Operations Research

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To

श्रासु नाम भवभेषज, हरण ताय त्रय शूल।
सो कृपाल मोंहि तोहि पर, सदा रहहू अनुकूल।।

उत्तर कांड १२४
(श्री राम चरित मानस)

[The Supreme Lord (Shri Rama) whose name is a (complete) medicine and is savior from birth-cycles and removes the three types of agonies and sufferings, be (He) gracious always - propitious on both - to me and to you.]

Also to

My family members, Vidit and Shivansh

Preface

A book on mathematics and related subjects is, as I have experienced, a physical form of collection of nature's grace in developing and sustaining individual's perspicacity in the subject, a sound experience of professing and excavating it, and art of presentation in an appealing way.

This book is targeted primarily at students, written strictly keeping their requirements in mind. My teaching experience with students for several years - from various disciplines - has made me realize many questions, problems, and mathematical intricacies that they would face while studying the subject of Operations Research. This book is an attempt to provide answers and solutions to all of these.

Each chapter begins with a set of learning objectives, followed by an introductory note, and then moves straight to discuss and explain various topics with the help of numerous and varied illustrations. These illustrations are carefully selected to ensure that no area or no special working approach is ignored or skipped. Important approaches, special techniques, special cases, applications, and important mathematical facts have been in-built within the text. Additional examples are, but with a little change, based on different examination papers from universities and institutions across India.

Each chapter ends with a summary of key points to remember, and is followed by rich exercises. Multiple choice questions are the real checks that will help students assess their correct insights into the fundamentals of the subject. Numerical problems help testing and reinforcing the understanding of the topics by way of solving problems. Answers have been provided for checking accuracy by the students themselves.

What I teach is what you will find in the pages of this book. I have worked hard, now it is the time for you to read, to understand, and to solve problems; all these when combined together can put you on the path to success.

I request, very politely, to all the readers to draw my attention towards any technical or printing mistakes that would have skipped our attention. Suggestions for improvements in the future editions of the book will also be welcome.

PRADEEP J JHA

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Their nobility reflects in their words. I am really thankful to them.

Pradeep J Jha

Brief Contents

Contents

Introduction

This section of the book aims at providing readers or students (especially from a non-technical background) a polite feel of comfort about the subject dealt with. We begin by introducing 'Operations Research', its characteristics, working methodology, its dominant role in representing real-life situation problems, and classical approach of finding optimal feasible solution.

Origin of Operations Research

The term 'Operations Research' (OR) was coined during World War-II as a result of solving problems of allocation of resources like arms, ammunitions, war-planes, frigates, and human resources employed on military operations. Since the war involved highly complicated strategic and tactical problems, expecting adequate and feasible solutions from individuals or specialists in a single discipline was unrealistic. Therefore, decision-making authorities planned to form a team of individuals, who were specialists in mathematics, economics, statistics, engineering, and military sciences, as special units within the armed forces to tackle with strategic and tactical problems arising from time to time in carrying out various military operations.

Both, British Air Force and American armed forces, independently adopted same policies of forming specialist groups discharging duties of analyzing military problems and finding effective and optimal solutions. The group formed under the leadership of Professor P M S Blackett was assigned to the Radar Operational Research unit and was engaged in solving the problem of coordination of radar equipment at gun sites. The plans and war strategies designed and recommended proved successful in carrying out military operations. Adopting this policy, allied nations designed teams involving experts of various branches.

After the end of the world war, some scientists decided to share their experiences on the problems of business, industrial research, and development. The economic and industrial boom after World War-II resulted in continuous mechanization, automation, decentralization of operations and division of management functions. Rapidly changing and progressive industrialization changed the scenario of the nations. Global business and sharing of technology within friendly nations totally changed human life by making progress in developing new gadgets, new drugs, and automatically designed machineries. Operations research teams, working in every area, made enormous contributions to perceive a new-era on global age.

It would be unjust not to mention the role of George B Dantzig who continued post-war development of OR. He developed the concept of linear programming. The use of computers made it possible to apply many OR techniques for practical decision analysis.

In 1949, the influences of applications of OR in different areas, and the impacts of successful results knocked Indian doors and the subject was given proper recognition. Regional Research Laboratory, Hyderabad for planning and organizing research in OR came into existence and formed an OR unit. Professor R S Verma set up an OR team at Defence Science Laboratory to solve problems of designing, planning, and allocating military sources. In 1953, Professor P C Mahalanobis established an OR team in the Indian Statistical Institute (ISI) Kolkata. It was through efforts of OR team under his leadership that could design the Five-Year Plan and budget and we could see successful results through industrial development and business globalization. In 1957, OR Society of India (ORSI) was founded, and with a view to make the subject popular it started publishing its journal, *OPSEARCH*, which was well accepted and reliable in imparting educational growth.

Some Definitions of Operations Research

Operations research is the application of the methods of science to complex problems in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific model of the system incorporating measurements of factors such as chance and risk, with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management in determining its policy and actions scientifically.

—Operational Research Society, UK

Operations research is concerned with scientifically deciding how to best design and operate man-machine systems usually requiring the allocation of scarce resources.

—Operations Research Society, America

Operations research is the systematic application of quantitative methods, techniques and tools to the analysis of problems involving the operation of systems.

—Daellenbach and George

Operations research is the art of winning wars without actually fighting them.

—Auther Clark

Operations research is the art of finding bad answers to problems to which otherwise worse answers are given.

—T L Saaty

Therefore, fundamentally as understood and interpreted, OR is an applied decision science, which effectively performs screening and diagnosing of real-life situation problems, and on finding effective and better solutions, suggests steps to achieve optimality and post-optimality justification, if necessary.

Characteristics of Operations Research Approach

A process following an 'OR approach' is in fact an 'OR team approach' to a real-life system.

The very first point is to make a search for possible existence of some problem or a problem of modification in a system, or a problem of improving upon current plans of operating system by adding some additional features and probably new objectives.

The first leading characteristic is that 'OR approach' being a team approach, is capable of understanding many real situations that arise from time to time or some virtual situations likely to shape its form in real life. Once the nature of the problem is understood, it becomes a challenging task of accepting the problem as a short-term project.

The characteristics of OR can be summarized as under.

1. It is a 'team approach' working, remaining within given constraints and specified directives/ objectives, on a project.
2. A team having experienced persons capable of handling any type of problem makes a collective study and generates a report that can serve as a system summary / primary report.
3. As the team involves experts in different areas, the problem can be viewed from different angles and more than one feasible solution can be recommended. This allows the solution capable of incorporating fluctuations in resources or constraints of the problem. This is known as coordination between various departments and subjects.
4. Mathematization – the process of converting the given problem, variables of the system, and objective function in mathematical terminology and applying mathematical and statistical methods to derive feasible solution – makes 'OR approach' more scientific than being guided by emotions or based on past experience for finding or approximating the solution. This enhances reliability in implementing the solution.
5. A completely transparent system, right from the identification of the problem until the total solution and sensitivity analysis, has always remained a decency of the system on working pattern and procedures.

Phases of OR Approach

As we have already discussed and addressed that 'OR Approach' is a team approach; it is multi-directional in area, i.e. inter-disciplinary in subjects and system, there is always a transparent system from beginning to end. We must have on us classical routines from analysis to implementation stage and also a further analysis, and justification of the feasible solution.

(1) Judgment Phase: This phase includes:

(a) *Identification of the real-life problem:* The very first point is to make a search for possible existence of some problem or a problem of modification in a system, or a problem of improving upon current plans of operating system by adding some additional features and probably new objectives.

(b) *Mathematization*: Once the problem is identified with most of the known features, the immediate stage comes of Mathematization or conversion of the problem in mathematical terminology by identifying the system variables, resources, and objectives to be achieved on getting a feasible solution. Mathematical representation describing the given problem is called a mathematical model. The next important task is to examine the capability, and efficiency to solve the system within a time interval. This brings the system on a real platform of acting and managing resources for solving. There is an MOU between competent authorities of OR group and in-charge of the system.

(c) *Determining appropriate scale, units of measurement of variables, and approximating the range of variables likely to take-up the values:* This is a very important analytical stage. At this stage, before designing the mathematical model of the problem, the person-in-charge normalizes the units of resources and decision variables so that a tentative judgement on the range, in which the decision variables are likely to be estimated. (Suppose, 200 man-hours of given resources can be spared for making chairs and tables (decision variables) taking 20 and 30 minutes of time respectively, then the programmer is required to convert hours into minutes or vice-versa and estimate the maximum or minimum number of chairs and tables. On comparing the estimated values with actual market demand or lot-size production plan, the programmer can predict likely shortages or surplus of the resources (man-hours). This early planning and design helps uninterrupted production.

(d) *Constructing appropriate model with constraints and range of variables:* In order to understand the problem and identify objective function in accordance with the given constraints on resources, the best compact and precise option is to express the given real situation in mathematical notations model. If a model is designed keeping all above-mentioned points in consideration, it will help write algorithmic routines and application of suitable mathematical logic for the feasible solution.

(2) Research Phase: Next follows, in sequence, is the research phase. The objective of this stage is to look for various methods of finding feasible solution to a given problem. At this stage, theoretical sound knowledge of experts will be rightly used and will contribute largely to the solution. This phase is very critical and important because all possible methods to solutions are to be checked and assessed at this stage by taking trial runs on the new values of the quantitative variables. This is essential to check the soundness of feasible solution.

This phase is the one that correctly examines soundness of 'OR Team' in two ways:

(a) If the solution of the system is found using different mathematical or statistical tools/methods, it is important to determine the method or routine which is sound, speedy, and open-ended (i.e. capable of accepting small changes in resources, and constraints and yet remain steady).

(b) It is possible that in order to find a feasible solution using known resources, one may have to add or relax certain constraints. In such cases, job of the 'OR Team' becomes critical in establishing coordination between the values of variables and feasible solutions, which are derived on implementing the new solution values of the variables and the results are as expected by system operations in ideal situations.

There are some problems which need special attention; as such problems, in their own nature, are of the types different than found in routine; e.g. system updating, changing and setting up of new machineries, shortage of raw material, sudden strike, delayed payment and many like these. We need trial and error method to find the feasible solution in every changing situation. This phase is relatively longer in time because of its operating characteristics.

Working of this stage may need past records; useful in making primary study of the tendency of the system. In some cases, extrapolation on these records may be useful to predict the behaviour of the system.

(3) Action Phase: Once the 'OR Team', after careful calculations and taking trials to check the stability of the solution, arrives at a conclusion that makes different recommendations like purchase quantity of raw material, fixing sale price, recruitment of new staff, etc., the recommendations, based on solutions, are implemented and results are studied over a reasonable period and compared with the previous results. Fluctuations are bound to appear and accordingly new variations in suggestions are made. This is 'Sensitivity Analysis'; a routine necessary for achieving steady state or stable state condition of the system.

Models in Operations Research

Model means a representative structure of reality. It possesses mostly all descriptive qualities or features of the real situation; the difference being of only physical properties in both. If the representative model assumes all physical properties then the model is a reality. For Example, a symbolic model of a multi-storied building is able to demonstrate all descriptive features, in proportion, with the actual dimensions, shape, elevation, and other features but not the physical features of materials, like cement, sand, concrete, and iron used.

To depict a real system on a piece of paper, i.e. replica of reality, we need to identify the variables and components. Then we need to determine the best fit that describes the reality.

The importance of model lies in the fact that, it gives the estimation of the behaviour of the system when subjected to external forces. The ideal quality of the model is measured based on minimum deviation from reality on its performance. There are different types of models available which closely describe the behaviour of the system. Some models are subject to a little variation and depend on the prevailing mental state of the designer. We learn from models and make them include more and more features of reality.

Some types of models are as follows:

(1) Physical models: These models, either reduced in size or scaled-up in size, make you conceive about the actual shape and size of the actual reality. Proper scaling shown in the model will help review a better picture of the reality. For example:

(a) Perfectly designed and scaled down physical model (made-up using card board and wooden sticks) of a high-rise building helps civil engineers and fellow person to approximate and understand real situation.

(b) Perfectly designed and scaled up physical model (made-up using metal or plastic balls and sticks) of an atom will clearly explain construction of an atom.

Physical models are easy to envisage, build and describe. For example, once the automobile engineers and mechanical and electronics engineers design production of new cars; they sketch different models proportionally equivalent to actual size, dimension, and all other features. Then, the next step is to prepare a replica of this graphical model into a small toy-type model, which gives completely proportional outlines of the shape, size, space and many other features of actual version going to be in the market. Based on the study and different reports by patrons, some changes are made and conclusions are drawn.

(2) Iconic Models: Iconic models are very close to physical models but there is a conceptual difference between these. The best example would be a photograph of a person. The photograph depicts apparent features of a person but it is not the person itself; it is a scaled-down photograph of the person. When a biology teacher draws the picture of a cell describing all inherent facts, it is a scaled-up version of an actual cell in real life.

Blueprints of a home plans, maps, machine-drawings, etc., are good examples of iconic models. Using iconic models, one can clearly explain the inner details and dominant properties lying in or associated with an object.

(3) Analogue Models: These models represent (in some other set-up or system) technically equivalent and on-line system to a current status of a real system being modeled. For example, on line status or current position of a space craft orbiting around the globe can be correctly and continuously viewed on a computer screen. The former one is a real system while the latter one that describes the status is an analogue model.

Consider for a while what happens when a plane prepares for landing. If there is no analogue model describing continuously its current altitude from the ground level; then what would be the result? Now, we feel importance and correct depiction of the reality.

(4) Symbolic Models: These models use symbols (letters, numbers) and functions describing inter-relationship among the variables. Letters and symbols stand for specific meaning and express inter-relationship or the type and the direction of association between. For example:

$$\Delta = \pi r2$$

$$\frac{\Delta y}{\Delta x} = -\frac{1}{x}$$

These are the symbolic representations and convey meaning and the relationship between independent and dependent relationships in a symbolic form. Similarly, in calculating the ratio of a circumference to the diameter of the same circle, we know for any circle of any radius, having its centre anywhere, this ratio is always constant. This ratio is denoted by p, which we all know, is a symbolic model of the ratio.

(5) Mathematical Models: Finally, we take up the most important and not easy to describe and handle, known as mathematical models. First, to describe the actual problem of real-life situation, we use numerically equivalents symbolic models: For example:

Let x be = number of chairs, Y be = number of tables etc.

Then a problem of identification of what is given and what is to be achieved arises. What are the given resources; how different variables control their usage, etc. These are important points to study before mathematical functions and expressions represent the problem for feasible and optimal solutions.

The next part is to search for the suitable application of mathematical or statistical theories and standard results that are capable of deriving feasible values of decision variables.

It is not necessary that in all cases, there must be a representative model and each model must have a solution.

One should adopt the most practical approach to get a near-by solution, which on application, leads near-to-optimality solution. As the situation changes, decision variables are added; some of these are dropped, resources may increase, decrease, or new resources in new environment may be added; all these require timely updating.

There is a cycle stage; a stage where post study on new result is done and by incorporating the new entrants in the system model, the model changes. As a result of this the solution methods may change and finally the values of the decision variables are likely to change. This is called sensitivity analysis.

If the model does not provide a solution, then we relax certain constraints and try solving them or seek for some other procedure like 'Simulation' for alternate approach to optimal solution.

Readers would appreciate that development of the subject and treatment of each real-life situation problem in each chapter of the book follows the classical approach, as described above.

1 Linear Programming–I

—Every creation has been a linear function of time.

Learning Objectives

AFTER STUDYING THIS CHAPTER, THE STUDENTS WILL BE ABLE TO:

- **construct linear mathematical models of some cases of real life situations**
- **find graphical solution of linear problems of two variables**
- **solve LP problems using simplex algorithm**
- **study different cases and interpret**
- **shape thinking attitude towards the applications of LPP**

INTRODUCTION

In 1939, Russian mathematician, L.V. Kantorovitch published a monograph exploring possibilities of applying linear mathematical models in organizations and production planning, but his efforts did not sound effective. In 1947, in the US, during the period of end of World War II, a project called SCOOP (Scientific Computation of Optimum Programs) was set up under the leadership of G.B. Dantzig. The Simplex algorithm and much of the related work was developed and applied successfully in real situations—allocation of existing resources in critically competing activities with an objective of optimization of certain goals. In the last three decades, Linear Programming (LP) has established its importance in all the areas—commerce, applied sciences like engineering, biology, and many others.

The principal role of operations research (OR) is to observe, understand, analyze, and to construct a generalized and open-ended mathematical model of the situations, to apply all available resources to arrive at an optimal situation; and finally to make a sensitivity analyses in order to evaluate the strength of the solution.

Following on the same lines, we have some real situations that follow a linear pattern in its flow. The ***resources*** like money, work force, and raw material, etc., which serve as basic components. We assume that they are easily available but in a prevailing situation, we have a limited stock or a pre-determined amount, which is allocated to manufacture different items. The items under question are called ***decision variable*** and short-term goal of maximizing the total profit on sale of all such items or to minimize the cost of manufacturing of each item is known as ***objective function***. The conditions imposed on actual utilization of resources by the decision variables in process and the available resources are called ***constraints***. In this chapter, the mathematical model describing all the above terms becomes linear in nature of the decision variable. Hence the name ***Linear Programming***—is truly justified.

Focusing on the application part, linear programming model in its very nature, or has proved to be a power fool that some of the major applications in OR have been cast in the format of linear programming. Transportation models, transshipment models, assignment models, etc., are immediate reflections of the linear models. Some branches based on the format of LPP are integer LPP, Goal Programming, and Dynamic Programming. Allowing minor structural changes it can take up the quadratic programming form or non-linear programming problem.

1.1 LINEAR MODELS

We begin study of linear models by taking certain illustrations. They will help you find the basic concepts of mathematical models.

ILLUSTRATION 1

A contractor has received an order for making tables and chairs for a newly established school. He has enough amount of raw materials available for operation. There are two departments, *A* and *B*. There are three persons in department A and each works for 7 hours per day. In department B, there are 2 persons and each works for 9 hours per day. There are exactly 4 days available. The following table shows distribution of time per unit in each department.

Time in Hours		
Item	*Department A*	*Department B*
Table	3	2
Chair	2	2
Available time/day	21	18

Sale of one table brings a profit of ₹20 while one chair brings ₹15. You are required to find the product mix that brings the maximum profit to the contractor.

Solution

Mathematical Model

Let us assume that the contractor can produce x number of tables and y number of chairs.

According to the table, x number tables will consume $3x$ hours and y number of chairs will consume $2y$ hours of the department A. In turn, the department A has $21 \times 4 = 84$ hours available. So we have $3x + 2y \leq 84$ as a constraint on department A.

It takes 2 hours for a table and 2 hours for a chair in department B. This consumes $2x + 2y$ hours of $18 \times 4 = 72$ available hours of department B.

So we have $2x + 2y \leq 72$ as a constraint on department B.

Focusing on profit factor, x tables will fetch ₹$20x$ and y chairs will fetch ₹$15y$ profit. This makes a total profit of ₹$20x + 15y$.

In any case, one may think of not producing anything or else, what we mean by this is $x \geq 0$ and $y \geq 0$.

Matching all these together, we have the problem of producing x number of tables and y number of chairs.

Find x (number of tables) and y (number of chairs) with an objective of maximizing the total profit, i.e. $z = 20x + 15y$.

This is under the constraints of the two departments A and B.

$3x + 2y \leq 84$ as a constraint on department A.

$2x + 2y = 72$ Constraint on department B.

In addition, we have $x, y, \geq 0$.

This is the mathematical model of the problem. Here x and y are decision variables.

ILLUSTRATION 2

A dietician plans diet menu for a group of students. She concentrates on three basic components—fat, carbohydrate, and protein. She has two main foods A and B. Each 100 gram of A has 2 units of fat, 1 unit of carbohydrate and 5 units of protein. Each 100 gram of food B has 3 units of fat, 2 units of carbohydrate and 3 units of protein. She wants that the diet must contain at least 18 units of fat, 20 units of carbohydrate, and 24 units of protein. The basic cost of 100 gram of food A is ₹10 and ₹12 for that of food B. Her problem is to make the proportionate combination of these types of food that satisfies the basic needs of the diet and minimizes the total cost on food.

Solution

Mathematical Model

Let X and Y denote the amount (in some multiples of 100 gram) of food A and the food B to prepare balanced diet. Now, we deal with each constraint.

Fat constraint: Actual utilization = $2x + 3y$ units,
Minimum requirement = 18 units,
so we write $2x + 3y \geq 18$.

Carbohydrate Constraint: Actual utilization = $1x + 2y$ units,
Minimum requirement = 20 units,
so we write $1x + 2y \geq 20$

Protein constraint: Actual utilization = $5x + 3y$ units,
Minimum requirement = 24 units,
so we write $5x + 3y \geq 24$,

The total cost for this combination = $10x + 12y$. We put all these facts together.

Let x (grams in multiple of 100 grams) amount of food A and y (grams in multiple of 100 grams) amount of food B be the combination.

This will minimize $z = 10x + 12y$

subject to the constraints $2x + 3y \geq 18$,

$$1x + 2y \geq 20,$$
$$5x + 3y \geq 24,$$

with $x, y \geq 0$.

ILLUSTRATION 3

Mr Hanry has ₹80,000 and he wants to invest the same amount in at least one of the two companies A and B. He wants to invest at least 20,000 in company A which pays 10% interest per year. He is also interested to invest some of the amount in company B but the amount must be at most 40,000. The company B pays 9% annual interest. He is interested in maximizing return on his investment. Write a mathematical model.

Solution

Mathematical Model

Let us suppose that he invests ₹x in company A and ₹y in the company B.

As he has to maximize his return, we have an objective function of maximization.

Find x and y to maximize $z = 0.1x + 0.09y$

subject to the constraints $x + y = 80{,}000$ (i.e. total investment).

Also he wants that $x \geq 20{,}000$ investment in company A.

Again he wants that $y \leq 40{,}000$ investment in company B

with x and $y \geq 0$.

ILLUSTRATION 4

A manufacturing company has two plants, located in cities A and B. There are three warehouses for distribution, one in city P, second in city Q and the third in city R. The plant in city A can supply 90 tons of products per month whereas the city B can supply 150 tons per month. The warehouses in cities P, Q and R need 100 tons, 60 tons, and 80 tons respectively. You are to schedule the transportation setup that minimizes the total cost of transportation. The following table gives transportation cost per unit between plants and cities.

	To		
From	*City P*	*City Q*	*City R*
Plant (A)	100	200	150
Plant (B)	200	250	250

Solution

Mathematical Model

Let the amount going out from plant A to cities P, Q, and R respectively be x_{11}, x_{12}, and x_{13} units. In the same way, what goes out from plant B to cities P, Q, and R respectively be x_{21}, x_{22}, and x_{23} units.

Also, we note the fact that the total production at these two plants amounts to 90 + 150 = 240 tons.

The sum demand for the cities P, Q, and R is 100 + 60 + 80 = 240 tons.

So we have equality constraints on product and demand.

$x_{11} + x_{12} + x_{13} = 90$ (Supply from plant A)
$x_{21} + x_{22} + x_{23} = 150$ (Supply from plant B)
$x_{11} + x_{21} = 100$ (demand of city P)
$x_{12} + x_{22} = 60$ (demand of city Q)
$x_{13} + x_{23} = 80$ (demand of city R)

Thus, we have five constraints (two on supply and three on demand.) Now, the cost of transportation associated with the corresponding units of supply is

$100x_{11} + 200x_{12} + 150x_{13} + 200x_{21} + 250x_{22} + 250x_{23}$ (cost function)

The objective is to determine the six variables x_{ij}, $i = 1, 2$ and $j = 1, 2, 3$ that minimizes the above cost function. In addition to the two supply constraints and three demand constraints shown above, we have non-negative condition on variables, i.e. $x_{ij} \geq 0$ for $i = 1, 2$ and $j = 1, 2, 3$

ILLUSTRATION 5

ABC printing press has two departments. Both are capable enough to print and bind both hardcover and paperback books. The printing department A can produce 100 hardcover books in 2 hours or 100 paperback books in one hour. The printing department B can bind 100 hardcover books in 1 hour or 100 paperback books in 2 hours. The operational capacity of department A is to work for at least 80 hours and the department B is 60 hours. The printing cost of one hardcover book is ₹10 and ₹8 per copy for the paperback. You are supposed to plan the printing schedule to minimize the cost.

Solution

Mathematical Model

Let x copies of hardcover books and y copies of paperback books per hour be printed within the limitations of the resources of printing departments A and B. The total printing cost of x hardcover and y paperback books is $10x + 8y$. We are interested in finding such values of x and y that will minimize this cost $10x + 8y$. We want to achieve this within the constraints of the printing departments.

Time Matrix

Department A	For x hardcover copies $\frac{2x}{100}$	For y paperback copies $\frac{1y}{100}$
Department B	For x hardcover copies $\frac{1x}{100}$	For y paperback copies $\frac{2y}{100}$

Considering the operational time (hours), we have

$$\frac{2x}{100} + \frac{1y}{100} \geq 80 \text{ and } \frac{1x}{100} + \frac{2y}{100} \geq 60 \text{ with } x, y \geq 0.$$

The final mathematical model is

Find x (hardcover copies) and y (paperback copies) to minimize the total printing cost ₹$10x + 8y$,
subject to the constraints $2x + y \geq 8000$ and $x + 2y \geq 6000$, with $x, y \geq 0$

ILLUSTRATION 6

A newly developed dairy has started producing cheese, butter, and milk candy. There are three departments: one is the manufacturing department and the other two are pasteurization and packing departments respectively. The following table shows the labor hours spent by one unit (kg) in each department.

Time/kg.			
Department	*Cheese*	*Butter*	*Milk candy*
I Manufacturing	10	1	2
II Pasteurization	7	2	3
III Packing	2/5	4/5	2/5

The minimum working capacity of each plant is 100, 75, and 80 hours respectively. The profit on sale of one (kg) of cheese, butter, and milk candy is ₹12, ₹10, and ₹8 respectively. You have to plan the schedule that maximizes the total profit.

Solution

Mathematical Model

Let x_1 (kg) of cheese, x_2 (kg) of butter, and x_3 (kg) of milk candies be produced. The total profit generated from the sale of these products is $z = 12x_1 + 10x_2 + 8x_3$. There are three departments with their minimum working capacities in hours.

Department I: Actual time utilized = $10x_1 + 1x_2 + 2x_3$ which can be minimum 100 working hours so $10x_1 + 1x_2 + 2x_3 \geq 100$.

Department II: Actual pasteurization time = $7x_1 + 2x_2 + 3x_3$ which can be minimum 75 hours so $7x_1 + 2x_2 + 3x_3 \geq 75$.

Department III: Actual packing time = $(2/5)x_1 + (4/5)x_2 + (2/5)x_3$ which can be minimum 80 hours, i.e. $\frac{2}{5}X_1 + \frac{4}{5}X_2 + \frac{2}{5}X_3 \geq 80$. (i.e. $2X_1 + 4X_2 + 2X_3 \geq 400$).

It is obvious that x_1, x_2, and $x_3 \geq 0$ putting all these together, we have the mathematical model of the problem.

Find x_1 (kg) of cheese, x_2 (kg) of butter and x_3 (kg) of milk candies so as to maximize the total profit $= 12x_1 + 10x_2 + 8x_3$

subject to the constraints $10x_1 + 1x_2 + 2x_3 \geq 100$,

$$7x_1 + 2x_2 + 3x_3 \geq 75$$
$$2x_1 + 4x_2 + 2x_3 \geq 400, \text{ with } x_1, x_2, \text{ and } x_3 \geq 0.$$

ILLUSTRATION 7

A firm is producing two items I_1 and I_2. Each unit of I_1 requires 4 units of raw material and requires 6 hours processing time in the machining department. While the item I_2 needs 3 units of raw material and 3 hours in processing department. The factory has 90 units of raw material as a stock for the current week and 96 hours in the processing department. The profit per unit from the sale of I_1 and I_2 is ₹60 and ₹45 respectively. Remaining within the limits of resources, we are required to find the number of units of I_1 and I_2 to be produced to maximize the profit on sale.

Solution

Mathematical Model

Let x units of I_1 and y units of I_2 produced to maximize the total profit. Profit $P(I_1, I_2) = 60x + 40y$.

We consider the constraints on resources $4x + 3y$ = total raw material required. This cannot exceed available 90 units, i.e. $4x + 3y \leq 90$.

For the required production, the time spent in the processing department is $3x + 3y$ which can be at the most 96 hours available i.e. we have $3x + 3y \leq 96$.

We understand that in any case $x, y \geq 0$.

Putting all these things together, the model is

Find x (unit of I_1) and y (unit of I_2) to maximize

$Z = 60x + 40y$ subject to the constraints $4x + 3y \leq 90$,

$3x + 3y \leq 96$, with $x, y \geq 0$.

ILLUSTRATION 8

The zeta company is engaged in preparing lawn products. It uses phosphate and nitrate as fertilizers. The company has 50 units (metric tons) and 80 metric tons of phosphate and nitrate as the current inventory. Three types of lawns (regular, super, and garden) require some of these two fertilizers. The profit on sale and service per 1000 bags is given in the table. You are required to plan the different types of lawns that can maximize the profit on sale.

Requirement (metric tons, 1000 bags for each type of lawn)

Type of Lawn	*Phosphate*	*Nitrate*	*Profit (₹1000 Bag)*
Regular Lawn	2	4	300
Super Lawn	3	4	500
Garden Lawn	2	2	400

Solution

Mathematical Model

Let x_1 units of regular lawn products, x_2 units of super lawn products, and x_3 units of garden products be the requirement per 1000 bags of each type (Unit – number of bags in thousands).

The profit on sale $Z = 300x_1 + 500x_2 + 400x_3$

Total amount of phosphate required = $2x_1 + 3x_2 + 2x_3$ which can be less than or equal to the existing inventory 50 metric tons of phosphate,

i.e. $2x_1 + 3x_2 + 2x_3 \leq 50$ with x_1, x_2 and $x_3 \geq 0$

Similarly, for the nitrate, we have $4x_1 + 4x_2 + 2x_3 \leq 80$ with x_1, x_2, and $x_3 \geq 0$.

The required model is to find x_1, $x_{2,}$ and x_3 so as to maximize $Z = 300x_1 + 500x_2 + 400x_3$,

subject to the constraints

$$2x_1 + 3x_2 + 2x_3 \leq 50,$$
$$4x_1 + 4g_2 + 2x_3 \leq 80, \text{ with } x_1, x_{2,} \text{ and } x_3 \geq 0$$

1.1(A) Observations from the Illustrations

In the above illustrations, we observe the following characteristics:

1. In each one there are some variables to be determined (found\evaluated) with some objectives. In the case of profit, it is maximization and in the case of cost time, etc., it is minimization.
2. The given resources are to be allocated/used in the production of total number of units of the variable under consideration.
3. The variables, which appear in the profit/cost function and constraints on the resources, are in first degree only.
4. The constraints are exactly of any one of the three types, ≤, =, or ≥.
5. In all the illustrations, the variables to be determined, are always non-negative.

1.1(B) General Format of Linear Programming Problem

Find values of the variables $x_1, x_2, x_3, \ldots, x_n$ which will maximize or minimize

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{1}$$

subject to the restrictions

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq, (\geq), (=)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq, (\geq), (=)\ b_2 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq, (\geq), (=)\ b_m \end{array}\right\} \tag{2}$$

With all $\quad x_1, x_2, \ldots x_n \geq 0 \qquad (3)$

The variables $x_1, x_2, \ldots x_n$ are called ***decision variables.***

The linear function given by (1) is called the ***objective function.***

The conditions (inequality or equality) imposed on the resources given by (2) are called ***constraints.***

Also, we note that in each of (m) constraints exactly one of the signs $\leq$, $\geq$, or = (given in (2)) holds true at a time.

We note that the objective function (1) and the constraint (2) appear as a linear function of the decision variables. At last, we have the condition given by (3), which is called a ***non-negativity condition*** on the decision variables.

There are m constraints in n variables. (generally $m \leq n : m, n \in N$.)

1.1(C) Standard Form and Canonical Form

Standard Form

The linear programming problem in the ***standard form*** appears as follows.
Find values of $x_1, x_2, \ldots x_n \geq 0$ which will maximize

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{4}$$

subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \end{array}\right\} \tag{5}$$

with all $\quad x_i \geq 0\ ;\ i = 1, 2, \ldots n \qquad (6)$

(Maximization type of linear programming problem with all constraints of $\leq$ type and all decision variables are non-negative.)

Note: We call this a standard form of LPP, because using this form, it enables one to write down the dual of the given problem. This dual form, you will see in the next chapter.

Canonical Form

The linear programming problem in the canonical form has the following features.
Find value of $x_1, x_2, \ldots, x_n \geq 0$, which will maximize

$$z = c_1\, x_1 + c_2\, x_2 + \ldots + c_n\, x_n \tag{7}$$

subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}\right\} \tag{8}$$

with all variables $x_1, x_2, \dots, x_n \geq 0$ (9)

(Maximization type of linear programming problem with all the constraints of = type. All decision variables are non-negative)

We observe that the system given by (1), (2) and (3) is the most ***general form*** of LPP.

What is given by the set of conditions (4), (5) and (6), stand for the ***standard form*** of LPP.

Given by the system equations (7), (8) and (9) is the ***canonical form*** of LPP.

[This should not leave any chance of confusion.]

Note that the following mathematical models are ***neither*** in standard form ***nor*** in canonical form.

1. Maximize $Z = 3x + 2y$
   Subject to $2x + 3y \leq 5$; $3x - 2y \leq 11$; $-x + y \geq -8$; $x, y \geq 0$
2. Maximize $Z = 3x_1 + 2x_2 + 3x_3$
   Subject to $3x_1 + 2x_2 + x_3 = 11$; $2x_1 - x_2 + 5x_3 = 9$; $x_1 + x_2 - x_3 \leq 9$; $x_1, x_2 \geq 0$
   (Give proper justification for the same.)

1.1(D) Some Mathematical Facts

(1) Minimization Problem as Maximization and Vice Versa

Every maximization problem can be viewed as a minimization problem and conversely also.

i.e. Maximize $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ is mathematically same as
$-1x$ minimize $(-Z = -c_1x_1 - \dots - c_nx_n) = -1x \,.\, (-z) = z$

(2) Reversing an Inequality

An inequality of $\leq$ type can be converted to an inequality of $\geq$ (or conversely), by multiplying each term of the inequality by -1.

i.e. $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$ is mathematically same as
$-a_{11}x_1 - a_{12}x_2 - \dots -a_{1n}x_n \geq -b_1$

Thus maximize $z = 3x_1 + 2x_2 + 3x_3$
subject to $-1x_1 + 2x_2 - x_3 \geq -4$, $x_1 + 3x_2 + 1x_3 \leq 2$, $x_1, x_2, x_3 \geq 0$.

is not in the standard form, the inequality in the first constraint is of $\geq$ type. We multiply by each term of it by -1 and get $1x_1 - 2x_2 + x_3 \leq 4$.

Now, substituting it in the place of first constraint, the problem is maximize $Z = 3x_1 + 2x_2 + 3x_3$ subject to $1x_1 - 2x_2 + x_3 \leq 4$, $x_1 + 3x_2 + 1x_3 \leq 2$ with $x_1, x_2, x_3 \geq 0$ is a linear programming problem in the standard form.

(3) Changing Equality to an Inequality

Some mathematical equation, say $x = c$ is mathematically equivalent to $x \leq c$ and $x \geq c$ both true at a time.

In general $\sum_{j=1}^{j=n} a_{ij} = b_i$ for $i = 1, 2, \ldots, m$ is equivalent to $\sum_{j=1}^{j=n} a_{ij} \le b_i$ and $\sum_{j=1}^{j=n} a_{ij} \ge b_i$ for each value of $i = 1, 2, \ldots, m$. and $j = 1$ to n.

Maximize $Z = 2x + 3y$ subject to $x + y \le 9$ and $2x + 3y = 15$ with $x, y \ge 0$ is not in the standard form.

The second constraint $2x + 3y = 15$ is equivalent to $2x + 3y \le 15$ and $2x + 3y \ge 15$. It now can be shown as $2x + 3y \le 15$ and $-2x - 3y \le -15$.

We introduce this fact in the problem.

Now the problem maximize $Z = 2x + 3y$ subject to $x + y \le 9$,

$2x + 3y \le 15$, and $-2x - 3y \le -15$. With $x, y \ge 0$ is a problem in the standard form.

1.1(E) LPP in Matrix Notation

In this section, we use matrix notations in the standard and canonical form of LPP. In the remaining part of this chapter and the next one, we will use matrix form of LPP. We consider the following pattern of matrices.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{mn} \end{pmatrix} \text{ which is matrix of order } m \times n \text{ (} m \text{ rows and } n \text{ columns)}$$

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \\ b_m \end{pmatrix}_{m\times 1}, \ \mathbf{C} = (c_1, c_2, c_3, \ldots, c_n)_{1\times n} \text{ where } \mathbf{C} \text{ is a row vector.}$$

And finally $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1}$ is a null vector.

Using these matrices in a suitable form in the standard LPP,

$$\begin{aligned} &\text{Maximize} && Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n && (1) \\ &\text{subject to} && \left.\begin{aligned} &a_{11}x_1 + a_{12}\,x_2 + \ldots + a_{1n}\,x_n \le b_1 \\ &a_{21}x_1 + a_{22}\,x_2 + \ldots + a_{2n}\,x_n \le b_2 \\ &\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ &a_{m1}x_1 + a_{m2}\,x_2 + \ldots + a_{mn}\,x_n \le b_m \end{aligned}\right\} && (2) \\ &\text{with all} && x_1, x_2, \ldots x_n \ge 0 && (3) \end{aligned}$$

takes up the following form:

Find a vector $\mathbf{X} \in R_{n\times 1}$ to maximize the objective function $Z = \mathbf{CX}$ (4)

Subject to $A\mathbf{X} \le \mathbf{b}$ (5)

With $\mathbf{X} \ge \mathbf{0}$ (6)

The system in inequalities (1) (2) and (3) and the one in inequalities (4), (5), and (6) are equivalent system in the standard form of LPP.

As an illustration to this point,

Maximize $Z = 2x + 3y$,
subject to $x + y \le 9$
$2x + 3y \le 15$
$-2x - 3y \le -15$
$x, y \ge 0$

is

Maximize

$$Z = \begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

subject to

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} 9 \\ 15 \\ -15 \end{pmatrix}$$

With

$$\begin{pmatrix} x \\ y \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0} = \text{a null vector}$$

1.1(F) Some Definitions

The most general format of LPP is: find $\mathbf{X} \in \mathbf{R}_{n\times 1}$, so as to optimize (maximize or minimize) subject to m constraints given by the matrix system as

$$Z = \mathbf{CX} \tag{1}$$

$$A\mathbf{X} \le, (\ge), (=)\ \mathbf{b} \tag{2}$$

$$\text{with } \mathbf{X} \ge \mathbf{O} \tag{3}$$

In the set **(2)**, there are m constraints in n variables ($m \le n$); also in ***each constraint*** exactly any one of three signs ($\le, \ge, =$) holds true at a time

1. Any set of n values, which satisfies all the m given constraints of the system given by **(2)** is called a ***solution***.
2. Any set of n values of $\mathbf{X} \in \mathbf{R}_{n\times 1,}$ which simultaneously satisfies the conditions **(2)** and **(3)**, is called a ***feasible solution***.
3. Any set of n values of $X \in \mathbf{R}_{n\times 1}$ which satisfies **(1), (2)** and **(3)** is called an ***optimal*** (feasible) ***solution***, i.e. a feasible solution which satisfies the condition **(1)**, i.e. which optimizes $Z = \boldsymbol{CX}$ is called an *optimal solution*. Where $C_{1\times n}$ is a row vector.

1.2 Graphical Solution and Fundamentals

Now, we look forward to finding feasible solution to the problems. First, we recall all fundamentals necessary to build-up the algorithmic thinking.

1.2(A) Fundamentals of Line and Corresponding Regions in $X\,0\,Y$ Plane

An equation in two variables, say x and y, having a general linear form $ax + by = c$ with $a^2 + b^2 \ne 0$ represents a **line** in two-dimensional plane, i.e. x-y plane, i.e. the different forms of a line in the above format are as follows:

1. $2x + 3y = 0$
2. $2x + 3y = 5$
3. $3x = 6$
4. $4y = 9$

Any equation of line has any one of the above forms in x-y system (R^2 space).

In x-y plane (x 0 y Plane), a line is determined by two points, i.e. given any two distinct points, exactly one line can pass through these two points. Once the equation of a line has been determined, it means that all infinite points lying on the line will satisfy that equation. Such a line divides the R^2 space in exactly three open regions/parts.

The set of points below the line is given by $ax + by < c$, the set of points on the line is given by $ax + by = c$, and the set of points above the line is given by $ax + by > c$ (with all necessary conditions on a, b, and c.)

In x–y plane, we have the equation of x–axis given by $y = 0$ and similarly the equation of y–axis is given by $x = 0$. This means that in the process of determining the line, we put $y = 0$ to find the point of intersection of line with x–axis.

For example, for a line $2x + 3y = 12$, put $y = 0$.

This gives $2x + 3(0) = 12$, i.e. $x = 6$.

It means that the line cuts the x-axis in the point (6, 0). Similarly for the point of intersection of line with y-axis; we put $x = 0$.

This gives $2(0) + 3y = 12$ or $y = 4$, i.e. the line cuts $2x + 3y = 12$ the y-axis at the point (0, 4).

Thus we have two point (6, 0) on x-axis and (0, 4) on y-axis. The said line $2x + 3y = 12$ passes through these two points or joining two points throughout the plane/continuously on both the sides, we get the position of the line. This is one of the easiest methods to draw a line.

1.2(B) Lines and Regions

Figure 1.1 will provide you with the total idea about the regions and quadrant in x-y plane.

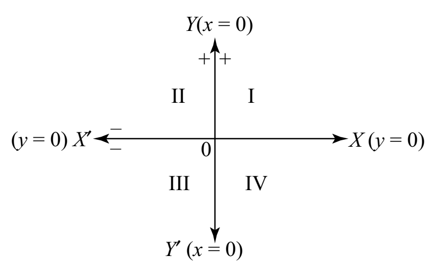


Figure 1.1

x-y plane and four quadrant point 0—origin denoted as (0, 0).
For the points like (x, y) in first quadrant (I), we have (x, y) having (+, +) signs,
Points (x, y) in second quadrant (II) have (–, +) signs,
Points (x, y) in third quadrant (III) have (–, –) signs, and
Points (x, y) in fourth quadrant (IV) have (+, –) signs.

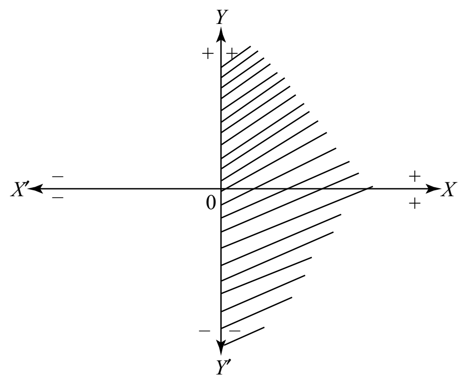


Figure 1.2

Shaded region indicate $x \geq 0$ {Including y-axis ($x = 0$) also}

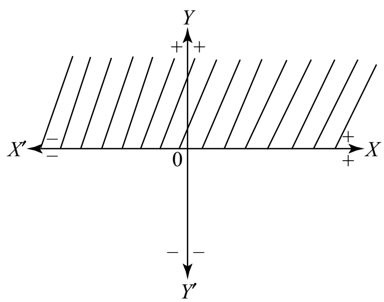


Fig. 1.3 Shaded region indicate $y \geq 0$ (Including x-axis ($y = 0$) also)

In the most of the cases in LPP; we have $x \geq 0$ and $y \geq 0$ simultaneously. From the Figures 1.2 and 1.3, we conclude that the shaded region (first quadrant including the positive x-axis part and positive part of y-axis) shows $x \geq 0$ and $y \geq 0$.

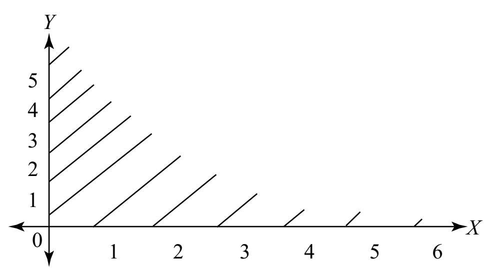


Fig. 1.4

Line in *x*-*y* plane and corresponding regions

Let us consider the equation of a line, say $2x + 3y = 12$. To plot this line we have to find at least two points, which are on the line. To determine the points is to find intersection of the line with x- and y-axis.

For finding the point of intersection with x-axis, put $y = 0$. This brings $2x + 3(0) = 12$; giving $x = 6$. The point is (6, 0) on x-axis. Similarly the point on y-axis is (0, 4) (putting $x = 0$).

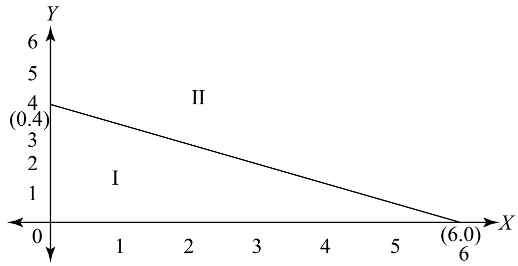


Fig. 1.5

In most of cases in LPP, we have the restrictions $x \geq 0$ and $y \geq 0$. Considering these restrictions, we take care of the points in the ***first quadrant only*** corresponding to the ***line segment*** of $2x + 3y = 12$ in the first quadrant, we have following three regions:

(a) Set of points below the line and above the corresponding portion of $x \geq 0$ and $y \geq 0$. It is denoted by (I) in Figure 1.5. In this region, we have set of points like (x, y) for which $2x + 3y < 12$.

(b) Set of points on the line segment for which $2x + 3y = 12$

(c) Set of points like (x, y) above the line segment for which $2x + 3y > 12$. It is denoted by (II) in Figure 1.5.

1.2(C) Graphical Solution of LPP

In the case where the decision variables **X** has only two components,

i.e. $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ or $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, it is possible to solve the problem graphically. The LPP problem looks as follows.

Find $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix} \in R^2$

so as to optimize (maximize or minimize)

$$Z = \mathbf{CX} = c_1 x + c_2 y \text{ (or } c_1 x_1 + c_2 x_2) \quad (1)$$

subject to the constraints $\quad \mathrm{A}\mathbf{X} \leq \text{ or } (=) \text{ or } (\geq) \boldsymbol{b} \quad (2)$

with $\quad x \geq 0 \text{ and } y \geq 0 \quad (3)$

The system given by inequality (2) has equations in two variables only. Looking at the definitions, if

$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m\times 1} \geq \mathbf{0}$ then we have, because $x \geq 0$ and $y \geq 0$, a line segment showing its corresponding region in first quadrant.

With all these mathematical facts on hand, we will follow the following steps to find the optimal solution of a linear programming problem described above.

Steps

1. Consider any type of inequality constraint (≤ or ≥) as = one.
2. Now, draw the line corresponding to equality constraint and corresponding to $x \geq 0$, $y \geq 0$ and consider the segment of the line in the first quadrant.
3. Once this is done; consider the original constraint (either ≤ or ≥) and shade the region corresponding to the segment in the first quadrant.
4. Repeat the steps (1), (2) and (3) above for **all** the constraints given in the problem.
5. Find the region (in the first quadrant) which is **the most common** to all the constraints. This region, **if it exists,** is the **feasible region**.
6. If either the feasible region is convex and bounded or has a lower bound (bounded below) then the optimal solution to the problem exists on at least one of the vertices. (Practically, we find all the vertices of the convex region, evaluate the objective function at these vertices, and pick up the optimum value and the corresponding vertex\vertices.

1.2(D) Illustrations: Graphical Solution

ILLUSTRATION 9

We consider some examples that we have formulated in the previous unit. We consider mathematical model of Example 1. The problem is

A contractor has received an order for making tables and chairs for a newly established school. He has enough amount of raw materials available for operation. There are two departments, *A* and *B*. There are three persons in department A and each works for 7 hours per day. In department B, there are 2 persons and each works for 9 hours per day. There are exactly 4 days available. The following table shows distribution of time per unit in each department.

Maximize $Z = 20x + 15y$, subject to the constraints $3x + 2y \leq 84$

$2x + 2y \leq 72$ with $x, y, \geq 0$.

Solution

1. Consider $3x + 2y = 84$, putting $x = 0$, we have $y = 42$. The point on y-axis is (0, 42). Similarly putting $y = 0$ we have $x = 28$. The point on x-axis is (28, 0). We join these two points (28, 0) and (0, 42) to get the line $3x + 2y = 84$. We consider the segment of this line in the first region only.
2. Now consider the corresponding region of the line segment of the constraint $3x + 2y \leq 84$. Identify this region by shading it by separated line segments in the region (Figure 1.6).

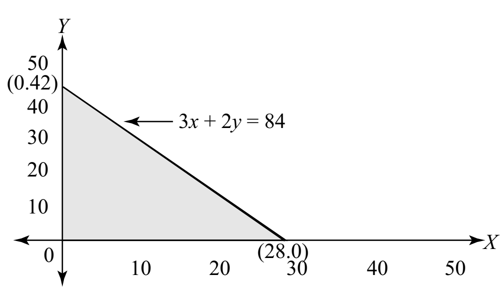


Fig. 1.6

3. Now we consider the second constraint in equality form $2x + 2y = 72$ for $x = 0$, $y = 36$. The point on y-axis is (0, 36) also consider $y = 0$ and so $x = 36$. Corresponding to this on the X-axis is (36, 0). Plot these points and join them by a line—this line itself is the set of points which satisfies $2x + 2y = 72$ corresponding to $x \geq 0$ and $y \geq 0$; there is a segment of this line in the first quadrant.
4. Now consider $2x + 2y \geq 72$, find the corresponding region in the first quadrant.

(Shaded region including points of x-axis, y-axis and the line segment is the region $3x + 2y \leq 84$; $x \geq 0$, $y \geq 0$)

Note: Convexity of the feasible region and existence of optimality at vertices only—these two points are well supported by theorems that we will prove.

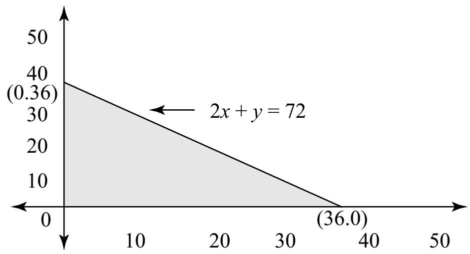


Fig. 1.7

(Shaded portion shows the region corresponding to $2x + 2y \leq 72$; $x \geq 0$, $y \geq 0$)

Now we find the most common region to both these constraints, this is our feasible region for $x \geq 0$, $y \geq 0$.

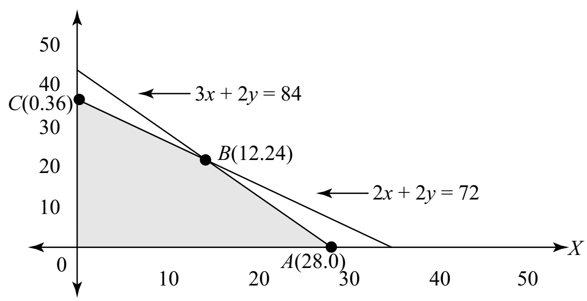


Fig. 1.8

The shaded region is 0–ABC0, it is a convex region (polygon).

Evaluation of value for objective function.

1. 0(0, 0) 20(0) + 15(0) = 0
2. A(28, 0) 20(28) + 15(0) = 560
3. B(12,24) 20(12) + 15(24) =1140 * Maximum
4. C(0, 36) 20(0) + 15(36) =540

For $x = 12$ tables and $y = 24$ chairs will fetch a profit of ₹1140.

ILLUSTRATION 10

Find the graphical solution of the following LP problem.
Maximize $Z = 5x + 10y$
subject to $2x + y \geq 8$; $3x + 4y \leq 24$; $y \geq 2$; $x, y \geq 0$.

Solution

Step 1

Consider $2x + y = 8$ and find two points to draw the line.

For $x = 0, y = 8,$

point (0, 8) and $y = 0$ gives $x = 4$; the point is (4, 0). Plot the points (0, 8) and (4, 0) and draw a line passing through them. Now $x, y \geq 0$, draw or shade (or indentify) the corresponding region.

Similarly for $3x + 4y = 24$, the two points are (8, 0) and (0, 6) (for $y = 0$ and $x = 0$) plot the points; join them and identify the corresponding region corresponding to $3x + 2y \leq 24$.

At last, consider $y = 2$ which is a line parallel to x-axis and passing through (0, 2). Now identify the region for $y \geq 2$.

Step 2

After identifying the three regions, we find the region that is the most common to all these constraints, this region is shown in Figure 1.9 as shaded. It is a triangular region with vertices A, B and C.

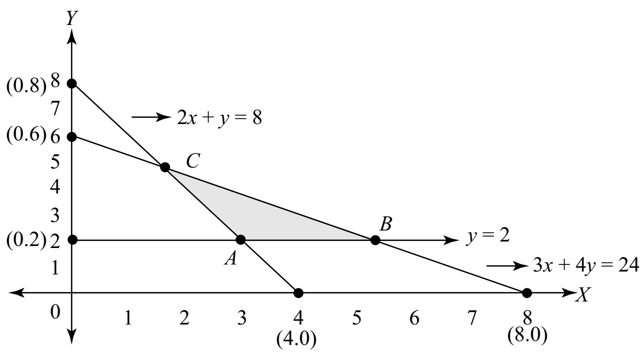


Fig. 1.9

(In Figure 2.9, the feasible region is ABC–A—a closed triangular region.)

Step 3

We find the coordinates of the vertices A, B and C as $A = A(3, 2)$, $B = B(16/3, 2)$ and $C = C$ (8/5, 24/5), we evaluate the objective function $5x + 10y$ at these vertices.

Vertex	*Value of 5x + 10y*
A(3, 2)	5(3) +10(2) = 35
B(16/3, 2)	5(16/3) + 10(2) = 140/3
C(8/5, 24/5)	5(8/5) + 10(24/5) = 56 *

The value of the objective function at C(8/5, 24/5) is maximum (= 56). Thus at $x = 8/5$, $y = 24/5$, the maximum value of the objective function $5x + 10y$ is found and it is 56.

ILLUSTRATION 11

Find the graphical solution.
Maximize $Z = 2x + 5y$
Subject to the constraints $x + 2y \leq 2$ and $2x + 3y \geq 6$; $x, y \geq 0$.

Solution

Step 1

Consider $x + 2y = 2$ and find two points to draw the line segment in the first quadrant ($x \geq 0, y \geq 0$). For $x = 0, y = 1$, the point is (0, 1) and for $y = 0, x = 2$; the point is (2, 0). Join these two points (0, 1) and (2, 0); which gives the line segment in the first quadrant. Now identify the region corresponding to $x + 2y \leq 2$.

Step 2

Now consider the line $2x + 3y = 6$. We find (0, 2) for $x = 0$ and (3, 0) for $y = 0$.

To draw the line segment in the first quadrant $x, y \geq 0$, we now consider $2x + 3y \geq 6$ and identify the region for the same. (This region is above the line $2x + 3y = 6$; not toward the origin.)

Step 3

In this step, we find the most common region to these two regions corresponding to the given constraints. As it is clear from Figure 1.10 that there is **no** common region (The set of points in the common region is ϕ—null set.) to both the regions.

The solution to this problem is an ***infeasible solution.***

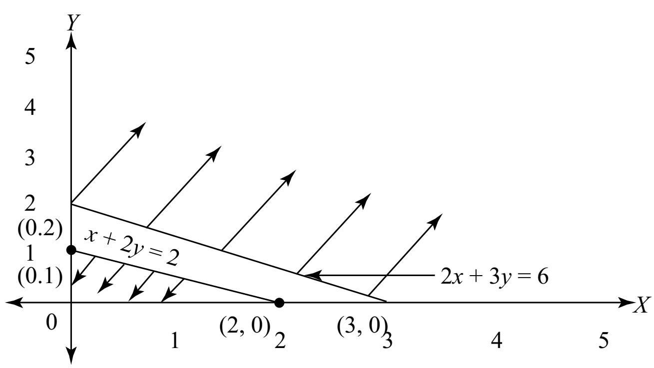


Fig. 1.10

The arrow shows the region corresponding to each constraint. There is no common region to them. The solution set to the problem is a null set—It is a case of ***infeasible solution***.

ILLUSTRATION 12

Find the graphical solution to the problem
Minimize $Z = 14x + 28y$,
subject to the constraints $2x + 3y \geq 6$;

$$6x + 2y \geq 9;$$
$$x, y \geq 0.$$

Solution

Step 1

Consider $2x + 3y \geq 6$ and find two points to get the line segment corresponding to $x, y \geq 0$. For $y = 0$, $x = 3$; the point is (3, 0) and for $x = 0$, $y = 2$; the point is (0, 2). Now identify the region for $2x + 3y \geq 6$.

Step 2

Consider $6x + 2y \geq 9$ and find two points to get the line segment corresponding to $x, y \geq 0$. For $x = 0$ gives $y = 9/2$ so the point is (0, 4.5) and $y = 0$ gives $x = 9/6 = 3/2$, so the point is (3/2, 0). Now, identify the region corresponding to the inequality $6x + 2y \geq 9$.

Step 3

In this step, we find the region that is most common to the two regions, if the most common region is bounded (at least on one side) then we evaluate the objective function at its vertices.

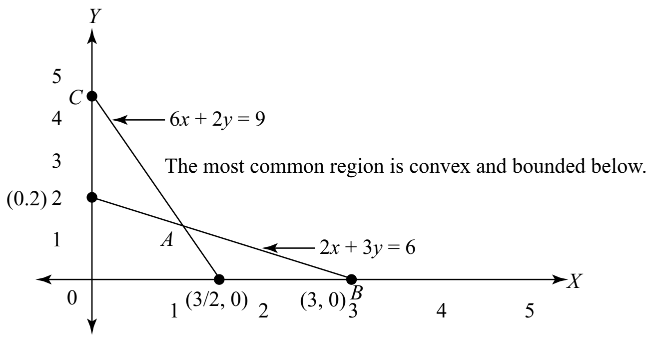


Fig. 1.11

The unbounded region has the points A (15/14, 9/7) (by solving the two equations of lines) B (3, 0) and (0, 9/2) and x- and y-axis of first quadrant as its lower boundary.

Step 4

We evaluate the objective function at these vertices (Fig. 1.11).

Vertex	*Value of 14x + 28y*
A (3, 0)	14(3) +28(0) = 42
B (15/14, 9/7)	14(15/14) + 28(9/7) = 51
C (0, 9/2)	14(0) + 28(9/2) = 126 *

The maximum of all these three values is 126 and this occurs at $x = 0$, $y = 9/2$. It means that as we move away from the origin, with the objective function, the profit increases. You can check this from the figure.

This is a case of unbounded solution.

ILLUSTRATION 13

Find the graphical solution of the problem

maximize $Z = 2x + 3y$;

Subject to the constraints $4x + 6y \leq 24$;

$$5x + 3y \leq 15;$$
$$x + y \geq 2 \text{ and } x, y, \geq 0.$$

Solution

Step 1

As we have done in all the previous examples, we perform the operations in brief. For $4x + 6y \le 24$; we consider $4x + 6y = 24$. For $y = 0$, $x = 6$ and so the point is (6, 0);

For $x = 0$, $y = 4$ and the point is (0, 4). We draw the line segment and identify the region for $4x + 6y \le 24$; $x \ge 0$, $y \ge 0$. Similarly for $5x + 3y \le 15$; we consider $5x + 3y = 15$. For $y = 0$ $x = 3$, and so the point is (3, 0); for $x = 0$, $y = 5$ so the point is (0, 5). Now, we join these two points and then identify the corresponding region in the first quadrant.

Finally for the constraint $x + y \ge 2$; we consider $x + y = 2$. For $x = 0$, $y = 2$ so the point is (0, 2); for $y = 0$, $x = 2$ so the point is (2, 0) after joining these two points in $x, y \ge 0$, we find the region corresponding to $x + y \ge 2$.

Step 2

Now we find the most common region to all these regions. This region in Figure 1.12 is shown by $ABCDE - A$. It is a convex pentagon. The solution of the problem exists on at least one vertex of this most common feasible region.

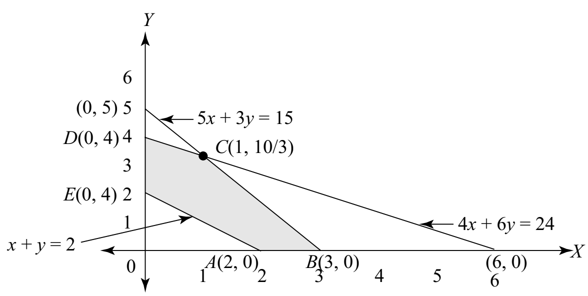


Fig. 1.12

Step 3

We evaluate the objective function at these vertices.

Vertex	*Value of 2x + 3y*
A (2, 0)	2(2) +3(0) = 4
B (3, 0)	2(3) + 3(0) = 6
C (1, 10/3)	2(1) + 3(10/3) = 12 *
D (0, 4)	2(0) + 3(4) =12 *
E (0, 2)	2(0) + 3(2) = 6

Comment

The maximum value 12 occurs at two different points C (1, 10/3) and D (0, 4). It means that there is no unique point giving this optimal solution to this maximization problem. In fact, the points on the line segment joining C (1, 10/3) and D (0,4) given by the set;

$\{(X, Y) \mid X = \lambda(0) + (1 - \lambda)(1);\ Y = \lambda(4) + (1 - \lambda)(10/3)$ for $0 \le \lambda \le 1\}$ are all such points giving maximum value = 12 to this objective function.

This is a case of **multiple optimal solutions**. [The last three Examples 11, 12 and 13 are the examples demonstrating special cases to the linear programming problems. We will study them in some of the remaining parts of this chapter.]

1.2(E) Mathematical Facts and Two Important Theorems

The theorems, we are going to prove in the following parts, are very important and key points. We have mentioned about these points in Section **1.2 (c)-6** of graphical solution to LPP.

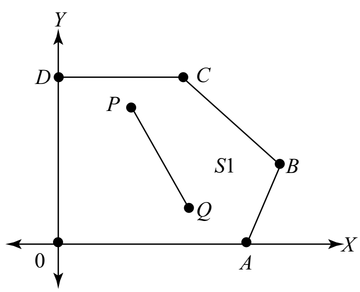
Before discussing the theorems, we state some mathematical facts, which will serve important helplines in order to understand the theorems.

1.3 Mathematical Facts

1.3(A) Convex Set

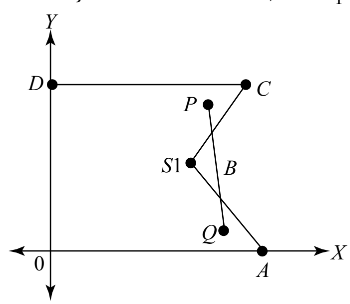
A region is a convex region if a line segment joining any two points of the region completely remains within the region.

Let $S = \{(x, y) \mid x, y \ge 0\}$. Then for $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two distinct points the line segment. $S_1 = \{(x, y) \mid x = \lambda x_2 + (1 - \lambda)x_1,\ y = \lambda y_2 + (1 - \lambda)\, y_1,\ 0 \le \lambda \le 1\}$ is contained in S; i.e. $S_1 \subset S$, so S is convex.



Region S is convex [$PQ = S_1$]

Fig. 1.13



S_1 is not a convex region [$PQ \not\subset S_1$]

Fig. 1.14

1. At this stage we accept that hyper plane is a convex set.
2. Intersection of a finite collection of convex sets is convex.
3. Let S be the convex set in R^n. A point P in S is an **extreme point** if and only if P is not a convex combination of other points of S.
4. Convex sets are of two types: bounded and unbounded.

1.3(B) Convex Function

A function f defined on convex set S in R^n is called a convex function if for any two points $\mathbf{X}_1$ and $\mathbf{X}_2$ if $f(\lambda \mathbf{X}_1 + (1 - \lambda)\, \mathbf{X}_2) \le \lambda f(X_1) + (1 - \lambda) f(\mathbf{X}_2)$ with $0 \le \lambda \le 1$)

1.3(C) Theorems

Theorem 1: The set S of all feasible solutions to a linear programming problem is convex.

Proof:

Let us consider the standard form of LPP in matrix notation.

Find $\mathbf{X} \in R_{n\times 1}$

so as to maximize $Z = \mathbf{CX}$ (1)

subject to $A\mathbf{X} \le \mathbf{b}$ (2) [There are m constraints in n variables]

with $\mathbf{X} \ge \mathbf{0}$ (3)

where $\mathbf{C} \in R_{1\times n}$ (a row vector)

Let $\mathbf{X}_1$ and $\mathbf{X_2}$ satisfy (2) and (3). The set of all such $\mathbf{X}$ is a feasible region S.

We have $AX_1 \le \mathbf{b}$; $\mathbf{X}_1 \ge \mathbf{0}$ and $A\mathbf{X}_2 \le \mathbf{b}$; $\mathbf{X}_2 \ge \mathbf{0}.$

Let $\mathbf{X} = \lambda\, \mathbf{X}_2 + (1 - \lambda)\mathbf{X}_1$ be a convex combination of $\mathbf{X_1}$ and $\mathbf{X_2}$ with $0 \le \lambda \le 1$, $\mathbf{X}_1, \mathbf{X}_2 \in$ S and we want to prove that $\mathbf{X} \in$ S.

Now $A\mathbf{X} = A(\lambda\, \mathbf{X}_2 + (1 - \lambda)\, \mathbf{X}_1) = \lambda\, A\mathrm{X}_2 + (1 - \lambda)\, A\mathbf{X}_1$

But $A\mathbf{X}_1 \le \mathbf{b}$ and $A\mathbf{X}_2 \le \mathbf{b}$**;** using these two conditions, we have

$\therefore$ $A\mathbf{X} = \lambda(A\mathbf{X}_2) + (1 - \lambda)(A\mathrm{X}_1) \le \lambda\, \mathbf{b} + (1 - \lambda)\, \mathbf{b} = \mathbf{b}$

$\therefore$ $A\mathrm{X} \le \mathbf{b} \Rightarrow \mathbf{X} \in$ S

This implies that S, set of all feasible solutions of LPP, is convex.

Theorem 2

Let S, the set of all feasible solutions to an LPP, be bounded. Then one of the extreme points is an optional solution.

Proof

Before proving the theorem we, once again, recall that an extreme point cannot be a convex combination of other points [Note 3 in the unit]. We consider the standard form of LPP.

The standard form of LPP (in matrix notation) is as follows.

Find $\mathbf{X} \in \mathrm{R}_{n\times 1}$

so as to maximize $Z = \mathbf{CX}$ (1)

subject to the constraints $A\mathbf{X} \le \mathbf{b}$ (2) [There are m constraint in n variables]

with $\mathbf{X} \ge \mathbf{0}$ (3)

where $\mathbf{C} \in \mathrm{R}_{1\times n}$ (a row vector)

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ be the extreme points of S. We have r values of the objective functions. $Z_1 = \mathbf{cu}_1$, $Z_2 = \mathbf{cu}_2, \ldots, Z_r = \mathbf{cu}_r$. Let one of these values; say $Z_m = \mathbf{cu}_m$ be the maximum value.

Let $\mathbf{X}$ be any possible solution of the LPP, i.e. $\mathbf{X}$ satisfies both $\mathrm{A}\mathbf{X} \le \mathbf{b}$ and $\mathbf{X} \ge \mathbf{0}$**,** Let $\mathbf{X} \le$ be written as a convex combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$

$$X = \lambda_1 \mathrm{u}_1 + \lambda_2 \mathbf{u}_2 + \ldots + \lambda_\mathrm{r}\, \mathbf{u}_\mathrm{r}$$

with each $\lambda_i \ge 0$ and

$\therefore$ $\mathbf{C}X = \lambda_1\, (\mathbf{cu}_1) + \lambda_2\, (\mathbf{cu}_2) + \cdots + \lambda_r\, (\mathbf{cu}_r)$

$\therefore$ $Z = \mathbf{CX} = \lambda_1 Z_1 + \lambda_2 Z_2 + \cdots + \lambda_r Z_r$

But each $Z_i \le Z_m$ (As Z_m is maximum)

$$\therefore \quad \mathbf{CX} = \lambda_1 Z_1 + \lambda_2 Z_2 + \cdots + \lambda_r Z_r \le \lambda_1\, Z_m + \lambda_2 Z_m + \cdots + \lambda_r Z_m$$
$$= (\lambda_1 + \lambda_2 + \cdots + \lambda_r) \cdot Z_m$$

but $$\sum_{i=1}^{i=r} \lambda_i = 1$$

$\therefore$ $$Z = \mathbf{CX} \le Z_m \text{ i.e } Z \le Z_m$$

This means that the object function takes the maximum value at $\overline{u}_m$, which is an extreme point.

Comment: This theorem is very important for finding the optimal solution in the case of extreme points of the closed and bounded feasible regions.

1.4 Fundamentals of Simplex and Examples

Now, on using the facts and fundamentals, we develop the main topic of this chapter. The facts and results will prove very important in some more chapters that follow this one. We request students pay high attention.

1.4(A) Basics

We have already studied the salient features of constructing the mathematical model in the cases where the decision variables strictly observe linearity in constraints and objective function. Also to the problems having only two variables, we have attempted on solving them graphically.

At this point, we note that in many cases on obtaining optimal solution, it is observed that all the resources are not completely utilized. Unutilized resources become dead investment and so we, before starting the production operation, take care of the amount of the unutilized resources. In some cases, we may run short of a little amount of resources or may need additional resource over and above the given resource; all these situations are liable to affect the final production and hence the profit on sales.

These two points can be, to some extent, taken care of when the problem has only two invariables but in the cases of more than two invariables, we, before designing the whole set up, must be aware of the final output, the profit the surplus resources and the expected shortages of resources.

In some cases, the number of decision variables exceed the number of linearly independent constraints; these constraints may be in equality forms also. The number of variables finally, in cases of problems with constraints involving many variables, it may not be a practical to find out the vertices of the feasible regions.

All points mentioned above, can be summarised as under.

1. Search of unutilized resources.
2. Search of additional resources.
3. Solving the system of equations.
4. Finding the possible number of vertices for evaluating the objective function.
5. Avoiding unwanted vertices that violate the non-negativity conditions on variables.
6. Determining the purchase price of raw materials or resources, which is essential in the production or planning.

The above points are the leading points that inspire us to search for other method which lead to finding optimal solution within the given constraints.

Before we slowly ascend on simplex method, need certain definition, which play important role in finding optimal solution and take crucial part to answering some of the above points 1 to 6 discussed above.

1.4(B) Some Definitions

1. Slack Variable

In a given LPP some constraints may be of ≤ type.
For example,

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$

The left-hand side of the inequality shows the actual usage while the right-hand side shows the given resource. To make both the sides equal (=), we need to add some non-negative variable in terms of the resource on left side; it is called a slack variable.

Let $s_1 \geq 0$ so that $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + s_1 = b_1$; $s_1 \geq 0$ is a slack invariable. We note that the profit contribution of a slack variable is zero.

i.e. It will not add any amount to the profit. In fact, the slack variable is the amount of unutilized portion of the given resource.

Suppose $3x + 2y \leq 7$; let $s_1 \geq 0$ be such that $3x + 2y + 1s_1 = 7$ with $s_1 \geq 0$

Here, the variable $s_1 \geq 0$ is a slack variable.

2. Surplus Variable

In a given LPP problem some constraints may be of ≥ type.

For example, $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$

For given values of $x_1, x_2, \ldots x_n$, the left side of the inequality shows the actual usage while the right side shows the minimum usage to be made. What we call a surplus variable is the absolute difference between left side and the right side, i.e. we introduce a variable, say $s_2 \geq 0$, so that

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n - s_2 = b_1;\ s_2 \geq 0.$$

It shows the additional amount of resource used within the given lower limit. Contribution of the surplus variable to the objective function is zero.

Suppose $2x + 3y \geq 8$; let $s_2 \geq 0$ be the surplus variable so that $2x + 3y - s_2 = 8$.

3. Three Musts

As we have seen that to search for all the six points, we need applying simplex program. This requires the given LP problem must have the following three features.

(M1) Every constraint must be converted to = type.
(M2) Right side of the constraint (resource amount) ≥ 0
(M3) Every equation must have a basic variable.

Comment

1. The very first point above can be met with introducing either a slack or a surplus variable according to the given inequality ≤ type (slack variable) or ≥ type (surplus variable)—so we are through with that.
2. For the resource side, it is obvious in most of the cases to be ≥ 0. (In case if it is negative, multiply the inequality by –1 and solve the purpose.) Say $-3x - 4y \geq -7$ then multiplying the inequality by –1, we get **3x + 4y ≤ 7** (note the change in equality sign.)
3. A variable is said to be a basic variable if it appears with coefficient **+1** in only one equation and in the remaining equation that variable has '0' as its coefficient.

For example, suppose we have an equality of ≤ type, say $2x + 3y \leq 7$.

By adding a **slack variable** $s_1 \geq 0$; we can write $2x + 3y + 1s_1 = 7$. This **slack variable** works as a **basic variable**.

Let us consider $x + 2y \geq 8$. To make equality we introduce a surplus variable $s_2 \geq 0$ so that

$$x + 2y - 1s_2 = 8.$$

This variable **cannot** work as a basic variable as its coefficient is –1.

This clarifies the third point also. Let us find a way out.

4 Artificial Slack Variable (ASV)

Again, we recall

(1) we need to have equality sign in each constraint (M1), and

(2) fundamental need of having a basic variable in each constraint (M3),

In the case of a constraint with ≥ sign; we satisfy M1 by introducing a surplus variable. This does not serve the purpose of a basic variable. At this stage, we introduce a variable, called artificial slack variable; say $A \geq 0$ to the equality constraint.

Suppose $3x + 2y \geq 7$

To make equality introduce $s \geq 0$, a surplus variable

∴ $3x + 2y - 1s = 7$, this s cannot be a basic variable as its coefficient here is –1. We introduce $A \geq 0$ — an artificial slack variable.

∴ $3x + 2y - 1s + 1A = 7$.

Characteristics of Artificial Slack Variable (ASV)*

1. ASV is introduced to satisfy the need of having a basic variable in every equation.
2. Artificial slack variable must be either equal (=) or greater than (>) 0, i.e. $A \geq 0$.
3. ASV is added in an equality constraint (see above example) and hence $A > 0$ will violate = condition. We must have, for feasible solution, the value of A equal to zero.
4. As it is an additional and probably temporary variable, its contribution to the objective function is the highest. For a maximization problem; we write $-M$, For a minimization problem; we write $+M$, where $M \to \infty$.

1.4(C) Preparing the Problem for Simplex Table

We, at this stage, start heading towards the main unit of this chapter—Simplex Table. This requires the problem to be in the standard format presentable in simplex table.

Say the given problem is; find x_1, x_2, and x_3 so as to maximize $Z = 2x_1 - 3x_2 + 3x_3$ subject to the constraints

$$x_1 - x_2 + 2x_3 \leq 9;$$
$$x_1 + x_2 - x_3 \geq 4;$$
$$2x_1 + x_2 = 8;$$
$$x_1, x_2, x_3 \geq 0.$$

We emphasize M1, M2 and M3 of the **three musts**. In the first constraint, there is ≤ sign. To reach M1 and M3; add a slack variable $s_1 \geq 0$ (This $s_1 \geq 0$ will serve as a basic variable also) its contribution to the objective function is zero. We have

$$x_1 - x_2 + 2x_3 + s_1 = 9.$$

In the next constraint, there is ≥ sign and it will call for a surplus variable $\mathbf{s_2 \geq 0}$ to satisfy M1 and an ASV = $\mathbf{A_1 \geq 0}$ to satisfy M3. So, we have

$$x_1 + x_2 - x_3 - 1s_2 + 1A_1 = 4$$

The last constraint is of = sign. We introduce only ASV at this stage, say $A_2 \geq 0$; (This $A_2 \geq 0$ serves as a basic variable) then $2x_1 + x_2 + A_2 = 8$. The problem is converted to, find x_1, x_2, x_3, s_1, s_2, A_1 and A_2 so as to maximize

$$Z = 2x_1 - 3x_2 + 3x_3 + 0s_1 + 0s_2 - MA_1 - MA_2$$

Subject to the constraints $x_1 - x_2 + 2x_3 + 1s_1 + 0s_2 + 0A_1 + 0A_2 = 9;$

$$x_1 + x_2 - 1x_3 + 0s_1 - 1s_2 + 1A_1 + 0A_2 = 9;$$
$$2x_1 + x_2 + 0x_3 + 0s_1 + 0s_2 + 0A_1 + 1A_2 = 4$$

with x_1, x_2, x_3, s_1, s_2, A_1 and $A_2 \geq 0$.

1. LPP in the Simplex Standard Form

We have already seen that the LPP in the matrix form:

Maximize	$Z = \mathbf{CX}$	(1)
Subject to	$A\mathbf{X} \leq, =, \geq \mathbf{b}$	(2)
with	$\mathbf{X} \geq \mathbf{0}$	(3)

Where, in each one of the given constraint one and only one sign, out of the three given signs, holds true at a time.

By adding slack variables (basic variable) in ≤ type, surplus and artificial slack variable in ≥ type, and only artificial slack variable in = type constraint, we will make mathematical format ready for simplex system.

Note: The variables, which are not basic at a given point of time, are non-basic variables. Now we have one more **'MUST';** call it M4.

M4 : Only **basic variables** attain values while **non-basic** variables are always assigned zero value.

In = type constraints, the given problem takes up the following standard form acceptable by simplex procedure to deal with (refer to above example).

Find $\mathbf{X} \in R_{n\times 1}$ so as to optimize	$Z = \mathbf{CX}$	(1)
subject to the constraints	$A\mathbf{X} = \mathbf{b}$	(2)
With	$\mathbf{X} \geq \mathbf{0}$	(3)

We accept that, in this system, there are ***n*** **variables** (including given variables, slack variables, surplus variables, and artificial slack variables) in m linearly independent equations

$$A = (a_{ij})_{m\times n},\ \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},\ \mathbf{C} = (c_1, c_2, \ldots, c_n)_{1\times n} \text{—A row vector}$$

and **0** is a null vector.

1.4(D) Some Definition and Approach to Solution: (in the context of the above form)

1. **A basic solution**: In the context of the above system, any m of n variables which satisfy (2), i.e. $A\mathbf{X} = \mathbf{b}$. With remaining $(n - m)$ non-basic variables assigned zero value, becomes **a *basic solution***. These m variables are called basic variables.
2. **A basic feasible solution:** With reference to the above system and definition (1), if all the basic variables attain ≥ 0 values satisfying condition (3), the solution is called a *basic feasible solution*.
3. **An optimal solution:** A basic feasible solution definition (2) above which optimizes (maximizes or minimizes) the given objective function is called an *optimal solution*.
4. **Degenerate solution**: In the context of the above system, there are m basic variables say $\mathbf{X}_{m\times 1}$ which are ≥ 0 (Number of equations equals number of basic variables—M3). The remaining $(n - m)$ variables $\mathbf{X}_{1\times n-m}$ are called non-basic variables which are always **zero;** they are **initialized**. If at least one of m **basic variables** becomes zero then the solution is a **degenerate solution**. (A degenerate basic feasible solution may be optimal also.)
5. **Towards the solution**:
   We take up standard simplex form;

find $\mathbf{X} \in R_{n\times 1}$ so as to maximize $Z = \mathbf{CX}$ (1)
with m equality constraints $A\mathbf{X} = \mathbf{b}$ (2)
and $\mathbf{X} \geq 0$ (3)

As there are m linearly independent equations and with M3 condition on the problem; we have m basic variables (all of them are ≥ 0) and $(n - m)$ non-basic variables (all equal to zero) we rearrange the matrix system keeping in view of basic and non-basic variables.

Let $X = \begin{pmatrix} \mathbf{X}_\mathrm{B} \\ \mathbf{X}_\mathrm{N} \end{pmatrix}$, $\mathbf{C} = (\mathrm{C_B\ C_N})$, $\mathrm{A} = (a_{ij})_{m\times n} = (\mathrm{A_B\ A_N})$

$$\mathbf{X_B} = \begin{pmatrix} x_1 \\ x_2 \\ \\ x_m \end{pmatrix}_{m\times 1}$$ A basic vector (each one of $\mathbf{X}_\mathrm{B} \geq 0$)

$$\mathrm{X_N} = \begin{pmatrix} x_{m+1} \\ x_{m+2} \\ \vdots \\ x_n \end{pmatrix}$$ A non-basic vector (each one of $\mathbf{X}_N = 0$, etc)

This changes $Z = \mathbf{CX} = (C_B\ C_N) \begin{pmatrix} \mathbf{X}_\mathrm{B} \\ \mathbf{X}_\mathrm{N} \end{pmatrix} = \mathbf{C}_B\mathbf{X}_B + \mathbf{C}_N\mathbf{X}_N$ but each $\mathbf{X}_N = 0$.

$$\left.\begin{aligned} \therefore \quad \mathrm{Z} &= \mathbf{CX} = \mathbf{C}_\mathrm{B}\mathbf{X}_\mathrm{B} = (C_1 \cdot C_2 \cdots\cdot C_\mathrm{m})\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \\ &= C_1x_1 + C_2X_2 + \cdots C_mx_m \end{aligned}\right\} \tag{1}$$

$$AX = b \Rightarrow (A_B, A_N) \begin{pmatrix} \mathbf{X}_\mathrm{B} \\ \mathbf{X}_\mathrm{N} \end{pmatrix} = \mathbf{b}$$

$\therefore$ $A_B\mathbf{X}_B = \mathbf{b}$ (As each one of X_N is zero) (2)

With all $\mathbf{X_B} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \geq 0$, A set of basic vectors

$$\mathbf{X_N} = \begin{pmatrix} x_{m+1} \\ x_{m+2} \\ x_{m+3} \\ \cdots \\ X_n \end{pmatrix}_{(n-m)\times 1} = 0,$$ A non-basic vector; each equals zero. (3)

If $\mathbf{A}_B$ is a non-singular matrix, then A_B^{-1} exists.

$\therefore \quad A_B X_B = \mathbf{b} \Rightarrow A_B^{-1}(A_B \mathbf{X}_B) = A_B^{-1}\mathbf{b}$

$\therefore \quad (A_B^{-1} A_B)X_B = A_B^{-1}\mathbf{b}$;

but $\quad A_B^{-1} A_B = I_{m\times m}$ = An identify matrix.

$\therefore \quad \mathbf{X}_B = A_B^{-1}\mathbf{b}$ which is a basic feasible solution with all non-basic variables = 0 and all entries of $\mathbf{X}_B \geq \mathbf{0}_{m\times 1}$.

$\therefore \quad Z = \mathbf{C}_B \mathbf{X}_B = \mathbf{C}_B (A_B^{-1}\mathbf{b})$

$\therefore \quad \mathbf{Z} = (\mathbf{C}_B A_B^{-1})\mathbf{b}$

This indicates the direction towards the basic feasible solution.

1.4(E) Simplex Procedure

Structure

(1) Algebraic Structure

We now land upon structural mathematics and try to develop it into algorithmic procedures. Looking to the enormous applications, we feel, it should be brought down to the reachable platform. Students of any branch, without fear of encrypted mathematical jargons, should be able to understand the fundamentals of simplex and be able to apply the same after a little practice on their own. We try but with one strong appeal to the students that they read the logic twice and solve (not read) the problems following the lines that we have followed.

We begin with a maximization example with all constraints of ≤ type.

ILLUSTRATION 14

Maximize $Z = 2x + 6y$; subject to the constraints

$x + y \leq 6$;
$4x + 3y \leq 12$;
$x, y \geq 0$.

Solution

Before we begin, let us recall all the important facts, necessary for solving this example.

(2) Check Points

1: M1: Equality of constraints.
2: M2: Right-hand side ≥ 0.
3: M3: Basic variables in each equation.
(This means number of linearly independent equations equals number of basic variables.)
4: Only basic variables attain the value from right-hand side
5: Non-basic variable, we deal with, are always zero.
6: Basic feasible solution is one which has all basic variables ≥ 0 and non-basic variable = 0.

We follow the above steps.

Step 1

As the constraints are of ≤ type; we add slack variables (to take the charge of basic variables) s_1 and s_2 both ≥ 0. We express the given system in simplex standard form.

Find x, y, s_1 and s_2 to maximize $z = 2x + 6y + 0s_1 + 0s_2$;

subject to the constraints $x + y + 1s_1 + 0s_2 = 6;$

$$4x + 3y + 0s_1 + 1s_2 = 12;$$
$$x, y, s_1 \text{ and } s_2 \geq 0.$$

We express the algebraic expressions in the table form.

At this stage, s_1 and s_2 are **basic variables** and x and y are non-basic variables.

Table 1.1

Maximize

$C_j \rightarrow$ VAR / $\downarrow$ Basis	2 X	6 y	0 s_1	0 s_2	X_B	R. R.
0 s_1	1	1	1	0	6	6
0 $\leftarrow$ s_2	4	3	0	1	12	4 $\rightarrow$
$C_j - Z_j$	2	6$\uparrow$	0	0	$Z = 0$	

2. The Characteristics of Table 1.1

1. Compare the given problem (objective function, equality constraints with the style of the table.)
2. VAR = variables (all given variables)
   Basis: set of basic variables in a given iteration. [s_1 and s_2 are basic variables and those variables $x = 0$, $y = 0$ which are not present here are non-basics.]
3. $\mathbf{X}_B$ = Right-hand side; resource side (As given in the equality constraints.)
4. $C_j - Z_j$ = Net evaluation row = Relative profit row.
   = C_j – Dot product of corresponding j^{th} column vector with corresponding C_j entries of basic vectors.

$$C_1 - Z_1 = 2 - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 2 - (0.1 + 0.4)$$
$$= 2 - 0 = 2;$$
$$C_2 - Z_2 = 6 - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 6 - (0.1 + 0.3)$$
$$= 6 - 0 = 6.$$
$$C_3 - Z_3 = 0 - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$
$$C_4 - Z_4 = 0 - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

3. Replacement Ratio (R.R.)

It is a ratio which decides the outgoing variable from the basis.

$$\text{R.R.} = \frac{\text{Right side vector } (X_B)}{\text{Positive entries of Pivot Column}}$$

The minimum of R.R. entries is selected and the corresponding variable is an outgoing variable. The variable corresponding to the highest $C_j - Z_j$ entry replaces the outgoing variable. If there is a tie in the highest $C_j - Z_j$ entry then the choice for the incoming variable is optional. (The outgoing variable is shown by ($\rightarrow$ arrow) in the column of R.R.)

(Additional relevant information is given in **M5** that immediately follows after the next topic.)

4. M(5)

For a maximization problem, the most positive entry of $C_j - Z_j$ entries is selected and the corresponding variable is an **incoming variable** in the next table, **i.e**. that variable will become a basic variable. For the minimization problem the most negative entry from all $C_j - Z_j$ entries is selected and the corresponding variable becomes an incoming variable in the next iteration.

5. Pivot (Pivot element-Key element)

Pivot is an entry, which is common to the outgoing variable (row) and the incoming variable (column). It is a key element. It conveys that in the next table (next iteration) the column corresponding to the incoming variable will look like the column corresponding to outgoing variable in the current table.

(For this example, in the next iteration y becomes a basic variable and column of the variable **y** will look like the column of the basic variable, i.e. y will become $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as s_2 looks like in the current table.)

Before we construct the entries of the next table, we would like to furnish the criteria for optimality. This will enable us to determine the termination process of on-going iterations.

6. Optimality Criteria

Optimality in a maximization problem is attained when all ($C_j - Z_j$) entries are ≤ 0.

The given solution is maximum.

The rationale behind this is, selection of most positive entry and hence the corresponding incoming variable, is the one, that will increase the profit faster then the other variables can increase the profit. The column that corresponds to the most positive $C_j - Z_j$ entry is the **pivot column** or key column.

Similarly for a minimization problem, the optimality criteria is that all $C_j - Z_j \geq 0$.

7. Constructing the next table

(a) Write the incoming variable in the place of outgoing variable from the basis. (indicated by outgoing arrow)

(b) Divide the **pivot** row by the **pivot value** and write the result. (New entry in the place of pivot entry is '1' now.)

(c) Make all the entries, above and below that '1', equal to zero. (Confirmation of a basic variable.) This can be done by elementary row operations. Remember that all the above points are important and need to be performed in any example requiring further iterations from a given stage.

- Based, on the above points, we write the next iteration as follows.
- Divide the pivot row by the pivot 3. (Check this in the new Table 1.2) and write in the new table where in the basis is the set of vectors $\mathbf{s}_1$ and **y.** Multiply the new row by 1 and subtract the **result** from the first row of Table 1.1. (Check this in the new Table 1.2.)

Table 1.2

Maximize

$C_j \rightarrow$ VAR / $\downarrow$ Basis		2 x	6 y	0 s_1	0 s_2	x_B	R. R.
0	s_1	–1/3	0	1	–1/3	2	
6	y	4/3	1	0	1/3	4	
$C_j - Z_j$		–6	0	0	–2	24	

$$C_j - Z_j \text{ entries for } \mathbf{x} = 2 - \begin{pmatrix} 0 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -1/3 \\ 4/3 \end{pmatrix} = 2 - (0 + 8) = -6$$

For $$\mathbf{s}_2 = 0 - \begin{pmatrix} 0 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} -1/3 \\ 1/3 \end{pmatrix} = 0 - (0 + 2) = -2;$$

For basic variables $\mathbf{y}$ and $\mathbf{s}_2$ are zero

$$Z \text{ value} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 0 \times 2 + 6 \times 4 = 24.$$

Since all $C_j - Z_j$ entries are ≤ 0, therefore the basic feasible solution is maximum at $x = 0$ (non-basic); $y = 4$ (basic), $s_1 = 2$ (basic) and $s_2 = 0$ (non-basic) with maximum value $Z = 24$. All the above steps prior to the Table 1.2, were for reading, understanding, and applying at a given stage of simplex table. We put both the Tables 1.1 and 1.2 together. Also the above explanation (unless essential) will not be repeated in further illustrations.

Table 1.3 Simplex Table

Maximize

$C_j \rightarrow$ VAR $\downarrow$ Basis	2 x	6 y	0 s_1	0 s_2	RHS x_B	R. R.
0 s_1 0 $\leftarrow s_2$	1 4	1 3	1 0	0 1	6 12	6 4 $\rightarrow$
$C_j - Z_j$	2	6↑	0	0	$Z = 0$	
0 s_1 6 y	–1/3 4/3	0 1	1 0	–1/3 1/3	2 4	
$C_j - Z_j$	–6	0	0	–2	$Z = 24$	

Optimum (maximum) value = 24 with $y = 4$; $s_1 = 2$ (basic variables); $x = 0$; $x_2 = 0$ (non-basic variables)

ILLUSTRATION 15

Solve the following problem.

Maximize $Z = 5x_1 + 7x_2$;

subject to the constraints $x_1 + x_2 \le 4$;

$$3x_1 + 8x_2 \le 24;$$
$$10x_1 + 7x_2 \le 35;$$
$$x_1, x_2 \ge 0.$$

Solution

All the three constraints are of $\le$ type, so we add slack variables $s_1, s_2, s_3 \ge 0$ in each one. (We follow steps M1, M2, M3, M4, M5, etc.) The standard simplex format is

Maximize $Z = 5x_1 + 7x_2 + 0s_1 + 0s_2 + 0s_3$;

subject to
$$x_1 + x_2 + 1s_1 + 0s_2 + 0s_3 = 4;$$
$$3x_1 + 8x_2 + 0s_1 + 1s_2 + 0s_3 = 24;$$
$$10x_1 + 7x_2 + 0s_1 + 0s_2 + 1s_3 = 35;$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

(s_1, s_2, and s_3 are basic variables)

Putting this in the simplex format.

Table 1.4

Maximize

	$C_j \rightarrow$ VAR / $\downarrow$ Basis		5 x	7 y	0 s_1	0 s_2	0 s_3	x_B	R. R.
	0	s_1	1	1	1	0	0	4	4
I	0 $\leftarrow$	s_2	3	8	0	1	0	24	3 $\rightarrow$
	0	s_3	10	7	0	0	1	35	5
	$C_j - Z_j$		5	7$\uparrow$	0	0	0	$Z = 0$	
	0 $\leftarrow$	s_1	5/8	0	1	–1/8	0	1	8/5 $\rightarrow$
II	7	y	3/8	1	0	1/8	0	3	24/3 = 8
	0	s_3	59/8	0	0	–7/8	1	14	112/59
	$C_j - Z_j$		19/8$\uparrow$	0	0	–7/8	0	21	
	5	x	1	0	8/5	–1/5	0	8/5	
III	7	y	0	1	–3/5	1/5	0	12/5	
	0	s_3	0	0	–59/5	3/5	1	11/5	
	$C_j - Z_j$		0	0	–19/5	–2/5	0	124/5	

Comments:

1. In the first part-I, y' is an incoming variable and s_2 is an outgoing variable, with $Z = 0$
2. In the second part-II, x is an incoming variable and s_1 is an outgoing variable with $Z = 21$
3. In the third part, all entries $c_j - z_j \leq 0$; the solution is maximum

All entries of $C_j - Z_j$ are ≤ 0 for this maximization problem. The basic feasible solution (**bfs**) is maximum = 124/5

Basic variable $x = 8/5$; $y = 12/5$; $s_3 = 11/3$; non-basic variable $s_1 = 0$; $s_2 = 0$.

ILLUSTRATION 16

Maximize $Z = 7x + 14y$
subject to the constraints

$$3x + 2y \leq 36$$
$$x + 4y \leq 10$$
$$x, y, \geq 0.$$

Solution

As both constraints are of $\leq$ type, so we add slack variable $s_1, s_2 \geq 0$ to each constraint. We know that contribution of slack variable to the objective function is zero. The problem in canonical form is as follows.

Find x, y, s_1 and s_2 to maximize $Z = 7x + 14y + 0s_1 + 0s_2$
subject to the constraints

$$3x + 2y + 1s_1 + 0s_2 = 36$$
$$1x + 4y + 0s_1 + 1s_2 = 10$$

with x, y, s_1, and $s_2 \geq 0$.
We put this form in the simplex table.

Table 1.5

Maximize

	$C_j \rightarrow$ VAR / $\downarrow$ Basis	7 X	14 y	0 s_1	0 s_2	X_b	R. R.
I	0 s_1	3	2	1	0	36	18
	0 ← s_2	1	4	0	1	10	5/2 →
	$C_j - Z_j$	7	14↑	0	0	Z = 0	
II	0 s_1	5/2	0	1	–1/2	31	62/5
	14 ← y	1/4	1	0	1/4	5/2	10 ←
	$C_j - Z_j$	7/2↑	0	0	–7/2	35	
III	0 s_1	0	–10	1	–3	6	
	7 x	1	4	0	1	10	
	$C_j - Z_j$	0	–14	0	–7	Z = 70	

As all entries of $C_j - Z_j$ row are ≤ 0, so we conclude that the solution is maximum at $x = 10$, $s_1 = 6$, are basic variables ; $y = 0$ and $s_2 = 0$ are non-basic variables.
The basic feasible solution is maximum = 70.

ILLUSTRATION: 17

Minimize $Z = -3x_1 + 2x_2$
subject to the constraints $-x_1 + 3x_2 \leq 10$

$$x_1 - x_2 \leq 2$$
$$x_1 + x_2 \leq 6; \text{ with } x_1, x_2, x_3 \geq 0.$$

Solution

All three constraints are of ≤ type. We add slack variables s_1, s_2 and $s_3 \geq 0$. They serve as basic variables.
We have minimize

$$Z = -3x_1 + 2x_2 + 0s_2 + 0s_2 + 0s_3$$

subject to

$$-1x_1 + 3x_2 + 1s_1 + 0s_2 + 0s_3 = 10$$
$$1x_1 - 1x_2 + 0s_1 + 1s_2 + 0s_3 = 2$$
$$1x_1 + 1x_2 + 0s_1 + 0s_2 + 1s_3 = 6$$

with $\quad x_1, x_2, s_1, s_2, s_3 \geq 0$

Table 1.6

Minimize

$C_j \rightarrow$ VAR / $\downarrow$ Basis	–3 x_1	2 x_2	0 x_1	0 s_2	0 s_3	X_b	R. R.
0 s_1	–1	3	1	0	0	10	**
0 ← s_2	1	–1	0	1	0	2	2 →
0 s_3	1	1	0	0	1	6	6
$C_j - Z_j$	–3↑	2	0	0	0	Z = 0	
0 s_1	0	2	1	1	0	12	6
–3 x_1	1	–1	0	1	0	2	**
0 s_3	0	2	0	–1	1	4	2 →
$C_j - Z_j$	0	–1↑	0	3	0	–6	

(*Contd.*)

(*Contd.*)

0	s_1	0	0	1	2	–1↑	8	
–3	x_1	1	0	0	1/2	1/2	4	
2	x_2	0	1	0	–1/2	1/2	2	
$C_j - Z_j$		0	0 ↑	0	5/2	1/2	–8	

** Undefined.

All $C_j - Z_j$ entries are ≥ 0 ∴, so the cost is minimum. Basic variable: $x_1 = 4$; $x_2 = 2$; $s_1 = 8$; Non-basic variables: $x_3 = 0$, $s_2 = 0$, $s_3 = 0$.
The basic feasible solution is minimum = –8.

ILLUSTRATION 18

Minimize $Z = -7x - 14y$; subject to the constraints

$$3x + 2y \leq 36$$
$$x + 4y \leq 10$$
$$x, y \geq 0.$$

Solution

Both the constraints are of ≤ type, so we add slack variables $s_1, s_2 \geq 0$ respectively to each constraint. Contribution of slack variable to the objective function is zero and this helps put the problem as follows. Find x, y, s_1 and s_2 so as to maximize $Z = -7x - 14y + 0s_1 + 0s_2$; subject to
$3x + 2y + 1s_1 + 0s_2 = 36$
$1x + 4y + 0s_1 + 1s_2 = 10$; with $x, y, s_1, s_2 \geq 0$. The corresponding simplex table is

Table 1.7

Maximize

$C_j \rightarrow$ / ↓ Basis	VAR	–7 x	–14 y	0 s_1	0 s_2	X_b	R. R.
0	s_1	3	2	1	0	36	18
0 ←	s_2	1	4	0	1	10	5/2 →
$C_j - Z_j$		–7	–14↑	0	0	$Z = 0$	
0	s_1	5/2	0	1	–1/2	31	62/5
←–14	y	1/4	1	0	1/4	5/2	10 →
$C_j - Z_j$		–7/2↑	0	0	7/2	–35	
0	s_1	0	–10	1	–3	6	
–7	x	1	4	0	1	10	
$C_j - Z_j$		0	14	0	7	–70	

As all the $C_j - Z_j$ entries are ≥ 0, so the solution for this minimization problem is optimal at $x = 10$; $y = 0$; $s_1 = 6$; $s_2 = 0$; with minimum value $Z = -70$.

1.5 SPECIAL CASES AND METHODS

There are some special cases of examples and special techniques to solve LP **Examples**. Now, in the above illustration and some more illustrations to follow in the sequence, we use some of the following procedures.

SCM (1) Big M Method (Method 1)
SCM (2) Infeasible Solution (Special Case 1)
SCM (3) Multiple Optimal Solution (Special Case 2)
SCM (4) Unbounded Solution (Special Case 3)
SCM (5) Bounded Variable Case (Method 2)
SCM (6) Unrestricted Variable (Method 3)
SCM (7) Two-phase Method (Method 4)
SCM (8) Degenerate Solution (Case 4)
SCM (9) Dual Simplex Method (Method 5)

SCM (1): Big M Method

The given LP problem, in the case of constraints ≥ type and = type constraints, needs special treatment. Following standard instructions M1 (= sign) and M3 (basic variable), we add a surplus variable to satisfy M1. To satisfy M3 (basic variable requirement), we introduce ASV in the equality constraint. For a maximization problem its cost coefficient of ASV in objective function = –M and for a minimization problem cost coefficient = +M. We will also study a two-phase method and Dual Simplex method on the same lines treating inequalities of ≥ type. –M or +M is the penalty in the case if ASV = $A > 0$. It reflects infeasibility of the problem in the case of $A > 0$ remains in the optimal table.

ILLUSTRATION 19

Solve Using Big M method

Maximize $Z = 2x_1 + 3x_2$
subject to the constraints $x_1 + x_2 \geq 2$

$$x_1 + 2x_2 \leq 8$$
$$x_1, x_2 \geq 0$$

Solution

$x_1 + x_2 \geq 2$; let $s_1 \geq 0$ (surplus variable) makes $x_1 + x_2 - s_1 = 2$; and with $A \geq 0$ (ASV)
$x_1 + x_2 - s_1 + A = 2$; $A \geq 0$ works as a basic variable; put it into simplex table.
$x_1 + 2x_2 \leq 8$; $s_2 \geq 0$ (slack variable = basic variable), makes it $x_1 + 2x_2 + s_2 = 8$. With objective function maximize $Z = 2x_1 + 3x_2 + 0s_1 - \text{MA} + 0s_2$.

Table 1.8

Maximize

$C_j \rightarrow$ VAR / $\downarrow$ Basis	2 x_1	3 x_2	0 s_1	–M A	0 s_2	x_B	R. R.
–M ← A	1	1	–1	1	0	2	2 →
0 s_2	1	2	0	0	1	8	4
$C_j - Z_j$	2 + M	↑3 + M	–M	0	0	Z = –2M	
3 x_2	1	1	–1	1	0	2	(Undefined R.R)
0 s_2	–1	0	2	–2	1	4	2 →
$C_j - Z_j$	–1	0	↑3	–M–3	0	Z = 6	
3 ← x_2	1/2	1	0	0	½	4	8 →
0 s_1	–1/2	0	1	–1	½	2	(Undefined R.R)
$C_j - Z_j$	1/2↑	0	0	–M	–3/2	Z = 12	
2 x_1	1	2	0	0	1	8	
0 s_1	0	1	1	–1	1	6	
$C_j - Z_j$	0	–1	0	–M	–2	Z =16	

All $C_j - Z_j$ entries are ≤ 0. This gives maximum solution.
Basic variables $x_1 = 8$; $s_1 = 6$; non-basic variables $x_2 = 0$; $s_2 = 0$; $A = 0$;
Maximum value = 16.

ILLUSTRATION 20

(Solve using Big M Method)
Minimize $Z = 36x_1 + 10x_2$
subject to the constraints $3x_1 + 1x_2 \geq 7$
$2x_1 + 4x_2 \geq 14$
and $x_1, x_2 \geq 0$.

Solution

As both constraints are of ≥ type; we introduce surplus variable s_1 and s_2 to make = constraints and then add artificial slack variables $A_1 \geq 0$ and $A_2 \geq 0$ to satisfy the need of basic variables in each constraint. This being a minimization problem; we have $+M$ as a cost coefficient in the objective functions. All these steps put the problem as follows.

Maximize $Z = 36x_1 + 10x_2 + 0s_1 + MA_1 + 0s_2 + MA_2$; subject to the constraints

$$3x_1 + 1x_2 - 1s_1 + 1A_1 + 0s_2 + 0A_2 = 7$$

$$2x_1 + 4x_2 + 0s_1 + 0A_1 - 1s_2 + 1A_2 = 14$$

with $x_1, x_2, s_1, A_1, s_2, A_2 \geq 0$.
This, represented in the simplex table form, is as follows.

Table 1.9

Minimize

$C_j \rightarrow$ ↓ Basis	VAR	36 x_1	10 x_2	0 s_1	M $A1$	0 s_2	M A_2	X_b	R. R.
+M ←	A1	3	1	–1	1	0	0	7	7/3 →
+M	A2	2	4	0	0	–1	1	14	7
$C_j - Z_j$		36–5M ↑	10–5M	+ M	0	+	M 0	21M	
36	x_1	1	1/3	–1/3	–	0	0	7/3	7
M ←	A_2	0	10/3	2/3	–	–1	1	28/3	14/5 →
$C_j - Z_j$		0	↑–2–10/3M	12–2/3M	–	M	0	84+28/3M	
36 ←	x_1	1	0	–2/5	–	1/10	–	7/5	14
10	x_2	0	1	1/5	–	–3/10	–	14/5	Undefined
$C_j - Z_j$		0	0	62/5	–	↑–3/5	–	322/5	
0	s_2	10	0	–4	–	1	–	14	
10	x_2	3	1	–1	–	0	–	7	
$C_j - Z_j$		6	0	10	–	0	–	70	

This is a minimization problem and all $C_j - Z_j$ entries are ≥ 0. The solution is minimum.
The basic feasible solution is
Basic variables: $x_2 = 7$, $s_2 = 14$, non-basic variables: $s_1 = 0$, $s_2 = 14$, $A_1 = A_2 = 0$. minimum $Z = 70$.

SCM (2): Infeasible Solution

We know that the optimal solution to LP problem exists at least on one vertex of the ***most common feasible*** region to all the constraints.

There are cases where there is **no** common feasible region and hence there **cannot** be the optimal solution to the problem. It is a case of ***infeasible solution***. We have studied this case in the graphical solution of LPP.

Working on the problem using Simplex procedure; we arrive at a point where the optimality criterion is satisfied but the **artificial slack variable** continues in the basis. This allows $-M$, in case of maximization problem or $+M$ in case of minimization problem rendering the solution not feasible. We illustrate this by an example.

ILLUSTRATION 21

Find x and y to maximize $Z = 2x + 5y$;
subject to the constraints $x + 2y \le 2$;

$$2x + 6y \ge 12$$

with $\quad x, y \ge 0.$

Solution

Add $s_1 \ge 0$ (slack variable as a basic variable) in $x + y \le 2$. Use surplus variable $s_2 \ge 0$ and $A \ge 0$ (ASV – basic variable with $-M$ as cost coefficient) in the second constraint. The problem shapes as follows.
Maximize $Z = 2x + 5y + 0s_1 + 0s_2 - \text{MA}$;
subject to the constraints

$$1x + 2y + 1s_1 + 0s_2 + 0A = 2$$
$$4x + 6y + 0s_1 - 1s_2 + 1A = 12$$
$$x, y, s_1, s_2, A \ge 0.$$

Table 1.10

Maximize

	$C_j \rightarrow$	VAR ↓ Basis	2 x	5 y	0 S_1	0 S_2	$-M$ A	X_B	R. R.
I	0 ←	s_1	1	2	1	0	0	2	1 →
	–M	A	2	6	0	–1	1	12	2
	$C_j - Z_j$		2 + 2M	5 + 6M ↑	0	– M	0	–12M	Z = –12M shows negative profit.
II	5	y	1/2	1	1/2	0	0	1	2 →
	–M	A	–1	0	–3	–1	1	6	undefined
	$C_j - Z_j$		↑ $\left(\frac{-1}{2} - M\right)$	0	$-3M - \frac{5}{2}$	–M	0	5–6M	
III	2	x	1	2	1	0	0	2	
	–M	A	0	2	–2	–1	1	8	
	$C_j - Z_j$		0	1– M	–2–2 M	–M	0	4–8M	

All $C_j - Z_j$ entries are ≤ 0; it is a sign of optimal solution. Continuation of ASV indicates infeasibility.

Basic variables $x = 2$, $\mathbf{A = 8}$; $Z = 4 - 8M$; originally this being a two-dimensional problem, we can graph the constraints.

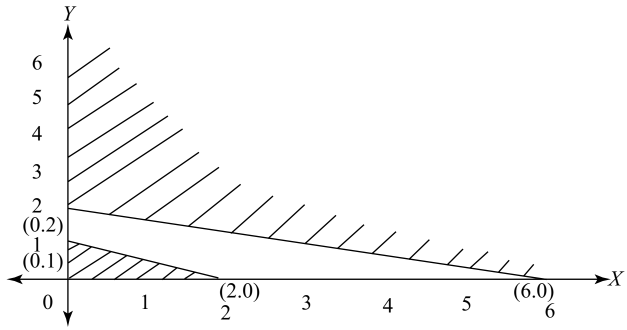


Fig. 1.15 The most common region to all the constraints is a null set.

SCM (3): Multiple Optimal Solutions

When the objective function retains its optimum value at more than one vertex and hence there are infinite points lying on the line segment joining those two points.

In two-dimensions, the slope of the objective function with convenient profit figure (selected) is ***same as that*** of one of the constraint lines which contains the vertex giving the optimal value. (The last iso-profit line coincides with one of the constraints—line segment.) In simplex table this case is identified in a slightly different way. We know that $C_j - Z_j$ entries corresponding to basic variables are always zero. In addition to this, if a $C_j - Z_j$ entry corresponding to a ***non-basic variable becomes zero*** then it is a sign of multiple optimal solution.

To get another basic feasible optimal solution we allow the non-basic variable with zero $C_j - Z_j$ entry, to become the basic variable in the next iteration. This will give another solution with the same optimal value.

Once two basic feasible (optimal) solutions are identified, we can find infinite points (giving optimal solution) on the line segment joining those two points. We illustrate this by an illustration.

ILLUSTRATION 22

Maximize $Z = 4x + 6y$; subject to the constraints

$$x + 2y \geq 2$$
$$x \leq 5$$
$$2x + 3y \leq 12$$

with $x, y \geq 0$.

Solution Set (1)

Adding surplus variable $s_1 \geq 0$ and $A \geq 0$ (ASV) as a basic variable in the first constraint, $s_2 \geq 0$ in the second one and $s_3 \geq 0$ (both slack variables s basic variables) the system can be written down as follows.

Maximize $Z = 4x + 6y + 0s_1 - MA + 0s_2 + 0s_3$

subject to the constraints $1x + 2y - 1s_1 + 1A + 0s_2 + 0s_3 = 2$

$$1x + 0y + 0s_1 + 0A + 1s_2 + 0s_3 = 5$$
$$2x + 3y + 0s_1 + 0A + 0s_2 + 1s_3 = 12$$

with $x, y, s_1, A, s_2, s_3 \geq 0$.

Table 1.11

Maximize

	$C_j \rightarrow$ / ↓ Basis	VAR	4 X	6 y	0 s_1	–M A	0 s_2	0 s_3	RHS XB	R. R.
I	–M ←	A	1	2	–1	1	0	0	2	1 →
	0	s_2	1	0	0	0	1	0	5	
	0	s_3	2	3	0	0	0	1	12	4
	$C_j - Z_j$		4 + M	6 + 2M↑	–M	0	0	0	$Z = -2M$	
II	6	y	1/2	1	–1/2	1/2	0	0	1	Undefined
	0	s_2	1	0	0	0	1	0	5	Undefined
	0 ←	s_3	1/2	0	3/2	–3/2	0	1	9	6 →
	$C_j - Z_j$		1	0	↑3	–M–3	0	0	6	
III	6	y	2/3	1	0	0	0	1/3	4	
	0	s_2	1	0	0	0	1	0	5	
	0	s_1	1/3	0	1	–1	0	2/3	6	
	$C_j - Z_j$		0	0	0	–M	0	–2	$Z = 24$	

All $C_j - Z_j \leq 0$; the solution is optimal.

Note:

$C_j - Z_j$ entry corresponding to the ***non-basic variable x*** is also zero. It is a sign of ***multiple optimal solution.*** Current optimal solution is

Basic variable: $y = 4$; $s_2 = 5$; $s_1 = 6$;

Non-basic variables: $x = 0$; $A = 0$; $s_3 = 0$;

Maximum $Z = 24$.

Solution Set (2)

Which one is another basic feasible optimal solution?

As we said earlier, allow additional zero corresponding to the non-basic variable to enter the basis. Make the table and observe. (We repeat Part **III** of Table 1.11 and extend it to find another optimal solution.)

Table 1.12

Maximize

	$C_j \rightarrow$ / ↓ Basis	VAR	4 x	6 y	0 s_1	–M A	0 s_2	0 s_3	RHS X_B	R. R.
III	6	y	2/3	1	0	0	0	1/3	4	6
	0 ←	s_2	1	0	0	0	1	0	5	5 →
	0	s_1	1/3	0	1	–1	0	2/3	6	18
	$C_j - Z_j$		0↑	0	0	–M	0	–2	24	
IV	6	y	0	1	0	0	–2/3	1/3	2/3	
	4	x	1	0	0	0	1	0	5	
	0	s_1	0	0	1	–1	–1/3	2/3	13/3	
	$C_j - Z_j$		0	0	0	–M	0	–2	$Z = 24$	

Optimal value = $6 \times 2/3 + 4 \times 5 + 0 \times 13/3 = 4 + 20 = 24$. (Same as the Z value of the previous table)
Basic variable: $x = 5$, $y = 2/3$, $s_1 = 13/3$
Non-basic variable: $s_2 = 0 = s_3 = A$.
Comment: On obtaining $(x, y, s_1, A, s_2, s_3) = (0, 4, 6, 0, 5, 0)$ form previous answer and $(x, y, s_1, A, s_2, s_3) = (5, 2/3, 13/3, 0, 0, 0)$ form last answer. We can find other basic solution giving the same optimal value = 24.

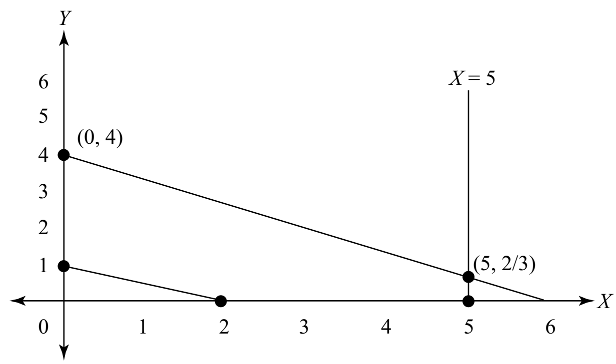


Fig. 1.16 Multiple Optimal Solution at (0, 4) and (5, 2/3)

We can clearly understand that the slope of the constraint line $2x + 3y \leq 12$, i.e. $-2/3$ is same as that of the objective function/line $4x + 6y$ (= 24; A convenient profit figure). This implies that the lines are parallel. In this type of situation, the iso-profit line coincides with the constraint line segment. As a result all the points lying on that line segment are the points that contribute the same optimality.

SCM (4): Unbounded Solution

In general, the convex set of feasible solution is closed. There are special cases indicating that the set of feasible solution, most common to all constraint, is convex and bounded below (***open*** on the counterpart). In the case of maximization, the iso-profit line can be extended upto any extent in that unbounded region, i.e. as we move the iso-profit line far away (from origin) in the open region, the profit increase. In two-dimensional cases, this can be plotted and can be easily justified. In the case of simplex procedure, this can be identified by the following characteristic of the table.

At some stage of iterations, a particular variable becomes ready to enter the basis in the next iteration (i.e. incoming variable is decided.) but the outgoing variable, which is determined based on R.R:, cannot be decided because all the R.R. entries are undefined.

$$\text{R.R.} = \frac{\text{RHS entries}}{\text{Corresponding positive entries of pivot column}}$$

This becomes a case of an unbounded solution. We take an example and graph the region. Which will agree to the simplex conditions described above.

ILLUSTRATION 23

Maximize $Z = 8x + 6y$
subject to the constraints $2x + 3y \geq 6$

$$5x + y \geq 5$$

with $x, y \geq 0$.

Solution

Both the constraints are of ≥ type and so we add surplus variable and ASV s_1, and $A_1 \geq 0$, in the first constraint and s_2, $A_2 \geq 0$ in the second constraint.

Maximize $\quad Z = 8x + 6y + 0s_1 - MA_1 + 0s_2 - MA_2$;

subject to the constraints $\quad 2x + 3y - 1s_1 + 1A_1 + 0s_2 + 0A_2 = 6$

$\quad 5x + 1y + 0s_1 + 0A_1 - 1s_2 + 1A_2 = 5$

with x, y, s_1, A_1, s_2 and $A_2 \geq 0$.

Simples Table 1.13

Table 1.13

Maximize

	$C_j \rightarrow$ VAR ↓ Basis	8 x	6 y	0 s_1	–M A_1	0 s_2	–M A_2	X_B	R. R.
I	–M A_1	2	3	–1	1	0	0	6	3
	–M ← A_2	5	1	0	0	–1	1	5	1 →
	$C_j - Z_j$	8 + 7M↑	6 + 4M	– M	0	–M	0	Z= –11M	
II	–M ← A_1	0	13/5	–1	1	2/5	–2/5	4	20/13 →
	8 x	1	1/5	0	0	–1/5	1/5	1	5
	$C_j - Z_j$	0	$\left(\frac{22}{5}+\frac{13}{5}\right)\uparrow$	–M	0	$\frac{8}{5}+\frac{2}{5}$ M	–7/5M–8 /5	8–4M	
III	6 y	0	1	–5/13	5/13	2/13	–2/13	20/13	– Undefined
	8 ← x	1	0	1/13	–1/13	–3/13	3/13	9/13	9 →
	$C_j - Z_j$	0	0	↑22/13	$-M - \frac{22}{13}$	$\frac{12}{13}$	$-M - \frac{12}{13}$	192/13	
	6 y	5	1	0	0	–1	1	5	Undefined
	0 s_1	13	0	1	–1	–3	3	9	Undefined
	$C_j - Z_j$	–22	0	0	–M	6↑	–M–6	Z = 30	

Comment: At this stage $C_j - Z_j$ entry for s_2 is +6 and s_2 becomes an incoming variable in the next stage of iteration. The outgoing variable form the basis, which is decided on the criteria of minimum entry of R.R. cannot be decided, as the replacement ratios are undefined. This is a case of unbounded solution.

Current Solution

Basic variable: $y = 5$, $s_1 = 9$

Non-basic variable:

$$x = 0 = A_1 = A_2 = s_2;$$
$$Z \text{ value} = 30.$$

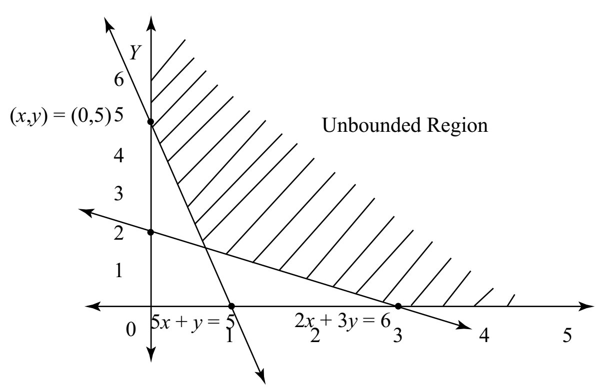


Fig. 1.17

Graphical Solution: The broken-line portion shown in Figure 1.17 is unbounded.

SCM (5): Bounded Variable

In some cases of LPP, the decision variables are subjected to some extreme conditions. Such problems can be simplified by incorporating the extreme condition on the variables. On attaining the optimal solution, one can find the values of the original variables.

ILLUSTRATION 24

Maximize $Z = 2x_1 + 3x_2 + 1x_3$

subject to the constraints $x_1 + 2x_2 + x_3 \le 9$

$$3x_1 + x_2 + x_3 \le 12$$

with $x_1 \ge 2, x_2 \ge 3, x_3 \ge 0.$

Solution

If we wish to express this problem in simplex standard form, it needs $s_1, s_2, s_3, A_1, s_4, A_2 \ge 0$. Six more variables to be introduced and on the top of that there are two more given variables x_1 and x_2. As $x_1 \ge 2$; we write

$x_1 = 2 + y_1$ and $x_2 = 3 + y_2$ with y_1 and $y_2 \ge 0$.

We substitute for x_1 and x_2 in all the terms of the given problem, we get a transformed set of problem.

Find y_1, y_2, and x_3 so as to maximize $Z = 2y_1 + 3y_2 + x_3 + 13$

subject to the constraints $y_1 + 2y_2 + x_3 \le 1$

$$3y_1 + y_2 + x_3 \le 3$$

$$y_1, y_2, x_3 \ge 0.$$

We construct corresponding simplex table.

Introducing slack variables $s_1, s_2 \ge 0$, the simplex table is as follows (Table 1.14).

Table 1.14

Maximize

	$C_j \rightarrow$ VAR / $\downarrow$ Basis	2 y_1	3 y_2	0 s_1	0 s_2	1 x_3	X_B	R. R.
I	0 $\leftarrow$ S_1	1	2	1	0	1	1	½ $\rightarrow$
	0 S_2	3	1	0	1	1	3	5
	$C_j - Z_j$	2	$\uparrow$3	0	0	1	0	
II	3 $\leftarrow$ y_2	1/2	1	1/2	0	1/2	1/2	1 $\rightarrow$
	0 s_2	5/2	0	–1/2	1	1/2	5/2	1
	$C_j - Z_j$	1/2$\uparrow$	0	–3/2	0	–1/2	3/2	
III	2 y_1	1	2	1	0	1	1	
	0 S_2	0	–5	–3	1	–2	0	
	$C_j - Z_j$	0	–1	–2	0	–1	$Z = 2$	

Since all $C_j - Z_j \le 0$ (Maximization problem), so we get optimal solution $Z^* = 2$; for basic variable $y_1 = 1$; $x_1 = y_1 + 2$

$\therefore$ $x_1 = 1 + 2 = 3$ and so $s_2 = 0.$

Non-basic variable $y_2 = 0$ but $x_2 = y_2 + 3 \therefore x_2 = 0 + 3 = 3$; with $s_2 = 0$ and $x_3 = 0$;

$\therefore \quad Z = 2(3) + 3(3) + 1(0) = 15.$

SCM (6): Unrestricted Variable

In some cases the decision variables may attain any real value—positive, zero or negative. In general, what we have been doing so far was to impose non-negativity conditions on the decision variables. Emphatically we mentioned, e.g. $x \geq 0$, $y \geq 0$, …, etc., but it cannot be taken for granted if such conditions are not given in the program. This means that the variable may take any real value. So far as the values are non-negative, the situation of treating the variables remains same as we have been doing so far. If such a variable takes up negative value then we treat it in a special way.

Let us consider an example.

Maximize $Z = 2x + 3y$; subject to the constraints $2x + 3y \geq 6$

$x + 2y \leq 4$; with $y \geq 0$; x is unrestricted in sign. (Sometimes only $y \geq 0$ is written and nothing more. This implies for the variable x to be unrestricted and it can take any real value.)

We express the unrestricted variable x as the difference of two **non-negative** variables.

Let $x = x_1 - x_2$; where x_1 and $x_2 \geq 0$;

(If $x_1 > x_2$, then $x > 0$

$x_1 = x_2$, then $x = 0$

$x_1 < x_2$, then $x < 0$)

Once we get the values of x_1 and x_2; we replace x_1 and x_2 to find the given variable x.

ILLUSTRATION 25

Maximize $Z = 2x + 6y$

subject to the constraints $x + y \leq 5$;

$x \geq 2$;

$y \leq 1$; with $y \geq 0$. Write that x is an unrestricted variable.

Solution

As x is unrestricted in sign; let us express $x = x_1 - x_2$ with $x_1, x_2 \geq 0$. With this, the problem shapes as

Maximize $Z = 2x_1 - 2x_2 + 6y$

subject to $x_1 - x_2 + y \leq 5$

$x_1 - x_2 \geq 2$; $y \leq 1$

$x_1, x_2, y \geq 0$.

Adding necessary slack surplus and artificial variable the simplex table is as follow (Table 1.15).

Table 1.15

Maximize

	$C_j \rightarrow$ / VAR / $\downarrow$ Basis		2 X_1	–2 x_2	6 y	0 s_1	0 s_2	–M A	0 s_3	X_B	R. R.
	0	s_1	1	–1	1	1	0	0	0	5	5
I	$-M \leftarrow$	A	1	–1	0	0	–1	1	0	2	$2 \rightarrow$
	0	s_3	0	0	1	0	0	0	1	1	–
	$C_j - Z_j$		$2 + M\uparrow$	$-2-M$	0	0	$-M$	0	0	$Z = -2M$	

(*Contd.*)

(*Contd.*)

II	0 ←	s_1	0	0	1	1	1	–1	0	3	3 →
	2	x_1	1	–1	0	0	–1	1	0	2	–
	0	s_3	0	0	1	0	0	0	1	1	–
	$C_j - Z_j$		0	0	0	0	↑2	$-M-2$	0	4	
III	0	S_2	0	0	1	1	1	–1	0	3	3
	2	x_1	1	–1	1	1	0	0	0	5	5
	0	S_3	0	0	1	0	0	0	1	1	1 →
	$C_j - Z_j$		0	0	↑4	–2	0	$-M$	0	10	
IV	0	S_2	0	0	0	1	1	–1	–1	2	
	2	x_1	1	–1	0	1	0	0	–1	4	
	6	y	0	0	1	0	0	0	1	1	
	$C_j - Z_j$		0	0	↑0	–2	0	$-M$	–4	$Z = 14$	

All $C_j - Z_j$ entries are ≤ 0 and the problem has reached an optimal stage.

Basic variable: $x_1 = 4$; $y = 1$; $s_2 = 2$; Non-basic variables: $x_2 = 0$; $s_1 = 0$; $s_3 = 0$, $A = 0$; with maximum $Z = 14$. Hence $x = x_1 - x_2$ gives $x = 4 - 0 = 4$; $y = 1$, $s_2 = 2$.

SCM (7): Two-Phase Method

We know that artificial slack variables are just added to fulfil the requirement of having a basic variable in each equality constraint. The prime objective is to remove the existence of ASV in the basis so they become non-basic variables and hence become zero. In examples with many constraints of different types, it is most likely that either they (ASV) are eliminated from basis at a very late stage after many iterations. Again, in all such cases it is at any stage of iterations, not known that the problem shall end up in a case of infeasible solution. A method that targets at possible removal of all ASV **in the *first phase*** and then only treating with regular simplex procedure in the ***second phase*** is called a **Two-Phase Method**. Some points to be taken care of, before applying the Two-Phase Method, are as follows.

Phase I

1. The problem may be in the form of maximization or minimization. It does not matter much; the point is only to concentrate on how we deal with $+M$ and $-M$ arising from $\geq$ or $=$ sign given in the constraints.
2. Cost coefficients of the all the variables, except **ASV** (After following three steps $M1$, $M2$ and $M3$), will be taken zero; those of ASV should be taken as +1 or –1 according to the type of problem — minimization or maximization.
3. This fixes up the prime objective of removal of **ASV** in the initial iteration.
4. Now, express the problem into simplex table form and start dealing on it with regular procedures as we do in dealing with simplex tables.
5. At the end, when all ASVs are removed, $C_j - Z_j$ entries (other than ASV) will be all zero and the Z value will also be zero.

(Note: If ASVs are not removed then it is a case of ***infeasible solution***)

Now, move to Phase II.

Phase II

6. As the example is now free from ASV, our basic purpose is solved and this assures that the solution, at the end, will be a feasible solution. We assign the given cost coefficient to the decision variables and make the new simplex table with the same entries (except C_j cost coefficient entries) those found in the last table of **Phase I**.
7. Now, we operate the example as we regularly do, following simplex method. Finally, we get the optimality criterion satisfied.

Solve the problem by two-phase method:

ILLUSTRATION 26

Minimize $Z = x_1 + 2x_2 + 3x_3$
subject to the constraints $x_2 - x_3 \geq 2$

$$x_1 + x_2 + 2x_3 \leq 8$$
$$-2x_1 + x_2 - x_3 \leq -4;$$

with $x_1, x_2, x_3 \geq 0$.

Solution

On observation, first, we change the third constraint. (Its right side is negative.) We get $2x_1 - x_2 + x_3 \geq 4$. First and third constraints are $\geq$ type, which need surplus and artificial slack variable while the second one needs only slack variable. The problem on introduction of all such variables is
Minimize $Z = 1x_1 + 2x_2 + 3x_3 + 0s_1 + MA_1 + 0s_2 + 0s_3 + MA_2$;
Subject to the constraints $0x_1 + 1x_2 - 1x_3 - 1s_1 + 1A_1 + 0s_2 + 0s_3 + 0A_2 = 2$;

$$1x_1 + 1x_2 + 2x_3 + 0s_1 + 0A_1 + 1s_2 + 0s_3 + 0A_2 = 8;$$
$$2x_1 - 1x_2 + 1x_3 + 0s_1 + 0A_1 + 0s_2 - 1s_3 + 1A_2 = 4; \text{ with all variables} \geq 0.$$

There are 3 equations with 8 variables.
We apply Two-Phase Method.

Phase I

In Phase I, we take cost coefficient of all variables, except ASV, as zero and those of ASV as +1. (It is a minimization case.)

Table 1.16

Minimize

	$C_j \rightarrow$	VAR	0	0	0	0	1	0	0	1		
	$\downarrow$ Basis		x_1	x_2	x_3	s_1	A_1	s_2	s_3	A_2	X_B	R. R.
	1	A_1	0	1	–1	–1	1	0	0	0	2	–
I	0	S_2	1	1	2	0	0	1	0	0	8	8
	1	A_3	2	–1	1	0	0	0	–1	1	4	2 →
	$C_j - Z_j$		–2↑	0	0	1	0	0	1	0	6	

This is a minimization problem and hence the most negative of $C_j - Z_j$ entry will be selected and the corresponding variable will become an incoming variable. We select x_1 as an incoming variable and find R.R. (Table 1.17).

Table 1.17

Minimize

	$C_j \rightarrow$ / VAR / $\downarrow$ Basis	0 x_1	0 x_2	0 x_3	0 s_1	1 A_1	0 s_2	0 s_3	0 A_2	X_B	R. R.
II	1 ← A1	0	①	–1	–1	1	0	0	0	2	2 →
	0 s_2	0	3/2	3/2	0	0	1	1/2	–1/2	6	4
	0 x_1	1	–1/2	1/2	0	0	0	–1/2	1/2	2	–
	$C_j - Z_j$	0	–1↑	1	1	0	0	0	0	2	
III	0 x_2	0	1	–1	–1	1	0	0	0	2	
	0 s_2	0	0	3	3/2	–3/2	1	1/2	–1/2	3	
	0 x_1	1	0	0	–1/2	+1/2	0	–1/2	1/2	3	
	$C_j - Z_j$	0	0	0	0	1	0	0	1	Z = 0	

- All the ASV (A_1 and A_3) have become non-basic variables (i.e. $A_1 = 0 = A_2$)
- All $C_j - Z_j = 0$ (other than ASV).
- $Z = 0$.

We are allowed to move to Phase II with given cost coefficients, and operate with usual simplex method.

Phase: II

Table 1.17

Minimize

$C_j \rightarrow$ / VAR / $\downarrow$ Basis	1 x_1	2 x_2	3 x_3	0 s_1	0 s_2	0 s_3	$\mathbf{X}_B$	R. R.
2 ← x_2	0	1	–1	–1	0	0	2	
0 s_2	0	0	3	3/2	1	1/2	3	
1 x_1	1	0	0	–1/2	0	–1/2	3	
$C_j - Z_j$	0	0	5	5/2	0	1/2	7	

This is a minimization problem and all $C_j - Z_j$ entries are ≥ 0. This is optimality criterion. Minimum value = 7; basic variable: $x_1 = 3$; $x_2 = 2$, $s_2 = 3$; Non-basic variable: $x_3 = 0$; $s_1 = 0 = s_3$.

ILLUSTRATION 27

Solve the following problem using **Two-Phase** Method. Maximize $z = 4x + 6y$

subject to the constraints $2x + 3y \geq 6$

$$4x + y \geq 4$$

$$x \leq 5$$

With; $x, y \geq 0$.

This example, on following steps of Phase I, comply with format. (Surplus $s_1 \geq 0$ and ASV $A_1 \geq 0$ in first constraint; surplus $s_2 \geq 0$ and ASV $A_2 \geq 0$ in the second constraint; and $s_3 \geq 0$ as slack variable in the third constraint and all cost coefficient = 0; those for A_1 and A_2 to be –1.)

Phase I

Table 1.18

Maximize

	$C_j \rightarrow$ / $\downarrow$ Basis	VAR	0 x	0 y	0 s_1	–1 A_1	0 s_2	–1 A_2	0 s_3	$\mathbf{X}_B$	R. R.
I	–1	A_1	2	3	–1	1	0	0	0	6	3
	–1 ←	A_2	4	1	0	0	–1	1	0	4	1→
	0	s_3	1	0	0	0	0	0	1	5	5
	$C_j - Z_j$		6↑	4	–1	0	–1	0	0	–10	
II	–1 ←	A_1	0	5/2	–1	1	1/2	–1/2	0	4	8/5→
	0	x	1	1/4	0	0	–1/4	1/4	0	1	4
	0	s_3	0	–1/4	0	0	1/4	–1/4	1	4	–
	$C_j - Z_j$		0	5/2↑	–1	0	–1/2	–3/2	0	–4	
III	0 ←	y	0	1	–2/5	2/5	1/5	–1/5	0	8/5	
	0	x	1	0	1/10	–1/10	–3/10	3/10	0	3/5	
	0	s_3	0	0	–1/10	1/10	3/10	–3/10	1	22/5	
	$C_j - Z_j$		0	0	0	–1	0	–1	0	$Z = 0$	

All $C_j - Z_j$ are ≤ 0. The problem has fulfilled the necessary criteria and Phase I ended; also $Z = 0$. Now we go to Phase II with given coefficient.

Phase: II

Table 1.19

Maximize

	$C_j \rightarrow$ / $\downarrow$ Basis	VAR	4 x	6 y	0 s_1	0 s_2	0 s_3	$\mathbf{X}_B$	R. R.
I	6	y	0	1	–2/5	1/5	0	8/5	–
	4 ←	x	1	0	1/10	–3/10	0	3/5	6 →
	0	s_3	0	0	–1/10	3/10	1	22/5	–
	$C_j - Z_j$		0	0	2↑	0	0	12	
II	6	y	4	1	0	–1	0	4	undefined
	0	s_1	10	0	1	–3	0	6	undefined
	0	s_3	1	0	0	0	1	5	undefined
	$C_j - Z_j$		–20	0	0	6↑	0	$Z = 24$	

At this stage, the variable $s_2 \geq 0$ wants to enter the basis but the outgoing variable cannot be decided. (R.R. becomes undefined.) It is a case of unbounded solution.
Now the problem has reached an optimal stage.
Basic variable: $y = 4$; $s_1 = 6$; $s_2 = 5$;
non-basic variable: $x = 0$; $s_2 = 0$

Maximum value = 24.
This clarifies our standpoint that Two-Phase Method has a strict concern with removal of ASV and not the type of problem.

SCM (8): Degenerate Solution

This is a very important characteristic in the simplex procedure. It is given high importance in most of the cases. First, we define degenerate solution.

We know that a basic feasible solution has two parts—***A set of basic variables*** which are extracted (taken) from the simplex table and, we hope, they are non-negative (≥ 0). and ***a set of non-basic variables***. (which are not present in the table of vectors in the basis; they are always zero)

If ***at least one basic variables*** attains zero value, then the basic feasible solution is called a ***degenerate solution***. The early sign of degeneracy is identified in the R.R. column of the simplex table; when two or more minimum replacement ratios are equal.

At this state, the selection of the outgoing variable is questionable. If a variable corresponding to the minimum R.R. is an ASV, then it is obvious that we select that one only as an outgoing variable (as we expect the basic feasible solution to be free from ASV). In the ***next iteration***, the competing outgoing variable of the basis attains zero value in the right-hand side.

This is ***degeneracy*** as one of the basic variables has become zero.

Now, if the solution is not optimal, then continue next iteration, the minimum replacement ratio becomes zero. ($\because$ RHS for that variable = 0.) In the next iteration if the solution is optimal then fine; we get a final optimal basic feasible solution. If it is not optimal then we are dragged into a case—called *Cycling*. This clarifies the concept that the selection of outgoing variable, in the stage when two or more minimum replacement ratios are equal, is really a critical job. There is a method—*Charne's Method* that allows proper selection of outgoing variable so that simplex procedure may not end up in cycling.

ILLUSTRATION 28(A)

Maximize $Z = 2x + 6y$;
subject to the constraints $2x + 4y \geq 8$

$$2x + y \geq 2$$
$$3x + 4y \leq 12$$
with $x, y \geq 0$.

Solution

Adding surplus $s_1 \geq 0$; ASV $A_1 \geq 0$ in the first constraint, surplus $s_2 \geq 0$ and ASV $A_2 \geq 0$ in the second, and a slack variable $s_3 \geq 0$ in the third constraint; we write the simplex table as follows (Table 1.20).

Table 1.20

Maximize

	$C_j \rightarrow$ VAR / $\downarrow$ Basis	2 x	6 y	0 s_1	$-M$ A_1	0 s_1	$-M$ A_2	0 s_3	$\mathbf{X}_B$	R. R.
	$-M \leftarrow A_1$	2	4	-1	1	0	0	0	8	$2 \rightarrow$
I	$-M$ A_2	2	1	0	0	-1	1	0	2	2
	0 s_3	3	4	0	0	0	0	1	12	3
	$C_j - Z_j$	$2 + 4M$	$6 + 5M\uparrow$	$-M$	0	$-M$	0	0		

Comment: Two minimum R.R corresponding to the two ASV; A_1 and A_2, replacement ratios are equal. We let any of them to be out from basis; say A_1 leaves the basis making a room for the variable y to enter the basis and to become a basic variable in the next turn. Also, note that the right side for A_2 will be zero in the next table.

Table 2.21

6 y	1/2	1	–1/4	1/4	0	0	0	2	4
–M ← A_2	3/2	0	1/4	–1/4	–1	1	0	0	0 →
0 s_3	1	0	1	–1	0	0	1	4	4
$C_j - Z_j$	$\left(-1+\frac{3}{2}M\right)\uparrow$	0	$\frac{3}{2}+\frac{1}{4}M$	$\frac{3}{2}+\frac{3}{4}M$	–M	0	0	12	

Table 1.22

$C_j \rightarrow$ VAR ↓ Basis	2 x	6 y	0 s_1	–M A_1	0 s_1	–M A_2	0 s_3	X_B	R. R.
6 y	0	1	–1/3	1/3	1/3	–1/3	0	2	–
2 ← x	1	0	1/6	–1/6	–2/3	2/3	0	0	0 →
0 s_3	0	0	5/6	–5/6	2/3	–2/3	1	4	24/5
$C_j - Z_j$	0	0	↑5/3	–M–5/3	–2/3	2/3–M	0	12	

Table 1.23

$C_j \rightarrow$ VAR ↓ Basis	2 x	6 y	0 s_1	–M A_1	0 s_1	–M A_2	0 s_3	$\mathbf{X}_B$	R. R.
6 y	2	1	0	0	–1	1	0	2	–
0 s_1	6	0	1	–1	–4	4	0	0	–
0 ← S_3	–5	0	0	0	4	–4	1	4	1 →
$C_j - Z_j$	–10	0	0	–M	6↑	–M–6	0	12	

The iterations may continue and we may terminate the process as incoming and outgoing process of variables may not terminate at any stage; i.e. **cycling**.

We take up a turn from the first table and let A_2 out from the basis. (From the table 1.20).

ILLUSTRATION 28 (B)

Table 1.24

Maximize

	$C_j \rightarrow$ VAR ↓ Basis	2 x	6 y	0 s_1	–M A_1	0 s_1	–M A_2	0 s_3	$\mathbf{X}_B$	R. R.
	–M A_1	2	4	–1	1	0	0	0	8	2
I	–M ← A_2	2	1	0	0	–1	1	0	2	2 →
	0 s_3	3	4	0	0	0	0	1	12	3
	$C_j - Z_j$	2	$+4M$	$6+5M$	$-M$	0	$-M$	0 0	$-10M$	

(*Contd.*)

(*Contd.*)

II	$-M$	$\leftarrow A_1$	-6	0	-1	1	4	-4	0	0	$0 \rightarrow$
	6	y	2	1	0	0	-1	1	0	2	–
	0	s_3	-5	0	0	0	4	-4	1	4	1
	$C_j - Z_j$		$-6M-10$	0	$-M$	0	$4M$	$-5M$	0	$Z = 12$	
III	0	s_2	$-3/2$	0	$-1/4$	$1/4$	1	-1	0	0	–
	6	y	$1/2$	1	$-1/4$	$1/4$	0	0	0	2	–
	0	$\leftarrow s_3$	1	0	1	-1	0	0	1	4	$4 \rightarrow$
	$C_j - Z_j$		-1	0	$\uparrow 3/2$	$-M-3/2$	0	$-M$	0	$Z = 12$	
IV	0	s_2	$-5/4$	0	0	0	1	-1	$1/4$	1	
	6	y	$3/4$	1	0	0	0	0	$1/4$	3	
	0	s_1	1	0	1	-1	0	0	1	4	
	$C_j - Z_j$		$-5/2$	0	0	$-M$	$0\uparrow$	$-M$	$-3/2$	$Z = 18$	

All $C_j - Z_j \le 0$ for this maximization problem and so the solution is optimal.
Basic variable: $y = 3$; $S_1 = 4$; $S_2 = 1$. Non-basic variable: $x = 0$; $s_3 = 0 = A_1 = A_2$.
Maximum $Z = 18$. Trying both the ways by removing any one of ASV, we get the optimal solution.

Comment:
In both the examples solved above, we meet with degenerate solution but a few more iterations remove degeneracy.

ILLUSTRATION 29

Maximize $Z = 2x + 6y$;
subject to the constraints $3x + 4y \le 8$

$$2x + 5y \le 15$$
$$x + y \ge 2$$
$$x, y \ge 0.$$

Solution

We add slack $s_1 \ge 0$; $s_2 \ge 0$ in the first and second constraints respectively and $s_3 \ge 0$ surplus variables and corresponding AVS $A \ge 0$ in the third constraint.

Table 1.25

Maximize

	$C_j \rightarrow$	VAR ↓ Basis	2 x	6 y	0 s_1	0 s_2	0 s_3	$-M$ A	X_B	*R. R.*
I	0	s_1	3	4	1	0	0	0	8	2*
	0	s_2	2	5	0	1	0	0	15	3
	$-M$	A	1	1	0	0	-1	1	2	$2^* \rightarrow$
	$C_j - Z_j$		$2 + M$	$6 + M\uparrow$	0	0	$-M$	0	$-2M$	Equal RR

(*Contd.*)

(*Contd.*)

II	0	s_1	–1	0	1	0	4	–4	0	0 →
	0	s_2	–3	0	0	1	5	–5	5	1
	6	y	1	1	0	0	–1	1	2	Undefined
	$C_j - Z_j$		–4	0↑	0	0	6↑	$-M+6$	$Z = 12$	
III	0	s_2	–1/4	0	1/4	0	1	–1	0*	
	0	s_3	–7/4	0	–5/4	1	0	0	5	
	6	y	3/4	1	–1/4	0	0	0	2	
	$C_j - Z_j$		–5/2	0	–3/2	0	0	$-M$	12	

All entries of $C_j - Z_j$ are ≤ 0. It is an optimal stage.
Basic variable: $s_2 = 5$; $(s_3 = 0)$*
Non-basic variable: $x = 0$; $s_1 = 0 = \text{A}$; with Maximum $z = 12$.
*S_3 is a basic variable and has a zero value. This is a degenerate solution.

Comment:
In the first table if we make s_1 an outgoing variable instead of all then continuing further. We get the final table as follows. It also gives a basic feasible ***degenerate*** solution.

Table 1.26

Maximize

	$C_j \rightarrow$ VAR / ↓ Basis		2 x	6 y	0 s_1	0 s_2	0 s_3	$-M$ A	X_B
I	6	y	0	1	1	0	3	–3	2
	0	s_2	0	0	–3	1	–7	7	5
	2	x	1	0	–1	0	4	4	0
	$C_j - Z_j$		0	0	–4	0	–10	$-M+10$	12

Basic variable: $(x = 0)$; $y = 2$; $s_2 = 5$
Non-basic variable: $s_1 = 0 = s_3 = A$, with Maximum $z = 12$.

Comment:
In some cases, during the process of iterations, we get a basic variable having zero value, i.e. degenerate solution; but continuing further the degeneracy is removed. Thus the final optimal solution is free from degeneracy. This is called a case of ***removable degeneracy***.

SCM (9): Dual Simplex Method

We know that treating a ≥ type constraint, we use a surplus and artificial slack variable (as a basic variable); this calls for Big M Method. (M for minimization problem and $-M$ for maximization.) This involves more variables and increases complicacy. Initially, the solution is an ***infeasible*** one and gradually as an ASV in each step is removed, the solution may tend to a basic feasible and then to an optimal one. In turn, making right-hand side negative violates (M2 the fundamental essentiality of simplex algorithm); the solution at this initial stage is ***infeasible***.

Dual simplex method takes care of these two points. The standard points to solve the problem are as follows.

1. The original given problem must have at least one constraints of ≥ type or = type.

2. Transform the problem to a maximization problem with all constraints of ≤ type. This, in the case of inequality of ≥ type is achieved by multiplying each term of the constraint by –1. As a result, the inequality will be of ≤ type and the right-hand side will be negative in sign.(Here lies the difference between **Big-M** Method and the **Dual Simplex** Method.)
3. Add slack variables on the left hand side of the ≤ type inequalities. (RHS may be positive negative or zero)

We first follow the above steps and then study the remaining steps.

ILLUSTRATION 30

Maximize $Z = 2x + 6y$;
subject to the constraints $2x + 4y \geq 8$;

$$2x + y \geq 2;$$
$$3x + 4y \leq 12;$$
$$x, y \geq 0.$$

Solution

We worry for only ≥ type constraints, we treat them as follows.
$2x + y \geq 8$, $\therefore$ $-2x - 4y \leq -8$; adding slack variable $s_1 \geq 0$; $-2x - 4y + s_1 = -8$.
Also,
$2x + y \geq 2$ gives $-2x - y \leq -2$ and introduction of s_2 – a slack variable gives
$-2x - y + s_2 = -2$. The third constraint is already of ≤ type and it needs a slack $s_3 \geq 0$. $\therefore$ $3x + 4y + S_3 = 12$.
Now, we combine all to form the simplex table.

Table 1.27

Maximize

$C_j \rightarrow$ VAR ↓ Basis		2 x	6 y	0 s_1	0 s_2	0 s_3	$\mathbf{X}_B$	
0	s_1	–2	–4	1	0	0	–8	
0	s_2	–2	–1	0	1	0	–2	
0	s_3	3	4	0	0	1	12	
$C_j - Z_j$		2	6↑	0	0	0		

4. If the right-hand side column (X_B) has some negative entries; it violates the basic condition of simpltex algorithm. We follow a special routine.
5. From the X_B column entries, select the most negative value. The corresponding variable in the basis is the outgoing variable. [*This is just opposite of the simplex; the choice of outgoing variable is prior to the one of incoming variables*]
6. Find replacement ratios below the $C_j - Z_j$ entries as indicated below.

   R.R. = $C_j - Z_j \div$ negative entries of the outgoing row.
7. The minimum of R.R. is selected and the corresponding vector is an incoming variable in the next step.
8. Continue this process, till all the entries of X_B (RHS) are ≥ 0; follow the regular simplex method.

Maximize

	$C_j \rightarrow$ VAR / $\downarrow$ Basis		2 x	6 y	0 s_1	0 s_2	0 s_3	$\mathbf{X}_B$
I(B)	0	s_1	–2	–4	1	0	0	–8→
	0	s_2	–2	–1	0	1	0	–2
	0	s_3	3	4	0	0	1	12
	$\mathbf{C_j - Z_j}$		2	6	0	0	0	
R.R. →	$\frac{C_j - Z_j}{aij < 0}$		$\frac{2}{-2} = -1$	$\frac{6}{-4} = -3/2\uparrow$				

Table 1.28

	$C_j \rightarrow$ VAR / $\downarrow$ Basis		2 x	6 y	0 s_1	0 s_2	0 s_3	X_B	R.R.
II	6	y	1/2	1	–1/4	0	0	2	Undefined
	0	s_2	–3/2	0	–1/4	1	0	0	Undefined
	0	s_3	1	0	1	0	1	4	4 →
	$C - Z_j$		–1	0	3/2↑	0	0	12	

As all entries X_B are non-negative, we follow regular simplex method.

Table 1.29

	$C_j \rightarrow$ VAR / $\downarrow$ Basis		2 x	6 y	0 s_1	0 s_2	0 s_3	X_B	
III	6	y	3/4	1	0	0	1/4	3	
	0	s_2	–5/4	0	0	1	1/4	1	
	0	s_1	1	0	1	0	1	4	
	$C_j - Z_j$		–5/2	0	0	0	–3/2	$Z = 18$	

All $C_j - Z_j$ entries are ≤ 0. $\therefore$ The solution is optimal.
Basic variable: $y = 3$; $s_2 = 1$; $s_1 = 4$;
Non-basic variable: $x = 0$; $s_3 = 0$;
Maximum $Z = 18$.

ADDITIONAL QUESTIONS FOR PRACTICE (WITH HINTS AND ANSWERS)

Question 1

Maximize $Z = 7x + 14y$;
subject to the constraints

$$3x + 2y \leq 16;$$
$$x + 4y \leq 10;$$

with $x, y \geq 0$.

(Hint: Add slack $s_1 \geq 0$ and $s_2 \geq 0$ in the first and second constraints respectively and solve.
Answer: $x = 22/5$; $y = 7/5$ Maximum $Z = 252/5$.)

Question 2

Maximize $Z = x_1 - x_2 + 3x_3$;
subject to the constraints $x_1 + x_2 + x_3 \leq 10$;

$$2x_1 - x_3 \leq 2;$$
$$2x_1 - 2x_2 + 3x_3 \leq 0$$

with $x_1, x_2, x_3 \geq 0$.

Solution

Adding slack variable $s_1, s_2, s_3 \geq 0$, the simplex table is as follows.

Table 1.30

Maximize

	$C_j \rightarrow$	VAR / ↓ Basis	*1* x_1	*–1* x_2	*3* x_3	*0* s_1	*0* s_2	*0* s_3	X_B	*R. R.*
I	0	s_1	1	1	1	1	0	0	10	10
	0	s_2	2	0	–1	0	1	0	2	–
	0 of ← the type	s_3	2	–2	3	0	0	1	0	0→
	$C_j - Z_j$		1	–1	3↑	0	0	0	$Z = 0$	
II	0 ←	s_1	1/3	5/3	0	1	0	–1/3	10	6→
	0	s_2	8/3	–2/3	0	0	1	1/3	2	Undefined
	3	x_3	2/3	–2/3	1	0	0	1/3	0	Undefined
	$C_j - Z_j$		–1	1↑	0	0	0	–1		Undefined
III	–1	x_2	1/5	1	0	3/5	0	–1/5	6	
	0	s_2	14/5	0	0	2/5	1	1/5	6	
	3	x_3	4/5	0	1	2/5	0	0	4	
	$C_j - Z_j$		–11/5	0	0	–3/5	0	0	$Z = 6$	

Maximize
All $C_j - Z_j \leq 0$; the solution is optimal
Basic variable: $x_2 = 6$; $x_3 = 4$; $s_2 = 6$
non-basic variable: $x_1 = 0$; $s_1 = 0$; $s_3 = 0$, Maximum $Z = 6$.

Question 3

Solve the problem using Big-M method. Minimize $Z = -2x_1 - x_2 - 3x_3$
subject to the constraints $2x_1 + 3x_2 + 4x_3 = 12$;

$$x_1 + x_2 + 2x_3 \leq 5;$$

with $x_1, x_2, x_3 \geq 0$.

Solution

Adding ASV in the first constraint and a slack variable in the second constraint, the simplex table is as follows.

Table 1.31

Minimize

	$C_j \rightarrow$ VAR $\downarrow$ Basis	–2 x_1	–1 x_2	–3 x_3	M A_1	0 s_1	X_B	R.R.
I	M A_1	2	3	4	1	0	12	3
	0 ← s_1	1	1	2	0	1	5	5/2 →
	$C_j - Z_{jj}$	–2–2M	–1–3M	↑–3–4M	0	0	Z = 12M	
II	M ← A_1	0	1	0	1	–2	2	2 →
	–3 x_3	1/2	1/2	1	0	1/2	5/2	5
	$C_j - Z_j$	3/2	½–M↑	0	0	2M+3/2	2M–15/2	
III	–1 x_1	0	1	0	1	–2	2	–
	–3 ← x_3	1/2	0	1	–1/2	3/2	3/2	1 →
	$C_j - Z_j$	–1/2↑	0	0	M–1/2	5/2	–13/2	
IV	–1 x_2	0	1	0	1	–2	2	
	–2 ← x_1	1	0	2	–1	3	3	
	$C_j - Z_j$	0	0	4	M–1	4	–8	

All $C_j - Z_j$ entries for this minimization problem are ≥ 0. This gives a minimum solution,
Basic variable: $x_2 = 2$; $x_1 = 3$
Non-basic variable: $x_3 = 0 = A_1 = s_1$;
Minimum value = –8.

Question 4

Solve the following LP problem using simplex method. Maximize $Z = 4x_1 + 3x_2$ (Total profit on sale), subject to the constraints $x_1 \le 400$, and $x_2 \le 700$; $2x_1 + x_2 \le 1000$;

$$x_1 + x_2 \le 800 \text{ with } x_1, x_2 \ge 0.$$

Solution:

Let us find this solution using restricted variable case. Let $x_1 = 400 - y_1$ and $x_2 = 700 - y_2$ where $y_1, y_2 \ge 0$. Introducing these new conditions the problem is;
Maximize $Z = 4(400 - y_1) + 3(700 - y_2) = 3700 - 4y_1 - 3y_2$
subject to the constraints $2(400 - y_1) + (700 - y_2) \le 1000$
and

$$(400 - y_1) + (700 - y_2) \le 800.$$

i.e. Maximize $Z = -4y_1 - 3y_2 + 3700$;
subject to the constraints $-2y_1 - y_2 \le -500$; $-y_1 - y_2 \le -300$ with $y_1, y_2 \ge 0$. Constraints can be viewed as $2y_1 + y_2 \ge 500$; $y_1 + y_2 \ge 300$; $y_1, y_2 \ge 0$.

We introduce s_1, A_1, s_2, and A_2 (Surplus & ASV) all ≥ 0; the problem in the simplex is as follows.

Table 1.32

Maximize

$C_j \rightarrow$	VAR / $\downarrow$ Basis	-4 y_1	-3 y_2	0 s_1	$-M$ A_1	0 s_1	$-M$ A_2	$+3700$ X_B	R.R.
$-M$	A_1	2	1	-1	1	0	0	500	250 $\rightarrow$
$-M$	A_2	1	1	0	0	-1	1	300	300
$C_j - Z_j$		$-4 + 3M$ $\uparrow$	$-3 + 2M$	$-M$	0	$-M$	0	$-800M + 3700$	
-4	y_1	1	1/2	$-1/2$	1/2	0	0	250	500
$-M \leftarrow$	A_2	0	1/2	1/2	$-1/2$	-1	1	50	100 $\rightarrow$
$C_j - Z_j$		0	$-1+1/2M$ $\uparrow$	$-2+1/2M$	$2-3/2M$	$-M$	0	$-1000 -50M+3700$	
-4	y_1	1	0	-1	1	1	-1	200	–
-3	y_2	0	1	1	-1	-2	2	100	1
$C_j - Z_j$		0	0	-1	$-M+1$	-2	$-M+2$	$-1100+3700=2600$	

All $C_j - Z_j$ entries are ≤ 0. So the solution is optimal

$$y_1 = 200 \text{ and } y_2 = 100;$$

$x_1 = 400 - y_1$ gives $\mathbf{x_1 = 200}$; $x_2 = 700 - y_2$ gives $\mathbf{x_2 = 600}$ and profit $= -1100 + 3700 = 2600$.

POINTS TO REMEMBER

In this section, some important points and useful hints are given for the students to read, understand, and apply them.

1. The fundamental objectives are
   (a) to construct (if possible) a linear model of a real or given situation,
   (b) to solve it possibly for feasible solution, and [c] to interpret the solution.
2. In case of two decision variables, one may find graphical solution. Either for finding details about the residual resources in case of two variables or in case of problems having more than two variables, we apply simplex algorithm to arrive at feasible solution if it exists. We have already mentioned *Three musts* as a key point towards the construction of table.
   (M1) Every constraint must be converted in = type.
   (M2) Right side of the constraint (resource amount) ≥ 0
   (M3) Every equation must have a basic variable.
3. In case you find $C_j - Z_j$,
   (a) entries corresponding to basic variables are always zero.
   (b) for a maximization problem, the variable that corresponds to the most positive $C_j - Z_j$ entry, becomes an incoming variable in the next iteration.
   (c) for a minimization problem, the variable that corresponds to the most negative $C_j - Z_j$ entry, becomes an incoming variable in the next iteration.
   (d) For a maximization problem, if $C_j - Z_j$ entries for all non-basic variables are ≤ 0, it is an indication for optimal solution.

(e) For a minimization problem, if $C_j - Z_j$ entries for all non-basic variables are ≥ 0, it is an indication for optimal solution.

(f) Optimality criteria satisfied but the artificial slack variable persists in the basis (set of basic variables), is a sign of infeasible solution.

(g) A ($C_j - Z_j$) entry corresponding to a non-basic variable becoming zero, is a sign of multiple optimal solution.

4. Replacement ratio (R.R.) is an indicator for the outgoing variable.
X_B = resource vector entries are always ≥ 0,
R.R = Entries of X_B ÷ positive entries of pivot column.
   (a) If all pivot column entries are ≤ 0, it is a sign of unbounded solution.
   (b) If two or more minimum R.R. values are equal, then it is an indication of degenerate solution in the next iteration.
   (c) Either it be a maximization or a minimization problem, the criteria for the selection of outgoing variable is the choice of minimum of R.R. entries.

EXERCISES

OBJECTIVE TYPE QUESTIONS

I. State True or False:

1. Every problem of real life situation when formulated in a mathematical model assumes a linear form.
2. Every LP problem can be solved graphically.
3. If the most common region to all the constraints is bounded then there exists an optimal solution.
4. In a standard LP problem (ready to write in simplex table), the number of basic variables equals the number of equality constraints.
5. The early sign that the problem may end-up in degeneracy, any two minimum replacement ratios are equal.
6. Artificial slack variables are successively removed in each simplex iteration.
7. If there is no common feasible region to all the constraints then the solution becomes a case of multiple optimal solution.
8. In the final table of simplex, all $c_j - z_j$ entries corresponding to non-basic variables are always zero.
9. During simplex procedures, if at any stage, an ASV leaves the basis then the same variable will never enter the basis in further iterations.
10. The last simplex table of Phase I has all $c_j - z_j$ entries are zero.
11. Only Big-M method or two phase method is used when at least some constraints are of $\geq$ type.
12. Pivot element is the entry which is common to incoming variable and outgoing variable.

ANSWERS

1. false.	2. false.	3. false.	4. true.	5. true.
6. false.	7. false	8. false.	9. true.	10. false.
11. false.	12 true.			

II. Multiple Choice Questions

Question 1: In a linear programming problem of maximization type, there are three linearly independent constraints; two of them are of ≤ type and one is = type. Answer the following questions.

(a) How many are total number of variables there in the simplex table?
   1. 6   2. 5   3. 7   4. cannot be said

(b) How many are artificial slack variables there in the table?
   1. 1   2. 2   3. 4   4. cannot be any

(c) How many are the basic variables in the simplex table at any stage?
   1. 2   2. 3   3. 4   4. none of these

(d) How many are the non-basic variables in the optimal table?
   1. 3   2. 4   3. 5   4. cannot be said

(e) Is it possible to find the graphical solution of the problem?
   1. yes   2. no   3. cannot be said   4. after certain changes

(f) If the problem has a positive optimal value then at least how many variables attain positive values in the optimal table?
   1. 1   2. 2   3. 3   4. all

Question 2: Study the following simplex table for a maximization problem and answer the following questions.

$C_j \rightarrow$ VAR / $\downarrow$ Basis	*12* x_1	*15* x_2	*16* x_3	*0* s_1	*0* s_2	*0* s_3	x_B	*R.R*
12 x_1	A	P	C	1/2	d	e	8	
0 s_2		Q					3	
0 s_3		R					2	
$C_j - Z_j$	0	–9	–20	–6	0	0		

(a) Is it an optimal table?
   1. no   2. yes   3. cannot be said   4. needs one more iteration

(b) From the second row of the table if we write
$ax_1 + Px_2 + cx_3 + (½)\, s_1 + ds_2 + es_3 = 8$; then

(i) Value of a is = _______.
   1. 1   2. 2   3. 3   4. cannot be said

(ii) Values of both d and e are
   1. zero   2. any positive values
   3. any negative values   4. one positive and the other negative

(iii) Values of $P + c$ is
   1. zero   2. 5
   3. any negative value   4. one positive and the other negative

(iv) What is the maximum value of the objective function?
   1. 0   2. 96   3. –35   4. 13

(v) What is the value of P?
   1. 2   2. 3   3. 1   4. cannot be determined

Answers

Q:1 (a) ->(3).7; (b)->(1). 1; (c)->(2) 3; (d)-> (2). 4; (e)-> (3). Cannot be said. (f)-> (1). 1.

