change of origin but not of scale.

Proof Let $d_x = \dfrac{x-a}{h}, \; d_y = \dfrac{y-b}{k}$

$$x = a + hd_x, \; y = b + kd_y$$

where a, b, h (> 0) and k(> 0) are constants.

$$r_{d_x d_y} = r_{xy}, \sigma_{d_x}^2 = \frac{1}{h^2}\sigma_x^2, \sigma_{d_y}^2 = \frac{1}{k^2}\sigma_y^2$$

$$\begin{aligned} b_{d_x d_y} &= r_{d_x d_y} \frac{\sigma_{d_x}}{\sigma_{d_y}} \\ &= r_{xy} \frac{\sigma_x}{h} \frac{k}{\sigma_y} \\ &= \frac{k}{h} r_{xy} \frac{\sigma_x}{\sigma_y} \\ &= \frac{k}{h} b_{xy} \end{aligned}$$

Similarly, $b_{d_y d_x} = \dfrac{h}{k} b_{yx}$

5. Both regression coefficients will have the same sign i.e., either both are positive or both are negative.

6. The sign of correlation is same as that of the regression coefficients, i.e., $r > 0$ if $b_{xy} > 0$ and $b_{yx} > 0$; and $r < 0$ if $b_{xy} < 0$ and $b_{yx} < 0$.

4.16 PROPERTIES OF LINES OF REGRESSION (LINEAR REGRESSION)

1. The two regression lines x on y and y on x always intersect at their means $(\bar{x}, \bar{y})$.
2. Since $r^2 = b_{yx}\, b_{xy}$, i.e., $r = \sqrt{b_{yx}\, b_{xy}}$, therefore, r, b_{yx}, b_{xy} all have the same sign.
3. If $r = 0$, the regression coefficients are zero.
4. The regression lines become identical if $r = \pm 1$. It follows from the regression equations that $x = \bar{x}$ and $y = \bar{y}$. If $r = 0$, these lines are perpendicular to each other.

Example 1

The regression lines of a sample are $x + 6y = 6$ *and* $3x + 2y = 10$*. Find*
(i) sample means $\overline{x}$ and $\overline{y}$, *and*
(ii) the coefficient of correlation between x and y.
(iii) Also estimate y when $x = 12$*.*

Solution

(i) The regression lines pass through the point $(\overline{x}, \overline{y})$.

$$\overline{x} + 6\overline{y} = 6 \qquad \text{...(1)}$$

$$3\overline{x} + 2\overline{y} = 10 \qquad \text{...(2)}$$

Solving Eqs (1) and (2),

$$\overline{x} = 3, \quad \overline{y} = \frac{1}{2}$$

(ii) Let the line $x + 6y = 6$ be the line of regression of y on x.

$$6y = -x + 6$$

$$y = -\frac{1}{6}x + 1$$

$$\therefore \qquad b_{yx} = -\frac{1}{6}$$

Let the line $3x + 2y = 10$ be the line of regression of x on y.

$$3x = -2y + 10$$

$$x = -\frac{2}{3}y + \frac{10}{3}$$

$$\therefore \qquad b_{xy} = -\frac{2}{3}$$

$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{\left(-\frac{1}{6}\right)\left(-\frac{2}{3}\right)} = \frac{1}{3}$$

Since b_{yx} and b_{xy} are negative, r is negative.

$$r = -\frac{1}{3}$$

Estimated value of y when $x = 12$ is

$$y = -\frac{1}{6}(12) + 1 = -1$$

Example 2

If the two lines of regression are $4x - 5y + 30 = 0$ and $20x - 9y - 107 = 0$, which of these are lines of regression of x on y and y on x? Find r_{xy} and σ_y when $\sigma_x = 3$.

Solution

For the line $4x - 5y + 30 = 0,$

$$-5y = -4x - 30$$

$$y = 0.8x + 6$$

$$\therefore \quad b_{yx} = 0.8$$

For the line $20x - 9y - 107 = 0$

$$20x = 9y + 107$$

$$x = 0.45y + 5.35$$

$$\therefore \quad b_{xy} = 0.45$$

Both b_{yx} and b_{xy} are positive.

Hence, line $4x - 5y + 30 = 0$ is the line of regression of y one x and line $20x - 9y - 107 = 0$ is the line of regression of x on y.

$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{(0.8)(0.45)} = 0.6$$

$$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

$$0.8 = 0.6\left(\frac{\sigma_y}{3}\right)$$

$$\therefore \quad \sigma_y = 4$$

Example 3

The following data regarding the heights (y) and weights (x) of 100 college students are given:

$$\sum x = 15000, \quad \sum x^2 = 2272500, \quad \sum y = 6800$$

$$\sum y^2 = 463025, \quad \sum xy = 1022250$$

Find the coefficient of correlation between height and weight and also the equation of regression of height and weight.

Solution

$$n = 100$$

$$b_{yx} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}$$

$$= \frac{1022250 - \frac{(15000)(6800)}{100}}{2272500 - \frac{(15000)^2}{100}}$$

$$= 0.1$$

$$b_{xy} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}$$

$$= \frac{1022250 - \frac{(15000)(6800)}{100}}{463025 - \frac{(6800)^2}{100}}$$

$$= 3.6$$

$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{(0.1)(3.6)} = 0.6$$

$$\bar{x} = \frac{\sum x}{n} = \frac{15000}{100} = 150$$

$$\bar{y} = \frac{\sum y}{n} = \frac{6800}{100} = 68$$

The equation of the line of regression of y on x is

$$y - \bar{y} = b_{yx}\,(x - \bar{x})$$

$$y - 68 = 0.1(x - 150)$$

$$y = 0.1x + 53$$

The equation of the line of regression of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 150 = 3.6(y - 68)$$

$$x = 3.6y - 94.8$$

Example 4

For a bivariate data, the mean value of x is 20 and the mean value of y is 45. The regression coefficient of y on x is 4 and that of x on y is $\frac{1}{9}$. *Find*

(i) the coefficient of correlation, and

(ii) the standard deviation of x if the standard deviation of y is 12.

(iii) Also write down the equations of regression lines.

Solution

$$\bar{x} = 20, \quad \bar{y} = 45, \quad b_{yx} = 4, \quad b_{xy} = \frac{1}{9}$$

(i) $$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{(4)\left(\frac{1}{9}\right)} = \frac{2}{3} = 0.667$$

(ii) $$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

$$4 = \frac{2}{3}\left(\frac{12}{\sigma_x}\right)$$

$$\therefore \quad \sigma_x = 2$$

(iii) The equation of the regression line of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 45 = 4(x - 20)$$

$$y = 4x - 35$$

The equation of the regression line of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 20 = \frac{1}{9}(y - 45)$$

$$x = \frac{1}{9}y + 15$$

Example 5

From the following results, obtain the two regression equations and estimate the yield when the rainfall is 29 cm and the rainfall, when the yield is 600 kg:

	Yield in kg	Rainfall in cm
Mean	508.4	26.7
SD	36.8	4.6

The coefficient of correlation between yield and rainfall is 0.52.

Solution

Let rainfall in cm be denoted by x and yield in kg be denoted by y.

$$\bar{x} = 26.7, \quad \bar{y} = 508.4, \quad \sigma_x = 4.6, \quad \sigma_y = 36.8, \quad r = 0.52$$

$$\begin{aligned} b_{yx} &= r\frac{\sigma_y}{\sigma_x} \\ &= 0.52\left(\frac{36.8}{4.6}\right) \\ &= 4.16 \\ b_{xy} &= r\frac{\sigma_x}{\sigma_y} \\ &= 0.52\left(\frac{4.6}{36.8}\right) \\ &= 0.065 \end{aligned}$$

The equation of the line of regression of y on x is

$$\begin{aligned} y - \bar{y} &= b_{yx}(x - \bar{x}) \\ y - 508.4 &= 4.16(x - 26.7) \\ y &= 4.16x + 397.328 \end{aligned}$$

The equation of the line of regression of x on y is

$$\begin{aligned} x - \bar{x} &= b_{xy}(y - \bar{y}) \\ x - 26.7 &= 0.065(y - 508.4) \\ x &= 0.065y - 6.346 \end{aligned}$$

Estimated yield when the rainfall is 29 cm is

$$y = 4.16\,(29) + 397.328 = 517.968 \text{ kg}$$

Estimated rainfall when the yield is 600 kg is

$$x = 0.065\,(600) - 6.346 = 32.654 \text{ cm}$$

Example 6

Find the regression coefficients b_{yx} and b_{xy} and hence, find the correlation coefficient between x and y for the following data:

x	4	2	3	4	2
y	2	3	2	4	4

Solution

$n = 5$

x	y	x^2	y^2	xy
4	2	16	4	8
2	3	4	9	6
3	2	9	4	6
4	4	16	16	16
2	4	4	16	8
$\Sigma x = 15$	$\Sigma y = 15$	$\Sigma x^2 = 49$	$\Sigma y^2 = 49$	$\Sigma xy = 44$

$$b_{yx} = \frac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}}$$

$$= \frac{44 - \dfrac{(15)(15)}{5}}{49 - \dfrac{(15)^2}{5}}$$

$$= -0.25$$

$$b_{xy} = \frac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sum y^2 - \dfrac{\left(\sum y\right)^2}{n}}$$

$$= \frac{44 - \dfrac{(15)(15)}{5}}{49 - \dfrac{(15)^2}{5}}$$

$$= -0.25$$

$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{(-0.25)(-0.25)} = 0.25$$

Since b_{yx} and b_{xy} are negative, r is negative.

$$r = -0.25$$

Example 7

The following data give the experience of machine operators and their performance rating as given by the number of good parts turned out per 100 pieces.

Operator	1	2	3	4	5	6
Performance rating (x)	23	43	53	63	73	83
Experience (y)	5	6	7	8	9	10

Calculate the regression line of performance rating on experience and also estimate the probable performance if an operator has 11 *years of experience.* **[Summer 2015]**

Solution

$n = 6$

x	y	y^2	xy
23	5	25	115
43	6	36	258
53	7	49	371
63	8	64	504
73	9	81	657
83	10	100	830
$\sum x = 338$	$\sum y = 45$	$\sum y^2 = 355$	$\sum xy = 2735$

$$b_{xy} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}$$

$$= \frac{2735 - \frac{(338)(45)}{6}}{355 - \frac{(45)^2}{6}}$$

$$= 11.429$$

$$\bar{x} = \frac{\sum x}{n} = \frac{338}{6} = 56.33$$

$$\bar{y} = \frac{\sum y}{n} = \frac{45}{6} = 7.5$$

The equation of regression line of x on y is

$$x - \bar{x} = b_{xy}\ (y - \bar{y})$$

$$x - 56.33 = 11.429(y - 7.5)$$

$$x = 11.429\ y - 29.3875$$

Estimated performance if $y = 11$ is

$$x = 11.429(11) - 29.3875 = 96.3315$$

Example 8

The number of bacterial cells (y) per unit volume in a culture at different hours (x) is given below:

x	0	1	2	3	4	5	6	7	8	9
y	43	46	82	98	123	167	199	213	245	272

Fit lines of regression of y on x and x on y. Also, estimate the number of bacterial cells after 15 *hours.*

Solution

$n = 10$

x	y	x^2	xy	y^2
0	43	0	0	1849
1	46	1	46	2116
2	82	4	164	6724
3	98	9	294	9604
4	123	16	492	15129
5	167	25	835	27889
6	199	36	1194	39601
7	213	49	1491	45369
8	245	64	1960	60025
9	272	81	2448	73984
$\sum x = 45$	$\sum y = 1488$	$\sum x^2 = 285$	$\sum xy = 8924$	$\sum y^2 = 282290$

$$b_{yx} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}$$

$$= \frac{8924 - \frac{(45)(1488)}{10}}{285 - \frac{(45)^2}{10}}$$

$$= 27.0061$$

$$b_{xy} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}$$

$$= \frac{8924 - \frac{(45)(1488)}{10}}{282290 - \frac{(1488)^2}{10}}$$

$$= 0.0366$$

$$\bar{x} = \frac{\sum x}{n} = \frac{45}{10} = 4.5$$

$$\bar{y} = \frac{\sum y}{n} = \frac{1488}{10} = 148.8$$

The equation of the line of regression of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 148.8 = 27.0061\,(x - 4.5)$$

$$y = 27.0061x + 27.2726$$

The equation of the line of regression of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 4.5 = 0.0366\,(y - 148.8)$$

$$x = 0.366\,y - 0.9461$$

At $x = 15$ hours,

$y = 27.0061\ (15) + 27.2726 = 432.3641$

Example 9

Find the regression coefficient of y on x for the following data:

x	1	2	3	4	5
y	160	180	140	180	200

Solution

$$n = 5$$

$$\bar{x} = \frac{\sum x}{n} = \frac{15}{5} = 3$$

$$\bar{y} = \frac{\sum y}{n} = \frac{860}{5} = 172$$

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(x-\bar{x})(y-\bar{y})$
1	160	–2	–12	4	24
2	180	–1	8	1	–8
3	140	0	–32	0	0
4	180	1	8	1	8
5	200	2	28	4	56
$\sum x = 15$	$\sum y = 860$	$\sum(x-\bar{x}) = 0$	$\sum(y-\bar{y}) = 0$	$\sum(x-\bar{x})^2 = 10$	$\sum(x-\bar{x})(y-\bar{y}) = 80$

$$b_{yx} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$$

$$= \frac{80}{10}$$

$$= 8$$

Example 10

Calculate the two regression coefficients from the data and find correlation coefficient.

x	7	4	8	6	5
y	6	5	9	8	2

Solution

$$n = 5$$

$$\bar{x} = \frac{\sum x}{n} = \frac{30}{5} = 6$$

$$\bar{y} = \frac{\sum y}{n} = \frac{30}{5} = 6$$

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
7	6	1	0	1	0	0
4	5	–2	–1	4	1	2
8	9	2	3	4	9	6
6	8	0	2	0	4	0
5	2	–1	–4	1	16	4
$\sum x = 30$	$\sum y = 30$	$\sum(x-\bar{x}) = 0$	$\sum(y-\bar{y}) = 0$	$\sum(x-\bar{x})^2 = 10$	$\sum(y-\bar{y})^2 = 30$	$\sum(x-\bar{x})(y-\bar{y}) = 12$

$$\begin{aligned} b_{yx} &= \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2} \\ &= \frac{12}{10} \\ &= 1.2 \end{aligned}$$

$$\begin{aligned} b_{xy} &= \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(y-\bar{y})^2} \\ &= \frac{12}{30} \\ &= 0.4 \end{aligned}$$

$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{(1.2)(0.4)} = 0.693$$

Example 11

Obtain the two regression lines from the following data and hence, find the correlation coefficient.

x	6	2	10	4	8
y	9	11	5	8	7

[Summer 2015]

Solution

$$n = 5$$

$$\bar{x} = \frac{\sum x}{n} = \frac{30}{5} = 6$$

$$\bar{y} = \frac{\sum y}{n} = \frac{40}{5} = 8$$

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
6	9	0	1	0	1	0
2	11	–4	3	16	9	–12
10	5	4	–3	16	9	–12
4	8	–2	0	4	0	0
8	7	2	–1	4	1	–2
$\sum x = 30$	$\sum y = 40$	$\sum(x-\bar{x}) = 0$	$\sum(y-\bar{y}) = 0$	$\sum(x-\bar{x})^2 = 40$	$\sum(y-\bar{y})^2 = 20$	$\sum(x-\bar{x})(y-\bar{y})^2 = -26$

$$b_{yx} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2} = \frac{-26}{40} = -0.65$$

$$b_{xy} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(y-\bar{y})^2} = \frac{-26}{20} = -1.3$$

The equation of regression line of y on x is

$$y-\bar{y} = b_{yx}(x-\bar{x})$$
$$y-8 = -0.65(x-6)$$
$$y = -0.65\,x + 11.9$$

The equation of regression line of x on y is

$$x-\bar{x} = b_{xy}(y-\bar{y})$$
$$x-6 = -1.3(y-8)$$
$$x = -1.3\,y + 16.4$$
$$r = \sqrt{b_{yx}\,b_{xy}} = \sqrt{(-0.65)\,(-1.3)} = 0.9192$$

Since b_{yx} and b_{xy} are negative, r is negative.

$$r = -0.9192.$$

Example 12

Calculate the regression coefficients and find the two lines of regression from the following data:

x	57	58	59	59	60	61	62	64
y	67	68	65	68	72	72	69	71

Find the value of y when x = 66.

Solution

$$n = 8$$

$$\bar{x} = \frac{\sum x}{n} = \frac{480}{8} = 60$$

$$\bar{y} = \frac{\sum y}{n} = \frac{552}{8} = 69$$

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
57	67	–3	–2	9	4	6
58	68	–2	–1	4	1	2
59	65	–1	–4	1	16	4
59	68	–1	–1	1	1	1
60	72	0	3	0	9	0
61	72	1	3	1	9	3
62	69	2	0	4	0	0
64	71	4	2	16	4	8
$\sum x = 480$	$\sum y = 552$	$\sum(x-\bar{x}) = 0$	$\sum(y-\bar{y}) = 0$	$\sum(x-\bar{x})^2 = 36$	$\sum(y-\bar{y})^2 = 44$	$\sum(x-\bar{x})(y-\bar{y}) = 24$

$$b_{yx} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$$

$$= \frac{24}{36}$$

$$= 0.667$$

$$b_{xy} = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (y-\bar{y})^2}$$
$$= \frac{24}{44}$$
$$= 0.545$$

The equation of regression line of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$
$$y - 69 = 0.667(x - 60)$$
$$y = 0.667x + 28.98$$

The equation of regression line of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$
$$x - 60 = 0.545(y - 69)$$
$$x = 0.545y + 22.395$$

Value of y when $x = 66$ is

$$y = 0.667\ (66) + 28.98 = 73.002$$

Example 13

The following data represents rainfall (x) and yield of paddy per hectare (y) in a particular area. Find the linear regression of x on y.

x	113	102	95	120	140	130	125
y	1.8	1.5	1.3	1.9	1.1	2.0	1.7

Solution

Let $a = 120$ and $b = 1.8$ be the assumed means of x and y series respectively.

$$d_x = x - a = x - 120$$
$$d_y = y - b = y - 1.8$$
$$n = 7$$

x	y	d_x	d_y	d_y^2	$d_x d_y$
113	1.8	–7	0	0	0
102	1.5	–18	–0.3	0.09	5.4
95	1.3	–25	–0.5	0.25	12.5
120	1.9	0	0.1	0.01	0
140	1.1	20	–0.7	0.49	–14
130	2.0	10	0.2	0.04	2.0
125	1.7	5	–0.1	0.01	–0.5
$\sum x = 825$	$\sum y = 11.3$	$\sum d_x = -15$	$\sum d_y = -1.3$	$\sum d_y^2 = 0.89$	$\sum d_x d_y = 5.4$

$$b_{xy} = \frac{\sum d_x d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sum d_y^2 - \dfrac{\left(\sum d_y\right)^2}{n}}$$

$$= \frac{5.4 - \dfrac{(-15)(-1.3)}{7}}{0.89 - \dfrac{(-1.3)^2}{7}}$$

$$= 4.03$$

$$\bar{x} = \frac{\sum x}{n} = \frac{825}{7} = 117.86$$

$$\bar{y} = \frac{\sum y}{n} = \frac{11.3}{7} = 1.614$$

The equation of the regression line of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 117.86 = 4.03\,(y - 1.614)$$

$$x = 4.03\,y + 111.36$$

Example 14

Find the two lines of regression from the following data:

Age of husband (x)	25	22	28	26	35	20	22	40	20	18
Age of wife (y)	18	15	20	17	22	14	16	21	15	14

Hence, estimate (i) the age of the husband when the age of the wife is 19, *and (ii) the age of the wife when the age of the husband is* 30.

Solution

Let $a = 26$ and $b = 17$ be the assumed means of x and y series respectively.

$$d_x = x - a = x - 26$$
$$d_y = y - b = y - 17$$
$$n = 10$$

x	y	d_x	d_y	d_x^2	d_y^2	$d_x d_y$
25	18	–1	1	1	1	–1
22	15	–4	–2	16	4	8
28	20	2	3	4	9	6
26	17	0	0	0	0	0
35	22	9	5	81	25	45
20	14	–6	–3	36	9	18
22	16	–4	–1	16	1	4
40	21	14	4	196	16	56
20	15	–6	–2	36	4	12
18	14	–8	–3	64	9	24
$\sum x = 256$	$\sum y = 172$	$\sum d_x = -4$	$\sum d_y = 2$	$\sum d_x^2 = 450$	$\sum d_y^2 = 78$	$\sum d_x d_y = 172$

$$b_{yx} = \frac{\sum d_x d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sum d_x^2 - \dfrac{\left(\sum d_x\right)^2}{n}}$$

$$= \frac{172 - \dfrac{(-4)(2)}{10}}{450 - \dfrac{(-4)^2}{10}}$$

$$= 0.385$$

$$b_{xy} = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}$$

$$= \frac{172 - \frac{(-4)(2)}{10}}{78 - \frac{(2)^2}{10}}$$

$$= 2.227$$

$$\bar{x} = \frac{\sum x}{n} = \frac{256}{10} = 25.6$$

$$\bar{y} = \frac{\sum y}{n} = \frac{172}{10} = 17.2$$

The equation of the regression line of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$y - 17.2 = 0.385(x - 25.6)$$

$$y = 0.385x + 7.344$$

The equation of the regression line of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$x - 25.6 = 2.227(y - 17.2)$$

$$x = 2.227y - 12.704$$

Estimated age of the husband when the age of the wife is 19 is

$$x = 2.227\ (19) - 12.704 = 29.601 \text{ or } 30 \text{ nearly}$$

Age of the husband = 30 years

Estimated age of the wife when the age of the husband is 30 is

$$y = 0.385\ (30) + 7.344 = 18.894 \text{ or } 19 \text{ nearly}$$

Age of the wife = 19 years

Example 15

From the following data, obtain the two regression lines and correlation coefficient.

Sales (x)	100	98	78	85	110	93	80
Purchase (y)	85	90	70	72	95	81	74

Solution

Let $a = 93$ and $b = 81$ be the assumed means of x and y series respectively.

$$d_x = x - a = x - 93$$

$$d_y = y - b = y - 91$$

$$n = 7$$

x	y	d_x	d_y	d_x^2	d_y^2	d_xd_y
100	85	7	4	49	16	28
98	90	5	9	25	81	45
78	70	–15	–11	225	121	165
85	72	–8	–9	64	81	72
110	95	17	14	289	196	238
93	81	0	0	0	0	0
80	74	–13	–7	169	49	91
$\sum x = 644$	$\sum y = 567$	$\sum d_x = -7$	$\sum d_y = 0$	$\sum d_x^2 = 821$	$\sum d_y^2 = 544$	$\sum d_xd_y = 639$

$$b_{yx} = \frac{\sum d_xd_y - \dfrac{\sum d_x \sum d_y}{n}}{\sum d_x^2 - \dfrac{\left(\sum d_x\right)^2}{n}}$$

$$= \frac{639 - \dfrac{(-7)(0)}{7}}{821 - \dfrac{(-7)^2}{7}}$$

$$= 0.785$$

$$b_{xy} = \frac{\sum d_xdy - \dfrac{\sum d_x \sum d_y}{n}}{\sum d_y^2 - \dfrac{\left(\sum d_y\right)^2}{n}}$$

$$= \frac{639 - \dfrac{(-7)(0)}{7}}{544 - \dfrac{(0)^2}{7}}$$

$$= 1.1746$$

$$\bar{x} = \frac{\sum x}{n} = \frac{644}{7} = 92$$
$$\bar{y} = \frac{\sum y}{n} = \frac{567}{7} = 81$$

The equation of regression line of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$
$$y - 81 = 0.785(x - 92)$$
$$y = 0.785x + 8.78$$

The equation of regression line of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$
$$x - 92 = 1.1746(y - 81)$$
$$x = 1.1746 - 3.1426$$
$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{(0.785)(1.1746)} = 0.9602$$

EXERCISE 4.3

1. The following are the lines of regression $4y = x + 38$ and $9y = x + 288$. Estimate y when $x = 99$ and x when $y = 30$. Also, find the means of x and y.

[Ans.: $y = 43$, $x = 82$, $\bar{x} = 162$, $\bar{y} = 50$ **]**

2. The equations of the two lines of regression are $x = 19.13 - 0.87\,y$ and $y = 11.64 - 0.50\,x$. Find (i) the means of x and y, and (ii) the coefficient of correlation between x and y.

[Ans.: $\bar{x} = 15.79$, $\bar{y} = 3.74$, (ii) $r = -0.66$, $b_{yx} = -0.5$, $b_{xy} = 0.87$ **]**

3. Given var(x) = 25. The equations of the two lines of regression are $5x - y = 22$ and $64\,x - 45\,y = 24$. Find (i) $\bar{x}$ and $\bar{y}$, (ii) r, and (iii) σ_y.

[Ans.: $\bar{x} = 6$, $\bar{y} = 8$, (ii) $r = 1.87$ (iii) $\sigma_y = 0.2$ **]**

4. In a partially destroyed laboratory record of analysis of correlation data the following results are legible. Variance = 9, the equations of the lines of regression $4x - 5y + 33 = 0$, $20\,x - 9\,y - 107 = 0$. Find (i) the mean values of x and y, (ii) the standard deviation of y, and (iii) the coefficient of correlation between x and y

[Ans.: (i) $\bar{x} = 13$, $\bar{y} = 17$, (ii) $\sigma_y = 4$, (iii) $r = 0.6$ **]**

5. From a sample of 200 pairs of observation, the following quantities were calculated:

$$\sum x = 11.34, \sum y = 20.78, \sum x^2 = 12.16, \sum y^2 = 84.96, \sum xy = 22.13$$

From the above data, show how to compute the coefficients of the equation $y = a + bx$.

[Ans.: $a = 0.0005, b = 1.82$ **]**

6. In the estimation of regression equations of two variables x and y, the following results were obtained:

$$\bar{x} = 90, \bar{y} = 70, n = 10, \Sigma(x - \bar{x})^2 = 6360, \Sigma(y - \bar{y})^2 = 2860$$
$$\Sigma(x - \bar{x})(y - \bar{y}) = 3900$$

Obtain the two lines of regression.

[Ans.: $x = 1.361\, y - 5.27$, $y = 0.613\, x + 14.812$**]**

7. Find the likely production corresponding to a rainfall of 40 cm from the following data:

	Rainfall (in cm)	Output (in quintals)
mean	30	50
SD	5	10
$r = 0.8$		

[Ans.: 66 quintals**]**

8. The following table gives the age of a car of a certain make and annual maintenance cost. Obtain the equation of the line of regression of cost on age.

Age of a car	2	4	6	8
Maintenance	1	2	2.5	3

[Ans.: $x = 0.325\, y + 0.5$**]**

9. Obtain the equation of the line of regression of y on x from the following data and estimate y for $x = 73$.

x	70	72	74	76	78	80
y	163	170	179	188	196	220

[Ans.: $y = 5.31\, x - 212.57$, $y = 175.37$**]**

10. The heights in cm of fathers (x) and of the eldest sons (y) are given below:

x	165	160	170	163	173	158	178	168	173	170	175	180
y	173	168	173	165	175	168	173	165	180	170	173	178

Estimate the height of the eldest son if the height of the father is 172 cm and the height of the father if the height of the eldest son is 173 cm. Also, find the coefficient of correlation between the heights of fathers and sons.

[Ans.: (i) $y = 1.016\,x - 5.123$ (ii) $x = 0.476\,y + 98.98$

(iii) 169.97, 173.45 (iv) $r = 0.696$**]**

11. Find (i) the lines of regression, and (ii) coefficient of correlation for the following data:

x	65	66	67	67	68	69	70	72
y	67	68	65	66	72	72	69	71

[Ans.: (i) $y = 19.64 + 0.72\,x$, $x = 33.29 + 0.5\,y$, (ii) $r = 0.604$**]**

12. Find the line of regression for the following data and estimate y corresponding to $x = 15.5$.

x	10	12	13	16	17	20	25
y	19	22	24	27	29	33	37

[Ans.: $y = 1.21x + 7.71$, $y = 26.465$**]**

13. The following data give the heights in inches (x) and weights in lbs (y) of a random sample of 10 students:

x	61	68	68	64	65	70	63	62	64	67
y	112	123	130	115	110	125	100	113	116	126

Estimate the weight of a student of height 59 inches.

[Ans.: 126.4 lbs**]**

14. Find the regression equations of y on x from the data given below taking deviations from actual mean of x and y.

Price in rupees (x)	10	12	13	12	16	15
Demand (y)	40	38	43	45	37	43

Estimate the demand when the price is ₹20.

[Ans.: $y = -0.25\,x + 44.25$, $y = 39.25$**]**

Points to Remember

Karl Pearson's Coefficient of Correlation

(i) $$r = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y}$$

(ii) $$r = \frac{\sum (x-\overline{x})(y-\overline{y})}{\sqrt{\sum (x-\overline{x})^2}\sqrt{\sum (y-\overline{y})^2}}$$

(iii) $$r = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}\sqrt{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}}$$

(iv) $$r = \frac{\sum d_x d_y - \frac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \frac{\left(\sum d_x\right)^2}{n}}\sqrt{\sum d_y^2 - \frac{\left(\sum d_y\right)^2}{n}}}$$

Spearman's Rank Correlation Coefficient

$$r = 1 - \frac{6\sum d^2}{n(n^2-1)}$$

Spearman's Rank Correlation Coefficient for Tied Ranks

$$r = 1 - \frac{6\left[\sum d^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \cdots\right]}{n(n^2-1)}$$

Lines of Regression

Line of Regression of *y* on *x*

$$y - \overline{y} = r\frac{\sigma_y}{\sigma_x}(x - \overline{x})$$

It is also written as

$$y = a + bx$$

Line of Regression of *x* on *y*

$$x - \overline{x} = r\frac{\sigma_x}{\sigma_y}(y - \overline{y})$$

It is also written as

$$x = a + by$$

Regression Coefficients

$$b_{yx} = r\,\frac{\sigma_y}{\sigma_x}$$

$$b_{xy} = r\,\frac{\sigma_x}{\sigma_y}$$

Expressions for Regression Coefficients

(i) $$b_{yx} = \frac{\sum (x-\overline{x})\,(y-\overline{y})}{\sum (x-\overline{x})^2}$$

and $$b_{xy} = \frac{\sum (x-\overline{x})(y-\overline{y})}{\sum (y-\overline{y})^2}$$

(ii) $$b_{yx} = \frac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sum x^2 - \dfrac{\left(\sum x\right)^2}{n}}$$

and $$b_{xy} = \frac{\sum xy - \dfrac{\sum x \sum y}{n}}{\sum y^2 - \dfrac{\left(\sum y\right)^2}{n}}$$

(iii) $$b_{yx} = \frac{\sum d_x d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_x^2 - \dfrac{\left(\sum d_x\right)^2}{n}}}$$

and $$b_{xy} = \frac{\sum d_x d_y - \dfrac{\sum d_x \sum d_y}{n}}{\sqrt{\sum d_y^2 - \dfrac{\left(\sum d_y\right)^2}{n}}}$$

CHAPTER 5

Curve Fitting

Chapter Outline

5.1 Introduction
5.2 Least Square Method
5.3 Fitting of Linear Curves
5.4 Fitting of Quadratic Curves
5.5 Fitting of Exponential and Logarithmic Curves

5.1 INTRODUCTION

Curve fitting is the process of finding the 'best-fit' curve for a given set of data. It is the representation of the relationship between two variables by means of an algebraic equation. On the basis of this mathematical equation, predictions can be made in many statistical problems.

Suppose a set of n points of values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ of the two variables x and y are given. These values are plotted on a rectangular coordinate system, i.e., the xy-plane. The resulting set of points is known as a *scatter diagram* (Fig. 5.1). The scatter diagram exhibits the trend and it is possible to visualize a smooth curve approximating the data. Such a curve is known as an *approximating curve*.

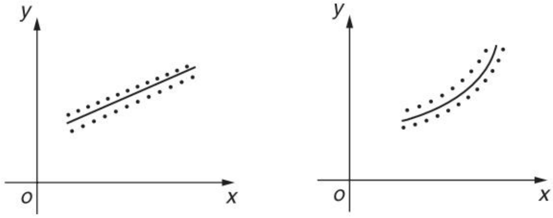


Fig. 5.1

5.2 LEAST SQUARE METHOD

From a scatter diagram, generally, more than one curve may be seen to be appropriate to the given set of data. The method of least squares is used to find a curve which passes through the maximum number of points.

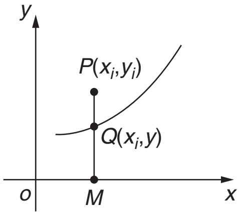


Fig. 5.2

Let P (x_i, y_i) be a point on the scatter diagram (Fig. 5.2). Let the ordinate at P meet the curve $y = f(x)$ at Q and the x-axis at M.

Distance
$$\begin{aligned} QP &= MP - MQ \\ &= y_i - y \\ &= y_i - f(x_i) \end{aligned}$$

The distance QP is known as *deviation*, *error*, or *residual* and is denoted by d_i. It may be positive, negative, or zero depending upon whether P lies above, below, or on the curve. Similar residuals or errors corresponding to the remaining $(n - 1)$ points may be obtained. The sum of squares of residuals, denoted by E, is given as

$$E = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} [y_i - f(x_i)]^2$$

If $E = 0$ then all the n points will lie on $y = f(x)$. If $E \neq 0, f(x)$ is chosen such that E is minimum, i.e., the best fitting curve to the set of points is that for which E is minimum. This method is known as the least square method. This method does not attempt to determine the form of the curve $y = f(x)$ but it determines the values of the parameters of the equation of the curve.

5.3 FITTING OF LINEAR CURVES

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be the set of n values and let the relation between x and y be $y = a + bx$. The constants a and b are selected such that the straight line is the best fit to the data.

The residual at $x = x_i$ is

$$\begin{aligned} d_i &= y_i - f(x_i) \\ &= y_i - (a + bx_i) \qquad i = 1, 2, \ldots, n \\ E &= \sum_{i=1}^{n} d_i^2 \\ &= \sum_{i=1}^{n} \left[y_i - (a + bx_i) \right]^2 \\ &= \sum_{i=1}^{n} (y_i - a - bx_i)^2 \end{aligned}$$

For E to be minimum,

(i) $\dfrac{\partial E}{\partial a} = 0$

$$\sum_{i=1}^{n} 2(y_i - a - bx_i)(-1) = 0$$

$$\sum_{i=1}^{n} (y_i - a - bx_i) = 0$$

$$\sum_{i=1}^{n} y_i = a\sum_{i=1}^{n} 1 + b\sum_{i=1}^{n} x_i$$

$$\sum y = na + b\sum x$$

(ii) $\dfrac{\partial E}{\partial b} = 0$

$$\sum_{i=1}^{n} 2(y_i - a - bx_i)(-x_i) = 0$$

$$\sum_{i=1}^{n} (x_i y_i - ax_i - bx_i^2) = 0$$

$$\sum_{i=1}^{n} x_i\, y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2$$

$$\sum xy = a\sum x + b\sum x^2$$

These two equations are known as *normal equations*. These equations can be solved simultaneously to give the best values of a and b. The best fitting straight line is obtained by substituting the values of a and b in the equation $y = a + bx$.

Example 1

Fit a straight line to the following data:

x	1	2	3	4	6	8
y	2.4	3	3.6	4	5	6

Solution

Let the straight line to be fitted to the data be

$$y = a + bx$$

The normal equations are

$$\sum y = na + b\sum x \qquad \ldots(1)$$

$$\sum xy = a\sum x + b\sum x^2 \qquad \ldots(2)$$

Here, $n = 6$

x	y	x^2	xy
1	2.4	1	2.4
2	3	4	6
3	3.6	9	10.8
4	4	16	16
6	5	36	30
8	6	64	48
$\sum x = 24$	$\sum y = 24$	$\sum x^2 = 130$	$\sum xy = 113.2$

Substituting these values in Eqs (1) and (2),

$$24 = 6a + 24b \qquad \ldots(3)$$

$$113.2 = 24a + 130b \qquad \ldots(4)$$

Solving Eqs (3) and (4),

$$a = 1.9764$$

$$b = 0.5059$$

Hence, the required equation of the straight line is

$$y = 1.9764 + 0.5059x$$

Example 2

Fit a straight line to the following data. Also, estimate the value of y at $x = 2.5$.

x	0	1	2	3	4
y	1	1.8	3.3	4.5	6.3

Solution

Let the straight line to be fitted to the data be

$$y = a + bx$$

The normal equations are

$$\sum y = na + b\sum x \qquad \ldots(1)$$

$$\sum xy = a\sum x + b\sum x^2 \qquad \ldots(2)$$

Here, $n = 5$

x	y	x^2	xy
0	1	0	0
1	1.8	1	1.8
2	3.3	4	6.6
3	4.5	9	13.5
4	6.3	16	25.2
$\sum x = 10$	$\sum y = 16.9$	$\sum x^2 = 30$	$\sum xy = 47.1$

Substituting these values in Eqs (1) and (2),

$$16.9 = 5a + 10b \qquad \ldots(3)$$

$$47.1 = 10a + 30b \qquad \ldots(4)$$

Solving Eqs (3) and (4),

$$a = 0.72$$

$$b = 1.33$$

Hence, the required equation of the straight line is

$$y = 0.72 + 1.33x$$

At $x = 2.5$,

$$y\ (2.5) = 0.72 + 1.33\ (2.5) = 4.045$$

Example 3

A simply supported beam carries a concentrated load P(lb) *at its midpoint. Corresponding to various values of P, the maximum deflection* Y(in) *is measured. The data is given below:*

P	100	120	140	160	180	200
Y	0.45	0.55	0.60	0.70	0.80	0.85

Find a law of the form Y = a + bP using the least square method.

[Summer 2015]

Solution

Let the straight line to be fitted to the data be

$$Y = a + bP$$

The normal equations are

$$\sum Y = na + b\sum P \qquad \ldots(1)$$

$$\sum PY = a\sum P + b\sum P^2 \qquad ...(2)$$

Here, $n = 6$

P	Y	P^2	PY
100	0.45	10000	45
120	0.55	14400	66
140	0.60	19600	84
160	0.70	25600	112
180	0.80	32400	144
200	0.85	40000	170
$\sum P = 900$	$\sum Y = 3.95$	$\sum P^2 = 142000$	$\sum PY = 621$

Substituting these values in Eqs (1) and (2),

$$3.95 = 6a + 900\,b \qquad ...(3)$$

$$621 = 900\,a + 142000\,b \qquad ...(4)$$

Solving Eqs (3) and (4),

$$a = 0.0476$$

$$b = 0.0041$$

Hence, the required equation of the straight line is

$$Y = 0.0476 + 0.0041\,P$$

Example 4

Fit a straight line to the following data. Also, estimate the value of y at x = 70.

x	71	68	73	69	67	65	66	67
y	69	72	70	70	68	67	68	64

Solution

Since the values of x and y are larger, we choose the origin for x and y at 69 and 67 respectively,

Let $X = x - 69$ and $Y = y - 67$

Let the straight line to be fitted to the data be

$$Y = a + bX$$

The normal equations are

$$\sum Y = na + b\sum X \qquad \ldots(1)$$

$$\sum XY = a\sum X + b\sum X^2 \qquad \ldots(2)$$

Here, $n = 8$

x	y	X	Y	X^2	XY
71	69	2	2	4	4
68	72	–1	5	1	–5
73	70	4	3	16	12
69	70	0	3	0	0
67	68	–2	1	4	–2
65	67	–4	0	16	0
66	68	–3	1	9	–3
67	64	–2	–3	4	6
		$\sum X = -6$	$\sum Y = 12$	$\sum X^2 = 54$	$\sum XY = 12$

Substituting these values in Eqs (1) and (2),

$$12 = 8a - 6b \quad \text{...(3)}$$
$$12 = -6a + 54b \quad \text{...(4)}$$

Solving Eqs (3) and (4),

$$a = 1.8182$$
$$b = 0.4242$$

Hence, the required equation of the straight line is

$$Y = 1.8182 + 0.4242X$$
$$y - 67 = 1.8182 + 0.4242(x - 69)$$
$$y = 0.4242x + 39.5484$$
$$y(x = 70) = 0.4242(70) + 39.5484 = 69.2424$$

Example 5

Fit a straight line to the following data taking x as the dependent variable.

x	1	3	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9

Solution

If x is considered the dependent variable and y the independent variable, the equation of the straight line to be fitted to the data is

$$x = a + by$$

The normal equations are

$$\sum x = na + b\sum y \quad \ldots(1)$$

$$\sum xy = a\sum y + b\sum y^2 \quad \ldots(2)$$

Here, $n = 8$

x	y	y^2	xy
1	1	1	1
3	2	4	6
4	4	16	16
6	4	16	24
8	5	25	40
9	7	49	63
11	8	64	88
14	9	81	126
$\sum x = 56$	$\sum y = 40$	$\sum y^2 = 256$	$\sum xy = 364$

Substituting these values in Eqs (1) and (2),

$$56 = 8a + 40b \quad \ldots(3)$$

$$364 = 40a + 256b \quad \ldots(4)$$

Solving Eqs (3) and (4),

$$a = -0.5$$

$$b = 1.5$$

Hence, the required equation of the straight line is

$$x = -0.5 + 1.5y$$

Example 6

If P is the pull required to lift a load W by means of a pulley block, find a linear law of the form $P = mW + c$ connecting P and W using the following data:

P	12	15	21	25
W	50	70	100	120

where P and W are taken in kg-wt. Compute P when $W = 150$ kg.

Solution

Let the linear curve (straight line) fitted to the data be

$$P = mW + c = c + mW$$

The normal equations are

$$\sum P = nc + mW \qquad ...(1)$$

$$\sum PW = c\sum W + m\sum W^2 \qquad ...(2)$$

Here, $n = 4$

P	W	W^2	PW
12	50	2500	600
15	70	4900	1050
21	100	10000	2100
25	120	14400	3000
$\sum P = 73$	$\sum W = 340$	$\sum W^2 = 31800$	$\sum PW = 6750$

Substituting these values in Eqs (1) and (2),

$$73 = 4c + 340\,m \qquad ...(3)$$

$$6750 = 340\,c + 31800\,m \qquad ...(4)$$

Solving Eqs (3) and (4),

$$c = 2.2759$$

$$m = 0.1879$$

Hence, the required equation of the straight line is

$$P = 0.1879\,W + 2.2759$$

When $W = 150$ kg,

$$P = 0.1879(150) + 2.2759 = 30.4609$$

EXERCISE 5.1

1. Fit the line of best fit to the following data:

x	0	5	10	15	20	25
y	12	15	17	22	24	30

$\left[\textbf{Ans.:}\ y = 0.7x + 11.28\right]$

2. The results of a measurement of electric resistance R of a copper bar at various temperatures t°C are listed below:

$t^\circ C$	19	25	30	36	40	45	50
R	76	77	79	80	82	83	85

Find a relation $R = a + bt$ where a and b are constants to be determined.

$\left[\textbf{Ans.:}\ R = 70.0534 + 0.2924t\right]$

3. Fit a straight line to the following data:

x	1.53	1.78	2.60	2.95	3.42
y	33.50	36.30	40.00	45.85	53.40

$$\left[\textbf{Ans.:}\ y = 19 + 9.7x\right]$$

4. Fit a straight line to the following data:

x	100	120	140	160	180	200
y	0.45	0.55	0.60	0.70	0.80	0.85

$$\left[\textbf{Ans.:}\ y = 0.0475 + 0.00407x\right]$$

5. Find the relation of the type $R = aV + b$, when some values of R and V obtained from an experiment are

V	60	65	70	75	80	85	90
R	109	114	118	123	127	130	133

$$\left[\textbf{Ans.:}\ R = 0.8071\text{V} + 61.4675\right]$$

5.4 FITTING OF QUADRATIC CURVES

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be the set of n values and let the relation between x and y be $y = a + bx + cx^2$. The constants a, b, and c are selected such that the parabola is the best fit to the data. The residual at $x = x_i$ is

$$\begin{aligned} d_i &= y_i - f(x_i) \\ &= y_i - \left(a + bx_i + cx_i^2\right) \end{aligned}$$

$$\begin{aligned} E &= \sum_{i=1}^{n} d_i^2 \\ &= \sum_{i=1}^{n}\left[y_i - \left(a + bx_i + cx_i^2\right)\right]^2 \\ &= \sum_{i=1}^{n}\left(y_i - a - bx_i - cx_i^2\right)^2 \end{aligned}$$

For E to be minimum,

(i) $\dfrac{\partial E}{\partial a} = 0$

$$\sum_{i=1}^{n} 2\left(y_i - a - bx_i - cx_i\right)(-1) = 0$$

$$\sum_{i=1}^{n}\left(y_i - a - bx_i - cx_i\right) = 0$$

$$\sum_{i=1}^{n} y_i = a\sum_{i=1}^{n} 1 + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2$$

$$\sum y_i = na + b\sum x + c\sum x^2$$

(ii) $$\frac{\partial E}{\partial b} = 0$$

$$\sum_{i=1}^{n} 2(y_i - a - bx_i - cx_i)(-x_i) = 0$$

$$\sum_{i=1}^{n}\left(x_i y_i - ax_i - bx_i^2 - cx_i^3\right) = 0$$

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i^3$$

$$\sum xy = na + b\sum x^2 + c\sum x^3$$

(iii) $$\frac{\partial E}{\partial c} = 0$$

$$\sum_{i=1}^{n} 2(y_i - a - bx_i - cx_i^2)(x_i^2) = 0$$

$$\sum_{i=1}^{n} x_i^2 y_i - ax_i^2 - bx_i^3 - cx_i^4 = 0$$

$$\sum_{i=1}^{n} x_i^2 y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^4$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4$$

These equations are known as *normal equations*. These equations can be solved simultaneously to give the best values of a, b, and c. The best fitting parabola is obtained by substituting the values of a, b, and c in the equation $y = a + bx + cx^2$.

Example 1

Fit a least squares quadratic curve to the following data:

x	1	2	3	4
y	1.7	1.8	2.3	3.2

Estimate $y(2.4)$.

Solution

Let the equation of the least squares quadratic curve (parabola) be $y = a + bx + cx^2$. The normal equations are

$$\sum y = na + b\sum x + c\sum x^2 \qquad \ldots(1)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \qquad \ldots(2)$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4 \qquad \ldots(3)$$

Here, $n = 4$

x	y	x^2	x^3	x^4	xy	x^2y
1	1.7	1	1	1	1.7	1.7
2	1.8	4	8	16	3.6	7.2
3	2.3	9	27	81	6.9	20.7
4	3.2	16	64	256	12.8	51.2
$\Sigma x = 10$	$\Sigma y = 9$	$\Sigma x^2 = 30$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$	$\Sigma xy = 25$	$\Sigma x^2 y = 80.8$

Substituting these values in Eqs (1), (2), and (3),

$$9 = 4a + 10b + 30c \qquad \ldots(4)$$

$$25 = 10a + 30b + 100c \qquad \ldots(5)$$

$$80.8 = 30a + 100b + 354c \qquad \ldots(6)$$

Solving Eqs (4), (5), and (6),

$$a = 2$$

$$b = -0.5$$

$$c = 0.2$$

Hence, the required equation of least squares quadratic curve is

$$y = 2 - 0\cdot 5x + 0\cdot 2x^2$$

$$y(2\cdot 4) = 2 - 0\cdot 5(2\cdot 4) + 0\cdot 2(2.4)^2 = 1\cdot 952$$

Example 2

Fit a second-degree polynomial using least square method to the following data:

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

[Summer 2015]

Solution

Let the equation of the least squares quadratic curve be $y = a + bx + cx^2$. The normal equations are

$$\sum y = na + b\sum x + c\sum x^2 \qquad ...(1)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \qquad ...(2)$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4 \qquad ...(3)$$

Here, $n = 5$

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	1.8	1	1	1	1.8	1.8
2	1.3	4	8	16	2.6	5.2
3	2.5	9	27	81	7.5	22.5
4	6.3	16	64	256	25.2	100.8
$\sum x = 10$	$\sum y = 12.9$	$\sum x^2 = 30$	$\sum x^3 = 100$	$\sum x^4 = 354$	$\sum xy = 37.1$	$\sum x^2y = 130.3$

Substituting these values in Eqs (1), (2), and (3),

$$12.9 = 5a + 10b + 30\,c \qquad ...(4)$$

$$37.1 = 10a + 30b + 100c \qquad ...(5)$$

$$130.3 = 30a + 100b + 354c \qquad ...(6)$$

Solving Eqs (4), (5), and (6),

$$a = 1.42$$

$$b = -1.07$$

$$c = 0.55$$

Hence, the required equation of the least squares quadratic curve is

$$y = 1.42 - 1.07\,x + 0.55\,x^2$$

Example 3

By the method of least squares, fit a parabola to the following data:

x	1	2	3	4	5
y	5	12	26	60	97

Also, estimate y at $x = 6$.

Solution

Let the equation of the parabola be $y = a + bx + cx^2$. The normal equations are

$$\sum y = na + b\sum x + c\sum x^2 \qquad ...(1)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \qquad ...(2)$$

$$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4 \qquad ...(3)$$

Here, $n = 5$

x	y	x^2	x^3	x^4	xy	x^2y
1	5	1	1	1	5	5
2	12	4	8	16	24	48
3	26	9	27	81	78	234
4	60	16	64	256	240	960
5	97	25	125	625	485	2425
$\sum x = 15$	$\sum y = 200$	$\sum x^2 = 55$	$\sum x^3 = 225$	$\sum x^4 = 979$	$\sum xy = 832$	$\sum x^2y = 3672$

Substituting these values in Eqs (1), (2), and (3),

$$200 = 5a + 15b + 55\,c \qquad ...(4)$$

$$832 = 15a + 55b + 225c \qquad ...(5)$$

$$3672 = 55a + 225b + 979c \qquad ...(6)$$

Solving Eqs (4), (5), and (6),

$$a = 10.4$$

$$b = -11.0857$$

$$c = 5.7143$$

Hence, the required equation of the parabola is

$$y = 10.4 - 11.0857\,x + 5.7143\,x^2$$

$$y(6) = 10.4 - 11.0857(6) + 5.7143(6)^2 = 149.6006$$

Example 4

Fit a second-degree parabolic curve to the following data.

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

Solution

Let
$$X = x - 5$$

$$Y = y - 10$$

Let the equation of the parabola be $Y = a + bX + cX^2$.

The normal equations are

$$\sum Y = na + b\sum X + c\sum X^2 \qquad ...(1)$$

$$\sum XY = a\sum X + b\sum X^2 + c\sum X^3 \quad \ldots(2)$$

$$\sum X^2Y = a\sum X^2 + b\sum X^3 + c\sum X^4 \quad \ldots(3)$$

Here, $n = 9$

x	y	X	Y	X^2	X^3	X^4	XY	X^2Y
1	2	–4	–8	16	–64	256	32	–128
2	6	–3	–4	9	–27	81	12	–36
3	7	–2	–3	4	–8	16	6	–12
4	8	–1	–2	1	–1	1	2	–2
5	10	0	0	0	0	0	0	0
6	11	1	1	1	1	1	1	1
7	11	2	1	4	8	16	2	4
8	10	3	0	9	27	81	0	0
9	9	4	–1	16	64	256	–4	–16
		$\Sigma X = 0$	$\Sigma Y = -16$	$\Sigma X^2 = 60$	$\Sigma X^3 = 0$	$\Sigma X^4 = 708$	$\Sigma XY = 51$	$\Sigma X^2Y = -189$

Substituting these values in Eqs (1), (2), and (3),

$$-16 = 9a + 60c \quad \ldots(4)$$

$$51 = 60b \quad \ldots(5)$$

$$-189 = 60a + 708c \quad \ldots(6)$$

Solving Eqs (4), (5), and (6),

$$a = 0.0043$$

$$b = 0.85$$

$$c = -0.2673$$

Hence, the required equation of the parabola is

$$Y = 0.0043 + 0.85X - 0.2673X^2$$

$$y - 10 = 0.0043 + 0.85(x - 5) - 0.2673(x - 5)^2$$

$$\begin{aligned} y &= 10 + 0.0043 + 0.85(x - 5) - 0.2673(x^2 - 10x + 25) \\ &= 10 + 0.0043 + 0.85x - 4.25 - 0.2673x^2 + 2.673x - 6.6825 \\ &= -0.9282 + 3.523x - 0.2673x^2 \end{aligned}$$

Example 5

Fit a second-degree parabola $y = a + bx^2$ to the following data:

x	1	2	3	4	5
y	1.8	5.1	8.9	14.1	19.8

Solution

Let the curve to be fitted to the data be

$$y = a + bx^2$$

The normal equations are

$$\sum y = na + b\sum x^2 \quad \text{...(1)}$$

$$\sum x^2 y = a\sum x^2 + b\sum x^4 \quad \text{...(2)}$$

Here, $n = 5$

x	y	x^2	x^4	x^2y
1	1.8	1	1	1.8
2	5.1	4	16	20.4
3	8.9	9	81	80.1
4	14.1	16	256	225.6
5	19.8	25	625	495
	$\sum y = 49.7$	$\sum x^2 = 55$	$\sum x^4 = 979$	$\sum x^2 y = 822.9$

Substituting these values in Eqs (1) and (2),

$$49.7 = 5a + 55b \quad \text{...(3)}$$

$$822.9 = 55a + 979\,b \quad \text{...(4)}$$

Solving Eqs (3) and (4),

$$a = 1.8165$$

$$b = 0.7385$$

Hence, the required equation of the curve is

$$y = 1.8165 + 0.7385\,x^2$$

Example 6

Fit a curve $y = ax + bx^2$ *for the following data:*

x	1	2	3	4	5	6
y	2.51	5.82	9.93	14.84	20.55	27.06

Solution

Let the curve to be fitted to the data be

$$y = ax + bx^2$$

The normal equations are

$$\sum xy = a\sum x^2 + b\sum x^3 \quad \text{...(1)}$$

$$\sum x^2 y = a\sum x^3 + b\sum x^4 \qquad \ldots(2)$$

x	y	x^2	x^3	x^4	xy	x^2y
1	2.51	1	1	1	2.51	2.51
2	5.82	4	8	16	11.64	23.28
3	9.93	9	27	81	29.79	89.37
4	14.84	16	64	256	59.36	237.44
5	20.55	25	125	625	102.75	513.75
6	27.06	36	216	1296	162.36	974.16
		$\Sigma x^2 = 91$	$\Sigma x^3 = 441$	$\Sigma x^4 = 2275$	$\Sigma xy = 368.41$	$\Sigma x^2 y = 1840.51$

Substituting these values in Eqs (1) and (2),

$$368\cdot 41 = 91a + 441b \qquad \ldots(3)$$

$$1840\cdot 51 = 441\,a + 2275\,b \qquad \ldots(4)$$

Solving Eqs (3) and (4),

$$a = 2.11$$

$$b = 0.4$$

Hence, the required equation of the curve is

$$y = 2\cdot 11x + 0\cdot 4x^2$$

EXERCISE 5.2

1. Fit a parabola to the following data:

x	–2	–1	0	1	2
y	1.0	1.8	1.3	2.5	6.3

[**Ans.:** $y = 1.48 + 1.13x + 0.55x^2$]

2. Fit a curve $y = ax + bx^2$ to the following data:

x	–2	–1	0	1	2
y	–72	–46	–12	35	93

[**Ans.:** $y = 41.1x + 2.147x^2$]

3. Fit a parabola $y = a + bx + cx^2$ to the following data:

x	0	2	5	10
y	4	7	6.4	–6

[**Ans.:** $y = 4.1 + 1.979x - 0.299x^2$]

4. Fit a curve $y = a_0 + a_1x + a_2x^2$ for the given data:

x	3	5	7	9	11	13
y	2	3	4	6	5	8

[**Ans.:** $y = 0.7897 + 0.4004x + 0.0089x^2$]

5.5 FITTING OF EXPONENTIAL AND LOGARITHMIC CURVES

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be the set of n values and let the relation between x and y be $y = ab^x$.

Taking logarithm on both the sides of the equation $y = ab^x$,

$$\log_e y = \log_e a + x \log_e b$$

Putting $\log_e y = Y$, $\log_e a = A, x = X$, and $\log_e b = B$,

$$Y = A + BX$$

This is a linear equation in X and Y. The normal equations are

$$\sum Y = nA + B\sum X$$

$$\sum XY = A\sum X + B\sum X^2$$

Solving these equations, A and B, and, hence, a and b can be found. The best fitting exponential curve is obtained by substituting the values of a and b in the equation $y = ab^x$.

Similarly, the best fitting exponential curves for the relation $y = ax^b$ and $y = ae^{bx}$ can be obtained.

