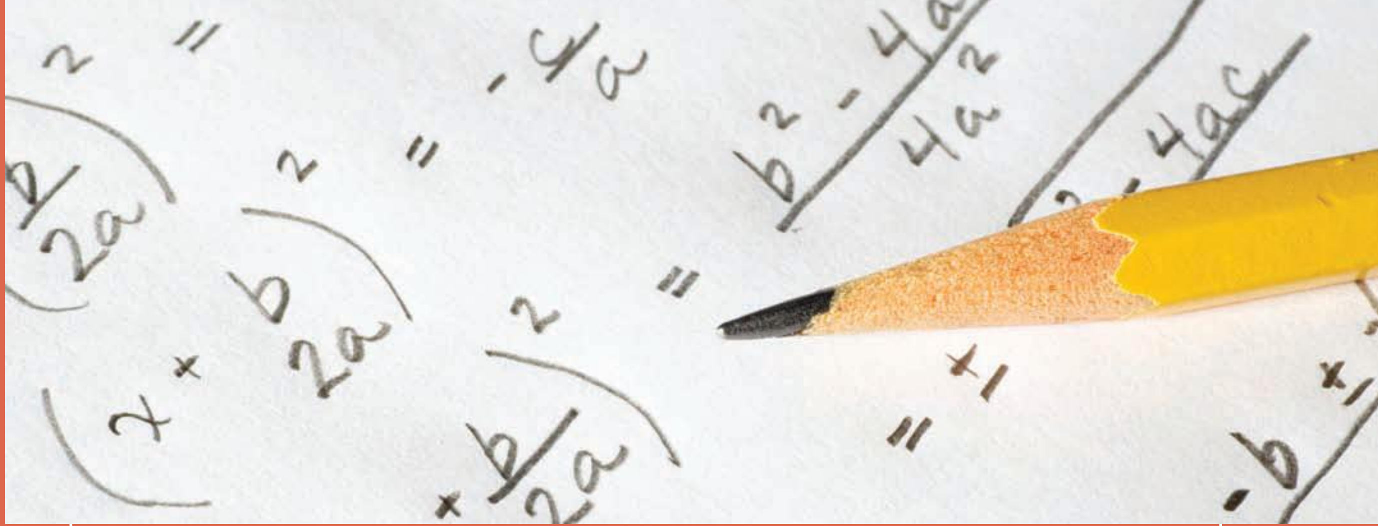

Beginning and Intermediate Algebra

third edition

Sherri Messersmith

College of DuPage

Mc Graw Hill

Connect Learn Succeed™

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BEGINNING AND INTERMEDIATE ALGEBRA, THIRD EDITION

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Message from the Author

Dear Colleagues,

Students constantly change—and over the last 10 years, they have changed a lot, therefore this book was written for today's students. I have adapted much of what I had been doing in the past to what is more appropriate for today's students. This textbook has evolved from the notes, worksheets, and teaching strategies I have developed throughout my 25-year teaching career in the hopes of sharing with others the successful approach I have developed.

To help my students learn algebra, I meet them where they are by helping them improve their basic skills and then showing them the connections between arithmetic and algebra. Only then can they learn the algebra that is the course. Throughout the book, concepts are presented in **bite-size pieces** because developmental students learn best when they have to digest fewer new concepts at one time. The **Basic Skills Worksheets** are quick, effective tools that can be used in the classroom to help strengthen students' arithmetic skills, and most of these worksheets can be done in 5 minutes or less. The **You Try** exercises follow the examples in the book so that students can practice concepts immediately. The **Fill-It-In exercises** take students through a step-by-step process of working multistep problems, asking them to fill in a step or a reason for a given step to prepare them to work through exercises on their own and to reinforce mathematical vocabulary. **Modern applications** are written with student interests in mind; students frequently comment that they have never seen "fun" word problems in a math book before this one. **Connect Mathematics hosted by ALEKS** is an online homework manager that will identify students' strengths and weaknesses and take the necessary steps to help them master key concepts.

The **writing style** is friendlier than that of most textbooks. Without sacrificing mathematical rigor, this book uses language that is mathematically sound yet easy for students to understand. Instructors and students appreciate its conversational tone because it sounds like a teacher talking to a class. The **use of questions** throughout the prose contributes to the conversational style while teaching students how to ask themselves the questions we ask ourselves when solving a problem. This friendly, less intimidating writing style is especially important because many of today's developmental math students are enrolled in developmental reading as well.

Beginning and Intermediate Algebra, third edition, is a compilation of what I have learned in the classroom, from faculty members nationwide, from the national conferences and faculty forums I have attended, and from the extensive review process. Thank you to everyone who has helped me to develop this textbook. My commitment has been to write the most mathematically sound, readable, student-friendly, and up-to-date text with unparalleled resources available for both students and instructors. To share your comments, ideas, or suggestions for making the text even better, please contact me at sherri.messersmith@gmail.com. I would love to hear from you.