Q:2 (a) -> (2). Yes.
(b) ->(1) -> (1). 1.
(2) -> (1). Zero.
(3) -> (2). 5.
(4) -> (2) 96.
(5) -> (1) 2.

NUMERICAL PROBLEMS

A – MATHEMATICAL MODELS AND GRAPHICAL SOLUTIONS

1. A new rose dust is being prepared by using two products: *A* and *B*. Each kg of *A* contains 30 grams of carbonyl and 40 grams of flower dust, while each kg of *B* contains 40 grams of carbonyl and 20 grams of flower dust. The final product must contain at least 120 grams of flower dust. If each kg of product A costs ₹30 and each kg of product *B* costs ₹25, how many kilograms of each product must be used to minimize the cost. [Formulate a mathematical model for the problem.]
2. In treating the cases of tactutis the physician has prescribed a combination of two brands—Speed-up and Go-well. The Speed-up costs, ₹4 per pill while Go-well costs ₹3 per pill. Each compound contains SND plus an activator. In the case of particular patient's case he needs at least 10 mg of SND per day. Speed-up contains 4 mg of SND while Go-well contains 2 mg of SND. activators in Excessive amount can be harmful. Keeping this view in his mind, the physician limits the total amount of activator to no more than 2 mg per day each pill of both Speed-up and Go-well contains 1⁄2 mg of activator per pill: Formulate the mathematical model to minimize the cost of medication and still observing the conditions within limits.
3. A production unit can produce at most 50, 25 and 30 units of three different items *A*, *B* and *C* respectively. As per the contract, it is supposed to supply at least 45 units of *A* and 50 units each of items *B* and *C*. The time producing the items *A*, *B* and *C* is 0.8, 0.6, and 2 hours respectively and the total availability of time is 100 hours in the department. The profits on sale of the items *A*, *B* and *C* are ₹10, ₹20, and ₹30 respectively. You are required to model this problem.
4. Saga Pharma produces two drugs *A* and *B* and as a result of leftover after producing the product *B* some amount of the product *C* can be separated at no cost. The remaining amount left is discarded as wastage by paying ₹2 per kg. Production of both *A* and *B* need to pass through two departments — I and II. The available time and other necessary data is supplied in the following table. You are required to construct a mathematical model of the situation.

Time Required

Product

	A	*B*	*C*	*Maximum*	*Available*
Department I	2	4	—	20	
Department II	3	2	—	24	
Production ratio per kg		1	2		

It is predicted that 6 units of the product *C* can be sold. The profits on sale of *A*, *B*, and *C* at the most are ₹10, ₹20, and ₹25 respectively.
5. A firm places an order of an item in the first week of a month and receives in the last week of the same month. The items received will be on sale on the next month. (Assuming that he has no opportunity loss in any month.) The receipts and the sale functioning is carried out from the

warehouse that has a maximum capacity of stocking 200 units. This being the seasonal business, activates for winter months only. The following table gives the idea about the operations [Assume that at the end of the season all items are completely sold.]

No	*Month*	*Operation*	*Cost*	*Salary Price*	*Item*
1	December	Order placed	₹50	–	
2	January	Order placed/sale	₹70	–	80
3	February	Order placed/sale	₹80	–	95
4	March	Order placed/sale		–	90

Formulate the problem to maximize the return function.

6. A dietician plans a menu for her patients. She has two types of main food: say P and Q. The ingredients required are carbohydrate, fat, and protein. The following table shows the content of the ingredients per 10 grams of each food P and Q. The corresponding costs are also given. She plans the food-mix in a way that minimizes the total cost of food.

Food type	*Fat*	*Carbohydrate*	*Protein*	*Cost*
P (10 grams)	2	3	5	20
Q (10 grams)	3	5	2	15
Minimum requirement	10	24	15	

Make a mathematical model of the problem. [Cost given is per 10 grams of each food.]

7. A company makes two types of shirts of the standard sizes: Type P and Type Q. One shirt of Type P needs 10 press-buttons per shirt and type Q requires one special design stripe for value addition. A pack of 10 press-buttons costs ₹5 and the cost of one designer stripe is ₹12. Labour charge for making one shirt of either type is ₹20. At the most 500 packs of press-buttons are available. The company has to buy at least 200 designer stripes per day. The production of maximum 800 shirts can be done on a working say. Cost of cloth per shirt is ₹100. The sale price for one Type P shirt is ₹180 and for Type Q is ₹200. Assuming the availability of the cloths, write a program for the maximization of the profit on sale per day. [Assume that all the shirts are sold at the end of the day.]

Find the **graphical solution** of the following problem.

8. Maximize $Z = 3x + 4y$

subject to the constraints $2x + 4y \leq 13$;

$$2x + 2y \geq 1;$$
$$6x - 45 \leq 15;$$
$$-2x + y \leq 2;$$
$$x, y \geq 0$$

9. Maximize $Z = 2x + y$

subject to the constraints $x + 2y \leq 10$;

$$x + y \leq 6;$$
$$x - 2y \leq 1;$$
$$x - y \leq 2;$$
$$x, y \geq 0$$

10. Maximize $Z = 20x + 40y$.
subject to the constraints $x + y \le 800$;
$$2x + y \le 1000;$$
$$x \le 400;$$
$$y \le 700$$
$$x, y \ge 0$$

11. Minimize $Z = 20x + 60y$
subject to the constraints $20x + 20y \le 160$;
$$12x + 6y \ge 48;$$
$$3x + 6y \ge 24;$$
$$x, y \ge 0$$

12. Maximize $Z = x_1 + x_2$
subject to the constraints $x_1 + x_2 \le 2$;
$$2x_1 + 3x_2 \ge 8;$$
With $x_1, x_2 \ge 0$

13. Maximize $Z = 6x + 8y$
subject to the constraints $x + 2y \ge 3$;
$$x - 2y \le 1;$$
With $x, y \ge 0$

14. Maximize $Z = 3x + 6y$
subject to the constraints $x + y \ge 2$;
$$x + 2y \le 8;$$
$$x \ge 1;$$
$$y \ge 1/4;$$
$$x, y \ge 0$$

15. Maximize $Z = 2x + 6y$
subject to the constraints $x + y \ge 2$;
$$-x + 2y \le 4;$$
$$3x - 2y \le 6;$$
$$x, y \ge 0$$

16. The production manager of a machine tool company is planning to make two types of tools: *A* and *B*. Both the tools are processed in two departments *X* and *Y*. Tools *A* take 2 hours in department *X* and 3 hours in department *Y*. The corresponding time for the Tool *B* is 3 and 2 hours respectively. Department *X* works 10 hours a day while *Y* works for 12 hours a day. At least one unit of each product must be produced. Assuming integral number of units of production that maximizes the total profit on sale, find the number of hours left unutilized in the department *X* and *Y*. Assume that the profit per unit of *A* is ₹20 and of *B* is ₹25 respectively.

17. The financial secretary of Aashahi Dyechem proposes her management to invest ₹5,00,000 in one or more investment firms offering the following schemes.

1. At least 30,000 must be invested in each of the four firms: *A*, *B*, *C* and *D*.
2. The sum total of investment in *B* and *D* must be between 30% and 50%.
3. Investment in the firm *A* must be at the most 40% of the total amount.
4. Investment in *D* must be greater than ₹70,000.

Make a mathematical model of the problem, assuming Companies A, B, C and D gives 8%, 8.25%, 9% and 8% investment per annum respectively.

18. Find the graphical solution of the following problem.
Maximize $Z = 4x_1 + 2x_2$;
subject to the constraints $x_1 + 3x_2 \geq 6$;

$$5x_1 + 2x_2 \geq 10;$$
$$x_1 \leq 4,\ x_2 \leq 5;$$

with $x_1, x_2 \geq 0$.

19. Find the graphical solution of maximize $Z = 5x + 4y$;
subject to the constraints $x + 3y \leq 8$;

$$x - y \leq 2;$$
$$y \leq 3;$$
$$-5x - 3y \geq -20$$
with $x, y \geq 0$.

20. A jeweler plans making three types of ornaments. The amount of gold and silver required to make them is given in the following table. Gold costs ₹3000 per gram while silver costs ₹60 per gram.

Ornament	*Gold (Gram)*	*Silver (Gram)*	*Profit on sale*
A	5	8	2000
B	4	10	4000
C	3	12	3000

The jeweler invests ₹2,00,000 (per day) to buy gold and silver for the ornaments. He has to buy both gold and silver. He cannot buy more than 36 grams of gold. In addition to this, he has to buy a minimum of 300 grams of silver.

Also, there must be equal number of ornaments of both the types A and B. The total number of ornaments made per day is at least 10. Formulate this as an LPP with an objective of maximizing the profit on sale.

B – Simplex Method and Cases

Solve the following examples as per given the instructions.

21. Maximize $Z = 5x_1 + 7x_2$,
subject to the constraints $x_1 + x_2 \leq 4$,

$$3x_1 + 8x_2 \leq 24,$$
$$10x_1 + 7x_2 \leq 35, \text{ with } x_1, x_2 \geq 0$$

22. Maximize $z = x_1 + x_2 + x_3$
subject to the constraints $2x_1 + x_2 - x_3 \leq 2$,

$$-2x_1 + x_2 - 5x_3 \geq -6$$
$$4x_1 + x_2 + x_3 \leq 6,$$
$$x_1, x_2, x_3 \geq 0$$

23. Minimize $Z = x_1 + x_2$
subject to the constraints $2x_1 + x_2 \geq 4$,

$$x_1 + 7x_2 \geq 7,$$

with $x_1, x_2 \geq 0$

24. Maximize $Z = 4x_1 + 10\,x_2$
subject to the constraints $2x_1 + x_2 \le 50,$

$$2x_1 + 5x_2 \le 100$$
$$x_1 + 3x_2 \le 90,$$

with $x_1, x_2 \ge 0$

25. Maximize $Z = 2x_1 + x_2$
subject to the constraints $4x_1 + 3x_2 \le 12$, $4x_1 + x_2 \le 8,$

$$4x_1 - x_2 \le 8,$$

with $x_1, x_2 \ge 0$

26. Minimize $Z = x_1 + x_2 + x_3$
subject to the constraints $x_1 - 3x_2 + 4x_3 = 5,$

$$x_2 - 2x_3 \le 3,$$
$$2x_2 - x_3 \ge 4,$$

with $x_1, x_2, x_3 \ge 0$

27. Maximize $Z = 5x + 3y$
subject to the constraints $3x + 5y \le 30,$

$$5x + 2y \le 20$$
$$x + y \ge 13/2,$$

with $x, y \ge 0$

28. Maximize $Z = 8x_1 + 9x_2 + 5x_3$
subject to the constraints $x_1 + x_2 + 2x_3 \le 2$,

$$2x_1 + 3x_2 + 4x_3 \le 3,$$
$$6x_1 + 6x_2 + 2x_3 \le 8,$$

with $x_1, x_2, x_3 \ge 0$

29. Maximize $Z = 5x_1 + 3x_2$
subject to the constraints $2x_1 + 4x_2 \le 12,$

$$2x_1 + 2x_2 = 10,$$
$$5x_1 + 2x_2 \ge 10,$$

with $x_1, x_2 \ge 0$

30. Minimize $Z = 2x_1 + x_2$
subject to the constraints $3x_1 + x_2 = 3$, $4x_1 + 3x_2 \ge 6,$

$$x_1 + 2x_2 \le 3,$$

with $x_1, x_2 \ge 0$

31. Minimize $Z = -5x_1 - 3\,x_2$
subject to the constraints $x_1 + x_2 \le 2,$

$$5x_1 + 2x_2 \le 10,$$
$$3x_1 + 8x_2 \le 12,$$

with $x_1, x_2 \ge 0$

C – Examples on Different Cases in LPP

Examples on Multiple Optimal Solutions [Find at least two optimal solutions.]

32. Maximize $Z = 6x_1 + 4x_2$

subject to the constraints $2x_1 + 3x_2 \le 30$,

$$3x_1 + 2x_2 \le 24$$
$$x_1 + x_2 \ge 3,$$

with $x_1, x_2 \ge 0$

33. Maximize $Z = 6x_1 + 3x_2$

subject to the constraints $2x_1 + x_2 \le 8$,

$$3x_1 + 3x_2 \le 18,$$
$$x_2 \le 3,$$

with $x_1, x_2 \ge 0$

Examples on Unbounded Solution

34. Maximize $Z = 3x_1 + 5x_2$

subject to the constraints $x_1 - 2x_2 \le 6$,

$$x_1 \le 10,$$
$$x_2 \ge 1,$$

with $x_1, x_2 \ge 0$

35. Maximize $Z = 6x_1 - 4x_2$

subject to the constraints $2x_1 - x_2 \le 2$,

$$x_1 \le 4$$
$$x_1, x_2 \ge 0$$

Examples on Infeasible Solution

36. Maximize $Z = 5x_1 + 3x_2$

subject to the constraints $3x_1 + 4x_2 \le 12$,

$$2x_1 + 2x_2 \ge 10$$
$$x_2 \ge 4,$$

with $x_1, x_2 \ge 0$

37. Minimize $Z = 5x_1 + 6x_2$

subject to the constraints $2x_1 + 4x_2 \le 8$,

$$2x_1 + 2x_2 = 10,$$
$$x_1 \ge 5,$$

with $x_1, x_2 \ge 0$

Examples on Big–M method

38. Minimize $Z = 2x_1 + x_2$

subject to the constraints $3x_1 + x_2 = 3$,

$$4x_1 + 3x_2 \ge 6,$$
$$x_1 + 2x_2 \le 3,$$

with $x_1, x_2 \ge 0$

39. Maximize $Z = -x_1 - x_2 - x_3$
subject to the constraints $x_1 - 3x_2 + 4x_3 = 5$,
$x_2 - 2x_3 \le 3$,
$2x_2 - x_3 \ge 4$,
with $x_1, x_2, x_3 \ge 0$

Examples on Two – Phase method

40. Maximize $Z = -x_1 - 2x_2 - 3x_3$,
subject to $x_1 - x_2 + x_3 \ge 4$,
$x_1 + x_2 + 2x_3 \le 8$
$x_2 - x_3 \ge 2$,
with $x_1, x_2, x_3 \ge 0$

41. Minimize $Z = 2x_1 + 2x_2$
subject to the constraints $x_1 + 2x_2 \ge 1$,
$2x_1 + x_2 \ge 1$,
with $x_1, x_2 \ge 0$

Examples on Bounded Variables

42. Maximize $Z = 4x_1 + 5x_2$
subject to the constraints $2x_1 + 3x_2 \le 24$,
$x_1 + x_2 \le 5$,
with $x_1 \ge 3$, $x_2 \ge 1$.

43. Maximize $Z = 5x_1 + 3x_2$; subject to the constraints $2x_1 + 4x_2 \le 5$,
$x_1 + x_2 = 5$,
$5x_1 + 2x_2 \ge 10$,
with $x_1 \ge 2$, $x_2 \ge 1$

Examples on Degenerate Solution

44. Maximize $Z = 5x_1 + 3x_2$
subject to the constraints $x_1 + x_2 \le 2$,
$5x_1 + 2x_2 \le 10$
$3x_1 + 8x_2 \le 12$,
With $x_1, x_2 \ge 0$

45. Maximize $Z = 4x_1 + 3x_2$, $3x_1 + 2x_2 \le 12$, $2x_1 + 3x_2 \le 8$, $x_1, x_2 \ge 0$

Answers to Numerical Problems

1. X = k.g. of product A; Y = k.g. of product B. Minimize the total cost $Z = 30X + 25Y$ subject to $30X + 40Y \ge 120$; $40X + 20Y \le 80$; $X, Y \ge 0$
2. X = Number of Speed-up pills and Y = number of Go-well pills. Minimize $Z = 4X + 3Y$; Subject to $4X + 2Y \ge 10$; $½X + ½Y \le 2$; $X, Y \ge 0$
3. Let X_1, X_2, and X_3 Number of units be product for items A, B, and C Maximize $Z = 10X_1 + 20X_2 + 30X_3$; Subject to $X_1 \le 50$, $X_2 \le 25$, $X_3 \le 30$, $0.8X_1 + 0.6X_2 + 2X_3 \le 100$ Also, $X_1 \ge 45$; $X_2 + X_3 \ge 50$; With $X_1, X_2, X_3 \ge 0$

4. X_1 units of A, x_2 units of B x_3 units of C and x_4 units of wastage. Maximize $Z = 10x_1 + 20x_2 + 25x_3 - 2x_4$; subjected to $2x_1 + 4x_2 \leq 20$; $3x_1 + 2x_2 \leq 24$ sale constraint for product C; $x_3 \leq 6$. Production for C; $2x_2 = 1x_3 + x_4$. (Ratio is 1 unit of C; $x_2 / x_3 = 1/2$ $\therefore$ $2x_2 = x + x$ wastage) and $x_1, x_2, x_3, x_4 \geq 0$.
5. X_1, X_2, and X_3 items ordered in January, February, and March. y_1, y_2, and y_3 are sold in February, March, and April $\therefore$ total profit on sale, $Z = (80y_1 + 95y_2 + 90y_3) - (50x_1 + 70x_2 + 80x_3)$ subject to $y_1 \leq x_1 \leq 200$; $0 \leq (x_2 + x_1 - y_1) - (y_2) \leq 200$ (or $y_2 \leq x_1 + x_2 - y_1, \leq 200$); And $y_3 \leq (x_1 + x_2 + x_3) - (y_1 + y_2) \leq 200$; $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$; All x_1, x_2, x_3, y_1, y_2, and $y_3 \geq 0$.
6. Let X grams of food p and y grams of food Q be determined. The objective is to minimize the total cost $Z = 20x + 15y$ subject to $2x + 3y \geq 10$; $3x + 5y \geq 24$; $5x + 2y \geq 15$ with $x, y \geq 0$.
7. Pay X number o type P and y number of type Q shirts be manufactured per day. Profit on one type P shirt = 180 – (20 + 12 + 100) = 48, profit on one type Q shirt = 200 – (20 + 5 + 100) = 55, Maximize $Z = 48x + 55y$, subject to $x + y \leq 800$; $x \leq 500$; $y \geq 200$; $x, y \geq 0$.
8. Maximum at $x = 7/2$; $y = 3/2$; maximum value = 33/2.
9. Maximum at $x = 4$, $y = 2$, maximum value = 10.
10. Maximum at $x = 100$, $y = 700$, ; Maximum value = 44,000
11. Minimum at $x = 8$, $y = 0$; minimum value = 160
12. Infeasible solution.
13. Unbounded solution.
14. Multiple optimal solution ($x = 1$, $y = 7/2$) and (15/2, 1/4) and the line segment joining them points on gives the maximum solution = 24.
15. Maximum at $x = 5$, $y = 9/2$. Maximum value = 37.
16. Maximum at $x = 16/5$ and $y = 6/5$; maximum value = 20(16/5) + 25(6/5) = 94, [(16/5, 6/5) = $\left(3\frac{1}{5}, 1\frac{1}{5}\right)$ = (3.2, 1.2) integer value = (3, 1). Value of objective function is 20(3) + 25(1) = 85.]
17. Let x_1, x_2, x_3 and x_4 be the amount of investments in companies A, B, C, and D respectively. Maximize (total return) $Z = 1/100\ (8x_1 + 8.25x_2 + 9x_3 + 8x_4)$; subject to the constraints $x_1 + x_2 + x_3 + x_4 = 5{,}00{,}000$; $1{,}50{,}000 \leq x_3 + x_4 \leq 2{,}50{,}000$; $x_1 \leq 2{,}00{,}000$; $x_4 \geq 7{,}70{,}000$ with each $x_1, x_2, x_3, x_4 \geq 30{,}000$.
18. Maximum at $x_1 = 4$, $x_2 = 5$; maximum value = 26.
19. Maximum at $x = 4$, $y = 0$; maximum value = 20.
20. x_1 number of ornaments type A
x_2 number of ornaments type B
x_3 number of ornaments type C
be made to maximize the total profit on sale $Z = 2000x_1 + 4000x_2 + 3000x_3$ subject to the constraints $3000(5x_1 + 4x_2 + 3x_3) + 60(8x_1 + 10x_2 + 12x_3) = 2{,}00{,}000$;
$5x_1 + 4x_2 + 3x_3 \leq 36$ (grams gold); $8x_1 + 10x_2 + 12x_3 \geq 300$ (grams silver); $x_1 = x_2$;
$x_1 + x_2 + x_3 \geq 10$; $x_1, x_2, x_3 \geq 0$.
21. Basic variables: $x_1 = 8/5$, $x_2 = 12/5$, $s_3 = 11/5$;
non-basic variables: $s_1 = s_2 = 0$; Max $Z = 124/5$
22. Basic variables: $x_2 = 4$, $x_3 = 2$, $s_2 = 0$
non-basic variables: $x_1 = s_1 = s_3 = 0$, Max $Z = 6$
23. Basic variables: $x_1 = 21/13$, $x_2 = 10/13$
non-basic variables: $s_1 = s_2 = 0$, Max $Z = 31/13$
24. Basic variables: $x_2 = 20$, $s_1 = 30$, $s_3 = 30$;
non-basic variables: $x_2 = s_2 = 0$, Max $Z = 200$

25. Basic variables: $x_1 = 3/2$, $x_2 = 2$, $s_3 = 4$;
non-basic variables: $s_1 = s_2 = 0$, Max $Z = 5$

26. Basic variables: $x_1 = 19/5$, $x_3 = 77/16$, $s_1 = 37/5$;
non-basic variables: $A_1 = A_2 = s_2 = 0$; minimum $Z = 689/80$

27. Basic variables: $x = 7/3$, $y = 25/6$, $s_3 = 13/6$;
non-basic variables: $s_2 = s_3 = 0 = A$; max $Z = 145/6$

28. Basic variables: $x_1 = 1$, $x_2 = 1/3$, $s_1 = 2/3$;
non-basic variables: $s_2 = s_3 = 0$; max $Z = 11$

29. Basic variables: $x_1 = 3/5$, $x_2 = 6/5$, $s_2 = 12$;
non-basic variables: $s_1 = s_3 = 0 = A_1 = A_2$; max $Z = 12/5$

30. Basic variables: $x_1 = 8/5$, $x_2 = 6/5$,
non-basic variables: $s_1 = s_2 = 0$; max $Z = 12/5$

31. Basic variables: $x_1 = 2$, $x_2 = 0$, $s_3 = 6$;
non-basic variables: $s_1 = s_2 = 0$; max $Z = 10$

32. SET 1, basic variables: $x_1 = 8$; $x_2 = 0$
SET 2, basic variables : $x_1 = 12/5$, $x_2 = 42/5$;
Max $Z = 48$

33. SET1, basic variables: $x_1 = 4$, $x_2 = 0$
SET2, basic variables : $x_1 = 5/2$, $x_2 = 3$;
Max $Z = 24$

34. Case of unbounded solution

35. Case of unbounded solution
(Some variable wants to leave the basis but the outgoing variable cannot be decided.)

36. and **37.** Both have infeasible solution.
(ASV does not leave the basis though optimality has been reached.)

38. Basic variables: $x_1 = 8/5$, $x_2 = 6/5$, non-basic variables: $s_1 = s_2 = 0$;
Max $Z = 12/5$

39. Basic variables: $x_1 = 19/5$, $x_3 = 77/16$, $s_1 = 37/5$;
non-basic variables: $A_1 = A_2 = s_2 = 0$; Max $Z = -689/80$

40. Basic variables: $x_1 = 6$, $x_2 = 2$, $x_3 = 0$;
non-basic variables: $s_1 = s_2 = s_2 = 0$; Max $Z = -10$

41. Basic variables: $x_1 = 1/3$, $x_2 = 1/3$,
non-basic variables: $s_1 = s_2 = 0$; Minimum $Z = 4/3$

42. Basic variables: $x_1 = 3$, $x_2 = 2$, $s_1 = 12$; Maximum $Z = 22$

43. Basic variables: $x_1 = 4$, $x_2 = 1$,
non-basic variables = 0; Maximum $Z = 23$

44. Basic Variables: $x_1 = 2$, $x_2 = 0$ (basic variable = 0, degenerate solution),
$s_3 = 6$; non-basic variables: $s_1 = s_2 = 0$; Min $Z = -10$

45. Basic Variables : $x_1 = 4$, $x_2 = 0$,(Shows Degeneracy)
non-basic variables $s_1 = s_2 = 0$, Max $Z = 16$

2 Linear Programming–II

—God induces a ray of thought, the rests are details.

Learning Objectives

AFTER STUDYING THIS CHAPTER, THE STUDENTS WILL BE ABLE TO:

- understand primal and dual problems and interpretations
- learn and use the variations and fluctuations in the LP model—Sensitivity Analysis
- learn a special property of LPP—Complementary slackness
- understand and apply the methodology of the most realistic extension of Linear Programming, i.e. Goal Programming.
- study and enjoy ***Master Problem***.

INTRODUCTION

The contents in this chapter speak on the basic applications on what we have discussed in the first chapter. There are varieties of ideas and each variety has many areas of applications to linear programming. As one looks into the details, he finds something of his own, which others may have known until date.

The topic like ***primal and dual*** gives a real insight and is very important. It is a perfect and complete problem and right from purchase price determination of raw materials up to the maximum profit on final sale of all units of the production. It is an useful tool for decision making.

The unit ***complementary slackness*** is the screwing topic that brings out the hidden mathematical treasure and so it has remained very important to mathematicians pursuing pure mathematics.

On the top of all the topics and different units that we have studied and understood reasonably, the topic ***sensitivity analysis*** is like a central nervous system that keeps a track, reports, alerts, and orders accordingly. It is an important tool guarding interest of the problem in cases of various fluctuations—on every fluctuation in any part of the problem.

Useful to businessperson of any class; right away from small organization to big multinational corporate, extremely important to persons closely associated with ***operations***, is ***goal programming***—a classical programming technique suggesting the best options and optimal solutions satisfying different goals that may be simultaneous requirements in a given situation (problem).

At last, the ***Master Problem***, is very interesting and useful. It may open some unknown areas and change the basic mathematical format of reviewing the problem.

All the units discussed in this chapter are of prime importance and we emphasize that the students understand thoroughly and apply in real situations.

2.1 Primal and Dual

2.1(a) Duality: Introduction and Some Fundamentals

If we are given any problem of maximization, what we clearly understand and it is quite natural that maximization is possible only if something gets minimized. There is no existence of any single problem and that there is always a pair, one completely merged in the other. Though they are different problems in different variables, having different resources *m* and different objectives yet both look alike and are a single entity. Any problem has ***inherent*** features of its paired problem.

A maximization problem has an inbuilt problem of minimization and vice versa. Only the point is to extract the problem and interpret the variables, constraints, resources and finally the results. The results derived are very useful at the planning stage, purchase stage and operations stage, etc.

Let us try to observe every feature about the following two problems.

Illustration 1

Problem: A	*Problem: B*
Find x_1 and x_2 so as to	Find y_1 and y_2 so as to
maximize $2x_1 + 3x_2$	minimize $6y_1 + 9y_2$
subject to the constraints $4x_1 + 5x_2 \le 6$	subject to the constraints $4y_1 + 7y_2 \ge 2$
$7x_1 + 8x_2 \le 9$	$5y_1 + 8y_2 \ge 3$
with $x_1, x_2 \ge 0$.	with $y_1, y_2 \ge 0$.

Try to observe the coefficients and their arrangements in both the problems given; make a close study and write down the different results of your observations. This study and your work shall prove our asset in the near future.

Let us put these in standard format.

I		*II*	
Find $\mathbf{X} \in R_{nx1}$ so as to		Find $y \in R_{m \times 1}$ so as to	
maximize $Z = \mathbf{CX}$	(1)	minimize $Z' = \boldsymbol{b}^T\boldsymbol{y}$	(1)
subject to the constraints $Ax \le b$	(2)	subject to the constraints $\boldsymbol{A}^T\boldsymbol{y} \ge C^T$	(2)
with $x \ge 0$	(3)	With $\boldsymbol{y} \ge \mathbf{0}$	(3)
with **C**—A row vector $(1 \times n)$		With b^T—*A* related row vector	
$\mathbf{A} = (a_{ij})_{m \times n}$		A^T—Transpose of *A*	
B—A resource vector $(m \times 1)$		C^T—A related column	
0—A null vector $(n \times 1)$		**0**—A null vector	

These two are the inter-connected problems.

You may call **(I)** as a primal problem and its associated problem **(II)** as its dual problem.

(To clarify further, let us take the problem)

ILLUSTRATION 2

Primal Problem:

Find $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so as to maximize $Z = [2 \quad 3] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

subject to the constraints $\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 1 \\ 4 \\ 10 \end{bmatrix}$

with $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Its companion (dual) problem is:

Find $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ so as to minimize $\acute{Z} = [1 \quad 4 \quad 10] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

subject to the constraints $\begin{bmatrix} 2 & 4 & 5 \\ -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \ge \begin{bmatrix} 2 \\ 3 \end{bmatrix}$; $\quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Students are requested to go through about the type, the number of variables, number of constraints, and the method of putting down the dual of the given problem.

2.1(b) Facts

1. Each problem has its unique dual.
2. There are exactly two problems; we mean that the "Dual of any dual problem is a primal problem."
3. Of course, the meaning of the variable (what they stand for) and the resources of primal problem and the same for the dual problems are different. They have economic interpretation and we will discuss all these later on.

(The variables of dual problem are called ***shadow variables***.)

Now, it is the time to understand the details and characteristics about these two problems.

Before this, we clarify one very important point. To write the dual of any given problem, the given problem must be in the standard form, **i.e.** a maximization problem with all constraints of ≤ or = type. (Note: The very fact that we have included = type constraint in standard format has a special meaning or else ≤ inequality can also work equally well.)

We show here salient features about writing down the dual of a given problem.

(Note that you can consider any one of the problems on either side and the other side will be its dual problem.)

	Side I (Primal side or Dual side)	Side II (Dual side or Primal side)
(1)	Maximization	Minimization
(2)	Number of variables	Number of constraints
(3)	Number of constraints	Number of variables
(4)	Coefficients of objective function	Right-hand side of constraints
(5)	Right-hand side of constraints	Coefficients of objective function.
(6)	i^{th} constraint $\le$ type	i^{th} variable ≥ 0.
(7)	i^{th} constraint = type	i^{th} variable unrestricted.
(8)	j^{th} variable ≥ 0	j^{th} constraint $\ge$ type.
(9)	j^{th} variable unrestricted	j^{th} constraint = type.
(10)	coefficient $\boldsymbol{a_{ij}}$ of constraint	Coefficient $\boldsymbol{a_{ji}}$ of the corresponding constraint.

(Though we have said above yet we have clarified some points by duplicating them in the above points 1 to 10. In fact, we need not write the points (3), (5), (8) and (9) as they are merely the duplication. Yet – for the clarity we care.)

We substantiate, all that we have said in the above paragraphs, by examples.

2.1(c) Illustration and Analysis

ILLUSTRATION 3

Find x_1, x_2 and x_3 so as to maximize $Z = 3x_1 - 4x_2 + 6x_3$

subject to the constraints $2x_1 + 3x_2 - 5x_3 \le 7$

$-x_1 + x_2 + 3x_3 \ge 5$

$2x_1 + 4x_3 = 9$

$3x_1 + 2x_2 + x_3 \le 10$. with $x_2, x_3 \ge 0$ (x_1 is unrestricted in sign.)

Analysis

1. There are three variables. (So there will be ***three*** constraints in the dual.)
2. It is a ***maximization*** *problem*. (So the dual is of minimization.)
3. There are ***four*** constraints. (So the dual has four variables.)
4. About the types of constraints.
   (a) First and the fourth constraints are $\le$ type. (They are fine and we can use them.)
   (b) Second constraint is $\ge$ type. (We will write it as $1x_1 - 1x_2 - 3x_3 \le -5$ obtained by multiplying -1.)
   (c) Third constraint is = type. (This will do but in the dual the third variable will be unrestricted.)
5. First variable is unrestricted. (The first constraint of the dual will be with = sign.)
   We write the primal again to help us write its dual.

Primal

Find x_1, x_2 and x_3 so as to maximize $Z = 3x_1 - 4x_2 + 6x_3$

subject to the constraints $2x_1 + 3x_2 - 5x_3 \le 7$;

$1x_1 - 1x_2 - 3x_3 \le -5$;

$2x_1 + 0x_2 + 4x_3 = 9$;

$3x_1 + 2x_2 + 1x_3 \le 10$.

with x_2 and $x_3 \ge 0$ and x_1 being unrestricted in sign

Dual

Find y_1, y_2, y_3 and y_4, so as to minimize $Z^* = 7y_1 - 5y_2 + 9y_3 + 10y_4$

subject to the constraints $2y_1 + 1y_2 + 2y_3 + 3y_4 = 3$;

$3y_1 - 1y_2 + 0y_3 + 2y_4 \geq -4$;

$-5y_1 - 3y_2 + 4y_3 + 1y_4 \geq 6$;

with y_1, y_2, and $y_4 \geq 0$ and y_3 being unrestricted in sign.

[**Note:** The first constraint is = type and the third constraint is unrestricted in sign.].

We take one more illustration and discuss all the different aspects related to it.

2.1(d) All About Primal & Dual

ILLUSTRATION 4

A merchant has two types of tea—type A and type B. He makes two types of tea mix—regular type and special type. Each kilogram of regular type consumes 40% of type A and 60% of type B. While the special type requires 60% of type A and 40% of type B. The merchant has 48 kilograms of type A and 60 kilograms of type B. Regular type of tea fetches a profit of ₹24 per kg while special type fetches a profit of ₹18 kg. Perform the followings:

1. Make an LPP to earn maximum profit.
2. Find the graphical solution to the primal (given) problem.
3. Write the dual of the problem.
4. Find the graphical solution of the dual problem.
5. Find the simplex solution of the primal problem.
6. Find the simplex solution of the dual problem.
7. Compare the last simplex tables of both the problems.

Solution

1. Let x_1 units of regular type mix and x_2 units of special type mix be made from both the types A and B.

	Type A	*Type B*	*Profit*
Regular (1 kg)	0.4	0.6	24 ₹/kg
Special (1 kg)	0.6	0.4	18 ₹/kg
Available (kg)	48	60	

This allows us to write the primal problem as follows.

Primal

Find x_1 and x_2 so as to maximize $Z = 24x_1 + 18x_2$

subject to the constraints $0.4x_1 + 0.6x_2 \leq 48$

$0.6x_1 + 0.4x_2 \leq 60$

with $x_1, x_2 \geq 0$.

2. **Graphical Solution of Primal Problem**

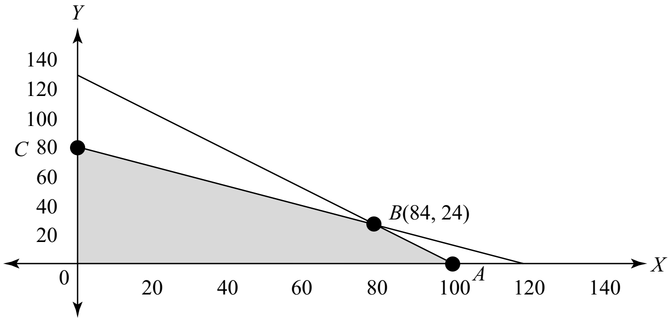


Fig. 2.1

Region – OABC

Vertex	*Value of Objective Function*
O (0, 0)	24(0) + 18(0) = 0.
A (100, 0)	24(100) + 18(0) = 2400.
B(84, 24)	24(84) + 18(24) = 2448. **
C (0, 80)	24(0) + 18(80) = 1440.

3. **The Dual Problem**

Find y_1 and y_2 to minimize $Z = 48y_1 + 60y_2$

subject to the constraints $0.4y_1 + 0.6y_2 \geq 24$

$0.6y_1 + 0.4y_2 \geq 18$

with $y_1, y_2 \geq 0$.

4. **Graphical Solution of Dual Problem**

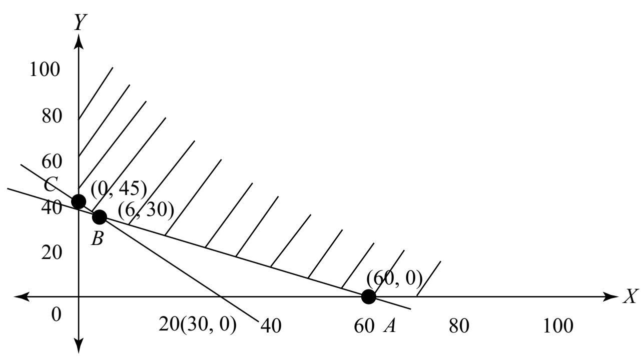


Fig. 2.2

Vertex	*Value of Objective Function*
A (60, 0)	48(60) + 60(0) = 2880.
B (6, 30)	48(6) + 60(36) = 2448. **
C(0, 45)	48(0) + 60(45) = 2700.

Optimal Solution:

Primal	*Dual*
Close Bounded Region	Unbounded Region
Maximum at (84, 24)	Minimum at (6, 36)
Maximum value = 2448	Minimum value = 2448

(5) Solving the Primal using Simplex Method

Adding slack variables s_1, $s_2 \geq 0$, simplex table and its iterations are as follows.

Maximize

Table 2.1

	$C_j \rightarrow$ VAR ↓ Basis	24 x_1	18 x_2	0 s_1	0 s_2	$\mathbf{X}_B$	R.R.
I	0 s_1	2/5	3/5	1	0	48	120
	0 $\leftarrow s_2$	3/5	2/5	0	1	60	100 →
	$C_j - Z_j$	24↑	18	0	0	$Z = 0$	
II	0 $\leftarrow s_1$	0	1/3	1	–2/3	8	24 →
	24 x_1	1	2/3	0	5/3	100	150
	$C_j - Z_j$	0	↑2	0	–40	2400	
III	18 x_2	0	1	3	–2	24	
	24 x_1	1	0	–2	3	84	
	$C_j - Z_j$	0	0	–6	–36	Z= 2448	

All entries of $C_j - Z_j$ are ≤ 0.

∴ This is an optimal stage,

Basic variables: $x_1 = 84$; $x_2 = 24$;

Non-basic variables: $s_1 = 0 = s_2$;

$C_j - Z_j$ entries for non-basic variable s_1 is –6 and that for s_2 is –36.

Maximum: 2448.

(6) Solving the Dual using Simplex Method

Adding surplus and ASV (s_1, s_2 and A_1, $A_2 \geq 0$) in constraints of the dual; the corresponding simplex table is:

Minimize

Table 2.2

	$C_j \rightarrow$ VAR ↓ Basis	48 y_1	60 y_2	0 s_1	M A_1	0 s_2	M A_2	$\mathbf{X}_B$	R.R.
I	M A_1	2/5	3/5	–1	1	0	0	24	60
	M A_2	3/5	2/5	0	0	–1	1	18	30 →
	$C_j - Z_j$	(48–M)↑	60–M	M	0	M	0	42M	
II	M$\leftarrow A_1$	0	1/3	–1	1	2/3	–2/3	12	18 →
	48 y_1	1	2/3	0	0	–5/3	5/3	30	–
	$C_j - Z_j$	0	$28 - \frac{1}{3}$M	M	0	$80 - \frac{2}{3}$M↑	$\frac{5}{3}$M–80	1440+12M	
III	0 $\leftarrow S_2$	0	1/2	–3/2	3/2	1	–1	18	36 →
	48 y_1	1	3/2	–5/2	5/2	0	0	60	40
	$C_j - Z_j$	0	–12↑	120	M–120	0	M	2880	
IV	60 y_2	0	1	–3	3	2	–2	36	
	48 y_1	1	0	2	–2	–3	3	6	
	$C_j - Z_j$	0	0	84	M–84	24	M–24	2448	

Since all entries of $C_j - Z_j$ are ≥ 0, so it is an optimal stage.
Basic variables: $y_1 = 6$; $y_2 = 36$;
Non-basic variables: $s_1 = 0 = s_2 = A_1 = A_2$;
$C_j - Z_j$ entries for non-basic variables s_1 is 84 and that of s_2 is 24.
Minimum z 2448.

(7) Comparison of Optimal Tables
A: For a primal problem of ***maximization*,**
Basic variables $x_1 = 84^*$ and $x_2 = 24^{**}$
$C_j - Z_j$ entries of non basic variables $s_1 = -6$ and $s_2 = -36$
Maximum value = 2448.
B: For a Dual problem of ***Minimization***,
Basic variable: y_1=6; y_2 =36;
$C_j - Z_j$ entries of corresponding non-basic variables $s_1 = 84^*$ and $s_2 = 24^{**}$ Minimum value = 2448.
Basic variables of primal problem becomes the corresponding non-basic variables in the dual problem—they appear in $\mathbf{C_j - Z_j}$ entries and vice versa.
Also, final/optimal values of both problems are the same.

2.1(e) Economic Interpretation of shadow variables

We know that the ***primal*** and its ***dual*** are non-separable problems but it is very interesting to know and understand the meaning—rather what they represent—of the variables.

Let us take an example.

ILLUSTRATION 5

A merchant decides to make tables and chairs. The raw materials are timber wood and man–hours. Available resources are 400 cubic units of wood and a team of 10 unskilled workers, each working for 9 hours a day for 5 days. A table needs 5 units of wood and 10 man–hours for the entire process upto the showroom where it can fetch a profit of ₹45 per unit. A chair consumes 20 units of wood and 15 man–hours and earns a profit of ₹80 per unit.

Plan the schedule to earn a maximum profit.

Solution

Let x_1 number of tables and x_2 number of chairs be made. The objective function is maximize $Z = 45x_1 + 80x_2$;
subject to the constraints $5x_1 + 20x_2 \leq 400$ (available units of timber)
and $10x_1 + 15x_2 \leq 450$ (available man–hours)
(10 men $\times$ 9 hours $\times$ 5 days = 450 man–hours) with $x_1, x_2 \geq 0$.

The merchant thinks of optimal production plan as described above.

Simultaneously, he thinks that what he should pay for additional units of resources. Answer to the second question lies in examining the values of the **dual** variables. In order to make tables and chairs timber-units of wood and man-hours to work upon are required and we call them resources.

Planning for Dual

Let y_1 and y_2 be the cost/worth per unit of these resources. Now, we write the **dual** of this problem and understand the details.

Find y_1 and y_2 to minimize

$$Z = 400y_1 + 450y_2;$$

subject to the constraints $5y_1 + 10y_2 \geq 45$; and
$20y_1 + 15y_2 \geq 80$; with y_1 and $y_2 \geq 0$.

The left–hand side of the first inequality means the ***worth of resources used to make (a) table***, i.e. $5y_1 + 10y_2$.

The merchant assesses that whether it is greater than or equal to 45.

Similarly $20y_1 + 15y_2$ is the worth of utilization of resources to make one chair. Is it greater than or equal to 80?

The objective function $Z^* = 400y_1 + 450y_2$ is the ***total worth*** of resources utilized and minimization plan means that the merchant is not overestimating (or underestimating) the value of resources.

It is interesting to note that the worth of resources utilized for a table, i.e. $5y_1 + 10y_2$ and if it is greater than 45, the production must ***not*** be done. If $5y_1 + 10y_2 = 45$, the production may be continued. Similarly we can interpret for the worth of resources for the chair.

2.1(f) Important Properties of Solution (Primal and Dual Problems)

Let the primal problem be find $\mathbf{Y} \in R_{n\text{x}1}$ so as to;

Maximize $\quad Z = \mathbf{CX}$ (1)

subject to the constraint $\quad \mathrm{A}\mathbf{X} \le \mathbf{b}$ (2)

$\mathbf{X} \ge \mathbf{0}$ (3)

Let $\mathbf{X}_0$ be a feasible solution.

The dual of the above problem is find $Y \in R_{m\text{x}1}$ so as to;

Maximize $\quad Z^* = \mathbf{b}^{\mathrm{T}}\mathbf{Y}$ (4)

subject to the constraints $\quad \mathrm{A}^{\mathrm{T}}\mathbf{Y} \ge \mathbf{C}^{\mathrm{T}}$ (5)

with $\quad \mathbf{Y} \ge \mathbf{0}$ (6)

Let $\mathbf{Y}_0$ be its feasible solution.

In context to the above problem, we have some properties.

1. $\mathbf{CX}_0 \le \mathbf{b}^{\mathrm{T}}\mathbf{Y}_0$
2. If the primal problem has an ***unbounded optimal*** solution, the dual problem has ***no feasible*** solution.
3. If $\mathbf{X}_0$ be an optimal solution of the primal problem and $\mathbf{y}_0$ be an optimal solution of the dual problem, $\mathbf{CX}_0 \le \mathbf{b}^{\mathrm{T}}\mathbf{Y}_0$
4. It is also possible that neither the primal problem nor its dual will have a feasible solution. For the above cases (3) and (4) we will cite examples.
5. ***Important Deduction*:** This corresponds to optimal table of **primal** and its **dual** problem.
   (A) Basic variables of {whose value appear in X_B (or RHS column} primal problem corresponds to respective non-basic variables of the dual problem. $C_j - Z_j$ entries of these corresponding non-basic variables are same as these corresponding basic variables of primal problem.
   (B) $C_j - Z_j$ entries of the non-basic variable of primal problem (Maximization type) are ≤ 0 and they, with the change in sign, appear in X_B (or RHS) to their corresponding basic variables of the dual problem.

   To clarify the above–mentioned two points; we take an illustration.

ILLUSTRATION 6 (Understand the relationship of variables)

Primal Problem

Find x_1 and x_2 so as to maximize $z = 24x_1 + 18x_2$

subject to the constraints $0.4x_1 + 0.6x_2 \le 48$

$0.6x_1 + 0.4x_2 \le 60$

with $\quad x_1, x_2 \ge 0$

It has an optimal table as follows.

Table 2.3

$C_j \rightarrow$ VAR / $\downarrow$ Basis	24 x_1	18 x_2	0 s_1	0 s_2	X_B	
18 x_2	0	1	3	–2	24	
24 x_1	1	0	–2	3	84	
$C_j - Z_j$	0	0	–6	–36	2448	

Basic variables: $x_1 = 84$; $x_2 = 24$
Non-variables: $s_1 = 0$; $s_2 = 0$.

Dual Problem

Find y_1 and y_2 so as to mnimize $Z^* = 48y_1 + 60y_2$
subject to the constraints
$0.4y_1 + 0.6y_2 \leq 24$; $0.6y_1 + 0.4y_2 \leq 18$
with $y_1, y_2 \geq 0$
It has an optimal table as follows.

Table 2.4

$C_j \rightarrow$VAR / $\downarrow$Basis	24 y_1	18 y_2	0 s_1	0 s_2	X_B	
60 y_2	0	1	–3	2	36	
48 y_1	1	0	2	–3	6	
$C_j - Z_j$	0	0	84	28	2448	

C: x_1, the first basic variable of primal = 84; this corresponds to the first non-basic variable s_1 of the dual $C_j - Z_j$ entry of s_1 in the **dual = 84**.

D: x_2, the second basic variable of primal problem = 24; this corresponds to the second non-basic variable s_2 of the dual $C_j - Z_j$ entry of s_2 in the dual = 84.

E: The first non-basic variable of primal is s_1 whose $C_j - Z_j$ entry is = –6; this will correspond to the first basic variable y_1 of the dual whose value from X_B column is = 6 (change in sign)

F: The second non-basic variable of the primal is s_2 whose $C_j - Z_j$ entry in the optimal table of primal = –36. With a change in sign, it is the value of the second basic variable y_2 of the dual which appears in the optimal table of the dual.

$\therefore y_2 = 36$.

2.1(g) Complementary Slackness

This is a very important and useful property. It is obviously a fundamental property and can be easily understood on the core principles of the above property (5).

In simple language, it is as follows.

If a slack variable (Say s_i) corresponding to an i^{th} constraint of the primal problem be a ***basic variable*** with positive value (form RHS) in the optimal table, its corresponding i^{th} ***non-basic variable*** of the dual has zero value.

For example, s_1—The first slack variable of the first constraint of the primal appears as a basic variable (in the optimal table) with a positive value, and its corresponding non-basic variable will be y_1, as it is a non-basic of the dual, we have $y_1 = 0$.

In fact, it is the extension of the Property (5). and is called *complementary slackness*.

We make this clear by taking an illustration.

ILLUSTRATION 7

Maximize $Z = 8x_1 + 2x_2$

subject to the constraints $x_1 + x_2 \leq 2$

$2x_1 + 6x_2 \leq 12$

$x_1 + 2x_2 \leq 3$

with $x_1, x_2 \geq 0$

Maximize

Table: 2.5

	$C_j \rightarrow$ VAR / $\downarrow$ Basis	8 / x_1	2 / x_2	0 / s_1	0 / s_2	0 / s_3	X_B	
I	0 s_1	1	1	1	0	0	2	2→
	0 s_2	2	6	0	1	0	12	6
	0 s_3	1	2	0	0	1	3	3
	$C_j - Z_j$	**8↑**	**2**	**0**	**0**	**0**	**$Z = 0$**	

Table: 2.6

	$C_j \rightarrow$ VAR / $\downarrow$ Basis	8 / x_1	2 / x_2	0 / s_1	0 / s_2	0 / s_3	X_B	
II	8 x_1	1	1	1	0	0	2	
	0 s_2	0	4	–2	1	0	8	
	0 s_3	0	1	–1	0	1	1	
	$C_j - Z_j$	0	–6	–8	0	0	$Z = 16$	

Basic variables: $x_1 = 2$, $s_2 = 8$, $s_3 = 1$

non-basic variables: $x_2 = 0$, $s_1 = 0$

Maximum value = 16

Now, we write the dual of the above problem and write its optimal table.

ILLUSTRATION 8

Find y_1, y_2, and y_3 to minimize $Z = 2y_1 + 12y_2 + 3y_3$

subject to the constraints $y_1 + 2y_2 + y_3 \geq 8$

$y_1 + 6y_2 + 2y_3 \geq 2$

with y_1, y_2, and $y_3 \geq 0$

The optimal table of the dual problem is as follows.

Minimize

Table 2.7

$C_j \rightarrow$ var / $\downarrow$ Basis	2 / y_1	12 / y_2	3 / y_3	0 / s_1	M / A_1	0 / s_2	M / A_2	X_B
0 s_2	0	–4	–1	–1	1	1	–1	6
2 y_1	1	2	1	–1	1	0	0	8
$C_j - Z_j$	0	8	1	2	$M - 2$	0	M	$Z = 16$

Now, we can compare the optimal tables.

It is seen from the optimal table of the primal table **that the slack variables s_2 and s_3 corresponding to the second and the third constraints are basic variables and non-zero also. ($s_2 = 8$ and $s_3 = 1$)**

These two basic variables will correspond to non-basic variables y_2 and y_3 and so they do not appear in the basis of the optimal tables of the dual.
So, we have $y_2 = y_3 = 0$
This establishes the complementary slackness.

2.2 SENSITIVITY ANALYSIS

2.2(a) Introduction

To formulate some real life problems, to solve them and implement the results obtained to test the feasibility, is the working order of the operations research. But on implementation the system may not react as theory expects; it may deviate in number of features. We may have the following types of fluctuations in the mathematical model of the problem and one or more types of such changes may or may not affect the optimal solution. Even sometimes, we may be interested to study the changes that could likely take place due to the changes in certain parameters.

Such study may enable us to pre-plan before actual operations are carried out.

In this unit, we shall study the impact of changes in

1. changes in the resource side/right-hand side
2. changes in the coefficients of the objective function
3. addition of a new constraint
4. addition of a new variable

Without involving into intricacy of mathematics, we would like to discuss the effects of each of the above-mentioned points. Just to be on the practical aspects, we begin with an example and try to incorporate the above points one by one.

ILLUSTRATION 9

Find x_1 and x_2 so as to maximize $Z = 24x_1 + 18x_2$ (1)

subject to the constraints $4x_1 + 6x_2 \leq 24$ (2)

$6x_1 + 3x_2 \leq 18$ (3)

with $x_1, x_2 \geq 0$.

Solution

Adding s_1 and $s_2 \geq 0$ as slack variables, we write down the simplex table.

Maximize

Table 2.8

	$C_j \rightarrow$ VAR / $\downarrow$ Basis	24 x_1	18 x_2	0 s_1	0 s_2	RHS	R.R.
I	0 s_1	4	6	1	0	24	6
	0 ← s_2	6	3	0	1	18	3→
	$C_j - Z_j$	24↑	18	0	0	$Z = 0$	
II	0 ← s_1	0	$\boxed{4}$	1	–2/3	12	3→
	24 x_1	1	$\boxed{1/2}$	0	1/6	3	6
	$C_j - Z_j$	0	↑6	0	4	72	
III	18 x_2	0	1	1/4	–1/6	3	
	24 x_1	1	0	–1/8	1/4	3/2	
	$C_j - Z_j$	0	0	–3/2	–3	$Z = 90$	

Since all entries of $C_j - Z_j$ are ≤ 0 and so the solution is optimal.
Basic variables: $x_1 = 3/2$, $x_2 = 3$
non-basic variables: $s_1 = 0 = s_2$.
Maximum value = 90.

2.2(b) Change in Resource Side

Case 1: Let the resource side of the problem, **i.e.** 24 and 18 be changed to 18 and 16.
Case 2: Let the first resource 24 be changed to 8 and the second does not change.
The final effect of this changes are considered as follows

$$\begin{bmatrix}\text{Basic variables in}\\ \text{the optical table}\end{bmatrix} = \begin{bmatrix}\text{Matrix corresponding to original}\\ \text{basic variables in the optimal table}\end{bmatrix} \cdot \begin{bmatrix}\text{New set}\\ \text{of resources}\end{bmatrix}$$

Apply this to **Case: 1,** i.e.

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} +1/4 & -1/6 \\ -1/8 & +1/4 \end{bmatrix} \begin{bmatrix} 18 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 11/6 \\ 7/4 \end{bmatrix}$$

i.e. new solution is $x_1 = 11/4$ and $x_2 = 11/6$ corresponding to this, the maximum value
= 24 (7/4) + 18(11/6)
= 42 + 33
= 75.

Case 2: Variables in the optimal table = $\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$ matrix in the optimal table which corresponding to the basic variable of the initial table = $\begin{bmatrix} +1/4 & -1/6 \\ -1/8 & +1/4 \end{bmatrix}$ and new set of resources = $\begin{bmatrix} 8 \\ 18 \end{bmatrix}$

$$\therefore \qquad \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} +1/4 & -1/6 \\ -1/8 & +1/4 \end{bmatrix} \begin{bmatrix} 8 \\ 18 \end{bmatrix} = \begin{bmatrix} -1 \\ 7/2 \end{bmatrix}$$

At this stage $x_2 = -1$ and $x_1 = 7/2$, the value of $x_2 = -1$ violate the condition that $x_2 \geq 0$. To get the feasible solution, we have to apply dual simplex method on the optimal table.

Table: 2.9

$C_j \rightarrow$ VAR / $\downarrow$ Basis	24 x_1	18 x_2	0 s_1	0 s_2	X_B
18 x_2	0	1	1/4	(–1/6)	–1→
24 x_1	1	0	–1/8	1/4	7/2
$C_j - Z_j$	0	0	–3/2	–1↑	66

Applying dual simplex method, we have $x_2 = -1$ as an outgoing variable. Also, there is only one negative entry in the row of x_2 and hence s_2 is an incoming variable. At this stage, –1/6 is the pivot and the operation gives the following result.

Table 2.10

$C_j \rightarrow$ VAR ↓ Basis	24 x_1	18 x_2	0 s_1	0 s_2	X_B
0 s_2	0	–6	–3/2	1	6
24 x_1	1	3/2	1/4	0	2
$C_j - Z_j$	0	–18	–6	0	$Z = 48$

All $C_j - Z_j \leq 0$. We have reached optimality.
Basic variables: $x_1 = 2$ and $s_2 = 6$
Non-basic variables: $s_1 = 0 = x_2$
Optimal $Z = 48$.
Two different types of effects can be seen on changing the components of original resource vector.

2.2(c) Changes in the co-efficient of objective function:

Sometimes during ongoing system set up, the cost coefficient (cost factor) changes.

We want to study its impact on the final solution. We have two different cases.

Case 1: A cost coefficient corresponding to a basic variable changes.

In this case we find the range within which the new cost coefficient, if remains, will not change the optimality.

Case 2: A cost coefficient corresponding to a non-basic variable changes, We take an example and clarify the consequences on changing the cost-coefficient.

We consider an example and understand the effect of changes, Case 1 and Case 2, on the optimal solution.

ILLUSTRATION 10

Maximize $Z = 4x_1 + 5x_2 + 2x_3$
subject to the constraints $2x_1 + x_2 + x_3 \leq 10$
$1x_1 + 3x_2 + 1x_3 \leq 12$
with $x_1, x_2 \geq 0$.

Solution

We introduce slack variables $s_1 \geq 0$ and $s_2 \geq 0$, and solve it using simplex algorithm. The optimal table is as follows.

Maximize:

Table 2.11

$C_j \rightarrow$ VAR ↓ Basis	4 x_1	5 x_2	2 x_3	0 s_1	0 s_2	X_B
4 x_1	1	0	2/5	3/5	–1/5	18/5
5 x_2	0	1	1/5	–1/5	2/5	14/5
Z_j	4	5	13/5	7/5	6/5	
$C_j - Z_j$	0	0	–3/5	–7/5	–6/5	$Z = 142/5$

Case 1: Cost coefficient corresponding to a basic variable changes.

Let $C_2 = 5$ change $C'_2 = 5 + \Delta$. This means $C_j - Z_j$ corresponding to the non-basic variables (x_3, s_1 and s_2) will change (Since $C_j - Z_j$ for basic variables is always zero.). If the final effect of this change makes any one or more $C_j - Z_j \geq 0$, then the corresponding variable becomes an incoming variable. So long as $C_j - Z_j$ for non-basic variables remain negative; we are through.

Now let us find $C_j - Z_j$ for non-basic variables.

$$
\begin{aligned}
C_j - Z_j \text{ for } x_3 &= 2 - (4 \quad 5 + \Delta\cdot) \cdot \begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix} \\
&= 2 - (8/5 + 1 + \Delta/5) \\
&= 2 - 13/5 - \Delta/5 \\
C_3 - Z_3 &= -3/5 - \Delta/5.
\end{aligned}
$$

$$
\begin{aligned}
C_j - Z_j \text{ for } s_1 &= 0 - (4 \quad 5 + \Delta) \cdot \begin{pmatrix} +3/5 \\ -1/5 \end{pmatrix} \\
&= 0 - (12/5 - 1 - \Delta/5) \\
C_4 - Z_4 &= -7/5 + \Delta/5.
\end{aligned}
$$

$$
\begin{aligned}
C_j - Z_j \text{ for } s_2 &= 0 - (4 \quad 5 + \Delta) \cdot \begin{pmatrix} -1/5 \\ +2/5 \end{pmatrix} \\
&= 0 - (-4/5 + 2 + 2\Delta/5) \\
C_5 - Z_5 &= -6/5 - 2\Delta/5.
\end{aligned}
$$

We wish, for optimality to be preserved, $C_j - Z_j$ for non-basic variables to be ≤ 0, **i.e.**

$$C_3 - Z_3 = -3/5 - \Delta/5 \leq 0 \Rightarrow -3 - \Delta \leq 0 \;\Rightarrow\; -3 \leq \Delta.$$

$$C_4 - Z_4 = -7/5 + \Delta/5 \leq 0 \Rightarrow \Delta/5 \leq 7/5 \Rightarrow 7 \geq \Delta.$$

$$C_5 - Z_5 = -6/5 - 2\Delta/5 \leq 0 \Rightarrow -6 - 2\Delta \leq 0 \Rightarrow -3 \leq \Delta.$$

Combining the above facts for Δ, we have $-3 \leq \Delta \leq 7$.

i.e. for C_2 to be $C_2 + \Delta$ ($C'_2 = 5 + \Delta$)

i.e. $-3 + 5 \leq \Delta + 5 \leq 7 + 5$

i.e. $2 \leq C'_2 \leq 12$

i.e. Δ can have any value in the interval in $[-3, 7]$ or new value of $C_2 = 5$ can we have any value so that $2 \leq C'_2 \leq 12$, will not change the value of the basic variables, i.e. $x_1 = 18/5$ and $x_2 = 14/5$. (Please note that the optimal value of Z changes depending on Δ; i.e. $5 + \Delta$)

Case 2: Cost coefficient of a non-basic variable change.

It is simple as it concerns with $C_j - Z_j$ entry which must be ≤ 0. (for a maximization problem)

The logic is the change in C_j for the non-basic variable is allowed till it keeps the corresponding $C_j - Z_j \leq 0$. (for optimal values of basic variables)

Once $C_j - Z_j > 0$ then that non-basic variable will, in the next iteration, be a basic variable, and the table changes correspondingly and hence everything!

In our case, x_3 is a non-basic variable and its cost factor $C_3 = 2$, let $C'_3 = 2 + \Delta$

Now $C'_3 - Z_3 = (2 + \Delta) - (13/5) = -3/5 + \Delta \leq 0.$

This implies that $\Delta \leq 3/5$ i.e. C'_3 can be at the most $C_3 + \Delta = 2 + 3/5 = 13/5$.

Any value of $C_3 > 13/5$ will make $C_3 - Z_3 > 0$ and then x_3 becomes a basic variable, i.e. an increment up to an amount 3/5 in the cost factor $C_3 = 2$ will not change the optimality of the variables and hence the optimal value also.

2.2 (d) Adding a New Constraint

In real life situations, as the time passes, certain changes are bound to come. Looking at the constraints, we may like to add a new constraint to allow us to keep up the optimality.

By introducing a new constraints there are two possible cases.

Case 1: If the new constraint ***satisfies*** the values of the basic variables, then it is not necessary to include that new constraint, i.e. it becomes redundant.

Case 2: If the new constraint does not satisfy the values of the basic variables of the optimal table, then it requires due attention to be paid. We have to introduce the new constraint (with proper change) in the optimal table in a way that it does not change the current values of the basic variables. We again ***rewrite*** the Illustration 6 with its final table.

ILLUSTRATION 11

Find x_1 and x_2 so as to maximize $Z = 24x_1 + 18x_2$ (1)

subject to the constraints $4x_1 + 6x_2 \le 24$ (2)

$6x_1 + 3x_2 \le 18$ (3)

with $x_1, x_2 \ge 0$.

Solution

Adding s_1 and $s_2 \ge 0$ (as slack variables, we write the simplex table 2.12).

Maximize

Table 2.12

	$C_j \rightarrow$ VAR / $\downarrow$Basis	24 x_1	18 x_2	0 s_1	0 s_2	RHS	R.R.
I	0 s_1	4	6	1	0	24	6
	0 $\leftarrow s_2$	6	3	0	1	18	3$\rightarrow$
	$C_j - Z_j$	24$\uparrow$	18	0	0	$Z = 0$	
II	0 $\leftarrow s_1$	0	4	1	–2/3	12	3$\rightarrow$
	24 x_1	1	1/2	0	1/6	3	6
	$C_j - Z_j$	0	$\uparrow$6	0	4	$Z = 72$	
III	18 x_2	0	1	1/4	–1/6	3	
	24 x_1	1	0	–1/8	1/4	3/2	
	$C_j - Z_j$	0	0	–3/2	–3	$Z = 90$	

Since all entries of $C_j - Z_j$ are ≤ 0, so the solution is optimal.
Basic variables: $x_1 = 3/2$, $x_2 = 3$
Non-basic variables: $s_1 = 0 = s_2$.
Maximum value = 90.

Case 1: Now we add a new constraint, say $2x_1 + 3x_2 \le 13$ with optimal $x_1 = 3/2$ and $x_2 = 3$; Putting these values in the new constraint, we have

$$2(3/2) + 3(3) = 12 \le 13$$

These values satisfy new constraint. This means that if we write $2x_1 + 3x_2 \le 13$, i.e. any value, which is more than 12, then it is fine and there is no need for updating. This deals with **Case 1**.

Case 2: Let us introduce a new constraint $2x_1 + 5x_2 \leq 10$; now $x_1 = 3/2$; $x_2 = 3$ gives $2x_1 + 5x_2 = 18$ which does not satisfy the constraint.

Now from the optimal table $1x_2 + \frac{1}{4}s_1 - \frac{1}{6}s_2 = 3$

and

$$1x_1 - \frac{1}{8}s_1 + \frac{1}{4}s_2 = 3/2.$$

The new constraint, before introducing in the final table, is $2x_1 + 5x_2 + s_3 = 10$; $s_3 \geq 0$ being a slack variable.

If this is to be introduced in the optimal table, then it has to be made ready with necessary changes.

From the above written two equations $x_2 = 3 - \frac{1}{4}s_1 + \frac{1}{6}s_2$ and $x_1 = \frac{3}{2} + \frac{1}{8}s_1 - \frac{1}{4}s_2$; then

$2x_1 + 5x_2 + s_3 = 10$ becomes $2\left(\frac{3}{2} + \frac{1}{8}s_1 - \frac{1}{4}s_2\right) + 5\left(3 - \frac{1}{4}s_1 - \frac{1}{6}s_2\right) + s_3 = 10$

on simplification it is $-s_1 + \frac{1}{3}s_2 + s_3 = -8$.

(Note that s_3 is a basic variable. Do not try to change the sign.)

When introduced in the optimal table, it takes the new form as follows (Table 2.13).

Maximize

Table 2.13

$C_j \rightarrow$ VAR / $\downarrow$Basis	24 x_1	18 x_2	0 s_1	0 s_2	0 s_3	X_B
18 x_2	0	1	1/4	–1/6	0	3
24 x_1	1	0	–1/8	1/4	0	3/2
0 s_3	0	0	–1	1/3	1	–8 →
$C_j - Z_j$	0	0	↑–3/2	–3	0	

In this table $s_3 = -8$, i.e., the right-hand side is negative and hence it calls for **dual simplex method**.

Solution by dual simplex method

s_3 is an outgoing variable and s_1 is an incoming variable (Since there is only one negative entry in the last row.) and so –1 is a **pivot**. We perform the simplex procedure and write the resulting table as follows (Table 2.14).

Maximize:

Table 2.14

$C_j \rightarrow$ VAR / $\downarrow$Basis	24 x_1	18 x_2	0 s_1	0 s_2	0 s_3	X_B
18 x_2	0	1	0	–1/12	1/4	1
24 x_1	1	0	0	–1/6	–1/8	5/2
0 s_1	0	0	1	–1/3	–1	8
$C_j - Z_j$	0	0	0	–5/2	–3/2	78

Since all $C_j - Z_j \leq 0$, so the optimal value is $Z = 78$.

Basic variables: $x_1 = 5/2$; $x_2 = 1$; $s_1 = 8$.

Non-basic variables: $s_2 = 0 = s_3$.

It explains the effect of introducing a new constraint.

2.2(e) Adding a New Variable

In real life situation when the need of the market calls for change—a change in size, specifications, then over and above of continuing with the old product, one may wish to introduce a new product. It requires, for new set-up, new conditions, and new environment. We have to, before implementing or introducing a new variable, justify possible changes and equip the existing set-up, accordingly. We want to study the structural changes in the table. We introduce a new variable x_3 in the **Illustration 6** which is as follows.

Find x_1 and x_2 so as to maximize $Z = 24x_1 + 18x_2$ (1)

subject to the constraints $4x_1 + 6x_2 \le 24$ (2)

$6x_1 + 3x_2 \le 18$ (3)

with $x_1, x_2 \ge 0.$

We write the new problem, by introducing a new variable $x_{3,}$ as follows.

ILLUSTRATION 12

Maximize $Z = 24x_1 + 18x_2 + C_3x_3$ (x_3 is a new product and its profit factor is C_3)

subject to the constraints $4x_1 + 6x_2 + 1x_3 \le 24$

$6x_1 + 3x_2 + 2x_3 \le 18$

with $x_1, x_2, x_3 \ge 0$

Now we observe our original problem and study the effect of x_3.

Find x_1 and x_2 so as to maximize $Z = 24x_1 + 18x_2$ (1)

subject to the constraints $0.4x_1 + 6x_2 \le 24$ (2)

$6x_1 + 3x_2 \le 18$ (3)

with $x_1, x_2 \ge 0.$

Solution

Adding s_1 and $s_2 \ge 0$ as slack variables, we write the simplex table (Table 2.15).

Maximize:

Table 2.15

	$C_j \to$ VAR / $\downarrow$Basis	24 x_1	18 x_2	0 s_1	0 s_2	RHS	R.R.
I	0 s_1	4	6	1	0	24	6
	$0 \leftarrow s_2$	6	3	0	1	18	$3 \rightarrow$
	$C_j - Z_j$	$24\uparrow$	18	0	0	$Z = 0$	
II	$0 \leftarrow s_1$	0	4	1	–2/3	12	$3 \rightarrow$
	24 x_1	1	1/2	0	1/6	3	6
	$C_j - Z_j$	0	$\uparrow 6$	0	4	$Z = 72$	
III	18 x_2	0	1	1/4	–1/6	3	
	24 x_1	1	0	–1/8	1/4	3/2	
	$C_j - Z_j$	0	0	–3/2	–3	$Z = 90$	

The study shows

basic variables $x_1 = 3/2$; $x_2 = 3$.

non-basic variables $s_1 = 0 = s_2$; with maximum value = 90.

In addition to this, the basic variables of the dual problem (as derived from the optimal table $y_1 = 3/2$ and $y_2 = 3$. Basic variables of the dual problem are the $C_j - Z_j$ of the corresponding non-basic variables with a change in sign.)

Now, we write the third constraint of the corresponding dual problem.

The corresponding constraint of the dual is $\mathbf{1y_1 + 2y_2 \geq C_3}$; with $y_1 = 3/2$ and $y_2 = 3$; we have $\frac{3}{2} + 2(3) \geq C_3$, i.e. $C_3 \leq 15/2$ (= 7.5), i.e. so long as $C_3 \leq 15/2$, introducing the new product x_3 in the original system will not affect the optimality. If $C_3 > 15/2$, we have to modify the existing table.

The basic variables of primal problem correspond to the non-basic variables of dual problem.

2.3 GOAL PROGRAMMING

2.3(a) Introduction

Up till now, we have studied the LP problems having a single objective; profit maximization or cost minimization In short, problems of any one of the types—maximization or minimization. In real situations, the system having certain constraints on resources, do impose certain constraints on the output also. In addition to this the maximization (or minimization) as one of the objective, there are other objectives also, like production of a particular item to a desired quantity, total working hours of a team of skilled workers to exact number of hours, investment in costly raw materials fixed up to a particular amount, etc.

Simultaneous satisfaction of all the objectives and that too remaining within the given constraints on resources force us down to think about some other techniques. If all the new objectives and constraints are linear, then we think of additions in features of LP problems. It requires, in addition to techniques of solving LP problems, some additional techniques.

The different types of objectives are known as *goals*—short-term goals.

On the top of that, some person from management may intervene and list the priorities of goals to be achieved. This will slightly complicate the problem but we, any way, conceptualize all these in a mathematical model and try to find a feasible and optimal solution in the prevailing situation of indecisive state of reality.

We have a better way to tackle such situation and it is dealt with goal programming. We start with an example and slowly keep an adding our goals and keep on modifying the solution.

Some goals are, for some further uses/applications, to be met with precision by justifying amount of fluctuation if any. Let us understand it as follows.

In a case, say we want that production at the end of the day in a machining unit of a factory must be 30 units, which is a goal.

If it is less than 30; say 28 units, then 2 units are under achieved. Then it may be denoted as D^- or D_{up}, i.e. production is under achieved.

D_{up} = 2 units = 30 – 28 = Desired production – actual production

Quite opposite to that, if the production is over achieved, i.e. say there is a production of 33 units, then three units are over achieved (Over achieved production).

i.e. D_{op} = 3 units = 33 – 30

= Actual production – target production

We state this as follows.

Objective: Production = P = 30 units.

The constraint depicting the same is $P + D_{up} - D_{op} = 30$ units.
We note that both $\boldsymbol{D_{up}}$ ***and*** $\boldsymbol{D_{op} \geq 0}$ ***and both cannot exist at a time*** unless both are zero; saying $P = 30$ units. This means that for $P = 30$ units; an algebraic sum of fluctuations must be zero.

i.e. $D_{up} - D_{op} = 0$

The goals are treated in the same way.

e.g. Goal 1 : profit = p = ₹2000 and

Goal 2 : production = x = 200 units

Then with equal priorities $P + D_{up} - D_{op} = 2000$; $x + D_{ux} - D_{ox} = 200$

2.3(b) Some Facts

1. However, some goals may be conflicting with each other but the best can be done is to find a compromise solution.
2. $D_{up} - D_{op}$; may be denoted as D^- and D^+ are known as *deviational variables* and they are always ≥ 0.

   These variables are, by definition, dependent and hence cannot work as basic variables simultaneously in a simplex program.

   Let us take two goals (conditions) of different types and understand the situation.

   (a) Product 1 utilizes 3 units of raw materials and Product 2 utilizes 4 units of the same material. If we have a production of x_1 units of Production 1 and x_2 units of production 2, then the total utilization of raw material is $3x_1 + 4x_2$.

   If the availability of the raw material is restricted to a limit of 50 units, then we have to write;

   $$3x_1 + 4x_2 \leq 50.$$

   Now, considering deviational variables D^- (D_{up}) and D^+ (D_{op}); we write,

   $$3x_1 + 4x_2 + D^- - D^+ = 50; \text{ with } D^- \text{ and } D^+ \geq 0.$$

   If $D^+ > 0$, then it violates the condition. In fact, we want $D^+ = 0$. We state this by putting as ***Minimize*** $\boldsymbol{D^+}$.

   (b) Similarly, for a profit of ₹50 per unit of x_1 and ₹80 per unit of x_2, we have total profit = $\mathbf{50}\boldsymbol{x_1} + \mathbf{80}\boldsymbol{x_2}$.

   If the goal is to generate a profit of at least ₹800, then we write

   $$50x_1 + 80x_2 \geq 800$$

   Again, introducing D_1^- and D_1^+; we write

   $$50x_1 + 80x_2 + D_1^- - D_1^+ = 800;$$
   $$D_1^-, D_1^+ \geq 0.$$

   In this case, a profit lower than 800, is a violation of the condition and so, we want that $D_1^- = 0$, i.e. Minimize D_1^-.

   In short, all constraints of $\geq$ types will have a goal of minimizing D^- (D_{up}) and similarly for the constraints of $\leq$ types there will be a goal of minimizing $D^+(D_{op})$.
3. In goal programming problem, the objective function involves only deviational variables and not the decision variables of the given problem. The target is to meet the different goals remaining within the given limits.
4. Some goals, to be achieved, are according to the pre–determined priorities.

2.3 (c) Difference between LP and GP problems, models and solution

ILLUSTRATION 13

Alpha power supply is engaged in making two types of remote control switches *A* and *B*. Each type of switch has to pass through two departments—*X* and *Y*. Department *x* takes 2 hours in assembling *A* and 3 hours for *B*. Department *Y* is a testing unit where it takes 1 hour for *A* and 2 hours for *B* for testing under strict constraints of quality control. The availability of man-hours per day in *X* and *Y* departments is 30 hours and 16 hours respectively. The profit on unit sale of Type *A* switch is ₹20 and ₹10 on the second Type *B*.

1. Construct a linear programming model.
2. If there is a goal of making profit of at least ₹360, then what are the necessary changes to be done.

General Assumptions

Let x_1 and x_2 represent the number of first and second type of switches—*A* and *B* respectively. Assembly unit has 30 man-hours and these switches (x_1 and x_2) will consume $2x_1 + 3x_2$ hours and we have a constraint $2x_1 + 3x_2 \leq 30$.

Testing unit has 16 hours available and total-time consumed by x_1 and x_2 switches is $1x_1 + 2x_2$ man-hours.

We have $1x_1 + 2x_2 \leq 16$.

The non-negativity constraint on variables is $x_1, x_2 \geq 0$. Now, we attempt the LP problem.

1. Making profit of ₹20 on each of first Type *A* and ₹10 on each one of second Type *B*, the total profit on production of both types of switches is $Z = 20x_1 + 10x_2$.
   The corresponding LP problem is
   find x_1 (number of first type of switches) and x_2 (number of second type of switches) so as to maximize the total profit
   $$\left.\begin{aligned} &Z = 20x_1 + 10x_2. \\ \text{subject to the constraints } &2x_1 + 3x_2 \leq 30 \\ &1x_1 + 2x_2 \leq 16 \\ \text{with } &x_1, x_2 \geq 0. \end{aligned}\right\} \quad (1)$$
2. The only goal of the company is to make profit of at least ₹360. This means that the company wishes total profit $Z = 20x_1 + 10x_2 \geq 360$.
   Adding the deviational variables, say d^- and d^+, we get
   $$20x_1 + 10x_2 + d_1^- - d_1^+ = 360.$$
   As discussed earlier, we will see that d_1^- is minimized—this becomes our objective function with the two rigid constraints
   $$\begin{aligned} &2x_1 + 3x_2 \leq 30 \text{ and} \\ &1x_1 + 2x_2 \leq 16 \\ \text{with } &x_1, x_2 \geq 0. \end{aligned}$$
   Making all these in one basket, we have goal–programming problem as follows.
   Find x_1, x_2, d_1^- and d_1^+ (with slack variables $s_1, s_2 \geq 0$ also) which minimizes $Z = 1\, d_1^-$.
   $$\left.\begin{aligned} \text{subject to the constraints } &20x_1 + 10x_2 + d_1^- - d_1^+ = 360 \\ &2x_1 + 3x_2 + s_1 = 30 \\ &1x_1 + 2x_2 + s_2 = 16 \\ \text{with } &x_1, x_2 \geq 0. \end{aligned}\right\} \quad (2)$$

We can solve this problem using simplex method. Note that ₹20 and ₹10 which were the ***profit factors*** in the problem (1) in LP form are ***no more*** in this goal-programming problem (2) shown above.

Minimize:

Table 2.17

$C_j \to$ VAR / $\downarrow$Basis	0 x_1	0 x_2	1 d_1^-	0 d_1^+	0 s_1	0 s_2	X_B	R.R.
1 d_1^-	20	10	1	–1	0	0	360	360/20 = 18
0 s_1	(2)	3	0	0	1	0	30	30/2 = 15 $\to$
0 s_2	1	2	0	0	0	1	16	16/1 = 16
$C_j - Z_j$	–20$\uparrow$	–10	0	1	0	0	360	

(As it is a minimization problem, the most negative of $C_j - Z_j$ entry is selected and in this case it being –20; the corresponding variable x_1 will be an incoming variable in the next iteration. The outgoing variable is decided based on R.R.)

Minimize

Table 2.18

$C_j \to$ VAR / $\downarrow$Basis	0 x_1	0 x_2	1 d_1^-	0 d_1^+	0 s_1	0 s_2	X_B	
1 d_1^-	0	–20	1	–1	–10	0	60	
0 $\leftarrow x_1$	1	3/2	0	0	1/2	0	15	
0 s_2	0	1/2	0	0	–1/2	1	1	
$C_j - Z_j$	0	20	0	1	10	0	60	

Since all $C_j - Z_j$ entries are ≥ 0 for this minimization problem, we have attained optimality.
Basic variables: $d_1^- = 60$; $x_1 = 15$; $s_2 = 15$;
Non-basic variables: $x_2 = 0 = s_1 = d_1^+$;
Minimum value = 60.
This means that;

1. Production of type A switches $x_1 = 15$ units and, the production of type B switches $x_2 = 0$
2. As $d_1^- = 60$; with $d_1^+ = 0$ and $x_2 = 0$; we have shortage—under achievement of ₹60 in achieving the goal of profit = 360, i.e. there will be a profit of ₹360 – ₹60 = 300.

Comment: Problems having single goal can be easily solved by simplex method. If there are two or more goals, our method of solving the problem is yet the same—simplex method.

ILLUSTRATION 14

In the above problem, we add one more equally ranked goal. Say, we add one more goal—the production of x_1 must at the most 5 units.
i.e. $x_1 \leq 5$.
Now, we make it in the language fit for goal programming, i.e. introducing deviational variables d_2^- and d_2^+; we have $x_1 + d_2^- - d_2^+ = 5$ and we wish that d_2^+ must be minimize, i.e. any over achievement d_2^+ will violate the condition.
In this connection the goal programming (GP) problem is
Find $x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, s_1$, and s_2 so as to

Minimize $\quad Z = d_1^- + d_2^+$

subject to the constraints $20x_1 + 10x_2 + \mathrm{d}_1^- - \mathrm{d}_1^+ = 360$

$$2x_1 + 3x_2 + s_1 = 30$$
$$1x_1 + 2x_2 + s_2 = 10$$

Moreover, a new goal (equally ranked) $x_1 + d_2^- - d_2^+ = 5$ with all variables ≥ 0.

Solution

We solve it by simplex method.

Minimize

Table 2.19

$C_j \rightarrow$ VAR / $\downarrow$Basis	0 x_1	0 x_2	1 d_1^-	0 d_1^+	0 d_2^-	1 d_2^+	0 s_1	0 s_2	$\mathbf{X}_B$	R.R
1 d_1^-	20	10	1	–1	0	0	0	0	360	18
1 ← d_2^-	1	0	0	0	1	–1	0	0	5	5 →
0 s_1	2	3	0	0	0	0	1	0	30	15
0 s_2	1	2	0	0	0	0	0	1	10	10
$C_j - Z_j$	–21 ↑	–10	0	1	–1	2	0	0	60	

Table 2.20

	$C_j \rightarrow$ VAR / $\downarrow$Basis	0 x_1	0 x_2	1 d_1^-	0 d_1^+	0 d_2^-	1 d_2^+	0 s_1	0 s_2	$\mathbf{X}_B$	R.R
II	1 d_1^-	0	10	1	–1	–20	20	0	0	260	13
	0 x_1	1	0	0	0	1	–1	0	0	5	–
	0 s_1	0	3	0	0	–2	2	1	0	20	10
	0 ← s_2	0	2	0	0	–1	1	0	1	5	5 →
	$C_j - Z_j$	0	–10	0	1	20	↑–19	0	0		

Table 2.21

	$C_j \rightarrow$ VAR / $\downarrow$Basis	0 x_1	0 x_2	1 d_1^-	0 d_1^+	0 d_2^-	1 d_2^+	0 s_1	0 s_2	X_B	R.R
III	1 d_1^-	0	–30	1	–1	0	0	0	–20	160	
	0 x_1	1	2	0	0	0	0	0	1	10	
	0 s_1	0	–1	0	0	0	0	1	–2	10	
	1 ← d_2^+	0	2	0	0	–1	1	0	1	5	
	$C_j - Z_j$	0	28	0	1	1	0	0	19		

Since all $C_j - Z_j \geq 0$, this minimization problem has reached an optimal state.
Basic variables: $x_1 = 10$; $d_1^- = 160$; $s_1 = 10$; $d_2^+ = 5$
Non-basic variables: $x_2 = 0$; $s_2 = 0$; $d_1^+ = 0$; $d_2^- = 0$.

Comment

1. As $d_1^- = 160$, it means that there is a short of ₹160 in order to satisfy the goal of profit ₹360, i.e. actual profit is ₹360 – ₹160 = ₹200.
2. $x_1 + d_2^- - d_2^+ = 5$ where $x_1 = 10$; $d_2^- = 0$ and $d_2^+ = 5$.
   It means that the production of x_1 is over achieved by 5 units (as $d_2^+ = 5$)
   $x_1 = 10$; $d_2^- = 0$; $d_2^+ = 5$; (This goal is over achieved and it is controlled by d_2^+)
   *Note that like what we did previously—finding the minimum value of the objective function, cannot be done at this stage, the reason being simply that d_1^- and d_2^+ both are different variables of different type.

   D_1^- being underachieved amount of profit (in ₹) and d_2^+ being the over achievement of x_1 (number of switches of Type A).

2.3(d) Priority Based Goals—Introduction and Illustrations

There are real life situations where optimality based linear programming may not be sufficient. There are many situations where different goals and that too with priority orders are given. These goals are to be satisfied in order and that too with desired priorities. The objective is to carry out the program—may be sale, production, etc. which try to satisfy the priorities in the expected hierarchy.

Goals are the elements of a set of pre–given objectives to be achieved. In some cases, not all the goals can be satisfied completely. Some goals, in the presence of other goals, may not be fully satisfied.

Also, there are some goals which consume less time and use less resources to be satisfied if some related or associated goals (to these goals) have already been established. All these factors do call for developing goal-programming techniques in which the goals are given a priority in the execution of goals.

Priority are numbers given to the different goals and it is required that the goals are satisfied according to the order of the priorities. If the first priority is satisfied then only the program will look for the avenues for the second priority goals. To understand all these, we take an example.

Goal: Some equally ranked goals are,

1. Production of second type of switches must not exceed 20 units.
2. Workers "*A*" and "*B*" should not be allotted production work.
3. Packing department members cannot work more than 60 hours in a 5 day-week.
4. The profit from the sale must be at least ₹2000.
5. Total production of both the items should be at the most 40 units.

Now, in addition to the above goals, achievement plans are assigned priorities by the senior manager operations. These are as follows.

Priorities:

(P1) Profit from sale ≥ ₹2000 (Goal 4)

(P2) Total production $x_1 + x_2 \leq 40$ units (Goal 5)

(P3) Production of second type of switches $x_2 \leq 10$ (Goal 1) and so on;

These priorities are represented by numeric subscriptions $P_1, P_2, \ldots$, etc. They are called ***Preemptive priority factors*** (like cost factors in LP).

They strictly stand in order of importance and dominate over the priorities of lower order of occurrence.

For example, P2 cannot be achieved unless P1 is achieved.

We take one illustration–a modified form of the previous illustration.

ILLUSTRATION 15

The previous illustration is,

Minimize: d_1^-

subject to the constraints $20x_1 + 10x_2 + d_1^- - d_1^+ = 360$

$$2x_1 + 3x_2 + s_1 = 30$$

$$1x_1 + 2x_2 + s_2 = 16$$

with $x_1, x_2 \geq 0$.

An optimal solution to this illustration is;

Basic variables: $x_1 = 10, d_1^- = 160, s_1 = 10, d_1^+ = 5,$

Non-basic variables: $x_2 = 0 = s_2 = d_1^+ = d_1^-$

To this, we add two more goals with priorities as follows.

P_1: profit ≥ 220.

P_2: $x_2 \leq 3$.

Based on this information, we make a mathematical model as follows.
There are two priorities P_1 and P_2 to be satisfied in order.
Introducing the deviational variables d_1^-, d_1^+, d_2^-, and d_2^+ ; the problem is as follows.
Minimize: $P_1 d_1^- + P_2 d_2^+$
Subject to the constraints $20x_1 + 10x_2 + d_1^- - d_1^+ = 220$; (profit goal)
$0x_1 + 2x_2 + d_2^- - d_2^+ = 3$; (switches type two goal)
$2x_1 + 3x_2 + s_1 = 30$;
$1x_1 + 2x_2 + s_2 = 16$
with all variables being non-negative.
Putting all these in simplex table form, we have

Minimize

Table 2.22

$C_j \rightarrow$ VAR / $\downarrow$Basis	0 x_1	0 x_2	P_1 d_1^-	0 d_1^+	0 d_2^-	P_2 d_2^+	0 s_1	0 s_2	X_B	R.R.
P_1 d_1^-	20	10	1	–1	0	0	0	0	220	11→
$P_2 \leftarrow d_2^+$	0	1	0	0	1	–1	0	0	3	–
0 s_1	2	3	0	0	0	0	1	0	30	15
0 s_2	1	2	0	0	0	0	0	1	16	16
$C_j - Z_j$	$-20P_1\uparrow$	$-10P_1 - P_2$	0	P_1	$-P_2$	$2P_2$	0	0		
P_2	0	–1	0	0	–1	2	0	0		Read C(1) and
P_1	–20	–10	0	1	0	0	0	0		C(2) below
0 x_1	1	1/2	1/20	–1/20	0	0	0	0	11	22
$P_2 \leftarrow d_2^-$	0	1	0	0	1	–1	0	0	3	3→
0 s_1	0	2	–1/10	1/10	0	0	1	0	8	4
0 s_2	0	3/2	–1/20	1/20	0	0	0	1	5	10/3
	0	$-P_2$	P_1	0	0	$2P_2$	0	0		
P_2	0	$-1\uparrow$	0	0	0	0	0	0		Read C(1) and
P_1	0	0	1	0	0	2	0	0		C(2) below
0 x_1	1	0	1/20	–1/20	–1/2	1/2	0	0	19/2	
P_2 x_2	0	1	0	0	1	–1	0	0	3	
0 s_1	0	0	–1/10	1/10	–2	2	1	0	2	
0 s_2	0	0	–1/20	1/20	–3/2	3/2	0	1	1/2	
$C_j - Z_j$	0	0	P_1	0	0	P_2	0	0		
P_2	0	0	0	0	0	1	0	0		
P_1	0	0	1	0	0	0	0	0		

All $C_j - Z_j$ entries are ≥ 0 ; it has reached an optimal stage.
From the available information from the optimal table, we can write

(1) Basic variables : $x_1 = 19/2$, $x_2 = 3$, $s_1 = 2$, $s_2 = 4$
(2) Non-basic variables: All deviational variables are zero.

This clearly means that with $x_1 = 19/2$ and $x_2 = 3$, the profit goal (priority P_1) and $x_2 = 3$ itself is the second priority and which is satisfied. Slack variables do exist but they affect the third and the fourth constraints. At this stage, we like to make ***comments*** on the construction of table and the $C_j - Z_j$ entries in the simplex table.

C(1) In such a simplex table, in fact, we have different goals and they have different units; so it is rather not justified to write $C_j - Z_j$ as one entry. We find $C_j - Z_j$ and put them separately as multiples of P_1 and P_2.

For example, in Table 2.22 the second entry in $C_j - Z_j$ row is $-10P_1 - P_2$; which we write as -10 under the row of P_1 and -1 under the row of P_2. *Please verify this and all such remaining entries.* Do not think much on looking at different rows of priorities. As priorities are in order and they stand for different units of resources; we have to show them separately. Putting them together becomes meaningless.

C(2) The highest priority is P_1 and the corresponding row is written in the lowest order in the table below the $C_j - Z_j$ entry row. Above that the row comes is for the priority P_2. This distinctly means that ***once P_1 is satisfied*** then only P_2 will be considered. The criterion for selection of incoming variable remains the same, i.e. the most negative entry is selected; while the outgoing is determined on the basis of the replacement ratio. When all the $C_j - Z_j$ entries are ≥ 0, then only we move for considering the next priority in order.

2.4 Master Problems—Primal and Dual

In this section, we, for the sake of possible extension, study and show some linear programming problems having amazing mathematical properties. We request students to understand and analyze the mathematical facts and develop some details.

Illustration 16

Problem 1

Find y_1 and y_2 to minimize $z_1 = 5y_1 + 12y_2$

subject to the constraints $y_1 + 3y_2 \geq 6$

$1y_1 + 2y_2 \geq 5$

with y_1 and $y_2 \geq 0$

Problem 2

Find y_3 and y_4 to minimize $z_1 = 8y_3 + 10y_4$

subject to the constraints $1y_3 + 2y_4 \geq 3$

$2y_3 + 1y_4 \geq 4$

with y_3 and $y_4 \geq 0$

At this stage, we add up the above two problems and write the Problem 3

Problem 3

Find y_1, y_2, y_3, and y_4 to minimize $Z = 5y_1 + 12y_2 + 8y_3 + 10y_4$

subject to the constraints $y_1 + 3y_2 \geq 6$

$1y_1 + 2y_2 \geq 5$

$1y_3 + 2y_4 \geq 3$

$2y_3 + 1y_4 \geq 4$

with y_1 and y_2, y_3, and $y_4 \geq 0$

We have solved the above problems but we give you only the optimal table of each one.

Optimal Table – Problem 1

$C_j \rightarrow$ ↓	5	12	0	M	0	M	X_B
Basis \ Var →	y_1	y_2	s_1	A_1	s_2	A_2	
12 y_2	0	1	–1	1	1	–1	1
5 y_1	1	0	2	–2	–3	3	3
$C_j - Z_j$	0	0	2	M – 2	3	M – 3	$Z = 27$

Comment:
All $C_j - Z_j \geq 0$; it is an optimality criteria for a minimization problem.
Basic variables: $y_1 = 3$, $y_2 = 1$
Non-basic variables: $s_1 = 0 = s_2$
Optimal $Z = 27$
[Students should try writing the optimal table of the dual.]
We, now, write the optimal table of the problem 2.

Optimal Table—Problem 2

$C_j \rightarrow$ ↓	8	10	0	M	0	M	X_B
Basis \ Var →	y_3	y_4	s_3	A_1	s_4	A_2	
10 y_4	0	1	–2/3	2/3	1/3	–1/3	2/3
8 y_3	1	0	1/3	–1/3	–2/3	2/3	5/3
$C_j - Z_j$	0	0	4	M – 4	2	M – 2	20

Comment
All $C_j - Z_j \geq 0$; it is an optimality criteria for a minimization problem.
Basic variables: $y_3 = 5/3$, $y_4 = 2/3$
Non-basic Variables: $s_3 = 0 = s_4$
Optimal $Z = 20$
[Students should, using the data of corresponding optimal tables, try writing the optimal table of the dual of Problem 1 and Problem 2.]

The beauty of the present situation is that the basic feasible solution of the Problem 3 is the concatenation of the basic feasible solution of the Problem 1 and that of Problem 2.

Basic Feasible Solution of Problem 3
Basic variables: $y_1 = 3$, $y_2 = 1$, $y_3 = 5/3$, $y_4 = 2/3$
Non-basic variables: $s_1 = 0 = s_2$, $s_3 = 0 = s_4$
Optimal $Z = Z_1 = 20 + Z_2 = 27 = 47$

Comment
We can write the duals of master Problems 1, 2, and 3 and can find the basic feasible solution of each one of it. These problems also exhibit the same pattern of properties in their primal described above.

Additional Questions for Practice (with Hints and Answers)

Question 1

Under what circumstances do we prefer to solve the dual of the given problem? Write the dual of the following problem and solve it. Also, write down the optimal solution of the primal from the optimal table of the dual.

Maximize $Z = 5x - 2y + 3z$

subject to the constraints $2x + 2y - z \geq 2$

$3x - 4y \leq 3$

$y + 3z \leq 5$

$x, y, z \geq 0$

Solution

In order to write the dual of the given problem; we put it in the standard form and as follows.

Find x, y, and z so as to maximize $Z = 5x - 2y + 3z$

subject to the constraints $-2x - 2y + z \leq -2$,

$3x - 4y + 0z \leq 3$

$0x + 1y + 3z \leq 5$

with x, y, and $z \geq 0$

Its dual is as follows.

Let x_1, x_2, and x_3 be the variables of the dual problem and we find them to

minimize $Z^* = -2x_1 + 3x_2 + 5x_3$

subject to the constraints $-2x_1 + 3x_2 + 0x_3 \geq 5$

$-2x_1 - 4x_2 + 1x_3 \geq -2$

$1x_1 + 0x_2 + 3x_3 \geq 3$

with x_1, x_2, and $x_3 \geq 0$

To make the calculation easier, we write the second constraint as $2x_1 + 4x_2 - 1x_3 \leq 2$

We have to add surplus variables s_1, s_3, and ASV A_1 and A_3 (to work for basic variables) in the first and the last constraints and a slack variable s_2 in the second constraint. Now, students can make the simplex table and solve as required.

Question 2

Write the dual of the following problem.

Minimize $Z = 2x_1 + 3x_2 + 4x_2$

subject to the constraints $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + 1x_2 + 7x_3 = 3$

$1x_1 + 4x_2 + 6x_3 \leq 5$

with x_1, $x_2 \geq 0$ and x_3 is unrestricted.

Solution

To write the dual of the above problem, we have to change the last constraint and make it having $\geq$ sign. We do so by multiplying it by -1. The primal problem is as follows.

Minimize $Z = 2x_1 + 3x_2 + 4x_2$

subject to the constraints $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + 1x_2 + 7x_3 = 3$

$-1x_1 - 4x_2 - 6x_3 \geq -5$

with x_1, $x_2 \geq 0$ and x_3 is unrestricted.
The dual is as follows.
Maximize $Z^* = 2y_1 + 3y_2 - 5y_3$
subject to the constraints $2y_1 + 3y_2 - 1y_3 \leq 2$
$3y_1 + 1y_2 - 4y_3 \leq 3$
$5y_1 + 7y_2 - 6y_3 = 4$
with y_1, $y_3 \geq 0$ and y_2 is unrestricted in sign.

Question 3

Obtain the dual of the following problem.
Maximize $Z = 4x_1 + 2x_2$
subject to the constraints $x_1 - 2x_2 \geq 2$
$x_1 + 2x_2 = 8$
$x_1 - x_2 \leq 10$
with $x_1 \geq 0$.

Solution

In this case, no information is given about the variable x_2. It means that it is unrestricted in sign. Also, we have to change the sign of the constraint. We write it as

$$-1x_1 + 2x_2 \leq -2$$

Now, the problem is,
Maximize $Z = 4x_1 + 2x_2$
subject to the constraints $-1x_1 + 2x_2 \leq -2$
$x_1 + 2x_2 = 8$
$x_1 - x_2 \leq 10$
$x_1 \geq 0$. x_2 is unrestricted in sign.

Its dual is
Find y_1, y_2, and y_3 so as to minimize $Z = -2y_1 + 8y_2 + 10y_3$
subject to the constraints $-1y_1 + 1y_2 + 1y_3 \geq 4$
$2y_1 + 2y_2 - 1y_3 = 2$
with $y_1 \geq 0$ and y_2 is unrestricted in sign.

Question 4

Construct the dual of the following problem.
Maximize $Z = 3x_1 + 17x_2 + 9x_3$
subject to the constraints $x_1 - x_2 + x_3 \geq 3$
$-3x_1 + 2x_3 \leq 1$
$2x_1 + x_2 - 5x_3 = 1$ with all $x_i \geq 0$ for $i = 1, 2, 3$

Solution

In this case, we have to change the sign of the first constraint. We write it as $-1x_1 + x_2 - 1x_3 \leq -3$; Remaining two constraints can be dealt easily.
The problem is as follows.
Maximize $Z = 3x_1 + 17x_2 + 9x_3$
subject to the constraints $-1x_1 + x_2 - 1x_3 \leq -3$
$-3x_1 + 0x_2 + 2x_3 \leq 1$
$2x_1 + x_2 - 5x_3 = 1$

with all $x_i \geq 0$ for $i = 1, 2, 3$.
Now, we write its dual as follows.
Find variables y_1, y_2, and y_3 so as to minimize $Z = -3y_1 + 1y_2 + 1y_3$
subject to the constraints $-1y_1 - 3y_2 + 2y_3 \geq 3$
$1y_1 + 0y_2 + 1y_3 \geq 17$
$-1y_1 + 2y_3 - 5y_3 \geq 9$
with y_1, y_2, and $y_3 \geq 0$

Question 5

Solve the dual of the problem. From the optimal table of the dual find the values of the basic variables of the primal.
Minimize $Z = 24x_1 + 30x_2$
subject to the constraints $2x_1 + 3x_2 \geq 10$
$6x_1 + 6x_2 \geq 20$
$4x_1 + 9x_2 \geq 15$
with x_1 and $x_2 \geq 0$

Solution

As all the constraints are in tune with standard minimization problem; we can write its dual.
Find y_1, y_2, and y_3 so as to maximize $z = 10y_1 + 20y_2 + 15$
subject to the constraints $2y_1 + 6y_2 + 4y_3 \leq 24$; $3y_1 + 6y_2 + 9y_3 \leq 30$
with $y_1, y_2 \geq 0$
We have to add two slack variables s_1 and $s_2 \geq 0$, one to each constraint.

Maximize:

Table 2.23

$C_j \rightarrow$ VAR $\downarrow$Basis	10 y_1	20 y_2	15 y_3	0 s_1	0 s_2	R.H.S. X_B	R.R.
0 s_1	2	(6)	4	1	0	24	4→
0 s_2	3	6	9	0	1	30	5
$C_J - Z_J$	10	20↑	15	0	0	$Z = 0$	
20 y_2	1/3	1	2/3	1/6	0	4	12
0 ← s_2	(1)	0	5	–1	1	6	6→
$C_j - Z_j$	10/3 ↑	0	5/3	–10/3	0	$Z = 80$	
20 y_2	0	1	–1	½	–1/3	2	
10 y_1	1	0	5	–1	1	6	
$C_j - Z_j$	0	0	–15	0	–10/3	**Z = 100**	

Since all $C_j - Z_j$ are ≤ 0; this maximization problem has reached optimality.

We write the corresponding values of the basic variables of the dual. This becomes the corresponding values of the $C_j - Z_j$ entries of the non-basic variables in the primal problem. (As we have solved the dual.)

The first basic variable $x_1 = 0$; The second basic variable $x_2 = 10/3$

Minimum value of the objective function $Z = 100$.

POINTS TO REMEMBER

In this section, we give some important points necessary to review before students take up their final examination in the subject. These are the points we have collected and compiled in a finer way. We hope that students read them carefully.

(a) Before one writes the dual of a given problem, it is very important to check whether a given problem is in the standard form.
For a maximization problem all constraints are of the type ≤ or = type and for a minimization problem constraints are of the type ≥ or = type.
We take care to determine the number of variables and constraints of the dual by checking the number of constraints and the number of variables of the given primal problem. They have one–one correspondence with each other.
In addition to this, if an i^{th} variable unrestricted or a j^{th} constraint is equality type then the corresponding dual has i^{th} constraint of = type and j^{th} variable unrestricted in sign.

(b) An optimal table of any problem is capable enough to convey nearly all the information about the dual of its problem and viceversa. You understand and verify the following facts by looking and comparing optimal tables of both the problems keeping them side by side. (This is important, please you do not wave it.)
Basic variables of the primal problem are in correspondence with the non-basic variables of the dual and non-basics of the primal problem have correspondence with basic variables of the dual problem. $C_j - Z_j$ **entries** corresponding to the non-basic variables of a primal problem (may be with a change in sign) are the values of the corresponding basic variables of the dual problem and these entries appear in RHS (X_B) column in the optimal table of the dual.

(c) As an output to this property, we have the important property called *complementary slackness.*

(d) The topic sensitivity analysis is an extremely important one and in real life situations, it plays a very important role. A real consultant having managerial capabilities cannot afford ignore the critical details of this topic. Students are advised to study these points in details.

(e) Now, what remains is about goal programming and peeping into the realities faced aspects; one may not always be after maximizing the profit or minimizing the cost of the production. There are, in addition to this, many objectives and one always tries to look into the matter of finding the realistic solution that may be or may not be capable enough of satisfying all the goals at a given time. Some goals, when not equally ranked, have their priorities, which are pre–determined in their order of occurrence.
Goal-programming problems have their objective function in terms of minimizing the sum total of deviational variables—some of them control excess or over achievement from the target values (d^+) and some control the under achievement (d^-) from the target in terms of their original units. Goal programming is a multi-unit and multi-task optimization problem. Here we have the true meaning of the word ‘Optimization’.

EXERCISES

OBJECTIVE TYPE QUESTIONS

I. State True or False.

1. If a given primal problem has 3 variables and 4 linearly independent constraints then its dual has also 3 variables and 4 constraints.

2. If all the constraints in a primal of maximization type is 'equality' type then only its corresponding dual is of maximization type with all variables unrestricted in sign.
3. We can obtain the optimal table of the dual of the given problem from the optimal table of the primal problem.
4. If second constraint of the primal is of = sign then the second constraint of the dual has = sign.
5. If second variable of the primal is unrestricted in sign then the second constraint of the dual has = sign.
6. An objective function in a goal programming problem is always in terms of deviational variables.
7. Priorities are subjective and determined by the goal setters.
8. If the primal problem has a feasible solution but the dual problem has no feasible solution, then the primal problem has an unbounded optimal solution.
9. Variables of the dual problems act as prices or costs of one unit of each of the inputs (resources) of the primal problem.
10. If the i^{th} slack variable of the primal problem is not zero, then the i^{th} dual variable is also not zero.
11. Goal programming problem can be solved as a minimization problem but finally the solution optimizes the deviational variables having different units.

ANSWERS

1. false.	2. false.	3. true.	4. false.	5. true.	6. true.
7. true.	8. true.	9. true.	10. false.	11. true.	

II. Multiple Choice Questions

Study the optimal table below and answer the following questions.

Optimal Table (Primal Problem)

Table 2.26

$C_j \rightarrow$ VAR / ↓ Basis	5 x_1	10 x_2	(a) x_3	(b) s_1	(c) s_2	$\mathbf{X}_B$
5 x_1	1	2	(d)	–3	1/5	(h)
(a) x_3	0	2/5	(e)	(f)	(g)	20
$C_j - Z_j$	(k)	–8	(i)	–5	–5	525

1. What is the value of a? **(20, –20, 0, 10)**
2. What is the value of $k + i$? **(0, 10, 15, 5)**
3. What is the value of h? **(20, 15, 30, 25)**
4. What is the value of $2x_2 + x_1$? **(20, 10, 25, 35)**
5. What is Value of $b - (af - 15)$? **(5, 10, –5, 0)**
6. What is the value of g? **(4/25, –4/25, 5, –4)**
7. How many constraints are there in the dual of the problem? **(2, 3, 4, 5)**
8. What is the algebraic sum of values obtained by basic variables of the dual? **(–18, 45, –45, 18)**
9. What is sum of all $C_j - Z_j$ entries of dual problem? **(–18, 45, –45, 18)**
10. If objective function of the dual is $45y_1 + by_2$, then what is the value of b? **(60, 45, 50, 70)**

ANSWERS

(1) 20	(2) 0	(3) 25	(4) 25	(5) –5
(6) 4/25	(7) 3	(8) 18	(9) 45	(10) 60

NUMERICAL PROBLEMS

1. Write the dual of the following problems.
   (a) Minimize $Z = 6x_1 + 6x_2 + 8x_3 + 9x_4$
   subject to the constraints $-1x_1 - 2x_2 - 1x_3 - 1x_4 \leq -3$
   $2x_1 + 1x_2 + 4x_3 + 9x_4 \geq 8$
   with $x_1, x_2, x_3 \geq 0$ (x_4 unrestricted)
   (b) Minimize $Z = 3x_1 + 2x_2 + 5x_3 + 7x_4$
   subject to the constraints $3x_1 + 2x_2 + x_3 \leq 80$
   $5x_1 + 1x_2 + 2x_3 + 4x_4 \leq 70$
   $-4x_1 - 1x_3 + 2x_4 \geq -12$
   with $x_1, x_3 \geq 0$
   (c) Maximize $Z = 2x_1 + x_3$
   subject to the constraints $x_1 + x_2 - x_3 \leq 9$
   $2x_1 + x_2 = 9$
   $x_1 + 2x_2 + x_3 \geq 10$
   $x_1 \geq 0$ and (x_2 and x_3 are unrestricted in sign.)
2. Write the dual of the following problem and solve the dual. Draw the graph of the primal comment on the final solution.
   Maximize $Z = 2x_1 + 3x_2$
   subject to the constraints $x_1 \leq 4$
   $x_2 \geq 1$
   $x_1 + 2x_2 \geq 2$
   with $x_1, x_2 \geq 0$
3. The optimal table for the following primal problem is
   Maximize $Z = 5x + 3y$
   subject to the constraints $x - y \leq 2$
   $2x + y \leq 4$
   $-3x + 2y \leq 6$
   with $x, y \geq 0$.

Table 2.24

$C_j \rightarrow$ VAR ↑Basis	5 x	3 y	0 s_1	0 s_2	0 s_3	$\mathbf{X}_B$
5 x	1	0	0	2/7	–1/7	2/7
3 y	0	1	0	3/7	2/7	24/7
0 s_1	0	0	1	1/7	3/7	36/7
$C_j - Z_j$	0	0	0	–19/7	–1/7	82/7

Answer the following questions:
(a) Make the optimal table of the dual.
(b) From the entry $\boldsymbol{s_1}$ of the above primal table (Table 2,24), write the value of the first variable of the dual? Will it be a basic variable?
(c) If it {in connection of (b) above} is non-basic variable of the dual then find the value of its $C_j - Z_j$ (NER) entry.
(d) In connection of the above points (b) and (c); name the principle that is used to explain the criteria.

4. The optimal table of the problem
Maximize $Z = 6y_1 + 8y_2$
Subject to the constraints $4y_1 + 4y_2 \le 40$
$5y_1 + 10y_2 \le 60$ is given as follows (Table 2.25):

Table 2.25

$C_j \rightarrow$ VAR / $\uparrow$Basis	6 y_1	8 y_2	0 s_1	0 s_2	$\mathbf{X}_B$
6 y_1	1	0	–1/5	(a)	8
8 y_2	0	1	(b)	–1/4	(c)
$C_j - Z_j$	0	0	–2/5	–1	64/Z

Based on Table 2.25; answer the following questions.
(a) What is the value of $(a) + (b)$?
(b) What is the value of (c)?
(c) If x_1 and x_2 are the corresponding basic variables of the dual problem, then what is the value of $x_1 + x_2$?

5. If the original resources given in constraints of the above example 4 (which are 40 and 60) are changed to 20 and 40 respectively, then will the dual give a feasible solution?

6. In the given constraints of Example 4 above, if a new constraint $y_1 + y_2 \le 11$ is added, then find the corresponding change in the solution.

7. In the given constraints of Example 4 above, if a new constraint $2y_1 + 3y_2 \le 6$ is added, then find the corresponding change in the profit.

8. In the problem, Maximize $Z = 5x + 3y$
subject to the constraints $3x + 5y \le 30$
$5x + 2y \le 10$
with x and $y \ge 0$; if we add a new constraint $2x + y \le 8$, will the optimal solution change?

9. Draw the graph of the above constraint and add the line for the new constraint and find the most common feasible region; observe and write your comment.

10. **Consider** the LP problem
Maximize $Z = x_1 + 2x_2 + x_3 + x_4$
subject to the constraints $2x_1 + x_2 + 3x_3 + x_4 \le 8$
$2x_1 + 3x_2 + 4x_4 \le 12$
$3x_1 + x_2 + 2x_3 \le 14$
with $\boldsymbol{x_i \ge 0, = 1, 2, 3, 4}$
Solve the problem and verify that $x_2 = 4$, $x_3 = 1$, and a slack $s_3 = 34/3$.
For the cost factors c_2 and c_3, find the range of values for which the above solution remains optimal.

11. For the second resource, find the range of values for which the above solution remains optimal.

12. The given problem is
Maximize $Z = 3y_1 + 3y_2 + 4y_3 + 7y_4$
subject to the constraints $y_1 + y_2 + y_3 + y_4 \le 9$
$5y_1 + 3y_2 + 2y_3 + y_4 \le 60$
$y_1 + 3y_2 + 5y_3 + 8y_4 \le 50$
with all $y_i \ge 0$; If a new variable y_5 is to be added with the certain changes in the constraints, then find the range of the corresponding cost factor, i.e. c_5 up to which y_5 cannot be a part of product mix.

The constraints with necessary changes are as follows.
$y_1 + y_2 + y_3 + y_4 + (7/4)\, y_5 \leq 9;$
$5y_1 + 3y_2 + 2y_3 + y_4 + 3y_5 \leq 60;$
$y_1 + 3y_2 + 5y_3 + 8y_4 + 7y_5 \leq 50;$
with all $y_i \geq 0$

13. Study the problem. Draw a graph and find the optimality. Also solve it by simplex method and verify your answer. Find p_1 and p_2 so as to
maximize $z = 20p_1 + 80p_2$
subject to the constraints $8p_1 + 6p_2 \leq 100$
$2p_1 + 3p_2 \leq 45$
with p_1 and $p_2 \geq 0$.
Now, we introduce a constraint $(p_1 / 16) + (p_2 / 80) \leq 1$
Will this additional constraint change the solution?

14. In the above Problem 13 if we introduce the additional constraint $p_1 + p_2 \geq 10$; will the feasible region be changed?

15. Consider the following LP problem.
Maximize $Z = 3y_1 + 2y_2 - 5y_3$ subject to the constraints
$y_1 - y_2 + 3y_3 = 0$
$2y_1 + y_2 + 6y_3 \leq 6$
$y_1 + y_2 \leq 2$
with all $y_i \geq 0$.
If the corresponding resources are changed to 5, 10 and 2 respectively, will the new solution be changed?

16. **A** company manufactures two products x and y. It takes 4 hours for each of x and 5 hours for each of y to be ready for sale. The production capacity in hours of the department is 80 hours in a week. Maximum allowable overtime is 20 hours per week. In any situation, it is required that the total production of items must not be more than 12 items. The profit on sale of each one of x type is ₹40 and on each of y is ₹30.

(a) Formulate the problem as a linear programming model. Find the amount of production of each type that maximizes the profit.

(b) Programming problem, if the **two equally ranked** goals are given as follows. Formulate a linear.
Goal 1: There must be a profit of ₹600
Goal 2: Minimize the overtime in the department.

(c) **Two goals are given with the following priorities.**
P_1: A minimum profit of ₹600.
P_2: Production of x type must be at least 8 units.
Formulate the problem as prioritized goal-programming problem, and solve it.

17. Write the dual of the Master Problems 1, 2 and 3 given below. Solve the dual problems and compare their results to find the relation between their optimal values.
Problem 1
Find y_1 and y_2 to minimize $Z_1 = 5y_1 + 12y_2$
subject to the constraints $y_1 + 3y_2 \geq 6$
$1y_1 + 2y_2 \geq 5$
with $y_1, y_2 \geq 0$

Problem 2

Find y_3 and y_4 to minimize $Z_1 = 8y_3 + 10y_4$

subject to the constraints $1y_3 + 2y_4 \geq 3$

$2y_3 + 1y_4 \geq 4$

with $y_3, y_4 \geq 0$

Problem 3

Find y_1, y_2, y_3, and y_4 to minimize $z = 5y_1 + 12y_2 + 8y_3 + 10y_4$

subject to the constraints $y_1 + 3y_2 \geq 6$

$1y_1 + 2y_2 \geq 5$

$1y_3 + 2y_4 \geq 3$

$2y_3 + 1y_4 \geq 4$

with y_1, y_2 **and** $y_3, y_4 \geq 0.$

Answers to Numerical Problems

1 (a) Find y_1 and y_2 so as to minimize $Z = 3y_1 + 8y_2$

subject to the constraints $y_1 + 2y_2 \leq 6$

$2y_1 + y_2 \leq 6$

$y_1 + 4y_3 \leq 8$

$y_1 + 9y_2 \leq 9$

with all $y_i \geq 0$.

(b) Find y_1, y_2 and y_3 so as to minimize $Z = 80y_1 + 70y_2 + 12y_3$;

subject to the constraints $3y_1 + 5y_2 + 4y_3 \geq 3$

$2y_1 + 1y_2 = 2$

$1y_1 + 2y_2 + 1y_3 \geq 5$

$0y_1 + 4y_2 - 2y_3 = 7$

with $y_1, y_2, y_3, y_4 \geq 0$.

(c) Find y_1, y_2 and y_3 so as to minimize $Z = 9y_1 + 9y_2 - 10y_3$

subject to the constraints $1y_1 + 2y_2 - 1y_3 \geq 2$

$y_1 + y_2 - 2y_3 = 0$

$-1y_1 + 0y_2 - 1y_3 = 1$

with $y_1, y_3 \geq 0$

y_2 unrestricted in sign.

2. Dual is as follows.

Find y_1, y_2 and y_3 so as to minimize $Z = 4y - 1y_2 - 2y_3$;

subject to the constraints $y_1 - y_3 \geq 2$

$-y_2 - y_3 \geq 3$

with $y_1, y_2, y_3, \geq 0$

The primal has an unbounded solution and the dual has infeasible solution.

3. This is a relatively simple example and you can verify the answers by putting the values in the left-hand side equations of the primal and dual problems.

4. (a) 7/10, (b) 2, (c) 7/5

5. No feasible solution. (x_2 is a basic variable and its value = –6)
6. No change
7. Original value = 64
   New value = 18 Change in profit by ₹46

8 and 9

There will be no change in the optimality. There is a degenerate solution at $(x, y) = (0, 5)$. By introducing the new constraint, the feasible region does not change. As an effect to this, the optimality is not affected.

10. $\mathbf{-7/6 \leq \delta c_2 \leq \infty, \; -1 \leq \delta c_3 \leq 5}$
11. $\mathbf{-12 \leq \delta b_2 \leq 12}$
12. $\mathbf{C_5 \leq 33/4}$
13. **$y_1 = 0, y_2 = 15$, there is no change by adding the new constraint**
14. There is a change by adding a new constraint. Yes, the feasible region changes.
15. **The solution after the change is $y_1 = 2, y = 0, y_3 = 1$**
16. **(a)** Let x_1 items of x type and x_2 items be of y type.
    Find x_1 and x_2 to maximize the profit $Z = 40x_1 + 30x_2$
    subject to the constraints $4x_1 + 5x_2 \leq 80$
    $x_1 + x_2 \leq 12$
    with $x_1, x_2 \geq 0$
16. **(b)** Goal 1; $40x_1 + 30x_2 = 600$
    We write $40x_1 + 30x_2 + d_1^- - d_1^+ = 600$
    At this stage we keep a view to minimize d_1^-.
    Goal 2: Let d stands for overtime
    we want $d \leq 20$ hours
    Introducing deviational variables d_2^- and d_2^+
    we have $d + d_2^- - d_2^+ = 20$
    At this stage we minimize $d_1^- + d_2^+$
    subject to the constraints $40x_1 + 30x_2 + d_1^- - d_1^+ = 600$
    $d + d_2^- - d_2^+ = 20$
    The rigid condition $x_1 + x_2 \leq 12$, with all variables ≥ 0.
16. **(c)** Considering the first priority P_1
    Profit generated = $40x_1 + 30x_2 \geq 600$
    So, we have $40x_1 + 30x_2 + d_1^- - d_1^+ = 600$
    Minimize $P_1\, d_1^-$
    Now, we consider the second priority P_2 :
    The production of type x: amounting to x_1 units ≥ 8
    $x_1 + d_2 - d_2 = 8$. with minimize $P_2 d_2^-$
    Minimize $P_1\, d_1^- + P_2 d_2^-$

Minimize

Table 2.15

	$C_j \to$ VAR / $\downarrow$Basis		0 x_1	0 x_2	P_1 d_1^-	0 d_1^+	P_2 d_2^-	0 d_2^+	0 s_1	X_B	R.R.
I	P_1	d_1^-	40	30	1	–1	0	0	0	600	15
	$P_2 \leftarrow$	d_2^+	1	0	0	0	1	–1	0	8	8 $\to$
	0	s_1	1	1	0	0	0	0	1	12	12
		P_2	0	0	0	0	1	1	0		
		P_1	$\uparrow$–40	–30	0	1	0	0	0		
II	P_1	d_1^-	0	30	1	–1	–40	40	0	280	7
	0	x_1	1	0	0	0	1	–1	0	8	–
	0	s_1	0	1	0	0	–1	1	1	4	4 $\to$
		P_2	0	0	0	0	1	0	0		
		P_1	0	–30	0	1	40	$\uparrow$–40	0		
III	P_1	d_1^-	0	–10	1	–1	0	0	–40	120	
	0	x_1	1	1	0	0	0	0	1	12	
	0	d_2^+	0	1	0	0	–1	1	1	4	
		P_2	0	0	0	0	1	0	0		
		P_1	0	10	0	1	0	$\uparrow$0	40		

17. Dual of problem –1 is as follows.

Maximize $Z = 6x_1 + 5x_2$

subject to the constraints $1x_1 + 1x_2 \le 5$

$3x_1 + 2x_2 \le 12$

with $x_1, x_2 \ge 0$

Dual of problem –2 is as follows.

Maximize $Z = 3x_3 + 4x_4$

subject to the constraints $1x_3 + 2x_4 \le 8$

$2x_3 + 1x_4 \le 10$

with $x_3, x_4 \ge 0$

Dual of Problem –3 is as follows.

Maximize $Z = 6x_1 + 5x_2 + 3x_3 + 4x_4$

subject to the constraints $1x_1 + 1x_2 \le 5$

$3x_1 + 2x_2 \le 12$

$1x_3 + 2x_4 \le 8$

$2x_3 + 1x_4 \le 10$

with $x_1, x_2, x_3, x_4 \ge 0$

Basic Feasible Solutions:

Problem 1 $x_1 = 2, x_2 = 3$, **Max.** $Z = 27$

Problem 2 $x_3 = 4, x_4 = 2$, **Max.** $Z = 20$

Problem 3 $x_1 = 2, x_2 = 3, x_3 = 4, x_4 = 2$, **Max.** $Z = 27 + 20 = 47$

3 Transportation Problem

We cannot solve our problems with the same thinking
we used when we created them.
Albert Einstein

Learning Objectives

AFTER STUDYING THIS CHAPTER, THE STUDENTS WILL BE ABLE TO

- understand applications of LPP in real life situations
- identify resources, constraints on allocations (pre-known storage capacities of destinations), and objective function that minimizes transportation cost.
- understand three different methods—
  (a) North–West Corner rule/method (NWCM)
  (b) Minimum cost method or Matrix Minima method and
  (c) Vogel's Approximation method (VAM) or Penalty method of finding basic feasible solutions (BFS) of the transportation method.
- learn the techniques of verifying optimality of the BFS of transportation cost using Modified Distribution (MODI) method.
- understand the basic concepts of trans-shipment problem and solves it using VAM.

INTRODUCTION

Operations research is an applied decision theory and we attempt to bring feasible solutions to real situation problems and keep on adding extra features to include all possible cases foreseen within the bounds of prevailing realities. The different cases and various algorithms to arrive at feasible solutions date back to many decades.

In 1941, the transportation problem was first studied by F.L. Hitchcock in an attempt to find some feasible solutions of dispatch of a product to different localities. In 1949, T.C. Koopman studied the same type of problems and he added an important feature of finding an economical way to the total cost of transportation.

Such problems of finding feasible and optimal solutions to transportation conform or can be made to conform to the mathematical structure of linear programming problems.

In this chapter, we introduce a generalized transportation model and after giving some mathematical facts; we explain some practical methods of finding feasible and optimal solutions.

3.1 Mathematical Model and Characteristics of Transportation Problem

First, we construct a mathematical model of the transportation problem (TP). This mathematical model will allow us to understand the variables, constraints on resources, and the objective function.

3.1.1 Mathematical Model

(a) Notations

Before designing the model, we introduce certain notations.

Let $F_1, F_2, ..., F_m$ indicate m origins/factories.

Let $a_i(i = 1$ to $m)$ denote the production or available units at each of the m factories.

Let $W_1, W_2,..., W_n$ indicate n warehouses.

Let $d_1, d_2,..., d_n$ denote the storage capacities at the corresponding warehouses.

Let X_{ij} denote the number of units transported from i^{th} factory (F_i) to the j^{th} warehouse (W_j). $(i = 1$ to m and $j = 1$ to $n)$.

Let C_{ij} stand for the cost of transporting **one** unit from i^{th} factory (F_i) to j^{th} warehouse (W_j).

Now, at this stage, we can construct the mathematical model of transportation problem.

(b) Model

Find X_{ij} ($i = 1$ to m; $j = 1$ to n ; total mn variables)

so as to minimize $Z = \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} C_{ij} X_{ij}$ = Total cost of transportation (1)

subject to the conditions

$$\sum_{j=1}^{j=n} X_{ij} = a_i = \text{available units at } i^{th} \text{ factory } (i = 1 \text{ to } m). \tag{2}$$

$$\sum_{i=1}^{i=m} X_{ij} = b_j = \text{demand at } j^{th} \text{ warehouse } (j = 1 \text{ to } n) \tag{3}$$

$$\text{With all } X_{ij} \geq 0 \tag{4}$$

3.1.2 Special Features of the Model

1. The mathematical model of TP is a particular case of linear programming problem. In the above model, we have $m \times n = mn$ unknown variables.
   In (2) above, we have m row equations and in (3) we have n column equations.
2. $\sum_{i=1}^{i=m} a_i = \sum_{j=1}^{i=n} b_j$ is the rim condition for a balanced TP.
3. The case where $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$ or $\sum_{i=1}^{i=m} a_i < \sum_{j=1}^{j=n} b_j$, we have the cases of unbalanced TP.
4. The m row equations and n column equations are ***linearly dependent***. In fact, if we subtract of any $(n - 1)$ column equations from the sum of corresponding sides all m row equations, the result is the remaining n^{th} column equation.
   We have the total number of $(\boldsymbol{m + n - 1})$ ***linearly independent*** equations.

5. The above conditions clearly convey that by solving an independent system of $(m + n - 1)$ linear equations, we can find the values of only $(m + n - 1)$ variables from the mn variables.
6. The condition that $x_{ij} \geq 0$ implies that all mn variables attain non-negative values.

3.1.3 The Nature of the Solution

It is clear from the nature of the mathematical model and its features as discussed above, that the $m + n - 1$ of mn variables are related to $m + n - 1$ linearly independent equations. All these $m + n - 1$ variables are called ***basic*** *variables*.

The remaining variables, i.e. $mn - (m + n - 1)$ are called ***non-basic*** *variables*. Non-basic variables are always zero.

The basic feasible solution vector **X,** has the following nature.

1. $\mathbf{X} = \{x_{ij} / i = 1 \text{ to } m; j = 1 \text{ to } n\}$
2. **X** satisfies all the conditions of the model.
3. $(m + n - 1)$ values of X_{ij}, i.e. basic variables are ≥ 0.
4. $mn - (m + n - 1)$ values of X_{ij}, i.e. non-basic variables = 0.

Now, we take up an illustration and understand the basics that we have discussed.

ILLUSTRATION 1

There are three factories and four warehouses owned by xyz corporation. The manufacturing capacities of factories and holding capacities of warehouses are given below. The entries in the different cells are the cost/unit (c_{ij}) of transporting one unit from i^{th} factory to j^{th} warehouse. We have an immediate problem of making allocations. (Finding a bfs.)

Cost Matrix

To / *From*	W_1	W_2	W_3	W_4	*AVAILABLE UNITS= ai*
F_1	23	27	16	18	30
F_2	12	17	20	51	40
F_3	22	23	12	32	53
CAPACITY = b_j	22	35	25	41	

Primary analysis of the problem

Factory	*Production*	*Warehouse*	**Storage capacity**
F_1	30	D_1	22
F_2	40	D_2	35
F_3	53	D_3	25
	Total = 123	D_4	41
			Total = 123

$$\sum_{i=1}^{3} a_i = 123 = \sum_{j=1}^{4} b_j \text{ i.e. Total production = total storage capacity = 123 units}$$

It is a balanced TP m = no. of rows = 3, n = no. of column = 4
Total no. of cells = $3 \times 4 = 12$ (unknown variables)
BFS will have $\boldsymbol{m + n - 1} = 3 + 4 - 1 = 6$ allocated cells
$mn - (m + n - 1)$ = **non-allocated cells** $= 12 - (3 + 4 - 1) = 6$

3.2 Methods of Finding Basic Feasible Solution (BFS)

There are three methods, widely known and popular, to find a basic feasible solution of any given transportation problem.
These methods are
1. North-West Corner Method
2. Minimum Cost Method
3. Vogel's Approximation Method (Penalty Method)

In the next section, we take up these methods and show their algorithm substantiated by illustrations.

3.2.1 North-West Corner Method (NWCM)

To understand its algorithm, we first take a theoretical table and understand some basic terms.

Cost Matrix

From \ *To*	W_1	W_2	W_3	W_i	W_n	*Available units* = a_i
F_1	C_{11}	C_{12}	C_{13}	–	C_{1n}	a_1
F_2	C_{21}	C_{22}	C_{23}	–	C_{2n}	a_2
F_3	C_{31}	C_{32}	C_{33}	–	C_{3n}	a_3
–	–	–	–	–	–	–
–	–	–	–	–	–	–
F_m	C_{m1}	C_{m2}	C_{m3}	–	C_{mn}	a_m
Demand b_j	b_1	b_2	b_3	–	b_n	

Storage capacity
1. Before we start, check that it is a balanced problem, i.e. whether. $\Sigma a_i = \Sigma b_j$ (Method for solving a non-balanced problem a balanced one is shown in next units.)
2. At any stage, the top-most corner on your left-hand side is the North-West corner. All we need is to make allocation in that corner cell.

Algorithm
1. The top-most left-hand corner cell (1, 1) or (F_1, W_1) with the unit cost C_{11} is our first target.
   The available units in that row are from the factory F_1 which shows maximum supply from it = a_1 units. The demand in that column is b_1 units.
   We have three cases
   $$a_1 < b_1,\ a_1 = b_1 \text{ or } a_1 > b_1$$
   Allocate in that cell with the minimum of a_1 or b_1.
   [If $a_1 < b_1$ then allocation $x_{11} = a_1$
   If $a_1 = b_1$ then allocation $x_{11} = a_1 = b_1$
   If $a_1 > b_1$ then allocation $x_{11} = b_1$]

2. If $a_1 < b_i$, then $x_{11} = a_1$ will leave no available units in first row and hence, it is removed from further calculations. It implies that the first warehouse still needs $(b_1 - a_1)$ units to attain its maximum storage capacity.
   If $a_1 = b_1$, then $x_{11} = a_1 = b_1$ will leave no units in the first row and first column as they are both fully emptied on making allocation. We remove the first row and first column both, from further calculations.
   If $a_1 > b_1$, then $x_{11} = b_1$ will leave no demand units in first column and hence it is removed from further calculations. It is implied that the first row still has $(a_1 - b_1)$ units available for further dispatch.
3. In any one of the above cases, the top-most corner cell is now an allocated one.
   The next cell to be focused is again the top-most corner one from the modified matrix after the first allocation.
   In the case $a_1 < b_1$, the first row having been cancelled, the target cell is (2, 1), i.e. (F_2, W_1) with cost C_{21}.
   In the case $a_1 = b_1$, the first row and first column having been cancelled, the target cell is (2, 2), i.e. (F_2, W_1) with cost C_{22}.
   In the case $a_1 > b_1$, the first column having been cancelled, the target cell for allocation is (1, 2), i.e. (F_1, W_1) with cost C_{12}.
4. Considering the new target cell, repeat the procedure from Step (1) of the algorithm and modify the matrix accordingly.
5. Repeat the chain until you reach the last cell. Make allocation in that cell. Solution of Example 1 by North-West Corner method;

ILLUSTRATION 2

Cost Matrix

To / From	W_1	W_2	W_3	W_4	a_i
F_1	23	27	16	18	30
F_2	12	17	20	51	40
F_3	22	22	12	32	51
b_j	22	35	25	41	

The primary analysis reflects that it is a balanced TP, i.e.

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{j=4} b_j = 123$$

Step I

North-West Corner is the cell in the first row and first column.

$$a_1 = 30 \text{ and } b_1 = 22$$

We have $a_1 < b_1$ and we allocated 22 units in that cell.
We have $x_{11} = 22$ units. $b_1 = 0$ (drop the first column)
Available units in the first row = 30 – 22 = 8 units
i.e. New $a_1 = 8$ units.

Diminished matrix is now as follows:

To / *From*	W_1	W_2	W_3	W_4	*Available Units*
F_1	$\boxed{22}$ 23	27	16	18	30/8
F_2	12	17	20	51	40
F_3	22	22	12	32	53
b_j	22/0	35	25	41	

Step 2

Now consider the North-West corner. (The cell in the first row and second column)

For this cell, available units = 8 (= 30 – 22) and demand = 35 = b_2

Allocate 8 units in the cell

∴ $x_{12} = 8$; the matrix after allocation looks like demand in $W_2 = 35 - 8 = 27$.

Drop first row.

$\boxed{22}$ 23	$\boxed{8}$ 27	16	18	30/8/0
12	17	20	51	40
22	22	12	32	53
22/0	35/27	25	41	

Step 3

The diminished matrix does not consider the first row and the first column. Now consider the left-most corners. (the cell in the second row and the second column)

It still needs 35 – 8 = 27 units, which can be fetched from the availability = $a_2 = 40$ units.

Allocate 27 units in that cell. We have $x_{22} = 27$ units.

This results into $b_2 = 0$, $x_{22} = 27$, and available units in second row = 40 – 27 = 13 units.

				a_i
$\boxed{22}$	$\boxed{8}$			30/8/0
	$\boxed{27}$	$\boxed{13}$		40/13/0
			32	53
	35/27	25/12	41	

Step 4

In the diminished matrix, the North-West corner is the cell a_{23}. The demand of the third warehouse is $b_3 = 25$ units.

Now the available units in the second row is 13 units.

We make an allocation $x_{23} = 13$ units. Because of this, $a_2 = 0$, $x_{23} = 13$, unfulfilled demand for the third warehouse = $b_3 = 25 - 13 = 12$ units. These make now the second row out from the procedure. We have, only the last two cells of the third row, in calculation.

Step 5

In the North-West corner cell in the diminished matrix is a_{33}. It needs 12 more units to satisfy the demand of W_3, i.e. x_{33} = 12 units.

Allocate 12 units from the available units 53 = a_3 = availability of third row

The matrix of allocations is as follows:

$\boxed{22}$ 23	$\boxed{8}$ 27	16	18	30/8/0
12	$\boxed{27}$ 17	$\boxed{13}$		40/13/0
		$\boxed{12}$ 12	32	53/41/0
22/0	35/27/0	25/12/0	41	

Step 6

The result is $b_3 = 0$, $x_{33} = 12$ and the third column is out from the calculation. The only N-W cell left is a_{34}. The available units in the last row are 41. This balances the demand b_4 = 41 units of the fourth column.

Now make the last allocation x_{34} = 41 units.

This satisfies the demand of the fourth warehouse = 41 units = x_{34} = remaining available units in the third row.

The final matrix (row/column of allocation) looks as follows:

$\boxed{22}$ 23	$\boxed{8}$ 27			30/8/0
	$\boxed{27}$ 17	$\boxed{13}$ 51		40/13/0
		$\boxed{12}$ 12	$\boxed{41}$ 32	53/41/0
22/0	35/27/0	25/12/0	41/0	

Analysis of the resultant matrix

m = number of rows = 3

n = number of columns = 4

mn = total number of cells = $m \times n = 3 \times 4 = 12$

$m + n - 1$ = number of basic cells/variable = 3 + 4 – 1 = 6

	Allocation Cost/unit	*Applicable cost*
$X_{11} = 22$	23	$22 \times 23 = 506$
$X_{12} = 8$	27	$8 \times 27 = 216$
$X_{22} = 27$	17	$27 \times 17 = 459$
$X_{23} = 13$	51	$13 \times 51 = 663$
$X_{33} = 12$	12	$12 \times 12 = 144$
$X_{34} = 41$	32	$41 \times 32 = 1312$
		Total = 3300

3.2.2 Minimum Cost Method (Matrix Minima Method)

This is also a method to get a basic feasible solution of the given problem.

(The basic objectives are minimization of transportation cost and hence select the cell having minimum cost entry.)

As the name suggests, we find the minimum cost entry from all cost entries in the given TP. We fill that cell with units, which are minimum of the demand units in that column and available units in that row.

If there is a tie amongst the minimum cost entries, then we select the cell which can assimilate the maximum supply.

Reframe the matrix with the new supply-demand. Search for the next minimum cost entry and make maximum possible allocation to that cell from the available units in that row. Make allocation and reframe the matrix of remaining supply and demand.

Continue in this way till demands of all warehouses are fulfilled from the available supply.

We consider the same example and using this method to find the cost of allocation.

ILLUSTRATION 3

We consider the following transportation problem and solve it using matrix minima method.

Cost Matrix

To / *From*	W_1	W_2	W_3	W_4	*Available Units* = a_i
F_1	23	27	16	18	30
F_2	12	17	20	51	40
F_3	22	28	12	32	53
b_j	22	35	25	41	

Analysis:

$$a_1 = 30,\ a_2 = 40,\ a_3 = 53,\ \sum_{i=1}^{i=3} a_i = 123 \text{ units,}$$

$b_1 = 22$, $b_2 = 35$, $b_3 = 25$, $b_4 = 41$, $\Sigma b_j = 123$ units.

It is a balanced transportation problem.

m = number of rows = 3

n = number of columns = 4

mn = number of cells = $m \times n = 3 \times 4 = 12$

We solve the problem using matrix minima method.

Step I

The least cost within all the different costs is 12. This minimum appears in two different cells—a_{21} and a_{33}.

W_1 has demand of 22 units and can be fulfilled through $a_2 = 40$ units. The warehouse W_3 has a demand of 25 units which can be pushed from $a_3 = 53$ units. To bring the cost down effectively we must fill the cell a_{33}; the reason being W_3 has more demand than that of W_1 has (which can be satisfied).

Allocate 25 units in the cell a_{33} from $a_3 = 53$ units, i.e.

$$x_{33} = 25 \text{ units}$$

Now we have demand of $W_3 = b_3 = 0$ and the third column is out of further consideration.

Available units in the third row is 53 – 25 = 28 units;

The diminished matrix looks like as follows:

From \ To	W_1	W_2	W_3	W_4	Available Units
F_1	23	27	16	18	30
F_2	12	17	20	51	40
F_3	22	28	$\boxed{25}$ 12	32	53/28
b_j	22	35	25/0	41	

Step 2

We have the least cost C_{21} = 12, still open for further proceedings. We allocate x_{21} = 22 units from available a_2 = 40 units.

The diminished matrix looks like

From \ To	W_1	W_2	W_3	W_4	Available Units
F_1	23	27	16	18	30
F_2	$\boxed{22}$ 12	17	20	51	40/18
F_3	22	28	$\boxed{25}$ 12	32	53/28
b_j	22/0	35	25/0	41	

At this stage, W_1 and W_3 are the warehouses with satisfied demands and hence they are not the parts of the remaining procedures.

Step 3

The next lowest cost, in the diminished matrix above, is C_{22} = 17. It is a cell in W_2 and F_2 — cell a_{22}. W_2 needs 35 units but F_2 now can supply only 18 units.

Allocate 18 units in a_{22}. The diminished matrix is as follows:

From \ To	W_1	W_2	W_3	W_4	Available Units
F_1	23	27	16	18	30
F_2	$\boxed{22}$ 12	$\boxed{18}$ 17	20	51	40/18/0
F_3	22	28	$\boxed{25}$ 12	32	53/28
b_j	22/0	35/17	25/0	41	

Now, W_1 and W_3 fully satisfied while F_2 is completely empty.

Step 4

The next least cost is now C_{14} = 18 which appears in the cell a_{14}. The cell a_{14} is a cell in the column for W_4 which needs 41 units and the cell a_{14} can get the units from a_1 = 30 units. Allocate all 30 units, i.e. x_{14} = 30 units.

This will make a_1 = 0 (as available units in F_1, i.e. a_1 = 30 units are allocated). W_1 still needs 41 – 30 = 11 units.

Now, W_1, W_3, F_2 and F_1 are out of remaining proceedings.

The diminished matrix looks like as follows:

To / *From*	W_1	W_2	W_3	W_4	*Available Units*
F_1	23	27	16	[30] 18	30/0
F_2	[22] 12	[18] 17	20	51	40/18/0
F_3	22	28	[25] 12	32	53/28
b_j	22/0	35/17	25/0	41/11	

Step 5

The above matrix shows that there are only two cells left—a_{32} and a_{34}. These fall in the column for W_2 and W_4 respectively.

The remaining requirements in these columns are 17 units and 11 units respectively. Also, these two cells can take their demands from F_3 with 28 units left with it.

There is no further choice left with us; we make allocations according to demand and supply.

We allocate x_{32} = 17 units and finally x_{34} = 11 units from 28 units available in F_3.

This method finally terminates giving following chain of supply and demand.

To / *From*	W_1	W_2	W_3	W_4	*Available Units*
F_1	23	27	16	[30] 18	30/0
F_2	[22] 12	[18] 17	20	51	40/18/0
F_3		[17] 28	[25] 12	[11] 32	53/28/17/11/0
b_j	22/0	35/17/0	25/0	41/11/0	

Observations:

1. Check that the sum of allocations in each row or column equals the available units or demand of that row or column.
2. m = 3, n = 4, number of allocated cells are $m + n - 1 = 3 + 4 - 1 = 6$
3. Transportation cost

Allocation	*Cost/unit*	*Total*
$x_{14} = 30$	18	540
$x_{21} = 22$	12	264
$x_{23} = 18$	17	296
$x_{32} = 17$	28	476
$x_{33} = 25$	12	300
$x_{34} = 11$	32	352
		Total cost = 2228

3.2.3 Vogel's Approximation Method (VAM) OR (Penalty Method)

This is an *iterative method* of finding a basic feasible solution (BFS) to the given TP. This method is also known as *penalty method*. It is observed that in most of the situations, the BFS obtained by using this method, becomes an optimal BFS. The ***algorithmic*** *procedure* is as follows:

1. We calculate the absolute difference between the lowest and the next lowest cost entries for each row and each column. We have m penalty figures—one for each row and n penalty figures—one for each column.
2. Now, select the figure of the highest penalty, and search for the least cost cell in that row or column. On finding that, we make allocation in that cell with the minimum of available (a_i) units or demand (b_j).
   We need to clarify some points at this stage.
   (a) If there is a tie in the highest penalty figures, then compare the least costs in corresponding row or column. You select the cell in which the least cost is minimum. In addition to this, if there is a further tie in the least costs, select the cell in which higher supply can be made. We will mention this reference in illustration.
   (b) In the case of an unbalanced TP, the initial allocations will be in the row with the cell of zero cost.
3. On making allocation in a cell at a given time, either a row or a column or both are satisfied. Cancel that row or the column or both and do not consider that one for further calculations.
4. Again, calculate penalties for the remaining rows and columns and make allocation in the least cost cell.

Repeat this process (Steps 1 to 4) until all columns and rows are fully satisfied.

We take up the same example and apply Vogel's approximation method.

ILLUSTRATION 4

Cost Matrix

To / *From*	W_1	W_2	W_3	W_4	a_i
F_1	23	27	16	18	30
F_2	12	17	20	51	40
F_3	22	28	12	32	53
b_j	22	35	25	41	

Primary Analysis

$$\sum_{i=1}^{i=3} a_i = 123 = \sum_{j=1}^{j=4} b_j$$

It is a balanced problem.
Now $m = 3$, $n = 4$, $mn = 12$
Number of allocated cells should be less than or equal to $m + n - 1 = 3 + 4 - 1 = 6$
We solve it using VAM (or penalty method).

From \ *To*	W_1	W_2	W_3	W_4	a_i	*Penalty*
F_1	23	27	16	18	30	2
F_2	12	17	20	51	40	5
F_3	22	28	12	32	53	10
b_j	22	35	25	41		
Penalty	10	10	4	14↑		

Step 1

Select the highest penalty = 14. Mark it as ↑. Choose the least cost cell $C_{14} = 18$, $a_1 = 30$, $b_4 = 41$. Allocate 30 units.

First row, now, has no available units. The warehouse W_4 still needs $41 - 30 = 11$ units; the final matrix look is as follows:

From \ *To*	W_1	W_2	W_3	W_4	a_i
F_1	22	27	16	$\boxed{30}$ 18	30/0
F_2	12	17	20	51	40
F_3	22	28	12	32	53
b_j	22	35	25	41/11	

Step 2

First row is dropped. Find the new set of penalties. Select the highest penalty = 19 units and mark it by ↑ sign.

From \ *To*	W_1	W_2	W_3	W_4	a_i	*Penalty*
F_1				$\boxed{30}$ 18	30/0	
F_2	12	17	20	51	40	5
F_3	22	28	12	32	53	10
b_j	22	35	25	41/11		
Penalty	10	11	8	19↑		

The least cost cell in the fourth column is the cell with $C_{34} = 32$. At this stage $a_3 = 53$ units and improved $b_4 = 11$.

Give 11 units.
The fourth column is dropped. The third row has 53 – 11 = 42 units left.
The result of combined effect of Steps 1 and 2 is shown as follows:

To / *From*	W_1	W_2	W_3	W_4	a_i
F_1				$\boxed{30}$ 18	30/0
F_2	12	17	20		40
F_3	22	18	12	$\boxed{11}$ 32	53/42
b_j	22	35	25	41/11/0	

Step 3

The first row and the fourth column are not included in the remaining calculation. Calculate penalties. Select the highest penalty = 10 and mark it by ↑ sign.

To / *From*	W_1	W_2	W_3	W_4	a_i	*Penalty*
F_1				$\boxed{30}$ 18	30/0	–
F_2	12	17	20		40	5
F_3	22	18	12	$\boxed{11}$ 32	53/42	6
b_j	22	35	25	41/11/0		
Penalty	10↑	1	8	–		

Find the minimum cost = C_{21} = 12 in the first column. Here a_2 = 40 and b_1 = 22 units. Allocate 22 units. This satisfies the need of W_1. Now available units in F_2 is 40 – 22 = 18 units. The combined result of all the three steps is shown as follows.

To / *From*	W_1	W_2	W_3	W_4	a_i
F_1				$\boxed{30}$ 18	30/0
F_2	$\boxed{22}$ 12	17	20		40/18
F_3		18	12	$\boxed{11}$ 32	53/42
b_j	22/0	35	25	41/11/0	

Step 4

The first row, the fourth column, and the first column are free from further calculations. Now, find the penalties. The highest penalty is = 8 units. Mark this by ↑. Allocate $b_3 = 25$ and improved $a_3 = 42$ units. Search for the least cost cell. It is $C_{33} = 12$ units.

To / From	W_1	W_2	W_3	W_4	a_i	*Penalty*
F_1				$\boxed{30}$ 18	30/0	
F_2	$\boxed{22}$ 12	17	20		40/18	3
F_3		18	12	$\boxed{11}$ 3	53/42/6	
b_j	22/0	35	25	41/11/0		
Penalty		1	8↑			

Allocate 25 units, i.e. $x_{33} = 25$ units. This satisfies the need of W_3. We drop third column. The third row has 42 – 25 = 17 units. The combined effect of all these Steps, 1, 2, 3 and 4 is as follows:

To / From	W_1	W_2	W_3	W_4	a_i
F_1				$\boxed{30}$ 18	30/0
F_2	$\boxed{22}$ 12	17			40/18
F_3		18	$\boxed{25}$ 12	$\boxed{11}$ 32	53/42/17
b_j	22/0	35	25/0		

Step 5

Looking at the above matrix, we have $C_{22} = 17$ and improved $a_2 = 18$ units.
Also $C_{32} = 18$ and $b_2 = 35$ units. The only one entry for the penalty is 18 – 17 = 1 unit.
The least cost cell is $C_{22} = 17$, $b_2 = 35$ and improved $a_2 = 18$ units.

Allocate all 18 units, i.e. $x_{22} = 18$ units. This makes second row empty and hence we drop it. The combined result is shown as follows:

To / From	W_1	W_2	W_3	W_4	a_i	*Penalty*
F_1				$\boxed{30}$ 18	30/0	–
F_2	$\boxed{22}$ 12	$\boxed{18}$ 17			40/18/0	–
F_3		18	$\boxed{25}$ 12	$\boxed{11}$ 32	53/42/17	–
b_j	22/0	35/17	25/0	41/11/0		
Penalty	–	–	1↑	–	–	

Step 6

We have allocations at five different cells. The first row, the second row, the first column, the third column, and the fourth column are out of further consideration.

We have $b_2 = 17$ units and improved = 17 units. There is only one entry and that is $C_{32} = 18$. There is no question of finding any penalty and we have the matrix of allocation as follows:

Matrix of Allocation

To / From	W_1	W_2	W_3	W_4	a_i
F_1				30	30/0
F_2	22	18			40/18/0
F_3		18	25	11	53/42/17
b_j	20/0	35/17	25/0	41/11/0	

$C_{32} = 18$, improved $b_2 = 17$ units, improved $a_3 = 17$ units.
Allocate 17 units in the cell, i.e. $x_{32} = 17$ units.

This will make all availabilities and all demands fully satisfied. The final matrix of allocation with corresponding cost entries is as follows:

To / From	W_1	W_2	W_3	W_4	a_i
F_1				30 / 18	30
F_2	22 / 12	18 / 17			40
F_3		17 / 18	25 / 12	11 / 32	53
b_j	22	35	25	41	

Let us now calculate the transportation cost

$x_{14} = 30$	$C_{14} = 18$	$30 \times 18 = 540$
$x_{21} = 22$	$C_{21} = 12$	$22 \times 12 = 264$
$x_{22} = 18$	$C_{22} = 17$	$18 \times 17 = 306$
$x_{32} = 17$	$C_{32} = 18$	$17 \times 18 = 306$
$x_{33} = 25$	$C_{33} = 12$	$25 \times 12 = 300$
$x_{34} = 11$	$C_{34} = 32$	$11 \times 32 = 352$

The sum total of transportation cost = 2068
There are $m + n - 1 = 3 + 4 - 1 = 6$ allocations.

Comparison of Costs:

We have solved one example using three different methods. The o*bj*ectives are

1. to find a basic feasible solution, and

2. to check it for its optimality of cost. (If the cost is not minimum then we have to make necessary reshuffling of allocations and see that the transportation cost is minimized.)
   For the problem we have solved,

Method	Transportation cost
1. North-West Corner Method	3300
2. Minimum Cost Method	2228
3. Vogel's Approximation Method	2068

It is likely that the most effective method is the penalty method (VAM).

The solution obtained has $m + n - 1 = 3 + 4 - 1 = 6$ allocated cells. Also allocated units are non-negative.

The question, still pending, is about the optimality of the cost. We want to verify that this cost is the minimum cost of transportation. If it is not minimum then we have to make another search for finding a procedure of reallocations of some units to satisfy optimality.

3.3 Unbalanced Transportation Problem

We have studied the mathematical model of transportation problem. The rim condition for the existence of the basic feasible solution is:

$$\sum_{i=1}^{i=m} a_i = \text{Total available units} = \text{Total demand} = \sum_{j=1}^{j=n} b_j$$

The case when $\sum_{i=1}^{i=m} a_i \neq \sum_{j=1}^{j=n} b_j$; we have a balanced TP

There are two cases.

1. $\sum_{i=1}^{i=m} a_i > \sum_{j=1}^{j=n} b_j$, i.e. $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j > 0$

   i.e. Total of available units exceeds the total of demands. This is Type I unbalanced problem.

2. $\sum_{i=1}^{i=m} a_i < \sum_{j=1}^{j=n} b_j$, i.e. $\sum_{i=1}^{i=m} a_i - \sum_{j=1}^{j=n} b_j < 0$

   It means that the total of the demands exceeds the total of available units. This is Type II unbalanced problem.
   Illustration for the Case 1, i.e. $\Sigma a_i > \Sigma b_j$

Illustration 5

Type I (Unbalanced TP)

Cost Matrix

From \ *To*	W_1	W_2	W_3	a_i
F_1	10	8	12	100
F_2	12	9	7	200
b_j	50	60	70	

Note that $\Sigma a_i = 100 + 200 = 300$ total of available units.

$\Sigma b_j = 50 + 60 + 70 = 180$ total of demands.

$\Sigma a_i > \Sigma b_j$

It is an unbalanced problem of Type I.

We take one more problem and show unbalanced problem of Type II.

ILLUSTRATION 6

Type II (unbalanced TP)

Cost Matrix

From \ To	W_1	W_2	W_3	W_4	a_i
F_1	2	5	8	9	150
F_2	3	5	7	12	50
b_j	30	40	70	90	

In this case $\sum_{i=1}^{i=2} a_i$ = 200 units = Total of available units

and

$$\sum_{j=1}^{3} b_j = 230 \text{ units = Total of demands.}$$

We see that $\Sigma a_i < \Sigma b_j$.

i.e. $\Sigma a_i - \Sigma b_j = 200 - 230 = -30 < 0$.

How do we tackle these situations?

In both the cases that we have seen above, we have

$$\Sigma a_i \neq \Sigma b_j.$$

We know that before solving any given TP, we have the necessary and sufficient condition $\sum_{i=1}^{i=m} a_i = \sum_{j=1}^{j=n} b_j$ must be satisfied.

In the Type I unbalanced TP, we have,

$$\Sigma a_i - \Sigma b_j > 0.$$

This means that the total demand or capacity of storage units fall short than the total number of available units.

To satisfy the condition $\Sigma a_i = \Sigma b_j$, we have to generate an additional warehouse having its demand or storage capacity which equals $(\Sigma a_i - \Sigma b_j)$ units.

In this case $\Sigma b_j = 230 > \Sigma a_i = 200$

and $$\Sigma b_j - \Sigma a_i = 230 - 200 = 30 \text{ units.}$$

We have the basic condition that $\Sigma a_i = \Sigma b_j$. We generate an additional supplier to provide $(\Sigma a_i - \Sigma b_j)$ units to anyone or more of the existing warehouses. Thus, the problem on introduction of additional supplier, (say F_3), looks as follows.

Cost Matrix

From \ To	W_1	W_2	W_3	W_4	a_i
F_1	2	5	8	9	150
F_2	3	5	7	12	50
F_3	0	0	0	0	30
b_j	30	40	70	90	

With $\Sigma a_i = 230$ units $= \Sigma b_j = 30 + 40 + 70 + 90 = 230$ units; this problem can be solved using anyone of the standard methods.

Now, let us study the following unbalanced transportation problem of Type II.

From \ To	W_1	W_2	W_3	a_i
F_1	10	8	12	100
F_2	12	9	7	200
b_j	50	60	70	

Total available units $\Sigma a_i = 300$ units and the total demand $= \Sigma b_j = 280$ units.

We need to store $300 - 180 = 120$ units. We create a dummy warehouse W_4 with $\Sigma a_i - \Sigma b_j = 300 - 180 = 120$ units of storage capacity.

This process solves the problem making $\Sigma a_i = \Sigma b_j$ condition satisfied. We note one more point that the cost coefficient entries of all the cells under the virtual warehouse column, will be taken as zero. This is so because, this warehouse has only one job and that is to absorb the additional amount $\Sigma a_i - \Sigma b_j$ and render its work making $\Sigma a_i = \Sigma b_j$. This problem, when applied the above criteria, looks as follows.

From \ To	W_1	W_2	W_3	W_4	a_i
F_1	10	8	12	0	100
F_2	12	9	7	0	200
b_j	50	60	70	120	

we have a balanced problem.

$$\Sigma a_i = 100 + 200 = 300 = 50 + 60 + 70 + 120 = \Sigma b_j.$$

The above problem can be solved using any one of the standard algorithms. Now we take an example of an unbalanced transportation problem and solve it using the penalty method.

ILLUSTRATION 7

This problem will show all the different important features of a transportation problem.

Cost Matrix

From \ To	Warehouses W_1	W_2	W_3	W_4	Available units a_i
F_1	3	5	2	2	90
F_2	8	9	2	5	80
F_3	2	8	9	8	90
b_j	50	65	65	70	

Primary Analysis

$$\Sigma a_i = 260 \text{ units},$$
$$\Sigma b_j = 250 \text{ units},$$
$$\Sigma a_i \neq \Sigma b_j; \text{ it is an unbalanced TP.}$$
$$\Sigma a_i - \Sigma b_j = 260 - 250 = 10 \text{ units.}$$

We generate an additional warehouse W_5 having storage capacity of 10 units;

Cost Matrix

From \ To	W_1	W_2	W_3	W_4	W_5	Available units a_i
F_1	3	5	2	2	0	90
F_2	8	9	2	5	0	80
F_3	2	8	9	8	0	90
Demand b_j	50	65	65	70	10	

Step I

We find penalty for each row and each column. (Penalty is the difference or opportunity loss. It is the absolute difference between the lowest cost and the next lowest cost). Select the highest penalty and make an allocation in the least cost cell. The following matrix shows the working.

Cost Matrix

From \ To	W_1	W_2	W_3	W_4	W_5	Available units a_i	Penalty
F_1	3	5	2	70 2	0	90/20	2
F_2	8	9	2	5	0	80	2
F_3	2	8	9	8	0	90	2
Demand b_j	50	65	65	70/0	10		
Penalty	1	3	0	↑3	0		

Important Note

1. There is a tie in the highest penalty = 3.

2. Column 2 corresponds to W_2 and column 4 corresponds to W_4. They need 65 units and 70 units respectively. To meet the purpose of cost reduction, we choose to allocate 70 units. This can be done in cell (1, 4) through. a_1 = 90 units.
3. This will satisfy the demand of W_4 and hence it will be out of consideration. New $a_1 = 90 - 70 =$ 20 units.

Step 2

We repeat this process and make remaining allocations.

To / *From*	W_1	W_2	W_3	W_4	W_5	a_i	*Penalty*				
F_1	3	$\boxed{20}$ 5	2	$\boxed{70}$ 2	0	90/20/0	2	2	–	–	–
F_2	8	$\boxed{5}$ 9	$\boxed{65}$ 2	5	$\boxed{10}$ 0	80/15/5/0	2	2	2	← 8	1
F_3	$\boxed{50}$ 2	$\boxed{40}$ 8	9	8	0	90/40/0	2	2	2		6
Demand b_j	50/0	65/45 /40/0	65/0	70/0	10/0						
Penalty	1	3	0	↑3	0						
	1	3↑	0	–	0						
	6	1	7↑	–	0						
	6	1	–	–	0						
	6↑	1	–	–	–						

Note the effects of steps as follows:

1. C_4 dropped.
2. R_1 dropped.
3. C_3 dropped.
4. C_5 dropped.
5. C_1 dropped.
6. The second column remains with only two cells with C_{22} = 9 units and C_{32} = 8 units.
7. We allocate 5 units in the cell (2, 2) from improved a_2 = 5 units. Also we allocate 40 units in the cell (3, 2) from improved a_3 = 40 units.

This completes the procedure. Now we calculate the cost of transportation.

Allocation x_{ij}	*Unit Cost C_{ij}*	*Cost*
$x_{12} = 20$	5	100
$x_{14} = 70$	2	140
$x_{22} = 05$	9	045
$x_{23} = 65$	2	130
$x_{25} = 10$	0	000
$x_{31} = 50$	2	100
$x_{32} = 40$	8	320
		Total cost = 835

Comment

(i) All allocations $x_{ij} > 0$.
(ii) There are $m + n - 1 = 3 + 5 - 1 = 7$ allocation.
(iii) It is a basic feasible solution.

3.4 DEGENERACY IN TRANSPORTATION PROBLEM

Let us consider the following problem and solve it using different methods that we have studied so far.

ILLUSTRATION 8

Solve this problem using different methods.

Cost Matrix

To / *From*	W_1	W_2	W_3	a_i
F_1	2	4	7	50
F_2	5	2	4	100
b_j	50	80	20	

(a) Analysis: For the above problem, we have m = number of rows = 2; n = number of columns = 3; there are $m \times n = 2 \times 3 = 6$ cells.

$$\Sigma a_i = 50 + 100 = 150 \text{ units;}$$
$$\Sigma b_j = 50 + 80 + 20 = 150 \text{ units}$$

and so we have $\Sigma a_i = \Sigma b_j = 150$ units. It is a balanced TP.

Solution by North-West Corner Method

Step I

To / *From*	W_1	W_2	W_3	a_i
F_1	[50] 2	4	7	50/0
F_2	5	2	4	100
b_j	50/0	80	20	

(Allocation of 50 units is done in North-West corner. Row 1 and column 1 are fully satisfied and so they are no more in further calculations.)

Step 2

To / *From*	W_1	W_2	W_3	a_i
F_1	[50] 2			50/0
F_2		[80] 2	4	100/20
b_j	50/0	80/0	20	

(Allocation of 80 units is done, i.e. $X_{22} = 80$ units. Column 2 has acquired its total requirement from $a_2 = 100$. There are $100 - 80 = 200$ units left with F_2. The final matrix look is as follows.)

Step 3

To / From	W_1	W_2	W_3	a_i
F_1	[50] 2			50/0
F_2		[80] 2	[20] 4	100/20/0
b_j	50/0	80/0	20/0	

(We allocate 20 units in the only cell left and hence $X_{23} = 20$ units.)
We now calculate the total cost of transportation.