Example 1

Find the law of the form $y = ab^x$ to the following data:

x	1	2	3	4	5	6	7	8
y	1	1.2	1.8	2.5	3.6	4.7	6.6	9.1

Solution

$$y = ab^x$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + x \log_e b$$

Putting $\log_e y = Y$, $\log_e a = A$, $x = X$ and $\log_e b = B$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + B\sum X \quad \ldots(1)$$

$$\sum XY = A\sum X + B\sum X^2 \quad \ldots(2)$$

Here, $n = 8$

x	y	X	Y	X^2	XY
1	1	1	0.0000	1	0.0000
2	1.2	2	0.1823	4	0.3646
3	1.8	3	0.5878	9	1.7634
4	2.5	4	0.9163	16	3.6652
5	3.6	5	1.2809	25	6.4045
6	4.7	6	1.5476	36	9.2856
7	6.6	7	1.8871	49	13.2097
8	9.1	8	2.2083	64	17.6664
		$\sum X = 36$	$\sum Y = 8.6103$	$\sum X^2 = 204$	$\sum XY = 52.3594$

Substituting these values in Eqs (1) and (2),

$$8.6103 = 8A + 36B \quad \ldots(3)$$

$$52.3594 = 36A + 204B \quad \ldots(4)$$

Solving Eqs (3) and (4),

$$A = -0.3823$$

$$B = 0.3241$$

$$\log_e a = A$$

$$\log_e a = -0.3823$$

$$a = 0.6823$$

$$\log_e b = B$$

$$\log_e b = 0.3241$$

$$b = 1.3828$$

Hence, the required law is

$$y = 0.6823\ (1.3828)^x$$

Example 2

Fit a curve of the form $y = ab^x$ to the following data by the method of least squares:

x	1	2	3	4	5	6	7
y	87	97	113	129	202	195	193

Solution

$$y = ab^x$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + x \log_e b$$

Putting $\log_e y = Y$, $\log_e a = A$, $x = X$ and $\log_e b = B$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + B\sum X \qquad ...(1)$$

$$\sum XY = A\sum X + B\sum X^2 \qquad ...(2)$$

Here, $n = 7$

x	y	X	Y	X^2	XY
1	87	1	4.4659	1	4.4659
2	97	2	4.5747	4	9.1494
3	113	3	4.7274	9	14.1822
4	129	4	4.8598	16	19.4392
5	202	5	5.3083	25	26.5415
6	195	6	5.2730	36	31.6380
7	193	7	5.2627	49	36.8389
		$\sum X = 28$	$\sum Y = 34.4718$	$\sum X^2 = 140$	$\sum XY = 142.2551$

Substituting these values in Eqs (1) and (2),

$$34.4718 = 7A + 28\ B \qquad ...(3)$$

$$142.2551 = 28\ A + 140\ B \qquad ...(4)$$

Solving Eqs (3) and (4),

$$A = 4.3006$$

$$B = 0.156$$

$$\log_e a = A$$
$$\log_e a = 4.3006$$
$$a = 73.744$$
$$\log_e b = B$$
$$\log_e b = 0.156$$
$$b = 1.1688$$

Hence, the required curve is

$$y = 73.744\ (1.1688)^x$$

Example 3

Fit a curve of the form $y = ax^b$ to the following data:

x	20	16	10	11	14
y	22	41	120	89	56

Solution

$$y = ax^b$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + b \log_e x$$

Putting $\log_e y = Y$, $\log_e a = A$, $b = B$ and $\log_e x = X$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + B\sum X \qquad \ldots(1)$$

$$\sum XY = A\sum X + B\sum X^2 \qquad \ldots(2)$$

Here, $n = 5$

x	y	X	Y	X^2	XY
20	22	2.9957	3.0910	8.9742	9.2597
16	41	2.7726	3.7136	7.6873	10.2963
10	120	2.3026	4.7875	5.3019	11.0237
11	89	2.3979	4.4886	5.7499	10.7632
14	56	2.6391	4.0254	6.9648	10.6234
		$\sum X = 13.1079$	$\sum Y = 20.1061$	$\sum X^2 = 34.6781$	$\sum XY = 51.9663$

Substituting these values in Eqs (1) and (2),

$$20.1061 = 5A + 13.1079\,B \qquad \ldots(3)$$

$$51.9663 = 13.1079\,A + 34.6781\,B \qquad \ldots(4)$$

Solving Eqs (3) and (4),

$$A = 10.2146$$
$$B = -2.3624$$
$$\log_e a = A$$
$$\log_e a = 10.2146$$
$$a = 27298 \cdot 8539$$
$$\text{and } \; b = B = -2.3624$$

Hence, the required equation of the curve is

$$y = 27298.8539\, x^{-2.3624}$$

Example 4

Fit a curve of the form $y = ae^{bx}$ *to the following data*:

x	1	3	5	7	9
y	115	105	95	85	80

Solution

$$y = ae^{bx}$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + bx \log_e e$$
$$= \log_e a + bx$$

Putting $\log_e y = Y$, $\log_e a = A$, $b = B$ and $x = X$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + B\sum X \qquad \ldots(1)$$

$$\sum XY = A\sum X + B\sum X^2 \qquad \ldots(2)$$

Here, $n = 5$

x	y	X	Y	X^2	XY
1	115	1	4.7449	1	4.7449
3	105	3	4.6539	9	13.9617
5	95	5	4.5539	25	22.7695
7	85	7	4.4427	49	31.0989
9	80	9	4.3820	81	39.438
		$\sum X = 25$	$\sum Y = 22.7774$	$\sum X^2 = 165$	$\sum XY = 112.013$

Substituting these values in Eqs (1) and (2),

$$22.7774 = 5A + 25B \qquad \ldots(3)$$

$$112.013 = 25A + 165B \qquad \ldots(4)$$

Solving Eqs (3) and (4),

$$A = 4.7897$$

$$B = -0.0469$$

$$\log_e a = A$$

$$\log_e a = 4.7897$$

$$a = 120.2653$$

and

$$b = B = -0.0469$$

Hence, the required equation of the curve is

$$y = 120.2653\, e^{-0.0469x}$$

Example 5

Fit the exponential curve $y = ae^{bx}$ *to the following data*:

x	0	2	4	6	8
y	150	63	28	12	5.6

[Summer 2015]

Solution

$$y = ae^{bx}$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + bx \log_e e$$

$$= \log_e a + bx$$

Putting $\log_e y = Y$, $\log_e a = A$, $b = B$ and $x = X$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + b\sum X \qquad ...(1)$$

$$\sum XY = A\sum X + B\sum X^2 \qquad ...(2)$$

Here, $n = 5$

x	y	X	Y	X^2	XY
0	150	0	5.0106	0	0
2	63	2	4.1431	4	8.2862
4	28	4	3.3322	16	13.3288
6	12	6	2.4849	36	14.9094
8	5.6	8	1.7228	64	13.7824
		$\sum X = 20$	$\sum Y = 16.6936$	$\sum X^2 = 120$	$\sum XY = 50.3068$

Substituting these values in Eqs (1) and (2),

$$16.6936 = 5A + 20B \quad \text{...(3)}$$
$$50.3068 = 20A + 120B \quad \text{...(4)}$$

Solving Eqs (3) and (4),

$$A = 4.9855$$
$$B = -0.4117$$
$$\log_e a = A$$
$$\log_e a = 4.9855$$
$$a = 146.28$$

and $\quad b = B = -0.4117$

Hence, the required equation of the curve is

$$y = 146.28\, e^{-0.4117\, x}$$

Example 6

The pressure and volume of a gas are related by the equation $PV^\gamma = c$. Fit this curve to the following data:

P	0.5	1.0	1.5	2.0	2.5	3.0
V	1.62	1.00	0.75	0.62	0.52	0.46

Solution

$$PV^\gamma = c$$

Taking logarithm on both the sides,

$$\log_e P + \gamma \log_e V = \log_e c$$

$$\log_e V = \frac{1}{\gamma}\log_e c - \frac{1}{\gamma}\log_e P$$

Putting $\log_e V = y, \frac{1}{\gamma}\log_e c = a, \log_e P = x, -\frac{1}{\gamma} = b,$

$$y = a + bx$$

The normal equations are

$$\sum y = na + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

Here, $n = 6$

P	V	x	y	x^2	xy
0.5	1.62	–0.6931	0.4824	0.4804	–0.3343
1.0	1.00	0	0	0	0
1.5	0.75	0.4055	–0.2877	0.1644	–0.1166
2.0	0.62	0.6931	–0.4780	0.4804	–0.3313
2.5	0.52	0.9163	–0.6539	0.8396	–0.5992
3.0	0.46	1.0986	–0.7765	1.2069	–0.8531
		$\sum x = 2.4204$	$\sum y = -1.7137$	$\sum x^2 = 3.1717$	$\sum xy = -2.2345$

Substituting these values in Eqs (1) and (2),

$$-1.7137 = 6a + 2.4204\,b \qquad \text{...(3)}$$
$$-2.2345 = 2.4204a + 3.1717\,b \qquad \text{...(4)}$$

Solving Eqs (3) and (4),

$$a = -0.002$$
$$b = -0.7029$$
$$-\frac{1}{\gamma} = b$$
$$\gamma = 1.4227$$
$$\frac{1}{\gamma}\log_e c = a$$
$$\frac{1}{1.4227}\log_e c = -0.002$$
$$c = 0.9972$$

Hence, the required equation of the curve is

$$PV^{(1.4227)} = 0.9972$$

EXERCISE 5.3

1. Fit the curve $y = ab^x$ to the following data:

x	2	3	4	5	6
y	144	172.3	207.4	248.8	298.5

[Ans.: $y = 100\,(1.2)^x$**]**

2. Fit the curve $y = ae^{bx}$ to the following data:

x	0	2	4
y	5.012	10	31.62

[Ans.: $y = 4.642e^{0.46x}$**]**

3. Fit the curve $y = ax^b$ to the following data:

x	1	2	3	4
y	2.50	8.00	19.00	50.00

[Ans.: $y = 2.227x^{2.09}$**]**

4. Estimate γ by fitting the ideal gas law $PV^{\gamma} = c$ to the following data:

P	16.6	39.7	78.5	115.5	195.3	546.1
V	50	30	20	15	10	5

[Ans.: $\gamma = 1.504$**]**

Points to Remember

Fitting of Linear Curves

(i) The normal equations for the straight line $y = a + bx$ are

$$\sum y = na + b\sum x$$

$$\sum xy = a\sum x + b\sum x^2$$

(ii) The normal equations for the straight line $x = a + by$ are

$$\sum x = na + b\sum y$$

$$\sum xy = a\sum y + b\sum y^2$$

Fitting of Quadratic Curves

(i) The normal equations for the least squares quadratic curve (parabola) $y = a + bx + cx^2$ are

$$\sum y = na + b\sum x + c\sum x^2$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4$$

(ii) The normal equations for the curve $y = a + bx^2$ are

$$\sum y = na + b\sum x^2$$

$$\sum x^2 y = a\sum x^2 + b\sum x^4$$

(iii) The normal equations for the curve $y = ax + bx^2$ are

$$\sum xy = a\sum x^2 + b\sum x^3$$

$$\sum x^2 y = a\sum x^3 + b\sum x^4$$

Fitting of Exponential and Logarithmic Curves

For the curve $y = ab^x$,

Taking logarithm on both the sides of the equation $y = ab^x$,

$$\log_e y = \log_e a + x\log_e b$$

Putting $\log_e y = Y$, $\log_e a = A$, $x = X$, and $\log_e b = B$,

$$Y = A + BX$$

This is a linear equation in X and Y. The normal equations are

$$\sum Y = nA + B\sum X$$

$$\sum XY = A\sum X + B\sum X^2$$

Similarly, the best fitting exponential curves for the relation $y = ax^b$ and $y = ae^{bx}$ can be obtained.

CHAPTER 6

Finite Differences and Interpolation

Chapter Outline

6.1 Introduction
6.2 Finite Differences
6.3 Different Operators and their Relations
6.4 Interpolation
6.5 Newton's Forward Interpolation Formula
6.6 Newton's Backward Interpolation Formula
6.7 Central Difference Interpolation
6.8 Gauss's Forward Interpolation Formula
6.9 Gauss's Backward Interpolation Formula
6.10 Stirling's Formula
6.11 Interpolation with Unequal Intervals
6.12 Lagrange's Interpolation Formula
6.13 Divided Differences
6.14 Newton's Divided Difference Formula
6.15 Inverse Interpolation

6.1 INTRODUCTION

Interpolation is the process of reading between the lines of a table. It is the process of computing intermediate values of a function from a given set of tabular values of the function. Extrapolation is used to denote the process of finding the values outside the given interval.

In the interpolation process, the given set of tabular values are used to find an expression for $f(x)$ and then using it to find its required value for a given value of x. But it is difficult to find an exact form of $f(x)$ using the limited values in the table. Hence, $f(x)$ is replaced by another function $\phi(x)$, which matches with $f(x)$ at the discrete values in the table. This function $\phi(x)$ is known as the *interpolating function*.

When the interpolating function is a polynomial function, the process is known as *polynomial interpolation*. Polynomial interpolations are preferred because of the following reasons:

(i) They are simple forms of functions which can be easily manipulated.
(ii) Polynomials are free from singularities whereas rational functions or other types have singularities.

6.2 FINITE DIFFERENCES

Let the function $y = f(x)$ be tabulated for the equally spaced values $y_0 = f(x_0)$, $y_1 = f(x_0 + h)$, $y_2 = f(x_0 + 2h)$, …, $y_n = f(x_0 + nh)$, as

x	x_0	$x_0 + h$	$x_0 + 2h$	....	$x_0 + nh$	....
$y = f(x)$	y_0	y_1	y_2	....	y_n	....

To determine the values of $f(x)$ for some intermediate values of x, the following three types of differences can be used.

6.2.1 Forward Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the first forward differences of y and are denoted by $\Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1}$ respectively.

$$\begin{aligned}
\Delta y_0 &= y_1 - y_0 \\
\Delta y_1 &= y_2 - y_1 \\
\vdots \quad & \quad \vdots \\
\Delta y_{n-1} &= y_n - y_{n-1}
\end{aligned}$$

where Δ is called the *forward difference operator*. The differences of the first forward differences are called *second forward differences* and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots, \Delta^2 y_{n-1}$. Similarly, third forward differences, fourth forward differences, etc., can be defined.

$$\begin{aligned}
\Delta^2 y_0 &= \Delta (\Delta y_0) \\
&= \Delta (y_1 - y_0) \\
&= \Delta y_1 - \Delta y_0 \\
&= y_2 - y_1 - (y_1 - y_0) \\
&= y_2 - 2y_1 + y_0 \\
\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 \\
&= (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) \\
&= y_3 - 3y_2 + 3y_1 - y_0
\end{aligned}$$

$$\begin{aligned}\Delta^4 y_0 &= \Delta^3 y_1 - \Delta^3 y_0\\ &= (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0)\\ &= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0\end{aligned}$$

Since the coefficients occurring on the right-hand side are the binomial coefficients, the general form is

$$\Delta^n y_0 = y_n - {}^nc_1\, y_{n-1} + {}^nc_2\, y_{n-2} - \cdots + (-1)^n\, y_0$$

Forward Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		$\Delta y_0 = y_1 - y_0$			
$x_0 + h = x_1$	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
$x_0 + 2h = x_2$	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
$x_0 + 3h = x_3$	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
		$\Delta y_3 = y_4 - y_3$			
$x_0 + 4h = x_4$	y_4				

In a difference table, x is called the *argument*, and y is called the *function* or *entry*.

Note When $(n + 1)$ values are given, the highest-order difference is n, e.g., when 5 values are given, the highest-order difference is 4.

6.2.2 Backward Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the *first backward differences* of y and are denoted by $\nabla y_1, \nabla y_2, \ldots., \nabla y_n$, respectively.

$$\begin{aligned}\nabla y_1 &= y_1 - y_0\\ \nabla y_2 &= y_2 - y_1\\ \vdots\ \ &\quad \vdots\quad \vdots\\ \nabla y_n &= y_n - y_{n-1}\end{aligned}$$

where ∇ is called the *backward difference operator.* Similarly, backward differences of higher order can be defined.

$$\nabla^2 y_2 = \nabla\,(\nabla y_2)$$

$$
\begin{aligned}
&= \nabla (y_2 - y_1) \\
&= \nabla y_2 - \nabla y_1 \\
&= y_2 - y_1 - (y_1 - y_0) \\
&= y_2 - 2y_1 + y_0 \\
\nabla^3 y_3 &= \nabla^2 y_3 - \nabla^2 y_2 \\
&= y_3 - 3y_2 + 3y_1 - y_0 \text{ etc.}
\end{aligned}
$$

Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		$\nabla y_1 = y_1 - y_0$			
x_1	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$		
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$	
x_2	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$		$\nabla^4 y_4 = \nabla^3 y_4 - \nabla^3 y_3$
		$\nabla y_3 = y_3 - y_2$		$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$	
x_3	y_3		$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$		
		$\nabla y_4 = y_4 - y_3$			
x_4	y_4				

6.2.3 Central Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the *central differences* of y and are denoted by $\delta y_{\frac{1}{2}}, \delta y_{\frac{3}{2}}, \ldots, \delta y_{n-\frac{1}{2}}$ respectively.

$$
\begin{aligned}
\delta y_{\frac{1}{2}} &= y_1 - y_0 \\
\delta y_{\frac{3}{2}} &= y_2 - y_1 \\
\vdots \quad & \quad \vdots \quad \quad \vdots \\
\delta y_{n-\frac{1}{2}} &= y_n - y_{n-1}
\end{aligned}
$$

where δ is called the *central difference operator*. Similarly, higher-order central differences can be defined.

$$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}}$$
$$= (y_2 - y_1) - (y_1 - y_0)$$
$$= y_2 - 2y_1 + y_0$$
$$\delta^3 y_{\frac{3}{2}} = \delta^2 y_2 - \delta^2 y_1 \text{ etc.}$$

Central Difference Table

x	y	δ	δ^2	δ^3	δ^4
x_0	y_0				
		$\delta y_{\frac{1}{2}} = y_1 - y_0$			
x_1	y_1		$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}}$		
		$\delta y_{\frac{3}{2}} = y_2 - y_1$		$\delta^3 y_{\frac{3}{2}} = \delta^2 y_2 - \delta^2 y_1$	
x_2	y_2		$\delta^2 y_2 = \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}}$		$\delta^4 y_2 = \delta^3 y_{\frac{5}{2}} - \delta^3 y_{\frac{3}{2}}$
		$\delta y_{\frac{5}{2}} = y_3 - y_2$		$\delta^3 y_{\frac{5}{2}} = \delta^2 y_3 - \delta^2 y_2$	
x_3	y_3		$\delta^2 y_3 = \delta y_{\frac{7}{2}} - \delta y_{\frac{5}{2}}$		
		$\delta y_{\frac{7}{2}} = y_4 - y_3$			
x_4	y_4				

From the central difference table, it is clear that the central differences on the same horizontal line have the same suffix. Also, the differences of odd orders are known only for half values of the suffix and those of even orders, for only integral values of the suffix.

Note It is clear from the three difference tables that it is only the notations which change and not the differences.

$$y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{\frac{1}{2}}$$

6.3 DIFFERENT OPERATORS AND THEIR RELATIONS

1. **Forward Difference Operator** The forward difference operator is denoted by Δ and is defined as

$$\Delta f(x) = f(x+h) - f(x)$$

or
$$\Delta y_r = y_{r+1} - y_r$$

where h is known as the *interval of differencing*.

2. **Backward Difference Operator** The backward difference operator is denoted by ∇ and is defined as

$$\nabla f(x) = f(x) - f(x-h)$$

or
$$\nabla y_r = y_r - y_{r-1}$$

3. **Central Difference Operator** The central difference operator is denoted by δ and is defined as

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

or
$$\delta y_r = y_{r+\frac{1}{2}} - y_{r-\frac{1}{2}}$$

4. **Shift Operator** The shift operator is denoted by E and is defined as

$$E\,[f(x)] = f(x+h)$$

or
$$E\,y_r = y_{r+1}$$

Similarly,
$$E^{-1}\,[f(x)] = f(x-h)$$

or
$$E^{-1}\,y_r = y_{r-1}$$

5. **Averaging Operator** The averaging operator is denoted by μ and is defined as

$$\mu f(x) = \frac{f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)}{2}$$

6. **Differential Operator** The differential operator is denoted by D and is defined as

$$D\,f(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x)$$

6.3.1 Relations between Operators

1. Relation between Δ and E

[Summer 2015, Winter 2014, Summer 2014, Summer 2013]

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x)\\ &= Ef(x) - f(x) \qquad [\because Ef(x) = f(x+h)]\\ &= (E-1)f(x)\end{aligned}$$

$$\therefore \quad \Delta \equiv E - 1 \quad \text{or} \quad E \equiv 1 + \Delta$$

2. Relation between ∇ and E **[Winter 2014, Winter 2013]**

$$\begin{aligned}\nabla f(x) &= f(x) - f(x-h)\\ &= f(x) - E^{-1}f(x) \qquad [\because E^{-1}f(x) = f(x-h)]\\ &= (1-E^{-1})f(x)\end{aligned}$$

$$\therefore \quad \nabla \equiv 1 - E^{-1}$$

3. Relation between δ and E

$$\begin{aligned}\delta f(x) &= f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\\ &= E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x)\\ &= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) f(x)\end{aligned}$$

$$\therefore \quad \delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

4. Relation between μ and E

$$\begin{aligned}\mu f(x) &= \frac{f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)}{2}\\ &= \frac{E^{\frac{1}{2}} f(x) + E^{-\frac{1}{2}} f(x)}{2}\end{aligned}$$

$$\therefore \quad \mu \equiv \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}$$

5. Relation between *D* and *E*

[Winter 2014, Summer 2014]

$$
\begin{aligned}
E f(x) &= f(x+h) \\
&= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots \qquad \text{[By Taylor's series]} \\
&= f(x) + hD f(x) + \frac{h^2}{2!} D^2 f(x) + \cdots \\
&= \left(1 + hD + \frac{h^2}{2!} D^2 + \cdots\right) f(x) \\
&= e^{hD} f(x)
\end{aligned}
$$

$$\therefore \quad E \equiv e^{hD}$$

Also, $$hD \equiv \log E \equiv \log(1+\Delta)$$

Corollary $E^{-\frac{1}{2}} \equiv e^{-\frac{hD}{2}}$

Proof

$$
\begin{aligned}
E^{-\frac{1}{2}} f(x) &= f\left(x - \frac{h}{2}\right) \\
&= f(x) - \frac{h}{2} f'(x) + \frac{\left(\frac{h}{2}\right)^2}{2!} f''(x) - \cdots \\
&= \left[1 - \frac{h}{2} D + \frac{\left(\frac{h}{2}\right)^2}{2!} D^2 - \cdots\right] f(x) \\
&= e^{-\frac{hD}{2}} f(x)
\end{aligned}
$$

$$\therefore \quad E^{-\frac{1}{2}} \equiv e^{-\frac{hD}{2}}$$

6. Relation between μ and δ

$$
\begin{aligned}
\mu &\equiv \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2} \\
\mu^2 &\equiv \left(\frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}\right)^2 \\
&\equiv \frac{E + 2 + E^{-1}}{4}
\end{aligned}
$$

$$\equiv \frac{4+(E-E^{-1})^2}{4}$$

$$\equiv 1+\frac{\delta^2}{4}$$

$$\mu \equiv \sqrt{1+\frac{\delta^2}{4}}$$

Example 1

Prove that $(1 + \Delta)\ (1 - \nabla) = 1$

Solution

$$E \equiv 1+\Delta$$

$$\nabla \equiv 1-E^{-1}$$

$$E^{-1} \equiv 1-\nabla$$

$$(1+\Delta)(1-\nabla) \equiv EE^{-1} = 1$$

Example 2

Prove that $\delta \equiv 2 \sinh \dfrac{hD}{2}$

Solution

$$\delta \equiv f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

$$\equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

$$\equiv e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}$$

$$\equiv 2\sinh\left(\frac{hD}{2}\right)$$

Example 3

Prove that $hD \equiv \sinh^{-1}(\mu\delta)$

Solution

$$\mu\delta \equiv \left(\frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2}\right)\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)$$

$$\equiv \frac{1}{2}(E - E^{-1})$$

$$2\,\mu\delta \equiv E - E^{-1}$$

$$\equiv e^{hD} - e^{-hD}$$

$$\equiv 2\sinh(hD)$$

$\therefore$
$$hD \equiv \sinh^{-1}(\mu\delta)$$

Example 4

Prove that $\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$

Solution

$$\Delta \log f(x) = \log f(x+h) - \log f(x)$$

$$= \log \frac{f(x+h)}{f(x)}$$

$$= \log \frac{E\,f(x)}{f(x)}$$

$$= \log \frac{(1+\Delta)\,f(x)}{f(x)}$$

$$= \log\left[\frac{f(x) + \Delta f(x)}{f(x)}\right]$$

$$= \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

Example 5

Evaluate (a) $\Delta(x^2 + \sin x)$, and (b) $\Delta^2 \cos 3x$, the interval of differencing being h.

Solution

(i) $$\Delta(x^2+\sin x) = [(x+h)^2+\sin(x+h)]-(x^2+\sin x)$$
$$= h^2+2hx+\sin(x+h)-\sin x$$
$$= h^2+2hx+2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$$

(ii) $$\Delta^2\cos 3x = \Delta(\Delta\cos 3x)$$
$$= \Delta[\cos 3(x+h)-\cos 3x]$$
$$= \Delta\cos 3(x+h)-\Delta\cos 3x$$
$$= \cos 3[(x+h)+h]-\cos 3(x+h)-\cos 3(x+h)+\cos 3x$$
$$= \cos 3(x+2h)-2\cos 3(x+h)+\cos 3x$$
$$= \cos 3(x+2h)+\cos 3x-2\cos 3(x+h)$$

Example 6

Prove that $\left(\dfrac{\Delta^2}{E}\right)e^x\cdot\dfrac{E\,e^x}{\Delta^2\,e^x} = e^x$

Solution

$$\left(\frac{\Delta^2}{E}\right)e^x\cdot\frac{E\,e^x}{\Delta^2\,e^x} = \frac{(E-1)^2}{E}e^x\cdot\frac{e^{x+h}}{\Delta(e^{x+h}-e^x)}$$
$$= \left(\frac{E^2-2E+1}{E}\right)e^x\,\frac{e^{x+h}}{\left(e^{x+2h}-e^{x+h}-e^{x+h}+e^x\right)}$$
$$= (E-2+E^{-1})e^x\cdot\frac{e^{x+h}}{\left(e^{x+2h}-2e^{x+h}+e^x\right)}$$
$$= \frac{\left(e^{x+h}-2e^x+e^{x-h}\right)e^{x+h}}{\left(e^{x+2h}-2e^{x+h}+e^x\right)}$$

$$= \frac{e^x \left(e^{x+2h} - 2e^{x+h} + e^x\right)}{e^{x+2h} - 2e^{x+h} + e^x}$$

$$= e^x$$

Example 7

Prove that $\Delta\nabla \equiv (\Delta - \nabla)$

Solution:

$$\begin{aligned}
\Delta\nabla f(x) &= \Delta[f(x) - f(x-h)] \\
&= \Delta f(x) - \Delta f(x-h) \\
&= \Delta f(x) - [f(x) - f(x-h)] \\
&= \Delta f(x) - \nabla f(x) \\
&= (\Delta - \nabla) f(x)
\end{aligned}$$

$\therefore$ $$\Delta\nabla \equiv (\Delta - \nabla)$$

Example 8

Prove that $\Delta \equiv E\nabla \equiv \nabla E$

Solution

$$\Delta f(x) = f(x+h) - f(x) \qquad \text{...(1)}$$

$$\begin{aligned}
E\nabla f(x) &= E\left\{f(x) - f(x-h)\right\} \\
&= E f(x) - E f(x-h) \\
&= f(x+h) - f(x) \qquad \text{...(2)}
\end{aligned}$$

$$\begin{aligned}
\nabla E f(x) &= \nabla f(x+h) \\
&= f(x+h) - f(x) \qquad \text{...(3)}
\end{aligned}$$

From Eqs (1), (2), and (3),

$$\Delta \equiv E\nabla \equiv \nabla E$$

Example 9

Find the missing term in the following table:

x	1	2	3	4	5
y	7	–	13	21	37

Solution

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	7				
		$y_1 - 7$			
2	y_1		$20 - 2y_1$		
		$13 - y_1$		$3y_1 - 25$	
3	13		$y_1 - 5$		$38 - 4y_1$
		8		$13 - y_1$	
4	21		8		
		16			
5	37				

Since only four entries are given, the fourth-order difference will be zero.

$$\Delta^4 y_0 = 0$$
$$38 - 4y_1 = 0$$
$$y_1 = 9.5$$

Example 10

Obtain the estimate of missing terms in the following table:

x	1	2	3	4	5	6	7	8
y	2	4	8	–	32	–	128	256

Solution

Let $$f(4) = a, \quad f(6) = b$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1	2						
		2					
2	4		2				
		4		$a - 14$			
3	8		$a - 12$		$66 - 4a$		
		$a - 8$		$52 - 3a$		$10a + b - 222$	
4	a		$40 - 2a$		$b + 6a - 156$		$706 - 20a - 6b$
		$32 - a$		$b + 3a - 104$		$484 - 10a - 5b$	
5	32		$b + a - 64$		$328 - 4a - 4b$		$15b + 15a - 1196$
		$b - 32$		$224 - 3b - a$		$10b + 5a - 712$	
6	b		$160 - 2b$		$6b + a - 384$		
		$128 - b$		$3b - 160$			
7	128		b				
		128					
8	256						

Since only six values are given,

$$\Delta^6 y_0 = 0$$

$$20a + 6b = 706 \qquad \ldots(1)$$

$$15a + 15b = 1196 \qquad \ldots(2)$$

Solving Eqs (1) and (2),

$$a = 16.26$$

$$b = 63.48$$

Example 11

The following table gives the value of y which is a polynomial of degree five. It is known that y = f(3) is in error. Correct the error.

x	0	1	2	3	4	5	6
y	1	2	33	254	1025	3126	7777

Solution

Let $f(3) = a$

Since y is a polynomial of degree 5,

$$\Delta^6 y_0 = 0$$

$$(E-1)^6 y_0 = 0$$

$$(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)\, y_0 = 0$$

$$y^6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0 = 0$$

$$7777 - 6(3126) + 15(1025) - 20a + 15(33) - 6(2) + 1 = 0$$

$$-20a = -4880$$

$$a = 244$$

$$\text{Error} = 254 - 244 = 10$$

Example 12

If u_x is a function for which the fifth difference is constant and $u_1 + u_7 = -784$, $u_2 + u_6 = 686$, $u_3 + u_5 = 1088$, find u_4.

Solution

Since the fifth difference is constant,

$$\Delta^6 u_1 = 0$$

$$(E-1)^6 u_1 = 0$$

$$(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)\, u_1 = 0$$

$$u_7 - 6u_6 + 15u_5 - 20u_4 + 15u_3 - 6u_2 + u_1 = 0$$

$$(u_7 + u_1) - 6(u_6 + u_2) + 15(u_5 + u_3) - 20u_4 = 0$$

$$-784 - 6(686) + 15(1088) - 20u_4 = 0$$

$$20u_4 = 11420$$

$$u_4 = 571$$

6.3.2 Factorial Notation

A product of the form $x(x-1)(x-2)\ldots(x-n+1)$ is called a factorial polynomial or function and is denoted by $[x]^n$.

$$[x]^n = x(x-1)(x-2)\ldots(x-n+1)$$

If the interval of differencing is h then

$$[x]^n = x(x-h)(x-2h)\ldots\{x-(n-1)h\}$$

The factorial notation is of special utility in the theory of finite differences. It is useful in finding the successive differences of a polynomial directly by simple rule of differentiation.

Example 1

Write $f(x) = x^4 - 2x^3 + x^2 - 2x + 1$ *in factorial notation and find* $\Delta^4 f(x)$.

Solution

$$f(x) = x^4 - 2x^3 + x^2 - 2x + 1$$

Let $\qquad f(x) = A[x]^4 + B[x]^3 + C[x]^2 + D[x]^1 + E$

Using synthetic division,

1	1	–2	1	–2	1 = E
	0	1	–1	0	
2	1	–1	0	–2 = D	
	0	2	2		
3	1	1	2 = C		
	0	3			
	1 = A	4 = B			

$\therefore$
$$f(x) = [x]^4 + 4[x]^3 + 2[x]^2 - 2[x]^1 + 1$$
$$\Delta f(x) = 4[x]^3 + 12[x]^2 + 4[x] - 2$$
$$\Delta^2 f(x) = 12[x]^2 + 24[x] + 4$$
$$\Delta^3 f(x) = 24[x] + 24$$
$$\Delta^4 f(x) = 24$$

Example 2

Express $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$ *and its successive differences in terms of factorial polynomials. Also, find the function whose first difference is* $f(x)$.

Solution

$$f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$$

Let $\qquad f(x) = A[x]^4 + B[x]^3 + C[x]^2 + D[x]^1 + E$

Using synthetic division,

1	1	–12	42	–30	$9 = E$
	0	1	–11	31	
2	1	–11	31	$1 = D$	
	0	2	–18		
3	1	–9	$13 = C$		
	0	3			
	$1 = A$	$-6 = B$			

$\therefore$
$$f(x) = [x]^4 - 6[x]^3 + 13[x]^2 + [x]^1 + 9$$
$$\Delta f(x) = 4[x]^3 - 18[x]^2 + 26[x]^1 + 1$$
$$\Delta^2 f(x) = 12[x]^2 - 36[x]^1 + 26$$
$$\Delta^3 f(x) = 24[x]^1 - 36$$
$$\Delta^4 f(x) = 24$$
$$\Delta^5 f(x) = 0$$

By integrating $f(x)$, the function $\phi(x)$ whose first difference is $f(x)$, is obtained.

$$\phi(x) = \frac{[x]^5}{5} - \frac{6[x]^4}{4} + \frac{13[x]^3}{3} + \frac{[x]^2}{2} + 9[x]^1 + c$$

Example 3

Express $f(x) = 2x^3 - 3x^2 + 3x - 10$ *in factorial polynomial and, hence, show that* $\Delta^3 f(x) = 12$.

Solution

$$f(x) = 2x^3 - 3x^2 + 3x - 10$$

Let
$$f(x) = A[x]^3 + B[x]^2 + C[x]^1 + D$$

Using synthetic division,

1	2	–3	3	$-10 = D$
	0	2	–1	
2	2	–1	$2 = C$	
	0	4		
	$2 = A$	$3 = B$		

$$\therefore \quad f(x) = 2[x]^3 + 3[x]^2 + 2[x]^1 - 10$$

$$\Delta f(x) = 6[x]^2 + 6[x]^1 + 2$$

$$\Delta^2 f(x) = 12[x]^1 + 6$$

$$\Delta^3 f(x) = 12$$

EXERCISE 6.1

1. Prove the following identities:

(i) $\Delta\nabla \equiv \nabla\Delta$

(ii) $\nabla \equiv E^1 \Delta$

(iii) $E\nabla \equiv \nabla E$

(iv) $hD \equiv -\log(1 - \nabla)$

(v) $\Delta + \nabla \equiv \dfrac{\Delta}{\nabla} - \dfrac{\nabla}{\Delta}$

(vi) $\left(E^{\frac{1}{2}} - E^{\frac{-1}{2}}\right)(1-\nabla)^{\frac{1}{2}} \equiv \nabla$

2. Find

(i) $\Delta\left[\dfrac{2^x}{(x+1)!}\right]$

(ii) $\Delta \tan^{-1} x$

(iii) $\Delta^n e^{ax}$

(iv) $\Delta (x + \cos x)$

(v) $\Delta^4 (ax^4 + bx^2 + cx + d)$

$\left[\textbf{Ans.:} \text{ (i) } -\dfrac{x \cdot 2^x}{(x+2)!} \quad \text{(ii) } \tan^{-1}\left(\dfrac{h}{1 + hx + x^2}\right) \quad \text{(iii) } (e^a - 1)^n \cdot e^x \right.$

$\left. \text{(iv) } 1 - 2\sin\left(x + \dfrac{1}{2}\right)\sin x \quad \text{(v) } a^4 \cdot 4!\right]$

3. Evaluate $\left(\dfrac{\Delta^2}{E}\right)$ sin x, where the interval of difference is h.

[**Ans.:** $\sin(x + h) - 2\sin x + \sin(x - h)$]

4. Prove that

(i) $\Delta[x f(x)] = (x + h)\, \Delta f(x) + hf(x)$

(ii) $(\Delta + \nabla)^2 (x^2 + x) = 8h^2$

5. Prove that $f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$.

6. Find $\dfrac{\Delta^2(x^2)}{E(x^2)}$ and $\left(\dfrac{\Delta^2}{E}\right)x^2$.

$\left[\textbf{Ans.: } \dfrac{2}{(1+x)^2}, 2\right]$

7. If $y = a3^x + b(-2)^x$ and $h = 1$, prove that $(\Delta^2 + \Delta - 6)y = 0$.

8. Find the missing term from the following data:

x	0	1	2	3	4
y	1	3	9	–	81

[Ans.: 31]

9. From the following table, estimate y at $x = 2$.

x	4	6	8	10	12
y	6	7	13	32	77

[Ans.: 7]

10. If $u_0 = -10$, $u_1 = -6$, $u_2 = 2$, $u_3 = 12$, $u_4 = 26$, $u_5 = 42$, find u_6.

[Ans.: 46]

11. If $u_3 = 4$, $u_4 = 12$, $u_5 = 22$, $u_6 = 37$, $u_7 = 55$, find u_8.

[Ans.: 69]

12. From the following table, find $(15)^3$.

x	3	5	7	9	11
y	27	125	343	721	1331

[Ans.: 3375]

6.4 INTERPOLATION

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to the values $x_0, x_1, x_2, \ldots, x_n$ of x. The process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called *interpolation*. Thus, interpolation is a technique of finding the value of a function for any intermediate value of the independent variable. The process of computing the value of the function outside the range of given values of the variable is called *extrapolation*. The study of interpolation is based on the concept of finite differences which were discussed in the preceding section.

6.5 NEWTON'S FORWARD INTERPOLATION FORMULA

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$. corresponding to the values $x_0, x_1, x_2, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + rh$, where r is any real number.

$$\begin{aligned} y_r &= f(x_0 + rh) \\ &= E^r f(x_0) \\ &= (1+\Delta)^r f(x_0) \\ &= (1+\Delta)^r y_0 \\ &= \left[1 + r\Delta + \frac{r(r-1)}{2!}\Delta^2 + \frac{r(r-1)(r-2)}{3!}\Delta^3 + \dots\right] y_0 \end{aligned}$$

[Using Binomial theorem]

$$= y_0 + r\,\Delta\, y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots \qquad \text{...(6.1)}$$

Equation (6.1) is known as *Newton's forward interpolation formula*.

Note This formula is used for evaluating the value of y near the initial tabulated value of x, i.e., near x_0.

Example 1

Compute cosh (0.56) *using Newton's forward difference formula from the following table:*

x	0.5	0.6	0.7	0.8
$f(x)$	1.127626	1.185465	1.255169	1.337435

Solution

Let $x = 0.56,\ x_0 = 0.5,\ h = 0.1$

$$r = \frac{x - x_0}{h} = \frac{0.56 - 0.5}{0.1} = 0.6$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0.5	1.127626			
		0.057839		
0.6	1.185465		0.011865	
		0.069704		0.000697
0.7	1.255169		0.012562	
		0.082266		
0.8	1.337435			

By Newton's forward difference formula,

$$f(x_0 + rh) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \frac{r(r-1)(r-2)}{3!}\Delta^3 f(x_0) + \cdots$$

$$\begin{aligned}\cosh(0.56) &= 1.127626 + 0.6(0.057839) + \frac{0.6(0.6-1)}{2!}(0.011865) \\ &\quad + \frac{0.6(0.6-1)(0.6-2)}{3!}(0.000697) \\ &= 1.127626 + 0.034703 - 0.001424 + 0.000039 \\ &= 1.160944\end{aligned}$$

Example 2

Find the value of sin 52° *using Newton's forward interpolation formula from the following table:*

$\theta°$	45°	50°	55°	60°
sin $\theta°$	0.7071	0.7660	0.8192	0.8660

Solution

Let $x = 52°,\ x_0 = 45°,\ h = 5°$

$$r = \frac{x - x_0}{h} = \frac{52° - 45°}{5°} = 1.4°$$

Difference Table

$x = \theta°$	$y = \sin\theta°$	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0.7071			
		0.0589		
50°	0.7660		−0.0057	
		0.0532		−0.0007
55°	0.8192		−0.0064	
		0.0468		
60°	0.8660			

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

$$\sin 52° = 0.7071 + 1.4(0.0589) + \frac{1.4(1.4-1)}{2!}(-0.0057) + \frac{1.4(1.4-1)(1.4-2)}{3!}(-0.0007)$$
$$= 0.7071 + 0.0825 - 0.0016 + 0.00004$$
$$= 0.7880$$

Example 3

Using Newton's forward interpolation formula, find the value of f(1.6).

x	1	1.4	1.8	2.2
$f(x)$	3.49	4.82	5.96	6.5

Solution

Let $x = 1.6,\ x_0 = 1,\ h = 0.4$

$$r = \frac{x - x_0}{h} = \frac{1.6-1}{0.4} = 1.5$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	3.49			
		1.33		
1.4	4.82		−0.19	
		1.14		−0.41
1.8	5.96		−0.6	
		0.54		
2.2	6.5			

By Newton's forward interpolation formula,

$$f(x_0 + rh) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \frac{r(r-1)(r-2)}{3!}\Delta^3 f(x)_0 + \cdots$$
$$f(1.6) = 3.49 + 1.5(1.33) + \frac{1.5(1.5-1)}{2!}(-0.19) + \frac{1.5(1.5-1)(1.5-2)}{3!}(-0.41)$$
$$= 3.49 + 1.995 - 0.0713 + 0.0256$$
$$= 5.4393$$

Example 4

Use Newton's forward difference method to find the approximate value of $f(1.3)$ from the following data:

x	1	2	3	4
$f(x)$	1.1	4.2	9.3	16.4

Solution

Let $x = 1.3,\ x_0 = 1,\ h = 1$

$$r = \frac{x - x_0}{h} = \frac{1.3 - 1}{1} = 0.3$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1.1			
		3.1		
2	4.2		2	
		5.1		0
3	9.3		2	
		7.1		
4	16.4			

By Newton's forward interpolation formula,

$$f(x) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \frac{r(r-1)(r-2)}{3!}\Delta^3 f(x)_0 + \cdots$$

$$f(1.3) = 1.1 + 0.3(3.1) + \frac{0.3(0.3-1)}{2!}(2) + 0$$

$$= 1.1 + 0.93 - 0.21$$

$$= 1.82$$

Example 5

Use Newton's forward difference method to find the approximate value of $f(2.3)$ from the following data:

x	2	4	6	8
$f(x)$	4.2	8.2	12.2	16.2

Solution

Let $x = 2.3,\ x_0 = 2,\ h = 2$

$$r = \frac{x - x_0}{h} = \frac{2.3 - 2}{2} = 0.15$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
2	4.2		
		4	
4	8.2		0
		4	
6	12.2		0
		4	
8	16.2		

By Newton's forward interpolation formula,

$$f(x) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \cdots$$

$$\begin{aligned} f(2.3) &= 4.2 + 0.15(4) + 0 \\ &= 4.2 + 0.6 \\ &= 4.8 \end{aligned}$$

Example 6

Using Newton's forward interpolation formula, find the value of $f(218)$.

x	100	150	200	250	300	350	400
$f(x)$	10.63	13.03	15.04	16.81	18.42	19.90	21.27

[Summer 2014]

Solution

Let $x = 218,\quad x_0 = 100,\quad h = 50$

$$r = \frac{x - x_0}{h} = \frac{218 - 100}{50} = 2.36$$

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
100	10.63						
		2.4					
150	13.03		−0.39				
		2.01		0.15			
200	15.04		−0.24		−0.07		
		1.77		0.08		0.02	
250	16.81		−0.16		−0.05		0.02
		1.61		0.03		0.04	
300	18.42		−0.13		−0.01		
		1.48		0.02			
350	19.90		−0.11				
		1.37					
400	21.27						

By Newton's forward interpolation formula,

$$f(x) = f(x_0) + r\Delta f(x_0) + \frac{r(r-1)}{2!}\Delta^2 f(x_0) + \frac{r(r-1)(r-2)}{3!}\Delta^3 f(x_0)$$
$$+ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 f(x_0) + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 f(x_0)$$
$$+ \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 f(x_0) + \cdots$$

$$f(218) = 10.63 + 2.36(2.4) + \frac{2.36(2.36-1)}{2!}(-0.39) + \frac{2.36(2.36-1)(2.36-2)}{3!}(0.15)$$
$$+ \frac{2.36(2.36-1)(2.36-2)(2.36-3)}{4!}(-0.07)$$
$$+ \frac{2.36(2.36-1)(2.36-2)(2.36-3)(2.36-4)}{5!}(0.02)$$
$$+ \frac{2.36(2.36-1)(2.36-2)(2.36-3)(2.36-4)(2.36-5)}{6!}(0.02)$$
$$= 10.63 + 5.664 - 0.6259 + 0.0289 + 0.0022 + 0.0002 - 0.00009$$
$$= 15.6993$$

Example 7

From the following table, estimate the number of students who obtained marks between 40 *and* 45:

Marks	30–40	40–50	50–60	60–70	70–80
Number of students	31	42	51	35	31

[Summer 2015]

Solution

Cumulative Frequency Table

Marks less than (x)	40	50	60	70	80
Number of students (y)	31	73	124	159	190

Since $x = 45$ is nearer to the beginning of the table, Newton's forward interpolation formula is used.

Let $x = 45$, $x_0 = 40$, $h = 10$

$$r = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		–25	
60	124		–16		37
		35		12	
70	159		–4		
		31			
80	190				

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots$$

$$y(45) = 31 + 0.5(42) + \frac{0.5(0.5-1)}{2!}(9) + \frac{0.5(0.5-1)(0.5-2)}{3!}(-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}(37)$$

$$= 31 + 21 - 1.1250 - 1.5625 - 1.4453$$

$$= 47.8672$$

$$\approx 48$$

The number of students with marks less than 45 = 48
The number of students with marks less than 40 = 31
Hence, the number of students obtaining marks between 40 and 45 = 48 – 31 = 17

Example 8

Determine the polynomial by Newton's forward difference formula from the following table:

x	0	1	2	3	4	5
y	–10	–8	–8	–4	10	40

Solution

Let $x_0 = 0,\ h = 1$

$$r = \frac{x - x_0}{h} = \frac{x - 0}{1} = x$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	–10				
		2			
1	–8		–2		
		0		6	
2	–8		4		0
		4		6	
3	–4		10		0
		14		6	
4	10		16		
		30			
5	40				

By Newton's forward difference formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0$$

$$= -10 + x(2) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(6) + 0$$

$$= -10 + 2x - x(x-1) + x(x-1)(x-2)$$

$$= -10 + 2x - x^2 + x + x(x^2 - 3x + 2)$$

$$= -10 + 2x - x^2 + x + x^3 - 3x^2 + 2x$$

$$= x^3 - 4x^2 + 5x - 10$$

Example 9

Find a polynomial of degree 2 which takes the following values:

x	0	1	2	3	4	5	6	7
y	1	2	4	7	11	16	22	29

Solution

Let $x_0 = 0, h = 1$

$$r = \frac{x - x_0}{h} = \frac{x-0}{1} = x$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		1	
		2		0
2	4		1	
		3		0
3	7		1	
		4		0
4	11		1	
		5		0
5	16		1	
		6		
6	22		1	
		7		
7	29			

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1 + x(1) + \frac{x(x-1)}{2!}(1) + 0$$

$$= 1 + x + \frac{x^2 - x}{2}$$

$$= 1 + x + \frac{x^2}{2} - \frac{x}{2}$$

$$= \frac{1}{2}(x^2 + x + 2)$$

Example 10

Construct Newton's forward interpolation polynomial for the following data:

x	4	6	8	10
y	1	3	8	16

[Summer 2015]

Solution

Let $x_0 = 4,\ h = 2$

$$r = \frac{x - x_0}{h} = \frac{x-4}{2}$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1 + \left(\frac{x-4}{2}\right)(2) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2} - 1\right)}{2!}(3) + 0$$

$$= 1 + (x-4) + \frac{(x-4)(x-6)}{8}(3)$$

$$= x - 3 + \frac{(x^2 - 10x + 24)}{8}(3)$$

$$= x - 3 + \frac{3x^2}{8} - \frac{15x}{4} + 9$$

$$= \frac{3x^2}{8} - \frac{11x}{4} + 6$$

$$y(5) = \frac{3(25)}{8} - \frac{11(5)}{4} + 6 = \frac{13}{8}$$

6.6 NEWTON'S BACKWARD INTERPOLATION FORMULA

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots.$ corresponding to the values $x_0, x_1, x_2, \ldots.$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + rh$, where r is any real number.

$$\begin{aligned} y_r &= f(x_n + rh) \\ &= E^r f(x_n) \\ &= (E^{-1})^{-r} y_n \\ &= (1 - \nabla)^{-r} y_n \\ &= \left[1 + r\nabla + \frac{r(r+1)}{2!}\nabla^2 + \frac{r(r+1)(r+2)}{3!}\nabla^3 + \cdots\right] y_n \end{aligned}$$

[Using Binomial theorem]

$$= y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots \qquad \text{...(6.2)}$$

Equation (6.2) is known as *Newton's backward interpolation formula*.

Note This formula is used for evaluating the value of y near to the end of tabulated value of x, i.e., near x_n.

Example 1

Consider the following tabular values:

x	50	100	150	200	250
y	618	724	805	906	1032

Determine $y(300)$ *using Newton's backward interpolation formula.*

Solution

Let $x = 300,\ x_n = 250,\ h = 50$

$$r = \frac{x - x_n}{h} = \frac{300 - 250}{50} = 1$$

Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
50	618				
		106			
100	724		–25		
		81		45	
150	805		20		–40
		101		5	
200	906		25		
		126			
250	1032				

By Newton's backward interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

$$y(300) = 1032 + 1(126) + \frac{1(2)}{2!}(25) + \frac{1(2)(3)}{3!}(5) + \frac{1(2)(3)(4)}{4!}(-40)$$

$$= 1032 + 126 + 25 + 5 - 40$$

$$= 1148$$

Example 2

The area A of a circle of diameter d is given for the following values:

d	80	85	90	95	100
A	5026	5674	6362	7088	7854

Calculate the area of a circle of a diameter of 105 *units using Newton's interpolation formula.* **[Summer 2015]**

Solution

Since $x = d = 105$ is near to the end of the table, Newton's backward interpolation formula is used.

Let $x = 105,\ x_n = 100,\quad h = 5$

$$r = \frac{x - x_n}{h} = \frac{105 - 100}{5} = 1$$

Difference Table

$x = d$	$y = A$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		648			
85	5674		40		
		688		–2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

By Newton's backward interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

$$y(105) = 7854 + 1(766) + \frac{1(2)}{2!}(40) + \frac{1(2)(3)}{3!}(2) + \frac{1(2)(3)(4)}{4!}(4)$$

$$= 7854 + 766 + 40 + 2 + 4$$

$$= 8666$$

Example 3

From the following table, find P when t = 142°C and 175°C using appropriate Newton's interpolation formula.

Temperature t°C	140	150	160	170	180
Pressure P	3685	4845	6302	8076	10225

[Winter 2014]

Solution

Since $x = 142$ is nearer to the beginning of the table, Newton's forward interpolation formula is used.

Let $\quad x = 142, \quad x_0 = 140, \quad h = 10$

$$r = \frac{x - x_0}{h} = \frac{142 - 140}{10} = 0.2$$

Difference Table

$x = t°$	$y = P$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	3685				
		1169			
150	4854		279		
		1448		47	
160	6302		326		2
		1774		49	
170	8076		375		
		2149			
180	10225				

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots$$

$$P(142) = y(142) = 3685 + 0.2(1169) + \frac{0.2(0.2-1)}{2!}(279) + \frac{0.2(0.2-1)(0.2-2)}{3!}(47) + \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{4!}(2)$$

$$= 3685 + 233.8 - 22.32 + 2.256 - 0.0672$$

$$= 3898.6688$$

Since $x = 175$ is near to the end of the table, Newton's backward interpolation formula is used.

$$x = 175, \quad x_n = 180, \quad h = 10$$

$$r = \frac{x - x_n}{h} = \frac{175 - 180}{10} = -0.5$$

By Newton's backward interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

$$\begin{aligned} P(175) = y(175) &= 10225 + (-0.5)(2149) + \frac{(-0.5)(-0.5+1)}{2!}(375) \\ &\quad + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(49) \\ &\quad + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}(2) \\ &= 10225 - 1074.5 - 46.875 - 3.0625 - 0.0781 \\ &= 9100.4844 \end{aligned}$$

Example 4

The population of a town is given below. Estimate the population for the year 1895 and 1930 using suitable interpolation.

Year	x	1891	1901	1911	1921	1931
Population (in thousand)	y	46	66	81	93	101

[Summer 2015]

Solution

Since $x = 1895$ is near to the beginning of the table, Newton's forward interpolation formula is used.

Let $x = 1895, \quad x_0 = 1891, \quad h = 10$

$$r = \frac{x - x_0}{h} = \frac{1895 - 1891}{10} = 0.4$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots$$

$$y(1895) = 46 + 0.4(20) + \frac{0.4(0.4-1)}{2!}(-5) + \frac{0.4(0.4-1)(0.4-2)}{3!}(2) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!}(-3)$$

$$= 46 + 8 + 0.6 + 0.128 + 0.1248$$

$$= 54.8528 \text{ thousands}$$

Since $x = 1930$ is near to the end of the table, Newton's backward interpolation formula is used.

Let $x = 1930, \; x_n = 1931, \quad h = 10$

$$r = \frac{x - x_n}{h} = \frac{1930 - 1931}{10} = -0.1$$

By Newton's forward interpolation formula,

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \cdots$$

$$y(1930) = 101 + (-0.1)(8) + \frac{(-0.1(-0.1+1)}{2!}(-4) + \frac{(-0.1)(-0.1+1)(-0.1+2)}{3!}(-1) + \frac{(-0.1)(-0.1+1)(-0.1+2)(-0.1+3)}{4!}(-3)$$

$$= 101 - 0.8 + 0.18 + 0.0285 + 0.062$$

$$= 100.4705 \text{ thousands}$$

Example 5

In the table below, the values of y are consecutive terms of a series of which 23.6 *is the sixth term. Find the first and tenth terms of the series:*

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Solution

To find the first term, Newton's forward interpolation formula is used.

Let $x = 1, x_0 = 3, h = 1$

$$r = \frac{x - x_0}{h} = \frac{1-3}{1} = -2$$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

By Newton's forward interpolation formula,

$$y(x) = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots.$$

$$y(1) = 4.8 + (-2)(3.6) + \frac{(-2)(-3)}{2!}(2.5) + \frac{(-2)(-3)(-4)}{3!}(0.5)$$

$$= 4.8 + 7.2 + 7.5 - 2$$

$$= 3.1$$

To find the tenth term, Newton's backward interpolation formula is used.

Let $x = 10,\ x_n = 9,\ h = 1$

$$r = \frac{x - x_n}{h} = \frac{10-9}{1} = 1$$

By Newton's backward interpolation formula,

$$y(x) = y_n + r\,\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

$$y(10) = 73.9 + 1(21.1) + \frac{1(2)}{2!}(4.5) + \frac{1(2)(3)}{3!}(0.5)$$

$$= 73.9 + 21.1 + 4.5 + 0.5$$

$$= 100$$

EXERCISE 6.2

1. Find tan 67° 20′ from the table:

θ	65°	66°	67°	68°	69°
tan θ	2.1445	2.2460	2.3559	2.4751	2.6051

[Ans.: 2.3932**]**

2. Find $(5.5)^3$ from the following table:

x	3	5	7	9	11
$y = x^3$	27	125	343	729	1331

[Ans.: 166.375**]**

3. Calculate $e^{1.85}$ from the following table:

x	1.7	1.8	1.9	2	2.1	2.2	2.3
e^x	5.474	6.050	6.686	7.389	8.166	9.025	9.974

[Ans.: 6.36**]**

4. Find $\sqrt{x}$ at $x = 2.52$ and $x = 2.62$ from the table:

x	2.5	2.55	2.6	2.65	2.7	2.75
$\sqrt{x}$	1.58114	1.59687	1.61245	1.62788	1.64317	1.65831

[Ans.: 1.58745, 1.6186**]**

5. The values of specific heat (C_p) at constant pressure of a gas are tabulated below for various temperatures. Find the specific heat at 900°C.

Temperature °C	500	1000	1500	2000
C_p	31.23	35.01	39.18	43.75

[Ans.: 34.223]

6. *P* and *V* are connected by the following data:

V	10	20	30	40
P	1.1	2	4.4	7.9

Determine P when $V = 25$ and $V = 45$. **[Ans.:** 3.0375, 9.9375]

7. Find the number of persons getting wages less than ₹15 from the following data:

Wages in ₹	0–10	10–20	20–30	30–40
Number of persons	9	30	35	42

[Ans.: 24]

8. Find the number of students getting marks less than 70 from the following data:

Marks	0–20	20–40	40–60	60–80	80–100
Number of students	41	62	65	50	17

[Ans.: 196]

9. From the following table, estimate the profit in the year 1925.

Year	1891	1901	1911	1921	1931
Profit in lakhs	46	66	81	93	101

[Ans.: ₹ 96.8365 lakhs]

10. Find the polynomial of degree three which takes the same values as $y = 2^x + 2x + 1$ at $x = -1, 0, 1, 2$.

$$\left[\textbf{Ans.: } \frac{1}{12}(x^3 + 3x^2 + 32x + 24)\right]$$

11. Obtain the cubic polynomial which takes the values

x	0	1	2	3
y	1	2	1	0

and, hence, find $f(4)$. **[Ans.:** $2x^3 - 7x^2 + 6x + 1$, 41]

12. Find a polynomial of degree 4 which takes the values

x	2	4	6	8	10
y	0	0	1	0	0

$$\left[\textbf{Ans.: } \frac{1}{64}(x^4 - 24x^3 + 196x^2 - 624x + 640)\right]$$

13. Given $u_1 = 40$, $u_3 = 45$, $u_5 = 54$, find u_2 and u_4. **[Ans.:** 42, 49]
14. Given $y_0 = 3$, $y_1 = 12$, $y_2 = 81$, $y_3 = 200$, $y_4 = 100$, $y_5 = 8$. Without forming the difference table, find $\Delta^5 y_0$. **[Ans.:** 755]
15. Find the polynomial of least degree passing through the points (0,–1), (1,1), (2,1), and (3,–2).

$$\left[\textbf{Ans.:}\ -\frac{1}{6}(x^3 + 3x^2 - 16x + 6)\right]$$

6.7 CENTRAL DIFFERENCE INTERPOLATION

Central difference interpolation formulae are used for interpolation near the middle of the tabulated values. If x takes the values $x_0 - 2h$, $x_0 - h$, x_0, $x_0 + h$, $x_0 + 2h$ and the corresponding values of $y = f(x)$ are y_{-2}, y_{-1}, y_0, y_1, y_2, the difference tables in the two notations are given as follows:

x	y	First Difference	Second Difference	Third Difference	Fourth Difference
$x_0 - 2h$	y_{-2}				
		$\Delta y_{-2} = \delta y_{-\frac{3}{2}}$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} = \delta^2 y_{-1}$		
		$\Delta y_{-1} = \delta y_{-\frac{1}{2}}$		$\Delta^3 y_{-2} = \delta^3 y_{-\frac{1}{2}}$	
x_0	y_0		$\Delta^2 y_{-1} = \delta^2 y_0$		$\Delta^4 y_{-2} = \delta^4 y_0$
		$\Delta y_0 = \delta y_{\frac{1}{2}}$		$\Delta^3 y_{-1} = \delta^3 y_{\frac{1}{2}}$	
$x_0 + h$	y_1		$\Delta^2 y_0 = \delta^2 y_1$		
		$\Delta y_1 = \delta y_{\frac{3}{2}}$			
$x_0 + 2h$	y_2				

6.8 GAUSS'S FORWARD INTERPOLATION FORMULA

By Newton's forward interpolation formula,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots \qquad ...(6.3)$$

where $r = \dfrac{x - x_0}{h}$

$$\Delta^2 y_0 = \Delta^2 E y_{-1} = \Delta^2 (1+\Delta) y_{-1} = \Delta^2 y_{-1} + \Delta^3 y_{-1} \qquad ...(6.4)$$

$$\Delta^3 y_0 = \Delta^3 E y_{-1} = \Delta^3 (1+\Delta) y_{-1} = \Delta^3 y_{-1} + \Delta^4 y_{-1} \qquad ...(6.5)$$

$$\Delta^4 y_0 = \Delta^4 E y_{-1} = \Delta^4 (1+\Delta) y_{-1} = \Delta^4 y_{-1} + \Delta^5 y_{-1} \qquad ...(6.6)$$

Also,

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}, \text{ etc.} \qquad ...(6.7)$$

Substituting the values of $\Delta^2 y_0$ $\Delta^3 y_0$, ... in Eq. (6.3),

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{r(r-1)(r-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{r(r-1)(r-2)(r-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

[Using Eqs (6.4), (6.5), and (6.6)]

$$= y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \left[\frac{r(r-1)}{2!}\Delta^3 y_{-1} + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_{-1}\right] + \left[\frac{r(r-1)(r-2)}{3!}\Delta^4 y_{-1} + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_{-1} + \cdots\right]$$

$$= y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-1} + \cdots$$

$$= y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\left(\Delta^4 y_{-2} + \Delta^5 y_{-2}\right) + \cdots$$

[Using Eq. (6.7)]

$$= y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-2} + \cdots \qquad ...(6.8)$$

Equation (6.8) is known as *Gauss's forward interpolation formula*.

Corollary In the central difference notation,

$$y_r = y_0 + r\,\delta y_{\frac{1}{2}} + \frac{r(r-1)}{2!}\delta^2 y_0 + \frac{(r+1)r(r-1)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{(r+1)r(r-1)(r-2)}{4!}\delta^4 y_0 + \cdots$$

Notes

(i) This formula involves odd differences below the central line and even differences on the central line.

$$\begin{array}{cccccccccccccc}
y_0 & & \cdots & & \Delta^2 y_{-1} & & \cdots & & \Delta^4 y_{-2} & & \cdots & & \Delta^6 y_{-3} & & \cdots & \text{Central line} \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \\
& & \Delta y_0 & & & & \Delta^3 y_{-1} & & & & \Delta^5 y_{-2} & & & & \Delta^7 y_{-3} &
\end{array}$$

(ii) This formula is used to evaluate the values of y for r between 0 and 1.

Example 1

Find $y(32)$ from the following table:

x	25	30	35	40
$y = f(x)$	0.2707	0.3027	0.3386	0.3794

Solution

Let $x = 32,\ x_0 = 30,\ h = 5$

$$r = \frac{x - x_0}{h} = \frac{32 - 30}{5} = 0.4$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$
25	–1	0.2707			
			0.0320		
30	0	0.3027		0.0039	
			0.0359		0.0010
35	1	0.3386		0.0049	
			0.0408		
40	2	0.3794			

By Gauss's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \cdots$$

$$y(32) = 0.3027 + 0.4(0.0359) + \frac{0.4(0.4-1)}{2!}(0.0039) + \frac{(0.4+1)(0.4)(0.4-1)}{3!}(0.0010)$$

$$= 0.3027 + 0.0144 - 0.0005 - 0.0001$$

$$= 0.3165$$

Example 2

Use Gauss's forward interpolation formula to find $y(3.3)$ from the following table:

x	1	2	3	4	5
y	15.3	15.1	15	14.5	14

Solution

Let $x = 3.3,\ x_0 = 3,\ h = 1$

$$r = \frac{x - x_0}{h} = \frac{3.3 - 3}{1} = 0.3$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	–2	15.3				
			–0.2			
2	–1	15.1		0.1		
			–0.1		–0.5	
3	0	15		–0.4		0.9
			–0.5		0.4	
4	1	14.5		0		
			–0.5			
5	2	14				

By Gauss's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-2} + \cdots$$

$$y(3.3) = 15 + 0.3(-0.5) + \frac{0.3(0.3-1)}{2!}(-0.4) + \frac{(0.3+1)(0.3)(0.3-1)}{3!}(0.4) + \frac{(0.3+1)(0.3)(0.3-1)(0.3-2)}{4!}(0.9)$$

$$= 15 - 0.15 + 0.042 - 0.0182 + 0.0174$$

$$= 14.8912$$

Example 3

Find the polynomial which fits the data in the following table using Gauss's forward interpolation formula.

x	3	5	7	9	11
y	6	24	58	108	174

Solution

Let $x_0 = 7,\ h = 2$

$$r = \frac{x - x_0}{h} = \frac{x-7}{2}$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	–2	6				
			18			
5	–1	24		16		
			34		0	
7	0	58		16		0
			50		0	
9	1	108		16		
			66			
11	2	174				

By Gauss's forward interpolation formula,

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \cdots$$

$$= 58 + \left(\frac{x-7}{2}\right)(50) + \frac{1}{2}\left(\frac{x-7}{2}\right)\left(\frac{x-7}{2} - 1\right)(16)$$

$$= 58 + 25(x-7) + 2(x-7)(x-9)$$

$$= 58 + 25x - 175 + 2x^2 - 32x + 126$$

$$= 2x^2 - 7x + 9$$

6.9 GAUSS'S BACKWARD INTERPOLATION FORMULA

By Newton's forward interpolation formula,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots \quad ...(6.9)$$

where $r = \dfrac{x - x_0}{h}$

$$\Delta y_0 = \Delta E y_{-1} = \Delta(1+\Delta)y_{-1} = \Delta y_{-1} + \Delta^2 y_{-1} \quad ...(6.10)$$

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1} \quad ...(6.11)$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \quad ...(6.12)$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \quad ...(6.13)$$

Also, $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}, \text{ etc.} \quad ...(6.14)$$

Substituting the values of Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, …. in Eq. (6.9),

$$y_r = y_0 + r(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{r(r-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{r(r-1)(r-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{r(r-1)(r-2)(r-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

[Using Eqs (6.10), (6.11), and (6.12)]

$$= y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-1} + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^5 y_{-1} + \cdots$$

$$= y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}(\Delta^3 y_{-2} + \Delta^4 y_{-2})$$
$$+ \frac{(r+1)r(r-1)(r-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + ...$$

[Using Eq. (6.14)]

$$= y_0 + r\,\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\,\Delta^3 y_{-2}$$
$$+ \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + \cdots \qquad ...(6.15)$$

Equation (6.15) is known as *Gauss's backward interpolation formula*.

Corollary In the central difference notation,

$$y_r = y_0 + r\,\delta y_{-\frac{1}{2}} + \frac{(r+1)r}{2!}\delta^2 y_0 + \frac{(r+1)r(r-1)}{3!}\delta^3 y_{-\frac{1}{2}} + \frac{(r+2)(r+1)r(r-1)}{4!}\delta^4 y_0 + ...$$

Notes

(i) This formula involves odd differences above the central line and even differences on the central line.

$$\begin{tikzcd}
 & \Delta y_{-1} \arrow[dr] & & \Delta^3 y_{-2} \arrow[dr] & & \Delta^5 y_{-3} \arrow[dr] & \\
y_0 \arrow[ur] & \cdots & \Delta^2 y_{-1} \arrow[ur] & \cdots & \Delta^4 y_{-2} \arrow[ur] & \cdots & \Delta^6 y_{-3}
\end{tikzcd}$$

Central line

(ii) This formula is used to evaluate the values of y for r between –1 and 0.