Units	*Cost*	*Total*
$x_{11} = 50$	$C_{11} = 2$	$50 \times 2 = 100$
$x_{22} = 80$	$C_{22} = 2$	$80 \times 2 = 160$
$x_{23} = 20$	$C_{23} = 4$	$20 \times 4 = 80$
		Total = 340

(b) Comment on Degenerate Solution:
There are three allocated cells. In fact, the mathematics behind this conveys that there must be (at the most) $m + n - 1 = 2 + 3 - 1 = 4$ allocated cells. Actual allocations fall short by one cell. The only compromise that we make is to accept that there must be one more cell for which $x_{ij} = 0$. This contributes zero cost.

If the actual numbers of allocated cells are less than the number $\boldsymbol{m + n - 1}$, then it is a case of ***degenerate solution***. There is no way but to accept the mathematical fact of having $m + n - 1$ number of allocated cells, for all such cells $x_{ij} \geq 0$.

These $(m + n - 1)$ cells represent ***basic variables*** for which $x_{ij} \geq 0$. The remaining cells, $mn - (m + n - 1)$, i.e. total number of cells—basic variable cells, are the cells which represent the non-basic variables. Allocations in these non-basic cells are zero.
i.e. for basic cells $x_{ij} \geq 0$, $(m + n - 1)$ cells and for non-basic cells $x_{ij} = 0$, $mn - (m + n - 1)$ cells.

If at least one of the $(m + n - 1)$ basic cells has zero allocation, then the solution is called a ***degenerate solution***.

Solution by Minimum Cost Method

ILLUSTRATION 9

We consider the same problem and solve it by minimum cost method.

Cost Matrix

To / From	W_1	W_2	W_3	a_i
F_1	2	4	7	50
F_2	5	2	4	100
b_j	50	80	20	

We have already made analysis of this situation in Illustration 8.

Step I

Minimum cost is found in two different cells $C_{11} = 2 = C_{22}$. It is important to note that W_2 needs 80 units (which can be supplied also) than the units required by W_1 which requires 50 units.

We select $C_{22} = 2$ and allocate 80 units from $a_2 = 100$ units. This will remove C_2 from further consideration and new $a_2 = 100 - 80 = 20$ units.

The resultant matrix is as follows:

Cost Matrix

From \ To	W_1	W_2	W_3	a_i
F_1	2		7	50
F_2	5	$\boxed{80}$ 2	4	100/20
b_j	50	80/0	20	

Step 2

Now, the next minimum cost is also $C_{11} = 2$. W_1 needs $b_1 = 50$ units which can be supplied from $a_1 = 50$ units.

Making this favour, we have $x_{11} = 50$ units. First row and the first column both are satisfied with this process and so they are out from remaining calculation. The resultant matrix is as follows.

Cost Matrix

From \ To	W_1	W_2	W_3	a_i
F_1	$\boxed{50}$ 2			50/0
F_2		$\boxed{80}$ 2	4	100/20
b_j	50/0	80/0	20	

Step 3

There is only one cell left for which $C_{23} = 4$ units.

There is no option but to allocate 20 units in that cell. Finally we have the following matrix of allocation.

Cost Matrix

From \ To	W_1	W_2	W_3	a_i
F_1	$\boxed{50}$ 2			50/0
F_2		$\boxed{80}$ 2	$\boxed{20}$ 4	100/20/0
b_j	50/0	80/0	20/0	

All the cells are simultaneously satisfied, we have a degenerate solution. (This always happens when we make the last allocation; in that case it does not become degenerate.)

In order to solve degeneracy and to establish the current degenerate solution as a basic feasible non-degenerate solution, allocate very small unit 'θ' (Theta) to an independent least cost cell. To avoid complications and doubts, choose the least cost non-allocated cell, which is adjacent to the cell that caused degeneracy.

We now calculate the total cost of transportation. [**We will show this allocation of θ units in illustration 10 and then verify the optimality of cost.**] In the current context, we find the transportation cost.

Units	*Cost*	*Total*
$x_{11} = 50$	$C_{11} = 2$	$50 \times 2 = 100$
$x_{22} = 80$	$C_{22} = 2$	$80 \times 2 = 160$
$x_{23} = 20$	$C_{23} = 4$	$20 \times 4 = 80$
		Total = 340

(b) Comment on the degenerate solution:

Applying minimum cost method, we have the same allocations in the same cells as we have obtained in applying North-West corner method. Here also, it is a case of degenerate solution.

$$m = 2, n = 3, m + n - 1 = 4 \text{ – basic cells.}$$

Actual number of allocated cells is 3. It means that there exists one more basic cell with $x_{ij} = 0$ unit.

3.5 How to Solve Degeneracy?

In the next section, we will discuss two different methods to check the optimality of the transportation cost and under the case of non-optimality once established, will discuss the method of finding alternate solution by reshuffling the allocations to achieve optimality of the cost.

In order to apply these methods, we have the major requirement of a basic feasible non-degenerate solution.

In the process of making allocations using any one of the three methods, when the row and the column corresponding to the just allocated cell are simultaneously satisfied, we have a degenerate solution. (This always happens when we make the last allocation; in that case it does not become degenerate.)

In order to solve degeneracy and to establish the current degenerate solution as a basic feasible non-degenerate solution, allocate very small unit 'θ' (Theta) to an ***independent least cost cell***. To avoid complications and doubts, choose the least cost non-allocated cell which is adjacent to the cell which caused degeneracy.

3.6 Modified Distribution Method (MODI Method)

This method helps us determine the optimality of transportation cost of allocation. If not so then it also, by a finite number of iterations of reshuffling of allocations by internal loop/chain, brings cost to minimum.

The first important condition to begin with is to have $m + n - 1$ number of allocated cells. (In the case of degenerate solution, we have less than $(m + n - 1)$ number of cells. In addition to this, we have also discussed the method of solving degeneracy.

The following algorithm will be applied for solving the above-mentioned objectives.

1. There must be $(m + n - 1)$ number of allocated cells.
2. Find u_i and v_j ($i = 1$ to m, $j = 1$ to n) so that $C_{ij} = u_i + v_j$ for **allocated cells**.
   u_i or v_j (any one) which corresponds to a row or column of maximum allocated cells should be taken as any arbitrary value (generally zero.) If there is a tie in the selection then choice is optimal. Generally, u_i or $v_j = 0$ is taken.
3. Find $\Delta_{ij} = C_{ij} - (u_i + v_j)$ for non-allocated cells.
4. **Decision Criteria**
   (a) If all Δ_{ij} entries for non-allocated cells are positive, then the allocation cost is minimum.
   (If some Δ_{ij} for non-allocated cells are zero, then it is a case of multiple optimal solutions. It indicates that there exists some more sets of $(m + n - 1)$ number of allocated cells (basic feasible solutions) and each one of the multiple optimal solution has the same (minimum) transportation cost.)
   (b) If some Δ_{ij} entries (for non-allocated) cells are negatives, then the current basic feasible solution does not give minimum cost of transportation.
   It requires reshuffling some of the allocations forming loops with the most negative Δ_{ij} entry. We study this in the next units.

3.7 Loops and Optimization

Let us note the fact that all $m + n - 1$ number of allocated cells are the basic cells and they are independent. We cannot connect them in the form of a closed loop or chain.

3.7.1 Loop

1. A loop is a closed chain of line segments obtained by joining even number of cells greater than or equal to four.
2. A loop has exactly **one** non-allocated cell with the most negative Δ_{ij} entry and remaining ***odd number*** of cells from $m + n - 1$ ***allocated cells.***
3. In a given situation, i.e. there can be exactly one loop with the above described property.

Some of the loops may look like as follow:

(The cell with √ mark is non-allocated one. With the most negative Δ_{ij} entry)

Intersection of lines, as shown in Figure 3.1, line segment is not a cell.

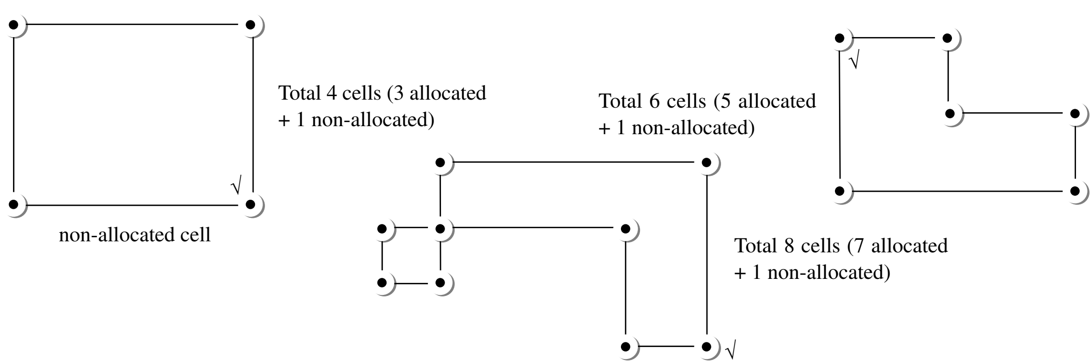


Figure 3.1

3.7.2 Optimization Technique

The following steps will enable us making reshufflings. It will make allocation in the non-allocated cell (cell with √ mark and most negative Δ_{ij} entry) and empty one allocated cell thus preserving $(m + n - 1)$ number of allocated cells.

1. The cell with √ mark; we call it an *acceptor cell* (A). The next cell either in the row or column is called a *donor cell* (D). The next that follows the donor cell is an acceptor cell and so on. (Study this pattern. A = Acceptor cell, D = Donor cell)

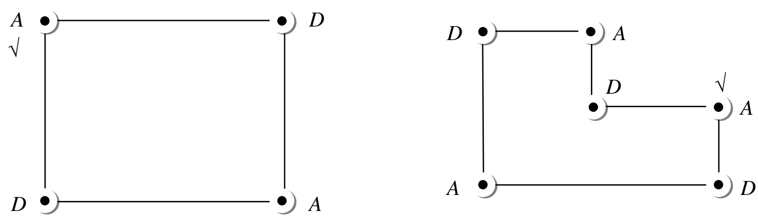


Figure 3.2

2. Check the availability with all the donors. Find the donor with minimum availability, if tie, choice is optional.
3. The fact is that the minimum of all the donors is the allocation that can be donated.
4. Beginning with the right marked (√) allocated cell make pairs of acceptor cells and the donor cells. (One can move in any direction, row-wise or column-wise to make pairs.) Each donor donates that minimum amount (minimum allocated amount in donor cells) to its paired acceptor cell. Thus make all pair-wise transactions.
5. As a result of this, the first √ marked non-allocated cells is row, an allocated cell and (any) one cell having minimum allocated amount becomes empty and thus, is row a non-basic (non-allocated) cell. This chain does not change the number $(m + n - 1)$ of allocated cell. We have a new matrix of allocation.
6. Again, the modified distribution method is applied on the new matrix of allocation. We find $\Delta_{ij} = C_{ij} - (u_i + v_j)$ for non-allocated cells.
7. The above process is continued until the optimality criterion—Step 4(a) of MODI method is satisfied.

ILLUSTRATION 10

In the previous unit, we have already given an Illustration 9. We solved the problem using North-West corner method and matrix minima method. The feasible solution obtained by both the methods is a basic degenerate one. We bring that problem in this unit and apply MODI method.

Cost Matrix

To / From	W_1	W_2	W_3	a_i
F_1	2	4	7	50
F_2	5	2	4	100
b_j	50	80	20	

Solution (Basic Degenerate)

To / From	W_1	W_2	W_3	a_i
F_1	$\boxed{50}$ 2	4	7	50/0
F_2	5	$\boxed{80}$ 2	20 4	100/20/0
b_j	50/0	80/0	20/0	

(m = rows = 2, n = column = 3, $m + n - 1 = 2 + 3 - 1 = 4$: actual number of allocated cells is 3. It is a degenerate solution. In the cell (1, 1), we have done an allocation of 50 units and on that, the first row and the first column entries of availability and demand (= 50) both are satisfied. Hence, both are dropped from the further calculations.)

We, now solve degeneracy. We allocate θ units in a least cost independent cell. ($\theta \rightarrow 0$)

To / From	W_1	W_2	W_3	a_i
F_1	$\boxed{50}$ 2	$\boxed{\theta}$ 4	7	50/0
F_2	5	$\boxed{80}$ 2	$\boxed{20}$ 4	100/20/0
b_j	50/0	80/0	20/0	

Now, there are 4 allocated cells. We apply MODI method of verification of optimality.

We find $\boldsymbol{C_{ij} = u_i + v_j}$ **for allocated cells.**

To / From	W_1	W_2	W_3	u_i
F_1	$\boxed{50}$ 2	$\boxed{\theta}$ 4		$u_1 = 0$
F_2		$\boxed{80}$ 2	$\boxed{20}$ 4	$u_2 = -2$
V_j	$v_1 = 2$	$V_2 = 4$	$V_3 = 6$	

Now, we find $\Delta_{Ij} = \boldsymbol{C_{ij} - (u_i + v_j)}$ **for non-allocated cells.**

To / From	W_1	W_2	W_3	u_i
F_1	$\boxed{50}$ 2	$\boxed{\theta}$ 4		$u_1 = 0$
F_2		$\boxed{80}$ 2	$\boxed{20}$ 4	$u_2 = -2$
v_j	$V_1 = 2$	$V_2 = 4$	$v_3 = 6$	

The first non-allocated cell is (F_1 , W_3) for which $\Delta_{13} = C_{13} - (u_1 + v_3)$

$$\Delta_{13} = 7 - (0 + 6) = 1$$

The second non-allocated cell is (F_2 , W_1) for which $\Delta_{21} = C_{21} - (u_2 + v_1)$

$$\Delta_{21} = 5 — (-2 + 2) = 5$$

Both, Δ_{13} and Δ_{21} are ≥ 0; so the transportation cost is minimum and the assignment is optimum feasible.

The cost is calculated as follows.

Units	*Cost*	*Total*
$x_{11} = 50$	$C_{11} = 2$	$50 \times 2 = 100$
$x_{22} = 80$	$C_{22} = 2$	$80 \times 2 = 160$
$x_{23} = 20$	$C_{23} = 4$	$20 \times 4 = 80$
		Total = 340

[More problems on MODI method are done in the Additional Questions part.]

3.8 TRANS-SHIPMENT PROBLEM

In real life situations, we face this problem in many situations. In a transportation problem; we have distinct origins and distinct/identified destination, where we send goods.

In some cases, it may not be possible to send goods directly to the destination or it can prove very costly. The transporter may not agree to spare the truck on a special route unless he has enough amounts of goods to carry. There are some cases in which the transportation from a particular origin may be cheaper also. In such cases the origins/factories themselves transfer the goods to a pre-determined origin/factory and, in turn, the receiver factory, after a sufficient collection of goods from all such dispatching factories, plans to send the goods to some pre-defined or identified destination. (i.e. some origins work as origins and as well as receivers/destination)

Also there are cases, in which one receiver or destination on receipt of goods play role as origin and dispatches the goods to the other destination which cannot receive the good directly from original source of origin. (i.e. some destination, on receipt of goods, play role as origins and transport goods to other destinations.)

All the facts described above play important role. We call a trans-shipment problem as an extended version of transportation problem.

Basics About the Problem

In a ***transportation problem*** there are m rows (sources) and n columns (destination) but in this case, in general, we can write that, there are $(m + n)$ sources and $(m + n)$ destinations. The basic feasible solution to such a problem will involve $[(m + n) + (m + n) - 1] = 2m + 2n - 1$ number of basic variables. Also the matrix of transportation is a square matrix of the size $(m + n)$. Hence there are $(m + n)$ entries in the leading diagonal. If we omit all such $(m + n)$ entries then the resultant matrix has $(m + n - 1)$ number of ***basic*** variables.

To understand the fundamentals concerning the problem, we take a transportation problem and extend it to explore the possibilities of converting it to a trans-shipment problem.

ILLUSTRATION 11

Origins A and B are given with $a_1 = 300$ units and $a_2 = 200$ units.

Destinations d_1 d_2 and d_3 are given with a demand of 100, 150 and 250 units respectively. In addition to this, we are given the cost matrix as follows.

Cost Matrix

		Origins		*Destinations*			
		A	*B*	d_1	d_2	d_3	a_i
Origins	A	0	6	7	8	9	300
	B	6	0	4	1	7	200
Destinations	d_1	7	2	0	8	5	
	d_2	1	4	1	0	3	
	d_3	5	2	5	3	0	
Demand = b_j				100	150	250	

Solution

To get a basic feasible solution ,we apply VAM.

For the trans-shipment problem,

buffer stock = total demand = total availability = 500 units.

We represent the cost matrix as follows.

Trans-shipment Cost Matrix

Origins				*Destinations*				*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*
		A	*B*	d_1	d_2	d_3	a_i							
Origins	A	$\boxed{500}$ 0	6	$\boxed{100}$ 7	$\boxed{200}$ 8	9	800/300 /200/0	← 6	1	1	1	1	1	1
	B	6	$\boxed{500}$ 0	4	$\boxed{200}$ 1	7	700 /200/0	1	1	1	3	3	← 3	–
Destinations	d_1	7	2	$\boxed{500}$ 0	8	5	500/0	2	2	2	← 5	–	–	–
	d_2	1	4	1	$\boxed{250}$ 0	250 3	500 /250/0	1	1	1	1	1	1	1
	d_3	5	2	5	3	$\boxed{500}$ 0	500/0	2	2	–	–	–	–	–
Demand = b_j		500/0	500/0	600/ 100/0	650/450 /200/0	750/ 250/0								
(1)		1	2	1	1	3								
(2)		–	2	1	1	↑3								
(3)		–	↑2	1	1	2								
(4)		–	–	1	1	2								
(5)		–	–	3	1	↑4								
(6)		–	–	3	1	–								
(7)		–	–	6	↑8	–								

Contd.

Operation	Result
1.	**C_1 vacated**
2.	**R_5 Vacated**
3.	**C_2 Vacated**
4.	**R_3 Vacated**
5.	**C_5 Vacated**
6.	**R_2 Vacated**
7.	**R_4 Vacated**

Remaining two operations are natural operations and carried out without selection of penalty.

Comment: The allocation x_{45} = 250 units is an interesting one. It implies that the destination d_2 sends 250 units to the destination d_3 incurring a cost of ₹250 × 3 = ₹750

Allocations	*Cost*	*Total*
$x_{13} = 100$	$c_{13} = 7$	700
$x_{14} = 200$	$c_{14} = 8$	1600
$x_{24} = 200$	$c_{24} = 1$	200
$x_{44} = 250$	$c_{44} = 0$	000
$x_{45} = 250$	$c_{45} = 3$	750

There are diagonal allocations also, but the cost in those cells being zero; have not shown here. The sum total of cost is ₹3200.

We have one more trans-shipment problem, which is as follows:

ILLUSTRATION 12

There are two factories F_1 and F_2. Available units at these two are 200 and 600 respectively. There are three destinations where the goods can be stored. These destinations d_1, d_2 and d_3 have storage capacities b_1 = 150 units, b_2 = 250 units, and b_3 = 400 units respective. From the transportation cost given below, construct a trans-shipment problem and find a basic feasible solution.

Cost Matrices:

1.

	d_1	d_2	d_3	a_i
F_1	5	9	8	200
F_2	7	6	10	600
b_j	150	250	400	

2.

	F_1	F_2
F_1	0	8
F_2	9	0

3.

	d_1	d_2	d_3
d_1	0	8	5
d_2	10	0	9
d_3	8	9	0

4.

	F_1	F_2
d_1	7	8
d_2	9	10
d_3	12	10

Solution

We first construct the trans-shipment cost matrix and then solve it.

The basic idea is,

1. factories **cannot** have demand/storage capacities
2. destinations **cannot** have production capacities.

There is a demand-supply chain in which the buffer stock = $\Sigma a_i = \Sigma b_j$ = 800 units will move. It can be shown as follows.

Trans-shipment Cost Matrix:

	F_1	F_2	d_1	d_2	d_3	a_i
F_1	0	8	5	9	8	200 + 800 = 1000
F_2	9	0	7	9	10	600 + 800 = 1400
d_1	7	8	0	8	5	800
d_2	9	10	10	0	9	800
d_3	12	10	8	9	0	800
Demand b_j	800	800	150 + 800 = 950	250 + 800 = 1050	400 + 800 = 1200	

[Once again, note that factories do not have storage capacities and warehouses do not have production capacities; bearing this concept, we have 800 units-pseudo allocations.]

Step I

We apply Vogel`s approximation method; **Penalty**

	F_1	F_2	d_1	d_2	d_3	a_i	1	2	3	4	5	6
F_1	800 0	8	(150) 5	9	(50) 8	1000/200 /50/0	5	5	5	5	3	←3
F_2	9	800 0	7	(250) 9	(350) 10	1400/600	7	7	7	2	2	2
d_1	7	8	800 0	8	5	800/0	5	5	5	5	5	–
d_2	9	10	10	800 0	9	800/0	←9	–	–	–	–	–
d_3	12	10	8	9	800 0	800/0	8	←8	–	–	–	–
	800/0	800/0	950/ 150/0	1050/ 250/0	1200/400 /350 / 0							

Penalty:						
	1	7	8	5	8	5
	2	7	8	5	1	5
	3	7	↑8	5	1	3
	4	7↑	–	5	1	3
	5	–	–	↑5	1	3
	6	–	–	2	0	2

Result of applying maximum penalty;

1. R_4 cancelled
2. R_5 cancelled
3. C_2 cancelled
4. C_1 cancelled
5. R_3 cancelled
6. C_3 cancelled

Remaining allocations are done as a part of the **VAM.**

Observations

1. All diagonal elements are allocated 800 units; this is not of any high importance.
2. Remaining procedure of allocations is a part of **VAM**.
3. Allocation cost is as follows;

$x_{13} = 150$	$C_{13} = 5$	Cost = 750
$x_{15} = 50$	$C_{15} = 8$	Cost = 400
$x_{24} = 250$	$C_{24} = 9$	Cost = 2250
$x_{25} = 350$	$C_{25}=10$	Cost = 3500
	Total cost of allocations = ₹6900.	

Additional Questions for Practice (with Hints and Answers)

Question 1

1. Obtain the initial basic feasible solution by Vogel`s method and optimal solution by MODI method.

Destinations

		1	*2*	*3*	*4*	a_i
	1	21	16	25	13	11
Source	2	17	18	14	23	13
	3	32	27	18	41	19
	Demand	6	10	12	15	

Solution

For the problem above;

$\Sigma a_i = 11 + 13 + 19 = 43$ units = Available units;

$\Sigma b_j = 6 + 10 + 12 + 15 = 43$ units = demand = $\Sigma a_i = \Sigma b_j = 43$ units.

It is a balanced transportation problem.

We solve it by **Vogel's** approximation method.

Destination **Penalty**

	d_1	d_2	d_3	d_4	a_i	*1*	*2*	*3*	*4*
s_1	21	16	25	$\boxed{11}$ 13	11/0	3	–	–	–
s_2	$\boxed{6}$ 17	$\boxed{3}$ 18	14	$\boxed{4}$ 23	13/9/3/0	3	3	3	4
s_3	32	$\boxed{7}$ 27	$\boxed{12}$ 18	↑ 41	19/7/0	9	9	9	←9
Demand	6/0	10/3/0	12/0	15/4/0					
Penalty									
1	4	2	4	10					
2	15	9	4	18↑					
3	↑15	9	4	–					
4	–	9	4	–					

[In the first round of allocation, the highest penalty is 10. (first row in the bottom) In the last column C_4 the minimum cost is $C_{14} = 13$. Requirement in D_4 is 15 units and the allocation can be done from available units $a_1 = 11$ from R_1. By making this allocation the first row has no more elements left. The first row is now out from further process of allocation. Also the last column d_4 has 15 – 11 = 4 units as its further requirement. This can be satisfied only via $a_2 = 23$ units or $a_3 = 41$ units.]

Following the above procedure the final allocation is shown below. (Procedural steps are shown in the matrix table.)

Resultant Matrix

	d_1	d_2	d_3	d_4	a_i
s_1	21	16	25	$\boxed{11}$ 13	**11**
s_2	$\boxed{6}$ 17	$\boxed{3}$ 18	14	$\boxed{4}$ 23	13
s_3	32	$\boxed{7}$ 27	$\boxed{12}$ 8	41	19
b_j	6	10	12	15	

We calculate the cost as follows.

Allocation	*Cost/Unit*	*Total cost*
$x_{14} = 11$	13	143
$x_{21} = 6$	17	102
$x_{22} = 3$	18	54
$x_{24} = 4$	23	92
$x_{32} = 7$	27	189
$x_{33} = 12$	18	216
		Total Cost = 796

Now we apply the optimality test (**MODI method**).

Step 1
There are $m = 3$ rows and $n = 4$ columns, there must be, by theory, $m + n - 1 = 3 + 4 - 1 = 6$ allocated cells. There are six allocated cells in the above matrix and so we can apply MODI method.

Step 2
$C_{ij} = u_i + v_j$ for allocated cells,

	d_1	d_2	d_3	d_4	u_1
S_1				13	$u_1 = 0$
S_2	17	18		23	$u_2 = 10$
S_3		27	18		$u_3 = 19$
v_1	$v_1 = 7$	$v_2 = 8$	$v_3 = -1$	$v_4 = 13$	

Step 3
$\Delta_{ij} = C_{ij} - (u_i + v_j)$ for allocated cells,

	d_1	d_2	d_3	d_4
S_1	14	8	26	0
S_2			5	0
S_3	6			

All $\Delta_{ij} \geq 0$; this implies that the solution is optimal.
$\Delta_{14} = 0$; $\Delta_{24} = 0$ implies that it is a case of multiple optimal solution.
So optimal allocation and optimal cost are as follows:

Allocation	*Cost/Unit*	*Total cost*
$x_{14} = 11$	$C_{14} = 13$	143
$x_{21} = 6$	$C_{21} = 17$	102
$x_{22} = 3$	$C_{22} = 18$	54
$x_{24} = 4$	$C_{24} = 23$	92
$x_{32} = 7$	$C_{32} = 27$	189
$x_{33} = 12$	$C_{33} = 18$	216
		Total Optimal cost = 796

Question 2
For the following transportation problem apply North-West Corner method and find the optimal solution.

	P	*Q*	*R*	*Supply*
A	6	4	1	50
B	3	8	7	40
C	4	4	2	60
Demand	20	95	35	150

Solution
We apply North-West Corner method and allocate 20 units in cell (1,1). Continuing allocation in North-West cell, the resultant matrix is as follows.

	P	Q	R	a_i
A	20 / 6	30 / 4		50
B		40 / 8		40
C		25 / 4	35 / 2	60
b_j	20	95	35	

Allocation	**Cost/Unit**	*Total cost*
$x_{11} = 20$	$C_{11} = 6$	120
$x_{12} = 30$	$C_{12} = 4$	120
$x_{22} = 40$	$C_{22} = 8$	320
$x_{32} = 25$	$C_{32} = 4$	100
$x_{33} = 35$	$C_{33} = 2$	70
		Total cost = 730

MODI method for optimality verification

Step 1

Number of rows = $m = 3$; Number of columns = $n = 3$, by theory, the number of allocated cells = $m + n - 1 = 3 + 3 - 1 = 5$. There are, in fact, 5 (five) allocated cells;

Step 2

$C_{ij} = u_i + v_j$ for allocated cells,

	P	Q	R	u_i
A	6	4		$u_1 = 0$
B		8		$u_2 = 4$
C		4	2	$u_3 = 0$
v_j	$v_1 = 6$	$v_2 = 4$	$v_3 = 2$	

Step 3

$\Delta_{ij} = C_{ij} - (u_i + v_j)$ for non-allocated cells,

	P	Q	R
A			–1
B	√–7		1
C	–2		

Some Δ_{ij} values are negative. This implies that the cost is not minimum; we have to use loop-method for reshuffling the allocations;

Loop:

	P	*Q*	*R*
A	20 D	30 A	
B	√ A	40 D	
C			

We select the most negative $\Delta_{ij} = \Delta_{21} = -7$ (√ mark) and make a loop connecting that cell with allocated cells. The minimum of the donors is 20 units. We reshuffle 20 units. The new allocations are shown as follows.

	P	*Q*	*R*
A		50 4	
B	20 3	20 8	
C		25 4	35 2

The new allocation and cost are as follows;

Allocation	*Cost/Unit*	*Total cost*
$x_{12} = 50$	$C_{12} = 4$	200
$x_{21} = 20$	$C_{21} = 3$	60
$x_{22} = 20$	$C_{22} = 8$	160
$x_{32} = 25$	$C_{32} = 4$	100
$x_{33} = 35$	$C_{33} = 2$	70
		Total Optimal cost = 590

[It is recommended that students verify the optimality of this cost, ₹590, by the same method.]

Question 3

What is degeneracy in a transportation problem? Explain how to resolve degeneracy in a transportation problem. Solve the following transportation for minimum cost;

From \ *To*	W_1	W_2	W_3	*Supply*
F_1	16	20	12	200
F_2	14	8	18	160
F_3	26	24	16	90
Demand	180	120	150	

Solution

We apply the minimum cost method and find the initial basic feasible solution. The minimum cost as seen in the matrix is 8 (= C_{22}). The demand in $W_2 = d_2 = 120$ units while it can be fulfilled from F_2 with $a_2 = 160$ units. We allocate 120 units in the cell (2, 2). This way we continue to allocate in minimum cost cell and find the following matrix of allocations.

Matrix of allocation

	W_1	W_2	W_3
F_1	50 / 16		150 / 12
F_2	40 / 14	120 / 8	
F_3	90 / 26		

Allocation	*Cost/Unit*	*Total cost*
$x_{11} = 50$	$C_{11} = 16$	800
$x_{13} = 150$	$C_{13} = 12$	1800
$x_{21} = 40$	$C_{21} = 14$	560
$x_{22} = 120$	$C_{22} = 8$	960
$x_{31} = 90$	$C_{31} = 26$	2340
		Total Cost = 6460

Verification of optimality (MODI method)

Step 1

Number of rows = $m = 3$;

Number of columns = $n = 3$;

According to the theory, number of allocated cells = $m + n - 1 = 5$. Since the actual number of allocated cells is 5, so MODI method is applicable.

Step 2

$C_{ij} = u_i + v_j$ for allocated cells (we take $u_1 = 0$)

	W_1	W_2	W_3	
F_1	16		12	$u_1 = 0$
F_2	14	8		$u_2 = -2$
F_3	26			$u_3 = 10$
	$v_1 = 16$	$v_2 = 10$	$v_3 = 12$	

Step 3

$\Delta_{ij} = C_{ij} - (u_i + v_j)$ for non-allocated cells only,

	W_1	W_2	W_3
F_1		10	
F_2			8
F_3		4	√–6

Some $\Delta_{ij} \leq 0$, the solution is not optimal and the cost ₹6460 is not minimum.

Step 4

$\Delta_{33} = -6 < 0$. It implies that there is way to reduce the cost. We make a loop from that cell with the negative entry $\Delta_{33} = -6$.

The cell (3, 3) is √ marked and it is going to be the first 'acceptor' cell. The remaining cells are the allocated cells in the loop.

There are two acceptor cells. The loop is as follows.

50 A		150 D
90 D		√ A

Minimum of the donors is 90 units, which is given to the first acceptor cell. This is given from either the cell (1, 3) or (3, 1).

After reshuffling, the final table of allocations is as follows.

40 16		60 12
40 14	120 8	
		90 16

We calculate the cost as follows:

Allocation	*Cost/Unit*	*Total cost*
$x_{11} = 140$	$C_{11} = 16$	2240
$x_{13} = 60$	$C_{13} = 12$	720
$x_{21} = 40$	$C_{21} = 14$	560
$x_{22} = 120$	$C_{22} = 8$	960
$x_{31} = 90$	$C_{31} = 16$	1440
		Total Cost = 5920

[Students are advised to verify this cost ₹5920 for its optimality.]

Question 4

Solve the following transportation problem for minimum cost:

Destination

	1	*2*	*3*	*4*	*5*	a_i
A	4	1	3	4	4	60
B	2	3	2	2	3	35
C	3	5	2	4	4	40
b_j	22	45	30	18	20	

b_j = demand; a_i = availability;

Solution

We solve this by minimum cost method. The minimum cost is 1 (= C_{12}). It belongs to column C_2 where it requires $b_2 = 45$ units which can be supplied from $a_1 = 60$ units. At the end of this process the resultant matrix is as follows.

4	$\boxed{45}$ 1	3	4	4	60/15
2	3	2	2	3	35
3	5	2	4	4	40
22	45/0	30	18	20	

Now, the next minimum cost is ₹2. Note that $C_{21} = C_{23} = C_{24} = C_{33} = 2$. At this point of time we select that for allocation which can absorb the most availability. Destination 3 needs 30 units which can be supplied through $a_2 = 35$ units or $a_3 = 40$ units. It becomes optional. We select $a_2 = 35$ units and fulfill the need of d_2 in the cell (2, 3).

At the end of this, the matrix of allocation is as follows.

4	$\boxed{45}$ 1	3	4	4	60/15
2	3	$\boxed{30}$ 2	2	3	35/5
3	5	2	4	4	40
22	45/0	30/0	18	20	

Making allocations in the same way, the final matrix of allocations is as follows.

Step 1

Number of rows = $m = 3$; number of column = $n = 3$ by theory, the number of allocated cells = $m + n - 1$ = $3 + 3 - 1 = 5$. There are in fact 5 (five) allocated cells;

Step 2

$C_{ij} = u_i + v_j$ for allocated cells,

6	4		$u_1 = 0$
	8		$u_2 = 4$
	4	2	$u_3 = 0$
$v_1 = 6$	$v_2 = 4$	$v_3 = 2$	

Step 3

$\Delta_{ij} = C_{ij} - (u_i + v_j)$ for non-allocated cells,

		–1
√ –7		1
–2		

Some Δ_{ij} values are negative. This implies that the cost is not minimum; we have to use loop-method for reshuffling the allocations;

Loop:

20 D	30 A	
√ A	40 D	

We select the most negative $\Delta_{ij} = \Delta_{21} = -7$ (√ mark) and make a loop connecting that cell with allocated cells. The minimum of the donors is 20 units. We reshuffle 20 units. The new allocations are shown as follows.

	50 4	
20 3	20 8	
	25 4	35 2

The new allocation and cost are as follows;

Allocation	*Cost/Unit*	*Total cost*
$x_{12} = 50$	$C_{12} = 4$	200
$x_{21} = 20$	$C_{21} = 3$	60
$x_{22} = 20$	$C_{22} = 8$	160
$x_{32} = 25$	$C_{32} = 4$	100
$x_{33} = 35$	$C_{33} = 2$	70
		Total optimal cost = 590

[It is recommended that students verify the optimality of this cost ₹590, by the same method.]

Question 5

What is degeneracy in a transportation problem? Explain how to resolve degeneracy in a transportation problem. Solve the following transportation for minimum cost;

From \ *To*	W_1	W_2	W_3	*Supply*
F_1	16	20	12	200
F_2	14	8	18	160
F_3	26	24	16	90
	180	120	150	

Solution

We apply the minimum cost method and find the initial basic feasible solution. The minimum cost as seen in the matrix is 8 (= C_{22}). The demand in $W_2 = d_2 = 120$ units while it can be fulfilled from F_2 with $a_2 = 160$ units. We allocate 120 units in the cell (2, 2). This way we continue to allocate in minimum cost cell and find the following matrix of allocations.

Matrix of allocation

50 16		150 12
40 14	120 8	
90 26		

Allocation	*Cost/Unit*	*Total cost*
$x_{11} = 50$	$C_{11} = 16$	800
$x_{13} = 150$	$C_{13} = 12$	1800
$x_{21} = 40$	$C_{21} = 14$	560
$x_{22} = 120$	$C_{22} = 8$	960
$x_{31} = 90$	$C_{31} = 26$	2340
		Total Cost: 6460

Verification of optimality (MODI method)

Step 1

Number of rows = $m = 3$;

Number of columns = $n = 3$;

According to the theory, number of allocated cells = $m + n - 1 = 5$. Since the actual number of allocated cells is 5 and so MODI method is applicable.

Step 2

$C_{ij} = u_i + v_j$ for allocated cells, (we take $u_1 = 0$)

16		12	$u_1 = 0$
14	8		$u_2 = -2$
26			$u_3 = 10$
$v_1 = 16$	$v_2 = 10$	$v_3 = 12$	

Step 3

$\Delta_{ij} = C_{ij} - (u_i + v_j)$ for non-allocated cells only,

	10	
		8
	4	$\sqrt{}-6$

Some $\Delta_{ij} \leq 0$, the solution is not optimal and the cost ₹6460 is not minimum.

Step 4

$\Delta_{33} = -6 < 0$, it implies that there is way to reduce the cost. We make a loop from that cell with the negative entry $\Delta_{33} = -6$.

The cell (3, 3) is √ marked and it is going to be the first 'acceptor' cell. The remaining cells are allocated cells in the loop.

There are two and two acceptor cells. The loop is as follows.

50 A		150 D
90 D		√ A

Minimum of the donors is 90 units, which is given to the first acceptor cell. This is given from either the cell (1, 3) or (3, 1).

After reshuffling, the final table of allocations is as follows:

140 16		60 12
40 14	120 8	
		90 16

We calculate the cost as follows:

Allocation	*Cost/Unit*	*Total cost*
$x_{11} = 140$	$C_{11} = 16$	2240
$x_{13} = 60$	$C_{13} = 12$	720
$x_{21} = 40$	$C_{21} = 14$	560
$x_{22} = 120$	$C_{22} = 8$	960
$x_{31} = 90$	$C_{31} = 16$	1440
		Total Cost = 5920

[Students are advised to verify this cost ₹5920 for its optimality.]

Question 6

Solve the following transportation problem for minimum cost;

Destination

	1	*2*	*3*	*4*	*5*	a_i
A	4	1	3	4	4	60
B	2	3	2	2	3	35
C	3	5	2	4	4	40
b_j	22	45	30	18	20	

b_j = demand; a_i = availability;

Solution

We solve this by minimum cost method. The minimum cost is 1 (= C_{12}). It belongs to column C_2 where it requires b_2= 45 units which can be supplied from a_1 = 60 units. At the end of this process the resultant matrix is as follows.

4	$\boxed{45}$ 1	3	4	4	60/15
2	3	2	2	3	35
3	5	2	4	4	40
22	45/0	30	18	20	

Now, the next minimum cost is ₹2. Note that $C_{21} = C_{23} = C_{24} = C_{33} = 2$. At this point of time we select that for allocation which can absorb the most availability. Destination 3 needs 30 units which can be supplied through a_2 = 35 units or a_3 = 40 units. It becomes optional. We select a_2 = 35 units and fulfill the need of d_2 in the cell (2, 3).

At the end of this, the matrix of allocation is as follows.

4	$\boxed{45}$ 1	3	4	4	60/15
2	3	$\boxed{30}$ 2	2	3	35/5
3	5	2	4	4	40
22	45/0	30/0	18	20	

Making allocations in the same way, the final matrix of allocations is as follows.

Matrix of allocation

4	$\boxed{45}$ 1	3	$\boxed{13}$ 4	$\boxed{2}$ 4	60/15/2/0
2	3	$\boxed{30}$ 2	$\boxed{5}$ 2	3	35/5/0
$\boxed{22}$ 3	5	2	4	$\boxed{18}$ 4	40/18/0
22/0	45/0	30/0	18/13/0	20/2/0	

The cost and allocations are given as follows;

Allocation	*Cost/Unit*	*Total cost*
$x_{12} = 45$	$C_{12} = 1$	45
$x_{14} = 13$	$C_{14} = 4$	52
$x_{15} = 2$	$C_{15} = 4$	08
$x_{23} = 30$	$C_{23} = 2$	60
$x_{24} = 5$	$C_{24} = 2$	10
$x_{31} = 22$	$C_{31} = 3$	66
$x_{35} = 18$	$C_{35} = 4$	72
		Total Cost = ₹313

[Students are requested to verify this cost for its optimality.]

Question 7

Solve the following transportation problem using North-West corner method. Check the optimality of the cost.

To / From	W_1	W_2	W_3	W_5	a_i
O_1	2	10	15	14	40
O_2	10	6	10	12	30
O_3	9	5	8	15	30
b_j	35	5	40	20	

Solution

In a transportation problem with *m* rows and *n* columns; we have ($m + n - 1$) number of linearly independent equality constraints. It means that there are ($m + n - 1$) number of allocated cells (basic variables). If at least one allocated cell/basic variable attains zero value (allocation), then the solution is a degenerate solution.

Applying North-West corner method, we get the following matrix of allocation;

To / From	W_1	W_2	W_3	W_5	a_i
O_1	35 2	5 10*	15	14	40/5/0
O_2	10	6	30 10	12	30/0
O_3	9	8	10 8	20 15	30/20
b_j **Demand**	35/0	5/0	40/10	20/0	

Allocation	*Cost/Unit*	*Total cost*
$x_{11} = 35$	$C_{11} = 2$	70
$x_{12} = 5$	$C_{12} = 10$	50
$x_{23} = 30$	$C_{23} = 10$	300
$x_{33} = 10$	$C_{33} = 8$	80
$x_{34} = 20$	$C_{34} = 15$	300
		Total Cost = ₹800

Now, we apply MODI method.

Step I

Number of rows = $m = 3$;

Number of columns = $n = 4$;

According to the theory, the number of allocated cells is $m + n - 1 = 3 + 4 - 1 = 6$. Actual number of allocated cells is 5. It is a case of degeneracy solution.

To apply MODI method, we need $m + n - 1$ number of allocated cells.

In order to resolve degeneracy, we allocate very small units = θ ($\theta \to 0$) in the least cost independent cell.

[Finding the least cost independent cell is, in fact, a matter of trial and error or selection and verification. To resolve this issue, refer to the Section 3.9.]

The cell (1, 2) where √ is the particular cell where from the degeneracy is initiated. Attach with that cell (1, 2), there are:

Cell (1, 1)—Already allocated;

Cell (1, 3)—Empty with cost $C_{13} = 15$;

Cell (2, 2)—Empty with cost $C_{22} = 6$;

We put θ units ($\theta \to 0$) in the cell (2, 2), the new matrix of allocation is as follows.

We can apply MODI method. There are, as required, 3 + 4 – 1 = 6 allocated cells [including the cell (2, 2) with θ units]. The basic criterion is satisfied and so we write Step 1.

New allocation matrix (resolution of degeneracy)

Step 1

35 2	5 10*		
	θ 6	30 10	
9		10 8	20 15

Step 2

$C_{ij} = u_i + v_j$ for allocated cells,

2	10			$u_1 = 0$
	6	10		$u_2 = -4$
		8	15	$u_3 = -6$
$v_1 - 2$	$v_2 - 10$	$v_3 - 14$	$v_4 = 21$	

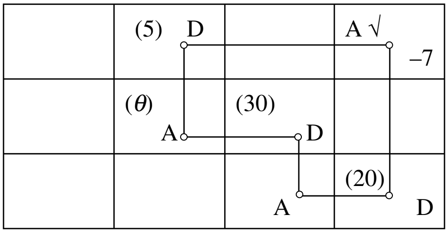
Step 3

$\Delta_{ij} = C_{ij} - (u_i + v_j)$ for non-allocated cells only,

		1	–7
12			–5
13	4		

$\Delta_{ij} = -7$ and $\Delta_{24} = -5$ both are ≤ 0. This means that the solution is not optimal and the cost is not minimum, we have to apply loop which, begins with –7 entry. [The most negative of the two negative entries is $\Delta_{14} = -7$; it is selected and it indicates that any amount of allocation in that cell will reduce the cost at the rate –7 per unit of allocation.]

The loop is as follows;



The √ cell is the first acceptor cell and that will take the allocation amount, which equals the minimum amount possessed by all different donors.

It involves 5 allocated cells and one non-allocated with –7 entry.

Donors have $x_{12} = 5$; $x_{23} = 30$; and $x_{34} = 20$; in unit capacity. You can donate 5 units (minimum of all)

New matrix after reshuffling is as follows.

35 / 2			5 / 14
	5 / 6	25 / 10	
		15 / 8	15 / 15

Now, we find the new cost;

Allocation	*Cost/Unit*	*Total cost*
$x_{11} = 35$	$C_{11} = 2$	70
$x_{14} = 5$	$C_{12} = 14$	70
$x_{22} = 5$	$C_{22} = 6$	30
$x_{33} = 15$	$C_{33} = 8$	120
$x_{34} = 15$	$C_{34} = 15$	225
		Total cost = ₹765

($\Delta_{14} = –7$ times allocation $x_{14} = 5 = –35$.)

It is necessary to verify the optimality of the new allocations and new cost ₹765 and it is left for the reader.

Points to Remember

Transportation problem is a special case of LP problem. It is a minimization problem.

1. With ***m*** origins (production units) and ***n*** destination (absorption centres) it has $m \times n = mn$ cells or locations.
2. It has ***m*** row conditions each one having availability resource on its right side.
3. It has ***n*** column conditions each one having demand or capacity indicator figure on the right side.
4. The fundamental condition that $\sum_{i=1}^{i=m} a_i = \sum_{i=1}^{i=n} b_j$ implies that all $m + n$ constraints are linearly dependent.

5. There are $(m + n - 1)$ numbers of linearly independent equations, i.e. $m + n - 1$ number of equality constraints, and so are $(m + n - 1)$ number of basic variables.
6. Total number of cells are $m \times n = mn$. Allocation to each one is represented as x_{ij} where $i = 1$ to m and $j = 1$ to n.
7. Total number of variables = total number of cells = $m \times n = mn$.
   Total number of basic variables = $(m + n - 1)$, i.e. out of mn number of x_{ij} values, $(m + n - 1)$ values stand for basic variables, i.e. $(m + n - 1)$ number of cells will have corresponding $x_{ij} \geq 0$.
8. Other than these $(m + n - 1)$, remaining cells $mn - (m + n - 1)$ stand for non-basic variables. It means that $mn - (m + n - 1)$ values of x_{ij} are zero; as they are for non-basic variables.
9. If any one or more out of $(m + n - 1)$ number of basic cells, i.e. x_{ij} values are equal to zero, then it is a degenerate solution.
10. A basic feasible solution of a transportation problem has at the most $(m + n - 1)$ basic cells with $x_{ij} \geq 0$ and $mn - (m + n - 1)$ non-basic cells with $x_{ij} = 0$. If the basic condition $\Sigma a_i = \Sigma b_j$ is satisfied then, any one of the following three methods may be used to find a feasible solution.
    1. North-West corner rule/method
    2. Minimum cost method or Matrix Minima method
    3. Vogel's approximation method (VAM) or Penalty method
11. Modified Distribution Method (MODI) is just to check the optimality of the cost for a given feasible solution having $(m + n - 1)$ number of allocated cells.
12. Loop consists of ***even*** number of cells;
    (a) Greater than or equal to four.
    (b) A loop is a closed circuit with ***one*** non-allocated cells and remaining allocated cells.
    (c) By working with loops, we ***reshuffle*** the ***allocations*** and finally allocate the minimum of all donors' cell allocation amount to the one non-allocated cell. One non-allocated cell gets allocation and one allocated cell becomes vacated of allocation (becomes non-allocated) and hence the total number of allocated cells remains $(m + n - 1)$.
13. If two penalties are the same, with two minimum cost cells then there is a criterion of selecting that cell which can absorb more amounts of allocations than that of the other one in competition is preferred. For example,

					a_i	**Penalty**
*2	1	5	8	7	40	1
12	16	12	8	9	20	1
8	14	10	11	2*	10	6←
12	2	8	7	1	40	1
b_j	20	35	25	20	10	
Penalty	↑6	1	3	1	1	

C_1 and R_3 show the highest penalty. Also minimum cost cells are also the same (cost = ₹2), then we select cell (1, 1) (* mark) instead of all (3, 5).

EXERCISES

OBJECTIVE TYPE QUESTIONS

I. State True or False:

1. In a balanced transportation problem having 4 rows and 6 columns; there are 10 allocated cells (provided the solution is a basic feasible solution).
2. A feasible solution to a transportation problem is always a basic feasible solution.
3. A basic feasible is a degenerate one if exactly one allocated cell has zero allocation.
4. The North-West corner method does not care for the cost; it works on the principle of allocation in the empty North-West cell.
5. Working with any method for allocation, during the process of allocation, a row and a column both get empty at a time then we have a degenerate solution.
6. All the three methods of finding a basic feasible solution to a transportation problem work on different working principles.
7. Minimum cost method when applied in comparison with **VAM** gives a better optimal solution.
8. In a loop, if two donor cells have the same allocated value, then after reshuffling both the cells get vacated.
9. In the process of finding $C_{ij} = u_i + v_j$ (during application of MODI method), for allocated cells, $i + j = m + n$ (m = number of rows; n = number of columns).
10. To form a loop any one non-allocated cell can be selected.
11. Once the loop process is carried out, then there is guarantee that the new allocations give the optimal solution.
12. As transportation problem is an LP problem of minimization type, we must have NER = $C_{ij} - (u_i + v_j) = 0$ for allocated cells.
13. The most negative entry of all $\Delta_{ij} = C_{ij} - (u_i + v_j)$ entries is selected for making a loop and reducing the cost very fast.
14. If working with penalty method, if penalty are the same for any row and any column, then the selection of incoming allocation is optional.
15. If two penalties are same with minimum cost (in the cell) also same, then the choice is to selection of higher value allocation.

ANSWERS

1. false.	2. false.	3. false.	4. true.	5. true.	6. true.
7. false	8. false.	9. true.	10. false.	11. false.	12. true.
13. true.	14. false.	15. true.			

II. Multiple Choice Questions

Data: In a transportation problem there are 4 rows and 5 columns.

1. The very first point to get a basic feasible solution is to check/apply.
   (a) Find $C_{ij} = u_i + v_j$ (b) North-West corner method
   (c) $\Sigma a_i = \Sigma b_j$ (d) MODI method
2. Cost-effective basic feasible solution is found by
   (a) applying MODI method (b) checking = $\Sigma a_i = \Sigma b_j$
   (c) North-West corner method (d) minimum cost method
3. In a transportation problem, there are 4 rows and 5 columns; the number of non-allocated cells are
   (a) 12 (b) 13 (c) 20 (d) 11

4. In a transportation problem, there are 4 rows and 5 columns; the number of allocated cells in a basic feasible solution are at most
   (a) 9 (b) 8 (c) 12 (d) 11
5. Once the most negative $\Delta_{ij} = C_{ij} - (u_i + v_j)$ entry concerning a non-allocated cell is found, then the number of loop/loops connecting to that cell with the number allocated cells is
   (a) 2 (b) 9 (c) 8 (d) 1
6. If some $\Delta_{ij} = C_{ij} - (u_i + v_j) = 0$ for non-allocated cells is/are zero; it is a sign of
   (a) infeasible solution (b) multiple-optimal solution
   (c) unique solution (d) unbounded solution
7. The number of entries of $\Delta_{ij} = C_{ij} - (u_i + v_j)$ to verify the optimality criterion are
   (a) $m + n$ (b) $m + n - 1$ (c) mn (d) $mn - (m + n - 1)$
8. If a row has the four cost entries given as p, q, r and s, then the penalty for that row is (given that $p \leq q < r \leq s$)
   (a) 0 if $p = q$ (b) $r - q$ if $p = q$ (c) $s - p$ (d) $s - r$
9. If two different cells with minimum cost correspond to the same maximum penalty then;
   (a) any one can be selected for allocation
   (b) the cell which requires less amount of allocation is selected
   (c) the cell which requires more amount of allocation is selected
   (d) search for next minimum penalty
10. In a transportation problem corresponding to (m) rows and (n) columns, the total number of $C_{ij} = (u_i + v_j)$ entries are
    (a) mn (b) $m + n$ (c) $m + n - 1$ (d) $m + n - 2$
11. In a transportation problem corresponding to (m) rows and (n) columns, the total number of $C_{ij} = (u_i + v_j)$ entries are
    (a) related to basic cells (b) related to non-basic cells
    (c) all cells (d) cells of last row and last column
12. Can we make a loop including some or all allocated cells only?
    (a) Yes (b) No (c) May be possible (d) Depends on number

ANSWERS

1. (c)	2. (d)	3. (d)	4. (b)	5. (d)	6. (b)
7. (d)	8. (a)	9. (c)	10. (b)	11. (a)	12. (b)

NUMERICAL PROBLEMS

1. Solve the transportation problem using North-West corner method.

TO / *FROM*	*P*	*Q*	*R*	*AVAILABILITY*
A	8	6	3	70
B	5	10	9	60
C	6	6	4	90
DRMAND	40	115	65	

2. Solve the transportation problem using minimum cost method.

FROM \ *TO*	*P*	*Q*	*R*	*AVAILABILITY*
A	8	6	3	70
B	5	10	9	60
C	6	6	4	90
Demand	40	115	65	

3. Solve the following transportation problem using Vogel's Approximation Method.

FROM \ *TO*	*P*	*Q*	*R*	*AVAILABILITY*
A	8	6	3	70
B	5	10	9	60
C	6	6	4	90
Demand	40	115	65	

4. Solve the following transportation problem using north-west corner rule

FROM \ *TO*	*P*	*Q*	*R*	*Availability*
A	8	6	3	70
B	5	10	9	60
C	6	6	4	90
Demand	70	75	75	

5. In the context of the Problem 4, answer the following questions.
   (a) How many allocated cells are there?
   (b) How many allocated cells are theoretically expected?
   (c) What do you call this situation?
   (d) Can you distinctly identify the reason for this situation?
   (e) Can you, without any modification, apply MODI method on the matrix of allocation?
6. Solve the transportation problem by VAM.

FROM \ *TO*	*P*	*Q*	*R*	*S*	*Availability*
A	7	3	5	5	34
B	5	5	7	6	15
C	8	6	6	5	12
D	6	1	6	4	19
Demand	21	25	17	17	

7. Solve the following transportation problem by VAM.

TO / FROM	P	Q	R	S	Availability
A	11	6	15	3	11
B	07	08	04	13	13
C	22	17	8	31	19
Demand	6	10	12	15	

8. Obtain an initial basic feasible solution by minimum cost method. Also, verify its optimality.

TO / FROM	P	Q	R	S	Availability
A	11	12	13	14	6
B	14	13	12	10	8
C	10	12	12	11	10
Demand	4	6	8	6	

9. Obtain an initial basic feasible solution by minimum cost method. Also, verify its optimality.

TO / FROM	P	Q	R	S	Availability
A	12	11	13	14	60
B	13	14	12	10	80
C	12	10	12	11	100
Demand	60	40	80	60	

10. Obtain an initial basic feasible solution by VAM method. Also, verify its optimality by MODI method

TO / FROM	P	Q	R	S	Availability
A	12	11	13	14	60
B	13	14	12	10	80
C	12	10	12	11	100
Demand	60	40	80	60	

11. Solve the following transportation problem using minimum cost rule. Apply MODI method to verify optimality.

Cost Matrix

To / From	P	q	r	s	t	Available units
O_1	6	6	3	9	2	90
O_2	7	8	5	7	5	70
O_3	3	4	8	2	7	110
O_4	6	0	0	2	9	150
Demand	100	40	100	60	120	

12. Solve the Problem 11 using VAM. Also find the absolute difference between the two types of costs.
13. How do we solve degeneracy in a transportation problem? Does the following problem give a degenerate solution?

TO / *FROM*	*P*	*Q*	*R*	*S*	*Availability*
A	21	22	23	24	6
B	24	23	22	20	8
C	20	22	22	21	10
Demand	4	6	8	6	

14. Solve the following transportation problem and find the minimum cost of transportation.

Cost Matrix

To / *From*	D_1	D_2	D_3	D_4	D_5	*Available units*
O_1	5	8	6	6	3	80
O_2	4	7	7	6	5	50
O_3	8	4	6	6	4	90
Demand	40	40	50	40	80	

15. Solve the following transportation problem for minimum cost of transportation.

Cost Matrix

To / *From*	D_1	D_2	D_3	D_4	D_5	*Available units*
O_1	40	20	30	20	60	8
O_2	50	40	50	20	10	12
O_3	60	50	40	70	30	14
Demand	4	4	6	8	8	

Answers to Numerical Problems

1.

Allocation	*Cost/ unit*	*Total cost*
$x_{11} = 40$	$C_{11} = 8$	320
$x_{12} = 30$	$C_{12} = 6$	180
$x_{22} = 60$	$C_{22} = 10$	600
$x_{32} = 25$	$C_{32} = 6$	150
$x_{33} = 65$	$C_{33} = 4$	260
	Total cost = ₹1510	

2. Solution using minimum cost method

Allocation	*Cost/ unit*	*Total cost*
$x_{12} = 05$	$C_{12} = 6$	30
$x_{13} = 65$	$C_{13} = 3$	195
$x_{21} = 40$	$C_{21} = 5$	200
$x_{22} = 20$	$C_{22} = 10$	200
$x_{32} = 90$	$C_{33} = 6$	540
	Total Cost = ₹1165	

3. Solution using VAM

Allocation	*Cost/ unit*	*Total cost*
$x_{12} = 05$	$C_{12} = 6$	30
$x_{13} = 65$	$C_{13} = 3$	195
$x_{21} = 40$	$C_{21} = 5$	200
$x_{22} = 20$	$C_{22} = 10$	200
$x_{32} = 90$	$C_{33} = 6$	540
	Total Cost = ₹1165	

4. Cost of allocation (a feasible solution) = ₹1550
5. (A) 4,
   (B) 5
   (C) degeneracy,
   (D) During the process of allocation in the first North-West corner, the demand = 70 units and availability = 70 units, both being the same, the first row and the first column were satisfied simultaneously. This is the basic reason for degeneracy.
   (E) no
6. A Basic feasible solution by VAM

Allocation	*Cost/unit*	*Total cost*
$x_{11} = 06$	$C_{11} = 7$	42
$x_{12} = 06$	$C_{12} = 3$	18
$x_{13} = 17$	$C_{13} = 5$	85
$x_{14} = 05$	$C_{14} = 05$	25
$x_{21} = 15$	$C_{21} = 05$	75
$x_{34} = 12$	$C_{34} = 05$	60
$x_{42} = 19$	$C_{42} = 1$	19
	Total Cost = ₹324	

7.

Allocation	*Cost/ unit*	*Total cost*
$x_{14} = 11$	$C_{14} = 3$	33
$x_{21} = 06$	$C_{21} = 7$	42
$x_{22} = 03$	$C_{22} = 8$	24
$x_{24} = 04$	$C_{24} = 13$	52
$x_{32} = 07$	$C_{32} = 17$	119
$x_{33} = 12$	$C_{34} = 08$	96
	Total Cost = ₹366	

8.

Allocation	*Cost/ unit*	*Total cost*
$x_{12} = 6$	$C_{12} = 12$	72
$x_{23} = 02$	$C_{23} = 12$	24
$x_{24} = 06$	$C_{24} = 10$	60
$x_{31} = 04$	$C_{31} = 10$	40
$x_{33} = 06$	$C_{33} = 12$	72
	Total Cost = ₹268	

It is a degenerate solution.
$m = 3$, $n = 4$ $m + n - 1 = 6$
Since actual numbers of allocated cells is 5.
and so it is an optimal solution. It is also a case of multiple optimal solution.

9.

Allocation	*Cost/ unit*	*Total cost*
$x_{11} = 60$	$C_{11} = 12$	720
$x_{23} = 20$	$C_{23} = 12$	240
$x_{24} = 60$	$C_{24} = 10$	600
$x_{32} = 40$	$C_{32} = 10$	400
$x_{33} = 60$	$C_{33} = 12$	720
	Total Cost = ₹2680	

It is a degenerate solution.
$m = 3$, $n = 4$ $m + n - 1 = 6$
Since actual number of allocated cells is 5, and so it is an optimal solution. It is also a case of multiple optimal solution.

10.

Allocation	*Cost/ unit*	*Total cost*
$x_{11} = 60$	$C_{11} = 12$	720
$x_{23} = 20$	$C_{23} = 12$	240
$x_{24} = 60$	$C_{24} = 10$	600
$x_{32} = 40$	$C_{32} = 10$	400
$x_{33} = 60$	$C_{33} = 12$	720
	Total Cost = ₹2680	

It is a degenerate solution.
$m = 3, n = 4$ $m + n - 1 = 6$
Since actual number of allocated cells is 5 and so it is an optimal solution. It is also a case of multiple optimal solution.

11. Solution by minimum cost method.

Allocations

				90
		70		
50			60	
50	40	30		30
			Total Cost = ₹1370	

12. Solution by VAM.

				90
			40	30
100			10	
	40	100	10	

Total cost = ₹950 Absolute difference between the two costs = ₹420

13. Method of solving degeneracy is to check that $m + n - 1$ number of allocated cells must have allocations values, which are ≥ 0. If actual number of allocated cells are less than $m + n - 1$; then we have to assign very small , say θ_1, θ_2, etc. ($\theta_1, \theta_2 \to 0$) allocations to least cost independent cells. Once this condition is satisfied only then MODI method can be further applied.
In this example, we have a degenerate solution. the cost is ₹508

14. It is an unbalanced transportation problem.

$$x_{11} = 80, x_{21} = 40, x_{24} = 10 , x_{32} = 40, x_{33} = 20 , x_{34} = 30, x_{43} = 30$$

Total optimal cost = ₹920. This is a case of degenerate solution.

15. It is a case of unbalanced transportation problem.

$$x_{11} = 4, x_{12} = 4\ x_{24} = 8, x_{25} = 4, x_{33} = 6, x_{35} = 4,$$

Transportation cost = ₹380

4 Assignment Problem

—Life is an assignment; justify your lot with simplicity and nobility.

Learning Objectives

AFTER STUDYING THIS CHAPTER, THE STUDENTS WILL BE ABLE TO:

- **understand assignment techniques of allocation of resources**
- **learn application of assignment problems. (Maximization Problem, Travelling Salesman's Problem)**

INTRODUCTION

Problems discussed in this chapter are based on the fundamental logic of transportation problem. Assignment problems, in its true nature, are problems of allocations and we try finding feasible solutions with an objective of minimizing the total cost of allocation. The fundamental difference between the transportation problem and assignment problem lies in the basic requirement of the nature of solution one expects as an output of assignment problem. The underlying square matrix of the assignment problem is designed to give a solution vector of the size ($n \times 1$); while this is not the case with transportation problem.

Assignment problem is a special case of a transportation problem and in turn, as we know, transportation problem is a special case of linear programming problem. This is the basic point that conveys that why do we use special techniques for finding feasible and optimal solution. Looking at the application point, there are many areas of real life situation problems (management decision field, allocation of resources with an objective of maximization of revenue, locating the shortest in a network, etc.) where we use fundamental logic of assignment problem. We begin with an assignment problem and stabilize what we have discussed above.

4.1 ILLUSTRATION 1

There are 5 technicians who can work on 5 different machines having the same configuration. The average time taken by each technician to make a job on each machine is given in the following table (Table 4.1); we require to assign one machine to a technician so that the total working time to make five jobs (one by each) is minimum.

Table 4.1 Time matrix

Technicians \ Machines	M_1	M_2	M_3	M_4	M_5
T_1	7	8	9	9	8
T_2	8	9	11	10	9
T_3	7	8	10	8	9
T_4	10	9	8	9	10
T_5	10	8	9	11	9

(One may try trial and error method and can find a feasible solution also but there may not be a classical theoretical guarantee for the optimality.)

4.2 Mathematical Model

The mathematical model of the **T**ransportation **P**roblem (TP) is as follows.

There are n members in one set, say A and exactly n items (members) in another set 'B'.

Set A = $\{a_1, a_2, ..., a_n\}$, Set B = $\{b_1, b_2,..., b_n\}$

Each a_i of the set A is to be associated with exactly one b_j of the set B. (with i and j on a range of through 1 up to n)

Let x_{ij} denote the assignment if a_i is associated with b_j. Let c_{ij} represent the cost of assigning a_i of A to some b_j of B.

As each one of a_i, $i = 1$ to n, is connected with exactly one b_j; there are exactly n assignments. If a_i is assigned to b_j; we write $x_{ij} = 1$ and $x_{ij} = 0$ if assignment is not made. This means that the cost of assignment c_{ij} is applicable only if $x_{ij} = 1$ or else it is zero for $x_{ij} = 0$.

The problem is to find x_{ij} ($n \times n = n^2$ variables) that minimizes the total cost

$$Z = \sum_{j=1}^{j=n} \sum_{i=1}^{i=n} c_{ij} x_{ij} \tag{1}$$

$$\text{subject to } \sum_{j=1}^{j=n} x_{ij} = 1 \text{ for each } i = 1 \text{ to } n \tag{2}$$

$$\sum_{i=1}^{i=n} x_{ij} = 1 \text{ for each } j = 1 \text{ to } n \tag{3}$$

$$\text{with any one either } x_{ij} = 1 \text{ or } x_{ij} = 0 \tag{4}$$

4.3 Some Additional Features of A.P.

In the introduction to A.P., we have seen some features that build up a level to formulate the A.P. Looking at its mathematical model, we identify and add more features.

1. The matrix of assignment is a square one. (there are n rows and n columns in the matrix).
2. A source is assigned to exactly one activity and *vice versa*.
3. There are n row equations and n column equations. If all $(n - 1)$ assignments are done then the last one has no choice. The number of linearly independent equations is $(2n - 1)$.

4. There are exactly n (out of $2n - 1$) places (cells) where allocations have been made.
5. Total number of cells = $n \times n$; number of basic cells = $2n - 1$
   number of actual allocations = n.

This means that the feasible solution to assignment problem is highly degenerate one.

4.4 Solution Method

In earlier sections, we have discussed that assignment problem is a particular case of transportation problem which, in turn, is a linear programming problem, we have two options:

1. As a special case of LPP (0, 1 class of integer programming problem) and
2. As a special case of transportation problem.

We know that the solution to this problem is always highly degenerate; so in practice, to avoid complications of finding a set of feasible solutions and selecting an optimal one and on the top of that verifying its optimality, we follow a special algorithm developed by Hungarian mathematician.

Hungarian Algorithm

The steps to be followed in Hungarian algorithm are:

Step 1
Subtract the minimum of each row from all the elements of that row.

Step 2
On the resultant matrix obtained as a result of Step 1, subtract the minimum of each column from all the elements of that column.

Step 3
Cover (or suppress) all the zeros of rows and columns of the resultant matrix of the Step 2 by drawing minimum number of lines.

Step 4

(a) If the number of lines covering the zeros equals the number of rows of the matrix, then the matrix obtained as a result of Step 2, is ready for assignment. Go to Step 5.

(b) If the number of lines covering all the zeros is *not* equal to the number of rows then, follow the following procedure.
   1. Find the **minimum** element of all the uncovered entries of the matrix obtained as a result of Step 3.
   2. Subtract the **minimum** from all the uncovered entries.
   3. Do not change the entries under single (zero covering) lines.
   4. Add the same **minimum** entry to the cell entry of two intersecting (zero covering) lines.
   5. Now, go to step 3.

Step 5
Assignment Technique

(A) Find any *row* or a *column* containing a single zero (0) entry. (You may always begin the search from first row; keep on checking each row one by one and if a single zero is not found then search for a single zero in the column.)

Mark (generally * mark) or encircle that zero and cancel that *column* or *row*. Repeat this process till **all** single entry zero containing rows and columns are considered.

If at any stage performing activities of Step 5, a single zero is not found, then select any arbitrary zero, **mark it** and draw two lines—a row covering line and a column covering line through that cell.

Repeat the Step 5. (In this case, we have multiple optimal solutions to the A.P.)

Using the techniques of the above algorithm, we now solve Illustration 1.

4.5 Solution of Illustration I

There are 5 technicians who can work on 5 different machines having the same configuration.

The average time taken by each technician to make a job on each machine is given in the following table (Table 4.1); we require to assign one machine to a technician so that the total working time to make five jobs (one by each) is minimum.

Table 4.1 Time Matrix

Technicians \ *Machines*	M_1	M_2	M_3	M_4	M_5
T_1	7	8	9	9	8
T_2	8	9	11	10	9
T_3	7	8	10	8	9
T_4	10	9	8	9	10
T_5	10	8	9	11	9

Step 1

Subtract the minimum of each row from all the entries of that row.

Technicians \ *Machines*	M_1	M_2	M_3	M_4	M_5
T_1	0	1	2	2	1
T_2	0	1	3	2	1
T_3	0	1	3	1	2
T_4	2	1	0	1	2
T_5	2	0	1	3	1

Step 2

Subtract the minimum of each column from all the entries of above matrix.

Technicians \ *Machines*	M_1	M_2	M_3	M_4	M_5
T_1	0	1	2	1	0
T_2	0	1	3	1	0
T_3	0	1	3	0	1
T_4	2	1	0	0	1
T_5	2	0	1	2	0

Step 3

Try to cover all zeros with minimum number of lines.

Technicians \ Machines	M_1	M_2	M_3	M_4	M_5
T_1	0	1	2	1	0
T_2	0	1	3	1	0
T_3	0	1	3	0	1
T_4	2	1	0	0	1
T_5	2	0	1	2	0

Step 4

There are five lines covering all zeros. The matrix obtained at the end of Step 3 is ready for assignment. We go to assignment procedure Step 5.

Step 5

We write the matrix and apply Step 5.

Technicians \ Machines	M_1	M_2	M_3	M_4	M_5
T_1	0	1	2	1	0
T_2	0	1	3	1	0
T_3	0	1	3	0	1
T_4	2	1	0	0	1
T_5	2	0*	1	2	0

1. There is no single zero in any row. We search for a single zero in columns. We get one in the second column [cell (T_5, M_2)]; mark that (* marked) and delete the row (here it is the fifth row).

Technicians \ Machines	M_1	M_2	M_3	M_4	M_5
T_1	0	1	2	1	0
T_2	0	1	3	1	0
T_3	0	1	3	0	1
T_4	2	1	0	0	1
T_5	~~2~~	~~0*~~	~~1~~	~~2~~	~~0~~

2. Again look for a single zero; we find in the third column. We mark it (*) and cancel the fourth row.

Technicians \ Machines	M_1	M_2	M_3	M_4	M_5
T_1	0	1	2	1	0
T_2	0	1	3	1	0
T_3	0	1	3	0	1
T_4	2	~~1~~	~~0*~~	~~0~~	~~1~~
T_5	2	~~0*~~	~~1~~	~~2~~	~~0~~

3. Again we have a single zero in fourth column [cell ($R_3 C_4$)]; mark it with * and cancel the third row.

Machines / *Technicians*	M_1	M_2	M_3	M_4	M_5
T_1	0	1	2	1	0
T_2	0	1	3	1	0
T_3	0	1	3	0*	1
T_4	2	1	0*	0	1
T_5	2	0*	1	2	0

4. Now, at this stage we do not get a single zero (the first and the second row, the first and the fifth column contain two zeros. We apply Step 5(B).

   Any zero can be arbitrarily selected. We select the cell (1, 5)—(R_1, C_5); we mark it and draw two lines passing through that cell.

Machines / *Technicians*	M_1	M_2	M_3	M_4	M_5
T_1	0	1	2	1	0*
T_2	0	1	3	1	0
T_3	0	1	3	0*	1
T_4	2	1	0*	0	1
T_5	2	0*	1	2	0

5. Finally, a single zero is found in the cell (2, 1) (or equally in the cell (2, 5)). We select the cell (2,1)—mark it and cancel the second row.

Machines / *Technicians*	M_1	M_2	M_3	M_4	M_5
T_1	0	1	2	1	0*
T_2	0*	1	3	1	0
T_3	0	1	3	0*	1
T_4	2	1	0*	0	1
T_5	2	0*	1	2	0

There are five rows (or five technicians) and five columns (or five machines) and five assignments are done; we have the final answer.

Technician	*Machine*	*Job time(hours)*	
T_1	M_5	8	
T_2	M_1	8	
T_3	M_4	8	
T_4	M_3	8	
T_5	M_2	8	Total 40 hours.

ILLUSTRATION 2

There are five workers and their work time to complete their jobs on different machines is given here (Table 4.2). Assign one machine to each worker that minimizes the total working time.

Table 4.2 Time Matrix

Machines / *Workers*	M_1	M_2	M_3	M_4	M_5
W_1	8	5	7	7	8
W_2	9	5	6	7	8
W_3	6	8	5	6	9
W_4	8	10	7	6	5
W_5	4	6	5	6	4

Step 1

Subtracting minimum of each row from all the elements of the row.

Machines / *Workers*	M_1	M_2	M_3	M_4	M_5
W_1	3	0	2	2	3
W_2	4	0	1	2	3
W_3	1	3	0	1	4
W_4	3	5	2	1	0
W_5	0	2	1	2	0

Step 2

Subtracting the minimum of each column from all the elements of that column. (This process will be done on the matrix obtained after the application of Step 1.)

Machines / *Workers*	M_1	M_2	M_3	M_4	M_5
W_1	3	0	2	1	3
W_2	4	0	1	1	3
W_3	1	3	0	0	4
W_4	3	5	2	0	0
W_5	0	2	1	1	0

Step 3

Suppress all the zeros in the rows and columns by drawing *minimum* number of lines. We draw lines in column 2, rows 3, 4 and 5. It is shown in the following table.

Machines / *Workers*	M_1	M_2	M_3	M_4	M_5
W_1	3	~~0~~	2	1	3
W_2	4	~~0~~	1	1	3
W_3	~~1~~	~~3~~	~~0~~	~~0~~	~~4~~
W_4	~~3~~	~~5~~	~~2~~	~~0~~	~~0~~
W_5	~~0~~	~~2~~	~~1~~	~~2~~	~~0~~

Step 4

Here, the minimum number of lines covering zeros = 4 ≠ number of rows = 5. So, we follow the *procedure Step 4*. The minimum entry of all uncovered elements in the above matrix is 1 [Cell (1, 4), (2, 3),and (2, 4)]. Subtract it from all uncovered entries. Add the same entry to the cell of intersection of two lines. In this case, such cells are (3, 2), (4, 2), and (5, 2). The matrix obtained, as a result of this procedure, is follows.

Workers \ Machines	M_1	M_2	M_3	M_4	M_5
W_1	2	0	1	0	2
W_2	3	0	0	0	2
W_3	1	4	0	0	4
W_4	3	6	2	0	0
W_5	0	3	1	2	0

Study the elements of this matrix and the previous matrix and note the changes made according to the instructions in Step 4. Now, it asks to go again to the Step 3—a step covering zeros. Suppressing zeros in the matrix above, by drawing minimum number of lines.

Workers \ Machines	M_1	M_2	M_3	M_4	M_5
W_1	2	0	1	0	2
W_2	3	0	0	0	2
W_3	1	1	0	0	4
W_4	3	6	2	0	0
W_5	0	3	1	2	0

There are chances that sometimes your efforts in suppressing zeros result into more than 5 lines.

You should be sure about drawing minimum lines. The number of lines covering all zeros is 5 = number of rows.

The matrix is ready for assignment and we move to Step 5 of algorithm. Assignment technique, applied on the matrix, gives allocations marked with circles.

Workers \ Machines	M_1	M_2	M_3	M_4	M_5
W_1	2	0*	1	0	2
W_2	3	0	0	**0	2
W_3	1	4	0*	0	4
W_4	3	6	2	0	0*
W_5	0*	3	1	2	0

Steps:

1. A single 0 found in column 1, it is marked and row 5 is cancelled.
2. A single zero is found in column 5, it is marked and row 4 is cancelled.
3. Now, we cannot find a single isolated zero. We follow instructions, a case of multiple optimal solution, given in Step 5 (b) of algorithm.

4. A single zero is found in the first row. Mark it, and cancel the second column.
5. A single zero found in the third row, mark it and cancel the third column.

Assignment

Worker	*Machine*	*Time (hours)*	
W_1	M_2	5	
W_2	M_4	7	
W_3	M_3	5	
W_4	M_5	5	
W_5	M_1	4	Total: 27 hours

4.6 Assignment Techniques and Maximization Problem

Assignment problem, in its basic nature, is a minimization problem. The fundamental conveys that there is a one on one assignment with minimum cost. Also, by nature, the feasible solution is highly degenerate. We cannot apply assignment algorithm on problems searching maximization of objective function. Let us look at the problem and then discuss how we do about it

Illustration 3

There are five salesmen and each one of them could work in any one of the given five districts to make sales for the company they work for. The following matrix (Table 3) gives the average revenue generated by each one of them working in these districts. The given revenue matrix (in ‘000 ₹) is as follows:

Salesmen \ *Districts*	D_1	D_2	D_3	D_4	D_5
S_1	125	99	103	110	105
S_2	120	110	98	104	106
S_3	130	120	98	110	110
S_4	120	125	97	104	100
S_5	120	110	99	100	102

The algorithm does not change but we change our approach of looking at the problem.

We will convert this problem into minimization (opportunity loss) problem. The highest opportunity is the highest revenue generated by any salesperson in any district. This is what is expected and if one fails to achieve then it is called an *opportunity loss*. One would like to minimize the loss. This logic works and we are now ready for application of the assignment algorithm on the matrix of opportunity loss derived from the given matrix seeking maximization. In order to do so, we find the highest entry from all the entries of the given matrix and subtract all the entries of the matrix from that highest entry.

The highest entry amongst all the entries is 130. We subtract all the entries from 130 and find the opportunity loss matrix.

Opportunity Loss Matrix

Salesmen \ Districts	D_1	D_2	D_3	D_4	D_5
S_1	5	31	27	20	25
S_2	10	20	32	26	24
S_3	0	10	32	20	20
S_4	10	5	33	26	30
S_5	10	20	31	30	28

Now we apply the standard algorithm.

Step 1

Subtracting the minimum of each row from all the elements of the row.

Salesmen \ Districts	D_1	D_2	D_3	D_4	D_5
S_1	0	26	22	15	20
S_2	0	10	22	16	14
S_3	0	10	32	20	20
S_4	5	0	28	21	25
S_5	0	10	21	20	18

Step 2

Subtracting the minimum of each column from all the entries of that column.

Salesmen \ Districts	D_1	D_2	D_3	D_4	D_5
S_1	0	26	1	0	6
S_2	0	10	1	1	0
S_3	0	10	11	5	6
S_4	5	0	7	6	11
S_5	0	10	0	5	4

Step 3

Now, suppress all the zeros with minimum number of lines.

Salesmen \ Districts	D_1	D_2	D_3	D_4	D_5
S_1	0	26	1	0	6
S_2	0	10	1	1	0
S_3	0	10	11	5	6
S_4	5	0	7	6	11
S_5	0	10	0	5	4

[First we suppress the first column C_1. The remaining four zeros can be cancelled arbitrarily by drawing row line or column line. We have done suppression in C_1, R_1, C_5, C_2 and R_4]

Step 4

The number of lines covering all the zeros = 5 = number of rows (= columns). The problem is ready for assignment.

Step 5

We find a single zero in a row or column, mark it and cancel the corresponding column or row.

Districts / *Salesmen*	D_1	D_2	D_3	D_4	D_5
S_1	0	26	1	0*	6
S_2	0	10	1	1	0*
S_3	0*	10	11	5	6
S_4	5	0*	7	6	11
S_5	0	10	0*	5	4

Now we list the actual figures and make final answer.

Salesman	*District*	*Revenue*	
S_1	D_4	110	
S_2	D_5	106	
S_3	D_1	130	
S_4	D_2	125	
S_5	D_3	99	Total = 570

∴ Maximum amount of revenue earned by this allocation is 570 (in '000 ₹).

Additional Questions for Practice (with Hints and Answers)

Question 1

You are required to make pairs of the numbers –3, 0, 5, 7, and 9 with the numbers –2, 5, 7, 8, and 10 in a way that the total distance between the assigned pairs is minimum. [The absolute distance between the pair $(a, b) = |a - b|$. For example, the distance between –3 and 7 is $|-3 - 7| = 10$ units, etc.]

Distance	–2	5	7	8	10
–3	1	8	10	11	13
0	2	5	7	8	10
5	7	0	2	3	5
7	9	2	0	1	3
9	11	4	2	1	1

Step 1

Subtracting the minimum of each row from all the elements of the row.

	–2	5	7	8	10
–3	0	7	9	10	12
0	0	3	5	6	8
5	7	0	2	3	5
7	9	2	0	1	3
9	10	3	1	0	0

Step 2

Subtract the minimum of each column from all the elements of that column.
[Each column has one 0 element and hence, we get the same matrix as a result of Step 2.]

	–2	5	7	8	10
–3	0	7	9	10	12
0	0	3	5	6	8
5	7	0	2	3	5
7	9	2	0	1	3
9	10	3	1	0	0

We apply Step 3 on this matrix.

Step 3

We suppress all the zeros with minimum number of lines.

0	7	9	10	12
0	3	5	6	8
7	0	2	3	5
9	2	0	1	3
10	3	1	0	0

Step 4

The number of lines covering zeros = 4 ≠ number of rows = 5.

Step 5

We find the minimum of all uncovered entries (it is 1 in this case). Subtract the same minimum from all uncovered entries. Add the same minimum to the cell—entry at which two lines intersect. The resultant matrix is as follows.

0	6	9	9	11
0	2	5	5	7
8	0	3	3	5
9	1	0	0	2
11	3	2	0	0

Now we move to apply Step 4 of standard algorithm (covering zeros) on this matrix.

Step 6

0	6	9	9	11
0	2	5	5	7
8	0	3	3	5
9	1	0	0	2
11	3	2	0	0

In this step there are four lines (C_1, R_3, R_4, and R_5) covering all zeros.

Step 7

Number of lines covering all the zeros in the matrix above is four ≠ the number of rows (= 5). It means that we have to again move to the Step 4 of standard algorithm.

The minimum of all the uncovered entries of the previous matrix is 2. Subtract 2 from all uncovered entries. Add 2 to the cell entry which is at the intersection of two lines. The resultant matrix obtained is as follows.

0	4	7	7	9
0	0	3	3	5
10	0	3	3	5
11	1	0	0	2
13	3	2	0	0

Again we try to cover all zeros and note that still we have all the zeros covered by the lines C_1, C_2, R_4, and R_5.

0	4	7	3	5
0	0	3	3	5
10	0	3	3	5
11	1	0	0	2
13	3	2	0	0

Step 8

Minimum number of lines suppressing all zeros = 4 ≠ number of rows = 5. Again, we find minimum 1 of all uncovered entries, which is 3. Subtract 3 from all uncovered entries. Add 3 to the cell of intersection of two lines. We get

0	4	4	0	2
0	0	0	0	2
10	0	0	0	2
14	4	0	0	2
16	6	2	0	0

Step 9

Suppress all the zeros. You will find five lines (C_4, C_3, C_1, C_2, and R_5).

0*	4	4	0	2
0	0	*0*	0	2
10	0*	0	0	2
14	4	0	0*	2
16	6	2	0	0*

0	4	4	0	2
0	0	0	0	2
10	0	0	0	2
14	4	0	0	2
16	6	2	0	0

Number of lines covering all zeros = 5 = number of rows = 5. We make assignment using assignment techniques (standard algorithm Step 5).

Step 10

1. First assignment in the cell (5, 5) it is found in C_5. We cancel the 5th row.
2. No single zero is found. We select any arbitrary zero. Selection of cell (2, 3) is done. Two lines are drawn from that cell.
3. Third selection of cell (1, 1) is done. R_1 is cancelled.
4. Fourth selection of cell (4, 4) is done.
5. The last selection of cell (2, 3) is done.

	–2	5	7	8	10
–3	0	4	4	0	2
0	0	0	0	0	2
5	10	0	0	0	2
7	14	4	0	0	2
9	16	6	2	0	0

Result of Assignment

Element	*Paired with*	*Distance*	
–3	–2	1	
0	7	7	
5	5	0	
7	8	1	
9	10	1	Total Distance = 10

Comment:

In the final assignment, we made arbitrary selection of zero Step 10 point (2). This implies that there are multiple solutions to this problem. There are different assignments but the sum of total distances is minimum = 10.

Question 2

Solve the following assignment problem for minimum solution.

Time Matrix

Machines / Workers	M_1	M_2	M_3	M_4	M_5
W_1	9	5	6	7	8
W_2	8	5	7	7	8
W_3	6	8	5	6	9
W_4	8	10	7	6	5
W_5	4	6	5	6	4

Step 1

We subtract the minimum of each row from all the elements of that row.

Machines / Workers	M_1	M_2	M_3	M_4	M_5
W_1	4	0	1	2	3
W_2	3	0	2	2	3
W_3	1	3	0	1	4
W_4	3	5	2	1	0
W_5	0	2	1	2	0

Step 2

subtracting the minimum of each column from all the entries of that column.

Machines / Workers	M_1	M_2	M_3	M_4	M_5
W_1	4	0	1	1	3
W_2	3	0	2	1	3
W_3	1	3	0	0	4
W_4	3	5	2	0	0
W_5	0	2	1	1	0

Step 3

We try to cover all the zeros in the above matrix with minimum number of lines.

Machines / Workers	M_1	M_2	M_3	M_4	M_5
W_1	4	0	1	1	3
W_2	3	0	2	1	3
W_3	1	3	0	0	4
W_4	3	5	2	0	0
W_5	0	2	1	1	0

Number of lines covering zeros = 4 ≠ 5 (Number of rows). The problem is not ready for assignment.

Step 4

The minimum of the uncovered entries is '1'

We subtract '1' from all uncovered entries. [Do not change entries under single line. Add the same minimum entry to the point of intersection of two lines.] The resultant marix is as follows.

Machines / Workers	M_1	M_2	M_3	M_4	M_5
W_1	3	0	0	0	2
W_2	2	0	1	0	2
W_3	1	4	0	0	4
W_4	3	6	2	0	0
W_5	0	3	1	1	0

Step 5

On the matrix obtained above, try to cover all zeros with minimum number of lines.

Machines / Workers	M_1	M_2	M_3	M_4	M_5
W_1	3	0	0	0	2
W_2	2	0	1	0	2
W_3	1	4	0	0	4
W_4	3	6	2	0	0
W_5	0	3	1	1	0

Number of lines covering all zeros = 5 = Number of rows/columns. The last matrix is ready for assignment.

Step 6

Assignment

Machines / Workers	M_1	M_2	M_3	M_4	M_5
W_1	3	0	0	0	2
W_2	2	0	1	0	2
W_3	1	4	0	0	4
W_4	3	6	2	0	0
W_5	0	3	1	1	0

(a) The very first assignment is done in the cell (5,1) = (W_5, M_1). Mark it by * and delete the last row.

(b) Now, we find a single zero in the last column. The cell (4,5) = (W_4, M_5). Mark that zero and delete the particular (fourth row.)

Let us observe impacts of these two operations.

Machines Workers	M_1	M_2	M_3	M_4	M_5
W_1	3	0	0	0	2
W_2	2	0	1	0	2
W_3	1	4	0	0	4
W_4	~~3~~	~~6~~	~~2~~	~~0~~	~~0*~~
W_5	~~0~~	~~3~~	~~1~~	~~1~~	~~0~~

Now, we cannot find single zero and hence we select an arbitrary zero; make an assignment and draw two lines passing through that cell.

Let us make an arbitrary choice of the cell (1,2) = (W_1, M_2). Let us see the result of this operation.

Machines Workers	M_1	M_2	M_3	M_4	M_5
W_1	3	0	0	0	2
W_2	2	0	1	0	2
W_3	1	4	0	0	4
W_4	~~3~~	~~6~~	~~2~~	~~0~~	~~0*~~
W_5	~~0*~~	~~3~~	~~1~~	~~1~~	~~0~~

Now, we cnanot find single zero and hence we select an arbitrary zero; make an assignment and draw two lines passing through that cell.

Let us make an arbitrary choice of the cell (1, 2) = (W_1, M_2). Let us see the result of this operation.

Machines Workers	M_1	M_2	M_3	M_4	M_5
W_1	~~3~~	~~0*~~	~~0~~	~~0~~	~~2~~
W_2	2	0	1	0	2
W_3	1	4	0	0	4
W_4	~~3~~	~~6~~	~~2~~	~~0~~	~~0*~~
W_5	~~0*~~	~~3~~	~~1~~	~~1~~	~~0~~

Now, identifying a single zero in the cell (2,4) = (W_2, M_4), make an asignment. After that, the only cell (3, 3) = (W_3, M_3) is left to choice.

The final table looks as follows.

Assignment Matrix

Machines Workers	M_1	M_2	M_3	M_4	M_5
W_1	3	**0***	0	0	2
W_2	2	0	1	**0***	2
W_3	1	4	**0***	0	4
W_4	3	6	2	0	0*
W_5	**0***	3	1	1	0

We conclude as follows.

Workers	*Machine*	*Time*
W_1	M_2	5
W_2	M_4	7
W_3	M_3	5
W_4	M_5	5
W_5	M_1	4

Total minimum working hours are 26. this is a case of multiple optimal solution. There are some allocations different than what we have done here. The minimum hours (26) remains same.

Question 3

Airlink agencies schedule regular flights between two airports X and Y. The time table showing their coded flight numbers is shown below. Each plane becomes fly-worthy after one hour from arrival to an airport.

Any time after one hour from arrival becomes costly and over-time is paid to the crewmembers. The Airlink management is interested in minimizing this over-time. You have to design schedule accordingly.

Airport X
X to Y

Flight	*Departure*	*Arrival*
101	7	8
102	9	10
103	12	13
104	14	15
105	17	18
106	20	21

Airport Y
Y to X

Flight	*Departure*	*Arrival*
201	7	8
202	10	11
203	13	14
204	15	16
205	16	17
206	21	22

Solution

From this information we construct a table which shows over time spent by a crew at a given airport. By this process we get two matrices showing over time at an airport. Finally, we apply assignment algorithm on each one of the two sub-matrices.

Waiting time showing flight X_i to be flight Y_j

	101	102	103	104	105	106	201	202	203	204	205	206
101							23	2	5	7	8	13
102							21	24	3	5	6	11
103							18	21	24	2	3	8
104							16	19	22	24	1	6
105							13	16	19	21	22	3
106							10	13	16	18	19	24
201	23	1	4	6	9	12						
202	20	22	1	3	6	9						
203	17	19	22	24	3	6						
204	15	17	20	22	1	4						
205	14	16	19	21	24	3						
206	9	11	14	16	19	22						

[Showing an example to interpret the entries of this table is as this If the flight 101 becomes the flight 201 then it will take 23 hours of waiting time. If the flight 201 becomes the flight 102 then it will take 1 hour of waiting time]

Part 1

In this part we solve the lower sub-matrix shown below.

23	1	4	6	9	12
20	22	1	3	6	9
17	19	22	24	3	6
15	17	20	22	1	4
14	16	19	21	24	3
9	11	14	16	19	22

We use assignment techniques to find feasible and minimum solution.
The optimal solutions are shown below.

Assignment or **Assignment**

Flight	*Flight*	*time*		*Flight*	*Flight*	*time*
201	102	1		201	102	1
202	103	1		202	103	1
203	105	3		203	106	6
204	106	4		204	105	1
205	101	14		205	104	21
206	104	16		205	101	9

In both the possibilities shown above it takes crew of airport Y 39 hours time out of the departing destination.(Remaining at airport X)

Part 2

In this part we solve the upper sub-matrix shown below. We use assignment techniques to find feasible and minimum solution.

23	2	5	7	8	13
21	24	3	5	6	11
18	21	24	2	3	8
16	19	22	24	1	6
13	16	19	21	22	3
10	13	16	18	19	24

Assignment

Flight	*Flight*	*time*
101	202	2
102	203	3
103	204	2
104	205	1
105	206	3
106	201	10

It takes a total of 21 hours for the crew of airport X to be at airport Y.

POINTS TO REMEMBER

Some short questions and definitions which are favorites to the examiners and paper-setters are given below. We need that the students read them to have strong foundations.

1. Both, transportation problem (TP) and assignment problem (AP) are nothing but special cases of LPP. To be more precise, AP is a special case of TP.
2. Cost matrix in TP can be rectangular or square but the same in an AP it is always a square one.
3. Optimal solution in a TP may be or may not be degenerate but in an AP it is always degenerate one.
4. In TP having the cost matrix of $n \times n$ size there are $(2n - 1)$ number of allocated cells with entries ≥ 0; while in an AP with cost matrix of $n \times n$ size, there are exactly n numbers of allocated cells.
5. In an AP, if assignment is done in (i, j) cell, we mention it by describing that stage as $x_{ij} = 1$; otherwise $x_{ij} = 0$
6. Mathematical model of an AP is like 0 – 1 programming problem.
7. In its basic nature assignment problem is a minimization type of LPP but maximization type of problems can be solved by converting them into opportunity loss problem.
8. If we add or subtract a constant k to all the cost entries of an AP problem, the optimal allocations does not change but total cost chanes by an amount $= k \times n$, where n stands for the numbers of rows or columns.
9. In an AP of size $n \times n$ the number of non-allocated cells are equal to $n(n - 1)$.

EXERCISES

OBJECTIVE TYPE QUESTIONS

I. State True or False

1. The assignment problem, at the time of applying the Hungarian algorithm must be an unbalanced one.
2. AP is a special case of TP.
3. Adding a constant to each entry of the cost matrix of an AP, then allocations changes.
4. AP may have multiple optimal solutions (different allocations and same cost).
5. Allocations process is carried out when number of zero suppressing lines equals the sum of number of columns and number of rows.
6. Allocation process cannot be continued when we get more than one zero in each row and each column.
7. $x_{ij} = 1$ (standard notation) means the cell is not assigned and hence it is a non-basic cell.
8. $\sum\sum C_{ij} X_{ij}$ shows the objective function in both TP and AP with $i = 1$ to m, $j = 1$ to n, with $m \neq n$.
9. Sometimes AP problem may not give a degenerate solution.
10. In a 5×5 AP, there are 5 allocated cells.
11. We can always arrange the matrix of assignment, after necessary permutation of columns, in the form of a unit matrix.
12. If the assignment is in chain [1→2, 2→3, 3→4, 4→5, and 5→1) then it gives the shortest path from first node 1 to the last node 5.

ANSWERS

1. false.	2. true.	3. false.	4. true.	5. false.	6. false.
7. false	8. false.	9. false.	10. true.	11. true.	

II. Multiple Choice Questions

	F	G	H	I	J
A	1	2	3	4	5
B	6	7	8	9	10
C	11	12	13	14	15
D	16	17	18	19	20
E	21	22	23	24	25

The following matrix concerns the data of problems 1, 2, and 3.

1. Which of the following assignment pairs are essential for the optimal assignment?
   (a) $A \to F$, $D \to I$ (b) $A \to G$, $D \to F$
   (c) $A \to I$, $D \to F$ (d) $A \to J$, $D \to F$
2. What do you estimate about the optimal cost of assignment?
   (a) 65 (b) 66 (c) 67 (d) 68
3. The final matrix of assignment, without any permutation of columns or rows will be
   (a) leading diagonal (b) non-leading diagonal
   (c) rectangular (d) null
4. In a 6×6 assignment problem, the number of non-basic cells is
   (a) 25 (b) 31 (c) 36 (d) 5
5. In a 6×6 assignment problem, the number of allocated basic cells are
   (a) 6 (b) 31 (c) 36 (d) 11
6. During the process of assignment when we do not find a single zero in any row or column, then it is an indication of
   (a) infeasible solution (b) multiple optimal solution
   (c) unbounded solution (d) degenerate solution
7. If we multiply all the entries of the cost matrix of an AP by a constant K, then the cost of assignment will be
   (a) same (b) will become double
   (c) K times (d) divided by K times than that of the original cost of assignment matrix.

Answers

1. (a) 2. (a) 3. (a) 4. (a) 5. (a) 6. (b)
7. (d)

Numerical Problems

Solve the following assignment problems: (Examples 1 to 6 are of the same type.)

1.

10	11	12	12	11
11	12	14	13	12
10	11	13	11	12
13	12	11	12	13
13	11	12	14	12

2.

2	3	4	4	3
3	4	6	5	4
2	3	5	3	4
5	4	3	4	5
5	3	4	6	4

3.

14	16	18	18	16
16	18	22	20	18
14	16	20	16	18
20	18	16	18	20
20	16	18	22	18

4.

Machines / *Workers*	M_1	M_2	M_3	M_4	M_5
W_1	16	10	14	14	16
W_2	18	10	12	14	16
W_3	12	16	10	12	18
W_4	16	20	14	12	10
W_5	8	12	10	12	8

5.

Machines / *Workers*	M_1	M_2	M_3	M_4	M_5
W_1	13	10	12	12	13
W_2	14	11	11	12	13
W_3	11	13	10	11	14
W_4	13	15	12	11	10
W_5	9	11	10	11	9

6.

Machines / *Workers*	M_1	M_2	M_3	M_4	M_5
W_1	4	1	3	3	4
W_2	5	2	2	3	4
W_3	2	4	1	2	5
W_4	4	6	3	2	1
W_5	0	2	1	2	0

7. Solve the following allocation problem for maximization.

Salesmen \ Districts	D_1	D_2	D_3	D_4	D_5
S_1	225	199	203	210	205
S_2	220	210	198	204	206
S_3	230	220	198	210	210
S_4	220	225	197	204	200
S_5	220	210	199	200	202

8. Solve the following assignment problem for maximization.

Salesmen \ Districts	D_1	D_2	D_3	D_4	D_5
S_1	45	19	23	30	25
S_2	40	30	18	24	26
S_3	50	40	18	30	30
S_4	40	45	17	24	20
S_5	40	30	19	20	22

9. Solve the assignment problem for maximization

Salesmen \ Districts	D_1	D_2	D_3	D_4	D_5
S_1	250	198	206	220	210
S_2	240	220	196	208	212
S_3	260	240	198	220	220
S_4	240	250	194	208	200
S_5	240	220	198	200	204

10. Solve the following assignment problem.

1	8	10	11	13
2	5	7	8	10
7	0	2	3	5
9	2	0	1	3
11	4	2	1	1

11. Solve the following assignment problem.

6	13	15	16	18
7	10	12	13	15
12	5	7	8	10
14	7	5	6	8
16	9	7	6	6

12. Solve the following assignment problem.

5	40	50	55	65
10	25	35	40	50
35	0	10	15	25
45	10	0	5	15
55	20	10	5	5

ANSWERS

(A) Assignments for Problems 1, 2, and 3 are as follows.
$1 \to 5, \quad 2 \to 1, \quad 3 \to 4, \quad 4 \to 3, \quad 5 \to 2$

Example	Assignment Cost
1	55
2	15
3	80

(B) Assignments for Problems 4, 5, and 6 are as follows.
$W_1 \to M_2, \quad W_2 \to M_4, \quad W_3 \to M_3, \quad W_4 \to M_5, \quad W_5 \to M_1$

Example	Assignment Cost
4	52
5	51
6	06

(C) Assignments for Problems 7, 8 and 9 are as follows.
$S_1 \to D_4, \quad S_2 \to D_5, \quad S_3 \to D_1, \quad S_4 \to D_2, \quad S_5 \to D_3$

Problems 7 assignment cost = 1070
Problems 8 assignment cost = 170
Problems 9 assignment cost = 1040

(D) Assignments for Problems 10, 11 and 12 are as follows.
$1 \to 1, \quad 2 \to 3, \quad 3 \to 2, \quad 4 \to 4, \quad 5 \to 5$

Example	Assignment Cost
Problem 10	assignment cost = 10
Problem 11	assignment cost = 35
Problem 12	assignment cost = 50

5 Integer Programming Problem

—Those who stand for nothing fall for anything.

Learning Objectives

AFTER STUDYING THIS CHAPTER, THE STUDENTS WILL BE ABLE TO:

- learn the basics of Integer Linear Programming Problems.
- learn and interpret the basic concepts of following methods of finding integer solution to some or all the decision variables of LPP.
  - Gomory's Cutting Plane Method
  - Zero-One Programming, and
  - Branch and Bound Method

INTRODUCTION

The contents of this chapter are extension of fundamentals of LPP with additional constraints on the variables appearing in the given problem.

There are certain types of problems which do fit in the model scheme of LPP but do not finally end up in the expected pattern. The problems involving the decision variables like tables, benches, number of buses, number of routes, number of persons to be hired, etc.; we expect, in such cases, that in the final optimal and meaningful feasible solution they appear in terms of positive integers only.

In the case where a feasible solution in terms of integral units is meaningful, fractional solution may not prove logical. In addition to that mathematical techniques of 'rounding off' on fractional values of decision variables, does never guarantee of optimality of the problem. We take such an example whose solution will efficiently explain the soundness of the above arguments. We have at this point of time following two issues.

1. If we do not get the integral solution of some necessary decision variables then **rounding off** process give the solution which satisfies all constraints. We mean that it will be a point in the feasible region satisfying all the constraints of the given problem.
2. Even if the rounded solution lies within the feasible region; does it guarantee optimality?

5.1 Graphical Solution and Discussion

We introduce the basic concepts by taking an illustration. This illustration, in its mathematical nature, is a complete one that conveys all that we wish to convey.

Illustration 1

Maximize $Z = 3x + 4y$;
subject to $2x + 4y \leq 13$;
$2x + 2y \geq 1$;
$6x - 4y \leq 15$;
$-2x + y \leq 2$;
with $x, y \geq 0$; also x, and y are integers.

Solution by Graphical Method:
As we have already done in the first chapter; we consider equality sign for each constraint, draw the line segments for each constraints , find the corresponding region for each constraint, and then find the most common feasible region.

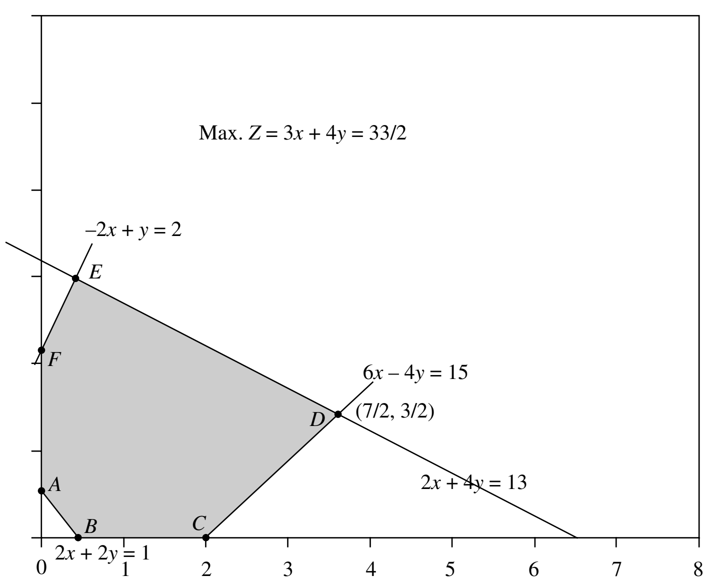
If it is a convex polygon then only we evaluate the objective function at all of its vertices and pick up the one (or may be more) at which the maximum value of the objective function is optimal. We follow the steps as said above.

1. Consider $2x + 4y = 13$; points on the axis are (0, 13/4), (13/2, 0)
2. Consider $2x + 2y = 1$; points on the axis are (0, 1/2), (1/2, 0)
3. Consider $6x - 4y = 15$; points on the axis are (0, –15/4), (5/2, 0)
4. Consider $-2x + y = 2$; points on the axis are (0, 2), (–1, 0)

Graph 1 shows the corresponding graph and the most common region corresponding to each constraints is the convex polygon ABCDEF—A.

The maximum value taken by the objective function $Z = 3x + 4y$ at (7/2, 3/2) is 33/2.

Graph 1



As said in the problem, we must have both x and y as integers ≥ 0. Our optimal value is at (7/2, 3/2); i.e at $x = 7/2$ and $y = 3/2$ and both are positive but non-integer values.

The way out at this juncture is to round off and $x = 7/2 = 3.5$, $y = 3/2 = 1.5$

In addition to this, we know that rounding off of 0.5 is done with equal justification on either sides. It gives four possibilities as follows.

Rounded integral points are (3, 1), (3, 2),(4, 1), and (4, 2).

The stags at these points are as follows.

1. Identification of those points which are within the feasible regions. In our example, the points (3, 2), (4, 1), and (4, 2) are not within the feasible region; so we discard all such points. The only point on which we base our hope of optimality is (3, 1).
   And the value of the objective function is $Z = 3(3) + 4(1) = 13$.
2. Verification of optimality at these points. Here, we have only one feasible point (3, 1) obtained by following *rounded off* process, and there is no guarantee that the objective function will take optimal value at that point.
3. If we make a search at all such integral points like (x, y) within the feasible region; evaluate the objective function then find that the objective function attains optimality at the point (2, 2). At this point, the objective function attains the value $Z = 3(2) + 4(2) = 14$.

This problem reveals all that we need to make a systematic study of integer programming problem in a classical way.

5.2 Gomory's Cutting Plane Method and Geometric Meaning

This unit is the dominating one and it helps solving problems of ILPP. It is required that the following important units should be well understood before we make any attempt on fundamentals of **Gomory's** cutting plane method.

5.2.1 Study of Some Important Points from the Simplex Table

Let us recall our standard format of LPP.

Find $\boldsymbol{X}$ so as to maximize $Z = \boldsymbol{CX}$

Subject to $\boldsymbol{AX = b}$ **and** $\boldsymbol{X \geq 0}$.

The optimal simplex table looks as follows.

Maximize

Table 5.1

$C_j \rightarrow$ $\downarrow$ Variables $\rightarrow$ Basis $\downarrow$	C_1 x_1	C_2 x_2	C_3 x_3	...	...	...	C_r x_r	C_{r+1} x_{r+1}	...	C_m x_m	C_{m+1} x_{m+1}	C_{m+2} x_{m+2}	... ...	C_n X_n	$X_B = RHS$ b
C_1 x_1	1	0	0				0	0	0	0					b_1
C_2 x_2	0	1	0				0	0	0	0					b_2
C_3 x_3	0	0	1				0	0	0	0					b_3
	...	...	...												
	...	...	...												
	...	...	...												
C_r x_r	0	0	0	0	0	0	1	0	0	0	$a_{r\,m+1}$	$a_{r\,m+2}$	...	a_{rn}	b_r
C_{r+1} x_{r+1}	0	0	0	0	0	0	0	1	0	0					b_{r+1}
...										0					
C_m x_m	0	0	0	0	0	0	0	0	0	1					b_m
$C_j - Z_j$	0	0	0	0	0	0	0	0	0	0	$\leftrightarrow$	negative	$\leftrightarrow$		

1. There are m linearly independent equations in n variables ($m \le n$).
2. As arranged, first m of the n variables are basic variables and the remaining $(n - m)$ are non-basic variables.
3. The optimal value of the objective function is $Z = C_1 b_1 + C_2 b_2 + \cdots + C_m b_m$
4. $C_j - Z_j$ entries for the x_1 up to x_m basic variables are zero and those for x_{m+1} to x_n are negative.
5. We can write equation for each row. The $\boldsymbol{r}^{\mathbf{th}}$ row equation written down from the table is as follows.

$$1\boldsymbol{x_r} + (\boldsymbol{a_{r\,m+1}})\,\boldsymbol{x_{r\,m+1}} + (\boldsymbol{a_{r\,m+2}})\,\boldsymbol{x_{r\,m+2}} + \cdots + (\boldsymbol{a_{r\,n}})\,\boldsymbol{x_{r\,n}} = \boldsymbol{b_r}$$

5.2.2 Integral part notation

In this chapter, we focus on integer values and to help the routines we introduce special notation.

For any real x, **$[x]$ is the integer in x which is not greater than x.**

For example, $[2.9] = 2$, $[0.78] = 0$, $[-1.23] = -2$, $[-0.52] = -1$, etc.

We have the given real value $x = [x] + f$; where $\boldsymbol{o < f < 1}$

For example, $2.9 = [2.9] + 0.9$, where $[2.9] = 2$

$-2.41 = [-2.41] + 0.59$, where $[-2.41] = -3$,

Note that in any case, $0 < f < 1$

5.2.3 Definition of Integer Linear Programming Problem (ILPP)

At this stage, we define classical structure of ILPP.

Find $X \in \boldsymbol{R}_{n \times 1}$ so as to maximize $Z = \boldsymbol{CX}$ **(1)**

with the m constraints $AX = b$ **(2)**

and $X \ge 0$ **(3)**

all X_i integers (i = 1 to m)
(All ILPP)

Some X_i integers (i = 1 to $k < m$)
(Mixed ILPP)

5.2.4 Gomory's Cutting Plane Method

Now, we are well equipped to establish the theoretical part of Gomory's cutting plane method. WE write the main problem of our continued discussion and proceed with discussion.

Find $X \in \boldsymbol{R}_{n \times 1}$ so as to maximize $Z = \boldsymbol{CX}$ **(1)**

with the m constraints $AX = b$ **(2)**

and $X \ge 0$ **(3)**

all X_i integers (i = 1 to m)
(All ILPP)

Some X_i integers (i = 1 to $k <$ m)
(Mixed ILPP)

We discuss a general procedure applicable for any one of two types of problems.

Step 1

We apply simplex procedure to find optimal feasible solution. If all or required decision variables are positive integers then the purpose is solved. If some variables are not integers then we apply Gomory's cutting plane method.

Step 2

From the set of ***non-integer values*** of the decision variables, identify the fractional part and select the ***variable*** with the highest fractional part. If there is a tie, choice is optional.

Let us assume that the basic variable $\boldsymbol{x_r}$ has the highest fractional part.

Step 3

From the simplex table, we write down the corresponding x_r row equation. (We have already written the equation in Section **5.2.1.(5)**)

It is $1\boldsymbol{x_r} + (\boldsymbol{a_{r\,m+1}})\,\boldsymbol{x_{r\,m+1}} + (\boldsymbol{a_{r\,m+2}})\,\boldsymbol{x_{r\,m+2}} + \cdots + (\boldsymbol{a_{r\,n}})\,\boldsymbol{x_{r\,n}} = \boldsymbol{b_r}$

The bracketed values like $(a_{r\,m+1})$, $(a_{r\,m+2})$, $(a_{r\,n})$, and (b_r) are real values.

We write them as $\boldsymbol{a_{r\,m+1} = [a_{r\,m+1}] + f_{r\,m+1}}$ **where** $\boldsymbol{0 < f_{r\,m+1} < 1}$,

$\boldsymbol{a_{r\,m+2} = [a_{r\,m+2}] + f_{r\,m+2}}$ **where** $\boldsymbol{0 < f_{r\,m+2} < 1}$

$\boldsymbol{a_{r\,n} = [a_{r\,n}] + f_{r\,n}}$ **where** $\boldsymbol{0 < f_{r\,n} < 1}$

and finally $\boldsymbol{b_r = [b_r] + f_r}$ **where** $\boldsymbol{0 < f_r < 1}$

We note that each one of the bracketed terms are integers. We write integer expression on one side and the fractional portion of the equation on the other side.

$$1x_r + [a_{r\,m+1}]\,x_{r\,m+1} + [a_{r\,m+2}]\,x_{r\,m+2} + \cdots + [a_{r\,n}]\,x_{r\,n} - [b_r] = f_r - (f_{r\,m+1} \cdot x_{r\,m+1} + f_{r\,m+2} \cdot x_{r\,m+2} \cdots + f_{r\,n} \cdot x_{r\,n})$$

As the left side portion represents integral value so must be right side too.

Also $0 < f_r < 1$ so the right side $= f_r - (f_{r\,m+1} \cdot x_{r\,m+1} + f_{r\,m+2} \cdot x_{r\,m+2} + \cdots + f_{r\,n} \cdot x_{r\,n})$ must give either **zero or a negative integer** only.

So $f_r - (f_{r\,m+1} \cdot x_{r\,m+1} + f_{r\,m+2} \cdot x_{r\,m+2} + \cdots + f_{r\,n} \cdot x_{r\,n}) \le 0$

$\therefore \quad (f_{r\,m+1} \cdot x_{r\,m+1} + f_{r\,m+2} \cdot x_{r\,m+2} + \cdots + f_{r\,n} \cdot x_{r\,n}) \ge f_r$

$\therefore \quad \boldsymbol{-(f_{r\,m+1} \cdot x_{r\,m+1} + f_{r\,m+2} \cdot x_{r\,m+2} + \cdots + f_{r\,n} \cdot x_{r\,n}) \le -f_r}$

or

$$-\sum_{j=m+1}^{j=n} f_{rj} \le -f_r \tag{2}$$

is called *Gomory's constraint*.

This is to be introduced in the main table after introducing **a basic variable, say $\boldsymbol{G \ge 0}$**.

$$\boldsymbol{-(f_{r\,m+1} \cdot x_{r\,m+1} + f_{r\,m+2} \cdot x_{r\,m+2} + \cdots + f_{r\,n} \cdot x_{r\,n}) + G = -f_r} \tag{3}$$

i.e.

$$-\sum_{j=m+1}^{j=n} f_{rj} + G \le -f_r$$

(At this point, we again note that $0 < f_r < 1$. In the constraint (3) shown above the right side value is $-f_r$ and this equality constraint when introduced in the optimal simplex table will not permit simplex table to work further. We know that for the simplex procedure right side or the resource side must be positive. To solve this, we apply ***dual simplex*** *method*.)

5.3 ILLUSTRATIONS

At this stage, we give some illustrations to understand the situations under consideration.

ILLUSTRATION 2

Maximize $Z = 3y$

subject to $-1x + y \le 2$,

$3x + 2y \le 7$,

$x, y \ge 0$,

x and y both are integers.

Solution

The given constraints are of $\le$ type and we introduce slack variables s_1 and s_2 both ≥ 0.
The constraint that both x and y are integers will be taken care of once the simplex solution is obtained.
The simplex table is as follows.

Table 5.2

$C_j \rightarrow$ ↓		0	3	0	0		
	Variables → / Basis ↓	X	Y	S_1	s_2	b	R.R.
0	$s_1 \leftarrow$	–1	(1)	1	0	2	2→
0	s_2	3	2	0	1	7	7/2
	$C_j - Z_j$	0	3↑	0	0	$Z = 0$	
3	Y	–1	1	1	0	2	–
0	s_2	(5)	0	–2	1	3	3/5→
	$C_j - Z_j$	3↑	0	–3	0	$Z = 6$	
3	y	0	1	3/5	1/5	13/5	
0	X	1	0	–2/5	1/5	3/5	
	$C_j - Z_j$	0	0	–9/5	–3/5	$Z = 39/5$	

All $C_j - Z_j$ entries are ≤ 0 shows optimality criterion for this maximization problem.

Basic variables are $x = 3/5$ and $y = 13/5$; non-basic variables s_1 and $s_2 = 0$.

$$Z = 39/5$$

As required by the constraints both x and y must be integers. We now apply Gomory's cutting plane method.

$y = 13/5 = 2 + 3/5$ and $x = 0 + 3/5$. Fractional parts in both x and y are the same and is = 3/5

We can select any one equation from the optimal table.

$$0x + 1y + (3/5)\, s_1 + (1/5)\, s_2 = 13/5$$

Now we separate integer part and fractional part from this equation.

$$0x + 1y + \{[3/5] + 3/5\}\, s_1 + \{[1/5] + 1/5\}\, s_2 = \{[13/5] + 3/5\}$$

Note that [3/5] = 0, [1/5] = 0 and [13/5] = 2; using these notations, we have

$$1y + 0\, s_1 + 3/5\, s_1 + 0s_2 + 1/5\, s_2 = 2 + 3/5$$

$$\therefore \quad 1y - 2 = 3/5 - (3/5\, s_1 + 1/5\, s_2)$$

Left side indicates integral value and so the right side also stand for the same.

We have $3/5 - (3/5\, s_1 + 1/5\, s_2) \leq 0$, an integer.

$\therefore$ $-(3/5\, s_1 + 1/5\, s_2) \leq -3/5$; which is *Gomory's constraint*.

To introduce this in the simplex table, we add a slack variable $G \geq 0$.

$-3/5\, s_1 - 1/5\, s_2 + G = -3/5$ with $G \geq 0$

Introducing this in the last part of the table, we have

Maximize:

Table 5.3

$C_j \rightarrow$ ↓	Variables → / Basis ↓	0	3	0	0	0		
		x	y	s_1	s_2	G	b	*R.R.*
3	y	0	1	3/5	1/5	0	13/5	
0	x	1	0	–2/5	1/5	0	3/5	
0	G	0	0	(–3/5)	–1/5	1	–3/5	
	$c_j - Z_j$	0	0	–9/5↑	–3/5	0	$Z = 39/5$	

As the basic variable $G = -3/5$ (a negative value); we cannot apply regular simplex method. So we apply **dual simplex** method. For that, one point is sure that G is an out-going variable. On the basis of replacement ratio, we decide the incoming variable. If at least one entry in the corresponding row is negative then only we can find the replacement ratio.

Replacement Ratio = $(C_j - Z_j)$ entries /Corresponding **Negative** entries of that row.
= minimum of { (–9/5)/ (–3/5), (–3/5) / (–1/5) }
= minimum of {3, 3} =3

(Both are same; and hence any one of these two variables can be selected as an incoming variable.)
Let us select s_1 as an incoming variable and hence **–3/5 is the pivot.**
Now we use regular procedure as we did for simplex method.

Maximize:

Table 5.4

$C_j \rightarrow$ ↓	Variables → / Basis ↓	0	3	0	0	0		
		x	y	s_1	s_2	G	b	*R.R.*
3	y	1	0	0	0	1	2	
0	x	0	1	0	1/3	–2/3	1	
0	s_1	0	0	1	1/3	–5/3	1	
	$C_j - Z_j$	0	0	0	0	–3	$Z = 6$	

Comment:

1. This is a case of multiple optimal solution. ($C_j - Z_j$ entry corresponding to a non-basic variable s_2 is zero.
2. All $C_j - Z_j$ entries are ≤ 0; so the solution is optimal.
   Basic Variables: $x = 1$, $y = 2$, $s_1 = 0$
   Non-basic variables: $s_2 = G = 0$ Optimal $Z = 6$
3. In the same way had we considered $x = 3/5$; we would have got different result.

5.3.1 Graphical Interpretation of the Cutting Plane Constraint

We have two constraints:

1. $-1x + y \leq 2$, i.e. $-1x + y + s_1 = 2$
2. $3x + 2y \leq 7$, i.e. $3x + 2y + s_2 = 7$

$\therefore$ $s_1 = 2 + x - y$ and $s_2 = 7 - 3x - 2y$

Gomory's constraint has come out to be $-(3/5\ s_1 + 1/5\ s_2) \leq -3/5$

$\therefore$ $(-3/5)\mathbf{(2 + x - y) - (1/5)\ (7 - 3x - 2y) \leq -3/5}$

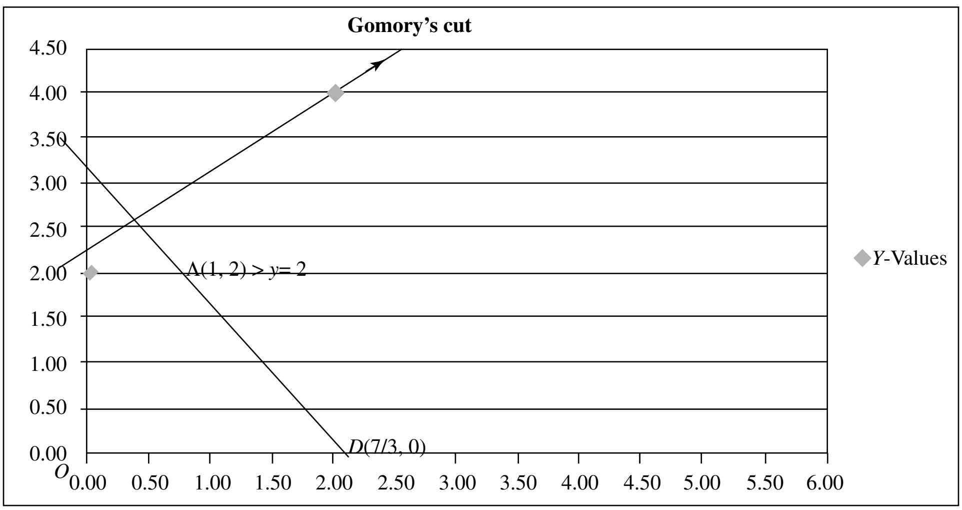
$\therefore$ $-6 - 3x + 3y - 7 + 3x + 2y \leq -3$

$5y \leq 10$

$\therefore$ $y \leq 2$

This is a cutting plane (line) to the original feasible region. This line, when introduced as an additional constraint, resizes the original feasible region.

Graph 2



The feasible region resized by the cut is **OBAD—O;** which is a convex region. AB is the line segment such that any point on that is a feasible optimal solution.

ILLUSTRATION 3

Maximize $Z = 26x_1 + 13x_2$

Subject to $1x_1 + 3x_2 \leq 10$

and $6x_1 + 5x_2 \leq 25$

With $x_1, x_2, \geq 0$ and integers.

Solution

In this problem both the given constrains are of less than ($\leq$) type and we add slack variables $s_1, s_2 \geq 0$ and introduce them in simplex table

Maximize:

Table 5.5

	$C_j \rightarrow$ VAR ↓Basis		26 x_1	13 x_2	0 s_1	0 s_2	x_B	R.R.
I	0	s_1	1	3	1	0	10	10
	0 ←	s_2	6	5	0	1	25	25/6 →
	$C_j - Z_j$		26↑	13	0	0	Z = 0	
II	0	S_1	0	13/6	1	–1/6	35/6	
	26	x_1	1	5/6	0	1/6	25/6	
	$C_j - Z_j$		0	–26/3	0	–13/3	Z = 325/3	

All $C_j - Z_j$ are ≤ 0 and so the solution is optimal.

Basic variables: $x_1 = 25/6$; $s_1 = 35/6$; $Z = 325/3$;

$x_1 = 25/6 = 4 + 1/6$; $s_1 = 35/6 = 5 + 5/6$;

The higher fraction value is 5/6 which in s_1. We make an equation for the first row.

$13/6\ x_2 + 1s_1 - 1/6\ s_2 = 5 + 5/6$;

Now, $13/6 = [13/6] + 1/6$; where $[13/6] = 2$;

$-1/6 = [-1/6] + 5/6$; where $[-1/6] = -1$

$\therefore \quad 2x_2 + 1/6\ x_2 + s_1 - 1s_2 + 5/6\ s_2 = 5 + 5/6$;

$\therefore \quad 2x_2 + s_1 - 5 = 5/6 - (1/6\ x_2 + 5/6\ s_2)$

By Gormory's cut, we have $5/6 - 1/6\ x_2 - 5/6\ s_2 \leq 0$, an integer;

$\therefore \quad -1/6\ x_2 - 5/6\ s_2 = -5/6$ which is Gormory's cut. We introduce this in the final simplex table.

Let $G_1 \geq 0$ be a slack variable. We introduce this constraint in the last simplex *Table 1*.

Maximize:

Table 5.6

$C_j \rightarrow$ VAR ↓Basis		13 x_1	26 x_2	0 s_1	0 s_2	0 G_1	x_B
0	s_1	0	13/6	1	–1/6	0	35/6
26	x_1	1	5/6	0	1/6	0	25/6
0	G_1	0	–1/6	0	–(5/6)	1	–5/6
$C_j - Z_j$		0	– 26/3	0	–13/3	0	

Here, we apply dual-simplex method. G_1 is an out-going variable, an incoming variable is decided based on replacement ratio. (R.R.)

$$\text{R.R. for } x_2 = \frac{-26/3}{-1/6} = 52$$

$$\text{R.R. for } s_2 = \frac{-13/3}{-5/6} = 26/5.$$

R.R. for s_2 being minimum: we select s_2 as an incoming variable; this gives the next table as follows.

Table 5.7

	$C_j \rightarrow$ VAR ↓ Basis	13 x_1	26 x_2	0 s_1	0 s_2	0 G_1	x_B	R. R.
I	0 ← s_1	0	**(11/5)**	1	0	–1/5	6	30/11→
	26 x_1	1	4/5	0	0	1/5	4	5
	0 s_2	0	1/5	0	1	–6/5	1	5
	$C_j - Z_j$	0	26/5↑	0	0	–26/5	104	
II	26 x_2	0	1	5/11	0	–1/11	30/11→	
	13 x_1	1	0	–4/11	0	3/11	20/11	
	0 s_2	0	0	–1/11	1	–13/11	5/11	
	$C_j - Z_j$	0	0	–78/11	0	–13/11	$Z = 1040/11$	

In this case, we again get fractional values of basic variables and it requires finding Gomory's cut to the variable with higher fractional value.

$x_2 = 30/11 = 2 + 8/11$; $x_1 = 20/11 = 1 + 9/11$ and $s_2 = 5/11 = 0 + 5/11$.

We select x_1 as it has the highest fractional value = 9/11.

We write equation for x_1;

$1x_1 - 4/11\ s_1 + 3/11\ G_1 = 20/11.$

Separating this into fractional and integer portion; we have

$1x_1 - 1s_1 + 7/11\ s_1 + 3/11\ G_1 = 1 + 9/11;$

$\therefore \quad 1x_1 - 1s_1 - 1 = 9/11 - (7/11\ s_1 + 3/11\ G_1)$

$\therefore \quad -7/11\ s_1 - 3/11\ G_1 = -9/11$ is a Gomory's cut.

We introduce slack variable $G_2 \geq 0$ and get; $-7/11\ s_1 - 3/11\ G_1 + G_2 = -9/11$;

Introducing this in the Table 3, we have,

Table 5.8

	$C_j \rightarrow$VAR ↓Basis	13 x_1	26 x_2	0 s_1	0 s_2	0 G_1	0 G_2	x_B	R. R.
I	26 x_2	0	1	5/11	0	–1/11	0	30/11	
	13 x_1	1	0	–4/11	0	3/11	0	20/11	
	0 s_2	0	0	–1/11	1	–13/11	0	5/11	
	0 ← G_2	0	0	–7/11	0	**(–3/11)**	1	–9/11→	
	$C_j - Z_j$	0	0	–78/11	0	–13/11	0		
		At this stage, we apply dual simplex method. G_2 is an outgoing variable and G_1 is an incoming variable.							
II	26 x_2	0	1	2/3	0	0	–1/3	3	
	13 x_1	1	0	–1	0	0	1	1	
	0 s_2	0	0	–8/3	1	0	–13/3	4	
	0 g_1	0	0	7/3	0	1	–11/3	3	
	$C_j - Z_j$	0	0	–13/3	0	0	–13/3	$Z = 91$	

All entries of $C_j - Z_j \leq 0$. This solution is optimal.

Basic variables: $x_1 = 1$; $x_2 = 3$; $s_2 = 4$, $G_1 = 3$

Non basic variables: $s_1 = G_2 = 0$;

Maximum $Z = 91$.

ILLUSTRATION 4

Solve the problem using Gomory's cutting plane method.

Maximise $z = 2x_1 + 4x_2$

Subject to $3x_1 + 4x_2 \leq 6$

$x_2 \leq 2$

$x_1, x_2 \geq 0$ x_1 and x_2 both are integers.

Solution

Adding slack variable $s_1 \geq 0$ and $s_2 \geq 0$, the problem can be represented in simplex table form as follows:

Maximize:

Table 5.9

	$C_j \rightarrow$ ↓ Basis \ VAR	2 x_1	4 x_2	0 s_1	0 s_2	b	R.R.
I	0 s_1	3	4	1	0	6	3/2 ←
	0 s_2	0	1	0	1	2	2
	$C_j - Z_j$	2	4 ↑	0	0	$Z = 0$	0
II	4 x_2	3/4	1	1/4	0	3/2	
	0 s_2	–3/4	0	–1/4	1	1/2	
	$C_j - Z_j$	–1	0	–1	0	6	

As all $C_j - Z_j$ entries are ≤ 0 for this maximization problem, the solution is optimal. But $x_2 = 3/2$ and $s_2 = 1/2$ are not integers.

The fraction in x_2 is 1/2 and that in s_2 is 1/2.

We write equation for the first row.

$3/4\ x_1 + 1\ x_2 + 1/4\ s_1 + 0\ s_2 = 3/2$

$\therefore$ $([3/4] + 3/4)x_1 + 1x_2 + ([1/4] + 1/4)s_1 = [3/2] + 1/2$

$\therefore$ $(0 + 3/4)x_1 + 1x_2 + (0 + 1/4)s_1 = 1 + 1/2$

$\therefore$ $1x_2 - 1 = 1/2 - (3/4\ x_1 + 1/4\ s_1)$

The left side represents integer value and so the right side must be ≤ 0 and integer.

$1/2 - (3/4\ x_1 + 1/4\ s_1) \leq 0$

$\therefore$ $1/2 \leq 3/4\ x_1 + 1/4\ s_1$

$\therefore$ $-3/4\ x_1 - 1/4\ s_1 \leq -1/2$; which is Gomory's constraint and by adding a slack variable $G_1 \geq 0$, it is introduced in the optimal table

Maximize:

Table 5.10

$C_j \rightarrow$ ↓ Basis \ VAR	2 x_1	4 x_2	0 s_1	0 s_2	0 G_1	b
4 x_2	3/4	1	1/4	0	0	3/2
0 s_2	–3/4	0	–1/4	1	0	1/2
0 G_1	–3/4	0	–1/4	0	1	–1/2 →
$C_j - Z_j - 1$	–1	–1	0	0	0	

We apply dual simplex method as the basic variable $G_1 = -1/2$

Maximize:

Table 5.11

	$C_j \rightarrow$ VAR / $\downarrow$ Basis	2 x_1	4 x_2	0 s_1	0 s_2	0 G_1	b
	4 x_2	0	1	0	0	1	1
I	0 s_2	0	0	0	1	–1	1
	2 x_1	1	0	1/3	0	–4/3	2/3
	$C_j - Z_j$	0	0	–2/3	0	–4/3	16/3

The solution obtained does not give integer values to the basic variables. The basic, variable $x_1 = 2/3$. We write its corresponding equation

$\therefore$ $x_1 + 1/3\ s_1 - 4/3\ G_1 = 2/3;$

$\therefore$ $x_1 + ([1/3] + 1/3)s_1 + ([-4/3] + 2/3)\ G_1 = [2/3] + 2/3.$

$\therefore$ $x_1 + (0 + 1/3)s_1 + (-2 + 2/3)\ G_1 = 0 + 2/3.$

$\therefore$ $x_1 - 2G_1 = 2/3 - 1/3\ s_1 - 2/3\ G_1$ and we must have $2/3 - 1/3\ s_1 - 2/3\ G_1 \leq 0$ and integer.

$\therefore$ $-1/3\ s_1 - 2/3\ G_1 \leq -2/3$

Gomory's constraint adding slack variable $G_2 \geq 0$; we introduced in the final table.

Maximize:

Table 5.12

$C_j \rightarrow$ VAR / $\downarrow$ Basis	2 x_1	4 x_2	0 s_1	0 s_2	0 G_1	0 G_2	b
4 x_2	0	1	0	0	1	0	1
0 s_2	0	0	0	1	–1	0	1
2 x_1	1	0	1/3	0	–4/3	0	2/3
0 G_2	0	0	(–1/3)	0	–2/3	1	–2/3
$C_j - Z_j$	0	0	–2/3	0	–4/3	0	16/3
R. R.	For $s_1 = (-2/3) / (-1/3)$, For $G_1 = (-4/3) / (-2/3)$						

We select s_1 to be an incoming variable. The next table is as follows.

Maximize:

Table 5.13

$C_j \rightarrow$ VAR / $\downarrow$ Basis	2 x_1	4 x_2	0 s_1	0 s_2	0 G_1	0 G_2	b
4 x_2	0	1	0	0	1	0	1
0 s_2	0	0	0	1	–1	0	1
2 x_1	1	0	0	0	–2	1	0
0 s_1	0	0	1	0	2	–3	2
$C_j - Z_j$	0	0	0	0	0	–2	4

Since all $C_j - Z_j \leq 0$ for the maximization problem, it has reached optimal value. Also all basic variables are integers.

$\therefore$ $x_1 = 0, x_2 = 1, s_1 = 2, s_2 = 1$ and optimal $Z = 4$.

5.4 Branch and Bound Method

We have already seen Gomory's cutting plane method of finding a linear cut in the feasible region of the given constraints. Every time we introduce Gomory's cut and find the new feasible solution applying dual simplex method.

This is continued until

1. all basic variables are positive integers, and
2. optimality criterion $C_j - Z_j \leq 0$ condition is satisfied. (For a maximization problem)

In addition to this, we have one analytical method called *Branch and Bound method*.

We have two clear points;

1. The given problem is an LPP.
2. All or some of the variables, as required, must be greater than or equal to zero and integers.

How do we work with this method?

We solve the given ILPP (Integer Linear Programming Problem) using the regular simplex procedure without considering integer constraint. If it comes, that all basic-variables in the optimal table are positive integers or zero and optimality criterion is satisfied; we are done with what we required. If some or all the basic variables are not integers (≥ 0) then we either follow *Gomory's cutting plane method* or *Branch and Bound method*.

The non-integer solution obtained in the optimal table is the starting node, and is the origin.

Consider the Problem:

Find $\boldsymbol{X}$ so as to maximize $Z = \boldsymbol{CX}$ (1)

With $A\boldsymbol{X} = \boldsymbol{b}$ (2)

And $\boldsymbol{X} \geq 0$ (3)

$X = \{x_j \mid x_j \geq 0 \text{ and integer}\}$ (4)

1. Node

Node is the first event shown in a circle and it shows the information regarding the status of basic variables and the optimal value of the objective function. (An upper bound or a lower bound as the case may be.)

Node 1 refers to the details of the original or given problem; denoted as P_1.

2. Selection of a basic variable

From the optimal solution, we select some variables denoted as $x_i \geq 0$ and which has the highest fractional value amongst all the fractional values of the non-integer positive fractional values of all the basic variables in the optimal solution.

There is no integer between $[X_i]$ and $[X_i] + 1$

$\therefore$ Integer solution has only two options.

$\therefore$ Either $X_i \leq [X_i]$ or we have $X_i \geq [X_i] + 1$ [Where $[X_i]$ integer in X_i not greater than X_i]

This division or bifurcation of $\mathbf{P_1}$ gives rise to two sub-problems, they are now $\mathbf{P_{11}}$ and $\mathbf{P_{12}}$ [P_1 is a parent problem.]

3. Branching

The process of replacing a problem by two sub-problems is called ***Branching***. The line-segment connecting P_1 to P_2 and P_1 to P_3 is called a *branch*.

[P_{11} problem is P_1 + an additional constraint $X_i \leq [X_i]$.

P_{12} problem is P_1 + an additional constraint $X_i \geq [X_i] + 1$]

3.1 We solve $\mathbf{P_{12}}$ and $\mathbf{P_{13}}$ by either simplex method or graphical method.

We note that each non-terminal node has two branches and each branch has a sub-problem for the solution.

Case 1

The sub-problem when solved has an infeasible solution. We do not or cannot branch further from this infeasible node.

This phase is called *pruning* of the current node.

Case 2

The sup-problem when solved gives a feasible solution in which **either** all the basic variables are, as required, integers **or** some basic variables possess fractional (non-integer positive) value.

(a) In case if all the basic variables are integers, then we terminate the process at this node. This event is, in integer programming terminology, called ***fathoming.*** The optimal value of the objective function is a new upper/lower bound and it is considered for final selection of the solution.

(b) In case, if some of the required basic variables do not attain integer values then, there is a further branching from this node into two more nodes.

(c) The basic idea of ***bound*** implies that any sub-problem at any node assures that further branching will not give a better solution to that what we have now on hand, then that criterion limits further branching.

This calls for two more sub-problems [Say P_{11} is of this type, then we have P_{111} and P_{112} as two sub-problems and we repeat the procedure.

Finally, when the nodes either are ***fathomed*** or ***pruned*** by infeasibility, we compare all the upper bounds (lower bounds) values and select the one which is the highest /lowest.

ILLUSTRATIONS 5

We take some illustrations to explain the working procedure of the branch and bound method.

Maximize $Z = 7x_1 + 9x_2$

Subject to $7x_1 + 1x_2 \leq 35$

$-1x_1 + 3x_2 \leq 6_j$

and both x_1 and $x_2 \leq 7$ and positive integers.

Solution

We solve the above problem using regular simplex method or graphical method.

We get $x_1 = 9/2$, and $x_2 = 7/2$ as basic variables and maximum $Z = 7(9/2) + 9(7/2) = 63$.

[This is the problem P_1 and first node.]

Now, we take up the integer restriction on basic variables and solve the problem using branch bound method.

$x_1 = 9/2 = 4 + 1/2$; and $x_2 = 7/2 = 3 + 1/2$; both x_1 and x_2 have the same fractional value = 1/2 and so the choice for any one basic variable to begin with is optional. We select $x_1 = 9/2$;

Now [9/2] = 4; We have two branches.

There are ***two*** branches from the first node. This gives rise to two more problems P_{11} and P_{12}.

In P_{11}, we have all of the problems P_1 plus an additional constraint $x_1 = 4$.

In P_{12}, we have all of the problems P_1 plus an additional constraint $x_1 = 5$.

[In both P_{11} and P_{12}; integer constraint for the basic variable continue uninterrupted.]

We have to solve P_{11} and P_{12}, [One may use simplex method or graphical method or in case of $\geq$ type inequality the dual simplex method may be used.]

Solution to these two sub-problems is given below.

1. Problem P_1: Maximize $Z = 7x_1 + 9x_2$

   Subject to $7x_1 + 1x_2 \leq 35$

   $-1x_1 + 3x_2 \leq 6_j$

   and both x_1 and $x_2 \leq 7$ and positive integers.

2. Problem P_{11}

   Basic variables $x_1 = 4$; $x_2 = 10/3$ and maximum value of $Z = 58$.

   [This will need further branching as $x_2 = 10/3$ which is not an integer as required.]

3. Problem P_{12}

   Basic variables $x_1 = 5$; $x_2 = 0$ and maximum value of $Z = 35$; this node is fathomed for comparison.

   [No further branching from this node.]

   From P_{11} we have two sub-problems.

   $x_2 = 10/3 = 3 + 1/3$

   $\therefore$ for P_{111}, $x_2 \le 3$ and for P_{112}, $x_2 \ge 4$. With all constraints of the problem P_{11} continued.

4. Problem P_{111}

   we have maximize $Z = 7x_1 + 9x_2$

   subject to $7x_1 + x_2 \le 35$

   $-1x_1 + 3x_2 \le 6$

   and both $x_1 \le 4$ and $x_2 \le 3$ and positive integers. [x_1, x_2 are positive integers]

5. Problem P_{112}

   Maximize $Z = 7x_1 + 9x_2$

   subject to $7x_1 + x_2 \le 35$

   $-1x_1 + 3x_2 \le 6$

   and both $x_1 \le 4$ and $x_2 \ge 4$ and positive integers.

   [x_1, x_2 are positive integers]

Solving P_{111} using simplex method, we have ***basic variables***: $x_1 = 4$ and $x_2 = 3$: maximum value of $Z = 55$. [Both basic variables are positive integers; this node is fathomed.]

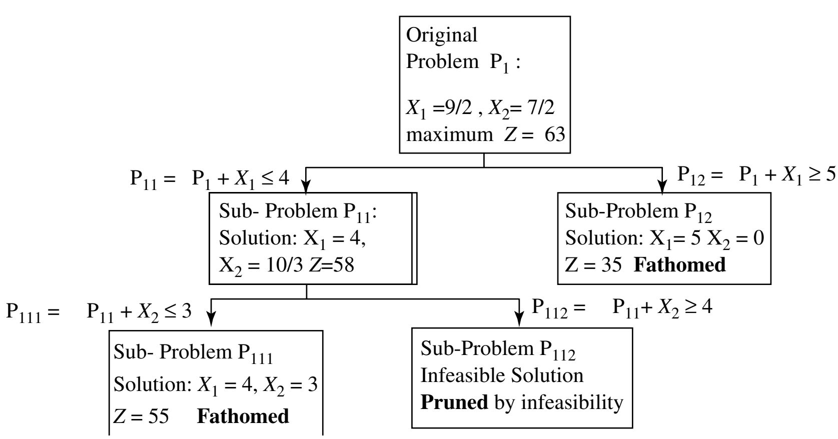
Solving P_{112}, using simplex method, we have ***infeasible*** solution. This node is ***pruned*** for infeasibility.

P_1: Maximize $Z = 7x_1 + 9x_2$

Subject to $7x_1 + 1x_2 \le 35$

$-1x_1 + 3x_2 \le 6_j$

and both x_1 and $x_2 \le 7$ and positive integers.



Final Selection

Now, we make final selection from the fathomed nodes, P_{12} and P_{111}.
P_{12} has basic variables $x_1 = 5$; $x_2 = 0$; maximum $Z = 35$.
P_{111} has basic variables $x_1 = 4$; $x_2 = 3$; maximum $Z = 55$.
Choice is clear, we select P_{111} and $x_1 = 4$; $x_2 = 3$; maximum value of objective function = 55.

5.5 Zero–One Programming

In this section, we discuss very important and widely useful concepts of zero-one programming.

Zero-one Integer Programming Problem

In the general class of integer programming problems, we have an important sub-class known as zero-one or binary (0–1) programming problem.

There are several real life situations in which representative variables are expected to be on either side—yes (1) or no (0). We will quote many important examples related to this important notion but just to begin with and to convince you upon this, we mention the example of assignment problem; we have taken the complete chapter on that, is the best example of 0-1 programming problem.

Problems on the lines of (1) capital budgeting), (2) assignment (3) allowing different research proposal in an R&D of a drug manufacturing company, (4) airline crew scheduling and many real life situations fall under this class.

If there is only one proposal, them either we accept (yes =1) or reject (no = 0)

If there are two proposals P and Q, then there are four alternatives. Remaining within the limitations and constraint **any one** is the selection.

Project P can have any one of the two states at a given level.

Project P → Accept = 1   Project Q → Accept = 1
Project P ↘ Reject = 0   Project Q ↘ Reject = 0

Set of alternatives = {(0, 0), (0, 1), (1, 0), (1, 1)}; Number of states = $2^2 = 4$

In the same way, if there are three projects then keeping all constraints within the limits of given resources, there are $2^3 = 8$ types of different alternatives.

In all these cases the fact is that we choose or select the one which, under the given constraints, can give the best result like minimum cost, minimum time, maximum return on investment, etc.

With this background, we give the most general mathematical model of 0-1 programming problem.

Let us take an illustration to explain what we have said above:

Knapsack Problem

Consider the problem faced by a hiker who cannot carry more than $\mathbf{L} = \sum_{j=1}^{j=n} b_j$ pounds of equipments. He has n items which he is considering bringing. To each item she assigns a relative value C_j with the most important items having the highest values. Let b_j be the weight of the jth item . The hiker's problem is to decide which of the n items to carry; she will choose those which maximize the total relative value subject to the weight limitation.

To construct the mathematical model, let $x_j = 1$ if the jth item is chosen and let $X_j = 0$ if the jth item is not chosen. Then the model is

$$\text{Maximize } Z = \sum_{j=1}^{j=n} C_j X_j$$

Subject to $\sum_{j=1}^{j=n} a_j X_j \leq \mathbf{L} = b_j$ for all $j = 1$ to n.

$x_j = 0$ or $1, j = 1,2, \ldots n$

Note that by limiting the value of x_j to 0 or 1, the left-hand side of the constraint represents just the weight of the items which are chosen. This type of an integer programming problem is called a **Zero-one programming problem.**

Comment:

1. There are 2^n types different solutions and we are in search of exactly one which optimizes our objective function. The method leading to feasible and optimal solution begins its operation by initializing all variables to zero value and then systematically allocating the next value (= 1) to the variable that raises the output comparatively and effectively at a faster rate.
2. We should understand and study certain techniques for solving zero-one or binary problems because certain types of non-linear integer programming problems, with certain constraints on variables to remain within a range, can be converted to binary problems.
3. Such types of binary problems are very useful in planning stage in different departmental activities for proposed acceptance or possible rejection of the plans and certain number of additional projects to which allocation of financial budget is a major problem.
4. The different stages of operations involved in finding feasible solutions are only additions and subtraction; this is also known as *additive algorithm*.

How do we plan to solve the binary problem?

1. The objective function should be of minimization type.
2. All the constraints must be in $\leq$ type format.
   [For $\geq$ type, multiply all the terms by –1; for = type, you can convert it into two weak inequality $\geq$ and simultaneously $\leq$ type.]
3. Define a new variable $y_j = x_j$ if $C_j \geq 0$ in the minimization type of problem; and $y_j = 1 - x_j$ if $C_j \leq 0$ in the minimization type of problem.
4. To each one of the n constraint, we add a slack variable s_j,
   i.e. $\sum_{j=1}^{j=n} a_j x_j \leq \mathrm{L} = b_j$ becomes $\sum_{j=1}^{j=n} a_j X_j + s_j = b_j$

ILLUSTRATION 6

Minimize $Z = 3y_1 - 4y_2$,
subject to $3y_1 + 5y_2 \geq 6$
$2y_1 + y_2 = 5$
with y_1 and y_2 either 0 or 1

Solution

As explained above about the plan of finding feasible solution,

1. Constraints are $-3y_1 - 5y_2 \leq -6$, $2y_1 + y_2 \leq 5$, $-2y_1 - y_2 \leq -5$, y_1 and y_2 can be 0 or 1.
2. Add slack variables s_1, s_2, and $s_3 \geq 0$. As coefficient of y_2 in the objective function is = –4, a negative value; we write $y_1 = x_1$ and $y_2 = 1 - x_2$
3. The problem is
   Minimize $Z = 3x_1 - 4(1 - x_2) = 3x_1 + 4x_2 - 4$
   subject to $-3x_1 - 5(1 - x_2) + s_1 = -6$
   $2x_1 + (1 - x_2) + s_2 = 5$ and $-2x_1 - (1 - x_2) + s_3 = -5$
   With x_1, x_2 taking values either 0 or 1 and s_1, s_2, and $s_3 \geq 0$

The problem now takes a form;
[Minimize $Z = 3x_1 - 4(1 - x_2) = 3x_1 + 4x_2 - 4$]
Minimize $Z^* = Z + 4 = 3x_1 + 4x_2$
subject to $\mathbf{-3x_1 + 5x_2 + s_1 = -1}$,
$\mathbf{2x_1 - x_2 + s_2 = 4}$,
$\mathbf{-2x_1 + x_2 + s_3 = -4}$
With **x_1, x_2 taking values either 0 or 1 and s_1, s_2, and $s_3 \geq 0$**]

Solution

Step 1

There are two variables and the initial solution is obtained by putting x_1 and $x_2 = 0$ With this $Z^* = -4$. and these two values make s_1 and s_3 negative; it is a contradiction. With x_1 and $x_2 = 0$; we have $y_1 = 0$ and $y_2 = 1 - 0 = 1$ do not satisfy the constraints. So this cannot be feasible.

At this stage, x_1 and x_2 are called free variables and they are free to take either 0 or 1 values.

Already slack variables are ≥ 0.

Step 2

Now, $s_1 = -1 + 3x_1 - 5x_2 \geq 0$
$s_2 = 4 - 2x_1 + x_2 \geq 0$, and
$s_3 = -4 + 2x_1 - x_2 \geq 0$

Assign $x_1 = 1$ and $x_2 = 0$, the slack variables $s_1 = 1$, $s_2 = 2$, $s_3 = -2$ (violates the condition).
Assign $x_2 = 1$ and $x_1 = 0$, again slack variables are undefined; so it is not possible.
Finally, both x_1 and $x_2 = 1$ does not change the situations for the slack variable but keep them negative.
We take one more example and give hints for branching to different options.

ILLUSTRATION 7

Minimize $Z = -y_1 - 2y_2 + 3y_3$
subject to $3y_1 + y_2 + y_3 \leq 6$
$12y_1 - 3y_2 - 4y_3 \leq 20$
$20y_1 + 15y_2 - y_3 \leq 10$
with y_1, y_2, y_3
taking the values either 0 or 1.

Solution

Since the coefficients of y_1 and y_2 are negative; we put $y_1 = 1 - x_1$, $y_2 = 1 - x_2$ and $y_3 = x_3$
We also add slack variables s_1, s_2, and $s_3 \geq 0$
With these changes, the problem is as follows.
Minimize $Z = -(1 - x_1) - 2(1 - x_2) + 3x_3$
subject to $3(1 - x_1) + (1 - x_2) + x_3 \leq 6$
$12(1 - x_1) - 3(1 - x_2) - 4x_3 \leq 20$
$20(1 - x_1) + 15(1 - x_2) - x_3 \leq 10$
with x_1, x_2, x_3 taking the values either 0 or 1.
Finally, we have
Minimize $Z^* = x_1 + 2x_2 + 3x_3$
subject to $-3x_1 - x_2 + x_3 + s_1 = 2$
$-12x_1 + 3x_2 - 4x_3 + s_2 = 11$
$-20x_1 - 15x_2 - 1x_3 + s_3 = -25$ with each x either 0 or 1.
[We define $Z = Z^* - 3$]

We have eight possible chances for getting a probable feasible solution. We begin with a vertex, taking each $x = 0$.
This gives s_1, s_2, feasible but $s_3 = -25$
This does not give a feasible solution. There is a branching from this point.

Each $x = 0$ Mini. $Z^* = 0$

Branching Node 1

$x_1 = 0$ $x_3 = 1$

$x_2 = x_3 = 1, Z^* = 5$ $x_2 = x_3 = 0, Z^* = 1$

In this way , following the principles of fathoming and branching in *brach and bound method*, we keep on assigning zero or one value to the free variable.
[Final solution is $x_1 = x_2 = 1$, $x_3 = 0$, with minimum $Z^* = 3$; now you can convert these in terms of y_1, y_2, and y_3 and the value of Z.]

ADDITIONAL QUESTIONS FOR PRACTICE (WITH HINTS AND ANSWERS)

Question 1
Solve the problem by *branch and bound method*.
Maximize $Z = 5x + 3y$
subject to $4x + 2y \leq 25$
$x \leq 5$
$y \leq 8$
$x, y \geq 0$ and Integers.

Solution
We solve this problem $\mathbf{P_1}$ using regular simplex method (dropping integer constraints).
The optimal feasible solution is
basic variables: $x = 9/4$, $y = 8$ and $z = 141/4$
$x = 9/4$ is not an integer. $x = 9/4 =$ **[9/ 4] + 1 /4 ;** There are two cases $\boldsymbol{x \leq 2}$ **and** $\boldsymbol{x \geq 3}$
From this, we have two sub-problems: $\mathbf{P_{11}}$ and $\mathbf{P_{12}}$; which we describe as follows.

$\mathbf{P_{11}}$: Maximize $Z = 5x + 3y$
subject to $4x + 2y \leq 25$
$x \leq 5$
$y \leq 8$
$x, y \geq 0$ and Integers. with $\boldsymbol{x \leq 2}$

with this, Maximize $Z = 5x + 3y$
subject to $4x + 2y \leq 25$
$\boldsymbol{x \leq 2}$; $y \leq 8$
$x, y \geq 0$ and Integers.

Solution of P_{11}: $x = 2$, $y = 8$ and $Z = 34$

$\mathbf{P_{12}}$: Maximize $Z = 5x + 3y$
subject to $4x + 2y \leq 25$
$x \leq 5$; $y \leq 8$
$x, y \geq 0$ and Integers. with $\boldsymbol{x \geq 3}$

with this, Maximize $Z = 5x + 3y$

subject to $4x + 2y \leq 25$

$\mathbf{x \leq 5, x \geq 3}$

$y \leq 8$

$x, y \geq 0$ and Integers.

Solution of P_{12}: $\mathbf{x = 3, y = 13/2}$ **and** $\mathbf{Z = 69/2}$

From this sub-problem P_{12}, we have two branches; each giving rise to one sub-problem for $y \leq 6$ and $y \geq 7$ [$y \leq 6$ makes $y \leq 8$ a redundant constraint.]

They are P_{121} which is equivalent to $P_{12} + y \leq 6$ and P_{122} equivalent to $P_{12} + y \geq 7$

This may be further continued.

Question 2

Solve the problem using branch and bound method.

Maximize $Z = 18x_1 + 20x_2$

subject to $2x_1 + 5x_2 \leq 15$

$x_1 \leq 3$

$x_1, x_2 \geq 0$ and integer.

(You are given that the optimal solution is $x_1 = 3$, $x_2 = 9/5$ and maximum $Z = 90$]

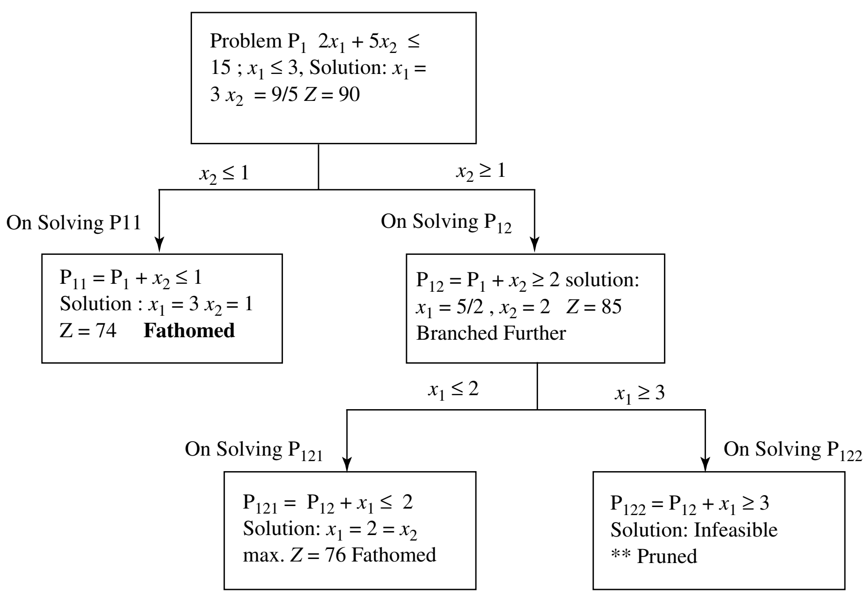
Solution

In this case, we are given the optimal solution of the initial problem; what we call $\mathbf{P_1}$.

$x_1 = 3$, $x_2 = 9/5$ and maximum $Z = 90$, $x_2 = 9/5$; so we write $x_2 \leq 1$ or $x_2 \geq 2$. This allow us to make two sub-problems $\mathbf{P_{11}}$ **and** $\mathbf{P_{12}}$.

P_{11} is the problem P_1 + a new constraint $x_2 \leq 1$.

P_{12} is the problem P_1 + a new constraint $x_2 \geq 2$.



** Solving P_{122}, we get $x_1 = 15/2$; which is not possible as $x_1 \leq 3$ is given in original constraint of the given problem P_1.

From two Fathomed nodes, we conclude that $x_1 = 2 = x_2$ and maximum value of $Z = 76$

POINTS TO REMEMBER

1. Gomory's cutting plane method, when applied to the basic variable with the highest fractional value, helps us finding a new constraint. This constraint, on introducing a basic variable, is introduced in the last simplex table.
2. This basic variable always becomes the out-going variable. At this stage, we apply dual simplex method. [The right side of this equality constraint is negative in numeric value.]
3. The procedure described above is an iterative one and gets terminated after a finite number of steps.
4. Each time a Gomory's constraint when added helps resizing the current feasible region. All the time at the end of each iteration, we have one basic variable taking up an integer value.
5. Branch and bound method works on the same line. In this method, you have to find a branch springing out from a feasible node having at least one basic variable with a non-integer value. Each branch is associated with old constraints plus a new constraint imposed on the non-integer basic variable.
6. If x_i is a non-integer basic variable then $[x_i] \le x_i \le [x_i] + 1$. This gives rise to two related sub-problems. We have three cases.
   (a) Either the node gives both integer value and it has the optimum value and is less than the optimum value of the objective function of the original problem. We consider this value for further consideration and comparison with same type of any branched sub-problem, if any, that may appear. (Fathoming—accepting the reality but left to further verification of optimality.)
   (b) The sub-problem ends up in an infeasible solution. (Pruning of the node from further branching)
   (c) The solution of a sub-problem, has yet some non-basic variable taking up a non-integer value. We have further branching from this node.

EXERCISES

OBJECTIVE TYPE QUESTIONS

I. State True or False.

1. Gomory's cutting plane constraint resizes the original feasible region.
2. The slack variable introduced in Gomory's constraint cannot be a basic variable in the final optimal table.
3. The slack variable of the Gomory's variable, just introduced in the last table, is always the out-going variable.
4. We have to apply dual simplex method on introducing Gomory's constraint in the optimal table.
5. The final simplex table gives $x_1 = 9/5$ and $x_2 = 11/5$; so we select x_2 as the with higher fractional value.
6. In the branch and bound method, at some stage, we have $x_1 = 9/2$, $x_2 = 2$, and maximum $Z = 80$ then we fathom this node by infeasibility.
7. In the branch and bound method, at some stage, we have $x_1 = 9/2$, $x_2 = 2$, and maximum $Z = 80$ then we cannot make further branching from this node.
8. In the branch and bound method, at some stage, we have $x_1 = 9/2$, $x_2 = 2$, and maximum $Z = 80$ then we fathom this node and move to the subsequent node to make further branching.
9. In the branch and bound method, at some stage, we have $x_1 = 9/2$, $x_2 = 2$, and maximum $Z = 80$ then we make further branching by taking $x_1 \le 4$ and $x_1 \ge 4$.

10. After writing Gomory's constraint, we introduce the ASV (artificial slack variable) in the constraint and make it a basic variable in the next iteration.

ANSWERS

1. true.	2. false.	3. true.	4. true.	5. false.
6. false.	7. false.	8. false.	9. false.	10. false.

II. Multiple Choice Questions

1. We get the integer values of the basic variables by applying
   (a) simplex method (b) dual simplex method
   (c) big-M method (d) cutting plane method
2. In Gomory's constraint, we introduce
   (a) surplus variable (b) ASV (c) slack variable (d) no variable
3. If the optimal simplex table gives values $(x_1, x_2) = (7/2, 3/2)$ then we can consider (x_1, x_2) then the optimal solution is at the point
   (a) (3, 2) (b) (4, 2) (c) (4, 1) (d) probably no one
4. In the optimal solution by simplex method, $(x_1, x_2) = (9/5, 11/5)$ then we consider
   (a) $x_1 < 1$ (b) $x_1 \leq 1$ (c) $x_1 \geq 1$ (d) $x_1 > 1$
5. In the optimal solution by simplex method, $(x_1, x_2) = (9/5, 11/5)$ then we consider
   (a) $x_1 < 1$ (b) $x_1 \leq 1$ (c) $x_2 \geq 2$ (d) $x_1 > 2$
6. In case of two basic variables having same fractional values, we consider
   (a) a non-basic variable
   (b) a slack variable
   (c) ASV
   (d) any one of the two basic variables for finding Gomory's cut.
7. At least one corner of reduced or resized region by Gomory's cut
   (a) has one integer point (b) no integer point
   (c) all integer points (d) no guarantee of integer points
8. Mixed integer programming problems require
   (a) all (b) some (c) first two
   (d) no one variable to take up integer values in the optimal solution.
9. When we apply Gomory's cutting method, in some cases, the optimal table (showing all integer values of the basic variables) may generate
   (a) the same value of objective function (b) lower value to that of objective function
   (c) higher than that of objective function (d) cannot be said
10. Introducing Gomory' cut as a constraint means
    (a) introducing the new variable to the existing set-up
    (b) resizing the original region
    (c) making at least one basic variable having integer value
    (d) all the three (a), (b), and (c)

ANSWERS

1. d	2. c	3. d	4. b	5. b
6. d	7. d*	8. b	9. b	10. d

[* There are cases showing that at the end of applying Gomory's cut; we get a solution in which one or more than one basic variables are non-integers.]

NUMERICAL PROBLEMS

(A) Solve the following examples by cutting plane method.

1. Maximize $Z = x_1 + x_2$
   subject to $-3x_1 + 8x_2 \leq 6$
   $6x_1 + 3x_2 \geq 7$
   $7x_1 - 6x_2 \leq 5$
   x_1 and x_2 being non-negative integers.
2. Maximize $Z = 2x_1 + 3x_2$
   subject to $2x_1 + 2x_2 \leq 7$
   $0 \leq x_1, x_2 \leq 2$
   with x_1 and x_2 being integers.
3. Maximize $Z = -3x_1 + x_2$
   subject to $14x_1 + 18x_2 \geq 63$
   $10x_1 - 6x_2 \geq -15$
   x_1 and x_2 being non-negative integers.
4. Maximize $Z = x_1 + 2x_2$
   subject to $5x_1 + 7x_2 \leq 21$
   $-1x_1 + 3x_2 \leq 8$
   x_1 and x_2 being non-negative integers.
5. Minimize $Z = 9x_1 + 10x_2$
   subject to $3x_1 + 5x_2 \geq 45$
   $0 \leq x_1 \leq 10$
   $0 \leq x_2 \leq 8$
   x_2 being non-negative integer.
6. Maximize $Z = 5x_1 + 8x_2$
   subject to $4x_1 + x_2 \leq 10$
   $x_1 + 2x_2 \leq 8$
   $x_1, x_2 \geq 0$ and integers.
7. Maximize $Z = 10x_1 + 20x_2$
   subject to $1x_1 + 3x_2 \leq 12$
   $6x_1 + 8x_2 \leq 48$
   $x_1, x_2 \geq 0$ and integers.

(B) Solve the following problems by branch and bound method.

8. Minimize $Z = 9x_1 + 10x_2$
   subject to $3x_1 + 5x_2 \geq 45$
   $0 \leq x_1 \leq 10$
   $0 \leq x_2 \leq 8$, x_2 being non-negative integer.
9. Maximize $Z = 7x_1 + 9x_2$
   subject to $-x_1 + 3x_2 \leq 6$
   $7x_1 + x_2 \leq 35$
   $0 \leq x_1, x_2 \leq 7$
   $x_1, x_2 \geq 0$ and integers.

10. Maximize $Z = 3x_2$
subject to $3x_1 + 2x_2 \leq 7$
$x_1 - x_2 \geq -2$
$x_1, x_2 \geq 0$ and integers.
11. Maximize $Z = 3x_1 + 3x_2 + x_3$
subject to $-x_1 + x_2 + 2x_3 \leq 4$
$-3x_2 + 4x_3 \leq 2$
$x_1 + 2x_2 - 3x_3 \leq 3$
$x_1, x_2 \geq 0$ and integers.

(C) Find the graphical solution to the following problems by plotting lattice points in the feasible region and evaluating the objective function at such points. In addition to this, verify your answer by applying Gomory's cutting plane method.

12. Maximize $Z = 15x_1 + 12x_2$
subject to $x_1 + 2x_2 \leq 90$
$2x_1 + x_2 \leq 80$, and
$3x_1 + 2x_2 \leq 150$
$x_1, x_2 \geq 0$, and integers
13. Using the following data, find the number of electronic toys (whole number) the production manager should manufacture that maximizes his profit from the day's production of two type of toys—type P and type Q. Type P toy requires 2 hours in the manufacturing unit and 6 hours on designing unit. Type Q toy requires 4 hours in the manufacturing unit and 5 hours on designing unit. Manufacturing department, because of labour constraints, can work for maximum 18 hours a day while the designing unit can spend 30 hours a day. Each toy can earn a profit of ₹100 on sale.
14. Maximize $Z = 5x_1 + 5x_2$
subject to $6x_1 + 5x_2 \leq 30$
$2x_1 + 5x_2 \leq 16$
$x_1, x_2 \geq 0$, and x_1 an integer.
15. Maximize $Z = 3x_1 + 4x_2$
subject to $3x_1 + 5x_2 \leq 15$
$x_1 + x_2 \leq 4$
$x_1, x_2 \geq 0$ and integers
16. Minimize $-7x_1 - 10x_2$
subject to $7x_1 + x_2 \leq 35$
$-x_1 + 3x_2 \leq 6$
$x_1, x_2 \geq 0$, and integers

ANSWERS TO NUMERICAL PROBLEMS

1. $(x_1, x_2) = (1, 1)$ and optimal $Z = 2$
2. $(x_1, x_2) = (1, 2)$ and optimal $Z = 2$
3. $(x_1, x_2) = (1, 4)$ and optimal $Z = 1$
4. $(x_1, x_2) = (1, 2)$ and optimal $Z = 5$
5. $(x_1, x_2) = (5/3, 8)$ and optimal $Z = 95$

6. $(x_1, x_2) = (0, 4)$ and optimal $Z = 32$
7. $(x_1, x_2, x_3) = (5, 2)$ and optimal $Z = 90$
8. $(x_1, x_2) = (5/3, 8)$ and optimal $Z = 95$
9. $(x_1, x_2) = (4, 3)$ and optimal $Z = 55$
10. $(x_1, x_2, s_1) = (0, 2, 3)$ and optimal $Z = 6$
11. $(x_1, x_2, x_3) = (5, 3, 8/3)$ and optimal $Z = 80/3$
12. $x_1 = 24$, $x_2 = 32$ and maximum value of $Z = 744$
13. Toy type P = 5 units and Type Q = 0 units. Maximum profit = ₹500
14. $x_1 = 4$, $x_2 = 6/5$ and maximum $Z = 26$
15. $(x_1, x_2) = (3, 1)$ and optimal $Z = 13$
16. $(x_1, x_2) = (4, 3)$ and optimal $Z = 58$

6 Special Types of Problems and Applications

A clear vision to goal, planned hard work with devotion, and continued persistence till achievement are the steps to total success.

Learning Objectives

AFTER STUDYING THIS CHAPTER, THE STUDENTS WILL BE ABLE TO:

- understand fundamentals of *graph theory*
- understand *travelling salesman's problem*, relate it with graph theory and find solution
- understand *chinese postman's problem*, relate it with graph theory and find solution
- understand *minimum spanning tree problem*, relate it with graph theory and find solution
- understand *shortest path problem*, relate it with graph theory and find solution
- understand *maximal flow problem*, relate it with graph theory and find solution

INTRODUCTION

Within the framework of this chapter, we will discuss very important topics, highly concerned to problems of real life situation and useful in describing the problem in a graphical pattern or network diagrams. May it be the area of computer science, information technology, medical or para-medical branch, business planning or production planning, what we have designed will prove you an asset in defining, representing the problem and apply the underlying principles to find feasible solution or optimal solution in a given problem. The problems discussed here have their origin from solving and interpreting the existing problems of real life situation at different times and in different given situations. The first unit ***graph theory*** will be a very useful instrument to understand the problems and to search for technical procedures of finding feasible solution if at all it exists.

6.1 GRAPH THEORY

In this section, we will discuss fundamental concepts in graph theory. It is our experiences and an unchallengeable fact that fundamental concepts of graph theory are very useful in describing real life situation problems and existing or planned set ups. Before we begin it will prove an interesting gesture to describe the problem known as *seven bridges problem* or *Konigsberg bridges problem*.

6.1.1 Seven Bridges Problem

This problem dates back to 1736 and it proved highly interesting in those days and very important to mathematicians to work with and develop a new branch in mathematics—graph theory. Every decade in time span since 1736 has contributed some depths in existing profiles and opened new avenues the problem is as follows.

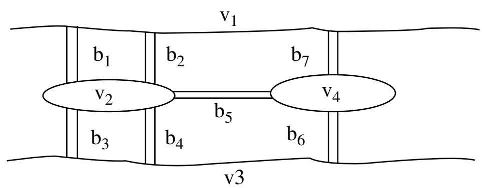


Figure 6.1

There are seven bridges connecting two islands v_2 and v_4 to the banks v_1 and v_3. The problems asks to make a route from v_1 (or v_3) round in a way that each bridge must be traversed one only exactly and in one direction. This problem can be described on plane paper with the same facts. This description, we call it a graph, has remained highly interesting to mathematical minded.

Set $\quad$ V = vertices/nodes = $\{v_1, v_2, v_3, v_4\}$

E = Edges/arcs (bridges) = $\{e_1, e_2, \ldots, e_7\}$

Figure 6.2 is called a graph. It narrates the real situation.

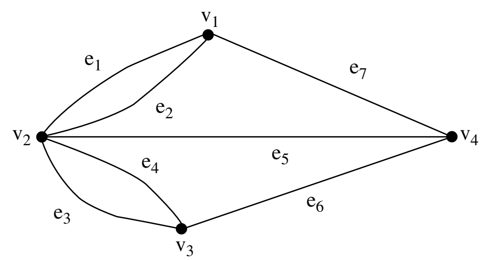


Figure 6.2

6.1.2 Some Definitions

Graph: A graph G = [V, E] is a possible two-dimensional structure with some members of the set V called *nodes* or *vertices* and all the members of the set E at the most; the members are called *edges/arcs*. Edges join the vertices we have the following figure (Figure 3) of graphs.

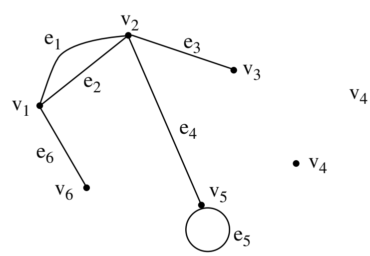


Figure 6.3

1. Two vertices are said to be *adjacent* if there is an edge connecting them.
2. An arc connects exactly two nodes, we say that the arc is *incident* on a vertex. For example, edge e_1 is incident on vertex v_1 (and v_2 also) An edge connecting two nodes *i*, and *j* can be written as (i, j), for example, $e_1 = (v_1, v_2)$

$$e_6 = (v_1, v_6)$$

3. Two edges are said to be *parallel* if both are incident on a same pair of vertices, for example, edges e_1 and e_2 are parallel edges (they are incident on vertices v_1, v_2).
4. If an edge connects a node to itself only then it is called a *loop* or a *self loop* on a vertex. For example, the edge e_5 is a loop on the vertex v_5.
5. If a vertex of a graph is not adjacent to any other vertex then it is called an *isolated* vertex, for example, the vertex v_4 is an isolated vertex.

6. If exactly one edge is incident on a vertex then it is called a *pendant vertex*, for example, the vertex v_3 and v_6 are pendant vertices.
7. The number of edge coming out or going into a vertex represents the *degree* of a vertex.
   A vertex with degree zero is a *null* vertex.
   A vertex with degree one is a *pendant* vertex.
   Degree of a vertex is a non-zero integer; it is dented as d(v).
   For example

$$\begin{array}{ll} d(v_1) = 3 & d(v_2) = 4 \\ d(v_3) = 1 & d(v_4) = 0 \\ d(v_5) = 3^* & d(v_6) = 1 \end{array}$$

   A vertex with even degree is called an *even* vertex. Here v_2 is an even vertex.
   A vertex with odd degree is called an *odd* vertex. Here v_1 is an odd vertex.
   An important point to note is that the degree of a vertex.
   $= 3 + 4 + 1 + 0 + 3 + 1 = 12 = 2 \times 6 = 2$ (number of edges)
   (if e stands for number of edges) = 2e
8. Adjacency matrix: An *adjacency matrix* in a graph denotes the relation between the vertices. Basically, the non-negative whole numbers like a_{ij} shows the number of edges between the vertex i and the vertex j.
   We, take an example of a graph and then write its adjacency matrix.

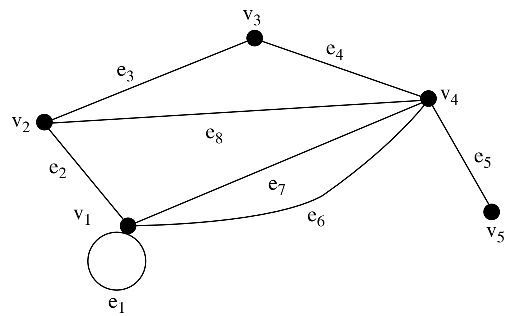


Figure 6.4

$$\text{Adjacency matrix A} = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array}\begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

9. Two vertices in a graph are said to be *connected* if there is at least one edge incident on both. For example Vertices v_1 and v_2 are connected by an edge e_1 (e_2 also).

9. A path between two nodes i and j in a graph is an *union ordered set* of edges.

$$(i, v_1), \cup (v_1, v_2) \cup \cdots \cup (v_n, j)$$

i = origin
j = terminus

$v_1, v_2 \cdots, v_n$ are internal vertices and each internal vertex is distinct.

No vertex is repeated if $i = j$ then it is called a *closed path* in a path the vertex and hence the edges are not repeated.

*Cycle: A *cycle* in a graph is a closed path joining a node to it.

For example $e_1 = (v_1, v_1)$, a self loop is also a cycle or order one.

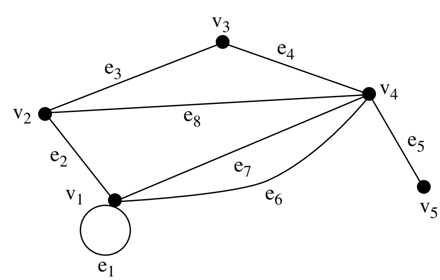


Figure 6.5

$(v_1, v_4) \cup (v_4, v_1)$
$= e_6\, e_7$
= a cycle of order 2

In Figure 5 a path P_1 between the vertices v_2 and v_5 is

$P_1 = (v_2, v_3) \cup (v_3, v_4) \cup (v_4, v_5)$

Some more paths are shown below.

$P_2 = (v_1, v_4) \cup (v_4, v_3) \cup (v_3, v_2)$
$= e_6 \cup e_4 \cup e_3$

or

$P_3 = (v_1, v_4) \cup (v_4, v_2) \cup (v_2, v_1)$, it is a closed path.
$= e_7 \cup e_8 \cup e_2$

or

$= e_6 \cup e_8 \cup e_2$

10. A graph in which all vertices are connected, *i.e.* if you find a path from any given vertex to any other vertex, then the graph is a connected one. In Figure 5, it is a connected graph.

11. Trail & Euler Graph: A *trail* in a graph G = <V, E> is a walk on the edges between the vertices so that the edges are not repeated.

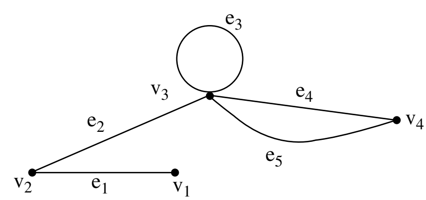


Figure 6.6

G = <V, E>
$V = \{v_1, v_2, v_3, v_4\}$
$E = \{e_1, e_2, e_3, e_4, e_5\}$
Trail $= (v_1, v_2) \cup (v_2, v_3) \cup (v_3, v_4)$
$= e_1 \cup e_2 \cup e_4$ (or $e_1 \cup e_2 U\, e_5$)
$T_2 = e_1 \cup e_2 \cup e_3 \cup e_4 U\, e_5$

(This is a trail which begins from the vertex v_1 and passing through v_2, v_3, v_4 and ends at the vertex v_3. It cannot be extended from v_3 to v_2 and v_2 to v_1. (we cannot repeat edges.)

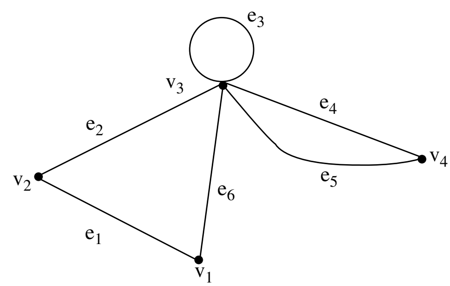


Figure 6.7

A *closed trail* in a graph in which all the edges are traversed exactly once is an *Euler trail* or *Euler tour*. A graph in which there is an Euler tour is called an *Euler graph.*

In Figure 6.7, there is an Euler tour. One can start from any vertex and traversing though each edge exactly once comes back to the same (starting) vertex.

A graph is said to be an *Euler graph* if it has an Euler tour. The above graph is an Euler graph.

One tour $= e_1\, e_2\, e_3\, e_4\, e_5\, e_6$
$= [(v_1, v_2) \cup (v_2, v_3) \cup (v_3, v_3)\, (v_3, v_4) \cup (v_4, v_3) \cup (v_3, v_1)]$

Comment

Euler graph plays a very important role in application aspects of graph theory.

12. **Tree:**

A graph G = <V, E>; $V \neq \phi$ is a *tree* if it is (i) acyclic and (ii) connected.

Some important properties of a tree are

- A tree on n vertices has exactly $(n - 1)$ edges.

- The end vertices of the longest path are of degree one only.

  [The longest path is from the vertex V_1, to V_7, i.e. $d(V_1) = d(V_7) = 1$]
- There are 7 vertices and 6 edges.

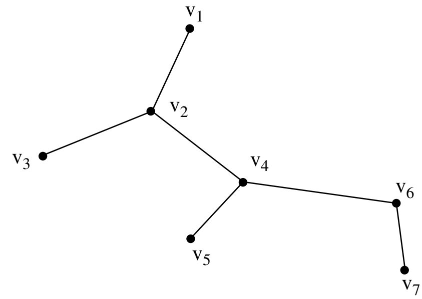


Figure 6.8

Result

An important result follows from it.
An undirected connected graph G = <V, E> with *n* vertices has more than $(n – 1)$ edges, then there must be at least one cycle in it.

Spanning Tree

A *spanning tree* of a given graph G = <V, E>; $V \neq \phi$ is a tree having the same vertices of the given graph.

This graph has six vertices and eight edges. A spanning tree of this graph has the same six vertices and five edges.

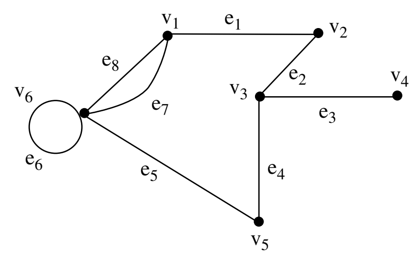


Figure 6.9

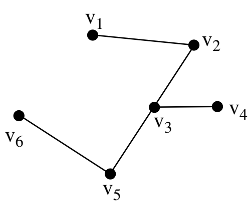


Figure 6.10A

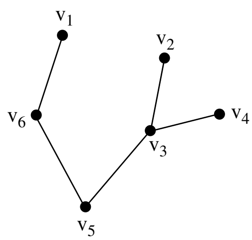


Figure 6.10B

The two graphs drawn above are spanning trees of graph shown in Figure 6.9.

Hamiltonian Graph

A closed path in a graph G = <V, E> is called a *Hamiltonian circuit* if it passes through every vertex exactly once.
In Figure 6.11, we cannot find a Hamiltonian circuit.
In Figure 6.12, there is a Hamiltonian circuit. $(v_1e_1v_2e_2v_3e_3v_4e_4v_5e_5v_1)$.
A graph is having a Hamiltonian circuit is called *Hamiltonian graph*.
The graph in Figure 6.12 is a Hamiltonian graph.

A graph may have more than one Hamiltonian circuits.

A Hamiltonian circuit on a graph having *n* vertices has *n* edges.

Every Hamiltonian graph may or may not have an Euler tour.

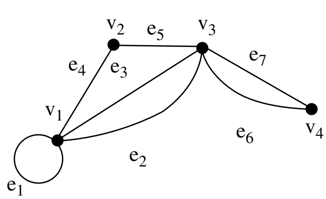


Figure 6.11

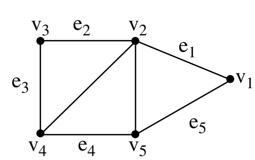


Figure 6.12

Diagraph

An arc of a graph is called *directed* or *oriented* if there is a sense of direction so that one node/vertex is considered as *origin* and the other node is a *terminus* or a *point of termination*. The directed arc connecting the origin (node) i to the terminus (node) j is denoted as (i, j). A graph in which every arc is directed is called a *directed graph*, a *diagraph* or an *oriented graph*.
Figure 6.13 is a diagraph.

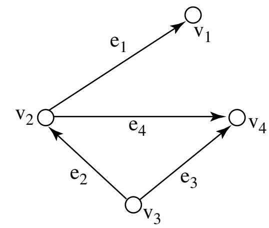


Figure 6.13

Network

A network is a *connected, directed* graph for which a non-negative number called *connectivity relation* or *weight* has been assigned to every arc.

This number may be thought as *distance between two nodes*, *amount of flow*, or the *capacity of the directed arc*.

6.2 Travelling Salesman's Problem

Statement

There are n cities and a salesman is expected to visit each city exactly once traversing on any given route passing through a city. In this case, we make a graphical presentation of the given situation; vertices showing the cities and the edges are weighted edges. The weight of an edge shows the distance between the two cities that are connected by the edge. We have to find a route or routes that minimizes the total distance from the first city to the last city and then that of coming back to the city of origin.

Mathematical Model of Travelling Salesman's Problem

Let there be n number of cities. We assume that mostly all the cities are inter-connected by some mean or transportation and we consider that distance as the distance between the two cities.

Let X_{ij} denote a situation showing that the salesman visits the j^{th} city from the i^{th} city.

If there is a visit then $X_{ij} = 1$ or else $X_{ij} = 0$.

Let C_{ij} represent the distance from the i^{th} city to j^{th} city (one can consider this as total expense or fare between the cities.)

The central idea or plan of this problem is to determine a route from the city of origin to the last city and coming back in such a way that each city is visited exactly once only.

Minimize $$Z = \sum_{j=1}^{j=n} \sum_{i=1}^{i=n} C_{ij} X_{ij}$$

subject to $$\sum_{j=1}^{j=n} X_{ij} = 1$$

and $$\sum_{i=1}^{i=n} X_{ij} = 1, \sum_{i=1}^{i=n} X_{ij} = 1$$

for all values of i and j from 1 to n.

How Do We Find Feasible Solution?

We take an example and discuss the complete procedure. In some cases, we will show the alternate approaches. [We suggest you that once you go through the topic on minimal spanning tree, then think on the same topic and try to implement it for solving travelling salesman's problem. First, make a minimal spanning tree and then try different alternatives, keeping the minimum distance concept in mind, to make a complete Hamiltonian circuit. You will appreciate this once you read both the topics—the above one and the minimal spanning tree.]

ILLUSTRATION 1

Figure 6.14 on the edges represent the distance between the two cities. We want to find a closed path passing through each city exactly once.

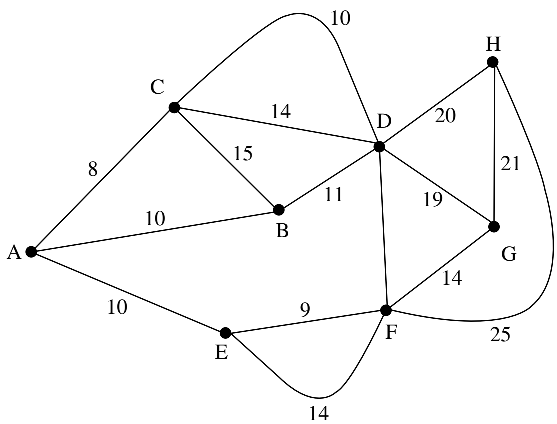


Figure 6.14

The graph (Figure 6.14) depicts between the cities denoted by the numbers A = 1 through H = 8 are connected by different intermediate routes between the cities. The figures on the arcs show the distances between cities.

We want to find ***shortest path*** connecting all the cities so that the salesman can visit each city exactly once and comes back to the city of origin from which he started.

What do we Plan?

The cities are taken as nodes, and routes showing the distances are the weighted arcs.

We want to find a Hamiltonian circuit. If there is more than one Hamiltonian circuit then we choose the one with the smallest total distance.

Solution

By observation and trial and error, we locate one Hamiltonian circuit.

It is A – E – F – H – G – D – B – C – A; there are two questions with this gesture.

1. Is this the only path?
2. Is the distance on this path or any path that we find is the shortest one?

In this Hamiltonian circuit, there are some parallel edges and so uniqueness of the path is not decided.

There are methods for finding the optimal solution to travelling salesman's problem. Methods like

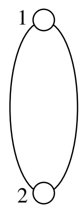
1. extension to the optimal assignment matrix,
2. closest inclusion algorithm, and
3. branch and bound method.

We will discuss the first technique.

The steps are as follows.

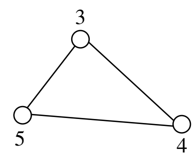
1. Solve the problem using the techniques of assignment problem.
2. Study the pattern of assignment, if the assignment forms a cycle from any node/vertex to the other node forming a continuous cycle then the cycle is a distance efficient Hamiltonian cycle.
   If such a cycle does not exists, then in most of the cases two or three internal cycles are obtained.
3. In case of two cycles, pick the smaller one. (In case of tie, let the selection be optimal.) For each entry of the distance matrix, say in a 5 × 5 matrix with each city showing distance to remaining 4 cities, we have two cycles like 1—2—1 (1—2 and 2—1) and 3—5—4—3 (3—5, 5—4 and 4—3).

Two cycles C_1 and C_2 as shown below.



1 – 2 + 2 – 1

Figure 6.15



3 – 5 + 5 – 4 + 4 – 3

Figure 6.16

Comment

In some cases, the distance matrix is asymmetrical. [We mean distance between the cities A and B depends on the route, i.e. distance between A to B is different from that from B to A.]

In case of symmetric matrix, the distance is the same either way.

Step 1

Keeping this point in continuous attention, our work is to join the two cycles. We join smaller cycle (C_1) with the other cycle C_2.

Step 2

We consider joining any one of the vertices either 1 or 2 with any one of the vertices 3, 4 or 5.

Consider pairing of 1—3, 1—4 and 1—5 or 2—3, 2—4 and 2—5. [Note this carefully.]

[In case of directed segments, consider pairs 1—3, 3—1, 1—4, 4—1 and 1—5 with 5—1.]

Now selecting each one of the pairs, find the distance between corresponding cities.

[Let 1—3 = a, 1—4 = b, and 1 — 5 = c, 2—3 = p, 2—4 = q and 2—5 = r]

From these distances whichever is the minimum, say 2—3, i.e. distance between the cities 2 and 3 is minimum amongst all other distances.

Step 3

By selecting a pair (on our case 2–3), we have broken both cycles.

Now, join the vertex 2 with vertex 3.

The vertex of one cycle is connected with the vertex of the another cycle by an arc. The distance between these two vertices is minimum. In our case, join the vertex 2 with the vertex 3.

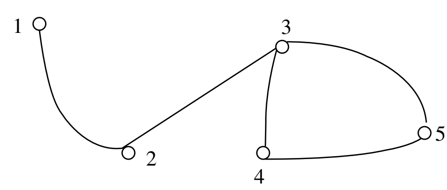


Figure 6.17

The figure with new set up is as given in Figure 6.17.

Step 4

One edge of smaller cycle is detached and connects to one vertex of the another cycle. Now, connect either 1 with 4 or 1 with 5 (joining 1 with 3 has no meaning as it forms a cycle 1—2—3—1)

Step 5

Find the distance of the pairs 1—4 and 1—5; whichever is minimum and forms a Hamiltonian cycle is the right choice of the *closed path*, the shortest Hamiltonian Path.

The final figures are as follows (Figure 6.17A).

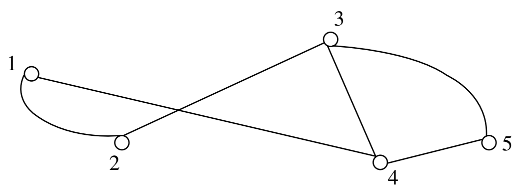


Figure 6.17A On joining 1—4

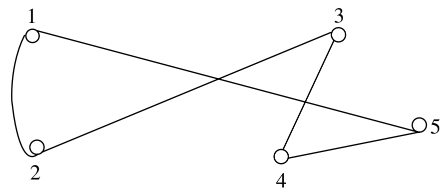


Figure 6.17B On joining 1—5

Now, we have two closed Hamiltonian paths.
In Figure 6.17A, the path is 1—2—3—5—4—1
In Figure 6.17B, the path is 1—2—3—4—5—1.
[Note the difference.]

Step 6

Find the distance along both the paths and find the one with minimum length.

Comment

As we have, in Step 3, joined vertex 2 of the smaller cycle with the longer cycle, we could have selected the vertex 1 also and perform the same procedure.

We take two illustrations to sound the concept.

ILLUSTRATION 2

The entries of the following matrix show the distance (in miles), between the cities A to E.

Distance Matrix

$$\mathbf{P} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 18 & 6 & 14 & 20 \\ 18 & \infty & 8 & \infty & 10 \\ 6 & 8 & \infty & 7 & 8 \\ 14 & \infty & 7 & \infty & 22 \\ \infty & 10 & 8 & 22 & \infty \end{bmatrix} \end{array}$$

where ∞ denotes 'not used'.

[This is not a symmetric matrix]

A man wants to take a round trip so that he passes through each city exactly once. Find the route.

Solution

Solving the problem using Hungarian method (assignment technique), the optimal solution can be located from the following optimal matrix.

Matrix for allocation,

$$\begin{bmatrix} \infty & 10 & 0 & 7 & 12 \\ 10 & \infty & 0 & \infty & 0 \\ 0 & 0 & \infty & 0 & 0 \\ 7 & \infty & 0 & \infty & 13 \\ \infty & 0 & 0 & 13 & \infty \end{bmatrix}$$

We make allocations and find the resultant matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 3 & \boxed{0} & 0 & 12 \\ 3 & \infty & 0 & \infty & \boxed{0} \\ 0 & 0 & \infty & \boxed{0} & 0 \\ \boxed{0} & \infty & 0 & \infty & 13 \\ \infty & \boxed{0} & 7 & 13 & \infty \end{bmatrix} \end{array}$$

A—C, B—E, C—D, D—A and E—B is the optimal solution that we have obtained using assignment technique.

This makes two cycles,

$$C_1 = A—C—D—A \quad \text{and} \quad C_2 = B—E—B$$

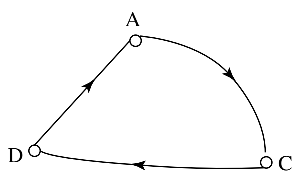


Figure 6.18

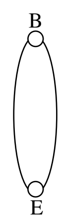


Figure 6.19

Now, we try to join these two cycles.

Step I

$$\text{We find } \begin{cases} BA = AB = 18 \\ BC = CB = 8 \\ BD = DB = \infty \text{ (not used)} \end{cases}$$

Also, $EA = \infty$ (not used); $AE = 20$
$EC = 8 = CE$
$ED = 22 = DE$

Let us select CE (= 8); as a result, we have

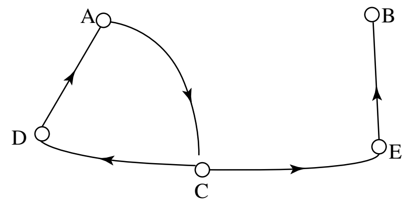


Figure 6.20

Step 2

Now, joining B with C will not give us a Hamiltonian path because in a Hamiltonian path except the first and the last vertex, no one of the internal vertex is repeated.

We consider, $BA = 18$ (because BD = ∞), so finally we have

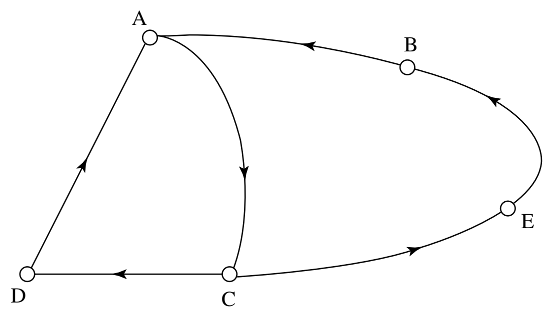


Figure 6.21

There is no Hamiltonian path, it can be easily seen from Figure 21.

Now, we think of vertex B to take part (repeating Step 1, 2, etc.)

Step 1

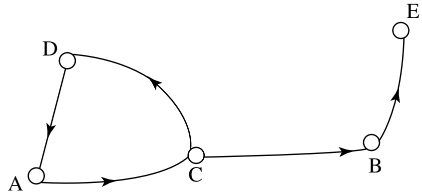


Figure 6.22

Instead of the vertex E, we have taken vertex B to take part as BE is already there.

Step 2

Now, $ED = DC = 22$ and $EC = 8$, will not make a Hamiltonian cycle. (There are two cycles with C as a common vertex and so it cannot be a Hamiltonian cycle.)

Join $ED = 22$; we have

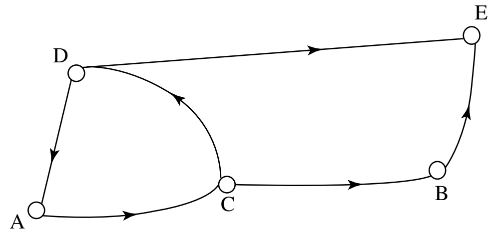


Figure 6.23

We have a cycle passing through every vertex exactly once.

A—C—B—E—D—A is a closed Hamiltonian path. This is the path the salesman should follow.

Illustration 3

Find the Hamiltonian path for the travelling salesman visiting five villages from A through E.

The distance matrix is as follows

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 8 & 11 & 7 & 8 \\ 8 & \infty & 10 & 7 & 8 \\ 11 & 10 & \infty & 11 & 9 \\ 7 & 7 & 11 & \infty & 11 \\ 8 & 8 & 9 & 11 & \infty \end{bmatrix} \end{array}$$

Solution

Apply all principles of assignment techniques, the matrix of allocation is as follows

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & \boxed{0} & 2 & 0 & 1 \\ 0 & \infty & 1 & \boxed{0} & 1 \\ 1 & 0 & \infty & 2 & \boxed{0} \\ \boxed{0} & 0 & 3 & \infty & 5 \\ 0 & 0 & \boxed{0} & 4 & \infty \end{bmatrix} \end{array}$$

Allocation is A—B—D—A and C—E—C, this is not a distance efficient Hamiltonian cycle. We make two cycles.

Step 1

Two cycles are

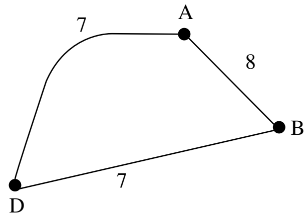


Figure 6.24

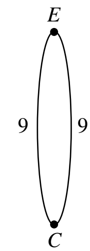


Figure 6.25

We take either the vertex C or E and connect to the any one of the vertices A, C, or D.

$$CA = 11 \text{ units},\ CB = 10 \text{ units, and } CD = 11 \text{ units}$$

We choose $CB = 10$ units as $CE = 9$ already exists.

Step 2

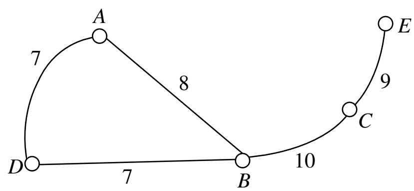


Figure 6.26

Now E can be connected to the vertex A or to the vertex D.

$$EA = 8 \text{ and } ED = 11.$$

We take both options.
Option 1: Join E to A.

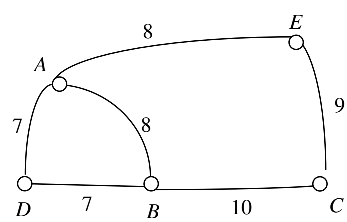


Figure 6.27

The required path is A—D—B—C—E—A of 7 + 7 + 10 + 9 + 8 = 41 units.
Option 2: Join E to D.

The closest path is A—B—C—E—D—A. [Find the sum of all distances and compare with the above result.]

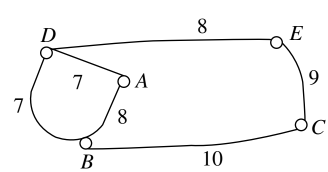


Figure 6.28

6.3 Chinese Postman's Problem

This problem is also a well-known problem and in olden days people tried to find the solution by drawing pictures of the routes and counting number of steps and making graphical presentation of the problem. This problem parallels to what we have seen Euler's seven bridges problem.

In this problem, we will discuss a real situation in which a postal worker or postman who has been allocated a work of delivering posts has to move in each street in the region allocated to him.

He begins from a vertex (may be the post office) and traversing through each street (connecting different vertices) comes back to the point of origin.

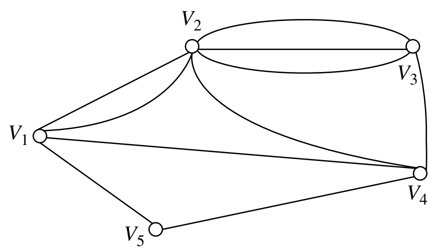


Figure 6.29

If he does not plan his tour or the route, he will have to pass through a hard time. If unplanned, he may have to pass through one street more than once and still doing nothing on that repeated streets.

What is the central idea?

The main point in this problem is to find an Euler tour. A closed path that begins from any vertex, traversing through every edge exactly once and comes back to the origin.

How do we plan to solve it?

1. If the given graph of the real situation is an Euler graph then we can find Eulerian tour which will guide us to describe the path on which the postman should follow. For searching the Euler path, one can apply *Fleury's algorithm.*
2. If the given graph is not an Euler graph then we have to develop some technique or algorithm to meet our objective.
   At this stage, we note the following two facts:
   (A) If a given graph is an Euler graph then the degree of each vertex is even.
   (B) An Euler graph can be sub-divided into number of closed cycles in a way that no cycle has a common edge.

In Figure 6.29, we have an Euler graph (you can note that the each vertex is of even degree.

$$d(v_1) = 4;\ d(v_2) = 6;\ d(v_3) = 4;\ d(v_4) = 4;\ d(v_5) = 2.$$

In this type of situation, we can find the solution to Chinese postman's problem.

Non-Euler Graph

If the given graph is a non-Eulerian one, then there are even number of odd vertices.

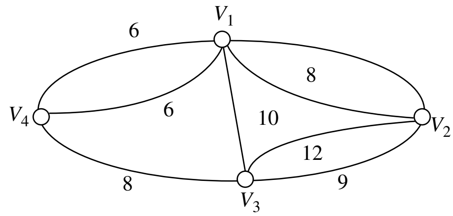


Figure 6.30

The above graph (Figure 6.30) is non-Euler graph.

$$d(v_1) = 5;\ d(v_4) = 3$$

The arcs of Figure 6.30 show the weight of the arcs or the distance between two nodes.

Duplication Process

The above process helps us making an Euler graph by introducing *duplicate arc* (*mirror arc*) between two odd-degree vertices.

The duplicate arc has the same weight as that of the arc to which it parallels between two vertices.

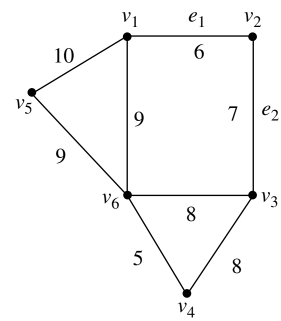


Figure 6.31

The graph (Figure 6.31) has

$$d\ (v_1) = 3;\ d\ (v_3) = 3 \text{ and } d\ (v_2) = 2.$$

There is an edge $e_1 = v_1v_2$; $w(e_1) = 6$ also $e_2 = v_2v_3$; $w(e_2) = 7$

We introduce duplicate arcs between v_1v_2 & $v_2\, v_3$.

These two arcs will have the same weight as that of e_1 and e_2.

We denote these arcs as e_1* and e_2*.

Figure 6.32, note that $w(e_1*) = 6$

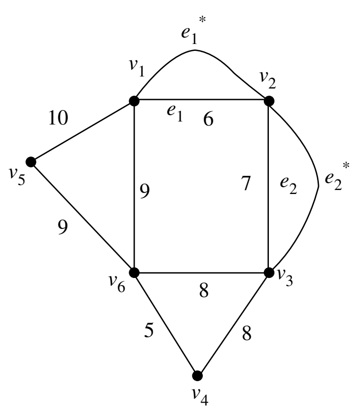


Figure 6.32 Graph G*.

e_1* and e_2* are duplicate arcs.

The resultant graph is G*(super graph of G).

Solution to the main problem

In a given graphical presentation G, if not an Eulerian graph then its super graph G*(obtained by adding duplicate edges and making all vertices of even degree) is an Eulerian graph. The problem is of introducing the duplicated edges to minimize their sum.

Add duplicated edges so that $\sum_{V \in V(G^*)-V(G)}^{\infty} v(e)$ is minimum; where v(e) is weight of a duplicate edge of G*.

Now the graph G* being an Euler graph; we find an Euler tour and if we do so, our work is done. We take illustrations.

ILLUSTRATION 4

Find a closed route of minimum weight beginning from the vertex v_1 and terminating at the same point.

Solution

The graph in Figure 6.33 is an Euler graph. (It is the union of three disjoint cycles with no common edge. Also, we can see that all the vertices are of even degree.)

All the edges are weighted and so from the origin, say v_1, we apply *Fleury's algorithm* and find the tour; this makes solution to the problem.

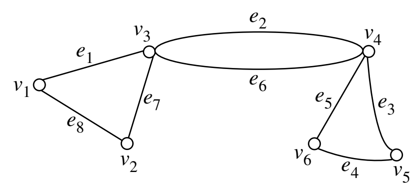


Figure 6.33

$$(v_1\ e_1\ v_3\ e_2\ v_4\ e_3\ v_5\ e_4\ v_6\ e_5\ v_4\ e_6\ v_3\ e_7\ v_2\ e_8\ v_1)$$

All the edges are covered exactly once.

ILLUSTRATION 5

Find the solution to the postman's problem given below. The set of vertices is v = $\{v_1 \ldots v_5\}$

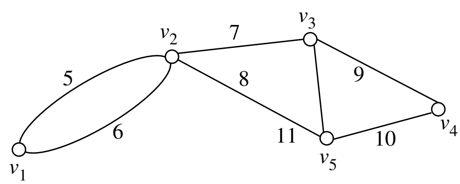


Figure 6.34 Graph G = <V, E>

The figures on the arcs, are distances.

Solution

The graph (Figure 6.34) is not an Euler graph.

$$d(v_3) = 3 = d(v_5).$$

We introduce duplicated edges between (v_3 and v_4) and (v_4 and v_5).

We note that

$$d(v_3, v_4) = e_1^* \text{ and } w(e_1^*) = 9$$

$$d(v_4, v_5) = e_2^* \text{ and } w(e_2^*) = 10$$

and we obtain a super graph G*.

$$G^* = G \cup \{e_1^*, e_2^*\}$$

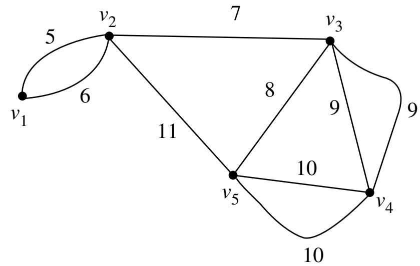


Figure 6.35

Graph $G^* = G \cup \{e_1^*, e_2^*\}$

This makes the graph G^* an Euler graph. We have to travel on $e_1^*$and e_2^* which is equivalent to travelling on edges v_3v_4 and v_4v_5 twice.

Now, one can find the total length by adding weights.

Comment

An Euler tour passes through each edge exactly once and so is the process of postman doing his duty on duplicate edges connecting odd degree vertices.

6.4 Minimal Spanning Tree (MST)

In the introductory portion, we have discussed the basic properties of tree. In addition to this, we have also seen the definition of spanning tree. The basic concept of spanning tree is very important in real life situation. It is useful in planning and designing stage. There are many applications of this concept. First, we give an illustration and then consider two different algorithms to do what we develop during the following illustration.

Illustration 6

The vertices in the graph (Figure 6.36) shows the villages and edge connecting any two villages shows the projected water canal for irrigation. The figure on the edge shows projected estimate in millions of rupees. We want to lay pipelines connecting all the villages at a minimum cost.

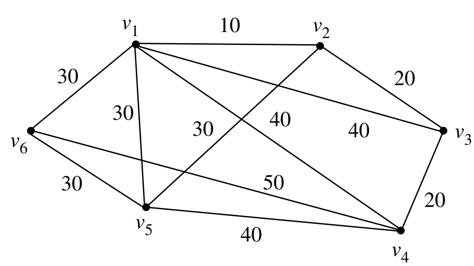


Figure 6.36

The problem is to provide water to every village designing a cost-efficient pipeline work**.**

Solution

We want to find a connected spanning sub-graph (tree) with minimum cost on the work.

We want a spanning tree of minimum weight. It is called *minimal spanning tree*.

We will discuss two different algorithms.

1. Kruskal's algorithm, and
2. Prim's algorithm

We discuss each one by one and sound the basic concepts by giving illustrations.

6.4.1 Krushkal`s Algorithm

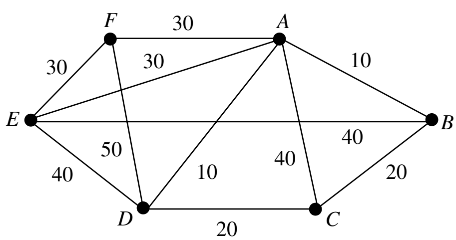
Let G be a weights connected graph. [The weights are real values ≥ 0.] We accept that if T is a sub-graph of G obtained by Kruskal`s algorithm, then 'T' is a minimal spanning tree.

Algorithm

1. We choose an edge $e_1 \in$ G so that w (e_1) is minimum and edge e_1 is not a loop in G.
2. Continue the above process of selection of edges $e_1, e_2, \ldots, e_j$,; $1 \le j < (n-1)$. Now, select an edge e_{j+1} so that
   (a) the sub-graph with these edges $e_1, e_2, \ldots, e_{j+1}$, is acyclic, and
   (b) $w(e_{j+1})$ is as small as possible and e_{j+1} is not a part of any cycle in G.
3. If G be a graph on n vertices, then the algorithm terminates after selecting $(n-1)$ edges following above steps.

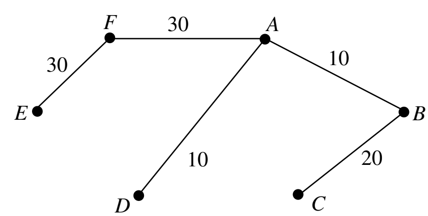
ILLUSTRATION 7

We consider the above example and apply Kruskal`s algorithm to find the minimal spanning tree.



Graph G (6 vertices)

Figure 6.37

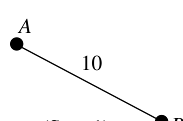


MST of G (6 vertices and 5 edges)

Figure 6.38

Step 1

Begin with selection of either AB or AD (weight is minimum and they are not loops [We select AB.]).



(Step 1)

Figure 6.39

Step 2

Next, in this line is the edge AD with weight = 10. AD is not a loop and does not form a cycle with the first edge AB.

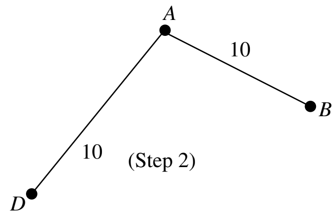
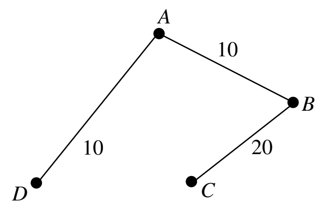


Figure 6.40

Step 3

Next in this line, we have edges with weights 20. w(BC) = 20 and w(CD) = 20. Choice is optional. We select BC.

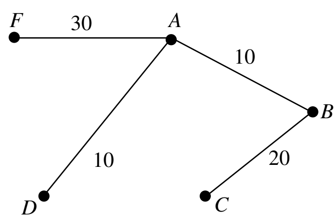


(Step 3)

Figure 6.41

Step 4

We cannot select CD; w (CD) = 20, because it forms a loop when considered in Figure 6.41. Now, the next edge with weight 30. AF, EF, and AE are in the line of selection. Select AF, it does not form a cycle in Figure 6.41.

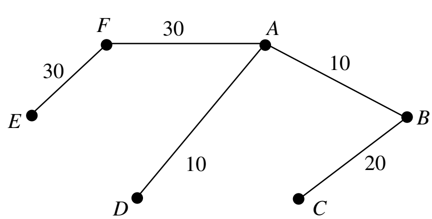


(Step 4)

Figure 6.42

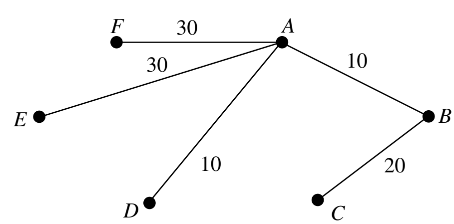
Step 5

Now, we have edge EF and AE with the same weight 30. Any one of these, when selected to contribute in Figure 6.42, will not form a cycle. We show both the cases.



(Step 5)

Figure 6.43



(Step 5)

Figure 6.44

Both are minimal spanning trees with sum of weights = 100. Both have 6 vertices and 5 edges.

6.4.2 Prim's Algorithm

Algorithm works on the same line. It is a recursive process of finding minimal spanning tree.

We begin with a weighted connected graph. It is a graph on n vertices and so its spanning tree must have all the same n vertices and $(n - 1)$ edges. In addition to that, being a tree, it is acyclic.

Step 1

Let G* be a minimal spanning tree and $v_1 \in$ G, select an edge $v_1 v_2 = e_1$ of G is that

(a) it is incident on v_1 with minimum weight.
(b) it is not a loop in G.
(c) it does not make a loop or cycle in G*.

Step 2

The set G* has two vertices v_1 and v_2 and one edge e_1. Select an edge incident on either v_1 or v_2 (not the edge $v_1 v_2 = e_1$ previously selected.). The edge in the selection is of minimum weight. This increases the set of vertices and edges of G*. Say $v_2 v_3 = e_2$ and w (e_2) is minimum.

Step 3

Now G* = $<\{v_1, v_2, v_3,\}, \{e_1, e_2\}>$

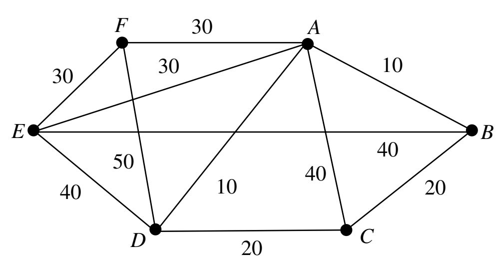
Continue the process of Step 2 of selecting an edge with minimum weight and incident on v_1, v_2, or v_3. Also in G* it does not make a cycle. In this way, we continue $(n - 1)$ times.

We get a minimum spanning tree with

(a) n vertices same that of G
(b) $(n - 1)$ edges of G
(c) the graph (spanning tree) is an acyclic, connected and the sum of weights is minimum.

ILLUSTRATION 8

We take the same example and find minimal spanning tree using Prim's algorithm.



Graph G (6 vertices)

Figure 6.45

Application of Prim's algorithm.

Step 1

We select the vertex C on C = v_1. On C there are three incident edges w (CA) = 40, w (CB) = 20, and w (CD) = 20. We take **w (CD) = 20**. This is considered as a part of the spanning tree, two vertices C and D with one edge CD; w (CD) = 20.

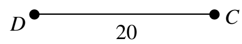


Figure 6.46

Step 2

With either the vertex C or D the non-loop edge incident with minimum weight is DA so that w (DA) = 10. This includes one more vertex A and one more edge DA.

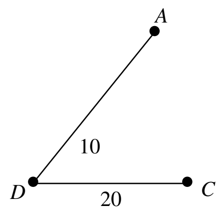


Figure 6.47

Step 3

Continuing on the same line w (AB) = 10 so that B comes with w (AB) = 20.

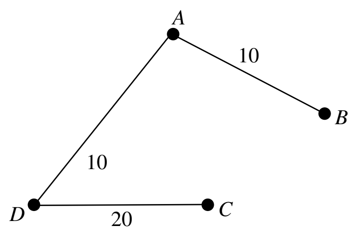


Figure 6.48

Step 4

Next on the same logic, we have AE and AF. w(AE) = 30 and w(AF) = 30 in addition, they do not make a cycle in this new setup of edges

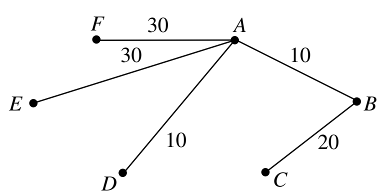


Figure 6.49

It is a minimal spanning tree with sum of weights = 100.

ILLUSTRATION 9

Make a minimal spanning tree from the following weighted connected graph G.

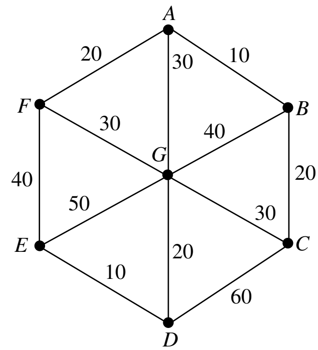


Figure 6.50

Solution

We apply Kruskal's algorithm.

Step 1

w (AB) = 10 and w (ED) = 10, again none of AB and ED is a loop. Both have the same weight, choice is optimal and select AB. In addition to this, we can select ED.

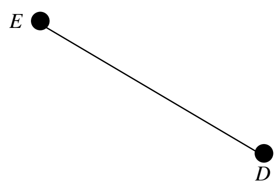


Figure 6.51

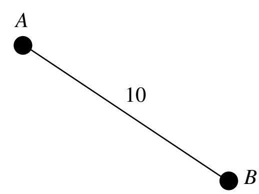


Figure 6.52

Step 2

We have edges AF, BC, and DG with the same weight 20. In addition, these when included in Figure 6.50, will not make a cycle. We consider all these three edges.

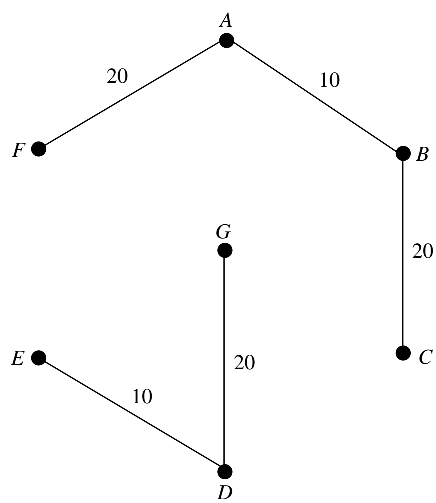


Figure 6.53

Step 3

We have edges AG, FG and CG with weight 30. There are 7 vertices in G so the minimal spanning tree must contain 6 edges. We include AG with w (AG) = 30 and it does not form a cycle. Final figure of minimum spanning tree is Figure 6.54. [Comment: Inclusion of either any one from edges FG or CG will give the same result in terms of sum of weights.]

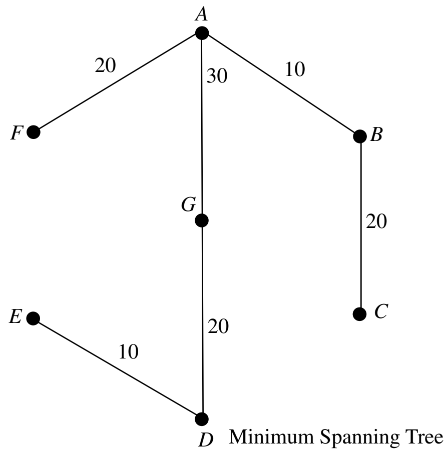


Figure 6.54 Total weight = 110

Applications of minimum spanning tree

This concept has extensive application area in real life cases. Some of them are:

1. Theoretically, we can show that every network problem having a feasible solution is a spanning tree.
2. It has a wide application in the area of supply chain management.
3. In the problems on inventory management and control, the minimal spanning tree has an important role in distribution of goods and normalization of demand.
4. Communication network and setup of interior nodes is the principal application of minimal spanning tree.

6.5 Shortest Path Problems

Introduction

Let G = <V, E> be a connected graph. The edges or the arcs are assigned non-negative real values called the weight. In case, if the edges are directed, then it has an order from vertex v_i to v_j shown as v_i v_j.

Two vertices S and T of G are given and we are interested to find a shortest path from S to T. there are different algorithms of finding the shortest path which are most commonly used for application purposes.

1. The back-tracking algorithm, and
2. Dijkstra algorithm

6.5.1 Dijkstra Algorithm

This method begins with labeling technique to the vertices beginning with the starting (first) vertex S in G.

Let L (S) = 0 [Some mention this as L (1) = 0]

To understand the application of Dijkstra algorithm step wise, we use some notations. We have the weighted graph G with vertices given as follows.

Set of vertices = T = {S, v_1, v_2, ..., v_n, t} and λ (v) shows the distance of the vertex v from all vertices connected to v. S is the first vertex. Set λ (S) = 0

Now

$$T = \{v_1, v_2, ..., v_n, t\}$$

From this vertex S, find the vertices which are directly connected by weighted arcs. Say they are v_1 and v_2.

Find $\lambda(v_1) = \lambda(S) + w(Sv_1)$ and $\lambda(v_2) = \lambda(S) + w(Sv_2)$

From these two (may be more like $\lambda(v_1), \lambda(v_2), \lambda(v_3), \ldots$), find the minimum. Say $\lambda(v_1)$ is minimum. Let v_1 = u. and make a label of the value $\lambda(v_1)$ on the vertex v_1 = u. [If there is a tie in $\lambda(v_1)$ and $\lambda(v_2)$; choice is optimal.]

This label is permanent on v_1 = u.

Now we have T = {$v_2, v_3, \ldots, v_n$, t}

permanently labeled vertices: (1) s (2) v_1 = u,

temporary unlabelled vertices are: {$v_3, v_4, \ldots, v_n$, t}

Act:

Now from the vertex v_1= u, find those vertices which are directly connected by weighted edges. Say; they are v_2, v_3, v_4, and v_5.

Find

$$\lambda(v_2)^* = d(u) + w(uv_2) * \text{new } \lambda(v_2)$$
$$\lambda(v_3) = d(u) + w(uv_3)$$
$$\lambda(v_4) = d(u) + w(uv_4) \text{ and}$$
$$\lambda(v_5) = d(u) + w(uv_5)$$

Decision:

If $\lambda(v_2)^*$ (new value) < $\lambda(v_2)$, then replace temporary label of $\lambda(v_2)$ by $\lambda(v_2)^*$.

If $\lambda(v_2)^*$ (new value) ≥ $\lambda(v_2)$, then do not replace and continue with old label.

Act:

Compare $\lambda(v_2)^*$, $\lambda(v_3)$, $\lambda(v_4)$ and $\lambda(v_5)$ from these real values, whichever is minimum, say $\lambda(v_4)$ is minimum.

Set u = v_4, the vertex v_4 gets a permanent label = $\lambda(v_4)$.

Permanent labeled vertices: (1)S (2) v_1 (3) v_4,

Temporarily labeled vertices: $\lambda(v_2)$, $\lambda(v_3)$, $\lambda(v_5)$

Unlabelled vertices: {$v_6, \ldots v_n$, t}

Act:

Now from u = v_4; find all vertices which are directly connected to v_4 = u. if u = t; algorithm terminates or else move to **Act 2** mentioned earlier.

We take an illustration and apply Dijkstra algorithm.

ILLUSTRATION 10

Find the shortest path in the following weighted graph.

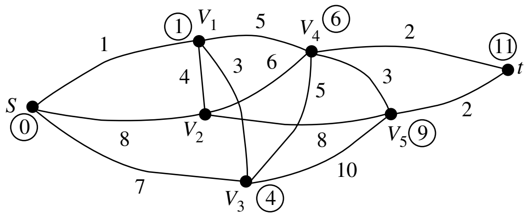


Figure 6.55

Solution

Analytical Method

Denote T = {S, v_1, v_2, v_3, v_4, v_5, t}

Weights are given on the arcs.

1. λ (S) = 0 (Set S = u)
   ∴ T = {S, v_1, v_2, v_3, v_4, v_5, t} connected to the vertex S are v_1, v_2, and v_3.
2. (a) $\lambda(v_1) = \lambda(S) + w(Sv_1) = 0 + 1 = 1^*$
   (b) $\lambda(v_2) = \lambda(S) + w(Sv_2) = 0 + 8 = 8$
   (c) $\lambda(v_3) = \lambda(S) + w(Sv_3) = 0 + 7 = 7$
3. Minimum is $\lambda(v_1) = 1$
   ∴ v_1 takes up permanently label.
   Let v_1 = u; $\lambda(v_1) = \lambda(u) = 1$
   [Temporary labels $\lambda(v_2) = 8$; $\lambda(v_3) = 7$]
4. Connected to v_1 are the vertices v_2, v_3, and v_4, and so T = {v_2, v_3, v_4, v_5, t}
5. $\lambda(v_2) = \lambda(v_1) + w(v_1 v_2) = 1 + 4 = 5$
   $\lambda(v_3) = \lambda(v_1) + w(v_1 v_3) = 1 + 3 = 4^*$
   $\lambda(v_4) = \lambda(v_1) + w(v_1 v_4) = 1 + 5 = 6$
6. Vertex v_3 takes up permanent label $\lambda(v_3) = 4$; let v_3 = u
   ∴ $\lambda(v_3) = \lambda(u)$ now $\lambda(v_2) = 5 <$ old $\lambda(v_2) = 8$; we change the label.
   Now $\lambda(v_2) = 5$ and $\lambda(v_4) = 6$ [Temporary labels are on v_2 and v_4.]
7. Connected directly to v_3 are the vertices v_4 and v_5. Now, T = {v_2 , v_4 ,v_5, t}
   $\lambda(v_4) = \lambda(v_3) + w(v_3 v_2) = 4 + 5 = 9$
   $\lambda(v_5) = \lambda(v_3) + w(v_3 v_3) = 4 + 10 = 14$
   $\lambda(v_2) = 5^*$
8. Vertex v_2 gets a permanent label. $\lambda(v_2) = 5$, let v_2 = u
   ∴ $\lambda(v_2) = \lambda(u) = 5$;
   [Temporary labels are on v_4, and v_5]
   T = {v_4, v_5, t}
9. Connected directly to the vertex v_2 are the vertices v_4 and v_5. $\lambda(v_2) = \lambda(u) = 5$
   $\lambda(v_4) = \lambda(v_2) + w(v_2 v_4) = 5 + 6 = 11$;
   $\lambda(v_5) = \lambda(v_2) + w(v_2 v_5) = 5 + 8 = 13$
   Old label of v_4 is 6 and new $\lambda(v_4) = 11$. We do not change the value $\lambda(v_4) = 6$;
   $\lambda(v_5) = \lambda(v_2) + w(v_2 v_5) = 5 + 8 = 13$;
   Minimum of $\lambda(v_4)$ and $\lambda(v_5)$ = minimum (6, 13) = 6 = $\lambda(v_4)$.
   $\lambda(v_4)$ is given permanent label. $\lambda(v_4) = \lambda(u) = 6$.

10. Connected with v_4 are the vertices v_5 and t. $\lambda(v_4) = \lambda(u) = 6$;
$$\lambda(v_5) = \lambda(v_4) + w(v_4 v_5) = 6 + 3 = 9$$
$$\lambda(t) = \lambda(v_4) + w(v_4 t) = 6 + 2 = 8$$
As t is the last vertex, we give permanent label to t, $\lambda(t) = 9$.
[Among the previous lists for $\lambda(v_5)$ are the values 14, 13, and 9, so $\lambda(v_5)$ is labeled with the values 9.]

Solution to Illustration (Tabular Method):

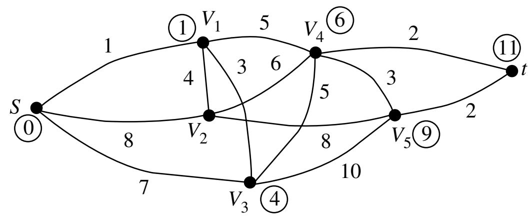


Figure 6.56

Vertex S	v_1	v_2	v_3	v_4	v_5	*t*	*Label*
0*	—	—	—	—	—	—	S = 0
—	1*	8	7	—	—	—	$v_1 = 1$
—	—	5	4*	6	—	—	$v_3 = 4$
—	—	5*	—	6	14	—	$v_2 = 5$
—	—	—	—	6*	13	—	$v_4 = 6$
—	—	—	—	—	9	8 *	t = 8

At this point, there is a classical technique of finding the shortest path from source to the sink node (= t). In Figure 6.55, try to follow and understand the direction of different arrows that we have drawn from sink = t to the source = S.

[Move upward from 8*, immediately on the above cell it shows – sign, now move in the row and find next * sign. It hits on 6; stop there, go upward; where you hit change, go row-wise, hit the next * marked cell, then go up and so on reach to the cell with star marked 0. Now, find the path.]

The shortest path is S—v_1—v_4—t with distance = 1 + 5 + 2 = 8.

ILLUSTRATION 11

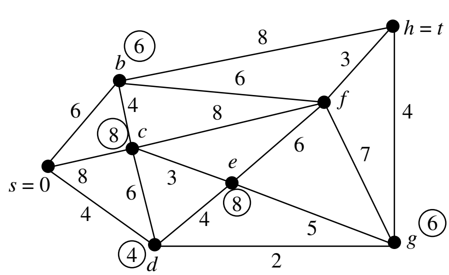


Figure 6.57

Find using Dijkstra algorithm, the shortest path from S = a to h = t.

Solution

Proceedings are shown in tabular form.

/ ƛ (S) Vertices

S = a	*b*	*c*	*d*	*e*	*f*	*g*	*h = t*	*Label*
0*	6	8	4	—	—	—	—	a = 0
—	6	8	4*	—	—	—	—	D = 4
—	6	8	—	8	—	6	—	B = 6
—	—	8	—	8	12	6*	14	G = 8
—	—	8*	—	8	12	—	10	G = 6
—	—	—	—	8*	12	—	10	E = 8
—	—	—	—	—	12	—	10*	T = 10

As we have found the shortest path in the previous case by moving in the direction of the arrow, the same approach is followed here to get the shortest path from origin S to the sink = t.

S = a → d → g → t (is the shortest path); with the total length = 4 + 2 + 4 = 10

Illustration 12

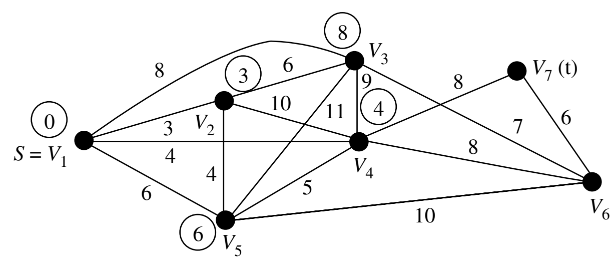


Figure 6.58

S = v_1	*v_2*	*v_3*	*v_4*	*v_5*	*v_6*	*v_7*	*Label*
∴ 0*	3	8	4	6	—	—	S = 0
—	3	8	4	6	—	—	$v_2 = 3$
—	—	8	4*	6	—	—	$v_4 = 4$
—	—	8	—	6	12	12	$v_5 = 6$
—	—	8	—	—	12	12	$v_3 = 8$
—	—	—	—	—	—	12	$v_6 = 12$
	—	—	—	—	—	12*	$v_7 = 12$

The shortest path is $v_1 = s \rightarrow v_4 \rightarrow v_7$

$v_1 v_4 = 4$; $v_4 v_7 = 8$

∴ $4 + 8 = 12$ = length of the shortest path.

6.6 Maximal Flow and Minimum Cut

There are some real life situations, which makes us plan to reschedule the given flow in a network: may be an electric network, electronic circuits, water lines or drainage line. All such situations demand for timely changes, modifications and expansions. Therefore, planning in advance, keeping future needs in mind, becomes highly essential. If we plan at an initial stage or establishment stage, then further planning of expansions using the same setup is feasible at a lower cost and faster rate. In addition to these, there are many useful features of the entire system.

We first plan the following.

1. What do we want to supply?
2. What is the maximum amount of feasible supply?
3. How do we establish a proper setup?
   (a) We collect data by making a survey
   (b) We draw plans and network, calculate flows, maximum capacities of lines.
   (c) We calculate the sizes of pipelines, maximum capacity. (Or wires of different sizes and types having different characteristics and properties to be used in the network.)
   (d) Cost, time, material to be used and the technical and non-technical labour force to be employed.

With this background, we show two different graphical presentations of a project.

We have some technical terms in this unit. We define these terms giving illustrations and become ready to understand a very important topic.

Some Definitions

1. Flow

A *flow* in a graph shown on an edge between two nodes indicates the amount or quantity that passes from one end of a node towards the other end of the node. We denote this non-negative quantity by f_{ij} or $f(i, j)$.

2. Capacity

It shows the maximum amount or the permissible quantity that can flow through the edge between two nodes. (We assume that the flow through that edge cannot exceed the capacity of the node or else the structure is liable to breakdown.) The capacity of the flow between two nodes is denoted by C_{ij} or $C(i, j)$.

From the two definitions given above, it is clear that $f_{ij} \le C_{ij}$. We write (f, C) on a node.

For example,

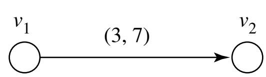


Figure 6.59

v_1 and v_2 are two nodes; $f(1, 2) = 3$ units and $C(1, 2) = 7$ units.

From this, we understand that the edge has a further capacity or still it can carry maximum $7 - 3 = 4$ units.

Conventions

In a given network, on a particular path the maximum amount that all the edges can sequentially carry is the minimum of the maximum additional capacities of all the edges on the path.

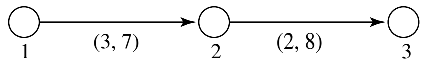


Figure 6.60

The edge 1—2 has an additional capacity to carry 7 – 3 = 4 units while the edge 2—3 has an additional capacity to carry 8 – 2 = 6 units. Minimum of these two excess (additional) capacities 4 and 6 is 4 units and the path from the node 1 through the node 2 to the node 3 can carry maximum 4 units and this will keep the system within the limits.

On doing this favour on the path, the change is shown in Figure 6.61.

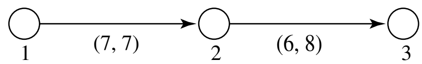


Figure 6.61

This implies that the path 1—2 has reached its flow limit and cannot carry any load further.

6.6.1 Flow-Augmenting Path

A path in a given graph on which every edge is capable of carrying a non-negative amount of flow, is a flow-augmenting path.

Here is an example of a flow-augmenting path in a graph.
It is a path that carries the minimum of the maximum additional flow capacities of all the edges on the path.

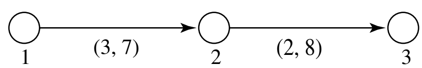


Figure 6.62

The path 1—2—3 is a flow-augmenting path as each edge on that path has an additional capacity of flowing extra flow. The edge 1—2 has of 7 – 3 = 4 units and that of 2—3 is 8 – 2 = 6 units.

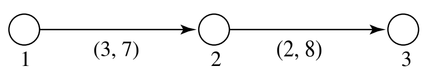


Figure 6.63

The path after rearranging the flow is as follows.

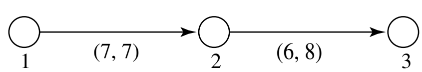


Figure 6.64

Now, the edge cannot carry any more flow and it is a saturated edge. The edge 2—3 is a non-saturated one.

We now consider the following graph.

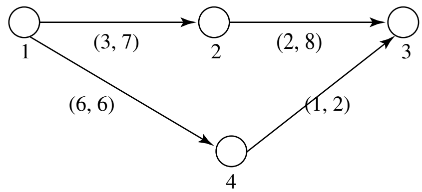


Figure 6.65

The path 1—2—3 can carry (we have just discussed) 4 units of load but the path 1—4—3 cannot do so as the edge 1—4 is already saturated.

Backward flow (Back-flow) in a graph on an edge

To understand an important concept of back-flow, we take an example and analyze the network.

We now consider the following graph.

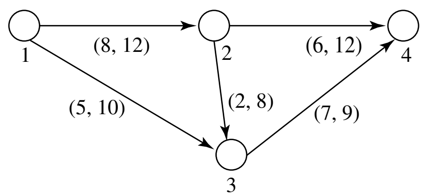


Figure 6.66

The graph (Figure 6.66) has four vertices. Along the path 1—2—4, we can arrange 4 units and 1—3—4 we can arrange a further flow of 2 units. At the end, we have the Figure 6.67.

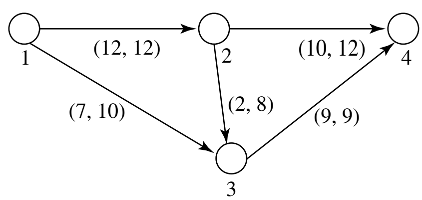


Figure 6.67

Now, the path 1—3—2—4 is also, by definition, a flow-augmenting path; with a flow (2,8) in opposite direction. We can augment a flow in this path also but we have to change the flow in the pipeline (edge) 2—3. What we do here is redirecting the flow and adjusting it in the direction of the next edge 2—4 and also making a settlement on the current flow on the edge 1—3.

Please observe the effect of doing so.

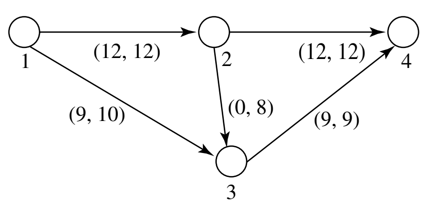


Figure 6.68

Observe that

1. total flow from node 1 is 12 + 9 = 21.
2. total receipt of flow in node 4 is 12 + 9 = 21
3. in-flow in node 2 is from the edge 1—2 which equals 12 units.
4. out-flow from the node 2 is 12 + 0 = 12 units.

This clearly explains the situation at the end of allowing back-flow.

6.6.2 Maximal Flow

This is the one of main points in this topic. In a given connected and weighted network, each edge represents the present flow and its maximum capacity. The amount of the present flow plus the maximum amount of flow that can be adjusted through all different paths from source to sink is called the *maximal flow* in a given graph.

We now consider the following graph.

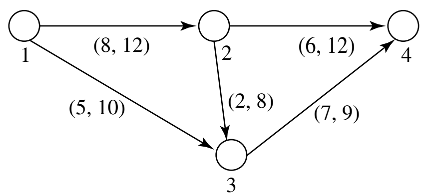


Figure 6.69

The graph (Figure 6.69), as we have seen in earlier example, present amount of flow from origin 1, is 8 + 5 = 13 units. The amount that goes to the sink node is also 6 + 7 = 13 units.

After allowing the feasible adjustments along different paths, the final graph is Figure 6.70.

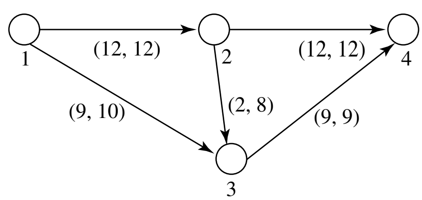


Figure 6.70

By adjustment of flow and capacity of different paths from source to sink, we have added 4 units along 1—2—4, 2 units on 1—3—4, and 2 units on 1—3—2—4 (backward path) and all these **additional flow** amounts to 4 + 2 +2 = 8 units.

Original given flow + additional flow = 13 + 8 = 21 units; this is called **maximal flow.**

Conclusion

In a given graph, the amount of flow entering a given node equals the amount of flow leaving the same node. This relationship is maintained at every point in a given graph.

6.6.3 Minimal Cut

Let us consider a given finite weighted connected graph. The set of vertices V contains the source node S and the sink node t. We find two disjoint sets S_1 and S_2 so that $S \in S_1$ and $t \in S_2$. The set containing minimum number of edges, joining any vertex of S_1 with any vertex of S_2 so that the connection between source node S and the sink node t is completely disassociated, is called a *minimal cut* in the set of vertices. There can be many cut-sets in a given graph.

The sum amount of flow on the edges of minimal cut has a strong relation with the amount of maximal flow. We will study this in the following illustrations.

ILLUSTRATION 13

We have a graph of six vertices and ten edges.

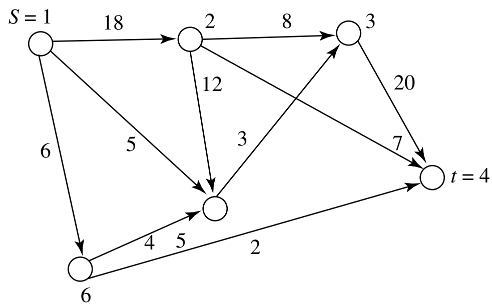


Figure 6.71

Set of vertices = {1, 2, 3, 4, 5, 6}

Weight of edge is indicated on the corresponding edges. Source or origin is the vertex 1 and sink or the terminal vertex is t.

Analysis of the graph (Figure 6.71)

1. From the source 1, there is a flow of 29 units (18 + 5 + 6),
2. To the sink 4, there is an in-flow of 29 units (20 + 7 + 2)

Distribution and allocation

Stage 1

1. Minimum that can be shared along the path 1—2—3—4 is 8 units.
2. Minimum that can be shared along the path 1—2—4 is 7 units.
3. Minimum that can be shared along the path 1—2—5—3—4 is 3 units.

At the end of this re-allocation, the resultant graph is Figure 6.72.

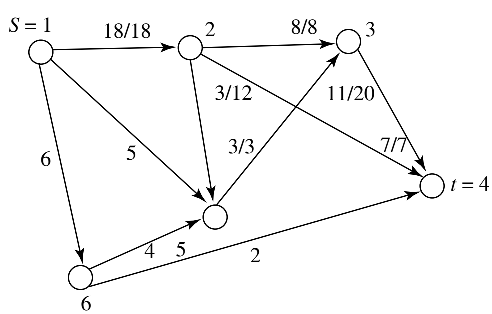


Figure 6.72

Stage 2

1. Maximum re-allocation of 5 units can be done on the path 1—5—3—4.
2. Maximum re-allocation of 4 units can be done on the path 1—6—5—4.

At the end of this plan the graph is Figure 6.73.

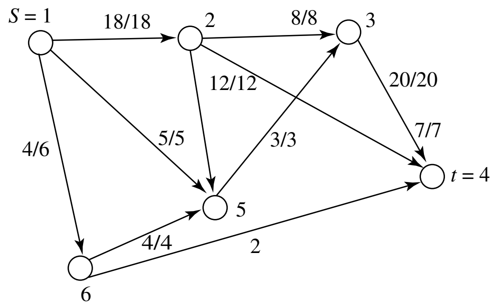


Figure 6.73

Stage 3

Finally, we look at the path 1—6—4; and make an adjustment of 2 units. Finally, we have

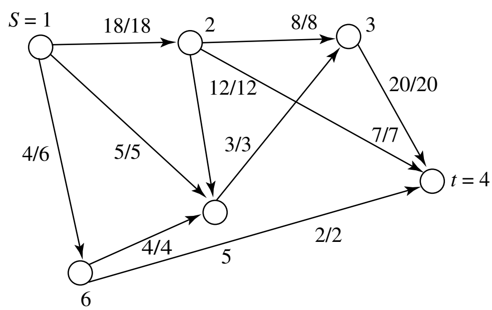


Figure 6.74

Final Look

1. All the edges are saturated. No edge can possess or pass any more flow.
2. The graph is completely saturated.

ILLUSTRATION 14

In the following graph (Figure 6.75), find the maximal flow, the minimal cut, and establish the relation.

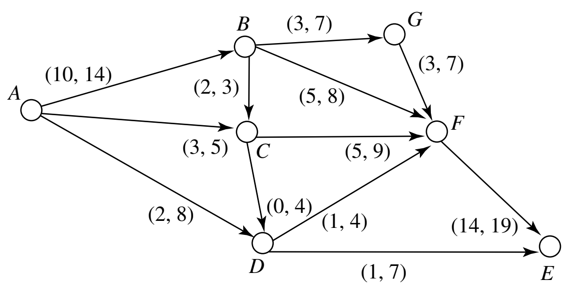


Figure 6.75

Solution

Analysis

In the given graph, there are 7 vertices (A through E) and 10 edges.

The source node is the node A and the sink node is the node E.

The present flow at the source vertex A is 10 + 3 + 2 = 15 units.

The present flow into the sink is 14 + 1 = 15 units.

Distribution and Allocation

1. On the path A—B—G—E, each edge has an additional capacity to carry 4 units of flow while the edge F—E has of 5 units. We adjust 4 units on this path.
2. On the path A—C—F—E, now we can allow a maximum flow of 1 unit. (As the edge FE has, now, the additional capacity to carry only one more unit.)

   As a result, of these two actions the resultant graph is depicted as Figure 6.76.

   In the following graph, find the maximal flow, the minimal cut, and establish the relation.

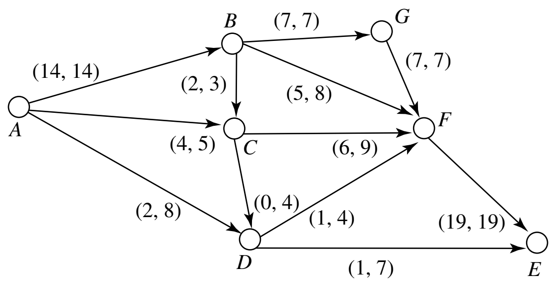


Figure 6.76

3. On the path A—C—D—E, we can allow a maximum flow of 1 unit.

   Now, we have the edges A—B, B—G, G—E, F—E, A—C completely saturated and we have to search other paths.
4. On the path A—D—E, we can, now allow a maximum flow of 5 units as the edge D—E has (2, 7) situation.

   Now, we observe the combined effect of actions (3) and (4) taken. The resultant graph is as follows.

   In the following graph (Figure 6.77), find the maximal flow, the minimal cut, and establish the relation.

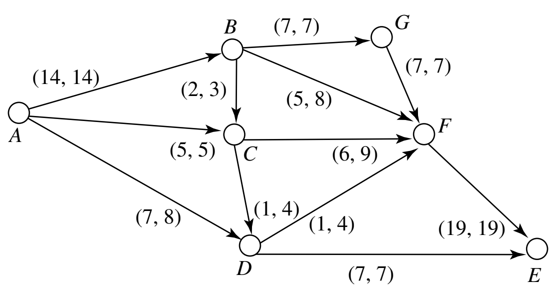


Figure 6.77

Conclusions and derivations:

1. Total flow from source node = 14 + 5 + 7 = 26 units.
2. Total amount of flow into the sink = 19 + 7 = 26 units
3. Maximal flow = 26 units.
4. Saturated edges are A—B, A—C, B—C, G—E, F—E, and D—E,
5. Se S_1 = { A, B, D, G, F} , S_2 = { C, E}
6. Minimal cut is the set with edges D—E and F—E. (Carrying capacities of these two saturated edges are 19 + 7 = 26 units.

This shows that maximal amount of flow equals the sum of flows on the edges of minimal cut.

Additional Questions for Practice (with Hints and Answers)

Question 1

(a) Explain basic concepts of maximal flow and minimum cut.
(b) In the following weighted graph (Figure 6.78), establish the fact 'maximum flow minimum cut' (a, b) is a = flow amount, b = capacity of edge.

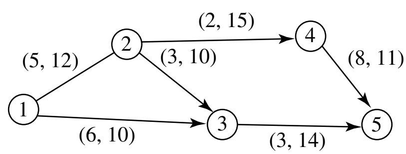


Figure 6.78

Solution

(a) This is discussed in our material; please go through this
(b) Adjust 3 units on 1—2—4—5 (Now 4—5 saturated)
Adjust 4 units on 1—2—3—5 (Now 1—2 saturated)
Adjust 4 units on 1—3—5 (Now 1—3 saturated)
Total flow = 12 + 10 = 22 units.

Question 2

Apply Dijkstra algorithm and find the shortest path. (The graph is in Figure 6.79.)

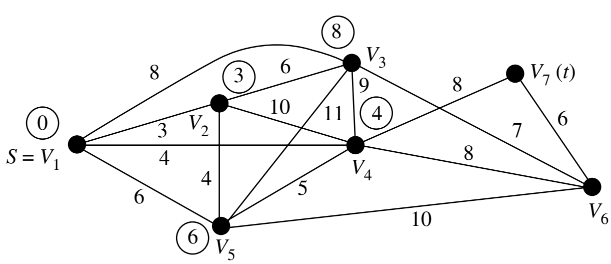


Figure 6.79

Solution

We apply Dijkstra algorithm and make a table showing the labeling process.

$S = v_1$	v_2	v_3	v_4	v_5	v_6	v_7	*Label*
∴ 0*	3	8	4	6	—	—	S = 0
—	3	8	4	6	—	—	$v_2 = 3$
—	—	8	4*	6	—	—	$v_4 = 4$
—	—	8	—	6	12	12	$v_5 = 6$
—	—	8	—	—	12	12	$v_3 = 8$
—	—	—	—	—	—	12	$v_6 = 12$
	—	—	—	—		12*	$v_7 = 12$

The shortest path is $v_1 = s \to v_4 \to v_7$

$v_1 v_4 = 4$; $v_4 v_7 = 8$

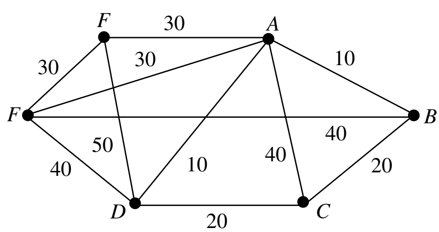
∴ $4 + 8 = 12$ = length of the shortest path.

Question 3

(a) Find the minimal spanning tree in the following graph.

ILLUSTRATION 11

We take up the same example and find minimal spanning tree using Prim's algorithm.



Graph G (6 vertices)

Figure 6.80

Solution

We can apply either Kruskal's algorithm or Prim's algorithm and find the minimal spanning tree.

We begin with the vertex A. We know that the spanning tree must contain all the vertices. As there are 6 vertices; the tree must have exactly 5 edges involved and the resultant being a tree must not contain any cycle.

The first vertex A has an edge A—B with the weight = 10. This is a part of the spanning tree. Now, take the edge B—D with the weight = 20 units.

Keep on doing this and on the other side draw the figure.

Finally, you will get the minimal spanning tree.

Points to Remember

1. To find a feasible solution in travelling salesman's problem, we find Hamiltonian circuit.
   We first solve it using assignment technique and then, if such a circuit is not found, apply some method of joining two different cycles with an objective of keeping the total distance of round trip minimum.
2. To find a feasible solution in Chinese postman's problem, we find an Euler tour from the origin vertex, making a possible tour of traversing on all edge (once along an edge) and come back to the origin. If the graph is not Euler, find odd degree vertices and use the concept of duplicated edges.
3. Prim's algorithm and Kruskal's algorithm are useful to find a minimal spanning tree in a given weighted connected graph.
   One algorithm begins with the first vertex and traverses to the other using the minimal weighted edge. The other one searches for minimum weighted edges sequentially and prepares a route that associated all edges with the same objective of finding a tree (acyclic and connected) with minimal weight.
4. Dijkstra algorithm is to find the shortest path in a weighted (distance depicting) connected graph. Labeling process begins with the source node and, all the time, searches the next (directly) connected node with minimum distance.
5. In 'maximal flow and minimum cut' unit, note that
   (a) on each path, we have to adjust the minimum additional capacity of all the edges on the route.
   (b) once an edge is saturated then it cannot flow any further amount.
   (c) at any node, all the time, the sum of amount of entering flow must equals the sum of amount of out-going flow.
   (d) In some cases, an edge or edges having flow in opposite direction (back flow) becomes very useful in making a flow on a flow-augmenting path.
   (e) Some of the saturated edges on a totally saturated graph helps determining the cut sets of vertices. These cut sets are disjoint and some edges having one end in each set ($V = S_1 \cup S_2$) make a minimal cut. The maximal flow that we adjust on paths and finally find (by adding adjusted flow plus the given flow) equals the sum of amount of the existing flow on edges of minimal cut.

Exercises

Objective Type of Questions

I. State Whether True or False

1. Prim's algorithm is useful in finding the shortest path between any two nodes (cities) in a graph showing the distant matrix between given five cities.
2. To find a Hamiltonian circuit in a given graph is equivalent to finding the feasible solution to Chinese postman's problem.
3. To find an Euler tour in a given graph is equivalent to finding the feasible solution to Chinese postman's problem.
4. Using Kruskal's algorithm we can solve the travelling salesman's problem.
5. In some cases the feasible solution to the problem becomes equivalent to finding a feasible solution to travelling salesman's problem.

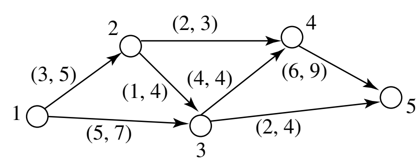
6. Dijkstra algorithm helps finding the shortest path from the origin to the terminus in a graph.
7. Dijkstra algorithm helps finding a minimal spanning tree in a given graph.
8. Maximal flow in a weighted network is the additional amount of flow, which is generated by the adjustment of excess capacity.
9. A minimal cut is to find two mutually disjoint sets so that both sets contain a source node and a sink node.
10. A bridge in a graph is a minimal cut which when removed divides the graph in two disconnected parts.
11. A minimum spanning tree in a weighted graph is the minimal cut in a graph.

Answers

1. false.	2. false.	3. true.	4. false.	5. true.
6. true.	7. false.	8. false.	9. false.	10. true.
11. false.				

II. Multiple Choice Questions

1. To find a feasible solution to the Chinese postman's problem, we use
   (a) Prim's algorithm (b) Euler tour and duplicated edges
   (c) assignment technique (d) Dijkstra algorithm
   (e) Floyd algorithm
2. To find a feasible solution to travelling salesman's problem, we use
   (a) Prim's algorithm (b) Euler tour and duplicated edges
   (c) assignment technique (d) Dijkstra algorithm
   (f) Floyd algorithm
3. To find the shortest path in a given directed network, we use
   (a) Prim's algorithm (b) Euler tour and duplicated edges
   (c) assignment technique (d) Dijkstra algorithm
   (f) Floyd algorithm
4. To find the minimal spanning tree in a given graph, we use
   (a) Prim's algorithm (b) Euler tour and duplicated edges
   (c) assignment technique (d) Dijkstra algorithm
   (f) Floyd algorithm



In the given weighted graph, answer the following multiple choice questions.

5. Maximum flow that can pass through the path 1—2—3—5 is
   (a) 0 (b) 3 (c) 2 (d) 6
6. Maximal flow that can pass through the path 1—2—3—5 is
   (a) 0 (b) 2 (c) 3 (d) 1
7. Maximal flow that can pass through the path 1—2—4—5 is
   (a) 6 (b) 1 (c) 2 (d) 3
8. In the given graph after applying Floyd–Fulkenson algorithm, the maximal flow amounts to
   (a) 7 (b) 12 (c) 10 (d) 11

9. The minimum cut on the edge
   (a) (1, 2), (1, 3)
   (b) (1, 3), (2, 4)
   (c) (2, 4), (3, 4), (3, 5)
   (d) (2, 3), (2, 4)
10. The minimal spanning tree on a graph on n vertices
   (a) has at heart one cycle
   (b) has $(n - 1)$ distinct edges and one cycle
   (c) has n edges and one cycle
   (d) has $(n - 1)$ edges and no cycle
11. Maximal flow on a weighted graph is the
   (a) additional amount of flow re-adjusted on all edges
   (b) given flow plus the maximum capacity of each edge
   (c) sum of maximum capacities of all edges
   (d) sum of flows on the minimal cut edges

ANSWERS

1. (b)	2. (c)	3. (d)	4. (a)	5. (a)
6. (b)	7. (a)	8. (d)	9. (c)	10. (d)
11. (d)				

NUMERICAL PROBLEMS

1. Solve the following travelling salesman's problem.
The distance matrix is given below.
Travelling salesman's time matrix:

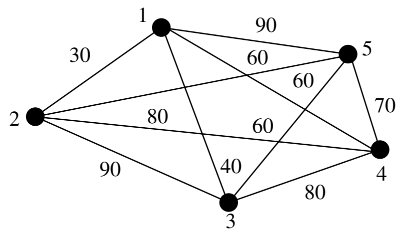
$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} \infty & 20 & 30 & 40 & 60 \\ 20 & \infty & 40 & 70 & 90 \\ 30 & 40 & \infty & 70 & 100 \\ 40 & 30 & 90 & \infty & 50 \\ 80 & 60 & 80 & 40 & \infty \end{pmatrix} \end{array}$$

2. Solve the following travelling salesman's problem.
The distance matrix is given below.

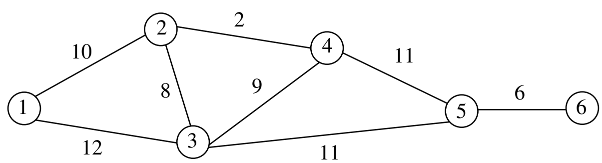
$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} \infty & 30 & 40 & 60 & 90 \\ 30 & \infty & 90 & 80 & 60 \\ 40 & 90 & \infty & 80 & 60 \\ 60 & 80 & 80 & \infty & 70 \\ 90 & 60 & 60 & 70 & \infty \end{pmatrix} \end{array}$$

3. Find the graphical presentation of the above matrix (Exercise 2) and find minimal spanning tree.

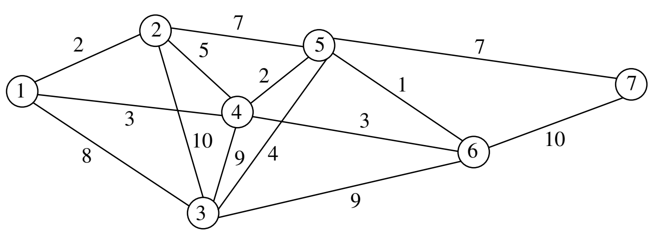
4. Find the minimal spanning tree of the following graph.



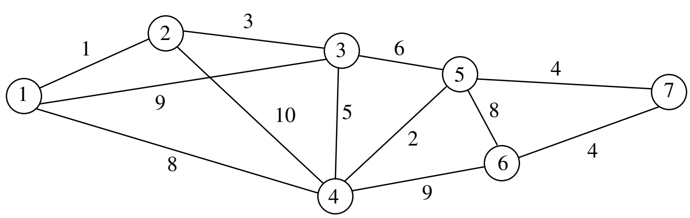
5. Find minimal spanning tree in the following graph.



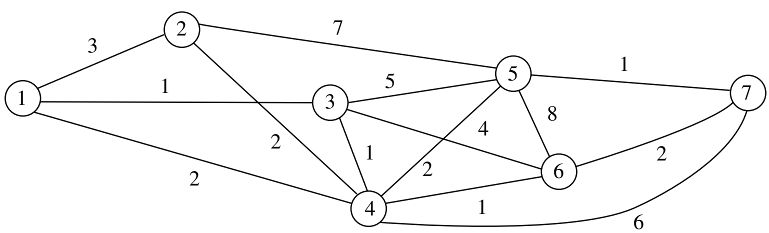
6. Find minimal spanning tree in the following graph.



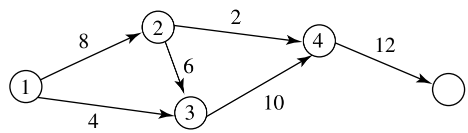
7. Find minimal spanning tree in the following graph.



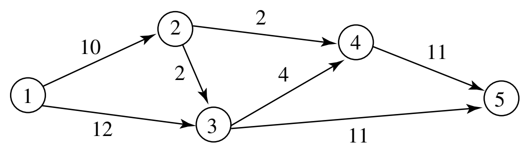
8. Find minimal spanning tree in the following graph.



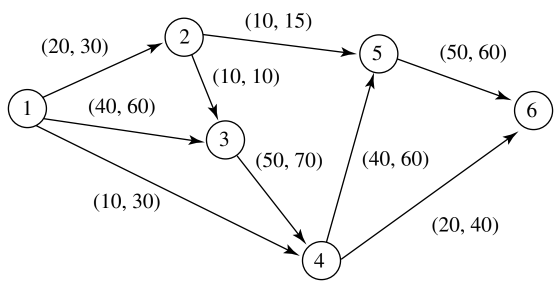
9. Find the maximal flow and minimum cut in the following graphical presentation.



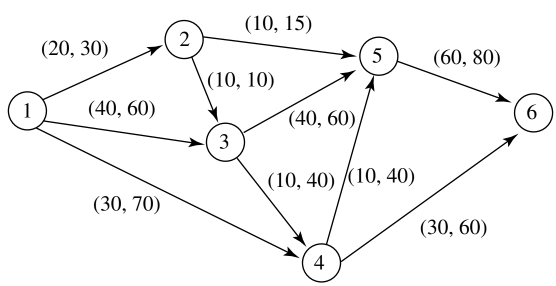
10. Find the maximal flow and minimum cut in the following graphical presentation.



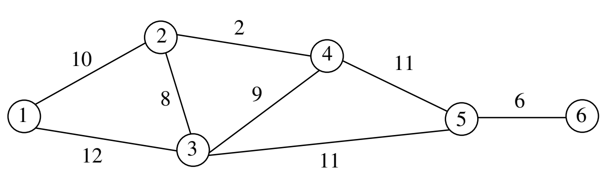
11. Find the maximal flow and minimum cut in the following graphical presentation.



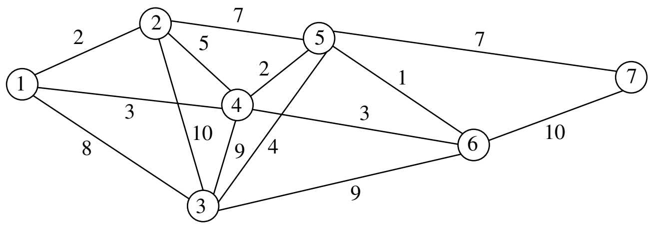
12. Find the maximal flow and minimum cut in the following graphical presentation.



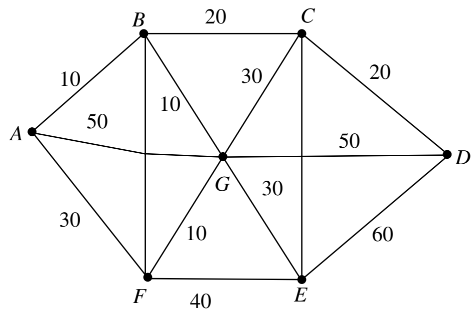
13. Apply Dijkstra algorithm and find the shortest distance.



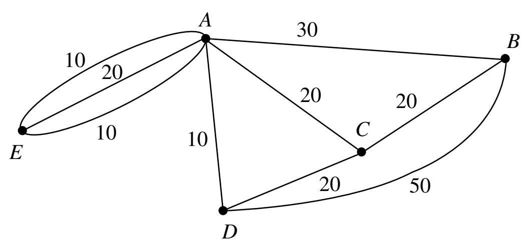
14. Apply Dijkstra algorithm and find the shortest distance.



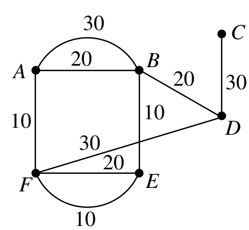
15. Find the solution to Chinese postman's problem and calculate the total path length in metre.



16. Find the solution to Chinese postman's problem and calculate the total path length in metre.



17. Find the solution to Chinese postman's problem and calculate the total path length in meter.



ANSWERS TO NUMERICAL PROBLEMS

1. Solving by assignment technique, we have two cycles 1—2—3—1 and 4—5—4. We combine these two by drawing loops, as explained.
Distant efficient route is 1—3—4—5—2—1 and the total distance = 230

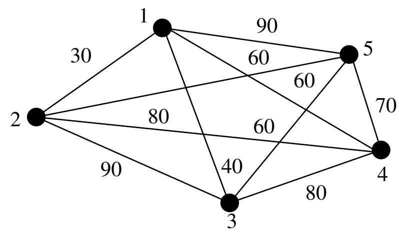
2. Solving by assignment technique, we have two cycles.

Combining these two cycles, we have 5—3—1—2—4—5

Total distance = 260.

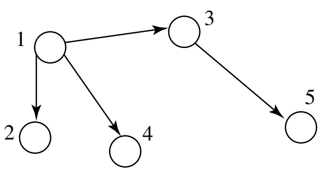
We find minimal spanning tree and finally make plan of joining edges with minimum distance.

3. Graphical Presentation

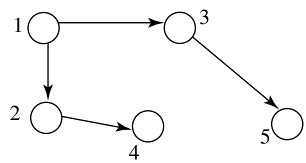


4. Minimal spanning trees are as follows.

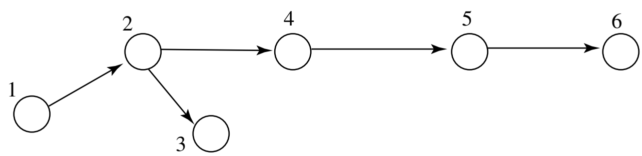
Total length = 190

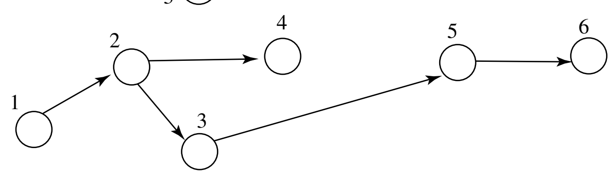


or



5.

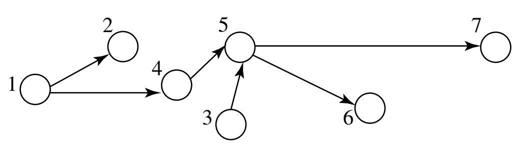




w(1, 2) = 10, w(2, 3)= 8, w(2, 4) = 2 w(4, 5) = 11, w(5, 6) = 6, w(3, 5) = 11

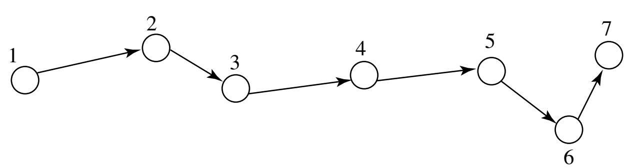
Using this information, check that the sum of weights on minimal path is the same.

6.

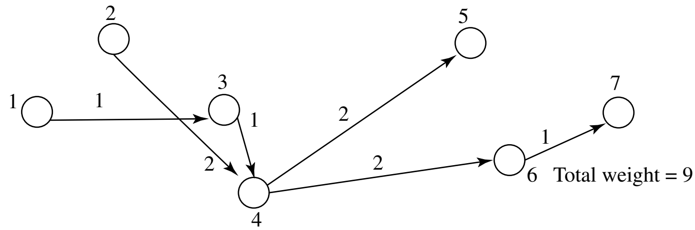


Total weight of the minimal path is 19 units.

7. One of the minimal path is shown in the answer. Total weight = 23 units



8. Minimal path is shown as follows.



9. Edge 1—2, 1—3, 2—3, 2—4, 3—4, 4—5

Flow 8/8, 4/4, 6/6, 2/2, 10/10, 12/12

10. Maximal flow = 16, cut edges are 2—4, 3—4, 3—5

11. Maximal flow = 100, cut edges are 2—5, 4—5, 4—6

12. Maximal flow = 140, cut edges are 4—6, 5—6

13. Shortest path = 1—2—4—5—6, length = 29

14. Shortest path = 1—4—5—7, length = 12

15. Vertices A and D are of odd degrees, introduce edges between D and C, C and D, B and A (as duplicate edges, this will make an Euler graph. additional edges added are of weight 20 + 20 + 10 = 50

16. Construct duplicate edges E—A, A—B, and C—D. This will make an Euler graph. Additional weight added is 20 + 20 + 10 = 50 units

17. The graph is not a Euler graph. Add duplicate edges between the vertices C and D, make (A—B and B—E) or (AF and FE) duplicate edges. By doing so you are adding an additional 30 + 20 + 10 = 60 units.

7 Sequencing Problems

Imagination is more important than knowledge.
Einstein

Learning Objectives

AFTER STUDYING THIS CHAPTER, THE STUDENTS WILL BE ABLE TO:

- understand the importance of sequencing problems.
- identify the controlling parameters of the sequencing problems.
- justify Johnson's algorithm
- apply the algorithm to different types of cases in sequencing problems
- extend the application area to queuing theory and simulation theory
- check whether fundamental theories of any other subject like simulation, queuing, etc., be useful in the development of some new application area.

INTRODUCTION

There are certain types of jobs, which require processing through different departmental activities. Each job consumes different times in different departmental activities. In most of the cases, jobs are of different types and the order on which they go for processing in different departments is technically fixed. (It cannot be interchanged.) A job requires two types of processes, say the first P_1 and upon completion of P_1 the process P_2 [strictly in the same order], then there is no choice but to carry out both processes. We take an example and try to derive what we exactly understand by **sequencing**.

ILLUSTRATION 1

Company X is planning production of two items—chairs and tables. There are two departments A and B. The following data shows the operation times in each department.

Job:	making one chair (1)	making one table (2)
Time (in hours) in Department A:	2	3
Time (in hours) in Department B:	4	5

A job after being completely processed in department A can be sent to the department B. The company has appointed labours in both departments. We also assume that there is no pending work in any department. You are required to study the consequences under two different orders of sequencing the jobs.

Case 1

If we plan to send the Job 1 (chair) and then Job 2 (table) to the departments A and B respectively, then we get the following matrix.

Processing–time Matrix

	Processing in department A		Processing in department B	
Job	*Time-in*	*Time-out*	*Time-in*	*Time-out*
1	00	02	02	06
2	02	05*	06*	11

Analysis of the result:

Total time for department A in operations is 5 hours*

Total time for department B in operations is 9 hours = (11 – 2) = 9 hours

Total idle time for department A: 11 – 5 = 6 hours

Total idle time for department B: 2 hours

Idle time taken by Job 2: 1 hour = 6 – 5 = 1 hour

Total elapsed time: 5 + 6 = 9 + 2 = 11 hours

Case 2

If we plan to send the Job 2 (table) and then Job 1 (chair) to the departments A and B respectively, then we get the following matrix.

Processing–time Matrix

	Processing on machine A		Processing on machine B	
Job	*Time-in*	*Time-out*	*Time-in*	*Time-out*
2	00	03	03	08
1	03	05	08	12

Analysis of the result:

Total time for department A in operations: 5 hours

Total time for department B in operations: 9 hours

Total idle time for department A: 12 – 5 = 7 hours

Total idle time for department B: 3 hours

Idle time taken by Job 1: 3 hours

Total elapsed time: 5 + 7 = 9 + 3 = 12 hours

We conclude from the analysis that in Case 2, total elapsed time is one hour more than that of in Case 1. In addition, in Case 2 idle time for both departments is greater than that of in Case 1. There is also waiting period for Job 1, which is out from department A and ready to move for processing in department B,

We have waiting time losses of departments and the job also.

Here, we have discussed a case of only two jobs—a chair and a table—but in practice, there may be many jobs, which need *sequential processing* in different departments and on the top of that, the different jobs require different times in different departments.

If the sequence of the jobs is not designed well in advance, then waiting time of the jobs being processed may cause partial break down of the entire set up causing serious consequences in different areas like, production, finance, etc. In addition to this, stand by or waiting of the department in any case, is a loss to the entire set up and business also.

Thus, in sequencing problems, we are mainly concerned with the effective measure of the feasible solution lies in identifying order or sequence of jobs; this sequence will eventually minimize the waiting time of the jobs.

7.1 IMPORTANT FEATURES OF SEQUENCING PROBLEM

We have discussed some of the basic points in the introductory part. Now, we are in a position of constructing mathematical model of the problem.

(A) Assumptions

To develop the basic model and construct an algorithm (known as Johnson's algorithm) for sequencing problems, we have some fundamental assumptions which are as follows.

1. All the jobs are of a common nature and can be handled by each department.
2. All the departments work equally efficiently for each job on the processing.
3. Jobs cannot make passover. It means that unless all the jobs, on hand, are over any department cannot lift the jobs from the previous department. At the same time, the jobs cannot break the sequence of passing through the departments in sequence of their working order and importance.
4. Time slot for a job in the department remains constant during the processing activity of the jobs.
5. The time for departmental transfer is negligible.
6. Departments or machines cannot repeat the job processing in case of inferior quality output.
7. Working capacity of department/machines remains constant during the complete operation of all jobs.
8. No part process of a job or partial transfer of some units of jobs is not allowed.

(B) Classification of the problems

The following classification, in general, covers all types of problems which are found in real situation.

(C_1) n jobs and 2 processing units (machines)
(C_2) n jobs and 3 processing units (machines)
(C_3) 2 jobs and n machines
(C_4) n jobs and m machines

(C) Notations and Terminologies

We shall use the following notations in this chapter;

(1) t_{ij} stands for the processing time of ith processing unit (machine) operating on jth job.
(2) T = Maximum of all times spent during the entire operation of all the jobs on all the machines; this is known as *elapsed time*.
(3) w_{ij} stands for the idle time of the Ith machine before processing the jth job.
(4) x_{ij} stands for the idle time period of the ith job completely processed by jth machine and accepted by $(j + 1)$th machine.

At this stage, we take one more illustration and understand the usage of the notations.

[In order to understand the fundamental features, it is a polite suggestion to understand all details of the following example. It is solved in two different ways and feasible solutions are found. The solution will finally take the reader to the root of the algorithms. Study it well].

ILLUSTRATION 2

There are three jobs and the final product will be only after they are taken for processing on two machines A and B in the same order. Processing time is given below, we are required to arrange the sequencing the jobs, in order, on both the machines A and B in order to minimize the total elapsed time.

Processing Time (in hours) Matrix

Machines	*Job 1*	*Job 2*	*Job 3*
A	5	6	9
B	7	4	2

Solution

Initially, we note that there are **3! = 6 different types of orders** in which we can plan these schedule of these jobs on these machines in different orders.

These orders are (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), and (3,2,1).

Case I

Selection of sequence (2,3,1)

These jobs are Job 2, Job 3, and Job 1. The following table shows the timings taken by each job on machines A and B. The time matrix is as follows.

Machines	*Job 1*	*Job 2*	*Job 3*
A	5	6	9
B	7	4	2

By notation, t_{A1} = time taken by Job 1 on machine A = 5

The process timings for the sequence (2,3,1) are shown below.

Job	*Machine A*		*Machine B*		*Waiting Time*		
	In	*Out*	*In*	*Out*	*Machine A*	*Machine B*	*Job*
2	00	06	**(06)**	10	00	**06**	00
3	06	15	15	17	00	**05**	00
1	15	20**	20	27**	00	**03**	00
					= **7****	= **14****	**00**

Waiting time of machine A = T_A = (27** – 20**) = 7 hours

Waiting time of machine B = 06 + (15 – 10) + (20 – 17) = 06 + 05 + 03 = 14 hours

$$T_B = W_{B2} + W_{B3} + W_{B1} = 6 + 5 + 3 = 14 \text{ hours}$$

Total time machine A works on jobs = $t_{A2} + t_{A3} + t_{A1}$

= Σt_{Ai} = 20 hours for all i = 1, 2, and 3 (This time 20 is fixed.)

Total time machine B works on jobs = $t_{B2} + t_{B3} + t_{B1}$

= Σt_{Bi} = 13 hours for all i = 1, 2, and 3 (This time 13 is fixed.)

In terms of machine A, total elapsed time = $T_A + \Sigma t_{Ai}$ = 7 + 20 = 27

In terms of machine B, total elapsed time = $T_B + \Sigma t_{Bi}$ = 14 + 13 = 27

Now, we consider the second case.

Case I

Selection of sequence (3,1, 2)

These jobs are Job 3, Job 1, and Job 2. The following table shows the timings taken by each job on machines A and B. The time matrix is as follows.

Machines	*Job 1*	*Job 2*	*Job 3*
A	5	6	9
B	7	4	2

By notation, t_{A1} = time taken by Job 1 on machine A = 5

The process timings for the sequence (2,3,1) are shown below.

Job	*Machine A*		*Machine B*		*Waiting Time*		
	In	*Out*	*In*	*Out*	*Machine A*	*Machine B*	*Job*
3	00	09	**(09)**	11	00	**09**	00
1	09	14	14	21	00	**03**	00
2	14	20**	21	25**	00	**00**	***(01)**
					= 5**	**= 12****	**00**

Waiting time of machine A = T_A = (25** – 20**) = 5 hours

Waiting time of machine B = 09 + (14 – 11) + (21 – 21) = 09 + 03 + 00 = 12 hours

$$T_B = W_{B2} + W_{B3} + W_{B1} = 9 + 3 + 0 = 12 \text{ hours}$$

Total time machine A works on jobs = $t_{A2} + t_{A3} + t_{A1}$

$= \Sigma t_{Ai}$ = 20 hours for all i = 1, 2, and 3 (This time **20** is fixed.)

Total time machine B works on jobs = $t_{B2} + t_{B3} + t_{B1}$

$= \Sigma t_{Bi}$ = 13 hours for all i = 1,2, and 3 (This time 13 is fixed.)

(D) Total elapsed time = Total time the unit/factory remains in operations either with machine A or with machine B or with both machines A and B working simultaneously.

This is the time slot between the starting time of the first job on the first machine A till the finish time of the last job on the second machine B. Total elapsed time can be defined in terms of machine A or machine B.

In terms of machine A, total elapsed time = $T_A + \Sigma t_{Ai} = 5 + 20 = 25$

In terms of machine B, total elapsed time = $T_B + \Sigma t_{Bi} = 12 + 13 = 25$

The important point here, is to note that, one cannot change the total operation time of both machines A and B but can surely, by proper sequencing, can control total *waiting time on machine B and hence eventually on machine A*. If this is done, then we can control the total *elapsed time*.

In Case 1, T_A = 7 and T_B = 14. We will design our algorithms in such a way that we can control T_B = Total waiting time of machine B.

In Case 2, T_A = 5 and T_B = 12. We will design our algorithms in such a way that we can control T_B = Total waiting time of machine B.

(E) Here Lies the Facts.

This is the dominant plan—we want to design our algorithms in such a way that we can control T_B = Total waiting time of machine B. This plan will effectively control total waiting time of machine A . This will reduce the total elapsed time.

All the different cases that we have seen here, have the above-mentioned logic in its centre. What we are going to study is ***Johnson's algorithms***; its basics are centred around these points only.

1. Waiting time for the second machine to process the first job out from machine A, must be controlled to its minimum and for this, processing time of the first job selected to be processed on the first machine must be the least amongst all such process timings on the first machine.
2. Processing time for the last job on the second machine must be the least amongst all such timings on the second machine.

These two are prime factors of this chapter.

7.3 DIFFERENT TYPES OF PROBLEMS

7.3.1 Type 1: C_1 type (n jobs and 2 processing units (machines))

A_1: Algorithm for C_1 type problems

(There are j jobs ($j = 1, 2, 3, \ldots, n$) and 2 machine (M_1 and M_2). There is a matrix format for such problems. Generally, the rows show the machine number/activity order/department number. The column shows the job number. The entries show the time (t_{ij}) to completely process a jth job on the ith machine.)

Step 1

Make a matrix presentation of the given problem (Generally 2 rows and n column matrix form is better.)

Step 2

From all t_{ij} entries find the minimum entry. If this minimum occurs in the first row (corresponding to the first machine M_1), then the corresponding job is given the top priority in the operations. If this minimum entry occurs in the second row (corresponding to the second machine M_2), then it is given the last priority. If there is a tie between two or more minimum entries of the first row or the second row, the choice of the order becomes optional in the order of operation plan. (Refer Step 6 for all possible cases.)

Step 3

Once the first selection is made, then the number of the corresponding job is written (either the first or the last) in the operation plan and the corresponding job is deleted from the list of jobs.

Step 4

Repeat the process of Step 2 followed by Step 3 until all the jobs are deleted.

Step 5

Make a processing chart of operations taken in order. This shows information regarding each job, its time (in and out) for each machine and based on these data one can find the total waiting period of each machines. In short, one can plan the entire activity and study related factors of operations.

Step 6

Optional—Priority cases

If there is a tie in the timings of machines, then we follow the following directives. Tie between the timings can be any one of the following four types.

1. When two or more jobs have the same processing time on the first machine, in this case the choice of their sequence in the operation plan is optional. Traditionally we keep that job ahead in the sequence whose serial number is higher in order then the order of the remaining jobs.

2. When two or more jobs have the same processing time on the second machine, in this case the choice of their sequence in the operation plan is optional. Traditionally the job with the higher serial number than that of the remaining jobs shall be kept last in the operation plan.
3. When the processing time of *i*th job on the first machine equals that of on the second machine , in this case, the job on the first machine will be placed ahead then one which runs parallel in time on the second machine.
4. When a job has same processing time on both machines, in this case, the placement of the job in the operation plan is optional, i.e. it can be placed in the higher order or lower order.

All these cases are shown in Illustration 3.

ILLUSTRATION 3

Time Matrix

Jobs / Machines	J_1	J_2	J_3	J_4	J_5	J_6
A	4	6	8	5	9	5
B	8	6	4	8	8	9

Using the time-record of the above matrix, you are required to sequence the jobs in order to minimize the total elapsed time.

Step I

We have, $t_{A1} = 4 = t_{B3}$; traditional to put the Job 1 on the first priority list and the Job 3 last on the priority plan.

Operational Plan 1

1					3

This process eliminates the Job 1 and the Job 3 from the list.

Now, we have,

Time Matrix

Job / Machines	J_1	J_2	J_3	J_4	J_5	J_6
A	4	6	8	5	9	5
B	8	6	4	8	8	9

Step 2

Job 4 and Job 6 have the same time on the machine A, i.e. $t_{A4} = t_{A6} = 5$. The choice of their placement is optional but Job 4 has the least processing time on machine B and so we like to put its order behind the order of Job 6.

On updating the operation plan, we will drop Job 4 and Job 6.

Operational Plan 2

1	6	4			3

Time Matrix:

Job / Machines	J_1	J_2	J_3	J_4	J_5	J_6
A	4	6	8	5	9	5
B	8	6	4	8	8	9

Step-3

The Job 2, now, is on the next list having next minimum time and we place it in the operation plan.

$$t_{A2} = 6 = t_{B2}$$

Conventionally, according to the serial order, we put them in order as follows.

This act will drop the Job 2 and there is only one Job 5 is left.

Operational Plan 3

4	6	4	2		3

Jobs / Machines	J_1	J_2	J_3	J_4	J_5	J_6
A	4	6	8	5	9	5
B	8	6	4	8	8	9

Now, we put the last job (Job 5) left for the placement in the operation plan.

Operational Plan 4 (final)

4	6	4	2	5	3

The above example explains the different cases and the systematic method of allocation of jobs.

ILLUSTRATION 4

Process- Time Matrix

Subjects / Departments	J_1	J_2	J_3	J_4	J_5
A	6	8	10	7	5
B	5	4	8	6	7

Time matrix shows the working days. Department A stands for composing and verification work of manuscripts. Department B stands for printing and arranging the printed materials.

The operations will take place in order. The problem is to make job allocation sequence that minimizes the total time. The final operation plan should be answerable to detail of each job.

Step 1

From the given time matrix (we read column wise), we find the minimum entry; i.e. 4 days which corresponds to Job 2 and concerns to department B. We put this as the last entry in the operation plan.

Operation plan 1 (Sequence of Jobs)

				2

Now the column for Job 2 stands, cancelled from further working.

Step 2

We repeat the process of finding the minimum entry from the remaining entries.

The corresponding matrix is

Subjects / *Departments*	*1*	*2*	*3*	*4*	*5*
A	6	—	10	7	5
B	5	—	8	6	7

Minimum entry is 5. It corresponds to Job 5 in department A and Job 1 in department B. We represent this in the operation plan as follows.

Operation Plan (Sequence of Jobs)

5			1	2

We drop these two jobs—Job 1 and Job 5 from the matrix. We are left with the following matrix.

Subjects / *Departments*	*1*	*2*	*3*	*4*	*5*
A	--	--	10	7	--
B	--	--	8	6	--

Step 3

Again we find the minimum entry, i.e. 6 and it corresponds to Job 4 in department B. once this is allocated, then there is only one entry for Subject 3 and that will fill the gap in the operation plan.

Operation Plan (Sequence of Jobs)

5	3	4	1	2

Step 4

Now, we make job assignment and operation schedule chart.

Jobs	*Department A*		*Department B*		*Waiting time Department B*
	Job In	*Job Out*	*Job In*	*Job Out*	
5	0	5	(5)	12	5
3	5	15	15	23	3
4	15	22	23	29	--
1	22	28	29	34	--
2	28	36	36	40	2

Total operation time is 40 days and waiting (idle time) for department A is 4 days (Job out time of the last job from department B – time when the last job is out department A = (40 – 36) = 4 days

waiting time (idle time) of department B = 5 + 3 + 2= 10 days.

Waiting time of jobs = (23 – 22) + (29 – 28) = 1 + 1 = 2 hours

ILLUSTRATION 5

There are six jobs to be processed sequentially on two machines A and B. The time, in hours, matrix is given below. Assign a sequence schedule for operation to minimize the total functional time. Also, find the idle time for each machines.

Process-time (hours) Matrix

Machines \ Jobs	J_1	J_2	J_3	J_4	J_5	J_6
M_1	5	9	4	7	6	5
M_2	9	5	4	9	8	8

Step I

We find the minimum entry of the time matrix. This is 4 (hours) and it concerns both the machines M_1 and M_2. It can be placed either in the beginning and at the end also. Let us verify both cases and make necessary calculations.

Operation Plan

Job	3						**Plan 1**
Job						3	**Plan 2**

This makes the Job 3 assigned and hence stands out from further part.

Machines* \ *Jobs	J_1	J_2	J_3	J_4	J_5	J_6
M_1	5	9	--	7	6	5
M_2	9	5	--	9	8	8

Step 2

Finding the minimum of the entries, bring us to the number 5. It concerns Job 1 and Job 6 for M_1 and Job 2 for M_2.

There is a tie for sequencing the Job 1 and Job 6. We show it as follows.

There is no problem for sequencing the Job 2, it goes in the last order.

Plan 1.1	3	1	6			2
Plan 1.2	3	6	1			2
Plan 2.1	1	6			2	3
Plan 2.2	6	1			2	3

(Plan numbers are the sub-plans of the first operation plan.)

Step 3

We have assigned four jobs. The residual matrix is as follows.

Machines \ Jobs	J_1	J_2	J_3	J_4	J_5	J_6
M_1	—	—	—	7	6	—
M_2	—	—	—	9	8	—

It is obvious that two minimum entries are 6 and 7; corresponding to Job 5 and Job 4 respectively. Also, both are for M_1 and they select the natural order of minimum time—Job 5 and Job 4.

Operation Plans:

Plan 1.1.1

3	1	6	5	4	2

Plan 1.2.2

3	6	1	5	4	2

Plan 2.1.3

1	6	5	4	2	3

Plan 2.2.4

6	1	5	4	2	3

We make table of job assignment sequence and related timings.

Table for plan 2.1.3

1	6	5	4	2	3

Now, based on the above plan we analyse the data.

Jobs	Machine M_1		Machine M_2		Waiting Time—M_2
	In	Out	In	Out	
1	0	5	(5)	14	5
6	5	10 ⟶	14	22	--
5	10	16 ⟶	22	30	--
4	16	23 ⟶	30	39	--
2	23	32 ⟶	39	44	--
3	32	36 ⟶	44	48	--

Total elapsed time = 48-hours,

Waiting time before M_2 = 5 hours

Waiting time for M_1 = 48 – 36 = 12 hours

Waiting time for the job = 4 + 6 + 7 + 7 + 8 = 32 hours

Table for plan 2.1.4

6	1	5	4	2	3

Jobs	Machine M_1		Machine M_2		Waiting Time—M_2
	In	Out	In	Out	
6	0	5	5	13	5
1	5	10 →	13	22	--
5	10	16 →	22	30	--
4	16	23 →	30	39	--
2	23	32 →	39	44	--
3	32	36 →	44	48	--

Total elapsed time = 48 hours

Waiting time before M_1 takes up = 48 – 36 = 12 hours

Waiting time for M_2 = 5 hours

Waiting Time for job = 3 + 6 + 7 + 7 + 8 = 31 hours

Table for plan 1.1.1

3	1	6	5	4	2

Jobs	M_1		M_2		Waiting Time Department M_2
	Time In	Time Out	Time In	Time Out	
3	0	4	(4)	8	4
1	4	9	9	18	1
6	9	14 →	18	26	--
5	14	20 →	26	34	--
4	20	27 →	34	43	--
2	27	36 →	43	52	--

Total elapsed time = 52 hours

Waiting time for M_1 = 52 – 36 = 16 hours.

Waiting time for M_2 = 4 + 1 = 5 hours

Waiting time for job = 4 + 6 + 6 + 7 + 7 = 30 hours

Table for plan 1.1.2

3	6	1	5	4	2

Jobs	M_1		M_2		Waiting Time Department M_2
	Time In	Time Out	Time In	Time Out	
3	0	4	4	8	4
6	4	→ 9	9	17	1
1	9	→ 14	17	26	--
5	14	→ 20	26	34	--
4	20	→ 27	34	43	--
2	27	→ 36	43	52	--

Total elapsed time = 52 hours
Waiting time for M_1 = 52 – 36 = 16 hours
Waiting time for M_2 = 4 + 1 = 5 hours
Waiting time for job = 5 + 5 + 6 + 7 + 9 = 32 hours

7.3.2 Type 2: C_2 Type (*n* Jobs on Three Machines)

There are j jobs (j = 1, 2, 3, . . ., n) and three machines (say M_1, M_2 and M_3).

There is a matrix form of such problems. Generally, the rows show the machine number/activity order/department. The columns show the job number. The entries show the operation time/processing time.

Step 1

Represent the given problem in matrix notation in 2 rows and n columns.

Step 2:

Before solving C_2 type problems , we have two conditions to be checked.

Condition 1

Minimum of $t_{1j} \geq$ maximum of t_{2j}

Condition 2

Minimum of $t_{3j} \geq$ maximum of t_{2j}

(Both the conditions refer to all j = 1 to n.)

Condition 3

If any one or both of the above conditions are satisfied, then only, the problem has feasible solutions obtained by following the steps given below.

Step 3

We consider two virtual machines G and H (say). We assign the processing time of jobs on G and H machines as follows.

We define processing time on these machines as follows.

Processing time on machine G = $\mathbf{t_{G\mathit{i}} = t_{A\mathit{i}} + t_{B\mathit{i}}}$

In the same way we define processing time for the machine H.

$$\mathbf{t_{G\mathit{i}} = t_{B\mathit{i}} + t_{C\mathit{i}}} \text{ for } i = 1 \text{ to n.}$$

As a result, of these two operations we have the problem of three machines converted into a problem of two machines. Now, it can be solved using the algorithm for C_1 type (Refer 7.3.1: *n* jobs and 2 machines.)

Step 3

Once the final sequence of jobs is derived for the two virtual machines G and H; we have to separate the timings for the three machines A, B, and C. This helps making the entire operation plan and answering many related questions.

(In case of tie, refer to Step 6 of the Section 7.3.1)

We take two examples and solve using above algorithm.

ILLUSTRATION 6

Textbooks on five different subjects (Jobs 1 to 5) are to be processed in order, in three departments. A: Compose and proof checking, B: Printing and arranging the pages, C: Binding, packing and forwarding taken in order of the activity.

The time, in days, for each department for processing the jobs is given here. You are required to find a time efficient sequence of the job and make analysis of the results.

Process-time Matrix

Jobs / *Departments*	J_1	J_2	J_3	J_4	J_5
A	3	6	8	5	7
B	7	5	6	5	6
C	8	7	9	8	9

Solution

Step I

Verification of condition 1

Minimum processing time in the **first** department ≥ maximum of the processing time in the second department in this case:

1. Minimum $t_{Aj} = 3$ for all $j = 1$ to 5, maximum $t_{Bj} = 7$, and $3 < 7$. In fact $t_{Ai} \geq t_{Bi}$, and so the first condition fails.
2. Minimum processing time in the third department is at least as great as maximum processing time in the second department.
   Minimum $t_{Cj} = 7$ for all $j = 1$ to 5, maximum $t_{Bj} = 7$, we have $t_{Cj} = 7 \geq t_{Bj}$ and so this condition is satisfied. We can apply the algorithm.

Step 2

In this step, we consider two virtual machines G and H with the time schedule defined as follows.

$$t_{Gj} = t_{Aj} + t_{Bj} \text{ and } t_{Hj} = t_{Bj} + t_{Cj}; \text{ for all } j = 1 \text{ to } 5.$$

This defines a new time matrix.

Time matrix (for all machines G and H)

Jobs / Machines	J_1	J_2	J_3	J_4	J_5
G	10	11	14	10	13
H	15	12	15	13	15

Step 3

We follow the method of Section 7.3.1 to search the assignment sequence.

The minimum entry is 10 for the Job 1 and Job 4 processed on the machine G, either of Job 1 or Job 4 takes first place in the operation plan.

Operation Plan

Job Plan 1.1	1	4			
Job Plan 1.2	4	1			

For the Plans 1.1, and 1.2, in the next iteration, Job 1 and Job 4 are not considered. The matrix for the remaining jobs is as follows.

Jobs / Machines	J_2	J_3	J_5
G	11	14	13
H	12	15	15

Step 4

Applying the same logic, the sequence can be set as follows.

Operation Plans

1.1.1	1	4	2	5	3
1.2.1	4	1	2	5	3

Operation Schedule

Jobs	Time on G		Time on H	
	In	Out	In	Out
1	0	10	10	25
4	10	20	25	38
2	20	31	38	50
5	31	44	50	65
3	44	58	65	80

Timing for G and H are further split into corresponding timings for machines A, B and C.

$$t_{Gj} = t_{Aj} + t_{Bj} \text{ and } t_{Hj} = t_{Bj} + t_{Cj}; \text{ for all } j = 1 \text{ to } 5.$$

i.e. $t_{Gj} = 10 = 3 + 7 = t_{Aj} + t_{Bj}$ and

$t_{H1} = 15 = 7 + 8 = t_{Bj} + t_{Cj}$

The operation schedule according to the above break-up is as follows.

Jobs	*Machine A*			*Machine B*			*Machine C*		
	In	*Out*		*In*	*Out*		*In*	*Out*	
1	0	3		(3)*	10		(10)**	18	
4	3	8		10	15		18	26	--
2	8	14		15	20		26	33	--
5	14	21		21	27		33	42	--
3	21	29		29	35		42	51	--
									10

Total elapsed time = 51 hours
Waiting time (idle time)

1. For machine A = 51 – 29 = 22 hours
2. For Job 4 and Job 2 in department A = 2 + 1 = 3 hours
3. For machine B = 3* + 1 + 2 + (51 – 35) = 22 hours
4. For Job 4, Job 2, Job 5 and Job 3 = 3 + 6 + 6 + 7 (in department B) = 22 hours
5. For machine C = 10** hours

ILLUSTRATION 7

Determine the optimal sequence of the jobs using the following time matrix so as to minimize the total elapsed time.

Machines \ *Jobs*	J_1	J_2	J_3	J_4	J_5	J_6	J_7
A	3	8	7	4	9	8	7
B	4	3	2	5	1	4	3
C	6	7	5	11	5	6	12

Solution

Step I

There are seven jobs and three machines. We have to check the condition for the optimal solution first.

Condition 1

Minimum time for machine A to process a given job ≥ maximum time for machine B to process a job, i.e. Minimum $t_{Aj} \geq$ maximum t_{Bj} for all j = 1 to 7; minimum $t_{Aj} = 3 \geq$ maximum $t_{Bj} = 5$ and so the condition 1 is not satisfied.

Condition 2

Minimum time for machine C to process a job ≥ maximum time for machine B to process a job maximum $t_{Cj} = 5$; maximum $t_{Bj} = 5$ for all j = 1 to 7, 5 = 5; i.e. the condition 2 is satisfied.

(Note: we want any one or both the given condition to be satisfied at a time.) This implies that the optimal solution can be found.

Step 2

We make two virtual machines G and H, with the following conditions:

$$t_{Aj} + t_{Bj} = t_{Gj} \text{ and } t_{Bj} + t_{Cj} = t_{Hj} \text{ for all } j = 1 \text{ to } 7.$$

This leads to the following matrix.

Jobs / Machines	J_1	J_2	J_3	J_4	J_5	J_6	J_7
G	7	11	9	9	10	12	10
H	10	10	7	16	6	10	15

Step 3

There are seven jobs and two machines. We apply the standard algorithm.

Plan 3.1: We find the minimum of all the entries, it is 6 and that corresponds to the Job 5 on machine H. The operation plan is as follows:

Operation Plan:

						5

Now this job is out of further calculation.

Plan 3.2: We have now the following matrix:

Jobs / Machines	J_1	J_2	J_3	J_4	J_5	J_6	J_7
G	7	11	9	9	--	12	10
H	10	10	7	16	--	10	15

Now applying the algorithm, the remaining minimum occurs for the Job 1 on G and the Job 3 on H (i.e. 7 hours).

Operation Plan:

1					3	5

Now Job 1 and Job 3 are also out from the further processing. On completion of this step, we have the following matrix.

Plan 3.3:

Jobs / Machines	J_1	J_2	J_3	J_4	J_5	J_6	J_7
G	—	11	—	9	—	12	10
H	—	10	—	16	—	10	15

Following the same logic, we complete the remaining formalities. This leads to the final plan.

Operation Plan:

1	4	7	2	6	3	5

Time-schedule matrix based on this plan for the two virtual machines is shown below.

Time Matrix

Jobs	*Machine G*		*Machine H*		
	Time In	*Time Out*	*Time In*	*Time Out*	*Waiting time Machine H*
1	0	7	(7)	17	7
4	7	16	17	33	--
7	16	26	41	56	8
2	26	37	56	66	--
6	37	49	66	76	--
3	49	58	76	83	--
5	58	68	83	89	--

We make the break-up for the machines A, B and C.

Jobs	*Machine A*		*Machine B*			*Machine C*		
	Time In	*Time Out*	*Time In*	*Time Out*	*Waiting Machine B*	*Time In*	*Time Out*	*Waiting Machine C*
1	0	3	(3**)	7	3	(7*)	13	7
4	3	7	7	12	0	13	24	0
7	7	14	14	17	2	24	36	0
2	14	22	22	25	5	36	43	0
6	22	30	30	34	5	43	49	0
3	30	37	37	39	3	49	54	0
5	37	46	46	47	7	54	59	0

Total = 25

Total elapsed time = 59 hours

Waiting time machine A = 59 – 46 = 13 hours

Machine B = 3** + 25 = 28; 59 – 47 = 12;   Machine C = 7*(initial waiting time)

7.3.3 Type 3: C_3 type: (Processing *n* jobs on *m* machines)

This is an extension of C_2 type. There are more than three machines and jobs to be performed in order on each one of them. Processing time of each job on each one is pre-known and fixed. We have the following **algorithm**.

Step 1

Make matrix presentation of the given matrix of *m* rows for machines and *n* columns *t* for jobs.

Step 2

Conditions for existence of the optimum solution are as follows:

Condition 1

Minimum of all the timings for all the jobs on the first machine (say M_1) is as great as the maximum of timings of all jobs on each one of remaining $M_2, M_3, \ldots, M_{m-1}$ machines.

Minimum $(t_{M1j}) \geq$ maximum (t_{ij}) where $i = 2, 3, 4, \ldots, m-1$, and $j = 1, 2, \ldots, n$.

Condition 2

Minimum of all the timings for all the jobs on the last machine (say M_m) is as great as the maximum of timings of all jobs on each one of remaining $M_2, M_3, \ldots M_{m-1}$, machines.

Minimum $(t_{Mm,j}) \geq$ maximum (t_{ij}) where $i = 2, 3, \ldots m - 1$; and $j = 1, 2, 3, \ldots, n$.

If any one or both of the above conditions are satisfied then only, the optimal solution to the above problem can be found.

Step 3

In the case of C_2 type of situation, we formulated two virtual machines G and H, in the same way we follow the procedure in this case.

We define timings for these two machines G and H as follows.

$t_{Gj} = t_{1j} + t_{2j} + \ldots t_{m-1j}$; and

$t_{Hj} = t_{2j} + t_{3j} + \ldots\ldots\ldots t_{mj}$; for all jobs $j = 1$ to n.

Step 4

On completion of Step 3, we have two machines and all the steps shown in C_1 type (n jobs and 2 machines) can be followed to solve the problem. Once the problem is solved then timings like elapsed time, idle time for each machines and jobs can be found.

ILLUSTRATION 8

There are 5 jobs to be processed on machines M_1, M_2, M_3 and M_4 in a sequential order of machines. The time duration for jobs on each machine are given in the following matrix. You are required to arrange jobs to minimize the total working hours of all machines.

Process-time Matrix

Jobs / *Machines*	J_1	J_2	J_3	J_4	J_5
A	10	8	9	9	8
B	3	5	7	4	6
C	4	8	5	7	3
D	10	11	9	10	9

We collect the following information from the matrix.

Minimum $(t_{Aj}) = 8$

Maximum $(t_{Bj}) = 7$

Maximum $(t_{Cj}) = 8$

Minimum $(t_{Dj}) = 9$

Condition 1

Minimum $(t_{Aj}) = 8 \geq$ Maximum $(t_{Bj}) = 7$

Minimum $(t_{Aj}) = 8 \geq$ Maximum $(t_{Cj}) = 8$

Condition 2

Minimum $(t_{Dj}) = 9 \geq$ Maximum $(t_{Bj}) = 7$

Minimum $(t_{Dj}) = 9 \geq$ Maximum $(t_{Cj}) = 8$

Both the conditions are satisfied. There exists a sequence of jobs and its optimal solution.

Step 1

We formulate two virtual machines, say G and H, their timings are defined as follows.

$$T_{Gj} = t_{Bj} + t_{Cj} \text{ for all jobs } j = 1 \text{ to n}$$
$$T_{Hj} = t_{Bj} + t_{Cj} \text{ for all jobs } j = 1 \text{ to } n.$$

Step 2

With the introduction of virtual machines the time matrix reduces as follows.

Machines \ *Jobs*	J_1	J_2	J_3	J_4	J_5
G	17	21	21	20	17
H	17	24	21	21	18

There are five jobs and two machines following the steps for C_1 type (missing) the operation plan is defined as follows.

Operation Plan

Jobs	1	5	4	3	2

Based on the above information, the layout of job activities and allocation timings on machines can be summarized as follows.

Jobs	*Machine A*		*Machine B*		*Machine C*		*Machine D*	
	Time In	*Time Out*	*Time In*	*Time Out*	*Time In*	*Time Out*	*Time In*	*Time Out*
1	0	10	(10)	13	(13)	17	(17)	27
5	10	18	18	24	24	27	27	36
4	18	27	27	31	31	35	36	43
3	27	36	36	43	43	48	48	57
2	36	44	44	49	49	57	57	68

Total elapsed time = 68 hours

Idle time for machine A = 68 – 44
= 24 hours

(Machine A is free after its total working for 44 hours.)

Idle time for machine B = 10 + (5 + 3 + 5 + 1)** + (68 – 49)
= 43 hours.

In addition to this, machine B has idle time during its operation. This time is the waiting time for the job yet to come to it for processing.

This time is **(18 – 13) + (27 – 24) + (36 – 31) + (44 – 43) = 5 + 3 + 5 + 1
= 14 hours.

In the same way, we can find for the machines C and D.

9.3.4 Type C4

Processing 2 jobs on *m* machines

In this case, there are exactly 2 jobs, which require processing on *m* different machines. We have certain assumptions here.

Assumptions

1. The processing time of each job on a machine is known.
2. A machine can process one job at a time.
3. The sequence of machines processing the jobs is pre-known.
4. The processing capacity of a machine remains constant for all the jobs.
5. Processing cannot be done by parts. A job started on a machine has to be completed. In this situation, our objective is to minimize the elapsed time by making a sequence of jobs to be processed on machines. We use graphical method to determine the sequence. The steps for the graphical method are as follows.

We take one example and understand the steps of graphical method for optimization of the elapsed time.

ILLUSTRATION 9

There are two jobs say Job 1 and Job 2. These jobs are to be processed on four different machines. A, B, C and D. The order in which the jobs are required to be processed and their processing timings are given. Find the optimal sequence and operation plan.

Process-timings (hours) of Jobs on machines

Job	*Machine*	*Machine*	*Machine*	*Machine*
Job 1	2 on A	4 on B	3 on C	1 on D
Job 2	6 on D	4 on B	2 on A	4 on C

In order to solve the problem, we have the following steps.

Step 1

We consider the first quadrant of the graph. We processing consider slots of times of Job 1 in the given order of the machines on x-axis. In the same way for Job 2 is considered on y-axis.

Step 2

In this step, we make rectangular time blocks of the machines. These blocks are non-overlapping on each other.

Step 3

In order to show working of jobs on machines, we begin with origin. The following points play important role.

1. A line segment parallel to x-axis indicates that the Job 1 is being executed and Job 2 is idle.
2. A line segment parallel to y-axis indicates that the Job 2 is being executed and Job 1 is idle.
3. A line at 45° angle from a point (or origin) indicates that both the jobs are being executed **on different machines**. This is called *parallel processing*.
   (It is a right isosceles triangle with both the non-hypotenuse sides being equal. These sides show time taken for processing.)

Step 4

We begin by drawing line segment subtending 45° angle. We try to move as much we can move along the 45°line segment and avoid rectangular blocks. We keep on moving on this line until we reach the finish point. If it is not possible to move along this line then only we will move horizontally or vertically. We again note that both machines cannot do one job and this implies that movement through the blocks is not allowed. We begin from origin, reach the last or the extreme right vertex of the last block. Finally, we calculate all the required points for the analysis and operation plan.

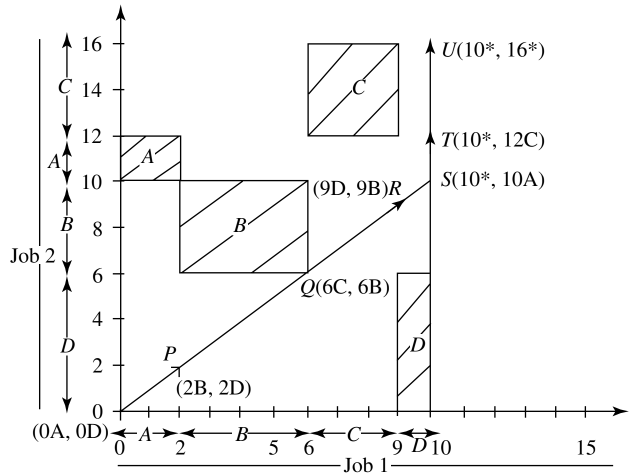


Figure 7.1

Hourly Work-chart

This chart corresponds to Figure 7.1.

	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9	9–10	10–11	11–12	12–13	13–14	14–15	15–16
Job 1	A	A	B	B	B	B	C	C	C	D	—	—	—	—	—	—
Job 2	D	D	D	D	D	D	B	B	B	B	A	A	C	C	C	C

It is enough to know that

1. you take Job 1 on x-axis with proper scale of time
2. you take Job 2 on y-axis with proper scale of time
3. Now, you draw a line $y = x$; extend it gradually. You should note that this line should not intersect (pass through) the rectangle. If the line $y = x$ is about to cut the rectangle then move along the edges of rectangle. Following this, complete the line segment until you reach the finish point.
4. Interpret the processing of Job 1 and Job 2 on x-axis and y-axis.
5. In order to interpret, assign coordinate values to the points where the line segment $y = x$ cuts the border lines of the rectangles.

ILLUSTRATION 10

There are two jobs—Job 1 and Job 2. These jobs are to be processed on each one of the four machines in a particular sequence as given below. We require to set up an optimal sequence of processing.

Job 1		*Job 2*	
Sequence of Machines	*Time (Hours)*	*Sequence of Machines*	*Time (Hours)*
A	2	D	6
B	4	B	4
C	5	A	2
D	1	C	3

Solution

We consider the sequence of machines along with the required timings of Job 1 on x-axis and similarly for Job 2 on y-axis. The length (horizontal) and height (vertical) of the rectangle show the process of the time of Job 1 and Job 2 in hours (Figure 7.2).

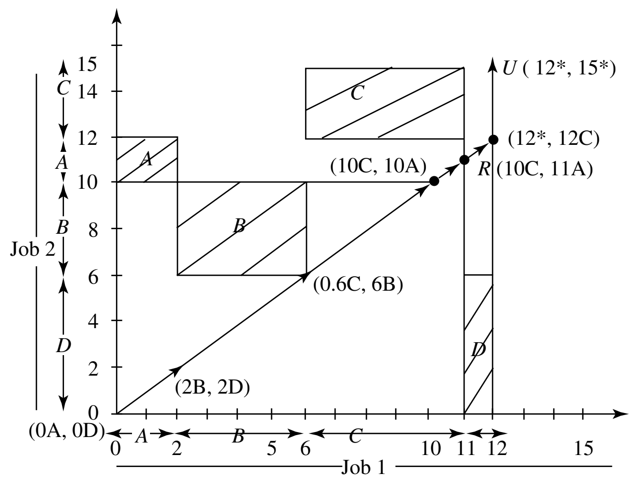


Figure 7.2

Analyzing the block diagrams, we conclude as follows.

Analysis

Hourly Work-chart

This chart corresponds to Figure 7.2 of the above example.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Job 1	A	A	B	B	B	B	C	C	C	D	--	--	--	--	--	--
Job 2	D	D	D	D	D	D	B	B	B	B	A	A	C	C	C	--

Using the above chart, we conclude that

1. total elapsed time = 15 hours
2. it takes 10 hours for Job 1 to complete all its operations
3. it takes 15 hours for Job 2 to complete all its operations

Step 1

The line segment (subtending an angle 45° with x-axis) implies that first 2 hours, the Job 1 on machine A and Job 2 on machine D is done at a time. (These are parallel activities.) At the end of that, machine A finishes the work of Job 1.

Step 2

From this point onwards, the next segment PQ has a role. It implies that Job 1 on machine B is done for 4 hours and simultaneously Job 2 is done on machine D.

Step 3

The next line segment has a role indicating that for 4 hours Job 1 is on machine C and Job 2 is on machine B.

Step 4

The next line segment indicates that for 1 hour, Job 1 is on machine C and being parallel in time Job 2 is on machine A.

Step 5

The next line segment indicates that Job 1 is on machine D and Job 2 on machine A. This completes the sequencing.

Step 6

The last line segment indicates that (being parallel to Job 2 means y-axis), Job 1 is done and Job 2 still needs 3 hours on machine C. This is done at the point U, i.e. final point.

ILLUSTRATION 11

For the two jobs—Job 1 and Job 2), the necessary processing sequence on five different machines and respective processing time in hours, is given as follow. you are required to make complete analysis of the entire process.

Job 1		*Job 2*	
Machines	*Time (Hours)*	*Machines*	*Time (Hours)*
A	3	B	2
B	4	C	3
C	2	A	3
D	3	D	2
E	2	E	3

Solution

First, we draw the time scaled blocks by taking Job 1 on x-axis and Job 2 on y-axis. The corresponding time blocks are shown Figure 7.3.

Students should read each stages and keep on continuous comparison and verification from Figure 7.3.

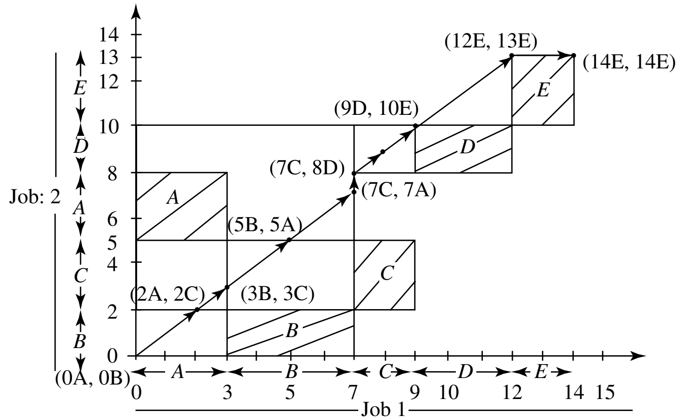


Figure 7.3

Read, Observe and Interpret.
Hourly Work-chart
This chart corresponds to Figure 7.3.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
Job 1	A	A	A	B	B	B	B	C	C	—	D	D	D	E	E	—
Job 2	B	B	C	C	C	A	A	A	D	D	E	E	E	—	—	—

Total time elapsed = 15 hours
Job 1 remains idle for 1 hour [going from machine C to machine D.]

ADDITIONAL QUESTIONS FOR PRACTICE (WITH HINTS AND ANSWERS)

Question 1
Two machines, P and Q in sequence, are required to process five jobs—1, 2, 3, 4 and 5. Their processing timings in hours are given in Process-time Matrix table. You are required to set up a sequence of jobs to be scheduled on these machines so that the total elapsed time is minimum.

Process-time Matrix (Hours)

Jobs / *Machines*	J_1	J_2	J_3	J_4	J_5
P	8	2	8	2	5
Q	6	5	9	8	3

Solution

Step I

Analyzing the data, we find that 2 hours is the minimum period; this relates to Job 2 and Job 3 on machine P. We make an operation plan and place them in first order. There are two options, we show both to justify each.

Operation Plan 1

2	4			

Operation Plan 2

4	2			

Now, these two jobs are omitted from the list of jobs. We work and apply the same logic for allocation.

Jobs / *Machines*	J_1	J_2	J_3	J_4	J_5
P	8	—	8	—	5
Q	6	—	9	—	3

Step 2

Now, comes the turn of the minimum entry 3 corresponding to Job 5 on the machine Q. We include this in the operation plans.

Operation Plan 1

2	4			5

Operation Plan 2

4	2			5

Step 3

Now, there is the next minimum entry 6 corresponding to the machine P and it is placed on the second last (before the last column). After placing it, there is only one entry left; which is to take in the centre cell of the block.

[Students can now complete the remaining part of the example.]

Answer

Operation Plan 1

2	4	3	1	5

Operation Plan 2

4	2	3	1	5

Question 2

For the following time matrix for the two jobs, allocate the time-efficient sequence of job activities on five machines.

Job 1		*Job 2*	
Sequence of Machines	*Time (Hours)*	*Sequence of Machines*	*Time (Hours)*
B	3	A	2
A	2	C	5
C	3	B	3
D	5	E	2
E	1	D	2

Solution

In this case, we take job 1 on x-axis and Job 2 on y-axis. We plot points showing position of machines B, A, C, D and E by taking appropriate scale (Fig. 7.4). In the same way, we do it for Job 2 on y-axis. As usual, we draw corresponding blocks or rectangles on XOY plane. Now, we draw a line $y = x$ (which subtends an angle of 45° with x-axis.)

To decide the time efficient sequencing, we move along this line so far as possible. We take care to see that the line does not cross the rectangular blocks.

Based upon these concepts and using these data, we get the required solution.

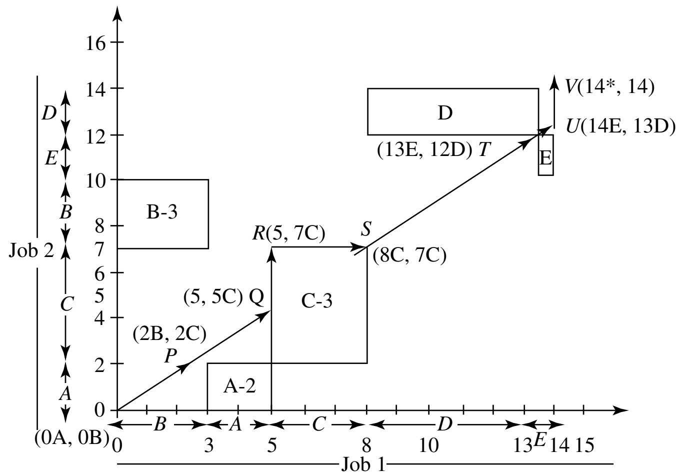


Figure 7.4

Total elapsed time = 17 hours

Waiting time of Job 1 = 2 + 1 = 3 hours

Waiting time of Job 2 = 3 hours

Hourly Work-chart

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Job 1	B	B	B	A	A	—	—	C	C	C	D	D	D	D	D	E	—
Job 2	A	A	C	C	C	C	C	—	—	—	B	B	B	E	E	D	D

Can we shift machine C in two blank spaces of Job 1? If you do so, you will find machine C working simultaneously on Job 1 and Job 2 at a time which is not possible.

Can we shift machine B in three blank spaces of Job 1? If you do so, you will find machine D working simultaneously on Job 1 and Job 2 at a time which is not possible.

Points to Remember

- Sequencing problem are the problems that arise in real life situations. These problems are connected with the different types of job when required to operate on different types of machines.
- In some cases, the order of machines is fixed for each job. The operation time of different jobs on different machines may be different. In some cases, the order in which the jobs are to be processed on different machines may be different.

- In the case of n jobs on 2 machines, the order of priority is either from beginning or from end depending upon the least operation time of a job on the first machine or the second machine respectively. In the case of more machines, the central idea is to convert the machine's timings into the timings of two virtual machines and apply the logic given in the previous paragraph.
- In the case of two jobs on different machines (the order of operation on machines may be different) we draw a graph by taking the first job and its operation timings on x-axis and for the second job on y-axis. Parallel operation of Job 1 and Job 2 on any machine is indicated by line-segment at 45º to x-axis. Vertical segments show that the Job 1 is waiting and Job 2 on some machine is in progress, horizontal line segments show that the Job 2 is waiting and Job 1 is in progress. At the end, we find the total elapsed time and a complete chart of work in progress.
- When solving the different cases we may face the tie in placing priorities. In the case of equal operation time on any two jobs on a machine, the job with the higher sequence in the sequence of order is selected. Also, we understand that a job cannot be operated on two different machines on a given time slot. In short, remember that you draw a working line $y = x$ and move as far as you can and see that this line does not cut or pass through the rectangular blocks. In such cases, move parallel to the edges of rectangles.

EXERCISES

OBJECTIVE TYPE QUESTIONS

I. State Whether True or False

1. If you process five items on a single machine then, by sequencing them we can save working time of the machine.
2. Sequencing four items on first machine A and then on the second machine B, has one objective
   (a) to keep machine A waiting as long as possible
   (b) to keep machine B waiting as long as possible
   (c) reducing total elapsed time
   (d) reducing total working time on jobs
3. By sequencing jobs on machines, we can reduce the actual operation hours of machines.
4. A completely processed first item on machine A and ready to go to machine B for processing cannot have waiting time.
5. Machine B, when free, can process some job not processed on machine A and can start processing; this is just total working time.
6. In processing plan of three machines A, B, and C, in order, we cannot impose any condition for sequencing these machines.
7. In processing plan of three machines A , B, and C, in order, we can combine timing of the first and the second machine and keep the timing for the third machine C without any change.
8. In processing plan of three machines A , B, and C, in order, we can combine timings of the second and the third machine and keep the timings of the first machine unchanged.
9. In processing plan of three machines A , B, and C, in order, there is a condition that the minimum working time of all the three machines must be same.
10. In processing plan of three machines A, B, and C, in order, there is a condition that the maximum working time of all the three machines must be the same.
11. Working on the line $y = x$ means two job, Job 1 and Job 2 are working simultaneously.

12. Working on the line $y = x$, when the line crosses the rectangular block we mean that the sequencing is done completely.
13. A line parallel to y-axis (Job 2) means that the Job 2 is waiting for processing.

ANSWERS

1.	false.	2A.	false.	2B.	false.	2C.	true.	2D.	false.
3.	false.	4.	false.	5.	false.	6.	false.	7.	false.
8.	false.	9.	false.	10.	false.	11.	true.	12.	false.
13.	false.								

II. Multiple Choice Questions

1. We use sequencing techniques
   (a) to increase working capacity of machines
   (b) to decrease working capacity of machines
   (c) to increase waiting time of jobs
   (d) to reduce waiting time of machines

Questions 2, 3, 4, 5, 6, 7and 8 share the common data.

Process-time (hours)

	Job 1	*Job 2*	*Job 3*	*Job 4*	*Job5*
Machine 1	5	9	8	3	2
Machine 2	7	4	7	2	9

2. What should be sequencing plan?
   (a) Put Job D first and Job E last in operation plan
   (b) Put Job E first and Job E last in operation plan
   (c) Put Job A and Job E last in operation plan
   (d) Put Job B and Job E last in operation plan
3. (a) Put Job E last in operation plan
   (b) Put Job D last in operation plan
   (c) Put Job A last in operation plan
   (d) Put Job B last in operation plan
4. (a) Put Job E first in operation plan
   (b) Put Job B last in operation plan
   (c) Put Job E last in operation plan
   (d) Put Job A first in operation plan
5. After proper sequencing machine B has an initial waiting time
   (a) 9 hours (b) 2 hours (c) 4 hours (d) 3 hours
6. On sequencing the second job to be processed is
   (a) D (b) E (c) A (d) B
7. The Time efficient sequence is
   (a) EABCD (b) AECBD (c) EACDB (d) EACBD
8. The total waiting time for machine B is
   (a) 2 hour (b) 3 hours (c) 8 hours (d) 31 hours

Question 9 and 10 share common data

Machines	*Jobs*				
	1	*2*	*3*	*4*	*5*
A	5	7	9	7	8
B	3	2	3	1	9
C	10	9	10	9	10

9. Can we apply sequencing principles on the above data?
   (a) Yes (b) No
   (c) Need more machines (d) Need more jobs
10. To find the two virtual machines in the above case we have to add timings of machines
   (a) A and B keep C as given (b) A and B, and B and C
   (c) Keep A as given and add for B and C (d) B and A, and C and A

Answers

1. (d) 2. (a) 3. (d) 4. (a) 5. (b)
6. (c) 7. (d) 8. (a) 9. (a) 10. (b)

Numerical Problems

1. Education corporation plans to print books on five subjects P, Q, R, S and T, it requires two-stage work plan. Stage A takes care of printing and binding while stage B takes care of marketing, stocking and accounting. The time period, necessary for processing in days, is given below and you are required to make an operation plan which minimizes the total time spent.

Time Matrix

Subjects / *Stages*	*P*	*Q*	*R*	*S*	*T*
A	8	9	10	8	6
B	5	4	3	7	9

2. Five Jobs J_1, J_2, J_3, J_4 and J_5 are to be processed on two machines P and Q. The processing time in hours is given. Suggest an operation plan that minimizes the idle time.

Time Matrix

Jobs / *Machines*	J_1	J_2	J_3	J_4	J_5
P	8	5	3	9	5
Q	5	8	7	4	8

3. Five machines are to be made compatible for the next batch of volatile liquid material. Two batches of servicemen are required to carry out the operation in order. The required time for the operations is given below. You are required to suggest an optimal plan which minimizes the elapsed time and idle time of machines.

Time Matrix

Machines / Servicemen	*P*	*Q*	*R*	*S*	*T*
Batch 1	6	9	5	1	3
Batch 2	6	3	2	5	5

4. Consider the following time matrix and plan a sequence of jobs that minimizes the total time.

Jobs / Machines	J_1	J_2	J_3	J_4	J_5
A	5	4	4	7	8
B	8	9	6	7	6
C	9	10	12	10	12

5. Four jobs are to be processed in order on each of the three machines, you are required to set up an operation plan that minimizes the elapsed time and idle time of the system.

Time Matrix

Jobs / Machines	*1*	*2*	*3*	*4*
P	12	10	10	11
Q	9	8	7	7
R	10	9	8	8

6. Suggest an optimal sequence for processing 6 jobs on 3 machines. Find the total idle time of jobs.

Time Matrix

Jobs / Machines	J_1	J_2	J_3	J_4	J_5	J_6
A	8	9	8	10	13	12
B	6	5	5	6	9	5
C	9	7	10	12	15	12

7. There are four jobs to be processed on four machines. The processing time of each job on each machine is given in time matrix. Find an optimal sequence of jobs that minimizes the total elapsed time.

Jobs / Machines	J_1	J_2	J_3	J_4
A	9	8	10	12
B	3	5	6	7
C	5	6	7	6
D	8	7	7	10

8. Consider the following matrix of the given job sequence on different machines using the given data, suggest an optimal sequence that minimizes the total elapsed time.

Process-time matrix

Jobs / *Machines*	J_1	J_2	J_3	J_4
A	8	10	9	8
B	2	3	5	7
C	7	5	6	4
D	6	7	8	5
E	9	10	10	9

9. There are two jobs Job 1 and Job 2 to be processed on four different machines A, B, C and D. The sequence of operations of jobs on the machines along with their functional time is given. You are required to search the optimal plan of activities order.

Job 1		*Job 2*	
Machines	*Time (Hours)*	*Machines*	*Time (Hours)*
A	3	C	3
B	5	D	2
C	8	A	5
D	3	B	3

10. From the following table suggest an operation plan of the two activities in different departments in the given order.

Activity 1		*Activity 2*	
Departments	*Time (Hours)*	*Departments*	*Time (Hours)*
A	3	B	3
C	5	C	2
D	2	A	1
E	1	D	5
B	2	E	3

Answers to Numerical Problems

1. Operation Plan

T	S	P	Q	R

Total elapsed time = 44 hours
Idle time for machine A = 3 hours
Idle time for machine B = 6 + 4 + 6 = 16 hours

2. Operation Plan

Plan 1

J_3	J_2	J_5	J_1	J_4

or

Plan 2

J_3	J_5	J_2	J_1	J_4

Total elapsed time = 35 hours
Waiting time for machine A = 5 hours
Waiting time for machine B = 3 hours

3. Operation Plan

S	T	P	Q	R

Total elapsed time = 30 hours
Waiting time for batch 1 = 2 hours
Waiting time for batch 2 = 2 + 6 = 8 hours

4. Operation Plans

1	3	1	2	4	5
2	3	1	2	5	4
3	3	2	1	4	5
4	3	2	1	5	4

With virtual machines G and H. $G_{ij} = t_{Aj} + t_{Aj}$; $H_{ij} = t_{Bj} + t_{Cj}$

5. Conditions satisfied, two virtual machines G and H, $G_{ij} = t_{Aj} + t_{Aj}$; $H_{ij} = t_{Bj} + t_{Cj}$
Operation Plans

1

1	2	3	5

or

2

1	2	4	3

6. Operation plans for two virtual machines G and H are as follows.

3	1	2	4	6	5

7. Minimum $(t_{Aj}) = 8 \geq$ Maximum $(t_{Bj}) = 7$;
Minimum $(t_{Aj}) = 8 \geq$ Maximum $(t_{Cj}) = 7$;
Condition satisfied.
Operation Plan.