Example 1

Using Gauss's backward interpolation formula, find the population for the year 1936 *given that*

Year (x)	1901	1911	1921	1931	1941	1951
Population in thousands (y)	12	15	20	27	39	52

Solution

Let $\quad x = 1936,\ x_0 = 1941,\ h = 10$

$$r = \frac{x - x_0}{h} = \frac{1936 - 1941}{10} = -0.5$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1901	−4	12					
			3				
1911	−3	15		2			
			5		0		
1921	−2	20		2		3	
			7		3		−10
1931	−1	27		5		−7	
			12		−4		
1941	0	39		1			
			13				
1951	1	52					

By Gauss's backward interpolation formula,

$$y(x) = y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \cdots$$

$$y(1936) = 39 + (-0.5)(12) + \frac{(-0.5+1)(-0.5)}{2!}(1) + \frac{(-0.5+1)(-0.5)(-0.5-1)}{3!}(-4)$$

$$= 39 - 6 - 0.1250 - 0.25$$

$$= 32.625 \text{ thousands}$$

Example 2

Find y(2.36) from the following table:

x	1.6	1.8	2	2.2	2.4	2.6
y	4.95	6.05	7.39	9.03	11.02	13.46

Solution

Let $x = 2.36,\ x_0 = 2.4,\ h = 0.2$

$$r = \frac{x - x_0}{h} = \frac{2.36 - 2.4}{0.2} = -0.2$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.6	–4	4.95					
			1.1				
1.8	–3	6.05		0.24			
			1.34		0.06		
2	–2	7.39		0.3		–0.01	
			1.64		0.05		0.06
2.2	–1	9.03		0.35		0.05	
			1.99		0.1		
2.4	0	11.02		0.45			
			2.44				
2.6	1	13.46					

By Gauss's backward interpolation formula,

$$y(x) = y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \cdots$$

$$y(2.36) = 11.02 + (-0.2)(1.99) + \frac{(-0.2+1)(-0.2)}{2!}(0.45) + \frac{(-0.2+1)(-0.2)(-0.2-1)}{3!}(0.1)$$

$$= 11.02 - 0.398 - 0.036 + 0.0032$$

$$= 10.5892$$

Example 3

From the following table, find y when x = 38.

x	30	35	40	45	50
y	15.9	14.9	14.1	13.3	12.5

Solution

Let $x = 38, \quad x_0 = 40, \quad h = 5$

$$r = \frac{x - x_0}{h} = \frac{38 - 40}{5} = -0.4$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
30	–2	15.9				
			–1			
35	–1	14.9		0.2		
			–0.8		–0.2	
40	0	14.1		0		0.2
			–0.8		0	
45	1	13.3		0		
			–0.8			
50	2	12.5				

By Gauss's backward interpolation formula,

$$y(x) = y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + \cdots$$

$$y(38) = 14.1 + (-0.4)(-0.8) + \frac{(-0.4+1)(-0.4)}{2!}(0) + \frac{(-0.4+1)(-0.4)(-0.4-1)}{3!}(-0.2) + \frac{(-0.4+2)(-0.4+1)(-0.4)(-0.4-1)}{4!}(0.2)$$

$$= 14.1 + 0.32 + 0 - 0.0112 + 0.0045$$

$$= 14.4133$$

6.10 STIRLING'S FORMULA

By Gauss's forward interpolation formula,

$$y_r = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)\,(r-2)}{4!}\Delta^4 y_{-2} + \cdots \qquad \text{...(6.16)}$$

By Gauss's backward interpolation formula,

$$y_r = y_0 + r\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + \cdots \qquad (6.17)$$

Adding Eqs (6.16) and (6.17) and then dividing by 2,

$$y_r = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right)$$

$$+\frac{r^2(r^2-1)}{4!}\Delta^4 y_{-2} + \cdots \qquad \text{...(6.18)}$$

Equation (6.18) is known as *Stirling's formula*.

Corollary In the central difference notation,

$$y_r = y_0 + r\,\mu\,\delta y_0 + \frac{r^2}{2!}\delta^2 y_0 + \frac{r(r^2+1^2)}{3!}\mu\,\delta^3 y_0 + \frac{r^2(r^2-1^2)}{4!}\delta^4 y_0 + \cdots$$

$$\frac{1}{2}(\Delta y_{-1} + \Delta y_0) = \frac{1}{2}\left(\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}}\right) = \mu\,\delta y_0$$

$$\frac{1}{2}(\Delta^3 y_{-2} + \Delta^3 y_{-1}) = \frac{1}{2}\left(\delta^3 y_{\frac{1}{2}} + \delta^3 y_{-\frac{1}{2}}\right) = \mu\,\delta^3 y_0, \quad \text{etc.}$$

Notes

(i) This formula involves means of the odd differences just above and below the central line and even differences on the central line.

$$y_0 \begin{pmatrix} \Delta y_{-1} \\ \Delta y_0 \end{pmatrix} \cdots \Delta^2 y_{-1} \cdots \begin{pmatrix} \Delta^3 y_{-2} \\ \Delta^3 y_{-1} \end{pmatrix} \cdots \Delta^4 y_{-2} \quad \cdots \text{Central line}$$

(ii) This formula gives fairly accurate values of y for r between –0.25 and 0.2 but can be used for r between –0.5 to 0.5.

Example 1

Using Stirling's formula, estimate the value of tan 16°.

x	0°	5°	10°	15°	20°	25°	30°
$y = \tan x$	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Solution

Let $\qquad x = 16°,\ x_0 = 15°,\ h = 5°$

$$r = \frac{x - x_0}{h} = \frac{16° - 15°}{5°} = 0.2$$

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0°	–3	0						
			0.0875					
5°	–2	0.0875		0.0013				
			0.0888		0.0015			
10°	–1	0.1763		0.0028		0.0002		
			0.0916		0.0017		–0.0002	
15°	0	0.2679		0.0045		0		0.0011
			0.0961		0.0017		0.0009	
20°	1	0.3640		0.0062		0.0009		
			0.1023		0.0026			
25°	2	0.4663		0.0088				
			0.1111					
30°	3	0.5774						

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right)$$

$$+ \frac{r^2(r^2-1)}{4!}\Delta^4 y_{-2} + \frac{r(r^2-1)(r^2-4)}{5!}\left(\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2}\right)$$

$$+ \frac{r^2(r^2-1)(r^2-4)}{6!}\Delta^6 y_{-3} + \cdots$$

$$y(16) = 0.2679 + 0.2\left(\frac{0.0916 + 0.0961}{2}\right) + \frac{(0.2)^2}{2!}(0.0045)$$

$$+ \frac{(0.2)(0.2^2-1)}{3!}\left(\frac{0.0017 + 0.0017}{2}\right) + 0$$

$$+ \frac{(0.2)(0.2^2-1)(0.2^2-4)}{5!}\left(\frac{-0.0002 + 0.0009}{2}\right)$$

$$+ \frac{(0.2)^2(0.2^2-1)(0.2^2-4)}{6!}(0.0011)$$

$$= 0.2679 + 0.0188 + (9 \times 10^{-5}) - (5.44 \times 10^{-5})$$

$$+ 0 + (2.2176 \times 10^{-6}) + (2.3232 \times 10^{-7})$$

$$= 0.2867$$

Example 2

Employ Stirling's formula to compute $y(35)$ from the following table:

x	20	30	40	50
y	512	439	346	243

Solution

Let $x = 35,\ x_0 = 30,\ h = 10$

$$r = \frac{x - x_0}{h} = \frac{35 - 30}{10} = 0.5$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	–1	512			
			–73		
30	0	439		–20	
			–93		10
40	1	346		–10	
			–103		
50	2	243			

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \cdots$$

$$y(35) = 439 + 0.5\left(\frac{-73-93}{2}\right) + \frac{(0.5)^2}{2!}(-20) + \frac{0.5(0.5^2-1)}{3!}\left(\frac{10}{2}\right)$$

$$= 439 - 41.5 - 2.5 - 0.3125$$

$$= 394.6875$$

Example 3

Let $f(40) = 836$, $f(50) = 682$, $f(60) = 436$, $f(70) = 272$. *Use Stirling's formula to find* $f(55)$.

Solution

Let $x = 55, \quad x_0 = 50, \quad h = 10$

$$r = \frac{x - x_0}{h} = \frac{55 - 50}{10} = 0.5$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$
40	–1	836			
			–154		
50	0	682		–92	
			–246		174
60	1	436		–82	
			–164		
70	2	272			

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2 - 1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \cdots$$

$$y(55) = 682 + 0.5\left(\frac{-154 - 246}{2}\right) + \frac{(0.5)^2}{2!}(-92) + \frac{0.5(0.5^2 - 1)}{3!}\left(\frac{174}{2}\right)$$

$$= 682 - 100 - 11.5 - 5.4375$$

$$= 565.0625$$

Example 4

Using Stirling's formula, find y(25) from the following table:

x	20	24	28	32
y	0.01427	0.01581	0.01772	0.01996

Solution

Let $x = 25, \quad x_0 = 24, \quad h = 4$

$$r = \frac{x - x_0}{h} = \frac{25 - 24}{4} = 0.25$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	–1	0.01427			
			0.00154		
24	0	0.01581		0.00037	
			0.00191		–0.00004
28	1	0.01772		0.00033	
			0.00224		
32	2	0.01996			

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \cdots$$

$$y(25) = 0.01581 + 0.25\left(\frac{0.00154 + 0.00191}{2}\right) + \frac{(0.25)^2}{2!}(0.00037)$$

$$+ \frac{0.25(0.25^2-1)}{3!}\left(\frac{-0.00004}{2}\right)$$

$$= 0.01581 + 4.3125\times10^{-4} + 1.15625\times10^{-5} + 7.8125\times10^{-7}$$

$$= 0.01625$$

Example 5

Find the value of $y(1.63)$ *from the following table using Stirling's formula:*

x	1.5	1.6	1.7	1.8	1.9
$y = f(x)$	17.609	20.412	23.045	25.527	27.875

Solution

Let $\quad x = 1.63, \quad x_0 = 1.6, \quad h = 0.1$

$$r = \frac{x - x_0}{h} = \frac{1.63 - 1.6}{0.1} = 0.3$$

Central Difference Table

x	r	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5	–1	17.609				
			2.803			
1.6	0	20.412		–0.17		
			2.633		0.019	
1.7	1	23.045		–0.151		–0.002
			2.482		0.017	
1.8	2	25.527		–0.134		
			2.348			
1.9	3	27.875				

By Stirling's formula,

$$y(x) = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \cdots$$

$$y(1.63) = 20.412 + 0.3\left(\frac{2.803 + 2.633}{2}\right) + \frac{(0.3)^2}{2!}(-0.17) + \frac{0.3(0.3^2-1)}{3!}\left(\frac{0.019}{2}\right)$$

$$= 20.412 + 0.8154 - 7.65\times 10^{-3} - 4.3225\times 10^{-4}$$

$$= 21.2193$$

EXERCISE 6.3

1. Use Gauss's interpolation formula to find y_{16}.

x	5	10	15	20	25
y	26.782	19.951	14.001	8.762	4.163

[Ans.: 12.901]

2. Find $e^{-1.7425}$ by Gauss's forward formula.

x	1.72	1.73	1.74	1.75	1.76
e^{-x}	0.17907	0.17728	0.17552	0.17377	0.17204

[Ans.: 0.17508]

3. Find $f(25)$ given $f(20) = 14$, $f(24) = 32$, $f(28) = 35$, and $f(32) = 40$ using Gauss's formula.

[Ans.: 33.41]

4. Apply Gauss's backward formula to find the population in 1926.

Year	x	1911	1921	1931	1941	1951
Population in lacs	y	15	20	27	39	52

[Ans.: 22.898 lacs]

5. Apply Gauss's backward interpolation formula to find sin 45°.

$x°$	20	30	40	50	60	70
sin $x°$	0.34202	0.50200	0.64279	0.76604	0.86603	0.93969

[Ans.: 0.705990]

6. Use Gauss's backward formula, find $f(5.8)$ given that $f(x)$ is a polynomial of degree four and $f(4) = 270$, $f(5) = 648$, $\Delta f(5) = 682$, $\Delta^2 f(4) = 132$

[Ans.: 1163]

7. Using Stirling's formula, find $y(5)$ from the following table:

x	0	4	8	12
y	14.27	15.81	17.72	19.96

[Ans.: 16.25]

8. Find $\sqrt{1.12}$ using Stirling's formula from the following table:

x	1.0	1.05	1.10	1.15	1.20	1.25	1.30
$f(x)$	1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.14017

[Ans.: 1.05830]

9. Use Stirling's formula to find tan 89° 26′ from the table:

x	89° 21′	89° 23′	89° 25′	89° 27′	89° 29′
tan x	88.14	92.91	98.22	104.17	110.90

[Ans.: 101.107]

6.11 INTERPOLATION WITH UNEQUAL INTERVALS

If the values of x are unequally spaced then interpolation formulae for equally spaced points cannot be used. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. There are two such formulae for unequally spaced values of x.

(i) Lagrange's interpolation formula
(ii) Newton's interpolation formula with divided difference

6.12 LAGRANGE'S INTERPOLATION FORMULA

Let $y = f(x)$ be a function which take the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x = x_0, x_1, x_2, \ldots, x_n$. Since there are $(n + 1)$ values of x and y, $f(x)$ can be represented by a polynomial in x of degree n.

$$\begin{aligned} y = f(x) = a_0 (x - x_1)(x - x_2) \ldots (x - x_n) + a_1 (x - x_0)(x - x_2) \ldots (x - x_n) + \ldots \\ + a_n (x - x_0)(x - x_1) \ldots (x - x_{n-1}) \end{aligned} \quad \ldots(6.19)$$

where $a_0, a_1, a_2, ..., a_n$ are constants.

Putting $x = x_0$, $y = y_0$ in Eq. (6.19),

$$y_0 = a_0 (x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)$$

$$a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}$$

Similarly, putting $x = x_1$, $y = y_1$ in Eq. (6.19),

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n)}$$

Proceeding in the same way,

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1) \ldots (x_n - x_{n-1})}$$

Substituting the values of $a_0, a_1, a_2 \ldots, a_n$ in Eq. (6.19),

$$\begin{aligned} f(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} y_1 + \ldots \\ + \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} y_n \end{aligned} \quad \ldots(6.20)$$

Equation (6.20) is known as *Lagrange's interpolation formula*.

Note This formula can also be used to split the given function into partial fractions. Dividing both sides of Eq. (6.20) by $(x - x_0)(x - x_1) \ldots (x - x_n)$,

$$\begin{aligned} \frac{f(x)}{(x - x_0)(x - x_1)\ldots(x - x_n)} &= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} \left(\frac{1}{x - x_0} \right) \\ &+ \frac{y_1}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} \left(\frac{1}{x - x_1} \right) + \ldots \\ &+ \frac{y_n}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} \left(\frac{1}{x - x_n} \right) \end{aligned}$$

Example 1

Compute f(9.2) by using Lagrange's interpolation method from the following data:

x	9	9.5	11
$f(x)$	2.1972	2.2513	2.3979

[Summer 2013]

Solution

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

$$f(9.2)=\frac{(9.2-9.5)(9.2-11)}{(9-9.5)(9-11)}(2.1972)+\frac{(9.2-9)(9.2-11)}{(9.5-9)(9.5\text{-}11)}(2.2513)+\frac{(9.2-9)(9.2-9.5)}{(11-9)(11-9.5)}(2.3979)$$

$$=1.1865+1.0806-0.048$$

$$=2.2191$$

Example 2

Find the value of y when x = 10 *from the following table:*

x	5	6	9	11
y	12	13	14	16

Solution

By Lagrange's interpolation formula,

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y(10)=\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12)+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14)+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$=2-4.3333+11.6666+5.3333$$

$$=14.6666$$

Example 3

Compute f(4) from the tabular values given:

x	2	3	5	7
$f(x)$	0.1506	0.3001	0.4517	0.6259

using Lagrange's interpolation formula. **[Winter 2012]**

Solution

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$f(4)=\frac{(4-3)(4-5)(4-7)}{(2-3)(2-5)(2-7)}(0.1506)+\frac{(4-2)(4-5)(4-7)}{(3-2)(3-5)(3-7)}(0.3001)$$
$$+\frac{(4-2)(4-3)(4-7)}{(5-2)(5-3)(5-7)}(0.4517)+\frac{(4-2)(4-3)(4-5)}{(7-2)(7-3)(7-5)}(0.6259)$$
$$=-0.0301+0.2251+0.2259-0.0313$$
$$=0.3896$$

Example 4

Compute f(2) by using Lagrange's interpolation method from the following data:

x	–1	0	1	3
$f(x)$	2	1	0	–1

[Winter 2013, Summer 2015]

Solution

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$f(2)=\frac{(2-0)(2-1)(2-3)}{(-1-0)(-1-1)(-1-3)}(2)+\frac{(2+1)(2-1)(2-3)}{(0+1)(0-1)(0-3)}(1)$$
$$+\frac{(2+1)(2-0)(2-3)}{(1+1)(1-0)(1-3)}(0)+\frac{(2+1)(2-0)(2-1)}{(3+1)(3-0)(3-1)}(-1)$$
$$=0.5-1+0-0.25$$
$$=-0.75$$

Example 5

By using Lagrange's formula, find y when x = 10.

x	5	6	9	11
y	12	13	14	16

[Summer 2015]

Solution

By Lagrange's interpolation formula,

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y(10)=\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12)+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$
$$+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14)+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$
$$=2-4.3333+11.6667+5.3333$$
$$=14.6667$$

Example 6

Evaluate f(9) by using Lagrange's interpolation method from the following data:

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

[Summer 2014]

Solution

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}f(x_0)$$
$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}f(x_2)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}f(x_3)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}f(x_4)$$

$$f(9)=\frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}(150)$$
$$+\frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}(392)$$
$$+\frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}(1452)$$
$$+\frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}(2366)$$
$$+\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}(5202)$$
$$=-16.6667+209.0667+1290.6667-788.6667+115.6$$
$$=810$$

Example 7

Determine $y(12)$ *by using Lagrange's interpolation method from the following data:*

x	11	13	14	18	20	23
y	25	47	68	82	102	124

[Winter 2014]

Solution

By Lagrange's interpolation formula,

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}y_0$$
$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}y_2$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}y_3$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}y_4$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}y_5$$

$$y(12) = \frac{(12-13)(12-14)(12-18)(12-20)(12-23)}{(11-13)(11-14)(11-18)(11-20)(11-23)}(25)$$
$$= \frac{(12-11)(12-14)(12-18)(12-20)(12-23)}{(13-11)(13-14)(13-18)(13-20)(13-23)}(47)$$
$$= \frac{(12-11)(12-13)(12-18)(12-20)(12-23)}{(14-11)(14-13)(14-18)(14-20)(14-23)}(68)$$
$$= \frac{(12-11)(12-13)(12-14)(12-20)(12-23)}{(18-11)(18-13)(18-14)(18-20)(18-23)}(82)$$
$$= \frac{(12-11)(12-13)(12-14)(12-18)(12-23)}{(20-11)(20-13)(20-14)(20-18)(20-23)}(102)$$
$$= \frac{(12-11)(12-13)(12-14)(12-18)(12-20)}{(23-11)(23-13)(23-14)(23-18)(23-20)}(124)$$
$$= 5.8201 + 70.9029 - 55.4074 + 10.3086 - 5.9365 + 0.7348$$
$$= 26.4225$$

Example 8

Find a second-degree polynomial passing through the points (0, 0), (1, 1) *and* (2, 20) *using Larange's interpolation.* **[Summer 2015]**

Solution

Let $x_0 = 0, \quad x_1 = 1, \quad x_2 = 2$

$f(x_0) = 0, \quad f(x_1) = 1, \quad f(x_2) = 20$

By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)}(0) + \frac{(x-0)(x-2)}{(1-0)(1-2)}(1) + \frac{(x-0)(x-1)}{(2-0)(2-1)}(20)$$

$$= 0 - x(x-2) + 10x(x-1)$$

$$= 9x^2 - 8x$$

Example 9

Using Lagrange's interpolation formula, find the interpolating polynomial for the following table:

x	0	1	2	5
$f(x)$	2	3	12	147

Solution

By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$

$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$= \frac{(x-1)(x^2-7x+10)}{-5} + \frac{3x(x^2-7x+10)}{4} + \frac{2x(x^2-6x+5)}{-1} + \frac{49x(x^2-3x+2)}{20}$$

$$= \frac{\left\{ \begin{array}{l} -4(x^3-8x^2+17x-10) + (15x^3-105x^2+150x) \\ \qquad\qquad -20(2x^3-12x^2+10x) + (49x^3-147x^2+98x) \end{array} \right\}}{20}$$

$$= \frac{20x^3+20x^2-20x+40}{20}$$

$$= x^3 + x^2 - x + 2$$

Example 10

Find the Lagrange interpolating polynomial from the following data:

x	0	1	4	5
$f(x)$	1	3	24	39

Solution

By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$=\frac{(x-1)(x-4)(x-5)}{(0-1)(0-4)(0-5)}(1)+\frac{(x-0)(x-4)(x-5)}{(1-0)(1-4)(1-5)}(3)$$
$$+\frac{(x-0)(x-1)(x-5)}{(4-0)(4-1)(4-5)}(24)+\frac{(x-0)(x-1)(x-4)}{(5-0)(5-1)(5-4)}(39)$$
$$=-\frac{(x-1)(x-4)(x-5)}{20}+\frac{x(x-4)(x-5)}{4}$$
$$-2x(x-1)(x-5)+\frac{39x(x-1)(x-4)}{20}$$
$$=-\frac{x^3-10x^2+29x-20}{20}+\frac{x^3-9x^2+20x}{4}$$
$$-(2x^3-12x^2+10x)+\frac{39(x^3-5x^2+4x)}{20}$$
$$=\frac{1}{20}(3x^3+10x^2+27x+20)$$

Example 11

Use Lagrange's formula to fit a polynomial to the data:

x	–1	0	2	3
y	8	3	1	12

and hence, find $y(2)$.

Solution

By Lagrange's interpolation formula,

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(8) + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3)$$

$$+ \frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1) + \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}(12)$$

$$= -\frac{2x(x-2)(x-3)}{3} + \frac{(x+1)(x-2)(x-3)}{2}$$

$$- \frac{(x+1)(x)(x-3)}{6} + (x+1)(x)(x-2)$$

$$= \frac{1}{3}(2x^3 + 2x^2 - 15x + 9)$$

$$y(2) = \frac{1}{3}\left[2(8) + 2(4) - 15(2) + 9\right] = 1$$

Example 12

Express the given rational function as a sum of partial fractions:

$$y = \frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)}$$

Solution

Let $f(x) = 3x^2 + x + 1$.
For $x = 1$, $x = 2$ and $x = 3$, the table is

x	1	2	3
$f(x)$	5	15	31

By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$

$$= \frac{5}{2}(x-2)(x-3) - 15(x-1)(x-3) + \frac{31}{2}(x-1)(x-2)$$

$$\therefore \qquad y = \frac{f(x)}{(x-1)(x-2)(x-3)}$$
$$= \frac{5}{2(x-1)} - \frac{15}{x-2} + \frac{31}{2(x-3)}$$

Example 13

Express the function $\dfrac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)}$ *as a sum of partial fractions using Lagrange's formula.*

Solution

Let $\quad f(x) = 3x^2 - 12x + 11$.
For $x = 1$, $x = 2$ and $x = 3$, the table is

x	1	2	3
$f(x)$	2	–1	2

By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$
$$= \frac{(x-2)(x-3)}{(1-2)(1-3)}(2) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(-1) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(2)$$
$$= (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)$$
$$\therefore \quad y = \frac{f(x)}{(x-1)(x-2)(x-3)}$$
$$= \frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3}$$

Example 14

The following values of the function $f(x)$ are given as $f(1) = 4$, $f(2) = 5$, $f(7) = 5$, $f(8) = 4$. Find the value of $f(6)$ and also the value of x for which $f(x)$ is maximum or minimum.

Solution

Tabular form of the data is

x	1	2	7	8
$f(x)$	4	5	5	4

By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$= \frac{(x-2)(x-7)(x-8)}{(1-2)(1-7)(1-8)}(4) + \frac{(x-1)(x-7)(x-8)}{(2-1)(2-7)(2-8)}(5) + \frac{(x-1)(x-2)(x-8)}{(7-1)(7-2)(7-8)}(5)$$

$$+ \frac{(x-1)(x-2)(x-7)}{(8-1)(8-2)(8-7)}(4)$$

$$= -\frac{2}{21}(x^3 - 17x^2 + 86x - 112) + \frac{1}{6}(x^3 - 16x^2 + 71x - 56)$$

$$- \frac{1}{6}(x^3 - 11x^2 + 26x - 16) + \frac{2}{21}(x^3 - 10x^2 + 23x - 14)$$

$$= -\frac{1}{6}x^2 + \frac{3}{2}x + \frac{8}{3}$$

$$f(6) = -\frac{1}{6}(6)^2 + \frac{3}{2}(6) + \frac{8}{3} = \frac{17}{3}$$

For extreme values, $f'(x) = 0$

$$-\frac{1}{3}x + \frac{3}{2} = 0$$

$$x = 4.5$$

Also, $f''(x) = -\frac{1}{3} < 0$

Since $f''(x)$ is negative, $f(x)$ is maximum at $x = 4.5$.

Example 15

A body moving with velocity v at any time t satisfies the data

t	0	1	3	4
v	21	15	12	10

Obtain the distance travelled in 4 seconds and acceleration at the end of 4 *seconds.*

Solutions

By Lagrange's interpolation formula,

$$\begin{aligned}
v &= \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)}v_0 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_1 \\
&\quad + \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)}v_2 + \frac{(t-t_0)(t-t_1)(t-t_2)}{(t_3-t_0)(t_3-t_1)(t_3-t_2)}v_3 \\
&= \frac{(t-1)(t-3)(t-4)}{(0-1)(0-3)(0-4)}(21) + \frac{(t-0)(t-3)(t-4)}{(1-0)(1-3)(1-4)}(15) \\
&\quad + \frac{(t-0)(t-1)(t-4)}{(3-0)(3-1)(3-4)}(12) + \frac{(t-0)(t-1)(t-3)}{(4-0)(4-1)(4-3)}(10) \\
&= \frac{1}{12}(-5t^3+38t^2-105t+252)
\end{aligned}$$

If s is the distance travelled in time t,

$$\begin{aligned}
v &= \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{12}(-5t^3+38t^2-105t+252) \\
s &= \int_0^4 v\,\mathrm{d}t \\
&= \frac{1}{12}\int_0^4 (-5t^3+38t^2-105t+252) \\
&= \frac{1}{12}\left|-\frac{5t^4}{4}+\frac{38t^3}{3}-\frac{105t^2}{2}+252t\right|_0^4 \\
&= \frac{1}{12}\left[-\frac{5}{4}\times 256+\frac{38}{3}\times 64-\frac{105}{2}\times 16+1008\right] \\
&= 54.88
\end{aligned}$$

Hence, the distance travelled in 4 seconds = 54.88

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{12}(-15t^2+76t-105)$$

At $t = 4$,

$$a = \frac{1}{12}(-15\times 16+76\times 4-105) = 3.416$$

EXERCISE 6.4

1. From the table given below, find $y(x = 2)$.

x	0	1	3	4
y	5	6	50	105

[Ans.: 19]

2. Use Lagrange's formula to find the velocity of the particle $v = f(t)$ at $t = 3.5$ from the following table:

t	0	1	2	3
v	21	15	12	10

[Ans.: 8.75]

3. Find $f(27)$ from the following table:

x	14	17	31	35
$f(x)$	68.7	64.0	44.0	39.1

[Ans.: 49.3]

4. Find $f(6)$ from the following table:

x	2	5	7	10	12
$f(x)$	18	180	448	1210	2028

[Ans.: 294]

5. Find $f(9)$ from the following table:

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

[Ans.: 809.997]

6. If $y_0 = 4.3315$, $y_1 = 7.4046$, $y_3 = 5.6713$, $y_5 = 7.1154$, find the curve passing through these points. Hence, find y_2 and y_4.

[Ans.: 5.1420, 6.3199]

7. If $f(1) = 3, f(2) = -5, f(-4) = 4$, find the three-point Lagrange's interpolation polynomial that takes the same values.

$$\left[\textbf{Ans.:}\ \frac{1}{20}(-39x^2 - 123x + 252)\right]$$

8. A third-degree polynomial passes through the points (0,–1), (1,1), (2,1), (3,–2). Find the polynomial.

$$\left[\textbf{Ans.: } \frac{1}{6}(-x^3 - 3x^2 + 16x - 6)\right]$$

9. If $y = a_0 + a_1x + a_2x^2 + a_3x^3$ passes through the points

x	1	3	5	7
y	0	50	236	654

find a_0, a_1, a_2 and a_3.

[**Ans.:** –4, 3, –1, 2]

10. Find the polynomial of degree 3 which takes the same values as $y = 2^x + 2x + 1$ at $x = -1, 0, 1, 2$.

$$\left[\textbf{Ans.: } \frac{1}{12}\left(x^3 + 3x^2 + 32x + 24\right)\right]$$

11. Find the polynomial which takes the values $f(1) = 1$, $f(2) = 9$, $f(3) = 25$, $f(4) = 55$, $f(5) = 105$.

[**Ans.:** $x^3 - 2x^2 + 7x - 5$]

12. Find $f(x)$ from the following table:

x	0	2	3	6
$f(x)$	659	705	729	804

$$\left[\textbf{Ans.: } \frac{1}{72}(-x^3 + 29x^2 + 1604x + 47448)\right]$$

13. Observe the following table:

x	1	3	4	6
$f(x)$	–3	9	30	132

Express $f(x)$ as a third-degree polynomial in x. Also, find $f'(x)$, $f''(x)$ at $x = 1$.

[**Ans.:** $x^3 - 3x^2 + 5x - 6$, 2, 0]

14. Using Lagrange's formula for unequal intervals, express the function $\dfrac{x^2 + 6x - 1}{(x^2 - 1)(x - 4)(x - 6)}$ as a sum of partial fractions.

$$\left[\textbf{Ans.: } \frac{1}{5(x-1)} + \frac{3}{35(x+1)} - \frac{13}{10(x-4)} + \frac{71}{70(x-6)}\right]$$

6.13 DIVIDED DIFFERENCES

In Lagrange's interpolation formula, if another interpolation value is added then the interpolation coefficients are required to be recalculated. To avoid this recalculation, Newton's general interpolation formula is used.

If (x_0, y_0), (x_1, y_1), (x_2, y_2) …. be given points then the first divided difference for x_0, x_1 is defined by the relation,

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

Similarly, $[x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1}$, etc.

The second divided difference for x_0, x_1, x_2 is defined as

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

The third divided difference for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$

Notes

(i) The divided differences are symmetrical in their arguments, i.e., independent of the order of arguments:

$$\begin{aligned}[x_0, x_1] &= \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0} \\ &= [x_1, x_0]\end{aligned}$$

$$\begin{aligned}[x_0, x_1, x_2] &= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)} \\ &= [x_1, x_2, x_0] \text{ or } [x_2, x_0, x_1]\end{aligned}$$

(ii) The n^{th} divided differences of a polynomial of the n^{th} degree are constant.

Let the arguments be equally spaced so that $x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$

$$\begin{aligned}[x_0, x_1] &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= \frac{\Delta y_0}{h}\end{aligned}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

$$= \frac{1}{2h}\left(\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right)$$

$$= \frac{1}{2!h^2}\Delta^2 y_0$$

In general,

$$[x_0, x_1, x_2 ..., x_n] = \frac{1}{n!h^n}\Delta^n y_0$$

If the tabulated function is an n^{th} degree polynomial, $\Delta^n y_0$ will be constant. Hence, the n^{th} divided differences will also be constant.

6.14 NEWTON'S DIVIDED DIFFERENCE FORMULA

Let the function $y = f(x)$ take values $y_0, y_1, y_2, \ldots, y_n$ corresponding to $x_0, x_1, x_2, \ldots, x_n$ respectively. According to the definition of divided differences,

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

$$y = y_0 + (x - x_0)[x, x_0] \qquad \text{...(6.21)}$$

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

$$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$$

Substituting the value of $[x, x_0]$ in Eq. (6.21),

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \qquad \text{...(6.22)}$$

Also, $$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

$$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$$

Substituting the value of $[x, x_0, x_1]$ in Eq. (6.22),

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]$$

Proceeding in the same manner,

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \ldots$$
$$+ (x - x_0)(x - x_1)\ldots(x - x_{n-1})[x, x_0, x_1, \ldots, x_n] \qquad \text{...(6.23)}$$

Equation (6.23) is known as *Newton's general interpolation formula* with divided differences.

Example 1

If $f(x) = \dfrac{1}{x}$, *find the divided difference* $[a, b]$ *and* $[a, b, c]$.

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference
a	$\dfrac{1}{a}$		
		$\dfrac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\dfrac{1}{ab}$	
b	$\dfrac{1}{b}$		$\dfrac{-\frac{1}{bc}+\frac{1}{ab}}{c-a} = \dfrac{1}{abc}$
		$\dfrac{\frac{1}{c}-\frac{1}{b}}{c-b} = -\dfrac{1}{bc}$	
c	$\dfrac{1}{c}$		

$$[a,b] = -\frac{1}{ab}$$

$$[a,b,c] = \frac{1}{abc}$$

Example 2

Find the second divided difference for the argument $x = 1, 2, 5,$ and 7 for the function $f(x) = x^2$. **[Summer 2015]**

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	1			
		$\frac{4-1}{2-1}=3$		
2	4		$\frac{7-3}{5-1}=1$	
		$\frac{25-4}{5-2}=7$		0
5	25		$\frac{12-7}{7-2}=1$	
		$\frac{49-25}{7-5}=12$		
7	49			

Example 3

Find the third divided difference with arguments 2, 4, 9, 10 *of the function* $f(x) = x^3 - 2x$.

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
2	4			
		$\frac{56-4}{4-2}=26$		
4	56		$\frac{131-26}{9-2}=15$	
		$\frac{711-56}{9-4}=131$		$\frac{23-15}{10-2}=1$
9	711		$\frac{269-131}{10-4}=23$	
		$\frac{980-711}{10-9}=269$		
10	980			

Example 4

Construct the divided difference for the data given below:

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335

[Summer 2015]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
-4	1245				
		$\frac{33-1245}{-1+4}=-404$			
-1	33		$\frac{-28+404}{0+4}=94$		
		$\frac{5-33}{0+1}=-28$		$\frac{10-94}{2+4}=-14$	
0	5		$\frac{2+28}{2+1}=10$		$\frac{13+14}{5+1}=13$
		$\frac{9-5}{2-0}=2$		$\frac{88-10}{5+1}=13$	
2	9		$\frac{442-2}{5-0}=88$		
		$\frac{1335-9}{5-2}=442$			
5	1335				

Example 5

Complete $f(9.2)$ from the following data by using Newton's divided difference interpolation formula.

x	8	9	9.5	11
$f(x)$	2.079442	2.197225	2.251292	2.397895

[Winter 2013]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
8	2.079442			
		0.117783		
9	2.197225		−0.006433	
		0.108134		0.000411
9.5	2.251292		−0.005200	
		0.097735		
11	2.397895			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$f(9.2) = 2.079442 + (9.2 - 8)(0.117783) + (9.2 - 8)(9.2 - 9)(-0.006433) + (9.2 - 8)(9.2 - 9)(9.2 - 9.5)(0.00041)$$

$$= 2.079442 + 0.141340 - 0.001544 - 0.000030$$

$$= 2.219208$$

Example 6

Using Newton's divided difference formula, compute $f(10.5)$ from the following data:

x	10	11	13	17
$f(x)$	2.3026	2.3979	2.5649	2.8332

[Summer 2013]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
10	2.3026			
		0.0953		
11	2.3979		–0.0039	
		0.0835		0.0002
13	2.5649		–0.0027	
		0.0671		
17	2.8332			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2]$$
$$+\ (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$f(10.5) = 2.3026 + (10.5 - 10)\,(0.0953) + (10.5 - 10)\,(10.5 - 11)\,(-0.0039)$$
$$+ (10.5 - 10)\,(10.5 - 11)\,(10.5 - 13)\,(0.0002)$$
$$= 2.3026 + 0.0477 + 0.00098 + 0.00013$$
$$= 2.3514$$

Example 7

Using Newton's divided difference interpolation, compute the value of $f(6)$ from the table given below:

x	1	2	7	8
$f(x)$	1	5	5	4

[Summer 2015]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	1			
		4		
2	5		$-\frac{2}{3}$	
		0		$\frac{1}{14}$
7	5		$-\frac{1}{6}$	
		–1		
8	4			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3]$$

$$f(6) = 1 + (6-1)(4) + (6-1)(6-2)\left(-\frac{2}{3}\right) + (6-1)(6-2)(6-7)\left(\frac{1}{14}\right)$$

$$= 1 + 20 - 13.3333 - 1.4286$$

$$= 6.2381$$

Example 8

Evaluate $f(9)$ using the following table:

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

[Summer 2014]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		42		
		709			
17	5202				

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)(x - x_2)(x - x_3)[x_0, x_1, x_2, x_3, x_4]$$

$$f(9) = 150 + (9-5)(121) + (9-5)(9-7)(24) + (9-5)(9-7)(9-11)(1) + 0$$

$$= 150 + 484 + 192 - 16$$

$$= 810$$

Example 9

Compute f(8) from the following values using Newton's divided difference formula:

x	4	5	7	10	11	13
$f(x)$	48	100	244	900	1210	2028

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
4	48				
		52			
5	100		15		
		97		1	
7	244		21		0
		202		1	
10	900		27		0
		310		1	
11	1210				
		409	33		
13	2028				

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$\begin{aligned} f(8) &= 48 + (8-4)(52) + (8-4)(8-5)(15) + (8-4)(8-5)(8-7)(1) + 0 \\ &= 48 + 208 + 180 + 12 \\ &= 448 \end{aligned}$$

Example 10

From the following table, find f(x) using Newton's divided difference formula:

x	1	2	7	8
$f(x)$	1	5	5	4

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	1			
		4		
2	5		$-\frac{2}{3}$	
		0		$\frac{1}{14}$
7	5		$-\frac{1}{6}$	
		–1		
8	4			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$= 1 + (x - 1)\,4 + (x - 1)\,(x - 2)\left(-\frac{2}{3}\right) + (x - 1)\,(x - 2)\,(x - 7)\left(\frac{1}{14}\right)$$

$$= 1 + 4x - 4 - \frac{2}{3}(x^2 - 3x + 2) + \frac{1}{14}(x^3 - 10x^2 + 23x - 14)$$

$$= \frac{1}{14}x^3 - \frac{29}{21}x^2 + \frac{107}{14}x - \frac{16}{3}$$

Example 11

Using Newton's divided difference formula, find a polynomial and also, find f(–1) and f(6).

x	1	2	4	7
$f(x)$	10	15	67	430

[Summer 2015]

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	10			
		5		
2	15		7	
		26		2
4	67		19	
		121		
7	430			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)\,[x_0, x_1] + (x - x_0)\,(x - x_1)\,[x_0, x_1, x_2] + (x - x_0)\,(x - x_1)\,(x - x_2)\,[x_0, x_1, x_2, x_3]$$

$$= 10 + (x-1)(5) + (x-1)(x-2)(7) + (x-1)(x-2)(x-4)2$$

$$= 10 + 5x - 5 + 7x^2 - 21x + 14 + 2x^3 - 14x^2 + 28x - 16$$

$$= 2x^3 - 7x^2 + 12x + 3$$

$$f(-1) = 2(-1)^3 - 7(-1)^2 + 12(-1) + 3 = -18$$

$$f(6) = 2(6)^3 - 7(6)^2 + 12(6) + 3 = 255$$

Example 12

Establish a cubic polynomial of the curve $y = f(x)$ passing through the points (1, –3), (3, 9), (4, 30), (6, 132). *Hence, find* $f(2)$.

Solution

Divided Difference Table

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1	–3			
		6		
3	9		5	
		21		1
4	30		10	
		51		
6	132			

By Newton's divided difference formula,

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3]$$

$$= -3 + (x - 1)(6) + (x - 1)(x - 3)(5) + (x - 1)(x - 3)(x - 4)(1)$$

$$= -3 + 6x - 6 + 5x^2 - 20x + 15 + x^3 - 8x^2 + 19x - 12$$

$$= x^3 - 3x^2 + 5x - 6$$

$$f(2) = (2)^3 - 3(2)^2 + 5(2) - 6 = 0$$

Example 13

The shear stress in kilopound per square foot (ksf) for 5 specimens in a clay stratum are as follows:

Depth (*m*)	1.9	3.1	4.2	5.1	5.8
Stress (*ksf*)	0.3	0.6	0.4	0.9	0.7

Use Newton's dividend difference interpolating polynomial to compute the stress at 4.5 m depth. **[Winter 2012]**

Solution

Divided Difference Table

Depth x	Stress y	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
1.9	0.3				
		0.25			
3.1	0.6		–0.1877		
		–0.1818		0.1739	
4.2	0.4		0.3687		–0.1295
		0.5556		–0.3313	
5.1	0.9		–0.5258		
		–0.2857			
5.8	0.7				

By Newton's divided difference formula,

$$y(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)(x - x_2)(x - x_3)[x_0, x_1, x_2, x_3, x_4]$$

$$\begin{aligned} y(4.5) &= 0.3 + (4.5 - 1.9)(0.25) + (4.5 - 1.9)(4.5 - 3.1)(-0.1877) \\ &\quad + (4.5 - 1.9)(4.5 - 3.1)(4.5 - 4.2)(0.1739) \\ &\quad + (4.5 - 1.9)(4.5 - 3.1)(4.5 - 4.2)(4.5 - 5.1)(-0.1295) \\ &= 0.3 + 0.65 - 0.6832 + 0.1899 + 0.0848 \\ &= 0.5415 \text{ ksf} \end{aligned}$$

EXERCISE 6.5

1. If $f(x) = \dfrac{1}{x^2}$, find the divided differences $f(a, b)$, $f(a, b, c)$, and $f(a, b, c, d)$.

$$\left[\textbf{Ans.: } -\frac{(a+b)}{a^2b^2}, \frac{ab+bc+ca}{a^2b^2c^2}, -\frac{(abc+bcd+acd+abd)}{a^2b^2c^2d^2}\right]$$

2. Find the third divided difference of $f(x)$ with arguments 2, 4, 9, 10 where $f(x) = x^3 - 2x$.

[Ans.: 1**]**

3. Obtain the value of $\log_{10} 656$ given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$ and $\log_{10} 666 = 2.8202$.

[Ans.: 2.8169**]**

4. Find $f(5)$ from the following table:

x	0	1	3	6
$f(x)$	1	4	88	1309

[Ans.: 636**]**

5. Find $y(x = 20)$ from the following table:

x	12	18	22	24	32
$y(x)$	146	836	19481	2796	9236

[Ans.: 1305.36**]**

6. Find a polynomial $f(x)$ of lowest degree which takes the values 3, 7, 9, and 19 when $x = 2, 4, 5, 10$.

[Ans.: $2x - 1$**]**

7. Using the divided difference table, find $f(x)$ which takes the values 1, 4, 40, 85 as $x = 0, 1, 3, 4$.

[Ans.: $x^3 + x^2 + x + 1$**]**

8. Find $f(x)$ as a polynomial by using Newton's formula:

x	–1	0	3	6	7
$f(x)$	3	–6	39	822	1611

[Ans.: $x^4 - 3x^3 + 5x^2 - 6$**]**

9. Find the polynomial $y = f(x)$ passing through (5, 1355), (2, 9), (0, 5), (–1, 33), and (–4, 1245).

[Ans.: $3x^4 - 5x^3 + 6x^2 + 14x + 5$]

10. Find the polynomial equation of degree 4 passing through the points (8, 1515), (7, 778), (5, 138), (4, 43), and (2, 3).

[Ans.: $x^4 - 10x^3 + 36x^2 - 36x - 5$]

11. Find the function $y(x)$ in powers of $(x - 1)$ given $y(0) = 8$, $y(1) = 11$, $y(4) = 68$, $y(5) = 123$.

[Ans.: $11 + 4(x - 1) + 2(x - 1)^2 + (x - 1)^3$]

12. Using the following table, find $f(x)$ as a polynomial in powers of $(x - 6)$.

x	–1	0	2	3	7	10
$f(x)$	–11	1	1	1	141	561

[Ans.: $73 + 54(x - 6) + 13(x - 6)^2 + (x - 6)^3$]

6.15 INVERSE INTERPOLATION

The process of evaluating the value of x for a value of y (which is not in the table) is called *inverse interpolation*. Lagrange's formula is a relation between two variables, either of which may be taken as the independent variable. On interchanging x and y in the Lagrange's interpolation formula,

$$x = \frac{(y-y_1)(y-y_2)\dots(y-y_n)}{(y_0-y_1)(y_0-y_2)\dots(y_0-y_1)}x_0 + \frac{(y-y_0)(y-y_2)\dots(y-y_n)}{(y_1-y_0)(y_1-y_2)\dots(y_1-y_n)}x_1 + \dots + \frac{(y-y_0)(y-y_1)\dots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\dots(y_n-y_{n-1})}x_n \qquad \dots(6.24)$$

Equation (6.24) is used for inverse interpolation.

Example 1

From the data given, find the value of x when y = 13.5.

x	93	96.2	100	104.2	108.7
y	11.38	12.80	14.70	17.07	19.91

Solution

By Lagrange's formula for inverse interpolation,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y-y_4)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_4)}x_1$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_3)(y-y_4)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_4)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_4)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)(y_3-y_4)}x_3$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_3)}x_4$$

$$= \frac{(13.5-12.80)(13.5-14.70)(13.5-17.07)(13.5-19.91)}{(11.38-12.80)(11.38-14.70)(11.38-17.07)(11.38-19.91)}(93)$$
$$+ \frac{(13.5-11.38)(13.5-14.70)(13.5-17.07)(13.5-19.91)}{(12.80-11.38)(12.80-14.70)(12.80-17.07)(12.80-19.91)}(96.2)$$
$$+ \frac{(13.5-11.38)(13.5-12.80)(13.5-17.07)(13.5-19.91)}{(14.70-11.38)(14.70-12.80)(14.70-17.07)(14.70-19.91)}(100)$$
$$+ \frac{(13.5-11.38)(13.5-12.80)(13.5-14.70)(13.5-19.91)}{(17.07-11.38)(17.07-12.80)(17.07-14.70)(17.07-19.91)}(104.2)$$
$$+ \frac{(13.5-11.38)(13.5-12.80)(13.5-14.70)(13.5-17.07)}{(19.91-11.38)(19.91-12.80)(19.91-14.70)(19.91-17.07)}(108.7)$$
$$x = -7.8137 + 68.4669 + 43.6076 - 7.2758 + 0.7711$$
$$= 97.7561$$

Example 2

Find the root of the equation $f(x) = 0$, *given that* $f(30) = -30$, $f(34) = -13$, $f(38) = 3$, *and* $f(42) = 18$.

Solution

Let $x_0 = 30,\quad x_1 = 34, x_2 = 38, x_3 = 42$

$y_0 = -30, y_1 = -13, y_2 = 3, y_3 = 18$

It is required to find x for $y = f(x) = 0$.
By Lagrange's formula for inverse interpolation,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1$$
$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3$$
$$= \frac{(0+13)(0-3)(0-18)}{(-30+13)(-30-3)(-30-18)}(30) + \frac{(0+30)(0-3)(0-18)}{(-13+30)(-13-3)(-13-18)}(34)$$
$$+ \frac{(0+30)(0+13)(0-18)}{(3+30)(3+13)(3-18)}(38) + \frac{(0+30)(0+13)(0-3)}{(18+30)(18+13)(18-3)}(42)$$
$$= -0.782 + 6.5323 + 33.6818 - 2.2016$$
$$= 37.2305$$

Hence, the root of $f(x) = 0$ is 37.2305.

EXERCISE 6.6

1. Find x given $y = 0.3887$ from the following data:

x	21	23	25
y	0.3706	0.4068	0.4433

[Ans.: 22]

2. Find x corresponding to $y = 85$ from the following table:

x	2	5	8	14
y	94.8	87.9	81.3	68.7

[Ans.: 6.5928]

3. Find x corresponding to $y = 100$ from the following table:

x	3	5	7	9	11
y	6	24	58	108	174

[Ans.: 8.656]

4. Find the value of θ given $f(\theta) = 0.3887$ where $f(\theta) = \int_0^\theta \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$ using the table:

θ	21°	23°	25°
$f(\theta)$	0.3706	0.4068	0.4433

[Ans.: 22.0020°]

5. Find the age corresponding to the annuity value 13.6 from the given table:

Age (x)	30	35	40	45	50
Annuity value (y)	15.9	14.9	14.1	13.3	12.5

[Ans.: 43]

Points to Remember

Forward Differences

$$\Delta y_{n-1} = y_n - y_{n-1}$$

Backward Differences

$$\nabla y_n = y_n - y_{n-1}$$

Central Differences

$$\delta y_{n-\frac{1}{2}} = y_n - y_{n-1}$$

Newton's Forward Interpolation Formula

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

Newton's Backward Interpolation Formula

$$y_r = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

Gauss's Forward Interpolation Formula

$$y_r = y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-1} + \frac{(r+1)r(r-1)(r-2)}{4!}\Delta^4 y_{-2} + \cdots$$

Gauss's Backward Interpolation Formula

$$y_r = y_0 + r\,\Delta y_{-1} + \frac{(r+1)r}{2!}\Delta^2 y_{-1} + \frac{(r+1)r(r-1)}{3!}\Delta^3 y_{-2} + \frac{(r+2)(r+1)r(r-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Stirling's Formula

$$y_r = y_0 + r\left(\frac{\Delta y_{-1} + \Delta y_0}{2}\right) + \frac{r^2}{2!}\Delta^2 y_{-1} + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right) + \frac{r^2(r^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

Lagrange's Interpolation Formula

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}y_1 + \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}y_n$$

Newton's Divided Difference Formula

$$\begin{aligned} y = y_0 &+ (x-x_0)\,[x_0, x_1] + (x-x_0)\,(x-x_1)\,[x_0, x_1, x_2] \\ &+ (x-x_0)\,(x-x_1)\,(x-x_2)\,[x_0, x_1, x_2, x_3] + \ldots \\ &+ (x-x_0)\,(x-x_1)\,\ldots\,(x-x_{n-1})\,[x, x_0, x_1, \ldots, x_n] \end{aligned}$$

Inverse Interpolation

$$x = \frac{(y - y_1)(y - y_2)\ldots(y - y_n)}{(y_0 - y_1)(y_0 - y_2)\ldots(y_0 - y_1)} x_0 + \frac{(y - y_0)(y - y_2)\ldots(y - y_n)}{(y_1 - y_0)(y_1 - y_2)\ldots(y_1 - y_n)} x_1 + \ldots$$

$$+ \frac{(y - y_0)(y - y_1)\ldots(y - y_{n-1})}{(y_n - y_0)(y_n - y_1)\ldots(y_n - y_{n-1})} x_n$$

CHAPTER 7

Numerical Integration

Chapter Outline

7.1 Introduction
7.2 Newton–Cotes Quadrature Formula
7.3 Trapezoidal Rule
7.4 Simpson's 1/3 Rule
7.5 Simpson's 3/8 Rule
7.6 Gaussian Quadrature Formulae

7.1 INTRODUCTION

The process of evaluating a definite integral from a set of tabulated values of $f(x)$ is called *numerical integration*. This process when applied to a function of a single variable is known as *quadrature*. In numerical integration, $f(x)$ is represented by an interpolation formula and then it is integrated between the given limits. In this way, the quadrature formula is derived for approximate integration of a function defined by a set of numerical values only.

7.2 NEWTON–COTES QUADRATURE FORMULA

Let the function $y = f(x)$ takes values $y_0, y_1, y_2 \ldots, y_n$ for $x_0, x_1, x_2 \ldots, x_n$ respectively (Fig. 7.1).

Let $I = \int_a^b f(x)\,dx$. Dividing the interval (a, b) into n sub-intervals of width h such that

$x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots, x_n = x_0 + nh = b$

$$\therefore \qquad \int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$$

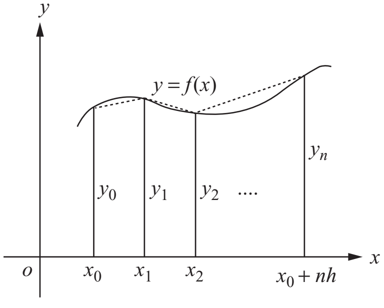


Fig. 7.1

Putting $x = x_0 + rh$, $dx = h dr$
When $x = x_0$, $r = 0$
When $x = x_0 + nh$, $r = n$

$$\int_a^b f(x)\,dx = h\int_0^n f(x_0 + rh)\,dr$$

$$= h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots \right] dr$$

[By Newton's forward interpolation formula]

$$= h\left| ry_0 + \frac{r^2}{2}\Delta y_0 + \left(\frac{\frac{r^3}{3} - \frac{r^2}{2}}{2} \right)\Delta^2 y_0 + \left(\frac{\frac{r^4}{4} - r^3 + r^2}{6} \right)\Delta^3 y_0 + \cdots \right|_0^n$$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right]$$

This equation is known as the *Newton–Cotes quadrature formula*.

7.3 TRAPEZOIDAL RULE

By the Newton–Cotes quadrature formula,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right] \quad \ldots(7.1)$$

Putting $n = 1$ in Eq. (7.1) and ignoring the differences of order higher than one,

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0 \right)$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0) \right]$$

$$= \frac{h}{2}(y_0 + y_1)$$

Similarly,

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}(y_1 + y_2)$$

$$\vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding all these integrals,

$$\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \cdots + y_{n-1}\right)\right]$$

$$= \frac{h}{2}[X + 2R]$$

where X = extreme terms, R = remaining terms
This is known as the *trapezoidal rule*.

Errors in the Trapezoidal Rule

Expanding $y = f(x)$ in the neighbourhood of $x = x_0$ by Taylor's series,

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \cdots \qquad \ldots(7.2)$$

where $y_0' = [y'(x)]_{x=x_0}$, and so on.

$$\int_{x_0}^{x_1} y\,\mathrm{d}x = \int_{x_0}^{x_1}\left[y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \cdots\right]\mathrm{d}x$$

$$= \left| y_0 x + \frac{(x - x_0)^2}{2!}y_0' + \frac{(x - x_0)^3}{3!}y_0'' + \cdots \right|_{x_0}^{x_1}$$

$$= y_0(x_1 - x_0) + \frac{(x_1 - x_0)^2}{2!}y_0' + \frac{(x_1 - x_0)^3}{3!}y_0'' + \cdots$$

$$= h\,y_0 + \frac{h^2}{2!}y_0' + \frac{h^3}{3!}y_0'' + \cdots \qquad \ldots(7.3)$$

where $\qquad x_1 - x_0 = h$

Also, $\displaystyle\int_{x_0}^{x_1} y\,\mathrm{d}x \approx \frac{h}{2}(y_0 + y_1)$ = Area of the first trapezium = A_1 ...(7.4)

Putting $x = x_1$ in Eq. (7.2),

$$y(x_1) = y_1 = y_0 + (x_1 - x_0)\,y_0' + \frac{(x_1 - x_0)^2}{2!}y_0'' + \cdots$$

$$= y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots \qquad \ldots(7.5)$$

Substituting Eq. (7.5) in Eq. (7.4),

$$A_1 \approx \frac{h}{2}\left[y_0 + y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots\right]$$

$$\approx h\,y_0 + \frac{h^2}{2}y_0' + \frac{h^3}{2(2!)}y_0'' + \cdots \qquad \ldots(7.6)$$

Subtracting Eq. (7.6) from Eq. (7.3),

$$\int_{x_0}^{x_1} y\,dx - A_1 = h^3 y_0''\left[\frac{1}{3!} - \frac{1}{2(2!)}\right] + \cdots$$

$$= -\frac{1}{12}h^3 y_0'' + \cdots$$

Hence, the error in the first interval (x_0, x_1), neglecting other terms, is $-\frac{1}{12}h^3\ y_0''$.

Similarly, the error in the interval (x_1, x_2) is $-\frac{1}{12}h^3\ y_1''$ and the error in the interval (x_{n-1}, x_n) is $-\frac{1}{12}h^3 y_{n-1}''$.

Hence, the total error is

$$E = -\frac{1}{12}h^3(y_0'' + y_1'' + \cdots + y_{n-1}'')$$

Let $y''(\xi)$ be the largest value of $y_0'', y_1'', ..., y_{n-1}''$ where $x_0 < \xi < x_n$.

$$E < -\frac{1}{12}nh^3\ y''\ (\xi)$$

$$< -\frac{(x_n - x_0)}{12}h^2 y''(\xi) \qquad [\because\quad nh = x_n - x_0]$$

Example 1

Find the area bounded by the curve and the x-axis from x = 7.47 to x = 7.52 from the following table, by using the trapezoidal rule.

x	7.47	7.48	7.49	7.50	7.51	7.52
$f(x)$	1.93	1.95	1.98	2.01	2.03	2.06

Solution

$$a = 7.47,\ b = 7.52,\ h = 0.01$$

$$\text{Area} = \int_{7.47}^{7.52} f(x)\,\mathrm{d}x$$

By the trapezoidal rule,

$$\int_{7.47}^{7.52} f(x)\,\mathrm{d}x = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\right]$$

$$= \frac{0.01}{2}\left[(1.93 + 2.06) + 2(1.95 + 1.98 + 2.01 + 2.03)\right]$$

$$= 0.0996$$

Example 2

Consider the following tabular values:

x	25.0	25.1	25.2	25.3	25.4	25.5	25.6
$f(x)$	3.205	3.217	3.232	3.245	3.256	3.268	3.280

Determine the area bounded by the given curve and the x-axis between $x = 25$ *and* $x = 25.6$ *by the trapezoidal rule.*

Solution

$$a = 25, \quad b = 25.6, \quad h = 0.1$$

By the trapezoidal rule,

$$\begin{aligned}\int_{25}^{25.6} y\,\mathrm{d}x &= \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right] \\ &= \frac{0.1}{2}\left[(3.205 + 3.280) + 2(3.217 + 3.232 + 3.245 + 3.256 + 3.268)\right] \\ &= 1.9461\end{aligned}$$

Example 3

Given the data below, find the isothermal work done on the gas if it is compressed from $v_1 = 22$ L to $v_2 = 2$ L.

Use $W = -\int_{v_1}^{v_2} p\,\mathrm{d}v$

v, L	2	7	12	17	22
P, atm	12.20	3.49	2.049	1.44	1.11

[Winter 2012]

Solution

$$v_1 = 22, v_2 = 2, h = 5$$

By the trapezoidal rule,

$$\begin{aligned}W &= -\int_{v_1}^{v_2} p\,\mathrm{d}v \\ &= -\int_{22}^{2} p\,\mathrm{d}v\end{aligned}$$

$$= \int_{2}^{22} p \, \mathrm{d}v$$

$$= \frac{h}{2}\left[\left(y_0 + y_4\right) + 2\left(y_1 + y_2 + y_3\right)\right]$$

$$= \frac{5}{2}\left[(12.20 + 1.11) + 2(3.49 + 2.049 + 1.44)\right]$$

$$= 68.17$$

Example 4

Use trapezoidal rule to evaluate $\int_0^2 \frac{x}{\sqrt{2+x^2}} \mathrm{d}x$, *dividing the interval into four equal parts.*

Solution

$$a = 0, b = 2, n = 4$$

$$h = \frac{x_n - x_0}{n} = \frac{2-0}{4} = 0.5$$

$$y = f(x) = \frac{x}{\sqrt{2+x^2}}$$

x	0	0.5	1	1.5	2
$y = f(x)$	0	0.3333	0.5774	0.7276	0.8165
	y_0	y_1	y_2	y_3	y_4

By the trapezoidal rule,

$$\int_0^2 \frac{x}{\sqrt{2+x^2}} \mathrm{d}x = \frac{h}{2}\left[\left(y_0 + y_4\right) + 2\left(y_1 + y_2 + y_3\right)\right]$$

$$= \frac{0.5}{2}\left[(0 + 0.8165) + 2(0.3333 + 0.5774 + 0.7276)\right]$$

$$= 1.0233$$

Example 5

Evaluate $\int_0^1 e^x \, \mathrm{d}x$, *with* $n = 10$ *using the trapezoidal rule.*

Solution

$$a = 0,\ b = 1,\ n = 10$$

$$h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$$

$$y = f(x) = e^x$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.10
$f(x)$	1	1.1052	1.2214	1.3499	1.4918	1.6487	1.8221	2.0138	2.2255	2.4596	2.7183
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By the trapezoidal rule,

$$\int_0^1 e^x \,\mathrm{d}x = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)\right]$$

$$= \frac{0.1}{2}\left[(1 + 2.7183) + 2(1.1052 + 1.2214 + 1.3499 + 1.4918 + 1.6487)\right.$$

$$\left. + 1.8211 + 2.0138 + 2.2255 + 2.4596)\right]$$

$$= 1.7196$$

Example 6

Calculate $\int_0^1 2\,e^x \,\mathrm{d}x$ *with* $n = 10$ *using the trapezoidal rule.*

[Winter 2014]

Solution

$$a = 0,\ b = 1,\ n = 10$$

$$h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$$

$$y = f(x) = 2e^x$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$f(x)$	2	2.2103	2.4428	2.6997	2.9836	3.2974	3.6442	4.0275	4.4511	4.9192	5.4365
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By the trapezoidal rule,

$$\int_0^1 2e^x \mathrm{d}x = \frac{h}{2}\left[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)\right]$$

$$= \frac{0.1}{2}\left[(2 + 5.4365) + 2(2.2103 + 2.4428 + 2.6997 + 2.9836 + 3.2974 + 3.6442 + 4.0275 + 4.4511 + 4.9192)\right]$$

$$= 3.4394$$

Example 7

Compute the integral $\int_{-1}^{1} e^x \,\mathrm{d}x$ *using the trapezoidal rule for* $n = 4$.