4	3	2	1

8. Operation plan for two virtual machines

1	4	2	3

9. and 10. Students are advised to draw graphs for Job 1 on x-axis and Job-2 on y-axis. Students should take care for

1. Draw a working line, $y = x$ and move as far as you can and see that this line does not cut or pass through the rectangular blocks. In such cases, move parallel to the edges of rectangles.

There are 5 different subjects; say **Physics, Chemistry, Biology, Mathematics**, and **English**. We call these as jobs. In order to print the books on these subjects, we need the first department (A) to be involved. Let this be composing and verification department. It is obvious that once the work of composing is done, then only it can move to printing department (B).

Once the printing is done, then only it can move to the binding and packing/ forwarding department (C). It is obvious that each subject job take different working hours in different departments. These departments work in absolute connection in order.

The manager of the unit thinks of making a sequence of allocation of jobs in a way that minimizes the total working time and also the idle time of all these departments.

Let us put the facts on the table:

(Job numbers through 1 up to 5 are subject codes as mentioned above.)

Department A is **Composing and verification.**

Department B is **Printing.**

Department C is **Binding and packing/forwarding.**

Work- Time Matrix

Job / subjects →		*1*	*2*	*3*	*4*	*5*
Departments ↓	*A*	6	8	10	7	5
	B	5	4	8	6	7
	C	8	7	10	8	11

8 Simulation

Two things are infinite: the universe and human stupidity; and
I'm not sure about the universe.
—Albert Einstein

Learning Objectives

AFTER STUDYING THIS CHAPTER, THE STUDENTS WILL BE ABLE TO:

- **understand the basic concepts of simulation**
- **apply the techniques of constructing models explaining different real situations**
- **apply the simulation techniques to understand and explain expected results**
- **justify the output and compare discrepancy with the results of actual events**

INTRODUCTION

The literary meaning of 'to simulate' is 'to imitate or to copy'. There are situations which cannot be mathematically modelled or if they can be modelled then there cannot be all sufficient mathematical techniques to solve the model and arrive at a feasible solution.

These are some situations where we need simulation techniques to find the feasible solution. Dr. John Von Neumann and Dr. Stanislaw Ulam were the first who conceived the idea and employed special techniques (since then known as *simulation*) to study the irregular and indescribable behaviour of neutrons in a nuclear-shielding problem. As it was not probable to construct a mathematical model describing the behaviour, simulation techniques were employed to study and interpret the output and identify some pattern in the behaviour of neutrons. In cases where it may prove highly risk-prone to test the response of the given unit like airplanes capable of sustaining against heavy turbulence in adverse weather conditions, one cannot wait for such disturbance to occur and then test the unit or even in such cases of adverse weather conditions if the experiments fail to be successful, the end result is loss of lives of crew members and costly airplanes too. This means that if we generate a replica of adverse weather condition in a wind tunnel and then test the performance of the airplane then, the effects and results can be studied. In addition to this, based on the rsults further modifications can also be made to construct the better and sturdy version of the airplanes. In such a way, we have examples of testing of electric or electronic equipment. Successful application opened many application areas in business and production engineering where such techniques proved highly effective.

Simulation techniques do not search for optimality but aim to search for finding feasible solution. In most of the cases—like earthquakes, cyclones, weather conditions, etc.—of natural phenomena the total parameters responsible for occurrence, the time span of the occurrence and time of the next cycle are not known and hence a mathematical model cannot be designed accurately. The ways to fix most of situations are the techniques of simulations.

8.1 What is Simulation?

There are some real situations which can be understood but its response in different situations and at different times cannot be predicted well in advance. This type of uncertainty is the result of imperfect mathematical model with some unidentified parameters.

There are some situations which can be modelled but mathematical tools become incapable of solving the system and arrive at a feasible solution. To explain what we have been writing; cases like (1) find the value of π (Do not wander by saying oho!! it is 3.142857 or 22/7—all and we too know it!!) (2) Think of a circle; take any three points on its circumference now find the probability that they are on a semi-circumference. Do not you find it very complex and imaginative situation? On the top of all this we are to find feasible solution.

Simulation techniques will help you to construct an analogue model that can help you finding solutions.

There are, we observe and in cases, experience some uncertainties in real life situations which have been identified but it has not remained possible to predict the behaviour of the system at a given time. We have a situation; a shopkeeper dealing with bakery items faces a problem. The amount of loaves he receives each morning for the sale on the whole day is uncertain; the numbers of customers inquiring for the loaves is uncertain and also the number of loaves bought by each one of the inquiring customer is uncertain. Our problem is to make planning for the next week and hence it is essential to understand the system and its operations. We want to interpret the probabilistic/stochastic process. Simulation techniques help you out of these complications.

Let us take the following examples.

1. A builder having an open land thinks of constructing a multi-storied building. He, being very selective, has conceived a special plan and has designed attractive features. To attract customers, he wants to explain and put forward his ideas.
2. An airplane manufacturing company has some new airplanes and it is obvious that before beginning the flying operations, all its systems and especially its performance against high turbulence must be tested in extreme weather conditions. How do we go about this?

To answer the first one, the builder prepares a small ***iconic model*** made from hard-board sheets and wooden chips keeping a proportional alignment with the actual size of all the measures that he has already planned in his mind. This will enable him to convey his ideas to the one who inquires about the size, dimensions, and other details pertaining to the apartments.

To answer the second question and resolve the conceived conditions, a long tunnel in which, different types of weather conditions can be generated is made and all testings are done. If testings results, in all extreme conditions, are up to their satisfactions, then only its actual operations are allowed by competent licencing authorities. This experiment was a miniature form of what could or would happen in actual reality.

In all the above cases we discussed, and tried to convey the concepts of some models.

8.2 DIFFERENT TYPES OF MODELS

The different types of models are constructed either to find the feasible solution of some real situations or to represent its analogue/equivalent solvable model.

We have some cases of models.

(1) Iconic Models

* **Scaled-down version**: Miniature form or symbolic structure (physical model) or diagrammatic presentation on scaled paper of the proposed building, a fabricated version (physical model) or a scaled diagram/picture drawn on paper describing the solar system.
* **Scaled-up version:** Graphical version or physical model depicting molecular structure.

(2) Analogue Models

The main vision behind this concept is to establish most equivalent system between two sets. In the plane, the monitor shows the current status or position above the ground and the barometer shows the equivalent numeric figure of the current temperature. The glucometer shows the figure which is, by some formula or result of some chemical reaction of radiation, equivalent to the percentage amount of sugar in the blood stream at the time of measure.

It is easy to construct devices representing equivalent systems but its accuracy is questionable.

(3) Symbolic Models

Symbolic models are the small figures or a set of figures or a group of numbers in any pre-defined and commonly forms that make sense and clarifies the concept or theme conveyed through the symbols. Different pictures found on gadgets show the message describing the methods of their operations. Mathematical symbols and formulae describe the rule connecting given systems.

 For example, the figure represents a triangular shape of a region.

$C = (5/9)\,(F - 32)$ represents a mathematical formula that converts the temperature given in Fahrenheit scale into Celsius scale. [It represents a real-life situation—temperature at a given time and given position—in terms of two different measuring systems. This may be called an ***abstract model***.]

When the decision variables connecting given resources, constraints and objective functions are known and fixed, we have ***deterministic*** results. The cases when any one or more variables are stochastic or random in a given situation then, the describing model is a ***probabilistic or stochastic*** model.

Model Validation

After designing the abstract model or probabilistic model describing the problem of the given real situation we employ probable methods of finding the feasible solution. Once the feasible solution is achieved; the most important criteria is to test all aspects—number of variables, constrains, and values of the objective functions, etc.

If the variable(s) or system parameters observe all the criteria formulating the given problem, then only system is said to obey and observe the model. This process or the procedure is called ***model validation*** on a given set of values. On the basis of the results of model validation, the underlying model with minor changes is accepted or rejected to describe the real situation.

8.3 Nature of Simulation

(1) Limitations of Simulations

Simulation is a process describing or imitating an equivalent system of the real situation and it is expected to give most likely results when the original or a given system is subjected to operations. The results, obtained after solving a simulation model, never guarantee that the given real system gives the same results when subjected to reality.

In addition to this, different types of models created by different persons and also solved using different techniques are liable to different variations in the solution. We mean that what one can do is to conceive about the likely results or output but should not totally rely upon the results obtained by solving a particular model.

In some cases, in order to solve the given abstract or a mathematical model, we have to relax or to omit or to add one or two constraints then in such cases we get an outline or an ill-conceived picture of the actual results that the system will give. In such cases, if all the controlling parameters pertaining to a given real system are not pre-known or not given properly weights in the simulation model then the results obtained may not be distracting and are subjected to a high variations from those of the real system.

In some cases where the process of solving the simulation model depends upon random numbers, the different trials show different results as the random numbers used in different trials are different to the previous trials. One has to perform many trials of the solution procedures before arriving at outlines of the results of actual system.

(2) Merits and Demerits of Simulation

Merits of applying simulation techniques are as follows.

(A) The real-life problem or the conceived situation can be explained easily. Either a physical model or an abstract model can convey the problem.

(B) As the model of the problem has been structured or mathematically formulated, one can find its solution by applying different techniques or different trial runs, finding the feasible solution can be iterated.

(C) By applying simulation process, we can save time, money, and usage of man-power to study feasibility.

(D) Simulation results, in most of the cases, are derived by using sets of unbiased random numbers and hence prove most agreeable to what comes out as a reality.

(E) In most of the cases, simulation runs are easy to perform and are programmable in simulation languages.

Demerits of application of Simulations are as follows.

(A) In some cases, there is a discrepancy between the simulation model and the existing reality. If latter has many components and all the components are not assigned due weightages in the mathematical model, then the result obtained on solving the model may not be accurate and flexible enough in general.

(B) In some cases, simulation process becomes a long and approximate procedure which finally end-up in non-feasible solutions.

(C) Different programmers and planners view and interpret real situations in their own ways and apply different types of routines in order to derive the feasible solutions. In most of the cases (like applying different sets of random numbers in different trials), the end results are different and hence prove misleading and do not allow to derive the justifiable conclusions.

8.4 Application of Simulation

We have already identified the fundamental principles and working methodology of simulation. We have, in some cases, suggested to make physical model, but in other cases, mathematical models or equivalent analogue models are solved either using mathematical technique here or Monte-Carlo techniques using random numbers.

Simulation techniques can be useful in many areas; some of them are described here. Each one of the systems described here has, in real cases, different problems to be identified and feasible solutions are to be found.

(1) Simulation in Education and Training

Using simulation techniques many different educational programs from primary to post-graduate and doctorate level can be implemented and knowledge can be shared.

(2) Computer science

Simulation techniques using computer-based higher-level simulation languages can further open many areas of applications.

(3) Clinical Healthcare and Related Industries

Many simulation models are very useful in making diagnosis based on tests of blood samples. Models for some critical surgery and online conferences are developed and have proved very useful. Many researches and new models are constantly developed and simulation techniques prove very useful in cases where immediate solutions are not found or feasible on a given time period.

(4) Business and Production Areas

In business, we have many situations where one has to think and plan about the future events most likely to occur; in such cases simulation models are very useful. Such cases are solved using past records and data obtained in parallel cases are very useful to serve as the main source to plan the action, some of them are:

1. introducing new products in the market
2. financial planning
3. solving the inventory models and hence taking the decisions about production of different items
4. solving the problems of waiting line theories and services that can be possibly offered by the companies to the customers
5. We list below some important situations of real life situations where the simulations become applicable and useful.
   (i) automobiles—designing and planning of future production (ii) planning of cities and residential areas (iii) satellite communication (iv) flight simulation—training and operations (v) marine simulation—training, planning and operations. (vi) weather condition simulation—this area is extremely important for public safety, scheduling the flights and further connectivity, designing operations of vessels troops, warships, and submarines in high sea area (vii) designing nuclear plants (viii) planning and setting up of radiation therapy for the patients.

8.5 Random Numbers

What is a Random Number?

A non-negative real integer number mathematically associated with some occurrence or event from a set of mutually exclusive and totally exhaustive finite or infinite occurrences is called a *random number*.

Take an unbiased die and roll it, see the number on the top face. It is a random number. It is an output of a chance variable and cannot be predicted well in advance. Mathematicians, scientists of different branches and gamblers too, want to simulate chances of occurrences and envisage the results of certain real life situations using different models.

The Origin

In 1927, L. Tippitt, a scientist working for a Biometric Laboratory, University College, London, compiled and published a table of 41,600 numbers. These numbers were obtained by taking middle digits of the measurements of the areas of the English parishes. He thought of using these numbers as **random numbers.**

In 1950, the Rand Corporation planned and constructed a special machine to generate random numbers, calling them as *pseudorandom binary* bits of 0 or 1. The machine produced a table of **one million random decimal digits.**

Sir John Von Neumann was an early pioneer in the development of computer-based random numbers. Some standard programs and routines were developed to generate random numbers. Different seeds were given in the program in order to generate different sets of random numbers. These numbers are called *pseudorandom numbers*.

Different Methods of Generating Random Numbers

Simulations require the use of random numbers and, therefore, some of the different methods that we discuss here will give the basic ideas to generate random numbers and we call these numbers as **pseudorandom numbers**. They are not random numbers because some logic or mathematical rule or formula works behind its generation and so they are random-like numbers. At this stage, we will discuss (1) Mid-square method, and (2) Congruence method of generating random numbers.

There are other methods also. Random numbers can be generated using any higher level language. In order to generate random numbers a program is written and executed. Initially, as we have discussed earlier, all the random numbers (pseudorandom) are between 0 and 1.

(1) Mid-square method

Consider a two-digit number, square it and make it a four-digit number. Omit the first and the last digit. The remaining two middle digits show a random number. Use this random number and perform the above process. A sequence of such procedure carried for a required number of times gives a set of random (pseudorandom) numbers.

For example, the initial number that we begin with is $\boldsymbol{N_1 = 35}$. Its square $(35)^2$ is 1225.

Omitting the first and the last digits we get the number, $\boldsymbol{N_2 = 22}$.

Again, $(N_2)^2 = (22)^2 = 484$ which when subjected to four digits is = 0484. Omit the first and the last digits, we get the next number, $\boldsymbol{N_3 = 48}$. One may continue this method to get a required set of random numbers(pseudorandom)

Comments

1. Some methods describe the beginning by taking the first number N_1 as a four-digit number and perform the same process and omit the first two and the last two digits. This gives a four-digit random number.
2. In some cases of initial selection, two different criteria may occur. Either we end up with middle digits as **00** or **0000** or we end up in a **cycle**.

We cite the two illustrations one for each case.

Case 1

$N_1 = 6000$, $(N_1)^2 = 36000000$ and omitting the first two and the last two digits we get 0000.

Case 2

The next example of **cycling** and the process is as follows. We begin with the initial number $N_1 = 79$ $\therefore$ $(N_1)^2 = 6241$

From this we write the next random number $\boldsymbol{N_2 = 24}$**.** So $(N_2)^2 = (24)^2 = 576$ which is now expressed as 0576, The next random number extracted from this is $N_3 = 57$; now finding $(N_3)^2$ we get 3249. The next random number from 3249 is $\boldsymbol{N_4 = 24}$ which is same as $\boldsymbol{N_2 = 24 = N_4}$. This is known as cycling.

We avoid such cases in generating random numbers.

(2) Congruence Method

In this method, we perform arithmetic operations using an iterative procedure. Just like previous procedure, it may terminate in two different cases but we may omit such cases.

Three different models are open to us to begin with the routine of generating pseudorandom numbers.

The following iterative procedure determines the sequence of pseudorandom numbers.

Let a, b, m be the three two-digit numbers and r_i and r_{i+1} be the two consecutive random numbers for $i = 0, 1, 2, 3, \ldots n$

For $i = 0$, r_0 is the first random number that we start with and r_{i+1} will be developed by using the formula, $\boldsymbol{r_{i+1} = (a \cdot r_i + b)}$ **(modulo** $\boldsymbol{m}$**)**

[For given values of $\boldsymbol{a, r_i, b}$, and $\boldsymbol{m}$ **(where** $\boldsymbol{m \neq 0}$**);** $\boldsymbol{r_{i+1}}$ **is the remainder when** $\boldsymbol{(a \cdot r_i + b)}$ **is divided by** $\boldsymbol{m}$**.** This is an additive and multiplicative model. We have cases in the above formula.

If $\boldsymbol{b = 0}$**, then it becomes a formula for multiplicative model.**

$$\boldsymbol{r_{i+1} = (a \cdot r_i)} \textbf{ (modulo } \boldsymbol{m}\textbf{)}$$

If $\boldsymbol{a = 0}$**, then it becomes a formula for additive model.**

$$\boldsymbol{r_{i+1} = (r_i + b)} \textbf{ (modulo } \boldsymbol{m}\textbf{)}$$

[For $\boldsymbol{a = 1 = b}$ **and** $\boldsymbol{r_i < m}$ **we have** $\boldsymbol{r_{i+1} = r_i}$**. If at any stage in this process,** $\boldsymbol{r_{i+1} = 0}$ **then we have** $\boldsymbol{b = r_{i+2}}$ **if** $\boldsymbol{b < m}$**; Such typical cases may be avoided or to be dealt with accuracy and flexibility.]**

We take an illustration.

Let $\boldsymbol{a = 12, b = 20,}$ and $\boldsymbol{m = 51}$**.** To begin with let us take the first random number $\boldsymbol{r_1 = 35}$

We have the next random number $\boldsymbol{r_2 = (12 * 35 + 20)}$ **(modulo 51)**

$\therefore$ $\boldsymbol{r_2}$ **= remainder when (420 + 20) = 440 is divided by 51**

$\therefore$ $\boldsymbol{r_2 = 32}$

$\boldsymbol{r_3 = (12 * 32 + 20)}$ **(modulo 51) = remainder when (384 + 20) = 404 is divided by 51.**

$\therefore$ $\boldsymbol{r_3 = 04}$ **; in this way we continue to operate for a finite number of iterations.**

8.6 Monte-Carlo Simulation

At this stage, we describe the **systematic procedure** to solve or to interpret a given real situation.

A Monte-Carlo simulation is defined as any method that utilizes sequences of random numbers to perform the simulation. We have the following steps.

1. Study the given problem with all its parameters, given resources, constraints, and finally the objective that we would like to achieve on solving this system.

2. It is essential to note the fact that we solve a replica of the real system or an equivalent mathematical system.
3. In most of the cases, on studying the system requirements one of the following methods is applied.

How to Start?

Make a physical model	Make a generalized mathematical model	Collect all past records and tabulate
(1) Observe, study, make changes and finalize.	(1) Find suitable mathematical methods; apply, find solutions, validate the model and envisage the system solution.	(1) Tabulate the records, find distribution, use random numbers, and derive conclusion.

Note: We will use the following numbers from the random number table. In most of the cases, in examination random numbers are given and the students should use only those numbers.

1248	**2235**	**9870**	**0808**	**9879**	**5733**	**2478**	**9753**
2356	**9096**	**4578**	**7853**	**0876**	**8532**	**2218**	**8864**
2267	**0907**	**7538**	**0876**	**7542**	**6799**	**3690**	**8933**
6790	**3795**	**0679**	**6559**	**9056**	**0097**	**1643**	**7044**

8.7 ILLUSTRATIONS

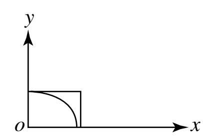
ILLUSTRATATION I

Find an approximation of π.

Solution

Consider the **first** quadrant of xoy plane. Draw a **unit** square with two adjacent sides on x- and y-axes. Draw a unit circle in the first quadrant.

Area of the quarter circle $= A_1 = \dfrac{\pi}{4}$, Area of the unit square $= A_2 = 1$



Any point like (x, y) where $0 < x, y < 1$ is a point inside the square. If $d(O, P) = OP = \sqrt{(x^2 + y^2)}$

if $0 < x^2 + y^2 \le 1$, then the point (x, y) is within the quarter circle or on the circumference of the quarter circle.

If $1 < x^2 + y^2 \le 2$, then the point (x, y) is outside the quarter circle but within or on the boundary of the square.

The above logic shows the plan of locating the position of the point like (x, y); where $0 < x$, $y < 1$

Now, we have to search for random numbers and pair them like (x, y). We develop a computer program that generates N **pairs like** (x, y)**. On using the above logic, we can find the number** (**say** n) **of the pairs which satisfy the criterion** $0 \le x^2 + y^2 \le 1$**. We have** $0 < n < N$**.**

[In other words, the probability that a pair like (x, y) with x and y being random numbers has its position within or on the circumference of the quarter circle is $\dfrac{n}{N}$.

If we take and actually plot its position in the figure above, then each time either it is a point within the quarter circle or outside the quarter circle but within the unit measure square.

As number of such observations (pairs), i.e. N becomes very large, i.e. $N \to \infty$, then in the limiting situation we have the result of $\lim \frac{n}{N}$ as a value of approximation to the division of the area of quarter circle to the area of the square.

i.e. as $N \to \infty$, $\lim \frac{n}{N} = \frac{A_1}{A_2} = (\pi/4)/\ 1 = \pi/4$.

From this we approximate π to be the limiting case of $4\frac{A_1}{A_2}$ as $N \to \infty$

ILLUSTRATION 2
(production planning and inventory management)

The following record shows the production of mopeds in a large automobile factory. The fluctuation is attributed to the availability of raw materials, shortage of man-power and other factors. The units are taken to the warehouse by a lorry having a space capacity of 95 mopeds at a time. The surplus units are taken to the warehouse on the next day. You are required to keep a record of next 8 days and find the average number of surplus mopeds per day.

Day	1	2	3	4	5	6	7	8	9	10	11
Production	94	98	96	95	94	98	97	96	95	98	94

Solution
Using the record, we associate probabilities of each occurrence as follows.

Total number of observations are 11; minimum amount = 94, maximum amount = 98. We make the initial analysis as follows.

Production	*Days*	*Probability*
94	3	3/11
95	2	2/11
96	2	2/11
97	1	1/11
98	3	3/11
	Total = 11	Total = 1

Now, from this, we make a table of less than type (≤) of cumulative probability distribution and making it proportional, we fit the random numbers.

Production	*Days*	*Cumulative probability*
= 94	3	3/11 = 0.2728
≤ 95	5	5/11 = 0.4546
≤ 96	7	7/11 = 0.6363
≤ 97	8	8/11 = 0.7272
≤ 98	11	1
	Total = 11	Total = 1

Fitting the following 8 random numbers from the given list,

1248 2235 9870 0808 9879 5733 2478 9753

Production	*Cumulative days*	*Cumulative probability*	*Range*	*Fitting (day number)*
= 94	3	3/11 = 0.2728	(0, 0.27279)	0.1248(1), 0.2235(2), 0.0808(4), 0.2478(7)
≤ 95	5	5/11 = 0.4546	(0.2728, 0.45459)	
≤ 96	7	7/11 = 0.6363	(0.4546 , 0.63629)	0.5733(6)
≤ 97	8	8/11 = 0.7272	(0.6363, 0.72719)	
≤ 98	11	1	(0.7272, 0.9999)	0.9870(3), 0.9879(5) ,0.9753(8)
	Total= 11	Total =1		

Result of Simulation

Day	1	2	3	4	5	6	7	8
Production	94	94	98	94	98	96	94	98
Moped over 95	—	—	3	—	3	1	—	3

The truck has a carrying capacity of 95 mopeds. The above table shows that out of 8 working days, there are 4 days on which there were 4 days [day number 3, 5, 6 and 8] on which all manufactured mopeds were not shifted by the lorry to the warehouse. Thus in 8 days, there are 10 mopeds which could not be sent same day of the production. This averages to 10/8 = 1.25 moped per day.

ILLUSTRATION 3

The average inter-arrival time between consecutive arrivals in a departmental store has the following distribution while the average service time also follows distribution.

Inter-arrival time (minutes)	*Probability*		*Inter-service time (minutes)*	*Probability*
1	0.10		1	0.10
2	0.25		1.5	0.25
3	0.30		2	0.45
4	0.20		2.5	0.15
5	0.15		3	0.05

The store starts its operations at 8.00 a.m. Simulate the both operations using the following given random numbers and simulate operations in accordance with beginning of operations time.

Set 1: 12 48 22 35 98 70 Set 2: 08 08 98 79 57 33

Solution

Using the records, we can construct cumulative probability table for each phase. The random numbers given by the Set 1 will be used to simulate arrival and the Set 2 for service time distribution.

The following table gives simulation for arrival of customers.

Simulated Arrival Timings

Inter-arrival (minutes)	*Probability*	*Cumulative values*	*Range*	*Fitting the random numbers*
1	0.10	0.10	(0.0, 0.09)	
2	0.25	0.35	(0.10, 0.34)	0.12(1), 0.22(3)
3	0.30	0.65	(0.35, 0.64)	0.48(2), 0.35(4)
4	0.20	0.85	(0.65, 0.84)	0.70(6)
5	0.15	1.00	(0.85, 0.99)	0.98(5)

Using the service time records and the given set of random numbers we can develop a table of simulated service time.

Simulated Service Timings

Inter-service time (minutes)	*Probability*	*Cumulative values*	*Range*	*Fitting the random numbers*
1	0.10	0.10	(0.0, 0.09)	0.08(1), 0.08(2)
1.5	0.25	0.35	(0.10, 0.34)	0.33(6)
2	0.45	0.80	(0.35, 0.79)	0.79(4), 0.57(5)
2.5	0.15	0.95	(0.80, 0.94)	
3	0.05	1.00	(0.95, 0.99)	0.98(3)

[**Note:** In both cases, we have taken two sets of 6 random numbers—the first set for arrivals and the second for the service time. Fitting of the numbers in the sequence is shown in the last column of each table. Now, we interpret the result.]

This is known as *simulation operation using random numbers*.

The store starts its operation at 8.00 a.m.

Customer	*Arrival*	*Waiting time before service*	*Service begins*	*Service Time (min)*	*Service complete*	*Server Free (min)*
1	8.02	00	8.02	01	8.03	02
2	8.05	00	8.05	01	8.06	02
3	8.07	00	8.07	03	8.10	01
4	8.11	00	8.11	02	8.13	01

In this way , we continue for the remaining customers . At the end, we can find the total waiting time from the server, the time system begins its operation.

ILLUSTRATION 4

PART (1): This Example is important—it interprets past events and helps making future planning. A shopkeeper keeps a record of the number of loaves he receives each morning. Assuming that a record of 25 days is sufficient enough to induce simulation process, you are required to carry out simulation for 5 days. Each loaf costs ₹12.00 and he sells ₹15 each. At the end of the day the left-over loaves are returned at ₹8.00 per loaf. Number of loaves received: 12, 15, 12, 11, 13, 12, 13, 14, 15, 11, 13, 12, 14, 15, 12, 13, 14, 15, 15, 15, 15, 12, 13, 12, 14

PART (2):

The shopkeeper keeps a record of 20 days of the number of customers those arrived for buying loaves and the number of loaves bought by each of them.

5 customers came to buy for 4 days; 6 customers for 10 days and 7 customers for 6 days.

PART (3)

The shopkeeper also keeps a record of number of loaves bought by each of 50 customers.

Number of loaves	*Number of customers*
1	10
2	25
3	10
4	05

You are required to simulate the entire phenomenon of all the **three parts** for the next 2 days and show the result.

Solution

Solution of PART (1)

Number of observations = n = 25; highest observation = 15, lowest observation = 11

We fit the following random numbers; **12 48 22 35 98 70**

Loaves	*Days*	*Cumulative frequency*	*Cumulative probability*	*Range*	*Fitting*
11	2	2	0.08	(0 0.0799)	
12	7	9	0.36	(0.08, 0.359)	0.12(1), 0.22(3), 0.35(4)
13	5	14	0.56	(0.36, 0.559)	0.48(2)
14	4	18	0.72	(0.56, 0.719)	0.70(6)
15	7	25	1.00	(0.72, 0.99)	0.98(5)
	Total = 25				

Simulation results are as follows.

Days	*Loaves*
1	12
2	13
3	12
4	12
5	15
6	14

The result will be used in the solution of Part (3).

Solution of PART (2):

The following random numbers are used to simulate the event.

Random numbers: 49, 76, 31, 14, 92

Customers	*Days*	*Cumulative frequency*	*Cumulative probability.*	*Range*	*Fitting of Random numbers*
5	04	04	0.2	(00, 19)	14(4)
6	10	14	0.7	(20, 69)	49(1), 31(3)
7	06	20	1.00	(70, 99)	76(2), 92(5)
	Total = 20				

Simulation results are as follows:

Days:	1	2	3	4	5
Customers:	6	7	6	5	7

[Note that this result will be used in the results of **Part 3.]**

Solution of PART (3):

In this simulation, we use the following random numbers.

47, 11, 35, 97, 75, 18, 86

Number of loaves	*Number of customers*	*Cumulative frequency*	*Cumulative probability*	*Range*	*Fitting*
1	10	10	0.2	(00, 19)	11(2), 18(6)
2	25	35	0.7	(20, 69)	47(1), 35(3)
3	10	45	0.9	(70, 89)	75(5), 86(7)
4	05	50	1.00	(90, 99)	97(4)

The result conveys that on the first customer bought 2 loaves; the second customer bought one loaf; the third customer bought 2 loaves, etc.

Now, we begin with very important part of the illustration.

We use simulated result obtained from each part and connect with the given data of the illustration.

Simulated Results of the First Day

From Part (1) : Number of loaves received = 12

From Part (2): Number of customers who bought loaves on the first day = 6

From Part (3): We make the following table showing the number of loaves bought by each customer on the first day.

Customers	*Loaves bought*
1	2
2	1
3	2
4	4
5	3
6	1

Total number of loaves bought by all 6 customers on the first day = **13 = Demand for the first day**

As the number of loaves received = 12, and so this is not possible.

This is called *opportunity loss*. The demand (= 13) exceeds the present stock (= 12).

Investment cost = 12 × 12 = ₹144; revenue on sale = 12 × 15 = ₹180

Profit on the first day = ₹180 – ₹144 = **₹36**

Simulated Results of the Second Day

From Part (1): Number of loaves received = 13

From Part (2): Number of customers who bought loaves on the second day = 7

From Part (3): We make the following table showing the number of loaves bought by each customer on the second day.

Customers	*Loaves bought*
1	2
2	1
3	2
4	4
5	3
6	1
7	3

Total number of loaves bought by all 7 customers on the second day = **16 = Demand for the second day**
As the number of loaves received = 13, and so this is not possible.
This is called *opportunity loss*. The demand (= 16) exceeds the present stock (= 13).
Investment cost = 13 × 12 = ₹156; Revenue on sale = 13 × 15 = ₹195
Profit on the second day = ₹195 – ₹156 = **₹39**
We continue following the same procedure/routine up to the required number of days.

ILLUSTRATION 5

A production unit has five machines that require timely inspection and disassembling the faulty units. The engineers take some time to put them in working condition. Repairs and assembling timings, as studied during last 25 observations are given as follows.

Inspection and disassembling timings (hours)	*Frequency*
0.5	5
1.00	8
1.5	6
2.00	6

Time required to repair and reassemble them depends on availability of the mechanics and the type of services required. The following table shows the distribution.

Repairs and reassembling (hours)	*Number of machines*
1	5
1.5	12
2.0	8

You are required to simulate the events for next four machines and find the total time these machines have remain idle (non-working condition). You may use the following random numbers: 39, 66 , 69 , 98 ,03 ,99 , 53, and 33.

Solution

In this situation, we find the inspection and disassembling timings and then we find the repair time required by the machines.

Part 1

Required inspection and disassembling timings for four machines

Inspection timings	*Frequency*	*Cumulative frequency (less than type)*	*Probabilty*	*Range*	*Fitting random numbers*
0.50	5	5	0.20	[00 , 19]	
1.00	8	13	0.52	[20 , 51]	39(1)
1.50	6	19	0.76	[52 , 75]	66(2), 69(3)
2.00	6	25	1.00	[76 , 99]	98(4)
	Total = 25				

Simulation results are as follows.

Machine number:	1	2	3	4
Inspection and disassembling timings:	1	1.50	1.50	2.00

Part 2

In this part, we find the simulated results of timings required for repairing the faulty machines and reassembling them.

Repair and reassembling timings (hours)	*Frequency*	*Cumulative frequency (less than type)*	*Probability*	*Range*	*Fitting random numbers*
1.00	5	5	0.20	[00, 19]	03(1)
1.50	12	17	0.68	[20, 67]	53(3), 33(4)
2.00	8	25	1.00	[68, 99]	99(2)
	Total = 25				

Simulation results are as follows.

Machine number:	1	2	3	4
Repair timings:	1	2	1.50	1.50

Part 3

We draw conclusion using the above two simulated results.

Machines	*Inspection time*	*Repair time*	*Total time*
1	1.00	1.00	2.00
2	1.50	2.00	3.50
3	1.50	1.50	3.00
4	2.00	1.50	3.50

Total estimated idle time = 12.00 hours.

ILLUSTRATION 6

The inter-arrival distribution of customers at a gas station during non-busy hours is as follows.

Time between two successive customers in arrivals (minutes)	*Number of customers*
02	15
03	20
04	40
05	20
06	05

The service time for the customers has the following distribution.

Service time	*Number of customers*
5	25
10	45
12	20
20	10

The gas station starts its operation at 6.00 a.m. You are required to simulate for first five customers and find the average waiting time.

You may use the following random numbers: 95, 79, 52, 37, 12, 01, 37, 49, 51, 93.

Solution

Part-I

First of all we find the arrival time of first 5 customers. The gas station starts its operations from 6.00 a.m. onwards.

We use the random numbers—95, 79, 52, 37 and 12.

Inter-arrival time (minutes)	*Frequency*	*Cumulative frequency*	*Probability*	*Range*	*Fitting of random numbers*
02	15	15	0.15	[00,14]	12(5)
03	20	35	0.35	[15,34]	
04	40	75	0.75	[35,74]	52(3), 37(4)
05	20	95	0.95	[75,94]	79(2)
06	05	100	1.00	[95,99]	95(1)

Results of simulation are as follows.

Customer arrival time	
1	6.06
2	6.11
3	6.15
4	6.19
5	6.21

Part II

When these customers go for the different types of service they need, the service time distribution is as follows.

We apply the random numbers—01, 37, 49, 51 and 93.

Service time (minutes)	*Frequency*	*Cumulative frequency*	*Probability*	*Range*	*Fitting of random numbers*
05	25	25	0.25	[00, 24]	01(1)
10	45	70	0.70	[25, 69]	37(2), 49(3), 51(4)
12	20	90	0.90	[70, 89]	
20	10	100	1.00	[90, 99]	93(5)

Simulated results are as follows.

Customers	*Service time (minutes)*	*Arrival time*	*Service begins at 6.00 a.m.*	*Service completion*	*Waiting time customers*	*Waiting time service units*
1	05	6.06	6.06	6.11	00	06
2	10	6.11	6.11	6.21	00	00
3	10	6.15	6.21	6.31	06	00
4	10	6.19	6.31	6.41	12	00
5	20	6.21	6.41	7.01	20	00

Total waiting time = 06 + 12 + 20 = 38 minutes.

Average waiting time per customer = 38/05 = 7.2 minutes.

ADDITIONAL QUESTIONS FOR PRACTICE (WITH HINTS AND ANSWERS)

Question 1

A dentist can perform five types of treatment at his dispensary. He begins his work at 9.00 a.m. on each day. His secretary schedules appointments of the patients at an interval of every 30 minutes beginning from 9.00 a.m. every day. In general, the routine takes the following time slots for the operations.

Types of treatments	*Average time (in minutes)*
Filling	40
Crowning	60
Cleaning	20
Extraction	30
Routine check-up	20

The secretary has maintained a record of last 100 patients and is as follows.

Types of operations	*Number of patients*
Filling	40
Crowning	15
Cleaning	15
Extraction	10
Routine check-up	20

Using the information from the above record, you are required to simulate the routine for following the first 6 patients and find the average free time the dentist can have to enjoy on a day. You may use the following random numbers given below.

Random numbers: 35, 55, 09, 69, 94, 76

Solution

First of all, we simulate the event on first 6 patients and then prepare the timing schedule. This will enable us to find both patients' waiting time and doctor's free time.

Types of treatment	*No. of patients*	*Cumulative frequency*	*Probability*	*Range*	*Fitting of random numbers (patient number)*
Filling	40	40	0.4	(00, 39)	35(1), 09(3)
Crowning	15	55	0.55	(40, 54)	
Cleaning	15	70	0.70	(55, 69)	55(2), 69(4)
Extracting	10	80	0.80	(70, 79)	76(6)
Routine check-up	20	100	1.00	(80, 99)	94(5)

Now, we generate a table for the patient's need and doctor's treatment time beginning from 9.00 a.m.

Patient no.	*Arrival time*	*Types of treatment*	*Waiting time (minutes)*	*Doctor starts treatment*	*Time taken for treatment (minutes)*	*Time-out*	*Doctor's free time*
1	9.00	Filling	00	9.00	40	9.40	00
2	9.30	Cleaning	10	9.40	20	10.00	00
3	10.00	Filling	00	10.00	40	10.40	00
4	10.30	Cleaning	10	10.40	20	11.00	00
5	11.00	Routine	10	11.00	20	11.20	10
6	11.30	Extracting	00	11.30	30	12.00	---

From the above table, the last column indicates that the doctor is free for 10 minutes before the arrival of the sixth patient.

Question 2

A furniture making factory has two processing departments, say A and B. Average distribution of processing time for the assembly line timing is given as below.

Time (minutes)	30	35	40	45
Frequency (Department. A)	20	30	40	10
Frequency (Department. B)	15	25	40	20

Production items go through Department A and then only to the Department B. You are required to simulate the work for next five items. Also, find the total waiting time for Department B. Use the following random numbers. (You may begin the simulation from 8.00 a.m.)

Random numbers:

For Department A—22, 87, 66, 87, 35, Department B—56, 87, 45, 94, 34

Solution

We want to simulate the events for both the departments. We have a condition that Department B cannot start the work unless the first item is sent from Department A. This means that, initially Department B has a waiting time till the first item is fully processed in Department A and it is out to move to Department B. We use the given random numbers for simulating the time for each production item spent in each department.

Simulation for Department A

Time in minutes	*Frequency*	*Cumulative frequency (less than type)*	*Probability*	*Range*	*Fitting of random numbers*
30	20	20	0.20	[00 , 19]	
35	30	50	0.50	[20 , 49]	22(1), 35(5)
40	40	90	0.90	[50 , 89]	87(2), 66(3), 87(4)
45	10	100	1.00	[90 , 99]	
	Total = 100				

Simulation results for first 5 items are as follows.

Item number	1	2	3	4	5
Time taken	35	40	40	40	35

Simulation for Department B

Time in minutes	*Frequency*	*Cumulative frequency (less than type)*	*Probability*	*Range*	*Fitting of random numbers*
30	15	15	0.15	[00 , 14]	
35	25	40	0.40	[15 , 39]	34(5)
40	40	80	0.80	[40 , 79]	56(1), 45(3)
45	20	100	1.00	[80 , 99]	87(2) , 94(4)
	Total = 100				

Simulation results for first 5 items are as follows.

Item number	1	2	3	4	5
Time taken	40	45	40	45	35

Simulation begins from 8.00 a.m. We use the above simulation results obtained for each department.

Simulated results beginning from **8.00 a.m.** are as follows.

Item number	*Process time in minutes for Department A*	*Time on which ready for Department B*	*Waiting time for Department B*	*Process time in Department B*	*Time when item is ready*
1	35	8. 35 a.m.	35	40	9.15 a.m.
2	40	9.15 (8.35 + 0.40)	00	45	10.00 a.m.
3	40	9.55 (9.15 + 0.40)	05	40	10.40 a.m.
4	40	10.35 (9.55 + 0.40)	05	45	11.25 a.m.
5	35	11.10	15	35	12.00

Question 3

ABC Corporation is planning its profit for the next month. Three important factors, viz., (1) cost of raw material, (2) processing cost, and (3) selling price acceptable in the market play vital role in the estimation. Past records in these cases are as follows.

Cost of raw material per kg. (₹)	4.00	4.50	5.00
Number of months	8	12	5
Processing cost (₹ per unit)	5	5.50	6
Number of months	5	12	8
Selling price (₹ per unit)	20	21	25
Number of months	10	8	7

You are required to simulate the profit for the next three months.

Random numbers for each simulation are as follows.

45, 67, 98, 37, 76, 87, 55, 39, 76

Solution

This example can be divided in the following three parts.

Part 1: In this part, we use the first set of records and find the simulated cost price.

Cost	*Frequency*	*Cumulative frequency (less than type)*	*Probability*	*Range*	*Fitting of random numbers*
4.00	8	8	0.32	[00, 31]	
4.50	12	20	0.80	[32, 79]	45(1), 67(2)
5.00	5	25	1.00	[80, 99]	98(3)

Simulated results

Month	1	2	3
Cost price	4.50	4.50	5

Now, we simulate for the processing cost.

Processing cost	*Frequency*	*Cumulative frequency (less than type)*	*Probability*	*Range*	*Fitting of random numbers*
5.00	5	5	0.2	[00 , 19]	
5.50	12	17	0.68	[20 , 67]	37(1)
6.00	8	25	1.00	[67 , 99]	76(2), 87(3)

Simulated results:

Month	1	2	3
Processing cost	5.50	6.00	6.00

Now, we process for the estimated selling price.

Estimated selling price	*Frequency*	*Cumulative frequency (less than type)*	*Probability*	*Range*	*Fitting of random numbers*
20	10	10	0.10	[00 , 09]	
21	8	18	0.72	[10 , 71]	55 (1) , 39(2)
25	7	25	1.00	[72 , 99]	76(3)

Simulated results

Month	1	2	3
Processing cost	21.00	21.00	25.00

Now, we conclude from the three types of simulated results and find the profit per unit per month.

Month	*Cost of raw material (A)*	*Processing cost (B)*	*Total cost, C = A + B*	*Selling Price = D*	*Profit per unit = D – C*
1	4.50	5.50	10	21	11
2	4.50	5.50	10	21	11
3	5.00	6.00	11	25	14

Points to Remember

1. Simulation is a model or a replica, which works to demonstrate the estimated behaviour of the system under given constraints.
2. In order to have very close idea of the behaviour of the system, we should take more and more trials and execute the process of simulation.
3. If all the constraints and parameters governing the system are not considered, then the results of simulation may prove very weak and misleading in some cases.
4. Different sets of random numbers be used in different trials.
5. All the same time, it is highly essential that, on realization of each events,
   (a) past records must be updated.
   (b) past constraints must be checked and if necessary, new constraints may be added.
   (c) Some of the ineffective constraints may be removed from the data.
   (d) In the same way, at the end of the realization of event, the number of parameters governing the system must be studied for its effectivity, and its expected range in further application of simulation.

Exercises

Objective Type Questions

I. State True or False.

1. The real life situation problems that cannot be easily modelled, are effectively understood by simulation.
2. Simulation helps us finding the optimum value of the objective function in the problem.
3. Simulation is a one-time process and will give total idea of the behaviour of the system.
4. Simulation is a process, which saves time and money in the case of testing the behaviour of the system where items under test fail or permanently destroyed.
5. Simulation is a process to understand the behaviour of the system and hence it has applications in many real life situations.
6. Some situations which cannot be mathematically modelled, simulation when applied, gives approximate/tentative picture of the behaviour.
7. Some situations which can be modelled but cannot be solved mathematically, simulation and repeated trials are not effective to study the pattern of the solution.

8. In order to apply simulation process, it is necessary to generate random number using standard computer programs.
9. As we keep on making more trials on the simulation process, the different results obtained in each trial, help taking a better idea about the performance of the system when it reacts in real life.
10. Simulation languages like SIMULA, GPSS, etc., are very useful in using the past records and making more trial runs for better conclusion.

Answers

1. true.	2. false.	3. false.	4. true.	5. true.
6. true.	7. false.	8. false.	9. true.	10. true.

II. Multiple Choice Questions

1. Large and complicated events are perceived and simulation models are built because
   (a) tentative outcomes are estimated and proper planning at a lower cost can be done in advance
   (b) past records are not available
   (c) it is difficult to find useful softwares to simulate
   (d) none of the above
2. The fundamental purpose of using simulation techniques' is
   (a) to save the cost of experiments
   (b) to forecast the estimated outcomes of events
   (c) to safeguard and make plans against expected outcomes
   (d) all the above
3. Simulation models fail in envisaging the replica of the events because
   (a) proper data for making trials is not available
   (b) proper model is not designed
   (c) all the parameters and constraints are not employed in the model
   (d) all the above
4. Simulation is an analytical model just prepared to perceive the reality and so the results are
   (a) considered as guidelines
   (b) exact results
   (c) misleading results
   (d) none of the above
5. Simulation is a process of
   (a) making a replica of an existing or a perceived situation of real life
   (b) solving the model
   (c) interpretation of the model
   (d) finding the outcome of an event
6. Simulation models are solved only using
   (a) past records
   (b) applying mathematical techniques on past records
   (c) computers
   (d) mathematical formulae
7. Simulation techniques can be applied in the area of
   (a) the problems of investment analysis only
   (b) the problems of planning and forecasting only
   (c) solving all the real life situation models of unpredicted outcomes
   (d) problems of the business situations only

8. Pseudorandom numbers are generated using
   (a) associating real values with results of random experiments
   (b) using pre-formulated computer programs
   (c) rolling a dice
   (d) rolling two fair dies
9. The main purpose of performing many trials is
   (a) to get each new result better than that of the previous one
   (b) to be satisfied by performing the most usage of data
   (c) make a sampling distribution of sample means and estimation of parameters
   (d) to be on the safer side for decision making and behaviour of the system
10. The discrepancy between the outcomes of the behaviour of the system in real life situation and that of simulated forecast
   (a) is useful for validation of the degree of accuracy of the model and the model-solving procedures
   (b) is not useful in any case
   (c) can be studied but does not play important role
   (d) non of the above

ANSWERS

1. (a)	2. (d)	3. (d)	4. (a)	5. (a)
6. (b)	7. (c)	8. (b)	9. (c)	10. (a)

NUMERICAL PROBLEMS

1. Consider any three points on the circumference of a circle. Find the probability that the points are on the circumference of the semi-circle.
   [This is a typical problem. It clearly explains the concept of constructing the logic to solve this problem using the techniques of simulation by Monte-Carlo technique. We construct a routine; select three random numbers (each less than 1), find the absolute distance between the two extreme points. The routine then checks whether the distance is less than that of the extreme points of the semi-circle.
2. There are five divisions in the central zone of a city. There are residential houses in each zone. The first division has 300 houses and then in each next division the number of houses increases by 50. (The last division has 500 houses.) National board of education surveying educational status of family members wants to select any six houses from all the houses of these five divisions. You are required to use random numbers given below and make the tabulated results showing the house number and the division in which it falls.

Random numbers:	32	98	56	45	23	33	58	87	02	45
	19	43	08	57	43	23	16	98	77	56
	04	45	42	67	23	46	25	67	15	92

   Hint: For using random numbers: You should select four digits at a time. e.g. 3298, 5645, 2333, etc., if the random number fits in your organized record then, keep it or else discard it and select the next random number.
3. Fast Ride Corporation is engaged in production of motorbikes. The daily production depends on many factors—like availability of raw materials, number of workers on the shift of the day, etc.

The motorbikes are, at the end of the day, are taken to the godown. The lorry engaged for this job has a capacity to carry 150 bikes at a time. A record of last 20 days production figures are given below. Using the records you are required to find, based on simulated results, the average number of empty spaces in a sequence of any week of **six working** days.

Production record of last 20 days is as follows.

149, 148, 152, 150, 152, 148, 149, 150, 148, 148, 150, 149, 149, 150, 151, 150, 148, 149, 152, 149

You may use the following random numbers 23, 45, 98, 67, 77, 64

4. A store keeper receives copies of newspaper for open market sales. The number of copies he receives is a random variable. Each copy costs him ₹5 and he sells it for ₹8 each. The leftover copies at the end of the day are returned at ₹4 per copy. He keeps a record of the number of copies he received in last 20 days. He also keeps a record of actual copies sold on the same 20 days.

Days	*No. of copies recieved*	*No. of copies sold*	*Days*	*No. of copies received*	*No. of copies sold*
1	20	20	11	22	21
2	18	15	12	22	21
3	20	15	13	20	16
4	22	18	14	18	17
5	20	19	15	19	19
6	19	19	16	20	18
7	19	18	17	20	18
8	20	20	18	19	19
9	22	18	19	20	18
10	21	20	20	21	19

You are required to generate a statement of account showing the details of the business (receipt and sales figure) and estimated sales for the next 5 days. You may use the following random numbers 12, 00, 74, 19, 25, 13, 40, 11, 38, 09.

5. Dense forest area of a state has the following records of the total rainfall in last 40 years . Simulate the record for the forecast of next 3 years.

Rainfall in inches	*Frequency (years)*
40	8
41	12
42	15
43	10
44	5

You may use the following random numbers 13, 56, 40.

6. A record of last 100 years rainfall is as follows.
Rainfall record

Status	*Frequency (years)*
Below average	20
Average	30
Good	50

Under different rainfall conditions, the production of the Crop A has the following frequency distribution.

Rain (below average)–production record

Production (kg)	*Years*
500	20
550	25
600	55

Rainfall(average)–production record

Production (kg)	*Years*
550	20
600	70
650	10

Rainfall (good)–production record

Production (kg)	*Years*
700	80
750	15
800	05

In the case of below average rainfall, the Crop A is sold at ₹25 per kg. In the case of average rainfall the Crop A is sold at ₹20 per kg. and in the case of good rainfall, the Crop A is sold at ₹18 per kg. You are required to simulate the expected rainfall, the total production and revenue from sale in next 2 years. You may use the following random numbers: 12, 14, 56, 13, 40, 23, 09, 25.

7. A doctor performs four types of minor operations—A, B, C, and D. The first two types of minor operations take 30 minutes while the last two (C and D) takes 50 minutes. Past records have shown the following distribution.

Types of operations:	A	B	C	D
Frequency	20	40	10	30

Doctor's clinic begins at 10.00 a.m. The doctor has given appointment to six patients at an interval of 30 minutes to each on first come first served basis. You are required to simulate doctor's schedule beginning from 10.00 a.m. You are given following six random numbers.

22, 59, 03, 67, 76, 50.

8. A record, showing time slot between two successive arrival of 50 customers at a remote-stationed inquiry booth is given below. There is a single service window serving on first come first served basis for all the customers approaching the booth. Customers demand for various types of services and it takes, on an average, different time for each customer being served at the booth. These two distributions are given below. You are required to simulate the event for the next three customers and find average waiting time of a customer. You may assume that the booth starts its operations at 10.00 a.m.

   Random numbers are 07, 89, 54, 72, 81, and 17.

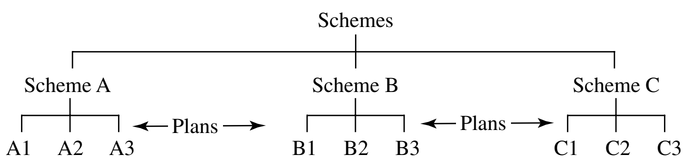
   Arrival time slot frequency table

Time (minutes) between two successive arrivals	*Frequency*
2	10
3	25
4	10
5	05

   Service time frequency table:

Average service time (minutes)	*Frequency*
5	35
6	10
7	05

9. A strategic planner has three schemes; A, B and C. Probability of selection of any one of the schemes A, B, or C is 0.5, 0.3, and 0.2 respectively. Each scheme has three further plans A_1, A_2, ... B_1, B_2, ... C_2, C_3 (nine plans) as shown below.



   Probability that plans A_1, A_2, and A_3 will be followed are 0.7, 0.2, and 0.1; for the plans B_1, B_2, and B_3 corresponding probabilities are 0.6, 0.3, and 0.1 while for plans C_1, C_2, and C_3 they are 0.2, 0.6. and 0.2 respectively. Selection of any scheme and the corresponding plans under that scheme are random variables and strictly chosen by the customers. Simulate the events for the first three customers. You can use the random numbers 77, 65, 12, 08, 97 and 45

10. A colony of 1000 houses is divided in four parts—A, B, C and D. Probabilities of selections of these parts are 0.1, 0.2, 0.3 and 0.4 respectively. Sun Energy Corporation wants to select houses for fitting solar panels. Simulate the event of selection for 10 customers and find the probability of selection of houses from Part C, sequential number of houses selected from Part C.

    You may use the following random numbers.

    12 34 00 98 08 23 89 12 23 45 59 74 54 12 34 56 65 38 45 56 87 23 21 03 56 37 91 02

ANSWERS TO NUMERICAL PROBLEMS

1. Consider a circle with radius $r = 1/(2\pi)$; as a result the circumference = $2\pi.\ (1/2\pi) = 1$.

   Random numbers generated by the system are non-negative and less than 1. Consider any triplet. say, A, B and C. [Each one has a non-negative measure less than 1] Now, find the absolute difference between the measures of the highest value and the lowest value.

   i.e. |A – C| or |B – C| or |A – B|. If this distance is **less than or equal** to 0.5, then those three points are on a semi-circle.

   In case the distance is greater than (>) 0.5 then, there are chances that it may be on a semi-circle or may not be.

   We take two cases.

   Case 1: A = 0.1, B = 0.2, and C = 0.9 Case 2: A = 0.1, B = 0.45, and C = 0.9

   For Case -1; **|C – A| = 0.8** and for the Case 2 , **|C – A| = 0.8**

   To ascertain the position, if you plot them; you will find that in Case 1, those three points are on a semi-circle while in Case 2 they are not on a semi-circle.

   [To solve and understand this situation, we follow a procedure. Add 1 to those numbers (values) whose measure is less than **0.5**

   Now, find the absolute difference between the highest and the lowest. If this is still less than or equal to 0.5, then they are on a semi-circle and otherwise.

   Case 1 A = 1.1 , B = 1.2 and C = 0.9 and so |1.2 – 0.9| = 0.3 < 0.5 and so Case 1 is of the points on a semi-circle.

   Case 2 A = 1.1 , B = 1.45 , and C = 0.9 and so |1.45 – 0.9| = 0.55 > 0.5 and so they are not on a semi-circle.

   In this, way, we take *N* number of triplets and find number of triplets (say *m*) which are on the semi-circle.

   In the limiting case as $n \to \infty$; the limit of (m/N) shows the probability.

2.

Range		*Random number/selection of house number*
001	300	0245(1)
301	650	0445(5)
651	1050	0857(3)
1051	1500	
1501	2000	1943(2), 1698(4)

3. Using the random numbers 23, 45, 98, 67, 77 and 64 we get the simulated results of estimated production figures as follows.

   Day number (Production): 1(148), 2(149), 3(152), 4(150), 5(150), 6(150)

   Using this information, one can calculate empty spaces in the lorry or surplus bikes left in the factory.

4. Using the random numbers 12, 00, 74, 19 and 25; we get simulated result for the number of copies received for the first five days. One more simulation, using the random numbers 13, 40, 11, 38 and 09, we get simulated results for number of copies sold (demanded).

Day number	1	2	3	4	5
Copies received	19	18	21	19	19
Copies sold	16	18	16	18	15

   Using these data, you can calculate profit/loss figures.
5. Years (Rainfall): 1(40) 2(42) 3(42)
6. Using the random numbers 12, and 14 for prediction, we have 'below average ' rain in the coming two years. Using the random numbers 56 and 13, the crop production amounts to 600 kg and 500 kg. Now you can find the revenue from this data.
7. Using the random numbers 22(1) and 59(2), the first patient comes for type B work and the doctor takes 30 minutes to treat him. As soon as the doctor is free from his work, according to the appointment given, the second patient arrives at 9.30 and the doctor immediately begins his work. According to the random number 59(2), the patient has come for type C work and that takes 50 minutes. Meanwhile, as scheduled, the third patient comes at 10.00 and he will have to wait as the doctor is busy on second patient's work. You may continue this way and make a table showing all these activities.
8. Fitting the random numbers in order, the first customer arrives at 10.02, the next comes after 4 minutes, i.e. at 10.06 and the third one at 10.09 a.m. The service time they use is 6, 6 and 5 minutes respectively. Now using this information, you can generate the time-table.
9. Customer 1—Selection of Scheme B , Customer 2—Selection of Scheme B, Customer 3—Scheme A [Within the scheme B, Customer 1 will select the Plan B_1, etc.]
10. Probability = 0.3, selection of houses 400, 455, 541 [We have selected random numbers 123, 400, 980, ... etc., from the given sequence of random numbers.]

9 Game Theory

Mathematics is a game played according to certain simple rules with meaningless marks on paper.
Hilbert, David (1862–1943)

Learning Objectives

AFTER STUDYING THIS CHAPTER, THE STUDENTS WILL BE ABLE TO:

- understand the basic principles of game theory
- understand the basic maximin and minimax principles and its application
- understand the different methods of finding feasible solution
- apply dominance principles
- understand the application of fundamental principles in competitive situations in diverse fields, economic interpretation of different real life situations.

INTRODUCTION

The very first attempt to envisage the competitive situation and formulate a 'mathematical theory of games of strategy' was done by Emile Borel during 1921–22.

In 1944, John Van Neumann and Morgenstern laid down the foundation stone of distinguished work in this area. His work appeared in the form of research paper **'Theory of games and economic behaviour'.** The work drew attention of many researchers. John Van Neumann made extensive contribution and as a result, he was known as the ***father of game theory***.

In competitive situation when decision making is very critical, the principles—minimax and maximin was extensively used by him in his work on the subject. Neumann played an important role. For example, if there are more than two or more intelligent opponents in which each opponent aspires to optimize his own decision at the cost of the other opponents. Typical examples of such situations include planning war tactics and fixing up war strategies, launching advertisement campaign against the competitor's advertisement, and especially in the interpretation of economic events, budget planning and related area.

9.1 Some Definitions Related to Game Theory

Before defining the terms, we understand that the competitive situation will arise if there are **two** or more participants playing with certain strategies and with pre-set and agreed amongst norms. The other class has no position here.

1. **Game:** It is an **activity** carried by two or more persons, having conflicting interest, who agreed to play abiding by certain pre-determined rules and limitations.
2. **Competitive games:** Games, by nature, are more or less competitive. In this context, we call them competitive if two or more intelligent persons or parties actively participate to dominate over the opponent's strategies and make attempts to win over.

   Each participant has a set of finite or infinite choice of actions available to him and any one of these choices can be used freely at a given time.

   Each possible combination of the strategies employed by the participating players will result in an outcome.
3. **Finite and infinite games:** The number of choice of actions/plans available to both the players is finite or infinite accordingly classify the games. [The different **choice** available to the players, who constantly aspires to dominate, is called ***strategy*** **on a given situation of game. Skilled plan with a winning objective employed on a given time may be known as strategy**. In some cases, the plans/strategies are called ***moves*** also. The context here may imply the next or the following situation also; hence we restrict its usage.]
4. **Two persons zero sum games:** In the competitive games when there are exactly **two** persons and the **outcome** of combinations of their strategies, calculated in terms of **monetary value,** is such that 'the loss of one player equals the gain of the other one'; describes the above definition, i.e. the total **sum value** of **gain** and **loss** to both the players at the end of results is zero.

   For example, two players, playing game following prefixed norms, adopt different strategies on a particular move, which results a winning situation for one player and a defeat for the other. The winning player evaluates his winning status in terms of monetary value (say ₹X) and claims the same amount ₹X from the other player. The winner gets ₹X amount while the opponent loses ₹X amount; means that the sum amount of gain of one player and the loss of the other player is zero. This is called *Two persons zero sum game*.
5. **Pay-off matrix:** In the case of two persons zero sum game, the matrix showing the monetary equivalent of different outcomes/results when each possible strategies are employed by both players, is called a *pay-off matrix*.
   * If there are ***m*** strategies available to Player A and ***n*** to the Player B, the related pay-off matrix is of order $m \times n$.

   [In this situation, the corresponding games are called *rectangular games*.]
   * It is customary to write the entries of the pay-off matrix in the **favour** of the first referred player, which appears on the **left side** to the pay-off matrix.

   [To write in favour of the other players, signs all entries of the pay-off matrix should be changed.]
   For example, for the two players A and B, each having two strategies, the pay-off matrix looks like

Player B

$$\begin{array}{ccc} & \text{Strategy} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \text{Player A} & \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{pmatrix} -3 & 5 \\ 8 & -2 \end{pmatrix} \end{array}$$

[The matrix is in favour of the Player A (as it appears on the left most part of the pay-off matrix.) The first entry '–3' means, if Player A decided the option a_1 and **without knowing that** if Player B opts for b_1, then the result is generated as a ***loss*** of ₹3 to the **Player A.** It implies that Player B receives ₹3. Similarly '5' means a gain of ₹5 to the Player A and loss of ₹5 to the Player B]

The general form of the pay-off matrix is shown here.

We note that a pay-off value that equals to gain of Player A equals the loss of Player B

General form of a pay-off matrix:

Pay-off matrix

Player B

$$\begin{array}{cc} & \begin{array}{ccccc} \text{Strategy} & b_1 & b_2 & \cdots & b_n \end{array} \\ \mathbf{A = Player\ A} \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_m \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & aij & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{m \times n} \end{array}$$

6. **Saddle point and value of the game:** In the context of the given pay-off matrix, we introduce following two terms.
   1. maximin, and
   2. minimax

Maximin is the value in context of the first Player A. The Player A has a finite set of all available strategies to him. When selected any one of these strategies and correspondingly reacted by the Player B, the combined act results into some output in terms of money value (or in terms of some physical quantity).

Player A can assure minimum amount by this choice of action. In this way, there is a set of minimum assured amount (we call it ***gain of A*** from a choice of action) corresponding to each action of the Player A.

The maximum value from this set of all minimum amounts is called ***maximin*** value. Maximin value is denoted as $\underline{\gamma}$

Player B, will always think and plan his strategy in such a way that in every choice of action, he can **minimize** his loss. Corresponding to selection of every choice of action of the Player B, when it is reacted by the Player A, in any one of possible acts available to him, the pay-off is called ***loss of the player B*.** We make a set of **maximum** amount of loss that can occur to the Player B corresponding to each one of available choice of actions. Player B would always aim at finding minimum of the set of **maximum loss**. He plays a defensive and relatively safer role. Minimum value of maximum losses is called *minimax* denoted as $\overline{\gamma}$.

A ***saddle point*** in the context of two persons zero sum game is that **position like (*x*, *y*)** in the pay-off matrix for which the cell showing maximum of the set of the row minimum is the same as that of showing minimum of the set of column maximum. Saddle point is a point of equilibrium of the game. It is a position or a cell with positional value that equals the maximin value or minimax value of the game. If there is a saddle point in the pay-off matrix then the value of corresponding position (entry in the corresponding cell) is called the ***value of the game***. It is denoted by γ .

At the saddle point, we have

maximin value = $\underline{\gamma}$ **= minimax value =** $\overline{\gamma}$ = γ **= value of the game**.

At this point, we note that some authors use the term ***strategic saddle point*.**

ILLUSTRATION 1

Find the following points from the following pay-off matrix.
(1) maximin value (2) minimax value (3) saddle point, if exists

Pay-off matrix

Player B

$$\begin{array}{ll} & \text{Strategy} \\ & a_1 \\ \text{A} = \textbf{Player A} & a_2 \\ & a_3 \\ & a_4 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{pmatrix} 4 & 6 & 8 & 5 \\ -4 & 11 & 10 & 9 \\ 9 & 7 & 10 & 6 \\ 6 & 3 & 8 & -3 \end{pmatrix} \end{array}$$

Part 1

If the Player A adopts the choice a_1, then he can assure of a pay-off of at least an amount worth, 4 = minimum of all the entries of the first row.

In the same way, we find the minimum entry of each row. At the end, we have a set of minimum of each row.

Set $S_1 = \{ 4, -4, 6, -3\}$ = each entry is the minimum output of corresponding row.

It is obvious that the Player A would like to maximize his minimum gains;

Maximum of the elements of the set S_1 = max. $\{4, -4, 6, -3\}$

$$\textbf{maximin value} = \underline{\gamma} = 6 \tag{1}$$

Part 2

If the Player B adopts the choice b_1, then he can assure of a pay-off of a maximum (loss) amount, 9 = maximum of all the entries of the first column.

In the same way, we find the maximum entry of each column. At the end, we have a set of maximum of each column.

Set $S_2 = \{9, 11, 10, 9\}$ = each entry is the maximum output of corresponding column.

It is obvious that the Player B would like to minimize his maximum losses, Minimum of the elements of the set S_2 = mini $\{9, 11, 10, 9\}$

$$\textbf{minimax value} = 9 = \overline{\gamma}\,. \tag{2}$$

From the Results (1) and (2), we conclude that maximin value is not equal to the minimax value.

Hence game has no saddle point.

ILLUSTRATION 2

From the following pay-off matrix, find the saddle point if it exists.

Pay-off matrix

Player B

$$\begin{array}{ll} & \text{Strategy} \\ & a_1 \\ \textbf{A = Player A =} & a_2 \\ & a_3 \\ & a_4 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{pmatrix} 4 & 6 & 8 & 5 \\ -4 & 11 & 10 & 4 \\ 12 & 7 & 10 & 6 \\ 6 & 3 & 8 & -3 \end{pmatrix} \end{array}$$

Solution

Part 1

For the Player A, we find the minimum guaranteed amount (gain) corresponding to each one of his strategy.

This set $S_1 = \{ 4, -4, 6, -3\}$ = row minimum

He would target to find the maximum of $\{4, -4, 6^*. -3\} = 6$ = maximin value = $\underline{\gamma}$

Part 2

For the Player B, we find the maximum amount (loss) corresponding to each one of his strategy.

This set $S_2 = \{12, 11, 10, 6\}$ = column maxima

He would target to find the minimum of $\{12, 11, 10, 6^*\} = 6$ = minimax value = $\overline{\gamma}$.

On comparing the last results of Part 1 and Part 2, we have

maximin value = $\underline{\gamma}$ = 6 = minimax value = $\overline{\gamma}$

This implies that the game has a saddle point.

Students should take care to note that the saddle point is the cell number like (x, y)

It is a point where maximin value = minimax value

In this case, saddle point = (3, 4) [We have marked this cell with 6 *]

Value of the game = the value in the cell (3, 4) = 6 = maximin = minimax

The procedure of finding a saddle point:

- We find minimum of each of each row, i.e. row minima.
- We find maximum of each column, i.e. column maxima.
- Now we find maximum of the row minima and call it maximin value = $\underline{\gamma}$
- We find minimum of the column maxima and call it miximax value = $\overline{\gamma}$
- If $\overline{\gamma} = \underline{\gamma} = \gamma$ then the game has a **saddle point**.
- The value at the saddle point is called the *value of the game* denoted as γ.
- If $\overline{\gamma} \neq \underline{\gamma}$ then, the game has no saddle point.
- In practice, students, probably as their teachers briefly find and show, mark maximum of each row values in the pay-off matrix itself. Then they mark maximum of column values in the pay-off matrix; and finally, where the two markings appear at a time, oh!, it is the answer. We show it but strongly suggest to follow the system what we have established in illustrations 1 and 2.

ILLUSTRATION 3

[We have solved the previous illustration; just followed the above discussed procedure. We put * for minimum row entry and # for maximum column entry.]

**Just watch sharply!!

$$\begin{array}{cc} & \textbf{Player B} \\ & \begin{array}{ccccc} \text{Strategy} & b_1 & b_2 & b_3 & b_4 \end{array} \\ \textbf{A = Player A} & \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{pmatrix} 4^* & 6 & 8 & 5 \\ -4^* & \#11 & \#10 & 4 \\ \#12 & 7 & \#10 & \#6^* \\ 6 & 3 & 8 & -3^* \end{pmatrix} \end{array}$$

* sign stands for minimum of each row; while # sign stands for maximum of each column.

Observe that at the cell **(3, 4)** two signs meet showing existence of the saddle point. The value of the game is **6.**

9.2 SADDLE POINT—A CLASSICAL APPROACH

We have another approach to define the **saddle point**.

Let $f(x, y)$ be a real function.

Pay-off matrix associated with a rectangular $m \times n$ game, we denote it as

$$A = (a_{ij})_{m \times n}; \text{ where } i \in \{1, 2, 3, \ldots, m\} \text{ and } j \in \{1, 2, 3, \ldots, n\}$$

This has a reference of m rows and n columns of the pay-off matrix.

Domain of definition of $f(\boldsymbol{X}, \boldsymbol{Y}) = D$

Where $D = \{(x, y) \mid (x, y) = (i, j) \text{ and } i \in \{1, 2, 3, \ldots, m\} \text{ and } j \in \{1, 2, 3, \ldots, n\}\}$

Now, keeping $i = 1$, change all j values from 1 to n.

Find all $f(1, j)$ and pick up minimum value. [This relates to minimum value for the first row of pay-off matrix.]

Now, $i = i + 1$, i.e. $i = 2$; find all $f(2, j)$ for $j = 1$ to n. Pick the minimum value.

[This relates to minimum value of the second row.]

We continue this process until all i values 1, 2, …, m are taken up.

At the end, we have a set of m values like $f(1, j), f(2, j), \ldots, f(m, j)$

Find maximum of $\{f(1, j), f(2, j), \ldots, f(m, j)\}$ = maximin value = $\underline{\gamma} = \boldsymbol{f(x_0, j)}$

[for some x_0 from 1 to m] (3)

Now, keeping $j = 1$, change all i values from 1 to m.

Find all $f(i, 1)$ and pick maximum value. [This relates to maximum value for the first column of pay-off matrix.]

Now, $j = j + 1$, i.e. $j = 2$; find all $f(i, 2)$ for $i = 1$ to m. Pick the maximum value. [This relates to maximum value of the second column.]

We continue this process until all j values 1, 2, …, n are taken up.

At the end, we have a set of n values like $f(i, 1), f(i, 2), \ldots, f(i, n)$

Find minimum of $\{f(i, 1), f(i, 2), \ldots, f(i, n)\}$ = minimax value = $\overline{\gamma} = \boldsymbol{f(i, y_0)}$

[for some y_0 from 1 to n.] (4)

If the game has a saddle point at (x_0, y_0) then, $\underline{\gamma} = \boldsymbol{f(x_0, y_0)} = \overline{\gamma}$

We have following two important deductions:

1. If maximin value and minimax value of pay-off matrix corresponding to a game exists then maximin value = $\underline{\gamma} = f(x_0, j) \leq$ minimax value = $\overline{\gamma} = f(i, y_0)$ for some x_0 from 1 to m and some y_0 from 1 to n.
2. If the game has a saddle point (x_0, y_0) then, maximin value = $\underline{\gamma} = f(x_0, j) = f(x_0, y_0)$ = minimax value = $\overline{\gamma} = f(i, y_0)$

ILLUSTRATION 4

Solve the following game to find the saddle point.

Player B

$$\textbf{Player A}\quad \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \\ \begin{pmatrix} 4 & 0 & 1 & 7 & -1 \\ 0 & -3 & -5 & -6 & 5 \\ 3 & 2 & 2 & 4 & 3 \\ -6 & 1 & -2 & 0 & -5 \end{pmatrix} \end{array}$$

Solution

We find row minima and column maxima.

$$\begin{array}{cc} & \textbf{Player B} & \textbf{Row Minima} \\ & \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} & \\ \textbf{Player A} \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} & \begin{pmatrix} 4 & 0 & 1 & 7 & -1 \\ 0 & -3 & -5 & -6 & 5 \\ 3 & 2 & 2 & 4 & 3 \\ -6 & 1 & -2 & 0 & -5 \end{pmatrix} & \begin{array}{c} -1 \\ -6 \\ 2 \\ -6 \end{array} \\ \text{Column Maxima} & \begin{array}{ccccc} 4 & 2 & 2 & 7 & 5 \end{array} & \end{array}$$

Now, maximin value $= \underline{\gamma}$ = Maxi [row minima]

= Maxi [–1, –6, 2, –6]

$\therefore$ $\underline{\gamma} = 2$ (which occurs in third row)

minimax value $= \overline{\gamma}$ = mini [column maxima]

= mini [4, 2, 2, 7, 5]

$\overline{\gamma} = 2$

(which occurs in **second** and **third** columns.)

Now $\underline{\gamma} = 2 = \overline{\gamma}$

$\therefore$ The game has a **saddle point** at **(3, 2)**.

The value of the game at the saddle point (3, 2) is **2**.

Also, the game has a **saddle point at (3, 3)**.

This game has **two** saddle points—the points **(3, 2)** and **(3, 3)**.

The value of the game at both of the points is 2.

9.3 Mixed Strategy and Pure Strategy

We consider two persons zero sum game. We have already observed that if the game has no saddle point then it implies that in order to find the value of the game we have to take some other mathematical approach. Let the Player A possess ***m*** different strategies denoted as $a_1, a_2, \ldots, a_m$. Let us assume that he plays only one at a time. He selects any one to play with probabilities $p_1, p_2, p_3, \ldots, p_m$; each $p_i \geq 0$ and $\sum_{i=1}^{m} p_i = 1$.

In this case, the vector $\mathbf{P} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix}$ is called a *mixed strategy* for the Player **A**.

In the same way, we think of the Player B having ***n*** different strategies $b_1, b_2, \ldots, b_n$ with corresponding probabilities, say $q_1, q_2, \ldots, q_n$. [each $q_j \geq 0$ and $\sum_{j=1}^{n} q_i = 1$.

In this case, the vector $\mathbf{Q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$ is called a *mixed strategy* for the Player **B**.

The pay-off matrix with above approach of mixed strategy is as follows:

Pay-off matrix

Player B

$$\begin{array}{cccccccc} & \text{Strategy} \rightarrow & & b_1 & b_2 & b_3 & \cdots & b_n \\ & \downarrow & \text{Probability} \rightarrow & q_1 & q_2 & q_3 & \cdots & q_n \\ & & \downarrow & & & & & \\ & a_1 & p_1 & \multicolumn{5}{c}{\multirow{5}{*}{$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$}} \\ \textbf{Player A} & a_2 & p_2 & \\ & a_3 & p_3 & \\ & \vdots & \vdots & \\ & a_m & p_m & \end{array}$$

At this stage, we note that

1. Player A has $a_1, a_2, \ldots, a_m$ strategies and he plays each strategy with probability $p_1, p_2, \ldots, p_m$ with $p_i \geq 0$ ($i = 1$ to m) and $\sum_{i=1}^{m} p_i = 1$.
2. Player B has $b_1, b_2, \ldots, b_n$ strategies and he plays each strategy with probability $q_1, q_2, q_3, \ldots, q_n$ with $q_i \geq 0$ ($j = 1$ to n) and $\sum_{i=1}^{n} q_j = 1$.
3. The matrix is a pay-off matrix written in favour of player A. This pay-off matrix $P = (a_{ij})_{m \times n}$

In the case of a pay-off matrix of the order 2×3, we show mixed strategy approach by the following example.

As a consequence of this definition, we add an important point. In the mixed strategy of the Player A, the vector P = $\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix}$ is such that some $p_j = 1$ and all other components are zero, then the vector P is called a pure strategy for the player A.

In the same way, we can define pure strategy for the player B.

Mixed strategy vector for the Player B is Q = $\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$, in this a particular component say, q_k, for only one k is one; i.e. $q_k = 1$ and remaining components of Q are zero, then the vector Q is called a pure strategy for the Player B.

ILLUSTRATION 5

This illustration shows that both players A and B adopt mixed strategies. The Player A has three strategies a_1, a_2, and a_3 with corresponding probabilities p_1,p_2, and p_3. The player B follows two strategies b_1, and b_2 with respective probabilities q_1 and q_2.

$$\begin{array}{lccc} & & \textbf{Player B} & \\ \text{Strategy} \rightarrow & & b_1 & b_2 \\ \downarrow \quad \text{Probability} \rightarrow & & q_1 & q_2 \\ & \downarrow & & \\ a_1 & p_1 & & \\ \textbf{Player A} \quad a_2 & p_2 & & \\ a_3 & p_3 & & \end{array} \begin{pmatrix} 3 & 2 \\ -1 & 4 \\ 7 & 8 \end{pmatrix}$$

for the Player A each $p_i \geq 0$ and $p_1 + p_2 + p_3 = 1$, and
for the Player B each q_j ³ 0 and $q_1 + q_2 + q_3 = 1$

9.3.1 Mathematical Expectation

Now, we introduce a very important and practically useful concept of *mathematical expectation*. In this case, we have already discussed the concept of mixed strategies adopted by both players A and B.

Let $\mathbf{A} = (a_{ij})_{m \times n}$ be a pay-off matrix. Let **X** and **Y** represent mixed strategies for the players A and B.

We define *mathematical expectation* as E(**X, Y**) as a real value associated with the pay-off matrix; it is given as follows.

$$E(\mathbf{X}, \mathbf{Y}) = \sum_{i=1}^{i=m} \sum_{j=1}^{i=n} x_i a_{ij} y_j = X'AY \text{ where } X' \text{ is the transpose matrix of } X \tag{6}$$

Once this is defined, we can technically write and prove also a mathematical fact that for a pay-off matrix corresponding to a game of m available strategies for the Player A and n available strategies for the player B,

maximin E(X, Y) and minimax E(X, Y) exist and they are equal.

This is known as fundamental principle of rectangular games.

ILLUSTRATION 6

For the given pay-off matrix, find the saddle point if it exists or else find the mixed strategies for the players.

$$\begin{array}{cc} & \textbf{Player B} \\ \textbf{Player A} & \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} \end{array}$$

Solution

Let us assume that the Player A has two choices of action a_1 and a_2 available to him and he plays each strategy with probability p_1 and p_2. Both p_1 and p_2 are ≥ 0 and $p_1 + p_2 = 1$

The Player B has two choices of action strategy b_1 and b_2; he plays them with corresponding probabilities q_1 and q_2. q_1, and $q_2 \geq 0$ and $q_1 + q_2 = 1$

$$\begin{array}{lccc} & & \textbf{Player B} & \\ \text{Strategy} \rightarrow & & b_1 & b_2 \\ \downarrow \quad \text{Probability} & & q_1 & q_2 \\ & \downarrow & & \\ a_1 & p_1 & & \\ \textbf{Player A} \quad a_2 & p_2 & & \end{array} \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$$

S_1 = set of row minima = {2, –1}

maximin of S_1 = max {2, –1} = 2 = $\underline{\gamma}$

S_2 = set of column maxima = {3, 4}

minimax of S_2 = mini {3, 4} = 3 = $\overline{\gamma}$

Note that, maximin = 2 ≤ minimax = 3

Thus the game has no saddle point.

Let the Player A plays a_1 and a_2 with probabilities p_1 and p_2. Both p_1 and p_2 are ≥ 0 and $p_1 + p_2 = 1$. Let the Player B plays b_1 and b_2 with probabilities q_1 and q_2.

$$q_1 \text{ and } q_2 \geq 0 \text{ and } q_1 + q_2 = 1$$

$$E(X, Y) = (p_1, p_2)\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$E(X, Y) = 3p_1q_1 - p_2q_1 + 2p_1q_2 + 4p_2q_2$, with $p_1 + p_2 = 1$, $q_1 + q_2 = 1$ and each of p and q are non-negative.

Solving this in terms of p_1 and q_1, we have

$$E(X, Y) = 6(p_1 - 5/6)\,(q_1 - 1/3) + 7/3$$

If the Player A makes a choice of the act a_1 with the probability $p_1 = 5/6$, then he can ensure the gain at least equal to 7/3. He cannot go beyond this amount as he remains in check by the Player B who selects the choice of action b_1 with probability $q_1 = 1/3$. As $p_1 + p_2 = 1$, $q_1 + q_2 = 1$, we have $p_2 = 1/6$ and $q_2 = 2/3$

From these values, we can write mixed strategy vector ***X*** and ***Y***.

$$X_0 = (p_1, p_2)' = (5/6, 1/6)' \quad Y_0 = (q_1, q_2)' = (1/3, 2/3)'$$

Saddle point is (X_0, Y_0) and the value of the game is = 7/3

9.4 Dominance Principles

These principles are very natural and intuitive that without knowing them technically, we would like to follow them with an optimal policy of winning over the opponent. Solving the examples, we will need to use these principles very often and hence we write them as P_1, P_2, P_{3+}, and P_4.

Immediate advantages of applying dominance principles are very useful in reducing the size of a given pay-off matrix. Any one or more successive application of these principles in most of the cases helps reducing the size of the pay-off matrix. We have shown an illustration wherein successive application of dominance principles ends up at the saddle point and it is the value of the game.

We state these principles and make them clear by giving illustrations.

P_1: In a $m \times n$ pay-off matrix, if all the elements of ith row are greater than or equal to the corresponding elements of jth row, then we say that the ith row dominates over jth row. As a result, we drop the jth row from the pay-off matrix.

P_2: In a $m \times n$ pay-off matrix, if all the elements of ith column are less than or equal to the corresponding elements of jth column, then we say that the ith column dominates over jth column. As a result , we drop the jth column from the pay-off matrix.

We give an illustration, which will intuitively clarify the two dominance principles.

Consider a pay-off matrix, $\mathbf{A} = \begin{bmatrix} 2 & 5 & -5 & 6 \\ 1 & 6 & 3 & 8 \\ 5 & 7 & 0 & 9 \end{bmatrix}$

Observe that the third row dominates over the first row; we drop the first row from the pay-off matrix. It is an application of principle the P_1 principle.

Now, the pay-off matrix is

$$\mathbf{A} = \begin{bmatrix} 1 & 6 & 3 & 8 \\ 5 & 7 & 0 & 9 \end{bmatrix}$$

Now, observe the first and the second columns; first column dominates over the second column; we drop the second column from the pay-off matrix. It is an application of P_2 principle.

The result is,

pay-off matrix, $\mathbf{A} = \begin{bmatrix} 1 & 3 & 8 \\ 5 & 0 & 9 \end{bmatrix}$

[You can still think of dropping the third column in the light of the first or the second column. We drop the third column, i.e.

pay-off matrix, $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 5 & 0 \end{bmatrix}$

Now, we introduce two more dominance principles P_3 and P_4.

P_3: In a pay-off matrix, if the average of corresponding entries of any two rows (say *p*th and *q*th) is greater than or equal to the corresponding entries of any other row (say *r*th), then those two rows p^{th} and *q*th dominate over the *r*th row; we drop *r*th row from the pay-off matrix.

***P_4*:** In a pay-off matrix, if the average of corresponding entries of any two columns (say *p*th and *q*th) is less than or equal to the corresponding entries of any other column (say *r*th), then those two columns *p*th and *q*th dominate over the *r*th column; we drop *r*th column from the pay-off matrix.

In practice, P_3, and P_4 are used most often. We give illustrations explaining each case.

ILLUSTRATION 7

Consider the pay-off matrix and apply dominance principles.

Pay-off matrix, $\mathbf{A} = \begin{bmatrix} 2 & 5 & -5 & 9 \\ 5 & 7 & 0 & 11 \\ 8 & 10 & 11 & -6 \end{bmatrix}$

*1 If you compare elements of Row 1 and Row 2; you will find that each element of Row 2 is greater than the corresponding elements of Row 1. (This is P_3 stated above)

As a result, we drop the first row from the pay-off matrix.

Now, the resultant matrix is $\begin{bmatrix} 5 & 7 & 0 & 11 \\ 8 & 10 & 11 & -6 \end{bmatrix}$

Now, P_1 is not applicable.

*2 Now, we compare the first and the second columns. Elements of the second column are greater than corresponding elements of the first column.

We apply P_4 ; and drop the second column.

Now, the resultant matrix is $\begin{bmatrix} 5 & 0 & 11 \\ 8 & 11 & -6 \end{bmatrix}$

We cannot apply any more dominance. Now, we look for finding saddle point if it exists.

ILLUSTRATION 8

Apply dominance principles and find the saddle point.

[This illustration will establish the importance of dominance principles and very effectively lead you to the saddle point.]

Pay-off matrix, $A = \begin{bmatrix} 2 & 3 & 7 & 10 \\ 1 & 4 & 10 & -1 \\ 3 & 1 & 5 & 11 \\ 3 & 7 & 12 & 11 \end{bmatrix}$

*1. Watch distinctly; the third row dominates over the second row. We drop the second row.
*2. Now, the second column dominates over the third column, We drop the third column.
*3. Now, the average of the second row and the third row dominates the first row. We drop the first row.
*4. Now, we have a pay-off matrix of two rows only.
In this matrix, the second row dominates over the first row. We drop the first row.
*5. Now, we have a single row with only three elements.

Pay-off matrix, = A = (3 7 11)

The second and the third columns dominate over the first column. We drop these two columns.

We have the pay-off matrix, **A** = (3)

This single element 3 has its position in the cell (4, 1); it is a saddle point of the game. The value of the game is at the saddle point.

Value of the game = 3.

[This can be verified by taking maximum of row minimum and minimum of column maximums.]

9.5 SOLUTION PROCEDURES

There are different methods to solve the game theory problems. We have one point plan—to determine the best strategies, for both players A and B, based upon the maximin and minimax criteria for optimality. We are interested in finding a **rationale** for each player. This will restrict the Player A to constantly win with a very high margin or equivalently the Player B to make a high loss.

There are different methods to solve the game theory problems. If the game has no saddle point then we have mixed strategies for each player. Once we are to use the notion of finding and applying mixed strategies; we first possibly use the concept of principles of dominance and try to reduce the pay-off matrix as low in order (size) as possible. We can possibly reduce the pay-off matrix in size in the order of;

1. 2×2
2. $m \times 2$
3. $2 \times n$

4. $m \times n$ size with m and n as least possible integers; may be $1 \times n$ or $m \times 1$.

In cases where it reduces to the size 2×2; we have;

1. algebraic method, and
2. graphical method

to solve the problem.

These two methods help us finding mixed strategy vectors for each player and the value of the game.

In the cases of $m \times 2$ or $2 \times n$ size pay-off matrix we can apply graphical method or sub-game methods.

There is also a known and classical approach; which converts the given matrix into its equivalent linear programming problem and finds the optimal value of the game.

We make a list here; but students generally follow the time-efficient and 'simple' method, they have practiced often.

1. Finding saddle point and value of game.
   or else; apply dominance and follow
2. algebraic method
3. sub-game method
4. graphical method
5. converting to equivalent LPP method

1. We have already seen the most known method of finding maximum of row-minimum and minimum of column maximum values and then in case of these values are equal; we find saddle point and the value of the game.

If ***saddle point does not exist then we try other methods.***

2. **Algebraic method:** Let us assume that after applying the dominance principles the pay-off matrix has reduced to a matrix of order 2×2.

$$\begin{array}{llll} & & & \textbf{Player B} \\ & \text{Probability} \rightarrow & & \begin{matrix} q_1 & q_2 \end{matrix} \\ & \downarrow & \text{Action} \rightarrow & \begin{matrix} b_1 & b_2 \end{matrix} \\ & & \downarrow & \\ \text{Pay-off matrix, } \mathbf{A} & p_1 & a_1 & \multirow{2}{*}{$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$} \\ \textbf{Player A} & p_2 & a_2 & \end{array}$$

1. We know that $p_1 + p_2 = 1$; $q_1 + q_2 = 1$ and $p_1, p_2, q_1, q_2, > 0$. Any one of these four variables when become zero means the corresponding player has a fixed policy.
2. Now, by the notion of expectation that we have seen earlier, expectation of the Player A when the Player B adopts the choice b_1 to react;

   i.e. $$E(A, b_1) = a_{11}\, p_1 + a_{21}\, p_2 \tag{1}$$

   Similarly, we define

   $$E(A, b_2) = a_{12}\, p_1 + a_{22}\, p_2 \tag{2}$$

   In the same way, we define expectation of the Player B when the Player A decides to react to the Player B's action by playing the choice a_1.

   i.e. $$E(B, a_1) = a_{11}\, q_1 + a_{21}\, q_2 \tag{3}$$

   Similarly,

   $$E(B, a_2) = a_{21}\, q_1 + a_{22}\, q_2 \tag{4}$$

   We know that maxi(min $E(\boldsymbol{X}, \boldsymbol{Y})$) = $\underline{\gamma}$ mini(max $E(\boldsymbol{X}, \boldsymbol{Y})$) = $\overline{\gamma}$ and $\underline{\gamma} \le \gamma \le \overline{\gamma}$ where γ is the value of the game.

We have, under this situation, E(A, b_1) and E(A, b_2) $\geq \gamma$ (5)

and E(B, a_1) and E(B, a_2) $\leq \gamma$ (6)

Having a reference to the above equation to find the mixed strategy vertex, let us consider inequalities to equalities.

From (5) $a_{11} p_1 + a_{21} p = \gamma = a_{12} p_1 + a_{22} p_2$

$\therefore \quad (a_{11} - a_{12}) p_1 = (a_{22} - a_{21}) p_2$

$\therefore \quad \dfrac{p_1}{p_2} = \dfrac{a_{22} - a_{21}}{a_{11} - a_{12}}$ With $a_{11} \neq a_{12}$ using; $p_1 + p_2 = 1$; we have

$$\frac{p_1}{1} = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \tag{7}$$

Now $p_2 = 1 - p_1$ and on simplification, we get

$$p_2 = \frac{a_{11} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \tag{8}$$

Similarly we can find q_1 and q_2 as

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \tag{9}$$

and

$$q_2 = \frac{a_{11} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \tag{10}$$

Substituting p_1 and p_2 (from [7] and [8]) or q_1 and q_2 (from [9] and [10]) in any one equation of the set (5) or (6); we can find the value of the game.

$$\text{Value of the game} = \gamma = \frac{a_{11}.a_{22} - a_{12}.a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \tag{11}$$

Comment: If you remember the order of elements in the pay-off matrix, then by close observation you will remember the pattern of results of p_1, p_2, q_1 and q_2 and finally the value of υ given by (11)].

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Now, observe the formula for p_1, q_1 and γ and remember the pattern.
(note that $p_2 = 1 - p_1$ and $q_2 = 1 - q_1$)

3. **Sub-game method:** This is a method which sub-divides the given pay-off matrix into all possible 2×2 sub-matrices . Then it finds the feasible solutions in each case. In some cases there are saddle points and it becomes simple to find the value of the game. In some cases where there is no saddle point, we follow algebraic method to find corresponding mixed strategies for each player and then finally the value of the sub-game on hand.

   Now, we have to make a comprehensive study of all the different types of values of the sub-games so far obtained by all possible sub-divisions of the original pay-off matrix.

From this study, we may, without any bias or reservation, take the decision about the sub-matrix of right decision and hence take our conclusions about the right act leading to the best choice of action for both players.

We will take illustrations to clear this concept.

4. **Graphical method of solving a game matrix**

Dominance principles are very useful and using the same, we can reduce the size of the game matrix. The point here is, if the pay-off matrix is of the size $m \times 2$ or $2 \times n$ or 2×2, then we can find the graphical solution.

In the game matrix of size $m \times 2$, the Player B has two available strategies b_1 and b_2 which he plays with probabilities q_1 and q_2 respectively. We have, $0 \le q_1, q_2 \le 1$ and $q_1 + q_2 = 1$.

In the game matrix of size $2 \times n$, the Player A has only two available strategies a_1 and a_2 which he plays with probabilities p_1 and p_2 respectively. We have, $0 \le p_1, p_2 \le 1$ and $p_1 + p_2 = 1$.

In the case of 2×2 game matrix, both the players A and B have two strategies (a_1 and a_2) and (b_1 and b_2) with probabilities (p_1 and p_2) and (q_1 and q_2) respectively. Well, in all these cases, we know that $p_1, p_2, q_1, q_2 \ge 0$ with $p_1 + p_2 = 1$ and $q_1 + q_2 = 1$.

How to find graphical solution? Given a pay-off matrix, we apply dominance principles and try to reduce the size of the pay-off matrix. Then we find (1) maximin, (2) minimax and if both are equal, saddle point exists and we can find the value of the game.

If there is no saddle point then we adopt mixed strategy approach to solve the game problem. The steps are as follows:

1. In the case 2×2 game matrix having no saddle point, players A and B have two strategies (a_1 and a_2) and (b_1 and b_2) with probabilities (p_1 and p_2) and (q_1 and q_2) respectively. Well, in all these cases, we know that $p_1, p_2, q_1, q_2 \ge 0$ with $p_1 + p_2 = 1$ and $q_1 + q_2 = 1$.
   Consider the following pay-off matrix.

$$
\begin{array}{lccc}
 & & & \textbf{Player B} \\
 & \text{Strategy} \rightarrow & & b_1 \quad b_2 \\
 & \downarrow & \text{Probability} \rightarrow & q_1 \quad q_2 \\
 & & \downarrow & \\
 & a_1 & p_1 & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\
\textbf{Player A} & a_2 & p_2 &
\end{array}
$$

2. Find E(A, b_1) ; expected gain of the Player A when the Player B plays the move b_1.
   E(A, b_1) $= a_{11} p_1 + a_{21} p_2$ and similarly E(A, b_2) $= a_{12} p_1 + a_{22} p_2$
   Now, $p_2 = 1 - p_1$.
3. Substitute $1 - p_1$ for p_2 in the two results for E(A, b_1) and E(A, b_2)
4. Now, E(A, b_1) $= a_{21} + (a_{11} - a_{21})p_1$, and E(A, b_2) $= a_{22} + (a_{12} - a_{22})p_1$ We know that $0 \le \pi_1 \le 1$.
5. Consider a scaled-horizontal line segment showing for $p_1 = 0$ to $p_1 = 1$.
   From the point showing $p_1 = 0$, draw a vertical scaled-line and the same way the another vertical line from the point showing $p_1 = 1$.
6. Put $p_1 = 0$ in E(A, b_1) and $p_1 = 1$ in E(A, b_1); find two points and by joining these two points ,we can draw **a line for $E(A, b_1)$**
   Similarly, we draw a line for E(A, b_2).
7. The point of intersection of these two lines E(A, b_1) and E(A, b_2) will give us a point which shows maximin value. Projecting this point on p_1 scale, we get the corresponding value of p_1; and projecting this point on vertical scale, we get value of the game.

Comment

1. We can follow the same procedure for the Player B. In this case, we take probabilities for the Player B. We find E(B, a_1) and E(B, a_2) and draw lines. The point of intersection of these two lines E(B, a_1) and E(B, a_2) will give us the minimax point.
   We can find the value of q_1 from the horizontal scale. [Instead, of p_1 scale, we have to take q_1 scale for the Player B.]
2. In case of $m \times 2$ pay-off matrix, we assume probabilities q_1 and $q_2 = (1 - q_1)$.
   Then we calculate E(A, b_1), E(A, b_2), … and follow the same procedure as described in Comment 1.
3. In case of $2 \times n$ pay-off matrix, we assume probabilities p_1 and $p_2 = (1 - p_1)$.
   Then we calculate E(B, a_1), E(B, a_2), …. and follow the same procedure as described in general procedure.
4. In case of $m \times 2$ type pay-off matrix, we find maximin value. It is the highest point of the lower envelope.
5. In case of $2 \times n$ type pay-off matrix, we find minimax value. It is the lowest point of the upper envelope.
6. A saddle point is the point of intersection of two lines, (In some cases more than two lines may be concurrent at the saddle point.) we can easily see and identify the additional lines which do not take part in searching or finding the saddle point.

We take some illustrations to understand these points.

ILLUSTRATION 9

Find the graphical solution of the following game. [Take both the players A and B into consideration.] The pay-off matrix is as follows.

Pay-off Matrix, $\mathbf{A} = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}$

Solution

Using the conventions, the game matrix can be written as follows (see that the game has **no** saddle point).

Part 1 Let us solve the problem for Player A

$$\text{E(A, } b_1) = 5p_1 + 1p_2$$
$$= 5p_1 + (1 - p_1) = 4p_1 + 1 \tag{1}$$

Also
$$\text{E(A, } b_2) = -2p_1 + 2p_2$$
$$= -2p_1 + 2(1 - p_1) = -4p_1 + 2 \tag{2}$$

In both the cases (1) and (2), we take $0 \le p_1 \le 1$.

For E(A, b_1) = $4p_1 + 1$, on $p_1 = 0$, E(A, b_1) = 1 (0, 1) is one point.

On $p_1 = 1$, E(A, b_1) = 5 (1, 5) is the another point to draw that line.

E(A, b_1) = $4p_1 + 1$ (1) passes through the points (0, 1) and (1, 5)

In the same way, for E(A, b_2) = $-4p_1 + 2$ gives

for $p_1 = 0$, E(a, b_2) = 2; point (0, 2)

for $p_1 = 1$, E(A, b_2) = –2; point (1, –2)

E(A, b_2) = $-4p_1 + 2$ (2) passes through the points (0, 2) and (1, –2)

Lines (1) and (2) intersect at the point P.

x-coordinate of the point P is $p_1 = 1/8$.

The highest point of the lower envelope, i.e. the point *P* is the point—**maximin** point.
The value of the game at *P* is 3/2 (shown in Figure 9.1).

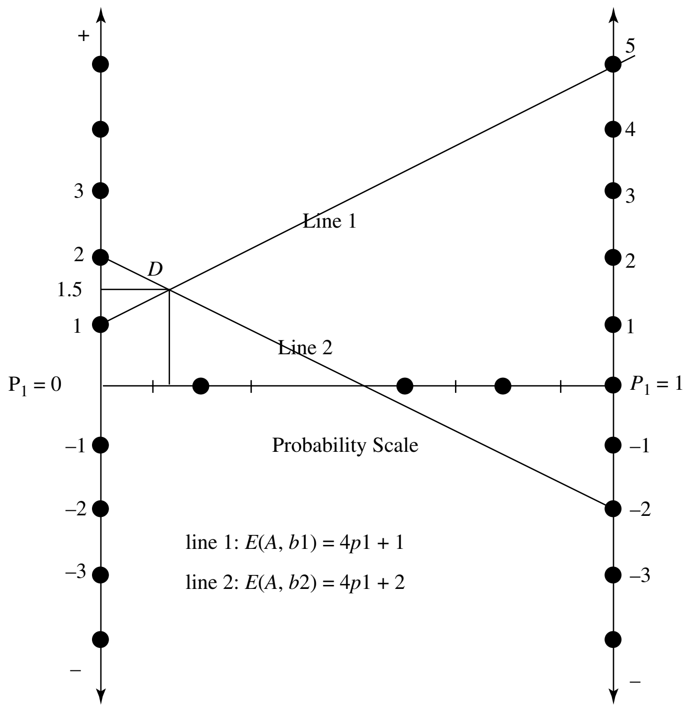


Fig. 9.1

Part 2

Now, we find graphical solution for Player B.

Problem for Player B is to find minimax value

$$E(B, a_1) = 5q_1 - 2q_2 = 5q_1 - 2\,(1 - q_1) = 7q_1 - 2 \tag{2}$$

for $q_1 = 0$, point is (0, –2)}

for $q_1 = 1$, point is (1, 5)}

Now $E(B, a_2) = 1q_1 + 2q_2 = 1q_1 + 2\,(1 - q_1) = 2 - q_1$ (3)

for $q_1 = 0$, point is (0, 2)}

for $q_1 = 1$, point is (1, 1)}

We draw graph for each E(B, a_1) and E(B, a_2). The set of points are in Eqs (2) and (3).

We have the following graph (Figure 9.2).

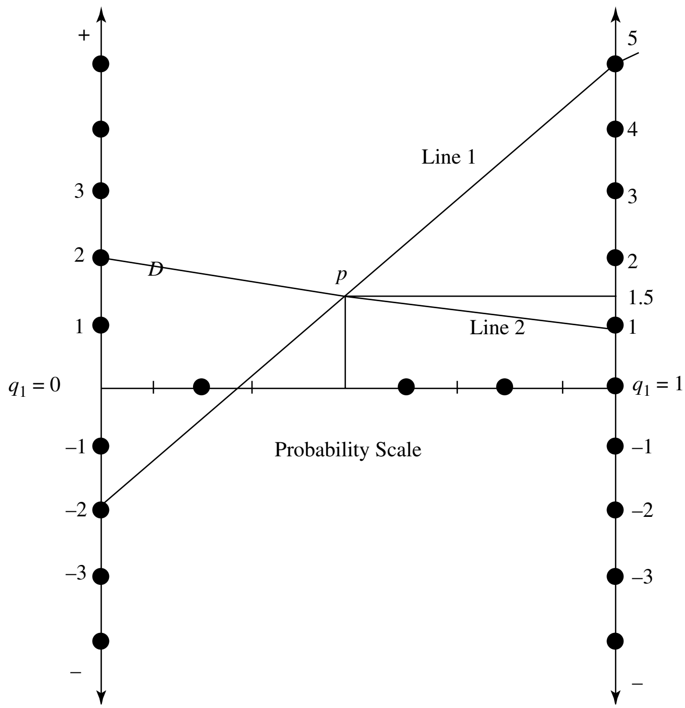


Fig. 9.2

From Figure 9.2 for the Player B, we can find the minimax value for the Player B.
We find the **lowest point** of the **upper envelope** (region).
The minimax point, the point P is the lowest point of upper region.

$q_1 = 1/2$ and the value of the game = 3/2.

ILLUSTRATION 10

Find the graphical solution of the following pay-off matrix,

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 5 & 2 \\ 1 & 3 \\ 2 & 1 \end{bmatrix}$$

Solution

The game matrix is of the size **4 × 2** having **no saddle point**. It means that Player B has two strategies b_1 and b_2 which he plays with probabilities q_1 and q_2. ($q_1, q_2 \geq 0$ and $q_1 + q_2 = 1$).

We find expected loss of the B under each move of the Player A.

$$\begin{aligned} E(B, a_1) &= 2q_1 - 1q_2 \\ &= 2q_1 - (1 - q_1) = 3q_1 - 1 \end{aligned} \tag{1}$$

$$\begin{aligned} E(B, a_2) &= 5q_1 + 2q_2 \\ &= 5q_1 + 2(1 - q_1) = 3q_1 + 2 \end{aligned} \tag{2}$$

$$E(B, a_3) = 1q_1 + 3q_2$$
$$= 1q_1 + 3\ (1 - q_1) = -2q_1 + 3 \quad (3)$$
$$E(B, a_4) = 2q_1 + 1q_2$$
$$= -2q_1 + 1\ (1 - q_1) = 1 - 3q_1 \quad (4)$$

The point for each line is obtained by putting $q_1 = 0$ and $q_1 = 1$.

The corresponding sets of points are (0, –1) and (1, 2), (0, 2) and (1, 5), (0, 3) and (1, 1) and the last (0, 1) and (1, –2).

Now we graph the points (Figure 9.3).

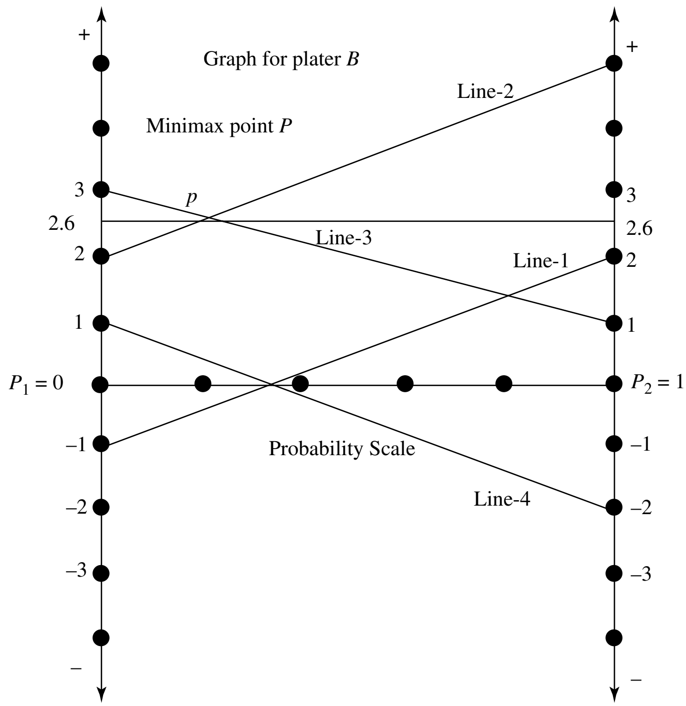


Fig. 9.3

The problem for the Player B is to find the minimum of the maximum expected losses, i.e. he aims for **minimax** value. This value is the **lowest point** of **upper region**. This point is P and corresponding values are $q_1 = 1/5$ and the value of y is 2.6.

$$q_1 = 1/5,\ q_2 = 4/5,\ y = 2.6.$$

[Lines (1) and (4) are **dominated by** other lines. Lines (1) and (4) do not contribute in making the decision].

ILLUSTRATION 11

Solve the following game graphically.

Pay-off matrix, $\mathbf{A} = \begin{bmatrix} -1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 7 \end{bmatrix}$

Solution

The game is of order **2 × 4** and it has **no saddle point**. Following the standard pattern of convention, let the Player A plays with strategies a_1 and a_2 with probabilities p_1 and p_2 with $p_1, p_2 \geq 0, p_1 + p_2 = 1$.

$$\text{Pay-off matrix, } \mathbf{A} = \begin{bmatrix} -1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 7 \end{bmatrix}$$

Now, we find the expected gain of A the Player under each strategy (b_1 and b_2) of the Player B.

$$E(\text{A}, b_1) = -1p_1 + 2p_2 = -1p_1 + 2(1 - p_1) = 2 - 3p_1 \quad (1)$$

$$E(\text{A}, b_2) = 2p_1 + 1p_2 = 2p_1 + 1(1 - p_1) = p_1 + 1 \quad (2)$$

$$E(\text{A}, b_3) = 3p_1 + 5p_2 = 3p_1 + 5(1 - p_1) = 5 - 2p_1 \quad (3)$$

$$E(\text{A}, b_4) = 4p_1 + 7p_2 = 4p_1 + 7(1 - p_1) = -3p_1 + 7 \quad (4)$$

Taking extreme values of $p_1 = 0$ and 1, all these four lines give the points (0, 2) and (1, –1), (0, 1) and (1, 2), (0, 5) and (1, 3) and (0, 7) and (1, 4).

These points can now be plotted to find **maximin** (Figure 9.4).

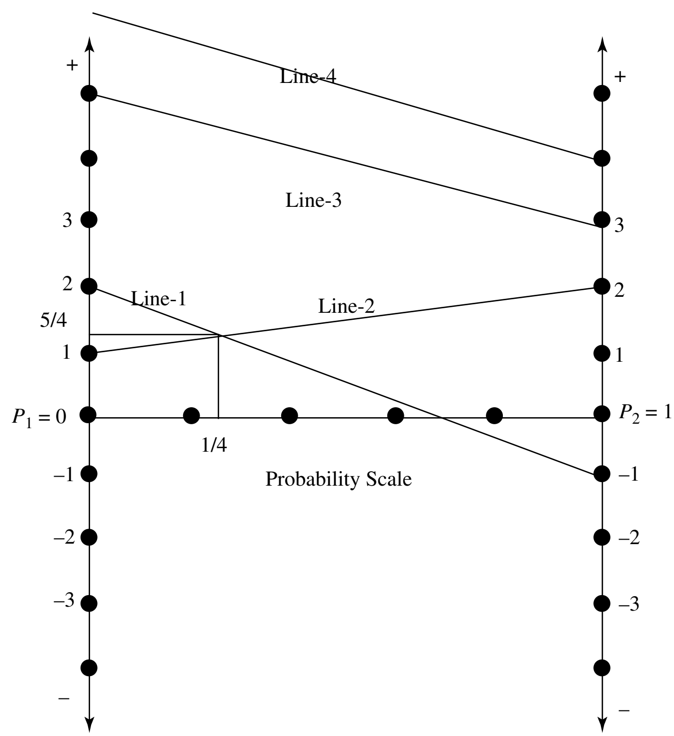


Fig. 9.4

Note that in Figure 9.4 lines (3) and (4) do not play any role in determining maximum value. Maximin point is the **highest point** of **lower envelope** (point P).

From the graph, $p_1 = 1/4$, $p_2 = 3/4$ and the value of the game is 5/4.

i.e. $p_1 = 1/4$, $y = 5/4$.

This can be obtained by comparing Eqs (1) and (2)

i.e. $2 - 3p_1 = p_1 + 1 \quad p_1 = 1/4$ and putting $p_1 = 1/4$ the value of the game $= 2 - 3\,(1/4) = 1/4 + 1 = 5/4$

5. Mixed Strategy Game Problem and its equivalence to a linear programming problem (LPP)

First, we show an equivalence of a system of a pay-off matrix corresponding to a game played by two players following mixed strategies. The game has no saddle point.

Let us assume that the Player A has m available choices of actions $a_1, a_2, ..., a_m$.

Let $p_1, p_2, \ldots, p_m$ represent corresponding probabilities of these actions. We have two facts; each $p_i \geq 0$, and $p_1 + p_2 + \cdot ... + p_m = 1$

Let us assume that the player B has 'n' available choices of actions $b_1, b_2, ..., b_n$.

Let $q_1, q_2, ..., q_n$ represent corresponding probabilities of these actions. We have two facts; each $q_i \geq 0$, and $q_1 + q_2 + \cdots + q_n = 1$

The complete system of the game problem described above is represented as.

$$\begin{array}{ccc} & & \textbf{Player B} \\ & & \begin{array}{cccc} b_1(q_1) & b_2(q_2) & \cdots & b_n(q_n) \end{array} \\ \textbf{Player A} & \begin{array}{c} a_1(p_1) \\ a_2(p_2) \\ \vdots \\ a_m(p_m) \end{array} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}$$

As we have defined

$$\begin{aligned} E(A, b_1) &= \text{expected gain of the Player A when the Player B plays } b_1 \\ &= a_{11} p_1 + a_{21} p_2 + \cdots + a_{m1} p_m \end{aligned}$$

$$\therefore \quad E(A, b_1) = \sum_{i=1}^{m} a_{i1} p_i \tag{B1}$$

Similarly, we have $E(A, b_2) = \sum_{i=1}^{m} a_{i2}\, p_i$ (B2)

And in this way finally we have $E(A, b_n) = \sum_{i=1}^{i=m} a_{in}\, p_i$ (Bn)

We have n such equations each corresponding to any one of b_i, $i = 1$ to n.

The Player A follows his maximin policy to maximize his minimum expected gain,

i.e. maxi [mini (E(A, b_1), E(A, b_2), ... E(A, b_n))] (1*)

Similarly, for the Player B we have the expected maximum loss under each move of the Player A, i.e.

$$\begin{aligned} E(B, a_1) &= \text{expected loss of the Player B when the Player A plays } a_1 \\ &= a_{11} q_1 + a_{12} q_2 + \cdots + a_{1n} q_n \end{aligned}$$

$$\therefore \quad E(B, a_1) = \sum_{j=1}^{n} a_{1j}\, q_j \tag{A1}$$

Similarly, $E(B, a_2) = \sum_{j=1}^{n} a_{2j}\, q_j$ (A2)

And this way, finally, $E(B, a_m) = \sum_{i=1}^{n} a_{mj}\, q_j$ (Am*)

We have m such equations each corresponding to any one of a_i, $i = 1$ to m.

The Player B follows minimax policy and by that he wants to minimize his expected loss.

i.e. mini [mix (E(B, a_1), E(B, a_2), ... E(B, a_m))] (2*)

Now, from (1*)

If minimum [E(A, b_1), E(A, b_2), ..., E(A, b_n)] = γ.

Then the Player A expects to maximize his minimum gain, i.e.

each E(A, b_1), E(A, b_2), ..., E(A, b_n) $\geq \gamma$.

This describes **Player A's problem**.

He has a problem to find such values of $p_1, p_2, ..., p_m$ that

Maximize $\quad Z = \gamma.$ (1)

subject to

$$\left.\begin{aligned} E(A, b_1) &= a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \geq \gamma \\ E(A, b_2) &= a_{21}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m \geq \gamma \\ &\vdots \\ E(A, b_n) &= a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_n \geq \gamma \end{aligned}\right\} \quad (2)$$

with each $\quad p_1, p_2, \dots, p_m \geq 0$ (3)

The set of Eq. (1), (2), and (3) represent a problem resembling to an **LPP**.

Now we try to simplify this problem. Let us assume that $\gamma < 0$. If $\gamma \leq 0$ then we have to make it > 0 by adding some constant to each element of the pay-off matrix.

Now we divide each value $p_1, p_2, ..., p_m$ by this $\gamma > 0$ and define $x_1 = \dfrac{p_1}{\gamma}$, $x_2 = \dfrac{p_2}{\gamma}$, $x_3 = \dfrac{p_3}{\gamma}$, $\cdots x_m = \dfrac{p_m}{\gamma}$ and put these values in the problem described by Eqs (1), (2) and (3).

Note that $\quad x_1 + x_2 \cdots + x_m = \dfrac{1}{\gamma}\,(p_1 + p_2 \cdots + p_m)$

$$= \frac{1}{\gamma} \quad (\because p_1 + p_2 \;\cdots\; + p_m = 1)$$

Now, we re-write the problem for the **Player A.**

Find $x_1, x_2, ..., x_m$ that maximizes = γ = minimizes = $\dfrac{1}{\gamma}$

$$= x_1 + x_2 \cdots + x_m \quad (4)$$

[Recall that $x_1 + x_2 \cdots + x_m = \dfrac{1}{\gamma}\,(p_1 + p_2 \cdots + p_m) = \dfrac{1}{\gamma}$]

subject to n constraints

$$\left.\begin{aligned} a_{11}\,x_1 + a_{21}\,x_2 + a_{31}\,x_3 + \cdots + a_{m1}\,x_m &\geq 1 \\ a_{21}\,x_1 + a_{22}\,x_2 + a_{32}\,x_3 + \cdots + a_{m2}\,x_m &\geq 1 \\ \vdots \qquad\quad \vdots \qquad\qquad\qquad\quad \vdots \\ a_{1n}\,x_1 + a_{2n}\,x_2 + a_{3n}\,x_3 + \cdots + a_{mn}\,x_n &\geq 1 \end{aligned}\right\} \quad (5)$$

with each $\quad x_i \geq 0$ (i = 1 to m) (6)

In the same way, we can write the problem for the **Player B.**

Let us define, $y_1 = \dfrac{q_1}{\gamma}$, $y_2 = \dfrac{q_2}{\gamma}$ $\cdots$, $y_n = \dfrac{q_n}{\gamma}$.

with, $\quad y_1 + y_2 \cdots + y_n = \dfrac{1}{\gamma}\,(q_1 + q_2 \cdots + q_n)$

$$= \frac{1}{\gamma} \quad \because q_1 + q_2 \cdots + q_n = 1$$

$\therefore \quad y_1 + y_2 \cdots + y_n = \dfrac{1}{\gamma}$

In all the m constraints we use $y_1 = \dfrac{q_1}{\gamma}$, $y_2 = \dfrac{q_2}{\gamma}$ $\cdots$, $y_m = \dfrac{q_n}{\gamma}$ after dividing each constraint by $\gamma \neq 0$.

The problem for the **Player B** is;

Find $y_1, y_2, ..., y_n$ that minimizes;

$$\gamma = \text{maximizes } \frac{1}{\gamma} = y_1 + y_2 \cdots + y_n \tag{7}$$

$$\left.\begin{aligned} a_{11}\, y_1 + a_{12}\, y_2 + \cdots + a_{1n}\, y_n &\le 1 \\ a_{21}\, y_1 + a_{22}\, y_2 + \cdots + a_{2n}\, y_n &\le 1 \\ \vdots \qquad \vdots \qquad\quad &\vdots \\ a_{m1}\, y_1 + a_{m2}\, y_2 + \cdots + a_{mn}\, y_n &\le 1 \end{aligned}\right\} \tag{8}$$

with each $\quad y_1, y_2, ..., y_n \ge 0 \qquad (9)$

Comments:

1. Problems for both the players A and B are **primal** and **dual** of each other.
2. In general, it becomes easier to solve the problem for the Player B. this is so as it has all constraints of $\le$ type and by adding slack variables (basic variables) all of them can be converted to equality constraints that can be readily used in simplex table.

Here, we introduce a systematic procedure that helps converting a game problem of mixed strategy into LPP.

Steps to solve pay-off matrix by LPP:

1. For the given pay-off matrix, find maximin value ($\underline{\gamma}$)
2. Now find the minimax value ($\overline{\gamma}$).
3. The value of the game γ is such that, $\underline{\gamma} \le \gamma \le \overline{\gamma}$. If $\underline{\gamma} = \overline{\gamma}$ then the game has a saddle point.
4. If $\underline{\gamma} < 0$ then add the maximum positive entry of the pay-off to each entry of pay-off matrix. If all entries are negative then change the sign of all entries. This will ensure that $\gamma > 0$. as $\underline{\gamma} > 0$.
5. We usually write and solve the problem for the **Player B**.
   This is so, as it turns out to be a maximization problem with all constraints $\le$ type.
   If there are n columns (assuming n different strategies available to the Player B) then let
6. Let $q_1, q_2, ..., q_n$ be the different probabilities with each $q_i \ge 0$ and $q_1 + q_2 + \cdots + q_n = 1$.

   Let **decision variables** be $y_1, y_2, ..., y_n$ where $x_1 = \dfrac{q_1}{\gamma}$, $x_2 = \dfrac{q_2}{\gamma}$ ..., $x_n = \dfrac{q_n}{\gamma}$

   Let $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and the problem for Player B is to find $x_1, x_2, ..., x_n$ so as maximize $Z = x_1 + x_2 + \cdots + x_n = \dfrac{1}{\gamma}$

$$\left.\begin{aligned} &\text{i.e.}\quad \text{Maximize } Z = \frac{1}{\gamma} = 1x_1 + 1x_2 + \cdots + 1x_n \\ &\qquad\quad \text{subject to } A\mathbf{X} \le 1 \\ &\qquad\qquad\quad \text{with } \mathbf{X} \ge 0 \end{aligned}\right\} \tag{10}$$

7. Apply simplex algorithm. It will give you the value of Z.

   Note that $Z = \dfrac{1}{\gamma}$ find $\gamma = \dfrac{1}{Z}$.

   Now form the values of basic variables $x_1, x_2, ...,$ etc., and
   find $q_1 = x_1 \cdot \gamma \quad q_2 = x_2 \cdot \gamma, ... \, q_n = x_n \cdot \gamma$
   [Note that,we find each x, then $1x_1 + 1x_2 \cdots + 1x_n = 1/\gamma$
   This will help finding each q_i where $i = 1$ to n.]

ILLUSTRATION 12

Solve the following game problem by converting it to LPP.

$$\begin{array}{c} \textbf{B} \\ A = \begin{pmatrix} 4 & 3 \\ -2 & 4 \\ 1 & 5 \end{pmatrix} \end{array}$$

Solution

The Player A has three strategies a_1, a_2, and a_3 which he plays with probabilities p_1, p_2, and p_3. The Player B has two strategies b_1, and b_2 which he plays with probabilities q_1, q_2 (where $q_1, q_2, ...$ 0 and $q_1 + q_2 = 1$)

$$\begin{array}{cc} & \text{Player } \mathbf{B} \\ & \begin{array}{cc} b_1(q_1) & b_2(q_2) \end{array} \\ \mathbf{A} = \begin{array}{c} a_1(p_1) \\ a_2(p_2) \\ a_3(p_3) \end{array} & \begin{pmatrix} 4 & 3 \\ -2 & 4 \\ 1 & 5 \end{pmatrix} \end{array}$$

1. Row minima = {3, –2, 1}

   $\therefore$ Maximin = 3 = $\underline{\gamma}$.

2. Column maxima = {4, 5}

   $\therefore$ Maximum = 4 = $\overline{\gamma}$

3. $\underline{\gamma} \neq \overline{\gamma}$

   $\therefore$ The game has no saddle point.

4. Let γ be the value of the game. As $\underline{\gamma} \leq \gamma \leq \overline{\gamma}$
   i.e. $3 \leq \gamma \leq 4 \therefore \gamma > 0$.

5. Let $y_1 = \dfrac{q_1}{\gamma}$ and $y_2 = \dfrac{q_2}{\gamma}$ $\gamma > 0$.

6. The problem for the Player B is: find y_1, y_2 so as to;

   Maximize $\dfrac{1}{\gamma} = y_1 + y_2$

   subject to $4y_1 + 3y_2 \leq 1$
   $-2y_1 + 4y_2 \leq 1$
   $1y_1 + 5y_2 \leq 1$
   with $y_1, y_2 \geq 0$.

Adding slack variables s_1, s_2, and $s_3 \geq 0$, the problem can be represented in simplex form as follows; the feasible solution is,

$$\text{Max } Z = \frac{5}{17} = \frac{1}{\gamma}$$

$$\therefore \quad \gamma = \frac{17}{5} \text{ is the value of the game.}$$

(Note that $4 < \gamma = \dfrac{17}{5} < 5$)

$$y_1 = \frac{2}{17} \text{ and } y_1 = \frac{q_1}{\gamma}$$

$$\therefore \quad \frac{2}{17} = \frac{q_1}{(17/5)}$$

$$\therefore \quad q_1 = \frac{2}{17} \times \frac{17}{5} = \frac{2}{5}$$

Also $$y_2 = \frac{3}{17} \text{ and } y_2 = \frac{q_2}{\gamma}$$

$$\therefore \quad \frac{3}{17} = \frac{q_2}{(17/5)}$$

$$\therefore \quad q_2 = \frac{3}{17} \times \frac{17}{5} = \frac{3}{5}$$

$$q_1 = \frac{2}{5}, q_2 = \frac{3}{5} \text{ and } \gamma = \frac{17}{5}.$$

9.6 APPLICATION OF GAME THEORY IN FINANCE AND BUSINESS

So far we have seen many illustrations that we would think ‘game theory as a simply the game but we depart from this and try to put before you some illustrations of real life situations. These illustrations will give you some glimpses of what we mean by application.

ILLUSTRATION 13

‘ABC pharma’ company has captured the market in three types of standard products—(1) ‘ Broncho syrup’ (2) ‘Broncho capsules’, and (3) ‘Clear Broncho’ inhalers. Against which ‘SAGA Pharma’ has well planed and studied all different strategies and plans to introduce two types of products—(1) ‘Coughno’, and (2) ‘clealungs’. Market movers and statisticians have carefully prepared a plan of impacts on revenue generated in case of fair and keen competition.

Revenue Generation (in ‘000 ₹) and Impacts
ABC Pharma

$$\begin{array}{lll} & & \begin{array}{ccc}\text{Broncho Syrup} & \text{Broncho Capsules} & \text{Broncho Inhalers}\end{array} \\ & \text{Coughno} & \multirow{2}{*}{$\begin{pmatrix} 50 & -25 & 5 \\ -24 & 15 & -12 \end{pmatrix}$} \\ \text{SAGA Pharma} & \text{Clealungs} & \end{array}$$

The problem is to determine the marketing strategies that companies should follow.

Solution

Row minima = {–25, –24}, Maximin = Maximum {–25, –24} = –24
Column Maxima = {50, 15, 5} , Minimax = Minimum {50, 15, 5} = 5

It is clear that

maximin = –24 ≠ minimax = 5

The game has no saddle point.

Dominance principles are not applicable in this matrix. And so we have to find graphical solution.

Let Saga Pharma adopts two strategies 'coughno' and 'clealungs' with probabilities p_1 and p_2. Mixed strategy vector $\mathbf{P} = \{p_1, p_2\}^t$ with $p_1 + p_2 = 1$ and each $p > 0$

We can find expected outcome of Saga Pharma corresponding to each strategy of 'ABC'

$$E\ (\text{SAGA, syrup}) = 50p_1 - 24p_2 = 50p_1 - 24(1 - p_1) = -24 + 74p_1$$
$$E(\text{SAGA, capsules}) = -25p_1 + 15p_2 = -25p_1 + 15\ (1 - p_1) = 15 - 40p_1$$
$$E(\text{SAGA, inhaler}) = 5p_1 - 12p_2 = 5p_1 - 12\ (1 - p_2) = -12 + 17p_1$$

Now draw the graph and find the highest point of the lower envelope.

This is the maximin point.

At this point, we will find that $p_1 = 9/19$ and so $p_2 = 10/19$

Value of the game = –3.94 .This means that SAGA Pharma in this case should be in a loss.

ILLUSTRATION 14

Labour union of the company is, through a mediator, making negotiation with the company secretary raising the issue of % rise in basic salary. This, in some cases, is opposed by the company. Negative sign shows that the company wants to reduce % reduction in the basic salary. The company has four policies against each one of the strategies of the union. You are supposed to take decision. The matrix, as is written is in the favour of union.

$$\begin{array}{cc} & \text{Company} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{Union policy} & \begin{bmatrix} 24 & 30 & 12 & 10 \\ -10 & 25 & 15 & 15 \\ -10 & -25 & 10 & 12 \\ 5 & 10 & -20 & -5 \end{bmatrix} \end{array}$$

Solution

We check whether we can apply dominance principles.

The first row dominates the 4th row and so we drop the 4th row.

$$\begin{bmatrix} 24 & 30 & 12 & 10 \\ -10 & 25 & 15 & 15 \\ -10 & -25 & 10 & 12 \end{bmatrix}$$

Second row dominates over the 3rd row; and so we drop the 3rd row.

In the reduced matrix, if you observe carefully, you will find that the 4th column dominates over the 3rd column. We drop the 3rd column.

We get $\begin{bmatrix} 24 & 30 & 12 \\ -10 & 25 & 15 \end{bmatrix}$

Again, the 3rd column dominates over the 2nd column; we drop the 2nd column.

We get a reduced pay-off matrix as follows.

$$A = \begin{bmatrix} 24 & 10 \\ -10 & 15 \end{bmatrix}$$

We can find maximin and minimax and see that there is no saddle point.
We may find either graphical or algebraic solution.

Value of the game = (24 × 15 – (–10)(10))/(24 + 15) – (10+10)
= 460/ 39

[One can find solution by equating the expectations.
$24p_1 - 10p_2 = 10p_1 + 15p_2$.
Put $p_2 = 1 - p_1$. Solving this we get $p_1 = 25/39$ and hence $p_2 = 14/39$

Additional Questions for Practice (with Hints and Answers)

Question 1
Find if the saddle point exists in the following pay-off matrix.

$$\begin{bmatrix} 2 & 3 & 5 & -6 \\ 5 & 8 & 11 & 2 \\ 6 & -5 & 4 & 4 \\ 1 & 4 & 8 & 3 \end{bmatrix}$$

Solution
We apply dominance principle.
Second row dominates over the 1st row; we drop the 1st row.
The resultant matrix is as follows.

$$\begin{bmatrix} 5 & 8 & 11 & 2 \\ 6 & -5 & 4 & 4 \\ 1 & 4 & 8 & 3 \end{bmatrix}$$

The 4th column dominates over the 3rd column; drop 3rd column.
The resultant matrix is as follows.

$$\begin{bmatrix} 5 & 8 & 2 \\ 6 & -5 & 4 \\ 1 & 4 & 3 \end{bmatrix}$$

Now, further application of dominance principle is not obvious.
Find maximin and minimax.

Maximin = maximum of {2, –5, 1} = 2
Minimax = minimum of {6, 8, 4} = 4

Maximin and minimax are not equal; the game has no saddle point.

Question 2

From the following pay-off matrix, find saddle point if it exists. Also write the equations of expectations.

$$\begin{bmatrix} 4 & 0 & 9 & 9 \\ 5 & 8 & 10 & 12 \\ 6 & -12 & 6 & 4 \\ 5 & 7 & 8 & -3 \\ 2 & 12 & 7 & 8 \end{bmatrix}$$

Solution

We use dominance principle.

1. Row 2 dominates over row 1; drop row 1.
2. Row 2 dominates over row 4; drop row 4.

   Now, let us write the resultant matrix.

$$\begin{bmatrix} 5 & 8 & 10 & 12 \\ 6 & -12 & 6 & 4 \\ 2 & 12 & 7 & 8 \end{bmatrix}$$

3. Column 1 dominates over column 3; drop column 3.

   The resultant matrix is as follows.

   Final/reduced pay-off matrix = $\begin{bmatrix} 5 & 8 & 12 \\ 6 & -12 & 4 \\ 2 & 12 & 8 \end{bmatrix}$

   We find maximin and minimax.

$$\text{Maximin} = \text{maximum of } \{5, -12, 2\} = 5$$
$$\text{Minimax} = \text{minimum of } \{6, 12, 12\} = 6$$

Since, maximin and minimax are not equal and so the game has no saddle point.

Now, on the reduced pay-off matrix,we apply mixed strategy.

It can be applied for both or any one of the player.

We know that the value of the game lies in [5 6]

Let the 2nd player plays his moves with probabilities q_1, q_2, and q_3 and each $q_i > 0$ and $q_1 + q_2 + q_3 = 1$

Expectation of the player B when the first Player A plays a_1.

$$E(B, a_1) = 5q_1 + 8q_2 + 12q_3 \le \gamma$$
$$E(B, a_2) = 6q_1 - 12q_2 + 4q_3 \le \gamma$$
$$E(B, a_3) = 2q_1 + 12q_2 + 8q_3 \le \gamma$$

We know that $5 \le \gamma \le 6$

Divide by γ and define $x_1 = q_1/\gamma$, $x_2 = q_2/\gamma$, $x_3 = q_3/\gamma$

$$5x_1 + 8x_2 + 12x_3 \le 1$$
$$6x_1 - 12x_2 + 4x_3 \le 1$$
$$2x_1 + 12x_2 + 8x_3 \le 1$$

with all $x_i \ge 0$,

Now, the problem is to find x_1, x_2, and x_3

so as to maximize $x_1 + x_2 + x_3 = (x_1 = q_1/\gamma + x_2 = q_2/\gamma + x_3 = q_3/\gamma)$

$= 1/\gamma$ (as $q_1 + q_2 + q_3 = 1$)

subject to above constraints.

Question 3

Solve the following pay-off matrix and find the value of the game.

$$\text{Pay-off matrix, A} = \begin{bmatrix} 5 & 4 & -6 & 8 \\ 3 & 7 & 1 & 2 \\ 5 & 9 & 4 & 6 \\ 2 & 1 & 4 & 7 \end{bmatrix}$$

Solution

First, we apply dominance principles and try to reduce the size of the pay-off matrix.

1. Row 3 dominates over row 2; drop row 2.
   The resultant matrix is as follows.
$$A = \begin{bmatrix} 5 & 4 & -6 & 8 \\ 5 & 9 & 4 & 6 \\ 2 & 1 & 4 & 7 \end{bmatrix}$$
2. Column 1 dominates over column 4; drop column 4.
$$A = \begin{bmatrix} 5 & 4 & -6 \\ 5 & 9 & 4 \\ 2 & 1 & 4 \end{bmatrix}$$
3. Second row dominates over the 3rd row; drop the 3rd row.
$$A = \begin{bmatrix} 5 & 4 & -6 \\ 5 & 9 & 4 \end{bmatrix}$$
4. Row 2 dominates over row 1; drop row 1.
   Resultant matrix is $[5 \quad 4 \quad -6]$
5. It is obvious that the first Player A has only one move and hence the second Player B will select the third move and win ₹6.
   (This amount i.e. ₹6 is a loss to the first plater.]

Points to Remember

1. In order to solve a game theory problem, consider the pay-off matrix.
   Find minimum of each row for each row. From the set of values obtained find maximum value. This is maximin value of the game.
   Now, for each column, find the maximum of each column and then find minimum of the set of values of column maximums. This is minimax value of the game.
   We must have maximin value ≤ minimax value
   If maximin value = minimax value, then the game has a saddle point.
   Saddle point is the position in the pay-off matrix and the value of the position is the value of the game.

2. If the game has no saddle point then the players adopt the policies to play the game and make selection of strategy with probabilities; this is called mixed strategy. Mixed strategies are vectors such that for each one, the elements like p_i and q_i are non-negative and less than 1.

   For the vector $X = (p_1 \quad p_1 \quad p_m)^t$; we have $p_1 + p_2 +, ..., + p_m = 1$ and similarly fop the mixed strategy of the other player.
3. Applying principles of dominance, we can reduce the size of a pay-off matrix. In general,
   (a) first apply dominance principles, then find maximin and minimax.
   (b) there is a saddle point, then you can find the value of the game.
   (c) there is no saddle point, then apply expectation approach of mixed strategy.
   (d) You can find **algebraic solution** in case of 2×2 matrix.
   You can find graphical solution in case 2×2, $m \times 2$, or $2 \times n$ pay-off matrix.
   You can find sub-games and solve the problems.
   (e) You can convert a game problem in linear programming problem.
   Generally it is easier and faster to solve a mixed strategy game problem in favour of the second player. Some hints are as follows.
   Let $q_1, q_2, ..., q_n$ be the different probabilities with each $q_i \geq o$ and $q_1 + q_2 + \cdots + q_n = 1$.
   Let **decision variables** be $y_1, y_2, ..., y_n$ where $x_1 = \dfrac{q_1}{\gamma}$, $x_2 = \dfrac{q_2}{\gamma}$ ..., $x_n = \dfrac{q_n}{\gamma}$

   Let $\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and the problem for the Player B is to find $x_1, x_2, ..., x_n$ so as maximize $Z =$

   $x_1 + x_2 + \cdots + x_n = \dfrac{1}{\gamma} \cdot \gamma$ i.e.

   Maximize $Z = \dfrac{1}{\gamma} = 1x_1 + 1x_2 \cdots + 1x_n$

   subject to $\text{A}\mathbf{X} \leq 1$ (10)

   with $\mathbf{X} \geq 0$

   Apply simplex algorithm. It will give you the value of Z.

   Note that $Z = \dfrac{1}{\gamma}$ and find $\gamma = \dfrac{1}{Z}$.

   Now form the values of basic variables $x_1, x_2, ...$ etc.,
   find $q_1 = x_1 \cdot \gamma$ $q_2 = x_2 \cdot \gamma$, ... $q_n = x_n \cdot \gamma$
   [Note that, we find each x, then $1x_1 + 1x_2 \cdots + 1x_n = 1/\gamma$
   This will help finding each q_i where $i = 1$ to n.]

EXERCISES

OBJECTIVE TYPE QUESTIONS

I. State True or False

1. In a two persons zero sum game; if one player wins ₹5. Then another player may win any positive amount in the same turn of the game.
2. In finding row minimum in a pay-off matrix, at least one value of row minimum for any row must be negative.

3. In finding column maximum related to the second player, at least one value of column minimum for any row is a non-negative value.
4. In general for any pay-off matrix, minimax $f(x, y) \geq$ maximin $f(x, y)$.
5. For a pay-off matrix of order 2×2, saddle point is always zero.
6. Saddle point means the value of the game.
7. The game has no saddle point means the maximin value = minimax value.
8. If we add a constraint k to each entry of a 2×2 pay-off matrix having a saddle point, then the value of the game becomes at least k^2 times greater than that of the original value of the game.
9. If $\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\boldsymbol{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ are the two mixed strategies of the Player A and player B respectively then always we have;
   (a) Each $x_i > 0$ and each $y_i < 0$ (b) $x_1 y_1 + x_2 y_2 = 0$ (c) $x_1 = y_1$ and $x_2 = y_2$
10. If all the entries of 2nd row in a pay-off matrix of order 2×3, are greater than the corresponding entries of 3rd column, then we drop the 3rd column using principle of dominance.
11. When we find the mixed strategy vector using linear programming techniques on a pay-off matrix, LPP of the player A and the LPP for the player B are primal and dual problems of each other.

ANSWERS

1. false.	2. false.	3. false.	4. true.	5. false.
6. false.	7. false.	8. false.	(9a) false.	(9b) false.
(9c) false.	10. false.	11. true.		

II. Multiple Choice Questions

The following pay-off matrix corresponds to fist six questions;

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{pmatrix} 4 & 3 & 5 & 7 & 4 \\ 11 & 2 & 1 & 0 & -3 \\ 8 & 2 & -4 & 7 & 9 \\ 1 & 1 & 3 & -9 & 2 \end{pmatrix} \end{array}$$

1. Maximin value is
   (a) –2 (b) 9 (c) 3 (d) 11
2. Minimax value is
   (a) 11 (b) 3 (c) –9 (d) 7
3. Saddle point is
   (a) (3, 3) (b) (1, 2) (c) (2, 1) (d) (4, 3)
4. Value of the game is
   (a) –3 (b) 3 (c) 11 (d) 2
5. Identify the row dominance
   (a) First row dominates over third row (b) First row dominates over fourth row
   (c) Third row dominates over first row (d) Forth row dominates over first row
6. Identify the column dominance
   (a) First column dominates over third column
   (b) Third column dominates over second column
   (c) Second column dominates over first column
   (d) First column dominates over second column

7. For the pay-off matrix $\begin{bmatrix} 6 & -3 \\ 5 & 7 \end{bmatrix}$, the value of the game is

(a) 6 (b) 57 (c) 57/11 (d) 27/11

8. For the pay-off matrix

$$\begin{array}{cc} & \text{B} \\ \text{A} & \begin{bmatrix} 7 & 3 \\ 2 & 5 \end{bmatrix} \end{array}$$

(a) there is no saddle point (b) there is a saddle point but
(c) value of the game cannot exist (d) value of the game is zero

9. Graphical solution cannot be found in a pay-off matrix of order 3×4;
   (a) cannot be said without applying dominance principles
   (b) find graphical solution by taking two rows
   (c) find the graphical solution of two matrices of 3×2 and 3×2 and add the result
   (d) find maximin and minimax and draw graph
10. By interchanging the first and second row of a 2×2 pay-off matrix
    (a) maximin value does not change
    (b) minimax = maximin value
    (c) value of the game changes
    (d) value of the game becomes reciprocal to the original value.
11. On multiplying each entry of a pay-off matrix by a constraint k, the value of the game (given $k \neq 0$)
    (a) becomes k times of the original value (b) becomes $4k$ times of the original value
    (c) becomes k^2 times of the original value (d) does not change
12. In a mixed strategy pay-off matrix of order 3×2 if a probability for any one of the three probability values (for the first Player A) is zero; then
    (a) it means that remaining values of probability are equal
    (b) the row with zero probability value dominates the other two rows
    (c) the row with zero probability value can be removed from the pay-off table
    (d) the saddle point does not exist.

Answers

1. (c)	2. (b)	3. (c)	4. (b)	5. (b)
6. (c)	7. (c)	8. (a)	9. (a)	10. (a)
11. (a)	12. (c)			

Numerical Problems

In Examples 1 to 5, find maximin value, minimax value and check the existence of saddle point.

1. Pay-off matrix, $A = \begin{bmatrix} -2 & 3 & 5 & 7 & 9 \\ 3 & 5 & 2 & 5 & 7 \\ 1 & 1 & 3 & 8 & 0 \\ 2 & -9 & 6 & 4 & -1 \end{bmatrix}$

2. Payoff Matrix, $A = \begin{bmatrix} 2 & -1 & 5 & 4 & 7 \\ 3 & 0 & 8 & 1 & 12 \\ 8 & 4 & 7 & 11 & 3 \\ -9 & 5 & 8 & 1 & 7 \\ 8 & 4 & 2 & 3 & -3 \end{bmatrix}$

3. Payoff Matrix, $A = \begin{bmatrix} -4 & -3 & -2 & -1 \\ 0 & -5 & -4 & -9 \\ -8 & -3 & -7 & -4 \\ -4 & -11 & -8 & -6 \end{bmatrix}$

4. Payoff Matrix, $A = \begin{bmatrix} 2 & -3 & 5 & 3 & -3 \\ 8 & -2 & 0 & 5 & 9 \\ -1 & -3 & 8 & -3 & 7 \\ 5 & -2 & 4 & 3 & -10 \end{bmatrix}$

5. Payoff Matrix, $A = \begin{bmatrix} 1 & 9 & 7 & 5 & 3 & 2 \\ 8 & 5 & 14 & 7 & 6 & 3 \\ 1 & 9 & 11 & 6 & 5 & 2 \\ -1 & 0 & 3 & -1 & 4 & 1 \end{bmatrix}$

Apply dominance principles to the following pay-off matrices and find saddle point if exists [Examples 6 to 10]

6. Payoff Matrix, $A = \begin{bmatrix} 7 & 2 & 10 & 8 & 2 \\ 13 & 3 & 5 & 10 & 14 \\ 4 & 2 & 13 & 2 & 12 \\ 10 & 3 & 9 & 8 & -5 \end{bmatrix}$

7. Payoff Matrix, $A = \begin{bmatrix} 5 & 13 & 11 & 9 & 7 & 6 \\ 12 & 9 & 18 & 11 & 10 & 7 \\ 5 & 13 & 15 & 10 & 9 & 6 \\ 3 & 4 & 7 & 3 & 8 & 5 \end{bmatrix}$

8. Payoff Matrix, $A = \begin{bmatrix} 10 & 3 & 9 & 8 & -5 \\ 4 & 2 & 13 & 2 & 12 \\ 13 & 3 & 5 & 10 & 14 \\ 7 & 2 & 10 & 8 & 2 \end{bmatrix}$

9. Payoff Matrix, A = $\begin{bmatrix} 2 & -1 & 5 & 4 & 7 \\ 3 & 0 & 8 & 1 & 12 \\ 8 & 4 & 7 & 11 & 13 \\ -9 & 5 & 8 & 1 & 7 \\ 8 & 4 & 6 & 3 & 3 \end{bmatrix}$

10. Payoff Matrix, A = $\begin{bmatrix} -5 & 8 & 9 & 3 & 10 \\ 12 & 2 & 13 & 2 & 4 \\ 14 & 10 & 5 & 3 & 13 \\ 2 & 8 & 10 & 2 & 7 \end{bmatrix}$

Apply dominance principles and reduce the pay-off matrix to the order 2 × 2 and solve the game algebraically (Examples 11 to 15)

11. $\begin{bmatrix} 2 & 1 & 2 \\ 5 & 4 & -2 \\ 6 & 3 & 7 \end{bmatrix}$

12. $\begin{bmatrix} 7 & 6 & 0 \\ 4 & 3 & 4 \\ 8 & 5 & 9 \end{bmatrix}$

13. $\begin{bmatrix} 2 & -1 & -8 \\ 0 & -3 & 7 \\ 5 & 4 & -6 \end{bmatrix}$

14. $\begin{bmatrix} 0 & 10 & 12 \\ 7 & -3 & 2 \\ -2 & 6 & 8 \end{bmatrix}$

15. $\begin{bmatrix} 6 & 4 & -3 \\ 7 & 1 & 5 \\ 5 & 0 & 2 \end{bmatrix}$

Find the graphical solution of the following games [Examples 16 to 20].

16. $\begin{bmatrix} -3 & 7 & 4 \\ 4 & -6 & -8 \end{bmatrix}$

17. $\begin{bmatrix} 4 & -2 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

18. $\begin{bmatrix} -3 & 7 \\ 4 & -6 \\ -4 & 4 \end{bmatrix}$

19. $\begin{bmatrix} 6 & 0 & 3 \\ 5 & 9 & 4 \end{bmatrix}$

20. $\begin{bmatrix} -1 & 9 \\ 6 & -4 \\ -2 & 6 \end{bmatrix}$

21. Convert the following matrix to LP problem and solve it.

$$A = \begin{pmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{pmatrix}$$

22. Sigma Corporation and Zydus Technology are the two different companies working on production of electronic circuits. If Sigma company adopts policy of continuing with its current policy of circuit S_1 then the respective output in three different z_1, z_2, and z_3 of Zydus are 25, 30, and –10 in terms of ₹10000. If the Sigma goes in the production of solar circuits then the corresponding output by following its three different policies z_1, z_2, and z_3 are –20, 15, and 10 in terms of ₹1000. You are required to find the optimal strategy.
[Hints: The pay-off matrix is as follows.

$$\begin{array}{ll} & \text{Zydus Technology} \\ & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \text{Sigma Corporation} \begin{array}{c} c_1 \\ c_2 \end{array} & \begin{pmatrix} 25 & 30 & -10 \\ -20 & 15 & 10 \end{pmatrix} \end{array}$$

Now, find maximin, and minimax and apply dominance principles.]

Answers to Numerical Problems

1. Maximin value = 2 Minimax value = 3
   No saddle point
2. Maximin value = 3 Minimax value = 5
   No saddle point
3. Maximin value = 4 Minimax value = –3
   No saddle point
4. Maximin value = –2 Minimax value = –2
   It has a saddle point at (2, 2); value of the game = –2
5. Maximin value = 3 Minimax value = 3
   It has a saddle point at (2, 6); value of the game = 3

6. Saddle point (2, 2); value of the game = 3
7. Saddle point (2, 6); value of the game = 7
8. Saddle point (3, 2); value of the game = 3
9. Final reduction $\begin{bmatrix} 4 & 11 \\ 5 & 1 \end{bmatrix}$; value of the game = 51/11
10. Saddle point (3, 4); value of the game = 3
11. Maximin = 3, Minimax = 4, final reduction $\begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}$ value of the game = 17/5
12. Maximin = 5, Minimax = 6, final reduction $\begin{bmatrix} 6 & 0 \\ 5 & 9 \end{bmatrix}$ value of the game = 27/5
13. No saddle point, $p_1 = 1/2$, $q_1 = 13/20$, value of the game = 1/2
14. No saddle point, $p_1 = 1/2$, $q_1 = 13/20$, value of the game = 7/2
15. Maximin = 1, Minimax = 4 final reduction $\begin{bmatrix} 4 & -3 \\ 1 & 5 \end{bmatrix}$ value of the game = 23/11 $p_1 = 4/11$
16. $p_1 = 1/2$, value of the game = 1/2
17. $p_1 = 2/5$, value of the game = 17/5
18. $q_1 = 13/20$, value of the game = 1/2
19. $p_1 = 2/5$, value of the game = 27/5
20. $q_1 = 13/20$, value of the game = 5/2
21. value of the game = 2

10 Replacement Theory

Trust, there is always a better replacement.

Learning Objectives

AFTER STUDYING THIS CHAPTER, THE STUDENTS WILL BE ABLE TO:

- justify the requirement of replacing the units
- find the replacement period in different cases
- replacement of items with known scrap value and maintenance cost
- replacement of items in consideration of present worth of future costs
- replacement of items, which permanently fail/(group replacement/individual replacement)
- staff replacement

INTRODUCTION

The basic notion of this chapter is to decide replacement of the different items, which are presently in working condition, or in proper functioning mode; but such continued operation need proper and timely maintenance and experienced handling.

It is also obvious that, the maintenance cost and handling charges keep on increasing with time. Increasing or continued and repetitive usage of machineries tear and wear increases which is responsible for the incremental cost of maintenance. In addition to this, there are new machineries more economic in cost, highly production-efficient, energy-efficient and possessing many additional features which replace the older ones.

In some cases, the companies discontinue production of the proto-type units and switch over to latest version; such circumstances create short supply of important parts required to change or repair often. On the top of that scrap value, as the time passes and usage increases or the model of the machine becomes older decreases.

In some cases, timely or pre-fixed replacement becomes cheaper to individual replacement.

In this chapter, we will discuss all possible cases. We will, in some cases, give due importance to the notion of net present worth of future amount.

To remain in cut-throat competition and to win-over, one must keep pace with time and current trends and increasing speed of modernization.

10.1 Replacement of Items with Increasing Maintenance Cost and Falling Scrap Value

First, we construct a mathematical model of the problem on hand.

Let C be the cost or purchase cost of the item. Let $M(n)$ stands for annual maintenance cost which is a function of time and generally assumed that it increases with time. (In some cases, as per the contract, it may remain fixed for a stipulated period of time.) Let $S(n)$ represents scrap value of the unit considered at the end of nth year.

The problem is to determine optimal period of replacement.

We consider the decision criterion that the period up to which the average cost of execution, remains lower than that considered up to the next period, is the most right year of making a replacement for a new one.

We construct a necessary formula explaining what we have discussed here.

Let $A(n)$ represents the average cost incurred up to the nth year.

The total cost incurred up to n years = $C - s(n) + \sum_{n=1}^{n} M(n)$

$$\therefore \qquad A(n) = \frac{C - s(n) + \sum_{n=1}^{n} M(n)}{n}$$

The decision criterion is $A(n-1) > A(n)$ and $A(n) < A(n+1)$

This implies that the unit should be replaced at the end of n years.

Comment

1. We have assumed that money value during the years of calculation is constant.
2. Scrap value either remains constant or diminishes with time.
3. Period of replacement is the end of nth year.

Illustration 1

For a new machine, costing ₹10000, the scrap value at any time is fixed and it is ₹3000.

Maintenance cost $M(n)$ (in the nth year) is variable and 500, 700, 800, 1200, 2000, 2500, 2800, and 2500 are the maintenance cost in ₹ per year for first 8 years in succession. Find the best period of replacement.

Solution

Referring to the above formula, the factor $C - S(n)$ is constant throughout this example.

$$C - S(n) = 10000 - 3000 = 7000$$

We make a table.

Table 10.1

Year = n	*C – S(n)*	*M(n)*	$\Sigma M(n)$	$\therefore \quad A(n) = \frac{C - s(n) + \sum_{n=1}^{n} M(n)}{n}$
1	7000	500	500	A(1) = (7000 + 500)/1 = 7500
2	7000	700	1200	A(2) = (7000 + 1200)/2 = 4100

(*Contd.*)

Year = n	*C – S(n)*	*M(n)*	*Σ M(n)*	$\therefore \quad A(n) = \dfrac{C - s(n) + \sum_{n=1}^{n} M(n)}{n}$
3	7000	800	2000	A(3) = (7000 + 2000)/3 = 3000
4	7000	1200	3200	A(4) = (7000 + 3200)/4 = 2550
5	7000	2000	5200	A(5) = (7000 + 5200)/5 = 2440**
6	7000	2500	7700	A(6) = (7000 +7700)/6 = 2450
7	7000	2800	10500	A(7) = (7000 + 10500)/7 = 2500
8	7000	2500	13000	A(8) = (7000 + 13000)/8 = 2500

From the last column, it is clearly observed that average cost at the end of 5 years is minimum.

Also as a check, we find that $A(4) = 2550 > A(5) = 2440$; $A(5) = 2440 < A(6) = 2500$ Period of replacement is the end of fifth year.

ILLUSTRATION 2

Find the optimal replacement policy in the following case.

Cost of the unit = ₹12000.

Year	1	2	3	4	5	6	7
Scrap Value in ₹	7000	5000	4000	3000	2500	2500	2500
Maintenance cost in ₹	800	1200	2000	2800	3000	4100	4000

Solution

We prepare a table.

Table 10.2

Year = n	*Cost = C*	*S(n) = S*	*M(n)*	*Sum = ΣM(n)*	*C – S + ΣM(n)*	$\therefore \quad A(n) = \dfrac{C - s(n) + \sum_{n=1}^{n} M(n)}{n}$
1	12000	7000	800	800	5800	A(1) = 5800
2	12000	5000	1200	2000	9000	A(2) = 9000/2 = 4500
3	12000	4000	2000	4000	12000	A(3) = 12000/3 = 4000
4	12000	3000	2800	6800	15800	A(4) = 15800/4 = 3950
5	12000	2500	3000	9800	19300	A(5) = 19300/5 = 3860**
6	12000	2500	4100	13900	23400	A(6) = 23400/6 = 3900
7	12000	2500	4000	17900	27400	A(7) = 27400/7 = 3915

From Table 2, it is clear that the average cost at the end of 5th year is minimum.

$$\text{Also } A(4) = 3950 > A(5) = 3860,\ A(6) = 3900 > A(5) = 3860$$

The right decision is to replace at the end of five years.

ILLUSTRATION 3

Find the optimal replacement policy in the following case.

Cost of the unit = ₹13000.

Year	1	2	3	4	5	6	7
Scrap Value in ₹	7000	5000	4000	3000	2500	2500	2500
Maintenance cost in ₹	1600	2000	3000	3800	4500	6000	4000

Solution

We make a table.

Table 10.3

Year = n	*Cost = C*	*S(n) = S*	*M(n)*	*Sum =* $\Sigma M(n)$	$C - S + \Sigma$ $M(n)$	$\therefore \quad A(n) = \dfrac{C - s(n) + \sum_{n=1}^{n} M(n)}{n}$
1	13000	7000	1600	1600	7600	A(1) = 7600
2	13000	5000	2000	3600	11600	A(2) = 11600/2 = 5800
3	13000	4000	3000	6600	15600	A(3) = 15600/3 = 5200
4	13000	3000	3800	10400	20400	A(4) = 20400/4 = 5100
5	13000	2500	4500	14900	25400	A(5) = 25400/5 =5080 **
6	13000	2500	6000	20900	31400	A(6) = 31400/6 = 5233

From Table 3 it is clear that the average cost at the end of 5th year is minimum.

$$\text{Also } A(4) = 5100 > A(5) = 5080,\ A(6) = 5233 > A(5) = 5080$$

The right decision is to replace at the end of five years.

ILLUSTRATION 4

Find the optimal replacement policy in the following case:

Cost of the unit = ₹14000; For $n = 1, 2, 3, \ldots$ shows complete years.

Let the scrap value be given by the formula: $4000 - 400(n)$

Let the maintenance cost be given by the formula: $1000 + 800(n)$

Solution

Using these data, we make the table.

Table 10.4

Year = n	*Cost = C*	*S(n) = S*	*M(n)*	*Sum =* $\Sigma M(n)$	$C - S + \Sigma$ $M(n)$	$\therefore \quad A(n) = \dfrac{C - s(n) + \sum_{n=1}^{n} M(n)}{n}$
1	14000	3600	1800	1800	12200	A(1) = 12200
2	14000	3200	2600	4400	15200	A(2) = 15200/2 = 7600
3	14000	2800	3400	7800	19000	A(3) = 19000/3 = 6333
4	14000	2400	4200	12000	23600	A(4) = 23600/4 = 5900
5	14000	2000	5000	17000	29000	A(5) = 29000/5 = 5800**
6	14000	1600	5800	22800	35200	A(6) = 35200/6 = 5867

From Table 4 it is clear that the average value at the end of 5th year is minimum.

$$\text{Also, } A(4) = 5900 > A(5) = 5800 \text{ and } A(6) = 5867 > A(5) = 5800$$

It is better to replace the unit at the end of 5 years.

ILLUSTRATION 5

Find the optimal replacement policy for the following:

Cost of the unit B = ₹9000
Scrap value in ₹ = 100

Maintenance cost is given by the formula: 200 + 2000(n – 1); for n = 1, 2, 3,... years

Solution

Using the above records, we make the table.

Table 10.5

Year = n	*Cost = C*	*S(n)= S*	*M(n)*	*Sum = ΣM(n)*	*C – S + ΣM(n)*	$\therefore\ A(n) = \dfrac{C - s(n) + \sum_{n=1}^{n} M(n)}{n}$
1	9000	100	200	200	9100	A(1) = 9100
2	9000	100	2200	2400	11300	A(2) = 11300/2=5650
3	9000	100	4200	6600	15500	A(3) = 15500/3=5167**
4	9000	100	6200	12800	21700	A(4) = 21700/4=5425
5	9000	100	8200	21000	29900	A(5) = 29900/5=5980

From Table 5, it is clear that average value at the end of 4th year is minimum.

$$\text{Also } A(2) = 5650 > A(3) = 5167 \text{ and } A(4) = 5425 > A(3) = 5167.$$

Thus, it is considered best to replace the unit at the end of 3 years.

ILLUSTRATION 6

Find the optimal replacement policy in the following case:

Cost of the Unit A is ₹10000, and scrap value is ₹100.

Maintenance cost is given by the formula: 400 + 800(n – 1); for n = 1, 2, 3, … years. Also, make a study of the previous example and compare both results and discuss the criterion of replacing the Unit B by this new Unit A.

Solution

Using the above records, we make the following table.

Table 10.6

Year = n	*Cost = C*	*S(n)= S*	*M(n)*	*Sum = ΣM(n)*	*C – S + ΣM(n)*	$\therefore\ A(n) = \dfrac{C - s(n) + \sum_{n=1}^{n} M(n)}{n}$
1	10000	100	400	400	10300	A(1) = 10300
2	10000	100	1200	1600	11500	A(2) = 11500/2 = 5750

(*Contd.*)

Year = n	Cost = C	S(n)= S	M(n)	Sum = $\Sigma M(n)$	$C - S +$ $\Sigma M(n)$	$\therefore \quad A(n) = \dfrac{C - s(n) + \sum_{n=1}^{n} M(n)}{n}$
3	10000	100	2000	3600	13500	A(3) = 13500/3 = 4500
4	10000	100	2800	6400	16300	A(4) = 16300/4 = 4075
5	10000	100	3600	10000	19900	A(5) = 19900/5 = 3980**
6	10000	100	4400	14400	24300	A(6) = 24300/6 = 4050
7	10000	100	5200	19600	29500	A(7)=29500/7 = 4214

From Table 6, it is clear that the average value at the end of 5 years is minimum.

$$\text{Also,} \quad A(4) = 4075 > A(5) = 3980 \text{ and } A(6) = 4050 > A(5) = 3980.$$

Thus, it is best to replace the unit at the end of 5 years.

Comment

With reference of the two Examples 5 and 6 solved here, we would like to clarify certain points.

1. Minimum average cost of operation of the machine in Example 6 is 3980 while the same for in Example 5 is 5167. From this, we conclude that we can replace the Machine B of Example 5 by the Machine A of the Example 6.
2. Running cost of Machine B of Example 5 in each year has been taken from Table 6 of Example 5. We study these costs and interpret the consequences.
   Difference in cost of years 2 and 1 = 11300 – 9100 = 2200
   Difference in cost of years 3 and 2 = 15500 – 11300 = 4200 **
   Running cost of this machine during the span of 3 years = 4200.

This amount ₹4200 is greater than that of the all time low average running cost ₹3980 of Unit A of Example 6.

So, we conclude that the Machine B should be replaced by the Machine A at the end of 3 years.

10.2 Replacement in the Case when Net Present Worth of Money Value is Considered

In this unit, we will consider the present worth of the future amount probably to be invested.

Net Present Value: In this unit, we discuss the important concept and also very appealing one. It is known that the worth or the importance of an amount what is today shall not be the same after some period; say one year. This is so because of many reasons, like inflation and easy and faster liquidity of money in the market. To balance this situation, we raise the amount by introducing the concept of interest. In general, by tradition, interest is calculated on the basic amount ₹100; whatsoever the denomination is. An interest of 10% simply mean that the amount ₹100 when deposited without any transaction on either way, shall become ₹110 at the end of one complete year.

We think in the opposite direction that the amount of maintenance during the second year calculated as on today, in the light of this concept is not justifiable at all.

If the rate of interest is 10 % per annum, then ₹100 is ₹110 at the end of one year. by this, we find what is ₹100 after one year is what amount today? By simple arithmetic, it is $100 \times \dfrac{100}{110} = 1000/11$.

This means that if this amount, **1000/11** (present worth) is invested today at 10% rate, will become **'100'** after one year (future worth of present worth.) **We** want to use this appealing concept in calculating lots of amount after four, five , and six years that we have already done previously!!!

In general, the rate of interest $r\%$, to find the present worth of ₹1 is taken as a mathematically equivalent factor, $\frac{1}{(1+r)}$ where $r = r\,\%/\,100$. [$r\,\% = 10\,\%$ implies $r = 10/\,100 = 0.1$.]

This means that what is one rupee after one year is today an amount worth $\frac{1}{(1+r)}$ with $r = 0.1$ Extending this concept for first 2 years in succession, what is rupee one after two years at $r\%$ rate of interest is today worth $\frac{1}{(1+r)^2}$. We can extend this process in the same way up to a period of ***n*** years.

With this tool on hand, we proceed to justify and remold the concept of replacement theory.

10.2.1 How do We Introduce this Notion?

1. There are certain notations;
   C = basic cost or purchase price of an item
   $S(n)$ = scrap value of an item in nth year
   $M(n)$ = maintenance cost in the nth year
   $r\,\%$ = rate of interest and so $\frac{1}{(1+r)}$ is the discount factor for finding NPV (net present value) of a rupee with $r = r\%\,/100$ [e.g. for $r\% = 10\%$, $r = 10/100 = 0.1 \therefore \frac{1}{(1+0.1)} = \frac{1}{1.1} = 0.9091$]
2. We assume that all the expenses to be incurred are estimated and done in advance.
   Discount factor of nth year = $\sum_{n=1}^{n} \frac{1}{(1+r)^{n-1}}$ for $k = 1, 2, 3, 4, 5, \ldots, n$. For example, for the first year (beginning) it is = 1/1, for the second year = $1 / (1 + r)$
   for the third year = $1/(1 + r)^2$ and so on.

10.2.2 Steps to Decision Making

1. Find present worth of maintenance cost of the nth year; apply $M(n)$ = given $M(n)$ * $\sum_{n=1}^{n} \frac{1}{(1+r)^{n-1}}$ with $k = 1, 2, 3, 4, \ldots, n$
2. Find cumulative $M(n)$ for each year.
3. Find C + cumulative $M(n)$
4. Find weight of the year on calculation = $w(n) = \sum_{n=1}^{n} \frac{1}{(1+r)^{n-1}}$
5. Find $A(n)$ = weighted average cost = C + Cumulative $M(n)/w(n)$
   [Result (3) divided by Result (4) shown above]
6. Decision Criterion: The year (n) for which, $A(n-1) > A(n)$, and $A(n) < A(n+1)$ is the year of replacement, i.e. at the end of that year the unit should be replaced.

ILLUSTRATION 7

Find the year of replacement. The facts are given as follows.

C = cost of the unit = 10000, $M(n) = 2000 + 500(n)$; $n = 1, 2, 3, 4, \ldots, r\% = 10\%$

Solution

We know that

$r\% = 10\%$ implies that $r = 0.1$ and hence the discount factor is $\dfrac{1}{(1+r)^{n-1}}$

We make a table (Table 7) of all terms.

Table 10.7

Year = n	$\dfrac{1}{(1+r)^{n-1}}$	*M(n) * given*	*M(n) new*	*Cum. M(n)*	*C + cum. M(n)*	*w(n)*	*A(n)*
1	1	2500	2500	2500	12500	1	12500
2	0.9091	3000	2727	5227	15227	1.9091	7976
3	0.8264	3500	2892	8119	18119	2.7355	6624
4	0.7513	4000	3005	11124	21124	3.4868	6058
5	0.6830	4500	3074	14198	24198	4.1698	5803
6	0.6209	5000	3105	17303	27303	4.7907	5699
7	0.5645	5500	3105	20408	30408	5.3552	5678**
8	0.5132	6000	3079	23487	33487	5.8684	5705

The weighted average for the 7th year is minimum. $A(7) = 5678$

According to the decision making criteria, $A(6) > A(7)$ and $A(8) > A(7)$

$\therefore$ End of 7th year is the economic time for replacement.

ILLUSTRATION 8

Based on the facts given here, find the year of replacement.

$$C = \text{cost of the unit} = 5000,\ M(n) = 1000 + 800(n-1);\ n = 1, 2, 3, 4, \ldots$$

Rate of interest $= r\ \% = 10\ \%$

Solution

We know that

$r\% = 10\ \%$ implies that $r = 0.1$ and hence the discount factor is $\dfrac{1}{(1+r)^{n-1}}$

We make a table (Table 8) of all terms.

Table 10.8

Year = n	$\dfrac{1}{(1+r)^{n-1}}$	*M(n) * given*	*M(n) new*	*Cum. M(n)*	*C + cum. M(n)*	*w(n)*	*A(n)*
1	1	1000	1000	1000	6000	1	6000
2	0.9091	1800	1636	2636	7636	1.9091	4000
3	0.8264	2600	2149	4785	9785	2.7355	3577
4	0.7513	3400	2555	7340	12340	3.4868	3542**
5	0.6830	4200	2869	10209	15209	4.1698	3647
6	0.6209	5000	3105	13314	18314	4.7907	3822

The weighted average for the 4th year is minimum. $A(4) = 3542$
According to decision making criteria, $A(3) > A(4)$ and $A(5) > A(4)$
The right decision is to replace at the end of the fourth year.

10.3 Replacement of Items that Suddenly Fail

We have studied the cases of replacement where there are two important features—(1) efficiency of the item depends upon the timely maintenance and (2) scrap value and normally it decreases with time. There are electrical and electronic items that do not require maintenance and there cannot be the scrap value of the unit. For example, electric bulb or tube lights; they serve as the best examples in this class. We need these units continuously in working or 'on' mode. Such electrical instruments have a life span determined by the material of the filaments used, and the environment under which they are manufactured.

The life span can be, by experimentation and statistical computation, can be calculated to some degree of accuracy. We use the given statistical measure of probabilities in our next point wherein we plan to decide replacement policies in such cases.

Before we begin, we remind you the important statistical measure known as *expectation* or *expected value*—it is an average on a long run basis. In case of discrete data, we find it by the formula connecting the real values associated with events and corresponding probabilities of occurrence of the events. It is denoted by $E(X)$; and very useful measure for estimation.

10.3.1 Expectation—Formula and Meaning

We take an illustration to understand the notion of estimation.

Illustration 9

At a game centre, a game is played with two fair dies. A man rolls two dies; conditions are as follows.

1. If the sum point on the two dies is less than or equal to 5, the player has to pay a penalty of ₹5.
2. If the sum is greater than 5 and less than 8, he will receive a sum of ₹2.
3. If the sum of points is greater than or equal to 8, he will get ₹4.

The question is if the game is played over a long period, what is the average amount of gain or loss to the player?

Solution

We know that when two fair dies are rolled, we have 36 different types of results.

The sample space = $S = \{(x, y): 1 \le x, y \le 6$ with x and y as positive integers$\}$ $\therefore$ $n(S) = 36$

In Condition (1), there are 10 different results whose sum of two points is ≤ 5
In Condition (2), there are 16 different results in favour of the event; $6 \le \text{sum} \le 9$
In condition (3), there are 10 different results in favour of the event; $9 \le \text{sum} \le 12$

Probability for event 1, i.e. p(1) = 10/36
Similarly p(2) = 16/ 36 and p(3) = 10/ 36

Real values associated with the events are x_1 – A penalty of ₹5 (Player has to give this amount)

x_2 = ₹2 in case of 2nd event, and
x_3 = ₹4 in case of 3rd event.

Now, the following table (Table 9) will make clear what we want to convey.

Table 10.9

Event	*Real Value = x_i*	*Probability = $P(x_i)$*	*$x_i - P(x_i)$*
Sum ≤ 5	–5	10/36	–50/36
6 ≤ sum ≤ 8	2	16/36	32/36
9 ≤ sum ≤ 12	4	10/36	40/36
			Sum = $\Sigma x_i \cdot P(x_i)$ = 22/36 = 11/18

This means that, if this game is played over a long time then the expected return is that the player will get per game is 11/18.

[If the player plays for 36 times, he is likely to get 36 × (11/18) = 22]

10.3.2 Main Point of the Unit

We begin with N_0 units in operations and working conditions. These units have a maximum life span. By extensive study, probability of rate of failure is estimated and it is given in terms of probability of failure. In addition, it is known that the price for bulk purchase and individual purchase are different and known in advance. Items or units need replacement as and when it fails.

Using the given records, we have to decide two points;

1. Should we go for individual replacement or group replacement?
2. If group replacement is the decision then what could be the optimal period of replacement?

We take an illustration for this and then proceed systematically.

ILLUSTRATION 10

In a factory, there are 1000 bulbs in operation. Continuously remaining on 'on' mode, humidity in the atmosphere, and sub-standard quality are the major reasons for early failures. A study indicates the failure rate in a given week. Bulk replacement will cost ₹3 per unit and ₹3 for individual replacement. Find the optimal strategy: should we go for individual replacement or group replacement?

Week:	1	2	3	4	5
% failure during the week:	20	40	20	15	5

10.3.3 Solution Procedure

Part 1 (Individual Replacement)

1. Collect the records and plan your approach:
2. Take x_i as different week number and $p(x_i)$ as probability of failure.
3. Find expectation by applying $\Sigma x_i \cdot p(x_i) = \mathrm{E(X)}$ for all i = months/week
4. Find the expected number of bulbs that fail during the span of given period.
5. Find the total cost of replacement = Number of bulbs x individual replacement cost

Part 2 (Group Replacement and Optimal Period)

The central idea is—we find average cost per week of group replacement in that week and sum total of amount spent for individual replacement till that week.

The minimum amount of this average is selected and the particular week is the one at the end of that group replacement should take place.

$$\frac{\text{Group replacement cost + Individual replacement cost till week}}{\text{Week number}} = \text{Average cost.}$$

Average failure in the 1st week, $N_1 = N_0 \times p_1$ = Number of bulbs replaced individually

Average failure in the 2nd week, $N_2 = N_0 \times p_2 + N_1 \times p_1$

Average failure in the 3rd week, $N_3 = N_0 \times p_3 + N_1 \times p_2 + N_2 \times p_3$

Average failure in the 4th week, $N_4 = N_0 \times p_4 + N_1 \times p_3 + N_2 \times p_2 + N_3 \times p_3$

Average failure in the 5th week, $N_5 = N_0 \times p_5 + N_1 \times p_4 + N_2 \times p_3 + N_3 \times p_2 + N_4 \times p_1$

Average cost = (group replacement cost + individual replacement cost in a week)/week no.

Part 3 (Individual Replacement or Group Replacement)

This minimum amount is compared with the total cost of individual replacement cost; whichever is less becomes our decision criterion.

Now, we come back to our Illustration 10.

We follow the above described three parts.

Solution

Given that N_0 = 1000, individual price ₹5 per unit, group replacement price ₹3 per unit

Part 1

We find the expected value and then total cost of individual replacement.

Week (x_i):	1	2	3	4	5
% failure during the week	20	40	20	15	5
Probability $p(x_i)$:	0.20	0.40	0.20	0.15	0.05

$\therefore$ Expectation = $\Sigma x_i \cdot P(x_i)$ = E(X) = (1) (0.2) + (2)(0.4) + (3)(0.2) + (4)(0.15) + (5)(0.05)
= 2.45

$\therefore$ Expected number of failures = (1000/2.45) = 408

$\therefore$ Total cost of individual replacement = $408 \times 5 = 2040$

Part 2

$N_0 = 1000$

N_1 = Failures during the 1st week, $N_0 \times p_1 = 1000 \times 0.2 = 200$

N_2 = Average failure in the 1st week, $N_2 = N_0 \times p_2 + N_1 \times p_1 = 1000 \times 0.4 + 200 \times 0.2$
$= 440$

N_3 = Average failure in the 3rd week, $N_3 = N_0 \times p_3 + N_1 \times p_2 + N_2 \times p_3$
$= 1000 \times 0.20 + 200 \times 0.4 + 440 \times 0.20$
$= 368$

N_4 = Average failure in the 4th week, $N_4 = N_0 \times p_4 + N_1 \times p_3 + N_2 \times p_2 + N_3 \times p_1$
$= 1000 \times 0.15 + 200 \times 0.20 + 440 \times 0.40 + 368 \times 0.20$
$= 440$

N_5 = Average failure in the fifth week, $N_5 = N_0 \times p_5 + N_1 \times p_4 + N_2 \times p_3 + N_3 \times p_2 + N_4 \times p_1$
$= 404$

Table 10.10

Week No. = n	*No. of failures*	*Failures till week*	*Group replacement cost = G*	*Individual replacement cost = C*	*Average cost A(n) = (G + C) / n*
1	200	200	$1000 \times 3 = 3000$	$200 \times 5 = 1000$	= 4000
2	440	640	3000	$640 \times 5 = 3200$	3100
3	368	1008	3000	$1008 \times 5 = 5040$	2680
4	440	1448	3000	$1448 \times 5 = 7240$	2560
5	404	1848	3000	$1848 \times 5 = 9240$	2448**

From Table 10, we conclude that in group replacement optimal period is the end of the 5th week.

This implies that, at the end of the 5th week all the units should be changed with the fact that individual units as and when fails to be changed.

Part 3

Cost of individual replacement = 2040 (Result of part 1)
Cost of group replacement = 2448 (Decision of part 2)
Cost of individual replacement being less than that of cost of group replacement; we decide for individual replacement.

ILLUSTRATION 11

A factory working round the clock has installed 2000 tube lights. The probability of failure till the end of 5 weeks is given below. An individual tube light costs ₹50 while in the group replacement scheme it costs ₹40 per unit. You are to choose the scheme of any one of the two types.

Week:	1	2	3	4	5
% failure till the end of week:	20	60	80	95	100

Solution

Part 1

From the given records, we find the probability in the individual week.

We find it by subtraction.

Week $\cdot$ (x_i):	1	2	3	4	5
% failure during the week:	20	40	20	15	5
Probability $p(x_i)$:	0.20	0.40	0.20	0.15	0.05

$\therefore$ Expectation $= \Sigma x_i \cdot p(x_i) = \text{E(X)} = (1)\,(0.2) + (2)(0.4) + (3)(0.2) + (4)(0.15) + (5)(0.05)$
$= 2.45$

$\therefore$ Expected number of failures = (2000/2.45) = 816

$\therefore$ Total cost of individual replacement $= 816 \times 50 = 40800$

Part 2

$N_0 = 2000$

N_1 = Failures during the first week, $N_1 = N_0 \times p_1 = 2000 \times 0.2 = 400$

N_2 = Average failure in the 1st week, $N_2 = N_0 \times p_2 + N_1 \times p_1 = 2000 \times 0.4 + 400 \times 0.2$
$= 880$

N_3 = Average failure in the 3rd week, $N_3 = N_0 \times p_3 + N_1 \times p_2 + N_2 \times p_3$
$= 2000 \times 0.20 + 400 \times 0.4 + 880 \times 0.20$
$= 736$

N_4 = Average failure in the 4th week, $N_4 = N_0 \times p_4 + N_1 \times \mathrm{p}_3 + N_2 \times p_2 + N_3 \times p_1$
$= 2000 \times 0.15 + 400 \times 0.20 + 880 \times 0.40 + 736 \times 0.20$
$= 880$

N_5 = Average failure in the fifth week, $N_5 = N_0 \times p_5 + N_1 \times p_4 + N_2 \times p_3 + N_3 \times p_2 + N_4 \times p_1$
$= 808$

Table 10.11

Week No. = n	*No. of failures*	*Failures till week*	*Group replacement cost = G*	*Individual replacement cost = C*	*Average cost A(n) = (G + C) / n*
1	400	400	2000 × 40 = 80000	400 × 50 = 20000	= 100000
2	880	1280	80000	640 × 50 = 32000	56000
3	736	2016	80000	1008 × 50 = 50400	43467
4	880	2896	80000	1448 × 50 = 72400	38100
5	808	3696	80000	1848 × 5 = 92400	34480 **

For the group replacement, optimal period is the end of the 5th week.

Part 3

Cost of individual replacement = 40800

Cost of group replacement = 34480

Cost of group replacement being less than that of the cost of individual replacement; we decide for group replacement at the end of 5 weeks.

10.4 Staff Replacement and Mortality Problems

In real life situations such problems are generally faced by big organizations having many interconnected departments. In some cases, there is a trend to change the job because of certain reasons like promotion to higher scale and designation, attraction of lucrative salary and some social or personal problems.

In order to meet this expected shortage, it has remained a policy to make a pool of aspirant job applicants who have been finalized for entry position job.

In multinational companies or big corporate houses staff recruitment, training, and promotion problems are handled by HR department. It has a longstanding and in some cases extrapolated record of job histories and past record which helps taking the decision for new recruitments.

These problems are attrition related problems and some extent and within reasonable limits, are necessary. New lot of fresh appointees work with enthusiasm and competition, which provides enough force for steady progress. At the same time, it is highly necessary to maintain a certain staff on the list. Any set has a hierarchy of staff members and this proportion is maintained by making regular appointments on yearly basis.

The records showing the number of persons leaving the job at different periods are derived from past records in reputed organization and it serves as a basic tool for making certain calculations.

Such problems are known as staffing and mortality problems. We assume that a person leaves a job, according to the pre-set and agreed contract on both sides, at the end of a year.

Let us give an illustration to understand the nature of such types of problems.

ILLUSTRATION 12

In a well organised organization, in general and broader base, the following records are known for the recruitees of the company.

Year:	0	1	2	3	4	5	6	7
% continued	100	80	60	50	40	15	5	0

If we want to continuously maintain a strength of 50 members on the staff, what should be the number of appointments each year?

In a staff of 50 members there are three sub-classes. Class III has 40 persons; Class II has a strength of (next in higher order) 8 employees and the remaining 2 are of Class I and they are in the cadre of VP. We want to find that, when a new appointee can be promoted to Class II?

At this stage, we clarify some points from the data given.

1. The figures are from the 100 employees who are appointed and resumed their duties. This is shown by writing year 0.
2. Out of these 100 employees appointed there are 20% employees who leave the job at the end of one year. This means that 80% continued at the end of one year.
3. From the total 100 appointed, 60% continued at the end of second year.
4. The contract being of 7 years, there is no employees or it is right to interpret in a way that those, who have been appointed and have served the company for complete 7 years, are no further continued on job. This means that if 100 employees are appointed each year then this set up is maintained.

To clarify this further, we make a list.

1. 100 employees appointed today, serve for at least one year = $N_0 = 100$
2. Out of those 100 employees, at the end of 1 year, 80 continue to serve; $N_1 = 80$
3. Out of those 100 employees, at the end of 2 years, 60 continue to serve; $N_2 = 60$
4. Out of those 100 employees, at the end of 3 years, 50 continue to serve; $N_3 = 50$
5. Out of those 100 employees, at the end of 4 years, 40 continue to serve; $N_4 = 40$
6. Out of those 100 employees, at the end of 5 years, 15 continue to serve; $N_5 = 15$
7. Out of those 100 employees, at the end of 6 years, 5 continue to serve; $N_6 = 5$
8. Out of those 100 employees, at the end of 7 years, 0 continue to serve; $N_7 = 0$

On a year (more than seven), there are 350 (summation of all those who continued in different years) presently on the staff role.

9. An intake of 100 fresh appointees each year, there are, on an average, 350 employees presently working, i.e. appointing 100 employees each year maintains a staff of 350 at any time of a year.

Now looking at the original table, we can find the proportion for 50 employees that we want to maintain on permanently basis.

The given table is

Year:	0	1	2	3	4	5	6	7
% continued	100	80	60	50	40	15	5	0

For 350 employees in the list, we appoint 100 employees each year then in order to maintain a figure of 50, we appoint,

$$(100 \times 50)/350 = 100/7 \approx 14.3 = 14 \text{ persons each year.}$$

This generates a table.

Year:	0	1	2	3	4	5	6	7
% continued	14	11	8	7	6	3	1	0
Sum at the end	14	25	33	$\boxed{40}$	46	49	50	

A new employee when crosses the 3rd year is likely to get the promotion as he reaches the Class II grade of above 40 and less than 48.

ILLUSTRATION 13

The same example discussed above can be represented in the following way. The approach to the data is slightly in a different way but the fundamentals do not change.

A company wants to hold a strength of 30 employees on a regular basis. It has two class of positions—Class B and Class A = 25 + 5 = 30. The top 5 employees being in the senior level. The following record shows the % of appointed persons leaving the job at the end of any year mentioned. (Assume that yearly appointment of 100 employees.)

Find 1. Number of persons appointed to maintain a staff of 30 employees.
2. When a newly appointed employee will get a raise on the senior level?

Year:	1	2	3	4	5	6	7
% left:	20	40	50	60	85	95	100

Solution

The data shows that at the end of 100 employees appointed in a current year, 20% leave the job; it means that 80% continue with the job. Keeping the understanding on the same line, we can convert the above table in the one of previous type and then make necessary calculation.

Year:	0	1	2	3	4	5	6	7
% continued	100	80	60	50	40	15	5	0

This means that keeping strength of 350 (Row sum of % continued) employees, we appoint 100 employees each year then to maintain a strength of 30 employees

$$(100 \times 30)/350 \approx 60/7 \approx 9 \text{ persons each year.}$$

Now we make a table.

Year:	0	1	2	3	4	5	6	7
% continued	9	7	5	4	4	1	0	0
Sum at the end	9	16	21	25	29	30	–	–

From this, we conclude that the employee who joined today and continues for a period more than 3 years is to get a promotion on the higher post.

ADDITIONAL QUESTIONS FOR PRACTICE (WITH HINTS AND ANSWERS)

Question 1

Explain the basic notion of replacement theory in case of items that suddenly fail.

For a new machine worth ₹8000 scrap value at any time is ₹2000 being the cost of copper material in it. The maintenance cost is given by $M(n) = 800 + 500\ (n - 1)$ per year. Find the optimal replacement policy.

Solution

From the given data, $C - S(n) = 6000$; this is constant.

We make a table.

Table 10.12

$Year = n$	$C - S(n)$	$M(n)$	$Sum = \Sigma M(n)$	$C - S(n) + \Sigma M(n)$	$\therefore\ A(n) = \dfrac{C - s(n) + \sum_{n=1}^{n} M(n)}{n}$
1	6000	800	800	6800	6800
2	6000	1300	2100	8100	4050
3	6000	1800	3900	9900	3300
4	6000	2300	6200	12200	3050
5	6000	2800	9000	15000	3000**
6	6000	3600	12600	18600	3050

From Table 10.12 we observe that the average cost of the item is the lowest in the 5th year.

$$A(5) < A(6) \quad \text{and} \quad A(5) < A(4)$$

The right year of replacement is $n = 5$. The unit should be replaced at the end of 5th year.

Question 2

For a new machine worth ₹10000 scrap value at a given time n is given by the following formula = $S(n) = 8000 - 1500 * n$ where $n = 1, 2, 3, 4, 5$ only. The maintenance cost is given by $M(n) = 1000 + 600\ (n - 1)$ per year. Find the optimal replacement policy.

Solution

Using the records given, we make a table.

Table 10.13

$Year = n$	$S(n)$	$C - S(n)$	$M(n)$	$Sum = \Sigma M(n)$	$C - S(n) + \Sigma M(n)$	$\therefore\ A(n) = \dfrac{C - s(n) + \sum_{n=1}^{n} M(n)}{n}$
1	6500	3500	1000	1000	4500	4500
2	5000	5000	1600	2600	7600	3800
3	3500	6500	2200	4800	11300	3767**
4	2000	8000	2800	7600	15600	3900
5	500	9500	3400	11000	20500	4100

From Table 10.13, it is observed that $A(3) = 3767$.

$$\text{Also,} \quad A(2) > A(3) \text{ and } A(4) > A(3)$$

The machine should be replaced at the end of 3rd year.

Question 3

A factory has installed 2000 bulbs of special category and expectancy of failure is given by the following distribution.

Week:	1	2	3	4	5
Probability of failure during the week:	0.1	0.1	0.2	0.3	0.3

Cost of individual replacement is ₹15 while the group replacement is ₹8. Using the records you are required to decide the replacement policy.

Solution

Part 1

We find expected life span = $E(X) = \Sigma x_i \cdot p(x_i) = 0.1 + 0.2 + 0.6 + 1.2 + 1.5 = 3.6$

From this, we find average number of failure = 2000/3.6 = 556

Individual cost of replacement = $556 \times 15 = 8340$

Part 2

In this part, we find group replacement cost at the end of each week.

$$N_0 = 2000$$
$$N_1 = N_0 \times p_1 = 2000 \times 0.1 = 200$$
$$N_2 = N_0 \times p_2 + N_1 \times p_2 = 2000 \times 0.1 + 200 \times 0.1 = 220$$

Similarly, if we keep on finding then $N_3 = 442$, $N_4 = 702$, and $N_5 = 820$

Now, we make the table (Table 10.14) and request students to fill in the remaining data.

Table 10.14

Week = n	*No. of failures*	*Failures till week*	*Group replacement cost = G*	*Individual replacement cost = C*	*Average cost A(n) = (G + C) / n*
1	200	200	$2000 \times 8 = 16000$	$200 \times 15 = 3000$	= 190000
2	220	420	16000	$420 \times 15 = 6300$	11150
3	442	862	16000	$862 \times 15 = 12930$	9643
4			16000		
5			16000		

Now, student should find and take decision in this part and the next Part 3 also.

POINTS TO REMEMBER

This chapter, with a little modification, is very useful and important. There are some important points that the students, we hope, take note and then practice examples on each unit.

1. You may be astonished to note that, in examination, may be the 'tension' surmounted on head, some students, though they know it, take value as that amount we save after discarding the machine. *Please do not make this error*.

   What you spent = Purchase cost – Scrap value

2. Discount factor is very deceiving factor. Students have, practiced by taking discount factor by considering 10% rate and sound practice makes them ignore to read what is the % rate and hence the discount factor given in the question; mostly they take it 10% and at the end realize it to be either 15% or 20%.
3. Very important creating illusion is when we solve examples on replacement items which suddenly fail—group replacement.
   After calculating the average number of failures in a given week, we forget to add those units who have failed and have been already replaced in the previous week. Do not count the cost of individual replacement in a particular week; you have to **add** the number of failure items of the previous weeks to the items of current week and then multiply the total number of failures by the **individual replacement** cost of the unit. [It is obvious that the cost of a unit in individual replacement is more than that of a unit in group replacement
   [I remember when I did the same mistake in counting number of failures and yes, students were very good and did not raise any doubts; and solving the next example of the same type I took the right approach; then it came to my mind and I rectified the mistake by polite confession in the class.]
4. In mortality tables, do make a good practice of reading and interpreting.
5. Make *part-wise practice*; as we have shown.

EXERCISES

OBJECTIVE TYPE QUESTIONS

I. State True or False

1. Replacement theory for any type of machineries is truly effective when net present worth of the amount incurred is estimated in advance and used in the formula.
2. Net present value depends upon the rate or the % determined at the time of making the contract at the time of purchase.
3. Multiplying the future maintenance cost by a discounting factor the resulting present worth is less than that of the the future maintenance cost.
4. As the rate of interest increases, the discounting factor increases in value.
5. If the scrap value is constant then the rate by which the maintenance cost increase is the important deterministic factor for finding the optimal period of replacement.
6. Probability of failure of an item in a week depends on the number of items to be replaced.
7. Probability of failure is a tentative measure of average life span of an item that fail suddenly.
8. In calculating the number of failures in a week, we calculate only the number of failures of the originally given amount of units.
9. In calculating average, we take common price to both types of units; replaced in a group and individually changed within a given period.
10. Total number of failure units is the sum of failed in a given week and the amount replaced in a group replacement.

ANSWERS

1. true.	2. true.	3. true.	4. false.	5. true.
6. false.	7. true.	8. false.	9. false.	10. false.

II. Multiple Choice Questions

1. Group replacement policy is always better than the individual replacement policy.
   (a) True (a) False
   (c) Depends on number of given units (d) Depends on complete data
2. If N_0 is the number of original units given then number of failures in the 2nd week, N_2 is
   (a) $N_1 \times p_1$ (b) $N_0 \times p_2$
   (c) $N_0 + N_1 \times p_1$ (d) $N_0 \times p_2 + N_1 \times p_1$
3. If N_0 is the number of original units given then the total number of failures during the 2nd week, N_2 is
   (a) $N_0 \times p_2 + N_1 \times p_1 + N_1$ (b) $N_0 \times p_2$
   (c) $N_0 + N_1 \times p_1$ (d) $N_0 \times p_2 + N_1 \times p_1$
4. Number of failures, in the case of finding individual replacement, is found by
   (a) finding failures per week (b) statistical expectation
   (c) distribution method (d) finding the sum of failures in each week
5. Sum of failures in the 1st week equals the sum of failures in
   (a) the last week (b) the 2nd week
   (c) the 3rd week (d) may be any other week
6. If r is the % rate, then the discount factor of finding the NPV of the 2nd year maintenance amount equals
   (a) $1/r$ (b) $1/r^2$
   (c) $1/(1 + r)$ (d) $1/(1 + r)^2$
7. In finding the average cost, we consider
   (a) discount rate (b) probability
   (c) total number of failures up to the week (d) total number of failures plus the total number of bulbs initially started with
8. Staff replacement problems are applied in maintaining
   (a) a constant staff (b) giving promotion to staff
   (c) finding probability of removal of staff (d) making group replacement of staff
9. If scrap values for each year decreases and are different in each case then the formula of finding average in a year is
   (a) $A(n) = \dfrac{C - s(n) + \sum_{n=1}^{n} M(n)}{n}$ (b) $A(n) = \dfrac{C - \sum_{n=1}^{n} S(n) + \sum_{n=1}^{n} M(n)}{n}$
   (c) $A(n) = \dfrac{C - s(n) + \sum_{n=1}^{n} M(n)}{n + s(n)}$ (d) $A(n) = \dfrac{\sum C_i - s(n) + \sum_{n=1}^{n} M(n)}{n}$
10. For finding the average cost per year the sum of maintenance costs up to the period of replacement is
    (a) subtracted from the purchase cost of the item
    (b) added to NPV of maintenance cost of the last year
    (c) added to the sum of scrap values
    (d) added to the difference of actual investment denoted as $C – S$ (where C is the cost and S is the scrap value.)

Answers

1. (d) 2. (d) 3. (a) 4. (b) 5. (d)
6. (d) 7. (d) 8. (a) 9. (a) 10. (d)

Numerical Problems

1. A fleet owner of ABC Corporation has bought a mini cab and he has estimated the maintenance cost per year as given in the table.

Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Maintenance cost in (₹)	1000	1200	1400	1800	2300	2800	3400
Scrap value in (₹)	3000	1500	750	375	200	200	200

Purchase price of the mini cab is ₹6000. Determine the right year for replacement.

2. A truck owner estimates that the running costs and the salvage value of trucks for various years will be as illustrated below.

Year	*Running Costs (₹)*	*Resale Price (₹)*
1	6000	60000
2	7000	40000
3	9000	30000
4	12000	30000
5	15000	25000
6	20000	20000
7	25000	20000
8	30000	20000

If the purchase price of a truck is ₹80,000 estimate the optimum replacement age for the truck. Take money's value as 15% per annum.

3. Find the cost per period of individual replacement policy of an installation of 300 light bulbs. Given the following:
   1. Cost of replacing an individual bulb is ₹2.
   2. Conditional probability of failure is given below:

Week	*0*	*1*	*2*	*3*	*4*
Probability of failure till the end of the week	0	0.1	0.3	0.7	1

Also, calculate the number of light bulbs that would fail during each of the 4 weeks.

4. Purchase price of a truck is ₹1,60,000. Interest rate is 10% per annum. Find the economic life of the truck.

Year n =	1	2	3	4	5	6	7
Scrap value =	140000	120000	100000	84000	78000	66000	50000
Maintenance =	16000	18000	20000	24000	28000	32000	36000

5. An electronic robot has 5000 integrated circuits. An integrated circuit is replaced as and when it fails. The cost of replacing a single circuit is ₹10. Failure rates per month is given below.

Month	*1*	*2*	*3*	*4*	*5*	*6*
% surviving at the end of each month	0.03	0.1	0.3	0.4	0.15	0.02

Obtain the optimal replacement policy.

6. A machine costs ₹500. Maintenance cost in the *n*th year is given by the following formula. $M(n) = 100(n - 1)$ with $n = 1, 2, 3, 4, 5 \ldots n$. If money is worth 5% every year, determine the optimal period by which machine should be replaced.
7. The probability of failure rate of tiny series bulbs just before age *n* is shown below. If individual replacement costs ₹1.25 and group replacement costs ₹0.50 per unit. Make a choice between individual replacement policy and group replacement policy.

Week (n)	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
Probability of failure	0.1	0.03	0.05	0.07	0.1	0.15	0.2	0.15	0.11	0.08	0.05

8. The maintenance cost of two machines P and Q per year of operation is given below.

Year	*1*	*2*	*3*
Machine P	900	600	700 ₹
Machine Q	1400	100	700 ₹

If the rate of interest is considered as 10% per annum; on the basis of the net present worth, determine the optimal strategy. You may assume that the costs are considered in the beginning of a year.

9. A computer company has 200 computers in regular use. The total number of capacitors used in all these computers is 10,000 resistors. When any resistor fails, it is replaced immediately. it costs ₹1 for the group replacement and for replacing an individual bulb, the cost is ₹0.35. You are required to decide the optimal policy for the replacement. In case of group replacement, decide the optimal period. Failure rates till the end of months are given below.

Month	*1*	*2*	*3*	*4*	*5*	*6*
% failure till the end of month	3	10	30	70	85	100

10. The following failure rates is observed for a certain type of light bulbs.

Week (*n*)	*1*	*2*	*3*	*4*	*5*
% failing till end of the week	10	25	50	80	100

There are 1000 bulbs in use and it costs ₹2 for the group replacement and for replacing an individual bulb, the cost is ₹0.50. You are required to decide the optimal policy for the replacement. In case of group replacement decide the optimal period.

11. A factory owner has purchased a machine for ₹6000. Maintenance cost and resale values are given in the table. You are required to find the optimal period of replacement.

Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Maintenance cost (₹)	1000	1200	1400	1800	2300	2800	3400	4000
Scrap value (₹)	3000	1500	750	375	200	200	200	200

12. A firm has just purchased a second owner truck and paid ₹17,500. Any time its scrap value is ₹500. Maintenance cost are found from experience as follows;

Year	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Maintenance cost (₹)	200	300	350	1200	1800	2400	3300	4500

What is the optimal period of replacement?

13. In a well organized organization, in general and in broader base, the following records are known for the recruited members of the companies.

Year:	0	1	2	3	4	5	6	7
% continued	100	80	60	50	40	15	5	0

14. A factory working round the clock has installed 2000 tube lights. The probability of failure till the end of 5 weeks is given below. An individual tube light costs ₹50 while in the group replacement scheme it costs ₹40 per unit. You are to choose the scheme of any one of the two types.
15. Find the year of replacement. The facts are given as follows.
C = cost of the unit = 5000, maintenance cost = $M(n) = 1000 + 800(n - 1)$; $n = 1, 2, 3, 4, \ldots$
$r\% = 10\ \%$ = rate of interest

Answers

1. End of 5th year.
2. 5 years.
3. Average number of failure per week = 300/2.9 = 103 (Approx)
Cost of individual replacement of bulbs = ₹206/week.
4. 7 years.
5. Average failure during the span = ₹1389, cost = ₹13890, find the failures per week. 1st week = 150, 2nd week = 505, etc. Find optimal policy.
6. 200 < 218 < 300. Hence replacement period is $n = 3$ years.
7. After every 6 weeks.
8. Total p = ₹2023.47, total Q = ₹2069.43.
9. Replace every 3 months.
10. 2 weeks.
11. Replacement age = 5 years, average yearly cost = ₹2700.
12. $n = 7$ years.
13. $n = 14$ employees per year.
14. Group replacement policy, at the end of 5 weeks replacement is advised.
15. At the end of four years ₹3542.

11 Queuing Theory

Big shots are small shots for those who keep on shooting..

Learning Objectives

AFTER STUDYING THIS CHAPTER, THE STUDENTS WILL BE ABLE TO

- **understand the basic components of queuing theory**
- **classify the different components and their modus operandi**
- **study different models of queuing theory**
- **apply the basic notion to different area of real life situation**

INTRODUCTION

The origin of the topics treated in the units of this chapter dates back to the years 1910. Mr A.K. Erlang, a Danish engineer working for Copenbagen Telephone Company, studied the 'delay period' when subscribers used to call the control exchange to connect to the calling destination. The operators being busy in attending to the work of calling customers were unable to prompty respond and attend to the customers' request.

The original problem, Erlang studied, was pertaining to the study of delay caused by one operator on the reason of being busy in responding to the customers who called earlier. The study continued over a long period and then in 1997, the study was extended for calculating delay period and probabilities for several customers. He published his academic work with the title, 'Solution to some problems in the theory of probabilities of significance in Automatic Telephone Exchanges'.

In real life situation, facing a queue has become a part of the entire process. Facing a queue is a gross wastage of time, money, resonances and agencies associated with the parties or persons.

Customers need particular types of services from the serving systems. Decide or fix-up the service unit. The time on which they reach the serving system, the servers at the system may be busy in attending the other customers. If they are not busy then new entrant gets immediate chance to be served as expected.

In real life situations, we find many cases where we have to make approach and join waiting lines to get services of the required types.

11.1 Components of 'Q' System

The most general structure of the 'Q' system is divided in following three parts and we discuss each mode.

1. Input mode
2. Service mode
3. Output mode

1. Input Mode

The basic elements of this type of mode is the customers.

Customers from any population as the case may be from finite or infinite population decide to get the services. They have some sources relying upon which they decide to get the services from a particular source.

Either they are well conversant with the system and its norms or come well prepared with the necessary and material essential to handle sense by servers.

If they are not well equipped, either they go back without getting the service or get delayed service.

Customers intending services form pattern to join the system. Either they go in a group or they make a sequence of individuals and join the system. Customers going to some hotels or attending some functions or visiting some tourist place go in a group of two or more or in a single family group. In cases of customers going to operate the safe-deposit lockers or ATM counter operates alone or hardly in a group of two.

This is the customers aspiring the types of services themselves decide the pattern of approach. What do they find?

2. Service Mode

Once planned, and well painted in mind, a customer entering the service system may find the required service unit. Either there are many service windows and each window being independently capable of satisfying the requirements of the customers. For example, this happens at railway ticket booking office. There are several windows capable of performing reservation or booking the tickets of the type and categories required—multi-channel service counters. In some cases, customers find a single line and a single server attending to the types of services demands by the customers. This you may find at the cash withdrawal counter of a small banking unit.

In some cases, we find that there is no queue and the facilities are waiting to provide the services. In such a situation, a happy entrant immediately rushes to the service counter and gets the job done.

One additional feature, which we find in joining either single channel or multi-channel service system, is that the customers' waiting times are different.

Some units, waiting in the queue, are disbursed at an average of constant rate. Waiting time of a customer and simultaneously the service time of a server is totally a function of types of services served at the service counter and demanded by the customer.

Facing the service window, at this junction, we have some important points to share with you when we find good entry point.

3. Output Mode

In many cases, some special care is taken for some special customers who, after receiving the required services, leave the system. In general, for input mode special care and also for providing services extra care is taken for executive class or priority class of customers but the output mode is not specially designed.

An output mode is, in many respect and features, is highly necessary in drawing further business and sustaining customer relationships. Customers feel priority of being attached specially to a special system.

11.2 Three Point Program

Looking to the important features of input mode and service mode, we have three important points to keep in mind.

1. *Arrival Rate and its mathematical form*

We have already discussed about pattern of thinking and customers' behavioural attitude to go to the serving unit. In some cases, through not much reflected in educational study is the 'selection of time internal' by the customer to go to the service centre. What important is for our purpose is the average time-interval between any two conservative arrivals at the service booth. A study over a long period of time and at different time intervals helps knowing and describing the average arrival rate of the customer.

Once the average arrival rate at different time intervals is known, then we can describe the statistical distribution to study its parameter like mean, and standard derivation.

In this part, we add one point and that is about the calling size of population. It is describing from the size of the population from which the customers are normally expected to come for taking the services. If the size is finite, then we call it as finite population and it is denoted by the letter N, and if the size is infinite, we call it as infinite population and is denoted by ∞.

We have, in general and popularly accepted symbol λ, describing the average number of customers per unit time arriving at a service centre.

For example, $\lambda = 20$ conveys that there are on an average 20 customers arriving at the service booth in an average per unit time.

From this, we conclude that $1/\lambda$ is the average time interval or time slot between two consecutive arrivals.

For, $\lambda = 20$ customers,

$1/\lambda = 1/20$ and taking $1 = 1$ hour $= 60$ minutes; we have $1/\lambda = 60/20 = 3$ minute/customer.

2. *Waiting Time in a queue*

Upon entering the service units, we have either single channel or multi-channel system and the expected services are rendered from the respective server at the booth. If there are no customers then it goes very simple and the job is over. In the case, where there are sizable customers and one or more channels then the customer is expected to join the *queue* unless he is a *priority customer*.

For example, banking system, we have a tradition of *priority account* for those customers who keep a constant behave of at least ₹2,00,000. Such customers are provided special services by the banking authorities of the particular branch.

Waiting time depends on following important factors:

1. number of customers standing ahead of a *n*th customer in the discussion,
2. the efficiency of the server, and
3. the types and the number of services required by each customer standing before the service counter.

Waiting time that a customer spends in a queue is a very important factor and leads to dissatisfaction and lack of trust in the operation of the server or operator.

3. Service rate and its mathematical forms

What important is for the service unit is increasing the number of customers by providing multiple facilities at a single counter. This will finally increase the total revenue and hence overall profit.

Increasing facilities to keep the customers impressively happy does cost much and reduces the profit margin. At the same time, keeping a small work force of servers, the company can reduce the total variable daily cost and on the other end it increases the waiting time of the customers and it results into dissatisfaction of customers and so results *switch over* concept and this ends up in a financial loss at the end.

This is why the fast-growing companies are providing two types of training to the employees.

1. Interaction with customers
2. Primary knowledge about all the aspects the company deals with.

All of these features can best be discussed and improved by constant feedback. At this point of time, we need some terms representing the reality.

Now we define the terms employed to describe the general status of the system for the service unit.

μ stands for describing the average number of customers served in a unit time at a service station. From this, we interpret $1/\mu$ as the average time to serve a customer.

We recall arrival rate and jointly with that discuss the above point.

λ: Arrival rate = Let the arrival rate is 20; It means that on an average per unit time there are 20 customers arriving at a service booth.

$\mu = 30$ means on an average the service operators can handle and serve 30 customers in a unit time.

$1/\mu = 1/30$ and taking unit time as one hour = 60 minutes;

We have, $1/\mu = 60/30 = 2$ minutes which shows average time taken by the server to serve one customer.

In general, we have just introduced three important key components in any queuing system.

1. arrival rate
2. waiting time
3. service rate

11.3 Additional Points

In some cases, it is very important to know about the psychology of the out-going customers, their responses, reaction at the time of departure and suggestions which are very important to study.

Some terms affecting the 'Q' mechanism

Bulk Arrival

When the customers arrive at a service unit in a large group or people. It is termed as *bulk arrival* [generally, customers arriving in a group of more than 5 or 6].

In many cases, if such events occur more than once on a given day then it is referred as *frequent bulk arrival*.

Some Basic Terms

1. During the course work of this chapter, we will follow the following terminologies.

   1. λ: Number of customers arrived at a service booth in a unit time.
   2. μ: Number customers served by the operators on a unit time working.

3. ρ: $\lambda/\mu = \rho$ is traffic intensity
4. p_n = probability that there are n customers in the queue. $n = 0, 1, 2, 3,...$
5. $E(W_q)$: Average waiting time of a customer in the queue
6. $E(W_s)$: Average waiting time of a customer in the system
7. $E(L_q)$: Average number of customers waiting in the queue to get the service
8. $E(L_s)$: Average number of customers in the system (waiting + being served)
9. M: Poisson Distribution

In addition to this, there are some more notations, we will mention them when we need them.

2. Balking

Customers, in most of the cases, do not have long period patience to join and wait for the turn; may be having lack of time. On seeing customers' waiting in a queue for the service, they may not like to join the queue. This is called *balking*.

This inspires other customers in the queue to follow the same attitude in the future.

3. Reneging

There are some customers who have already joined the queue but on account of slow service rate or some other reasons, break the queue and leave the system without getting the service. This is called *reneging*.

Such incidences are very impulsive and provoke other customers to follow the same behaviour. Finally, it leaves following two impressions.

1. Poor service rate: This affects the server and hence to the system operations.
2. Poor facilities in the system wastes more time and insufficient means to serve the customers.

4. Jockeying

Some service units have more channels to serve the customers. This is multi-channel facilities. All the channels are busy serving the customers but either some servers operate speedily or customers in the queue demand the type of services which consume less time and as a result the potential queue decreases very fast.

In case of such events, customers in the other queue break their queue and join the shorter queue. This is called *jockeying*. This has an impact on the efficiency of the slow server.

11.4 Why do we Study the 'Q' System?

There are some important aspects to make a sound study of the system.

1. Study results can suggest about the *bulk arrival* and poor arrival time period. This, if known, can arrange more or less servers on a given time. This saves time spent on waiting in the queue and establishes the effectiveness and trust on the system.
2. Some customers do not have patience and cannot spend more time beyond limit waiting in a queue.

   They are dissatisfied, provoke the other customers, also, and pass comments to the servers. Continued impacts of such events tend to decrease the number of customers and hence loss in revenue.

   We can control this by making and managing more channels as and when the waiting time increases beyond limit.
3. In order to react with the establishments and recurrence expenses (electricity bill, installment on the bank loan, rent of the building and gadgets, salary of the persons employed, etc.)

There is always a trend of

1. attracting more and more customers,
2. providing them dignified services,
3. minimizing waiting time,
4. planning,
5. managing the service units, and
6. establishing customer or services department relation to deal with grievances and making follow ups.

All these are the major factors in establishing credibility in market by winning trust and securing high return, i.e. reducing competition by branding the service pattern.

Expressing all the above said factors in a couple of statements, we can say

***'Q' is a result of a situation when $\lambda > \mu$**

***Solve this by managing μ.**

11.5 Empirical 'Q' Models

The basic queuing models follow a particular pattern of describing them. Terminology involved in each expression signifies properties of the concerned parameter.

The most general format is: $(a/b/c)$:$(e/f/g)$

where

'a' describes arrival rate distributors,
'b' describes service rate distributors,
'c' describes number of servers,
'e' describes general discipline of the system,
'f' describes capacity of the system, and
(Maximum number of customers it can include on a time)
'g' describes the calling population from which customers come to get the services.

11.6 Operating Characteristic of a 'Q' System

As we have discussed earlier, there are two main points.

1. waiting time cost, and
2. idle time cost.

1. We know that λ is number of arrival per unit time and μ is the number of units served in a unit time. If λ increases and μ remains constant, then customers have to wait for a comparatively longer period. This event is liable to dissatisfaction from customer point of view and finally has an impact on total revenue as dissatisfied customers change to some other sources.
2. As the service rate increases, the cost of providing services also increases. So long as the flow of customers increases, it does keep a balance. In the non-peak hours or idle time, the system bears a cost for maintenance of service units and operators pay for remaining present on their duties. This is an additional cost that reduces the profitability.

   Operating characteristics of a 'Q' system involves analysis of the system which are governed and controlled by the following parameters.

1. System utilization factor

(i) Traffic Intensity

It is the ratio of the arrival rate λ and the average number of customers μ and it is denoted by the symbol ρ.

i.e. $$\rho = \frac{\lambda}{\mu}$$

if $\mu > \lambda$ then $0 < \rho < 1$ and $1 - \rho$ = idle time factor of the system.

(ii) Queue Length (Lq)

It is the average number of customers waiting to get the service. (Number of persons who are being served by the system should not be counted.)

(iii) System Length (L_s)

It is the total number of customers in the system. In other words, it is the sum of the waiting customers and customers being served.

(iv) Waiting time in Queue (W_q)

It shows an average time that a customer spends waiting in a queue.

(v) Waiting time in System (W_s)

It shows the average sum total of the time spent by the customer in waiting and the time spent by the customer during the service. It is the average time that a customer spent in the set up or the **system**.

(vi) Transient state and steady state of the system

If the operating characteristics (refer to the five points discussed here), i.e. behaviour of the system, changes with the time than the system is said to be in transient state situation. It is the initial time when the system starts its operations. The transient state depends on certain parameters, like number of customers in the system, number of servers present in the system, and they are about to begin the operations, etc.

A system *settles down*, i.e. reaches a *steady state* position when its behaviour becomes independent of the initial condition and the total elapsed time. The steady state condition plays important role in the queuing theory.

In this case, waiting time settles down and attains a steady state position under the condition that the behaviour becomes independent of time.

(vii) Some important models

Basically the classification of queue models falls under following three catagories.

1. deterministic model,
2. probabilistic model, and
3. mixed model.

1. Deterministic Model In the first type of general form of the model, deterministic model, is of the type (D/D/1):(FCFS/∞/∞)

Here,

D stands for deterministic type of time distribution for inter-arrival and inter-service rate.

FCFS: It is a status showing the system has a rule of 'first come first served type'. There is only one server in the system. The system capacity is infinite and the calling population is also infinite.

2. Probabilistic Model In the probabilistic type of model, the symbolic presentation is (M/M/1): (FCFS/∞/∞)

First M shows *Poisson arrival* or *exponential inter-arrival* distribution. There is only one server.

There are some other probabilistic models, in which there are variations in one or more of the six features in the basic model.

There are some models in which number of servers changes. In some cases the system's behaviour in different patterns, like FCFS, LCFS, SIRO or GD type.

Here,

FCFS: First come first served
LCFS: Last come first served
SIRO: Service in random order
GD: General service discipline

There are some models of the type like,

(M/M/1): (SIRO/∞/∞), (M/M/C): (FCFS/∞/∞),
General Erlang Model (M/M/1): (FCFS/∞/∞),
(M/M/1): (SIRO/N/∞), (M/M/C): (FCFS/N/∞), etc.

Now, we will discuss three of these models which are very useful and highly applicable in real life situations.

11.7 Single Channel; Poisson Arrival; Exponential Service Rate

Model Type: (a:b:1) (FCFS/∞/∞)

In this basic model, we have some standard assumptions and notations.

1. Customers arrival at a service booth and the arrival is approximated by Poisson distribution ($a = M$)
2. Exponential distribution for service rate means it is independent of number of customers waiting in the queue. Customers behavior and demand, in general, does not affect the service pattern. ($b = M$)
3. There is a single service. ($c = 1$)
4. The pattern in which the services are given is 'First come first served' (FCFS).
5. The service capacity to incorporate the incoming customers is very large.
6. The customers from any size of population can join the service unit.
7. Achieving a steady state; we have $0 < \mu$ and $\mu > \lambda$.
   $\lambda\Delta t$ shows the probability that a new customer joins the system between the time t and $t + \Delta t$.
   $\mu\ \Delta\lambda$ shows the probability of completion of a service between the time t and $t + \Delta t$.
   $p_n(t)$ shows the probability of having exactly n customers in the system at a time t.
   $p_n(t + \Delta t)$ shows probability of having exactly n customers in the system at a time $t + \Delta t$

Now we derive the properties of the single channel system

Fundamental results for a single channel (server)

1. if $p_n(t)$ represents the probability of having exactly n persons on a given time t, then the event that there are exactly n persons in the system at a time $t + \Delta t$ can occur in exactly one and only one way from exactly four possible cases.

Events	*Probability of n units at time 't'*	*Arrivals in time $t + \Delta t$*	*Service completed in time $t + \Delta t$*	*Units in the system at $t + \Delta t$*
E_1	p_n	0	0	n
E_2	p_{n+1}	0	1 (left)	n
E_3	p_{n-1}	1 (joined)	0	n
E_4	p_n	1	1	n

As these four events are mutually exclusive and exactly any one can occur, we find

$$p_n(t + \Delta t) = p(E_1) + p(E_2) + p(E_3) + p(E_4)$$

$$\therefore \quad p_n(t + \Delta t) = p_n(t) \cdot (1 - \lambda\Delta t)(1 - \mu\Delta t) + p_{n+1}(t) \cdot (1 - \lambda\Delta t) \cdot \mu \,\Delta t$$
$$+ p_{n-1}(t) \cdot (\lambda\Delta t)(1 - \mu\Delta t) + p_n(t) \cdot (\lambda\Delta t) \cdot (\mu\Delta t)$$

Neglecting the higher order terms like $\Delta t^2 > 0$; we write

$$\frac{p_n(t + \Delta t) - p_n(t)}{\Delta t} = -(\lambda + \mu)\; p_n(t) + \mu\, p_{n+1}(t) + \lambda\, p_{n-1}(t)$$

Now as $\Delta t > 0$,

$$\frac{d}{dt} p_n(t) = -(\lambda + \mu)\; p_n(t) + \mu p_{n+1}(t) + \lambda\, p_{n-1}(t) \text{ ... for } n \geq 1 \tag{1}$$

From result (1), for $n = 0$; Exactly two events (which are mutually exclusive and collectively exhaustive) can occur

Event 1: No unit ($n = 0$) at time t and no arrival during the time slot $t + \Delta t$

Event 2: 1 unit at time t, no new customer arrived in time $t + \Delta t$ and 1 unit who was being served, left during $t + \Delta t$

$$\therefore \quad p_0(t + \Delta t) = p_0(t) \cdot (1 - \lambda\Delta t) + p_1(t)\,(1 - \lambda\Delta t) \cdot (\mu\Delta t)$$

Neglecting higher order terms; putting $\Delta t^2 = 0$ and algebraic arrangement brings the Equation 1 to

$$\frac{p_0(t + \Delta t) - p_0(t)}{\Delta t} = -\lambda P_0(t) + \mu\, p_1(t)$$

As $\Delta t > 0$; we have

$$\frac{d}{dt} p_0(t) = -\lambda\, p_0(t) + \mu\, p_1(t) \quad \text{for} \quad n = 0 \tag{2}$$

Equations 1 and 2 describe the probability density functions for exactly n customers at a given time t for all $n \geq 0$

For the stable or steady state condition, we have

$$\frac{d}{dt} p_n(t) = 0$$

We have from Equations 1 and 2

$$0 = -(\lambda + \mu)p_n + \mu p_{n+1} + \lambda p_{n-1} \qquad \text{for} \quad n > 0 \tag{3}$$

and

$$0 = -\lambda p_0 + \mu p_1 \cdot \text{for } n = 0 \tag{4}$$

from which we write

$$p_1 = \frac{\lambda}{\mu} p_0 \tag{5}$$

$$\frac{\lambda}{\mu} = \text{traffic intensity} = \rho$$

$$\therefore \qquad p_1 = \rho \cdot p_0$$

We derive the following results by using Equation 4

$$0 = -(\lambda + \mu)p_n + \mu p_{n+1} + \lambda p_{n-1} \quad \text{for} \quad n > 0 \ (n \in N)$$

We put $n = 1$

$$\therefore \qquad (\lambda + \mu)p_1 - \lambda p_0 = \mu p_2$$

$$\therefore \qquad p_2 = \frac{\lambda + \mu}{\mu} p_1 - \frac{\lambda}{\mu} p_0$$

But using Relation 5,

$$p_1 = \frac{\lambda}{\mu} p_0$$

We write

$$p_2 = \left[\frac{\lambda + \mu}{\mu} \left(\frac{\lambda}{\mu} \right) p_0 - \frac{\lambda}{\mu} \right] p_0$$

$$\therefore \qquad p_2 = (\lambda/\mu)^2 \, p_0 \tag{6}$$

This can be extended and in general, we write

$$\therefore \qquad p_n = (\lambda/\mu)^n \, p_0 \tag{7}$$

But, sum of the probabilities = $\sum_{n=0}^{\infty} (p_0) = 1$

$\therefore$ summing up

$$\sum_{n=0}^{\infty} (p_n) = \sum_{n=0}^{\infty} (\lambda/\mu)^n \cdot p_0$$

$$1 = \frac{1}{1 - (\lambda/\mu)} p_0$$

[$\because \quad 0 < \lambda/\mu < 1$ and sum of infinite terms of G.P. = $\frac{a}{1-r}$]

$$\therefore \qquad p_0 = 1 - (\lambda/\mu) \tag{8}$$

Using this in Equation 7, we get for $n \geq 0$ (9)

We write, $\lambda/\mu = \rho$

$$\therefore \qquad p_n = \rho^n \cdot (1 - \rho) \tag{9}$$

We can derive a formula for n number of customers in the system. This can be done by using expectation formula

$$E(n) = \sum_{n=0}^{\infty} n \cdot p_n$$

We get
$$E(n) = \frac{\lambda}{\mu - \lambda} \tag{10}$$

In order to find average number of customers waiting in the system to be served, i.e. $E(m)$ can be found as follows

$$E(m) = E(n) - \frac{\lambda}{\mu}$$

[$\frac{\lambda}{\mu} = \rho$ shows the probability that the service unit is busy giving the service to one customer at a time. Using $E(n)$ from Result 10, we get

$$E(m) = \frac{\lambda}{\mu - \lambda} - \frac{\lambda}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)} \tag{11}$$

Now, we derive the result for an average time that a customer spends in the system i.e. $E(W_s)$

During the time $E(W_s)$, the average number of customers arrive in the system is $\lambda E(s)$ and we write

$$\lambda \cdot E(W_s) = E(n)$$

∴
$$E(W_s) = \frac{E(n)}{\lambda} \tag{12}$$

$$= \frac{\lambda}{\mu(\mu - \lambda)}$$

$$= \frac{1}{\mu - \lambda}$$

From this we can find the average time that a customer spends in a queue is

$$E(W_q) = E(W_s) - \frac{1}{\mu}$$

$$E(W_q) = \frac{\lambda}{\mu} \cdot \frac{1}{\mu - \lambda} \tag{13}$$

One more result we add to this list

$$E(m) = \text{average length of non-empty queue}$$

$$= \frac{\mu}{\mu - \lambda} \tag{14}$$

In addition to the standard notations and results derived, we make an exhaustive list of formulae or results.

11.8 LIST OF FORMULAS/DERIVED RESULTS

1. λ: Arrival rate per unit time.
2. μ: service rate per unit time. [for steady state position $\mu > \lambda$]
3. $\rho = \dfrac{\lambda}{\mu} < 1$; [$\rho$ = traffic intensity.]

   ρ shows system utilization factor and in general $0 < \rho$ **< 1**.
   so that $(1 - \rho)$ shows the time factor during which the system remains idle. It shows non-utilization factor.
   $(1 - \rho)$ shows the probability that the system is empty, i.e. there is no customer.
   $(1 - \rho)$ also shows the probability that a new customer (just entered in the system) will be immediately get the service. It can be interpreted that if the system is idle, then it shows the probability of having no customer in the system.

   $p_0 = 1 - \rho = 1 - \dfrac{\lambda}{\mu} = \dfrac{\mu - \lambda}{\mu}$]
4. Average arrival time and the time spent in the system = $W_s = \dfrac{1}{\mu - \lambda}$
5. Average arrival time and the time spent in the queue (before being served) = $W_q = \dfrac{\lambda}{\mu} \cdot \dfrac{1}{\mu - \lambda}$
6. Average number of customers in the system = $L_s = \lambda \cdot \dfrac{1}{\mu - \lambda} = \lambda \cdot W_s$
7. Average number of customers in the $(\rho)^{k+1}$ queue = Average queue length

   $$= L_q = \lambda \cdot [W_q] = \lambda \cdot \frac{\lambda}{\mu} \cdot \left(\frac{1}{\mu - \lambda}\right) = \frac{\lambda^2}{\mu} \cdot \left(\frac{1}{\mu - \lambda}\right)$$
8. Probability that there are k customers in the system $= \boldsymbol{(\rho)^k \cdot (1 - \rho)} = \dfrac{\lambda}{\mu} \cdot \left(\dfrac{1}{\mu - \lambda}\right)$
9. Probability that there are more than k customers = $\left(\dfrac{\lambda}{\mu}\right)^{k+1} = (\rho)^{k+1}$
10. Probability that the number of customers is less than $k = 1 - [(\rho)^k (1 - \rho) + (\rho)^{k+1}] = 1 - \rho^k$

ILLUSTRATION 1

For a standard queue system we have λ **= 12** and μ **= 15**.
1. Find the percentage non-utilization factor.
2. Find the probability that there are more than 8 customers in the system.

Solution

It is given that λ **= 12** and μ **= 15**

We know that traffic intensity, $\rho = \dfrac{\lambda}{\mu}$

$$= \frac{12}{15} = \frac{4}{5}$$

i.e. $1 - \rho = 1 - \dfrac{4}{5} = \dfrac{1}{5} = \mathbf{20\%}$

i.e. in general on a working period the system remains 20% idle.

ILLUSTRATION 2

For a queue system with single server and having Poisson distribution for arrival and service time following exponential distribution, we have $\lambda = 15$ and $\mu = 20$

1. Find average queue length formed from time to time.
2. If the customer are ready to remain in the queue for not more than 6 minutes, do you suggest to increase the service facility?

Solution

It is given that

$$\lambda = 15 \quad \text{and} \quad \mu = 20$$

$$= \text{traffic intensity}, \rho = \frac{\lambda}{\mu}$$

$$= \frac{15}{20} = \frac{3}{4}$$

1. Average queue length $= \dfrac{\lambda^2}{\mu} \cdot \left(\dfrac{1}{\mu - \lambda}\right)$

$$= \frac{15^2}{20} \cdot \left(\frac{1}{20 - 15}\right)$$

$$= \frac{9}{4} \cong \mathbf{2\ customers.}$$

1 Waiting time of a customer in a queue

$$W_q = \frac{15}{20} \cdot \left(\frac{1}{20 - 15}\right)$$

$$= \frac{3}{20} \,\mathbf{hours = 9\ minutes}$$

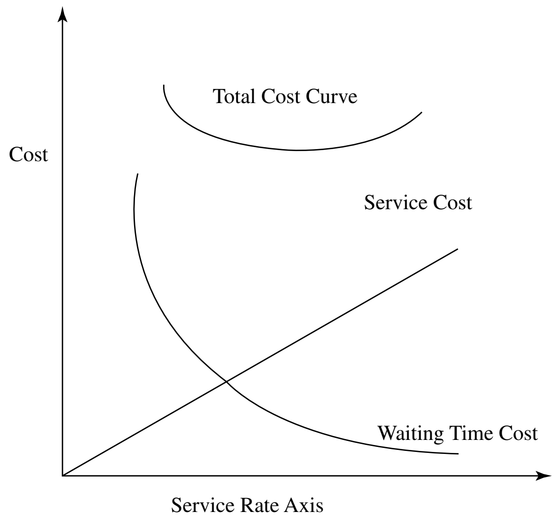
2. Customers are ready to wait for not more than 6 minutes and so the management must increase the service facilities so that average waiting time should not be more than 6 minutes.

11.9 SINGLE CHANNEL SYSTEM AND MINIMUM COST PLAN

As we have already discussed about the objectives of the study of queuing system, we write that

1. If μ is lower, then the waiting time cost affects the customers which indirectly results in loss of revenue in the long run, either the customers develop the tendency to shift or change to some other servers or they decrease the frequency to visit the service booth for acquiring some manageable types of services.
2. In turn, if the service facilities are increased and the flow of customers does not remain up to the mark, then there are overheads of the system facilities and operators which are more expensive.

We want to see that there is a rational balance between these two components.



G1. As the service rate increases, the waiting cost to the customers decreases as they have to wait for a short period.

G2. The service cost bear by the service providers is a linear function of the service rate.

G3. Total cost is the algebraic sum of these two types of costs. It is also a function of service rate. Hence it can be differentiated with respect to μ.

[Both the costs are considered per unit time]

$$\therefore \qquad \text{Total cost} = C_m + C_f \tag{15}$$

C_m = Waiting cost for m customers on an average who wait

C_f = Cost of facilities

C_w = The expected waiting cost is obtained by multiplying waiting cost of arrival per unit time by the number of customers in the system during the same period.

$$\therefore \qquad C_w = C_m \cdot E(n) = C_m * \frac{\lambda}{\mu - \lambda} \qquad \text{[using Equation 10]}$$

In the same way the expected service cost is obtained by multiplying service cost C_1 per unit by the number of customers (μ) served in a unit time.

$$\therefore \qquad C_f = C_1 \cdot \mu$$

$$\therefore \qquad \text{Total cost} = C_m \frac{\lambda}{\mu - \lambda} + C_1 \cdot \mu$$

$$\frac{d}{d\mu}(\text{Total cost}) = -\frac{\lambda \cdot C_m}{(\mu - \lambda)^2} + C_1$$

Hence, we get the formula

$$\mu = \lambda \pm \sqrt{\frac{C_{w \cdot \lambda}}{C_1}} \tag{16}$$

11.10 Multi-channel System: (M/M/K) Type

The theory starts with the main point of the assumption that number of parallel servers k are working simultaneously.

Arrival rate and the service rate are assumed to be approximated by Poisson distribution

The first impact comes on service utilization factor; denoted here by the symbol p_k

$$p_k = \frac{\lambda}{\mu K}$$

If there are n customers on a given time and there are k units serving (one customer at one point), then for $n > k$; there are $(n - k)$ units in the waiting line. This situation is continued until a server is free to attend the other customer.

For $n \geq k$, $\mu_n = k\mu$ and for $n < k$, $\mu_n = n\mu$

Here we have three cases

1. For $n = 0$
2. $1 \leq n < k$
3. $n \geq k$

We use Equations (1) and (2) in the above mentioned three cases.

Equations (1) and (2) are mentioned below with same number tag. Then we try for the further substitutions.

Equation (1) is

$$\frac{d}{dt} p_0(t) = -(\lambda + \mu)\, P_n(t) + \mu P_{n+1}(t) + \lambda P_{n-1}(t)$$

Equation (2) is

$$\frac{d}{dt} p_0(t) = -\lambda p_0(t) + \mu p_1(t)$$

Case 1

For $n = 0$, we have the same equation

$$\frac{d}{dt} P_0(t) = -\lambda P_0(t) + \mu P_1(t) \tag{17}$$

Case 2

For $1 \leq n < k$, we have

$$\frac{d}{dt} P_n(t) = -(\lambda + n\mu)\, P_n(t) + (n + 1)\mu\, P_{n+1}(t) + \lambda\, P_{n-1}(t) \tag{18}$$

Case 3

For $n \geq k$, we have

$$\frac{d}{dt} P_n(t) = -(\lambda + k\mu)\, P_n(t) + k\mu\, P_{n+1}(t) + \lambda\, P_{n-1}(t) \tag{19}$$

Now, $\lim_{t \to \infty} P_n(t) = P_n$

We derive $P_1 = \dfrac{\lambda}{\mu} P_0$; $(n = 0)$

$$(\lambda + n\mu)\, P_n = (n + 1)\mu\, P_{n+1} + \lambda P_{n-1} \text{ (For } 1 \leq n < k)$$

and

$$(\lambda + k\mu)\, P_n = k\mu\, P_{n+1} + \lambda\, P_{n-1};\ (n \geq k)$$

Also from these equations and recurrence relationship shapes the above equations as follows.

$$p_n = \frac{\mathbf{1}}{\mathbf{n!}}\left(\frac{\lambda}{\mu}\right)^n \cdot p_0 \qquad \text{with } P_0 = 1 - \rho \quad (\text{For } 1 \le n < k) \tag{20}$$

And

$$p_n = \frac{1}{k!\,k^{(n-k)}} \cdot (\lambda/\mu)^n\, p_0 \quad (\text{For } n \ge k) \tag{21}$$

Also, we add one more result for probability of having no customer in the queue.

$$p_0 = \frac{1}{\sum_{n=0}^{n=k-1} \frac{1}{n!}(\rho^n) + \frac{1}{k!}(\rho^k) \cdot \left(\frac{k}{k-\rho}\right)} \tag{22}$$

Which holds for $k > \rho$ where $\rho = \lambda/\mu$

For $k \le \rho$ the waiting line increases.

A list of some results related to this topic is given below. Students should understand and remember the style that will help them recalling the results faster.

$$\frac{\lambda}{\mu} = \rho$$

Average time a customer waits in the system

$$E(W_s) = \frac{\mu \cdot \rho^k}{(k-1)!\,(k\mu - \lambda)^2} \cdot p_0 + 1/\mu \tag{23}$$

Average time a customer waits in the queue

$$E(W_q) = \frac{\mu \cdot \rho^k}{(k-1)!\,(k\mu - \lambda)^2} \cdot p_0 \tag{24}$$

This shows the difference in the two results.

Average number of customers in the system

$$E(s) = \lambda \cdot \left[\frac{\mu \cdot \rho^k}{(k-1)!\,(k\mu - \lambda)^2} \cdot p_0\right] + \frac{\lambda}{\mu} \tag{25}$$

Compare this with the result for $E(w_s)$ and note the difference.

Average number of customers in the queue

$$E(q) = \lambda \cdot \left[\frac{\mu \cdot \rho^k}{(k-1)!\,(k\mu - \lambda)^2} \cdot p_0\right] \tag{26}$$

Compare this result with the result for $E(W_q)$ and note the difference.

Comments:

There are some points to take care about the system.

1. Find arrival rate λ, Service rate μ, and number of channels k or the number of servers.
   See that you do not make a mistake in finding μ by multiplying number of channels and average number of customers served per hour.
2. Sometimes, for faster calculation, you may keep the value of ρ ready for further use.
3. Initially, you have to find $\boldsymbol{p_0}$. This is the preliminary requirement of all related formulae.

4. In the above mentioned list, you may observe that the formulae related to finding expected waiting time in the system and expected waiting time in a queue are very closely connected to those of finding expected number of customers in the system and expected number of customers in the waiting line. The results are;

$$E(L_s) = \lambda \cdot E(W_s) \text{ and } E(L_q) = \lambda \cdot E(W_q)$$

Now, we take illustrations.

11.11 ILLUSTRATIONS

Now we take up illustrations depending upon two type of 'Q' system that we have discussed.

Type-1 ($m/m/1$): (FCFS/∞/∞)

ILLUSTRATION 3

Customers arrive at a service counter at a rate 15 in one hour time. Service provider serves them on the first come first served basis at the rate 20 customers in one hour. Assuming that the flow of customers follows Poisson distribution and service pattern follows exponential distribution; answer the following questions.

1. What is the probability that a new entrant immediately gets the service?
2. What is the probability that there is no customer in the queue?
3. How many customers are there in the system?
4. What is an average waiting time of a customer in the system?

Solution

As given, we have $\lambda = 15, \mu = 20$

$\therefore$ $\rho = \lambda/\mu = 15/20 = 3/4$

1. A new entrant can immediately get the service if the system is not busy.

$$p_0 = 1 - \rho = 1 - 3/4 = 1/4$$

2. Probability that there is no customer in the queue is when there is no customer in the system or there is only one customer is being served.

$\therefore$ We find $p_0 + p_1 = (1 - \rho) + (1 - \rho) \cdot p_0 = (1 - 3/4) + (1 - 3/4) \cdot 3/4 = 1/4$

3. Expected number of customers in the system $= \lambda/(\mu - \lambda) = 15/(20 - 15) = 3$
4. Average waiting time in the system $= 1/(\mu - \lambda) = 1/5 = 12$ minutes.

ILLUSTRATION 4

Customers arrive at a service counter at a rate of one customer in 4 minutes. Service provider serves them on the first come first served basis at the rate of 3 minutes per customer. Assuming that the flow of customers follows Poisson distribution and service pattern follows exponential distribution; answer the following questions.

1. What is the probability that a new entrant will have to wait for the service?
2. What is the probability that there are 5 customers in the system?
3. How many customers are there in the queue?
4. What is an average waiting time of a customer in the queue?

Solution

As given, we have $1/\lambda = 4$ minutes, so we have $\lambda = 15$, $\mu = 20$

$\therefore \quad \rho = \lambda/\mu = 15/20 = 3/4$

1. New entrant will have to wait for the service only if the system is busy.
   Probability that the system is busy is
   $$\rho = 3/4$$
2. Probability that there are 5 customers in the system $= p_{10} = \rho^5 \cdot p_0 = (3/4)^5 \cdot (1 - 3/4)$
   $= (3/4)^5 \cdot (1/4)$
3. Expected number of customers in the queue are $= \dfrac{\lambda^2}{\mu(\mu - \lambda)} = \dfrac{15^2}{20(20 - 15)} \approx 2.25 = 3$
4. Average waiting time of a customer in the queue, $E(W_q) = E(W_q) = \dfrac{\lambda}{\mu} \cdot (1/(\mu - \lambda))$
   $= (3/4) \cdot (1/5) = 3/20$ hour

ILLUSTRATION 5

Customers arrives at a single window counter at a rate of 4 customers per 12 minutes and they are served at a rate of 1 customer in 2 minutes. Answer the following questions:

1. What is the length of non-empty queue?
2. How many customers are there in the system?
3. Find the probability that there are at most 3 customers.
4. Find the probability that there are more than 3 customers.

Solution

From the given information, we deduce that $\lambda = 20$ and $\mu = 30$

1. Length of non-empty queues $= \dfrac{\mu}{\mu - \lambda} = \dfrac{30}{30 - 20} = 3$
2. Number of customers in the system $= \dfrac{\mu}{\mu - \lambda} = 20/(30 - 20) = 2$
3. Probability that there are at most 3 customers $= p_0 + p_1 + p_2 + p_3$
   $= 1 - \rho^4 = 1 - (20/30)^4$
   $= 1 - 16/81 = 65/81$
4. Probability that there are more than 3 customers $= 1 - (p_0 + p_1 + p_2 + p_3)$
   $= 1 - 65/81 = 16/81$

ILLUSTRATION 6

In a public call office there is a single international calling booth. Persons arriving at a booth, on study, found that it can be approximated by Poisson distribution with a mean equal to 6 customers in an hour. It is found that on an average a person makes a phone call lasting for 3 minutes. Based on above information answer the following questions.

1. What is the probability that a new entrant immediately gets the service?
2. What is an average waiting time of a customer in the queue?
3. How many customers are there in the system?

Solution

From the given information, we have $\lambda = 6$ customers in one hour, $\mu = 20$

1. It is possible only when the service unit is not busy.

   $$\rho = \lambda/\mu = (6/20) = 3/10 = 0.3$$

   Probability that a new entrant immediately gets the service = $1 - \rho = 1 - 0.3 = 0.7$
2. Average waiting time of a customer in the system = $1/(\mu - \lambda) = 1/(20 - 6) = 1/14$
3. Expected waiting time in a queue = $\dfrac{\lambda}{\mu} \cdot [1/(\mu - \lambda)] = (6/20) \cdot [1/(20 - 6)]$

   $$= (3/10) \cdot (1/14) = 3/140$$

   $$\approx 1.3 \text{ minutes}$$

ILLUSTRATION 7

Different customers arrive at a gas station requiring different types of services. They arrive following Poisson distribution with a mean time equal to 4 minutes between two consecutive arrivals. They are served at a single service point following an exponential distribution with mean equal to 20 customers.

1. Find the average waiting time that a customer has to spend in a queue.
2. A survey shows that customers will not change over if the waiting time is reduced to at most 5 minutes. You are required to find the incremental service resources that satisfies customers' demand.

Solution

As given, $\lambda = 60/4 = 15$, $\mu = 20$

1. Average waiting time, $E(W_q) = \dfrac{\lambda}{\mu(\mu - \lambda)} = \dfrac{15}{(20)\cdot(20-15)} = 3/20 = 9$ minutes
2. It is clear that average waiting time = $E(W_q) = \dfrac{\lambda}{\mu(\mu - \lambda)} \le 1/12$

   For a given value of $\lambda = 15, \dfrac{15}{\mu(\mu - 15)} \le \dfrac{1}{12}$

   $\therefore \qquad \mu^2 - 15\mu - 180 \ge 0,$

   Solving this as a quadratic equation, we get

   $$\mu = \frac{15 \pm \sqrt{225 + 4(1)\,(-180)}}{2} = \frac{15 \pm \sqrt{945}}{2} \approx 23$$

   This means that $\mu \ge 23$ customers per hour.

ILLUSTRATION 8

In a collage library, there are 4 computers having surfing facility. Students visit at a rate of 10 students in an hour. Each student, on an average, spends 20 minutes on the system. Assuming arrival rate following Poisson distribution and service rate an exponential one; find an average waiting time in the queue for a student for his/her turn on operation. How many students are there in the system?

Solution

Here, we have $\lambda = 10$, $\mu = 1/20 = 3$, and $k = 4$

We have to find p_0 and hence other results can be derived based on it.

$$p_0 = \frac{1}{\sum\limits_{n=0}^{n=k-1} \dfrac{1}{n!}(\rho^n) + \dfrac{1}{k!}(\rho^k) \cdot \left(\dfrac{k}{k-\rho}\right)}$$

with $\rho = \lambda/\mu = 10/3$, on substitution, we get $p_0 = 0.02131$

An average waiting time in the queue $E(W_q)$ for a student for his/her turn on operation

$$E(W_q) = \frac{\mu \cdot \rho^k}{(k-1)!\,(k\mu - \lambda)^2} \cdot p_0$$

By substitution we get,

$E(W_q) = 0.0329$ of 1 hour.

Number of students in the system, $E(L_s) = \lambda \cdot \left[\dfrac{\mu \cdot \rho^k}{(k-1)!\,(k\mu - \lambda)^2} \cdot p_0 \right] + \dfrac{\lambda}{\mu}$

Substituting the values, we get

$$E(L_s) = 6.622 \approx 7$$

11.12 Other Waiting Line Models

1. Poisson arrivals and Erlang Service Distribution ($M/E_K/1$)

In this type of model, we shall continue the basic assumption that the arrival rate follows Poisson distribution—a rare distribution which a limiting case of binomial distribution. We have, number of customers from infinite population and hence the mean of the distribution is = $np = \lambda$.

In some cases, it is not justified to approximate the service distribution only by Poisson distribution or exponential distribution. We tackle some real life situations and more justified distribution is known as *Erlang distribution*. This will incorporate both the exponential distributions and the situation where service time is constant. In the context of this book and rigorous mathematical calculations, we shall limit our discussion of this distribution favouring to only one server on a given time.

The exponential distribution describing service time distribution has the basis of a single parameter—mean (μ), handling all that the distribution can give. Keeping away exceptional cases, it is observed that larger service times are inherently less probable than smaller service times rendered to a customer. We generalize the exponential distribution to a two-parameter condition; called the Erlang service time distribution. It is very useful in handling many real life situations where we need two parameters to describe all necessary details of a distribution.

The Erlang distribution $g(t; \mu; k) = C_k \cdot t^{k-1} \cdot e^{-k_\mu t}$ for $0 < t < \infty$ (1)

where C_K are constants depending on given value of μ. Each member of this Erlang family is in itself a probability distribution over the range for $0 < t < \infty$. By taking $k = 1$, we get one parameter i.e. exponential distribution.

The constants C_K for $k = 1, 2, 3, \ldots$ are so assigned that the area under the curve becomes unity.

$$C_1 = \mu,\ C_2 = 4\mu^2,\ C_3 = (27/2) \cdot \mu^3 \ldots$$

In general, $$C_K = \frac{(k_\mu)^k}{(k-1)!} \quad (2)$$

[For $k = 1$, we have, $\int\limits_0^\infty C_1 \cdot t^{1-1} \cdot e^{-1\mu t}\, dt = \int\limits_{t=0}^\infty \mu \cdot 1 \cdot e^{-\mu t} dt = 1$ using the property of Poisson probability density function.

Erlang distribution has many interesting properties. The common mean is $1/\mu$.

The mode is located at $t = 0$ for $k = 1$;

The mode is found at $t = 1/2\mu$ for $k = 2$ in general the mode is located at $t = \dfrac{k-1}{k\mu}$

The variance for the kth member of the family is $1/k\mu^2$.

As k increases, the mode moves to the right towards $1/\mu$, and the variance is zero, so we may interpret $g(t; \mu, \infty)$ as the situation for which the service time is constant and has the value $1/\mu$.

The derivation of queue properties for the case of Erlang service is based on the use of state possibilities. An individual state is defined as the positive integer in the system together with the current servicing phase of the unit present in service.

Erlang distribution has an important property that if the random and independent variables are given as $x_1, x_2, \ldots, x_S$ and each variable has a common exponential distribution with mean $1/s\mu$, then the random variable $x_1 + x_2 + \cdots + x_S$ follows the sth Erlang distribution with parameter μ.

We shall consider the case of Poisson arrivals with mean arrival rate λ, and service time follows the Erlang distribution of order s and the mean service rate μ. The results are given below without proof. All of these results can be verified for the state of order $s = 1$; they conform the exponential service distribution.

Average waiting time of a customer in the system, $$E(W_s) = \frac{s+1}{2s} \cdot \frac{\lambda}{\mu(\mu - \lambda)} + \frac{1}{\mu} \tag{4}$$

Average waiting time of a customer in the queue, $$E(W_q) = \frac{s+1}{2s} \cdot \frac{\lambda}{\mu(\mu - \lambda)} \tag{5}$$

You may verify these results by substituting $s = 1$ and see that they are same as the ones of our results for ($M/M/1$): (FCFS/ ∞/∞)

In the previous cases, we have seen that the close relationship of multiplying these results by λ gives the corresponding relationships of finding number of customers in the system and waiting in queues.

ILLUSTRATION 9

The repair of a split AC machine requires 4 steps. These steps are carried out in a sequence of operations. The time taken to perform each step follows exponential distribution with a mean of 6 in one hour. It is independent of other steps . Average service time for servicing a state is 10 minutes. Answer the following

1. If there is only one mechanic available in the workshop then what is the expected idle time of the AC to remain in the system ?
2. What is the average waiting time of a breakdown machine in the queue?

Solution

We are given that $\lambda = 3$, $\mu = 4$ and being 4 stages so $s = 4$

We use Erlang distribution model.

1. The necessary formula is, $E(W_s)$ $\dfrac{s+1}{2s} \cdot \dfrac{\lambda}{\mu(\mu - \lambda)} + \dfrac{1}{\mu}$

   Substituting the above values, we get $E(W_s)$ = 16.25 minutes.

2. Expected waiting time in the queue, $E(W_s) = \dfrac{s+1}{2s} \cdot \dfrac{\lambda}{\mu(\mu - \lambda)}$

   By substitution, we have,

   $$E(w_q) = 6.25 \text{ minutes.}$$

ADDITIONAL QUESTIONS FOR PRACTICE (WITH HINTS AND ANSWERS)

Question 1

In a given system, the arrival rate is 20 customers per minute and the system follows Poisson distribution. The service rate is 30 customers per minute. Suddenly it happens that the arrival rate decreases by 10%. If the customers' waiting time in the system is constant, then by what percentage we can reduce the service rate?

Solution

It is given that $\lambda = 20$ and $\mu = 30$.

Now, we find

$$E(W_q) = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{20}{30(30-20)} = \frac{1}{15} \text{hour} = 4 \text{ minutes}$$

Now the arrival rate reduces by 10%. This means that the new arrival rate $\lambda = 20 - 20 \times 10\% = 18$. If the new service rate is μ^* and the customer waiting time is constant (as found) = 1/15 hour then

$$E(W_q) = \frac{\lambda^*}{\mu^*(\mu^* - \lambda^*)} = \frac{18}{\mu^*(\mu^* - 18)} = \frac{1}{15}$$

$$\frac{\lambda^*}{\mu^*(\mu^* - \lambda^*)} = \frac{18}{\mu^*(\mu^* - 18)} = \frac{1}{15}$$

This gives $\mu^{*2} - 18\mu^* - 270 = 0$

Solving the quadratic equation, we get $\mu^* = 28$. This means that we can reduce the service rate by 2 customers per hour, which is $(2/30) \times 100 = 6.66\%$.

If (in this context) arrival rate falls down by 10%, then we can decrease the service rate by 6.66%.

Question 2

There are 3 computers having surfing capacities in a college library. Students visit at a rate of 10 students in an hour. Each student, on an average, spends 15 minutes on the system. Assuming arrival rate following Poisson distribution and service rate an exponential one; find an average waiting time in the queue for a student for his/her turn on operation. How many students are there in the system?

Solution

Here, we have $\lambda = 10$, $\mu = \frac{1}{15} = 4$, and $k = 3$

We have to find p_0 and hence other results can be derived based on it.

$$p_0 = \frac{1}{\sum_{n=0}^{n=k-1} \frac{1}{n!}(\rho^n) + \frac{1}{k!}(\rho^k) \cdot \left(\frac{k}{k-\rho}\right)}$$

with $\rho = \lambda/\mu = 10/4$, on substitution, we get $p_0 = 0.02189$

An average waiting time in the queue for a student for his/her turn on operation

$$E(W_q) = \frac{\mu \cdot \rho^k}{(k-1)!\,(k\mu - \lambda)^2} \cdot p_0$$

By substitution, we get

$$E(W_q) = 0.057 \text{ of 1 hour.}$$

Points to Remember

As such, in the context of game theory, we may call important points to some useful clues.

1. The main pillars of game theory are
   (a) Arrival rate (λ) and associated distribution useful for describing the input mode
   (b) Service rate (μ) and its associated distribution useful for describing service mode
   (c) Number of servers, all equally efficient to render services as required
   (d) 'Q' discipline
2. Traffic Intensity (ρ) is a derived result, which is useful to many primary results and their interpretations.
3. Important factor responsible to determine the ups and downs in revenue and derived profit is waiting time cost of customers either in a 'Q' or in the system once known, can determine expected number of customers in 'Q' and in the system.
4. There are two types of costs (a) waiting time cost, and (b) cost of establishment and operating the system. The main aim of modeling a 'Q' system is to find a balance of these two types of costs.
5. Students should write down some formulae and then start solving the examples which are, sometimes mentioned *between the lines* and interpret the meaning conveyed by it.

Exercises

Objective Type Questions

I. State True or False:

1. Transient state of the system is one which tries to stabilize if we control the service mechanism.
2. If λ = 12 and μ = 20, then the probability that a new entrant immediately gets the sevice is zero.
3. $\left(\frac{1}{\mu}\right) - \left(\frac{1}{\lambda}\right) = 2$ means '2' more customers are served.
4. $(\mu - \lambda) = 5$ means that a customer waits for an average of 12 minutes in the system.
5. For a given λ and μ, p in % = 60 means 60 more customers can still be served by the system in one hour.
6. If a service unit serves 4 customers in one hour then inter-arrival time is 15 minutes.
7. For λ and $\mu > 1$; $\lambda\mu = 35$ means average waiting time in the system is 30 minutes.
8. If $\rho = 0.6$ then the probability of having 5 customers in the system is $(0.6)^5$.
9. A customer experiences that in 50% of the cases he, on entering the service unit, gets immediate service means the system remains busy for 50% of its working time.
10. Poisson distribution (is discrete) has only one parameter and that is the mean of the distribution.

Answers

1. true.	2. false.	3. false.	4. true.	5 false.
6. false.	7. true.	8. false.	9. true.	10. true.

II. Multiple Choice Questions

Use the following alternatives to make a selection of the right answer to Questions 1, 2 and 3.

(a) $\dfrac{1}{\mu-\lambda}$ (b) $\dfrac{\lambda}{\mu-\lambda}$ (c) $\dfrac{\lambda^2}{\mu(\mu-\lambda)}$ (d) $\dfrac{\lambda}{\mu^2}\dfrac{1}{(\lambda-\mu)}$

(e) $\dfrac{\lambda}{\mu}\dfrac{1}{(\mu-\lambda)}$ (f) $\dfrac{\lambda}{\mu^2}\dfrac{1}{(\mu-\lambda)}$

1. In $(M/M/1)$: (FCFS/∞/∞) queue system formula for expected number of customers in the system is
2. expected waiting time of customer in the queue is
3. expected waiting time of customer in the system is

The arrival follows Poisson distribution and the service time approximates to the exponential distribution. The system works using a single server with inter-arrival time equals 5 minutes and service given to 20 customers in a minute. Now answer Questions 4 to 9

4. The probability that there are 5 customers in a system is

 (a) $(3/5)^5$ (b) $(3/5)^6$ (c) $\dfrac{2}{5}\left(\dfrac{3}{5}\right)^5$ (d) $\dfrac{2}{5}\left(\dfrac{3}{5}\right)^6$
5. Percentage of time system in waiting for some customer to arrive

 (a) 40 (b) 60 (c) 50 (d) 20
6. The probability that there are more than 5 customers in the service is

 (a) $(3/5)^5$ (b) $(3/5)^6$ (c) $\dfrac{2}{5}\left(\dfrac{3}{5}\right)^5$ (d) $\dfrac{2}{5}\left(\dfrac{3}{5}\right)^6$
7. In the result $(1-\rho)$ we have

 (a) $\rho > 1$ (b) $\mu > \lambda$ (c) $\mu = 0$ (d) $0 < \lambda < \mu$
8. If x^2 is the arrival rate and y^2 is the service rate, then traffic intensity is

 (a) X^2/y^2 (b) x/y (c) $1-\dfrac{x}{y}$ (d) $\dfrac{x^2}{y^2-x^2}$
9. Balking in queue system refers to

 (a) breaking the queue and go (b) not to join in queue
 (c) same as jockeying (d) same as reneging
10. In a (M/M/1): (FCFS/∞, ∞) system, arrival rate = 10, customer waiting time in the system = 6 minutes, then μ is

 (a) 10 (b) 20 (c) 30 (d) 60

Answers

1. (b)	2. (e)	3. (a)	4. c	5. (a)
6. (b)	7. (d)	8. (a)	(9) b	10. (b)

NUMERICAL PROBLEMS

1. In a bank there is a single server who gives attention to every inquires. There is an average flow of customers at the rate of 10 customers in 1 hour. The server can attain 15 customers in 1 hour. You are required to answer the following.
   (i) Find the probably that the server is idle.
   (ii) How many customers are there at a given time in this system?
   (iii) What is the probability that there are 4 customers in the system?
2. Customers arrive at a telephone booth at the rate of 8 customers per hour. The expected time a customer takes is 5 minutes.
   Find the following.
   (i) What is the percentage of utilization factor of the system?
   (ii) What is an average waiting time of the system?
   (iii) What is the expected number of customers in the queue?
3. In an airline booking office there is a single counter reservation window, customers arrive at the rate of 8 customers per hour. The clerk at the desk can satisfy a customer at an average time of 5 minutes. Find the following:
   (i) What is the probability that there are more then 3 customers in the system?
   (ii) Find the number of customers in the system.
   (iii) What is an average waiting time in a queue?
4. In a post office there is one clerk serving all the customers. Customers arrive following Poison distribution at an average of 3 minutes per customer. The clerk can handle 25 customers in 1 hour. Find the following.
   (i) What the probability that the clerk is free?
   (ii) If there are 10 total working hours in a day, then during this period exactly how many hours he works?
   (iii) What is the expected number of customers in the system?
5. During night time on a gas station there is a single server attending all the customers, the inflow of the customers is 4 customers in 1 hour. The operator can serve 10 customers per hour. Assuming that the arrival flow follows Poison distribution and the service rate follows exponential distribution; answer the following questions.
   (i) What is the probability that there are less than 5 customers in the premises of the gas station?
   (ii) What is the probability that a new customer will immediately get the require service?
   (iii) What is the expected waiting time in the system?
6. The flow of customers on a single server operator system is 18 in hour and the operator can satisfy 20 customers in 1 hour. Assuming that the arrival flow follows Poisson distribution and the service rate follows exponential distribution, answer the following questions:
   (i) What is the utilization factor of the system?
   (ii) What is the expected number of the system?
   (iii) What is the probability that there are less than 20 customers in the system?
7. The flow of customers on a single server operator system is 18 in 1 hour and the operator can satisfy 20 customers in 1 hour. Assuming that the arrival flow follows Poisson distribution and the service rate follows exponential distribution, answer the following questions.
   (i) What is the expected waiting time in the system?
   (ii) What is the expected number of customers in the queue?
   (iii) What is the probability that there are exactly 20 customers in the system?

8. People arrive at a web-browsing centre in which there is a single computer for the surfers. Inflow of the surfers is 8 customers in 1 hour and on an average each customer takes 5 minutes to see his email. Assuming that the arrival flow follows Poisson distribution and the service rate follows exponential distribution, answer the following questions.
   (ii) Find the probability that a new customer has not to wait for the computer.
   (iii) How many customers are there in the queue?
9. People arrive at a web-browsing centre in which there are 4 computers for the surfers. Inflow of the surfers is 8 customers in one hour and on an average each customer takes 20 minutes to see his email. Assuming that arrival flow follows Poisson distribution and the service rate follows exponential distribution, answer the following questions.
   (i) Find, in hours, the expected waiting time of customer in the system.
   (ii) Find the expected number of customers in the system.
10. During the high busy hours, on an average 20 customers visit a bank. There are 3 'single service' windows in operations. Number of customers cleared in one hour by one window is 7. Assuming that the arrival flow follows Poisson distribution and the service rate follows exponential distribution, answer the following questions:
   (i) Find, in hours, the expected waiting time of customer in the system.
   (ii) Find the expected number of customers in the system.
11. People arrive at railway booking office at the rate of 8 customers per hour. There are 4 operators satisfying their requirements. Each operator takes, on an average, 20 minutes to satisfy customers' demand. Assuming that the arrival flow follows Poisson distribution and the service rate follows exponential distribution, answer the following questions.
   (i) Find the expected waiting time of customer in a queue.
   (ii) Find the expected number of customers in a queue.
12. During the high busy hours, on an average 20 customers visit a bank. There are 3 'single service' windows in operations. Number of customers cleared in one hour by one window is 7. Assuming that the arrival flow follows Poisson distribution and the service rate follows exponential distribution, answer the following questions.
   (i) Find, in hours, the expected waiting time of customer in the queue.
   (ii) Find the expected number of customers in the queue.
13. Patients arrive at a doctor's clinic at the rate of 4 patients per hour following Poisson distribution. The doctor can treat 10 patients per hour. If the doctors' clinic remains open for 8 hours, then find the following.
   (i) Find the expected free time of the doctor in 8 hours.
   (ii) What is the expected waiting time, in hours, of a patient in the clinic?
14. Assuming that the arrival flow follows Poisson distribution and the service rate follows exponential distribution, there is a flow of 25 customers in one hour and the operator, on an average, takes 1.5 minutes to serve a customer, answer the following.
   (i) Find the non-utilization factor of the system.
   (ii) Find the waiting time of the system.
   (iii) Find the expected number of customers in the system
15. In a photoshop near the passport office, customers flow is 15 customers per hour. The photoshop operator can serve a customer in 2 minutes' time.
   (i) Find the expected number of customers in the system.
   (ii) Find the expected waiting time, in hours, of customers in a queue.

ANSWERS TO NUMERICAL PROBLEMS

1.	(i) 0.3333	(ii) 2	(iii) 0.198
2.	(i) 66.67	(ii) 15 minutes	(iii) 1.33
3.	(i) 0.198	(ii) 2	(iii) 0.167
4.	(i) 0.2 = 12 minutes	(ii) 8 hours	(iii) 4
5.	(i) 0.986	(ii) 0.6	(iii) 0.1667
6.	(i) 0.9	(ii) 9	(iii) 0.876
7.	(i) 30 minutes	(ii) 8.1	(iii) 0.122
8.	(i) 1/3	(ii) 2	(iii) 1.33
9.	(i) 0.428	(ii) 3.42	
10.	(i) 0.255	(ii) 5.11	
11.	(i) 0.09	(ii) 0.76	
12.	(i) 0.11	(ii) 2.25	
13.	(i) 4.8 hours	(ii) 0.1667	
14.	(i) 3/8	(ii) 0.0667	(iii) 1.67
15.	(i) 1	(ii) 0.03	

12 Inventory Management

Learn to Obey, Then You Can Command.

Learning Objectives

AFTER STUDYING THIS CHAPTER, THE STUDENTS WILL BE ABLE TO:

- **understand the basic features of inventory management.**
- **design the production of units keeping the demand trends.**
- **design the purchase policy keeping in mind the production schedule**
- **keep a track of purchase, production, storage and sale.**
- **control the additional cost and hence keeping lowest possible cost of units**
- **design a trend in lead time demand, safety stock and reproduction level.**

INTRODUCTION

Inventory is basically the physical stock of items having monetary value and which are perfectly maintained to remain perfectly fit for sale in the open competitive market.

It is how you look on the marketing or sales department, it is a liability of quick disposal or sale while for accounts department, the existing inventory on a particular day reflects in the annual report as an asset. It does matter much how the things have sharpened our mind.

As the known term JIT (Just-in-time) comes to mind, it does not reflect zero inventory all the time. It allows to preserve and continue the routine operations with the bare minimum stock. In addition to this, the idea of material requirement planning (MRP) is possible only if the demand is predicted with utmost certainty and which is dependent on most of the factors pre-known to the manufacturer.

We basically keep a stock of items for some fundamental reasons which as follows:

1. If there is a reasonable amount of stock in the inventory, then one can meet the upheavals in the demand.
2. A reasonably good stock can control the short supply and hence allows the business to run steadily.
3. Where there is a sudden rise in the price, the stockist can take advantage of what he has invested.
4. Looking to the production point of view, a reasonably good amount of inventory of raw materials can keep the production unit run smoothly without being much affected during the period of short supply or no supply.

12.1 Types of Inventory

To a manufacturer, he/she has on to his/her mind inventory as

1. raw materials or unprocessed goods or units.
2. inventory or semi-processed or processed goods but not completely ready for despatch.
3. inventory of fully processed items just ready to move out for transition.
4. inventory of items on transit state.

To a buyer (purchaser), he/she has on to his/her mind inventory as

1. number of items taken on cardex file and ready to move (either from warehouse directly to other party to whom it is sold as per contract or ready to move to the shop-floor vendor)
2. the number of items of the same type, quality and make/brand lying unsold on the total stock on a given time.

12.2 What Inventory can Cause?

There are always two aspects to a given situation, as to the inventory are merits and de-merits of holding inventory of items; any time demand can arise.

Merits:

Following are the main points on which a good stockist can win over the competitors on the same line.

1. Keeping a reasonable stock of items helps running the business smooth, efficient and competitive.
2. Customers can have wide range of choice and selection, this feature is important and enhances the business and attracts more customers.
3. During the short-supply state, stockist can manage proper planning and distribution of sale that stabilizes the business on a long term.
4. The stockist can get the benefit of sudden price rise in case of items moving on a positive price trend.

Demerits:

1. Any type of inventory, so long it remains unsold, reflects the block of capital towards the basic investment on the production.
2. On the top of that there are some additional costs associated with the inventory items:
   (a) rent of warehouse,
   (b) human resources—persons associated with overall maintenance and despatch,
   (c) electricity charges,
   (d) insurance charges,
   (e) security charges,
   (f) obsolesce cost, and
   (g) pilferage cost.
3. Cost of moving inventory items from warehouse to the shop-floor.

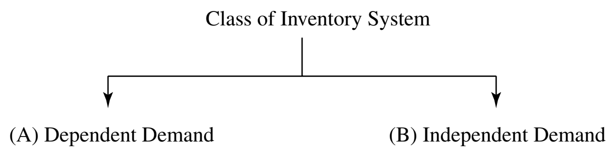
All the costs mentioned above are essential costs and are incremental cost to the basic purchase cost/ manufacturing cost.

12.3 Classes of Inventory System

In any case the decision to make production of the item (for manufacturer) or decision to make a pre-planned or determined stock of items to a decision maker forces him to think first about the demand of the items are expected demand on a short-term or long-term basis. This may vary depending upon

1. capacity to invest in the production or towards the stock,
2. demand pattern during a year or a time cycle,
3. availability of the items in the open market, and
4. comparative cost component of open market items.

Depending upon all the factors, we classify the inventory system on a wide span.



(A) - A(1) Just-in-time (JIT)
- A(2) Material requirements planning (MRP)
- A(3) Manageable-in-time (MIT)
(B) - B(1) Periodic review system (P-System)
- B(2) Fixed order system (Q-System)
B(2) - (1) EOQ Models
(2) EOQ Models (shortage allowed)
(3) EOQ Price breaks
(4) Recorder level—Safety level/buffer stock

There are parallel types of models for manufacturers. The different criteria are like short supply of raw material, price breaks offered to purchase raw materials, minimum stock to maintain.

The following problems may be broad for buyers and manufacturers:

1. When to order and how much to order,
2. When to produce and how much to produce.

12.4 Demand Pattern—An Overview

This explains very distinctly that the demand factor is the only major factor, which controls the decision maker, who, in turn, designs the system within human limitations.

(a) We do not, or probably cannot, know all the factors responsible for controlling the behaviour of demand function. As the word *demand* in its true sense is an undefined word (like a set and its elements in mathematics are). In some cases, individual parties/customers calculate their requirements and on completion of formalities, place their orders to the manufacturer or the dealer, who, in turn, is concerned to calculate the sum total of all such demands received from different parties or group of customers. This makes them feel sound to take logical decision about the purchase or sale or production.

To the highest percentage if demand is known and remains known over a reasonable period of time, then the concepts like **JIT**, **MRP**, and manageable-in-time, are very useful and prove effective on a given time-slot.

But saving on this aspect may cost you for being constantly in touch with the market, the buyers, and keep on making constant study of market trend.

12.4.1 Independent Demand

When most of the major factors generating the demand of particular items are not known or not completely studied and assumed to remain nearly constant over a period of time, such demands are called *independent demands*.

Whatever the situation may be, at one stage there is a time to place an order or there is a time on which one must take a decision to begin production in the factory. In such cases the decision makers rely, in practical situation, mostly on past records and past trend. In cases, simulation technologies become useful in predicting future demands. By nature, the forecast in such cases become probabilistic. Once the independent demand is determined then the demand of items/units-subcomponents of the main units/items is fixed. This becomes *dependent demand*.

For example, a manufacturer decides to make a production of 5000 scooters then the dependent demands, say of 10,000 tyres and tubes, 5000 head-light bulbs are automatically fixed.

Now we discuss some terms associated with inventory control system.

12.4.2 Lead Time

It is a time-slot between the point of time of placement of an order (and confirmed by the supplier) and the point of time on which items are received and taken to the inventory system.

12.4.3 Lead time—Demand

As such, we know that the demand is never precisely known or cannot be predicted. It is a stochastic variable. On the top of this, the demand of items during the lead time has remained highly uncertain. There are many external circumstances, which govern the demand pattern during the lead time. It may fluctuate in either direction—if it goes up, then there are chances that one may run out of stock situation or even can invite shortages by taking back-orders. If the demand goes low, then one may have to bear the holding cost for the period and on the top of that when the ordered stock arrives, the stock level rises up then what it was planned.

In extreme cases, we have variations in lead time also. Once a manufacturer commits for despatch of items and he may run out of stock in raw material, then it is quite obvious that he cannot keep up his commitment. It is called *delayed lead time*. If you depend upon some agency to supply the goods and on the other side you too have committed to some other agency for timely delivery, then there are chances that you may have to undergo a penalty either in monetary terms or you may lose the business.

Lead time demand is denoted by D_{LT} = Demand per day (during lead time) and we take the lead-time in days.

12.4.4 ROL (Re-Order Level)

(a) In some systems, it is pre-planned or pre-determined to place an order on about a fixed time interval, this point of time is a reorder level and in a time-based inventory system it is a strategic point to be determined. This time level is constantly fluctuating depending upon the existing stock. In general cases it becomes a problem on deciding the reorder point level in terms of time scale.

(b) On the other end, some system designers plan to place a new order when the existing inventory level crosses down the pre-determined inventory level .Working on the parallel lines, determining the reorder stock level for placing new order is a decision variable of inventory problem.

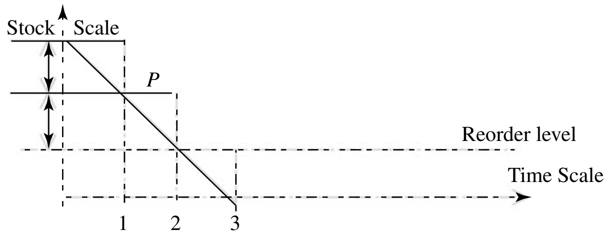


Figure 12.1

Constant demand per unit time depletes the existing inventory level. Falling inventory level is shown on stock axis (y-axis). The point p on the graph is an indicator of time scale on time axis (x-axis) and inventory level on stock-axis placing a new order.

If there is no lead time, then safety-stock level itself is a reorder level.

If there is a lead time, then the sum of the demand during the lead time (if lead time is 100% sure) and the safety stock is the reorder level.

$$\text{ROL} = \text{Maximum consumption per day} \times \text{Maximum lead-time}$$

12.4.5 Safety Stock (SS) or Buffer Stock

A well-planned past study of pattern of ***lead time demand*** suggests that it is highly necessary to maintain extra/additional stock of items over and above the items at the time of placing the new order.

This additional stock is called *safety stock*. It is necessary to safeguard against the sudden fall in inventory level; this may happen if the demand during the lead time rises suddenly. Keeping and preserving a safety stock protects against the shortages.

Number of items in the safety stock is a part of (fraction of) order size.

We will learn more about safety stock in the 'Q' system.

Reorder level fluctuates depending on the safety stock. Some known results for safety stock are as follows.

1. Safety stock (SS) = Average demand during the lead time.
2. **Maximum stock level**
   = (Maximum consumption per day × maximum lead time) – (minimum consumption per day × Minimum lead time)
3. **Minimum stock level = ROL – (normal consumption × normal lead time)**

12.5 Different Types of Additional Costs

Manufacturing cost or the purchase cost of the items are the fundamental or the basic costs. There are additional costs to these items before they reach technically and officially on to the consumers for actual functioning or operation.

These costs are classified as carrying cost, shortage cost and procurement cost. They are termed as type 1, type 2 and type 3 costs.

(A) Type 1 Cost: Carrying Cost

It is also known as *holding cost*. In general, it is denoted as C_1. It is the cost towards keeping, maintaining the items in the warehouse.

The purchase bears this cost and adds the same to the cost of the item.

In most of the cases, it is calculated as the fraction or percentage of cost of the item per unit time on a total period of maintenance. In case of high valuables it is counted on the total value/cost of items [diamond or gold/silver articles]. In some cases the carrying cost is calculated on the size or area it occupies the warehouse. [big size of machineries but not so costly as the ornaments are.]

Carrying Cost

Carrying cost includes the following types of costs.

1. rent of warehouse
2. electricity charges
3. insurance charges
4. security charges
5. clerical staff working on [stock takers + house keeping]
6. pilferage costs
7. obsolescence cost

Over and above bearing proportionate cost of all the above costs, there are following two additional costs.

1. Cost of transferring the goods from warehouse to the shop/offices/buyer's destination.
2. Octroi Charges: All these costs when counted proportionally on the basic cost of an item make the item costlier.

(B) Type 2 Cost: Shortage Cost

Basically, these costs are notional costs associated with the unit cost of an item. If vendor runs *out of stock* state at a point of time, then in order to sustain his credibility in the market he has to adopt the policy of *booking the orders* (taking *back orders*) at a lower cost giving benefit of waiting period to the customer. If not done so, he is liable to lose credibility in the market. Such events of shortages is liable to lower down the established goodwill of the buyer.

This cost is denoted by C_2 and it is calculated per unit of items short fallen during the lead time.

One additional point is noteworthy. Once the back orders are taken, then on arrival of new stock the vendor has to satisfy back order and as a result his inventory in the beginning of new cycle remains low and causes re-planning and re-designing his order policy .This is again loss to an organization.

(C) Type 3 Cost: Procurement Cost

Some popularly call this as *ordering cost* or *acquisition cost*.

Everytime one places an order he has to observe certain technical formalities. It takes time and also consumes human resources and cost of communications using electronic media. Also formalities like declaring the letter of intent, inviting tenders, fixing the parties by negotiations and final formality of closing the tenders – all these take time, planning and system study.

This cost is denoted as C_3; it is solely attached with the placement of each order. Finally it can be divided per unit on arrival of saleable items in inventory.

Comment

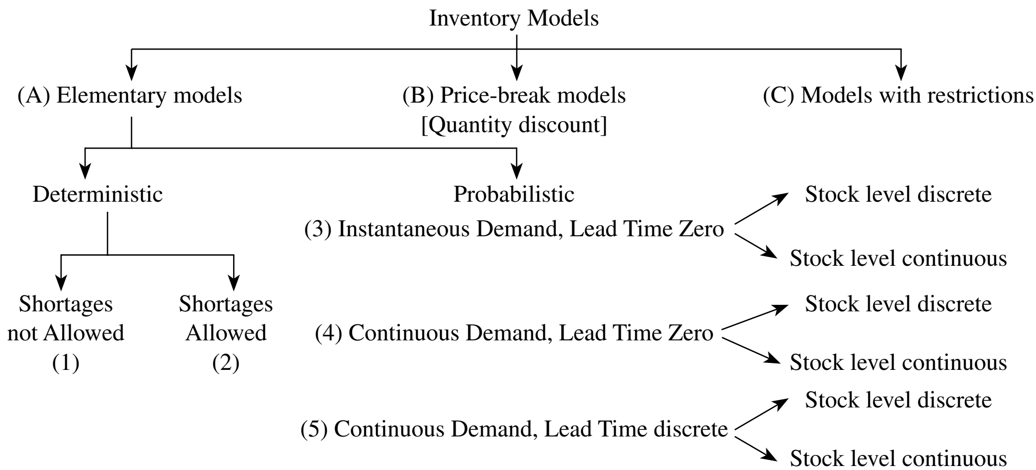
We add a few comments before we conclude this topic.

1. Carrying cost per unit decreases as number of items taken to inventory decreases and vice versa.
2. Total procurement cost is proportional to the number of orders placed.
3. If the number of orders increase
   (a) ordering cost increases
   (b) as number of items in each order decreases the carrying cost also decreases.
   (c) all the time C_1 and C_3 and shortage cost may be applicable; as per given conditions.

12.6 CLASSIFICATION OF INVENTORY SYSTEM

Classification of a system is done on some independent parameters. As we know that the common feature in all the models an of inventory system is to determine *economic order quantity* (EOQ), which minimizes the sum of incremental cost. As a result, the sum total of the basic purchase price and the incremental costs becomes minimum. This is an important feature, which allows to make fair competition in the market.

Inventory models are classified using the following properties.



(1) – (A) Uniform Demand, Replenishment Rate immediate and infinite
(1) – (B) Non-Uniform Demand, Replenishment Rate immediate and infinite
(1) – (C) Demand Rate Uniform, Supply Rate Finite

(2) – (A) Demand Rate Uniform, Replenishment Rate immediate and infinite
(2) – (B) Demand Rate Uniform, Replenishment Rate immediate and infinite, Fixed Time interval
(2) – (C) Demand Rate uniform, Supply Rate Finite

12.7 DIFFERENT MODELS

As in Section 12.7, we have seen the names of different types of models, now we begin to discuss mathematical models making different assumptions. We will derive necessary formulae in most of the cases.

Model 1: Saw Teeth Model (Fundamental EOQ Model)

In order to establish the first fundamental model popularly known as *EOQ model*, we make certain assumptions. The assumptions are as follows.

1. Cost of one item and ordering cost per order remains constant over a period of time—pre-defined or pre-fixed. (Generally one year.)
2. Demand rate is pre-known and remains fixed over a period of time.

3* Lead time is known and remains constant over a period of time. The demand during the lead time remains the same as the current demand rate.
Alternatively, if the lead time is zero, the demand rate is instantaneous and infinite.

4. Annual demand is known and remains constant.
5. Supply rate is instantaneous and infinite. [as per the requirement of an order size]

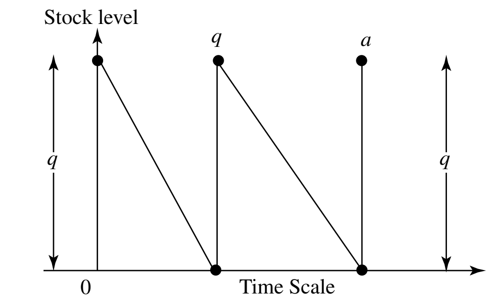


Figure 12.2

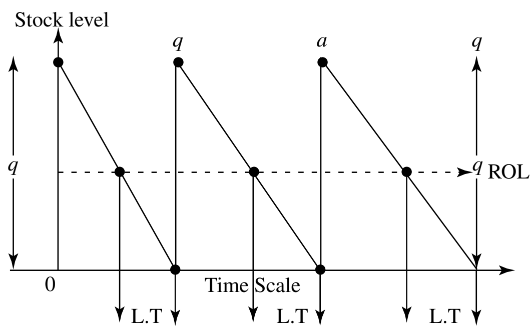


Figure 12.3

Objective of the Model: To determine the value of order size q, which will minimize the additional inventory costs.

Total incremental cost = $TIC(q)$ = Carrying cost + ordering cost

[Shortages are not allowed.]

Now, carrying cost = Average inventory × Carrying period cost;

Average inventory = (Maximum inventory + Minimum inventory)/2;

$$= \frac{q+0}{2} = q/2.$$

[Each time a lot size q is received on realization of order.]

$\therefore$ Carrying cost = $(q/2) \times C_1$

As there are n cycles in a year, there is n time average inventory level remains $q/2$.

Carrying cost in one cycle = $q/2 \times C_1 \cdot (1/n)$; Time period for which carrying cost in a year is applicable = $1/n$.

n cycles make total annual carrying = $n\,[q/2 \times C_1 \cdot 1/n]$ where ($1/n$ = one cycle time in year)

Carrying cost = $q/2 \times C_1$ (2)

Ordering cost = Number of orders × Ordering cost/year = $\left(\dfrac{A}{q}\right) \cdot (C_3)$ (3)

Putting Result 2 and 3 in Result 1,

$$TIC(q) = q/2 \cdot C_1 + \left(\frac{A}{q}\right) \cdot C_3 \qquad (4)$$

To optimize this cost, we differentiate with respect to q and equate the result to zero.

$$\therefore \quad \frac{d}{dq}[TIC\,(q)] = \frac{C_1}{2} - \frac{A}{q^2}C_3 = 0;$$

$$\therefore \quad q = \sqrt{\frac{2AC_3}{C_1}} = q^* \tag{5}$$

This value of q is called *economic ordering quantity* (EOQ) and is denoted as q^*;

[Also $\dfrac{d^2(TI(q))}{dq^2} = \dfrac{2A}{q^3}\ C_3 > 0$.]

It implies that $TIC\,(q)$ is minimum at $q = q^*$;

$\therefore$ Putting $q^* = \sqrt{\dfrac{2AC_3}{C_1}}$ in Result 4, simplification gives

$$TIC\,(q^*) = \sqrt{2AC_3C_1} \tag{6}$$

Using the Result 5, we get the following results.

- Number of orders in one year $= \dfrac{A}{q^*} = \dfrac{A}{\sqrt{\dfrac{2AC_3}{C_1}}} \times C_3$ (7)

- Total ordering cost $= \sqrt{\dfrac{AC_3}{2C_1}} \times C_3$ [Number of orders $\times$ Ordering cost per order]

  $\therefore$ Total ordering cost $= \sqrt{\dfrac{AC_1C_3}{2}}$ (8)

- Total holding cost $= \dfrac{q}{2}\ C_1 = \dfrac{\sqrt{\dfrac{2AC_3}{C_1}}}{2} \times C_1$;

  Total ordering cost $= \sqrt{\dfrac{AC_1C_3}{2}}$ (9)

  [Note that at EOQ (q^*), both costs are equal.]

Graphical Presentation of EOQ

We draw a graph of (1) carrying cost, (2) ordering cost, and (3) Total incremental cost. We take q—lot ordering size on x-axis, and cost on y-axis.

Curve (1) carrying cost during a period $= \dfrac{q}{2} \times C_1$ ($C = \dfrac{qC_1}{2}$ = line graph)

Curve (2) Ordering cost during a period $= \dfrac{A}{q} \times C_3$ (Hyperbolic graph)

Curve (3) Total incremental cost $= \dfrac{q}{2} \times C_1 + \dfrac{A}{q} \times C_3$

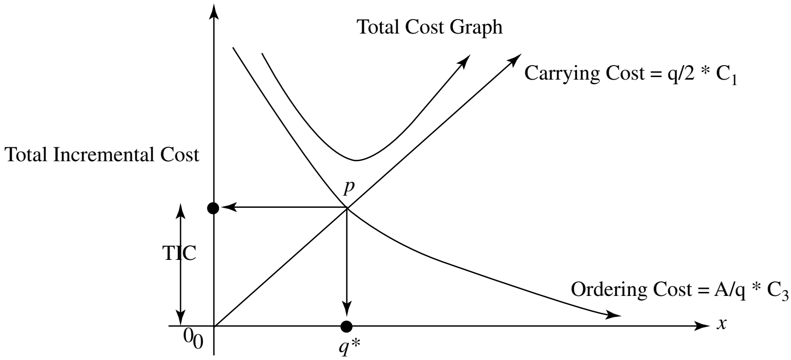


Figure 12.4

At the point P, both the costs are equal q^* = EOQ = $\sqrt{\dfrac{2AC_3}{C_1}}$; $TIC\ (q^*) = \sqrt{2AC_3C_1}$

The model discussed above is the most fundamental model and minor changes in the fundamental assumptions give rise to different models. We will discuss all such models in details. We take one illustration to understand applications of the results derived above.

ILLUSTRATION 1

ABC Corporation has an annual demand of 73000 empty capsules. The cost of one capsule is ₹1.50 and holding cost of one capsule is 10% of its basic cost. Capsules being an important component of daily usage, bulk stock are maintained by placing orders in advance. Each order costs ₹730 for all its legal formalities. Find the economic order quantity and total number of orders placed. If it is decided to place only 5 orders in a year, then what are the consequences?

Solution

It is given that

A = annual demand = 73000 items.

C = cost = ₹1.50 = 3/2

C_1 = carrying cost = $1.50 \times 0.1 = 0.15$ per unit per year

C_3 = ordering cost per order = 730

$$\text{EOQ} = q^* = \text{EOQ} = \sqrt{\frac{2AC_3}{C_1}}$$

$$TIC\ (q^*) = \sqrt{2AC_3C_1}$$

$$= \sqrt{\frac{2 \times 73000 \times 730}{0.15}} = 26656 \text{ items}$$

$$TIC\ (q^*) = \sqrt{2 \times 73000 \times 730 \times 0.15} = ₹3999$$

Number of orders = A/q^* = 73000/26656 = 2.74 orders in one year.

Ordering frequency = 365 ÷ 2.74 ≈ 133 days.

Part 2

If it is decided to put 5 orders in a year then order size = 73000/5 = 14600 items in one order.

Carrying cost $= 0.5 \times 14600 \times 0.15 = 1095$

Ordering cost = number of orders $\times 730 = 5 \times 730 = 3650$

Total incremental cost = 1095 + 3650 = ₹4745 {This amount exceeds ₹3999}

Model 2: EOQ model with lead time and fixed stock on hand

This model will help determining the level of safely stock under a given demand rate. With known costs and demand, one can allow fluctuation in reserve stock and corresponding EOQ will be determined.

We have some important assumptions as follows.

1. Demand is uniform and known over a period of time.
2. There is a lead time—pre-known and constant.
3. Shortages are not allowed.
4. Supply is immediate and as per demand.

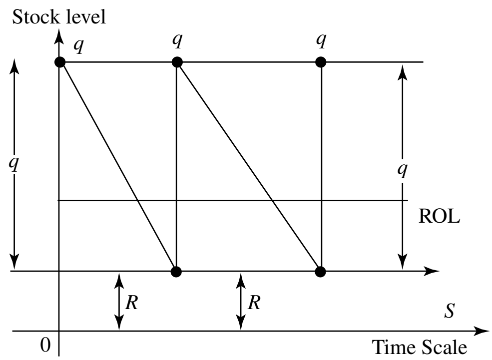


Figure 12.5

We begin with on hand stock of $(q + R)$ units. With the demand rate D; the stock depletes and on reaching a level R,

$$R = f \times Q;\text{ where, } 0 < f > 1.\ [R \text{ is a fraction of } q]$$

An order of q units is placed. As there is a known lead time and supply is immediate and as per demand, a new stock again reaches the level $q + R$.

Now, the next cycle begins.

R remains constant over a period of time.

$$\text{Average inventory } = R + \frac{q}{2}$$

Carrying cost of all n such cycles over a given period of one year $= C_1 \left(R + \frac{q}{2}\right)$

Ordering cost = Number of orders × Ordering cost per order

$$\therefore \quad \text{Ordering cost } = \left(\frac{A}{q}\right) C_3, \text{ where } A \text{ is the annual demand.}$$

$$\therefore \text{ Total incremental cost } = TIC\,(q) = C_1 \times \left(R + \frac{q}{2}\right) + \frac{1}{q} \cdot (AC_3) \qquad (1)$$

Put this $R = f \times q$ with $0 < f < 1$, differentiating Result 1 with respect to q and equating to zero; we get

$$\frac{dTIC(q)}{dq} = 0$$

$$C_1\left(f+\frac{1}{2}\right)-\frac{1}{q^2}-(AC_3)=0 \quad \left[\text{Also } \frac{d^2TIC(q)}{dq^2}>0\right]$$

Simplifying we get

$$q = \sqrt{\frac{2AC_3}{C_1(1+2f)}}$$

This value is the EOQ when one wants to maintain a reserve stock of R units.

$$\text{EOQ} = q^* = \sqrt{\frac{2AC_3}{C_1(2f+1)}} \tag{2}$$

$$R = \text{Reserve stock} = f \times q;\ 0 < f < 1.$$

Comments

1. Carrying cost = $C_1 \cdot (f \times q + q/2)$
2. Ordering cost = $\dfrac{A}{q^*} \cdot C_3$
3. One can adjust R by determining the fraction f.

Model 3: Finite Rate of Replenishment

We want to find EOQ in the case when the supply rate is immediate but not to the number of items ordered. With the following notations, we establish the EOQ model and related formulae.

Let:

p = Supply rate per unit time.
d = Demand rate per unit time.
q = Lot size/order size.
T = One cycle time.
C_1 = Inventory holding rate per unit per unit time.
C_3 = Ordering cost per order.

[Since shortages are not allowed and so $C_2 = 0$.]

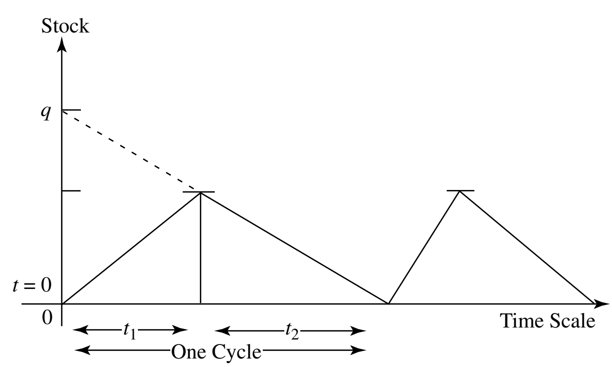


Figure 12.6

Cycle begins at $t = 0$. As the supply rate, p is not up to the order size, q $(p > q)$

We have, during the period t_1, both processes work simultaneously.

Supply at the rate p per unit time and demand at the rate d per unit time with clear assumption that $(p > d)$.

Order size, $$q = pt_1 \tag{1}$$

(If there is no demand during the period t_1)

Stock on hand, just at the end of time $t_1 = p \times t_1 - d \times t_1 = (p - d)t_1$;

This stock is consumed during the period $t = t_2$.

$$\therefore \quad (p - d)t_1 = d \times t_2$$

$$\therefore \quad \frac{p-d}{d} = \frac{t_2}{t_1}$$

using componendo,

$$\frac{p}{d} = \frac{t_1 + t_2}{t_1} = \frac{T}{t_1}$$

$$\therefore \quad \frac{p}{d} = \frac{T}{t_1} \tag{2}$$

We have two results.

$$q = p \times t_1 \text{ and}$$

$$\frac{p}{d} = \frac{T}{t_1} \text{ where } T = t_1 + t_2 \tag{3}$$

Inventory carrying cost during the period, T = Average inventory carrying cost during the period t_1 + Average inventory carrying cost during the period t_2.

$$= [(1/2) \cdot (p - d)t_1 + 1/2\,(p - d)t_2] \times C_1$$

$$= (1/2) \cdot (p - d)\, T \times C_1$$

But from Result 3,

$$T = \frac{pt_1}{d} = \frac{q}{d}$$

$$\therefore \quad \text{Total carrying cost during one cycle} = 1/2(p - d)\left(\frac{q}{d}\right) C_1 \tag{4}$$

$$\text{Now we work with total ordering cost} = \frac{C_3}{T}$$

but $$T = \frac{pt_1}{d} = \frac{q}{d}$$

[T = One cycle time and total number of cycle in one unit time = $\frac{1}{T}$]

$$\therefore \quad \text{Total ordering cost} = \frac{C_3}{T} = \frac{dC_3}{q} \tag{5}$$

$$\therefore \quad \text{Total incremental cost} = TIC(q) = \left(\frac{1}{2}\right)(p - d)\left(\frac{q}{d}\right) C_1 + \frac{dC_3}{q} \tag{6}$$

To find optimal q, we put $\frac{d}{dq}\ TIC(q) = 0$ and $\frac{d^2TIC\ (q)}{dq^2} > 0;$

$$\therefore \qquad \frac{d}{dq}\ [TIC(q)] = (1/2) \times \left(\frac{P-d}{d}\right) \times C_1 - d \times C_3\left(\frac{1}{q^2}\right) = 0$$

Simplifying it, we get

$$q = q^* = \text{EOQ} = \sqrt{\frac{2C_3 d}{C_1}}\ \sqrt{\frac{p}{P-d}} = \sqrt{\frac{2C_3 d}{C_1}}\ \sqrt{\frac{1}{\left(1-\frac{d}{p}\right)}} \tag{7}$$

For this value of $q = q^*$, we can find various results.

$$TIC(q^*) = \sqrt{2C_1C_3d}\ \sqrt{\frac{p-d}{p}} \tag{8}$$

Model 4: Planned Shortages

We take a situation where the shortages are planned and allowed to occur for a constant slot of time. Other main assumptions are the same.

- Demand rate is constant over a period
- Supply rate is immediate and infinite.
- There is no lead time.

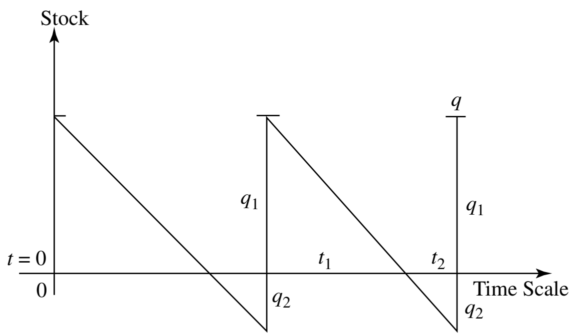


Figure 12.7

Notations: We take the following notations.

As usual C_1 is the carrying cost per unit time.

We consider c_1 = cost of the item × fraction of a rupee per year = $c \times I$

C_2 is the shortage cost per unit time

C_3 is the ordering cost per order.

q is the order size.

q_1 is the maximum inventory

q_2 is the maximum stock-out units

t_1 is the time until stock on hand becomes zero

t_2 is the time till there is a stock-out situation.

As the replenishment rate is immediate and infinite; items arrive and taken to current stock after fulfilling the back-orders of total amount = q_2. This makes at the same time the current inventory levels to q_1.

The cycle begins at the time $t = 0$ with stock on hand = q_1. As d is the demand rate, we have

$$q_1 = d \times t_1.$$

When the inventory level falls to zero, a shortage period t_2 begins. It continues for the time $t = t_2$

∴ Shortage units $\quad q_2 = d \times t_2$

∴ $\quad q_1 = d \times t_1$ and $q_2 = d \times t_2$.

Also, when an order of q realizes, q_2 units are taken away (back orders) and $q_1 = q - q_2$ are left on hand.

$$\therefore \quad q = q_1 + q_2$$

$$= d \times (t_1 + t_2) = d \times T \quad \text{where } T = t_1 + t_2 = \text{one cycle time} \tag{1}$$

$$q_1 = dt_1 \text{ and } q_2 = dt_2$$

$$\therefore \quad \frac{q_1}{q_2} = \frac{t_1}{t_2}.$$

and
$$\frac{t_1}{t_2 + t_1} = \frac{q_1}{q_2 + q_1} \tag{2}$$

Total incremental cost function is a function of two variables q_1 and $q \times (q_2 = q - q_1)$

Total incremental cost $(q \times q_1)$ = Average holding cost over the period t_1 + Average shortage cost over the period t_2 = Total ordering cost.

[All these costs are over a period—normally taken as 1 year span.]

When we write C_1 = Cost of one unit × % per annum. Then each t_1 and t_2 appear $\left(\frac{d}{q}\right)$ = number of order times making it 1 year period; so in the formula we do not consider t_1 and t_2;

$$\therefore \quad TIC\ (q, q_1) = (1/2C_1)\ \frac{q_1^2}{q} + (1/2 \times C_2)\left(\frac{(q - q_1)^2}{q}\right) + \frac{d}{q} C_3 \tag{3}$$

Differentiating partially with respect to q and q_1 and equating with zero, we get

$$q^* = \sqrt{\frac{2 \times C_3 \times d}{C_1}} \cdot \sqrt{\frac{C_2 + C_1}{C_2}} = \text{EOQ} \tag{4}$$

and

$$q_1^* = \sqrt{\frac{2 \times C_3 \times d}{C_1}} \cdot \sqrt{\frac{C_2}{C_2 + C_1}} \tag{5}$$

Using these two results, we can find q_2^* = Optimal shortage.

$$q_2^* = q^* - q_1^*$$

We can find the optimal time slots t_1^* and t_2^*

Model 5: Partial but Uniform Supply with Shortages

In this model, we have the following assumptions.

1. The demand rate is uniform and pre-known.
2. Supply rate is partial, uniform, and until the order size is reached.
3. There is no lead time.
4. Shortages are allowed; back-orders are fulfilled on delivery.

With the above assumptions one complete cycle of the model is described here.

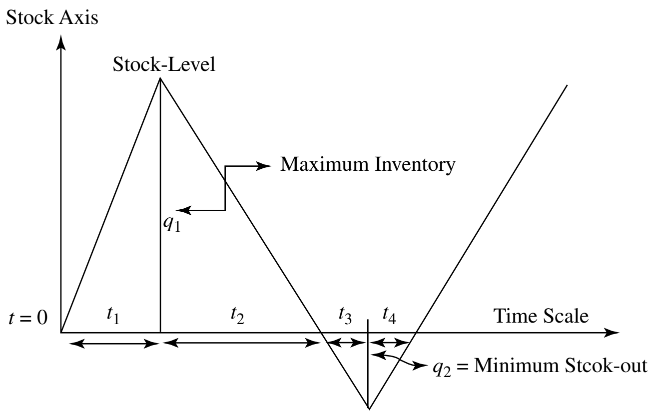


Figure 12.8

Description

Supply and demand both begin at the time $t = 0$. Supply rate per unit time = p is greater than the demand rate per unit time = d.

1. For the period $t = t_1$; both processes receiving the supply and fulfilling the demand continues.
2. Ordered quantity, q has been completely supplied but parallel demand during the same time period $t = t_1$ allows only $(p - d) \times t_1$ amount to accumulate at the time $t = t_1$
3. At the end of $t = t_1$, there is no supply and the existing stock = $(p - d) \times t_1$ diminishes at the rate d up to the time $t = t_2$.
4. Now, begins the period of shortages and back-orders are taken up to the period t = Total amount consumed in back-orders = $d \times t_3$
5. Just at the end of $t = t_3$ begins the period of partial replenishment and up to the period $t = t_4$ back-orders are fulfilled. Then onwards also the same first slot type behaviour of supply at the rate p and satisfying demand at the rate d continues and we get the same cycle with the same time intervals repeated.

These steps, in order, describe the patterns of cycles.

Our objective is to find an economic order quantity (EOQ) that minimizes the inventory incremental costs.

There are three types of costs involved in this model.

C_1 = Carrying cost

C_2 = Shortage cost

C_3 = Ordering cost per order.

Simplifying the inventory incremental cost function; we get

$$\text{EOQ} = q^* = \sqrt{\frac{2 \times C_3 \times d}{C_1} \cdot \frac{1}{\left(1 - \dfrac{d}{P}\right)} \cdot \frac{C_1 + C_2}{C_2}}$$

$$\text{Maximum inventory} = \sqrt{\frac{2 \times C_3 \times d}{C_1} \cdot \frac{\left(1 - \dfrac{d}{P}\right)}{1} \cdot \frac{C_2}{C_1 + C_2}}$$

$$\text{Minimum stock-out} = \sqrt{\frac{2 \times C_3 \times C_1 \times d}{(C_1 + C_2) \cdot C_2} \cdot \frac{\left(1 - \frac{d}{P}\right)}{1}}$$

These three results help us deriving the remaining formulae.

12.8 ILLUSTRATIONS

After making study of assumptions, setting equations of incremental cost functions, and necessary formulae, we see their applications by taking illustrations.

ILLUSTRATION 2

A factory needs 1500 units of 6″ bar (Raw material) every month. The cost of one such bar is ₹28. Ordering cost per order is ₹150. The inventory carrying cost is 0.2 fraction of a rupee per year. Find the following.

1. EOQ
2. Total inventory cost per year.
3. Number of orders per year

Solution:

There are no shortages and no lead time.

Annual requirement = 1500 × 12 = 18000 units.

Ordering cost = ₹150 per order.

Carrying cost = Cost of one item × 0.2 = 28 × 0.2 = 5.6

1. $\text{EOQ} = q^* = \sqrt{\dfrac{2AC_3}{C_1}}$ where A = Annual demand

$$= \sqrt{\frac{2 \times 18000 \times 150}{5.6}} = 982 \text{ items}$$

2. Total incremental cost = $\sqrt{2AC_1C_3} = \sqrt{2 \times 18000 \times 150 \times 5.6} = 5499$
3. Number of orders = A/q^* = 18000/982 = 18.32 orders.

ILLUSTRATION 3

Special type of integrated circuits, manufactured by an IT Solution Company needs 750 units of capacitors per month. Carrying cost is ₹12/5 per unit per year and ordering cost per order is ₹100. Also, it is given that the cost of one unit is ₹3.

Find (1) EOQ

(2) Ordering frequency

Solution

We have neither lead time nor the shortages in the given data.

Annual demand = 750 × 12 = 9000 units.

C_3 = 100 and C_1 = 12/5

We use the standard formula;

1. EOQ = $q^* = \sqrt{\dfrac{2AC_3}{C_1}}$ where A = Annual demand. Substituting all the values, we get

   EOQ = 867 units.

   Placing an order of this size will minimize the additional costs.
2. Ordering frequency = q^*/d = 9000/867 = 10.38 orders in a year.

ILLUSTRATION 4

In the context of example 3, it is given that

1. Total number of working days is 300.
2. There is a lead time of 12 days and the company maintains a safety stock of 200 items.

Find

1. Maximum and minimum inventory level
2. Reorder level
3. Actual unit cost after calculating additional costs
4. Minimum total additional cost.

Solution

1. Maximum inventory = q^* + Safety stock = 867 + 200 = 1067 items

   Minimum inventory = Safety stock = 200 units.
2. Reorder level = Safety stock + Demand during the lead time.

   = 200 + (9000/300) × 12 [∵ There are 300 working days.]

   = 200 + 360 = 560 items
3. Minimum additional cost = Total incremental cost

   $= \sqrt{2AC_1C_3} = 2078.5$
4. Total cost = 9000 × 3 + 2078.5 = 29078.5

ILLUSTRATION 5

A company has a demand rate of 25 items per day and the supply rate is 40 items per day. Ordering cost per order is ₹60 and carrying cost is ₹73 in one year. Find the EOQ and the total additional cost. It is given that an item costs ₹20.

Solution

It is given that

p = Supply rate = 40 items/day

d = Demand rate = 25 items/day

C_3 = ₹60 per order.

Now,

Carrying cost per unit per day = 73/365 = 1/5

$$\text{EOQ} = \sqrt{\frac{2C_3d}{C_1\left(1-\frac{d}{p}\right)}} = \sqrt{\frac{2\times 60\times 25}{0.2\times\left(1-\frac{25}{40}\right)}} = 200 \text{ items}$$

$$\text{Total additional cost} = \sqrt{2C_3d\times C_1\left(1-\frac{d}{p}\right)}$$

We put corresponding values and get

Total additional cost = ₹15 per day.

On a purchase of 30 items per day @ $40 \times 20 = 800$, there is an additional cost of ₹15.

Total cost = 815 for 40 items.

ILLUSTRATION 6

A firm has experienced different types of demand situations and studied that it follows a normal distribution law. Standard distribution of this normal demand pattern is 18 units. The company is clientage maintenance oriented and wants to have a service level = 95%. Average lead-time is 8 days and average demand is 20 units. Find the ROL.

Solution

It is given that

$\sigma = 18$ units

Safety stock = $k \times \sigma$,

ROL = Demand during the lead time + $k \times \sigma$

Demand during the lead time = Average lead time × average demand = $8 \times 20 = 160$ units.

Safety stock = $k \times (18)$

Value of k at 95% level of service is 1.64 [from the normal table]

Safety stock = $(1.64) \times (18) = 29.52 \approx 30$ units

ROL = 160 + 30 = 190 units.

ILLUSTRATION 7

Using the facts of the following data of Sapsun Corporation, answer the following questions.

Find

1. Lead time demand
2. Safety stock

Average Demand = 2400 units

Ordering cost per order = ₹250

Cost of one unit = ₹12.5

Carrying Cost = 8% of the unit cost

Number of Working Days = 300 per year

Lead time = 12 days. (Take this as average on a span)

Std. deviation of demand = 80 units

Confidence level = 95%.

Solution

Lead time demand = 12 days × Demand per day during lead time

$= 12 \times (2400/300) = 96$ units.

Safety stock = $k \times \sigma = (1.64) \times (80) = 131.2 = 132$ units.

ROL. = Lead time demand + Safety stock = 96 + 132 = 228 units.

ILLUSTRATION 8

Annual demand of a fast running item is 9000 units. Carrying cost is ₹50 per year per unit. Shortage cost is ₹1000 per unit per year. Ordering cost is ₹800 per order. Find the optimal order quantity and maximum shortage quantity.

Solution

This is a planned shortage model and we will, apply the following results using standard notations.

$$q^* = \sqrt{\frac{2 \times C_3 \times d}{C_1}} \times \sqrt{\frac{C_2 + C_1}{C_2}} = \text{EOQ}$$

and
$$q_1^* = \sqrt{\frac{2 \times C_3 \times d}{C_1}} \times \sqrt{\frac{C_2}{C_2 + C_1}}$$

$$q^* = \sqrt{\frac{2 \times 800 \times 9000}{50}} \sqrt{\frac{1000 + 50}{1000}} = 550 = \text{EOQ}$$

and
$$q_1^* = \sqrt{\frac{2 * 800 * 9000}{50_1}} \times \sqrt{\frac{1000}{1000 + 50}} = 524 \text{ items}$$

From these two, we can write that,
optimum stock out = 550 – 524 = 26 items.

12.9 ESSENTIALS OF 'Q' SYSTEM: FIXED ORDER QUANTITY SYSTEM—REORDER LEVEL, LEAD TIME DEMAND, RESERVE STOCK, AND SAFETY STOCK

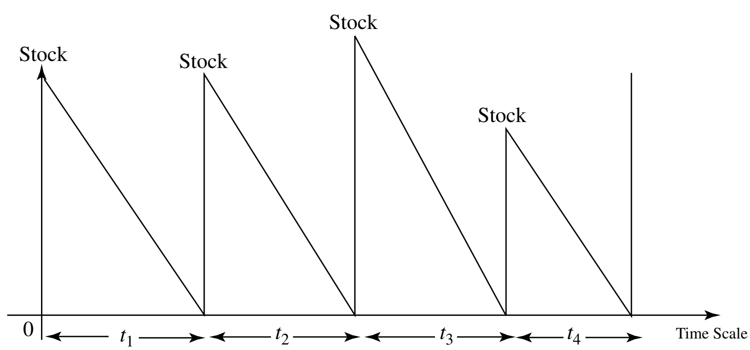


Figure 12.9

The system described above makes a pattern of intuitive system. The decision maker on his own mindset (probably study of the past pattern of demand) takes a decision about placing an order of some x amount of units.

At this stage, one point is certain that the replenishment rate is immediate and infinite (satisfying any order-size). This is a great favour to the decision maker.

If the supplier warns him about a fixed time (days) between placement of order and despatch of goods, then the decision maker will additionally calculate the transition time and stocking the goods into existing inventory level.

For example, if supplier gives 8 days' time then the dealer will add, say 2 days for transportation period and stocking it to warehouse. Thus, it makes $8 + 2 = 10$ days period. This is called lead time = L.

Let us, at the moment, think of constant demand rate over a period of time, say d units per day.

This quantity d is called the *demand rate during lead time*, say 5 units/day.

The wise decision-maker will calculate $L \times d$ = Total demand during lead time.

[$L = 10$ and $d = 5$ so we have $L \times d = 50$ units demand during the lead time.] The decision maker places an order when there is $L \times d = 50$ units of items left in the stock. This is called ***reorder point***.

$\therefore$ Reorder point = $L \times d$ of stock (1)

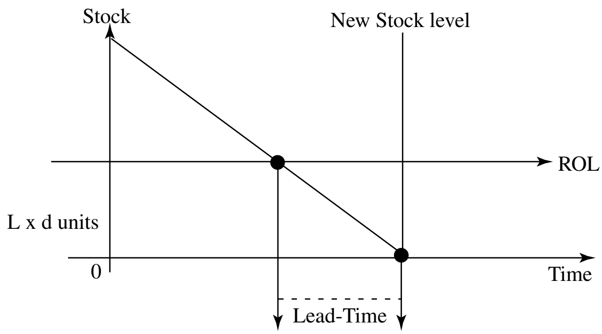


Figure 12.10

The above formula of reorder level (point) has following two assumptions.

1. The lead time is pre-known and fixed.
2. The demand during lead time is constant.

If any one or both fluctuate, then we have different type of situations as follows.

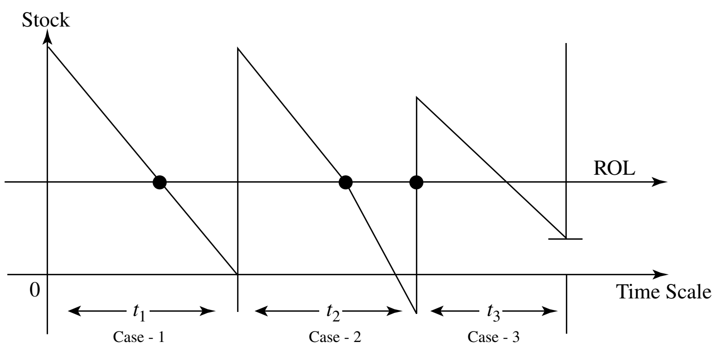


Figure 12.11

Case 1

In the first case, time slot t_1,

Everything okay, stock is zero, items arrive and no change in the status.

Case 2

In the second case, time slot t_2,

Lead-time increases, assuming steady demand, the dealer runs out of stock. Observe that the stock level crosses time scale. It generates shortages and back-orders are taken, shortage costs are applicable.

Case 3

In the third case, the time slot t_3

Lead time decreases and assuming steady demand, the goods arrive before the existing stock is completely to a zero level, i.e. stock is exhausted. The dealer is liable to bear carrying cost of new stock for some additional period till the existing stock is not exhausted.

To safeguard against all such events, we add a small fraction of quantity to the reorder level quantity, which is called a *safety stock*.

Now the Formula 1 above is ROL = Lead time demand + Safety stock.

$$\text{Safety Stock (SS)} = k \times \sigma$$

where

σ is the standard deviation of demand pattern distribution.

k is the standard normal statistical value service level. Service level 1 means complete satisfaction. This distribution follows a normal distribution law.

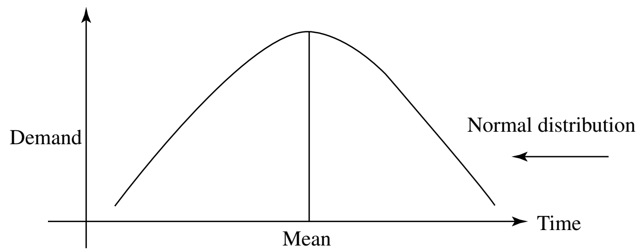


Figure 12.12

Reserve Stock

Again, it comes to the same fundamentals that the lead time is never constant. It depends on supplier's circumstances. We make a study and make an approximation for the lead time.

One more fact is equally true that the demand during the lead time is also not constant.

It always happens in cut-throat competition time that the competitors test your stocking capacities and competency to supply during lead time.

In some cases, suppliers make intentionally late supply by extending the lead time.

The average demand during the lead time is called the *reserve stock*; denoted as R.

Now, we put all things together: Demand during committed lead time + average demand during lead time fluctuation (delivery delay) + safety stock is our actual reorder level.

Figure 12.13 clarifies all these points.

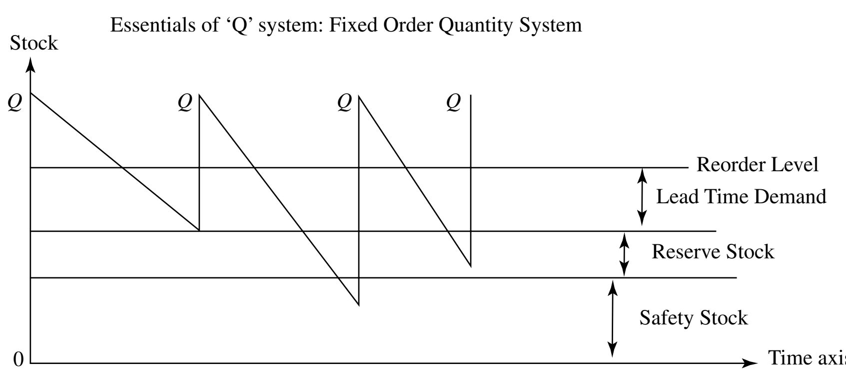


Figure 12.13

12.10 Periodic Review System

In this type, it is intuitively decided to review the status existing inventory at a fixed time interval. In most of the cases, an order is placed at that point of time. The review period is approximately decided by calculating

Review period = Total annual demand/EOQ

This, in fact, gives the time interval between two consecutive intervals which may be, without much hesitation can be accepted as periodic review system.

In fact, this does not allow fluctuations in demand and lead time demand. In addition to this, before placing an order of a fixed size, inventory on hand is calculated and an order is placed of the size that after arrival of the new stock, the total amount of inventory reaches to a pre-determined level.

Normally, this pre-determined inventory level is equal to average demand per day average lead time plus the demand between two consecutive review periods. If there remains delay in supply then accordingly reserve stock items are added to this, the result narrated above and the final value of the upper level is decided.

We take an illustration to explain this concept.

Illustration 9

Following data relate an existing inventory system.

Monthly demand = 3000 items

Cost of one unit = ₹15

Ordering cost per order = ₹400

Carrying cost fraction = ₹0.2 per unit per year

Average lead-time = 3 weeks with probable delay period = 1 week. Probability for this delay is 0.2 Assuming the demand distribution to be normal; standard deviation of the demand = 150 units.

Solution

First we apply standard EOQ formula

$$\text{EOQ} = q^* = \sqrt{\frac{2AC_3}{C_1}} \text{ where } A \text{ is annual demand.}$$

$$q^* = \sqrt{\frac{2 \times 36000 \times 400}{15 \times 0.2}} = 3099 \text{ units.}$$

Demand during lead time = (Monthly demand/4) × 3 (lead time in weeks)

= (3000/4) × 3 = 2250 units = Lead time demand (1)

Now, we find the reserve stock.

Reserve stock = Demand per week × number of weeks delay × Probability of maximum delay.

= (3000/4) × 1 × 0.2 = 150 units = Reserve stock (2)

We need standard deviation of demand during lead time

= √(Lead-time) × standard deviation of the demand during lead time.

Standard deviation of demand = √(3) × 150 = 260 units.

If we take service level = 95% then

$k = 1.64$ (from normal distribution table)

Safety stock = $k \times \sigma = (1.64) \times 260 = 427$ units (3)

Using Results 1, 2, and 3, we find

$$\text{Reorder level} = \text{Lead time demand} + \text{Reserve stock} + \text{Safety stock}$$
$$= 2250 + 150 + 427 = 2827 \text{ units}$$
$$\text{Review period} = \text{EOQ/Annual demand}$$
$$= 3099/(12 \times 3000)$$
$$= 0.086 = 0.086 \times 365 \text{ days} = 32 \text{ days.}$$

12.11 Inventory Model with Price Breaks

In some cases, manufacturers and/or some dealers announce the sales promotion schemes. Under the plan of the scheme, basic price or the sale price of the items decrease as the number of items bought by the retailers increase. Such offers look very lucrative, as one would like to purchase more items as the basic cost price decreases which finally ends in the lowest value of one unit. This, in turn, is thought of fetching more profit. We justify the case by taking one illustration and then finally describe the procedure of dealing with such cases.

Illustration 10

A manufacturer announces a scheme for his distributors. Set up cost is ₹100 per set up and monthly demand is 200 units. Holding cost is 2% of the cost of the item. Find the optimal order quantity.

Scheme	Quantity	Unit Price (Part 1)	Part 2
(1)	$0 < q < 500$	10	100
(2)	$500 \le q < 750$	9	90
(3)	$750 \le q$	8	80

Solution

Part 1

As given, set up cost = C_3 = 100, annual demand = $200 \times 12 = 2400$

C_1 = holding cost = $.02 \times$ basic cost

Let us consider cost = ₹8 and find EOQ.

$C_1 = 0.02 \times 8 = 0.16$

$$q^* = \sqrt{\frac{2AC_3}{C_1}} \text{ where } A \text{ is annual demand}$$

$$q^* = \sqrt{\frac{2 \times 2400 \times 100}{0.16}} = 1732 \text{ units.} = \text{EOQ}$$

As this amount, 1732 is greater than 750 units; we accept this as optimal order quantity.

At this level of EOQ, incremental cost is $\sqrt{2AC_1C_3} = \sqrt{2 \times 2400 \times 100 \times 0.16}$

$= ₹278$

Part 2

We, consider a change in basic price.

Scheme	Quantity	Unit Price
(1)	$0 < q < 500$	100
(2)	$500 \le q < 750$	90
(3)	$750 \le q$	80

Now, we apply EOQ formula.

$$q^* = \sqrt{\frac{2AC_3}{C_1}} \text{ where } A \text{ is annual demand}$$

$$q^* = \sqrt{\frac{2 \times 2400 \times 100}{1.6}} = 548 \text{ units} = \text{EOQ}.$$

This amount is less than 750 so it falls in the second part of the scheme. (2).

At this level of EOQ, incremental cost is $TIC\ (548) = (0.5)(548 \times 1.8) + (2400/548) \times 100$
$= ₹931.15$

If we place an order of 750 items then, the total additional inventory cost will be the sum of carrying charges and ordering cost.

$$TIC\ (750) = (0.5)\ (750 \times 1.8) + (2400/750) \times 100$$
$$= 675 + 320 = ₹995$$

If we place an order of 500 items then, the total additional inventory cost will be the sum of carrying charges and ordering cost.

$$TIC\ (500) = (0.5)\ (500 \times 1.8) + (2400/500) \times 100$$
$$= 450 + 480 = ₹930$$

Now, we compare,

$$TIC\ (548) = (0.5)\ (548 \times 1.8) + (2400/548) \times 100 = ₹931.15$$
$$TIC\ (750) = (0.5)\ (750 \times 1.8) + (2400/750) \times 100 = ₹995$$
$$TIC\ (500) = (0.5)\ (500 \times 1.8) + (2400/500) \times 100 = ₹930 \ ***$$

The difference between the values at $TIC(548)$ and $TIC(500)$ is very small; but anyway, placing an order of 500 items is better.

Working Procedure

Let us represent the problem and then working procedure.

Quantity Breaks	Price per unit
(1) $0 \le q_1 < b_1$	p_1
(2) $b_1 \le q_2 < b_2$	p_2
...	...
...	...
(n) $b_{n-1} \le q_n$	p_n

In the case, of price break or (quantity discounts); we follow,

1. find EOQ for the last price break. [Call this EOQ as q_n*.]
   [Take cost of the item = p_n and order price, etc., as given, which shall remain common to all the cases. What changes here is the price for a purchase in an amount ranging in a particular interval.]

2. If this $q_n^* \geq b_{n-1}$, then this EOQ = q_n^* is optimal and minimizes the incremental cost.
   If it is not so, then go to Step 3.
3. Now, find the interval in which this EOQ remains.
   Let us assume that it lies in the second interval, i.e. $b_1 \leq q_n^* < b_2$.
4. In this interval, you have come with order quantity q_n^* and price p_n.
   Now, for this quantity q_n^*, find total cost on all such q_n^* items. This is the sum of basic cost (purchase cost) and total incremental cost [take cost = p_2, as you are in the second interval. We have, EOQ = q_n^*.
   Total cost = $T(q_n^*) = q_n \times p_2$ + incremental cost.
5. It is essential that, we also find total cost for all b_2 items. The quantity b_2 is the opening quantity of the third interval.
6. In some cases, it is found that as EOQ of intervals are all different, the total cost for all limiting quantities (all $b_1, b_2, \cdots$) should be found and mutually compared before making decision about final interval and EOQ.

ILLUSTRATION 11

Let us consider a plan announced by a manufacturer to his customers.

Scheme	Quantity	Unit Price
(1)	$0 < q < 500$	100
(2)	$500 \leq q$	90

Unit ordering cost to one of his buyers is ₹200 and holding cost is 20% of the cost per unit per year. He has an annual requirement of 2400 units. You, as a buyer, are required to analyze the given scheme and take a decision.

Solution

1. First, we find EOQ considering ₹90 – unit cost

$$\text{EOQ} = q^* = \sqrt{\frac{2AC_3}{C_1}} \quad \text{where, } A \text{ is annual demand}$$

$$= q^* = \sqrt{\frac{2 \times 2400 \times 200}{0.2 \times 90}} = 231 \text{ units.}$$

   Comment: This cannot be acceptable because the price ₹90 is only permissible if the order quantity equals or exceeds 500 units.

2. Now, we find EOQ if the cost is considered ₹100.

$$\text{EOQ} = \sqrt{\frac{2 \times 2400 \times 200}{0.2 \times 100}} = 220 \text{ units.}$$

3. Total incremental cost of placing an order of 500 units.
   $= (1/2) \times 500 \times (0.2 \times 90) + (2400/500) \times 200$
   $= 4500 + 960 = 5460$
4. Let us find incremental cost for EOQ = 220
   Total incremental cost = ₹4382
   We compare incremental costs for order sizes 220 and 500.
   As calculated above, the costs are ₹4382 for 220 items and ₹5460 for 500 items.
   Incremental cost per unit (for order size 500) is ₹2.275
   Incremental cost per unit (for order size 220) is ₹1.825
   We conclude to place an order for 220 items.

12.12 Inventory Models with Probabilistic Demand

We have discussed all types of inventory models with a little variation in assumptions.

All the assumptions were feasible in reality except one and that is about the demand.

Demand is always a random variable and it can be approximated by using demand of the past records and applying probability theory. What we find in this section is expectation of demand occurring in a time slot.

Assumptions:

1. Set up cost is zero
2. Demand is instantaneous
3. Stock level is discrete.
4. Lead time is zero.

Notations

1. D is for discrete demand rate and probability of occurrence is p_D.
2. I_m shows discrete stock level during the time interval t.

Case 1

When the number of units supplied is less than the number of units in the inventory during that interval, i.e. $D \leq I_m$, then the manufacturer has to bear a holding cost on number of items lying surplus in the stock. He has carrying cost = $C_1 \times (I_m - D)$

Case 2

When the supply is committed more than what exists in the inventory, the supplier bears a shortage cost. He has a shortage cost = $C_2 \times (D - I_m)$

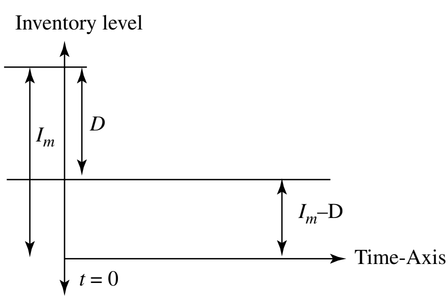


Figure 12.14

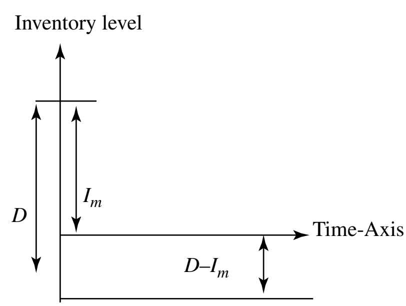


Figure 12.15

In the cases, we find the expected costs.

Expected costs are $= C_1 \times (I_m - D)^* \, p_D$ if $D \leq I_m$

$= C_2 \times (D - I_m) \times p_D$ if $I_m \geq D$

$= 0$ if $I_m = D$

Total expected cost per unit time is the algebraic sum of all these costs over all such intervals.

$$\text{Total expected cost} = TIC(I_m) = C_1 \cdot \sum_{D=0}^{D=I_m} (I_m - D) \cdot P_D + C_2 \cdot \sum_{D=I_{m+1}}^{\infty} (D - I_m) \cdot P_D \qquad (1)$$

We want to find the value of I_m, which minimizes total expected incremental cost per unit time.

What we have found here (Result 1) is the total expected incremental cost for I_m items.

Now, we find incremental cost for $(I_m - 1)$ items; i.e. $TIC(I_m - 1)$
[This is found by putting $(I_m - 1)$ for I_m in Result 1]

We get $\quad TIC(I_m - 1) = TIC(I_m) - (C_1 + C_2) \cdot p_D + C_2 \quad$ where $D \leq I_m - 1$ (2)

In the same way on putting $(I_m + 1)$ for I_m, we get a result for $TIC(I_m + 1)$

We get $\quad TIC(I_m + 1) = TIC(I_m) + (C_1 + C_2) \cdot p_D - C_2 \quad$ where $D \leq I_m$ (3)

For $TIC(I_m)$ minimum, we should have $TIC(I_m - 1) \geq TIC(I_m)$ and $TIC(I_m) \leq TIC(I_m + 1)$

It means that I_m is an optimal value which makes $TIC(II_m)$ minimum.

Again, for the discrete case, we have $I_m - 1 \leq I_m \leq I_m + 1$

$$P(D \leq I_m - 1) < \frac{C_2}{C_2 + C_1} < P(D \leq I_m)$$

If the oversupply cost C_1 and the shortage cost C_2 are known, the optimum quantity I_m is determined when the value of the cumulative probability distribution exceeds the ratio of holding cost to the sum of holding cost and shortage costs.

Note: In the case of perishable items, like newspapers, regularly delivered sweet breads, milk bags, etc., there cannot be a fixed pattern of distribution but the probability distribution can be established; it may be a discrete distribution.

Demand	d_1	d_2	d_3	d_4	d_5	d_6	d_7
Probability	p_1	p_2	p_3	p_4	p_5	p_6	p_7

Then under such cases, the optimal order quantity d_i* can be determined by using the following relation.

$$p_{(i-1)} < \frac{C_2}{C_2 + C_1} < p_i$$

ILLUSTRATION 12

A newspaper boy aided by a statistician keep a track on the actual sale of papers.

He brings the newspapers at a cost of ₹3.00 each and sells it at ₹5.50 per copy.

Assume that each day's demand is independent of the previous one, how many papers he should bring each day?

The record is as follows.

Demand	20	21	22	23	24	25	26	27	28
Probability	0.4	0.5	0.11	0.18	0.20	0.22	0.12	0.5	0.3

Solution

We have holding cost = C_1 = ₹3,

Shortage cost per copy = C_2 = 5.50 – 3 = ₹5/2

We make cumulative probability distribution of demand.

Demand	20	21	22	23	24	25	26	27	28
Probability	0.4	0.5	0.11	0.18	0.20	0.22	0.12	0.5	0.3
Cumulative Probability	0.4	0.9	0.20	0.38	0.58	0.80	0.92	0.97	1.00

The desired optimum value is determined by the following formula.

$$p_{(I-1)} < \frac{C_2}{C_2 + C_1} < p_i$$

Now, the factor, $\dfrac{C_2}{C_2 + C_1}$ = (5/2)/(3 + 5/2) = 5/11 = 0.455

This value is between $P(d = 23) = 0.38$ and $P(d = 24) = 0.58$

The ratio $\dfrac{C_2}{C_2 + C_1}$ = 0.455 being closer to 0.38, a decision of purchasing 23 newspapers will minimize his incremental cost.

ILLUSTRATION 13

A storekeeper, in morning, goes to the wholesale market and brings a lot of loaves. The amount of loaves he will get, depends on the total stock remaining with the wholeseller. Each loaf costs him ₹10 and he sells each one at ₹17 each. He maintains a complete record of his sale of last 100 days.

The record is as follows. You are required to determine the number of loaves he brings to sell.

Number of loaves sold:	22	23	24	25	26	27	28
Number of Days:	0.12	0.15	0.18	0.21	0.27	0.05	0.02

Solution

For the vendor, C_1 = ₹10

C_2 shortage cost per unit = 17 – 10 = 7

Now, we make cumulative probability distribution.

Number of loaves sold:	22	23	24	25	26	27	28
Number of Days:	0.12	0.15	0.18	0.21	0.27	0.05	0.02
Cumulative Probability	0.12	0.27	0.45	0.66	0.93	0.98	1.00

Now, we find $\dfrac{C_2}{C_2 + C_1} = \dfrac{17}{17+10} = \dfrac{17}{27} = 0.6296$

This falls between 0.45 and 0.66, 0.6296 being closer to 25, we take a decision for 35 items.

12.13 MULTIPLE ITEMS WITH STORAGE LIMITATIONS

We have so far discussed about finding economic order quantity and additional cost of keeping and ordering for one inventory item. That is a good beginning but at the same time it is a noticeable fact that in any business there are many items involved. Some items are related to raw materials that we need for production. When the production is ready there is an inventory of units produced. In addition to this, there are inventories of related items. As an example, we need to have inventories of packing materials and boxes too. Any shortage in any area will immediately reflect on the work efficiency of the system as a whole.