Solution

$$a = -1,\ b = 1,\ n = 4$$

$$h = \frac{x_n - x_0}{n} = \frac{1-(-1)}{4} = 0.5$$

$$y = f(x) = e^x$$

x	–1	–0.5	0	0.5	1
$f(x)$	0.3679	0.6065	1	1.6487	2.7183
	y_0	y_1	y_2	y_3	y_4

By the trapezoidal rule,

$$\int_{-1}^{1} e^x \,\mathrm{d}x = \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]$$

$$= \frac{0.5}{2}\ [(0.3679 + 2.7183) + 2(0.6065 + 1 + 1.6487)]$$

$$= 2.39916$$

Example 8

Evaluate $\int_0^1 e^{-x^2} \,\mathrm{d}x$ *with* $n = 10$ *using the trapezoidal rule.*

Solution

$$a = 0,\ b = 1,\ n = 10$$

$$h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$$

$$y = f(x) = e^{-x^2}$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.10
$f(x)$	1	0.99	0.9608	0.9139	0.8521	0.7788	0.6977	0.6126	0.5273	0.4449	0.3679
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By the trapezoidal rule,

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx &= \frac{h}{2}\left[\left(y_0 + y_{10}\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9\right)\right] \\ &= \frac{0.1}{2}\left[(1 + 0.3679) + 2(0.99 + 0.9608 + 0.9139 + 0.8521 + 0.7788 \right. \\ &\qquad \left. + 0.6977 + 0.6126 + 0.5273 + 0.4449)\right] \\ &= 0.7462\end{aligned}$$

7.4 SIMPSON'S 1/3 RULE

By the Newton–Cotes quadrature formula,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right] \qquad \ldots(7.7)$$

Putting $n = 2$ in Eq. (7.7) and ignoring the differences of order higher than 2,

$$\begin{aligned}\int_{x_0}^{x_0+2h} f(x)\,dx &= 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right] \\ &= 2h\left[y_0 + \left(y_1 - y_0\right) + \left(\frac{y_2 - 2y_1 + y_0}{6}\right)\right] \\ &= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)\end{aligned}$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\left(y_2 + 4y_3 + y_4\right)$$

$$\vdots$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right)$$

Adding all these integrals,

$$\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{h}{3}\Big[\big(y_0+y_n\big)+4\big(y_1+y_3+\cdots+y_{n-1}\big)+2\big(y_2+y_4+\cdots+y_{n-2}\big)\Big]$$

$$= \frac{h}{3}\big[X+4O+2E\big]$$

where X = extreme terms, O = odd terms, E = even terms
This is known as Simpson's 1/3 rule.

Note To apply this rule, the number of sub-intervals must be a multiple of 2.

Errors in Simpson's 1/3 Rule

Expanding $y = f(x)$ in the neighbourhood of $x = x_0$ by Taylor's series,

$$y(x) = y_0 + (x-x_0)\,y_0' + \frac{(x-x_0)^2}{2!}\,y_0'' + \frac{(x-x_0)^3}{3!}\,y_0''' + \frac{(x-x_0)^4}{4!}\,y_0^{\text{iv}} + \cdots \qquad \text{...(7.8)}$$

where $y_0' = [y'(x)]_{x=x_0}$, and so on.

$$\int_{x_0}^{x_2} y\,\mathrm{d}x = \int_{x_0}^{x_2}\left[y_0 + (x-x_0)\,y_0' + \frac{(x-x_0)^2}{2!}\,y_0'' + \frac{(x-x_0)^3}{3!}\,y_0''' + \frac{(x-x_0)^4}{4!}\,y_0^{\text{iv}} + \cdots\right]\mathrm{d}x$$

$$= \left| y_0 x + \frac{(x-x_0)^2}{2!}\,y_0' + \frac{(x-x_0)^3}{3!}\,y_0'' + \frac{(x-x_0)^4}{4!}\,y_0''' + \frac{(x-x_0)^5}{5!}\,y_0^{\text{iv}} + \cdots \right|_{x_0}^{x_2}$$

$$= y_0(x_2-x_0) + \frac{(x_2-x_0)^2}{2!}\,y_0' + \frac{(x_2-x_0)^3}{3!}\,y_0''$$

$$+ \frac{(x_2-x_0)^4}{4!}\,y_0''' + \frac{(x_2-x_0)^5}{5!}\,y_0^{\text{iv}} + \cdots$$

$$= 2h\,y_0 + \frac{4h^2}{2!}\,y_0' + \frac{8h^3}{3!}\,y_0'' + \frac{16h^4}{4!}\,y_0''' + \frac{32h^5}{5!}\,y_0^{\text{iv}} + \cdots$$

$$= 2h\,y_0 + 2h^2 y_0' + \frac{4h^3}{3}\,y_0'' + \frac{2h^4}{3}\,y_0''' + \frac{4h^5}{15!}\,y_0^{\text{iv}} + \cdots \qquad \text{...(7.9)}$$

where $x_2 - x_0 = 2h$

Also, $\displaystyle\int_{x_0}^{x_2} y\,\mathrm{d}x = \frac{h}{3}\big(y_0+4y_1+y_2\big)$ = Area in the interval $\big(x_0, x_2\big) = A_1$ ...(7.10)

Putting $x = x_1$ in Eq. (7.8),

$$y(x_1) = y_1 = y_0 + (x_1 - x_0)\, y_0' + \frac{(x_1 - x_0)^2}{2!} y_0'' + \frac{(x_1 - x_0)^3}{3!} y_0''' + \frac{(x_1 - x_0)^4}{4!} y_0^{\text{iv}} + \cdots$$

$$= y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{\text{iv}} + \cdots \qquad \text{...(7.11)}$$

Putting $x = x_2$ in Eq. (7.8),

$$y(x_2) = y_2 = y_0 + 2h y_0' + \frac{4h^2}{2!} y_0'' + \frac{8h^3}{3!} y_0''' + \frac{16h^4}{4!} y_0^{\text{iv}} + \cdots \qquad \text{...(7.12)}$$

Substituting Eq. (7.11) and (7.12) in Eq. (7.10),

$$A_1 = 2h\, y_0 + 2h^2 y_0' + \frac{4h^3}{3} y_0'' + \frac{2h^4}{3} y_0''' + \frac{5h^5}{18} y_0^{\text{iv}} + \cdots \qquad \text{...(7.13)}$$

Subtracting Eq. (7.13) from Eq. (7.9),

$$\int_{x_0}^{x_2} y\,dx - A_1 = \left(\frac{4}{15} - \frac{5}{18}\right) h^5\, y_0^{\text{iv}} + \cdots$$

$$= -\frac{1}{90} h^5 y_0^{\text{iv}} + \cdots$$

Hence, the error in the interval (x_0, x_2), neglecting higher powers of h, is $-\frac{1}{90} h^5\, y_0^{\text{iv}} + \cdots$

Similarly, the error in the interval (x_2, x_4) is $-\frac{1}{90} h^5 y_2^{\text{iv}}$.

Hence, the total error is

$$E = -\frac{1}{90} h^5 \left(y_0^{\text{iv}} + y_2^{\text{iv}} + \cdots \right)$$

Let $y^{\text{iv}}(\xi)$ be the largest value of $y_0^{\text{iv}}, y_2^{\text{iv}}, \ldots, y_{2n-2}^{\text{iv}}$ where $x_0 < \xi < x_{2n}$.

$$E < -\frac{1}{90} n h^5\, y^{\text{iv}}(\xi)$$

$$< -\frac{(x_{2n} - x_0)}{180} h^4\, y^{\text{iv}}(\xi) \qquad [\because\ 2nh = x_{2n} - x_0]$$

Example 1

Consider the following values:

x	10	11	12	13	14	15	16
y	1.02	0.94	0.89	0.79	0.71	0.62	0.55

Find $\int_{10}^{16} y\,\mathrm{d}x$ *by Simpson's* 1/3 *rule.*

Solution

$$a = 10, \quad b = 16, \quad h = 1$$

By Simpson's 1/3 rule,

$$\int_{10}^{16} y\,\mathrm{d}x = \frac{h}{3}\left[\left(y_0 + y_6\right) + 4\left(y_1 + y_3 + y_5\right) + 2\left(y_2 + y_4\right)\right]$$

$$= \frac{1}{3}\left[(1.02 + 0.55) + 4(0.94 + 0.79 + 0.62) + 2(0.89 + 0.71)\right]$$

$$= 4.7233$$

Example 2

A rocket is launched from the ground. Its acceleration is registered during the first 80 *seconds and is given as follows:*

t (s)	0	10	20	30	40	50	60	70	80
a (m/s^2)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

By Simpson's 1/3 *rule, find the velocity at t = 80 s.*

Solution

$$a = 0, \quad b = 80, \quad h = 10$$

By Simpson's 1/3 rule,

$$\text{Velocity} = \int_{0}^{80} a\,\mathrm{d}t$$

$$= \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{10}{3}\left[(30 + 50.67) + 4(31.63 + 35.47 + 40.33 + 46.69) + 2(33.34 + 37.75 + 43.25)\right]$$

$$= 3086.1 \text{ m/s}$$

Example 3

A river is 80 metres wide. The depth 'd' in metres at a distance x metres from one bank is given by the following table. Calculate the area of cross section of the river using Simpson's 1/3 rule. **[Summer 2015]**

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	7

Solution

$$a = 0, \quad b = 80, \quad h = 10$$

$$A = \int_0^{80} y\,\mathrm{d}x$$

By Simpson's 1/3 rule,

$$\int_0^{80} y\,\mathrm{d}x = \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{10}{3}\left[(0+7) + 4(4+9+15+8) + 2(7+12+14)\right]$$

$$= 723.33 \text{ m}^2$$

Example 4

Evaluate $\int_0^6 \frac{1}{1+x}\mathrm{d}x$ *taking* $h=1$ *using Simpson's* 1/3 *rule. Hence, obtain an approximate value of* log 7. **[Winter 2013]**

Solution

$$a = 0, \quad b = 6, \quad h = 1$$

$$n = \frac{b-a}{h} = \frac{6-0}{1} = 6$$

$$y = f(x) = \frac{1}{1+x}$$

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.3333	0.25	0.2	0.1667	0.1429
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_0^6 \frac{1}{1+x}\,\mathrm{d}x = \frac{h}{3}\left[\left(y_0+y_6\right)+4\left(y_1+y_3+y_5\right)+2\left(y_2+y_4\right)\right]$$

$$= \frac{1}{3}\left[(1+0.1429)+4(0.5+0.25+0.1667)+2(0.3333+0.2)\right]$$

$$= 1.9588 \qquad \text{...(1)}$$

By direct integration,

$$\int_0^6 \frac{1}{1+x}\,\mathrm{d}x = \left|\log(1+x)\right|_0^6 = \log 7 \qquad \text{...(2)}$$

From Eqs (1) and (2),

$$\log 7 = 1.9588$$

Example 5

Evaluate $\int_0^5 \frac{\mathrm{d}x}{4x+5}$ *by using Simpson's* 1/3 *rule, taking* 10 *equal parts. Hence, find the approximate value of* $\log_e 5$.

Solution

$$a = 0, \quad b = 5, \quad n = 10$$

$$h = \frac{b-a}{n} = \frac{5-0}{10} = 0.5$$

$$y = f(x) = \frac{1}{4x+5}$$

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	0.2	0.1428	0.1111	0.0910	0.0769	0.0667	0.0588	0.0526	0.0476	0.0435	0.04
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By Simpson's 1/3 rule,

$$\int_0^5 \frac{\mathrm{d}x}{4x+5} = \frac{h}{3}\left[\left(y_0+y_{10}\right)+4\left(y_1+y_3+y_5+y_7+y_9\right)+2\left(y_2+y_4+y_6+y_8\right)\right]$$

$$= \frac{0.5}{3}\left[(0.2+0.04)+4(0.1428+0.0910+0.0667+0.0526+0.0435)\right.$$

$$\left.+\,2(0.1111+0.0769+0.0588+0.0476)\right]$$

$$= 0.4026 \qquad \text{...(1)}$$

By the direct method,

$$\int_0^5 \frac{dx}{4x+5} = \left|\frac{\log_e(4x+5)}{4}\right|_0^5$$

$$= \frac{1}{4}(\log_e 25 - \log_e 5)$$

$$= \frac{1}{4}\log_e \frac{25}{5}$$

$$= \frac{1}{4}\log_e 5 \qquad \ldots(2)$$

Equating Eqs (1) and (2),

$$\frac{1}{4}\log_e 5 = 0.4026$$

$$\log_e 5 = 1.6104$$

Example 6

Evaluate the integral $\int_{-2}^{6} (1+x^2)^{\frac{3}{2}}\, dx$ *by Simpson's* 1/3 *rule with taking* 6 *sub-intervals. Use four digits after the decimal point for calculations.*

[Winter 2012]

Solution

$$a = -2, \quad b = 6, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{6-(-2)}{6} = \frac{4}{3}$$

$$y = f(x) = (1+x^2)^{\frac{3}{2}}$$

x	-2	$-\frac{2}{3}$	$\frac{2}{3}$	2	$\frac{10}{3}$	$\frac{14}{3}$	6
$f(x)$	11.1803	1.7360	1.7360	11.1803	42.1479	108.7094	225.0622
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_{-2}^{6}(1+x^2)^{\frac{3}{2}}\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{4}{9}\left[(11.1803+225.0622)+4(1.7360+11.1803+108.7094) + 2(1.7360+42.1479)\right]$$

$$= 360.2280$$

Example 7

Using Simpson's 1/3 *rule, find* $\int_0^{0.6} e^{-x^2}\,dx$ *by taking* $n = 6$.

[Summer 2015]

Solution

$$a = 0, \quad b = 0.6, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1$$

$$y = f(x) = e^{-x^2}$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	1	0.99	0.9608	0.9139	0.8521	0.7788	0.6977
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_0^{0.6} e^{-x^2}\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{0.1}{3}\left[(1+0.6977)+4(0.99+0.9139+0.7788)+2(0.9608+0.8521)\right]$$

$$= 0.5351$$

Example 8

Estimate $\int_0^3 \cos^2 x\,dx$ *by using Simpson's* 1/3 *rule with* 6 *intervals.*

Solution

$$a = 0, \quad b = 3, \quad n = 6$$
$$h = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$$
$$y = f(x) = \cos^2 x$$

x	0	0.5	1.0	1.5	2.0	2.5	3.0
$f(x)$	1	0.9999	0.9996	0.9993	0.9988	0.9981	0.9973
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_0^3 \cos^2 x\,\mathrm{d}x = \frac{h}{3}\left[\left(y_0 + y_6\right) + 4\left(y_1 + y_3 + y_5\right) + 2\left(y_2 + y_4\right)\right]$$
$$= \frac{0.5}{3}\left[(1 + 0.9973) + 4(0.9999 + 0.9993 + 0.9981) + 2(0.9996 + 0.9988)\right]$$
$$= 2.9978$$

Example 9

Compute the integral $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,\mathrm{d}x$ *for* $n = 6$ *with an accuracy to five decimal places using Simpson's* 1/3 *rule.*

Solution

$$a = 0, \quad b = \frac{\pi}{2}, \quad n = 6$$
$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$
$$y = f(x) = \sqrt{\sin x}$$

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$f(x)$	0	0.5087	0.7071	0.8409	0.9306	0.9828	1.0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 1/3 rule,

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{\pi}{36}\left[(0+1) + 4(0.5087 + 0.8409 + 0.9828) + 2(0.7071 + 0.9306)\right]$$

$$= 1.1873$$

Example 10

The speed v metres per second, of a car, t seconds after it starts, is shown in the following table:

t	0	12	24	36	48	60	72	84	96	108	120
v	0	3.60	10.08	18.90	21.60	18.54	10.26	4.50	4.5	5.4	9.0

Using Simpson's 1/3 *rule, find the distance travelled by the car in* 2 *minutes.*

Solution

Let s (metres) distance be travelled in t (seconds).

$$\frac{ds}{dt} = v$$

$$\int ds = \int v\, dt$$

$$s = \int v\, dt$$

The distance travelled in 2 minutes i.e., 120 seconds is

$$s = \int_0^{120} v\, dt$$

Also, $h = 12$ seconds

By Simpson's 1/3 rule,

$$\int_0^{120} v\, dt = \frac{h}{3}\left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$= \frac{12}{3}\left[(0 + 9.0) + 4(3.60 + 18.90 + 18.54 + 4.50 + 5.4) + 2(10.08 + 21.60 + 10.26 + 4.5)\right]$$

$$= 1222.56 \text{ metres}$$

7.5 SIMPSON'S 3/8 RULE

By the Newton–Cotes quadrature formula,

$$\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right] \quad \ldots(7.14)$$

Putting $n = 3$ in Eq. (7.14) and ignoring the differences of order higher than 3,

$$\int_{x_0}^{x_0+3h} f(x)\,\mathrm{d}x = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right)$$

Similarly,
$$\int_{x_0+3h}^{x_0+6h} f(x)\,\mathrm{d}x = \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$

$$\vdots$$

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{3h}{8}\left(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)$$

Adding all these integrals,

$$\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{3h}{8}\left[\left(y_0 + y_n\right) + 2\left(y_3 + y_6 + \cdots + y_{n-3}\right) + 3\left(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}\right)\right]$$

$$= \frac{3h}{8}\left[X + 2T + 3R\right]$$

where X = extreme terms, T = multiple of three terms, R = remaining terms
This is known as Simpson's 3/8 rule.

Note To apply this rule, the number of sub-intervals must be a multiple of 3.

Errors in Simpson's 3/8 Rule

Expanding $y = f(x)$ in the neighbourhood of $x = x_0$ by Taylor's series,

$$y(x) = y_0 + (x - x_0)\,y_0' + \frac{(x-x_0)^2}{2!}\,y_0'' + \frac{(x-x_0)^3}{3!}\,y_0''' + \frac{(x-x_0)^4}{4!}\,y_0^{\text{iv}} + \cdots \quad \ldots(7.15)$$

where $y_0' = [y'(x)]_{x=x_0}$, and so on.

$$\int_{x_0}^{x_3} y\,dx = \int_{x_0}^{x_3}\left[y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0^{\text{iv}} + \cdots\right]$$

$$= \left| y_0 x + \frac{(x-x_0)^2}{2!}y_0' + \frac{(x-x_0)^3}{3!}y_0'' + \frac{(x-x_0)^4}{4!}y_0''' + \frac{(x-x_0)^5}{5!}y_0^{\text{iv}} + \cdots \right|_{x_0}^{x_3}$$

$$= y_0(x_3 - x_0) + \frac{(x_3-x_0)^2}{2!}y_0' + \frac{(x_3-x_0)^3}{3!}y_0'' + \frac{(x_3-x_0)^4}{4!}y_0''' + \frac{(x_3-x_0)^5}{5!}y_0^{\text{iv}} + \cdots$$

$$= 3h\,y_0 + \frac{9h^2}{2!}y_0' + \frac{27h^3}{3!}y_0'' + \frac{81h^4}{4!}y_0''' + \frac{243h^5}{5!}y_0^{\text{iv}} + \cdots \qquad \text{...(7.16)}$$

where $x_3 - x_0 = 3h$

Also, $$\int_{x_0}^{x_3} y\,dx = \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3) = \text{Area in the interval } (x_0, x_3) = A_1 \qquad \text{...(7.17)}$$

Putting $x = x_1$ in Eq. (7.15),

$$y(x_1) = y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{\text{iv}} + \cdots \qquad \text{...(7.18)}$$

Putting $x = x_2$ in Eq. (7.16),

$$y(x_2) = y_2 = y_0 + 2hy_0' + \frac{4h^2}{2!}y_0'' + \frac{8h^3}{3!}y_0''' + \frac{16h^4}{4!}y_0^{\text{iv}} + \cdots \qquad \text{...(7.19)}$$

Putting $x = x_3$ in Eq. (7.17),

$$y(x_3) = y_3 = y_0 + 3h\,y_0' + \frac{9h^2}{2!}y_0'' + \frac{27h^3}{3!}y_0''' + \frac{81h^4}{4!}y_0^{\text{iv}} + \cdots \qquad \text{...(7.20)}$$

Substituting Eqs (7.18), (7.19) and (7.20) in Eq. (7.17),

$$A_1 = 3h\,y_0 + \frac{9h^2}{2!}y_0' + \frac{27h^3}{3!}y_0'' + \frac{81h^4}{4!}y_0''' + \frac{33h^5}{16}y^{\text{iv}} + \cdots \qquad \text{...(7.21)}$$

Subtracting Eq. (7.21) from Eq. (7.16),

$$\int_{x_0}^{x_3} y\,dx - A_1 = \left(\frac{81}{40} - \frac{33}{16}\right)h^5\,y_0^{\text{iv}} + \cdots$$

$$= -\frac{3}{80}h^5\,y_0^{\text{iv}} + \cdots$$

Hence, the error in the interval (x_0, x_3), neglecting higher powers of h, is $-\frac{3}{80}h^5\,y_0^{\text{iv}}$.

Similarly, the error in the interval (x_3, x_6) is $-\frac{3}{80}h^5\,y_3^{\text{iv}}$.

Hence, the total error is

$$E = -\frac{3}{80} h^5 \left(y_0^{\text{iv}} + y_3^{\text{iv}} + \cdots + y_{3n-3}^{\text{iv}}\right)$$

Let $y^{iv}(\xi)$ be the largest value of $y_0^{\text{iv}}, y_3^{\text{iv}}, \ldots, y_{n-3}^{\text{iv}}$ where $x_0 < \xi < x_{3n}$.

$$E < -\frac{3}{80} nh^5\, y^{\text{iv}}(\xi)$$

$$< -\frac{(x_{3n} - x_0)}{80} h^4 y^{\text{iv}}(\xi) \qquad \left[\because \quad 3nh = x_{3n} - x_0\right]$$

Example 1

Evaluate $\int_0^3 \frac{1}{1+x}\,\mathrm{d}x$ *with* $n = 6$ *by using Simpson's* 3/8 *rule and, hence, calculate* log 2. **[Summer 2014]**

Solution

$$a = 0, \quad b = 3, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$$

$$y = f(x) = \frac{1}{1+x}$$

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	1	0.6667	0.5	0.4	0.3333	0.2857	0.25
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^3 \frac{1}{1+x}\,\mathrm{d}x = \frac{3h}{8}\left[\left(y_0 + y_6\right) + 2\left(y_3\right) + 3\left(y_1 + y_2 + y_4 + y_5\right)\right]$$

$$= \frac{3(0.5)}{8}\left[(1 + 0.25) + 2(0.4) + 3(0.6667 + 0.5 + 0.3333 + 0.2857)\right]$$

$$= 1.3888 \qquad \ldots(1)$$

By direct integration,

$$\int_0^3 \frac{1}{1+x}\,\mathrm{d}x = \left|\log(1+x)\right|_0^3$$

$$= \log 4$$

$$= \log (2)^2$$

$$= 2\log 2 \qquad \ldots(2)$$

From Eqs (1) and (2),

$$2 \log 2 = 1.3888$$

$$\log 2 = 0.6944$$

Example 2

Evaluate $\int_0^{\pi} \frac{\sin^2 x}{5+4\cos x}\, dx$ *by using Simpson's 3/8 rule.*

Solution

$$a = 0, \quad b = \pi$$

Dividing the interval into six equal parts, i.e., $n = 6$,

$$h = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

$$y = f(x) = \frac{\sin^2 x}{5 + 4\cos x}$$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$f(x)$	0	0.02954	0.10714	0.2	0.25	0.16277	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^{\pi} \frac{\sin^2 x}{5+4\cos x}\, dx = \frac{3h}{8}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\right]$$

$$= \frac{3}{8}\left(\frac{\pi}{6}\right)\left[(0+0) + 2(0.2) + 3(0.02954 + 0.10714 + 0.25 + 0.16277)\right]$$

$$= \frac{\pi}{16}(2.04835)$$

$$= 0.40219$$

Example 3

Find $\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{1}{2}\sin^2 t}\, dt$ *using one of the methods of numerical integration.*

Solution

Dividing the interval $\left[0, \frac{\pi}{2}\right]$ into six equal parts and applying Simpson's 3/8 rule,

$$a = 0, \quad b = \frac{\pi}{2}, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

$$y = f(t) = \sqrt{1 - \frac{1}{2}\sin^2 t}$$

t	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$f(x)$	1	0.9831	0.9354	0.8660	0.7906	0.7304	0.7071
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{1}{2}\sin^2 t}\; dt = \frac{3h}{8}\left[\left(y_0 + y_6\right) + 2\left(y_3\right) + 3\left(y_1 + y_2 + y_4 + y_5\right)\right]$$

$$= \frac{3}{8}\left(\frac{\pi}{12}\right)\left[(1 + 0.7071) + 2(0.8660) + 3(0.9831 + 0.9354 + 0.7906 + 0.7304)\right]$$

$$= 1.3496$$

Example 4

Find $\int_0^{\frac{\pi}{2}} e^{\sin\theta} d\theta$ *by Simpson's 3/8 rule, dividing the interval* $\left[0, \frac{\pi}{12}\right]$ *into six equal parts.*

Solution

$$a = 0, \quad b = \frac{\pi}{2}, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

$$y = f(\theta) = e^{\sin\theta}$$

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$f(\theta)$	1	1.2953	1.6487	2.0281	2.3773	2.6247	2.7182
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^{\frac{\pi}{2}} e^{\sin\theta}\mathrm{d}\theta = \frac{3h}{8}\left[\left(y_0+y_6\right)+2\left(y_3\right)+3\left(y_1+y_2+y_4+y_5\right)\right]$$

$$= \frac{3}{8}\left(\frac{\pi}{12}\right)\left[(1+2.7182)+2(2.0281)\right.$$

$$\left.+\,3(1.2953+1.6487+2.3773+2.6247)\right]$$

$$= 3.1012$$

Example 5

By Simpson's 3/8 rule, evaluate $\int_0^1 \frac{\sin x}{x}$ *taking* $h=\frac{1}{6}$.

Solution

$$a=0,\quad b=1,\quad h=\frac{1}{6}$$

$$n=\frac{b-a}{h}=\frac{1-0}{\frac{1}{6}}=6$$

$$y=f(x)=\frac{\sin x}{x}$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$f(x)$	1	0.9954	0.9816	0.9589	0.9276	0.8882	0.8415
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^1 \frac{\sin x}{x}\,\mathrm{d}x = \frac{3h}{8}\left[\left(y_0+y_6\right)+2\left(y_3\right)+3\left(y_1+y_2+y_4+y_5\right)\right]$$

$$= \frac{3}{8}\left(\frac{1}{6}\right)\big[(1 + 0.8415) + 2(0.9589)$$
$$+ 3(0.9954 + 0.9816 + 0.9276 + 0.8882)\big]$$
$$= 0.9461$$

Example 6

The velocity of a train which starts from rest is given by the following table, the time being reckoned in minutes from the start and speed in km/h.

Time	3	6	9	12	15	18
Velocity	22	29	31	20	4	0

Estimate approximately the distance covered in 18 minutes by Simpson's 3/8 *rule.*

Solution

Let s km distance be covered in t minutes.

$$\frac{\mathrm{d}s}{\mathrm{d}t} = v$$
$$\int \mathrm{d}s = \int v\,\mathrm{d}t$$
$$s = \int v\,\mathrm{d}t$$

The distance covered in 18 minutes is

$$s = \int_0^{18} v\mathrm{d}t$$

Since the train starts from rest, at $t = 0$, $v = 0$ $\quad \therefore \; y_0 = 0$

Time (t)	0	3	6	9	12	15	18
Velocity (v)	0	22	29	31	20	4	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Also, $h = 3$ minutes $= \dfrac{3}{60} = \dfrac{1}{20}$ hours

By Simpson's 3/8 rule,

$$\int_0^{18} v\,\mathrm{d}t = \frac{3h}{8}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\right]$$

$$= \frac{3}{8}\left(\frac{1}{20}\right)\left[(0+0)+2(31)+3(22+29+20+4)\right]$$

$$= 5.38125$$

Example 7

Find the volume of a solid of revolution formed by rotating about the x-axis the area bounded by the lines $x = 0$, $x = 1.5$, $y = 0$, *and the curve passing through the following points:*

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50
y	1.00	0.9826	0.9589	0.9089	0.8415	0.7624	0.7589

Solution

Volume is given by

$$V = \int \pi y^2 \, dx$$

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50
y^2	1.00	0.9655	0.9195	0.8261	0.7081	0.5812	0.5759
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$h = 0.25$$

By Simpson's 3/8 rule,

$$\int y^2 \, dx = \frac{3h}{8}\left[\left(y_0 + y_6\right) + 2\left(y_3\right) + 3\left(y_1 + y_2 + y_4 + y_5\right)\right]$$

$$= \frac{3(0.25)}{8}\left[(1.00 + 0.5759) + 2(0.8261) + 3(0.9655 + 0.9195 + 0.7081 + 0.5812)\right]$$

$$= 1.1954$$

$$\text{Volume} = \pi \int y^2 \, dx$$

$$= \pi(1.1954)$$

$$= 3.7555$$

Example 8

Evaluate $\int_4^{5.2} \log x$ *using the trapezoidal rule and Simpson's* 3/8 *rule, take* $h = 0.2$.

Solution

$$a = 4, \quad b = 5.2, \quad h = 0.2$$

$$n = \frac{b-a}{h} = \frac{5.2-4}{0.2} = 6$$

$$y = f(x) = \log x$$

x	4	4.2	4.4	4.6	4.8	5.0	5.2
$f(x)$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By the trapezoidal rule,

$$\int_4^{5.2} \log\ x\,dx = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{0.2}{2}\left[(1.3863 + 1.6487)\right.$$

$$\left. + 2(1.4351 + 1.4816 + 1.5261 + 1.5686 + 1.6094)\right]$$

$$= 1.8277$$

By Simpson's 3/8 rule,

$$\int_4^{5.2} \log\,dx = \frac{3h}{8}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\right]$$

$$= \frac{3(0.2)}{8}\left[(1.3863 + 1.6487) + 2(1.5261)\right.$$

$$\left. + 3(1.4351 + 1.4816 + 1.5686 + 1.6094)\right]$$

$$= 1.8278$$

Example 9

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ *taking* $h = \frac{1}{6}$ *using Simpson's 3/8 rule and the trapezoidal rule.*

Solution

$$a = 0, \quad b = 1, \quad h = \frac{1}{6}$$

$$n = \frac{b-a}{h} = \frac{1-0}{\frac{1}{6}} = 6$$

$$y = f(x) = \frac{1}{1+x^2}$$

x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$f(x)$	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\begin{aligned}\int_0^1 \frac{dx}{1+x^2} &= \frac{3h}{8}\left[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)\right] \\ &= \frac{3}{8}\left(\frac{1}{6}\right)\left[\left(1+\frac{1}{2}\right)+2\left(\frac{4}{5}\right)+3\left(\frac{36}{37}+\frac{9}{10}+\frac{9}{13}+\frac{36}{61}\right)\right] \\ &= 0.7854\end{aligned}$$

By the trapezoidal rule,

$$\begin{aligned}\int_0^1 \frac{dx}{1+x^2} &= \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right] \\ &= \frac{1}{12}\left[\left(1+\frac{1}{2}\right)+2\left(\frac{36}{37}+\frac{9}{10}+\frac{4}{5}+\frac{9}{13}+\frac{36}{61}\right)\right] \\ &= 0.7842\end{aligned}$$

Example 10

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ *by using* (*i*) *trapezoidal rule,* (*ii*) *Simpson's* 1/3 *rule,* (*iii*) *Simpson's* 3/8 *rule.* **[Summer 2014]**

Solution

$$a = 0, b = 6$$

Dividing the interval into six equal parts, i.e., $n = 6$,

$$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$$

$$y = f(x) = \frac{1}{1+x^2}$$

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By the trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027)+2(0.5+0.2+0.1+0.0588+0.0385)\right]$$

$$= 1.4108$$

(ii) By Simpson's 1/3 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(1+0.027)+4(0.5+0.1+0.0385)+2(0.2+0.0588)\right]$$

$$= 1.3662$$

(iii) By Simpson's 3/8 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3}{8}\left[(1+0.027)+2(0.1)+3(0.5+0.2+0.0588+0.0385)\right]$$

$$= 1.3571$$

EXERCISE 7.1

1. Evaluate $\int_1^2 \frac{dx}{1+x^2}$ taking h = 0.2, using trapezoidal rule.

[Ans.: 0.3228**]**

2. Evaluate the value of $\int_0^{0.3} \sqrt{1-8x^3}\, dx$ using Simpson's 3/8 rule.

[Ans.: 0.2916**]**

3. Evaluate $\int_0^{\frac{\pi}{2}} e^{\sin x} dx$ by Simpson's 3/8 rule.

[Ans.: 3.1044]

4. Evaluate $\int_0^1 \frac{dx}{1+x}$ by using (i) trapezoidal rule, (ii) Simpson's 1/3 rule, and (iii) Simpson's 3/8 rule. Take h = 0.25.

[Ans.: (i) 0.6970 (ii) 0.6932 (iii) 0.6932]

5. Calculate $\int_0^{\frac{\pi}{2}} \sin x \, dx$ by dividing the interval into ten equal parts, using the trapezoidal rule and Simpson's 1/3 rule. **[Ans.:** 0.9981, 1.0006]

6. Find the value of $\log 2^{\frac{1}{3}}$ from $\int_0^1 \frac{x^2}{1+x^3}$ using Simpson's 1/3 rule with h = 0.25. **[Ans.:** 0.2311]

7. Compute the value of $\int_{0.2}^{1.4} (\sin x - \log x + e^x) \, dx$ taking h = 0.2 and using the trapezoidal rule, and Simpson's rule. **[Ans.:** 4.0715, 4.0521]

8. Evaluate $\int_{0.5}^{0.7} \sqrt{x}\, e^{-x} dx$ using Simpson's 3/8 rule.

[Ans.: 0.0841]

9. Evaluate $\int_0^1 \frac{dx}{x^3+x+1}$ using Simpson's 1/3 rule, taking h = 0.25.

[Ans.: 0.6305]

10. A curve is drawn to pass through the points given by the following table:

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Obtain the area bounded by the curve, the x-axis, and the lines x = 1 and x = 4 by any method.

[Ans.: 7.7833]

7.6 GAUSSIAN QUADRATURE FORMULAE

An n-point Gaussian quadrature formula is a quadrature formula constructed to give an exact result for polynomials of degree $2n - 1$ or less by a suitable choice of the points x_i and weights w_i for $i = 1, 2, ..., n$. Gauss quadrature formula can be expressed as

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i) \qquad ...(7.22)$$

7.6.1 One-point Gaussian Quadrature Formula

Consider a function $f(x)$ over the interval $[-1, 1]$ with sampling point x_1 and weight w_1. The one-point Gaussian quadrature formula is

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) \qquad ...(7.23)$$

This formula will be exact for polynomials of degrees up to $2n - 1 = 2(1) - 1 = 1$, i.e., it is exact for $f(x) = 1$ and x.

Substituting $f(x)$ in Eq. (7.23) successively,

$$\int_{-1}^{1} 1\,dx = w_1$$

$$\left| x \right|_{-1}^{1} = w_1$$

$$2 = w_1 \qquad ...(7.24)$$

$$\int_{-1}^{1} x\,dx = w_1 x_1$$

$$\left| \frac{x^2}{2} \right|_{-1}^{1} = w_1 x_1$$

$$0 = w_1 x_1 \qquad ...(7.25)$$

Solving Eqs (7.24) and (7.25),

$$w_1 = 2$$

$$x_1 = 0$$

Hence, $$\int_{-1}^{1} f(x)\,dx = 2f(0) \qquad ...(7.26)$$

Equation (7.26) is known as *one-point Gaussian quadrature formula*. This formula is exact for polynomials up to degree one.

7.6.2 Two-Point Gaussian Quadrature Formula

Consider a function $f(x)$ over the interval [–1, 1] with sampling points x_1, x_2 and weights w_1, w_2 respectively. The two-point Gaussian quadrature formula is

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = w_1 f(x_1) + w_2 f(x_2) \qquad \text{...(7.27)}$$

This formula will be exact for polynomials of degrees up to $2n - 1 = 2(2) - 1 = 3$, i.e., it is exact for $f(x) = 1, x, x^2$ and x^3.

Substituting $f(x)$ in Eq. (7.27), successively,

$$\int_{-1}^{1} 1\,\mathrm{d}x = w_1 + w_2$$

$$|x|_{-1}^{1} = w_1 + w_2$$

$$2 = w_1 + w_2 \qquad \text{...(7.28)}$$

$$\int_{-1}^{1} x\,\mathrm{d}x = w_1 x_1 + w_2 x_2$$

$$\left|\frac{x^2}{2}\right|_{-1}^{1} = w_1 x_1 + w_2 x_2$$

$$0 = w_1 x_1 + w_2 x_2 \qquad \text{...(7.29)}$$

$$\int_{-1}^{1} x^2\,\mathrm{d}x = w_1 x_1^2 + w_2 x_2^2$$

$$\left|\frac{x^3}{3}\right|_{-1}^{1} = w_1 x_1^2 + w_2 x_2^2$$

$$\frac{2}{3} = w_1 x_1^2 + w_2 x_2^2 \qquad \text{...(7.30)}$$

$$\int_{-1}^{1} x^3\,\mathrm{d}x = w_1 x_1^3 + w_2 x_2^3$$

$$\left|\frac{x^4}{4}\right|_{-1}^{1} = w_1 x_1^3 + w_2 x_2^3$$

$$0 = w_1 x_1^3 + w_2 x_2^3 \qquad \text{...(7.31)}$$

Solving Eqs (7.28), (7.29), (7.30), and (7.31),

$$w_1 = w_2 = 1$$
$$x_1 = -\frac{1}{\sqrt{3}},\ x_2 = \frac{1}{\sqrt{3}}$$

Hence, $$\int_{-1}^{1} f(x)\,\mathrm{d}x = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \qquad ...(7.32)$$

Equation (7.32) is known as the two-point Gaussian quadrature formula. This formula is exact for polynomials up to degree three.

7.6.3 Three-Point Gaussian Quadrature Formula

Consider a function $f(x)$ over the interval [–1, 1] with sampling points x_1, x_2, x_3 and weights w_1, w_2, w_3 respectively. The three-point Gaussian Quadrature formula is

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) \qquad ...(7.33)$$

This formula will be exact for polynomials of degrees up to $2n - 1 = 2(3) - 1 = 5$, i.e., it is exact for $f(x) = 1, x, x^2, x^3, x^4$ and x^5.
Substituting $f(x)$ in Eq. (7.33) successively,

$$\int_{-1}^{1} 1\,\mathrm{d}x = w_1 + w_2 + w_3$$
$$\left|x\right|_{-1}^{1} = w_1 + w_2 + w_3$$
$$0 = w_1 + w_2 + w_3 \qquad ...(7.34)$$

$$\int_{-1}^{1} x\,\mathrm{d}x = w_1 x_1 + w_2 x_2 + w_3 x_3$$
$$\left|\frac{x^2}{2}\right|_{-1}^{1} = w_1 x_1 + w_2 x_2 + w_3 x_3$$
$$0 = w_1 x_1 + w_2 x_2 + w_3 x_3 \qquad ...(7.35)$$

$$\int_{-1}^{1} x^2\,\mathrm{d}x = w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2$$
$$\left|\frac{x^3}{3}\right|_{-1}^{1} = w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2$$
$$\frac{2}{3} = w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2 \qquad ...(7.36)$$

$$\int_{-1}^{1} x^3\,dx = w_1x_1^3 + w_2x_2^3 + w_3x_3^3$$

$$\left|\frac{x^4}{4}\right|_{-1}^{1} = w_1x_1^3 + w_2x_2^3 + w_3x_3^3$$

$$0 = w_1x_1^3 + w_2x_2^3 + w_3x_3^3 \qquad \text{...(7.37)}$$

$$\int_{-1}^{1} x^4\,dx = w_1x_1^4 + w_2x_2^4 + w_3x_3^4$$

$$\left|\frac{x^5}{5}\right|_{-1}^{1} = w_1x_1^4 + w_2x_2^4 + w_3x_3^4$$

$$\frac{2}{5} = w_1x_1^4 + w_2x_2^4 + w_3x_3^4 \qquad \text{...(7.38)}$$

$$\int_{-1}^{1} x^5\,dx = w_1x_1^5 + w_2x_2^5 + w_3x_3^5$$

$$\left|\frac{x^6}{6}\right|_{-1}^{1} = w_1x_1^5 + w_2x_2^5 + w_3x_3^5$$

$$0 = w_1x_1^5 + w_2x_2^5 + w_3x_3^5 \qquad \text{...(7.39)}$$

Solving Eqs (7.34), (7.35), (7.36), (7.37), (7.38), and (7.39),

$$w_1 = \frac{5}{9}, \quad w_2 = \frac{8}{9}, \quad w_3 = \frac{5}{9}$$

$$x_1 = -\sqrt{\frac{3}{5}},\ x_2 = 0,\ x_3 = \sqrt{\frac{3}{5}}$$

Hence,

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) \qquad \text{...(7.40)}$$

Equation (7.40) is known as the three-point Gaussian quadrature formula. This formula is exact for polynomials up to degree 5.

Example 1

Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ *by one-point, two-point, and three-point Gaussian formulae.*

Solution

$$f(x) = \frac{1}{1+x^2}$$

By the one-point Gaussian formula,

$$\begin{aligned}\int_{-1}^{1} \frac{dx}{1+x^2} &= 2f(0)\\ &= 2\left(\frac{1}{1+0}\right)\\ &= 2\end{aligned}$$

By the two-point Gaussian formula,

$$\begin{aligned}\int_{-1}^{1} \frac{dx}{1+x^2} &= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\\ &= \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}}\\ &= 1.5\end{aligned}$$

By the three-point Gaussian formula,

$$\begin{aligned}\int_{-1}^{1} \frac{dx}{1+x^2} &= \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)\\ &= \frac{5}{9}\left(\frac{1}{1+\frac{3}{5}}\right) + \frac{8}{9}\left(\frac{1}{1+0}\right) + \frac{5}{9}\left(\frac{1}{1+\frac{3}{5}}\right)\\ &= 1.5833\end{aligned}$$

Example 2

Evaluate $\int_0^1 \frac{dt}{1+t}$ *by one-point, two-point, and three-point Gaussian formula.*

Solution

Let $$t = \frac{b-a}{2}x + \frac{b+a}{2}$$

Here, $a = 0, \quad b = 1$

$$t = \frac{1}{2}x + \frac{1}{2} = \frac{1}{2}(x+1)$$

$$dt = \frac{1}{2}\,dx$$

When $t = 0, \quad x = -1$

When $t = 1, \quad x = 1$

$$\int_0^1 \frac{dt}{1+t} = \frac{1}{2}\int_{-1}^{1} \frac{dx}{1+\frac{1}{2}(x+1)}$$

$$= \int_{-1}^{1} \frac{dx}{x+3}$$

$$f(x) = \frac{1}{x+3}$$

By the one-point Gaussian formula,

$$\int_0^1 \frac{dt}{1+t} = \int_{-1}^{1} \frac{dx}{x+3}$$

$$= 2f(0)$$

$$= 2\left(\frac{1}{0+3}\right)$$

$$= 0.6667$$

By the two-point Gaussian formula,

$$\int_0^1 \frac{dt}{1+t} = \int_{-1}^{1} \frac{dx}{x+3}$$

$$= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{-\sqrt{\frac{1}{3}}+3} + \frac{1}{\sqrt{\frac{1}{3}}+3}$$

$$= 0.6923$$

By the three-point Gaussian formula,

$$\int_0^1 \frac{dt}{1+t} = \int_{-1}^{1} \frac{dx}{x+3}$$

$$= \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)$$

$$= \frac{5}{9}\left(\frac{1}{-\sqrt{\frac{3}{5}}+3}\right) + \frac{8}{9}\left(\frac{1}{0+3}\right) + \frac{5}{9}\left(\frac{1}{\sqrt{\frac{3}{5}}+3}\right)$$

$$= 0.6931$$

Example 3

Evaluate the integral $\int_{-2}^{6} (1+x^2)^{\frac{3}{2}} \mathrm{d}x$ *by the Gaussian formula for* $n = 3$.

[Winter 2012]

Solution

Let $$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

Here, $a = -2, \quad b = 6$

$$x = 4t + 2$$

$$\mathrm{d}x = 4\mathrm{d}t$$

When $x = -2, \quad t = -1$

When $x = 6, \quad t = 1$

$$\int_{-2}^{6} (1+x^2)^{\frac{3}{2}}\, \mathrm{d}x = \int_{-1}^{1} \left[1+(4t+2)^2\right]^{\frac{3}{2}} 4\, \mathrm{d}t$$

$$= 4\int_{-1}^{1} (16t^2 + 16t + 5)^{\frac{3}{2}}\, \mathrm{d}t$$

$$f(t) = (16\,t^2 + 16t + 5)^{\frac{3}{2}}$$

By the three-point Gaussian formula,

$$\int_{-2}^{6} (1+x^2)^{\frac{3}{2}}\, \mathrm{d}x = 4\int_{-1}^{1} (16t^2 + 16t + 5)^{\frac{3}{2}}\, \mathrm{d}t$$

$$= 4\left[\frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= 4\left[\frac{5}{9}\left\{16\left(\frac{3}{5}\right) + 16\left(-\sqrt{\frac{3}{5}}\right) + 5\right\}^{\frac{3}{2}} + \frac{8}{9}(5)^{\frac{3}{2}} + \frac{5}{9}\left\{16\left(\frac{3}{5}\right) + 16\left(\sqrt{\frac{3}{5}}\right) + 5\right\}^{\frac{3}{2}}\right]$$

$$= 358.6928$$

Example 4

Evaluate $\int_0^1 e^{-x^2}\,\mathrm{d}x$ *by using the Gaussian quadrature formula with* $n = 3$. **[Winter 2014, Summer 2015]**

Solution

Let $$x = \frac{b-a}{2}t + \frac{b+a}{2}$$

Here, $a = 0, \quad b = 1$

$$x = \frac{1}{2}t + \frac{1}{2} = \frac{1}{2}(t+1)$$

$$\mathrm{d}x = \frac{1}{2}\,\mathrm{d}t$$

When $x = 0, \quad t = -1$

When $x = 1, \quad t = 1$

$$\int_0^1 e^{-x^2}\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{1} e^{-\frac{1}{4}(t+1)^2}\,\mathrm{d}t$$

$$f(x) = e^{-\frac{1}{4}(t+1)^2}$$

By the three-point Gaussian quadrature formula,

$$\begin{aligned}
\int_0^1 e^{-x^2}\,\mathrm{d}x &= \frac{1}{2}\int_{-1}^{1} e^{-\frac{1}{4}(t+1)^2}\,\mathrm{d}t \\
&= \frac{1}{2}\left[\frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)\right] \\
&= \frac{1}{2}\left[\frac{5}{9}e^{-\frac{1}{4}\left(-\sqrt{\frac{3}{5}}+1\right)^2} + \frac{8}{9}e^{-\frac{1}{4}(0+1)^2} + \frac{5}{9}e^{-\frac{1}{4}\left(\sqrt{\frac{3}{5}}+1\right)^2}\right] \\
&= 0.746815
\end{aligned}$$

Example 5

Evaluate $\int_0^{\frac{\pi}{2}} \sin t\,\mathrm{d}t$ *by the two-point Gaussian formula.*

Solution

Let $$t = \frac{b-a}{2}x + \frac{b+a}{2}$$

Here, $a = 0, \quad b = \frac{\pi}{2}$

$$t = \frac{\pi}{4}x + \frac{\pi}{4} = \frac{\pi}{4}(x+1)$$

$$\mathrm{d}t = \frac{\pi}{4}\,\mathrm{d}x$$

When $t = 0, \quad x = -1$

When $t = \frac{\pi}{2}, \quad x = 1$

$$\int_0^{\frac{\pi}{2}} \sin t \,\mathrm{d}t = \frac{\pi}{4}\int_{-1}^{1} \sin\frac{\pi}{4}(x+1)\,\mathrm{d}x$$

$$f(x) = \sin\frac{\pi}{4}(x+1)$$

By the two-point Gaussian formula,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin t \,\mathrm{d}t &= \frac{\pi}{4}\int_{-1}^{1} \sin\frac{\pi}{4}(x+1)\,\mathrm{d}x \\ &= \frac{\pi}{4}\left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right] \\ &= \frac{\pi}{4}\left[\sin\frac{\pi}{4}\left(-\frac{1}{\sqrt{3}}+1\right) + \sin\frac{\pi}{4}\left(\frac{1}{\sqrt{3}}+1\right)\right] \\ &= 0.99847\end{aligned}$$

EXERCISE 7.2

Evaluate the following integrals by using Gaussian quadrature formulae:

1. $\int_0^1 e^x \,\mathrm{d}x$ (2 points)

[Ans.: 2.342696**]**

2. $\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^4}}$ (2 points)

[Ans.: 1.311028**]**

3. $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\sin^2\theta}}$ (2 points)

[**Ans.:** 1.226]

4. $\int_0^{\frac{\pi}{2}} \log(1+x)\, dx$ (2 points)

[**Ans.:** 0.858]

5. $\int_0^{3} x^2 \cos x\, dx$ (3 points)

[**Ans.:** –4.936]

6. $\int_1^{2} e^x\, dx$ (3 points)

[**Ans.:** 4.67077]

Points to Remember

Newton–Cotes Quadrature Formula

$$\int_{x_0}^{x_0+nh} f(x)\,dx = hn\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

Trapezoidal Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \cdots + y_{n-1}\right)\right]$$

Simpson's 1/3 Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[\left(y_0 + y_n\right) + 4\left(y_1 + y_3 + \cdots + y_{n-1}\right) + 2\left(y_2 + y_4 + \cdots + y_{n-2}\right)\right]$$

Simpson's 3/8 Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\Big[\left(y_0 + y_n\right) + 2\left(y_3 + y_6 + \cdots + y_{n-3}\right)$$
$$+3\left(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}\right)\Big]$$

Gaussian Quadrature Formulae

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \sum_{i=1}^{n} w_i f(x_i)$$

1. One-point Gaussian Quadrature Formula

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = 2f(0)$$

2. Two-Point Gaussian Quadrature Formula

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

3. Three-Point Gaussian Quadrature Formula

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

CHAPTER 8

Solutions of a System of Linear Equations

Chapter Outline

8.1 Introduction
8.2 Solutions of a System of Linear Equations
8.3 Elementary Transformations
8.4 Numerical Methods for Solution of a System of Linear Equations
8.5 Gauss Elimination Method
8.6 Gauss Elimination Method with Partial Pivoting
8.7 Gauss–Jordan Method
8.8 Gauss–Jacobi Method
8.9 Gauss–Siedel Method

8.1 INTRODUCTION

A system of m nonhomogenous linear equations in n variables $x_1, x_2, ..., x_n$ or simply a linear system, is a set of m linear equations, each in n variables. A linear system is represented by

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Writing these equations in matrix form,

$$A\mathbf{x} = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is called the *coefficient matrix* of order $m \times n$,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ is any vector of order } n \times 1$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \text{ is any vector of order } m \times 1$$

8.2 SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS

For a system of m linear equations in n variables, there are three possibilities of the solutions to the system:

(i) The system has a unique solution.
(ii) The system has infinite solutions.
(iii) The system has no solution.

When the system of linear equations has one or more solutions, the system is said to be consistent, otherwise it is inconsistent.

The matrix
$$[A:B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the *augmented matrix* of the given system of linear equations.

To solve a system of linear equations, elementary transformations are used to reduce the augmented matrix to echelon form.

8.3 ELEMENTARY TRANSFORMATIONS

Elementary transformation is any one of the following operations on a matrix.

(i) The interchange of any two rows (or columns)
(ii) The multiplication of the elements of any row (or column) by any nonzero number
(iii) The addition or subtraction of k times the elements of a row (or column) to the corresponding elements of another row (or column), where $k \neq 0$

Symbols to be used for elementary transformation:

(i) R_{ij}: Interchange of i^{th} and j^{th} row
(ii) kR_i: Multiplication of i^{th} row by a nonzero number k
(iii) $R_i + kR_j$: Addition of k times the j^{th} row to the i^{th} row

The corresponding column transformations are denoted by C_{ij}, kC_i, and $C_i + kC_j$ respectively.

8.3.1 Elementary Matrices

A matrix obtained from a unit matrix by subjecting it to any row or column transformation is called an elementary matrix.

8.3.2 Equivalence of Matrices

If B be an $m \times n$ matrix obtained from an $m \times n$ matrix by elementary transformation of A then A is equivalent to B. Symbolically, we can write $A \sim B$.

8.3.3 Echelon Form of a Matrix

A matrix A is said to be in echelon form if it satisfies the following properties:

(i) Every zero row of the matrix A occurs below a nonzero row.

(ii) In a nonzero row the first nonzero number from the left is 1. This is called a leading 1.

(iii) For each nonzero row, the leading 1 appears to the right of any leading 1 in preceding rows.

The following matrices are in echelon form:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

8.4 NUMERICAL METHODS FOR SOLUTION OF A SYSTEM OF LINEAR EQUATIONS

There are two methods to solve linear algebraic equations:

(i) Direct methods

(ii) Iterative methods

8.4.1 Direct Methods

Direct methods transform the original equations into equivalent equations that can be solved easily. The transformation of the original equations is carried out by applying elementary row transformations to the augmented matrix of the system of equations.

We will discuss two direct methods:

(i) Gauss elimination method

(ii) Gauss–Jordan method

8.4.2 Iterative Methods

The direct methods lead to exact solutions in many cases but are subject to errors due to roundoff and other factors. In the iterative method, an approximation to the true solution is assumed initially to start the method. By applying the method repeatedly, better and better approximations are obtained. For large systems, iterative methods are faster than direct methods and round-off errors are also smaller. Any error made at any stage of computation gets automatically corrected in the subsequent steps.

We will discuss two iterative methods.

(i) Gauss–Jacobi method

(ii) Gauss–Seidel method

8.5 GAUSS ELIMINATION METHOD

This method solves a given system of equations by transforming the augmented matrix to an echelon form. The corresponding linear system of equations is then solved for the unknowns by back substitution.

Consider the system of equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The matrix form of the system is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Reducing the augmented matrix to echelon form by using elementary row transformations,

$$[A:B] \xrightarrow[\text{row transformations}]{\text{elementary}} \left[\begin{array}{ccc:c} c_{11} & c_{12} & c_{13} & d_1 \\ 0 & c_{22} & c_{23} & d_2 \\ 0 & 0 & c_{33} & d_3 \end{array}\right]$$

The corresponding system of equations is

$$c_{11}x + c_{12}y + c_{13}z = d_1$$

$$c_{22}y + c_{23}z = d_2$$
$$c_{33}z = d_3$$

The solution of the system is obtained by solving these equations by back substitution.

Working Rule

(i) Write the matrix form of the system of equations.
(ii) Write the augmented matrix.
(iii) Obtain the echelon form of the augmented matrix by using elementary row transformations.
(iv) Write the corresponding linear system of equations from the echelon form.
(v) Solve the corresponding linear system of equations by back substitution.

Example 1

Solve the following system of equations:

$$\begin{aligned} x + 3y + 2z &= 5 \\ 2x + 4y - 6z &= -4 \\ x + 5y + 3z &= 10 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 10 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 1 & 3 & 2 & 5 \\ 2 & 4 & -6 & -4 \\ 1 & 5 & 3 & 10 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$R_2 - 2R_1, R_3 - R_1$$

$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & 3 & 2 & 5 \\ 0 & -2 & -10 & -14 \\ 0 & 2 & 1 & 5 \end{array}\right]$$

$$\left(-\frac{1}{2}\right)R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 3 & 2 & 5 \\ 0 & 1 & 5 & 7 \\ 0 & 2 & 1 & 5 \end{array}\right]$$

$$R_3 - 2R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 3 & 2 & 5 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & -9 & -9 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x + 3y + 2z &= 5 \\ y + 5z &= 7 \\ -9z &= -9 \end{aligned}$$

Solving these equations by back substitution,

$$\begin{aligned} &z = 1 \\ &y = 7 - 5z = 7 - 5(1) = 2 \\ &x = 5 - 3y - 2z = 5 - 3(2) - 2(1) = -3 \end{aligned}$$

Hence, the solution is

$$x = -3,\ y = 2,\ z = 1$$

Example 2

Solve the following system of equations:

$$\begin{aligned} 2x + y + z &= 10 \\ 3x + 2y + 3z &= 18 \\ x + 4y + 9z &= 16 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \\ 16 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B]=\left[\begin{array}{ccc:c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$R_{13}$$

$$[A:B]\sim\left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 3 & 2 & 3 & 18 \\ 2 & 1 & 1 & 10 \end{array}\right]$$

$$R_2-3R_1, R_3-2R_1$$

$$\sim\left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & -10 & -24 & -30 \\ 0 & -7 & -17 & -22 \end{array}\right]$$

$$\left(-\frac{1}{10}\right)R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & 1 & \frac{24}{10} & 3 \\ 0 & -7 & -17 & -22 \end{array}\right]$$

$$R_3+7R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & 1 & \frac{24}{10} & 3 \\ 0 & 0 & -\frac{1}{5} & -1 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x+4y+\quad 9z &= 16 \\ y+\frac{24}{10}z &= \ \ 3 \\ -\ \ \frac{1}{5}z &= -1 \end{aligned}$$

Solving these equations by back substitution,

$$z = 5$$

$$y = 3 - \frac{24}{10}z = 3 - \frac{24}{10}(5) = -9$$

$$x = 16 - 4y - 9z = 16 - 4(-9) - 9(5) = 7$$

Hence, the solution is

$$x = 7,\ y = -9,\ z = 5$$

Example 3

Solve the following system of equations:

$$\begin{aligned} 6x - y - z &= 19 \\ 3x + 4y + z &= 26 \\ x + 2y + 6z &= 22 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 6 & -1 & -1 \\ 3 & 4 & 1 \\ 1 & 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 19 \\ 26 \\ 22 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 6 & -1 & -1 & 19 \\ 3 & 4 & 1 & 26 \\ 1 & 2 & 6 & 22 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$R_{13}$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & 6 & 22 \\ 3 & 4 & 1 & 26 \\ 6 & -1 & -1 & 19 \end{array}\right]$$

$$R_2 - 3R_1,\ R_3 - 6R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 6 & 22 \\ 0 & -2 & -17 & -40 \\ 0 & -13 & -37 & -113 \end{array}\right]$$

$$\left(-\frac{1}{2}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 6 & 22 \\ 0 & 1 & \frac{17}{2} & 20 \\ 0 & -13 & -37 & -113 \end{array}\right]$$

$$R_3 + 13R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 6 & 22 \\ 0 & 1 & \frac{17}{2} & 20 \\ 0 & 0 & \frac{147}{2} & 147 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x + 2y + \quad 6z &= 22 \\ y + \frac{17}{2}z &= 20 \\ \frac{147}{2}z &= 147 \end{aligned}$$

Solving these equations by back substitution,

$$z = 2$$

$$y = 20 - \frac{17}{2}z = 20 - \frac{17}{2}(2) = 3$$

$$x = 22 - 2y - 6z = 22 - 2(3) - 6(2) = 4$$

Hence, the solution is

$$x = 4,\ y = 3,\ z = 2$$

Example 4

Solve the following system of equations:

$$\begin{aligned} 5x + 5y + 2z &= 12 \\ 2x + 4y + 5z &= 2 \\ 39x + 43y + 45z &= 74 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 5 & 5 & 2 \\ 2 & 4 & 5 \\ 39 & 43 & 45 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 2 \\ 74 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 5 & 5 & 2 & 12 \\ 2 & 4 & 5 & 2 \\ 39 & 43 & 45 & 74 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$\left(\frac{1}{5}\right)R_1$$

$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & 1 & \frac{2}{5} & \frac{12}{5} \\ 2 & 4 & 5 & 2 \\ 39 & 43 & 45 & 74 \end{array}\right]$$

$$R_2 - 2R_1, R_3 - 39R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & \frac{2}{5} & \frac{12}{5} \\ 0 & 2 & \frac{21}{5} & -\frac{14}{5} \\ 0 & 4 & \frac{147}{5} & -\frac{98}{5} \end{array}\right]$$

$$\left(\frac{1}{2}\right)R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 1 & \frac{2}{5} & \frac{12}{5} \\ 0 & 1 & \frac{21}{10} & -\frac{14}{10} \\ 0 & 4 & \frac{147}{5} & -\frac{98}{5} \end{array}\right]$$

$R_3 - 4R_2$

$$\sim\left[\begin{array}{ccc:c} 1 & 1 & \frac{2}{5} & \frac{12}{5} \\ 0 & 1 & \frac{21}{10} & -\frac{14}{10} \\ 0 & 0 & 21 & -14 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x + y + \frac{2}{5}z &= \frac{12}{5} \\ y + \frac{21}{10}z &= -\frac{14}{10} \\ 21z &= -14 \end{aligned}$$

Solving these equations by back substitution,

$$\begin{aligned} z &= -\frac{14}{21} = -\frac{2}{3} \\ y &= -\frac{14}{10} - \frac{21}{10}z = -\frac{14}{10} - \frac{21}{10}\left(-\frac{2}{3}\right) = 0 \\ x &= \frac{12}{5} - y - \frac{2}{5}z = \frac{12}{5} - \frac{2}{5}\left(-\frac{2}{3}\right) = \frac{8}{3} \end{aligned}$$

Hence, the solution is

$$x = \frac{8}{3},\ y = 0,\ z = -\frac{2}{3}$$

Example 5

Use the Gauss elimination method to solve the following equations:

$$\begin{aligned} x + 4y - z &= -5 \\ x + y - 6z &= -12 \\ 3x - y - z &= 4 \end{aligned}$$

[Summer 2015]

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 4 & -1 & -5 \\ 1 & 1 & -6 & -12 \\ 3 & -1 & -1 & 4 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$R_2 - R_1,\ R_3 - 3R_1$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & -7 \\ 0 & -13 & 2 & 19 \end{array}\right]$$

$$\left(-\frac{1}{3}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 4 & -1 & -5 \\ 0 & 1 & \frac{5}{3} & \frac{7}{3} \\ 0 & -13 & 2 & 19 \end{array}\right]$$

$$R_3 + 13R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 4 & -1 & -5 \\ 0 & 1 & \frac{5}{3} & \frac{7}{3} \\ 0 & 0 & \frac{71}{3} & \frac{148}{3} \end{array}\right]$$

The corresponding system of equations is

$$x + 4y - z = -5$$

$$y + \frac{5}{3}z = \frac{7}{3}$$

$$\frac{71}{3}z = \frac{148}{3}$$

Solving these equations by back substitution,

$$z = \frac{148}{71}$$

$$y = \frac{7}{3} - \frac{5}{3}z = \frac{7}{3} - \frac{5}{3}\left(\frac{148}{71}\right) = -\frac{81}{71}$$

$$x = -5 - 4y + z = -5 - 4\left(-\frac{81}{71}\right) + \frac{148}{71} = \frac{117}{71}$$

Hence, the solution is

$$x = \frac{117}{71}, \quad y = -\frac{81}{71}, \quad z = \frac{148}{71}$$

Example 6

Solve the following system of linear equations:

$$8y + 2z = -7$$
$$3x + 5y + 2z = 8$$
$$6x + 2y + 8z = 26$$

[Summer 2014]

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 0 & 8 & 2 \\ 3 & 5 & 2 \\ 6 & 2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 8 \\ 26 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc|c} 0 & 8 & 2 & -7 \\ 3 & 5 & 2 & 8 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$[A : B] \overset{R_{12}}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

$$\left(\frac{1}{3}\right)R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{5}{3} & \frac{2}{3} & \frac{8}{3} \\ 0 & 8 & 2 & -7 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

$$R_3 - 6R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{5}{3} & \frac{2}{3} & \frac{8}{3} \\ 0 & 8 & 2 & -7 \\ 0 & -8 & 4 & 10 \end{array}\right]$$

$$\left(\frac{1}{8}\right)R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{5}{3} & \frac{2}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{4} & -\frac{7}{8} \\ 0 & -8 & 4 & 10 \end{array}\right]$$

$$R_3 + 8R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & \frac{5}{3} & \frac{2}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{4} & -\frac{7}{8} \\ 0 & 0 & 6 & 3 \end{array}\right]$$

The corresponding system of equations is

$$x + \frac{5}{3}y + \frac{2}{3}z = \frac{8}{3}$$

$$y + \frac{1}{4}z = -\frac{7}{8}$$

$$6z = 3$$

Solving these equations by back substitution,

$$z = \frac{1}{2}$$

$$y = -\frac{7}{8} - \frac{1}{4}z = -\frac{7}{8} - \frac{1}{4}\left(\frac{1}{2}\right) = -1$$

$$x = \frac{8}{3} - \frac{5}{3}y - \frac{2}{3}z = \frac{8}{3} - \frac{5}{3}(-1) - \frac{2}{3}\left(\frac{1}{2}\right) = 4$$

Hence, the solution is

$$x = 4, \quad y = -1, \quad z = \frac{1}{2}$$

8.6 GAUSS ELIMINATION METHOD WITH PARTIAL PIVOTING

For a large system of linear equations, the Gaussian elimination method can involve a large number of arithmetic computations, each of which can produce rounding errors. This is due to the fact that every computation is dependent on previous results.

Consequently, an error in the early step will tend to propagate, i.e., it will cause errors in subsequent steps, and the final solution will become inaccurate. The rounding error can be reduced by the Gaussian elimination method with partial pivoting.

Consider the system of equations:

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

The matrix form of the system is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

For the partial pivoting process, the left column is searched for the largest absolute-value entry. This entry is called the *pivot*. The row interchange is performed, if necessary, to bring the pivot in the first row. The first row is divided by the pivot and elementary row operations are used to reduce the remaining entries in the first column to zero. The completion of these steps is called a *pass*. After performing the first pass, the first row and first column are ignored and the process is repeated on the remaining submatrix. This process is continued until the matrix is in the row echelon form.

The term *partial* in partial pivoting refers to the fact that in each pivot search, only entries in the left column of the matrix or submatrix are considered. This search can be extended to include every entry in the coefficient matrix or submatrix. The resulting method is called the *Gaussian elimination method with complete pivoting*. Generally, partial pivoting is preferred because complete pivoting becomes very complicated.

Example 1

Solve the following system of equations using partial pivoting by the Gauss elimination method:

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &= 6 \\ 4x_1 + 2x_2 + 3x_3 &= 4 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

[Summer 2015]

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 2 & 2 & 1 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 2 & 2 & 1 & 6 \\ 4 & 2 & 3 & 4 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

In the left column, 4 is the pivot because it is the entry that has the largest absolute value.

$$R_{12}$$

$$[A : B] \sim \left[\begin{array}{ccc:c} 4 & 2 & 3 & 4 \\ 2 & 2 & 1 & 6 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\left(\frac{1}{4}\right) R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & \frac{1}{2} & \frac{3}{4} & 1 \\ 2 & 2 & 1 & 6 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 - 2R_1,\ R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{4} & 1 \\ 0 & 1 & -\frac{1}{2} & 4 \\ 0 & \frac{1}{2} & \frac{1}{4} & -1 \end{array}\right]$$

This completes the first pass. For the second pass, the pivot is 1 in the submatrix formed by deleting the first row and first column.

$$R_3 - \frac{1}{2}R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{4} & 1 \\ 0 & 1 & -\frac{1}{2} & 4 \\ 0 & 0 & \frac{1}{2} & -3 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{3}{4}x_3 &= 1 \\ x_2 - \frac{1}{2}x_3 &= 4 \\ \frac{1}{2}x_3 &= -3 \end{aligned}$$

Solving these equations by back substitution,

$$\begin{aligned} x_3 &= -6 \\ x_2 &= 4 + \frac{1}{2}x_3 = 4 + \frac{1}{2}(-6) = 1 \\ x_1 &= 1 - \frac{1}{2}x_2 - \frac{3}{4}x_3 = 1 - \frac{1}{2}(1) - \frac{3}{4}(-6) = 5 \end{aligned}$$

Hence, the solution is

$$x_1 = 5, \qquad x_2 = 1, \qquad x_3 = -6$$

Example 2

Solve the following system of equations using the Gauss elimination method with partial pivoting.

$$\begin{aligned} x + y + z &= 7 \\ 3x + 3y + 4z &= 24 \\ 2x + y + 3z &= 16 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 24 \\ 16 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 7 \\ 3 & 3 & 4 & 24 \\ 2 & 1 & 3 & 16 \end{array}\right]$$

In the left column, 3 is the pivot because it is the entry that has largest absolute value.

$$R_{12}$$

$$[A : B] \sim \left[\begin{array}{ccc:c} 3 & 3 & 4 & 24 \\ 1 & 1 & 1 & 7 \\ 2 & 1 & 3 & 16 \end{array}\right]$$

$$\left(\frac{1}{3}\right) R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & \frac{4}{3} & 8 \\ 1 & 1 & 1 & 7 \\ 2 & 1 & 3 & 16 \end{array}\right]$$

$$R_2 - R_1, \ \ R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & \frac{4}{3} & 8 \\ 0 & 0 & -\frac{1}{3} & -1 \\ 0 & -1 & \frac{1}{3} & 0 \end{array}\right]$$

This completes the first pass. For the second pass, the pivot is –1 in the submatrix formed by deleting the first row and first column.

$$\overset{R_{23}}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & \frac{4}{3} & 8 \\ 0 & -1 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} & -1 \end{array}\right]$$

$$\overset{(-1)\,R_2}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & \frac{4}{3} & 8 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} & -1 \end{array}\right]$$

The corresponding system of equations is

$$x + y + \frac{4}{3}z = 8$$

$$y - \frac{1}{3}z = 0$$

$$-\frac{1}{3}z = -1$$

Solving these equations by back substitution,

$$z = 3$$

$$y = \frac{1}{3}z = \frac{1}{3}(3) = 1$$

$$x = 8 - y - \frac{4}{3}z = 8 - 1\frac{4}{3}(3) = 3$$

Hence, the solution is

$$x = 3, \qquad y = 1, \qquad z = 3$$

EXERCISE 8.1

Solve the following systems of equations by the Gauss elimination method:

1. $x - y + z = 1$

$-3x + 2y - 3z = -6$

$2x - 5y + 4z = 5$

$\left[\textbf{Ans.:}\ x = -2, y = 3, z = 6\right]$

2. $x + 3y - 2z = 5$
$2x + y - 3z = 1$
$3x + 2y - z = 6$

$$\left[\textbf{Ans.:}\ x = 1, y = 2, z = 1\right]$$

3. $6x + 3y + 6z = 30$
$2x + 3y + 3z = 17$
$x + 2y + 2z = 11$

$$\left[\textbf{Ans.:}\ x = 1, y = 2, z = 3\right]$$

4. $2x + y + z = 4$
$3y - 3z = 0$
$-y + 2z = 1$

$$\left[\textbf{Ans.:}\ x = 1, y = 1, z = 1\right]$$

5. $2x + 2y + z = 12$
$3x + 2y + 2z = 8$
$5x + 10y - 8z = 10$

$$\left[\textbf{Ans.:}\ x = -12.75, y = 14.375, z = 8.75\right]$$

6. $3x + 4y + 5z = 18$
$2x - y + 8z = 13$
$5x - 2y + 7z = 20$

$$\left[\textbf{Ans.:}\ x = 3, y = 1, z = 1\right]$$

7. $2x + 6y - z = -12$
$5x - y + z = 11$
$4x - y + 3z = 10$

$$\left[\textbf{Ans.:}\ x = \frac{113}{69}, y = -\frac{172}{69}, z = \frac{22}{69}\right]$$

8.7 GAUSS–JORDAN METHOD

This method is a modification of the Gauss elimination method. This method solves a given system of equations by transforming the coefficient matrix into a unit matrix.

Consider the system of equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The matrix form of the system is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Applying elementary row transformations to augmented matrix to reduce coefficient matrix to unit matrix,

$$[A : B] \xrightarrow[\text{row transformations}]{\text{elementary}} \left[\begin{array}{ccc:c} 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & d_3 \end{array}\right]$$

The corresponding system of equations is

$$x = d_1$$
$$y = d_2$$
$$z = d_3$$

Hence, the solution is

$$x = d_1,\ y = d_2,\ z = d_3$$

Working Rule

(i) Write the matrix form of the system of equations.
(ii) Write the augmented matrix.
(iii) Reduce the coefficient matrix to unit matrix by applying elementary row transformations to the augmented matrix.
(iv) Write the corresponding linear system of equations to obtain the solution.

Example 1

Solve the following system of equations:

$$x + 3y + 2z = 17$$

$$x + 2y + 3z = 16$$

$$2x - y + 4z = 13$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ 16 \\ 13 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 3 & 2 & 17 \\ 1 & 2 & 3 & 16 \\ 2 & -1 & 4 & 13 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$R_2 - R_1, R_3 - 2R_1$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 3 & 2 & 17 \\ 0 & -1 & 1 & -1 \\ 0 & -7 & 0 & -21 \end{array}\right]$$

$$(-1)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 3 & 2 & 17 \\ 0 & 1 & -1 & 1 \\ 0 & -7 & 0 & -21 \end{array}\right]$$

$$R_1 - 3R_2, R_3 + 7R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 5 & 14 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -7 & -14 \end{array}\right]$$

$$\left(-\frac{1}{7}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 5 & 14 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_1 - 5R_3, R_2 + R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x &= 4 \\ y &= 3 \\ z &= 2 \end{aligned}$$

Hence, the solution is

$$x = 4, y = 3, z = 2$$

Example 2

Solve the following system of equations:

$$\begin{aligned} 3x - 2y + 5z &= 2 \\ 4x + y + 2z &= 4 \\ 2x - y + 4z &= 7 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 3 & -2 & 5 \\ 4 & 1 & 2 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 3 & -2 & 5 & 2 \\ 4 & 1 & 2 & 4 \\ 2 & -1 & 4 & 7 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$R_1 - R_3$$

$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & -5 \\ 4 & 1 & 2 & 4 \\ 2 & -1 & 4 & 7 \end{array}\right]$$

$$R_2 - 4R_1, R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & -5 \\ 0 & 5 & -2 & 24 \\ 0 & 1 & 2 & 17 \end{array}\right]$$

$$\underset{\sim}{R_{23}}\left[\begin{array}{ccc:c} 1 & -1 & 1 & -5 \\ 0 & 1 & 2 & 17 \\ 0 & 5 & -2 & 24 \end{array}\right]$$

$$R_1 + R_2, R_3 - 5R_2$$
$$\sim\left[\begin{array}{ccc:c} 1 & 0 & 3 & 12 \\ 0 & 1 & 2 & 17 \\ 0 & 0 & -12 & -61 \end{array}\right]$$

$$\left(-\frac{1}{12}\right)R_3$$
$$\sim\left[\begin{array}{ccc:c} 1 & 0 & 3 & 12 \\ 0 & 1 & 2 & 17 \\ 0 & 0 & 1 & \frac{61}{12} \end{array}\right]$$

$$R_1 - 3R_3, R_2 - 2R_3$$
$$\sim\left[\begin{array}{ccc:c} 1 & 0 & 0 & -\frac{13}{4} \\ 0 & 1 & 0 & \frac{41}{6} \\ 0 & 0 & 1 & \frac{61}{12} \end{array}\right]$$

The corresponding system of equations is

$$x = -\frac{13}{4}$$
$$y = \frac{41}{6}$$
$$z = \frac{61}{12}$$

Hence, the solution is

$$x = -\frac{13}{4},\ y = \frac{41}{6},\ z = \frac{61}{12}$$

Example 3

Solve the following system of equations:

$$\begin{aligned} x - 2y \quad\quad &= -\ 4 \\ -\ 5y + \ z &= -\ 9 \\ 4x \quad\quad -\ 3z &= -\ 10 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -5 & 1 \\ 4 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -9 \\ -10 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 1 & -2 & 0 & -4 \\ 0 & -5 & 1 & -9 \\ 4 & 0 & -3 & -10 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$R_3 - 4R_1$$

$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & -2 & 0 & -4 \\ 0 & -5 & 1 & -9 \\ 0 & 8 & -3 & 6 \end{array}\right]$$

$$\left(-\frac{1}{5}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & -2 & 0 & -4 \\ 0 & 1 & -\frac{1}{5} & \frac{9}{5} \\ 0 & 8 & -3 & 6 \end{array}\right]$$

$$R_1 + 2R_2, R_3 - 8R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -\frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{9}{5} \\ 0 & 0 & -\frac{7}{5} & -\frac{42}{5} \end{array}\right]$$

$$\left(-\frac{5}{7}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -\frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{9}{5} \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$R_1 + \left(\frac{2}{5}\right)R_3, R_2 + \left(\frac{1}{5}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

The corresponding system of equations is

$$x = 2$$
$$y = 3$$
$$z = 6$$

Hence, the solution is

$$x = 2, y = 3, z = 6$$

Example 4

Solve the following system of equations:

$$\begin{aligned} 2x - 6y + 8z &= 24 \\ 5x + 4y - 3z &= 2 \\ 3x + y + 2z &= 16 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 2 & -6 & 8 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \\ 16 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc:c} 2 & -6 & 8 & 24 \\ 5 & 4 & -3 & 2 \\ 3 & 1 & 2 & 16 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$\left(\frac{1}{2}\right)R_1$$

$$[A : B] \sim \left[\begin{array}{ccc:c} 1 & -3 & 4 & 12 \\ 5 & 4 & -3 & 2 \\ 3 & 1 & 2 & 16 \end{array}\right]$$

$$R_2 - 5R_1, R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & -3 & 4 & 12 \\ 0 & 19 & -23 & -58 \\ 0 & 10 & -10 & -20 \end{array}\right]$$

$$\left(\frac{1}{19}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & -3 & 4 & 12 \\ 0 & 1 & -\frac{23}{19} & -\frac{58}{19} \\ 0 & 10 & -10 & -20 \end{array}\right]$$

$$R_1 + 3R_2, R_3 - 10R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & \frac{7}{19} & \frac{54}{19} \\ 0 & 1 & -\frac{23}{19} & -\frac{58}{19} \\ 0 & 0 & \frac{40}{19} & \frac{200}{19} \end{array}\right]$$

$$\left(\frac{19}{40}\right)R_3$$

$$\sim\left[\begin{array}{ccc:c} 1 & 0 & \frac{7}{19} & \frac{54}{19} \\ 0 & 1 & -\frac{23}{19} & -\frac{58}{19} \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$R_1-\left(\frac{7}{19}\right)R_3, R_2+\left(\frac{23}{19}\right)R_3$$

$$\sim\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

The corresponding system of equations is

$$x = 1$$
$$y = 3$$
$$z = 5$$

Hence, the solution is

$$x = 1, y = 3, z = 5$$

Example 5

Solve the following system of linear equations:

$$\begin{aligned} 2x + 5y - 3z &= 1 \\ 5x + y + 4z &= 2 \\ 7x + 3y + z &= 4 \end{aligned}$$

Solution

The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 2 & 5 & -3 \\ 5 & 1 & 4 \\ 7 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B]=\left[\begin{array}{ccc|c} 2 & 5 & -3 & 1 \\ 5 & 1 & 4 & 2 \\ 7 & 3 & 1 & 4 \end{array}\right]$$

Applying elementary row transformations to the augmented matrix,

$$[A:B] \underset{\left(\frac{1}{2}\right)R_1}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 5 & 1 & 4 & 2 \\ 7 & 3 & 1 & 4 \end{array}\right]$$

$$\underset{R_2-5R_1,\ R_3-7R_1}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{23}{2} & \frac{23}{2} & -\frac{1}{2} \\ 0 & -\frac{29}{2} & \frac{23}{2} & \frac{1}{2} \end{array}\right]$$

$$\underset{\left(-\frac{2}{23}\right)R_2}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & -1 & \frac{1}{23} \\ 0 & -\frac{29}{2} & \frac{23}{2} & \frac{1}{2} \end{array}\right]$$

$$\underset{R_1-\frac{5}{2}R_2,\ R_3+\frac{29}{2}R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{9}{23} \\ 0 & 1 & -1 & \frac{1}{23} \\ 0 & 0 & -3 & \frac{26}{23} \end{array}\right]$$

$$\left(-\frac{1}{3}\right)R_3$$

$$\sim\begin{bmatrix} 1 & 0 & 1 & \vdots & \frac{9}{23} \\ 0 & 1 & -1 & \vdots & \frac{1}{23} \\ 0 & 0 & 1 & \vdots & -\frac{26}{69} \end{bmatrix}$$

$$R_1 - R_3,\ \ R_2 + R_3$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{53}{69} \\ 0 & 1 & 0 & \vdots & -\frac{1}{3} \\ 0 & 0 & 1 & \vdots & -\frac{26}{69} \end{bmatrix}$$

The corresponding system of equations is

$$x = \frac{53}{69}$$

$$y = -\frac{1}{3}$$

$$z = -\frac{26}{69}$$

Hence, the solution is

$$x = \frac{53}{69},\ y = -\frac{1}{3},\ z = -\frac{26}{69}$$

EXERCISE 8.2

Solve the following systems of equations by using the Gauss–Jordan method:

1. $x + 2y + z = 3$

$2x + 3y + 3z = 10$

$3x - y + 2z = 13$

$$\left[\textbf{Ans.: } x = 2, y = -1, z = 3\right]$$

2. $$\begin{aligned} 2x + 3y - z &= 5 \\ 4x + 4y - 3z &= 3 \\ 2x - 3y + 2z &= 2 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x = 1, y = 2, z = 3\right]$$

3. $$\begin{aligned} 10x + y + z &= 12 \\ 2x + 10y + z &= 13 \\ x + y + 5z &= 7 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x = 1, y = 1, z = 1\right]$$

4. $$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 11 \\ 4x_1 - 2x_2 + 3x_3 &= 8 \\ -2x_1 + 2x_2 - x_3 &= -6 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x_1 = 3, x_2 = -1, x_3 = -2\right]$$

5. $$\begin{aligned} 2x_1 + 6x_2 + x_3 &= 7 \\ x_1 + 2x_2 - x_3 &= -1 \\ 5x_1 + 7x_2 - 4x_3 &= 9 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x_1 = 10, x_2 = -3, x_3 = 5\right]$$

6. $$\begin{aligned} 2x + y + 4z &= 12 \\ 8x - 3y + 2z &= 20 \\ 4x + 11y - z &= 33 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x = 3, y = 2, z = 1\right]$$

7. $$\begin{aligned} x + y + z &= 1 \\ 4x + 3y - z &= 6 \\ 3x + 5y + 3z &= 4 \end{aligned}$$

$$\left[\textbf{Ans.:}\ x = 1, y = \frac{1}{2}, z = -\frac{1}{2}\right]$$

8.8 GAUSS—JACOBI METHOD

This method is applicable to the system of equations in which leading diagonal elements of the coefficient matrix are dominant (large in magnitude) in their respective rows.

Consider the system of equations

$$\left.\begin{aligned} a_{11}\, x + a_{12}\, y + a_{13}\, z &= b_1 \\ a_{21}\, x + a_{22}\, y + a_{23}\, z &= b_2 \\ a_{31}\, x + a_{32}\, y + a_{33}\, z &= b_3 \end{aligned}\right\} \qquad \ldots (8.1)$$

where $|a_{11}|$, $|a_{22}|$, $|a_{33}|$ are large as compared to the other coefficients in the corresponding row and satisfy the condition of convergence as follows:

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

Rewriting the equations for x, y, and z respectively,

$$\left.\begin{aligned} x &= \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z) \\ y &= \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z) \\ z &= \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y) \end{aligned}\right\} \qquad \ldots (8.2)$$

Iteration 1

Assuming $x = x_0$, $y = y_0$, $z = z_0$ as initial approximation and substituting in Eq. (8.2),

$$x_1 = \frac{1}{a_{11}}\left(b_1 - a_{12}y_0 - a_{13}z_0\right)$$

$$y_1 = \frac{1}{a_{22}}\left(b_2 - a_{21}x_0 - a_{23}z_0\right)$$

$$z_1 = \frac{1}{a_{33}}\left(b_3 - a_{31}x_0 - a_{32}y_0\right)$$

Again substituting these values of x, y, z in Eq. (8.2), the next approximation is obtained.

The above iteration process is continued until two successive approximations are nearly equal.

Working Rule

(i) Arrange the equations in such a manner that the leading diagonal elements are large in magnitude in their respective rows satisfying the conditions

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.

(iii) Start the iteration 1 by assuming the initial values of (x, y, z) as (x_0, y_0, z_0) and obtain (x_1, y_1, z_1).

(iv) Start the iteration 2 by putting $x = x_1$, $y = y_1$, $z = z_1$ in equations of x, y, z and obtain (x_2, y_2, z_2).

(v) The above process is repeated for the next iterations and it continues until two successive approximations are nearly equal.

Example 1

Solve the following system of equations:

$$\begin{aligned} 6x + 2y - z &= 4 \\ x + 5y + z &= 3 \\ 2x + y + 4z &= 27 \end{aligned}$$

Solution

Rewriting the equations,

$$\left.\begin{aligned} x &= \frac{1}{6}(4 - 2y + z) \\ y &= \frac{1}{5}(3 - x - z) \\ z &= \frac{1}{4}(27 - 2x - y) \end{aligned}\right\} \qquad \ldots (1)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and putting in Eq. (1),

$$x_1 = \frac{2}{3} = 0.67$$

$$y_1 = \frac{3}{5} = 0.6$$

$$z_1 = \frac{27}{4} = 6.75$$

Iteration 2: Putting x_1, y_1, z_1 in Eq. (1),

$$x_2 = \frac{1}{6}\left[4 - 2(0.6) + 6.75\right] = 1.59$$

$$y_2 = \frac{1}{5}\left[3 - 0.67 - 6.75\right] = -0.884$$

$$z_2 = \frac{1}{4}\left[27 - 2(0.67) - 0.6\right] = 6.265$$

Iteration 3: Putting x_2, y_2, z_2 in Eq. (1),

$$x_3 = \frac{1}{6}\left[4 - 2(-0.884) + 6.265\right] = 2.005$$

$$y_3 = \frac{1}{5}\left[3 - 1.59 - 6.265\right] = -0.971$$

$$z_3 = \frac{1}{4}\left[27 - 2(1.59) - (-0.884)\right] = 6.176$$

Iteration 4: Putting x_3, y_3, z_3 in Eq. (1),

$$x_4 = \frac{1}{6}\left[4 - 2(-0.971) + 6.176\right] = 2.01$$

$$y_4 = \frac{1}{5}\left[3 - 2.005 - 6.176\right] = -1.03$$

$$z_4 = \frac{1}{4}\left[27 - 2(2.005) - (-0.971)\right] = 5.99$$

Iteration 5: Putting x_4, y_4, z_4 in Eq. (1),

$$x_5 = \frac{1}{6}\left[4 - 2(-1.03) + 5.99\right] = 2.00$$

$$y_5 = \frac{1}{5}\left[3 - 2.01 - 5.99\right] = -1.00$$

$$z_5 = \frac{1}{4}\left[27 - 2(2.01) - (-1.03)\right] = 6.00$$

Since the fourth and fifth iteration values are nearly equal, the approximate solution is

$$x = 2, y = -1, z = 6$$

Example 2

Solve the following system of equations:

$$\begin{aligned} 8x - y + 2z &= 13 \\ x - 10y + 3z &= 17 \\ 3x + 2y + 12z &= 25 \end{aligned}$$

Solution

Since absolute values of all diagonal elements are large as compared to absolute values of other coefficients, rewriting the equations,

$$\left.\begin{aligned} x &= \frac{1}{8}(13 + y - 2z) \\ y &= -\frac{1}{10}(17 - x - 3z) \\ z &= \frac{1}{12}(25 - 3x - 2y) \end{aligned}\right\} \qquad \ldots(1)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as first approximation and putting in Eq. (1),

$$x_1 = \frac{13}{8} = 1.625$$

$$y_1 = -\frac{17}{10} = -1.7$$

$$z_1 = \frac{25}{12} = 2.08$$

Iteration 2: Putting x_1, y_1, z_1 in Eq. (1),

$$x_2 = \frac{1}{8}[13 - 1.7 - 2(2.08)] = 0.8925$$

$$y_2 = -\frac{1}{10}[17 - 1.625 - 3(2.08)] = -0.9135$$

$$z_2 = \frac{1}{12}[25 - 3(1.625) - 2(-1.7)] = 1.9604$$

Iteration 3: Putting x_2, y_2, z_2 in Eq. (1),

$$x_3 = \frac{1}{8}[13 - 0.9135 - 2(1.9604)] = 1.0207$$

$$y_3 = -\frac{1}{10}[17 - 0.8925 - 3(1.9604)] = -1.0226$$

$$z_3 = \frac{1}{12}[25 - 3(0.8925) - 2(-0.9135)] = 2.0124$$

Iteration 4: Putting x_3, y_3, z_3 in Eq. (1),

$$x_4 = \frac{1}{8}[13 - 1.0226 - 2(2.0124)] = 0.9941$$

$$y_4 = -\frac{1}{10}[17 - 1.0207 - 3(2.0124)] = -0.9942$$

$$z_4 = \frac{1}{12}\left[25 - 3(1.0207) - 2(-1.0226)\right] = 1.9985$$

Since the third and fourth iteration values are nearly equal, the approximate solution is

$$x = 1, y = -1, z = 2$$

The above method can also be represented in tabular form as follows:

Iteration number	$x = \frac{1}{8}(13 + y - 2z)$	$y = -\frac{1}{10}(17 - x - 3z)$	$z = \frac{1}{12}(25 - 3x - 2y)$
1	$x_0 = 0$ $x_1 = 1.625$	$y_0 = 0$ $y_1 = -1.7$	$z_0 = 0$ $z_1 = 2.08$
2	$x_2 = 0.8925$	$y_2 = -0.9135$	$z_2 = 1.9604$
3	$x_3 = 1.0207$	$y_3 = -1.0226$	$z_3 = 2.0124$
4	$x_4 = 0.9941$	$y_4 = -0.9942$	$z_4 = 1.9985$

EXERCISE 8.3

Solve the following system of equations by using the Gauss–Jacobi method:

1.
$$\begin{aligned} 4x + y + 3z &= 17 \\ x + 5y + z &= 14 \\ 2x - y + 8z &= 12 \end{aligned}$$

$\left[\textbf{Ans.: } x = 3, y = 2, z = 1\right]$

2.
$$\begin{aligned} 10x + y + 2z &= 13 \\ 2x + 10y + 3z &= 15 \\ x + 3y + 10z &= 14 \end{aligned}$$

$\left[\textbf{Ans.: } x = 1, y = 1, z = 1\right]$

3.
$$\begin{aligned} 10x - 2y - 3z &= 205 \\ 2x - 10y + 2z &= -154 \\ 2x + y - 10z &= -120 \end{aligned}$$

$\left[\textbf{Ans.: } x = 32, y = 26, z = 21\right]$

4.
$$\begin{aligned} 12x + 2y + z &= 27 \\ 2x + 15y - 3z &= 16 \\ 2x - 3y + 25z &= 23 \end{aligned}$$

$\left[\textbf{Ans.: } x = 2.0148, y = 0.9731, z = 0.8756\right]$

5. $$\begin{aligned} 20x + y - 2z &= 17 \\ 3x + 20y - z &= -18 \\ 2x - 3y + 20z &= 25 \end{aligned}$$

$$\left[\textbf{Ans.: } x = 1, y = -1, z = 1\right]$$

6. $$\begin{aligned} 10x - 5y - 2z &= 3 \\ 4x - 10y + 3z &= -3 \\ x + 6y + 10z &= -3 \end{aligned}$$

$$\left[\textbf{Ans.: } x = 0.342, y = 0.285, z = -0.505\right]$$

7. $$\begin{aligned} 8x - 3y + 2z &= 20 \\ 4x + 11y - z &= 33 \\ 6x + 3y + 12z &= 35 \end{aligned}$$

$$\left[\textbf{Ans.: } x = 3.0168, y = 1.9859, z = 0.9118\right]$$

8. $$\begin{aligned} x + y + 54z &= 110 \\ 27x + 6y - z &= 85 \\ 6x + 15y + 2z &= 72 \end{aligned}$$

$$\left[\textbf{Ans.: } x = 2.425, y = 3.573, z = 1.926\right]$$

8.9 GAUSS—SIEDEL METHOD

This method is applicable to the system of equations in which leading diagonal elements of the coefficient matrix are dominant (large in magnitude) in their respective rows.

Consider the system of equations

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \qquad \ldots(8.3)$$

where $|a_{11}|$, $|a_{22}|$, $|a_{33}|$ are large as compared to the other coefficients in the corresponding row and satisfy the condition of convergence as follows:

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

Rewriting the equations for x, y, and z respectively,

$$\left.\begin{aligned} x &= \frac{1}{a_{11}}\left(b_1 - a_{12}y - a_{13}z\right) \\ y &= \frac{1}{a_{22}}\left(b_2 - a_{21}x - a_{23}z\right) \\ z &= \frac{1}{a_{33}}\left(b_3 - a_{31}x - a_{32}y\right) \end{aligned}\right\} \qquad \ldots(8.4)$$

Iteration 1

Assuming $x = x_0$, $y = y_0$, $z = z_0$ as initial approximations and substituting in the equation of x,

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}y_0 - a_{13}z_0)$$

Now, substituting $x = x_1$, $z = z_0$ in the equation of y,

$$y_1 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}z_0)$$

Substituting $x = x_1$, $y = y_1$ in the equation of z,

$$z_1 = \frac{1}{a_{33}}(b_3 - a_{31}x_1 - a_{32}y)$$

Iteration 2

Substituting $y = y_1$, $z = z_1$ in the equation of x,

$$x_2 = \frac{1}{a_{11}}(b_1 - a_{12}y_1 - a_{13}z_1)$$

Substituting $x = x_2$, $z = z_1$ in the equation of y,

$$y_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_2 - a_{23}z_1)$$

Substituting $x = x_2$, $y = y_2$ in the equation of z,

$$z_2 = \frac{1}{a_{33}}(b_3 - a_{31}x_2 - a_{32}y_2)$$

The above iteration process is continued until two successive approximations are nearly equal.

Working Rule

(i) Arrange the equations in such a manner that the leading diagonal elements are large in magnitude in their respective rows such that

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.
(iii) Start the iteration 1 by assuming the initial values of (x, y, z) as (x_0, y_0, z_0).
(iv) In the iteration 1, put $y = y_0$, $z = z_0$ in the equation of x to obtain x_1, put $x = x_1$, $z = z_0$ in the equation of y to obtain y_1, put $x = x_1$, $y = y_1$ in the equation of z to obtain z_1.
(v) The above process is repeated for the next iterations and it continues until two successive approximations are nearly equal.

Example 1

Solve the following system of equations:

$$\begin{aligned} 3x - 0.1y - 0.2z &= 7.85 \\ 0.1x + 7y - 0.3z &= -19.3 \\ 0.3x - 0.2y + 10z &= 71.4 \end{aligned}$$

Solution

Since diagonal elements are largest, the Gauss–Siedel method can be applied.
Rewriting the equations.

$$\left.\begin{aligned} x &= \frac{1}{3}(7.85 + 0.1y + 0.2z) \\ y &= \frac{1}{7}(-19.3 - 0.1x + 0.3z) \\ z &= \frac{1}{10}(71.4 - 0.3x + 0.2y) \end{aligned}\right\} \quad \ldots(1)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{3}(7.85) = 2.6167$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{7}(-19.3 - 0.1x_1 + 0.3z_0) \\ &= \frac{1}{7}\left[-19.3 - 0.1(2.6167) + 0.3(0)\right] \\ &= -2.7945 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$z_1 = \frac{1}{10}(71.4 - 0.3x_1 + 0.2y_1)$$

$$= \frac{1}{10}[71.4 - 0.3(2.6167) + 0.2(-2.7945)]$$

$$= 7.0056$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$x_2 = \frac{1}{3}(7.85 + 0.1y_1 + 0.2z_1)$$

$$= \frac{1}{3}[7.85 + 0.1(-2.7945) + 0.2(7.0056)]$$

$$= 2.9906$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$y_2 = \frac{1}{7}(-19.3 - 0.1x_2 + 0.3z_1)$$

$$= \frac{1}{7}[-19.3 - 0.1(2.9906) + 0.3(7.0056)]$$

$$= -2.4996$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$z_2 = \frac{1}{10}(71.4 - 0.3x_2 + 0.2y_2)$$

$$= \frac{1}{10}[71.4 - 0.3(2.9906) + 0.2(-2.4996)]$$

$$= 7.0003$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$x_3 = \frac{1}{3}(7.85 + 0.1y_2 + 0.2z_2)$$

$$= \frac{1}{3}[7.85 + 0.1(-2.4996) + 0.2(7.0003)]$$

$$= 3.000$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$y_3 = \frac{1}{7}(-19.3 - 0.1x_3 + 0.3z_2)$$

$$= \frac{1}{7}[-19.3 - 0.1(3) + 0.3(7.0003)]$$

$$= -2.4999$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$z_3 = \frac{1}{10}(71.4 - 0.3x_3 + 0.2y_3)$$

$$= \frac{1}{10}[71.4 - 0.3(3) + 0.2(-2.4999)]$$

$$= 7.0000$$

Since the second and third iteration values are nearly equal, the approximate solution is

$$x = 3,\ y = -2.5,\ z = 7$$

Example 2

Solve the following system of equations:

$$5x + y - z = 10$$
$$2x + 4y + z = 14$$
$$x + y + 8z = 20$$

Solution

Since diagonal elements are largest, the Gauss–Siedel method can be applied.
Rewriting the equations,

$$x = \frac{1}{5}(10 - y + z)$$

$$y = \frac{1}{4}(14 - 2x - z)$$

$$z = \frac{1}{8}(20 - x - y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{5}(10) = 2$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$y_1 = \frac{1}{4}(14 - 2x_1 - z_0)$$

$$= \frac{1}{4}[14 - 2(2) - 0]$$

$$= 2.5$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= \frac{1}{8}(20 - x_1 - y_1) \\ &= \frac{1}{8}(20 - 2 - 2.5) \\ &= 1.9375 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{5}(10 - y_1 + z_1) \\ &= \frac{1}{5}(10 - 2.5 + 1.9375) \\ &= 1.8875 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$\begin{aligned} y_2 &= \frac{1}{4}(14 - 2x_2 - z_1) \\ &= \frac{1}{4}\left[14 - 2(1.8875) - 1.9375\right] \\ &= 2.0719 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned} z_2 &= \frac{1}{8}(20 - x_2 - y_2) \\ &= \frac{1}{8}(20 - 1.8875 - 2.0719) \\ &= 2.0050 \end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$\begin{aligned} x_3 &= \frac{1}{5}(10 - y_2 + z_2) \\ &= \frac{1}{5}(10 - 2.0719 + 2.0050) \\ &= 1.9866 \end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned} y_3 &= \frac{1}{4}(14 - 2x_3 - z_2) \\ &= \frac{1}{4}\left[14 - 2(1.9866) - 2.005\right] \\ &= 2.0055 \end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$\begin{aligned} z_3 &= \frac{1}{8}\left(20 - x_3 - y_3\right) \\ &= \frac{1}{8}\left(20 - 1.9866 - 2.0055\right) \\ &= 2.0009 \end{aligned}$$

Iteration 4: Putting $y = y_3$, $z = z_3$ in the equation of x,

$$\begin{aligned} x_4 &= \frac{1}{5}\left(10 - y_3 + z_3\right) \\ &= \frac{1}{5}\left(10 - 2.0055 + 2.0009\right) \\ &= 1.9991 \end{aligned}$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$\begin{aligned} y_4 &= \frac{1}{4}\left(14 - 2x_4 - z_3\right) \\ &= \frac{1}{4}\left[14 - 2(1.9991) - 2.0009\right] \\ &= 2.0002 \end{aligned}$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$\begin{aligned} z_4 &= \frac{1}{8}\left(20 - x_4 - y_4\right) \\ &= \frac{1}{8}\left(20 - 1.9991 - 2.0002\right) \\ &= 2.0001 \end{aligned}$$

Since the third and fourth iteration values are nearly equal, the approximate solution is

$$x = 2,\ y = 2,\ z = 2$$

Example 3

Solve the following system of linear equations:

$$\begin{aligned} 8x + y + z &= 5 \\ x + 8y + z &= 5 \\ x + y + 8z &= 5 \end{aligned}$$

[Summer 2015, Winter 2013]

Solution

Since diagonal elements are largest, the Gauss–Seidel method can be applied.
Rewriting the equations,

$$x = \frac{1}{8}(5 - y - z)$$

$$y = \frac{1}{8}(5 - x - z)$$

$$z = \frac{1}{8}(5 - x - y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x.

$$x_1 = \frac{1}{8}(5) = 0.625$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{8}(5 - x_1 - z_0) \\ &= \frac{1}{8}(5 - 0.625 - 0) \\ &= 0.5469 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= \frac{1}{8}(5 - x_1 - y_1) \\ &= \frac{1}{8}(5 - 0.625 - 0.5469) \\ &= 0.4785 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{8}(5 - y_1 - z_1) \\ &= \frac{1}{8}(5 - 0.5469 - 0.4785) \\ &= 0.4968 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$\begin{aligned} y_2 &= \frac{1}{8}(5 - x_2 - z_1) \\ &= \frac{1}{8}(5 - 0.4968 - 0.4785) \\ &= 0.5031 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned} z_2 &= \frac{1}{8}(5 - x_2 - y_2) \\ &= \frac{1}{8}(5 - 0.4968 - 0.5031) \\ &= 0.5 \end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation x,

$$\begin{aligned} x_3 &= \frac{1}{8}(5 - y_2 - z_2) \\ &= \frac{1}{8}(5 - 0.5031 - 0.5) \\ &= 0.4996 \end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned} y_3 &= \frac{1}{8}(5 - x_3 - z_2) \\ &= \frac{1}{8}(5 - 0.4996 - 0.5) \\ &= 0.5001 \end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$\begin{aligned} z_3 &= \frac{1}{8}(5 - x_3 - y_3) \\ &= \frac{1}{8}(5 - 0.4996 - 0.5001) \\ &= 0.5 \end{aligned}$$

Since the second and third iteration values are nearly equal, the approximate solution is

$$x = 0.5,\ y = 0.5,\ z = 0.5$$

Example 4

Use the Gauss–Siedel method to solve

$$\begin{aligned} 6x + y + z &= 105 \\ 4x + 8y + 3z &= 155 \\ 5x + 4y - 10z &= 65 \end{aligned}$$

[Summer 2015]

Solution

Since diagonal elements are largest, the Gauss–Seidel method can be applied.
Rewriting the equations,

$$x = \frac{1}{6}(105 - y - z)$$

$$y = \frac{1}{8}(155 - 4x - 3z)$$

$$z = -\frac{1}{10}(65 - 5x - 4y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{6}(105) = 17.5$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{8}(155 - 4x_1 - 3z_0) \\ &= \frac{1}{8}\left[155 - 4(17.5) - 3(0)\right] \\ &= 10.625 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= -\frac{1}{10}(65 - 5x_1 - 4y_1) \\ &= -\frac{1}{10}\left[65 - 5(17.5) - 4(10.625)\right] \\ &= 6.5 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{6}(105 - y_1 - z_1) \\ &= \frac{1}{6}(105 - 10.625 - 6.5) \\ &= 14.6458 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$\begin{aligned} y_2 &= \frac{1}{8}(155 - 4x_2 - 3z_1) \\ &= \frac{1}{8}\left[155 - 4(14.6458) - 3(6.5)\right] \\ &= 9.6146 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned} z_2 &= -\frac{1}{10}(65 - 5x_2 - 4y_2) \\ &= -\frac{1}{10}[65 - 5(14.6458) - 4(9.6146)] \\ &= 4.6687 \end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$\begin{aligned} x_3 &= \frac{1}{6}(105 - y_2 - z_2) \\ &= \frac{1}{6}(105 - 9.6146 - 4.6687) \\ &= 15.1195 \end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned} y_3 &= \frac{1}{8}(155 - 4x_3 - 3z_2) \\ &= \frac{1}{8}[155 - 4(15.1195) - 3(4.6687)] \\ &= 10.0645 \end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$\begin{aligned} z_3 &= -\frac{1}{10}(65 - 5x_3 - 4y_3) \\ &= -\frac{1}{10}[65 - 5(15.1195) - 4(10.0645)] \\ &= 5.0856 \end{aligned}$$

Iteration 4: Putting $y = y_3$, $z = z_3$ in the equation of y,

$$\begin{aligned} x_4 &= \frac{1}{6}(105 - y_3 - z_3) \\ &= \frac{1}{6}(105 - 10.0645 - 5.0856) \\ &= 14.975 \end{aligned}$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$\begin{aligned} y_4 &= \frac{1}{8}(155 - 4x_4 - 3z_3) \\ &= \frac{1}{8}[155 - 4(14.975) - 3(5.0856)] \\ &= 9.9804 \end{aligned}$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$\begin{aligned} z_4 &= -\frac{1}{10}(65 - 5x_4 - 4y_4) \\ &= -\frac{1}{10}[65 - 5(14.975) - 4(9.9804)] \\ &= 4.9797 \end{aligned}$$

Iteration 5: Putting $y = y_4$, $z = z_4$ in the equation of x,

$$\begin{aligned} x_5 &= \frac{1}{6}(105 - y_4 - z_4) \\ &= \frac{1}{6}(105 - 9.9804 - 4.9797) \\ &= 15.0067 \end{aligned}$$

Putting $x = x_5$, $z = z_4$ in the equation of y,

$$\begin{aligned} y_5 &= \frac{1}{8}(155 - 4x_5 - 3z_4) \\ &= \frac{1}{8}[155 - 4(15.0067) - 3(4.9797)] \\ &= 10.0043 \end{aligned}$$

Putting $x = x_5$, $y = y_5$ in the equation of z,

$$\begin{aligned} z_5 &= -\frac{1}{10}(65 - 5x_5 - 4y_5) \\ &= -\frac{1}{10}[65 - 5(15.0067) - 4(10.0043)] \\ &= 5.0051 \end{aligned}$$

Since the fourth and fifth iteration values are nearly equal, the approximate solution is

$$x = 15, y = 10, z = 5$$

Example 5

Solve the following system of equations:

$$\begin{aligned} 25x + 2y - 3z &= 48 \\ 3x + 27y - 2z &= 56 \\ x + 2y + 23z &= 52 \end{aligned}$$

starting with (1, 1, 0).

Solution

Since diagonal elements are largest, the Gauss–Siedel method can be applied. Rewriting the equations,

$$x = \frac{1}{25}(48 - 2y + 3z)$$

$$y = \frac{1}{27}(56 - 3x + 2z)$$

$$z = \frac{1}{23}(52 - x - 2y)$$

Iteration 1: It is given that $x_0 = 1$, $y_0 = 1$, $z_0 = 0$. Putting $y = y_0$, $z = z_0$ in the equation of x,

$$\begin{aligned} x_1 &= \frac{1}{25}(48 - 2y_0 + 3z_0) \\ &= \frac{1}{25}[48 - 2(1) + 3(0)] \\ &= 1.84 \end{aligned}$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{27}(56 - 3x_1 + 2z_0) \\ &= \frac{1}{27}[56 - 3(1.84) + 2(0)] \\ &= 1.8696 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= \frac{1}{23}(52 - x_1 - 2y_1) \\ &= \frac{1}{23}[52 - 1.84 - 2(1.8696)] \\ &= 2.0183 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$, in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{25}(48 - 2y_1 + 3z_1) \\ &= \frac{1}{25}[48 - 2(1.8696) + 3(2.0183)] \\ &= 2.0126 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$y_2 = \frac{1}{27}(56 - 3x_2 + 2z_1)$$

$$= \frac{1}{27}\left[56 - 3(2.0126) + 2(2.0183)\right]$$
$$= 1.9999$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$z_2 = \frac{1}{23}(52 - x_2 - 2y_2)$$
$$= \frac{1}{23}\left[52 - 2.0126 - 2(1.9999)\right]$$
$$= 1.9994$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$x_3 = \frac{1}{25}(48 - 2y_2 + 3z_2)$$
$$= \frac{1}{25}\left[48 - 2(1.9999) + 3(1.9994)\right]$$
$$= 1.9999$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$y_3 = \frac{1}{27}(56 - 3x_3 + 2z_2)$$
$$= \frac{1}{27}\left[56 - 3(1.9999) + 2(1.9994)\right]$$
$$= 1.9999$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$z_3 = \frac{1}{23}(52 - x_3 - 2y_3)$$
$$= \frac{1}{23}\left[52 - 1.9999 - 2(1.9999)\right]$$
$$= 2.0000$$

Since the second and third iteration values are nearly equal, the approximate solution is

$$x = 2, y = 2, z = 2$$

Example 6

Solve the following system of equations, by the Gauss–Seidel method:

$$2x + y + 6z = 9$$
$$8x + 3y + 2z = 13$$
$$x + 5y + z = 7$$

[Summer 2015]

Solution

Since diagonal elements are not largest in their respective rows, rearranging the equations, we have

$$\begin{aligned} 8x + 3y + 2z &= 13 \\ x + 5y + z &= 7 \\ 2x + y + 6z &= 9 \end{aligned}$$

Now, diagonal elements are largest. Rewriting the equations,

$$x = \frac{1}{8}(13 - 3y - 2z)$$

$$y = \frac{1}{5}(7 - x - z)$$

$$z = \frac{1}{6}(9 - 2x - y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{8}(13) = 1.625$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{5}(7 - x_1 - z_0) \\ &= \frac{1}{5}(7 - 1.625 - 0) \\ &= 1.075 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= \frac{1}{6}(9 - 2x_1 - y_1) \\ &= \frac{1}{6}\left[9 - 2(1.625) - 1.075\right] \\ &= 0.7792 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{8}(13 - 3y_1 - 2z_1) \\ &= \frac{1}{8}\left[13 - 3(1.075) - 2(0.7792)\right] \\ &= 1.0271 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$\begin{aligned} y_2 &= \frac{1}{5}(7 - x_2 - z_1) \\ &= \frac{1}{5}(7 - 1.0271 - 0.7792) \\ &= 1.0387 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned} z_2 &= \frac{1}{6}(9 - 2x_2 - y_2) \\ &= \frac{1}{6}[9 - 2(1.0271) - 1.0387] \\ &= 0.9845 \end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$\begin{aligned} x_3 &= \frac{1}{8}(13 - 3y_2 - 2z_2) \\ &= \frac{1}{8}[13 - 3(1.0387) - 2(0.9845)] \\ &= 0.9894 \end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned} y_3 &= \frac{1}{5}(7 - x_3 - z_2) \\ &= \frac{1}{5}(7 - 0.9894 - 0.9845) \\ &= 1.0052 \end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$\begin{aligned} z_3 &= \frac{1}{6}(9 - 2x_3 - y_3) \\ &= \frac{1}{6}[9 - 2(0.9894) - 1.0052] \\ &= 1.0027 \end{aligned}$$

Iteration 4: Putting $y = y_3$, $z = z_3$ in the equation of x,

$$\begin{aligned} x_4 &= \frac{1}{8}(13 - 3y_3 - 2z_3) \\ &= \frac{1}{8}[13 - 3(1.0052) - 2(1.0027)] \\ &= 0.9974 \end{aligned}$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$\begin{aligned} y_4 &= \frac{1}{5}(7 - x_4 - z_3) \\ &= \frac{1}{5}(7 - 0.9974 - 1.0027) \\ &= 1 \end{aligned}$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$\begin{aligned} z_4 &= \frac{1}{6}(9 - 2x_4 - y_4) \\ &= \frac{1}{6}[9 - 2(0.9974) - 1] \\ &= 1.0009 \end{aligned}$$

Since the third and fourth iteration values are nearly equal, the approximate solution is

$$x = 1,\ y = 1,\ z = 1$$

Example 7

Solve the following system of equations:

$$\begin{aligned} x + 2y + z &= 0 \\ 3x + y - z &= 0 \\ x - y + 4z &= 3 \end{aligned}$$

starting with (1, 1, 1).

Solution

Since diagonal elements are not largest in their respective rows, rearranging the equations,

$$\begin{aligned} 3x + y - z &= 0 \\ x + 2y + z &= 0 \\ x - y + 4z &= 3 \end{aligned}$$

Now, diagonal elements are largest. Rewriting the equations,

$$x = \frac{1}{3}(-y + z)$$

$$y = \frac{1}{2}(-x - z)$$

$$z = \frac{1}{4}(3 - x + y)$$

Iteration 1: Assuming $x_0 = 1$, $y_0 = 1$, $z_0 = 1$ as initial approximation and substituting in the equation of x,

$$\begin{aligned} x_1 &= \frac{1}{3}(-y_0 + z_0) \\ &= \frac{1}{3}(-1+1) \\ &= 0 \end{aligned}$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= \frac{1}{2}(-x_1 - z_0) \\ &= \frac{1}{2}(-0-1) \\ &= -0.5 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= \frac{1}{4}(3 - x_1 + y_1) \\ &= \frac{1}{4}(3-0-0.5) \\ &= 0.625 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$ in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{3}(-y_1 + z_1) \\ &= \frac{1}{3}\left[-(-0.5)+0.625\right] \\ &= 0.375 \end{aligned}$$

Putting $x = x_2$, $z = z_1$, in the equation of y,

$$\begin{aligned} y_2 &= \frac{1}{2}(-x_2 - z_1) \\ &= \frac{1}{2}(-0.375-0.625) \\ &= -0.5 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned} z_2 &= \frac{1}{4}(3 - x_2 + y_2) \\ &= \frac{1}{4}(3-0.375-0.5) \\ &= 0.5313 \end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$ in the equation of x,

$$\begin{aligned} x_3 &= \frac{1}{3}(-y_2 + z_2) \\ &= \frac{1}{3}[-(-0.5) + 0.5313] \\ &= 0.3438 \end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned} y_3 &= \frac{1}{2}(-x_3 - z_2) \\ &= \frac{1}{2}(-0.3438 - 0.5313) \\ &= -0.4376 \end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$\begin{aligned} z_3 &= \frac{1}{4}(3 - x_3 + y_3) \\ &= \frac{1}{4}(3 - 0.3438 - 0.4376) \\ &= 0.5547 \end{aligned}$$

Iteration 4: Putting $y = y_3$, $z = z_3$ in the equation of x,

$$\begin{aligned} x_4 &= \frac{1}{3}(-y_3 + z_3) \\ &= \frac{1}{3}[-(-0.4376) + 0.5547] \\ &= 0.3307 \end{aligned}$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$\begin{aligned} y_4 &= \frac{1}{2}(-x_4 - z_3) \\ &= \frac{1}{2}(-0.3307 - 0.5547) \\ &= -0.4427 \end{aligned}$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$\begin{aligned} z_4 &= \frac{1}{4}(3 - x_4 + y_4) \\ &= \frac{1}{4}(3 - 0.3307 - 0.4427) \\ &= 0.5566 \end{aligned}$$

Iteration 5: Putting $y = y_4$, $z = z_4$ in the equation of z,

$$\begin{aligned} x_5 &= \frac{1}{3}(-y_4 + z_4) \\ &= \frac{1}{3}[-(-0.4427)+0.5566] \\ &= 0.3331 \end{aligned}$$

Putting $x = x_5$, $z = z_4$ in the equation of y,

$$\begin{aligned} y_5 &= \frac{1}{2}(-x_5 - z_4) \\ &= \frac{1}{2}(-0.3331-0.5566) \\ &= -0.4449 \end{aligned}$$

Putting $x = x_5$, $y = y_5$ in the equation of z,

$$\begin{aligned} z_5 &= \frac{1}{4}(3 - x_5 + y_5) \\ &= \frac{1}{4}(3-0.3331-0.4449) \\ &= 0.5555 \end{aligned}$$

Iteration 6: Putting $y = y_5$, $z = z_5$ in the equation of z,

$$\begin{aligned} x_6 &= \frac{1}{3}(-y_5 + z_5) \\ &= \frac{1}{3}[-(-0.4449)+0.5555] \\ &= 0.3335 \end{aligned}$$

Putting $x = x_6$, $z = z_5$ in the equation of y,

$$\begin{aligned} y_6 &= \frac{1}{2}(-x_6 - z_5) \\ &= \frac{1}{2}(-0.3335-0.5555) \\ &= -0.4445 \end{aligned}$$

Putting $x = x_6$, $y = y_6$ in the equation of z,

$$\begin{aligned} z_6 &= \frac{1}{4}(3 - x_6 + y_6) \\ &= \frac{1}{4}(3-0.3335-0.4445) \\ &= 0.5555 \end{aligned}$$

Since the fifth and sixth iteration values are nearly equal, the approximate solution is

$$x = 0.333,\ y = -0.444,\ z = 0.555$$

Example 8

Solve the following system of equations:

$$\begin{aligned} 2x - 15y + 6z &= 72 \\ -x + 6y - 27z &= 85 \\ 54x + y + z &= 110 \end{aligned}$$

Solution

Since diagonal elements are not largest in their respective rows, rearranging the equations, we have

$$\begin{aligned} 54x + y + z &= 110 \\ 2x - 15y + 6z &= 72 \\ -x + 6y - 27z &= 85 \end{aligned}$$

Now, diagonal elements are largest. Rewriting the equations,

$$x = \frac{1}{54}(110 - y - z)$$

$$y = -\frac{1}{15}(72 - 2x - 6z)$$

$$z = -\frac{1}{27}(85 + x - 6y)$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ as initial approximation and substituting in the equation of x,

$$x_1 = \frac{1}{54}(110) = 2.037$$

Putting $x = x_1$, $z = z_0$ in the equation of y,

$$\begin{aligned} y_1 &= -\frac{1}{15}(72 - 2x_1 - 6z_0) \\ &= -\frac{1}{15}\left[72 - 2(2.037) - 6(0)\right] \\ &= -4.5284 \end{aligned}$$

Putting $x = x_1$, $y = y_1$ in the equation of z,

$$\begin{aligned} z_1 &= -\frac{1}{27}(85 + x_1 - 6y_1) \\ &= -\frac{1}{27}[85 + 2.037 - 6(-4.5284)] \\ &= -4.2299 \end{aligned}$$

Iteration 2: Putting $y = y_1$, $z = z_1$, in the equation of x,

$$\begin{aligned} x_2 &= \frac{1}{54}(110 - y_1 - z_1) \\ &= \frac{1}{54}[110 - (-4.5284) - (-4.2299)] \\ &= 2.1992 \end{aligned}$$

Putting $x = x_2$, $z = z_1$ in the equation of y,

$$\begin{aligned} y_2 &= -\frac{1}{15}(72 - 2x_2 - 6z_1) \\ &= -\frac{1}{15}[72 - 2(2.1992) - 6(-4.2299)] \\ &= -6.1987 \end{aligned}$$

Putting $x = x_2$, $y = y_2$ in the equation of z,

$$\begin{aligned} z_2 &= -\frac{1}{27}(85 + x_2 - 6y_2) \\ &= -\frac{1}{27}[85 + 2.1992 - 6(-6.1987)] \\ &= -4.6071 \end{aligned}$$

Iteration 3: Putting $y = y_2$, $z = z_2$, in the equation of x,

$$\begin{aligned} x_3 &= \frac{1}{54}(110 - y_2 - z_2) \\ &= \frac{1}{54}[110 - (-6.1987) - (-4.6071)] \\ &= 2.2371 \end{aligned}$$

Putting $x = x_3$, $z = z_2$ in the equation of y,

$$\begin{aligned} y_3 &= -\frac{1}{15}(72 - 2x_3 - 6z_2) \\ &= -\frac{1}{15}[72 - 2(2.2371) - 6(-4.6071)] \\ &= -6.3446 \end{aligned}$$

Putting $x = x_3$, $y = y_3$ in the equation of z,

$$z_3 = -\frac{1}{27}(85 + x_3 - 6y_3)$$

$$= -\frac{1}{27}[85 + 2.2371 - 6(-6.3446)]$$

$$= -4.6409$$

Iteration 4: Putting $y = y_3$, $z = z_3$, in the equation of x,

$$x_4 = \frac{1}{54}(110 - y_3 - z_3)$$

$$= \frac{1}{54}[110 - (-6.3446) - (-4.6409)]$$

$$= 2.2405$$

Putting $x = x_4$, $z = z_3$ in the equation of y,

$$y_4 = -\frac{1}{15}(72 - 2x_4 - 6z_3)$$

$$= -\frac{1}{15}[72 - 2(2.2405) - 6(-4.6409)]$$

$$= -6.3576$$

Putting $x = x_4$, $y = y_4$ in the equation of z,

$$z_4 = -\frac{1}{27}(85 + x_4 - 6y_4)$$

$$= -\frac{1}{27}[85 + 2.2405 - 6(-6.3576)]$$

$$= -4.6439$$

Iteration 5: Putting $y = y_4$, $z = z_4$, in the equation of x,

$$x_5 = \frac{1}{54}(110 - y_4 - z_4)$$

$$= \frac{1}{54}[110 - (-6.3576) - (-4.6439)]$$

$$= 2.2408$$

Putting $x = x_5$, $z = z_4$ in the equation of y,

$$y_5 = -\frac{1}{15}(72 - 2x_5 - 6z_4)$$

$$= -\frac{1}{15}[72 - 2(2.2408) - 6(-4.6439)]$$

$$= -6.3588$$

Putting $x = x_5$, $y = y_5$ in the equation of z,

$$z_5 = -\frac{1}{27}(85 + x_5 - 6y_5)$$
$$= -\frac{1}{27}[85 + 2.2408 - 6(-6.3588)]$$
$$= -4.6442$$

Iteration 6: Putting $y = y_5$, $z = z_5$, in the equation of x,

$$x_6 = \frac{1}{54}(110 - y_5 - z_5)$$
$$= \frac{1}{54}[110 - (-6.3588) - (-4.6442)]$$
$$= 2.2408$$

Putting $x = x_6$, $z = z_5$ in the equation of y,

$$y_6 = -\frac{1}{15}(72 - 2x_6 - 6z_5)$$
$$= -\frac{1}{15}[72 - 2(2.2408) - 6(-4.6442)]$$
$$= -6.3589$$

Putting $x = x_6$, $y = y_6$ in the equation of z,

$$z_6 = -\frac{1}{27}(85 + x_6 - 6y_6)$$
$$= -\frac{1}{27}[85 + 2.2408 - 6(-6.3589)]$$
$$= -4.6442$$

Since the fifth and sixth iteration values are nearly equal, the approximate solution is

$$x = 2.2408,\ y = -6.3589,\ z = -4.6442$$

The above method can also be represented in tabular form as follows:

Iteration number	$x = \frac{1}{54}(110 - y - z)$	$y = -\frac{1}{15}(72 - 2x - 6z)$	$z = -\frac{1}{27}(85 + x - 6y)$
1	$x_0 = 0$ $x_1 = 2.037$	$y_0 = 0$ $y_1 = -4.5284$	$z_0 = 0$ $z_1 = -4.2299$
2	$x_2 = 2.1992$	$y_2 = -6.1987$	$z_2 = -4.6071$
3	$x_3 = 2.2371$	$y_3 = -6.3446$	$z_3 = -4.6409$
4	$x_4 = 2.2405$	$y_4 = -6.3576$	$z_4 = -4.6439$
5	$x_5 = 2.2408$	$y_5 = -6.3588$	$z_5 = -4.6442$
6	$x_6 = 2.2408$	$y_6 = -6.3589$	$z_6 = -4.6442$

EXERCISE 8.4

Solve the following system of equations by using the Gauss–Seidel method:

1. $$\begin{aligned} 54x + y + z &= 110 \\ 2x + 15y + 6z &= 72 \\ -x + 6y + 27z &= 85 \end{aligned}$$

$\left[\textbf{Ans.:}\ x = 1.92,\ y = 3.57,\ z = 2.42\right]$

2. $$\begin{aligned} 20x + y - 2z &= 17 \\ 3x + 20y - z &= -18 \\ 2x - 3y + 20z &= 25 \end{aligned}$$

$[\textbf{Ans.:}\ x = 1,\ y = -1,\ z = 1]$

3. $$\begin{aligned} 10x + y + z &= 12 \\ 2x + 10y + z &= 13 \\ 2x + 2y + 10z &= 14 \end{aligned}$$

$\left[\textbf{Ans.:}\ x = 1,\ y = 1,\ z = 1\right]$

4. $$\begin{aligned} 27x + 6y - z &= 85 \\ 6x + 15y + 2z &= 72 \\ x + y + 54z &= 110 \end{aligned}$$

$\left[\textbf{Ans.:}\ x = 2.43,\ y = 3.57,\ z = 1.92\right]$

5. $$\begin{aligned} 28x + 4y - z &= 32 \\ 2x + 17y + 4z &= 35 \\ x + 3y + 10z &= 24 \end{aligned}$$

$[\textbf{Ans.:}\ x = 0.99,\ y = 1.51,\ z = 1.85]$

Points to Remember

Gauss Elimination Method

(i) Write the matrix form of the system of equations.

(ii) Write the augmented matrix.

(iii) Obtain the echelon form of the augmented matrix by using elementary row transformations.

(iv) Write the corresponding linear system of equations from the echelon form.

(v) Solve the corresponding linear system of equations by back substitution.

Gauss–Jordan Method

(i) Write the matrix form of the system of equations.
(ii) Write the augmented matrix.
(iii) Reduce the coefficient matrix to unit matrix by applying elementary row transformations to the augmented matrix.
(iv) Write the corresponding linear system of equations to obtain the solution.