Cases may arise about limitations of resources. One may not have enough space for storage of raw materials or the storage of packing materials. In some cases it is possible that some items are produced as by-product of the main item of production. In some cases there may be many different departments making different products from the same raw material.

Important for us is to develop some mathematical model that can handle all varied situations of production of economic order quantities for different items under different limitations of resources, like money, man power, and material.

We consider general method of dealing with such situations.

We shall use the same notations for this description except the fact that each notation used at this, will carry sub-script i (i = 1 to n)

If we deal in n different items, then as derived in the Basic Model 1, we write

$$\text{EOQ}_i = q_i^* = \sqrt{\frac{2 \times d_i \times C_{3i}}{C_{1i}}} \tag{1}$$

Let S_i stand for the space engaged by one i^{th} item. Let q_i denote the quantity of the i^{th} item.

The total space occupied by all these items is $q_i \times s_i$.

This implies that the total space required for all such i(i = 1 to n) items is

$$\sum_{i=1}^{i=n} s_i \times q_i \tag{1}$$

Let us consider limitation on the available space. If the available space is k square units, then we have

$$\sum_{i=1}^{i=n} s_i \times q_i \le k$$

So, we have

$$\sum_{i=1}^{i=n} s_i \times q_i - k \le 0 \tag{2}$$

In this case we have the objective function to minimize the total incremental cost of all such n items.

This cost is the sum of holding costs and ordering costs of all such n items

$$\text{Minimize } Z = \sum_{i=1}^{i=n} \left(\frac{1}{2} \times q_i \times C_{1i} + \frac{d_i}{q_i} \times C_{3i} \right) \tag{3}$$

We put all these points together to construct a mathematical model for optimization.

Find q_i so as to minimize $Z = \sum_{i=1}^{i=n} \left(\frac{1}{2} \times q_i \times C_{1i} + \frac{d_i}{q_i} \times C_{3i} \right)$

subject to $\sum_{i=1}^{i=n} s_i \times q_i - k \le 0$

This problem has n variables and one constraint; it can be solved by introducing Lagranage's undetermined multipliers (or by Kuhn Tucker conditions).

Lagrange function,

$$L\,(q_1, q_2, \cdots, q_n, \lambda) = \sum_{i=1}^{i=n} \left(\frac{1}{2} \times q_i \times C_{1i} + \frac{d_i}{q_i} \times C_{3i} \right) - \lambda \left\{ \sum_{i=1}^{i=n} s_i \times q_i - k \right\} \tag{4}$$

We know that for optimality partial derivatives should be zero.

$$\frac{\delta L}{\delta q_i} = \frac{1}{2} \times C_{1i} - \frac{1}{q_i^2} \times d_i \times C_{3i} - \lambda \times S_i = 0 \tag{5}$$

and

$$\frac{\delta L}{\delta \lambda} = -\left(\sum_{i=1}^{i=n} s_i \times q_i - K\right) = 0 \tag{6}$$

The above equations will be useful to serve our purpose.

$$\text{Economic order quantity} = q_i^* \sqrt{\frac{2 \times d_i \times C_{3i}}{C_{1i} - 2 \times \lambda^* * S_i}} \tag{7}$$

Equation 6 clearly signifies maximum usage of space = k = available units.

12.14 Inventory Classification System

It is very important to make and apply classification of inventories in a given system. In any given system, may it be your home or a big international corporation, there are thousands of items, some are utilized on daily basis, some are weekly, periodically, monthly, etc. This calls for classification of the items in the inventory.

(A) ABC Analysis

It is also known as *Always Better Control* (*ABC*).

It is an analysis based on levels of importance. It is also called *proportional value analysis*.

We have to keep and maintain stock of the items that you use. Senior level persons manage important ornaments or costly items.

Class A

A group of very costly items makes one division and it is a high value class. It requires high attention for maintenance. Items in this class are worth more than 85% or 90% of the total cost of all items in inventory. Such items are 5% to 10% in the total number of items in inventory.

Class B

It is a class of items having 15% to 20% contribution in the total cost of inventory. Approximately, 10% to 20% of total numbers of items in inventory fall in this class.

Class C

It is a class of items having 5% to 10% of the total value of all items in inventory. Approximately, 70% to 80% of total numbers of items in inventory fall in this class.

Plot of *ABC* curve is called *Pareto Curve* or *Lorenz Curve*.

Curve showing ABC analysis

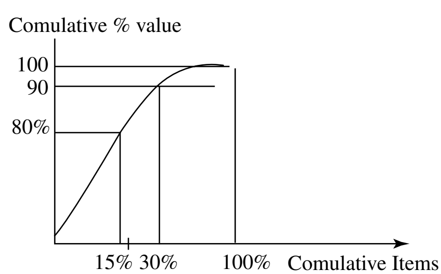


Figure 12.16

Some More Methods of Analysis:

1. XYZ Analysis
   When classification of inventory is done on the basis of closing stock value is called XYZ analysis.
2. HML Analysis
   Inventories are classified based on unit price of the product it is called HML Analysis.
   H = High value items
   M = Medium value items
   L = Low value items
3. VED Analysis
   This analysis classifies the inventory items into 3 catagories on the basis of criticality of the components
   V = Vital items—most attention is paid to such items. Such items must not be out of stock.
   E = Essential items
   D = Desirable items
4. FSND Analysis
   This classification of inventory items is done on the demand rate and primary usage.
   F = Fast moving items
   S = Slow moving items
   N = Non-moving items
   D = Dead items—these items do not have future demands.
5. SDE Analysis
   This analysis is done on the basis of lead time
   S = Scarce to obtain
   D = Difficult to obtain
   E = Easy to obtain
   This analysis is useful for designing
   (1) planning strategies and (2) purchasing strategies

ADDITIONAL QUESTIONS FOR PRACTICE (WITH HINTS AND ANSWERS)

Question 1

A stationary store sells 2400 calculators in one year. Holding cost the storekeeper bears is 10% of the cost of one calculator which is ₹600. When he places an order he has to incur a cost of ₹200 per order. Find the EOQ. If there is a lead time of 5 weeks then find the reorder level.

Solution

It is given that

$C_3 = ₹200$,

$C_1 = 0.1 \times 600 = 60$

A = annual requirement = 2400

We know that,

1. $\text{E.O.Q.} = q^* = \sqrt{\dfrac{2AC_3}{C_1}}$ where A = annual demand

$$= \sqrt{\frac{2 \times 2400 \times 200}{60}} = 127 \text{ units.}$$

2. Total incremental cost $= \sqrt{2AC_1C_3}$

$$= \sqrt{2 \times 2400 \times 200 \times 60} = ₹7590$$

3. Demand during the lead time = Lead time in weeks × demand per week during the lead time = 5 × (2400/52) = 231 units

   Reorder level = 231 items.

Question 2

The production rate of an item is 50 units per day. The set up cost for each run is ₹120 per set up. The holding cost is ₹0.02 per day. The demand rate is considered constant and it is 25 units per day. Find EOQ and the cycle time.

Solution

We are given,

$p = 50$ units

$d = 25$ units per day

Ordering cost per order = ₹120

Holding cost = 0.02 × 365 = 7.30

$$\text{EOQ} = \sqrt{\frac{2C_3 d}{C_1\left(1-\frac{d}{p}\right)}} = \sqrt{\frac{2 \times 120 \times 25}{7.30 \times \left(1-\frac{25}{50}\right)}} = 41 \text{ units}$$

Number of orders = Annual demand/EOQ = (25 × 365)/41 = 222.56

Annual demand = 25 × 365 = 9125 units.

Question 3

For an inventory system, it is given that annual demand is 9984 units. Cost of an item is ₹200 per unit. Carrying charges are 2% of the cost of one unit. Ordering cost is ₹300 per order. Find the per unit cost including incremental costs.

Solution

It is given that

$A = 9984$ units

$C_3 = 300$

$C_1 = 0.02 \times 200 =$ ₹4 per unit per year

Apply EOQ $= q^* = \sqrt{\frac{2AC_3}{C_1}}$ where A = annual demand

Total Incremental Cost $= \sqrt{2AC_1C_3}$

Incremental cost per unit = Basic cost + Total incremental cost/annual demand
[Substitute the figures and find the answers.]

POINTS TO REMEMBER

1. General Notations and Important Points:

   We consider C_1 as inventory carrying cost per unit per unit time.

   $C_1 = C \times I$ [where I is the fraction value of a rupee per year.]

   C_2 = shortage cost per unit per year

   C_3 = ordering cost per order.

   d = demand rate per unit time.

   q = lot size or order size.

   q^* = Economic order quantity

   EOQ is the amount which minimizes the additional (all types of inventory] costs.

   Total incremental cost = carrying cost + holding cost + ordering cost (1)

   Number of orders = (d/q^*); once this is found, one can find the average time between two successive intervals in a given period or unit time. You may take this as one year.

   Reorder level = Safety stock + Total consumption during lead time. (2)

   We repeat, lead time duration is an estimation; it cannot be constant. The fluctuation in production is an important factor causing short or partial supply or no supply state which ultimately results into lead time.

   Total consumption during lead time = demand during the lead time = lead-time demand

   Maximum inventory = EOQ + safety stock (3)

   Minimum inventory = Safety stock (4)

   Average inventory = (Maximum inventory + Minimum inventory)/2 (5)

   Total cost = Basic cost of items + Total incremental cost (6)

2. When demand is pre-known and it is considered as constant over a given period of time and assuming that the supply rate is practically immediate and as per demand, then in such cases, we have standard EOQ model. (Shortages are not allowed.)

   $$\text{EOQ} = q^* = \sqrt{\frac{2AC_3}{C_1}} \quad \text{where } A = \text{annual demand} \tag{1}$$

   Using this EOQ formula, one can find the total incremental cost for all the items.

   $$\text{Total incremental cost} = \sqrt{2AC_1C_3} \tag{2}$$

   If the supply rate is not infinite (or not completely according to the demand), then in this case if demand is constant, we have two results as follows.

   $$\text{EOQ} = \sqrt{\frac{2C_3d}{C_1\left(1-\frac{d}{p}\right)}}$$

   $$\text{Total additional cost} = \sqrt{2C_3d \times C_1\left(1-\frac{d}{p}\right)}$$

3. For planned shortages, we use the following results.

$$q^* = \sqrt{\frac{2 \times C_3 \times d}{C_1}} \cdot \sqrt{\frac{C_2 + C_1}{C_2}} = \text{E.O.Q}$$

and $$q_1^* = \sqrt{\frac{2 \times C_3 \times d}{C_1}} \cdot \sqrt{\frac{C_2}{C_2 + C_1}}$$

4. When there is an inventory of multiple items and there is an additional constraint on the total amount or worth of order size with an objective to minimize the total of incremental costs then you should apply the following formula.

$$\text{Optimum } Q = \sqrt{\frac{2 \times C_{3i} \times A_i}{C_{1i} + 2 \times \lambda \times C_i}}$$

Where A_i annual demand of ith item. C_i is the cost of ith item.
C_{1i} is the holding cost of ith item. C_{3i} the ordering cost of ith item.
λ is Lagrange's multiplier; you may take its value = 2, 3, 4, etc.]

EXERCISES

OBJECTIVE TYPE QUESTIONS

I. State True or False:

1. Inventory carrying cost is a part of basic cost of an item.
2. Ordering cost depends on total annual demand.
3. Carrying cost is a linear function of ordering quantity.
4. When demand is dependent and numbers of set ups (runs) are more, JIT is applicable.
5. Reorder level quantity is normally a part or fraction of order size.
6. In planned shortage model, the total incremental cost function is a function of two quantities.
7. As the time between two consecutive orders increases, the ordering cost increases.
8. Lead time is always constant and depends on shortage cost.
9. Fictitious shortages, in some cases, are planned to check the real demand of the items in market.
10. Set up cost for a manufacturer is technically parallel to ordering cost.
11. EOQ is the amount of safety stock which is kept for critical period.

ANSWERS

1. false.	2. false.	3. true.	4. true.	5. true.
6. true.	7. false.	8. false.	9. true.	10. true.
11. false.				

II. Multiple Choice Questions

1. Total incremental inventory cost is a combination of
   (a) carrying cost (b) shortage cost + basic cost
   (c) A and B (d) A + shortest cost + ordering cost.
2. Carrying cost is always
   (a) equal to shortage cost
   (b) dependent on orders
   (c) proportional to number of items in inventory
   (d) as per the policy, either number of items or cost of item
3. An item cost ₹300 and carrying cost is 5% per annum. Inventory carrying cost of 10 items for a period of 4 months is
   (a) ₹5 (b) ₹50 (c) ₹500 (d) ₹100
4. For pre-known demand of 30 items in a year time, and per order cost ₹200, in order to know the number of orders
   (a) it cannot be decided (b) you need to give order size
   (c) you need to give shortage cost (d) you need to give cost of items
5. One cycle time is 2 months. Replenishment rate is 18 items per day for a period of days. The annual demand rate is
   (a) 30 items (b) 360 items (c) 120 items (d) 60 items
6. In the above example, during the days of only sale (no supply) days, what is the demand per day?
   (a) 30 items (b) 3 items
   (c) cannot be found; insufficient data. (d) 18 items
7. The total ordering cost for a period of one year is ₹240. If per order cost is ₹80 then the ordering interval between two consecutive orders.
   (a) 1/3 year (b) 1/4 year
   (c) 3 years (d) cannot be decided

For questions 8, 9, 10 and 11 use the following data.

Cost of an item is ₹60. Carrying charge is 12% of the cost of an item per year. If there is an inventory of 240 items during the period of 6 months. Ordering cost is ₹ 400 per order then find

8. How many orders are there in a year?
   (a) 3 (b) 2 (c) 6 (d) 1
9. What is the total ordering cost?
   (a) ₹800 (b) ₹834 (c) ₹1200 (d) ₹400
10. What is the total carrying cost in ₹ for the period in days?
    (a) 144 (b) 72 (c) 864 (d) 288
11. What is the total cost per item?
    (a) 6.93 (b) 63.46 (c) 65.13 (d) 5.13

Answers

1. (d)	2. (d)	3. (b)	4. (b)	5. (b)
6. (b)	7. (a)	8. (b)	9. (b)	10. (c)
11. (b)				

NUMERICAL PROBLEMS

In the following examples, from 1 to 17, shortages are not allowed. Demand rate is fixed and known. Supply rate is instantaneous and infinite unless specified.

1. Following information are given for an inventory system. Find the economic order quantity. Also find the total incremental cost.
cost of an item = ₹40, annual carrying cost = 5% of the cost of an item.
annual demand = 600 units
ordering cost = ₹100 per order.

2. ABC Corporation is handling its important inventory. Using following facts; determine EOQ and the total incremental cost.
cost of an item = ₹80
annual carrying cost = 8% of the cost of an item
annual demand = 2400 units
ordering cost = ₹250 per order.

3. In the data of the Example 2, it is given that there is a fixed lead time of 10 days and the demand during the lead time is 25 items per day. What is the reorder point?

4. For the Vinsi Corporation, annual demand of sockets is 600 units; each one costs ₹40.
Ordering cost is ₹100 per order and carrying cost is 2% of the cost of an item. There are 300 working days and a fixed lead time of 5 days for every order. Find EOQ and ROI.

5. In a production unit, there has been regular usage of packing boxes and each costs ₹80. Ordering cost per order is ₹250 and the carrying cost is 8% of the cost. A supplier declares a fixed lead time of 12 days and it is an experience that demand during the lead time is 5 items per day. After consumption of how many items one must place an order?

6. For an export oriented division, there has been a regular need of boxes. Each one costs ₹80. Ordering cost per order is ₹250 per order and the carrying cost is 8% of the cost of an item. A fixed lead time is pre-known and it is 10 days. Average demand during the lead-time is 5 items with a standard deviation of 4 units. If service satisfaction factor is 95% then determine the EOQ and reorder level.

7. For an inventory system, the storage cost is 5% of the cost ₹40 of an item. Ordering cost is ₹100 per order. An annual demand is of 600 units; calculate the ordering frequency and ordering interval. Also calculate the difference if an order of size 20% more than the order of inventory is placed.

8. For an inventory system, the supplier has agreed to furnish 80 items per month against the demand of 50 items per month. An item costs ₹40 and ordering cost is ₹100 per order. Holding cost is 5% of the cost of an item. Find the EOQ and the additional inventory costs.

9. In a factory the supplier has agreed to supply at the rate 12000 items in 12 months' time. The demand rate is 9000 items per year. A fixed lead time of 10 days is known. There are 300 working days in 12 months period of 365 days. Cost of an item is ₹400, holding cost is 10% of the cost and an ordering cost is ₹200. Find the reorder level if the safety stock is 10% of EOQ.

10. For an inventory system, the supply rate is 90 items per day and there has been a demand of 50 items per day, cost of an item is ₹100 and the holding cost is 10% of the cost of an item. Ordering cost is ₹100 per order. Calculate EOQ and additional inventory cost.

11. For a manufacturing unit, there are 320 working days in a year and it consumes 32000 items in this period. Cost of one item is ₹400 and ordering cost is ₹200 with holding cost amounting to 10% of the cost of an item, find EOQ and reorder level if there is a fixed lead time of 10 days and safety stock is 20% of the total EOQ size.

12. In the Example 11, if the standard deviation of the demand during the lead time is 10 items and the company wants to preserve service level of satisfaction of 95%, find the reorder level.

13. In a factory the supplier has agreed to supply at the rate 12000 items in 12 months' time. The demand rate is 9000 items per year. A fixed lead time of 10 days is known. There are 300 working days in 12 months period of 365 days. Cost of an item is ₹400, holding cost is 10% of the cost and an ordering cost is ₹200. Find the reorder level if the safety stock is 20% of EOQ. Find the total cost (basic cost + additional cost).

14. A vendor announces a scheme for his dealers. If he receives an order of any amount less than 500 items then he will charge at the rate of ₹10 per item and if the number of items equals or exceeds 500 then he will charge per item at the rate of ₹8 per item. Considering the ordering cost ₹200 and the holding cost 10% of the cost, what decision as a dealer you will take?

15. A vendor announces a scheme for his dealers. If he receives an order of any amount less than 200 items then he will charge at the rate of ₹16 per item and if the number of items equals or exceeds 200 then he will charge per item at the rate of ₹14 per item. Considering the ordering cost ₹300 and the holding cost 8% of the cost, what decision as a dealer you will take?

16. A wholeseller announces the following sales promotion scheme. Considering ordering cost ₹200 per order and carrying charges 8% of the prevailing cost, calculate the most economical order size.

Order Size (q)	Price per Unit
$0 < q < 200$	10
$200 \leq q < 300$	9
$300 \leq q$	8.50

17. Given that the production rate is 36000 units in one year and the demand rate is 24000; calculate the EOQ and period of consumption. Set up cost is ₹500 per order and the carrying cost is ₹15 per unit per year.

18. For the following data: monthly demand = 3000 items, cost of one unit = ₹20, ordering cost per order = Rs 200, carrying cost fraction = ₹0.2 per unit per year, average lead time = weeks with probable delay period = 1 week. Probability for this delay is 0.2 and assuming the demand distribution to be 'normal' ; standard deviation of the demand = 150 units.

19. A newspaper boy aided by a statistician keep a track on the actual sale of papers. He brings the newspapers at a cost of ₹3.00 each and sells it at ₹5.50 per copy. Assume that each day's demand is independent of the previous one, how many papers he should bring each day?
The record is as follows.

Demand	20	21	22	23	24	25	26	27	28
Probability	0.3	0.5	0.11	0.21	0.24	0.20	0.12	0.4	0.1

How many newspapers he should purchase each day?

20. A company deals in three items. The dealer of the company decides that inventory level of these three items together should not cross worth ₹1000. Determine approximate the economic order quantity.

	Items		
	A	B	C
Cost of the item	6	7	5
Holding cost	20	20	20
Ordering cost	50	40	60

[Hint : Use the formula, Optimum $Q = \sqrt{\dfrac{2 \times C_{3i} \times A_i}{C_{1i} + 2 \times \lambda \times C_i}}$

Where A_i annual demand of ith item. C_i is the cost of ith item.
C_{1i} is the holding cost of ith item. C_{3i} the ordering cost of ith item.
$\therefore$ is Lagrange's multiplier; you may take its value = 2, 3, 4, etc.]

Answers to Numerical Problems

1. 244, 490
2. 433, 2772
3. 433, 250
4. 244, 10
5. 433, 60
6. 433, 57
7. 244, 149 days new order size = 293, *TIC* (293) = 538
8. 116
9. ROL = 490
10. 302
11. ROL = 1114
12. 1131
13. 680
14. EOQ = 500, cost = ₹8 per unit
15. EOQ = 79, total cost of 300 items at EOQ = 1631
16. EOQ = 420 units, applicable cost = ₹8.50
17. 2190 units
18. EOQ = 1898, review period is 20 days, ROL =2827 units
19. he should buy 24 newspapers.

13 Critical Path Method (CPM) and Program Evaluation and Review Technique (PERT)

We plan days by lessons from CPM and learn PERT by accidents.

Learning Objectives

AFTER STUDYING THIS CHAPTER, THE STUDENTS WILL BE ABLE TO UNDERSTAND:

- fundamental components of any project
- classification of the type of project
- arrangement of all the activities in their order
- analysis of the network—floats, critical path, time estimation, etc.
- application of crashing to compress project duration

INTRODUCTION

Initially and most fundamentally, any project is a vision and any research is an accident. Once, there is a vision and it is supplemented by hard work then it becomes a reality. Other projects on the same line are additions, deletions and modification of features and facilities. A visionary person prepares a team division in

(i) planning and organization, and
(ii) controlling the time and cost parameters and coordinating with the higher authorities.

For planning and extension of activities, in-charge of project operation takes care while for evaluation and reporting part the project-control in-charge takes care.

We will discuss here following three types of projects and their details.

1. critical path method
2. program evaluation and review technique
3. crashing project duration

We take each portion and make elaborative discussion.

(1) Critical Path Method (CPM)

This was developed jointly by E.I. Duport Company and Remington Rand Univac Division. The aim behind the work undertaken was to make better planning and controlling the overhaul and designing maintenance of a chemical plant, which was in operation since years and the old machines and spare parts were either to be replaced or to be serviced and make functioning properly. Their operations techniques and procedure were made public in **1959**.

(2) Program Evaluation and Review Technique (PERT)

It was developed during the same years. There was a research team working on development of Polaris Missile programme handled by US Navy department. It was a unique project and undertaken for possible realization. Its success was highly responsive.

13.1 Main Features of Project

Project planning and project control are the two dominant components of any project.

Every project whether it falls in any one of the above two categories—CPM or PERT need un-divided attention to the two main features; **project planning and project control**.

(1) Project Planning

On introducing the phrase, project planning, we mean the group of each of following important activities

(a) site selection and logistic planning
(b) manpower planning—this involves resource levelling.*
(c) procurement planning
(d) financial planning
(e) operation planning
(f) contract & marketing planning
(g) project evaluation and sensitivity analysis

All these features are to be properly taken care of any aspect, and if when any feature starts functioning improperly then it paralyses the whole system.

Once, you have hired skilled workers to carry out different activities then it becomes impractical to discontinue them. It may be a situation that again there is a requirement of the same workers in continuation of the work they have started with. For the period between these two slots, either the management cannot assign them other type of work or they cannot be relieved too. Paying them for the simple cause of the future requirement, increases the project cost. On the other end, if such work force is not made available at the time when we need them, will result either extension of project duration which may be liable to penalty cause or paying higher charges to get the required workers and perhaps it may be at the cost of quality compromise. In addition to this, there is also a technique of resources smoothening which is very useful and very efficiently solves both the issues discussed above.

We need resource planning and managing so that there may not be any consequence of either issue.

(2) Project Control

This is a system department but equally important and responsible one. The main features are:

(a) it maintains human resource, plans and manages the structure.
(b) it designs the overall quality policy and departmental quality policies. It keeps a check whether proper steps are taken to observe what is expected in lines of quality policy statement.
    1. making general evaluation of functioning of each and every department.

*This concept is very useful and important one. Availability of necessary class of skilled and unskilled manpower and that too at the time the projects requires, is really a big challenge. Delay in procurement and necessary raw materials is equally critical.

2. resolve the grievances and updating the higher authorities.
3. it makes a policy and design WBS (work breakdown structure). This is the most important component of any project.

13.2 Basic Differences—CPM & PERT

Both are in their very nature type or systematic methods of changing thoughts into an exiting reality. The main differences between them are as follows:-

(1) Critical Path Method (CPM)

Any type of project viewed and decided to undertake falls under CPM class if some has already and executed a parallel type of project.

The leading features are

(a) All the activities associated to this are known in advance or at the beginning of the project. This makes both the project completion duration and the project cost duration, plans easy and feasible. It is an **activity-oriented project**.
(b) It has a discrete beginning and discrete end. (Activity timing are pre-known.)
(c) Basic components of a diagram, i.e. activities are shown nodes. It is called activity-on-Node (AON).
(d) A project network is drawn by joining the nodes by arrows known as activity-on-arrow (AOA). The arrows between the nodes depict precedence relationship of activities.

Examples are:

1. Construction of a new bridge
2. Organizing the workshop
3. Preparing for university examination to be conducted.

2. Program Evaluation and Review Technique (PERT)

When any project is conceived for the first time and there is no such project has been previously done or undertaken; such projects fall under PERT class.

The main features are as follows:

1. First time project.
2. An exhaustive list of activities is not known previously.
3. It involves departmental activities and the progress of the project is reviewed at the constant interval. **It is event oriented**.
4. New activities at times are added and may some of the events already executed might be suspended.
5. Discrete beginning and discrete end cannot be predicted well in advance.
6. Activities are shown on arrow (A0A) and nodes represent the events. Nodes are numbered in sequence and so it becomes easy to identify the activities.
7. Activity timings are not known and hence three types of time estimate for each one of the activities are done.

13.3 Activity

1. It is a part of a total project plan to which a known resource is applied. It consumes time and cost of resources using.

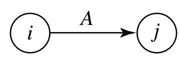
It is shown by an arrow diagram, and size and shape of the arrow does not indicate activity time or cost paid for the resources.

An activity operates between two nodes only. There can be exactly one activity between two nodes; this is the main logic applicable in a network.

The nodes are called *events*. An activity is carried out between two events.

Activity Relationship

1. An activity, generally denoted using capital letters like A, B, C, …, etc., is a continuous process of some act between two nodes . If one node is *i* and the next node is *j* then we have an activity, A shown as

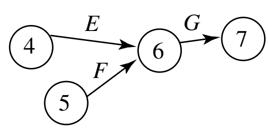


Activity *A* is denoted as *i*-*j* where, $i < j$

2. When a particular activity is completely done, then we say that event has taken place.

   If an activity A shows construction work of a wall, then on completion of that activity, we can say that the wall has been constructed. It is an event in the sequence of the events of the projects.
3. To an activity there is always a predecessor activity, except those activities which begin from initial node. If A is an activity and only after the completion of activity B can be started then we write this relationship as A < B OR B > A.

   An activity may have more than predecessor activities. You will find such points during the network diagram. If E and F are two activities and G is such an activity which can begin only on completion of E and F then we write E, F < G

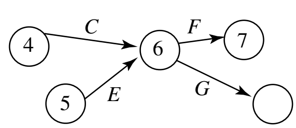


A part of some network

4. To an activity, there is always a follower or a successor activity. The last event is no like such one. In a way putting the activity like E < F

   Activity E proceeds activity F is at this point same as F is a follower activity to only the activity E.

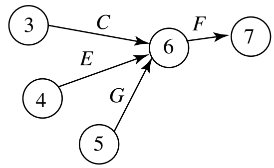
   It may be true that activity F followed by activities C and E also.



A part of a network

   You may find a situation that there may be more than one activities acting as follower activities to more than one activities. In the above diagram this point is clear.
5. **Parallel Activities:** At some stage in a network you may have more than activities beginning from one node; all such are called parallel activities. These parallel activities may have different time durations. In the above diagram F and G are parallel activities.

6. **Concurrent Activities:** When more than one activities coming out from different nodes end up in a common node, all such activities are called *concurrent activities*.

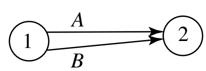


A part of network diagram

Activities C, E, and G are concurrent in the node (6).

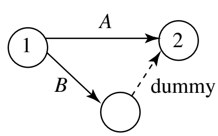
7. **Dummy Activity:** One of the fundamental principles of drawing a network are
   1. An activity from a node can be started only if all the different activities converging to that node are over.
   2. There can be one and only one activity between any two nodes.

      This conveys that, though you may have a situation like (1) ⇔ (2) may exist but its diagrammatic presentation does not following the point (2)—only one activity between two nodes.



To circumvent the situation, we introduce an extra activity and an extra node and see that both (i) the Rule (2), and (ii) our plan of converging that both the activities A and B have common nodes.

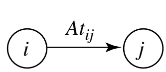
The extra/additional activity is called a dummy (= D) activity. **It is shown by a dotted arrow in a diagram.**



Dummy activity consumes no time and no resource. It is just introduced to set the logic right. This figure conveys the same meaning as what is conveyed by the above figure.

Time Duration of an Activity

Let A be an activity running between two nodes *i* and *j*. We denote it as:



With $i < j$; i and j are node numbers.

All activities except the dummy activity consumes time, the time is denoted as t_{ij}.

$\boldsymbol{t_{ij}}$ is the time taken by an activity A between the nodes i and j.

13.4 NETWORK

We have seen that the beginning and completion of an activity is a node—called the event. Each activity is either a preceding activity or a successor activity to any other activity or activities.

The diagram showing the inter-relationship of occurrence of all the activities in a given situation is called a *network diagram*.

Network Construction:

The following points should be paid due importance in drawing a network.

1. Make an exhaustive list of all the activities.
2. Make a table indicating predecessor activities to all the activities.
3. Remember that the initial activity has no predecessor one and the last activity has no successor activity.
4. We remember the golden rule that between any two nodes there can be exactly one activity.
5. Draw a network showing inter-dependency relationship.
6. Avoid looping in the network.
7. Do not go back—no backward-going back arrows.

We take an illustration.

ILLUSTRATION 1

We are given the following relationship amongst the activities.

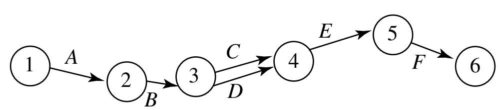
A < B, B < C, D C, D < E, E < F

Activity	*Predecessor*
A	—
B	A
C	B
D	B
E	C, D
F	*E*

* Activity A has no predecessor. Activity A is the first one.
* Activity F has no successor and so it is the last one.
* Activity can be between nodes.

Is the following diagram according to set rules?

Diagram



It conveys what you mean but at the same time it violates the basic golden rule**—Between two nodes there can be only one activity.**

We introduce dummy activity and show it by dotted arrow.

Diagram:

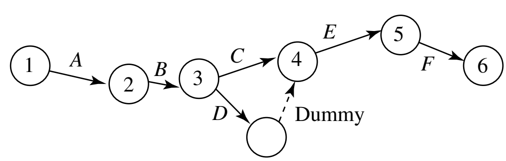


ILLUSTRATION 2

We take an illustration and draw the corresponding network. There are seven activities given as the following relationship

1. $A < C, D$
2. $B < E$
3. $C < F$
4. $D, E, F < G$

The activity times are also given in the solution.

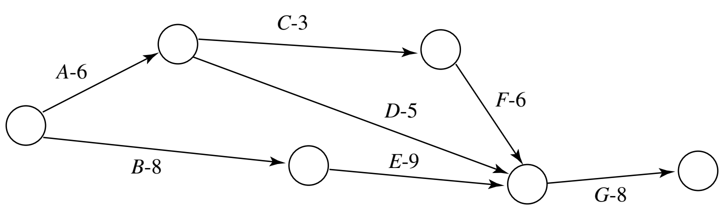
Solution

From the given relationship, we make the order-relation.

Activity	*Predecessor*	*Time*
A	—	6
B	—	8
C	A	3
D	A	5
E	B	9
F	C	6
G	D, E, F	8

From the above information we conclude that activities A and B are initial activities while activity G is the last one.

We draw a diagram and we show 'arrow on an activity' pattern of drawing this network. As being generally followed, the timing are shown either near to the activity letter (or below the activity letter) written on an arrow.



Here nodes are not numbered. Our purpose is that we study activity dependency relationships and draw a network. Also, drawing the network care is taken to see that two golden rules are remembered.

What now remains? Numbering the nodes—Events.

There is Fulkerson Rule: In general, we number the nodes in the network diagram following this rule.

We suggest you

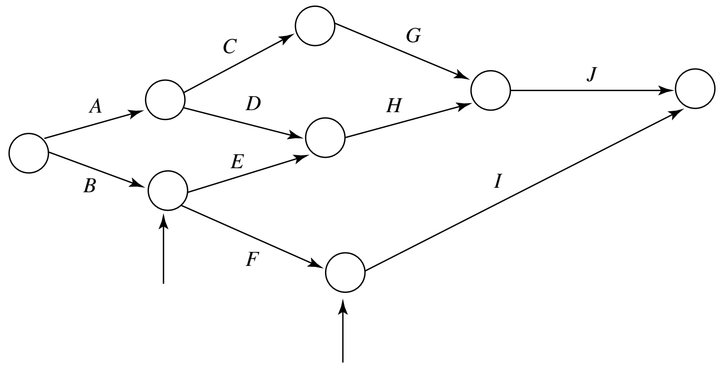
(i) To draw network in a proper dependency relation.

(ii) Avoid drawing nodes coming exactly *above* or *below* to any other node. This is always possible—except some non—logical way.

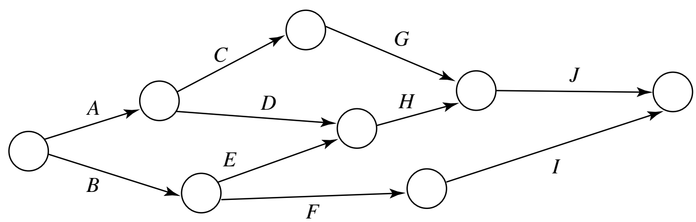
(iii) **Remember** that your node number advances as your figure on time scale advances from left to right.

You will never make any mistake if you follow the above rules.

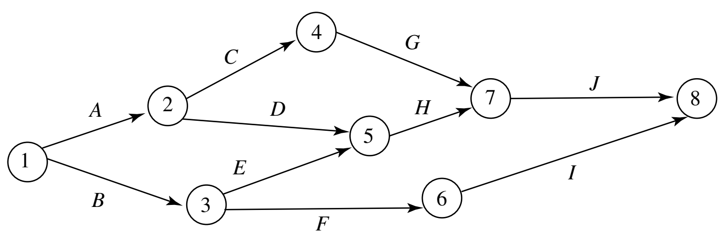
Do not draw figures like this.



You will be double minded in numbering such nodes. Well, draw it as follows.



Now beginning with the left most node and assign the number 1 there. As you move from left to right (on the positive x-axis) and as you see a node; number it and proceed. The same figure above with proper numbers is as follows.



Follow this—No mistakes.

13.5 Forward Pass and Backward Pass

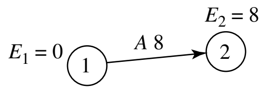
These two are the different activities and having perfectly opposite nature.

Forward pass (E)

The process of making forward pass begins at the initial node and passing through activity and remembering only one rule that an activity from a node can be started only if all the previous activities converging to that node are done, reaches the last node. It is associated with the earliest start time of an activity. It is denoted by E.

The E-earliest start time of the initial node, call it node 1 obviously it is zero, i.e $E_1 = 0$ at node 1. The earliest start time of the next node 2 is E_2. Hence $E_2 = E_1 + t_{12.}$

where the convention t_{ij} = activity time between the nodes i and j



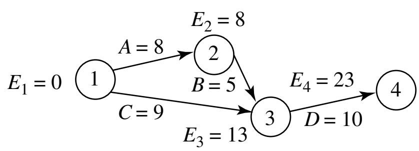
$E_1 = 0$; Activity A of duration 8 days is in operation and is completed at node 2.

$E_2 = E_1 + t_{12}$. Therefore $E_2 = 0 + 8 = 8$. This E_2 shows the earliest start time of the next availability emitting from node 2.

If there are more than one activities converging to a node, then we find timings for each activity and then consider maximum time.

This is because of the only one reason 'that an activity from a node can be started if all the activities converging to that node are done'. We take as example to clarify this point.

ILLUSTRATION 3



1. $E_1 = 0$
2. $E_2 = E_1 + t_{12} = 0 + 8 = 8$
3. E_3 = maximum of (8 + 5, 0 + 9)
   = maximum of (13, 9)
   = 13

   Therefore in this case $E_3 = E_2 + t_{23}$
   $= 8 + 5 = 13$

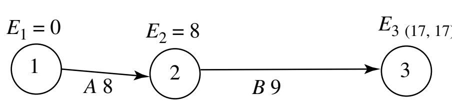
   Though the activity 1-3 can be completed in 9 days but for starting from node 2, you will have to wait because activity 2-3 is not completely done.
4. $E_4 = E_3 + t_{34} = 13 + 10 = 23$

   In this fashion, you make forward pass in a network diagram.

Backward Pass (L)

This is the procedure, which finds, *at a node*, the 'latest finish' time of all the previous activities. It is denoted as L. At the last node $E_{\text{last}} = L_{\text{last.}}$ $[E = L]$

To go back to the previous mode, subtract the activity time of a particular activity on which you are going in the opposite direction of an arrow.

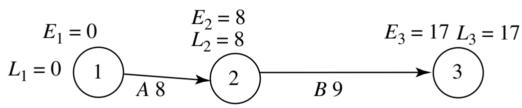


In this cae $L_3 = 17$

Therefore, $L_2 = L_3 - t_{23} = 17 - 9 = 8$

Therefore $L_2 = 8$

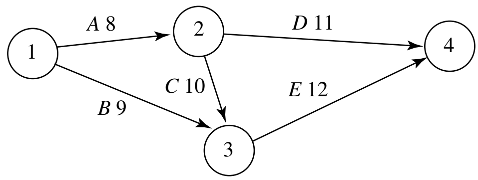
$L_1 = L_2 - t_{12} = 8 - 8 = 0$



In case of more than one activities emerging from a node then in making backward pass, you have to subtract the timings of both the activities and on reaching to that node write the *least* time.

ILLUSTRATION 4

Write least start time and latest finish time at each node in the above figure.



Solution

First, at the node 1, $E_1 = 0$, we write the earliest start tie at each node.

$$E_2 = E_1 + t_{12} = 8$$
$$E_3 = \text{Max } (8 + 10, 9) = 18$$
$$E_4 = \text{Max}(8 + 11, 18 + 12) = 30$$

At the last node $E_4 = L_4 = 30$

Now we turn on to making backward pass

$$L_4 = 32$$
$$L_3 = L_4 - t_{34} = 30 - 12 = 18$$
$$L_2 = \text{minimum of } (30 - 11), (18 - 10) = \text{minimum of } (19, 8).$$
$$\therefore \quad L_2 = 8$$
$$L_1 = \text{minimum of } (8 - 8, 18 - 9) = 0$$
$$L_1 = 0$$

If we write at the node 2, $L = 19$ which means that an activity from node 2 can begin latest on day 19.

Then, we take activity 2 – 3 of 10 days.

Therefore 19 + 10 = 29 days (at node 3)

Finally from node 3, we perform the activity 3-4 taking 12 days. This will put us on node 4 on 29 + 12 = 41 days.

We wanted to reach there by latest or earliest on day 30.

So we have a contradiction.

Hence we *conclude* to write the least time at a node for a backward pass.

Slack of an Event

At a node, we have two types of timings:

1. Earliest start—for an activity going out from that node, and
2. Latest finish time—The time on which all the activities converging to that node must be finished. (Else the project duration is extended.)

   Slack of an event is the difference between L and E at a node.

   This difference is always ≥ 0.

We take one more illustration to understand some more technical details of network diagram. Draw the network diagram in the following case. Make forward pass and backward pass.

ILLUSTRATION 5

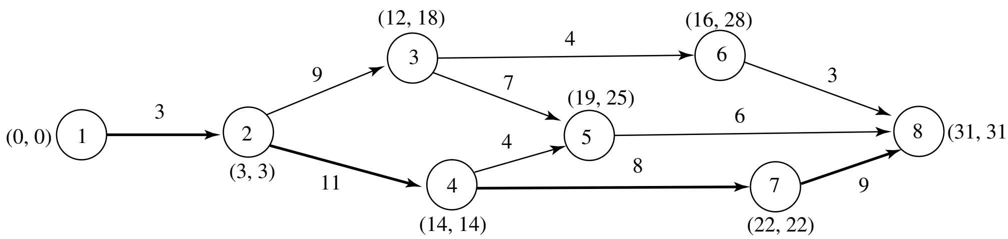
Draw network diagram in the following case and make forward and backward passes.

Activity	*Duration*	*Activity*	*Duration*
1–2	3	4–7	8
2–3	9	5–8	6
2–4	11	6–8	3
3–5	7	7–8	9
3–6	4		
4–5	4		

Solution

The network is as follows.

We make forward pass, backward pass, and put all at each node. Total project duration is 31 days. At each node we have introduced parenthesis like (E, L); where E indicates earliest start time of an event which is the result of forward pass and L shows the result of backward pass.



We now make a table showing earliest start time and latest finish time at each node. It is done simply by putting the above data in the table form.

(1)	(2)	(3)	(4)	(5)	(6)
Activity	*Duration*	*Start*		*Finish*	
		E_S	L_S	E_F	L_E
1–2	3	0	0	3	3
2–3	9	3	10	12	19
2–4	11	3	3	14	14
3–5	7	12	19	19	26
3–6	4	12	24	16	28
4–5	4	14	22	18	26
4–7	8	14	14	22	22
5–8	5	19	26	24	31
6–8	3	16	28	19	31
7–8	9	22	22	31	31

Note: E_S + activity time = E_F
Column (3) + Column (2) = Column (5)
L_F – activity time = L_S
Column (6) – Column (2) = Column (4)

We have Column (1) and Column (2)

We made forward pass and made Column (3)
We made backward pass and made Column (6)

We derived Column (5) = Column 2 + Column 3
We derived Column (4) = Column 6 – Column 2

We want to derive some more facts from the above figure and related table.

13.6 Floats and Critical Path

Now, at this stage, we introduce very important and useful concept of floats of an activity in the network.
There are four types of floats. We have described them with definition and examples.

1. Total Float

Total float of an activity is obtained as follows.
Total float of an activity = T

$$T = L_S - E_S = L_F - E_F = L(E + T_{ij})$$

Total float is the *maximum float* by which the activity can be delayed.

Example

Total float of activity 1 – 2 is 3 – 3 = 0 – 0 = 0

Total float of activity 2 – 3 is 19 – 12 = 10 – 3 = 7

2. Free Float

It is a part of total float. It is found by subtracting slack of head event from total float.

Free float = T – (Slack of head event)

It is that float which when enjoyed (without doing a particular activity) then the free float of subsequent activities is not disturbed. The total duration with free floats enjoyed will not extend the total duration of the project.

If $T = 0$, then be free float = 0

For example, free float activity 2–3

= Total float of 2–3 – slack of event 3

= 7 – (19 – 12) = 7 – 7 = 0

3. Independent Float

It is a part or portion of free float which will not disturb the floats of predecessor activities.

Independent float = free float – Slack of tail event

Example: Independent float of activity is taken equal to zero and its value is negative.

Example: Independent float of activity 2–3 = Free float of 2–3 – slack of event 2

= 0 – (3 – 3)

= 0

4. Interfering Float

Interfering float is a part of the total float. It is the slack in the head event and is the difference of the finish time (latest) of all previous activities and the earliest start time of successor activities. If it is fully consumed then it affects the duration time of the project

Interfering float of activity 4–5 is the slack at node (5). Therefore interfering float of 4–5 is 26 – 16 = 10

Critical Path

Critical path in a network diagram is a path on which all the activities have **zero float** (or no float). All such activities, having zero float, put in an increasing sequence of nodes make a critical path.

All the activities on the critical path are zero float activities. No time can be delayed in execution of any activity and can be allowed or else the total project duration is extended.

Critical path is the path having longest duration. Sum of activity times on critical path is the value = $E = L$ of the last node.

As a procedure, the activities on the critical path are shown by **dark of double** line arrows.

We now take one illustration and show all floats and critical path.

ILLUSTRATION 6

Draw network diagram in the following case. Make forward and backward passes.

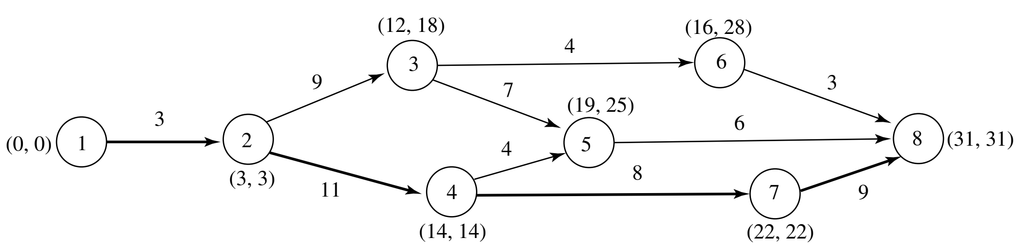
Activity	*Duration*	*Activity*	*Duration*
1–2	3	4–7	8
2–3	9	5–8	6
2–4	11	6–8	3
3–5	7	7–8	9
3–6	4		
4–5	4		

Solution

The network is as follows.

Diagram:

We make forward pass, backward pass, and put all at each node. Total project duration is 31 days. At each node we have introduced parenthesis like (E, L); where E indicates earliest start time of an event which is the result of forward pass and L shows the result of backward pass.



We now make a table showing earliest start time and latest finish time at each node. [Simply putting the above data in the table form].

(1)	(2)	(3)	(4)	(5)	(6)	(7)
Activity	*Duration*	*Start*		*Finish*		*Total Float*
		E_S	L_S	E_F	L_E	
1–2	3	0	0	3	3	0*
2–3	9	3	10	12	19	7
2–4	11	3	3	14	14	0*
3–5	7	12	19	19	26	7
3–6	4	12	24	16	28	12
4–5	4	14	22	18	26	8
4–7	8	14	14	22	22	0*
5–8	5	19	26	24	31	7
6–8	3	16	28	19	31	12
7–8	9	22	22	31	31	0*

Notes: E_S + activity time = E_F
Column (3) + Column (2) = Column (5)
L_F – activity time = L_S
Column (6) – Column (2) = Column (4)

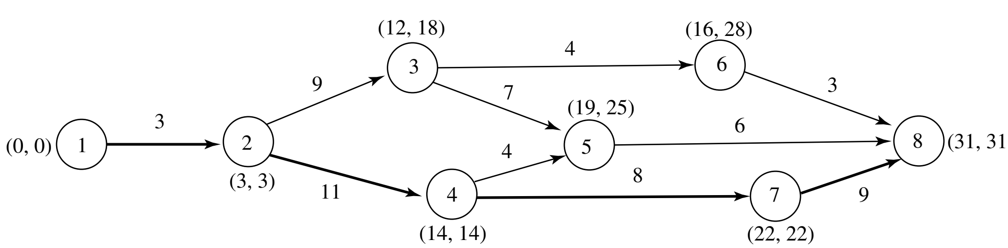
We have Column (1) and Column (2).
We made forward pass and made Column (3)
We made backward pass and made Column (6)

We derived Column (5) = Column 2 + Column 3
We derived Column (4) = Column 6 – Column 2

Activities having zero float are called *critical activities*. We cannot afford wasting time on critical activities, doing so will extend the project duration. In the above network 1–2–4–7–8 is the critical path. It is the longest path having a 31 days duration.

ILLUSTRATION 7

In the table showing all the details, write (1) free float, (2) independent float (3) interfering float.



(1) Activity	(2) Duration	(3) Start E_S	(4) Start L_S	(5) Finish E_F	(6) Finish L_E	(7) Total float
1–2	3	0	0	3	3	0*
2–3	9	3	10	12	19	7
2–4	11	3	3	14	14	0*
3–5	7	12	19	19	26	7
3–6	4	12	24	16	28	12
4–5	4	14	22	18	26	8
4–7	8	14	14	22	22	0*
5–8	5	19	26	24	31	7
6–8	3	16	28	19	31	12
7–8	9	22	22	31	31	0*

Solution

We repeat the figure and the table. This will help us to derive these floats.

As we have defined earlier, free float is a part of total float that we have found in Column (7).

Free float = Total float – slack of head event.

For example,

Free float of activity 2 – 3 = Total float of activity 2 – 3 – slack of event 3
= 7 – (18 – 12) = 7 – 6 = 1

Free float of activity 5 – 8 = Total float of activity 5 – 8 – slack of event 8
= 7 – (31 – 31) = 7 – 0 = 7

In this way, we can find free floats of all activities.

Independent float is defined as the difference between free float and slack of tail event.

If it is negative, then it is taken as zero.

For example,

Independent float of activity 2 – 3 = free float of 2 – 3 – slack of event 2
= 1 – (3 – 3) = 1 – 0 = 1

Independent float of activity 5 – 8 = free float of 5 – 8 – slack of event 5
= 7 – (25 – 19) = 7 – 6 = 1

Finally, we find the interfering float of an activity. For an activity $i - j$, we are concerned about the timings at the node j. At the node j the earliest time indicates the time on which all the activities coming out of the node j can start. The second component at the node j indicates that by what time all the activities converging to that node j must be completely finished or the latest time on which the activity or activities from emerging the node j must be started. In short, it is nothing but the slack of head event.

Based on the above explanation and illustrations explaining the floats, we complete the table of floats of the above example. We repeat that what we need for finding the critical activity is the total float of an activity.

Activity	*Duration*	*Start*		*Finish*		*Total Float*	*Free Float*	*Indep. Float*	*Interfering Float*
		E_S	L_S	E_F	L_E				
1–2	3	0	0	3	3	0*	0	0	0
2–3	9	3	10	12	19	7	7	1	6
2–4	11	3	3	14	14	0*	0	0	0
3–5	7	12	19	19	26	7	1	0**	6
3–6	4	12	24	16	28	12	6	0**	12
4–5	4	14	22	18	26	8	2	2	2
4–7	8	14	14	22	22	0*	0	0	0
5–8	5	19	26	24	31	7	7	1	0
6–8	3	16	28	19	31	12	12	0	0
7–8	9	22	22	31	31	0*	0	0	0

In the last second column, we have shown some entries with ** sign; it indicates that the independent float in that cell takes negative value and so we have taken those values equal to zero.

13.7 PERT

We have seen the fundamental properties of PERT class projects.

All the procedures of finding start and finish timings, finding floats and critical activities, etc., that we have followed in CPM are blindly carried out in the case of PERT routines also.

The main point of difference is the **three types of time estimates**, instead of a single time duration as it is given in CPM, of each one of the activities are given.

Time Estimates

In PERT, by convention, the average (technically known as *beta average*) is found. We use the following formula.

Average time of an activity = $(t_o + 4t_n + t_p)/6$

Where t_o is the optimistic time. It means that if all the circumstances favouring the occurrences of an event are possible then what is the expected earliest period during which the activity can be completed.

(1) t_m or t_n

t_n shows the normal or moderate, or regular time period or usual time slot of complete execution of an activity.

$\underline{t_o}$: It is an optimistic approach about completion time of an activity. If all the circumstances, natural and physical, remain in favour, then the estimated completion time is denoted as t_o.

t_p : It is the time given for completion of an activity in which all the adverse cases are considered. It is a pessimistic approach for activity timings.

Average time = Expected time for a complete execution of an activity.

$$T = (t_0 + 4t_n + t_p)/6$$

In addition to this average time, as we know that there can be some fluctuation in this time slot, we allow some percentage fluctuation on either side of expected time duration.

It is the ***one unit*** of standard deviation time. It is denoted by the letter s.

S is called *standard deviation* in time.

Square of the standard deviation is called the *variance*; denoted as S^2.

Steps for PERT Network and derivation of results.

1. From the given activity list, make a dependency relationships and draw a network diagram.
2. Find the estimated timings for each activity and its variances = S^2.
3. Find (E, L) for each event.
4. Find floats and critical path.
5. Find the standard deviation of the critical path.

Standard deviation of the critical path = square root of the sum of variances of critical activities.

Standard deviation = $(t_p - t_0)/6 = S$

n = number of activities on critical path.

Probabilistic Results

We know that for a normal distribution the variable is a continuous variable with two parameters. These two parameters are **mean** = $\bar{X}$ and **standard deviation** = s units.

For a normal distribution, mean = mode = median. Each random variable X when is employed, with proper scale $Z = \dfrac{X - \bar{X}}{s}$. The resulting figure is a standard normal curve with area = 1. This is sufficient back grouped to answer few more questions.

ILLUSTRATION 8

Using the records of the following data, draw a PERT network. Find the critical path and standard deviation of the critical path.

Activity	t_0	t_n	t_p
1–2	2	6	10
1–3	2	3	4
1–4	7	11	15
2–5	6	14	16
3–4	00	00	00
3–6	6	7	14
4–6	2	6	10
5–6	7	8	15

Solution

We know that the average time estimate is found by finding the average time by applying the formula:

$$i = (t_0 + t_n + t_p)/6$$

and

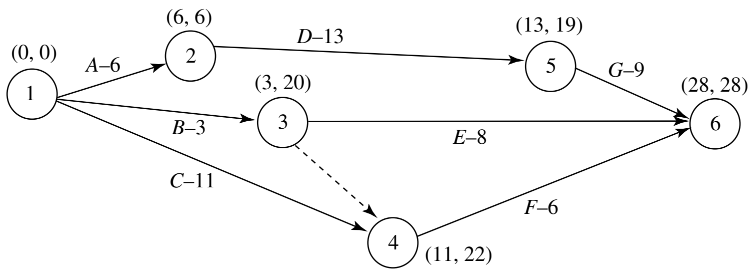
variance of an activity by finding

$$S^2 = [(t_p - t_o)/6]^2$$

1. Taking all the three estimates; we find average time (= t) and variance (= s^2) of each activity. This is given in the following table.

Activity	t_0	t_n	t_p	t	s^2
1–2	2	6	10	6	16/9
1–3	2	3	4	3	1/9
1–4	7	11	15	11	16/9
2–5	6	14	16	13	25/9
3–4	00	00	00	00	00
3–6	6	7	14	8	16/9
4–6	2	6	10	6	16/9
5–6	7	8	15	9	16/9

2. We draw network diagram. We then make a forward pass and backward pass.



Using the information, we make the table showing floats of activities. This will help determining the critical path.

Activity (1)	*Time 2*	*3 Start* *E*	*4* *L*	*5 Finish* *E*	*6* *L*	*Float*
1–2	6	0	0	6	6	0**
1–3	3	0	17	3	20	17
1–4	11	0	11	11	22	11
2–5	13	6	6	19	19	0**
3–4	0	3	22	3	22	19
3–6	8	3	20	11	28	17
4–6	6	11	22	17	28	11
5–6	9	19	19	28	28	0**

It is clear that activities having zero floats are 1 – 2, 2 – 5, and 5 – 6.

Critical path is 1 – 2 – 5 – 7 of 28 days duration.

Standard deviation of the critical path is obtained by finding the square root of sum of variances of critical activities.

Standard deviation of critical path = Square root of (16/9 + 25 /9 + 16/9)
= Square root of (57/9)
≈ 2.52 days.

ILLUSTRATION 9

From the given activity table and three types of estimates, find

1. Draw the network diagram.
2. The critical path.
3. Standard deviation of the critical path.
4. Find the probability of completion the project in due time.
5. What is the probability of completing the project earlier by 10% of the time?
6. What is the probability of completing the project by allowing 15% more time?

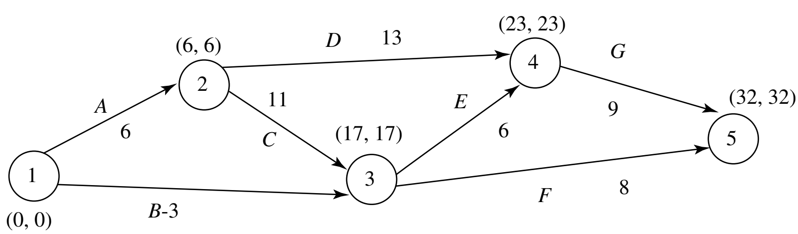
Activity	t_0	t_n	t_p
1–2	2	6	10
1–3	2	3	4
2–3	7	11	15
2–4	6	14	16
3–4	6	7	14
3–5	6	7	14
4–5	2	6	10

Solution

1. Taking all the three estimates; we find average time (= t) and variance (= s^2) of each activity given in the following table.

Activity	t_0	t_n	t_p	t	S^2
1–2	2	6	10	6	16/9
1–3	2	3	4	3	1/9
2–3	7	11	15	11	16/9
2–4	6	14	16	13	25/9
3–4	2	6	10	6	16/9
3–5	6	7	14	8	16/9
5	7	8	15	9	16/9

2. We draw network diagram. We then make a forward pass and backward pass.



Using the information, we make the table showing floats of activities. This will help determining the critical path.

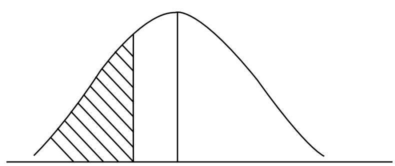
Activity (1)	Time 2	3 Start E	4 L	5 Finish E	6 L	Float
1–2	6	0	0	6	6	0*
1–3	3	0	14	3	17	14
2–3	11	6	6	17	17	0*
2–4	13	6	10	19	23	4
3–4	6	17	17	23	23	0*
3–5	8	17	24	25	32	7
4–5	9	23	23	32	32	0*

critical path is 1–2–3–4–5 of 32 days duration.

3. Standard deviation of critical path = Square root of (sum of variances of activities on the critical path)

   = Square root of ($V_A + V_C + V_E + V_G$)
   = Square root of (16/9 + 4 + 49/9 +4)
   = Square root of (8 + 55/9)
   ≈ 3.76 days

   Critical path of 32 days and it has 3.76 days standard deviation.
4. Probability of completion of project on the expected project duration is 50%.
5. Assuming that the total of project completion is normally distributed with a mean = 32 days and standard deviation 3.76 days.
   We can answer some time questions in connection of project completion.



x = 28.8 mean = 32

   shaded region gives probability.
   we find that corresponding to $Z = -8510$ the area is 0.8023.
   ∴ area = 0.8023 – 0.5 = 0.3023
   Probability of completion is 0.3023.
6. It takes 15% more time.
   We want to find probability completion up to period (32 + 15% of 32) = 36.8 days.
   now, $\mu = 32$, $s = 3.76$ days,
   $Z = (36.8 - 32)/3.76 = 4.8/3.76 = 1.28$
   Area for $Z = 1.28$ is 0.8997 (from the Z-table).

13.8 CRASHING

In real life situation there is a cut-throat competitions. The management possessing longer scale production units always face competitions of introducing and pushing their products in the market.

Each new product that is introduced first in the market will capture the market and the other that follows will have hard time to do. The management that introduces the product late will have to bear additional expenses, like

1. continuous advertisements
2. higher profit margin to the stockist and retailers

All these factors force the management to expedite the project work by spending more money on the activities and applying more resources.

What do we mean by crashing?

There are some activities whose regular time/duration for completion can be shortened by applying additional resources and one has to pay for such extra resources.

There are two impacts on compressing an activity.

(i) Total project duration decreases
(ii) Cost of project increases (As we have paid extra amount for compression.)

(A) Crash time and crash cost

For every activity there is a natural time of completion denoted as $t_n = t$

For performing activity task, resources are to be employed. One has to pay cost for this.

Normal cost = C_n

By employing extra resources, the activity duration can be shortened. This process of shortening has an upper limit, i.e. there is always a time beyond which one cannot shortened the time.

The **minimum** time to do an activity is called *crash time* and is denoted ac t_c.

The cost one incurs for reducing the time slot from t_n to t_d is called the crash cost. It is denoted as C_c. This is over and above the *regular cost*. It is denoted as C_c.

Compression period = $t_n - t_c$

Compression cost = $C_c - C_n$

Once again, we note that t_c = crash time = minimum time to do an activity

C_c = total cost of completing an activity up to the time slot t_c.

Therefore, what extra is paid = $C_c - C_n$

The time we saved is $t_n - t_c$, We have $C_c > C_n$; $t_n > t_c$

The ratio $\dfrac{C_c - C_n}{t_n - t_c}$ is called *cost time ratio* or *cost slope*.

It shows an average cost per day of reducing the time.

Note: At this point, we have assumed that the cost of crashing reduces linearly; this may not be a case in all cases.

Reducing the time beyond a certain limit may prove very costly.

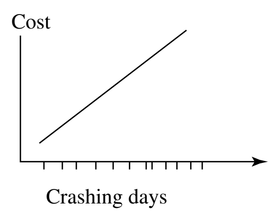
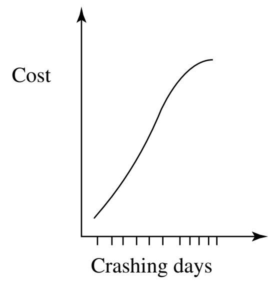
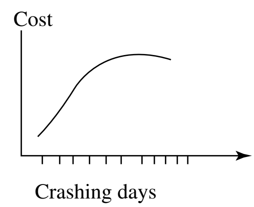


Figure indicates that the cost increases linearly to time.

One can see that in the initial period as the time decreases the cost increases moderately or following some rule. Then after some time, cost increases at a faster rate.



We can have the opposite type of case also. As we start reduction in time, the initial costs are high and then after some time becomes steady.



Finding a compromise between these two, we have assumed linearity.

(B) Cost—direct cost and indirect cost

With every projects there are two types of associated costs—direct and indirect.

Direct costs are also known as *cost at site*, the cost of labours, cost of resources, general expenses, and cost of crashing also. These all are the cost at site.

Indirect cost: This cost involves office overheads, cost of visits by supervisors or concerned higher authorities. It is calculated for the duration of the entire project duration.

If the project duration decreases then office cost also decreases. It is a linear function of time.

Direct cost increases as the compression cost increases and indirect cost decreases as the total project duration decreases.

Our objective may be

1. maximum possible compression of all the activities on the critical path
2. finding a balance between these two costs or finding the optimal time period on which the sum of these two costs is minimum.

Both criteria mentioned above may satisfy jointly but no assurance may be given.

(C) How do we perform Crashing?

1. Study the crash cost and the normal cost.
2. Find cost slope for those activities which can be crashed.
3. Main purpose of crashing is to decrease the total project duration. This is achieved by crashing the activities on the critical path.
4. That activity on the critical path can be compressed for which the crashing cost per day is minimum.
5. Be technical and systematic about it; compress only one least cost activity on the critical path.

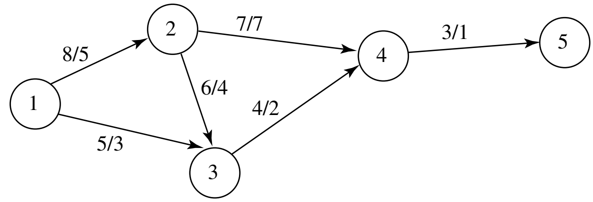
6. Study the impact of crashing. This will change the time durations of each activity. If you change the earliest start and finish timings then you can take better judgement about
   (a) path length of all paths
   (b) normal cost, crash cost, and total (decreased) indirect cost.

Note that if all the activities on critical paths are compressed, further compression on critical path is not possible. Then, any further compression on the critical path will not decrease the critical path length. Any further compression can be done on non-critical path.

We can save a non-critical activity time but it would **not** decrease total length of critical path.

ILLUSTRATION 10

From the given figure, identify the critical path. study the compression table and compress the total duration of project. Office overheads is ₹250/day.



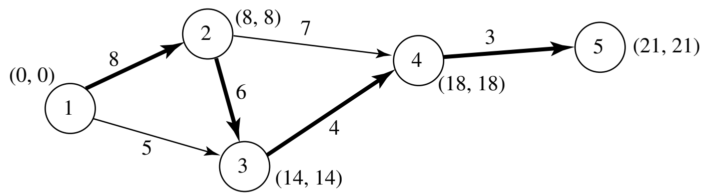
For this diagram, the primary details will be shown in the table. What concerns us most immediately is crash time and crash cost.

Activity	t_n=*t*	t_c	C_n	C_c	*cost / slope*
1–2	8	5	1600	2500	300 / day
1–3	5	3	1500	1800	150 / day
2–3	6	4	1800	2000	100 / day
2–4	7	7	1400	1400	---------
3–4	4	2	2000	3200	600 / day
4–5	3	1	900	2000	550 / day

Solution

First, we make the regular CPM table, find the critical path and then consider the other factors.

1	*2*	*3*	*4*	*5*	*6*	*Float*
Activity	*Duration*	*Start*		*Finish*		
		E	*L*	*E*	*L*	
1–2	8	0	0	8	8	0 *
1–3	5	0	9	5	14	9
2–3	6	8	8	14	14	0 *
2–4	7	8	11	15	18	3
3–4	4	14	14	18	18	0 *
4–5	3	18	18	21	21	0 *



Critical path is 1–2, 2–3, 3–4, 4–5 i.e.
1–2–3–4–5 of (8 + 6 + 4 + 3 = 21 days)
Now from the given table we would like to make compression of timings.
How should we compress the activities on the network diagram?

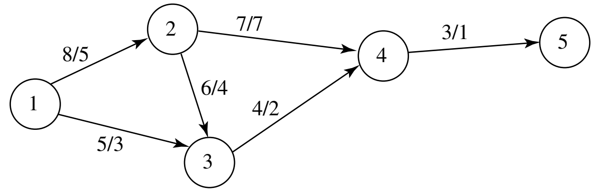
Step 1

The different paths are:
1. 1–2–4–5 of duration 18 days
2. 1–2–3–4–5 of duration 21 days ***
3. 1–3–4–5 of duration 12 days.

The path 1–2–3–4–5 is a critical path of duration 21 days.
Also, note that the activity 2–4 cannot be compressed. [it is of 7 days.]
There are four activities on the critical path.
The above cost slope, i.e., cost of crashing per day of activities on the critical path are;
₹300, ₹100, ₹600, and ₹550 per day of activities 1–2, 2–3, 3–4, and 4–5 respectively.

Step 2

The least cost is ₹100 per day and it is related to the activity 2–3.
It is an activity for 6 days and can be compressed for 2 days. [crash time; minimum time to do i.e. four days.]
We compress 2–3 by one day.
This will reduce the length of critical path by 1 day and is now of 20 days



The other paths do not involve critical activity; it means that the length of other paths is not disturbed.
The different paths are;
1. 1–2–4–5 of duration 18 days
2. 1–2–3–4–5 of duration 21 days ***/ now, 20 days
3. 1–3–4–5 of duration 12 days

The earliest start times are changed. We have paid extra ₹100 only. Critical path now is 20 days.
Indirect cost reduces by ₹250 per day.

Step 3

We are still in a position to reduce the same path 2–3 by 1 day at an extra cost of ₹100.
We do so and the results and corresponding impacts are as follows.
The different paths are:
1. 1–2–4–5 of duration 18 days
2. 1–2–3–4–5 of duration 21 days ***/ 20 days/now, 19 days****
3. 1–3–4–5 of duration 12 days

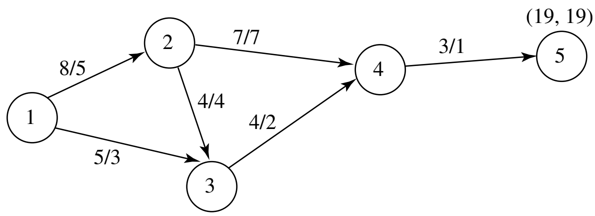
The earliest start times are changed. We have paid extra ₹100 only. Critical path now is 19 days.

Indirect cost reduces by ₹250 per day. The total reduction in indirect cost is ₹250 × 2 = ₹500

We spent ₹200 and saved 2 days.

Our saving is ₹500 – ₹200 = ₹300 in total cost and 2 days.

The new figure is



There is no further compression possible on the path 2–3. It has reached its crash time.

We have shown a circle on 2–3 indicating that this path is no more for further contraction of duration.

Next path for contraction having minimum cost is 1–2 .at the cost of ₹300 per day. It can be compressed for maximum 3 days (8/5).

Step 4

The different paths are:

1. 1–2–4–5 of duration 18 days
2. 1–2–3–4–5 of duration 21 days ***/ 20 days/now, 19 days****
3. 1–3–4–5 of duration 12 days

Immediate effect of compression is on two paths which share a common activity 1–2.

We, at this time, think of crashing the path by 3 days. This will not affect the last path 1–3–4–5, as it is of duration 12 days.

Compress 1–2 by 3 days (from 8 days now, we make it for 5 days.)

We pay extra amount ₹300 × 3 = ₹900 and get the things done.

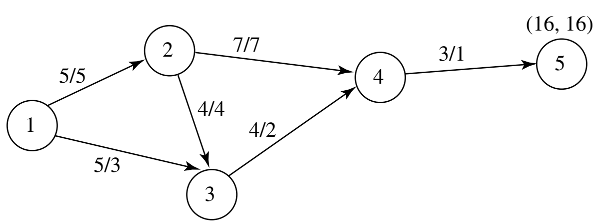
Let us study the impacts of doing such compression.

The different paths are

1. 1–2–4–5 of duration 18 days/ **15 days**
2. 1–2–3–4–5 of duration 21 days/19 days/**16 days.******
3. 1–3–4–5 of duration 12 days

Indirect cost decreases by ₹250 × 3 = ₹750. This is achieved at a cost ₹900 for reducing the time duration for 3 days.

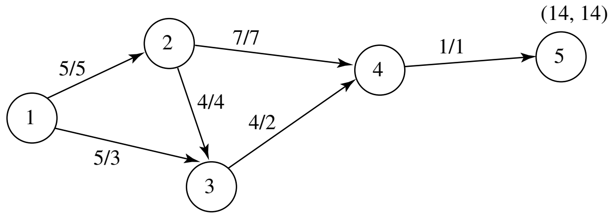
The activity 1–2 cannot be compressed further. We draw the figure.



Step 5

At this stage, we make a selection of compressing the activity 4–5 (compression cost = ₹550 per day). It can be compressed for 2 days. It is a common activity for all the paths.

We draw a new diagram and show it.



The different paths are;

1. 1–2–4–5 of duration 18 days/ **15 days/ 13 days**
2. 1–2–3–4–5 of duration 21 days/16 days/**14 days./** ****
3. 1–3–4–5 of duration 12 days/ **10 days.**

Indirect cost decreases by ₹250 × 2 = ₹500 days. This is achieved at a cost ₹550 × 2 = ₹1100 for reducing the time duration for 2 days.

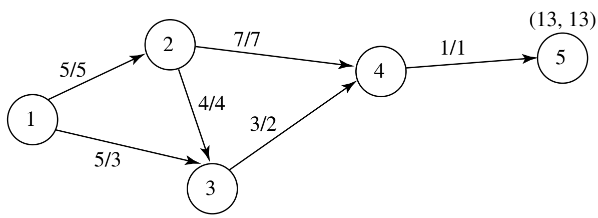
The activity 4–5 cannot be compressed further.

Step 6

Now, there is only one activity 3–4, which can be compressed. Cost of compression is ₹600 per day.

We can compress it by 1 day. We observe that if 3–4 is compressed by 1 day then critical path and the path 1–3–4–5 both are reduced by 1 day. The effect of reduction on critical path is that, it will make two critical paths. The path 1–2–4–5, and the path 1–2–3–4–5 both will be of duration 13 days.

We see the next figure and understand.



The path 1–2–4–5, and the path 1–2–3–4–5 both are of duration 13 days.

This clearly shows that there are two critical paths: each of 13 days.

{Add the present time, in days, and check this.]

We saved 1 day and an indirect cost ₹250. We paid ₹600 to achieve this.

Any further compression, with a view to reduce the time must be done so that both the critical paths have a joint effect of reduction and both should reduce form 13 days to 12 or lower days.

Simultaneous Crashing (Parallel Crashing)

This is done only in the case when we have two or more paths behaving as critical paths. We cannot compress any one path and keep the other path without crashing.

We first find whether there exists a common path such that compression on any one will compress both the paths.

If not so, then we search for the best economical combination of compressible activities on each path so that the impact of reduction is on both the paths under consideration.

For this example, we have paths (2–4 and 1–3) or paths or paths (2–4 and 3–4).

The restriction that you can see on the path is 7/7, i.e. this path is not further compressible. So we are at a dead end.

Let us put all that we have done so far in a tabular form.

Crash Time—Cost Table

Number	*Activity Crashed*	*Number of days*	*Cost*	*Cumulative cost = C_c*	*Direct Cost $C_c + C_n$*	*Project Duration*	*Indirect Cost*	*Total Cost*
1	–	–	–	–	00 + 8200	21	5250	13450
2	2–3	2	100 × 2 = 200	200	200 + 8200	19	4750	13150**
3	1–2	3	300 × 3 = 900	1100	1100 + 8200	16	4000	13300
4	4–5	2	550 × 2 = 1100	2200	2200 + 8200	14	3500	13900
5	3–4	1	600 × 1 = 600	2800	2800 + 8200	13	3250	14250
–	–	—	–	–	–	–	–	–

Minimum cost of the project is ₹13150 (project duration = 19 days)
Minimum time of the project is 13 days (project cost = ₹14250)

Comment: We have discussed the above example showing every detail of thinking procedure and taking a decision of crashing activities. The table shown above gives the entire picture of steps taken at each point of crashing. Students are requested to understand this by reading and working simultaneously.

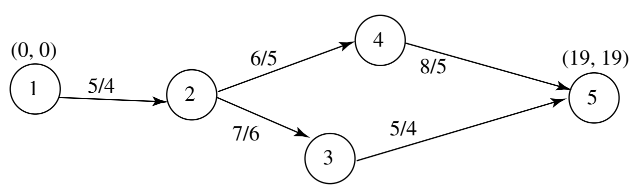
ILLUSTRATION 11

Draw the network from the following table. Find the critical path from the table. Crash the time for maximum possible period without affecting the critical path.

Activity	t_n	t_c	C_n	C_c	*Cost Slope*
1–2	5	4	1000	1600	600
2–3	7	6	100	170	70
2–4	6	5	200	400	2==
3–5	5	4	100	5==	4==
4–5	8	5	300	600	100

Solution

We draw network diagram.

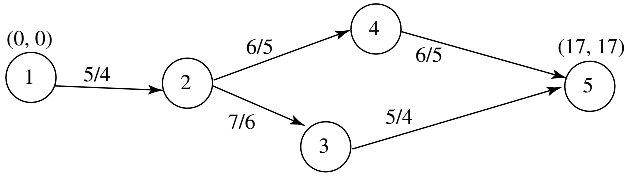


There are two paths.
1–2–4–5 of duration 19 days and the second one is 1–2–3–5 of duration 17 days.

Critical path being the longest one; we compress least cost activity on the critical path.
The activity is 4—5 which has a lowest cost of compression the critical path.
We compress it by 2 days. We incur a cost of ₹200 for the compression.
We save 2 days.

As a result, there are two critical paths. Each path has a duration of 17 days.

The new diagram is as follows.



We have to find different combinations of paths. As there are two paths; we have to take care in compression.

Combination of 2–3 and 2–4 costs ₹200 + 70 = ₹270

Combination of 2–3 and 4–5 costs ₹100 + 70 = ₹170

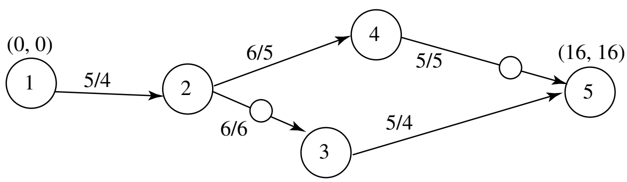
Combination of 2–4 and 3–5 costs ₹200 + 400 = ₹600

Combination of 3–5 and 4–5 costs ₹100 + 400 = ₹500

Crashing on 2–3 and 4–5 will be the best one. We have to pay an extra amount of ₹170 for 1 day compression on the critical path.

As a result both the critical paths will be of 16 days duration.

Figure is as follows.

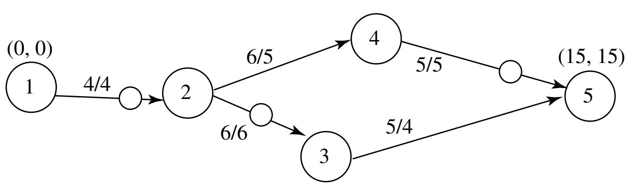


Extra circles on the path 2–3 and 4–5 show that further compression is not possible.

Now there is only one path to be compressed and that is 1–2.

Compression cost is ₹600 per day and also it can be compressed by 1 day.

We compress 1–2.



We have compressed in three different stages.

Total cost of compression = ₹200 + ₹170 + ₹600 = ₹970.

We have saved 4 days in the total project duration. It is now 15 days.

Normal cost of the project = ₹1700.

Total cost of the project for completion in 15 days is ₹1700 + ₹970 = ₹2670

Additional Questions for Practice (with Hints and Answers)

Question 1

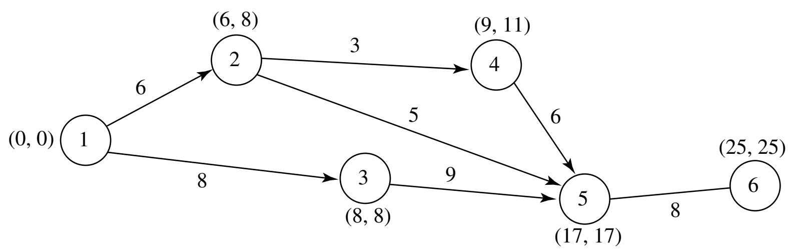
There are activities A through G.

Their timing and inter-relationships are given below. Draw a network diagram and find the critical path.

Activity	Predecessor	Time (days)
A	—	6
B	—	8
C	A	3
D	A	5
E	B	9
F	C	6
G	D, E, F	8

Solution

Following the dependency relation, we draw the network diagram. Then we make forward and backward passes and find the critical path.



Remember that critical path is the longest path in the entire network. Now make the table and find the critical path.

First, we make the regular CPM table, find the critical path and then consider the other factors.

1	*2*	*3*	*4*	*5*	*6*	*Float*
Activity	*Duration*	*Start*		*Finish*		
		E	*L*	*E*	*L*	
1–2	6	0	2	6	8	2
1–3	8	0	0	8	8	0**
2–4	3					
2–5	5					
3–5	9					
4–5	6					
5–6	8					

Complete the above table.

Question 2

For the following activities, dependency relations and three types of time estimates are given.

You are required to;

1. draw a network diagram.
2. find the critical path.
3. find the standard deviation of the critical path.
4. find the probability of the event that it takes 15% more time then the time length of critical path.
5. Find the probability of the event that it takes 10% less time then the time length of critical path.

Activity	*Predecessor*	t_o	t_n	t_p
A	–	4	6	20
B	–	5	7	15
C	A	6	6	6
D	A	2	3	10
E	B, C, D	3	3	3
F	B	1	4	13

Solution

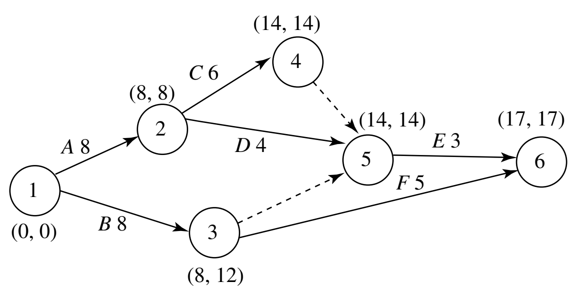
From the table above, we find time estimate for each activity by applying the formula

$$t = (t_o + t_n + t_p)/6 \text{ and variance} = S^2 = \left(\frac{t_p - t_o}{6}\right)^2$$

We make the table of estimated timings and variance.

Activity	*Predecessor*	t_o	t_n	t_p	t	S^2
A	–	4	6	20	8	64/9
B	–	5	7	15	8	25/9
C	A	6	6	6	6	0
D	A	2	3	10	4	16/9
E	B, C, D	3	3	3	3	0
F	B	1	4	13	5	4

Network diagram is as follows.



Now, students make the table to find the float and the critical path.

POINTS TO REMEMBER

To any project, the important points are

1. estimation of completion time
2. cost

In order to know about the time estimate of the project, one should know the

(A) number of activities involved.
(B) their precedence relationship.

In order to know about the cost estimation, one should know the

(A) cost associated with each activity involved in the project.
(B) crashing cost and crash time study.

- *CPM* is activity oriented. Types of activities, related cost and actual time taken, resources used—are all known components.
- *PERT* is event oriented. An exhaustive list of all the activities involved is ill-conceived. As a result completion time and associated cost remains unknown. It becomes a point of probabilistic estimation.
- *Forward Pass* One can begin an activity or activities from a *node* if all the activities converging to that node are completely done. This is not applicable to initial activity or activities.

  You can start an activity or activities from a node earliest by the highest/maximum of completion time of all the activities converging to that node. Then we add the activity time to that highest time and write it as the first component. The procedure following this order is called forward pass.
- *Backward Pass* In this format, we have to move in the opposite direction of an arrow and subtract the activity time and write that time as the second component at a node. We reach a node in the reverse way and write the least time. This called backward pass.
- *Critical Path* It is a path on the network diagram and contains only those activities having *no float*. It is the longest path in the entire network.
- In PERT, three types of time estimates are given t_o = optimistic, t_n = normal and t_p = pessimistic. As a time estimate, we find beta average.

$$t = (t_o + 4t_n + t_p)/6$$

$$\bar{x} \text{ variance} = s^2 = \left(\frac{t_P - t_o}{6}\right)^2$$

  The standard deviation of the critical path is found by taking square root of the sum of variances of the critical path.

  It is assumed that mean (critical path duration) and S = standard deviation of the critical path, follow a normal law with these two as parameters. We have a standard normal distribution which answers all the questions regarding the project completion.
- *Crashing* t_n : It is the regular or normal time taken by the system to complete an activity.

  c_n : It is the normal cost of performing an activity.

  In some cases, by employing additional resources by incurring extra cost, the activity time can be compressed up to a certain extent.

  t_c : It is the minimum time/crash time for completing an activity.

  c_c : Crash cost

  It is the total cost of doing an activity in a crash time.

$$\text{Cost slope} = \frac{C_c - C_n}{t_n - t_c} = \text{cost of compressing the activity period by 1 day.}$$

EXERCISES

OBJECTIVE TYPE QUESTIONS

I. State True or False:

1. Crash time is the time that shows number of days by which the activity period can be compressed.
2. There may be more than one critical path in a given time in a network diagram.
3. CPM is activity oriented and PERT is in some cases activity oriented.
4. Techniques of making forward pass and backward pass in CPM and PERT is the same.
5. Standard deviation of the critical path is the sum of standard deviation of all activities on the critical path.
6. Probability of competition of a project during the time period $[X^{-} - S, X^{-} + S]$ is 0.6827
   Where X^{-} = critical path length and S = standard deviation.
7. It is possible that some activities on a network may not be compressed by paying extra amount.
8. All activities on network are bound to have some float.
9. If the independent float is negative then it shows a critical activity.
10. During crashing process sometimes, it happens that all the paths are critical paths.

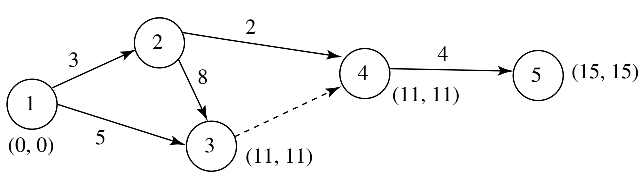
ANSWERS

1. false.	2. true.	3. false.	4. true.	5. false.
6. true.	7. true.	8. true	9. false.	10. true.

II. Multiple Choice Questions

1. If the float of an activity is zero than
   (a) free float is zero (b) independent float is zero
   (c) total float is zero (d) all the three
2. If an activity has all the three time estimates equal, then it is
   (a) critical activity (b) non-critical activity
   (c) dummy activity (d) (a) or (b)
3. The absolute difference between the earliest start and latest start at a node is called
   (a) free float (b) activity duration
   (c) independent float (d) slack of an event.

Questions (4), (5), (6) & (7) are related to following figure



4. Which is the timing pair at node 3?
   (a) (5, 11) (b) (9, 11) (c) (11, 9) (d) (11, 11)
5. Which one is the timing pair at node(4)
   (a) (9, 11) (b) (10, 11) (c) (16, 11) (d) (11, 11)
6. Critical path is
   (a) 1–2–3–5 (b) 1–2–3–4–5 (c) 1–5 (d) 1–2–4–5
7. Performing activity 1–3 there is a time slack of
   (a) at least 7 days (b) at least 12 days (c) at most 6 days (d) no days

8. _____ float is a part of _____ float and _______ float is a part of _____ float.
   (a) total, free, free, independent
   (b) free, total, free, independent
   (c) free, total, independent, free
   (d) free, free, total, independent.
9. Independent float is zero so we can conclude that
   (a) free float is zero
   (b) total float is zero
   (c) it is a critical activity
   (d) all may be true
10. Project duration extends, it must be true that
    (a) cost of some activity increases
    (b) performance of a non-critical activity increases by 1 day
    (c) total float of an activity increases from 1 to 3 days
    (d) no one of above is a must.

ANSWERS

1. (d) 2. (d) 3. (d) 4. (d) 5. (d)
6. (b) 7. (c) 8. (c) 9. (d) 10. (d)

NUMERICAL PROBLEMS

1. From the following activity table, draw a network diagram and find the critical path.

Activity	*Duration*
1–2	6
1–3	8
2–4	3
2–5	5
3–5	9
4–5	6
5–6	8

2. From the following activity table, draw a network diagram and find the critical path.

Activity	*Duration*
1–2	6
1–3	8
1–4	10
2–3	7
2–5	7
3–4	10
3–5	6
4–5	Dummy
4–7	6
5–6	4
6–8	7
7–8	2

3. From the following activity table, draw a network diagram and find the critical path.

Activity	*Duration*
1–2	8
1–3	14
1–4	8
2–3	10
2–5	12
3–4	Dummy
3–5	9
4–6	3
5–6	5
6–7	7

4. From the following activity table, draw a network diagram and find the critical path.

Activity	*Duration*
1–2	3
1–3	8
1–4	11
2–3	4
2–5	7
3–4	10
3–5	6
4–5	10
5–6	13
5–7	12
6–7	6

5. From the following activity table, draw a network diagram and find the critical path.

Activity	*Duration*
1–2	8
1–3	2
1–4	10
2–3	6
2–5	4
3–5	Dummy
3–6	14
4–6	8
4–7	10
5–7	7
6–7	7

6. Using the following predecessor relationship, draw a network diagram and find the critical path.

Activity	*Predecessor*	*Duration*
A	–	6
B	–	10
C	–	12
D	A	6
E	A	4
F	B	8
G	C	6
H	C	9
I	D, E, F, G	7
J	I	6
K	H	15
L	H	5

7. From the following activity table, draw a network diagram and find the critical path.

Activity	*Duration*
1–2	3
1–3	9
1–4	10
2–3	7
2–5	12
3–4	Dummy
3–5	14
4–5	16
4–6	12
4–7	8
5–7	10

8. From the following activity table, draw a network diagram and find the critical path.

Activity	*Duration*
1–2	6
1–3	8
1–5	10
2–3	Dummy
2–4	14
3–4	10
3–5	12
4–6	11
5–6	10
5–7	9
6–7	8

9. From the following activity table, draw a network diagram and find the critical path.

Activity	*Duration*
1–2	10
1–3	12
1–4	8
2–3	Dummy
2–4	13
3–5	8
4–5	6
4–6	14
5–6	4
5–7	3
6–7	10

10. Using the following predecessor relationship, draw a network diagram and find the critical path.

Activity	*Predecessor*	*Duration*
A	–	8
B	–	10
C	A	10
D	A	6
E	B, D	12
F	C	10
G	C	5
H	G, E	12
I	E, G, F	12
J	G, E, F	13
K	H, I	7

11. From the following activity table, draw a network diagram and find the critical path.

Activity	*Duration*
1–2	6
1–3	12
1–5	10
2–4	8
3–4	8
3–5	10
4–5	14
4–6	10
4–7	8
5–6	8
5–8	20
6–7	5
7–8	13

For Exercises 12 to 17 follow the instructions given below.

1. Find estimated durations using three types of estimates.
2. Draw the project network diagram.
3. Find Floats and determine the critical path.
4. Find the standard deviation of the critical path.
5. Use the following table of time estimates for each of the exercises.

[This is given just to reduce your long working on answering all the questions.]

Table of Time Estimates of Activities

Activity	*to*	*tm*	*tp*
A	2	6	10
B	3	4	5
C	7	10	19
D	10	12	20
E	1	5	15
F	2	7	18
G	5	8	17
H	8	12	22
I	5	5	5
J	2	6	22
K	4	7	10
L	2	4	12
M	3	3	3

12. 1. Find estimated durations using three types of estimates.
    2. Draw the project network diagram.
    3. Find floats and determine the critical path.
    4. Find the standard deviation of the critical path.
    5. Use the following table of time estimates for each one of the example.

Activity	*Predecessor*
A	—
B	—
C	—
D	A
E	B, D
F	B, D
G	C, F

13. For the following activities, dependency relations and three types of time estimates are given; you are required to:
    1. draw a network diagram.
    2. find the critical path.
    3. find the standard deviation of the critical path.
    4. find the probability of the event that it takes 15% more time then the time length of critical path.
    5. find the probability of the event that it takes 10% less time then the time length of critical path.

Activity	*Predecessor*
A	–
B	–
C	A
D	A
E	B, C, D
F	B

14. For the following activities, dependency relations and three types of time estimates are given, you are required to:
    1. draw a network diagram.
    2. find the critical path.
    3. find the standard deviation of the critical path.
    4. find the probability of the event that it takes 15% more time then the time length of the critical path.
    5. find the probability of the event that it takes 10% less time then the time length of the critical path.

Activity	*Predecessor*
A	–
B	–
C	A
D	A
E	B, C
F	B, C
G	D, E
H	G, F

15. For the following activities, dependency relations and three types of time estimates are given, you are required to:
    1. draw a network diagram.
    2. find the critical path.
    3. find the standard deviation of the critical path.

4. if one wants give an assurance of 95%,then how many days should he give for project completion.

Activity	*Predecessor*
A	–
B	–
C	A
D	A, B
E	A, B, C
F	E

16. For the following activities, dependency relations and three types of time estimates are given, you are required to:
    1. draw a network diagram.
    2. find the critical path.
    3. find the standard deviation of the critical path.
    4. find the probability of the event that it takes 15% more time then the time length of the critical path.
    5. find the probability of the event that it takes 10% less time then the time length of the critical path.

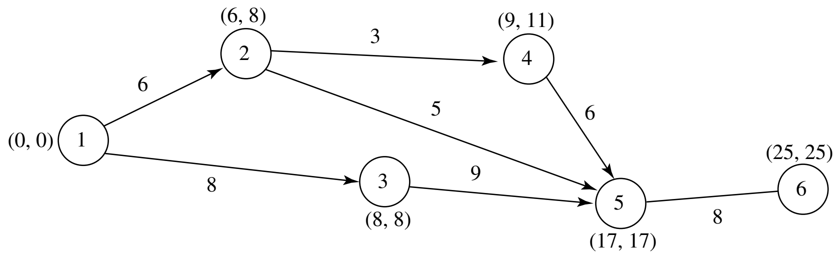
Activity	*Predecessor*
A	–
B	–
C	A
D	A
E	B, C
F	B, C
G	E, D

17. Draw the network and the critical path from the following table. Crash the time to a maximum possible period without affecting the critical path.

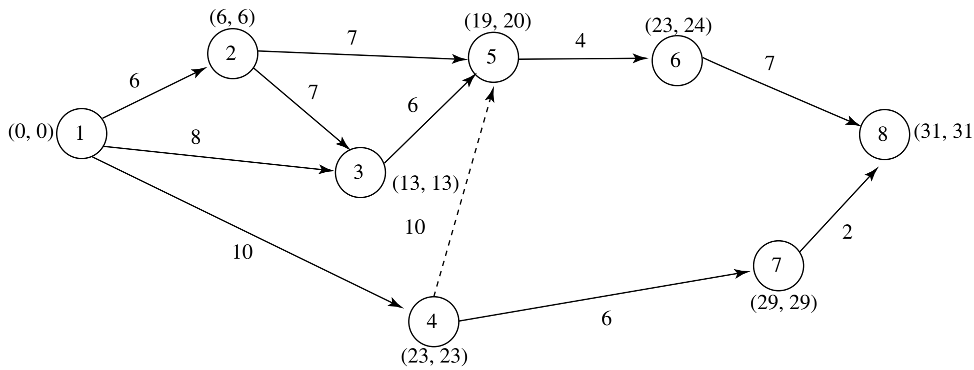
Activity	t_n	t_c	C_n	C_c	*Cost Slope*
1–2	5	4	1000	1600	600
2–3	7	6	100	200	100
2–4	6	5	200	400	200
3–5	5	4	200	600	400
4–5	8	5	300	570	90

ANSWERS TO NUMERICAL PROBLEMS

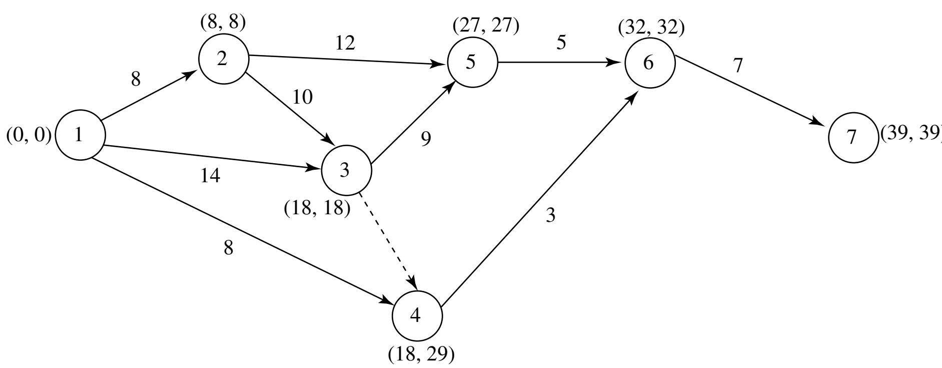
1. Network diagram is as follows. The critical path is 1–3–5–6 of 25 days.



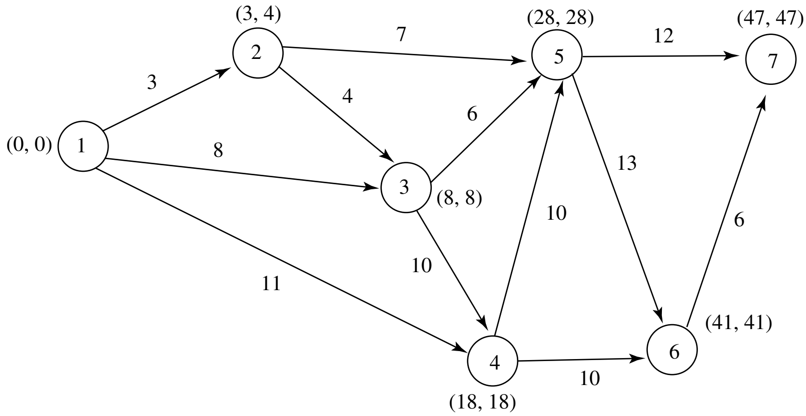
2. Network diagram is as follows. The critical path is 1–2–3–4–7–8 of 31 days.



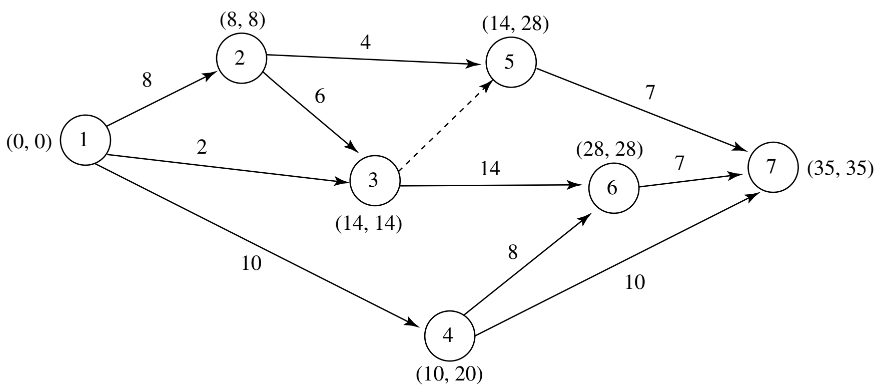
3. Network diagram is as follows. The critical path is 1–2–3–5–6–7 of 39 days.



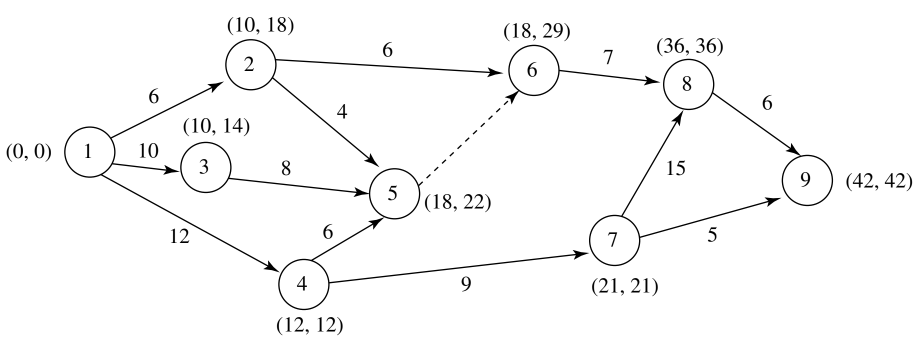
4. Network diagram is as follows. The critical path is 1–3–4–5–6–7 of 47 days.



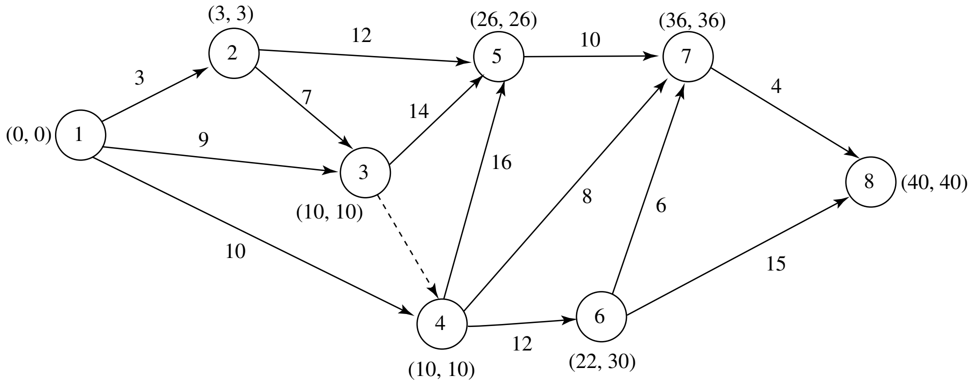
5. Network diagram is as follows. The critical path is 1–2–3–6–7 of 35 days.



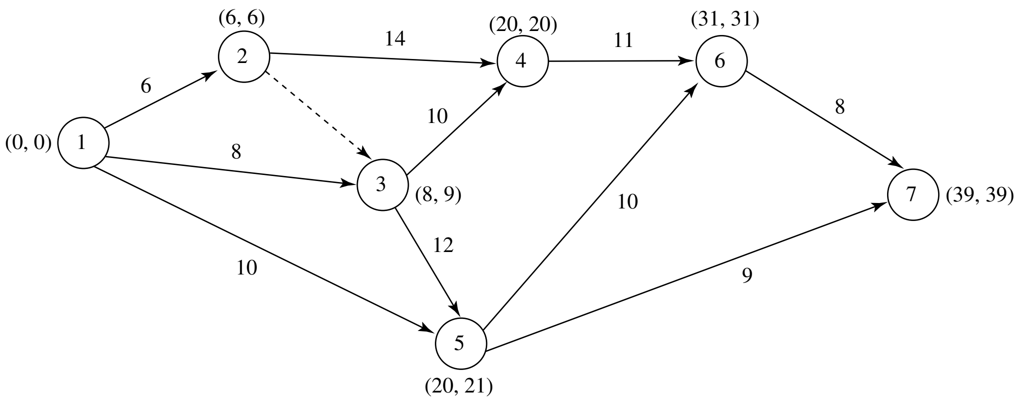
6. Network diagram is as follows. The critical path is 1–4–7–8–9 of 42 days.



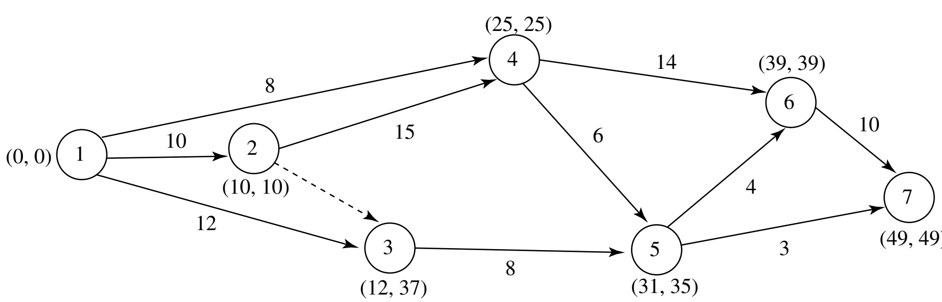
7. Network diagram is as follows. The critical path is 1–2–3–4–5–7–8 of 40 days.



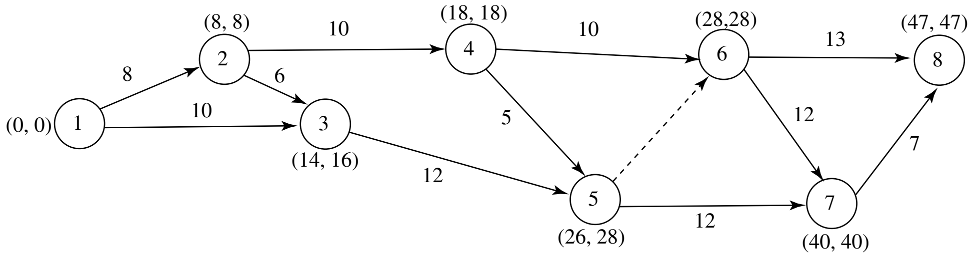
8. Network diagram is as follows. The critical path is 1–2–4–6–7 of 39 days.



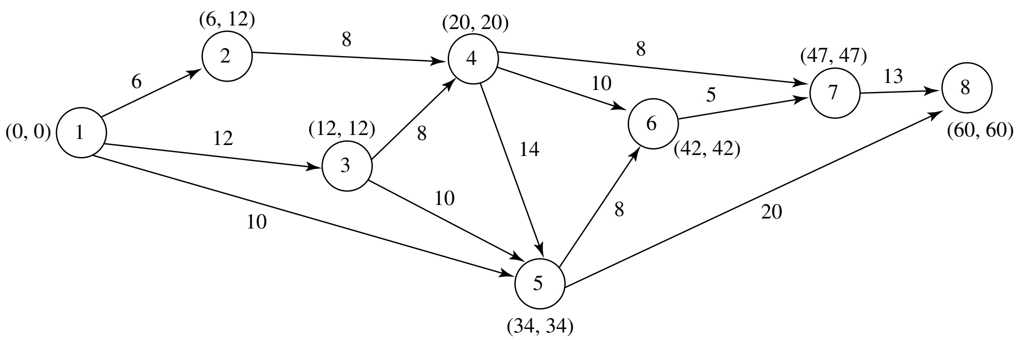
9. Network diagram is as follows. The critical path is 1–2–4–6–7 of 49 days.



10. Network diagram is as follows. The critical path is 1–2–4–6–7 of 47 days.



11. Network diagram is as follows. The critical path is 1–3–4–5–6–7–8 of 60 days.



	Critical Path	Duration	Standard Deviation
12.	A–D–F–G	36	3.96
13.	A–D–E	25	3.16
14.	A–C–E–G–H	45	4.546
15.	A–C–E–F	31	4.28
16.	A–C–E–G	32	3.90

17. Critical path of 19 days. After crashing, it is 15 days. Cost of crashing = ₹970

Note: Corresponding probabilities can be found by using Normal Distribution Table.

For the random variable X with mean equal to length of the critical path and standard deviation of the critical path = S; we use the formula to find the value of the standard normal variate Z.

$$Z = (X - \text{length of critical path})/\text{standard deviation}.$$

[After finding the value of Z, we have to refer to 'normal distribution table'.
Area under a standard normal curve is equal to 1. Area is symmetrical about the mean.
If the days are greater than the days of the critical path then the probability will be greater than 0.5.
If the number of days are less than the days of the critical path then the area will be less than 0.5.
From this, it is obvious that the probability of completion of the project up to the days of the critical path is 0.5.]

14 Dynamic Programming

'Human observe natural events; call them result of some events—— not within control'

Learning Objectives

AFTER STUDYING THIS CHAPTER, THE STUDENTS WILL BE ABLE TO UNDERSTAND:

- understand to sub-divide a given complex problem in possibly multi-level sub-problems
- understand either ascending or descending order hierarchy and formulate sub-problems
- understand the principles of optimality
- design stage-wise solution methods of finding optimal solution to each stage.
- connect all intermediate solution by some recursive formula and get the integrated solution

INTRODUCTION

In early 1950, dynamic programming technique was designed and successfully developed by Professor Richard Bellman. The technique was based on his own principle known as 'principle of optimality'. The principle of optimality is a natural law and once you read it and understand it, you will find it a throbbing reality.

Principle of Optimality

An optimal policy has an inherent tendency that, whatever the initial decisions are, the remaining decisions must constitute, in each stage, optimal policies in every situations which are states of outcomes of initial decisions already taken—may be right or partially or totally wrong.

The basic nature and procedure of solving a large and multi-stage problem is to break the problem into sub-problems and these are known as stages of a given problem. For example, an year-based profit or business or sales problem may be sub-divided into quarterly problem or monthly problems. This is called sub-division of a problem into stages. Further pulling each sub-problem into reality, we have different variables and constraints on resources—called states of stages.

14.1 DYNAMIC PROGRAMMING AND ITS CHARACTERISTICS

There are certain class of business problems in which variation in variables over a given time period becomes characterization of the problems. Variables in each stage are in different states of interval and keep on changing their state from stage to stage. Whatever the initial decisions are, either favourable, misleading, or deviating from the goal, the principle of optimality says that a natural behaviour makes a natural pattern of the solution in each stage, steadily improves upon the solution and leads to optimality.

The original problem is divided into a number of as many sub-problems as there are number of variables in the given problem. Each sub-problem is treated as an individual problem and optimal solution is found. Now, all such solutions are integrated together to find the best possible solution of the given problem.

Solution procedures have a hierarchical pattern beginning either from the first stage to the last stage or *vice versa*. This is carried out in a series of recursive steps or from the first stage to the last stage or *vice versa*.

While considering situations of optimal allocations using 'linear programming' methods; it was assumed that the values of decision variables are constants over the planning horizon and hence these problems are static type of problems by nature.

When the decision variables vary with time, the situations are considered **dynamic** in nature. Hence, a problem with multi-stage situation in which a series of decision-making steps are required is called a **dynamic programming problems** (DPP or simply DP).

If there are n stages then the objective function of the nth stage has n components.

Here there is a notable difference—number of state variables in all stages may be the same but the number of components of the objective function depends on the number of stages involved.

14.1.1 Some Technical Words and Explanation

Stage I: Each sub-problem of the given problem makes a stage. If the given problem has two variables then the corresponding DP has two stages. If Stage 1 is the last stage and then we come backward to the first variable—call it a Stage 2.

- **Alternative** m_i: In a given stage i, there may be more than one choices of carrying out a task. Each choice is called an *alternative* m_i.
- **State Variable**: A possible value of a resource within its permitted range at a given stage i is known as a *state variable* x_i.
- **A recursive function** $f_i(x_i)$: A function which associates the measure of performance of the current stage with the measure of performance of the previous stage or its succeeding stage is called a *recursive function* $[f_i(x_i)]$. For example, $f_1(x_1) = \max. (R(m_i))$
- Example 1; Maximize $Z = 10x_1 + 30x_2$
  subject to $3x_1 + 6x_2 \leq 168$
  $12x_2 \leq 240$
  with $x_1, x_2 \geq 0$

As there are two decision variables; there will be two stages—Stage 1 and Stage 2. Stage 1 is assigned to the decision variable x_1 and Stage 2 is assigned to the decision variable x_2.

Since we apply backward recursive process, Stage 2 is considered first.

Stage i	Decision Variable	Set of States
2	x_2	(b_{12}, b_{22})
1	x_1	(b_{11}, b_{21})

Recursive function for the Stage 2 with respect to x_2 is based on backward recursion. $f_2(b_{12}, b_{22}) = \max (30x_2)$

Hence a problem with multi-stage situation in which a series of decision-making steps are required is called a DPP.

14.2 Formulation of Dynamic Programming Problem

Four Steps/Stages Procedure

Step/Stage 1

Convert a real problem into a mathematical model. Identify the stages, state variable, and constraint on resources. Determine the stages and fix up working either from first stage to the last stage or *vice-versa*. Divide the given problem into a number of sub-problems—called stages.

Step/Stage 2

First, begin with either the first stage or last stage. Find the optimal solution of the objective function in terms of the state variable of that state only.

Step/Stage 3

Taking all state results together, you have to enter the next stage (either be it second from the origin or last one); study and find state variables (may be some more or some less or the same numbers of that of the first stage). Make possible combinations of alternatives and search for the optimal values under each of the combined states.

Step/Stage 4

This is a recursive process and you have to move to each stage either in forward or backward direction. When all the stages are completely dealt with optimization goal, the problem is done.

14.3 Application Area of Dynamic Programming

We have some illustrations showing applications of what dynamic programming stands for in real life situations. It is a useful tool helping the decision maker to take sequential decisions in each stage of the complete program. Different stages and different variables of each stage and step-wise systematic proceedings involving optimal decisions of previous stage and finally moving towards optimality, is a special feature of dynamic programming. The big and complicated problem is sub-divided into number of stages and each stage is taken into operations. Here are some areas where we find many applications of dynamic programming.

1. **Capital Budgeting Problem:** Distribution of capital budgets in competitive activities of different stages. It has an objective of maximizing the return on investment.
2. **Optimal Sub-division Problem:** We have problems of allocation of given resources to different systems and optimize either the time or profit as an output from the total set up.
3. **Cargo Loading Problems:** We have to manage the despatch of production units or raw materials or load the containers for export purpose. There are many companies, which provide the containers on hire and each container has its own loading and carrying capacities. We are required to plan the number of containers to hire with following two leading objectives:
   (a) maximum amount of goods can be despatched using minimum containers, and
   (b) the hiring cost is minimum.
4. **Production Smoothing Problems:** Production scheduling and inventory control are the major areas of dynamic programming problems. We face the problems of designing the best plan/the

best time to begin and complete production of seasonal goods or items. At this point of time there are following three issues:

(i) procurement of raw materials at the time when prices are in the low range.
(ii) when to start (with target to completion time) and how much to produce.
(iii) where to store and how much to store—this involves problems of minimizing carrying cost.

5. Linear and Non-linear Programming problems which are modeled but sometimes we need mathematical jargons to be applied and it may not be within the reach. Dividing the problem in numbers of stage technique is very useful and it proves more simple and easier.

ILLUSTRATION 1

A manufacturing company has three plans. Each plan has different projects for expansion. It is decided that the company shall invest a maximum amount of ₹80000. At the same time, it is not essential to invest the total allotted amount ₹80000 allotted. The following table shows investment amount in the plans and respective return. You are required to advise and show the most profitable sequence of alternative selecting exactly one from each plan.

Plan 1		*Plan 2*		*Plan 3*	
Investment	*Return*	*Investment*	*Return*	*Investment*	*Return*
0	00	0	00	0	00
10	20	20	30	20	28
20	30	30	40	30	40
30	35	40	48		

Solution

We use dynamic programming techniques. As there are three plans it, logically, equals three stages in the theory—S_1, S_2, and S_3.

Amount of investment done in each stage will be our state variables—S_{ij}, where *i* stands for the plan and *j* stands for the investment alternative numbered from the lowest one.

Stage 1

The following table provides detailed information of Stage 1 (in terms of ₹'1000)

State Variables	Investment	R_1 (S_{1j}) = First plan Return function of State
S_{11}	0	$R_1(S_{11}) = 0$
S_{12}	10	$R_1(S_{12}) = 20$
S_{13}	20	$R_1(S_{13}) = 30$
S_{14}	30	$R_1(S_{14}) = 35$

Stage 2

Now, the things would start getting a little complicated.

Matrix of Combined Returns

Stage 1

Variable →	$S_{11} = 0$	$S_{12} = 10$	$S_{13} = 20$	$S_{14} = 30$
Return →	$R_1 = 0$	$R_1 = 20$	$R_1 = 30$	$R_1 = 35$

Stage 2 Result of stage 1 and 2

↓	↓	↓	↓	↓	↓
Variables	Return	**00**	**20**	**30**	**35**
$S_{21} = 0$	**00** ⟶	00	20	30	35
$S_{22} = 20$	**30** ⟶	30	50	60	65
$S_{23} = 30$	**40** ⟶	40	60	70	75
$S_{24} = 40$	**48** ⟶	48	68	78	83

Stage 3

At this point, we have different combinations of investments of the first and second stages.

Now, to add to this, we have Stage 3. It has three investment proposals.

State variables are $S_{31} = 0$ $S_{32} = 20$ $S_{33} = 30$

Also, now it is a time to take investment limitation that total investment should not exceed ₹80000. This will invite proper combinations of alternatives of all the three investments.

We make a final table showing what we plan.

Combined Matrix of Return

Stage 1 + 2

Investment Variable	00	10	20	30	40	50	60	70
Return	00	20	30	50	60	70	78	88

Stage 3

Variables Return

$S_{31} = 0$	00	00	20	30	50	60	70	78	88
$S_{32} = 20$	30	28	48	58	78	88	98	106	–
$S_{33} = 30$	40	40	60	70	90	100	110**	–	–

The maximum of this matrix entries is 110, which corresponds to a maximum investment of 50 = 20 + 30 which corresponds to sum of the state variables S_{13} & S_{23} and a state variable $S_{33} = 30$.

$S_{13} + S_{23} + S_{33} = 20 + 30 + 30 = 80$, i.e. you will select third proposal in each plan.

Total return on the investment is 30 + 40 + 40 = 110.

ILLUSTRATION 2

A distributors has 5 full containers of eatable stuff. He has 4 buyers who are ready to purchase from him. (Assume that all the buyers possess necessary financial capacity to buy the stuff.) On the other end the distributor, according to his plan, can give none or all or any number of container loads to any buyer. The following table gives the profit on sale. You are required to use the dynamic programming technique to arrange the distribution of all the five load stuff containers to buyers so that the profit on sale is the maximum.

Profit (in ₹'000)

	Buyers			
Containers ↓	*1*	*2*	*3*	*4*
0	0	0	0	0
1	5	3	2	4
2	9	5	5	6
3	10	8	7	8
4	12	10	9	10
5	15	12	10	12

Solution

In this problem, it is obvious that each buyer is a stage. Within each stage there are 6 state variables number of containers 0, 1, 2, 3, 4, 5). We find stage-wise solution.

Stage 1

(S_1 = stage 1, with 6 containers – call them X_0, X_1, … X_5.

We have a profit function on sale. So we write it as $P_1(X_i)$; $i = 0, 1, \ldots, 5$

As shown in the table, we write the profit on sale.

State Variable (X_i)	0	1	2	3	4	5
Profit on sale $P_1(X_i)$	0	5	9	10	12	16

Since the maximum occurs at X_5, so we have

$$P_1(X_5) = 16$$

Stage 2

We denote this as S_2. State variables remain the same. State variables indicate number of containers. In this stage, we have a profit function with two variables. $P_1(X_i, X_j)$

$$i = 0, 1, 2, 3, 4 \text{ and } 5 \text{ and } j = 0, 1, 2, 3, 4 \text{ and } 5$$

But $0 \le i + j \le 5$ [as you cannot sale more than 5 containers.]

We make a combined profit table indicating all possibilities of state variables i and j (number of containers such that $0 \le i + j \le 5$

Combined profit = $P_1(X_i) + P_2(X_j)$

Stage 1 $X_i \to$		0	1	2	3	4	*5*
$P_1(X_i) \to$ *(given)*		0	5	9	10	12	*15*
Stage 2 X_i ↓	$P_2(X_i)$ ↓ *(given)*						
0	0	0*	5*	9*	10	12	15
1	3	3	8	12*	13	15	—
2	5	5	10	14*	15	—	—
3	8	8	13	17*	—	—	—
4	10	10	15	—	—	—	—
5	12	12	—	—	—	—	—

Explanation

1. If the distributor wants to give no container to any one, then the profit is denoted as 0^x.
2. If he wants is give 1 container then the choice is (1, 0) or (0, 1). At this point, (1, 0) means 1 container to the buyer and no to the second buyer.

State	Profit
(1, 0)	5
(0, 1)	3

Since 5 is maximum, so it is written as 5*.

3. If he wants to give 2 containers then the options are (2, 0), (1, 1), and (0, 2). The profit arising from this sale is 9, 8, and 5 respectively. We have maximum profit 9 at (2, 0). In this way, you have to make possible combination of sales of 0, 1, 2, 3, 4 and 5 containers. Note the direction of arrows and understand this procedure. Now mark (by *) the maximum at each state.
   [Read this again and again and try to understand the working of the table.]

Stage 3

We denote this stage by S_3. In this state, variables are the same (X_0 to X_5). Number of stages equals number of components (variables). The profit function is $P_3(X_i, X_j, X_k)$ with $0 \leq I + j + k \leq 5$.

We make a table which contains combination (Stage 2) of state variables and their optimal profit with the profit of possible variables in Stage 3.

Stage 2 →		*No. 0*	*1 = 1 + 0*	*2 = 2 + 0*	*3 = 2 + 1*	*4 = 2 + 2*	*5 = 2 + 3*
State $X_I + X_J$ →		*0*	*1*	*2*	*3*	*4*	*5*
Profit $P_1 + P_2$ →			*5*	*9*	*12*	*14*	*17*
Stage 3 State Profit ↓							
X_3 ↓	P_3 ↓						
0	0	0*	5*	9*	12*	14*	17*
1	2	2	7	11	14*	16	—
2	5	5	10	14*	17*	—	—
3	7	7	12	16	—	—	—
4	9	9	14	—	—	—	—
5	10	10	—	—	—	—	—

Stage 4 At this stage, we make a possible combination of all the results obtained in above table and associate the last stage with it to give you the final table.

Containers	*Buyers 1 2 3*	*Profit from Previous Stages*	*Container to Buyer 4*	*Profit from Sale*	*Total Profit*
0	0 + 0 + 0	00	5	12	12
1	1 + 0 + 0	05	4	10	15
2	2 + 0 + 0	09	3	8	17
3	2 + 1 + 0	12*	2	6	18**
4	2 + 2 + 0	17	1	4	18
	2 + 1 + 1	17	1	4	18
	2 + 0 + 2	17	1	4	18
5	2 + 3 + 0	17	0	0	17

The above table indicates the highest profit of ₹18000.

Decision	Buyers	1	2	3	4	
Containers		2	1	0	2	Total containers = 5

Total Profit = 18000

ILLUSTRATION 3

Capital Budgeting Problems

A manufacturing company has three independent units. Unit 1 makes furniture ancillaries, Unit 2 makes electrical spares and Unit 3 makes decoration items. Management has a budget of ₹22,000 for its extension in these units. One action/alternative can be chosen for expansion in each unit. You have to make division of capital budget in the activity of each unit with an objective of maximizing the return on investment.

Alternatives (Action)	*Furniture Ancillaries Investment*	*Returns*	*Electrical Parts Investment*	*Returns*	*Descriptive Parts Investment*	*Returns*
1. No action	0	0	0	0	0	0
2. Add new units	6000	8000	8000	12000	–	–
3. Replace some and modify some	10000	13000	15000	20000	3000	8000

Solution

This problem can be solved using the dynamic programming techniques.

There are three stages; we consider each unit as one stage. This means that the final stage can have 3 components. In each stage there are state variables, we will mention them on time.

Stage I

We call this the first stage, S_1. S_1 has 3 state variables; we can denote them as

S_{11}—No action.
S_{12}—Add new units.
S_{13}—Replace and modify.

The investment and returns are given in the following table. Return function: $R_1(S_{jj})$ i and $j = 1, 2, 3$

Stage 1 Variable	Unit 1 Investment	Returns
S_{11}	0	0
S_{12}	6000	8000
S_{13}	10000	13000

Now with the combination of actions and returns we move to the Unit 2; Stage 2.

Stage 2:

Return Function – $R_1 (S_{1j}) + R_2 (S_{2j})$
Act : $S_{1j} + S_{2j}$

Stage 2 Investment	*↓ Returns*	*Stage 1 →* *Investment →* *Returns →* $0 = S_{11}$, $0 = R_{11}$	$6000 = S_{12}$, $8000 = R_{12}$	$10000 = S_{13}$, $13000 = R_{13}$
$S_{21} = 0$	$R_{21} = 0$	0*	8000	13000
$S_{22} = 8000$	$R_{22} = 12000$	12000	20000	25000
$S_{23} = 15000$	$R_{23} = 20000$	20000	28000	Not possible; investment Exceeds the budget

We combine the above result in a compact form and then move to the Stage 3.

	$S_{11} + S_{21}$	$S_{12} + S_{21}$		...			$S_{13} + S_{22}$	$S_{12} + S_{23}$
** Investment:	0	6000	8000	10000	14000	15000	18000	21000
Return:	0	8000	12000	13000	20000	20000	25000	28000

** State variables are sum total of the two state variables.

Stage 3

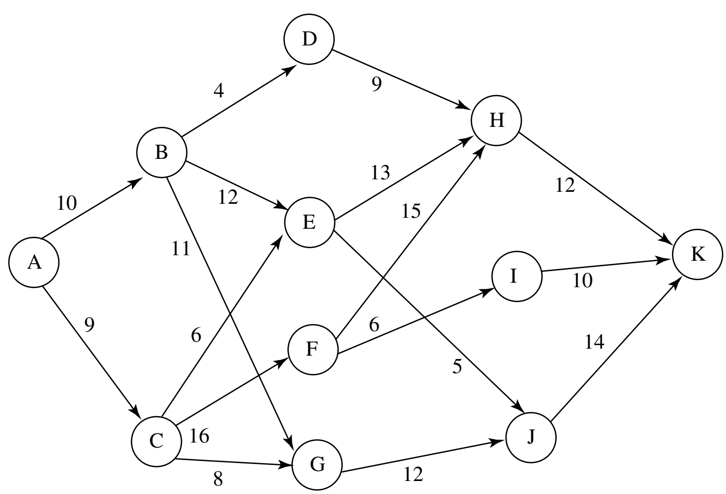
Stage 2		$S_{11} + S_{21}$	$S_{12} + S_{21}$			...		$S_{13} + S_{22}$	$S_{12} + S_{23}$
$S_{1j} + S_{2j}$ investment		0	6000	8000	10000	14000	15000	18000	21000
$R_{1j} + R_{2j}$ Return		0	8000	12000	13000	20000	20000	25000	28000
Act	Return								
$S_{31} = 0$	0	0							
–	–								
S_{33} 3000	8000	8000	16000	20000	21000	28000	28000	**33000**	* Not possible; exceeds investment limit

Optimal investment (Action)

$S_{13} + S_{22} + S_{33}$	Investment (in ₹)	Return (in ₹)
S_{13}: unit 1 : Replace and Modify	10000	13000
S_{22} : unit 2 : Add new units	8000	12000
S_{33} : unit 3 :Replace and Modify	3000	8000
		33000*

ILLUSTRATION 4 (Shortest Path Problem)

In this type of problem, possible routes joining the different cities are given. The expected outcome is to locate the shortest path from any of the cities to any other city. In this diagram, we have shown a directed graph between 11 cities. Here, we are required to find shortest-path from the city A to the last city K. The figures on the arrows show the distances in km between the cities.



Solution

We make four stages of the problem.

First Stage

It is sure that the vertex A is on the shortest path. The first stage begins at this point and extends its working up to vertices B and C. Distances are state variables. The distance from City A to City B = AB = 10 and distance from City A to City C = AC = 9 . We make a choice from City A to C. The decision of the first stage is City A to the City C which is shorter than that of from City A to B.

AB = 10 and AC = 9

Second Stage

In the second stage, we keep the first stage with us but do not declare it as the final decision.

With these two distances with us either on the route AB or the route AC, there lies a shortest path.

Let us work finding cities on the path AB; they are as follows with their distances shown by the side.

Routes on AB	Distance	Routes on AC	Distance
ABD	10 + 4 = 14 **	ACE	9 + 6 = 15
ABE	10 + 12 = 22	ACF	9 + 16 = 25
ABG	10 + 11 = 21	ACG	9 + 8 = 17

The decision should be that the shortest path contains the route ABD = 14

One important point that we just in the previous stage took a decision that the shortest path contains the route AC has to be corrected and there is a path ABD = 14.

Third Stage

In this stage, we have these six routes obtained above and bearing in mind the shortest amongst these, we work further. We move from these possible routes.

Routes on AB	Distance	Routes on AC	Distance
ABDH	14 + 9 = 23	ACEH	15 + 13 = 28
ABEH	22 + 13 = 35	ACFH	16 + 15 = 31
ABGI	16 + 12 = 28	ACGJ	17 + 12 = 29
ABEJ	22 + 15 = 37	ACFI	15 + 6 = 21***

Fourth Stage

Routes	Distance	**Distance up to K**
1. Ending on H		**HI = 12**
ABDH	23	23 + 12 = 35
ABEH	25	25 + 12 = 37
ACEH	28	28 + 12 = 40
ACFH	31	31 + 12 = 43
2. Ending on I		**IK = 10**
ABGI	26	26 + 10 = 36
ACFI	21	21 + 10 = 31**
3. Ending on J		**JK = 14**
ABEJ	27	27 + 14 = 41
ACGJ	29	29 + 14 = 44

Conclusion: The shortest path is ACFIK; a union of paths joining to the pair of cities AC + CF + FI + IK = 9 + 6 + 6 + 10 = 31 kms

ILLUSTRATION 5

Find u_1, u_2, and u_3 that maximizes $F(\mathbf{U}) = u_1 \cdot u_2 \cdot u_3$ subject to $u_1 + u_2 + u_3 = 10$

Solution

There are 3 stages to this problem.

We define variables to assist the given variables to reach the optimization.

$$X_3 = u_1 + u_2 + u_3$$
$$X_2 = u_1 + u_2$$
$$X_3 = u_1$$

Stage 1

We introduce a function which will work as a recursive function in the other stages.

$$f_1(X_1) = u_1 = X_2 - u_2$$

Stage 2

We extend the idea and let $f_2(X_2) = \text{Maximize } (u_2 \cdot u_1)$

$$= \text{Maximize } (u_2 \cdot (X_2 - u_2))$$

Maximization of a function of u_2 and X_2 is achieved by equating first order derivatives to zero. Differentiating $u_2 \cdot (X_2 - u_2) = u_2 \cdot X_2 - (u_2)^2$ with respect to u_2, and equating it to zero, we get

$$X_2 - 2.\ u_2 = 0 \text{ i.e. } u_2 = X_2/2 \qquad (1)$$

With this maximum value of $f_2(X_2)$ = maximum of $u_1.\ u_2 = (x_2 - u_2) \cdot u_2$

At $u_2 = X_2/2$; we have maximum value $= (X_2 - X_2/2) \cdot (X_2/2) = \left(\dfrac{X_2}{2}\right)^2 = \dfrac{X_2^2}{4}$ (2)

Stage 3

In the third stage we combine all the above activities with the latest one;

$$f_3\ (X_3) = \text{maximize } (u_3 \cdot (X_2)^2/4).$$

We substitute $X_2 = (X_3 - u_3)$

We get $f_3(X_3) = \text{maximize } (u_3 \cdot (X_3 - u_3)^2)/4$

Differentiating with respect to u_3 and equating the differentiated result to zero, we get

$$(X_3 - 3u_3) \cdot (X_3 - u_3) = 0$$
$$\therefore \quad X_3 = 3u_3 \text{ and } X_3 = u_3$$

Now we use the given condition that $u_1 + u_2 + u_3 = 10$; also we have three basic substitutions.

$$X_3 = u_1 + u_2 + u_3$$
$$X_2 = u_1 + u_2$$
$$X_3 = u_1$$

This gives $x_3 = 10$ but $x_3 = 3u_3$ so we have $3u_3 = 10$ giving $u_3 = 10/3$

This gives $u_1 + u_2 = 10 - 10/3 = 20/3$

but $u_1 + u_2 = X_2$ and $X_2 = 2 \cdot u_2$

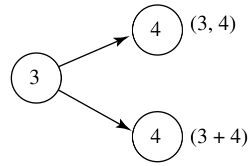
So we get $u_2 = 10/3$ and $u_1 = X_2 - u_2$ gives $u_1 = 10/3$

This gives maximum value of $u_1 \cdot u_2 \cdot u_3 = (10/3)^3 = 1000/27$

14.4 Useful Graph of Alternatives

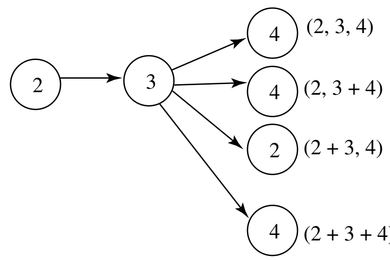
The note shows the order of different stage and its connectivity with previous stages.
The fourth process is done independently. 4

Either the processes 3 and 4 are done independently (3, 4) or the Process 4 is done with Process 3. (3 + 4)



Processes 2, 3 and 4 are done independently (2, 3, 4)

The Process 2 is done alone while 4 is done with 3 (2, 3 + 4)



Processes 2 and 3 jointly in the Process 2 and the Process 4 is done alone. (2 + 3, 4)
All three processes are done with the process 2. (2 + 3 + 4) (2 + 3 + 4) 4

Inventory Problems

Illustration 6

A manufacturing company makes a product from the surplus and waste materials at the end of finished goods. Assuming full availability of resources the company has to invest money in a small set up; the cost is ₹400. Goods unsold, can be for the next month's sale. There is no restriction on production of goods during any time in a month. If there is a carry over of surplus goods then the company has to bear the carrying charge for the warehouse. This cost is applicable and it is ₹5 per unit per month for a minimum period of one month till the day it is taken away for sale.

The company has agreed for a despatch of 10, 20, 30, and 50 units in the months of April, May, June, and July in order.

You are required to suggest the policy to make production so that the total cost (set up cost and carrying cost) is minimized.

Solution

We have, the carrying cost = ₹5 per unit per month
The set-up cost = ₹400 each time.

Months	April	May	June	July
	1	2	3	4
Demand	10	20	30	50

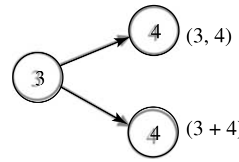
We have four months' plan and each month will be a stage for us. The state variables will be the cost of different alternatives.

Stage I

We start from the last month in operation, i.e. July (4)
There is only one deal; make a production of 50 units.
Set up cost = ₹400.

Stage 2

In the Stage 2, we have two options: either we make production of the month June and then make the production for the month July. As shown, it is (3, 4) or we make production for the month July in the month of June and bear a carrying cost for one month.



This is shown as (3 + 4)
The option (3, 4) conveys that production in each month.
The total production cost is two times the set up cost = 2 × 400 = ₹800
In the case (3 + 4), we have one set up cost and a carrying cost of ₹50 × 5 × 1 = ₹250
Total cost for (3 + 4) = ₹400 + 250 = ₹650

Comparing both costs for (3, 4) and (3 + 4); it is better to go for (3 + 4), i.e. to make the production of demand of July in the month of June.

Stage 3

Here, we have four alternatives. We consider months May (2), June (3), and July (4).
We make combinations. [This can be in tabular form also.)
Alternative 1: (2, 3, 4) production in each stage/month. Cost = ₹3 × 400 = ₹1200
Alternative 2 : (2, 3 + 4) production of May in May and that of June and July in June.
Cost = ₹400 for May + ₹400 (production in June) + (50 × 5 × 1 = 250) : Total = ₹1050
Alternative 3: (2 + 3, 4) production of June in May and that of July in July.
Set up in May (Cost = ₹400) + carrying cost of 30 items of June + production of July = ₹400 + 30 × 5 × 1 + 400 = Total = ₹950
Alternative 4: (2 + 3 + 4) production in May for the requirement of May + June + July
Cost = one time set up cost in May + carrying costs for June and July.
= ₹400 + (30 × 5 × 1) + (50 × 5 × 2) = ₹400 + 150 + 500 = ₹1050

Type:	(2, 3, 4)	(2, 3 + 4)	(2 + 3, 4)	(2 + 3 + 4)
Total cost:	1200	1050	950	1050

Stage 4

This stage is the last stage, there are four months
April(1), May (2), June (3), and July (4) and eight alternatives shown below.
We write the numbers and make calculations for all options.

1. Case (1, 2, 3, 4) Total cost = ₹400 × 4 = ₹1600
2. Case (1, 2, 3 + 4) Total cost = ₹400 + 400 + 400 + 250 = ₹1450
3. Case (1, 2 + 3, 4) Total cost = ₹400 + (400 + 150) + 400 = ₹1350
4. Case (1, 2 + 3 + 4) Total cost = ₹400 + (400 + 150 + 500) = ₹1450
5. Case (1 + 2, 3, 4) Total cost = ₹ (400 + 100) + 400 + 400 = ₹1300
6. Case (1 + 2, 3 + 4) Total cost = ₹(400 + 100) + (400 + 250) = ₹1150 **
7. Case (1 + 2 + 3, 4) Total cost = ₹(400 + 100 + 300) + 400 = ₹1200
8. Case (1 + 2 + 3 + 4) Total cost = ₹(400 + 100 + 300 + 750) = ₹1550

The decision is (1 + 2, 3 + 4), i.e. make the necessary production for April and May in the month of April and then make the necessary production of June and July in the month of June.

This will cost minimum to the manufacturer, i.e. minimum cost = ₹1150.

Storage Problem

ILLUSTRATION 7

A warehouse, hired by an engineering firm for storing the production items, has a maximum capacity to store M items. The firm has an initial stock of K items. It is also assumed that the variation in the cost and seasonal expected price are known for these K items. We want to frame an optimal policy of production, stocking, and sale over a period of time. The policy determines the optimality of the objective function.

Solution

We can consider each period of time as a stage. Our job is to frame an optimal policy in this connection keeping all these points in mind.

We will introduce certain notations to work with.

C_i = cost of one item at a given period i
S_i = selling price per unit on a given period i
X_1 = lot production just in the beginning of the period i
Y_i = amount sold during the period i

We have an objective function of maximizing the profit on sale;

Maximize Z = sum of $(S_i \times Y_i - C_i \times X_i)$ over all the period i from 1 to n. $\sum_{i=1}^{i=n} (S_iY_i - C_iX_i)$

$$\text{Maximize } Z = \text{Profit on sale over total period} = \sum_{i=1}^{i=n} (S_iY_i - C_iX_i) \tag{1}$$

The constraints are

$$K + \sum_{i=1}^{i=n} (X_i - Y_i) \le M \text{ [initial stock + sum of all leftover items} \le \text{maximum capacity]} \tag{2}$$

$$Y = \sum_{i=1}^{i=n} Y_i \le K + \sum_{i=1}^{i=n} (Y_i - X_i) \tag{3}$$

Sum of total sale over a period ≤ initial stock + sum of all leftover stocks till the end period non-negativity condition: X_i, $Y_i \ge 0$ at any time till the last period (4)

For the initial first period $i = 1$; (Stage 1)

We have, maximize Z = maximize $(S_i \times Y_\text{i} - C_i \times X_I)$

$$Y_i \le K \text{ ; } K + (X_i - Y_\text{i}) \le M; X_i, Y_\text{i} \ge 0$$

[Sale is less than or equal to the initial stock. Initial stock plus leftover amount is less than or equal to the maximum capacity.]

For $n \ge 2$, Total leftover stock $\le M - (K + (X_i - Y_i)$

i.e. $$\sum_{i=2}^{i=n} (X_i - Y_i) \le M - (K + (X_i - Y_i)$$

$$Y = \sum_{i=1}^{i=n} Y_i \le K + (X_1 - Y_1) + \sum_{i=2}^{i=n} (Y_i - X_i)$$

We use the principle of optimality and recurrence relation to get an objective function.

$$\text{maximize } \{(S_i \times Y_i - C_i \times X_I) + \text{Profit on sale of remaining period of all leftover items}$$
$$= \text{maximize } \{(S_i \times Y_i - C_i \times X_I) + F_{n-1} (K + (X_1 - Y_1)$$

Additional Questions for Practice (with Hints and Answers)

Question 1

Solve: Maximize $Z = 20X_1 + 30X_2$

subject to $4X_1 + 3X_2 \leq 96$

$2X_1 + 3X_2 \leq 60$

and non-negativity condition is applied on both the variables.

Solution

As there are two variables, we have two different stages for each variable.

Stage 1

Resources are 96 and 60. We make consumption from these two resources.

Let us first concentrate on $20X_1$ which is a part of objective function of maximization.

$F(X_1)$ can be maximized from the resources with only that value of X_1 which satisfies both the constraints.

If $X_2 = 0$, then from the first constraint $X_1 \leq 24$ and from the second constraint $X_1 \leq 30$

$X_1 = 30$ cannot be accepted as it cannot satisfy the first constraint.

We accept $X_1 \leq 24$. [For $X_1 = 24$, maximum $20X_1 = 20 \times 24 = 480$]

Stage 2

With the above information with us, we are in the second stage.

Maximize $20X_1 + 30X_2$ is equivalent to

maximize $20X_1$ + minimum of $\{(96 - 4X_1)/3, (60 - 2X_1)/3\}$

1. When $X_1 = 0$, then minimum will be a value from $\{32, 20\} = 20$
   So, we get a combination $X_1 = 0$ and $X_2 = 20$ (1)
2. When $X_1 = 24$, minimum of $\{(96 - 4X_1)/3, (60 - 2X_1)/3\}$ will be minimum of $\{0, 4\} = 0$
   so, we get a combination $X_1 = 24$ and $X_2 = 0$ (2)
   We are not treating with integers only, we work in a range of values.
3. Let us compare the two sets of values for X_2.
   $(96 - 4X_1)/3 = (60 - 2X_1)/3$ which gives $X_1 = 18$ and derived from the condition is $X_2 = 8$.
   So, we have one more combination $X_1 = 18$ and $X_2 = 8$ (3)

We enlist the results as follows.

1. (0, 20) with $Z = 20\ (0) + 30\ (20) = 600$** Maximum
2. (24, 0) with $Z = 20\ (24) + 30\ (0) = 480$
3. (18, 8) with $Z = 20\ (18) + 30\ (8) = 600$** Maximum

Question 2

A machine is built in three major circuits. The failure of any one circuit causes the failure of the machine. In order to avoid such failure, it is planned to connect one standby circuit to each in parallel connection to each major circuit. In the case of failure, the standby circuit will automatically start functioning and avoid machine failure. A budget of 9 units (1 unit = ₹20000) is approved for the project. The cost of each component and its reliability factor is given in the table. You are required to make the best selection of standby unit to each major unit that will maximize the reliability.

Major Circuit 1		Major Circuit 2		Major Circuit 3	
Cost/Unit	Reliability	Cost/Unit	Reliability	Cost/Unit	Reliability
1	0.749	3	0.83	2	0.799
2	0.879	4	0.94	3	0.909
4	0.94	6	0.97	5	0.90

Solution

The problem concerns about three stages, very distinctly each major circuit is a stage. The second point is about state variable; we put them as cost of circuits. We have following two constraints:

1. total cost approved is of 9 units. [1 unit = ₹20000]
2. each major unit must have one standby circuit

The objective function is to make a selection of standby [one to each major circuit] that will maximize the total reliability of the machine.

We begin with solving the third major circuit—call it Stage 1.

The state variable will be denoted as $S_{31} = 2$, $S_{32} = 3$, and $S_{33} = 5$

Reliability function is = $R(S_{3i})$, where $i = 1, 2, 3$

We are given reliability factors in the table and there is nothing much in this stage.

Major Circuit 3	
Cost/unit	Reliability
2	0.799
3	0.909**
5	0.90

We have $R(S_{32} = 3) = 0.909$ as a stage optimal decision (1)

Now, we move to Stage 2

Here, we construct a table making all possible combinations of the state variables and their values. We keep in mind that maximum allowable cost is of 9 units and one standby to each circuit.]

If we assume that we have spent 2 units in the above part then we have 7 units left to spend in the remaining two parts/stages, i.e. the state variable can take any value—either 3, 4 or 6. {This will restrict the last stage decision about the Major Circuit 1.}

If we assume that we have spent 3 units in the above part then we have 6 units left to spend in the remaining two stages, i.e. the state variable can take any value; either 3, or 4 {We cannot take a standby unit costing 6 units. If we do so then, we cannot save any amount for standby of Major Circuit 1. This will restrict the last stage decision about the Major Circuit 1.}

Wee make a table and understand distinctly.

Joint Reliability Table (Stages 1 & 2)

Stage 1 Cost Reliability / *Stage 2 Cost Reliability*	*2* *0.799*	*3* *0.909*	*5* *0.90*
3 0.83	0.799 × 0.83 = 0.663	0.909 × 0.83 = 0.754	0.90 × 0.83 = 0.747
4 0.94	0.799 × 0.94 = 0.751	0.909 × 0.94 = 0.854*	–
6 0.97	0.799 × 0.97 = 0.775	–	–

Current status decision: Stage 1, $S_{32} = 3$, $R(F(S_{32})) = 0\text{-}909$
Stage 2, $S_{22} = 4$, $R(F(S_{22})) = 0.94$
Optimal value = $R(F(S_{32})) = 0\text{-}909 \times F(S_{22})) = 0.94 = 0.854$
Cost = 3 + 4 = 7 units

Stage 3

Joint Reliability Table (Stages 1 & 2 + Stage 3)

Stages 1 + 2 Cost Reliability / *Stage 2 Cost Reliability*	*5* *0.663*	*6* *0.754*	*7* *0.854*	*8* *0.747*
1 0.749	0.663 × 0.749 = 0.496	0.754 × 0.749 = 0.564	0.854 × 0.749 = 0.639	0.559
2 0. 879	0.663 × 0.879 = 0.582	0.754 × 0.879 = 0.663***	–	–
4 0.94	0.663 × 0.94 = 0.623	–	–	–

Conclusion: From this table, we conclude that corresponding to the highest entry, 0.662, we have, from Stages 1 & 2, [cost 6 = (3 + 3) and reliability = 0.754] and from stage 3, cost = 2 Reliability = 0.879

Total cost = 2 – cost of standby for Major Circuit 1, Reliability = 0.879
+ 3 – cost of standby for Major Circuit 2, Reliability = 0.94
+ 3 – cost of standby for Major Circuit 3, Reliability = 0.909

Total Cost = 2 + 3 + 3 = 8 units

Points to Remember

1. What are the special or additional features of linear programming?
   It divides the given problem into sub-problems and it is, in most of the cases, to solve the problems of the initial stages.
2. Each sub-division of a problem is a stage.
   Within each stage, there are variables—some of them may be new and some continued from previous stages. They keep changing their values according to the mathematical coefficients of the state variable. This is why such programming is called dynamic programming.
3. What accumulates in each stage is a component of associated previous stages. If there are three stages of a given problem then, there are three components of the problem.
   Dynamic programming models can be solved either beginning from the last stage and gradually moving to first stage or *vice-versa*. It is such a technique of solving the problem through stages that all the possible alternatives are involved and so there are slim chance to make mistakes.

4. The principle of optimality assures that some decisions taken in initial stages may not prove effective but as the stages advance there is always a natural takeover by better decisions to improve the previous solutions.

Exercises

Objective Type Questions

I. State True or False.

1. The principle of optimality is a natural law of evolution towards optimality.
2. The values of state variables may change from stage to stage.
3. In dynamic programming problem, wrong decisions taken initially, highly affects the end result.
4. If the dynamic programming problem is solved beginning from the first stage or beginning from the last stage, the objective function gives different answers.
5. All the three points, stages, states, and optimal policy of the state variables, are important features.
6. In some cases, the number of state variables is constant but their values may keep on changing.
7. The dynamic programming problems can be applied only in mathematical programming where higher degree of the variable creeps in the problem.
8. In a dynamic programming problem, in general, number of stages equals the number of state variables.
9. Linear programming problems can be solved using the stage-wise techniques of dynamic programming problem.
10. Problems in which the variables have second or third degree cannot be solved using dynamic programming method.

Answers

Answers

1. true.	2. true.	3. false.	4. true.	5. true.
6. true.	7. false.	8. false.	9. true.	10. false.

II. Multiple Choice Questions

The following table shows two different plans, say Plan A and Plan B.
Investment and corresponding returns are shown below.

Plan A		Plan B	
Investments	Returns	Investment	Returns
00	00	00	00
10	12	10	20
30	36	30	35
40	50	40	80
80	100	80	90

[Each amount in hundreds of ₹, e.g. 10 = ₹1000]
At a time one alternative can be selected from each plan. Maximum amount funded for the plan is ₹11000.
Answer the following multiple choices questions.

1. Should we begin with Plan A or Plan B?
   (a) Plan A
   (b) Plan B

(c) You have to decide looking at profits
(d) Any plan to begin with

2. Can we invest 80 in first plan and 30 in second plan?
(a) No
(b) Yes
(c) calculate profit and decide
(d) state variables have free choice

3. How many state variables are there in the first stage?
(a) 2
(b) 5
(c) 4
(d) only one decision variable

4. Can it be a choice to select zero investment in the first plan and 80 investment in the second plan?
(a) No
(b) Yes
(c) You have to select a non-zero investment from first plan
(d) You have to take non-zero values from both plans

5. If you invest total allotted amount then what is your maximum return?
(a) 100 (b) 125 (c) 180 (d) 135

6. In case of optimal selection; what is your investment strategy?
(a) (30, 80) (b) (80, 40) (c) (80, 30) (d) (0, 0)

7. In some cases, dynamic programming becomes multi-stage linear programming.
(a) Yes
(b) No
(c) You cannot solve the problem
(d) State variables are not defined

8. Dynamic programming has multiple goals all in one stage.
(a) Yes
(b) No
(c) Goals are stages
(d) Goal remains constant in all stages

9. Dynamic programming will give a good feeling if net present worth of money value is considered
(a) True
(b) False
(c) Money value remains constant
(d) It is not traditional to calculate

10. If a decision maker takes some wrong decisions then in iterative situations, the decision gets improved.
(a) False
(b) Results cannot improve
(c) Results can improve
(d) Depends on state variables

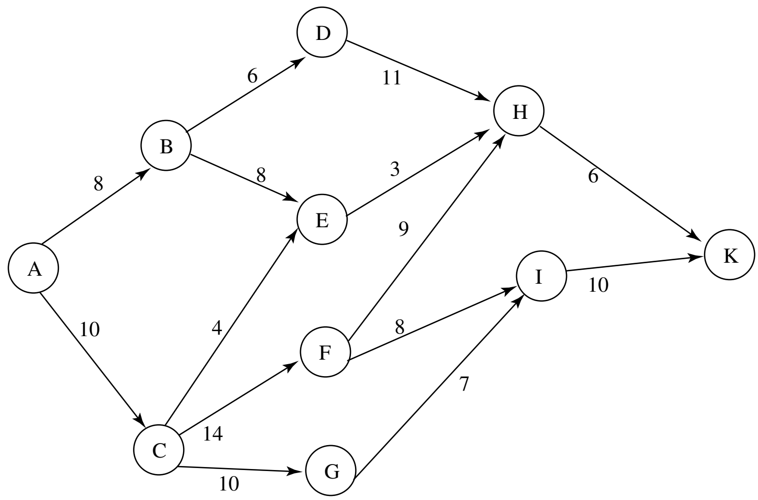
ANSWERS

1. (d)	2. (a)	3. (b)	4. (b)	5. (d)
6. (c)	7. (a)	8. (b)	9. (a)	10. (c)

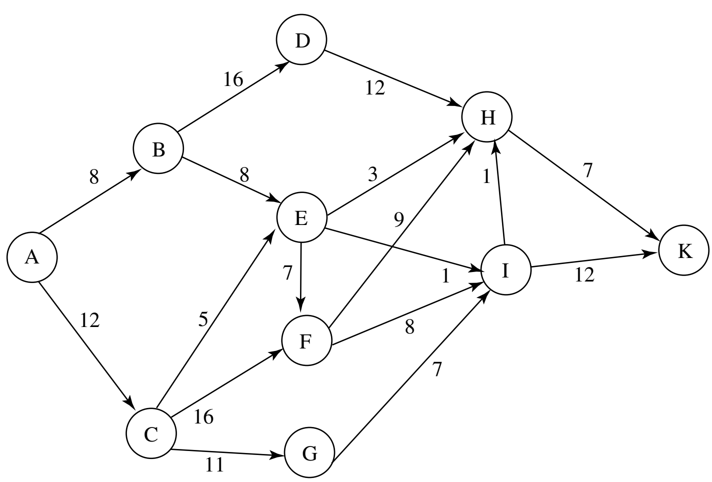
NUMERICAL PROBLEMS

Use the techniques of dynamic programming to solve the following problems:

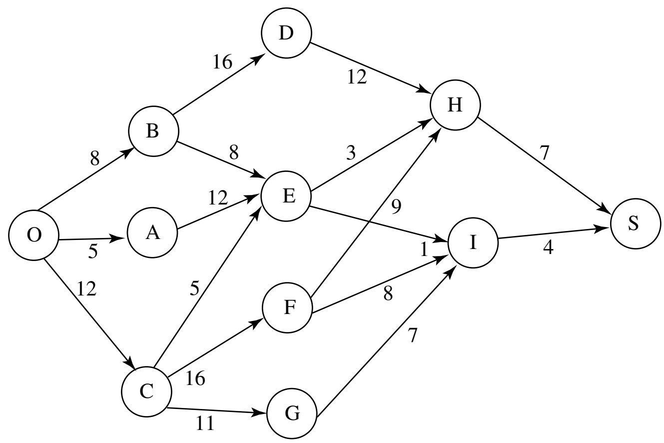
1. Maximize $Z = 2Y_1 + 5Y_2$ subject to $2Y_2 \leq 460$, $2Y_1 + Y_2 \leq 430$; $Y_1, Y_2 \geq 0$
2. Maximize $Z = 4X_1 + 14X_2$ subject to $2X_1 + 7X_2 \leq 21$, $7X_1 + 2X_2 \leq 21$; $X_1, X_2 \geq 0$
3. Maximize $Z = 3Y_1 + Y_2$ subject to $Y_2 \leq 4$, $Y_1 \leq 2$, $2Y_1 + Y_2 \leq 6$; $Y_1, Y_2 \geq 0$
4. Find X_1, X_2, X_3 so as to maximize $Z = X_1 \cdot X_2 \cdot X_3$ subject to $X_1 + X_2 + X_3 = 12$
5. Find X_1, X_2, X_3 so as to maximize $Z = X_1^2 + X_2^2 + X_3^2$ subject to $X_1 + X_2 + X_3 \geq 12$
6. Find the shortest path from City A to City J.



7. Find the shortest path from City A to City J.



8. Find the shortest path from Origin 'O' to Sink "S'



9. A merchant wants to send the goods of the same quality and stuff to his buyers. Depending up on the taxes applicable and the freights to different locations, he determines the amount of profit generated on his sale. It is given in the table. The merchant has to sell all his boxes to any buyer. You have to decide the distribution that can fetch him the highest profit.

Profit Matrix

Buyers → *Number of Boxes*	*I*	*II*	*III*	*IV*
0	0	0	0	0
1	8	12	10	8
2	16	18	14	12
3	18	20	16	14
4	20	30	20	16

10. Use Bellman's principle of optimality and the dynamic programming method, maximize the sum of n variables ΣX_i for all i = 1 to n subject to product of all such variables is a fixed positive number = P
11. A machine is built-up in three major circuits. The failure of any one circuit causes the failure of the machine. In order to avoid such failure, it is planned to connect one standby circuit to each in parallel connection to each major circuit. In the case of failure, the standby circuit will automatically start functioning and avoid machine failure. A budget of 9 units (1 unit = ₹10000) is approved for the project. Cost of each component and its reliability factor is given in the table. You are required to make the best selection of standby unit to each major unit that will maximize the reliability.

Major Circuit 1		Major Circuit 2		Major Circuit 3	
Cost/Unit	Reliability	Cost/Unit	Reliability	Cost/Unit	Reliability
1	0.75	3	0.83	2	0.801
2	0.881	4	0.94	3	0.911
4	0.94	6	0.97	5	0.90

12. Solve the following reliability problem.
In this case, you are required to find a standby one from each section; to support a main machine having four units. The manager of the unit has decided to allow a maximum budget of 14 units.

Section 1		Section 2		Section 3		Section 4	
Cost	Reliability	Cost	Reliability	Cost	Reliability	Cost	Reliability
4	0.70	2	0.599	3	0.899	3	0.799
5	0.75	3	0.699			5	0.811
7	0.85						

13. Maximize $Z = 8x + 7y$; subject to $3x + y \leq 8$, $5x + 2y \leq 15$; $x, y \geq 0$
14. A truck can carry a maximum weight load of maximum 4000 kg. There are three types of machine-packed wooden boxes. The following table shows the types of the boxes and corresponding amount of revenue generated by the transporters. The truck can be loaded with one or more of the three items.

Item number:	1	2	3
Weight in kg	2000	3000	1000
Revenue	3100	4700	4400

Answers to Numerical Problems

1. $Y_1 = 100$, $Y_2 = 230$, $Z = 1350$
2. $X_1 = 0$, $X_2 = 3$ or $X_1 = 7/3$, $X_2 = 7/3$ and $Z = 42$
3. $Y_1 = Y_2 = 2$ and $Z = 8$
4. (4, 4, 4)
5. (4, 4, 4)
6. Shortest path ABEHJ = 25 km
7. Shortest Path ABEIHK = 25 km
8. Shortest Path OAEIS = 22 km
9.

Buyers	1	2	3	4	
Boxes	2	1	1	0	
or Boxes	1	1	1	1	Total profit = 38

10. All variables take the same value. Each = $P^{1/n}$
11. Cost 2 + 4 + 3 = 9 units = ₹90000 Reliability = 0.754
12. Selection: second unit from first section, cost = 5
    second unit from second section, cost = 3
    first unit from third and fourth section, cost = 3 + 3 = 6
13. $x = 0$, $y = 15/2$ maximum $Z = 52.5$
14. Item 1 only and two boxes: total weight = 4000 kg revenue = ₹6200

15 Non-Linear Programming Problems

Tranquil at head, Tender at heart, and Tireless on hands, keeps Timeless go.

Learning Objective

AFTER STUDYING THIS CHAPTER, STUDENTS WILL BE ABLE TO

- **understand different types of non-linear programming problems.**
- **understand the different methods of solving non-linear programming problems.**
- **apply the concept to construct real life situation models falling in the category of non-linear programming problems with constraint or without constraints as the cases may be.**

INTRODUCTION

So far, we have studied many cases of real life situations and also constructed their mathematical models. We have solved such models using graphical methods and simplex procedures. In addition to these, there are certain linear programming problems, which demand integer solutions to some or all decision variables. We refer such problems in the class of integer programming problems. If the objective functions and all the constraints are linear, having first degree, we call them linear programming problems. If objective function is in second degree and the constraints are linear; we refer such problems as ***quadratic programming problems***. In general, we have non-linear programming problems of general form wherein, we have variables appearing anywhere in the model may take any degree. This class is known as ***general non-linear programming problems***. It is traditional to write such class as GNLPP.

It has always remained a subject of extreme joy to mathematicians of finding conditions under which the given function assumes local maximum or minimum and on the top to check global optimality of $f(x)$. As the number of variables increases and/or their degree in which they appear in the objective function and constraints increase, the solution become more mathematical and involves rigorous mathematical treatments.

15.1 DIFFERENT CLASSES

To our study and the study of general purposes, we, probably, need not higher than third degree of decision variables.

Optimization problems are classical problems in the area of mathematics. Rigorous mathematical treatments and different conditions make the problem either very complicated or simpler. The ultimate aim is either constrained or unconstrained optimization of the given function

1. Problems with one variable belong to class one, C—I, of general nature wherein optimization techniques depend on equating first order derivative to zero and finding the values of independent variables for which the function takes optimal values. Maximum or minimum value of the function depends on the sign of second order derivative. We assume that such functions are differentiable at least twice.
2. We switch over to the general class of optimization problems without constraints. Such problems are classified as class two (C—II). We accept that partial derivatives of these functions exist and they are continuous over a given interval.

What we have already studied is the necessary conditions for the function $f(\boldsymbol{X})$ having extreme points at $\boldsymbol{X}_0$. A sufficient condition is that the Hessian matrix evaluated at the point where first order partial derivatives vanish.

To a concave function if X_0 is a point giving maximum value of $f(\boldsymbol{X})$, then for the same **$\boldsymbol{X_0}$, there exists a family of infinite convex curves for all of which the same point** $\boldsymbol{X_0}$ is a minimum point. With maxima and minima, there is a close association of convex and concave functions.

15.2 Constrained Optimization Problem

In this type of problems, we have objective function to be optimized under certain given conditions. These conditions involve variables of the given function and given resources. Constraints involve any one of the three signs— =, ≤ or ≥ type. On a given time in a constraint, one and only one sign of the three signs, hold true.

For the problems, having two or more decision variables we have Lagrange's method of undetermined multipliers to find the extreme values of the function. Initially these multipliers are unknown and then eventually they become known when we solve the mathematical model.

The point (X_0, λ_0) is called the saddle point and if the saddle point exists then it will satisfy.

$$L(X_0, \lambda) \le L(X_0, \lambda_0) \le L(X, \lambda_0)$$

We have, in addition to Lagrange's method,

1. Kuhn-Tucker condition,
2. Wolfe's method or extended simplex method and many more methods in this area of classical optimization.

We will discuss, in the coming sections, some related points in this connection.

15.3 Separable Programming

Separable programming is a special case of convex programming. In addition to our definition of optimizing function, we have a condition that objective function can be expressed as algebraic sum of functions of different variables which appear in the function.

A function is called *separable* when it can be expressed as a sum of sub-functions each one of which is a function of only one variable. For example,

$$f(\bar{x}) = f(x, y) = x^2 + 2y^2 - 3x + 4y$$

then $\quad f(\bar{x}) = f_1(x) + f_2(y)$

where $\quad f_1(x) = x^2 - 3x$ and $f_2(y) = 2y^2 + 4y$

15.4 Solution Methods (Unconstrained optimization) Illustrations

In this section, we categorize the optimization techniques in the cases of functions of two variables and three variables.

15.4.1 Concavity and Convexity

First, we need to look at the basic concepts of concave and convex functions. At the present context, we have functions of one variable. For example,

$$y = f(x) = \sin x.$$

We know that sin ($x = 0$) = 0, sin($x = \pi/4$) = $1/\sqrt{2}$, sin ($\pi/2$) = 1

Functional value increases as x increases in the interval [0, $\pi/2$]—first Quadrant

In the same way as x increases from $x = \pi/2$ to $x = \pi$, the functional value decreases.

If you study the behaviour of the function in the interval from [0, π], then you would divide this interval in two parts: [0, π] = [0, $\pi/2$] U [$\pi/2$, π] to describe the pattern of behaviour.

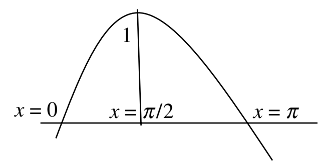


Figure 15.1

We call this type of graph as a concave curve. We may not like to do this procedure all times; finding the functional values, making a study and drawing graphs.

We find $f''(x)$, if for the values of x in the interval, if $f''(x) < 0$, then the graph has a concave nature for all values of x in that interval. This looks easier then what we have done previously.

For the convex functions, it shows opposite nature with respect to x-axis. We have $f''(x) > 0$, then the graph has a convex nature for all x in the interval. See Figure 15.2.