Gauss–Jacobi Method

(i) Arrange the equations in such a manner that the leading diagonal elements are large in magnitude in their respective rows satisfying the conditions

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.
(iii) Start the iteration 1 by assuming the initial values of (x, y, z) as (x_0, y_0, z_0) and obtain (x_1, y_1, z_1).
(iv) Start the iteration 2 by putting $x = x_1$, $y = y_1$, $z = z_1$ in equations of x, y, z and obtain (x_2, y_2, z_2).
(v) The above process is repeated for the next iterations and it continues until two successive approximations are nearly equal.

Gauss–Siedel Method

(i) Arrange the equations in such a manner that the leading diagonal elements are large in magnitude in their respective rows such that

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.
(iii) Start the iteration 1 by assuming the initial values of (x, y, z) as (x_0, y_0, z_0).
(iv) In the iteration 1, put $y = y_0$, $z = z_0$ in the equation of x to obtain x_1, put $x = x_1$, $z = z_0$ in the equation of y to obtain y_1, put $x = x_1$, $y = y_1$ in the equation of z to obtain z_1.
(v) The above process is repeated for the next iterations and it continues until two successive approximations are nearly equal.

CHAPTER 9

Roots of Algebraic and Transcendental Equations

Chapter Outline

9.1 Introduction
9.2 Bisection Method
9.3 Regula Falsi Method
9.4 Newton–Raphson Method
9.5 Secant Method

9.1 INTRODUCTION

An expression of the form $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$, where a_0, a_1, a_2, ..., a_n are constants and n is a positive integer, is called an *algebraic polynomial of degree n* if $a_0 \neq 0$. The equation $f(x) = 0$ is called an algebraic equation if $f(x)$ is an algebraic polynomial, e.g., $x^3 - 4x - 9 = 0$. If $f(x)$ contains functions such as trigonometric, logarithmic, exponential, etc., then $f(x) = 0$ is called a *transcendental equation*, e.g., $2x^3 - \log(x + 3) \tan x + e^x = 0$.

In general, an equation is solved by factorization. But in many cases, the method of factorization fails. In such cases, numerical methods are used. There are some methods to solve the equation $f(x) = 0$ such as

(i) Bisection method
(ii) Regula Falsi method
(iii) Newton–Raphson method
(iv) Secant method

9.2 BISECTION METHOD

Let $f(x) = 0$ be the given equation. Let x_0 and x_1 be two real values of x at P and Q respectively such that $f(x_1)$ is positive and $f(x_0)$ is negative or vice versa (Fig. 9.1). Then there is one root of the equation $f(x) = 0$ between x_0 and x_1. Now, this interval $[x_0, x_1]$ is divided into two sub-intervals $[x_0, x_2]$ and $[x_2, x_1]$, where $x_2 = \dfrac{x_0 + x_1}{2}$.

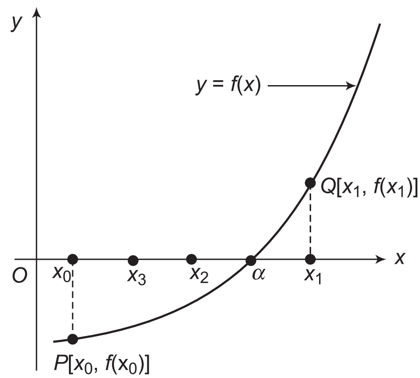


Fig. 9.1

If $f(x_0)$ and $f(x_2)$ are of opposite signs then the interval $[x_0, x_2]$ is divided into $[x_0, x_3]$ and $[x_3, x_2]$, where $x_3 = \dfrac{x_0 + x_2}{2}$. However, if $f(x_0)$ and $f(x_2)$ are of the same sign then $f(x_1)$ and $f(x_2)$ will be opposite signs and the interval $[x_1, x_2]$ is divided into $[x_1, x_3]$ and $[x_3, x_2]$, where $x_3 = \dfrac{x_1 + x_2}{2}$. This process is continued till the desired accuracy is obtained.

Example 1

Find the positive root of $x^3 - 2x - 5 = 0$, *correct up to two decimal places.*

Solution

Let $\quad f(x) = x^3 - 2x - 5$

$$f(1) = -6 \text{ and } f(2) = -1, f(3) = 16$$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

$$x_1 = \frac{2+3}{2} = 2.5$$

$$f(x_1) = f(2.5) = 5.625$$

Since $f(2.5) > 0$ and $f(2) < 0$, the root lies between 2.5 and 2.

$$x_2 = \frac{2.5+2}{2} = 2.25$$

$$f(x_2) = f(2.25) = 1.8906$$

Since $f(2.25) > 0$ and $f(2) < 0$, the root lies between 2.25 and 2.

$$x_3 = \frac{2.25+2}{2} = 2.125$$
$$f(x_3) = f(2.125) = 0.3457$$

Since $f(2.125) > 0$ and $f(2) < 0$, the root lies between 2.125 and 2.

$$x_4 = \frac{2.125+2}{2} = 2.0625$$
$$f(x_4) = f(2.0625) = -0.3513$$

Since $f(2.0625) < 0$ and $f(2.125) > 0$, the root lies between 2.0625 and 2.125.

$$x_5 = \frac{2.0625+2.125}{2} = 2.09375$$
$$f(x_5) = f(2.09375) = -0.0089$$

Since $f(2.09375) < 0$ and $f(2.125) > 0$, the root lies between 2.09375 and 2.125.

$$x_6 = \frac{2.09375+2.125}{2} = 2.109375$$
$$f(x_6) = f(2.109375) = 0.1668$$

Since $f(2.109375) > 0$ and $f(2.09375) < 0$, the root lies between 2.109375 and 2.09375.

$$x_7 = \frac{2.109375+2.09375}{2} = 2.10156$$

Since x_6 and x_7 are same up to two decimal places, the positive root is 2.10.

Example 2

Find a root of $x^3 - 5x + 3 = 0$ *by the bisection method correct up to four decimal places.* **[Summer 2015]**

Solution

Let $f(x) = x^3 - 5x + 3$

$f(0) = 3$ and $f(1) = -1$

Since $f(0) > 0$ and $f(1) < 0$, the root lies between 0 and 1.

$$x_1 = \frac{0+1}{2} = 0.5$$
$$f(x_1) = f(0.5) = 0.625$$

Since $f(0.5) > 0$ and $f(1) < 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5+1}{2} = 0.75$$
$$f(x_2) = f(0.75) = -0.3281$$

Since $f(0.75) < 0$ and $f(0.5) > 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75 + 0.5}{2} = 0.625$$
$$f(x_3) = f(0.625) = 0.1191$$

Since $f(0.625) > 0$ and $f(0.75) < 0$, the root lies between 0.625 and 0.75.

$$x_4 = \frac{0.625 + 0.75}{2} = 0.6875$$
$$f(x_4) = f(0.6875) = -0.1125$$

Since $f(0.6875) < 0$ and $f(0.625) > 0$, the root lies between 0.6875 and 0.625.

$$x_5 = \frac{0.6875 + 0.625}{2} = 0.65625$$
$$f(x_5) = f(0.65625) = 0.00137$$

Since $f(0.65625) > 0$ and $f(0.6875) < 0$, the root lies between 0.65625 and 0.6875.

$$x_6 = \frac{0.65625 + 0.6875}{2} = 0.67188$$
$$f(x_6) = f(0.67188) = -0.0561$$

Since $f(0.67188) < 0$ and $f(0.65625) > 0$, the root lies between 0.67188 and 0.65625.

$$x_7 = \frac{0.67188 + 0.65625}{2} = 0.66407$$
$$f(x_7) = f(0.66407) = -0.02750$$

Since $f(0.66407) < 0$ and $f(0.65625) > 0$, the root lies between 0.66407 and 0.65625.

$$x_8 = \frac{0.66407 + 0.65625}{2} = 0.66016$$
$$f(x_8) = f(0.66016) = -0.01309$$

Since $f(0.66016) < 0$ and $f(0.65625) > 0$, the root lies between 0.66016 and 0.65625.

$$x_9 = \frac{0.66016 + 0.65625}{2} = 0.65821$$
$$f(x_9) = f(0.65821) = -0.00589$$

Since $f(0.65821) < 0$ and $f(0.65625) > 0$, the root lies between 0.65821 and 0.65625.

$$x_{10} = \frac{0.65821 + 0.65625}{2} = 0.65723$$
$$f(x_{10}) = f(0.65723) = -0.0023$$

Since $f(0.65723) < 0$ and $f(0.65625) > 0$, the root lies between 0.65723 and 0.65625.

$$x_{11} = \frac{0.65723 + 0.65625}{2} = 0.65674$$
$$f(x_{11}) = f(0.65674) = -0.00044$$

Since $f(0.65674) < 0$ and $f(0.65625) > 0$, the root lies between 0.65674 and 0.65625.

$$x_{12} = \frac{0.65674 + 0.65625}{2} = 0.6565$$
$$f(x_{12}) = f(0.6565) = 0.00044$$

Since $f(0.6565) > 0$ and $f(0.65674) < 0$, the root lies between 0.6565 and 0.65674.

$$x_{13} = \frac{0.6565 + 0.65674}{2} = 0.6566$$
$$f(x_{13}) = f(0.6566) = 0.00075$$

Since $f(0.6566) > 0$ and $f(0.65674) < 0$, the root lies between 0.6566 and 0.65674.

$$x_{14} = \frac{0.6566 + 0.65674}{2} = 0.65667$$

Since x_{13} and x_{14} are same up to four decimal places, the root is 0.6566.

Example 3

Perform the five iterations of the bisection method to obtain a root of the equation $f(x) = x^3 - x - 1 = 0$.

Solution

Let $f(x) = x^3 - x - 1$

$$f(1) = -1 \text{ and } f(2) = 5$$

Since $f(1) < 0$ and $f(2) > 0$, the root lies between 1 and 2.

$$x_1 = \frac{1+2}{2} = 1.5$$
$$f(x_1) = f(1.5) = 0.875$$

Since $f(1.5) > 0$ and $f(1) < 0$, the root lies between 1.5 and 1.

$$x_2 = \frac{1.5+1}{2} = 1.25$$
$$f(x_2) = f(1.25) = -0.2968$$

Since $f(1.25) < 0$ and $f(1.5) > 0$, the root lies between 1.25 and 1.5.

$$x_3 = \frac{1.25+1.5}{2} = 1.375$$
$$f(x_3) = f(1.375) = 0.2246$$

Since $f(1.375) > 0$ and $f(1.25) < 0$, the root lies between 1.375 and 1.25.

$$x_4 = \frac{1.375 + 1.25}{2} = 1.3125$$
$$f(x_4) = f(1.3125) = -0.0515$$

Since $f(1.3125) < 0$ and $f(1.375) > 0$, the root lies between 1.3125 and 1.375.

$$x_5 = \frac{1.3125 + 1.375}{2} = 1.3438$$

Hence, the root is 1.3438 up to five iterations.

Example 4

Find the approximate solution of $x^3 + x - 1 = 0$ *correct to three decimal places.* **[Winter 2013]**

Solution

Let $\quad f(x) = x^3 + x - 1$

$$f(0) = -1 \text{ and } f(1) = 1$$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

$$x_1 = \frac{0 + 1}{2} = 0.5$$
$$f(x_1) = f(0.5) = -0.375$$

Since $f(0.5) < 0$ and $f(1) > 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5 + 1}{2} = 0.75$$
$$f(x_2) = f(0.75) = 0.1719$$

Since $f(0.75) > 0$ and $f(0.5) < 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75 + 0.5}{2} = 0.625$$
$$f(x_3) = f(0.625) = -0.1309$$

Since $f(0.625) < 0$ and $f(0.75) > 0$, the root lies between 0.625 and 0.75.

$$x_4 = \frac{0.625 + 0.75}{2} = 0.6875$$
$$f(x_4) = f(0.6875) = 0.01245$$

Since $f(0.6875) > 0$ and $f(0.625) < 0$, the root lies between 0.6875 and 0.625.

$$x_5 = \frac{0.6875 + 0.625}{2} = 0.6563$$
$$f(x_5) = f(0.6563) = -0.0644$$

Since $f(0.6563) < 0$ and $f(0.6875) > 0$, the root lies between 0.6563 and 0.6875.

$$x_6 = \frac{0.6563 + 0.6875}{2} = 0.6719$$

$$f(x_6) = f(0.6719) = -0.0248$$

Since $f(0.6719) < 0$ and $f(0.6875) > 0$, the root lies between 0.6719 and 0.6875.

$$x_7 = \frac{0.6719 + 0.6875}{2} = 0.6797$$

$$f(x_7) = f(0.6797) = -0.0141$$

Since $f(0.6797) < 0$ and $f(0.6875) > 0$, the root lies between 0.6797 and 0.6875.

$$x_8 = \frac{0.6797 + 0.6875}{2} = 0.6836$$

$$f(x_8) = f(0.6836) = 0.0031$$

Since $f(0.6836) > 0$ and $f(0.6797) < 0$, the root lies between 0.6836 and 0.6797.

$$x_9 = \frac{0.6836 + 0.6797}{2} = 0.6817$$

$$f(x_9) = f(0.6817) = -0.0015$$

Since $f(0.6817) < 0$ and $f(0.6836) > 0$, the root lies between 0.6817 and 0.6836.

$$x_{10} = \frac{0.6817 + 0.6836}{2} = 0.6827$$

$$f(x_{10}) = f(0.6827) = 0.00089$$

Since $f(0.6827) > 0$ and $f(0.6817) < 0$, the root lies between 0.6827 and 0.6817.

$$x_{11} = \frac{0.6827 + 0.6817}{2} = 0.6822$$

Since x_{10} and x_{11} are same up to three decimal points, the root is 0.682.

Example 5

Find a root of the equation $x^3 - 4x - 9 = 0$ *using the bisection method in four stages.*

Solution

Let $\quad f(x) = x^3 - 4x - 9$

$$f(2) = -9 \quad \text{and} \quad f(3) = 6$$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

$$x_1 = \frac{2+3}{2} = 2.5$$

$$f(x_1) = f(2.5) = -3.375$$

Since $f(2.5) < 0$ and $f(3) > 0$, the root lies between 2.5 and 3.

$$x_2 = \frac{2.5+3}{2} = 2.75$$

$$f(x_2) = f(2.75) = 0.7969$$

Since $f(2.75) > 0$ and $f(2.5) < 0$, the root lies between 2.75 and 2.5.

$$x_3 = \frac{2.75+2.5}{2} = 2.625$$

$$f(x_3) = f(2.625) = -1.4121$$

Since $f(2.625) < 0$ and $f(2.75) > 0$, the root lies between 2.625 and 2.75.

$$x_4 = \frac{2.625+2.75}{2} = 2.6875$$

Hence, the root is 2.6875 up to four stages.

Example 6

Find the negative root of $x^3 - 7x + 3$ by the bisection method up to three decimal places.

Solution

Let $f(x) = x^3 - 7x + 3$

$$f(-2) = 9 \quad \text{and} \quad f(-3) = -3$$

Since $f(-2) > 0$ and $f(-3) < 0$, the root lies between -2 and -3.

$$x_1 = \frac{-2-3}{2} = -2.5$$

$$f(x_1) = f(-2.5) = 4.875$$

Since $f(-2.5) > 0$ and $f(-3) < 0$, the root lies between -2.5 and -3.

$$x_2 = \frac{-2.5-3}{2} = -2.75$$

$$f(x_2) = f(-2.75) = 1.4531$$

Since $f(-2.75) > 0$ and $f(-3) < 0$, the root lies between -2.75 and -3.

$$x_3 = \frac{-2.75-3}{2} = -2.875$$

$$f(x_3) = f(-2.875) = -0.6387$$

Since $f(-2.875) < 0$ and $f(-2.75) > 0$, the root lies between –2.875 and –2.75.

$$x_4 = \frac{-2.875 - 2.75}{2} = -2.8125$$
$$f(x_4) = f(-2.8125) = 0.4402$$

Since $f(-2.8125) > 0$ and $f(-2.875) < 0$, the root lies between –2.8125 and –2.875.

$$x_5 = \frac{-2.8125 - 2.875}{2} = -2.8438$$
$$f(x_5) = f(-2.8438) = -0.0918$$

Since $f(-2.8438) < 0$ and $f(-2.8125) > 0$, the root lies between –2.8438 and –2.8125.

$$x_6 = \frac{-2.8438 - 2.8125}{2} = -2.8282$$
$$f(x_6) = f(-2.8282) = 0.1754$$

Since $f(-2.8282) > 0$ and $f(-2.8438) < 0$, the root lies between –2.8282 and –2.8438.

$$x_7 = \frac{-2.8282 - 2.8438}{2} = -2.836$$
$$f(x_7) = f(-2.836) = 0.0423$$

Since $f(-2.836) > 0$ and $f(-2.8438) < 0$, the root lies between –2.836 and –2.8438.

$$x_8 = \frac{-2.836 - 2.8438}{2} = -2.8399$$
$$f(x_8) = f(-2.8399) = -0.0246$$

Since $f(-2.8399) < 0$ and $f(-2.836) > 0$, the root lies between –2.8399 and –2.836.

$$x_9 = \frac{-2.8399 - 2.836}{2} = -2.838$$
$$f(x_9) = f(-2.838) = 0.0081$$

Since $f(-2.838) > 0$ and $f(-2.8399) < 0$, the root lies between –2.838 and –2.8399.

$$x_{10} = \frac{-2.838 - 2.8399}{2} = -2.8389$$

Since x_9 and x_{10} are same up to three decimal places, the negative root is –2.838.

Example 7

Perform three iterations of the bisection method to obtain the root of the equation $2 \sin x - x = 0$, *correct up to three decimal places.*

[Summer 2015]

Solution

Let $f(x) = 2\sin x - x$

$f(1) = 0.6829$ and $f(2) = -0.1814$

Since $f(1) > 0$ and $f(2) < 0$, the root lies between 1 and 2.

$$x_1 = \frac{1+2}{2} = 1.5$$

$$f(x_1) = f(1.5) = 0.4949$$

Since $f(1.5) > 0$ and $f(2) < 0$, the root lies between 1.5 and 2.

$$x_2 = \frac{1.5+2}{2} = 1.75$$

$$f(x_2) = f(1.75) = 0.2179$$

Since $f(1.75) > 0$ and $f(2) < 0$, the root lies between 1.75 and 2.

$$x_3 = \frac{1.75+2}{2} = 1.875$$

Hence, the root is 1.875 up to three iterations.

Example 8

Solve $x = \cos x$ by the bisection method correct to two decimal places.

[Summer 2014]

Solution

Let $f(x) = x - \cos x$

$f(0) = -1$ and $f(1) = 0.4597$

Since $f(0) < 1$ and $f(1) > 0$, the root lies between 0 and 1.

$$x_1 = \frac{0+1}{2} = 0.5$$

$$f(x_1) = f(0.5) = -0.3776$$

Since $f(0.5) < 0$ and $f(1) > 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5+1}{2} = 0.75$$

$$f(x_2) = f(0.75) = 0.0183$$

Since $f(0.75) > 0$ and $f(0.5) < 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75+0.5}{2} = 0.625$$

$$f(x_3) = f(0.625) = -0.186$$

Since $f(0.625) < 0$ and $f(0.75) > 0$, the root lies between 0.625 and 0.75.

$$x_4 = \frac{0.625 + 0.75}{2} = 0.6875$$
$$f(x_4) = f(0.6875) = -0.0853$$

Since $f(0.6875) < 0$ and $f(0.75) > 0$, the root lies between 0.6875 and 0.75.

$$x_5 = \frac{0.6875 + 0.75}{2} = 0.71875$$
$$f(x_5) = f(0.71875) = -0.0338$$

Since $f(0.71875) < 0$ and $f(0.75) > 0$, the root lies between 0.71875 and 0.75.

$$x_6 = \frac{0.71875 + 0.75}{2} = 0.7344$$
$$f(x_6) = f(0.7344) = -0.0078$$

Since $f(0.7344) < 0$ and $f(0.75) > 0$, the root lies between 0.7344 and 0.75.

$$x_7 = \frac{0.7344 + 0.75}{2} = 0.7422$$
$$f(x_7) = f(0.7422) = 0.0052$$

Since $f(0.7422) > 0$ and $f(0.7344) < 0$, the root lies between 0.7422 and 0.7344.

$$x_8 = \frac{0.7422 + 0.7344}{2} = 0.7383$$
$$f(x_8) = f(0.7383) = -0.0013$$

Since $f(0.7383) < 0$ and $f(0.7422) > 0$, the root lies between 0.7383 and 0.7422.

$$x_9 = \frac{0.7383 + 0.7422}{2} = 0.74025$$
$$f(x_9) = f(0.74025) = 0.00195$$

Since $f(0.74025) > 0$ and $f(0.7383) < 0$, the root lies between 0.74025 and 0.7383.

$$x_{10} = \frac{0.74025 + 0.7383}{2} = 0.7393$$
$$f(x_{10}) = f(0.7393) = 0.0004$$

Since $f(0.7393) > 0$ and $f(0.7383) < 0$, the root lies between 0.7393 and 0.7383.

$$x_{11} = \frac{0.7393 + 0.7383}{2} = 0.7388$$

Since x_{10} and x_{11} are the same up to two decimal places, the root is 0.73.

Example 9

Find a real root between 0 *and* 1 *of the equation* $e^{-x} - x = 0$, *correct up to three decimal places.*

Solution

Let
$$f(x) = e^{-x} - x$$
$$f(0) = 1 \text{ and } f(1) = -0.63$$

Since $f(0) > 0$ and $f(1) < 0$, the root lies between 0 and 1.

$$x_1 = \frac{0+1}{2} = 0.5$$
$$f(x_1) = f(0.5) = 0.1065$$

Since $f(0.5) > 0$ and $f(1) < 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5+1}{2} = 0.75$$
$$f(x_2) = f(0.75) = -0.2776$$

Since $f(0.75) < 0$ and $f(0.5) > 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75+0.5}{2} = 0.625$$
$$f(x_3) = f(0.625) = -0.0897$$

Since $f(0.625) < 0$ and $f(0.5) > 0$, the root lies between 0.625 and 0.5.

$$x_4 = \frac{0.625+0.5}{2} = 0.5625$$
$$f(x_4) = f(0.5625) = 7.28 \times 10^{-3}$$

Since $f(0.5625) > 0$ and $f(0.625) < 0$, the root lies between 0.5625 and 0.625.

$$x_5 = \frac{0.5625+0.625}{2} = 0.5938$$
$$f(x_5) = f(0.5938) = -0.0416$$

Since $f(0.5938) < 0$ and $f(0.5625) > 0$, the root lies between 0.5938 and 0.5625.

$$x_6 = \frac{0.5938+0.5625}{2} = 0.5782$$
$$f(x_6) = f(0.5782) = -0.0173$$

Since $f(0.5782) < 0$ and $f(0.5625) > 0$, the root lies between 0.5782 and 0.5625.

$$x_7 = \frac{0.5782+0.5625}{2} = 0.5704$$
$$f(x_7) = f(0.5704) = -5.1007 \times 10^{-3}$$

Since $f(0.5704) < 0$ and $f(0.5625) > 0$, the root lies between 0.5704 and 0.5625.

$$x_8 = \frac{0.5704 + 0.5625}{2} = 0.5665$$

$$f(x_8) = f(0.5665) = 1.008 \times 10^{-3}$$

Since $f(0.5665) > 0$ and $f(0.5704) < 0$, the root lies between 0.5665 and 0.5704.

$$x_9 = \frac{0.5665 + 0.5704}{2} = 0.5685$$

$$f(x_9) = f(0.5685) = -2.1256 \times 10^{-3}$$

Since $f(0.5685) < 0$ and $f(0.5665) > 0$, the root lies between 0.5685 and 0.5665.

$$x_{10} = \frac{0.5685 + 0.5665}{2} = 0.5675$$

$$f(x_{10}) = f(0.5675) = -5.5898 \times 10^{-4}$$

Since $f(0.5675) < 0$ and $f(0.5665) > 0$, the root lies between 0.5675 and 0.5665.

$$x_{11} = \frac{0.5675 + 0.5665}{2} = 0.567$$

Since x_{10} and x_{11} are the same up to three decimal places, the root is 0.567.

Example 10

Find the root of $\cos x - xe^x = 0$ *in four steps.*

Solution

Let $f(x) = \cos x - xe^x$

$f(0) = 1$ and $f(1) = -2.18$

Since $f(0) > 0$ and $f(1) < 0$, the root lies between 0 and 1.

$$x_1 = \frac{0 + 1}{2} = 0.5$$

$$f(x_1) = f(0.5) = 0.0532$$

Since $f(0.5) > 0$ and $f(1) < 0$, the root lies between 0.5 and 1.

$$x_2 = \frac{0.5 + 1}{2} = 0.75$$

$$f(x_2) = f(0.75) = -0.8561$$

Since $f(0.75) < 0$ and $f(0.5) > 0$, the root lies between 0.75 and 0.5.

$$x_3 = \frac{0.75 + 0.5}{2} = 0.625$$

$$f(x_3) = f(0.625) = -0.3567$$

Since $f(0.625) < 0$ and $f(0.5) > 0$, the root lies between 0.625 and 0.5.

$$x_4 = \frac{0.625 + 0.5}{2} = 0.5625$$

$$f(x_4) = f(0.5625) = -0.1413$$

Since $f(0.5625) < 0$ and $f(0.5) > 0$, the root lies between 0.5625 and 0.5.

$$x_5 = \frac{0.5625 + 0.5}{2} = 0.53125$$

Hence, the root is 0.53125 in four steps.

EXERCISE 9.1

Find a positive root of the following equations correct to four decimal places using the bisection method:

1. $x^3 - 4x - 9 = 0$

   [Ans.: 2.7065**]**

2. $x^3 + 3x - 1 = 0$

   [Ans.: 0.3222**]**

3. $x^3 + x^2 - 1 = 0$

   [Ans.: 0.7549**]**

4. $x^4 - x^3 - x^2 - 6x - 4 = 0$

   [Ans.: 2.5528**]**

5. $3x = \sqrt{1 + \sin x}$

   [Ans.: 0.3918**]**

6. $3x = \cos x + 1$

   [Ans.: 0.6071**]**

7. $x - \cos x = 0$

   [Ans.: 0.7391**]**

8. $xe^x = 1$

   [Ans.: 0.5671**]**

9. $x \log_{10} x = 1.2$ lying between 2 and 3

   [Ans.: 2.7406**]**

9.3 REGULA FALSI METHOD

This method resembles the bisection method. In this method, two points x_0 and x_1 are chosen such that $f(x_0)$ and $f(x_1)$ are of opposite signs, i.e., the graph of $y = f(x)$ crosses the x-axis between these points. Hence, a root lies between x_0 and x_1 and $f(x_0)f(x_1) < 0$ (Fig.9.2).

The equation of the chord joining the points $P[x_0, f(x_0)]$ and $Q[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

In this method, the curve PQ is replaced by the chord PQ and the point of intersection of the chord with the x-axis is taken as an approximation to the root.

If x_2 is the point of intersection of the x-axis and the line joining $P[x_0, f(x_0)]$ and $Q[x_1, f(x_1)]$ then x_2 is closer to the root α than x_0 and x_1.

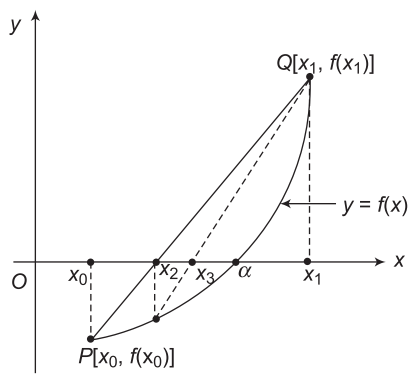


Fig. 9.2

Using the slope formula,

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_2) - f(x_0)}{x_2 - x_0} = \frac{0 - f(x_0)}{x_2 - x_0}$$

$$x_2 - x_0 = -\frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

which is an approximation to the root.

If $f(x_0)$ and $f(x_2)$ are of opposite signs, the root lies between x_0 and x_2, and the next approximation x_3 is obtained as

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

If the root lies between x_1 and x_2, the next approximation x_3 is obtained as

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

This process is repeated till the root is obtained to the desired accuracy. This iteration process is known as the method of false position or *regula falsi method*.

Example 1

Find a positive root of $x^3 - 4x + 1$ *correct up to three decimal places.*

[Summer 2015]

Solution

Let $f(x) = x^3 - 4x + 1$

$f(0) = 1$ and $f(1) = -2$

Since $f(0) > 0$ and $f(1) < 0$, the root lies between 0 and 1.

Let $x_0 = 0$, $x_1 = 1$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 0 - \frac{1-0}{-2-1}(1)$$

$$= 0.3333$$

$$f(x_2) = f(0.3333) = -0.2962$$

Since $f(0.3333) < 0$ and $f(0) > 0$, the root lies between 0.3333 and 0, i.e., x_2 and x_0.

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

$$= 0 - \frac{0.3333 - 0}{-0.2962 - 1}(1)$$

$$= 0.2571$$

$$f(x_3) = f(0.2571) = -0.0114$$

Since $f(0.2571) < 0$ and $f(0) > 0$, the root lies between 0.2571 and 0, i.e., x_3 and x_0.

$$x_4 = x_0 - \frac{x_3 - x_0}{f(x_3) - f(x_0)} f(x_0)$$

$$= 0 - \frac{0.2571 - 0}{-0.0114 - 1}(1)$$

$$= 0.2542$$

$$f(x_4) = f(0.2542) = -0.0004$$

Since $f(0.2542) < 0$ and $f(0) > 0$, the root lies between 0.2542 and 0, i.e., x_4 and x_0.

$$x_5 = x_0 - \frac{x_4 - x_0}{f(x_4) - f(x_0)} f(x_0)$$

$$= 0 - \frac{0.2542 - 0}{-0.0004 - 1}(1)$$

$$= 0.2541$$

Since x_4 and x_5 are same up to three decimal places, a positive root is 0.254.

Example 2

Find the root of the equation $2x - \log_{10}x = 7$, *which lies between* 3.5 *and* 4, *correct up to five places of decimal.*

Solution

Let $$f(x) = 2x - \log_{10}x - 7$$

$$f(3.5) = -0.54407 \quad \text{and} \quad f(4) = 0.39794$$

Since $f(3.5) < 0$ and $f(4) > 0$, the root lies between 3.5 and 4.

Let $x_0 = 3.5, \quad x_1 = 4$

$$\begin{aligned} x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 3.5 - \frac{4 - 3.5}{0.39794 + 0.54407}(-0.54407) \\ &= 3.78878 \end{aligned}$$

$$f(x_2) = f(3.78878) = -0.00094$$

Since $f(3.78878) < 0$ and $f(4) > 0$, the root lies between 3.78878 and 4, i.e., x_2 and x_1.

$$\begin{aligned} x_3 &= x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2) \\ &= 3.78878 - \frac{4 - 3.78878}{0.39794 + 0.00094}(-0.00094) \\ &= 3.78928 \end{aligned}$$

$$f(x_3) = f(3.78928) = 0.000003$$

Since $f(3.78928) > 0$ and $f(3.78878) < 0$, the root lies between 3.78928 and 3.78878, i.e., x_3 and x_2.

$$\begin{aligned} x_4 &= x_2 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_2) \\ &= 3.78878 - \frac{3.78928 - 3.78878}{0.000003 + 0.00094}(-0.00094) \\ &= 3.78928 \end{aligned}$$

Since x_3 and x_4 are same up to five decimal places, the root is 3.78928.

Example 3

Find a real root of the equation $x \log_{10} x = 1.2$ *by the regula falsi method.*

[Summer 2015]

Solution

Let $f(x) = x \log_{10} x - 1.2$

$$f(2) = -0.5979 \quad \text{and} \quad f(3) = 0.2314$$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

Let $x_0 = 2$, $x_1 = 3$

$$\begin{aligned} x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\ &= 2 - \frac{3-2}{0.2314 + 0.5979}(-0.5979) \\ &= 2.721 \\ f(x_2) &= f(2.721) = -0.0171 \end{aligned}$$

Since $f(2.721) < 0$ and $f(3) > 0$, the root lies between 2.721 and 3, i.e., x_2 and x_1.

$$\begin{aligned} x_3 &= x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2) \\ &= 2.721 - \frac{3 - 2.721}{0.2314 + 0.0171}(-0.0171) \\ &= 2.7402 \\ f(x_3) &= f(2.7402) = -0.0004 \end{aligned}$$

Since $f(2.7402) < 0$ and $f(3) > 0$, the root lies between 2.7402 and 3, i.e., x_3 and x_1.

$$\begin{aligned} x_4 &= x_1 - \frac{x_3 - x_1}{f(x_3) - f(x_1)} f(x_1) \\ &= 3 - \frac{2.7042 - 3}{-0.0004 - 0.2314}(0.2314) \\ &= 2.7406 \end{aligned}$$

Since x_3 and x_4 are same up to three decimal places, a real root is 2.740.

Example 4

Solve the equation $x \tan x = -1$, *starting with* $x_0 = 2.5$ and $x_1 = 3$, *correct up to three decimal places.*

Solution

Let $f(x) = x \tan x + 1$

$f(2.5) = -0.8676$ and $f(3) = 0.5724$

Since $f(2.5) < 0$ and $f(3) > 0$, the root lies between 2.5 and 3.

Let $x_0 = 2.5$, $x_1 = 3$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.5 - \frac{3 - 2.5}{0.5724 + 0.8676}(-0.8676)$$

$$= 2.8013$$

$$f(x_2) = f(2.8013) = 0.0082$$

Since $f(2.8013) > 0$ and $f(2.5) < 0$, the root lies between 2.8013 and 2.5, i.e., x_2 and x_0.

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

$$= 2.5 - \frac{2.8013 - 2.5}{0.0082 + 0.8676}(-0.8676)$$

$$= 2.7985$$

$$f(x_3) = f(2.7985) = 0.0003$$

Since $f(2.7985) > 0$ and $f(2.5) < 0$, the root lies between 2.7985 and 2.5, i.e., x_3 and x_0.

$$x_4 = x_0 - \frac{x_3 - x_0}{f(x_3) - f(x_0)} f(x_0)$$

$$= 2.5 - \frac{2.7985 - 2.5}{0.0003 + 0.8676}(-0.8676)$$

$$= 2.7984$$

Since x_3 and x_4 are same up to three decimal places, the root is 2.798.

Example 5

Find the real root of the equation $\log_{10} x - \cos x = 0$, *correct to four decimal places.*

Solution

Let $f(x) = \log_{10} x - \cos x$

$f(1) = -0.5403$ and $f(1.5) = 0.10535$

Since $f(1) < 0$ and $f(1.5) > 0$, the root lies between 1 and 1.5.

Let $x_0 = 1$, $\quad x_1 = 1.5$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1 - \frac{1.5 - 1}{0.10535 + 0.5403}(-0.5403)$$

$$= 1.41842$$

$$f(x_2) = f(1.41842) = 0.00002$$

Since $f(1.41842) > 0$ and $f(1) < 0$, the root lies between 1.41842 and 1, i.e., x_2 and x_0.

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

$$= 1 - \frac{1.41842 - 1}{0.00002 + 0.5403}(-0.5403)$$

$$= 1.41840$$

Since x_2 and x_3 are same up to four decimal places, the real root is 1.4184.

Example 6

Find the smallest root of an equation $x - e^{-x} = 0$ correct to three significant digits. **[Summer 2015]**

Solution

Let $\quad f(x) = x - e^{-x}$

$$f(0) = -1 \quad \text{and} \quad f(1) = 0.6321$$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 0$, $\quad x_1 = 1$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 0 - \frac{1 - 0}{0.6321 + 1}(-1)$$

$$= 0.6127$$

$$f(x_2) = f(0.6127) = 0.0708$$

Since $f(0.6127) > 0$ and $f(0) < 0$, the root lies between 0.6127 and 0, i.e., x_2 and x_0.

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

$$= 0 - \frac{0.6127 - 0}{0.0708 + 1}(-1)$$

$$= 0.5722$$

$$f(x_3) = f(0.5722) = 0.0079$$

Since $f(0.5722) > 0$ and $f(0) < 0$, the root lies between 0.5722 and 0, i.e., x_3 and x_0.

$$
\begin{aligned}
x_4 &= x_0 - \frac{x_3 - x_0}{f(x_3) - f(x_0)} f(x_0) \\
&= 0 - \frac{0.5722 - 0}{0.0079 + 1}(-1) \\
&= 0.5677
\end{aligned}
$$

$$f(x_4) = f(0.5677) = 0.0009$$

Since $f(0.5677) > 0$ and $f(0) < 0$, the root lies between 0.5677 and 0, i.e., x_4 and x_0.

$$
\begin{aligned}
x_5 &= x_0 - \frac{x_4 - x_0}{f(x_4) - f(x_0)} f(x_0) \\
&= 0 - \frac{0.5677 - 0}{0.0009 + 1}(-1) \\
&= 0.5672
\end{aligned}
$$

Since x_4 and x_5 are same up to three significant digits, a positive root is 0.567.

Example 7

Find the root of the equation $\cos x - xe^x = 0$ *correct up to three decimal places, lying between* 0.5 *and* 0.7.

Solution

Let $\quad f(x) = \cos x - xe^x$

$$f(0.5) = 0.0532 \quad \text{and} \quad f(0.7) = -0.6448$$

Since $f(0.5) > 0$ and $f(0.7) < 0$, the root lies between 0.5 and 0.7.

Let $x_0 = 0.5, \quad x_1 = 0.7$

$$
\begin{aligned}
x_2 &= x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \\
&= 0.5 - \frac{0.7 - 0.5}{-0.6448 - 0.0532}(0.0532) \\
&= 0.5152
\end{aligned}
$$

$$f(x_2) = f(0.5152) = 0.0078$$

Since $f(0.5152) > 0$ and $f(0.7) < 0$, the root lies between 0.5152 and 0.7, i.e., x_2 and x_1.

$$
\begin{aligned}
x_3 &= x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2) \\
&= 0.5152 - \frac{0.7 - 0.5152}{-0.6448 - 0.0078}(0.0078) \\
&= 0.5174
\end{aligned}
$$

$$f(x_3) = f(0.5174) = 0.0011$$

Since $f(0.5174) > 0$ and $f(0.7) < 0$, the root lies between 0.5174 and 0.7, i.e., x_3 and x_1.

$$\begin{aligned} x_4 &= x_3 - \frac{x_1 - x_3}{f(x_1) - f(x_3)} f(x_3) \\ &= 0.5174 - \frac{0.7 - 0.5174}{-0.6448 - 0.0011}(0.0011) \\ &= 0.5177 \end{aligned}$$

Since x_3 and x_4 are same up to three decimal places, the root is 0.517.

EXERCISE 9.2

Find a real root of the following equations correct to three decimal places using the regula falsi method:

1. $x^3 + x - 1 = 0$

[Ans.: 0.682]

2. $x^3 - 4x - 9 = 0$

[Ans.: 2.707]

3. $x^3 - 5x - 7 = 0$

[Ans.: 2.747]

4. $xe^3 = 3$

[Ans.: 1.050]

5. $e^{-x} - \sin x = 0$

[Ans.: 0.5885]

6. $2x = \cos x + 3$

[Ans.: 1.524]

7. $x^2 - \log_e x = 12$

[Ans.: 3.646]

8. $e^x = 3x$

[Ans.: 1.512]

9.4 NEWTON—RAPHSON METHOD

Let $f(x) = 0$ be the given equation and x_0 be an approximate root of the equation. If $x_1 = x_0 + h$ be the exact root then $f(x_1) = 0$.

i.e., $$f(x_0 + h) = 0$$

$$f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\cdots=0 \qquad \text{[By Taylor's series]}$$

Since h is small, neglecting h^2 and higher powers of h,

$$f(x_0)+h\,f'(x_0)=0$$

$$h=-\frac{f(x_0)}{f'(x_0)}$$

$$\therefore \qquad x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly, starting with x_1, a still better approximation x_2 is obtained.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general,
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This equation is known as the *Newton–Raphson formula* or *Newton's iteration formula*.

9.4.1 Geometrical Interpretation

Let x_0 be a point near the root α of the equation $f(x) = 0$ (Fig. 9.3). The equation of the tangent at $P_0[x_0, f(x_0)]$ is

$$y-f(x_0)=f'(x_0)(x-x_0)$$

This line cuts the x-axis at x_1.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

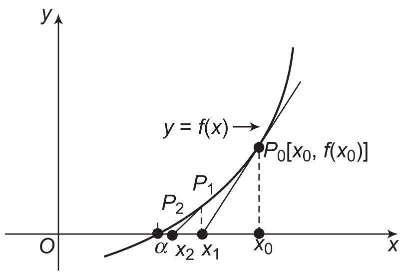


Fig. 9.3

which is a first approximation to the root α.

If P_1 is the point corresponding to x_1 on the curve then the tangent at P_1 will cut the x-axis at x_2 which is nearer to α and is the second approximation to the root. Repeating this process, the root α is approached quite rapidly. Thus, this method consists of replacing the part of the curve between the point P_0 and the x-axis by means of the tangent to the curve at P_0.

9.4.2 Convergence of the Newton—Raphson Method

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \phi(x_n) \qquad \text{...(9.1)}$$

The Newton–Raphson method converges if $|\phi'(x)| < 1$.

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

$$\phi'(x) = 1 - \left[\frac{[f'(x)]^2 - f(x)\, f''(x)}{[f'(x)]^2}\right] = \frac{f(x)\, f''(x)}{[f'(x)]^2}$$

$$|\phi'(x)| = \left|\frac{f(x)\, f''(x)}{[f'(x)]^2}\right|$$

Hence, the Newton–Raphson method converges if

$$\left|\frac{f(x)\, f''(x)}{[f'(x)]^2}\right| < 1$$

$$|f(x)\, f''(x)| < [f'(x)]^2 \qquad \text{...(9.2)}$$

If α is the actual root of $f(x) = 0$, a small interval should be selected in which $f(x)$, $f'(x)$ and $f''(x)$ are all continuous and the condition given by Eq. (9.2) is satisfied.

Hence, the Newton–Raphson method always converges provided the initial approximation x_0 is taken very close to the actual root α.

9.4.3 Rate of Convergence of the Newton–Raphson Method

Let α be exact root of $f(x) = 0$ and let x_n, x_{n+1} be two successive approximations to the actual root. If $\in_n$ and $\in_{n+1}$ are the corresponding errors then

$$x_n = \alpha + \in_n$$

$$x_{n+1} = \alpha + \in_{n+1}$$

Substituting in Eq. (9.1),

$$\alpha + \in_{n+1} = \alpha + \in_n - \frac{f(\alpha + \in_n)}{f'(\alpha + \in_n)}$$

$$\in_{n+1} = \in_n - \frac{f(\alpha + \in_n)}{f'(\alpha + \in_n)}$$

$$= \in_n - \frac{f(\alpha) + \in_n f'(\alpha) + \frac{\in_n^2}{2!} f''(\alpha) + \cdots}{f'(\alpha) + \in_n f''(\alpha) + \cdots} \qquad \text{[By Taylor's series]}$$

$$= \in_n - \frac{\in_n f'(\alpha) + \frac{\in_n^2}{2} f''(\alpha) + \cdots}{f'(\alpha) + \in_n f''(\alpha)} \qquad [\because \; f(\alpha) = 0]$$

Neglecting the derivatives of order higher than two,

$$\begin{aligned}
\in_{n+1} &= \in_n - \frac{\in_n f'(\alpha) + \dfrac{\in_n^2}{2} f''(\alpha)}{f'(\alpha) + \in_n f''(\alpha)} \\
&= \frac{1}{2}\left[\frac{\in_n^2 f''(\alpha)}{f'(\alpha) + \in_n f''(\alpha)}\right] \\
&= \frac{\in_n^2}{2}\left[\frac{\dfrac{f''(\alpha)}{f'(\alpha)}}{1 + \in_n \dfrac{f''(\alpha)}{f'(\alpha)}}\right] \\
&\approx \frac{\in_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \qquad \text{...(9.3)}
\end{aligned}$$

Equation (9.3) shows that the error at each stage is proportional to the square of the error in the previous stage. Hence, the Newton–Raphson method has a quadratic convergence and the convergence is of the order 2.

Example 1

Find the root of the equation $x^3 + x - 1 = 0$, *correct up to four decimal places.*

Solution

Let $\quad f(x) = x^3 + x - 1$

$f(0) = -1 \quad \text{and} \quad f(1) = 1$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 1$

$$f'(x) = 3x^2 + 1$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_0) = f(1) = 1$$

$$f'(x_0) = f'(1) = 4$$

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
&= 1 - \frac{1}{4} \\
&= 0.75
\end{aligned}$$

$$f(x_1) = f(0.75) = 0.171875$$
$$f'(x_1) = f'(0.75) = 2.6875$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 0.75 - \frac{0.171875}{2.6875}$$
$$= 0.68605$$
$$f(x_2) = f(0.68605) = 0.00894$$
$$f'(x_2) = f'(0.68605) = 2.41198$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 0.68605 - \frac{0.00894}{2.41198}$$
$$= 0.68234$$
$$f(x_3) = f(0.68234) = 0.000028$$
$$f'(x_3) = f'(0.68234) = 2.39676$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$= 0.68234 - \frac{0.000028}{2.39676}$$
$$= 0.68233$$

Since x_3 and x_4 are same up to four decimal places, the root is 0.6823.

Example 2

Find a root of $x^4 - x^3 + 10x + 7 = 0$, *correct up to three decimal places between –2 and –1 by the Newton–Raphson method.*

Solution

Let $f(x) = x^4 - x^3 + 10x + 7$
The root lies between –2 and –1.
Let $x_0 = -2$

$$f'(x) = 4x^3 - 3x^2 + 10$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$f(x_0) = f(-2) = 11$$

$$f'(x_0) = f'(-2) = -34$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= -2 - \frac{11}{(-34)}$$

$$= -1.6765$$

$$f(x_1) = f(-1.6765) = 2.8468$$

$$f'(x_1) = f'(-1.6765) = -17.2802$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= -1.6765 - \frac{2.8468}{(-17.2802)}$$

$$= -1.5118$$

$$f(x_2) = f(-1.5118) = 0.561$$

$$f'(x_2) = f'(-1.5118) = -10.6777$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= -1.5118 - \frac{0.561}{(-10.6777)}$$

$$= -1.4593$$

$$f(x_3) = f(-1.4593) = 0.0497$$

$$f'(x_3) = f(-1.4593) = -8.8193$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= -1.4593 - \frac{0.0497}{(-8.8193)}$$

$$= -1.4537$$

$$f(x_4) = f(-1.4537) = 0.0008$$

$$f'(x_4) = f'(-1.4537) = -8.6278$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$$

$$= -1.4537 - \frac{0.0008}{(-8.6278)}$$

$$= -1.4536$$

Since x_4 and x_5 are same up to three decimal places, a root is –1.453.

Example 3

Find the root of $x^4 - x - 10 = 0$, *correct up to three decimal places.*

Solution

Let $f(x) = x^4 - x - 10$

$$f(1) = -10, \quad \text{and} \quad f(2) = 4$$

Since $f(1) < 0$ and $f(2) > 0$, the root lies between 1 and 2.

Let $x_0 = 2$

$$f'(x) = 4x^3 - 1$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_0) = f(2) = 4$$

$$f'(x_0) = f'(2) = 31$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{4}{31}$$

$$= 1.871$$

$$f(x_1) = f(1.871) = 0.3835$$

$$f'(x_1) = f'(1.871) = 25.1988$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.871 - \frac{0.3835}{25.1988}$$

$$= 1.8558$$

$$f(x_2) = f(1.8558) = 5.2922 \times 10^{-3}$$

$$f'(x_2) = f'(1.8558) = 24.5655$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.8558 - \frac{5.2922 \times 10^{-3}}{24.5655}$$

$$= 1.8556$$

Since x_2 and x_3 are same up to three decimal places, the root is 1.855.

Example 4

Find the real root of $x \log_{10} x - 1.2 = 0$, *correct up to three decimal places.* **[Summer 2015]**

Solution

Let $\quad f(x) = x \log_{10} x - 1.2$

$$f(1) = -1.2,\ f(2) = -0.5979 \quad \text{and} \quad f(3) = 0.2314$$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

Let $\quad x_0 = 3$

$$f'(x) = \log_{10} x + x\frac{1}{x\log_e 10} = \log_{10} x + \log_{10} e = \log_{10} x + 0.4343$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_0) = f(3) = 0.2314$$

$$f'(x_0) = f'(3) = 0.9114$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{0.2314}{0.9114} = 2.7461$$

$$f(x_1) = f(2.7461) = 4.759\times10^{-3}$$

$$f'(x_1) = f'(2.7461) = 0.8730$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.7461 - \frac{4.759\times10^{-3}}{0.8730} = 2.7406$$

$$f(x_2) = f(2.7406) = -4.0202\times10^{-5}$$

$$f'(x_2) = f'(2.7406) = 0.8721$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7406 - \frac{(-4.0202\times10^{-5})}{0.8721} = 2.7406$$

Since x_2 and x_3 are the same up to three decimal places, the real root is 2.7406.

Example 5

Find a root between 0 *and* 1 *of the equation* $e^x \sin x = 1$, *correct up to four decimal places.*

Solution

Let $f(x) = e^x \sin x - 1$

$f(0) = -1$ and $f(1) = 1.28$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 0$

$$f'(x) = e^x (\cos x + \sin x)$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_0) = f(0) = -1$$

$$f'(x_0) = f'(0) = 1$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0 - \frac{(-1)}{1}$$

$$= 1$$

$$f(x_1) = f(1) = 1.2874$$

$$f'(x_1) = f'(1) = 3.7560$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1 - \frac{1.2874}{3.7560}$$

$$= 0.6572$$

$$f(x_2) = f(0.6572) = 0.1787$$

$$f'(x_2) = f'(0.6572) = 2.7062$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.6572 - \frac{0.1787}{2.7062}$$

$$= 0.5912$$

$$f(x_3) = f(0.5912) = 6.6742 \times 10^{-3}$$
$$f'(x_3) = f'(0.5912) = 2.5063$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$= 0.5912 - \frac{6.6742 \times 10^{-3}}{2.5063}$$
$$= 0.5885$$
$$f(x_4) = f(0.5885) = -8.1802 \times 10^{-5}$$
$$f'(x_4) = f'(0.5885) = 2.4982$$
$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$$
$$= 0.5885 - \frac{-8.1802 \times 10^{-5}}{2.4982}$$
$$= 0.5885$$

Since x_4 and x_5 are the same up to four decimal places, the root is 0.5885.

Example 6

Find the real root of the equation $3x = \cos x + 1$, *correct up to four decimal places.*

Solution

Let $f(x) = 3x - \cos x - 1$

$f(0) = -2$ and $f(1) = 1.4597$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 1$

$$f'(x) = 3 + \sin x$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$f(x_0) = f(1) = 1.4597$$
$$f'(x_0) = f'(1) = 3.8415$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 1 - \frac{1.4597}{3.8415}$$
$$= 0.62$$

$$f(x_1) = f(0.62) = 0.0461$$
$$f'(x_1) = f'(0.62) = 3.5810$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 0.62 - \frac{0.0461}{3.5810}$$
$$= 0.6071$$
$$f(x_2) = f(0.6071) = -5.8845 \times 10^{-6}$$
$$f'(x_2) = f'(0.6071) = 3.5705$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 0.6071 - \frac{-5.8845 \times 10^{-6}}{3.5705}$$
$$= 0.6071$$

Since x_2 and x_3 are the same up to four decimal places, the real root is 0.6071.

Example 7

Find the real positive root of the equation $x \sin x + \cos x = 0$, *which is near* $x = \pi$ *correct up to four significant digits.* **[Summer 2015]**

Solution

Let $f(x) = x \sin x + \cos x$

Let $x_0 = \pi$

$$f'(x) = x \cos x + \sin x - \sin x = x \cos x$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$f(x_0) = f(\pi) = -1$$
$$f'(x_0) = f'(\pi) = -\pi$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= \pi - \frac{(-1)}{(-\pi)}$$
$$= 2.82328$$

$$f(x_1) = f(2.82328) = -0.06618$$
$$f'(x_1) = f'(2.823287) = -2.68145$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 2.82328 - \frac{(-0.06618)}{(-2.68145)}$$
$$= 2.7986$$
$$f(x_2) = f(2.7986) = -0.00056$$
$$f'(x_2) = f'(2.7986) = -2.63559$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 2.7986 - \frac{(-0.00056)}{(-2.63559)}$$
$$= 2.79839$$
$$f(x_3) = f(2.79839) = -0.0001$$
$$f'(x_3) = f'(2.79839) = -2.63519$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$= 2.79839 - \frac{(-0.0001)}{(-2.63519)}$$
$$= 2.79839$$

Since x_3 and x_4 are same up to four decimal point, the root is 2.7983.

Example 8

Find the positive root of $x = \cos x$ using Newton's method correct to three decimal places.

Solution

Let $\quad f(x) = x - \cos x$

$$f(0) = -1 \quad \text{and} \quad f(1) = 0.4597$$

Since $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $\quad x_0 = 1$

$$f'(x) = 1 + \sin x$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x_0) = f(1) = 0.4597$$

$$f'(x_0) = f'(1) = 1.8415$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{0.4597}{1.8415}$$

$$= 0.7504$$

$$f(x_1) = f(0.7504) = 0.019$$

$$f'(x_1) = f'(0.7504) = 1.6819$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.7504 - \frac{0.019}{1.6819}$$

$$= 0.7391$$

$$f(x_2) = f(0.7391) = 0.00002$$

$$f'(x_2) = f'(0.7391) = 1.6736$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.7391 - \frac{0.00002}{1.6736}$$

$$= 0.7391$$

Since x_2 and x_3 are same up to three decimal places, the root is 0.739.

Example 9

Derive the iteration formula for $\sqrt{N}$ *and, hence, find*

(i) $\sqrt{28}$ **[Summer 2015]**

(ii) $\sqrt{65}$ **[Winter 2014]**

(iii) $\sqrt{3}$ **[Winter 2014]**

correct up to three decimal places.

Solution

Let $$x = \sqrt{N}$$
$$x^2 - N = 0$$

Let $$f(x) = x^2 - N$$
$$f'(x) = 2x$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^2 - N}{2x_n}$$
$$= \frac{x_n^2 + N}{2x_n}$$

This is the iteration formula for $\sqrt{N}$.

(i) For $N = 28$, $f(x) = x^2 - 28$

$$f(5) = -3 \quad \text{and} \quad f(6) = 8$$

Since $f(5) < 0$ and $f(6) > 0$, the root lies between 5 and 6.
Let $x_0 = 5$

$$x_{n+1} = \frac{x_n^2 + 28}{2x_n}$$
$$x_1 = \frac{x_0^2 + 28}{2x_0} = 5.3$$
$$x_2 = \frac{x_1^2 + 28}{2x_1} = 5.2915$$
$$x_3 = \frac{x_2^2 + 28}{2x_2} = 5.2915$$

Since x_2 and x_3 are same up to three decimal places,

$$\sqrt{28} = 5.2915$$

(ii) For $N = 65, f(x) = x^2 - 65$

$$f(8) = -1 \quad \text{and} \quad f(9) = 16$$

Since $f(8) < 0$ and $f(9) > 0$, the root lies between 8 and 9.

Let $x_0 = 8$

$$x_{n+1} = \frac{x_n^2 + 65}{2x_n}$$

$$x_1 = \frac{x_0^2 + 65}{2x_0} = 8.0625$$

$$x_2 = \frac{x_1^2 + 65}{2x_1} = 8.0623$$

Since x_1 and x_2 are same up to three decimal places,

$$\sqrt{65} = 8.0623$$

(iii) For $N = \sqrt{3},\ f(x) = x^2 - 3$

$$f(1) = -2 \quad \text{and} \quad f(2) = 1$$

Since $f(1) < 0$ and $f(2) > 0$, the root lies between 1 and 2.

Let $x_0 = 2$

$$x_{n+1} = \frac{x_n^2 + 3}{2x_n}$$

$$x_1 = \frac{x_0^2 + 3}{2x_0} = 1.75$$

$$x_2 = \frac{x_1^2 + 3}{2x_1} = 1.7321$$

$$x_3 = \frac{x_2^2 + 3}{2x_2} = 1.7321$$

Since x_2 and x_3 are same up to three decimal places,

$$\sqrt{3} = 1.7321$$

Example 10

Find an iterative formula for $\sqrt[k]{N}$ *, where N is a positive number and hence, evaluate (i)* $\sqrt[3]{11}$*, and (ii)* $\sqrt[3]{58}$ **[Summer 2015]**

Solution

Let $x = \sqrt[k]{N}$

$$x^k - N = 0$$

Let $$f(x) = x^k - N$$
$$f'(x) = kx^{k-1}$$

By the Newton–Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^k - N}{k\,x_n^{k-1}}$$
$$= \frac{(k-1)x_n^k + N}{k\,x_n^{k-1}}$$

This is the iterative formula for $\sqrt[k]{N}$.

(i) When $N = 11$ and $k = 3$,

$$f(x) = x^3 - 11$$
$$f(2) = -3 \quad \text{and} \quad f(3) = 16$$

Since $f(2) < 0$ and $f(3) > 0$, the root lies between 2 and 3.

Let $x_0 = 3$

$$x_{n+1} = \frac{2x_n^3 + 11}{3x_n^2}$$
$$x_1 = \frac{2x_0^3 + 11}{3x_0^2} = 2.4074$$
$$x_2 = \frac{2x_1^3 + 11}{3x_1^2} = 2.2376$$
$$x_3 = \frac{2x_2^3 + 11}{3x_2^2} = 2.2240$$
$$x_4 = \frac{2x_3^3 + 11}{3x_3^2} = 2.2240$$

Since x_3 and x_4 are same up to four decimal places,

$$\sqrt[3]{11} = 2.2240$$

(ii) When $N = 58$ and $k = 3$,

$$f(x) = x^3 - 58$$
$$f(3) = -31 \quad \text{and} \quad f(4) = 6$$

Since $f(3) < 0$ and $f(4) > 0$, the root lies between 3 and 4.

Let $x_0 = 4$

$$x_{n+1} = \frac{2x_n^3 + 58}{3x_n^2}$$

$$x_1 = \frac{2x_0^3 + 58}{3x_0^2} = 3.875$$

$$x_2 = \frac{2x_1^3 + 58}{3x_1^2} = 3.8709$$

$$x_3 = \frac{2x_2^3 + 58}{3x_2^2} = 3.8709$$

Since x_2 and x_3 are same up to four decimal places,

$$\sqrt[3]{58} = 3.8709$$

EXERCISE 9.3

I. Find the roots of the following equations using the Newton–Raphson method:

1. $x^3 - x - 1 = 0$

[Ans.: 1.3247]

2. $x^3 + 2x^2 + 50x + 7 = 0$

[Ans.: –0.1407]

3. $x^3 - 5x + 3 = 0$

[Ans.: 0.6566]

4. $x^4 - x - 9 = 0$

[Ans.: 1.8134]

5. $\cos x - xe^x = 0$

[Ans.: 0.5177]

6. $x \log_{10} x = 4.772393$

[Ans.: 6.0851]

7. $x - 2\sin x = 0$

[Ans.: 1.8955]

8. $x \tan x = 1.28$

[Ans.: 6.4783]

9. $\cos x = x^2$

[Ans.: 0.8241]

II. Find the values of the following:

1. $\sqrt{35}$

[Ans.: 5.916]

2. $\sqrt[3]{24}$

[Ans.: 2.884]

3. $\dfrac{1}{\sqrt{14}}$

[Ans.: 0.2673]

9.5 SECANT METHOD

The Newton–Raphson method requires the evaluation of two functions (the function and its derivative) per iteration. For complicated expressions, the method takes a large amount of time. Hence, it is desirable to have a method that converges as fast as the Newton–Raphson method but involves only evaluation of the function.

Let $f(x) = 0$ be the given equation. Let x_0 and x_1 be the approximate roots of the equation $f(x) = 0$ and $f(x_0)$ and $f(x_1)$ are their function values respectively. If x_2 is the point of intersection of the x-axis and the line joining points $P[x_0, f(x_0)]$ and $Q[x_1, f(x_1)]$ then x_2 is closer to the root α than x_0 and x_1 (Fig. 9.4).

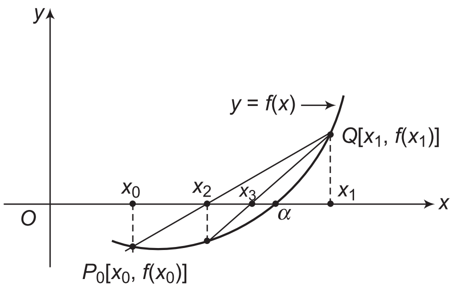


Fig. 9.4

Using the slope formula,

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0 - f(x_1)}{x_2 - x_1}$$

$$x_2 - x_1 = -\frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

Using x_1 and x_2, the process is repeated to obtain x_3.

In general,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n), \qquad n \geq 1$$

This method is similar to the regula falsi method. This method starts with two initial approximations x_0 and x_1 and calculates x_2 by the same formula as in the regula falsi method but proceeds to the next iteration without considering any root bracketing, i.e., the condition $f(x_0)\, f(x_1) < 0$.

Convergence of the Secant Method

By the Secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \qquad \text{...(9.4)}$$

Let α be the exact root of $f(x) = 0$ and let x_n, x_{n+1} be two successive approximations to the actual root.

If $\in_n$, $\in_{n-1}$, are the corresponding error then

$$x_n = \alpha + \in_n$$

$$x_{n-1} = \alpha + \in_{n-1}$$

$$x_{n+1} = \alpha + \in_{n+1}$$

Substituting in Eq. (9.4),

$$\alpha + \in_{n+1} = \alpha + \in_n - \frac{\in_n - \in_{n-1}}{f(\alpha + \in_n) - f(\alpha + \in_{n-1})} f(\alpha + \in_n)$$

$$\in_{n+1} = \in_n - \frac{\in_n - \in_{n-1}}{f(\alpha + \in_n) - f(\alpha + \in_{n-1})} f(\alpha + \in_n)$$

$$= \in_n - \frac{(\in_n - \in_{n-1})\left[f(\alpha) + \in_n f'(\alpha) + \frac{\in_n^2}{2} f''(\alpha) + \cdots\right]}{(\in_n - \in_{n-1}) f'(\alpha) + \frac{1}{2}(\in_n^2 - \in_{n-1}^2) f''(\alpha) + \cdots} \qquad \text{[By Taylor's series]}$$

$$= \in_n - \frac{\left(\in_n - \in_{n-1}\right)\left[\in_n f'(\alpha) + \frac{\in_n^2}{2} f''(\alpha)\right]}{\left(\in_n - \in_{n-1}\right) f'(\alpha) + \frac{1}{2}\left(\in_n^2 - \in_{n-1}^2\right) f''(\alpha)} \qquad \left[\begin{array}{l} \because\ f(\alpha) = 0 \\ \text{and neglecting} \\ \text{higher order} \\ \text{derivatives of } f(\alpha) \end{array}\right]$$

$$= \in_n - \frac{\left(\in_n - \in_{n-1}\right) \in_n f'(\alpha)\left[1 + \frac{\in_n}{2} \frac{f''(\alpha)}{f(\alpha)}\right]}{\left(\in_n - \in_{n-1}\right) f'(\alpha) + \left[1 + \frac{\left(\in_n + \in_{n-1}\right)}{2} \frac{f''(\alpha)}{f(\alpha)}\right]}$$

$$= \in_n - \frac{\in_n \left(1 + c \in_n\right)}{1 + c\left(\in_n + \in_{n-1}\right)}$$

$$\text{where } c = \frac{1}{2} \frac{f''(\alpha)}{f(\alpha)} \qquad \text{...(9.5)}$$

$$\in_{n+1} = \frac{\in_n + c\in_n^2 + c\in_n\in_{n-1} - \in_n - c\in_n^2}{1 + c\left(\in_n + \in_{n-1}\right)}$$

$$= \frac{c\in_n\in_{n-1}}{1 + c\left(\in_n + \in_{n-1}\right)}$$

$$\approx c\in_n\in_{n-1} \qquad \left[\because\ 1 + c\left(\in_n + \in_{n-1}\right) \approx 1\right] \qquad \text{...(9.6)}$$

Equation (9.5) is a nonlinear difference equation which can be solved by letting $\in_{n+1} = A\in_n^p$ or $\in_n = A\in_{n-1}^p$.

$$\therefore \qquad \in_{n-1} = \in_n^{\frac{1}{p}} A^{-\frac{1}{p}}$$

Substituting in Eq. (9.6),

$$A\in_n^p = c\in_n \in_n^{\frac{1}{p}} A^{-\frac{1}{p}}$$

$$\in_n^p = c\,A^{-\left(1+\frac{1}{p}\right)} \in_n^{1+\frac{1}{p}}$$

Equating the power of $\in_n$ on both the sides,

$$p = 1 + \frac{1}{p}$$

$$p^2 - p - 1 = 0$$

$$p = \frac{1}{2}(1 \pm \sqrt{5})$$

Taking the positive sign only,

$$p = 1.618$$

$$\in_{n+1} = A\in_n^{1.618}$$

Hence, the rate of convergence of the secant method is 1.618 which is lesser than the Newton–Raphson method. The secant method evaluates the function only once in each iteration, whereas the Newton–Raphson method evaluates two functions $f(x)$ and $f'(x)$ in each iteration. Hence, the secant method is more efficient than the Newton–Raphson method.

Example 1

Find the approximate root of $x^3 - 2x - 1 = 0$, *starting from* $x_0 = 1.5$, $x_1 = 2$, *correct upto three decimal places.*

Solution

Let $\qquad f(x) = x^3 - 2x - 1$

$$x_0 = 1.5, x_1 = 2$$

$$f(x_0) = f(1.5) = -0.625 \text{ and } f(x_1) = f(2) = 3$$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$= 2 - \frac{2 - 1.5}{3 + 0.625}(3)$$

$$= 1.5862$$

$$f(x_2) = f(1.5862) = -0.1815$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 1.5862 - \frac{1.5862 - 2}{-0.1815 - 3}(-0.1815)$$

$$= 1.6098$$

$$f(x_3) = f(1.6098) = -0.0479$$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$= 1.6098 - \frac{1.6098 - 1.5862}{-0.0479 + 0.1815}(-0.0479)$$

$$= 1.6183$$

$$f(x_4) = f(1.6183) = 0.0016$$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4)$$

$$= 1.6183 - \frac{1.6183 - 1.6098}{0.0016 + 0.0479}(0.0016)$$

$$= 1.6181$$

Since x_4 and x_5 are same up to three decimal places, the root is 1.618.

Example 2

Find the approximate root of the equation $x^3 + x^2 - 3x - 3 = 0$, *correct up to five decimal places.*

Solution

Let $f(x) = x^3 + x^2 - 3x - 3 = 0$

Let $x_0 = 1, x_1 = 2$

$f(x_0) = f(1) = -4$ and $f(x_1) = f(2) = 3$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$= 2 - \frac{2-1}{3+4}(3)$$

$$= 1.57143$$

$$f(x_2) = f(1.57143) = -1.36442$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 1.57143 - \frac{1.57143 - 2}{-1.36442 - 3}(-1.36442)$$

$$= 1.70541$$

$$f(x_3) = f(1.70541) = -0.24775$$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$= 1.70541 - \frac{1.70541 - 1.57143}{-0.24775 + 1.36442}(-0.24775)$$

$$= 1.73514$$

$$f(x_4) = f(1.73514) = 0.0293$$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4)$$

$$= 1.73514 - \frac{1.73514 - 1.70541}{0.0293 + 0.24775}(0.0293)$$

$$= 1.732$$

$$f(x_5) = f(1.732) = -0.00048$$

$$x_6 = x_5 - \frac{x_5 - x_4}{f(x_5) - f(x_4)} f(x_5)$$

$$= 1.732 - \frac{1.732 - 1.73514}{-0.00048 - 0.0293}(-0.00048)$$

$$= 1.73205$$

$$f(x_6) = f(1.73205) = -0.000008$$

$$\begin{aligned}
x_7 &= x_6 - \frac{x_6 - x_5}{f(x_6) - f(x_5)} f(x_6) \\
&= 1.73205 - \frac{1.73205 - 1.732}{-0.000008 + 0.00048}(-0.000008) \\
&= 1.73205
\end{aligned}$$

Since x_6 and x_7 are same up to five decimal places, the root is 1.73205.

Example 3

Find the root of $x \log_{10} x - 1.9 = 0$, *correct up to three decimal places with* $x_0 = 3$ *and* $x_1 = 4$.

Solution

Let $$f(x) = x \log_{10} x - 1.9$$

$$x_0 = 3,\ x_1 = 4$$

$$f(x_0) = f(3) = -0.4686 \text{ and } f(x_1) = f(4) = 0.5082$$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$\begin{aligned}
x_2 &= x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) \\
&= 4 - \frac{4-3}{0.5082 + 0.4686}(0.5082) \\
&= 3.4797
\end{aligned}$$

$$f(x_2) = f(3.4797) = -0.0156$$

$$\begin{aligned}
x_3 &= x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) \\
&= 3.4797 - \frac{3.4797 - 4}{-0.0156 - 0.5082}(-0.0156) \\
&= 3.4952
\end{aligned}$$

$$f(x_3) = f(3.4952) = -0.0005$$

$$\begin{aligned}
x_4 &= x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) \\
&= 3.4952 - \frac{3.4952 - 3.4797}{-0.0005 + 0.0156}(-0.0005) \\
&= 3.4957
\end{aligned}$$

Since x_3 and x_4 are same up to three decimal places, the root is 3.495.

Example 4

Find the positive solution of $x - 2\sin x = 0$, *correct up to three decimal places starting from* $x_0 = 2$ *and* $x_1 = 1.9$. **[Summer 2014]**

Solution

Let $f(x) = x - 2\sin x$

$$x_0 = 2,\ x_1 = 1.9$$

$$f(x_0) = f(2) = 0.1814 \text{ and } f(x_1) = f(1.9) = 0.0074$$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$\begin{aligned} x_2 &= x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) \\ &= 1.9 - \frac{1.9 - 2}{0.0074 - 0.1814}(0.0074) \\ &= 1.8957 \end{aligned}$$

$$f(x_2) = f(1.8957) = 0.00034$$

$$\begin{aligned} x_3 &= x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) \\ &= 1.8957 - \frac{1.8957 - 1.9}{0.00034 - 0.0074}(0.00034) \\ &= 1.8955 \end{aligned}$$

Since x_2 and x_3 are same up to three decimal places, the positive root is 1.895.

Example 5

Solve $xe^x - 1 = 0$, *correct up to three decimal places between* 0 *and* 1.

Solution

Let $f(x) = xe^x - 1$

Let $x_0 = 0,\ x_1 = 1$

$$f(x_0) = f(0) = -1 \text{ and } f(x_1) = f(1) = 1.7183$$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$\begin{aligned}
x_2 &= x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1) \\
&= 1 - \frac{1-0}{1.7183+1}(1.7183) \\
&= 0.3679 \\
f(x_2) &= f(0.3679) = -0.4685 \\
x_3 &= x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) \\
&= 0.3679 - \frac{0.3679-1}{-0.4685-1.7183}(-0.4685) \\
&= 0.5033 \\
f(x_3) &= f(0.5033) = -0.1675 \\
x_4 &= x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) \\
&= 0.5033 - \frac{0.5033-0.3679}{-0.1675+0.4685}(-0.1675) \\
&= 0.5786 \\
f(x_4) &= f(0.5786) = 0.032 \\
x_5 &= x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4) \\
&= 0.5786 - \frac{0.5786-0.5033}{0.032+0.1675}(0.032) \\
&= 0.5665 \\
f(x_5) &= f(0.5665) = -0.0018 \\
x_6 &= x_5 - \frac{x_5 - x_4}{f(x_5) - f(x_4)} f(x_5) \\
&= 0.5665 - \frac{0.5665-0.5786}{-0.0018-0.032}(-0.0018) \\
&= 0.5671 \\
f(x_6) &= f(0.5671) = -0.0001 \\
x_7 &= x_6 - \frac{x_6 - x_5}{f(x_6) - f(x_5)} f(x_6) \\
&= 0.5671 - \frac{0.5671-0.5665}{-0.0001+0.0018}(-0.0001) \\
&= 0.5671
\end{aligned}$$

Since x_6 and x_7 are same up to three decimal places, the root is 0.567.

Example 6

Find the root of $\cos x - xe^x = 0$, *correct up to three decimal places.*

Solution

Let $f(x) = \cos x - xe^x$

Let $x_0 = 0, x_1 = 1$

$$f(x_0) = f(0) = 1 \text{ and } f(x_1) = f(1) = -2.178$$

By the secant method,

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

$$= 1 - \frac{1-0}{-2.178-1}(-2.178)$$

$$= 0.3147$$

$$f(x_2) = f(0.3147) = 0.5198$$

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2)$$

$$= 0.3147 - \frac{0.3147-1}{0.5198+2.178}(0.5198)$$

$$= 0.4467$$

$$f(x_3) = f(0.4467) = 0.2036$$

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3)$$

$$= 0.4467 - \frac{0.4467-0.3147}{0.2036-0.5198}(0.2036)$$

$$= 0.5317$$

$$f(x_4) = f(0.5317) = -0.0429$$

$$x_5 = x_4 - \frac{x_4 - x_3}{f(x_4) - f(x_3)} f(x_4)$$

$$= 0.5317 - \frac{0.5317-0.4467}{-0.0429-0.2036}(-0.0429)$$

$$= 0.5169$$

$$f(x_5) = f(0.5169) = 0.0026$$

$$x_6 = x_5 - \frac{x_5 - x_4}{f(x_5) - f(x_4)} f(x_5)$$

$$= 0.5169 - \frac{0.5169 - 0.5317}{0.0026 + 0.0429}(0.0026)$$

$$= 0.5177$$

$$f(x_6) = f(0.5177) = 0.0002$$

$$x_7 = x_6 - \frac{x_6 - x_5}{f(x_6) - f(x_5)} f(x_6)$$

$$= 0.5177 - \frac{0.5177 - 0.5169}{0.0002 - 0.0026}(0.0002)$$

$$= 0.5178$$

Since x_6 and x_7 are same up to three decimal places, the root is 0.517.

EXERCISE 9.4

Find a real root of the following equations correct upto three decimal places using the secant method:

1. $x^3 - 2x^2 + 3x - 4 = 0$

[Ans.: 1.650**]**

2. $x^3 + 3x^2 - 3 = 0$

[Ans.: 0.879**]**

3. $e^x - 4x = 0$

[Ans.: 0.357**]**

4. $\sin x = e^x - 3x$

[Ans.: 0.360**]**

5. $2x - 7 - \log_{10} x = 0$

[Ans.: 3.789**]**

6. $e^x \tan x = 1$

[Ans.: 3.183**]**

7. $3x - 6 = \log_{10} x$

[Ans.: 2.108**]**

Points to Remember

Bisection Method

In this method, two points x_0 and x_1 are chosen such that $f(x_0)$ and $f(x_1)$ are of opposite signs. The first approximation to the root is

$$x_2 = \frac{x_0 + x_1}{2}$$

If $f(x_0)$and $f(x_2)$ are of opposite signs, the root lies between x_0 and x_2 and the next approximation x_3 is obtained as

$$x_3 = \frac{x_0 + x_2}{2}$$

This process is repeated till the root is obtained to the desired accuracy.

Regula Falsi Method

In this method, two points x_0 and x_1 are chosen such that $f(x_0)$ and $f(x_1)$ are of opposite signs.

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

which is an approximation to the root.

If $f(x_0)$ and $f(x_2)$ are of opposite signs, the root lies between x_0 and x_2, and the next approximation x_3 is obtained as

$$x_3 = x_0 - \frac{x_2 - x_0}{f(x_2) - f(x_0)} f(x_0)$$

If the root lies between x_1 and x_2, the next approximation x_3 is obtained as

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

This process is repeated till the root is obtained to the desired accuracy.

Newton–Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The Newton–Raphson method has a quadratic convergence and the convergence is of the order 2.

Secant Method

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$$

The rate of convergence of the secant method is 1.618.

CHAPTER 10

Numerical Solutions of Ordinary Differential Equations

Chapter Outline

10.1 Introduction
10.2 Taylor's Series Method
10.3 Euler's Method
10.4 Modified Euler's Method
10.5 Runge–Kutta Methods
10.6 Milne's Predictor-Corrector Method

10.1 INTRODUCTION

Many problems in science and engineering can be reduced to the problem of solving differential equations satisfying certain given conditions. The analytical method of solutions of differential equations can be applied to solve only a selected class of differential equations. In many physical and engineering problems, these methods cannot be used and, hence, numerical methods are used to solve such differential equations.

Consider the first-order differential equations

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition $y(x_0) = y_0$

A number of numerical methods yield solutions either as a power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y. Picard's and Taylor's series methods belong to the former class of solutions, whereas those of Euler, Runge–Kutta, Milne, etc., belong to the latter class. In these later methods, the values of y are calculated in short steps for equal intervals of x and are, therefore, termed step-by-step methods. In the Euler and Runge–Kutta methods, the interval length h should be kept small and, hence, these methods can be applied for tabulating y over a limited range only. If, however, the function values are desired over a wide range, the Milne method may be used. These later methods require starting values which are found by Picard's or Taylor series or Runge–Kutta methods.

10.2 TAYLOR'S SERIES METHOD

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{...(10.1)}$$

with the initial condition $y(x_0) = y_0$.
If $y(x)$ is the exact solution of Eq. (10.1) then the Taylor's series for $y(x)$ around $x = x_0$ is given by

$$y_1 = y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots \qquad \text{...(10.2)}$$

Putting $x - x_0 = h$ in Eq. (10.2),

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \qquad \text{...(10.3)}$$

Similarly, Taylor series for $y(x)$ around $x = x_1$ is given by

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \qquad \text{...(10.4)}$$

Proceeding in the same way,

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \cdots$$

Example 1

Solve $\frac{dy}{dx} = x + y$ *by the Taylor's series method. Start from* $x = 1$, $y = 0$, and carry to $x = 1.2$ with $h = 0.1$. **[Summer 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = x + y$$

(i) Given: $x_0 = 1$, $y_0 = 0$, $h = 0.1$, $x_1 = x_0 + h = 1 + 0.1 = 1.1$

$y' = x + y \qquad y_0' = 1 + 0 = 1$

$y'' = 1 + y' \qquad y_0'' = 1 + 1 = 2$

$$y''' = y'' \qquad y_0''' = 2$$

$$y^{\text{iv}} = y''' \qquad y_0^{\text{iv}} = 2$$

By Taylor's series,

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{\text{iv}} + \cdots$$

$$y_1 = y(1.1) = 0 + 0.1(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2) + \cdots$$

$$= 0.1103$$

(ii) Now, $x_1 = 1.1$, $y_1 = 0.1103$, $h = 0.1$, $x_2 = x_1 + h = 1.1 + 0.1 = 1.2$

$$y_1' = 1.1 + 0.1103 = 1.2103$$

$$y_1'' = 1 + 1.2103 = 2.2103$$

$$y_1''' = 2.2103$$

$$y_1^{\text{iv}} = 2.2103$$

By Taylor's series,

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{\text{iv}} + \cdots$$

$$y_2 = y(1.2) = 0.1103 + 0.1(1.2103) + \frac{(0.1)^2}{2!}(2.2103) + \frac{(0.1)^3}{3!}(2.2103) + \frac{(0.1)^4}{4!}(2.2103) + \cdots$$

$$= 0.2428$$

Example 2

Solve $\dfrac{dy}{dx} = 2y + 3e^x$ *with initial conditions* $x_0 = 0$, $y_0 = 1$ *by the Taylor's series method. Find the approximate value of y for* $x = 0.1$ and $x = 0.2$.

Solution

$$\frac{dy}{dx} = f(x, y) = 2y + 3e^x$$

(i) Given: $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $h = x_1 - x_0 = 0.1 - 0 = 0.1$

$$y' = 2y + 3e^x \qquad y_0' = 2(1) + 3e^0 = 5$$

$$y'' = 2y' + 3e^x \qquad y_0'' = 2(5) + 3e^0 = 13$$

$$y''' = 2y'' + 3e^x \qquad y_0''' = 2(13) + 3e^0 = 29$$

$$y^{\text{iv}} = 2y''' + 3e^x \qquad y_0^{\text{iv}} = 2(29) + 3e^0 = 61$$

By Taylor's series,

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{\text{iv}} + \cdots$$

$$y_1 = y(0.1) = 1 + 0.1(5) + \frac{(0.1)^2}{2!}(13) + \frac{(0.1)^3}{3!}(29) + \frac{(0.1)^4}{4!}(61) + \cdots$$

$$= 1.5700$$

(ii) Now, $x_1 = 0.1, y_1 = 1.5700, x_2 = 0.2, h = x_2 - x_1 = 0.2 - 0.1 = 0.1$

$$y_1' = 2(1.5700) + 3e^{0.1} = 6.4555$$

$$y_1'' = 2(6.4555) + 3e^{0.1} = 16.2265$$

$$y_1''' = 2(16.2265) + 3e^{0.1} = 35.7685$$

$$y_1^{\text{iv}} = 2(35.7685) + 3e^{0.1} = 74.8525$$

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{\text{iv}} + \cdots$$

$$y_2 = y(0.2) = 1.5700 + 0.1(6.4555) + \frac{(0.1)^2}{2!}(16.2265) + \frac{(0.1)^3}{3!}(35.7685) + \frac{(0.1)^4}{4!}(74.8525) + \cdots$$

$$= 2.303$$

Example 3

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2$ *with initial conditions* $x_0 = 0$, $y_0 = 0$ *by the Taylor's series method. Find the approximate value of y for* $x = 0.2$ *and* $x = 0.4$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = 1 + y^2$$

(i) Given: $x_0 = 0, \quad y_0 = 0, \quad x_1 = 0.2, \quad h = x_1 - x_0 = 0.2 - 0 = 0.2$

$$y' = 1 + y^2 \qquad y_0' = 1 + 0 = 1$$

$$y'' = 2yy' \qquad y_0'' = 0$$

$$y''' = 2yy'' + 2(y')^2 \qquad y_0''' = 0 + 2(1)^2 = 2$$

$$y^{\text{iv}} = 2y'y'' + 2yy''' + 4y'y'' = 2yy''' + 6y'y'' \qquad y_0^{\text{iv}} = 0$$

By Taylor's series,

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{\text{iv}} + \cdots$$

$$y_1 = y(0.2) = 0 + 0.2(1) + 0 + \frac{(0.2)^3}{3!}(2) + 0 + \cdots$$

$$= 0.2027$$

(ii) Now, $x_1 = 0.2$, $y_1 = 0.2027$, $x_2 = 0.4$, $h = x_2 - x_1 = 0.4 - 0.2 = 0.2$

$$y_1' = 1 + (0.2027)^2 = 1.0411$$

$$y_1'' = 2(0.2027)(1.0411) = 0.4221$$

$$y_1''' = 2(0.2027)(0.4221) + 2(1.0411)^2 = 2.3389$$

$$y_1^{\text{iv}} = 2(0.2027)(2.3389) + 6(1.0411)(0.4221) = 3.5849$$

By Taylor's series,

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{\text{iv}} + \cdots$$

$$y_2 = y(0.4) = 0.2027 + 0.2(1.0411) + \frac{(0.2)^2}{2!}(0.4221) + \frac{(0.2)^3}{3!}(2.3389) + \frac{(0.2)^4}{4!}(3.5849) + \cdots$$

$$= 0.4227$$

Example 4

Use the Taylor's series method to solve $\frac{dy}{dx} = x^2 y - 1$, $y(0) = 1$. *Also find* $y(0.03)$.

Solution

$$\frac{dy}{dx} = f(x, y) = x^2 y - 1$$

Given: $x_0 = 0$, $y_0 = 1$, $x = 0.03$, $h = x - x_0 = 0.03 - 0 = 0.03$

$$y' = x^2 y - 1 \qquad y_0' = 0 - 1 = -1$$

$$y'' = 2xy + x^2 y' \qquad y_0'' = 0$$

$$y''' = 2y + 4xy' + x^2 y'' \qquad y_0''' = 2(1) + 0 + 0 = 2$$

$$y^{\text{iv}} = 6y' + 6xy'' + x^2 y''' \qquad y_0^{\text{iv}} = 6(-1) + 0 + 0 = -6$$

By Taylor's series.

$$y(x) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{\text{iv}} + \cdots$$

$$y(0.03) = 1 + 0.03(-1) + 0 + \frac{(0.03)^3}{3!}(2) + \frac{(0.03)^4}{4!}(-6) + \cdots$$

$$= 0.970009$$

Example 5

Using the Taylor's series method, find correct to four decimal places, the value of $y(0.1)$*, given* $\frac{dy}{dx} = x^2 + y^2$ *and* $y(0) = 1$.

Solution

$$\frac{dy}{dx} = f(x, y) = x^2 + y^2$$

Given: $x_0 = 0$, $y_0 = 1$, $x = 0.1$, $h = x - x_0 = 0.1 - 0 = 0.1$

$$y' = x^2 + y^2 \qquad y_0' = 0 + 1 = 1$$

$$y'' = 2x + 2yy' \qquad y_0'' = 2(0) + 2(1)(1) = 2$$

$$y''' = 2 + 2yy'' + 2(y')^2 \qquad y_0''' = 2 + 2(1)(2) + 2(1)^2 = 8$$

$$y^{\text{iv}} = 6y'y'' + 2yy''' \qquad y_0^{\text{iv}} = 6(1)(2) + 2(1)(8) = 28$$

By Taylor's series,

$$y(x) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{\text{iv}} + \cdots$$

$$y(0.1) = 1 + 0.1(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(8) + \frac{(0.1)^4}{4!}(28) + \cdots$$

$$= 1.1115$$

Example 6

Using the Taylor's series method, find $y(1.1)$ *correct to four decimal places given that* $\frac{dy}{dx} = xy^{\frac{1}{3}}$, $y(1) = 1$, $h = 0.1$.

Solution

$$\frac{dy}{dx} = f(x, y) = xy^{\frac{1}{3}}$$

Given: $x_0 = 1$, $y_0 = 1$, $h = 0.1$, $x = x_0 + h = 1 + 0.1 = 1.1$

$$y' = xy^{\frac{1}{3}} \qquad y_0' = 1(1)^{\frac{1}{3}} = 1$$

$$y'' = \frac{1}{3}xy^{-\frac{2}{3}}y' + y^{\frac{1}{3}} = \frac{1}{3}x^2y^{-\frac{1}{3}} + y^{\frac{1}{3}} \qquad y_0'' = \frac{1}{3}(1)(1)^{-\frac{2}{3}} + (1)^{\frac{1}{3}} = \frac{4}{3}$$

$$y''' = \frac{1}{3}x^2\left(-\frac{1}{3}\right)y^{-\frac{4}{3}}y' + \frac{2}{3}xy^{-\frac{1}{3}} + \frac{1}{3}y^{-\frac{2}{3}}y' \qquad y_0''' = \frac{1}{3}(1)^2\left(-\frac{1}{3}\right)(1)^{-\frac{4}{3}}(1) + \frac{2}{3}(1)(1)^{-\frac{1}{3}} + \frac{1}{3}(1)^{-\frac{2}{3}}(1)$$

$$= -\frac{1}{9} + \frac{2}{3} + \frac{1}{3} = \frac{8}{9}$$

By Taylor's series,

$$y(x) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$y(1.1) = 1 + 0.1(1) + \frac{(0.1)^2}{2!}\left(\frac{4}{3}\right) + \frac{(0.1)^3}{3!}\left(\frac{8}{9}\right) + \cdots$$

$$= 1.1068$$

Example 7

Evaluate $y(0.1)$ *correct to four decimal places using the Taylor's series method if* $\frac{dy}{dx} = y^2 + x$, $y(0) = 1$.

[Summer 2015]

Solution

$$\frac{dy}{dx} = f(x, y) = y^2 + x$$

Given: $x_0 = 0$, $y_0 = 1$, $x = 0.1$, $h = x - x_0 = 0.1 - 0 = 0.1$

$$y' = y^2 + x \qquad y_0' = (1)^2 + 0 = 1$$

$$y'' = 2yy' + 1 \qquad y_0'' = 2(1)(1) + 1 = 3$$

$$y''' = 2yy'' + 2(y')^2 \qquad y_0''' = 2(1)(3) + 2(1)^2 = 8$$

$$y^{iv} = 2yy''' + 2y'y'' + 4y'y'' = 2yy''' + 6y'y'' \qquad y_0^{iv} = 2(1)(8) + 6(1)(3) = 34$$

By Taylor's series,

$$y(x) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y^{\text{iv}} + \cdots$$

$$y(0.1) = 1 + 0.1(1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(8) + \frac{(0.1)^4}{4!}(34) + \cdots$$

$$= 1.1165$$

EXERCISE 10.1

Solve the following differential equations:

1. $\frac{dy}{dx} = x^2 + y^2$ with $x_0 = 0, y_0 = 0$ at $x = 0.4$

[**Ans.:** 0.0215]

2. $\frac{dy}{dx} = y - xy$ with $x_0 = 0, y_0 = 2$

$\left[\textbf{Ans.: } 2 + 2x - \frac{2x^3}{3} - \frac{x^4}{6} + \cdots\right]$

3. $\frac{dy}{dx} = x - y^2$ with $x_0 = 0, y_0 = 1$ at $x = 0.1$

[**Ans.:** 0.9138]

4. $\frac{dy}{dx} = y \sin x + \cos x$ with $x_0 = 0, y_0 = 0$

$\left[\textbf{Ans.: } x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\right]$

5. $\frac{dy}{dx} = xy - 1$ with $x_0 = 1, y_0 = 2$ at $x = 1.02$

[**Ans.:** 2.0206]

6. $\frac{dy}{dx} = \frac{1}{x^2 + y^2}$ with $x_0 = 4, y_0 = 4$ at $x = 4.1$

[**Ans.:** 4.0031]

7. $\frac{dy}{dx} = 3x + \frac{1}{2}y$ with $x_0 = 0, y_0 = 1$ at $x = 0.1$

[**Ans.:** 1.0065]

8. $\frac{dy}{dx} = 3x + y^2$ with $y(0) = 1$ at $x = 0.1$

[**Ans.:** 1.1272]

9. $\frac{dy}{dx} = e^x - y^2$ with $y(0) = 1$ at $x = 0.1$

[**Ans.:** 1.005]

10. $\dfrac{dy}{dx} = -xy$ with $x_0 = 0, y_0 = 1$

$$\left[\textbf{Ans.:}\ 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \cdots\right]$$

10.3 EULER'S METHOD

Consider the differential equation

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition $y(x_0) = y_0$.

The solution of the differential equation is represented by the curve as shown in Fig. 10.1. The point $P_0(x_0, y_0)$ lies on the curve.

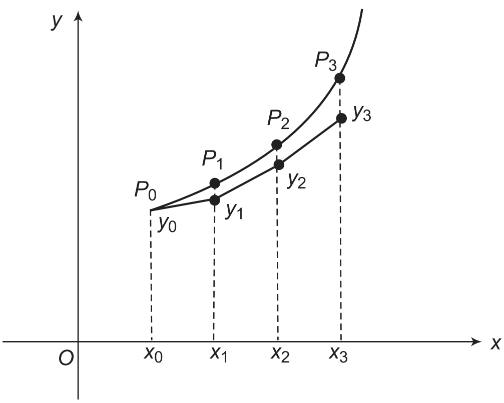


Fig. 10.1

At $x = x_0$, $$\left.\frac{dy}{dx}\right|_{x=x_0} = f(x_0, y_0)$$

The equation of the tangent to the curve at the point (x_0, y_0) is given by

$$y - y_0 = \left(\left.\frac{dy}{dx}\right|_{x=x_0}\right)(x - x_0)$$

$$= f(x_0, y_0)\,(x - x_0)$$

$$y = y_0 + f(x_0, y_0)\,(x - x_0)$$

If the point x_1 is very close to x_0, the curve is approximated by the tangent line in the interval (x_0, x_1). Hence, the value of y on the curve is approximately equal to the value of y on the tangent at the point (x_0, y_0) corresponding to $x = x_1$.

$\therefore$ $$y_1 = y_0 + f(x_0, y_0)\,(x_1 - x_0)$$

$$= y_0 + h\,f(x_0, y_0) \qquad \text{where } h = x_1 - x_0$$

At $x = x_1$, $$\left.\frac{dy}{dx}\right|_{x=x_1} = f(x_1, y_1)$$

Again the curve is approximated by the tangent line through the point (x_1, y_1).

$$y_2 = y_1 + h\,f(x_1, y_1)$$

Hence, $$y_{n+1} = y_n + h\,f(x_n, y_n)$$

This formula is known as *Euler's formula*. In this method, the actual curve is approximated by a sequence of short straight lines. As the step size h increases, the straight line deviates much from the actual curve.

Hence, accuracy cannot be obtained.

Example 1

Using Euler's method, find $y(0.2)$ *given* $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$ *with* $h = 0.1$.

Solution

$$\frac{dy}{dx} = f(x, y) = y - \frac{2x}{y}$$

Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad x = 0.2$

$$n = \frac{x - x_0}{h} = \frac{0.2 - 0}{0.1} = 2$$

$$x_1 = 0.1$$

$$\begin{aligned} y_1 &= y_0 + h\,f(x_0, y_0) \\ &= 1 + 0.1 f(0, 1) \\ &= 1 + 0.1\left[1 - \frac{2(0)}{1}\right] \\ &= 1.1 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + h\,f(x_1, y_1) \\ &= 1.1 + 0.1 f(0.1, 1.1) \\ &= 1.1 + 0.1\left[1.1 - \frac{2(0.1)}{1.1}\right] \\ &= 1.1918 \end{aligned}$$

Hence, $y_2 = y(0.2) = 1.1918$

Example 2

Find the value of y for $\frac{dy}{dx} = x + y$, $y(0) = 1$ *when* $x = 0.1, 0.2$ *with step size* $h = 0.05$. **[Summer 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = x + y$$

Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.05, \quad x = 0.2$

$$n = \frac{x - x_0}{h} = \frac{0.2 - 0}{0.05} = 4$$

$$x_1 = 0.05, \quad x_2 = 0.1, \quad x_3 = 0.15$$

$$\begin{aligned}
y_1 &= y_0 + h\, f(x_0, y_0)\\
&= 1 + 0.05 f(0, 1)\\
&= 1 + 0.05(0 + 1)\\
&= 1.05\\
y_2 &= y_1 + h\, f(x_1, y_1)\\
&= 1.05 + 0.05 f(0.05, 1.05)\\
&= 1.05 + 0.05(0.05 + 1.05)\\
&= 1.105\\
y_3 &= y_2 + h\, f(x_2, y_2)\\
&= 1.105 + 0.05 f(0.1, 1.105)\\
&= 1.105 + 0.05(0.1 + 1.105)\\
&= 1.16525\\
y_4 &= y_3 + h\, f(x_3, y_3)\\
&= 1.16525 + 0.05 f(0.15, 1.16525)\\
&= 1.16525 + 0.05(0.15 + 1.16525)\\
&= 1.231
\end{aligned}$$

Hence, $y_2 = y(0.1) = 1.105$

$y_4 = y(0.2) = 1.231$

Example 3

Solve the initial-value problem $\frac{dy}{dx} = x\sqrt{y},\ y(1) = 1$ *and, hence, find* $y(1.5)$ *by taking* $h = 0.1$ *using Euler's method.* **[Summer 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = x\sqrt{y}$$

Given: $x_0 = 1, \quad y_0 = 1, \quad h = 0.1, \quad x = 1.5$

$$n = \frac{x - x_0}{h} = \frac{1.5 - 1}{0.1} = 5$$

$$x_1 = 1.1, \quad x_2 = 1.2, \quad x_3 = 1.3, \quad x_4 = 1.4$$

$$
\begin{aligned}
y_1 &= y_0 + h\,f(x_0, y_0) \\
&= 1 + 0.1 f(1,1) \\
&= 1 + 0.1(1\sqrt{1}) \\
&= 1.1 \\
y_2 &= y_1 + h\,f(x_1, y_1) \\
&= 1.1 + 0.1 f(1.1, 1.1) \\
&= 1.1 + 0.1(1.1\sqrt{1.1}) \\
&= 1.2154 \\
y_3 &= y_2 + h\,f(x_2, y_2) \\
&= 1.2154 + 0.1 f(1.2, 1.2154) \\
&= 1.2154 + 0.1(1.2\sqrt{1.2154}) \\
&= 1.3477 \\
y_4 &= y_3 + h\,f(x_3, y_3) \\
&= 1.3477 + 0.1 f(1.3, 1.3477) \\
&= 1.3477 + 0.1(1.3\sqrt{1.3477}) \\
&= 1.4986 \\
y_5 &= y_4 + h\,f(x_4, y_4) \\
&= 1.4986 + 0.1 f(1.4, 1.4986) \\
&= 1.4986 + 0.1(1.4\sqrt{1.4986}) \\
&= 1.67
\end{aligned}
$$

Hence, $y_5 = y(0.5) = 1.67$

Example 4

Using Euler's method, find the approximate value of y at x = 1.5 taking $h = 0.1$. Given $\dfrac{dy}{dx} = \dfrac{y-x}{\sqrt{xy}}$ and $y(1) = 2$.

Solution

$$\frac{dy}{dx} = f(x, y) = \frac{y-x}{\sqrt{xy}}$$

Given: $x_0 = 1, \quad y_0 = 2, \quad h = 0.1, \quad x = 1.5$

$$n = \frac{x - x_0}{h} = \frac{1.5-1}{0.1} = 5$$

$$x_1 = 1.1, \quad x_2 = 1.2, \quad x_3 = 1.3, \quad x_4 = 1.4$$

$$\begin{aligned}
y_1 &= y_0 + h\,f(x_0, y_0)\\
&= 2 + 0.1f(1, 2)\\
&= 2 + 0.1\left[\frac{2-1}{\sqrt{1(2)}}\right]\\
&= 2.0707\\
y_2 &= y_1 + h\,f(x_1, y_1)\\
&= 2.0707 + 0.1(f(1.1, 2.0707)\\
&= 2.0707 + 0.1\left[\frac{2.0707 - 1.1}{\sqrt{1.1(2.0707)}}\right]\\
&= 2.1350\\
y_3 &= y_2 + h\,f(x_2, y_2)\\
&= 2.1350 + 0.1f(1.2, 2.1350)\\
&= 2.1350 + 0.1\left[\frac{2.1350 - 1.2}{\sqrt{1.2(2.1350)}}\right]\\
&= 2.1934\\
y_4 &= y_3 + h\,f(x_3, y_3)\\
&= 2.1934 + 0.1f(1.3, 2.1934)\\
&= 2.1934 + 0.1\left[\frac{2.1934 - 1.3}{\sqrt{1.3(2.1934)}}\right]\\
&= 2.2463\\
y_5 &= y_4 + h\,f(x_{4,}\ y_4)\\
&= 2.2463 + 0.1f\,(1.4, 2.2463)\\
&= 2.2463 + 0.1\left[\frac{2.2463 - 1.4}{\sqrt{1.4(2.2463)}}\right]\\
&= 2.2940
\end{aligned}$$

Hence, $y_5 = y(1.5) = 2.2940$

Example 5

Using Euler's method, find the approximate value of y at $x = 1$ taking $h = 0.2$. Given $\frac{dy}{dx} = x^2 + y^2$ and $y\,(0) = 1$.

Solution

$$\frac{dy}{dx} = f(x, y) = x^2 + y^2$$

Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.2, \quad x = 1$

$$n = \frac{x - x_0}{h} = \frac{1-0}{0.2} = 5$$

$$x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8$$

$$\begin{aligned}
y_1 &= y_0 + h\, f(x_0, y_0) \\
&= 1 + 0.2 f(0, 1) \\
&= 1 + 0.2[(0)^2 + (1)^2] \\
&= 1.2 \\
y_2 &= y_1 + h\, f(x_1, y_1) \\
&= 1.2 + 0.2 f(0.2, 1.2) \\
&= 1.2 + 0.2\left[(0.2)^2 + (1.2)^2\right] \\
&= 1.496 \\
y_3 &= y_2 + h\, f(x_2, y_2) \\
&= 1.496 + 0.2 f(0.4, 1.496) \\
&= 1.496 + 0.2\left[(0.4)^2 + (1.496)^2\right] \\
&= 1.9756 \\
y_4 &= y_3 + h\, f(x_3, y_3) \\
&= 1.9756 + 0.2 f(0.6, 1.9756) \\
&= 1.9756 + 0.2\left[(0.6)^2 + (1.9756)^2\right] \\
&= 2.8282 \\
y_5 &= y_4 + h\, f(x_4, y_4) \\
&= 2.8282 + 0.2 f(0.8, 2.8282) \\
&= 2.8282 + 0.2\left[(0.8)^2 + (2.8282)^2\right] \\
&= 4.5559
\end{aligned}$$

Hence, $y_5 = y(1) = 4.5559$

Example 6

Given $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y-x}{y+x}$ *with the initial condition* $y = 1$ *at* $x = 0$. *Find* y *at* $x = 0.1$ *in five steps.*

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = \frac{y-x}{y+x}$$

Given: $x_0 = 0, \quad y_0 = 1, \quad n = 5, \quad x = 0.1$

$$h = \frac{x - x_0}{n} = \frac{0.1 - 0}{5} = 0.02$$

$$x_1 = 0.02, \quad x_2 = 0.04, \quad x_3 = 0.06, \quad x_4 = 0.08$$

$$\begin{aligned}
y_1 &= y_0 + h\, f(x_0, y_0) \\
&= 1 + 0.02 f(0, 1) \\
&= 1 + 0.02\left(\frac{1-0}{1+0}\right) \\
&= 1.02 \\
y_2 &= y_1 + h\, f(x_1, y_1) \\
&= 1.02 + 0.02 f(0.02, 1.02) \\
&= 1.02 + 0.02\left(\frac{1.02 - 0.02}{1.02 + 0.02}\right) \\
&= 1.0392 \\
y_3 &= y_2 + h\, f(x_2, y_2) \\
&= 1.0392 + 0.02 f(0.04, 1.0392) \\
&= 1.0392 + 0.02\left(\frac{1.0392 - 0.04}{1.0392 + 0.04}\right) \\
&= 1.0577 \\
y_4 &= y_3 + h\, f(x_3, y_3) \\
&= 1.0577 + 0.02 f(0.06, 1.0577) \\
&= 1.0577 + 0.02\left(\frac{1.0577 - 0.06}{1.0577 + 0.06}\right) \\
&= 1.0756 \\
y_5 &= y_4 + h\, f(x_4, y_4) \\
&= 1.0756 + 0.02 f(0.08, 1.0756) \\
&= 1.0756 + 0.02\left(\frac{1.0756 - 0.08}{1.0756 + 0.08}\right) \\
&= 1.0928
\end{aligned}$$

Hence, $y_5 = y(0.1) = 1.0928$

EXERCISE 10.2

Solve the following differential equations using Euler's method:

1. $\frac{dy}{dx} = xy$ with $y(0) = 2$, $h = 0.2$ at $x = 1$

[Ans.: 2.9186**]**

2. $\dfrac{dy}{dx} = \dfrac{y-x}{x}$ with $y(1) = 2$ at $x = 2$ taking $h = 0.2$

[Ans.: 2.6137**]**

3. $\dfrac{dy}{dx} = y^2 - \dfrac{y}{x}$ with $y(1) = 1$ taking $h = 0.1$ at $x = 1.3$ and $x = 1.5$

[Ans.: 1.0268, 1.0889**]**

4. $\dfrac{dy}{dx} = x + y^2$ with $y(0) = 1$ taking $h = 0.1$ at $x = 0.2$

[Ans.: 1.231**]**

5. $\dfrac{dy}{dx} = 1 - 2xy$ with $y(0) = 0$ taking $h = 0.2$ at $x = 0.6$

[Ans.: 0.5226**]**

6. $\dfrac{dy}{dx} = x + \sqrt{y}$ with $y(2) = 4$ taking $h = 0.2$ at $x = 3$

[Ans.: 8.7839**]**

7. $\dfrac{dy}{dx} = x + y + xy$ with $y(0) = 1$ taking $h = 0.025$ at $x = 0.1$

[Ans.: 1.1117**]**

8. $\dfrac{dy}{dx} = 1 - y^2$ with $y(0) = 0$ taking $h = 0.2$ at $x = 1$

[Ans.: 0.8007**]**

10.4 MODIFIED EULER'S METHOD

The Euler's method is very easy to implement but it cannot give accurate solutions. A very small step size is required to get any meaningful result. Since the starting point of each sub-interval is used to find the slope of the solution curve, the solution would be correct only if the function is linear. In the modified Euler's method, the arithmetic average of the slopes is used to approximate the solution curve.

In the modified Euler's method, $y_1^{(0)}$ is first calculated from the Euler's method.

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0)$$

This value is improved by making use of average slopes at (x_0, y_0) and $(x_1, y_1^{(0)})$. The first approximation to y_1 is written as

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

This value of $y_1^{(1)}$ is further improved by the equation

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

which is the second approximation to y_1.

In general,

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right], n = 0, 1, 2,...$$

where $y_1^{(n)}$ is the n^{th} approximation to y_1.

The procedure will be terminated depending on the accuracy required. If two consecutive values of $y_1^{(k)}$ and $y_1^{(k+1)}$ are equal, $y_1 = y_1^{(k)}$.

Now, $y_2^{(0)}$ is calculated from the Euler's method.

$$y_2^{(0)} = y_1 + h f(x_1, y_1)$$

Better approximation to y_2 is obtained as

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

This procedure is repeated till two approximation to y_2 are equal. Proceeding in the same manner, other values, i.e., y_3, y_4, etc., can be calculated.

Example 1

Determine the value of y when $x = 0.1$ *correct up to four decimal places by taking* $h = 0.05$. *Given that* $y(0) = 1$ *and* $\dfrac{dy}{dx} = x^2 + y$.

Solution

$$\frac{dy}{dx} = f(x, y) = x^2 + y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.05, \quad x_1 = 0.05$

$$f(x_0, y_0) = 0 + 1 = 1$$

$$y_1^{(0)} = y_0 + h f(x_0, y_0) = 1 + 0.05(1) = 1.05$$

First approximation to y_1

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \\ &= 1 + \frac{0.05}{2}\left[1 + f(0.05, 1.05)\right] \\ &= 1 + \frac{0.05}{2}\left[1 + \{(0.05)^2 + 1.05\}\right] \\ &= 1.0513 \end{aligned}$$

Second approximation to y_1

$$
\begin{aligned}
y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \\
&= 1 + \frac{0.05}{2}[1 + f(0.05, 1.0513)] \\
&= 1 + \frac{0.05}{2}\left[1 + \{(0.05)^2 + 1.0513\}\right] \\
&= 1.0513
\end{aligned}
$$

Since the values of $y_1^{(1)}$ and $y_1^{(2)}$ are equal,

$$y_1 = y(0.05) = 1.0513$$

(ii) Now, $x_1 = 0.05, \quad y_1 = 1.0513, \quad h = 0.05, \quad x_2 = 0.1$

$$f(x_1, y_1) = (0.05)^2 + 1.0513 = 1.0538$$

$$y_2^{(0)} = y_1 + h\,f(x_1, y_1) = 1.0513 + 0.05(1.0538) = 1.1040$$

First approximation to y_2

$$
\begin{aligned}
y_2^{(1)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right] \\
&= 1.0513 + \frac{0.05}{2}[1.0538 + f(0.1, 1.1040)] \\
&= 1.0513 + \frac{0.05}{2}\left[1.0538 + \{(0.1)^2 + 1.1040\}\right] \\
&= 1.1055
\end{aligned}
$$

Second approximation to y_2

$$
\begin{aligned}
y_2^{(2)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] \\
&= 1.0513 + \frac{0.05}{2}[1.0538 + f(0.1, 1.1055)] \\
&= 1.0513 + \frac{0.05}{2}\left[1.0538 + \{(0.1)^2 + 1.1055\}\right] \\
&= 1.1055
\end{aligned}
$$

Since the values of $y_2^{(1)}$ and $y_2^{(2)}$ are equal,

$$y_2 = y(0.1) = 1.1055$$

Example 2

Using the modified Euler's method, solve $\dfrac{dy}{dx} = 1 - y$ *with the initial condition* $y(0) = 0$ *at* $x = 0.1, 0.2$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = 1 - y$$

(i) Given: $x_0 = 0, \quad y_0 = 0, \quad h = x_1 - x_0 = 0.1, \quad x_1 = 0.1$

$$f(x_0, y_0) = 1 - 0 = 1$$

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0) = 0 + 0.1(1) = 0.1$$

First approximation to y_1

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \\ &= 0 + \frac{0.1}{2}\left[1 + f(0.1, 0.1)\right] \\ &= 0 + \frac{0.1}{2}\left[1 + (1 - 0.1)\right] \\ &= 0.095 \end{aligned}$$

Second approximation to y_1

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \\ &= 0 + \frac{0.1}{2}\left[1 + f(0.1, 0.095)\right] \\ &= 0 + \frac{0.1}{2}\left[1 + (1 - 0.095)\right] \\ &= 0.0953 \end{aligned}$$

Third approximation to y_1

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \\ &= 0 + \frac{0.1}{2}\left[1 + f(0.1, 0.0953)\right] \\ &= 0 + \frac{0.1}{2}\left[1 + (1 - 0.0953)\right] \\ &= 0.0952 \end{aligned}$$

Fourth approximation to y_1

$$\begin{aligned} y_1^{(4)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right] \\ &= 0 + \frac{0.1}{2}\left[1 + f(0.1, 0.0952)\right] \\ &= 0 + \frac{0.1}{2}\left[1 + (1 - 0.0952)\right] \\ &= 0.0952 \end{aligned}$$

Since the values of $y_1^{(3)}$ and $y_1^{(4)}$ are equal,

$$y_1 = y(0.1) = 0.0952$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 0.0952, \quad h = 0.1, \quad x_2 = 0.2$

$$f(x_1, \quad y_1) = 1 - 0.0952 = 0.9048$$

$$y_2^{(0)} = y_1 + h\,f(x_1, y_1) = 0.0952 + 0.1(0.9048) = 0.1857$$

First approximation to y_2

$$\begin{aligned} y_2^{(1)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right] \\ &= 0.0952 + \frac{0.1}{2}\left[0.9048 + f(0.2, 0.1857)\right] \\ &= 0.0952 + \frac{0.1}{2}\left[0.9048 + (1 - 0.1857)\right] \\ &= 0.1812 \end{aligned}$$

Second approximation to y_2

$$\begin{aligned} y_2^{(2)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] \\ &= 0.0952 + \frac{0.1}{2}\left[0.9048 + f(0.2, 0.1812)\right] \\ &= 0.0952 + \frac{0.1}{2}\left[0.9048 + (1 - 0.1812)\right] \\ &= 0.1814 \end{aligned}$$

Third approximation to y_2

$$\begin{aligned} y_2^{(3)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right] \\ &= 0.0952 + \frac{0.1}{2}\left[0.9048 + f(0.2, 0.1814)\right] \\ &= 0.952 + \frac{0.1}{2}\left[0.9048 + (1 - 0.1814)\right] \\ &= 0.1814 \end{aligned}$$

Since the values of $y_2^{(2)}$ and $y_2^{(3)}$ are equal,

$$y_2 = y(0.2) = 0.1814$$

Example 3

Apply the modified Euler's method to solve the initial-value problem $y' = x + y$ with $y(0) = 0$ choosing $h = 0.2$ and compute y for $x = 0.2$, $x = 0.4$. **[Winter 2014]**

Solution

$$\frac{dy}{dx} = f(x, y) = x + y$$

(i) Given: $x_0 = 0, \quad y_0 = 0, \quad h = 0.2, \quad x_1 = 0.2$

$$f(x_0, y_0) = 0 + 0 = 0$$

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0) = 0 + 0.2(0) = 0$$

First approximation to y_1

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \\ &= 0 + \frac{0.2}{2}\left[0 + f(0.2, 0)\right] \\ &= 0 + \frac{0.2}{2}\left[0 + (0.2 + 0)\right] \\ &= 0.02 \end{aligned}$$

Second approximation to y_1

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \\ &= 0 + \frac{0.2}{2}\left[0 + f(0.2, 0.02)\right] \\ &= 0 + \frac{0.2}{2}\left[0 + (0.2 + 0.02)\right] \\ &= 0.022 \end{aligned}$$

Third approximation to y_1

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \\ &= 0 + \frac{0.2}{2}\left[0 + f(0.2, 0.022)\right] \\ &= 0 + \frac{0.2}{2}\left[0 + (0.2 + 0.022)\right] \\ &= 0.0222 \end{aligned}$$

Fourth approximation to y_1

$$\begin{aligned} y_1^{(4)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right] \\ &= 0 + \frac{0.2}{2}\left[0 + f(0.2, 0.0222)\right] \\ &= 0 + \frac{0.2}{2}\left[0 + (0.2 + 0.0222)\right] \\ &= 0.0222 \end{aligned}$$

Since the values of $y_1^{(3)}$ and $y_1^{(4)}$ are equal,

$$y_1 = y(0.2) = 0.0222$$

(ii) Now, $x_1 = 0.2, \quad y_1 = 0.0222, \quad h = 0.2, \quad x_2 = 0.4$

$$f(x_1, y_1) = 0.2 + 0.0222 = 0.2222$$

$$y_2^{(0)} = y_1 + h\,f(x_1, y_1) = 0.0222 + \frac{0.2}{2}(0.2222) = 0.0444$$

First approximation to y_2

$$\begin{aligned} y_2^{(1)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + f(0.4, 0.0444)\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + (0.4 + 0.0444)\right] \\ &= 0.0889 \end{aligned}$$

Second approximation to y_2

$$\begin{aligned} y_2^{(2)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + f(0.4, 0.0889)\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + (0.4 + 0.0889)\right] \\ &= 0.0933 \end{aligned}$$

Third approximation to y_2

$$\begin{aligned} y_2^{(3)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + f(0.4, 0.0933)\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + (0.4 + 0.0933)\right] \\ &= 0.0938 \end{aligned}$$

Fourth approximation to y_2

$$\begin{aligned} y_2^{(4)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(3)})\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + f(0.4, 0.0938)\right] \\ &= 0.0222 + \frac{0.2}{2}\left[0.2222 + (0.4 + 0.0938)\right] \\ &= 0.0938 \end{aligned}$$

Since the value of $y_2^{(3)}$ and $y_2^{(4)}$ are equal,

$$y_2 = y(0.4) = 0.0938$$

Example 4

Use modified Euler's method to find the value of y satisfying the equation $\frac{dy}{dx} = \log(x+y)$ *for* $x = 1.2$ *and* $x = 1.4$, *correct up to four decimal places by taking* $h = 0.2$. *Given that* $y(1) = 2$.

Solution

$$\frac{dy}{dx} = f(x, y) = \log(x+y)$$

(i) Given: $x_0 = 1, \quad y_0 = 2, \quad h = 0.2, \quad x_1 = 1.2$

$$f(x_0, y_0) = \log(1+2) = 1.0986$$

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0) = 2 + 0.2(1.0986) = 2.2197$$

First approximation to y_1

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \\ &= 2 + \frac{0.2}{2}\left[1.0986 + f(1.2, 2.2197)\right] \\ &= 2 + \frac{0.2}{2}\left[1.0986 + \log(1.2 + 2.2197)\right] \\ &= 2.2328 \end{aligned}$$

Second approximation to y_1

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \\ &= 2 + \frac{0.2}{2}\left[1.0986 + f(1.2, 2.2328)\right] \\ &= 2 + \frac{0.2}{2}\left[1.0986 + \log(1.2 + 2.2328)\right] \\ &= 2.2332 \end{aligned}$$

Third approximation to y_1

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \\ &= 2 + \frac{0.2}{2}\left[1.0986 + f(1.2, 2.2332)\right] \end{aligned}$$

$$= 2 + \frac{0.2}{2}[1.0986 + \log(1.2 + 2.2332)]$$

$$= 2.2332$$

Since the values of $y_1^{(2)}$ and $y_1^{(3)}$ are equal,

$$y_1 = y(1.2) = 2.2332$$

(ii) Now, $x_1 = 1.2, \quad y_1 = 2.2332, \quad h = 0.2, \quad x_2 = 1.4$

$$f(x_1, y_1) = \log(1.2 + 2.2332) = 1.2335$$

$$y_2^{(0)} = y_1 + h\,f(x_1, y_1) = 2.2332 + 0.2(1.2335) = 2.4799$$

First approximation to y_2

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 2.2332 + \frac{0.2}{2}[1.2335 + f(1.4, 2.4799)]$$

$$= 2.2332 + \frac{0.2}{2}[1.2335 + \log(1.4 + 2.4799)]$$

$$= 2.4291$$

Second approximation to y_2

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 2.2332 + \frac{0.2}{2}[1.2335 + f(1.4, 2.4921)]$$

$$= 2.2332 + \frac{0.2}{2}[1.2335 + \log(1.4 + 2.4921)]$$

$$= 2.4924$$

Third approximation to y_2

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$= 2.2332 + \frac{0.2}{2}[1.2335 + f(1.4, 2.4924)]$$

$$= 2.2332 + \frac{0.2}{2}[1.2335 + \log(1.4 + 2.4924)]$$

$$= 2.4924$$

Since the values of $y_2^{(2)}$ and $y_2^{(3)}$ are equal,

$$y_2 = y(1.4) = 2.4924$$

Example 5

Solve $\frac{dy}{dx} = 2 + \sqrt{xy}$ *with* $x_0 = 1.2$, $y_0 = 1.6403$ *by Euler's modified method for* $x = 1.6$, *correct up to four decimal places by taking* $h = 0.2$.

Solution

$$\frac{dy}{dx} = f(x, y) = 2 + \sqrt{xy}$$

(i) Given: $x_0 = 1.2, \quad y_0 = 1.6403, \quad h = 0.2, \quad x_1 = 1.4$

$$f(x_0, y_0) = 2 + \sqrt{(1.2)\,(1.6403)} = 3.4030$$

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0) = 1.6403 + 0.2\,(3.4030) = 2.3209$$

First approximation to y_1

$$\begin{aligned} y_1^{(1)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + f\,(1.4, 2.3209)\right] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + \left\{2 + \sqrt{(1.4)(2.3209)}\right\}\right] \\ &= 2.3609 \end{aligned}$$

Second approximation to y_1

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + f\,(1.4, 2.3609)\right] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + \left\{2 + \sqrt{(1.4)(2.3609)}\right\}\right] \\ &= 2.3624 \end{aligned}$$

Third approximation to y_1

$$\begin{aligned} y_1^{(3)} &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + f\,(1.4, 2.3624)\right] \\ &= 1.6403 + \frac{0.2}{2}\left[3.4030 + \left\{2 + \sqrt{(1.4)(2.3624)}\right\}\right] \\ &= 2.3625 \end{aligned}$$

Fourth approximation to y_1

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 1.6403 + \frac{0.2}{2}\left[3.4030 + f(1.4, 2.3625)\right]$$

$$= 1.6403 + \frac{0.2}{2}\left[3.4030 + \left\{2 + \sqrt{(1.4)(2.3625)}\right\}\right]$$

$$= 2.3625$$

Since the values of $y_1^{(3)}$ and $y_1^{(4)}$ are equal,

$$y_1 = y(1.4) = 2.3625$$

(ii) Now, $\quad x_1 = 1.4, \quad y_1 = 2.3625, \quad h = 0.2, \quad x_2 = 1.6$

$$f(x_1, y_1) = 2 + \sqrt{(1.4)\,(2.3625)} = 3.8187$$

$$y_2^{(0)} = y_1 + h\,f(x_1, y_1) = 2.3625 + 0.2\,(3.8187) = 3.1262$$

First approximation to y_2

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + f\,(1.6, 3.1262)\right]$$

$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + \left\{2 + \sqrt{(1.6)(3.1262)}\right\}\right]$$

$$= 3.1680$$

Second approximation to y_2

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + f(1.6, 3.1680)\right]$$

$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + \left\{2 + \sqrt{(1.6) + 3.1680)}\right\}\right]$$

$$= 3.1695$$

Third approximation to y_2

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + f(1.6, 3.1695)\right]$$

$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + \left\{2 + \sqrt{(1.6)(3.1695)}\right\}\right]$$

$$= 3.1696$$

Fourth approximation to y_2

$$y_2^{(4)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(3)})\right]$$

$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + f(1.6, 3.1696)\right]$$

$$= 2.3625 + \frac{0.2}{2}\left[3.8187 + \left\{2 + \sqrt{(1.6)(3.1696)}\right\}\right]$$

$$= 3.1696$$

Since the values of $y_2^{(3)}$ and $y_2^{(4)}$ are equal,

$$y_2 = y(1.6) = 3.1696$$

EXERCISE 10.3

Solve the following differential equations by the modified Euler's method:

1. $\frac{dy}{dx} = x + 3y$ with $x_0 = 0, y_0 = 1$ taking $h = 0.05$ at $x = 0.1$

[Ans.: 1.3548**]**

2. $\frac{dy}{dx} = x - y^2$ with $x_0 = 0, y_0 = 1$ taking $h = 0.05$ at $x = 0.1$

[Ans.: 0.9137**]**

3. $\frac{dy}{dx} = x + y$ with $x_0 = 0, y_0 = 1$ taking $h = 0.05$ at $x = 0.1$

[Ans.: 1.1104**]**

4. $\frac{dy}{dx} = -xy^2$ with $y(0) = 2$ for $x = 0.2$ by taking $h = 0.1$

[Ans.: 1.9238**]**

5. $\frac{dy}{dx} = 1 + \frac{y}{x}$ with $y(1) = 2$ for $x = 1.2$

[Ans.: 2.6182**]**

6. $\frac{dy}{dx} = x + \sqrt{y}$ with $y(0) = 1$ for $x = 0.2$

[Ans.: 1.2309**]**

7. $\frac{dy}{dx} = y^2 - \frac{y}{x}$ with $y(1) = 1$ for $x = 1.1$ taking $h = 0.05$

[Ans.: 1.0073**]**

8. $\frac{dy}{dx} = y - x$ with $y(0) = 2$ for $x = 0.2$

[Ans.: 2.4222**]**

10.5 RUNGE—KUTTA METHODS

Runge–Kutta methods do not require the determination of higher order derivatives. These methods require only the function values at different points on the sub-interval. The main advantage of Runge–Kutta methods is the self-starting feature and, consequently, the ease of programming. One disadvantage of Runge–Kutta methods is the requirement that the function must be evaluated at different values of x and y in every step of the function. This repeated determination of the function may result in a less efficient method with respect to computing time than other methods of comparable accuracy in which previously determined values of the dependent variable are used in the subsequent steps.

10.5.1 First-Order Runge—Kutta Method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition $y(x_0) = y_0$

By Euler's method,

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

Expanding LHS by Taylor's series,

$$y_{n+1} = y_n + h\, y_n' + \frac{h^2}{2!} y_n'' + \cdots$$

Euler's method is known as the first-order Runge–Kutta method.