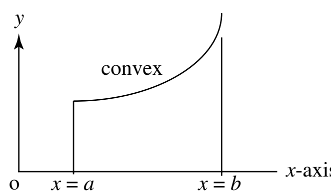


Figure 15.2

We have shown the graphs of concave and convex types of functions, we revise this concept by some illustrations.

Illustration 1

Discuss the nature of concavity or convexity for the following functions:

1. $f(x) = x^3 - 18x^2 + 5x - 4$
2. $f(x) = 2x^3 - 8x + 1$

Solution

1. Given that $f(x) = x^3 - 18x^2 + 5x - 4$

$$\therefore \quad f'(x) = 3x^2 - 36x + 5$$

$$\text{and} \quad f''(x) = 6x - 36 = 6(x - 6)$$

$$f''(x) > 0 \text{ for } x > 6$$

$\therefore$ for values of $x > 6$, we have the graph of $f(x)$ convex in nature.

2. Given that $2x^3 - 8x + 1$

$\therefore$ $f'(x) = 6x^2 - 8$

$\therefore$ $f''(x) = 12x$

$f''(x) > 0$ for $x > 0$

$\therefore$ for $x > 0$, the graph of $f(x)$ is convex in nature.

$f''(x) < 0$ for $x < 0$

$\therefore$ for $x < 0$, the graph of $f(x)$ is concave in nature.

15.4.2 Local maxima/minima of $f(x)$ for classical optimization

We have the following conditions:

1. Find $f'(x)$ and put $f'(x) = 0$
   solve $f'(x) = 0$ to find the values of x, say $x = a$, $x = b$ makes $f'(\text{a}) = 0 = f'(\text{b})$
2. Find $f''(x)$
   Put $x = a$ and $x = b$ in $f''(x)$.
   If $f''(\text{a}) < 0$, then $f(x)$ has a local maximum in neighbourhood of the point a.
   maximum value $= f(a)$
   if $f''(\text{b}) > 0$ then $f(x)$ has a local minimum in neighbourhood of the point b.
   Minimum value $= f(b)$.

ILLUSTRATION 2

Discuss extreme values of $f(x) = 2x^3 - 3x^2 - 36x + 30$

Solution

We have $f(x) = 2x^3 - 3x^2 - 36x + 30$

Differentiating with respect to x,

$$f'(x) = 6x^2 - 6x - 36 \text{ and } f''(x) = 6(2x - 1)$$

We put $f'(x) = 0$ to find the values of x giving optimum.

$$f'(x) = 6(x^2 - x - 6)$$
$$= 6(x - 3)(x + 2) = 0$$

Gives $x = 3$ and $x = -2$

Now $f''(x) = 6(2x - 1)$

For $x = 3, f''(x) = 30 > 0$

$\therefore$ $f(x)$ has a local minimum $x = 3$

Minimum value $= f(3) = 2(3)^3 - 3(3)^2 - 36(3) + 30$

$\therefore$ $f(3) = -51$ = minimum value.

For $x = -2, f''(x) = -30 < 0$

$\therefore$ $f(x)$ has a local minimum at $x = -2$

Minimum value $= f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 30$

$\therefore$ $f(-2) = 74$ = Maximum value.

15.4.3 Maxima/Minima of real functions of two variables

Note: we consider real function $f : S \subset R \times R \to R$ for $(x, y) \in S, f(x, y) \in R$

In general cases, we treat $f(x, y)$

Let h, k be the small real values (positive or negative) and $(x + h, y + k) \in S$
For (x, y) and $(x + h, y + k) \in S$

1. If $f(x, y) > f(x + h, y + k)$, then f is maximum at (x, y) in the circular neighbourhood of (x, y) and the maximum value $= f(x, y)$
2. If $f(x, y) < f(x + h, y + k)$, then f is minimum at (x, y) in the circular neighbourhood of (x, y) and the minimum value of f is $f(x, y)$.

Now, we study sufficient conditions for finding the point of extreme value which are as follows.

1. For the given function $f(x, y)$ find partial derivatives of $f(x, y)$ with respect to x and y, i.e. find f_x and f_y.
   Now, compare each one, just found, to zero.
   Solve those two equations for x and y.
   Find all possible combinations of values of x and y.
   Say, we have values like (x_1, y_1), (x_2, y_2), …
2. Find f_{xx}, f_{xy} and f_{yy}, i.e: partial derivatives of second order.
3. Take one point, say (x_1, y_1); evaluate f_{xx}, f_{xy} and f_{yy} at (x_1, y_1)
   Let $f_{xx}(x_1, y_1) = r$, $f_{xy}(x_1, y_1) = s$ and $f_{yy}(x_1, y_1) = t$
4. Find $\begin{vmatrix} r & s \\ s & t \end{vmatrix} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$
   i.e. find $rt - s^2$

We have following points to check the types of optimality.

1. If $(r \cdot t - s^2) > 0$ and $r < 0$, at (x_1, y_1), then $f(x_1, y_1)$ takes the maximum value at (x_1, y_1), i.e. (x_1, y_1) is a point of local maxima of the function in some neighbourhood of the point (x_1, y_1)
2. If $(r \cdot t - s^2) > 0$ and $r > 0$, at (x_1, y_1), then the function takes the minimum value at (x_1, y_1), i.e. (x_1, y_1) is a point of local minima of function in some neighbourhood of the point (x_1, y_1).
3. If $rt - s^2 < 0$ at (x_1, y_1), then $f(x_1, y_1)$ has no extreme value at (x_1, y_1). This point (x_1, y_1) is called a saddle point of $f(x, y)$
4. If $(rt - s^2) = 0$, then it requires further investigation for determining the extreme value.

At saddle point (x_1, y_1), $f(x_1, y_1)$ and $f(x_1 + h, y_1 + k)$ does **not** keep the same sign for all small values of h and k.

ILLUSTRATION 3

Find the extreme value of the following function.

$$f(x, y) = x^3 + 3x^2 - y^2.$$

Part I

1. $f_x = 3x^2 + 6x$ 2. $f_y = -2y$ 3. $f_{xx} = 6x + 6$
4. $f_{xy} = 0$ 5. $f_{yy} = -2$

Part II

Let $f_x = 0$ and $f_y = 0$

$\therefore$ $3x^2 + 6x = f_x = 0 \quad \Rightarrow \quad 3x(x + 2) = 0$

$\therefore$ $x = 0$ and $x = -2$

Let $f_y = 0 \quad \Rightarrow \quad -2y = 0 \quad \Rightarrow \quad y = 0$

The possible combinations at $x = 0$, $y = 0$, i.e. at $(0, 0)$
and $x = -2$, $y = 0$ i.e. $(-2, 0)$

Part III (A): Point (0,0)

$$r = f_{xx}(0, 0) = 6(0) + 6 = 6$$
$$s = f_{xy}(0, 0) = 0$$
$$t = f_{yy}(0, 0) = -2$$
$$rt - s^2 = (6)(-2) - (0)^2 = -12 < 0$$
$$rt - s^2 = -6 < 0$$

$\therefore$ the point (0, 0) is a saddle point. We cannot have either local maxima or local minima in the neighbourhood of this point.

$$\left[f\left(0+\frac{1}{2}, 0+\frac{1}{2}\right)\right] = f\left(\frac{1}{2}, \frac{1}{2}\right)$$
$$= \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$
$$= \frac{5}{8} > 0$$
$$f(0 + 0.1, 0 - 0.5) = f(0.1 - 0.5)$$
$$= (0.1)^3 + 3(0.1)^2 - (-0.5)2$$
$$= -219/250 < 0$$

This implies uncertain behaviour of $f(x, y)$ in the neighbourhood of (0, 0). Point (0, 0) is a saddle point.

Part III (B):

At the point (–2, 0)

$$r = f_{xx} = 6(-2) + 6 = -6 < 0$$
$$s = f_{xy} = 0$$
$$t = f_{yy} = -2$$
$$\therefore \quad rt - s^2 = (-6)(-2) - (0)^2 = 12$$
$$rt - s^2 = 12 \text{ and } r = -6 < 0$$

$\therefore$ $f(x, y)$ takes maximum value at (–2, 0).

Maximum value $= (-2)^3 + 3(-2)^2 - (0)^2$
$= 4$

$\therefore$ Maximum f at (–2, 0) = 4

ILLUSTRATION 4

Discuss the extreme value of $f(x, y) = x^3 + y^3 - 3axy$

Part I

$$f_x = 3x^2 - 3ay \qquad f_y = 3y^2 - 3ax$$
$$f_{xx} = 6x, \qquad f_{xy} = -3a, f_{yy} = 6y$$

Part II

Equate f_x and f_y with zero.

$$f_x = 3(x^2 - ay) = 0 \text{ gives } x^2 = ay$$

and $$f_y = 3(y^2 - ax) = 0 \text{ gives } y^2 = ax$$

$x^2 = ay$ gives $x^4 = a^2y^2$

$\therefore \quad x^4 = a^2(ax)$

$\therefore$ On simplification $x(x^3 - a^3) = 0$

This gives $x = 0$ or $x = a$

Corresponding to this, $y = 0$ and $a^2 = ay$

$\therefore \quad a(y - a) = 0$ giving either $a = 0$ or $a = y$

Possible combinations are the points (0, 0) and (a, a).

Part III (A)

At the point (0, 0)

$$r = f_{xx}(0, 0) = 6(0) = 0$$

$$s = f_{xy}(0, 0) = -3a$$

$$t = f_{yy}(0, 0) = 6(0) = 0$$

$$\therefore \quad rt - s^2 = (0)\,(0) - (-3a)^2$$
$$= -9a^2 < 0$$

$\therefore \quad f(x, y)$ has no extremum at (0, 0). It has a saddle point.

Part III (B)

at the point (a, a)

$$r = f_{xx}(a, a) = 6(a) = 6a$$

$$s = f_{xy}(a, a) = -3a$$

$$t = f_{yy}(a, a) = 6a$$

Now $\quad rt - s^2 = (6a)\,(6a) - (-3a)^2$
$$= 27a^2 > 0$$

Also $\quad r = 6a$

If $a > 0$ then $r > 0$ and $f(x, y)$ has local minimum at (a, a).

If $a < 0$ then $r < 0$ and $f(x, y)$ has a local maximum at (a, a).

15.4.4 Extreme Values of Functions of Three Variables

In this type, we find extreme values of functions of three variables.

We need some additional features to identify optimum as either maximum or minimum. It is about Hessian matrix which is evaluated at the point where the first order derivatives with respect to all the variables of the given function.

Hessian Matrix

1. In the case of the functions of one variable, we need to evaluate the second order derivative at the points where the first order derivative vanishes. The sign of $f''(x)$ decides the types of extremum. If $f''(x) < 0$ at $x = x_0$ then $f(x)$ is maximum at $x = x_0$.
   In the case of $f''(x) > 0$, then $f(x)$ is minimum at $x = x_0$
2. In the case of the functions of two variables, $Z = f(x_1, x_2)$, the steps towards locating points giving optimum values are as follows.
   (a) Find $\dfrac{\delta f}{\delta x_1}$ and $\dfrac{\delta f}{\delta x_2}$; equate them to zero.
   You have two equations in two variables; solve them and get the point or points like (x_0, y_0). There are chances to get different mathematical situations at such points.

(b) Now, find $\frac{\delta^2 f}{\delta x_1^2}, \frac{\delta^2 f}{\delta x_2^2}$, and $\frac{\delta^2 f}{\delta x_1 \delta x_2}$

Evaluate the above results at the points like (X_0, Y_0)

Let $\frac{\delta^2 f}{\delta x_1^2}$ evaluated at (X_0, Y_0) = say r

Let $\frac{\delta^2 f}{\delta x_1 \delta x_2}$ evaluated at (X_0, Y_0) = say s

Let $\frac{\delta^2 f}{\delta x_2^2}$ evaluated at (X_0, Y_0) = say t

(c) Now, we have the following points.

If $\frac{\delta^2 f}{\delta x^2} \cdot \frac{\delta^2 f}{\delta y^2} - \left(\frac{\delta^2 f}{\delta x \delta y}\right)^2 > 0$ and $r < 0$, the function has a maximum at (X_0, Y_0)

If $\frac{\delta^2 f}{\delta x^2} \cdot \frac{\delta^2 f}{\delta y^2} - \left(\frac{\delta^2 f}{\delta x \delta y}\right)^2 > 0$ and $r > 0$, the function has a minimum at (X_0, Y_0)

If $\frac{\delta^2 f}{\delta x^2} \cdot \frac{\delta^2 f}{\delta y^2} - \left(\frac{\delta^2 f}{\delta x \delta y}\right)^2 < 0$, the function has a saddle point at (X_0, Y_0)

If $\frac{\delta^2 f}{\delta x^2} \cdot \frac{\delta^2 f}{\delta y^2} - \left(\frac{\delta^2 f}{\delta x \delta y}\right)^2 = 0$, then it needs further investigation to take decision.

3. Now, we discuss the case of a function of three variables and then generalize it for a function of n variables.

There are three variables x_1, x_2, and x_3. Assuming that all the partial derivatives exist, necessary condition for a stationary point X_0, that makes the function the take extremum values, is that Hessuan matrix evaluated at X_0 have the following characteristics.

$$\begin{pmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 \delta x_2} & \frac{\delta^2 f}{\delta x_1 x_3} \\ \frac{\delta^2 f}{\delta x_2 \delta x_1} & \frac{\delta^2 f}{\delta x_2^2} & \frac{\delta^2 f}{\delta x_2 \delta x_3} \\ \frac{\delta^2 f}{\delta x_3 \delta x_1} & \frac{\delta^2 f}{\delta x_2 \delta x_2} & \frac{\delta^2 f}{\delta x_3^2} \end{pmatrix}$$

(a) It is positive and definite when the function attains minimum value at X_0.
(b) It is negative and definite when the function attains maximum value at X_0.

(c) Principal minors have the values when the following expressions are evaluated at the point obtained by equating first order partial derivatives to zero.

1. $\dfrac{\delta^2 f}{\delta x^2}$ is the value of the first principal minor
{Compare with $y = f(x)$ maximum at $x = x_1$ if $f''(x)$ is < 0, for $f(x)$ to be minimum $f''(x) > 0$]
2. $\dfrac{\delta^2 f}{\delta x^2} \cdot \dfrac{\delta^2 f}{\delta y^2} - \dfrac{\delta^2 f}{\delta x \delta y}$ is the value of the second principal minor
[Compare with (r).(t) – $(s^2) > 0$ and
(a) $r < 0$, then $f(x)$ is maximum
(b) $r > 0$, then f(x) is minimum
(c) $r \cdot t - (s^2) < 0$, then saddle point, etc.
3. The third principal minor is determinant of the third order matrix itself.
The signs of these minors will help you taking decisions about the type of optimality.

ILLUSTRATION 5

Find optimum value of $F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1 + 3x_2$

Solution

We find first order partial derivatives and equate them to zero.

$\dfrac{\delta f}{\delta x_1} - = 0$ gives $2x_1 + 2 = 0$, so we have $x_1 = -1$

$\dfrac{\delta f}{\delta x_2} = 0$ gives $2x_2 + 3 = 0$, so we have $x_2 = -3/2$

$\dfrac{\delta f}{\delta x_3} = 0$ gives $2x_3 = 0$, so we have $x_3 = 0$

The point is $(x_1, x_2, x_3) = (-1, -3/2, 0)$. It is a point where we expect extremum.
The Hessian matrix is as follows.

$$\begin{pmatrix} \dfrac{\delta^2 f}{\delta x_1^2} & \dfrac{\delta^2 f}{\delta x_1 \delta x_2} & \dfrac{\delta^2 f}{\delta x_1 x_3} \\ \dfrac{\delta^2 f}{\delta x_2 \delta x_1} & \dfrac{\delta^2 f}{\delta x_2^2} & \dfrac{\delta^2 f}{\delta x_2 \delta x_3} \\ \dfrac{\delta^2 f}{\delta x_3 \delta x_1} & \dfrac{\delta^2 f}{\delta x_2 \delta x_2} & \dfrac{\delta^2 f}{\delta x_3^2} \end{pmatrix}$$

All the partial derivatives to be evaluated at the point (–1, –3/2, 0)
which comes to

$$\text{H} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ which is positive and definite.}$$

The function attains minimum value at (–1, –3/2, 0).

15.5 CONSTRAINED OPTIMIZATION

As, we have discussed earlier, classical optimization techniques have been a constant interest of mathematicians.

In constrained optimization also, there are many methods which have come into lime light. No methods assure the best answers. As the degree of the variables increases and/or the number of constraints increase, the problem becomes more complicated.

We have, the very first in this class, a method known as Lagranges' undetermined multipliers for finding optimization of a given function having constraints imposed on the variables.

If the feasible region to all such constraints is a convex region, then there are chances that the extreme points give you the stationary points that makes the function optimum at these points.

15.5.1 Lagrange Method

Let us take Lagrange's function of two variables and equality type of constraints on the variables and it is required to find optimization under these constraints.

Let $f(x, y)$ be the given function of two variables

Let $g_1(x, y) = 0$, $g_2(x, y) = 0$, and $g_3(x, y) = 0$ be the three equality type constraints on the variables.

As there are three constraints, we attach them to the main function using real multiples λ_1 λ_2 and $\lambda_{3.}$ At this point of time, we do not know their values but latter on they will become known. This is the reason why we call these lambda values as undetermined multipliers.

Lagrange's function = $L(x, y, \lambda_1 \lambda_2, \lambda_3) = f(x, y) + \lambda_1 g_1(x, y) + \lambda_2 g_2(x, y) + \lambda_3 g_3(x, y)$

The *necessary conditions* for optimizations are as follows.

Partial derivatives with respect to x, y and each one of these λ must be zero.

The *sufficient conditions* are related to the values of principal minor values of the Hessian matrix.

15.5.2 Constraint Optimization and Kuhn-Tucker (K-T) conditions

We want to find X so that $z = f(X)$ is maximum. (1)

subject to m constraints $g_i(X) \le 0$ (2)

$(i = 1 \text{ to m}). \qquad X = (x_1, x_2, \ldots x_n)^T$

We assume that $f(X)$ and all $g_i(X)$ are differentiable twice.

At this stage, we have not imposed any condition on the decision variables. If we take $x_i \ge 0$, then it should be considered as one constraint and then it will need introduction of slack variable also.

[$x_i \ge 0, \therefore -x_i \le 0 \quad \therefore \quad -x_i^2 + s^2 = 0$ where $s^2 \ge 0$ is a slack variable.]

In general, we take Enequalities (1) and (2) as our main data set and work further with our objective.

As there are m conditions, as we expect equality constraints, we introduce m slack variables to convert the inequality constrains into equality.

We have $g_i(X) \le 0$ for all $i = 1$ to m.

And $\qquad X = (x_1, x_2, \ldots x_n)^T$

Converted in $g_i(X) + s_i^2 = 0 \quad i = 1$ to m (3)

This will prompt us to write the system (1) and (2) again in different form.

Find $X \in R_{n \times 1}$ so as to

Maximize $z = f(X)$ (1)

subject to $g_i(X) + s_i^2 = 0$ (3)

At this stage, we introduce $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_m)$

Lagrange's undetermined multipliers—to construct Lagrange's function denoted as L

Maximize $L(X, s, \lambda) = f(X) - \lambda(G(X) + s_i^2)$ (4)

$$\text{Where } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \qquad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}_{m\times 1} \qquad G(X) = \begin{bmatrix} g_{1(x)} \\ g_{2(x)} \\ \vdots \\ g_{m(x)} \end{bmatrix}_{m\times 1} \qquad S = \begin{bmatrix} s_1^2 \\ s_2^2 \\ \vdots \\ s_m^2 \end{bmatrix}_{m\times 1}$$

The *necessary condition* for optimization of Equation (4) are

$$\frac{\partial L}{\partial X} = 0; \ \frac{\partial L}{\partial \lambda} = 0; \text{ and } \frac{\partial L}{\partial S} = 0$$

For a point (X_0, λ_0, S_0) to be maximum or minimum.

$$\therefore \qquad \frac{\partial L}{\partial X} = \nabla f(X) - \lambda \nabla G(X) = 0$$

$$\frac{\partial L}{\partial \lambda} = -[G(X) + s^2] = 0$$

$$\text{and} \qquad \frac{\partial L}{\partial s} = -2\lambda s = 0$$

Kuhn-Tucker condition necessary to make X and λ to be stationary point for a maximization problem are

$$\nabla f(X) - \lambda \nabla G(X) = 0 \qquad (5)^*$$

[These are n conditions on m constraints.]

$$\lambda_i(g_i(X)) = 0 \qquad (6)$$

[These are m conditions.]

$$g_i(X) \le 0 \qquad \text{for all} \quad i = 1 \text{ to } m \qquad (7)$$

[These are m constraints given in Enequality (2).]

and $\lambda \ge 0$ (8)

Conditions (5), (6) ,(7) and (8) are known as Kuhn-Tucker conditions.

Note

1. Kuhn-Tucker conditions are same for a minimization problem. We have to take care to see that the pattern of expressing the constraints should not be changed at all.
2. For a minimization problem, same conditions may be applied but in that case, $\lambda \le 0$.
3. When we have m given conditions as equations all Lagrangean multipliers, i.e. λ values are unrestricted in sign.
4. If the objective function and constraints are convex functions, then the same conditions [(5) to (7)] become *sufficient conditions* for minimization.
5. If the objective function is concave and the feasible region of all the constraints is a convex set, then conditions [(5) to (7)] become *sufficient conditions* for maximization.
6. **We call condition** (5)* as primary condition and use them in quadratic programming problems.

15.5.3 Quadratic Programming Problem

Note: Quadratic programming problems make a special class of general non-linear programming problems. In this class, the objective function is quadratic and constraints are linear.

1. Quadratic programming problems are very close to linear programming problems. Only the point is that the objective function is quadratic; constraints are already up to the requirements. Only the point of quadraticity comes on the way—can we differentiate and make it linear? We will give a second thought to this idea.
2. In this section we develop a classical method of expressing a quadratic expression.

$$f(x, y) = ax^2 + by^2 + 2hxy + 2gx + 2fy + c$$

The quadratic terms are $ax^2 + 2hxy + by^2$.

This can be written as $\overline{X}^T A\overline{X}$

Where $\overline{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$ $\quad X^T = (x \;\; y)$

$$\therefore \qquad f(x, y) = (C_1, C_2)\begin{pmatrix} x \\ y \end{pmatrix} + X'A\ \overline{X}$$

$$f(\overline{x}) = C^T X + X^T A\overline{X} \tag{1}$$

is the most general format of a quadratic expression.

15.5.4 Wolfe's Method (Modified Simplex Method)

It is a very sharp method taking many turns and finally converging to the required result.
First, we describe our problem.
Find $\mathbf{X} \in R_{n \times 1}$ so as to maximize or minimize

$$Z = \mathrm{C^T\,X + X^T\,G\,X} \tag{1}$$

subject to $\mathrm{G(X)} \le b$ (2)

with $X \ge 0$ (3)

linear constraints are given by

$$\mathrm{G(X)} = \begin{pmatrix} g_1(\mathbf{X}) \\ g_2(\mathbf{x}) \\ \vdots \\ g_m(\mathbf{X}) \end{pmatrix}_{m\times 1} \le \mathbf{b} \text{ which can be converted into equality form.}$$

$\mathrm{G(X)} \le \mathbf{b}$ as $\mathrm{G(X)} - \mathbf{b} \le 0$; adding slack variables we write
$\mathrm{G(X)} - \mathbf{b} + \mathbf{s^2} = 0$

Also X ≥ 0 as –X ≤ 0 and it becomes $-X + R^2 = 0$

where $s^2 = \begin{pmatrix} s_1^2 \\ s_2^2 \\ \vdots \\ s_m^2 \end{pmatrix}_{m\times 1}$ and $R^2 = \begin{pmatrix} r_1^2 \\ r_2^2 \\ \vdots \\ r_n^2 \end{pmatrix}_{n\times 1}$

On the basis of this, we write the main problem.
Maximize (or Minimize)

$$Z = L(X, S, \lambda, \mu, r) = f(X) - \lambda(G(X) - b + s^2) - \mu(-X + r^2)$$

where $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix}_{m\times 1}$ and $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}_{n\times 1}$ are Lagrangean multipliers.

Primary Kuhn-Tucker conditions are

$$\nabla f(x) - (\lambda \ \ \mu)\, \nabla \begin{pmatrix} G(\boldsymbol{x}) \\ -\boldsymbol{X} \end{pmatrix} = 0 \text{ [where 0 is a null matrix]} \quad (4)$$

with μ and $\lambda \geq \mathbf{0}$

We will use these primary conditions along with the given conditions.
Some important points are as follows.
We will use these primary conditions along with linear constraints.

- All these primary conditions have linearity in terms of decision variables.
- The constraints are linear.
- Primary conditions are linear and do not have basic variables.
- We introduce artificial slack variables $A_1, A_2, \ldots$
- We use two phase method. In Phase I, the artificial slack variables will be removed while in Phase II, we get the feasible values of decision variables.
  [This is the reason why we said that this method takes sharp turns to reach to its destination]

Turn Steps

1. Express the given problem in the standard quadratic form.
2. Introduce slack/basic variables $S_i^{\,2} \geq 0$ in linear constraints of ≤ type (i = 1 to m)
3. Write $X \geq 0$ as $-X \leq 0$ and introduce $r_j^2 \geq 0$; n basic variables (j = 1 to n)
4. Introduce Lagrange's multipliers (λ)m multipliers with constraints and given constraints.
5. Introduce Lagrange's multipliers (μ) with non-negativity constraints on decision variables.
6. Apply Kuhn-Tucker conditions (take primary condition)
7. Primary condition need artificial slack variable (ASV) to be introduced.
8. Use two-phase simplex method with primary conditions and given constraints.
9. Once Phase I is clear, then in the last step of Phase II, you will get the feasible solution.
10. If the basis of Phase I maintains ASV, then it is a case of infeasible solution.
11. The values of the basic variables will maximize/minimize the given function and all the basic and non-basic values will also optimize Lagrange's function.

Now we take illustrations on these three methods.

ILLUSTRATION 6

Find the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the condition $x + y + z = p$ where p is any real value.

Solution

We are given $f(x, y, z,) = x^2 + y^2 + z^2$ and we want to find its minimum value subject to the condition $x + y + z = p$.

We make Lagrange's function $L(x, y, \lambda) = x^2 + y^2 + z^2 + \lambda(x + y + z - p)$ (1)

The necessary conditions for optimization are

$$\frac{\partial L}{\partial x} = 0 = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = \frac{\partial L}{\partial \lambda}$$

$\therefore$ Differentiating partially, we get $2x + \lambda = 0$; $2y + \lambda = 0$; $2z + \lambda = 0$ and $x + y + z - p = 0$

Solving each, we have $x = y = z = -\lambda/2$.

We get, $-\lambda/2 + -\lambda/2 + -\lambda/2 = p$

$$\lambda = -2p/3 \quad \therefore \quad x = y = z = p/3$$

$$x^2 + y^2 + z^2 = p^2/9 \text{ is an optimum value.}$$

[The Hessian matrix is $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ which is positive definite, so the optimum value is the minimum value]

ILLUSTRATION 7

Find the optimum value of $x^2 + y^2 + z^2$ subject to the constraint $2x + 3y + 4z = p$ where p is a real value.

Solution

We want to find the optimum value of the given function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $2x + 3y + 4z = p$.

We construct Lagrange's function

$$L(x, y, z, \lambda) = (x^2 + y^2 + z^2) + \lambda(2x + 3y + 4z - p)$$

The necessary condition for optimality is

$$\frac{\partial L}{\partial x} = 0 = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = \frac{\partial L}{\partial \lambda}$$

$\therefore \quad 2x + 2\lambda = 0,\ 3y + 2\lambda = 0,\ 4z + 2\lambda = 0$ and $2x + 3y + 4z = p$

From which we derive $x = -\lambda$, $y = -2\lambda/3$, $z = -\lambda/2$

$\therefore \quad 2(-\lambda) + 3(-2\lambda/3) + 4(-\lambda/2) = p$

$\therefore \quad \lambda = -p/6$ which gives $x = p/6$, $y = p/9$, and $z = p/12$

$\therefore \quad x^2 + y^2 + z^2 = 61p^2/1296$

[Students check the sufficiency conditions.]

ILLUSTRATION 8

Optimize $Z = 3x^2 + 5y^2$ subject to $x + 2y = 8$

Solution

We solve it using Lagrange's undetermined multipliers.

We make Lagrange's function; $L(x, y, \lambda) = Z = 3x^2 + 5y^2 + \lambda(x + 2y - 8)$ (1)

The necessary condition for optimization is

$$\frac{\partial L}{\partial x} = 0 = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda}$$

which gives $6x + \lambda = 0$, $10y + 2\lambda = 0$, $x + 2y - 8 = 0$

$\therefore \quad x = -\lambda/6$, and $y = -\lambda/5$,

Putting these values in the last condition, we get $-\lambda(1/6 + 2/5) = 8$

$\therefore \quad \lambda = -240/17$ which, in turn, gives $x = 40/17$ and $y = 48/17$

As Lagrange's conditions are necessary conditions, we need to check Hessian matrix at the point (40/17, 48/17)

In this case the Hessian is $(r).(t) - (s)^2$ and it is easy to verify the type of optimality.

ILLUSTRATION 9

Find the minimum value of $f(x) = (x_1 + 1)^2 + (x_2 - 2)^2$
subject to $g_1(x) = x_1 - 2 \le 0$; $g_2(\mathbf{x}) = x_2 - 1 \le 0$; $x_1, x_2 \ge 0$.

Solution

Language's function with λ_1 and λ_2 as Lagrangian multipliers is

$$f(x, y) = f(x_1, x_2, y_1, y_2) = [(x_1 + 1)^2 + (x_2 - 2)^2] + \lambda_1 [x_1 — 2] + \lambda_2 [x_2 - 1] \text{ as } x_1, x_2 \ge 0$$

The given function to be minimized is a convex function and for $c > 0$, it represents a circle with centre (–1, 2).

We write down Kuhn-Tucker conditions as follows:

$$\frac{\partial F}{\partial x_1} = 0 \text{ and } \frac{\partial F}{\partial x_2} = 0 \text{ gives } 2(x_1 + 1) + \lambda_1 = 0 \text{ and } 2(x_2 - 2) + \lambda_2 = 0 \quad (1)$$

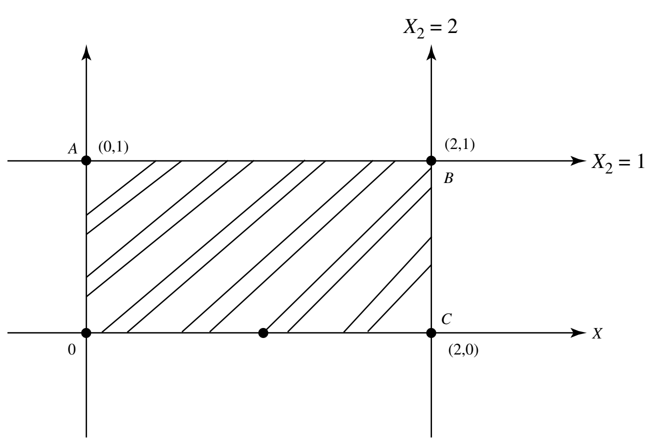
$$x_1 \frac{\partial F}{\partial x_1} = 0 \text{ and } x_2 \frac{\partial F}{\partial x_2} = 0 \text{ gives } x_1 [2(x_1 + 1) + y_1] = 0 \text{ and } x_2 [2(x_2 - 2) + y_2] = 0 \quad (2)$$

$$g_1(x) \le 0 \text{ and } g_2(x) \le 0 \text{ gives } x_1 - 2 \le 0 \text{ and } x_2 - 1 \le 0 \quad (3)$$

$$\lambda_1 \cdot g_1(x) = 0 \text{ and } \lambda_2 \cdot g_2(x) = 0 \text{ gives } \lambda_1 [x_1 - 2] = 0 \text{ and } \lambda_2 [x_2 - 1] = 0 \quad (4)$$

Also we have x_1, x_2, λ_1 and $\lambda_2 \ge 0$ (5)

Graphically $g_1(x) = x_1 - 2 \le 0$ and $g_2(x) = x_2 - 1 \le 0$ are both convex functions and $x_1, x_2 \ge 0$ implies the following region.



We have the circle with centre (–1, 2) with c = minimum radius that passes through the nearest point (0, 1) of the region O–ABC–O
So, (0, 1) is the point which minimizes the given function.
The minimum value is

$$f((0,1)) = (0 + 1)^2 + (1 - 2)^2 = 2$$

ILLUSTRATION 10

Maximize $Z = 2x_1 - x_1^2 + x_2$
subject to $2x_1 + 3x_2 \le 6$ and $2x_1 + x_2 \le 4$; $x_1, x_2 \ge 0$.

Solution

Lagrange's function is

$$L(x_1, x_2, \lambda_1, \lambda_2) = (2x_1 - x_1^2 + x_2) - \lambda_1 (2x_1 + 3x_2 - 6) - \lambda_2 (2x_1 + x_2 - 4)$$

K-T conditions are as follows:

$$\frac{\partial f}{\partial x_1} = -2x_1 + 2 - 2\lambda_1 - 2\lambda_2 = 0 \text{ and } \frac{\partial f}{\partial x_2} = 1 - 3\lambda_1 + \lambda_2 \ge 0$$

$$g_1(x) = 2x_1 + 3x_2 - 6 \le 0 \text{ and } \lambda_1 \cdot g_1(x) = 0$$

$$g_2(x) = 2x_1 + x_2 - 4 \le 0 \text{ and } \lambda_2 \cdot g_2(x) = 0$$

with $x_1, x_2, \lambda_1, \lambda_2 \ge 0$.
Now we focus our attention on

$$\lambda_1 \cdot g_1(x) = 0 \text{ and } \lambda_2 \cdot g_2(x) = 0. \ \frac{\partial f}{\partial x_1} = 0 \text{ and } \frac{\partial f}{\partial x_2} = 0$$

$$\therefore \quad [-2x_1 + 2 - 2\lambda_1 - 2\lambda_2] = 0; \text{ and } [1 - 3\lambda_1 + \lambda_2] = 0$$

$$\lambda_1 (2x_1 + 3x_2 - 6) = 0; \text{ and } \lambda_2 (2x_1 + x_2 - 4) = 0$$

We take different cases and derive the optimality.

1. $\lambda_1 = 0, \lambda_2 = 0 \Rightarrow x_2 = 0$ and $[-2x_1 + 2] = 0$ which again implies $x_1 = 1$.
2. $\lambda_1 = 0$ and $\lambda_2 \ne 0 \Rightarrow$ either $2x_1 + 3x_2 - 6 = 0$ or $2x_1 + 3x_2 - 6 \le 0$ and $2x_1 + x_2 - 4 \le 0$.
   Also $\lambda_1 = 0$ will mean $x_2 [1 + \lambda_2] = 0$ which implies $x_2 = 0$ or $\lambda_2 = -1$ which is not possible as $\lambda_2 \ge 0$.
3. $\lambda_1 \ne 0$ and $\lambda_2 = 0 \Rightarrow 2x_1 + 3x_2 - 6 = 0$; $2\lambda_1 + 2\lambda_2 = -2x_1 + 2$ and $3\lambda_1 - \lambda_2 = 1$ implies that

   $$3\lambda_1 = 1 \text{ so we have } \lambda_1 = 1/3 \ [\because \lambda_2 = 0]$$

   Putting $\lambda_1 = 1/3$ and $\lambda_2 = 0$, in the equation $2\lambda_1 + 2\lambda_2 = -2x_1 + 2$
   we have $x_1 = 2/3$ in the first equation $2x_1 + 3x_2 - 6 = 0$; $x_2 = 14/9$

   $$x_1 = 2/3 \text{ and } x_2 = 14/9$$

   These values will optimize the objective function.

ILLUSTRATION 11

Minimize $Z = 3.e^{2x_1+1} + 2 \cdot ex_2^{+5}$
subject to $x_1 + x_2 = 7$; $x_1, x_2 \ge 0$.

Solution

As the constraint has equality (=) sign, we take $-\lambda$ as Lagrange's multipliers with $\lambda \ge 0$.

This gives Lagrange's function as;

$$F(x_1, x_2, \lambda) = 3 \cdot e^{2x_1 + 1} + 2 \cdot e^{x_2 + 5} - \lambda (x_1 + x_2 - 7) \tag{1}$$

Kuhn-Tucker conditions are $\dfrac{\partial f}{\partial x_1} = 0$ and $\dfrac{\partial f}{\partial x_2} = 0$

$\therefore \quad 6 \cdot e_1^{2x+1} - \lambda = 0 \qquad (2)$

$\dfrac{\partial f}{\partial x_2} = 0$

$\therefore \quad 2 \cdot ex_2^{+5} - \lambda = 0 \qquad (3)$

$\lambda \cdot g(x) = 0$ and $g\,(x) = 0$

$\therefore \quad \lambda_1\,(x_1 + x_2 - 7) = 0$ and $x_1 + x_2 - 7 = 0 \qquad (4)$

Also gives that $x_1, x_2 \geq 0$ from (4) we conclude that $x_1 + x_2 - 7 = 0$ and $\lambda \neq 0$. Also, both x_1 and $x_2 \neq 0$.

From (2), $\lambda = 6 \cdot\ e^{2x_1+1} = 2 \cdot\ e^{x_2+5}$

Using $x_1 + x_2 = 7$, and simplifying

$$6 \cdot e^{2x_1+1} = 2 \cdot e^{7-x_1+5}$$

$\therefore \quad 3 \cdot e^{3x_1 - 11} = 1$

$\therefore \quad e^{3x_1 - 11} = 1/3$

$\therefore \quad 3x_1 - 11 = \log 1/3$

$\therefore \quad 3x_1 = 11 - \log 3 \ \therefore\ x_1 = 1/3\ [11 - \log 3].$

Using $x_1 + x_2 = 7$, we have $x_2 = 7 - 1/3\ (11) + 1/3 \log 3$

$\therefore \quad x_1 = 1/3 \cdot [11 - \log 3]$ and $x_2 = 1/3 \cdot [10 + \log 3].$

These values will minimize the objective function.

ILLUSTRATION 12

Using Kuhn-Tucker conditions, maximize $z = 2x_1^2 + 12x_1\,x_2 - 7x_2^2$
subject to $2x_1 + 5x_2 - 98 \leq 0$.

Solution

Lagrange's function is $F(x_1, x_2, y) = (2x_1^2 + 12x_1\,x_2 - 7x_2^2) - y\,(2x_1 + 5x_2 - 98) \qquad (1)$

Now K-T conditions are $\dfrac{\partial F}{\partial x_1} \geq 0$ and $x_1 \cdot \dfrac{\partial F}{\partial x_1} = 0$

$\therefore \quad 4x_1 + 12x_2 - 2y \geq 0$ and $x_1\,(4x_1 + 12x_2 - 2y) = 0 \qquad (2)$

$\dfrac{\partial F}{\partial x_2} \geq 0$ and $x_2 \cdot \dfrac{\partial F}{\partial x_2} = 0$

$\therefore \quad 12x_1 - 14x_2 - 5y \geq 0$ and $x_2\,(12x_1 - 14x_2 - 5y) = 0 \qquad (3)$

$y \cdot \text{g}\,(x) = 0$ and $\text{g}\,(x) \leq 0$

$\therefore \quad y\,(2x_1 + 5\,x_2 - 98) = 0$ and $2x_1 + 5x_2 - 98 \leq 0 \qquad (4)$

with $x_1, x_2, y \geq 0$

For $y \neq 0$, $2x_1 + 5\,x_2 - 98 = 0$

Taking $x_1, x_2 \neq 0$, $4x_1 + 12x_2 - 2y = 0$ and $12x_1 - 14x_2 - 5y = 0$

Now $\quad 2x_1 = 98 - 5x_2$

$\therefore \quad 196 - 10x_2 + 12x_2 - 2y = 0$

$\therefore \quad 2x_2 - 2y = -196 \Rightarrow x_2 - y = -98 \qquad (5)$

Also using $12x_1 - 14x_2 - 5y = 0$, we have, $588 - 44x_2 - 5y = 0$;
So we have, $44x_2 + 5y = 588$ (6)
Solving Equations (5) and (6), $5x_2 + 490 = 588 - 44x_2$
$\therefore$ $49x_2 = 98 \Rightarrow x_2 = 2$ Using $2x_1 = 98 - 5x_2$, $2x_1 = 88 \Rightarrow x_1 = 44$
Also $y = x_2 + 98$ gives $y = 100$
Using $x_1 = 44$, $x_2 = 2$ the maximum value is $2x_{12} + 12\,x_1\,x_2 - 7\,x_2^{\,2}$ is **4900.**

ILLUSTRATION 13

[Graphical solution and Verification by Kuhn-Tucker conditions and gradient search problem.]
Minimize $z = x_1^2 + x_2^2$
subject to $x_1 + x_2 \geq 4$, $2x_1 + x_2 \geq 5$; with $x_1, x_2 \geq 0$.

Solution

Lagrange's function is $F(x_1, x_2, y_1, y_2) = (x_1^2 + x_2^2) + y_1\,(x_1 - x_2 + 4) + y_2\,(-2x_1 - x_2 + 5)$ (1)
where y_1, and y_2 are Lagrange's undetermined multipliers.
with $y_1, y_2 \geq 0$
We apply Kuhn-Tucker conditions which are

$$\frac{\partial F}{\partial x_1} \geq 0 \text{ and } x_1 \cdot \frac{\partial F}{\partial x_1} = 0$$

$$\therefore \quad 2x_1 - y_1 - 2y_2 \geq 0 \text{ and } x_1\,(2x_1 - y_1 - 2y_2) = 0 \qquad (2)$$

$$\frac{\partial F}{\partial x_2} \geq 0 \text{ and } x_2 \cdot \frac{\partial F}{\partial x_2} = 0$$

$$\therefore \quad 2x_2 - y_1 - y_2 \geq 0 \text{ and } x_2\,(2x_2 - y_1 - y_2) = 0 \qquad (3)$$

$$y_1\,g_1\,(x) = 0 \text{ and } y_2 \cdot g_2\,(x) = 0$$

$$\therefore \quad y_1\,(-x_1 - x_2 + 4) = 0 \text{ and } y_2\,(-2x_1 - x_2 + 5) = 0 \qquad (4)$$

$$g_1\,(x) \leq 0 \text{ and } g_2\,(\mathbf{x}) \leq 0$$

$$\therefore \quad x_1 - x_2 + 4 \leq 0 \text{ and } -2x_1 - x_2 + 5 \leq 0 \qquad (5)$$

$$x_1, x_2, y_1, y_2 \geq 0.$$

1. Both y_1 and $y_2 = 0 \Rightarrow x_1 = 0 = x_2$ which violates condition (5).
2. $y_1 \neq 0$ and $y_2 = 0$, using these conditions in (4), $x_1 + x_2 = 4$ and $-2x_1 - x_2 + 5$ may or may not be zero.
   Now put $y_2 = 0$ in (2) and (3) with $x_2 = 4 - x_1$; so we get, $-2x_1 - y_1 = 0$; and $8 - 2x_1 - y_1 = 0$.
   $\therefore$ $-2x_1 - y_1 = 0$; and $2x_1 - y_1 = -8$ gives $y_1 = 4$ and so $x_1 = 2$ also $x_2 = 4 - x_2$ gives $x_2 = 2$;

Now, $y_2 = 0$ is already taken.
$\therefore$ Solution is $x_1 = 2$, $x_2 = 2$, $y_1 = 4$ and $y_2 = 0$.
This gives $Z = x_1^{\,2} + x_2^{\,2} = (2)^2 + (2)^2 = 8$.
Now, we find the algebraic solution of the problem
The constraint lines are $x_1 + x_2 = 4$ and $2x_1 + x_2 = 5$
We find the gradients of these lines.

$$\frac{\delta x_2}{\delta x_1} = -1 \text{ and } \frac{\delta x_2}{\delta x_1} - = -2$$

Considering $x_1^2 + x_2^2 = k$ where $k \in R_+$

$$2x_1 + 2x_2 \cdot \frac{\delta x_2}{\delta x_1} = 0 \therefore \frac{dx_2}{dx_1} = -\frac{x_2}{x_1}$$ (slope of tangent) comparing $\frac{-x_1}{x_2}$ with the slopes of given lines.

$$\frac{dx_2}{dx_1} = \frac{-x_1}{x_2} = -1 \text{ and } \frac{dx_2}{dx_1} = \frac{-x_1}{x_2} = -2$$

$\therefore$ $x_1 - x_2 = 0$ and $x_1 - 2x_2 = 0 \Rightarrow x_1 = x_2$ and $x_1 = 2x_2$

Now $x_1 = x_2$ is the relation corresponding to $x_1 + x_2 = 4$ and $x_1 = 2x_2$ is the relation corresponding to $-2x_1 - x_2 + 5$ for the first one we have (2, 2) as a point and the second one gives $x_2 = 1$, $x_2 = 2$ and the point is (2, 1) . The point (2, 1) is not in the feasible region.

This means that the point (2, 2) is the point which satisfies both the constraints and minimizes $z = x_1^2 + x_2^2$. The following graph explains the situation.

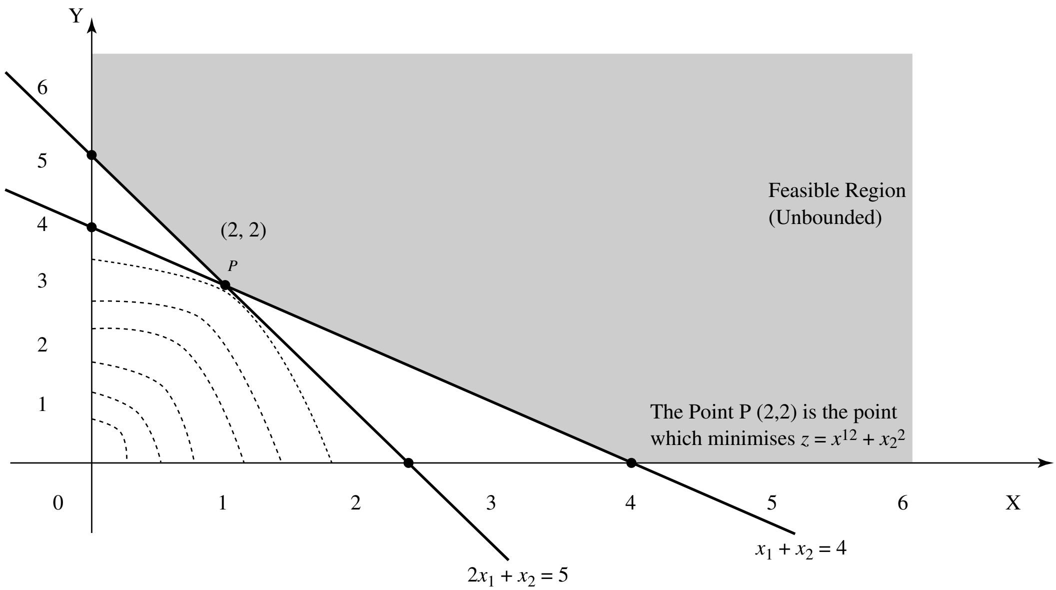


ILLUSTRATION 14

(Graphical Solution and Verification]
Maximize $Z = 2x + 3y$ subject to $x^2 + y^2 \leq 20$; $xy \leq 8$; $x, y \geq 0$.
Find the graphical solution and verify using Kuhn-Tucker conditions.

Solution

$x^2 + y^2 \leq 20$ is a **convex** region and the region shown by $xy \leq 8$ is not a convex region.

[(1, 8), (2,4), (4, 2), and (8, 1) are the integral points on the curve. We are interested in the region for which $xy < 8$.

Points in this region and the line segment joining them is not in the region $xy \leq 8$. For example, the mid-point of the segment is (9/2, 9/2) which does not satisfy $xy \leq 8$.]

Now, we obtain the points of intersection of $x^2 + y^2 \leq 20$ and $xy \leq 8$.

$\therefore$ $x^2 + (8/x)^2 = 20$

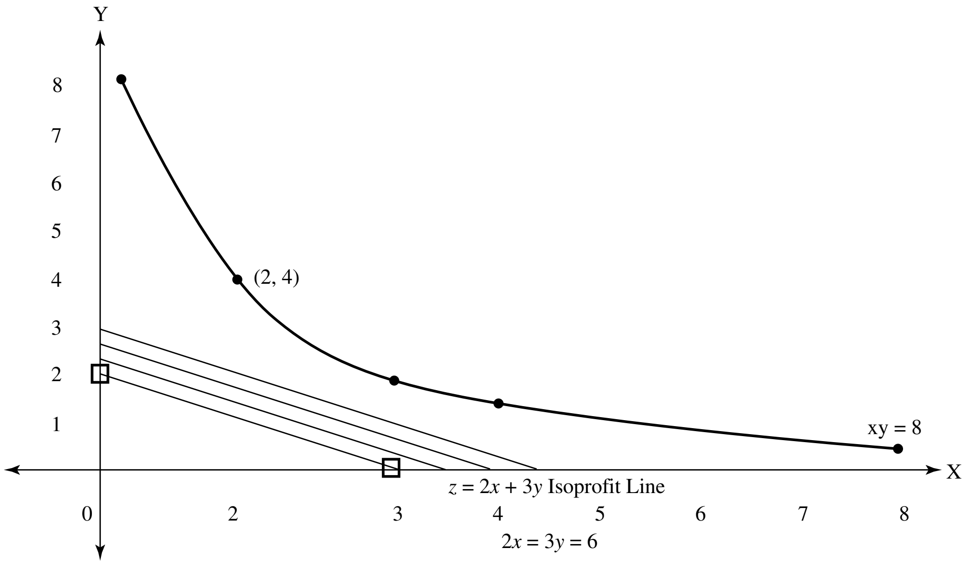
$\therefore$ $x^4 - 20x^2 + 64 = 0 \Rightarrow (x^2 - 16)(x^2 - 4) = 0.$

This gives $x^2 = 16 \Rightarrow x = 4$
and from $xy = 8$; $y = 2$, point (4, 2).

In the graph, point A = A (4, 2) and similarly solving $x^2 = 4$, we get the point B (2, 4).

The value of $2x + 3y$ for the point A is 2 (4) + 3 (2) = 14 and that for due point B is 2 (2) + 3 (4) = 16.

It implies that the point (2, 4) under the given constraints maximizes $2x + 3y$.



The Lagrange's function is $F(x, y, \lambda_1, \lambda_2) = (2x + 3y) + \lambda_1 (x^2 - y^2 - 20) + \lambda_2 (xy - 8)$ (1)

K-T conditions are $\dfrac{\partial F}{\partial x} = 0$ and $\dfrac{\partial F}{\partial y} = 0$

$\therefore \quad 2 + 2x\,\lambda_1 + \lambda_2\, y = 0$ and $3 - 2y\,\lambda_1 + \lambda_2\, x = 0$ (2)

$\lambda_1\, g_1(x) = 0$ and $\lambda_2\, g_2(x) = 0$

$\therefore \quad \lambda_1 (x^2 + y^2 - 20) = 0$ and $\lambda_2 (xy - 8) = 0$ (3)

$g_1(x) \le 0$ and $g_2(x) \le 0$

$\therefore \quad x_2 + y_2 - 20 \le 0$ and $xy - 8 \le 0$ (4)

with $x, y \ge 0$ since λ_1 and λ_2 are unrestricted as the region $xy - 8 \le 0$ is not convex, for $\lambda_1, \lambda_2 \ne 0$ we derive from (3) $(x^2 + y^2 - 20) = 0$ and $(xy - 8) = 0$ ($\therefore\ y = 8/x;\ x \ne 0$) gives $x_2 + 64/x^2 - 20 = 0$;

$\therefore \quad x^4 - 20x^2 + 64 = 0 \Rightarrow (x^2 - 16)(x^2 - 4) = 0$

$\therefore \quad x = 4$ or $x = 2 \Rightarrow y = 8/4 = 2$ and $y = 8/2 = 4$

The solutions are (4, 2) and (2, 4).

The point (4, 2) gives $2x + 3y = 2(4) + 3(2) = 14$;

The point (2, 4) gives $2x + 3y = 2(2) + 3(4) = 16$;

The point (2, 4) maximizes $Z = 2x + 3y$

ILLUSTRATION 15

[Separable Programming Problem]

Minimize $Z = 4x_1^2 - 8x_1 + 2x_2^2 - 3x_2 + 5x_3^2 + 5x_3 + 10$

Solution

This function can be separated in three different functions of one variable only.

$$f_1(x_1) = 4x_1^2 - 8x_1$$

$$f_2(x_2) = 2x_2^2 - 3x_2$$

$$f_3(x_3) = 5x_3^2 + 5x_3 + 10$$

$$\therefore \quad Z = f_1(x_1) + f_2(x_2) + f_3(x_3)$$

We treat each function separately. Also, the first look at the second order derivative of each function with its own respective variable conveys that all are convex functions.

(1) $\dfrac{\partial f_1}{\partial X_1} = 8x_1 - 8$ and $\dfrac{\partial^2 f_1}{\partial X_1^2} = 8 > 0$

(2) $\dfrac{\partial f_2}{\partial X_2} = 4x_2 - 3$ and $\dfrac{\partial^2 f_2}{\partial x_2^2} = 4 > 0$

(3) $\dfrac{\partial f_s}{\partial X_s} = 10x_3 + 5$ and $\dfrac{\partial^2 f_s}{\partial x_s^2} = 10 > 0$

Since the second order derivatives are > 0 which implies convexity. $Z = f(x_1, x_2, x_3)$ is the sum function of $f_1(x_1)$ and $f_2(x_2)$ and $f_3(x_3)$. This is defined on common domain of all the three functions.

Equating each first order partial derivatives to zero, we get

$$\frac{\partial f_1}{\partial X_1} = 8x_1 - 8 = 0 \quad \Rightarrow \quad x_1 = 1$$

$$\frac{\partial f_2}{\partial X_2} = 4x_2 - 3 = 0 \quad \Rightarrow \quad x_2 = \frac{3}{4}$$

$$\frac{\partial f_s}{\partial X_s} = 10x_3 + 5 = 0 \quad \Rightarrow \quad x_3 = \frac{-1}{2}$$

[Also for each second order, partial derivatives are positive]

These values, when we substitute in the function, we have

$$Z = 4(1)^2 - 8(1) + 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + 5\left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right) + 10$$

$$Z = \frac{29}{8}$$

ILLUSTRATION 16

Quadratic Programming Problem] Wolfe's Method
Solve the following problem by Wolfe's method.

Maximize $Z = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 6x_1 + 6$
subject to $x_1 + x_2 \le 2$ and $x_1, x_2 \ge 0$

Solution

Looking at the constraints, we make them feasible for working with simplex style.

$$s_1^2 \ge 0 \text{ makes } x_1 + x_2 + s^2 - 2 = 0$$
$$x_1 \ge 0 \text{ makes } -x_1 + r_1^2 = 0$$
$$x_2 \ge 0 \text{ makes } -x_2 + r_2^2 = 0$$

It is obvious that $s_1^2, r_1^2, r_2^2 \ge 0$.

The Lagrange's function is

$$L(x_1, x_2, \lambda_1, s_1, \mu_1, r_1, \mu_2, r_2)$$
$$= (2x_1^2 - 2x_1x_2 + 2x_2^2 - 6x_1 + 6) - \lambda(x_1 + x_2 + s^2 - 2) - \mu_1(-x_1 + r_1^2) - \mu_2(-x_2 + r_2^2)$$

[these are eight variables and three equality conditions]

The necessary condition will imply.

$$\frac{\partial L}{\partial x_1} = 4x_1 - 2x_2 - 6 - \lambda + \mu_1 = 0$$

$$\frac{\partial L}{\partial x_2} = -2x_1 + 4x_2 - \lambda + \mu_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 + s^2 - 2 = 0$$

$$\frac{\partial L}{\partial s} = 2\,\lambda s = 0$$

$$\frac{\partial L}{\partial \mu_1} = -x_1 + r_1^2 = 0$$

$$\frac{\partial L}{\partial \mu_2} = -x_2 + r_2^2 = 0$$

$$\frac{\partial L}{\partial r_1} = 2\mu_1 r_1 = 0 \text{ and}$$

$$\frac{\partial L}{\partial r_2} = 2\mu_2 r_2 = 0$$

First three of these are primary conditions on simplification they are

$$4x_1 - 2x_2 - \lambda + \mu_1 = 6$$
$$2x_1 - 4x_2 + \lambda - \mu_2 = 0$$
$$x_1 + x_2 + s^2 = 2$$

Now, we have to add artificial slack variable A_1 and A_2 in the first two (above) conditions. In the third conditions, $s^2 \ge 0$ can work as a basic variable.

The corresponding LPP is minimize $Z = A_1 + A_2$
subject to

$$4x_1 - 2x_2 - \lambda + \mu_1 + A_1 = 6$$
$$2x_1 - 4x_2 + \lambda - \mu_2 + A_2 = 0$$
$$1x_1 + 1x_2 + s^2 = 2$$

with all variables ≥ 0
Now, we apply two-phase method.
Phase 1

Minimize

Table 1

$C_j \rightarrow$ / VAR / $\downarrow$Basis	0 x_1	0 x_2	0 λ	0 μ_1	0 μ_2	0 s^2	1 A_1	1 A_2	X_B	R.R.
1 A_1	4	–2	–1	1	0	0	1	0	6	3/2
1 $\leftarrow A_2$	2	–4	1	0	–1	0	0	1	0	0 $\rightarrow$
0 s^2	1	1	0	0	0	1	0	0	2	2
$C_j - Z_j$	–6 $\uparrow$	6	0	0	1	0	0	0	6	

Comment: One basic variable is zero, it is a sign of degenerate solution.
The most negative of $C_j - Z_j$ entry is –6 which corresponds to the variable x_1; which is an entering variable in the next iteration.

Table 2

$C_j \rightarrow$ / VAR / $\downarrow$Basis	0 x_1	0 x_2	0 λ	0 μ_1	0 μ_2	0 s^2	1 A_1	X_B	R.R.
1 A_1	0	6	–1	1	2	0	1	6	1
0 $\leftarrow x_1$	1	–2	1/2	0	–1/2	0	0	0	---*-
0 s^2	0	3	–1/2	0	1/2	1	0	2	2/3 $\rightarrow$
$C_j - Z_j$	0	–6 $\uparrow$	3	–1	–2	0	0	Z = 6	

* R.R. is undefined.

Table 3

$C_j \rightarrow$ / VAR / $\downarrow$Basis	0 x_1	0 x_2	0 λ	0 μ_1	0 μ_2	0 s^2	1 A_1	X_B	R.R.
1 $\leftarrow A_1$	0	0	–2	1	1	–2	1	2	2 $\rightarrow$
0 x_1	1	0	1/6	0	–1/6	2/3	0	4/3	---*-
0 x_2	0	1	–1/6	0	1/6	1/6	0	2/3	4
$C_j - Z_j$	0	0	2	2	–1 $\uparrow$	2	0	2	

* R.R. is undefined.

Table-4

$C_j \rightarrow$ / VAR / $\downarrow$Basis	0 x_1	0 x_2	0 λ_1	0 μ_1	0 μ_2	0 s^2	1 A_1	X_B	R.R.
$0 \leftarrow \mu_2$	0	0	–2	1	1	–1	–	2	
0 x_1	1	0	–1/3	1/6	0	1/6	–	5/3	
0 x_2	0	1	1/3	–1/6	0	1/3	–	1/3	
$C_j - Z_j$	0	0	0	0	0	0	–	$Z = 0$	

Comment: All $C_j - Z_j$ entries are zero. Phase I is over.

Basic variables: $x_1 = \dfrac{5}{3}$; $x_2 = \dfrac{1}{3}$; $\mu_2 = 2$

Non-basic variables: $\lambda_2, \mu_2, \lambda_2, s^2 = 0$

$$\text{Maximum } Z = 2\left(\frac{5}{3}\right)^2 - 2\left(\frac{5}{3}\right)\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)^2 - 6\left(\frac{5}{3}\right) + 6$$

$$= \frac{50}{9} - \frac{10}{9} + \frac{2}{9} - 4 = 2/3$$

ADDITIONAL QUESTIONS FOR PRACTICE (WITH HINTS AND ANSWERS)

Question 1

Solve the following using Kuhn-Tucker conditions.

Find y_1, y_2, y_3 so as to minimize $Z = (y_1)^2 + 2(y_2)^2 + 3(y_3)^2$

subject to $-y_2 - y_3 + 6 \leq 0$, $-y_1 + 2 \leq 0$, and $-y_2 + 1 \leq 0$

Solution

We construct Lagrange's function L by introducing slack variables s_1^2, s_2^2, and s_3^2 all ≥ 0

$$L(y_1, y_2, y_3, \lambda_1, \lambda_2, \lambda_3, s_1^2, s_2^2, s_3^2)$$
$$= (y_1)^2 + 2(y_2)^2 + 3(y_3)^2 - \lambda_1(-y_2 - y_3 + 6 + s_1^2) - \lambda_2(-y_1 + 2 + s_2^2) - \lambda_3(-y_2 + 1 + s_3^2)$$

For optimization Kuhn-Tucker conditions are satisfied.

$$\frac{\delta L}{\delta X} = 0, \ \frac{\delta L}{\delta \lambda} = 0, \ \lambda G(X) = 0, \ G(X) \leq 0$$

Applying these conditions on Lagrangean function, we get the following conditions:

$$\frac{\delta L}{\delta x_1} = 2y_1 + \lambda_2$$

$$\frac{\delta L}{\delta x_2} = 4y_2 + \lambda_1 + \lambda 3 = 0$$

$$\frac{\delta L}{\delta x_3} = 6y_3 + y_1 = 0$$

$$\lambda_1 \cdot g_1(X) = 0 \quad \therefore \quad \lambda_1 \cdot (6 - y_2 - y_3) = 0$$
$$\lambda_2 \cdot g_2(X) = 0 \quad \therefore \quad \lambda_2 \cdot (2 - y_1) = 0$$
$$\lambda_3 \cdot g_3(X) = 0 \quad \therefore \quad \lambda_3 \cdot (1 - y_2) = 0$$

with all given constraints and very important is that all λ values are non-positive.{ This is required for a minimization problem.]

Some Hints:

If $\lambda_2 \neq 0$, then we have $y_1 = 2$ then from $2y_1 + \lambda_2 = 0$ we have $\lambda_2 = -4$ (1)

If $\lambda_3 \neq 0$, then $y_2 = 1$ then we have $4(-1) + \lambda_1 + \lambda_3 = 0 \Rightarrow \lambda_1 + \lambda_3 = -4$

If $\lambda_3 = 0$, then $4y_2 = -\lambda_1 = 6y_3$ from the main equations.

This gives a relation between y_2 and y_3. We have $y_2 = (3/2)y_3$

Now, either $\lambda_1 = 0$ or $y_2 + y_3 = 6$, using the above relation $y_2 = (3/2)y_3$ we get

$$y_2 = 12/5$$

From this we have $y_2 = 18/5$. Also this will give $\lambda_1 = -4y_2 = -14.4$

Finally we have $y_1 = 2$, $y_2 = 18/5$, and $y_3 = 12/5$ with $\lambda_1 = -14.4$ $\lambda_2 = -4$ and $\lambda_3 = 0$ and minimum $Z = 47.2$

Question 2

Find the graphical solution of the problem and verify by Kuhn-Tucker conditions.

Maximize $Z = -x_1^2 - x_2^2 + 8x_1 + 8x_2$;

subject to $x_1 + x_2 \leq 12$

$x_1 - x_2 \geq 4$

and $x_1, x_2 \geq 0$.

Solution

The problem can be represented as find x_1 and x_2 so as to minimize $Z' = x_1^2 + x_2^2 - 8x_1 - 8x_2$;

subject to $x_1 + x_2 \leq 12$

$-x_1 + x_2 + 4 \leq 0$

$x_1, x_2 \geq 0$.

The corresponding Lagrange's function is;

$$F = (x_1, x_2, y_1, y_2) = (x_1^2 + x_2^2 - 8x_1 - 8x_2) + y_1(x_1 + x_2 - 12) + y_2(-x_1 - x_2 + 4) \quad (1)$$

where y_1 and y_2 are Lagrange's undetermined multipliers. In the final maximization we get the value of each one a non-negative.

Now differentiating partially with respect to the given variables, we have

$$\frac{\partial F}{\partial x_1} = 2x_1 - 8 + y_1 - y_2 = 0;$$

$$\frac{\partial F}{\partial x_2} = 2x_2 - 8 + y_1 + y_2 = 0 \quad (2)$$

$$y_1 \, g_1(x) = 0; y_2 \, g_2(x) = 0 \quad \Rightarrow \quad y_1(x_1 + x_2 - 12) = 0 \quad \Rightarrow \quad y_2(-x_1 - x_2 + 4) = 0 \quad (3)$$

$g_1(x) = 0$; $g_2(x) = 0$ These are the given constraints.

$$\Rightarrow \quad x_1 + x_2 - 12 \leq 0; -x_1 - x_2 + 4 \leq 0 \quad (4)$$

with $x_1, x_2, y_1, y_2 \geq 0$.

Case 1

$y_1 \neq 0, y_2 \neq 0 \rightarrow x_1 + x_2 - 12 = 0$ and $-x_1 - x_2 + 4 = 0$ solving them we get $x_1 = 8, x_2 = 4$.

Now using these values in (1), $y_1 - y_1 = -8$ and $y_1 + y_1 = 0$ which give $y_1 = -4$. This is not possible as $y_1 \geq 0$.

Case 2

$y_1 \neq 0, y_2 = 0 \Rightarrow x_1 + x_2 - 12 = 0$ and from (1) we get $2x_1 + y_1 = 8$ and $2x_1 + y_1 = 8$

We have $2x_1 - 2x_2 = 0 \Rightarrow x_1 = x_2$ and $x_1 + x_2 - 12 = 0$ gives $x_1 = 6 = x_2$.

The point is (6, 6)

Again (1) gives, $2\,(6) - 8 + y_1 = 0 \Rightarrow y_1 = -4$ (as $y_2 = 0$) which is not feasible as $y_1 \geq 0$.

Case 3

Let $y_1 = 0, y_2 \neq 0$; this gives $-x_1 - x_2 + 4 = 0 \Rightarrow x_1 = x_2 + 4$

Also from (1), we have $2x_1 - 8 - y_2 = 0$ and $2x_2 - 8 + y_2 = 0$;

$\Rightarrow \quad 2x_1 - y_2 = 8$ and $2x_1 + y_2 = 8 \quad \Rightarrow \quad x_1 + x_2 = 8$.

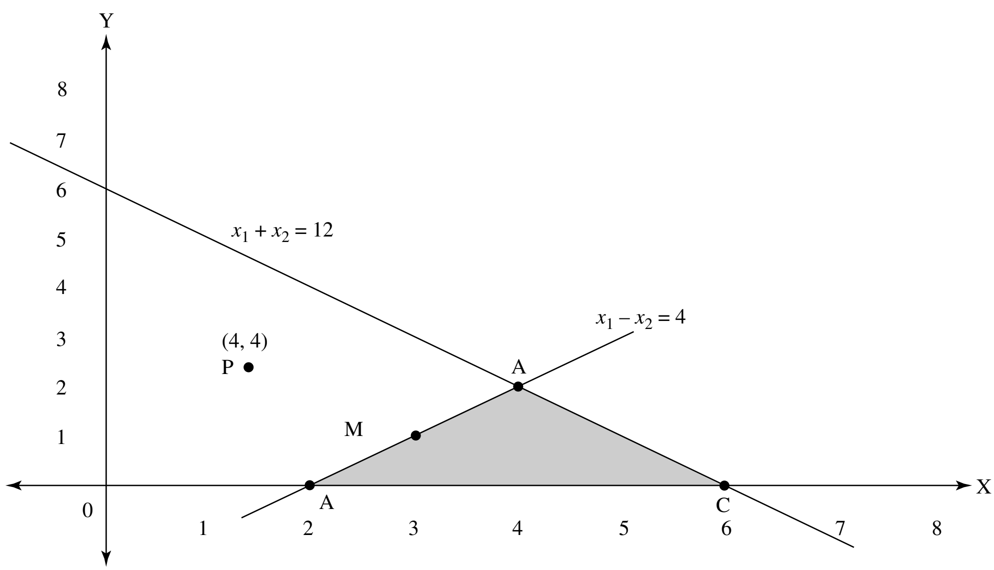
Using $x_1 = x_2 + 4$, we have $x_1 = 6$ and $x_2 = 2$.

These values, i.e. $x_1 = 6, x_2 = 2, y_1 = 0, y_2 = 4$ are the values satisfying all the constraints.

This will minimize $Z' = x_1^2 + x_2^2 - 8x_1 - 8x_2 \quad \Rightarrow \quad (6)\,2 + (2)\,2 - 8\,(6) - 8\,(2) = -24$

As required in the original problem, the maximum value of the objective function is **24**.

$[x_1 = 6, x_2 = 2, y_1 = 0, y_2 = 4]$



[Assuming point M, on the line $x_1 - x_2 = 4$, that makes PM minimum can be obtained using concepts of coordinate geometry. This point is M = M (6, 2)]

POINTS TO REMEMBER

General ideas and basics

In this chapter, there are classical optimization techniques. We try to simplify mathematical facts and try to express in a simpler way.

In a given problem/situation, we have some objective either maximization or minimization and you are required to achieve the same remaining within the constraints on the given resources.

If the objective function and the given constraint are linear functions of the decision variables then we have linear programming problems.

- If either the objective function or the constraints or both are non-linear then we call such problems as non-linear programming problems. General non-Linear Programming Problems denoted as **GNLPP**.
- If only the objective function is non-linear and all the constraints are linear then, it is called a *quadratic programming problem*.

Non-linear programming problems can be classified in two categories.

(A) Unconstrained GNLPP

(B) Constrained GNLPP

(A) Unconstrained GNLPP

In unconstrained GNLPP we have some standard techniques of finding optimum values.

We find first order partial derivatives of the function with respect to the given decision variables and equate them to zero. We have, because there are **n** variables in the given function; there are **n** number of such equations. We solve them and find the values of the variables which [like ($x_1, x_2, x_3, \ldots, x_n$)] are called *stationary points*. At these points the function can possess optimum values but you cannot classify them as maximum or minimum. Then we find second order derivatives of such functions and the Hessian matrix when evaluated at these points will let you know about the type of optimality.

In the case of a function of two variables $Z = f(x_1, x_2)$,

the steps towards locating points giving optimum values are as follows.

(a) find $\frac{\delta f}{\delta x_1}$ and $\frac{\delta f}{\delta x_2}$; equate them to zero.

You have two equations in two variables; solve them and get point or points like **(X_0, Y_0).** There are chances to get different mathematical situations at such points.

(b) Now, find $\frac{\delta^2 f}{\delta x_1^2}$, $\frac{\delta^2 f}{\delta x_2^2}$, and $\frac{\delta^2 f}{\delta x_1 \delta x_2}$

Evaluate the above results at the points like (X_0, Y_0)

Let $\frac{\delta^2 f}{\delta x_1^2}$ evaluated at (X_0, Y_0) = say r,

Let $\frac{\delta^2 f}{\delta x_1 \delta x_2}$ evaluated at (X_0, Y_0) = say s,

Let $\frac{\delta^2 f}{\delta x_2^2}$ evaluated at (X_0, Y_0) = say t,

(c) Now, we have the following points.

If $\frac{\delta^2 f}{\delta x^2} \cdot \frac{\delta^2 f}{\delta y^2} - \left(\frac{\delta^2 f}{\delta x \delta y}\right)^2 > 0$ and $r < 0$, the function has a maximum at (X_0, Y_0)

If $\frac{\delta^2 f}{\delta x^2} \cdot \frac{\delta^2 f}{\delta y^2} - \left(\frac{\delta^2 f}{\delta x \delta y}\right)^2 > 0$ and $r > 0$, the function has a minimum at (X_0, Y_0)

If $\frac{\delta^2 f}{\delta x^2} \cdot \frac{\delta^2 f}{\delta y^2} - \left(\frac{\delta^2 f}{\delta x \delta y}\right)^2 < 0$, then the function has a saddle point at (X_0, Y_0)

If $\frac{\delta^2 f}{\delta x^2} \cdot \frac{\delta^2 f}{\delta y^2} - \left(\frac{\delta^2 f}{\delta x \delta y}\right)^2 = 0$, then it needs further investigation to take decision.

(B) Constrained Optimization

Lagranges Undetermined Multipliers

You have to construct Lagrange's function by associating all constraints with undetermined multipliers.

1. $L(\boldsymbol{X}, \lambda_1, \lambda_2, \ldots, \lambda_m)$
   There are n components in $\boldsymbol{X}$ and m constraints on variables.
   The number of constraints equals the number of multipliers. Express all constraints in the form $g(\boldsymbol{X}) = 0$
   [If $g(\boldsymbol{X}) \leq b$, then $g(\boldsymbol{X}) - b + s^2 = 0$]
2. Find partial derivatives of $L(\boldsymbol{X}, \lambda_1, \lambda_2, \ldots, \lambda_m)$ with each variable.
   This gives n equations of original variables and m equations of the constraints.
   Solving all these, we get the values of the variables and multipliers (now they are determined).
   The conditions found and established are *necessary conditions* for optimality.

(C) Kuhn–Tucker Conditions

We have Kuhn-Tucker conditions, in some cases, the conditions are also sufficient conditions.

More or less we work on the lines Lagrange's undetermined multipliers techniques and do some additional work and derive Kuhn-Tucker conditions.

Maximize $L(X, S, \lambda) = f(X) - \lambda(G(X) + s_i^2)$

Kuhn-Tucker conditions necessary to make $\bar{X}$ and $\bar{\lambda}$ to be stationary point for a maximization problem are,

$$\nabla f(X) - \lambda \nabla G(X) = 0 \qquad (5)^*$$

[these are n condition on m constraints]

$$\lambda_i(g_i(X)) = 0 \qquad (6)$$

[these are m conditions]

$$g_i(X) \leq 0 \text{ for all } i = 1 \text{ to } m \qquad (7)$$

[These are m constraints given in (2)]

And
$$\lambda \geq 0 \qquad (8)$$

Conditions (5), (6) ,(7) and (8) are known as Kuhn-Tucker conditions.

Note:

1. Kuhn-tucker conditions are the same for a minimization problem. We have to take care to see that the pattern of expressing the constraints should not be changed at all.
2. For a minimization problem, same conditions may be applied but in that case, $\lambda \leq 0$.

(D) Wolfe's Method (Quadratic Programming)

This is, as it looks, is a very elegant methods and you can make a brief revision of important topics of all linear and non-linear programming.

Steps are as follows:

1. Confirm that it is a quadratic programming.
2. Construct Lagrange's function. Do not spare decision variables too. For example, $x \geq 0$, is $-x \leq 0$, and then adding slack variable, it is converted to equality; $-x + r^2 = 0$. This is for each given variable having non-negativity constraint.
3. Apply Kuhn-Tucker method and obtain primary conditions. They are in equality form. Add artificial slack variables (ASV) to each one of such equality constraints.
4. Write the simplex table with objective function as
   Minimize Z = sum of all such artificials slack variables just introduced.
5. Apply first phase (Phase-I) of the two-phase method by simplex algorithm.
6. Successful completion is when all ASV are removed. This stage will give you the values of all basic variables which will maximize the given function and Lagrange's function too.

EXERCISES

OBJECTIVE TYPE QUESTIONS

I. State True or False.

1. A linear function is always a concave function.
2. In the case of $y = f(x)$, when $f'(x) = 0$ we have points of inflexion.
3. In the function like $z = f(x, y)$, local maxima is also a global maxima.
4. Lagrange's undetermined multipliers are added to each constraints.
5. At the end of Lagrange's method when optimum is obtained the multipliers become known and real value.
6. In general non-linear programming problems, all constraints must be non-linear.
7. Lagrange's methods of optimization give only necessary conditions.
8. Hessian matrix in the case of function of two variations is $f_{xx} \cdot f_{yy} - (f_{xy})^2$.
9. When $f_{xx} \cdot f_{yy} - (f_{xy})^2 = 0$, at the point (x_0, y_0); the function has a saddle point.
10. We apply Woffe's method in the case of any non-linear programming problem.
11. Kuhn-Tucker method depends on the construction of Lagrange's function and in most of the cases it gives sufficient conditions also.

ANSWERS

1. false.	2. false.	3. true.	4. false.	5. true.	6. false.
7. true.	8. true.	9. false.	10. false.	11. true.	

II. Multiple Choice Questions

1. $y = f(x) = 2x^2 + 3$ is always a
   (a) concave (b) linear (c) convex (d) constraint

For Questions (2), (3) and (4) for a function $\dfrac{dy}{dx} = (x - 3)(2x + 1)$.

2. For $x \in (3, \infty)$ the function is
   (a) monotonic non-decreasing (b) increasing
   (c) non-differentiable (d) has tangents parallel to x-axis

3. The function is
   (a) concave (b) convex
   (c) having a saddle point (d) need more investigation of intervals of x
4. The function has optimum values
   (a) at $x = 3$ only (b) at $x = -1/2$ only
   (c) at $(3, -15/2)$ (d) at $(0, 0)$

For Questions (5), (6) and (7) for a function $f_x = 2x - y$ and $f_y = 2y - x$.

5. The functions has optimum at
   (a) $(1, 1)$ (b) $(0, 0)$ (c) $(1, 2)$ (d) $(2, 1)$
6. What can you decide about the extremum?
   (a) Cannot be decided as f_{xy} is not known (b) It is maximum
   (c) It is minimum (d) Requires further investingation
7. Does the original curve pass through (0, 0)?
   (a) Yes (b) No
   (c) Cannot be said (d) No but it passes through (1, 1)

   If $f(x_1, x_2, x_3) = 2x_1^2 - 3x_1x_2 + x_2 x_3 + x^2$ then
8. the principal minor of the first order is
   (a) $4x_1$ (b) 4 (c) $-3x_2$ (d) 0
9. the principal minor of second order is
   (a) $\begin{bmatrix} 4 & -2 \\ 1 & +1 \end{bmatrix}$ (b) $\begin{bmatrix} +4 & -3 \\ -3 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix}$ (d) $\begin{bmatrix} -3 & 4 \\ -3 & 4 \end{bmatrix}$

   If $f(x) = 2x_1^2 + 3x_2^2 + 4x_1 x_2$ to be optimized under $x_1 + x_2 - 7 = 0$;
10. Lagrange's function is
    (a) $L = 2x_1^2 + 3x_2^2 + x_1 + x$ (b) $L = 4x_1 + 6x_2 - 5 = 0$
    (c) $L = 2x_1^2 + 3x_2^2 + \gamma(x_1 + x_2)$ (d) $L = 2x_1^2 + 3x_2^2 + \gamma(x_1 + x_2 - 7 + s^2)$
11. Optimal solution in the case of a function of two variables
    (a) exists on the boundary of feasible region
    (b) exists inside the feasible region
    (c) no assurance can be given
    (d) given function is concave and so no optimal solution.

Answers

1. (c)	2. (a)	3. (d)	4. (c)	5. (b)	6. (d)
7. (c)	8. (b)	9. (b)	10. (d)	11. (c)	

Numerical Problems

Examine the following functions for **convexity;**
1 through 3

1. $f(x) = 3x^3 + 2x^2 - 8x + 4$ in the neighborhood of $x = 1$.
   $f(x) = -4x^3 + 3x^2 + 8x - 5$ in neighborhood of $x = -1$
2. $f(x_1, x_2) = x_1^2 + x_2^2 + 4x_1x_2 - 5x_1 + 6x_2$
3. $f(x_1, x_2) = 2x_1^2 + 2x_2^2 + x_1x_2$
4. Find three parts of 12 such that their product is maximum. [solve this as a function of two variables.]

5. Find three parts of 12 such that their product is maximum. [Use Lagrange's undetermined multipliers.]
6. Use **separable programming** for the following example.
   $f(x_1, x_2, x_3) = 2x_1^3 + 4x_1^2 + 3x_2^3 + 2x_2^2$

Solve the following problems using **Lagrange's multipliers**.

7. Optimize $Z = y_1^2 + 2y_2^2 + y_3^2$ subject to $2y_1 + y_2 + y_3 = 30$
8. Optimize $Z = 6y_1y_2 - 10y_3$ subject to $2y_1 + y_2 + 3y_3 = 10$
9. Optimize $Z = 4y_1 + 2y_2^2 + y_3^2 - 4y_1y_2$ subject to $y_1 + y_2 + y_3 = 15$; $2y_1 - y_2 + 2y_3 = 20$
10. Optimize $Z = -x_1^2 - x_2^2 + 8x_1 + 10x_2$ subject to $3x_1 + 2x_2 - 6 = 0$

Solve the following problems by applying **Kuhn-Tucker conditions**.

11. Optimize $Z = x_1^2 - x_2^2 + 8x_1 + 10x_2$ subject to $3x_1 + 2x_2 = 6$
12. Maximize $Z = 2x_1^2 - 6x_1 - 2x_1 + 2x_1x_2 + 2x_2^2 + 6$ subject to $x_1 + x_2 \le 2$; $x_1 + x_2 \ge 0$
13. Minimize $Z = -x_1^2 - 2x_1^2 + 2x_1 + 3x_2$ subject to $x_1 + 3x_2 \le 6$; $5x_1 + 2x_2 \le 10$
14. Maximize $Z = -x_1^2 + 2x_1 + x_2$ subject to $2x_1 + 3x_2 \le 6$; $2x_1 + x_2 \le 4$

Solve the following problems by **graphical method and verify by applying Kuhn-Tucker condition.**

15. Maximize $f(x, y) = x^2 + y^2$ subject to $x + y \ge 4$; $2x + y \le 5$.
16. Maximize $Z = 8x_1^2 + 2x_2^2$ subject to $x_1 \le 2$ and $x_1^2 + x_2^2 - 9 \le 0$

Solve the following by **Wolfe's method [Quadratic Programming].**

17. Maximize $Z = -2x_1^2 - 4x^2 + 15x + 30y + 4xy$ subject to $x + 2y \le 30$; $x, y \ge 0$.
18. Maximize $Z = -x^2 - {}^2 + 4x + 9y$ subject to $4x + 3y \le 15$; $3x + 5y \le 15$; $x, y \ge 0$.
19. Maximize $Z = 2x + 3y - x^2 - xy + y^2$ subject to $x + 2y \le 4$; $x, y \ge 0$.
20. Minimize $f(x, y) = x^2 - xy + 4y^2 - x - y$ subject to $2x + y \le 1$; $x, y \ge 0$.

ANSWERS TO NUMERICAL PROBLEMS

1. (1) convex (2) convex.
2. concave
3. convex
4. $x = 4 = y = 4$ product = 64
5. $x = y = z = 4$ product = 64
6. minimum = 0 at (0, 0); maximum = 32/9 at (–4/3, –4/3).
7. $y_1 = 120/17 = y_3; y_2 = 30/17$; $rt - s^2 > 0$; $r < 0$ $\therefore$ Maximum
8. $y_1 = -5/9$, $y_2 = 10/3$, $y_3 = 8$; $rt - s^2 > 0$; $r < 0$ $\therefore$ Minimize
9. $y_1 = 11/3$, $y_2 = 10/3$, $y_3 = 8$; $rt - s^2 > 0$; $r < 0$ $\therefore$ Minimize
10. $x_1 = 4/5$; $x_2 = 9/5$; $rt - s^2 > 0$; $r < 0$
11. $x_1 = 4/5$; $x_2 = 9/5$ satisfies K-T conditions
12. $x_1 = 1.5$, $x_2 = 0.5$, maximum $Z = 0.5$
13. $x_1 = 1$, $x_2 = 3/4$, minimum $Z = 17 / 8$
14. $x_1 = 2 /3$, $x_2 = 14/9$, maximum $Z = 14 /9$
15. $x = y = 2$, maximum $Z = 8$
16. $x_1 = 2$, $x_2 = \sqrt{5}$, maximum $Z = 42$
17. $x = 12$, $y = 9$, maximum $Z = 270$
18. $x = 3$, $y = 1$, maximum $Z = 11$
19. $x = 1/3$ $y = 4/3$, maximum $Z = 7/3$
20. $x = 4/11$ $y = 3/ 11$, minimum $Z = -5/11$

16 Decision Theory and Markov Chains

Logic will not change an emotion but action will.

Learning Objectives

After studying this chapter, the students will be able to:

- understand different procedures of decision-making.
- identify the right procedure and apply it to make decision under the prevailing situation.
- under the different criteria, like EMV, EVPI, and EOL.
- apply Bayesian principles to each branch of decision tree.
- understand Markov Chain, and Markov Process.
- apply principles of Markov Process to analyze branding problems and solve them for possible reframing of marketing policy.

Introduction

In real life, we take decisions based on our experience and on achieving sound and successful result to maximum degree of satisfaction; we say 'Thanks God'. Here we are to make systematic and rational approach to decision-making.

We have, as a primary source, maximum possible alternatives available to us and our limitations are confined in making the best selection of any one of the available and known or recommended alternatives.

Upon selection of any one, we wait and expect that the course of nature reacting upon it will finally approach to the best and probably predicted outcome. Events are external phenomena and it is not in human control. We can predict but it is not an assurance.

What we can possess is the analytical capacity to frame different strategies, analyse them under the different probable course of events or states and take decision hoping for the best.

We have

1. complete idea about how the things would shape: If we study a mathematical problem then we know, present status, then we take decision of applying some pre-known mathematical techniques and have surety that finally your efforts will bring to this result which was expected earlier.

2. some elements of taking risks: We may not be sure among any one of the two or three methods to apply to solve the problem and take a decision and attempt to solve the problem.
3. state of uncertainty: We are in a state of confusion and look for different alternatives without analyzing the given situation.
4. No Knowledge: We are supposed to solve a system of problems and make decision about the selection of immediate action. Probably, we may not faced such problems and hence cannot envisage the alternatives. This forces you to depend upon some other agency to help us or we blindly follow what others do and probably guide us; it is a state of ignorance.

16A DECISION THEORIES

16A.1 THREE STEPS PROGRAM

We have three steps program. Let us study the steps.

1. Decision Alternatives

This is what you can design, think and employ.

They are the different course of actions or strategies, which one can design to tackle a given situation. One may collect the different courses of actions by past experiences, self-thinking, or by seeking guidance from others.

2. States of Nature

They are also known as events, or future consequences, or role of nature. Their occurrence cannot be rightly predicted in advance. What and when, where such question words cannot be called for either to predict or to interpret occurrences of such events. They are not within human control. On the top of that, all parameters guiding them are not completely known.

Examples of such states of nature are

1. weather condition,
2. demand of certain commodity
3. inflation in the market.

States of nature are mutually exclusive events. One cannot say or claim about its exhaustivity until he has a complete set of different states.

3. Pay-off

When states of nature or the events react on the courses of action, will show some result called *outcome*. When an outcome, in order to understand or convey its affectivity, is evaluated in terms of monetary value is called *pay-off*. It is measured on an interval of time known as *decision-horizon*.

In some cases, decision-makers, instead of using pay-off (monetary values associated with outcomes), use the concept of ***utility*** of the outcome. It tries to associate a factor (real number) called utility or a factor representing usefulness of an outcome. In doing so, one cannot be always rational. There are always chances of subjectivity in associating numbers showing utility values. Different decision-makers associate different utility factors to evaluate the usefulness of the outcomes. It makes the decisions subjective hence not reliable for wide acceptance.

We take an illustration to understand all the three factors.

ILLUSTRATION 1

A farmer wants to grow, any one of the three crops—A, B or C.

Cost of seeds, and fertilizers plus labour charges for growing Crop A in his farm are estimated to be ₹3000. In the same way for Crops B and C estimated amounts are ₹4000 and ₹2500 respectively.

The yield depends on weather conditions and percentage of precipitation. As studied, we have three of such states.

1. heavy rain and damp weather
2. moderate rain and mixed (often dry and damp), and
3. scanty rain and dry weather.

The yield of each type of crop under different weather conditions is estimated as follows.

Yield in kilograms

Crop Type → / *Weather Condition ↓*	*A*	*B*	*C*
Damp	1000	900	850
Moderate	600	700	1100
Dry	500	900	700

In addition, market conditions are estimated as follows.

Crop Type:	A	B	C
Sale Price (In ₹/kg):	20	18	24

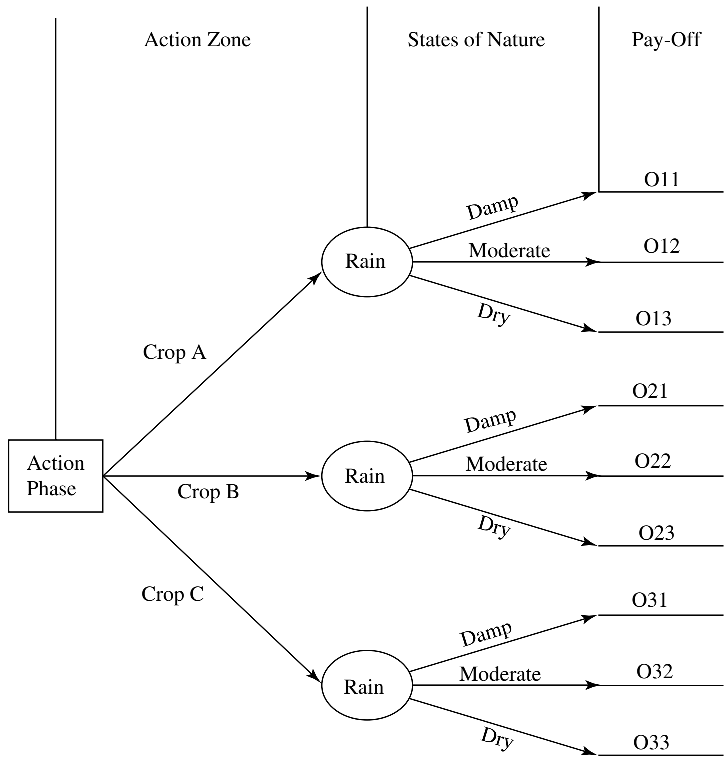
You are required to analyse the situation.

Solution

The revenue generated depends on quantity of the crop, type of crop, and sale price.

Weather Condition		Crop A	Crop B	Crop C
(1)	Damp	1000	900	850
	Revenue	20×1000	18×900	24×850
	Basic cost	3000	4000	2500
	Profit	17000	16200	17900**
(2)	Moderate	600	700	1100
	Revenue	20×600	18×700	24×1100
	Basic cost	3000	4000	2500
	Profit	9000	8600	23900**
(3)	Dry	500	900	700
	Revenue	20×500	18×900	24×700
	Basic cost	3000	4000	2500
	Profit	7000	12200	14300**

This makes complete analysis.



16A.2 DECISION–MAKING

What we have derived from above discussion is that we do not have control on the states of nature. They perform on their own and in many cases; we cannot predict about the time interval they take to perform their complete plan.

We have these points but they cannot stop to proceed statistically to decision-making process.

1. decision-making under certainty, and
2. decision-making in uncertainty

In addition to these points, we have an important notion, called Expected Value of Perfect Information; (EVPI).

16A.2.1 Decision-making under Certainty

Here, we have two important notions:

(A) Expected Monetary Value, and

(B) Expected Opportunity Loss.

The fundamental notion in the above-stated two notions is expected Value.

We have discussed about expected value in replacement theory.

We give in brief here.

If $X_1, X_2, \cdots, X_n$ are the real values associated with different outcomes and $P(X_1), P(X_2), \cdots, P(X_n)$ are corresponding probabilities then, the expected value of the system denoted as $E(X) = X_1 \cdot p(X_1) + X_2 \cdot p(X_2) + \cdots + X_n \cdot p(X_n) = \sum_{i=1}^{i=n} X_i \cdot p(X_i)$

In addition, at this state of our work, we assume that the probabilities of the occurrence of different events are known. We can think of this equivalent to rolling two fair dies or flipping of two or three fair coins.

(A) Expected Monetary Value

We are given the pay-offs for each act under the probable occurrence of each event. The probability of these events are pre-known.

We find expected values [here, known as expected monetary values] of each act and then choose the act with the highest EMV.

ILLUSTRATION 2

Using the following matrix, find the best act.

Probability → / *Acts* ↓	$p(E_1)$ 0.20	$p(E_2)$ 0.30	$p(E_3)$ 0.45	$p(E_4)$ 0.05
A	120	140	200	50
B	25	200	100	80
C	20	150	200	40

Solution

We find expected monetary value for each act. It is EMV(A), EMV(B), and EMV(C)

EMV ↓

$E(\text{A}) = 120 \times 0.20 + 140 \times 0.30 + 200 \times 0.45 + 50 \times 0.05 = 158.5$ **

$E(\text{B}) = 25 \times 0.20 + 200 \times 0.30 + 100 \times 0.45 + 80 \times 0.05 = 114$

$E(\text{C}) = 20 \times 0.20 + 150 \times 0.30 + 200 \times 0.45 + 40 \times 0.05 = 141$

The selection criteria for this type that the act whose expected monetary value is the highest is selected and it is the best act. It is to give you the best output.

(B) Expected Opportunity Loss

In this section, this is the second criteria for decision-making.

The basic logic behind this is to find expected opportunity loss (EOL) for each act.

Steps

*1. Under the probable occurrence of an **event,** find the act having the highest monetary value.
*2. Subtract other pay-off values of all the acts under the occurrence of that event; this is called ***opportunity loss***.
*3. Repeat the procedure for all the events.
*4. Now, we have a matrix called *opportunity loss matrix*.
*5. Find expected opportunity loss for all the acts.
*6. Selection criteria: Select the act whose expected opportunity loss is minimum.

We take the above illustration and apply principles of opportunity loss.

ILLUSTRATION 3

Using the following matrix, find the best act.

Probability → / Acts ↓	$p(E_1)$ 0.20	$p(E_2)$ 0.30	$p(E_3)$ 0.45	$p(E_4)$ 0.05
A	120	140	200	50
B	25	200	100	80
C	20	150	200	40

Apply opportunity loss criteria.

Step 1

Under the probable occurrence of Event 1, Act A is the best because it has the highest pay-off = 120.

Step 2

Subtract the other pay-offs of all the acts under the Event 1 from 120. We get under first column, the entries, 0, 95, and 100.

Step 3

Repeat this process for all the events.

Step 4

Find the opportunity loss matrix.

Opportunity Loss Matrix

Probability → / Acts ↓	$p(E_1)$ 0.20	$p(E_2)$ 0.30	$p(E_3)$ 0.45	$p(E_4)$ 0.05
A	00	60	00	30
B	95	00	100	00
C	100	50	00	40

Step 5

* Find expected opportunity losses under the probable occurrence of each event.

We find expected losses for each act. EOL ↓

EOL(A) = $00 \times 0.20 + 60 \times 0.30 + 00 \times 0.45 + 30 \times 0.05 = 19.5$ *

EOL(B) = $95 \times 0.20 + 00 \times 0.30 + 100 \times 0.45 + 00 \times 0.05 = 64$

EOL(C) = $100 \times 0.20 + 50 \times 0.30 + 00 \times 0.45 + 40 \times 0.05 = 37$

Step 6

The act A has the lowest opportunity loss amongst the remaining values of opportunity losses.

We select Act A the best choice of action.

(C) Expected Value of Perfect Information (EVPI)

This notion derives the amount that a person can spend on collecting all necessary data, and information pertaining to the different choices of actions. Past record are very important to estimate probabilities of different states of nature.

EVPI, in a way, it is the amount equivalent to one of minimum of expected opportunity losses, i.e. amount or the value of minimum of expected opportunity loss is the value of EVPI.

We have one more method of finding EVPI. EVPI is the difference between expected pay-off of perfect (the highest) information and the highest values of all expected monetary values.

We verify the truth value of the last statement by an illustration.

ILLUSTRATION 4

Find EVPI in the following case.

Probability → / *Acts* ↓	$p(E_1)$ 0.20	$p(E_2)$ 0.30	$p(E_3)$ 0.45	$p(E_4)$ 0.05
A	120	140	200	50
B	25	200	100	80
C	20	150	200	40

Solution

Part 1

We find the highest pay-off under the occurrence of each act for all the given acts.

Then find expectation of all such highest values.

Event	Probability	Highest Pay-off
E_1	$p(E_1) = 0.20$	120
E_2	$p(E_2) = 0.30$	200
E_3	$p(E_3) = 0.45$	200
E_4	$p(E_4) = 0.05$	80

Its expectation = $120 \times 0.20 + 200 \times 0.30 + 200 \times 0.45 + 80 \times 0.05 = 178$

Part 2

In this part, as we done in the Illustration 1, we find EMV for each act.

EMV ↓

EMV(A) = $120 \times 0.20 + 140 \times 0.30 + 200 \times 0.45 + 50 \times 0.05 = 158.5$ *

EMV(B) = $25 \times 0.20 + 200 \times 0.30 + 100 \times 0.45 + 80 \times 0.05 = 114$

EMV(C) = $20 \times 0.20 + 150 \times 0.30 + 200 \times 0.45 + 40 \times 0.05 = 141$

Part 3

In this part, we find EVPI by taking the difference of expectation of the highest value pay-off (Part 1) and highest of EMV (from Part 2) = 178 – 158.5 = 19.5

Therefore EVPI = 19.5 = the lowest of expected opportunity losses. (Illustration 3)

16A.3 DECISION-MAKING UNDER UNCERTAINTY

In this section, as the name suggests, we do not have much information about the probabilities about the states of nature. We have to develop criteria using pay-offs or utility values.

We have following criteria.

1. maximin
2. maximax
3. minimax regret
4. Hurwitz criterion
5. Jacob Bernoulli method

Let us discuss each.

1. Maximin Criterion

The word has a mathematical combination of maximum of row minimum. (Maxi + Min)

It suggests that for each act, we find minimum pay-off from the list of given pay-offs corresponding to each event. Performing this act, it gives a column vector representing minimums of each row. This conveys that the decision-maker is pessimistic and wants to be on the safer side for pay-offs resulting from each act.

Then we select the highest value from all the entries of minimums. In the list, we find the better one tending to the higher side.

2. Maximax Criterion

The word is a mathematical combination of maximum of maximums.

This criterion which reveals positive or optimistic tendency of the decision-maker.

We select the highest pay-off corresponding to each act. We pick-up the highest value from the entries of the column showing maximum pay-off entries, This is what maximax stands for.

We take illustration to show the relevance of the two thoughts.

ILLUSTRATION 5

Apply maximin and maximax criterion to the following pay-off matrix.

Pay-off Matrix

Events → / *Acts* ↓	E_1	E_2	E_3	E_4
A_1	150	200	50	–100
A_2	–160	–40	120	140
A_3	130	150	200	80

Solution

First, we find the result of maximin criterion.

1.

Minimum pay-off from Act 1 (A_1) = –100

Minimum pay-off from Act 2 (A_2) = –160

Minimum pay-off from Act 3 (A_3) = 80

Maximum of {–100, –160, 80} = 80

Maximin value = 80

This value corresponds to Act 3; hence according to maximin criterion Act A_3 is selected.

Now, we find the result of maximax criterion:

2.
Maximum pay-off from first Act 1 (A_1) = 200
Maximum pay-off from first Act 2 (A_2) = 140
Maximum pay-off from first Act 3 (A_3) = 200
Now, we find maximum of {200 140 200} = 200
Maximax value = 200; it corresponds to the Acts A_1 and A_3.
Following this, one can make a choice of any one of these two acts.

Comment:
If we consider maximin and maximax criteria, both at a time, then choice of Act A_3 satisfies both and therefore is a better choice.

3. Minimax regret

This criterion has a base of minimizing the maximum of opportunity losses.

Steps to applications are as follows.

*1. Find maximum pay-off corresponding to each event.
*2. Subtract all pay-offs under that event column from the highest pay-off just got in Step 1. This is called opportunity loss.
*3. Repeat this procedure for all the events. At the end of this procedure, you have a matrix of opportunity loss.
*4. From the entries of opportunity losses corresponding to each act, select the one with maximum loss. Repeat this procedure for all acts.
*5. Now, you have a column matrix whose entries are maximum opportunity losses. Select the minimum of these; which is 'minimax regret value.' Select the act corresponding to this entry.

ILLUSTRATION 6

Apply minimax regret criterion to the following pay-off matrix.

Pay-off Matrix

Events → / *Acts* ↓	E_1	E_2	E_3	E_4
A_1	150	200	50	–100
A_2	–160	–40	120	140
A_3	130	150	200	80

Solution

We find opportunity loss matrix.

Apply maximin and maximax criterion to the following pay-off matrix.

Opportunity-Loss Matrix

Events → / *Acts* ↓	E_1	E_2	E_3	E_4
A_1	00	00	150	240
A_2	310	240	80	00
A_3	20	50	00	60

The maximum regret for each act is shown adjacent to the matrix.

Events → / Acts ↓	E_1	E_2	E_3	E_4	Maximum Regret
A_1	00	00	150	240	240
A_2	310	240	80	00	310
A_3	20	50	00	60	60

Minimum of maximum regrets = minimum {240, 310, 60} = 60
Minimax regret value = 60.
This corresponds to the Act A_3; which, by this criterion is a choice.

4. Hurwitz's Criterion

Leonid Hurwitz developed this criterion. He tried to find a subjective balance between two extremes. Maximax and minimax are the two criteria which work in opposite directions. A decision-maker in the state of extreme mood swings can change his decision often. In an attempt to overcome this and find a rational Hurwitz suggested that only two extreme pay-offs play important role. A decision-maker has to associate his level of confidence, say $\propto$ where $0 < \propto < 1$ to the maximum pay-off and the counterpart $(1 - \propto)$ with the minimum pay-off of the same act.

Find H value, i.e. $H(A) = \propto \times$ maximum pay-off + $(1 - \propto) \times$ minimum pay-off.

Find H values for all the given acts.

The decision criterion is to select the act corresponding to maximum of all such H values.

Comment:

This criterion gives a chance to the decision-maker about selection of his confidence level indicator $\propto$. Both $\propto$ and $(1 - \propto)$ are the extreme correctors and rationalize the extremes. This criterion also, does not skip away subjectivity, as selection of $\propto$ is subjective and it is observed that different decision-makers possess different views to justify their selection of indicator value $\propto$.

ILLUSTRATION 7

Apply Hurwitz's criteria to the following pay-off matrix. Take confidence level $\propto = 0.4$ and $\propto = 0.6$ and verify if there is any change in the decision.

Pay-off Matrix

Events → / Acts ↓	E_1	E_2	E_3	E_4
A_1	150	200	50	–100
A_2	–160	–40	120	140
A_3	130	150	200	80

Solution

As said, we take $\propto = 0.4$

$$H(A) = \propto \times \text{maximum pay-off} + (1 - \propto) \times \text{minimum pay-off}$$
$$= 0.4 \times 200 + (1 - .4) \times (-100) = 80 - 60 = 20$$

$$H(B) = 0.4 \times 140 + 0.6 \times (-160) = 56 - 96 = -40$$

$H(C) = 0.4 \times 200 + 0.6 \times 80 = 128$

The choice is for the Act A_3. [$H(C) = 128 > H(A)$ and $H(C) = 128 > H(B)$]

Similarly on taking $\propto = 0.6$;

$H(A) = 120 + (-40) = 80$

$H(B) = 84 + (-64) = 20$

$H(C) = 120 + 32 = 152$ **

In this case, the choice is for the Act A_3. [$H(C) = 152 > H(A)$ and $H(C) = 152 > H(B)$]

5. Jacob Bernoulli Method

This is the simplest method amongst all the different methods of decision-making. If there is no special choice about selection of any act or a most favourable likeliness about occurrence of some events, then one can use this criterion.

Just take average of pay-offs corresponding to an act; denote the value by Ave (A). This clearly conveys that occurrence of each event is equally likely just as other events are likely to occur.

Repeat this procedure of finding averages for all the acts or choice of actions.

Decision criterion is the act, which corresponds to the highest *Average value* is the choice of action. It is also known as '***Laplace Principle***'

We take one illustration.

ILLUSTRATION 8

Apply Jacob Bernoulli criterion to make a selection of the act likely to yield the highest profit.

Pay-off Matrix

Events → / *Acts* ↓	E_1	E_2	E_3	E_4
A_1	150	200	50	–100
A_2	–160	–40	120	140
A_3	130	150	200	80

Solution

We assume that the occurrence of an event is equally likely.

Ave (A_1) = (150 + 200 + 50 – 100)/4 = 75

Ave (A_2) = (–160 – 40 + 120 + 140)/4 = 15

Ave (A_3) = (130 + 150 + 200 + 80)/4 = 140**

Ave (A_3) > Ave. (B), and Ave. (A)

We select, based on the logic of this criterion, the Act A_3.

16A.4 DECISION-MAKING THROUGH USE OF INCREMENTAL ANALYSIS

This criterion is also known as *marginal analysis*.

Using this procedure, one can avoid long routines and procedures involved in previous methods.

According to this criterion, any additional unit, if purchased is for an event that favours its sale. There are chances that it may not be sold. The decision-maker determines the probability, p of the new unit purchased and added to inventory (additional unit). If this new unit is sold, then this act results into increment in the current profit by some amount. We call this as an incremental profit (IP).

On the contrary, the probability that the new unit is not sold is $(1 - p)$. If it is not sold then this act of purchase will decrease the current profit by an amount, IL.

We say IP, and IL as marginal profit and marginal loss.

The expectation that the additional unit is sold and yields an incremental profit = $P \cdot IP$

The expectation that the additional unit is not sold and yields and decreases the current profit by an amount IL is = $(1 - p) \cdot IL$

Justifying both equally, we have

$$p \cdot IP = (1 - p) \cdot IL$$

We get,
$$p = (IL) / (IP + IL)$$

As mentioned above p stands for minimum necessary value of probability that the new unit is sold. Practically, the unit should be stocked so long as probability of likely sale is greater than or equal to (≥) p. One should make greater than or equal to type (≥) of cumulative frequency distribution and search for the variable which is just greater than the value of $p = (IL)/(IP + IL)$)

ILLUSTRATION 9

A newspaper carrier brings magazines at ₹12 per unit and sells one unit for ₹15. His record of last 25 days has the following probability distribution.

Magazines bought for sale:	12	13	14	15	16	17	18	19
Probability:	0.05	0.12	0.15	0.24	0.30	0.11	0.02	0.01

Solution

Basic cost = ₹12,

Sale price = ₹15

Profit (if sold) per unit = ₹3 = IP

IL (if not sold) = ₹12

$p = (IL) / (IP + IL) = 12/15 = 0.80$

So long as $p \geq 0.80$, one can afford making a stock of 1 unit.

Now, we make more than type (>) of frequency distribution.

Magazines sold more than demand ≥ k	Cumulative probability
12	1.00
13	0.95
14	0.83**
15	0.68
16	0.44
17	0.14
18	0.03
19	0.01

In the above table, it is clear that the probability of the highest value of demand which exceeds the critical value = $(IL)/(IP + IL) = 12/15 = 0.80$ is at $k = 14$.

The optimal act for the newspaper carrier is to buy 14 copies.

16A.5 Bayesian Approach to Decision Theory

This method of decision-making involves the use of probability theory at three stages. We take one illustration and analyse the data and after analysing, we will make our comments.

Illustration 10

Stage 1

A box (Box 1) contains 20 items; out of which 4 items have no defects and 10 items have one defect and remaining have two defects. There is a policy of accepting only 'no defective' items. The inspection cost and opportunity loss cost of rejecting a 'no defective' lot is ₹300 and the cost of accepting a 'defective lot' is ₹800.

Stage 2

The probability of getting one defective item from a lot of 'no defective' items, 'one defective' item, and 'two defective' items is 0.90, 0.60 and 0.20 respectively.

Stage 3

You have to take a decision regarding possible acceptance of the lot.

Solution

Part 1

Let B_1, B_2, and B_3 be the events showing 'no defective', 'one defective', and 'two defective' items respectively. These events are mutually exclusive and totally exhaustive and any one of them must occur.

$p(B_1) = p(\text{ no defective}) = 4/20 = 0.2$

$p(B_2) = p(\text{One defective}) = 10/20 = 0.5$

$p(B_3) = p(\text{two defectives}) = 6/20 = 0.3$

These are initial probabilities.

Let A be the event of selecting **one defective** item from a lot of 'no defective' items, 'one defective' items, and 'two defective' items.

Part 2

We find the following *conditional probabilities*.

$p(A/B_1)$ = Given that the lot has 'no defective' items; probability of getting 'no defective' items;
= 0.90 (given)

$p(A/B_2)$ = Given that the lot has 'one defective' items; probability of getting 'no defective' items;
= 0.60 (given)

$p(A/B_3)$ = Given that the lot has 'two defective' items; probability of getting 'no defective' items;
= 0.20 (given)

Part 3

Now, we apply Bayesian principle.

$$\begin{aligned} p(A) &= p(A/B_1) \cdot p(B_1) + p(A/B_2) \cdot p(B_2) + p(A/B_3) \cdot p(B_3) \\ &= (0.90) \times (0.2) + (0.60) \times (0.5) + (0.20) \times (0.3) \\ &= 0.54 \end{aligned}$$

Part 4

Now, we find the *posterior probabilities*.

$P(B_1/A)$ = Given that the item is non-defective; probability getting 'no defective' items
$= [p(A/B_1) \cdot p(B_1)]/p(A) = [(0.90) \times (0.2)]/0.54 = 0.33 = 1/3$

$P(B_2/A)$ = Given that the item is non-defective; probability getting 'one defective' items
$= [p(A/B_2) \cdot p(B_2)] / P(A) = [(0.60) \times (0.5)] / 0.54 = 0.555 = 5/9$

$P(B_3/A)$ = Given that the item is non-defective; probability getting 'two defective' items
$= [P(A/B_3) \cdot P(B_3)] / P(A) = [(0.20) \times (0.3)] / 0.54 = 0.111 = 1/9$

Part 5

Total cost of accepting $= (1/3) \times (0) + (5/9) \times (800) + (1/9) \times (800) = 2402$

Total cost of rejecting $= (1/3) \times (300) + (5/9) \times (0) + (1/9) \times (0) = 100$

Cost of accepting $= 2402 > 100 =$ cost of rejecting

Decision in this case is to reject the lot.

16A.6 Decision Tree Approach in Decision-making

This approach, in a way, is the same as the above topic on Bayesian approach is. In this approach, we identify different actions and then associate events with them. Pay-off values are immediate followers of the different states of nature. We apply Bayesian principles and make a 'roll-back', taking expected pay-off values, to the *action node*. We compare the pay-offs generated by each action branch and take the decision of selecting an act.

We take one illustration.

Illustration 11

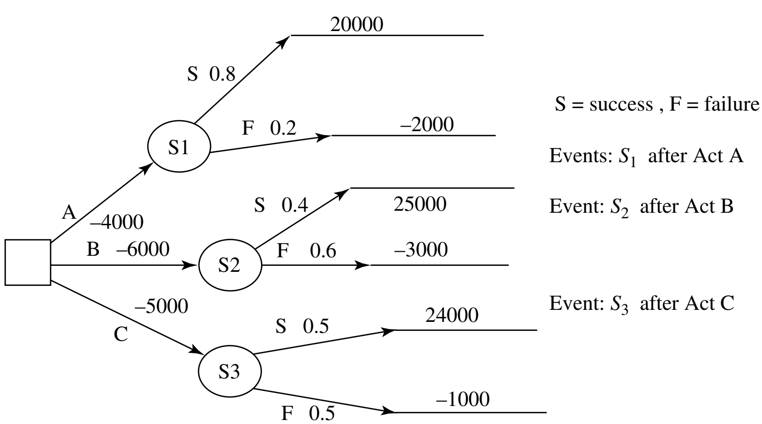
ABC Corporation plans going on with three subsidiary projects with minimum investments; Project A with an investment of ₹4000 and Projects B and C with ₹6000 and 5000 respectively. The projects are trial projects and have chances to be successful.

Projects	A	B	C
Probability of success	0.8	0.4	0.5
Profit if successful	20000	25000	24000
Loss if fails	2000	2000	1000

Draw a decision tree and take decision.

Solution

The decision tree diagram is as follows.



Output Analysis (Part A)

Basic investment**:** For Project A = ₹–4000

Expected outflow after Event S_1 = (20000) × (0.8) + (–2000) × (0.2) = 15600

Net profit = ₹15600 – ₹4000 = ₹11600 (1)

Output Analysis (Part B)

Basic investment**:** For Project B = ₹–6000

Expected outflow after Event S_1 = (25000) × (0.4) + (–2000) × (0.6) = 8800

Net profit = ₹8800 – ₹6000 = ₹2800 (2)

Output Analysis (Part C)

Basic investment: For Project C = ₹–5000

Expected outflow after Event S_1 = (24000) × (0.5) + (–1000) × (0.5) = 11500

Net profit = ₹11500 – ₹5000 = ₹6500 (3)

Maximum output corresponds to the Act A (₹11600); which is the best choice.

ILLUSTRATION 12

A man thinks of investing some amount initially in either Project A or Project B. For the Project 'A' he has to invest ₹8000 and ₹7000 for the Project 'B'.

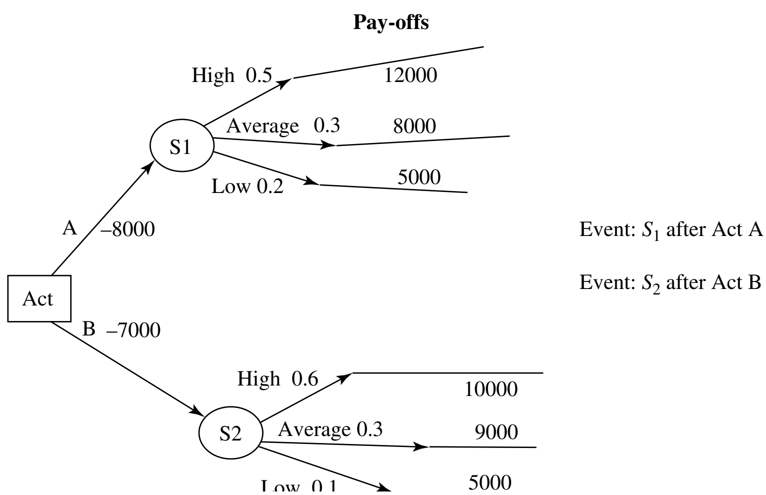
The production on completion of Project A has three states of nature; either there is a high demand (probability = 0.5) and yields a profit ₹12000 or an average demand (probability = 0.3) with a profit ₹8000 or a low demand giving a profit ₹5000.

The production on completion of Project B has three states of nature; either there is a high demand (probability = 0.6) and yields a profit ₹10000 or an average demand (probability = 0.3) with a profit ₹9000 or a low demand giving a profit ₹5000.

Draw the decision tree and take a choice giving maximum return.

Solution

The decision tree is drawn below.



Solution

Output Analysis (Part A)

Basic investment: For Project A = ₹–8000

Expected outflow after Event S_1 = (12000) × (0.5) + (8000) × (0.3) + (5000) × (0.2)
= 6000 + 2400 + 1000 = 9400

Net profit = 9400 – 8000 = 1400 (1)

Output Analysis (Part B)

Basic investment: For Project B = ₹(–7000)

Expected outflow after Event S_2 = (10000) × (0.6) + (9000) × (0.3) + (5000) × (0.1)
= 6000 + 2700 + 500 = 9200

Net profit = 9200 – 7000 = 2200 (2)

The return on Project B is higher than that from Project A; hence Project B is the best choice.

16B MARKOV CHAINS AND ANALYSIS

16B.1 INTRODUCTION

Andrei Markov and W. Feller were the pioneers to develop, analyse, and apply the general theory of Markov chain. This was initially to describe the pattern movement of gas particles in a closed container. A mathematical model was developed to describe transition from one variable from one state the other from finite number of available states. It is in an all time dynamic environment. The basic objective is to understand the pattern of current state of variables and forecast about future movement. One important point is, that we need not know the past data or past record; it only uses current state and makes forecast for the next point of time.

16B.2 MARKOV PROCESS AND MARKOV CHAIN

When we note the movement of variable X, at different time interval $t = 0, 1, 2 \cdots$, taking up the different values (or observations) as $X_0, X_1, X_2, \cdots$ then the process describing relationship is called a *stochastic process*.

This stochastic process has an analogy with Markov process. It is a discrete process beginning from, at a given point of time and a given state, passing through different time slots available. Observations regarding a current variable on hand are taken as it moves to end of different time zones. Markov process is discrete in current state (any from all possible and currently available states) and time (in a given time interval at different point of time). It is a process describing the current state of a random variable at different time slots.

On the other hand, Markov Chain—name itself suggests that it a continuous process based on finite countable set of states.

16B.3 APPLICATION AREA

Markov chains are very useful in real life situation. The basic point is to analyse a current state of a variable on a given point of time and use it to forecast the next state or status of the variable on the next slot or point of time. We have notes on some application areas.

1. Marketing

Most of the item has their aura of affluence. They sell on 'brand names'. People buy worth and on some change of test or just for a change, change brand. Customers have some unidentified or painted thinking of skipping from one brand to other. Marketing managers simply targeting of pushing their goods, are worried in making analysis and based on the result of either type, keep on changing marketing strategies. They need to know market movement. This is termed as *brand switching analysis*.

2. Politics

In general, we call population keep on showing their trend towards a political party. As time passes and circumstances change, some segment or unsatisfied fractions change their loyalty. We convey only two points:

(i) present (current time) status, and
(ii) selection from finitely available parties in some near future.

This study is effectively done on using Markov Analysis. This analysis does not depend upon what the previous records were, but what present records are.

3. Inventory Management

From the above discussion and the fundamental tenets it is clear that the stockists are concerned what they possess or hold at the moment and what should they do in the near future. They are simply concerned with effective results of inventory they hold at any time.

4. Movement of Stock Market

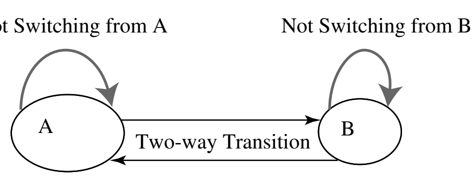
Markov analysis is very useful in prediction and hence has a strong influence on capital market.

16B.4 MARKOV CHAINS

We describe some patterns of Markov chains.

1. A Two–state Markov Chain

There are two brands. The figure depicts the chain of switching.



Let us study a transition matrix for a finite number of patrons of two brands.

Probability Matrix

To → / *From* ↓	*A*	*B*
A	0.8	0.2
B	0.5	0.5

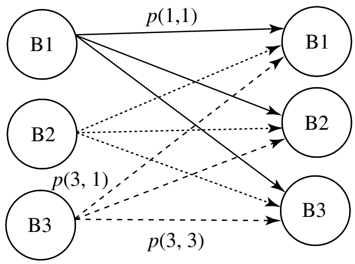
$p(A, A) = 0.8$ shows that from the total patrons of Brand A, there are 0.8 or 80% are still adhering to Brand A.

$p(A, B) = 0.2$ shows that from the total patrons of Brand A , 20% have switched to Brand B.

Similarly, we can derive meanings for other probabilities.

It should be noted that $p(A,A) + p(A, B) = 1$ and $p(B, A) + p(B, B) = 1$

2. A Three-state Transition



All $p(i, j)$ are not shown ($i = 1, 2, 3, j = 1, 2$)

As shown in the figure, a finite number of patrons either keeping the same brand or switching from one brand to the other.

An important point to note is that the sum of probabilities of all either keeping the brand or leaving the brand and joining the other brand is 1.

If $p_{ij} = p(i, j)$ shows the probability of switching from i to j ($i = 1, 2, 3$ also $j = 1, 2, 3$)

then, $\sum_{j=1}^{j=3} p_{i,j} = 1$, for $i = 1, 2, 3$

16B.5 Markov Process

Markov chain analysis is based on the following assumptions.

1. There is finite number of available states.

One of them is the current state. We also assume that the states are non-absorbing. Any customer currently stationed on one brand has no restriction to switch over to any other brand.

There always exists a path from ith state to move to jth state and vice-versa. Both the states are reachable—**(communicable states).**

2. Order of Markov Process

We have already stated that a current state is dependent only on the previous time state only. Putting in other words, the state of the next time depends only on the current state only. In this case, it is called ***first order Markov process***.

If a state on any given point of time depends on exactly two previous states then it is called ***second order Markov process***. We can extend this order notion to a finite number.

3. Constancy of Transition Matrix

If the transition matrix remains constant while making or achieving a transition from one state to the other, then we call it *state of constancy of probability matrix*. In this case *Markov Process* is known to acquire a ***stationary*** **or a** ***homogeneous state***.

4. Uniform Time Slot

The time span to achieve a steady state probability is considered to have a uniform interval. The transition matrix achieves constancy. After a fixed time interval the probability values in the transition matrix achieve a steady state.

Comment

Any process after a certain number of trials or iteration tends to normality and gets **stabilized**. This is taken into consideration by Markov analysis. A stabilized system is in the steady state or in the equilibrium. Then onwards the system becomes ***independent of time*.**

Illustrations

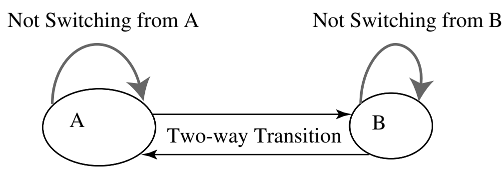
Now we take some illustrations to understand Markov Process.

ILLUSTRATION 1

We take the same example of two-state transition.

A Two-state Markov Chain

There are two brands. The figure depicts the chain of switching.



Probability Matrix

To → / From ↓	A	B
A	0.8	0.2
B	0.5	0.5

A market is of a fixed number of patrons. Also, it is assumed that initially share of patrons to accept one of the two brands was 50% each. Marketing strategies and promotion schemes have affected the present situation and a survey shows the new result of probabilities shown in the matrix above. As a result of this, you are required to determine market share of each brand.

Solution

1. The initial market share of each brand is assumed to be 50% (or it may be given). This is called the *current state*. [time $t = 0$]
2. In the next slot or period, we have the pulling effect on the transition matrix.

   The next share of the total market is $(50\% \quad 50\%)\begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} = (0.5 \quad 0.5)\begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix}$
   $= (0.65 \quad 0.35)$

   We interpret this. The current market share of each brand was 50% and the new strategies or schemes have increased Brand A's share from 50% to 65%. and for Brand B it has reduced to 35% from 50%. (time $t = 1$)

3. In the next period, the market share, assuming that the transition matrix remains the same can be found as follows. (time $t = 2$)

   The next share of total market is $(0.65 \quad 0.35)\begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} = (0.695 \quad 0.305)$

4. In the next period, the market share, assuming that the transition matrix remains the same can be found as follows. (time $t = 3$)

   The next share of the total market is $(0.695 \quad 0.305)\begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} = (0.7085 \quad 0.2915)$

5. In the next period, the market share, assuming that the transition matrix remains the same can be found as follows. (time $t = 4$)

   The next share of the total market is $(0.7085 \quad 0.2915)\begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} = (0.71255 \quad 0.28582)$

6. In the next period, the market share, assuming that the transition matrix remains the same can be found as follows. (time $t = 5$)

   The next share of total market is $(0.71255 \quad 0.28582)\begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} = (0.71295 \quad 0.28542)$

Comment

If we keep the transition matrix constant, then after a finite number of iterations taken at a fixed interval of time, forces the market share to tend to some constant.

We write the market share as per each time interval.

	Share in percentage of		
Period	A	and	B
$t = 0$ initial share (given)	50		50
$t = 1$	65		35
$t = 2$	69.5		30.5
$t = 3$	70.85		29.15
$t = 4$	**71.255**		**28.582**
$\boldsymbol{t = 5}$	**71.295**		**28.542**

In the last two observations, it can be seen that the market share gets stability.

Alternate Approach

Let (x, y) be the current as well as the optimum percentage of final market percentage; this holds true at equilibrium.

We have a system of simultaneous equations.

$$(x, y)\begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} = (x, y) \text{ and } x + y = 1$$

Solving this system, we get the final answer.

ILLUSTRATION 2

There are three different types of newspapers covering a small market of 1000 persons in a town. According to a survey, the Newspaper A captures a share of 200 copies, while the Newspaper B has a 50% share in 1000 and the Newspaper C has remaining portion. In an attempt to increase the market share, these three newspapers started marketing campaign. The result is shown below in matrix form.

Newspapers	*Readers on January 1, 2013*	*Received from*			*Readers on February 1, 2013*
		A	*B*	*C*	
A	200	**0**	**35**	**25**	220
B	500	**20**	**00**	**20**	490**
C	300	**20**	**15**	**00**	290
	Total share = 1000				**Total share = 1000**

You are required to find the transition matrix and make a prediction about market share of newspapers for next two time slots.

Solution

Data Analysis

Readers of the Newspaper A = 200, from this 20 + 20 = 40 switched over to the Newspapers B and C. The current contribution is $(200/1000 \quad 500/1000 \quad 300/1000) = (0.2 \quad 0.5 \quad 0.3)_{1\times 3}$

Now the Newspaper A has 200 – 40 = 160 readers who continued in A.

The Newspaper A received 35 + 25 = 60 customers from the Newspapers B and C.

What the Newspaper A has, is $\begin{pmatrix} 160 \\ 35 \\ 25 \end{pmatrix}_{3\times 1}$ or Percentage received from the Newspapers A, B, and C $\begin{pmatrix} 160/200 \\ 35/500 \\ 25/300 \end{pmatrix}_{3\times 1}$

** What the Newspaper B has now, (gone to the Newspaper A and C are 35, and 15 and = 500 – 50 = 450) plus (received from A and C are 20 and 20 = 40) = 500 – 50 + 40 = 490**

What the Newspaper B has in per cent value = $\begin{pmatrix} 20/200 \\ 450/500 \\ 20/300 \end{pmatrix}_{1\times 3}$

Similarly, what is the Newspaper C is now $\begin{pmatrix} 20/200 \\ 15/500 \\ 255/300 \end{pmatrix}_{1\times 3}$

These facts bring us to the transition matrix = $\begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.07 & 0.9 & 0.03 \\ 0.083 & 0.067 & 0.85 \end{pmatrix}_{3\times 3}$

We are now equipped with all first-hand information to compute share in the next iteration.

$$\text{The market share} = (0.2 \quad 0.5 \quad 0.3)_{1x3} \times \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.07 & 0.9 & 0.03 \\ 0.083 & 0.067 & 0.85 \end{pmatrix}_{3\times3}$$

$$= (0.2199 \quad 0.4901 \quad 0.29)$$

Based on the transition matrix, the market share of the Newspaper A is 21.99%, the market share of the Newspaper B is 49.01%, and the market share of the Newspaper C is 29%

We find one more iteration. This makes prediction regarding the market share of the Newspaper A, B, and of C for the next time period.

$$\text{The market share prediction} = (0.2199 \quad 0.4901 \quad 0.29) \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.07 & 0.9 & 0.03 \\ 0.083 & 0.067 & 0.85 \end{pmatrix}_{3\times3}$$

$$= (0.2342 \quad 0.4825 \quad 0.2832)$$

Based on the transition matrix, the market share of the Newspaper A is 23-42%, the market share of the Newspaper B is 48.25%, and the market share of C is 28.32%]

Alternate Method

We can solve the system using transition matrix. Let x be the market share for the Newspaper A, and y and z for the Newspapers B and C.

$$\text{We have } (x \quad y \quad z) \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.07 & 0.9 & 0.03 \\ 0.083 & 0.067 & 0.85 \end{pmatrix}_{3\times3} = (x \quad y \quad z)$$

with $x + y + z = 1$.

This is a system of four equations; we can solve it for x, y, and z.

Additional Questions for Practice (with Hints and Answers)

Question 1

The present market share of three dairies P, Q, and R, selling special quality milk pouches is 40%, 35%, and 25% respectively. There are 1000 customers buying these pouches. After a marketing campaign, another survey was made and the information revealed the following facts.

Let $p(i, j)$ represent probability of transition from state i to state j. ($i, j = 1, 2, 3$)

Transition from P to Q, and R is 0.15 and 0.10 fraction of total of who are in P.

Transition from Q to R and P is 0.03 and 0.12 fraction of total of who are in Q.

Transition from R to P and Q is 0.13 and 0.07 fraction of total of who are in R.

Make a prediction for the next period.

Solution

The initial distribution of market share of P, Q, and R is (0.4 0.35 0.25).

After the campaign, the transition matrix;

$$T = \begin{pmatrix} 0.75 & 0.15 & 0.10 \\ 0.12 & 0.85 & 0.03 \\ 0.13 & 0.07 & 0.80 \end{pmatrix}_{3\times3}$$

[Please take care to see that what goes out from P is in terms of fractions of P and hence their sum = 1. In this case,

Adhering to P = 75% of **P** + those who switched to Q is 15% of **P** + those who switched to R is 10% of **P**. The sum of what goes from is 100% or in terms of probability; it is 1.

We find the prediction for the next period is;

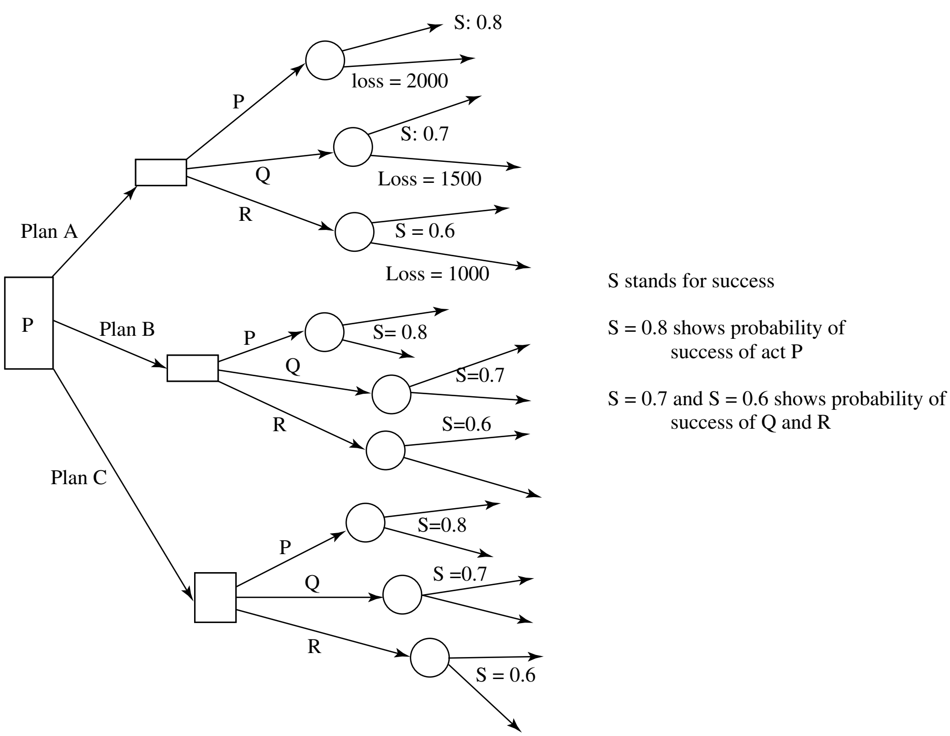
$$(0.4 \quad 0.35 \quad 0.25) \begin{pmatrix} 0.75 & 0.15 & 0.10 \\ 0.12 & 0.85 & 0.03 \\ 0.13 & 0.07 & 0.80 \end{pmatrix}_{3\times3} = (0.3745 \quad 0.375 \quad 0.2505)$$

The result (0.3745 0.375 0.2505) indicates that the market share in the next period is expected to be 37.45% for P, 37.5% of Q, and 25.05% of R.

Question 2

A financial advisor has three different plans A, B, and C for investments. He has complete details as follows.

In Plan A, there is an equal amount of investment in Proposals P, Q, and R from the total amount ₹45000.



In Plan B, the investment is 2 : 3 : 4 in proposals P, Q, and R from a total of ₹45000;

In Plan C, the investment is in the ratio 5 : 3 : 1 in proposals P, Q, and R from a total of ₹45000.

The probability that the proposal P becomes successful is 0.8 and it will give a return of 8% per annum. If it fails in execution, then there is a loss of ₹2000.

The probability that the proposal Q becomes successful is 0.7 and it will give a return of 7% per annum. If it fails in execution, then there is a loss of ₹1500.

The probability that the proposal R becomes successful is 0.6 and it will give a return of 6% per annum. If it fails in execution, then there is a loss of ₹1000.

Draw a decision tree diagram and analyse each plan. Give your final decision.

Solution

There are four stages of the total solution.

Stage 1

Plan A

It is for investing equal amount to each proposal. Total investment is ₹45000,
i.e. ₹15000 to each one P, Q, and R

*1. Amount generated (on implementing Proposal P)
$= 15000 \times 0.08 \times 8 + (-2000) \times 0.2 = 560$

*2. Amount generated (on implementing Proposal Q)
$= 15000 \times 0.07 \times 0.7 + (-1500) \times 0.3 = 285$

*3. Amount generated (on implementing Proposal R)
$= 15000 \times 0.06 \times 0.6 + (-1000) \times 0.4 = 140$

*4. Total expectation = 560 + 285 + 140 = 985

Stage 2

Plan B

Investment in the ratio 2 : 3 : 4 in P, Q, and R from a total of 45000
i.e. In Proposal P investment = ₹10000
In Proposal Q investment = ₹15000
In Proposal R investment = ₹20000

*1. Amount generated (on implementing Proposal P)
$= 10000 \times 0.08 \times 0.8 + (-2000) \times 0.2 = 240$

*2. Amount generated (on implementing Proposal Q)
$= 15000 \times 0.07 \times 0.7 + (-1500) \times 0.3 = 285$

*3. Amount generated (on implementing Proposal R)
$= 20000 \times 0.06 \times 0.6 + (-1000) \times 0.4 = 320$

*4. Total expectation = 240 + 285 + 320 = 845

Stage 3

Plan C

Investment in the ratio 5 : 3 : 1 in P, Q, and R from a total of ₹45000
i.e. In Proposal P investment = ₹25000
In Proposal Q investment = ₹15000
In Proposal R investment = ₹5000

*1. Amount generated (on implementing Proposal P)
$= 25000 \times 0.08 \times 0.8 + (-2000) \times 0.2 = 1200$

*2. Amount generated (on implementing Proposal Q)
$= 15000 \times 0.07 \times 0.7 + (-1500) \times 0.3 = 285$

*3. Amount generated (on implementing Proposal R)
$= 5000 \times 0.06 \times 0.6 + (-1000) \times 0.4 = -220$

*4. Total expectation = 1200 + 285 – 220 = 1265

Stage 4 Comparison of expected output from each plan:

Total expectation from Plan A = 560 + 285 + 140 = ₹985
Total expectation from Plan B = 240 + 285 + 320 = ₹845
Total expectation from Plan C = 1200 + 285 – 220 = ₹1265
The highest yield is from Plan C is ₹1265
Plan C is the choice of action.

EXERCISES

OBJECTIVE TYPE QUESTIONS

I. State True or False:

1. Expected opportunity loss value is same as EVPI.
2. EMV is the amount, which can be spent for obtaining the information.
3. The act corresponding to either maximum amount of EMV from different acts or minimum amount of all expected opportunity losses is the best choice.
4. Hurwitz criterion of decision under uncertainty is subjective for decision-makers.
5. In finding EMV, we use Bayesian theory of probability.
6. In decision-making through Bayesian principles, we use priori, conditional, and posterior probabilities.
7. In a brand-switching problem, there are a finite number of brands and the population in consideration for switching from or continuing on to the same is also finite.
8. After a finite number of iterations, when the system tends to a steady state, what keeps on changing is the market share of brands.
9. In a discrete stochastic process, the state of nature can be viewed at a fixed point of pre-fixed time interval.
10. Using the notion of Markov process, we can optimize the market share and optimize profit.
11. Markov process helps us finding state of nature of the next time interval using all the past records of states of nature.
12. The sum of probabilities of the probabilities of the persons' who leaves a particular brand is always one.

ANSWERS

1. false.	2. false.	3. true.	4. true.	5. false.
6. true.	7. true.	8. false.	9. true.	10. false.
11. false.	12. false.			

II. Multiple Choice Questions

The pay-off table given below is for the Questions 1, 2, 3, and 4.

Pay-off Matrix

Acts → / State of nature ↓	A_1	A_2	A_3	A_4
S_1	100	250	160	40
S_2	200	250	200	300
S_3	80	400	150	100
S_4	300	300	150	160

1. Maximin from the above pay-off table is
   (a) 250 (b) 80 (c) 40 (d) 150
2. Maximax value from the above pay-off table is
   (a) 300 (b) 200 (c) 250 (d) 400
3. What is the minimax regret value?
   (a) 320 (b) 50 (c) 250 (d) 300
4. In the above table, we may omit the Act A_3 in perspective of Act A_2 from pay-off table. (Comment)
   (a) Yes (b) No
   (c) We can drop the Act A_2 (d) Event S_3 should be dropped

   Questions 5, 6, 7, 8, and 9 are related to the pay-off table given below.

Pay-off Matrix

Acts → / State of Nature ↓	A_1	A_2	A_3	A_4
S_1	100	250	160	40
S_2	200	250	200	300
S_3	80	400	150	100
S_4	300	300	150	160

Probabilities of states of nature are equal for S_1 and S_2; while the probability of S_3 and S_4 are equal and four times that of Event S_1.

5. What is the expected monetary value of Act A_1?
   (a) 680 (b) 550 (c) 182 (d) 237
6. What is the expected monetary value of Act A_2?
   (a) 330 (b) 1200 (c) 950 (d) 200
7. Which act according to EMV criterion is the best one?
   (a) A_1 (b) A_2 (c) A_3 (d) A_4
8. What is the expected opportunity loss of Act A_2?
   (a) 320 (b) 50 (c) 250 (d) 300
9. What is the EVPI?
   (a) 320 (b) 50 (c) 250 (d) 300

The following data refers to Questions 10, and 11.
Present market share of two companies P and Q are 60% and 40% respectively. A survey shows that out of a combined total of 1000 persons, 30% using the Brand P wish to continue with and 70% wants to switch to Brand Q. On the other end, 40% using Brand Q wants to switch over to Brand P.

10. What is the next state in percentage of market share?
    (a) (48 66) (b) (60 40) (c) (54 46) (d) (34 66)
11. Based on the same survey result what is the second order state of market share?
    (a) (36.6 63.4) (b) (60 40) (c) (34 66) (d) (30 70)
12. The most optimistic criterion of decision-making is
    (a) opportunity loss (b) minimax loss (c) maximin (d) maximax
13. Expected opportunity loss is obtained
    (a) by finding lowest pay-off
    (b) by finding lowest EMV
    (c) subtracting all pay-offs of an act from the best pay-off under the same event
    (d) subtracting the highest pay-off of an event from all pay-off's under the same event.
14. This criterion of incremental analysis is also known as
    (a) marginal analysis (b) decision tree analysis
    (c) expected loss (d) Jacob Bernoulli criterion

Answers

1. (a)	2. (d)	3. (b)	4. (a)	5. (c)
6. (a)	7. (b)	8. (b)	9. (b)	10. (d)
11. (a)	12. (d)	13. (c)	14. (a)	

Numerical Problems

1. From the following pay-off table, find (1) maximin and (2) minimax values

Pay-off Matrix

Acts → / State of nature ↓	A_1	A_2	A_3	A_4
S_1	100	250	160	40
S_2	200	250	200	300
S_3	80	400	150	100
S_4	300	300	150	160

2. From the following pay-off table, find (1) maximax and (2) maximin values

Pay-off Matrix

Acts → / State of nature ↓	A_1	A_2	A_3	A_4
S_1	100	250	160	40
S_2	200	250	200	300
S_3	80	400	150	100
S_4	300	300	150	160

3. From the following pay-off table, find (1) maximin and (2) minimax values.

Pay-off Matrix

States of Nature → / *Acts* ↓	S_1	S_2	S_3	S_4
A_1	100	200	80	300
A_2	250	250	400	300
A_3	160	200	150	150
A_4	40	300	100	160

4. From the following pay-off table, find expected monetary values of all acts and make your decision.

Pay-off Matrix

States of Nature → / *Probability* → / *Acts* ↓	S_1 *0.1*	S_2 *0.1*	S_3 *0.4*	S_4 *0.4*
A_1	100	200	80	300
A_2	250	250	400	300
A_3	160	200	150	150
A_4	40	300	100	160

5. From the following pay-off table, find expected opportunity loss of all acts and make your decision.

Pay-off Matrix

State of Nature → / *Probability* → / *Acts* ↓	S_1 *0.1*	S_2 *0.1*	S_3 *0.4*	S_4 *0.4*
A_1	100	200	80	300
A_2	250	250	400	300
A_3	160	200	150	150
A_4	40	300	100	160

6. From the following pay-off table, find (1) EVPI, (2) EMV of the best act (3) expected value of perfect act under each state

Pay-off Matrix

State of Nature → / *Probability* → / *Acts* ↓	S_1 *0.1*	S_2 *0.1*	S_3 *0.4*	S_4 *0.4*
A_1	100	200	80	300
A_2	250	250	400	300
A_3	160	200	150	150
A_4	40	300	100	160

7. A newspaper carrier buys magazines at ₹10 per unit and sells one for ₹15. His record of last 25 days has the following probability distribution. Unsold newspaper has a recycling value of ₹8 per copy. You can also consider that opportunity loss is not a criterion for profit; find the best buy criterion on the basis of selecting the highest EMV.

Magazines bought for sale:	$A_1 = 12$	$A_2 = 13$	$A_3 = 14$	$A_4 = 15$	$A_5 = 16$	$A_6 = 17$	$A_7 = 18$
Probability:	0.05	0.12	0.15	0.24	0.30	0.11	0.03

8. A newspaper carrier brings magazines at ₹10 per unit and sells one for ₹15. His record of last 25 days has the following probability distribution. Unsold newspaper has a recycling value of ₹8 per copy. You can also consider that opportunity loss is not a criterion for profit; find the best buy criterion on the basis of selecting minimum opportunity loss. Also Find EVPI.

Magazines bought for sale:	$A_1 = 12$	$A_2 = 13$	$A_3 = 14$	$A_4 = 15$	$A_5 = 16$	$A_6 = 17$	$A_7 = 18$
Probability:	0.05	0.12	0.15	0.24	0.30	0.11	0.03

9. Apply Jacob Bernoulli criterion to find the choice for the best act.

Pay-off Matrix

States of Nature → / *Acts* ↓	S_1	S_2	S_3	S_4
A_1	100	200	80	300
A_2	250	250	400	300
A_3	160	200	150	150
A_4	40	300	100	160

10. Apply Hurwitz's criterion to select te best act. Take

1. probability of occurrence of the event giving the best pay-off = $\propto$ = 0.6
2. probability of occurrence of the event giving the best pay-off = $\propto$ = 0.8

Do you have the same decision in each case?

Pay-off Matrix

States of Nature → / *Acts* ↓	S_1	S_2	S_3	S_4
A_1	100	200	80	300
A_2	250	250	400	300
A_3	160	200	150	150
A_4	40	300	100	160

11. Mr. Hemant has two plans, A and B; both require initial investment of ₹1000. There are 0.3 chances to get successful operations for Plan A. This will fetch him ₹10000 and failure will cause him ₹2000 to pay penalty. For the Plan B, there are 0.8 chances to get success and he will get ₹5000. If it fails then there is a penalty of ₹500.

Draw a decision tree and make a choice of any one.

12. Miss Silvi has three plans, A, B, and C; each needs an initial investment of ₹1000. Probabilities of success of Plans A, B, and C are 0.6, 0.7, and 0.9 respectively. Successful result of Plan A, Plan B, and Plan C will fetch profit of ₹10000, ₹9000, and ₹8000 respectively while failure will cause penalty of ₹1500, ₹2000, and ₹1000 respectively. Draw a decision tree and make a choice of any one plan.

13. Some facts of the decision tree are given in the given figure. Each Plan A and Plan B need an initial investment of ₹3000.

Probability that Plan A is successful is 0.8 and it assures a profit of ₹10000; while failure causes a penalty of ₹2000. On successful completion of Plan A, there are two further plans P and Q with an initial cost of ₹1000 on either Plan P or Plan Q. Probability of success for Plans P and Q are respectively 0.9 and 0.7 and will give a net profit of ₹5000 and ₹8000 respectively. On failure of Plan P there is a penalty of ₹1000 and for Plan Q there is no penalty on failure.

Probability that Plan A is successful is 0.7 and it assures a profit of ₹12000; while failure causes a penalty of ₹3000. On successful completion of Plan A, there are two further Plans P and Q with an initial cost of ₹2000 on either Plan P or Plan Q. Probability of success for Plans P and Q are respectively 0.8 and 0.6 and will give a net profit of ₹8000 and ₹10000 respectively. On failure of Plan P there is a penalty of ₹3000 and for Plan Q it is ₹4000. Draw a decision tree and choose a better plan.

14. A retailer brings a gold-plated bangle that costs him ₹500 each and he can sell it for ₹800 each. If unsold, he has to return on the next day and pays ₹100 as penalty, as a part of the agreement. The following table shows his probable chances for sale.

Bangles:	10	11	12
Probability:	0.5	0.3	0.2

How many bangles he should bring to maximize his expected profit?

15. It appeared that the market share of two companies P and Q in a finite population of 1200 was equal in proportion.

A survey revealed that 80 % of the Company P want to continue their association with the Company P while other 20% plan to join the Company Q. In addition to this, it was learnt that 40% from Company Q want to switch to Company P while other were to continue with the Company Q. From this data, make your next interval prediction.

16. In Question 15, if the transition matrix continues to remain same, find the steady state probability of market share of both the companies.

17. The current contribution of a fixed group of 1000 persons to the three brands B_1, B_2, and B_3 is 50%, 30%, and 20%. Using the following transition matrix, find the incremental share of Brand B_3.

To → / *From ↓*	*P*	*Q*	*R*
P out of 500	400	00	100
Q out of 300	60	210	30
R out of 200	20	00	180

18. We are given a transition matrix obtained from reliable source. When it attains a steady state position; find the stable values in percentage.

$$\text{Transition matrix} = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$$

19. There are three IT companies; professionals working as on today are 200, 200, and 100 in the companies P, Q, and R respectively. Better chances, promotions and other social factors liable to make internal switching is given by the following transition matrix. Calculate the percentage after two intervals.

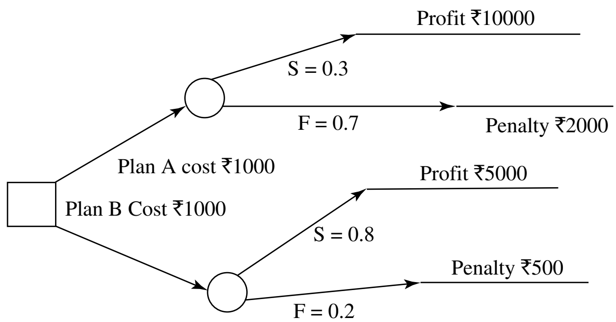
To → / *From* ↓	*P*	*Q*	*R*
P out of 200	120	40	40
Q out of 200	60	140	00
R out of 100	60	10	30

20. Sales figures are independent and it is established that there is either no sale state or a state of 5 items or 10 items. If there is no sale today, then probabilities having no sale, or a sale of 5 items, or a sale of 10 items are 0.2, 0.7, and 0.1 respectively. In the case of a sale of 5 items today, probabilities of sale being either of no item, or 5 items, or 10 items is 0.3, 0.6, and 0.1 respectively. In the case of a sale of 10 items today, probabilities of sale being either of no item, or 5 items, or 10 items is 0.4, 0.4, and 0.2 respectively. From the given records, write steady state equations of obtaining sale status after three intervals.

Answers to Numerical Problems

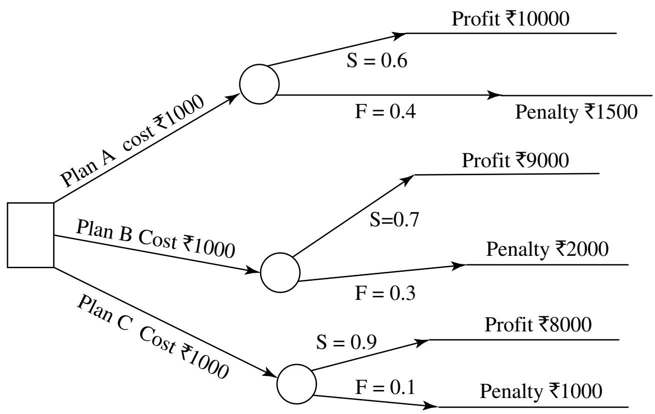
1. maximin = 250, minimax = 50
2. maximax = 400, maximin = 250
3. maximin = 250, minimax = 50
4. EMV(A_1) = 182, EMV(A_2) = 330, EMV(A_3) = 156, EMV(A_4) = 138, choice act is A_2
5. EOL(A_1) 153, EOL(A_2) = 5, EOL(A_3) = 179, EOL(A_4) = 197, Choice act is A_2
6. EVPI = 5, highest EMV = 330, expected perfect probabilistic information value = 335
7. EMV for Act (A_1) = 60, EMV for Act (A_2) = 64.65, EMV for Act (A_3) = 68.46
   EMV for Act (A_4) = 71.22, EMV for Act (A_5) = 71.50, EMV for Act (A_6) = 71.28
   EMV for Act (A_7) = 69.49
   Best act is that gives highest EMV. The best act is the choice of Act A_5; EMV = 71.50
8. EOL for Act A_1 = 15.35, EOL for Act A_2 = 13.2, EOL for Act A_3 = 6.89
   EOL for Act A_4 = 4.13, EOL for Act A_5 = 3.05, EOL for Act A_6 = 4.07
   EOL for Act A_7 =5.86
   The act corresponding to minimum opportunity loss is selected.
   The best act is A_5; minimum EOL = 3.05 EVPI = 3.05
9. Average pay-off from A_1 = 170, average pay-off from A_2 = 300,
   average pay-off from A_3 = 165, average pay-off from A_4 = 150,
   The highest pay-off corresponds to Act A_2. Choice is Act A_2.
10. For $\propto$ = 0.6, H(A) = 212, H(B) = 340, H(C) = 184, H(D) = 196; selection of Act B
    For $\propto$ = 0.8, H(A) = 256, H(B) = 370, H(C) = 190, H(D) = 248; selection of Act B

11.

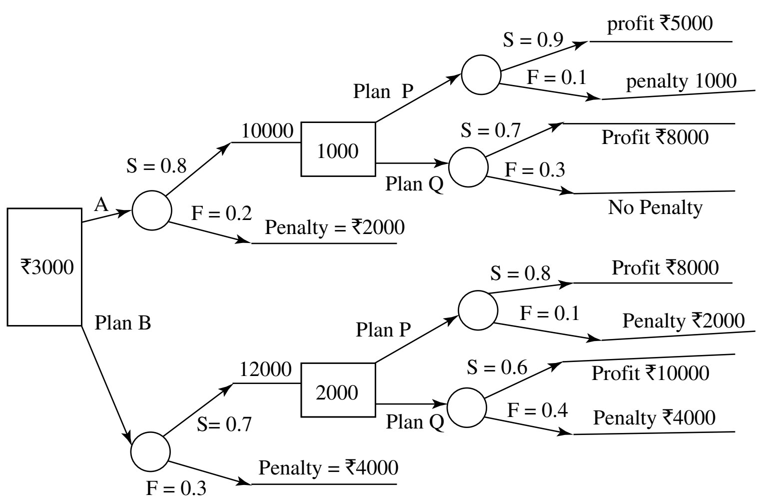


Plan B is better; it gives ₹2900

12.



cash-flow from Plan A is ₹5400 – ₹1000 = ₹4400
cash-flow from Plan B is ₹5700 – ₹1000 = ₹4700
cash-flow from plan C is ₹7100 – ₹1000 = ₹6100 ***
This Plan C is the choice of action.



Cash-flow from plan A:
expected cash-flow from Plan P = 5000 × 0.9 – 1000 × 0.1 = ₹4400
expected cash-flow from Plan Q = 8000 × 0.7 = ₹5600 * Selected project cost = ₹1000,
residual amount = ₹5600 – ₹1000 = ₹4600
and on successful completion of P = ₹10000;
total = ₹4600 + ₹10000 = ₹14600
expectation = 14600 × 0.8 + (–2000 × 0.2) = 11680 – 400 = ₹11280
project cost = ₹3000
net profit = ₹11280 – 3000 = ₹8280
Similarly from Plan B, net profit is ₹6860
Plan A is better than do the Plan B.

14. Incremental profit = IP = ₹800 – ₹500 = ₹300
incremental loss = IL = ₹125
probability = p = IL / (IP + IL) = 125/ (300 + 125) = 0.294
cumulative distribution (more than type)

Bangles	Probability	Cumulative Probability
10	0.5	1.0
11	0.3	0.5
12	0.2	0.2

$P = 0.294 > 0.2$; He can bring 11 bangles to sell.

15. New proportion will be 60% and 40% of the market share will be with Company P and the Company Q respectively.
16. The market share of Company P tends to 2/3 of the total population and for the Company Q, it tends to 1/3 of total population.
17. From 20% it becomes 31%; an increment of 11%
18. x = 60% and y = 40%
19. P 48.6%, Q 37.6 %, and R 13.8%
20. Current status (X, Y, Z),

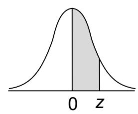
$$\text{Transition matrix } T = \begin{pmatrix} 0.2 & 0.7 & 0.1 \\ 0.3 & 0.6 & 0.1 \\ 0.4 & 0.4 & 0.2 \end{pmatrix}$$

Status of states after next 3 iterations is $(X, Y, Z)\, T^3$.

APPENDIX

Statistical and Other Tables

TABLE I Areas under the Standard Normal Curve from 0 to z



z	0	1	2	3	4	5	6	7	8	9
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	4495	4505	.4515	.4525	.4535	4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890

(*Contd.*)

(*Contd.*)

2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

TABLE II Random Numbers

22 17 68 65 84	68 95 23 92 35	61 09 43 95 06	87 02 22 57 51	58 24 82 03 47
19 36 27 59 46	13 79 93 37 55	85 52 05 30 62	39 77 32 77 09	47 83 51 62 74
16 77 23 02 77	09 61 87 25 21	16 71 13 59 78	28 06 24 25 93	23 05 47 47 25
78 43 76 71 61	20 44 90 32 64	46 38 03 93 22	97 67 63 99 61	69 81 21 99 21
03 28 28 26 08	73 37 32 04 05	88 69 58 28 99	69 30 16 09 05	35 07 44 75 47
93 22 53 64 39	07 10 63 76 35	08 13 13 85 51	87 03 04 79 88	55 34 57 72 69
78 76 58 54 74	92 38 70 96 92	82 63 18 27 44	52 06 79 79 45	69 66 92 19 09
23 68 35 26 00	99 53 93 61 28	56 65 05 61 86	52 70 05 48 34	90 92 10 70 80
15 39 25 70 99	93 86 52 77 65	22 87 26 07 47	15 33 59 05 28	86 96 98 29 06
58 71 96 30 24	18 46 23 34 27	49 18 09 79 49	85 13 99 24 44	74 16 32 23 02
57 35 27 33 72	24 53 63 94 09	44 04 95 49 66	41 10 76 47 91	39 60 04 59 81
48 50 86 54 48	22 06 34 72 52	33 29 94 71 11	82 21 15 65 20	15 91 29 12 02
61 96 48 95 03	07 16 39 33 66	77 21 30 27 12	98 56 10 56 79	90 49 22 23 62
36 93 89 41 26	29 70 83 63 51	87 09 41 15 09	99 74 20 52 36	98 60 16 03 03
18 87 00 42 31	57 90 12 02 07	54 08 01 88 63	23 47 37 17 31	39 41 88 92 10
88 56 53 27 59	33 35 72 67 47	08 18 27 38 90	77 34 55 45 70	16 95 86 70 75
09 72 95 84 29	49 41 31 06 70	64 84 73 31 65	42 38 06 45 18	52 53 37 97 15
12 96 88 17 31	65 19 69 02 83	24 64 19 35 51	60 75 86 90 68	56 61 87 39 12
85 94 57 24 16	92 09 84 38 76	29 81 94 78 70	22 00 27 69 85	21 94 47 90 12
38 64 43 59 98	98 77 87 68 07	40 98 05 93 78	91 51 67 62 44	23 32 65 41 18
53 44 09 42 72	00 41 86 79 79	35 55 31 51 51	68 47 22 00 20	00 83 63 22 55
40 76 66 26 84	57 99 99 90 37	37 40 13 68 97	36 63 32 08 58	87 64 81 07 83
02 17 79 18 05	12 59 52 57 02	28 14 11 30 79	22 07 90 47 03	20 69 22 40 98
95 17 82 06 53	31 51 10 96 46	56 11 50 81 69	92 06 88 07 77	40 23 72 51 39
35 76 22 42 92	96 11 83 44 80	33 42 40 90 60	34 68 35 48 77	73 96 53 97 86
26 29 13 56 41	85 47 04 66 08	82 43 80 46 15	34 72 57 59 13	38 26 61 70 04
77 80 20 75 82	72 82 32 99 90	89 73 44 99 05	63 95 73 76 63	48 67 26 43 18
46 40 66 44 52	91 36 74 43 53	78 45 63 98 35	30 82 13 54 00	55 03 36 67 68
37 56 08 18 09	77 53 84 46 47	24 16 74 11 53	31 91 18 95 58	44 10 13 85 57
61 65 61 68 66	37 27 47 39 19	53 21 40 06 71	84 83 70 07 48	95 06 79 88 54
93 43 69 64 07	34 18 04 52 35	61 85 53 83 45	56 27 09 24 86	19 90 70 99 00
21 96 60 12 99	11 20 99 45 18	18 37 79 49 90	48 13 93 55 34	65 97 38 20 46
95 20 47 97 97	27 37 83 28 71	45 89 09 39 84	00 06 41 41 74	51 67 11 52 49
97 86 21 78 73	10 65 81 92 59	04 76 62 16 17	58 76 17 14 97	17 95 70 45 80
69 92 06 34 13	59 71 74 17 32	23 71 82 13 74	27 55 10 24 19	63 52 52 01 41

(*Contd.*)

(*Contd.*)

04	31	17	21	56	33	73	99	19	87	53	77	57	68	93	26	72	39	27	67	60	61	97	22	61
61	06	98	03	91	87	14	77	43	96	45	60	33	01	07	43	00	65	98	50	98	99	46	50	47
85	93	85	86	88	72	87	08	62	40	23	21	34	74	97	16	06	10	89	20	76	38	03	29	63
21	74	32	47	45	73	96	07	94	52	25	76	16	19	33	09	65	90	77	47	53	05	70	53	30
15	69	53	82	80	79	96	23	53	10	45	33	02	43	70	65	39	07	16	29	02	87	40	41	45
02	89	08	04	49	20	21	14	68	86	11	29	01	95	80	87	63	93	95	17	35	14	97	35	33
87	18	15	89	79	85	43	01	72	73	89	74	39	82	15	08	61	74	51	69	94	51	33	41	67
98	83	71	94	22	59	97	50	99	52	87	80	61	65	31	08	52	85	08	40	91	51	80	32	44
10	08	58	21	66	72	68	49	29	31	59	73	19	85	23	89	85	84	46	06	65	09	29	75	25
47	90	56	10	08	88	02	84	27	83	66	56	45	65	79	42	29	72	23	19	20	71	53	20	25
22	85	61	68	90	49	64	92	85	44	50	14	49	81	06	16	40	12	89	88	01	82	77	45	12
67	80	43	79	33	12	83	11	41	16	77	02	54	00	52	25	58	19	68	70	53	43	37	15	26
27	62	50	96	72	79	44	61	40	15	27	31	58	50	28	14	53	40	65	39	11	39	03	34	25
33	78	80	87	15	38	30	06	38	21	54	96	87	53	32	14	47	47	07	26	40	36	40	96	76
13	13	92	66	99	47	24	49	57	74	10	97	11	69	84	32	25	43	62	17	99	63	22	32	98
10	27	53	96	23	71	50	54	36	23	04	14	12	15	09	54	31	04	82	98	26	78	25	47	47
28	41	50	61	88	64	85	27	20	18	39	71	65	09	62	83	36	36	05	56	94	76	62	11	89
34	21	42	57	02	59	19	18	97	48	05	24	67	70	07	80	30	03	30	98	84	97	50	87	46
61	81	77	23	23	82	82	11	54	08	44	07	39	55	43	53	28	70	58	96	42	34	43	39	28
61	15	18	13	54	16	86	20	26	88	14	53	90	51	17	90	74	80	55	09	52	01	63	01	59

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