10.5.2 Second-Order Runge—Kutta Method (Heun Method)

The second order Rungta-Kutta method is given by the equations

$$k_1 = h\, f(x_n, y_n)$$

$$k_2 = h\, f\,(x_n + h, y_n + k_1)$$

$$k = \frac{1}{2}(k_1 + k_2)$$

$$y_{n+1} = y_n + k$$

10.5.3 Third-Order Runge—Kutta Method

The third-order Runge–Kutta method is given by the equations

$$k_1 = h\, f(x_n, y_n)$$

$$k_2 = h\, f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2} \right)$$

$$k_3 = h\,f(x_n + h,\, y_n + 2k_2 - k_1)$$
$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$
$$y_{n+1} = y_n + k$$

10.5.4 Fourth-Order Runge—Kutta Method

This method is mostly used and is often referred to as the Runge–Kutta method only without reference of the order. The fourth-order Runge–Kutta method is given by the equations

$$k_1 = h\,f(x_n, y_n)$$
$$k_2 = h\,f\left(x_n + \frac{h}{2},\, y_n + \frac{k_1}{2}\right)$$
$$k_3 = h\,f\left(x_n + \frac{h}{2},\, y_n + \frac{k_2}{2}\right)$$
$$k_4 = h\,f(x_n + h,\, y_n + k_3)$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$y_{n+1} = y_n + k$$

Example 1

Given that $y = 1.3$ *when* $x = 1$ *and* $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x + y$. *Use the second-order Runge–Kutta method (i.e., Heun method) to approximate* y *when* $x = 1.2$. *Use a step size of* 0.1. **[Winter 2012]**

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = 3x + y$$

(i) Given: $x_0 = 1, \quad y_0 = 1.3, \quad h = 0.1, \quad n = 0$

$$\begin{aligned} k_1 &= h\,f(x_0, y_0) \\ &= 0.1\,f(1, 1.3) \\ &= 0.1\big[3(1) + 1.3\big] \\ &= 0.43 \end{aligned}$$

$$\begin{aligned} k_2 &= h\,f(x_0 + h,\, y_0 + k_1) \\ &= 0.1\,f(1 + 0.1, 1.3 + 0.43) \end{aligned}$$

$$= 0.1\, f(1.1, 1.73)$$
$$= 0.1\left[3(1.1) + 1.73\right]$$
$$= 0.503$$
$$k = \frac{1}{2}(k_1 + k_2)$$
$$= \frac{1}{2}(0.43 + 0.503)$$
$$= 0.4665$$
$$y_1 = y_0 + k$$
$$= 1.3 + 0.4665$$
$$= 1.7665$$

(ii) Now, $x_1 = 1.1, \quad y_1 = 1.7665, \quad h = 0.1, \quad n = 1$

$$k_1 = h\, f(x_1, y_1)$$
$$= 0.1\, f(1.1,\ 1.7665)$$
$$= 0.1\ [3(1.1) + 1.7665]$$
$$= 0.5067$$
$$k_2 = h\, f(x_1 + h,\ y_1 + k_1)$$
$$= 0.1\, f(1.1 + 0.1, 1.7665 + 05067)$$
$$= 0.1\, f(1.2,\ 2.2732)$$
$$= 0.1\left[3(1.2) + 2.2732)\right]$$
$$= 0.5873$$
$$k = \frac{1}{2}(k_1 + k_2)$$
$$= \frac{1}{2}(0.5067 + 0.5873)$$
$$= 0.5470$$
$$y_2 = y_1 + k$$
$$= 1.7665 + 0.5470$$
$$= 2.3135$$

Hence, $\quad y_2 = y(1.2) = 2.3135$

Example 2

Use the second-order Runge–Kutta method to find an approximate value of y given that $\dfrac{dy}{dx} = x - y^2$ *and* $y(0) = 1$ *at* $x = 0.2$ *taking* $h = 0.1$.

Solution

$$\frac{dy}{dx} = f(x, y) = x - y^2$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$\begin{aligned} k_1 &= h\, f(x_0, y_0) \\ &= 0.1\, f(0, 1) \\ &= 0.1\left[0 - (1)^2\right] \\ &= -0.1 \end{aligned}$$

$$\begin{aligned} k_2 &= h\, f(x_0 + h, y_0 + k_1) \\ &= 0.1\, f[0 + 0.1, 1 + (-0.1)] \\ &= 0.1\, f(0.1, 0.9) \\ &= 0.1\left[0.1 - (0.9)^2\right] \\ &= -0.071 \end{aligned}$$

$$\begin{aligned} k &= \frac{1}{2}(k_1 + k_2) \\ &= \frac{1}{2}(-0.1 - 0.071) \\ &= -0.0855 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + k \\ &= 1 - 0.0855 \\ &= 0.9145 \end{aligned}$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 0.9145, \quad h = 0.1, \quad n = 1$

$$\begin{aligned} k_1 &= h\, f(x_1, y_1) \\ &= 0.1\, f(0.1, 0.9145) \\ &= 0.1\left[0.1 - (0.9145)^2\right] \\ &= -0.0736 \end{aligned}$$

$$\begin{aligned} k_2 &= h\, f(x_1 + h, y_1 + k_1) \\ &= 0.1\, f\left[0.1 + 0.1, 0.9145 - 0.0736\right] \\ &= 0.1\, f\,(0.2, 0.8408) \\ &= 0.1\left[0.2 - (0.8408)^2\right] \\ &= -0.0507 \end{aligned}$$

$$\begin{aligned} k &= \frac{1}{2}(k_1 + k_2) \\ &= \frac{1}{2}(-0.0736 - 0.0507) \\ &= -0.0622 \\ y_2 &= y_1 + k \\ &= 0.9145 - 0.0622 \\ &= 0.8523 \end{aligned}$$

Hence, $y_2 = y(0.2) = 0.8523$

Example 3

Obtain the values of y at $x = 0.1, 0.2$ *using the Runge–Kutta method of third order for the differential equation* $\frac{dy}{dx} + y = 0,\ y(0) = 1$.

Solution

$$\frac{dy}{dx} = f(x, y) = -y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$\begin{aligned} k_1 &= h\, f(x_0, y_0) \\ &= 0.1\, f(0, 1) \\ &= 0.1\,(-1) \\ &= -0.1 \\ k_2 &= h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\ &= 0.1\, f\left(0 + \frac{0.1}{2}, 1 - \frac{0.1}{2}\right) \\ &= 0.1\, f(0.05, 0.95) \\ &= 0.1(-0.95) \\ &= -0.095 \\ k_3 &= h\, f(x_0 + h, y_0 + 2k_2 - k_1) \\ &= 0.1\, f[0 + 0.1, 1 + 2(-0.095) + 0.1] \\ &= 0.1\, f(0.1, 0.91) \\ &= 0.1(-0.91) \\ &= -0.091 \end{aligned}$$

$$\begin{aligned}
k &= \frac{1}{6}(k_1 + 4k_2 + k_3)\\
&= \frac{1}{6}[-0.1 + 4(-0.095) - 0.091]\\
&= -0.0952\\
y_1 &= y_0 + k\\
&= 1 - 0.0952\\
&= 0.9048
\end{aligned}$$

Hence, $y_1 = y(0.1) = 0.9048$

(ii) Now, $x_1 = 0.1, \quad y_1 = 0.9048, \quad h = 0.1, \quad n = 1$

$$\begin{aligned}
k_1 &= h\,f(x_1, y_1)\\
&= 0.1\,f(0.1, 0.9048)\\
&= 0.1(-0.9048)\\
&= -0.0905\\
k_2 &= h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)\\
&= 0.1\,f\left(0.1 + \frac{0.1}{2}, 0.9048 - \frac{0.0905}{2}\right)\\
&= 0.1\,f(0.15, 0.8596)\\
&= 0.1(-0.8596)\\
&= -0.086\\
k_3 &= h\,f(x_1 + h, y_1 + 2k_2 - k_1)\\
&= 0.1\,f[0.1 + 0.1, 0.9048 + 2(-0.086) + 0.0905]\\
&= 0.1\,f(0.2, 0.8233)\\
&= 0.1\,(-0.8233)\\
&= -0.0823\\
k &= \frac{1}{6}(k_1 + 4k_2 + k_3)\\
&= \frac{1}{6}[-0.0905 + 4(-0.086) - 0.0823]\\
&= -0.0861\\
y_2 &= y_1 + k\\
&= 0.9048 - 0.0861\\
&= 0.8187
\end{aligned}$$

Hence, $y_2 = y(0.2) = 0.8187$

Example 4

Apply the third-order Runge–Kutta method to the initial-value problem $\frac{dy}{dx} = x^2 - y,\ y(0) = 1$ *over the interval* (0, 0.2) *taking* h = 0.1.

Solution

$$\frac{dy}{dx} = f(x, y) = x^2 - y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$\begin{aligned}
k_1 &= h\,f(x_0, y_0) \\
&= 0.1\,f(0, 1) \\
&= 0.1(0-1) \\
&= -0.1 \\
k_2 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\
&= 0.1\,f\left(0 + \frac{0.1}{2}, 1 - \frac{0.1}{2}\right) \\
&= 0.1\,f(0.05, 0.95) \\
&= 0.1\left[(0.05)^2 - 0.95\right] \\
&= -0.0948 \\
k_3 &= h\,f(x_0 + h, y_0 + 2k_2 - k_1) \\
&= 0.1\,f\left[0 + 0.1, 1 + 2(-0.0948) + 0.1\right] \\
&= 0.1\,f(0.1, 0.9104) \\
&= 0.1\left[(0.1)^2 - 0.9104\right] \\
&= -0.09 \\
k &= \frac{1}{6}(k_1 + 4k_2 + k_3) \\
&= \frac{1}{6}\left[-0.1 + 4(-0.0948) - 0.09\right] \\
&= -0.0949 \\
y_1 &= y_0 + k \\
&= 1 - 0.0949 \\
&= 0.9051
\end{aligned}$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 0.9051, \quad h = 0.1, \quad n = 1$

$$\begin{aligned} k_1 &= h\, f(x_1, y_1) \\ &= 0.1\, f(0.1, 0.9051) \\ &= 0.1\left[(0.1)^2 - 0.9051\right] \\ &= -0.0895 \\ k_2 &= h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) \\ &= 0.1\, f\left(0.1 + \frac{0.1}{2}, 0.9051 - \frac{0.0895}{2}\right) \\ &= 0.1\, f(0.15, 0.8604) \\ &= 0.1\left[(0.15)^2 - 0.8604\right] \\ &= -0.0838 \\ k_3 &= h\, f(x_1 + h, y_1 + 2k_2 - k_1) \\ &= 0.1\, f\left[0.1 + 0.1, 0.9051 + 2(-0.0838) + 0.0895\right] \\ &= 0.1\, f(0.2, 0.827) \\ &= 0.1\left[(0.2)^2 - 0.827\right] \\ &= -0.0787 \end{aligned}$$

$$\begin{aligned} k &= \frac{1}{6}(k_1 + 4k_2 + k_3) \\ &= \frac{1}{6}\left[-0.0895 + 4(-0.0838) - 0.0787\right] \\ &= -0.0839 \\ y_2 &= y_1 + k \\ &= 0.9051 - 0.0839 \\ &= 0.8212 \end{aligned}$$

Example 5

Solve the differential equation $\dfrac{dy}{dx} = x + y$, *with the fourth-order Runge–Kutta method, where* $y(0) = 1$, *with* $x = 0$ *to* $x = 0.2$ *with* $h = 0.1$.

[Winter 2012]

Solution

$$\frac{dy}{dx} = f(x, y) = x + y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$\begin{aligned}
k_1 &= h\, f(x_0, y_0) \\
&= 0.1\, f(0, 1) \\
&= 0.1\,(0+1) \\
&= 0.1 \\
k_2 &= h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\
&= 0.1\, f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right) \\
&= 0.1\, f(0.05, 1.05) \\
&= 0.1\,(0.05 + 1.05) \\
&= 0.11 \\
k_3 &= h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\
&= 0.1\, f\left(0 + \frac{0.1}{2}, 1 + \frac{0.11}{2}\right) \\
&= 0.1\, f(0.05, 1.055) \\
&= 0.1(0.05 + 1.055) \\
&= 0.1105 \\
k_4 &= h\, f(x_0 + h, y_0 + k_3) \\
&= 0.1\, f(0 + 0.1, 1 + 0.1105) \\
&= 0.1\, f(0.1, 1.1105) \\
&= 0.1(0.1 + 1.1105) \\
&= 0.1211 \\
k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&= \frac{1}{6}\left[0.1 + 2(0.11) + 2(0.1105) + 0.1211\right] \\
&= 0.1103 \\
y_1 &= y_0 + k \\
&= 1 + 0.1103 \\
&= 1.1103
\end{aligned}$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 1.1103, \quad h = 0.1, \quad n = 1$

$$\begin{aligned}
k_1 &= h\, f(x_1, y_1) \\
&= 0.1\, f(0.1, 1.1103) \\
&= 0.1(0.1 + 1.1103) \\
&= 0.1210
\end{aligned}$$

$$
\begin{aligned}
k_2 &= h\,f\left(x_1+\frac{h}{2},\, y_1+\frac{k_1}{2}\right)\\
&= 0.1\,f\left(0.1+\frac{0.1}{2},\, 1.1103+\frac{0.1210}{2}\right)\\
&= 0.1\,f(0.15, 1.1708)\\
&= 0.1\,(0.15+1.1708)\\
&= 0.1321\\
k_3 &= h\,f\left(x_1+\frac{h}{2},\, y_1+\frac{k_2}{2}\right)\\
&= 0.1\,f\left(0.1+\frac{0.1}{2},\, 1.1103+\frac{0.1321}{2}\right)\\
&= 0.1\,f(0.15, 1.1764)\\
&= 0.1\,(0.15+1.1764)\\
&= 0.1326\\
k_4 &= h\,f(x_1+h,\, y_1+k_3)\\
&= 0.1\,f(0.1+0.1,\, 1.1103+0.1326)\\
&= 0.1\,f(0.2, 1.2429)\\
&= 0.1(0.2+1.2429)\\
&= 0.1443\\
k &= \frac{1}{6}(k_1+2k_2+2k_3+k_4)\\
&= \frac{1}{6}\left[0.1210+2(0.1321)+2(0.1326)+0.1443\right]\\
&= 0.1325\\
y_2 &= y_1+k\\
&= 1.1103+0.1325\\
&= 1.2428
\end{aligned}
$$

Example 6

Using the Runge–Kutta method of fourth-order, solve $10\dfrac{dy}{dx}=x^2+y^2$, $y(0)=1$ *at* $x=0.1$ *and* $x=0.2$ *taking* $h=0.1$. **[Summer 2015]**

Solution

$$\frac{dy}{dx}=f(x,y)=\frac{x^2+y^2}{10}=0.1(x^2+y^2)$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$\begin{aligned}
k_1 &= h f(x_0, y_0)\\
&= 0.1\, f(0, 1)\\
&= 0.1(0.1)(0+1)\\
&= 0.01\\
k_2 &= h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)\\
&= 0.1\, f\left(0 + \frac{0.1}{2}, 1 + \frac{0.01}{2}\right)\\
&= 0.1\, f(0.05, 1.005)\\
&= 0.1(0.1)\left[(0.05)^2 + (1.005)^2\right]\\
&= 0.0101\\
k_3 &= h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)\\
&= 0.1\, f\left(0 + \frac{0.1}{2}, 1 + \frac{0.0101}{2}\right)\\
&= 0.1\, f(0.05, 1.0051)\\
&= 0.1(0.1)\left[(0.05)^2 + (1.0051)^2\right]\\
&= 0.0101\\
k_4 &= h f(x_0 + h, y_0 + k_3)\\
&= 0.1\, f(0 + 0.1, 1 + 0.0101)\\
&= 0.1\, f(0.1, 1.0101)\\
&= 0.1(0.1)\left[(0.1)^2 + (1.0101)^2\right]\\
&= 0.0103\\
k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\\
&= \frac{1}{6}[0.01 + 2(0.0101) + 2(0.0101) + 0.0103]\\
&= 0.0101\\
y_1 &= y_0 + k\\
&= 1 + 0.0101\\
&= 1.0101
\end{aligned}$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 1.0101, \quad h = 0.1, \quad n = 1$

$$\begin{aligned}
k_1 &= h\,f(x_1, y_1) \\
&= h\,f(0.1, 1.0101) \\
&= 0.1(0.1)\left[(0.1)^2 + (1.0101)^2\right] \\
&= 0.0103 \\
k_2 &= h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) \\
&= 0.1\,f\left(0.1 + \frac{0.1}{2}, 1.0101 + \frac{0.0103}{2}\right) \\
&= 0.1\,f(0.15, 1.0153) \\
&= 0.1(0.1)\left[(0.15)^2 + (1.0153)^2\right] \\
&= 0.0105 \\
k_3 &= h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) \\
&= 0.1\,f\left(0.1 + \frac{0.1}{2}, 1.0101 + \frac{0.0105}{2}\right) \\
&= 0.1\,f(0.15, 1.0154) \\
&= 0.1(0.1)\left[(0.15)^2 + (1.0154)^2\right] \\
&= 0.0105 \\
k_4 &= h\,f(x_1 + h, y_1 + k_3) \\
&= 0.1\,f(0.1 + 0.1, 1.0101 + 0.0105) \\
&= 0.1\,f(0.2, 1.0206) \\
&= 0.1\,(0.1)\left[(0.2)^2 + (1.0206)^2\right] \\
&= 0.0108 \\
k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&= \frac{1}{6}\left[0.0103 + 2(0.0105) + 2(0.0105) + 0.0108\right] \\
&= 0.0105 \\
y_2 &= y_1 + k \\
&= 1.0101 + 0.0105 \\
&= 1.0206
\end{aligned}$$

Example 7

Use the fourth order Runge–Kutta method to find the value of y at $x = 1$, *given that* $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$ *with* $h = 0.5$. **[Summer 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = \frac{y-x}{y+x}$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.5, \quad n = 0$

$$\begin{aligned}
k_1 &= h\,f(x_0, y_0) \\
&= 0.5\,f(0, 1) \\
&= 0.5\left(\frac{1-0}{1+0}\right) \\
&= 0.5 \\
k_2 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\
&= 0.5\,f\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2}\right) \\
&= 0.5\,f(0.25, 1.25) \\
&= 0.5\left(\frac{1.25-0.25}{1.25+0.25}\right) \\
&= 0.3333 \\
k_3 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\
&= 0.5\,f\left(0 + \frac{0.5}{2}, 1 + \frac{0.3333}{2}\right) \\
&= 0.5\,f(0.25, 1.1667) \\
&= 0.5\left(\frac{1.1667-0.25}{1.1667+0.25}\right) \\
&= 0.3235 \\
k_4 &= h\,f(x_0 + h, y_0 + k_3) \\
&= 0.5\,f(0 + 0.5, 1 + 0.3235) \\
&= 0.5(0.5, 1.3235) \\
&= 0.5\left(\frac{1.3235-0.5}{1.3235+0.5}\right) \\
&= 0.2258
\end{aligned}$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}\left[0.5 + 2(0.3333) + 2(0.3235) + 0.2258\right]$$
$$= 0.3399$$
$$y_1 = y_0 + k$$
$$= 1 + 0.3399$$
$$= 1.3399$$

(ii) Now, $x_1 = 0.5, \quad y_1 = 1.3399, \quad h = 0.5, \quad n = 1$

$$k_1 = h\, f(x_1, y_1)$$
$$= 0.5\ f(0.5, 1.3399)$$
$$= 0.5\left(\frac{1.3399 - 0.5}{1.3399 + 0.5}\right)$$
$$= 0.2282$$
$$k_2 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.5 f\left(0.5 + \frac{0.5}{2}, 1.3399 + \frac{0.2282}{2}\right)$$
$$= 0.5\ f(0.75, 1.454)$$
$$= 0.5\left(\frac{1.454 - 0.75}{1.454 + 0.75}\right)$$
$$= 0.1597$$
$$k_3 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.5\ f\left(0.5 + \frac{0.5}{2}, 1.3399 + \frac{0.1597}{2}\right)$$
$$= 0.5\ f(0.75, 1.4198)$$
$$= 0.5\left(\frac{1.4198 - 0.75}{1.4198 + 0.75}\right)$$
$$= 0.1543$$
$$k_4 = h\, f(x_1 + h, y_1 + k_3)$$
$$= 0.5\ f(0.5 + 0.5, 1.3399 + 0.1543)$$
$$= 0.5\ f(1, 1.4942)$$
$$= 0.5\left(\frac{1.4942 - 1}{1.4942 + 1}\right)$$
$$= 0.0991$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.2282 + 2(0.1597) + 2(0.1543) + 0.0991]$$

$$= 0.1592$$

$$y_2 = y_1 + k$$

$$= 1.3399 + 0.1592$$

$$= 1.4991$$

Hence, $y_2 = y(1) = 1.4991$

Example 8

Using the fourth order Runge–Kutta method, find y at x = 0.1 for differential equation $\frac{dy}{dx} = 3e^x + 2y,\ y(0) = 0$ *by taking* $h = 0.1$.

[Summer 2015]

Solution

$$\frac{dy}{dx} = f(x, y) = 3e^x + 2y$$

(i) Given: $x_0 = 0, \quad y_0 = 0, \quad h = 0.1, \quad n = 0$

$$k_1 = h\,f(x_0, y_0)$$

$$= 0.1\,f(0, 0)$$

$$= 0.1(3e^0 + 0)$$

$$= 0.3$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1\,f\left(0 + \frac{0.1}{2}, 0 + \frac{0.3}{2}\right)$$

$$= 0.1\,f(0.05, 0.15)$$

$$= 0.1\left[3e^{0.05} + 2(0.15)\right]$$

$$= 0.3454$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\,f\left(0 + \frac{0.1}{2}, 0 + \frac{0.3454}{2}\right)$$

$$= 0.1\,f(0.05, 0.1727)$$

$$= 0.1\left[3e^{0.05} + 2(0.1727)\right]$$
$$= 0.3499$$
$$k_4 = h\,f(x_0 + h,\, y_0 + k_3)$$
$$= 0.1\, f(0 + 0.1,\, 0 + 0.3499)$$
$$= 0.1\, f(0.1,\, 0.3499)$$
$$= 0.1\left[3e^{0.1} + 2(0.3499)\right]$$
$$= 0.4015$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}\left[0.3 + 2(0.3454) + 2(0.3499) + 0.4015\right]$$
$$= 0.3487$$
$$y_1 = y_0 + k$$
$$= 0 + 0.3487$$
$$= 0.3487$$

Hence, $y_2 = y(0.1) = 0.3487$

Example 9

Determine $y(0.1)$ *and* $y(0.2)$ *correct to four decimal places from* $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + y,\; y(0) = 1$ *with* $h = 0.1$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = 2x + y$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$k_1 = h\,f(x_0,\, y_0)$$
$$= 0.1\, f(0, 1)$$
$$= 0.1\left[2(0) + 1\right]$$
$$= 0.1$$
$$k_2 = h\,f\left(x_0 + \frac{h}{2},\, y_0 + \frac{k_1}{2}\right)$$
$$= 0.1\, f\left(0 + \frac{0.1}{2},\, 1 + \frac{0.1}{2}\right)$$
$$= 0.1\, f(0.05,\, 1.05)$$

$$= 0.1[2(0.05)+1.05]$$
$$= 0.115$$
$$k_3 = h\,f\left(x_0+\frac{h}{2},\, y_0+\frac{k_2}{2}\right)$$
$$= 0.1\,f\left(0+\frac{0.1}{2}, 1+\frac{0.115}{2}\right)$$
$$= 0.1\,f(0.05, 1.0575)$$
$$= 0.1[2(0.05)+1.0575]$$
$$= 0.11575$$
$$k_4 = h\,f(x_0+h,\, y_0+k_3)$$
$$= 0.1\,f(0+0.1, 1+0.11575)$$
$$= 0.1\,f(0.1, 1.11575)$$
$$= 0.1[2(0.1)+1.11575]$$
$$= 0.13158$$
$$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$
$$= \frac{1}{6}[0.1+2(0.115)+2(0.11575)+0.13158]$$
$$= 0.1155$$
$$y_1 = y_0 + k$$
$$= 1+0.1155$$
$$= 1.1155$$

Hence, $y_1 = y(0.1) = 1.1155$

(ii) Given: $x_1 = 0.1, \quad y_1 = 1.1155, \quad h = 0.1, \quad n = 1$

$$k_1 = h\,f(x_1, y_1)$$
$$= 0.1\,f(0.1, 1.1155)$$
$$= 0.1[2(0.1)+1.1155]$$
$$= 0.13165$$
$$k_2 = h\,f\left(x_1+\frac{h}{2},\, y_1+\frac{k_1}{2}\right)$$
$$= 0.1\,f\left(0.1+\frac{0.1}{2}, 1.1155+\frac{0.13165}{2}\right)$$
$$= 0.1\,f(0.15, 1.1813)$$
$$= 0.1[2(0.15)+1.1813]$$
$$= 0.14813$$

$$\begin{aligned}
k_3 &= h\,f\left(x_1+\frac{h}{2},\,y_1+\frac{k_2}{2}\right)\\
&= 0.1\,f\left(0.1+\frac{0.1}{2},\,1.1155+\frac{0.14813}{2}\right)\\
&= 0.1\,f(0.15, 1.18965)\\
&= 0.1\big[2(0.15)+1.18965\big]\\
&= 0.149\\
k_4 &= h(x_1+h,\,y_1+k_3)\\
&= 0.1\,f(0.1+0.1, 1.1155+0.149)\\
&= 0.1\,f(0.2, 1.2645)\\
&= 0.1\big[2(0.2)+1.2645\big]\\
&= 0.16645\\
k &= \frac{1}{6}[k_1+2k_2+2k_3+k_4)\\
&= \frac{1}{6}[0.13165+2(0.14813)+2(0.149)+0.16645]\\
&= 0.1487\\
y_2 &= y_1+k\\
&= 1.1155+0.1487\\
&= 1.2642
\end{aligned}$$

Hence, $y_2 = y(0.2) = 1.2642$

Example 10

Apply the Runge–Kutta method of fourth order to find an approximate value of y at x $= 0.6$ $\dfrac{dy}{dx} = \sqrt{x+y},\ y(0.4) = 0.41$ *in two steps.*

Solution

$$\frac{dy}{dx} = f(x,y) = \sqrt{x+y}$$

(i) Given: $x_0 = 0.4, \quad y_0 = 0.41, \quad h = 0.1, \quad n = 0$

$$\begin{aligned}
k_1 &= hf(x_0, y_0)\\
&= 0.1\,f(0.4, 0.41)\\
&= 0.1\sqrt{0.4+0.41}\\
&= 0.09
\end{aligned}$$

$$
\begin{aligned}
k_2 &= hf\left(x_0+\frac{h}{2},\, y_0+\frac{k_1}{2}\right)\\
&= 0.1f\left(0.4+\frac{0.1}{2}, 0.41+\frac{0.09}{2}\right)\\
&= 0.1\, f(0.45, 0.455)\\
&= 0.1\sqrt{0.45+0.455}\\
&= 0.0951\\
k_3 &= hf\left(x_0+\frac{h}{2},\, y_0+\frac{k_2}{2}\right)\\
&= 0.1\, f\left(0.4+\frac{0.1}{2}, 0.41+\frac{0.0951}{2}\right)\\
&= 0.1\, f(0.45, 0.4576)\\
&= 0.1\sqrt{0.45+0.4576}\\
&= 0.0953\\
k_4 &= hf(x_0+h,\, y_0+k_3)\\
&= 0.1\, f(0.4+0.1, 0.41+0.0953)\\
&= 0.1\, f(0.5, 0.5053)\\
&= 0.1\sqrt{0.5+0.5053}\\
&= 0.1003\\
k &= \frac{1}{6}(k_1+2k_2+2k_3+k_4)\\
&= \frac{1}{6}\big[0.09+2(0.0951)+2(0.0953)+0.1003\big]\\
&= 0.0952\\
y_1 &= y_0+k\\
&= 0.41+0.0952\\
&= 0.5052
\end{aligned}
$$

(ii) Now, $x_1 = 0.5, \quad y_1 = 0.5052, \quad h = 0.1, \quad n = 1$

$$
\begin{aligned}
k_1 &= h\, f(x_1, y_1)\\
&= 0.1\, f(0.5, 0.5052)\\
&= 0.1\sqrt{0.5+0.5052}\\
&= 0.1003\\
k_2 &= h\, f\left(x_1+\frac{h}{2},\, y_1+\frac{k_1}{2}\right)\\
&= 0.1\, f\left(0.5+\frac{0.1}{2}, 0.5052+\frac{0.1003}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 0.1\, f(0.55, 0.5554) \\
&= 0.1\sqrt{0.55 + 0.5554} \\
&= 0.1051 \\
k_3 &= h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) \\
&= 0.1\, f\left(0.5 + \frac{0.1}{2}, 0.5052 + \frac{0.1051}{2}\right) \\
&= 0.1\, f(0.55, 0.5578) \\
&= 0.1\sqrt{0.55 + 0.5578} \\
&= 0.1053 \\
k_4 &= h\, f(x_1 + h, y_1 + k_3) \\
&= 0.1\, f(0.5 + 0.1, 0.5052 + 0.1053) \\
&= 0.1\, f(0.6, 0.6105) \\
&= 0.1\sqrt{0.6 + 0.6105} \\
&= 0.1100 \\
k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&= \frac{1}{6}\left[0.1003 + 2(0.1051) + 2(0.1053) + 0.1100\right] \\
&= 0.1052 \\
y_2 &= y_1 + k \\
&= 0.5052 + 0.1052 \\
&= 0.6104
\end{aligned}
$$

Hence, $y_2 = y(0.6) = 0.6104$

Example 11

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x+y}$, $x_1 = 0$, $y_1 = 1$ *for the interval* (0, 1) *choosing* $h = 0.5$ *by the Runge–Kutta method of fourth order.*

Solution

$$\frac{dy}{dx} = f(x, y) = \frac{1}{x+y}$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.5, \quad n = 0$

$$k_1 = h\,f(x_0, y_0)$$
$$= 0.5 f(0, 1)$$
$$= 0.5\left(\frac{1}{0+1}\right)$$
$$= 0.5$$
$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.5\ f\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2}\right)$$
$$= 0.5\ f(0.25, 1.25)$$
$$= 0.5\left(\frac{1}{0.25 + 1.25}\right)$$
$$= 0.3333$$
$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.5\ f\left(0 + \frac{0.5}{2}, 1 + \frac{0.3333}{2}\right)$$
$$= 0.5\ f(0.25, 1.1666)$$
$$= 0.5\left(\frac{1}{0.25 + 1.1666}\right)$$
$$= 0.3529$$
$$k_4 = h\,f(x_0 + h, y_0 + k_2)$$
$$= 0.5\ f(0 + 0.5, 1 + 0.3529)$$
$$= 0.5\ f(0.5, 1.3529)$$
$$= 0.5\left(\frac{1}{0.5 + 1.3529}\right)$$
$$= 0.2698$$
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}\left[0.5 + 2(0.3333) + 2(0.3529) + 0.2698\right]$$
$$= 0.3570$$
$$y_1 = y_0 + k$$
$$= 1 + 0.3570$$
$$= 1.3570$$

(ii) Now, $x_1 = 0.5, \quad y_1 = 1.3570, \quad h = 0.5, \quad n = 1$

$$
\begin{aligned}
k_1 &= h\,f(x_1, y_1)\\
&= 0.5\ f(0.5, 1.3570)\\
&= 0.5\left(\frac{1}{0.5+1.3570}\right)\\
&= 0.2692\\
k_2 &= h\,f\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right)\\
&= 0.5\ f\left(0.5+\frac{0.5}{2}, 1.3570+\frac{0.2692}{2}\right)\\
&= 0.5\ f(0.75, 1.4916)\\
&= 0.5\left(\frac{1}{0.75+1.4916}\right)\\
&= 0.2230\\
k_3 &= h\,f\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right)\\
&= 0.5\ f\left(0.5+\frac{0.5}{2}, 1.3570+\frac{0.2230}{2}\right)\\
&= 0.5\ f(0.75, 1.4685)\\
&= 0.5\left(\frac{1}{0.75+1.4685}\right)\\
&= 0.2253\\
k_4 &= h\,f(x_1+h, y_1+k_3)\\
&= 0.5\ f(0.5+0.5, 1.3570+0.2253)\\
&= 0.5\ f(1, 1.5823)\\
&= 0.5\left(\frac{1}{1+1.5823}\right)\\
&= 0.1936\\
k &= \frac{1}{6}(k_1+2k_2+2k_3+k_4)\\
&= \frac{1}{6}\big[0.2692+2(0.2230)+2(0.2253)+0.1936\big]\\
&= 0.2265\\
y_2 &= y_1+k\\
&= 1.3570+0.2265\\
&= 1.5835
\end{aligned}
$$

Example 12

Apply the Runge–Kutta method of fourth order to find an approximate value of y at $x = 0.2$ *if* $\frac{dy}{dx} = x + y^2$, *given that* $y = 1$ *when* $x = 0$ *in steps of* $h = 0.1$. **[Summer 2014]**

Solution

$$\frac{dy}{dx} = f(x, y) = x + y^2$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad n = 0$

$$\begin{aligned}
k_1 &= h\,f(x_0, y_0) \\
&= 0.1\,f(0, 1) \\
&= 0.1(0 + 1^2) \\
&= 0.1 \\
k_2 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\
&= 0.1\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right) \\
&= 0.1\,f(0.05, 1.05) \\
&= 0.1\left[0.05 + (1.05)^2\right] \\
&= 0.1152 \\
k_3 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \\
&= 0.1\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1152}{2}\right) \\
&= 0.1\,f(0.05, 1.0576) \\
&= 0.1\left[0.05 + (1.0576)^2\right] \\
&= 0.1168 \\
k_4 &= h\,f(x_0 + h, y_0 + k_3) \\
&= 0.1\,f(0 + 0.1, 1 + 0.1168) \\
&= 0.1\,f(0.1, 1.1168) \\
&= 0.1\left[0.1 + (1.1168)^2\right] \\
&= 0.1347
\end{aligned}$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.1 + 2(0.1152) + 2(0.1168) + 0.1347]$$
$$= 0.1164$$
$$y_1 = y_0 + k$$
$$= 1 + 0.1164$$
$$= 1.1164$$

(ii) Now, $x_1 = 0.1, \quad y_1 = 1.1164, \quad h = 0.1, \quad n = 1$

$$k_1 = h\,f(x_1, y_1)$$
$$= 0.1\,f(0.1, 1.1164)$$
$$= 0.1\left[0.1 + (1.1164)^2\right]$$
$$= 0.1346$$
$$k_2 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= 0.1\,f\left(0.1 + \frac{0.1}{2}, 1.1164 + \frac{0.1346}{2}\right)$$
$$= 0.1\,f(0.15, 1.1837)$$
$$= 0.1\left[0.15 + (1.1837)^2\right]$$
$$= 0.1551$$
$$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= 0.1\,f\left(0.1 + \frac{0.1}{2}, 1.1164 + \frac{0.1551}{2}\right)$$
$$= 0.1\,f(0.15, 1.1939)$$
$$= 0.1\left[0.15 + (1.1939)^2\right]$$
$$= 0.1575$$
$$k_4 = h\,f(x_1 + h, y_1 + k_3)$$
$$= 0.1\,f(0.1 + 0.1, 1.1164 + 0.1575)$$
$$= 0.1\,f(0.2, 1.2739)$$
$$= 0.1\left[0.2 + (1.2739)^2\right]$$
$$= 0.1822$$

$$\begin{aligned} k &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ &= \frac{1}{6}\left[0.1346 + 2(0.1551) + 2(0.1575) + 0.1822\right] \\ &= 0.157 \\ y_2 &= y_1 + k \\ &= 1.1164 + 0.157 \\ &= 1.2734 \end{aligned}$$

Hence, $y_2 = y(0.2) = 1.2734$

EXERCISE 10.4

Solve the following differential equations by the Runge–Kutta method:

1. $\dfrac{dy}{dx} = x + y$ with $x_0 = 0, y_0 = 1$ at $x = 0.2$

[**Ans.:** 1.2424]

2. $\dfrac{dy}{dx} = xy$ with $y(1) = 2$ at $x = 1.2, x = 1.4$

[**Ans.:** 2.4921, 3.2311]

3. $\dfrac{dy}{dx} = x^2 + y^2$ with $x_0 = 1,\ y_0 = 1.5,\ h = 0.1$ at $x = 1.2$

[**Ans.:** 2.5043]

4. $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$, at $x = 0.2$ and $x = 0.4$

[**Ans.:** 1.8310, 2.0214]

5. $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$ with $x_0 = 0, y_0 = 1$ at $x = 0.2$

[**Ans.:** 1.1678]

6. $\dfrac{dy}{dx} = 1 + y^2$ with $x_0 = 0, y_0 = 0$ at $x = 0.2, 0.4$ and 0.6

[**Ans.:** 0.2027, 0.4228, 0.6891]

7. $\dfrac{dy}{dx} = xy^2$ with $x_0 = 2, y_0 = 1$ for $x = 2.2$ taking $h = 0.2$

[**Ans.:** −1.7241]

8. $\dfrac{dy}{dx} = x - y^2$ with $x_0 = 0, y_0 = 1$ at $x = 0.2$ taking $h = 0.1$

[**Ans.:** 0.8512]

9. $\dfrac{dy}{dx} = \dfrac{x-y}{xy}$ with $x_0 = 1, y_0 = 1$ at $x = 1.1$

[Ans.: 1.0045**]**

10. $\dfrac{dy}{dx} = \dfrac{y^2 - 2x}{y^2 + x}$ with $x_0 = 0, y_0 = 1$ at $x = 0.1, 0.2, 0.3$, and 0.4

[Ans.: 1.0911, 1.1677, 1.2352, 1.2902**]**

10.6 MILNE'S PREDICTOR-CORRECTOR METHOD

Consider the differential equation

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition $y(x_0) = y_0$.

By Taylor's series method,

$$y_1 = y(x_0 + h)$$
$$y_2 = y(x_0 + 2h)$$
$$y_3 = y(x_0 + 3h)$$

Also,
$$f_0 = f(x_0, y_0)$$
$$f_1 = f(x_0 + h, y_1)$$
$$f_2 = f(x_0 + 2h, y_2)$$
$$f_3 = f(x_0 + 3h, y_3)$$

By Newton's forward interpolation formula,

$$f(x, y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 f_0 + \cdots$$

Now,
$$y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x, y)\, dx$$

$$= y_0 + \int_{x_0}^{x_0+4h} \left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 f_0 + \cdots \right) dx$$

Putting $x = x_0 + nh$, $\quad dx = h\,dn$

When $x = x_0$, $\quad n = 0$

When $x = x_0 + 4h$ $\quad n = 4$

$$y_4 = y_0 + h\int_0^4 \left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 f_0 + \cdots \right) dn$$

$$= y_0 + h\left| f_0 n + \frac{n^2}{2}\Delta f_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 f_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 f_0 + \cdots \right|_0^4$$

$$= y_0 + h\left[4f_0 + 8\Delta f_0 + \frac{1}{2}\left(\frac{64}{3} - 8\right)\Delta^2 f_0 + \frac{1}{6}(64 - 64 + 16)\Delta^3 f_0 + \cdots\right]$$

$$= y_0 + h\left[4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + \cdots\right]$$

Neglecting fourth and higher order differences and expressing Δf_0, $\Delta^2 f_0$ and $\Delta^3 f_0$ in terms of the function values,

$$y_{4p} = y_0 + h\left[4f_0 + 8(f_1 - f_0) + \frac{20}{3}(f_2 - 2f_1 + f_0) + \frac{8}{3}(f_3 - 3f_2 + 3f_1 - f_0)\right]$$

$$= y_0 + h\left[\left(4 - 8 + \frac{20}{3} - \frac{8}{3}\right)f_0 + \left(8 - \frac{40}{3} + 8\right)f_1 + \left(\frac{20}{3} - 8\right)f_2 + \frac{8}{3}f_3\right]$$

$$= y_0 + h\left(\frac{8}{3}f_1 - \frac{4}{3}f_2 + \frac{8}{3}f_3\right)$$

$$= y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

This equation is known as *predictor*.

In general,

$$y_{(n+1)p} = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n)$$

From y_4, a first approximation to $f_4 = f(x_0 + 4h, y_4)$ is obtained.

A better value of y_4 is obtained by Simpson's rule.

$$y_{4c} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

This equation is known as *corrector*.

In general,

$$y_{(n+1)c} = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1})$$

Then an improved value of f_4 is calculated using y_{4c} and again the corrector is applied to find a still better value of y_{4c}. This step is repeated till two consecutive values of y_{4c} are same.

Once y_4 and f_4 are obtained to the desired degree of accuracy, the next value of y is obtained from predictor-corrector equations.

This method is known as *Milne's predictor-corrector* method.

Example 1

Given $\dfrac{dy}{dx} = \dfrac{1}{2}(1+x^2)y^2$ *and* $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$, *evaluate* $y(0.4)$.

Solution

$$x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, x_4 = 0.4$$

$$y_0 = 1, y_1 = 1.06, y_2 = 1.12, y_3 = 1.21, h = 0.1$$

$$\frac{dy}{dx} = f(x, y) = \frac{1}{2}(1+x^2)\,y^2$$

$$f_1 = \frac{1}{2}(1+x_1^2)\,y_1^2 = \frac{1}{2}\left[1+(0.1)^2\right](1.06)^2 = 0.5674$$

$$f_2 = \frac{1}{2}(1+x_2^2)\,y_2^2 = \frac{1}{2}\left[1+(0.2)^2\right](1.12)^2 = 0.6523$$

$$f_3 = \frac{1}{2}(1+x_3^2)\,y_3^2 = \frac{1}{2}\left[1+(0.3)^2\right](1.21)^2 = 0.7979$$

By Milne's predictor method,

$$y_{4p} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.5674) - 0.6523 + 2(0.7979)\right]$$

$$= 1.2771$$

$$f_4 = \frac{1}{2}(1+x_4^2)\,y_{4p}^2$$

$$= \frac{1}{2}\left[1+(0.4)^2\right](1.2771)^2$$

$$= 0.9460$$

By Milne's corrector method,

$$y_{4c} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 1.12 + \frac{0.1}{3}\left[0.6523 + 4(0.7979 + 0.9460\right] = 1.2797$$

Again,

$$f_4 = \frac{1}{2}(1+x_4^2)\,y_{4c}^2$$

$$= \frac{1}{2}\left[1+(0.4)^2\right](0.2797)^2$$

$$= 0.9498$$

By Milne's corrector method,

$$y_{4c} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 1.12 + \frac{0.1}{3}\left[0.6523 + 4(0.7979) + 0.9498\right]$$

$$= 1.2798$$

$\therefore$ $y(0.4) = 1.2798$

Example 2

Find $y(4.4)$ *given* $5xy' + y^2 - 2 = 0$ *with* $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$.

Solution

$$x_0 = 4,\ x_1 = 4.1,\ x_2 = 4.2,\ x_3 = 4.3,\ x_4 = 4.4$$

$$y_0 = 1,\ y_1 = 1.0049,\ y_2 = 1.0097,\ y_3 = 1.0143,\ h = 0.1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) = \frac{2 - y^2}{5x}$$

$$f_1 = \frac{2 - y_1^2}{5x_1} = \frac{2 - (1.0049)^2}{5(4.1)} = 0.0483$$

$$f_2 = \frac{2 - y_2^2}{5x_2} = \frac{2 - (1.0097)^2}{5(4.2)} = 0.0467$$

$$f_3 = \frac{2 - y_3^2}{5x_3} = \frac{2 - (1.0143)^2}{5(4.3)} = 0.0452$$

By Milne's predictor method,

$$y_{4p} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.0483) - 0.0467 + 2(0.0452)\right]$$

$$= 1.0187$$

$$f_4 = \frac{2 - y_{4p}^2}{5x_4} = \frac{2 - (1.0187)^2}{5(4.4)} = 0.0437$$

By Milne's corrector method,

$$y_{4c} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 1.0097 + \frac{0.1}{3}\left[0.0467 + 4(0.0452) + 0.0437)\right]$$

$$= 1.0187$$

$\therefore$ $y(4.4) = 1.0187$

Example 3

Given $y' = x(x^2 + y^2)e^{-x}$, $y(0) = 1$, find y at 0.1, 0.2, and 0.3 by Taylor's series method and compute $y(0.4)$ by Milne's method.

Solution

$$\frac{dy}{dx} = f(x, y) = x(x^2 + y^2)e^{-x}$$

(i) Given: $x_0 = 0$, $y_0 = 1$, $h = 0.1$, $x_1 = x_0 + h = 0 + 0.1 = 0.1$

$$y' = x(x^2 + y^2)e^{-x} \qquad y_0' = 0$$

$$y'' = \left[(x^3 + xy^2)(-e^{-x}) + 3x^2 + y^2 + x(2y)\right]e^{-x}$$

$$= e^{-v}(-x^3 - xy^2 + 3x^2 + y^2 + 2xyy') \qquad y_0'' = 1$$

$$y''' = -e^{-x}\begin{bmatrix} -x^3 - xy^2 + 3x^2 + y^2 + 2xyy' + 3x^2 + y^2 \\ +2xyy' - 6x - 2yy' - 2x(y')^2 - 2xyy' \end{bmatrix} \qquad y_0''' = -2$$

By Taylor's series,

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$y_1 = y(0.1) = 1 + 0.1(0) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(-2) + \cdots$$

$$= 1.0047$$

(ii) Given: $x_1 = 0.1$, $y_1 = 1.0047$, $h = 0.1$, $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$$y_1' = 0.0922$$

$$y_1'' = 0.849$$

$$y_1''' = -1.247$$

By Taylor's series,

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots$$

$$y_2 = y(0.2) = 1.0047 + 0.1(0.0922) + \frac{(0.1)^2}{2!}(0.849) + \frac{(0.1)^3}{3!}(-1.247) + \cdots$$

$$= 1.018$$

(iii) Given: $x_2 = 0.2$, $y_2 = 1.018$, $h = 0.1$, $x_3 = x_2 + h = 0.2 + 0.1 = 0.3$

$$y_2' = 0.176$$

$$y_2'' = 0.77$$

$$y_2''' = 0.819$$

By Taylor's series,

$$y_3 = y(x_3) = y_2 + hy_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \cdots$$

$$y_3 = y(0.3) = 1.018 + 0.1(0.176) + \frac{(0.1)^2}{2!}(0.77) + \frac{(0.1)^3}{3!}(0.819) + \cdots$$

$$= 1.04$$

For Milne's method,

$$\begin{array}{lll} x_0 = 0 & y_0 = 1 & \\ x_1 = 0.1 & y_1 = 1.0047 & f_1 = 0.092 \\ x_2 = 0.2 & y_2 = 1.018 & f_2 = 0.176 \\ x_3 = 0.3 & y_3 = 1.04 & f_3 = 0.26 \end{array}$$

By Milne's predictor method,

$$y_{4p} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.092) - 0.176 + 2(0.26)\right]$$

$$= 1.09$$

$$x_4 = 0.4,\ y_{4p} = 1.09$$

$$f_4 = x_4\left(x_4^2 + y_{4p}^2\right)e^{-x_4}$$

$$= 0.4\left[(0.4)^2 + (1.09)^2\right]e^{-0.4}$$

$$= 0.3615$$

By Milne's corrector method,

$$y_{4c} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 1.018 + \frac{0.1}{3}\left[0.176 + 4(0.26) + 0.3615\right]$$

$$= 1.071$$

$\therefore \quad y(0.4) = 1.071$

Example 4

Determine the value of $y(0.4)$ *using the predictor-corrector method, given* $\frac{dy}{dx} = xy + y^2,\ y(0) = 1.$ *Use Taylor series to get the values of* $y(0.1),\ y(0.2),\ y(0.3)$. *Take* $h = 0.1$. **[Summer 2013, 2015]**

Solution

$$\frac{dy}{dx} = f(x, y) = xy + y^2$$

(i) Given: $x_0 = 0, \quad y_0 = 1, \quad h = 0.1, \quad x_1 = x_0 + h = 0 + 0.1 = 0.1$

$$y' = xy + y^2 \qquad y_0' = 0 + (1)^2 = 1$$

$$y'' = xy' + y + 2yy' \qquad y_0'' = 0 + 1 + 2(1)(1) = 3$$

$$y''' = xy'' + 2y' + 2yy'' + 2(y')^2 \qquad y_0''' = 0 + 2(1) + 2(1)(3) + 2(1)^2 = 10$$

By Taylor's series,

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$y_1 = y(0.1) = 1 + 0.1(1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(10) + \cdots$$

$$= 1.1167$$

(ii) Given: $x_1 = 0.1, \quad y_1 = 1.1167, \quad h = 0.1, \quad x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$$y_1' = 0.1(1.1167) + (1.1167)^2 = 1.3587$$

$$y_1'' = 0.1(1.3587) + 1.1167 + 2(1.1167)(1.3587) = 4.2871$$

$$y_1''' = 0.1(4.2871) + 2(1.3587) + 2(1.1167)(4.2871) + 2(1.3587)^2$$

$$= 16.4131$$

By Taylor's series,

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots$$

$$y_2 = y(0.2) = 1.1167 + 0.1(1.3587) + \frac{(0.1)^2}{2!}(4.2871) + \frac{(0.1)^3}{3!}(16.4131)$$

$$= 1.2767$$

(iii) Given: $x_2 = 0.2, \quad y_2 = 1.2767, \quad h = 0.1, \quad x_3 = x_2 + h = 0.2 + 0.1 = 0.3$

$$y_2' = 0.2(1.2767) + (1.2767)^2 = 1.8853$$

$$y_2'' = 0.2(1.8853) + 1.2767 + 2(1.2767)(1.8853) = 6.4677$$

$$y_2''' = 0.2(6.4677) + 2(1.8853) + 2(1.2767)(6.4677) + 2(1.8853)^2$$

$$= 28.6875$$

By Taylor's series,

$$y_3 = y(x_3) = y_2 + hy_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \cdots$$

$$y_3 = y(0.3) = 1.2767 + 0.1(1.8853) + \frac{(0.1)^2}{2!}(6.4677) + \frac{(0.1)^3}{3!}(28.6875) + \cdots$$

$$= 1.5023$$

For Milne's method,

$$x_0 = 0 \qquad y_0 = 1$$

$$x_1 = 0.1 \qquad y_1 = 1.1167 \qquad f_1 = (0.1)(1.1167) + (1.1167)^2 = 1.3587$$

$$x_2 = 0.2 \qquad y_2 = 1.2767 \qquad f_2 = (0.2)(1.2767) + (1.2767)^2 = 1.8853$$

$$x_3 = 0.3 \qquad y_3 = 1.5023 \qquad f_3 = (0.3)(1.5023) + (1.5023)^2 = 2.7076$$

By Milne's predictor method,

$$y_{4p} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 1 + \frac{4(0.1)}{3}\left[2(1.3587) - 1.8853 + 2(2.7076)\right]$$

$$= 1.833$$

$$x_4 = 0.4, \qquad y_{4p} = 1.833$$

$$f_4 = (0.4)(1.833) + (1.833)^2 = 4.093$$

By Milne's corrector method,

$$y_{4c} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 1.2767 + \frac{0.1}{3}\left[1.8853 + 4(2.7076) + 4.093\right]$$

$$= 1.83699$$

$\therefore \qquad y(0.4) = 1.83699$

Example 5

Using Taylor's series method, compute the approximate values of y at x = 0.2, 0.4, *and* 0.6 *for the differential equation* $\frac{dy}{dx} = x - y^2$ *with the initial condition* $y(0) = 0$. *Now, apply Milne's predictor-corrector method to find y at x =* 0.8. **[Winter 2012]**

Solution

$$\frac{dy}{dx} = f(x, y) = x - y^2$$

(i) Given: $x_0 = 0, \quad y_0 = 0, \quad h = 0.2, \quad x_1 = x_0 + h = 0 + 0.2 = 0.2$

$$y' = x - y^2 \qquad y_0' = 0$$

$$y'' = 1 - 2yy' \qquad y_0'' = 1 - 2(0)(-1) = 1$$

$$y''' = -2yy'' - 2(y')^2 \qquad y_0''' = -2(0)(3) - 2(0)^2 = 0$$

By Taylor's series,

$$y_1 = y(x_1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$y_1 = y(0.2) = 0 + 0.2(0) + \frac{(0.2)^2}{2!}(1) + \frac{(0.2)^3}{3!}(0) + \cdots$$

$$= 0.02$$

(ii) Given: $x_1 = 0.2$, $y_1 = 0.02$, $h = 0.2$, $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$$y_1' = 0.2 - (0.02)^2 = 0.1996$$

$$y_1'' = 1 - 2(0.02)(0.1996) = 0.9920$$

$$y_1''' = -2(0.02)(0.9920) - 2(0.1996)^2 = -0.1194$$

By Taylor's series,

$$y_2 = y(x_2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots$$

$$y_2 = y(0.4) = 0.02 + 0.2(0.1996) + \frac{(0.2)^2}{2!}(0.9920) + \frac{(0.2)^3}{3!}(-0.1194) + \cdots$$

$$= 0.0796$$

(iii) Given: $x_2 = 0.4$, $y_2 = 0.0796$, $h = 0.2$, $x_3 = x_2 + h = 0.4 + 0.2 = 0.6$

$$y_2' = 0.4 - (0.0796)^2 = 0.3937$$

$$y_2'' = 1 - 2(0.0796)(0.3937) = 0.9373$$

$$y_2''' = -2(0.0796)(0.9373) - 2(0.3937)^2 = -0.4592$$

By Taylor's series,

$$y_3 = y(x_3) = y_2 + hy_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \cdots$$

$$= 0.0796 + 0.2(0.3937) + \frac{(0.2)^2}{2!}(0.9373) + \frac{(0.2)^3}{3!}(-0.4592) + \cdots$$

$$= 0.1765$$

For Milne's method,

$x_0 = 0$	$y_0 = 0$	
$x_1 = 0.2$	$y_1 = 0.02$	$f_1 = 0.2 - (0.02)^2 = 0.7996$
$x_2 = 0.4$	$y_2 = 0.0796$	$f_2 = 0.4 - (0.0796)^2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1765$	$f_3 = 0.6 - (0.1765)^2 = 0.5688$

By Milne's predictor method,

$$y_{4p} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 0 + \frac{4(0.2)}{3}\left[2(0.1996) - 0.3937 + 2(0.5688)\right]$$

$$= 0.3048$$

$$x_4 = 0.8, \qquad y_{4p} = 0.3048$$

$$f_4 = 0.8 - (0.3048)^2 = 0.7071$$

By Milne's corrector method,

$$y_{4c} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 0.0796 + \frac{0.2}{3}\left[0.3937 + 4(0.5688) + 0.7071\right]$$

$$= 0.3047$$

$$\therefore \quad y(0.8) = 0.3047$$

EXERCISE 10.4

1. Find $y(2)$ if $y(x)$ is the solution of $\frac{dy}{dx} = \frac{1}{2}(x+y)$ given $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$ and $y(1.5) = 4.968$.

[Ans.: 6.8732]

2. Find $y(0.8)$ given $y' = y - x^2$, $y(0) = 1$, $y(0.2) = 1.12186$, $y(0.4) = 1.46820$, $y(0.6) = 1.73790$.

[Ans.: 2.01105]

3. Given $y' = x^2 - y$, $y(0) = 1$, $y(0.1) = 0.9052$, $y(0.2) = 0.8213$, find $y(0.3)$ by Taylor series. Also, find $y(0.4)$ and $y(0.5)$.

[Ans.: 0.6897, 0.6435]

4. If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$, $y(0.3) = 2.090$, find $y(0.4)$ and $y(0.5)$.

[Ans.: 2.1621, 2.546]

5. Given $y' = \frac{1}{x+y}$, $y(0) = 2, y(0.2) = 2.0933, y(0.4) = 2.1755, y(0.6) = 2.2493$, find $y(0.8)$.

[Ans.: 2.3164]

Points to Remember

Taylor's Series Method

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''_n + \cdots$$

Euler's Method

$$y_{n+1} = y_n + h\,f(x_n, y_n)$$

Modified Euler's Method

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0)$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right], n = 0, 1, 2, \ldots$$

$$y_2^{(0)} = y_1 + h\,f(x_1, y_1)$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

Runge–Kutta Methods

1. First-Order Runge–Kutta Method

$$y_{n+1} = y_n + h\,y'_n + \frac{h^2}{2!} y''_n + \cdots$$

2. Second-Order Runge–Kutta Method (Heun Method)

$$k_1 = h\,f(x_n, y_n)$$

$$k_2 = h\,f\,(x_n + h, y_n + k_1)$$

$$k = \frac{1}{2}(k_1 + k_2)$$

$$y_{n+1} = y_n + k$$

3. Third-Order Runge–Kutta Method

$$k_1 = h\,f(x_n, y_n)$$

$$k_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h\,f(x_n + h, y_n + 2k_2 - k_1)$$

$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$y_{n+1} = y_n + k$$

4. Fourth-Order Runge–Kutta Method

$$k_1 = h\,f(x_n, y_n)$$

$$k_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h\,f(x_n + h, y_n + k_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{n+1} = y_n + k$$

Milne's Predictor-Corrector Method

$$y_{(n+1)p} = y_{n-3} + \frac{4h}{3}\left(2f_{n-2} - f_{n-1} + 2f_n\right)$$

$$y_{(n+1)c} = y_{n-1} + \frac{h}{3}\left(f_{n-1} + 4f_n + f_{n+1}\right)$$

Index

A

Additive law of probability (addition theorem) 1.15
Arithmetic mean 3.2
Arithmetic mean by the step-deviation method 3.8
Arithmetic mean from assumed mean 3.5
Arithmetic mean of grouped data 3.3
Augmented matrix 8.2
Averaging operator 6.6
Axiomatic definition of probability 1.3

B

Backward difference operator 6.6
Backward differences 6.3
Bayes' theorem 1.47
Binomial distribution 2.76
Binomial frequency distribution 2.79
Bisection method 9.2

C

Central difference interpolation 6.39
Central difference operator 6.6
Central differences 6.4
Classical definition of probability 1.3
Coefficient of variation 3.62
Conditional probability theorem 1.25
Conditions for binomial distribution 2.77
Conditions of Poisson distributions 2.103
Constants of the binomial distribution 2.77
Constants of the normal distribution 2.128
Constants of the Poisson distribution 2.103
Continuous distribution function 2.33
Continuous probability distribution 2.33
Continuous random variable 2.2
Convergence of the Newton—Raphson method 9.23
Convergence of the Secant method 9.40
Correlation 4.2
Cumulative distribution function 2.4
Cumulative probability distribution 2.4

D

Differential operator 6.6
Discrete distribution function 2.4
Discrete probability distribution 2.3
Discrete random variables 2.2
Divided differences 6.70

E

Echelon form of a matrix 8.3
Elementary matrices 8.3
Elementary transformations 8.2
Empirical or statistical definition of probability 1.3
Equally likely events 1.2
Equivalence of matrices 8.3
Errors in Simpson's 1/3 rule 7.10
Errors in Simpson's 3/8 rule 7.19
Errors in the Trapezoidal rule 7.3
Euler's method 10.9
Event 1.2, 1.4
Exhaustive event 1.2
Expressions for regression coefficients 4.32

F

Factorial notation 6.15
Favourable events 1.3
Finite differences 6.2
First-order Runge—Kutta method 10.28
Fitting a normal distribution 2.149
Fitting of exponential and logarithmic curves 5.18
Fitting of linear curves 5.2
Fitting of quadratic curves 5.10
Forward difference operator 6.6
Forward differences 6.2
Fourth-order Runge–Kutta method 10.29

G

Gauss elimination method 8.4
Gauss elimination method with partial pivoting 8.15
Gauss's backward interpolation formula 6.44
Gauss's forward interpolation formula 6.40
Gauss–Jacobi method 8.31
Gauss–Jordan method 8.20
Gauss–Siedel method 8.37
Geometric mean 3.45

H

Harmonic mean 3.47
Heun method 10.28
High degree of negative correlation 4.4
High degree of positive correlation 4.4

I

Independent events 1.2
Interpolation 6.19
Interpolation with unequal intervals 6.55
Inverse interpolation 6.84

K

Karl Pearson's coefficient of correlation 4.5
Karl Pearson's coefficient of skewness 3.72

L

Lagrange's interpolation formula 6.56
Least square method 5.2
Line of regression of *x* on *y* 4.31
Line of regression of *y* on *x* 4.31
Linear correlation 4.3
Linear regression 4.30

M

Mean 2.18, 2.56
Mean of the binomial distribution 2.77
Measures of central tendency 3.2
Measures of central tendency for continuous probability distribution 2.56
Measures of central tendency for discrete probability distribution 2.18
Measures of skewness 3.72
Median 2.56, 3.24
Median for continuous frequency distribution 3.26
Merits of a scatter diagram 4.4
Method of least squares 4.31
Method of scatter diagram 4.30
Methods of studying regression 4.30
Milne's predictor-corrector method 10.53
Mode 2.56, 3.34
Mode by method of grouping 3.34
Mode for a continuous frequency distribution 3.38
Mode of the binomial distribution 2.78
Modified Euler's method 10.16
Multiple correlation 4.3
Multiple regression 4.30
Multiplicative law or compound law of probability 1.25
Multiplicative theorem for independent events 1.25
Mutual independence 1.26
Mutually exclusive events 1.2

N

Negative correlation 4.2
Newton's backward interpolation formula 6.30
Newton's divided difference formula 6.71
Newton's forward interpolation formula 6.19
Newton–Cotes quadrature formula 7.1
Newton–Raphson method 9.22
No correlation 4.4
Nonlinear correlation 4.3
Nonlinear regression 4.30
Normal distribution 2.127
Normal equations 5.3
Numerical integration 7.1

O

One-point gaussian quadrature formula 7.31
Outcome 1.2

P

Pairwise independence 1.26
Partial correlation 4.3
Perfect negative correlation 4.4
Perfect positive correlation 4.4
Poisson distribution 2.102
Positive correlation 4.2
Probability 1.1
Probability density function 2.4
Probability function 2.4
Probability mass function 2.4
Probability of a normal random variable in an interval 2.132
Properties of arithmetic mean 3.17
Properties of coefficient of correlation 4.6
Properties of distribution function 2.33
Properties of lines of regression 4.35
Properties of probability density function 2.33
Properties of the normal distribution 2.127

Q

Quadrature 7.1

R

Random experiment 1.1
Random variables 2.2
Rate of convergence of the Newton–Raphson method 9.24
Recurrence relation for a Poisson distribution 2.105
Recurrence relation for the binomial distribution 2.78

Regression 4.29
Regression coefficients 4.31
Regula falsi method 9.15
Relation between arithmetic mean, geometric mean, and harmonic mean 3.50
Relations between operators 6.7
Runge–Kutta methods 10.28

S

Sample space 1.3
Scatter diagram 4.4, 5.1
Secant method 9.39
Second-order Runge–Kutta method 10.28
Shift operator 6.6
Simple correlation 4.3
Simple graph 4.5
Simple regression 4.30
Simpson's 1/3 rule 7.9
Simpson's 3/8 rule 7.19
Skewness 3.71
Solutions of a system of linear equations 8.2
Spearman's rank correlation coefficient 4.22
Standard deviation 2.19, 2.56, 3.52
Standard deviation by step-deviation method 3.56
Standard deviation from the assumed mean 3.55
Standard deviation of the binomial distribution 2.78
Statistics 3.1
Stirling's formula 6.48

T

Taylor's series method 10.2
Theorems on probability 1.13
Third-order Runge–Kutta method 10.28
Three-point Gaussian quadrature formula 7.33
Tied ranks 4.27
Total correlation 4.3
Trapezoidal rule 7.2
Trial 1.2
Two-point Gaussian quadrature formula 7.32
Types of correlations 4.2
Types of regression 4.30

U

Uses of normal distribution 2.136

V

Variance 2.18, 2.56, 3.53
Variance of the binomial distribution 2.77

W

Weighted arithmetic mean 3.15