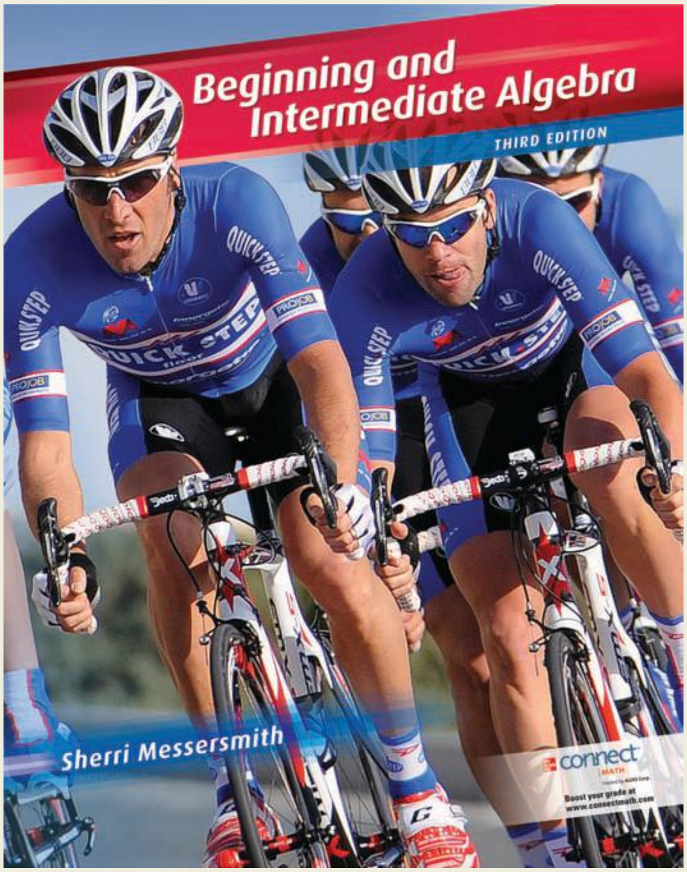
Sherri Messersmith

About the Author

Sherri Messersmith has been teaching at College of DuPage in Glen Ellyn, Illinois, since 1994. She has over 25 years of experience teaching many different courses from developmental mathematics through calculus. She earned a bachelor of science degree in the teaching of mathematics at the University of Illinois at Urbana-Champaign and went on to teach at the high school level for two years. Sherri returned to UIUC and earned a master of science in applied mathematics and stayed on at the university to teach and coordinate large sections of undergraduate math courses. This is her third textbook, and she has also appeared in videos accompanying several McGraw-Hill texts.

Sherri lives outside of Chicago with her husband, Phil, and their daughters, Alex and Cailen. In her precious free time, she likes to read, play the guitar, and travel—the manuscripts for this and her previous books have accompanied her from Spain to Greece and many points in between.

About the Cover



In order to be successful, a cyclist must follow a strict training regimen. Instead of attempting to compete immediately, the athlete must practice furiously in smaller intervals to build up endurance, skill, and speed. A true competitor sees the connection between the smaller steps of training and final accomplishment. Similarly, after years of teaching, it became clear to Sherri Messersmith that mastering math for most students is less about the memorization of facts and more of a journey of studying and understanding what may seem to be complex topics. Like a cyclist training for a long race, as pictured on the front cover, students must build their knowledge of mathematical concepts by connecting and applying concepts they already know to more challenging ones, just as a cyclist uses training and hard work to successfully work up to longer, more challenging rides. After following the methodology applied in this text, like a cyclist following a training program, students will be able to succeed in their course.

Brief Contents

Preface

Building a Better Path to Success

Beginning and Intermediate Algebra, third edition, helps students build a better path to success by providing the tools and building blocks necessary for success in their mathematics course. The author, Sherri Messersmith, learned in her many years of teaching that students had a better rate of success when they were connecting their knowledge of arithmetic with their study of algebra. By **making these connections between arithmetic and algebra** and **presenting the concepts in more manageable, "bite-size" pieces,** Sherri better equips her students to learn new concepts and strengthen their skills. In this process, students practice and build on what they already know so that they can understand and **master new concepts** more easily. These practices are integrated throughout the text and the supplemental materials, as evidenced below:

Connecting Knowledge

- **Examples** draw upon students' current knowledge while connecting to concepts they are about to learn with the positioning of arithmetic examples before corresponding algebraic examples.
- Inclusion of a **geometry review** in Section 1.3 allows students to practice several geometry concepts without variables *before* they have to write algebraic equations to solve geometry problems.
- The very popular author-created **worksheets** that accompany the text provide instructors with additional exercises to assist with overcoming potential stumbling blocks in student knowledge and to help students see the connections among multiple mathematical concepts. The worksheets fall into three categories:
  - Worksheets to Strengthen Basic Skills
  - Worksheets to Help Teach New Concepts
  - Worksheets to Tie Concepts Together

Presenting Concepts in Bite-Size Pieces

- The **chapter organization** help break down algebraic concepts into more easily learned, more manageable pieces.
- New **Fill-It-In exercises** take a student through the process of working a problem step-by-step so that students have to provide the reason for each mathematical step to solve the problem, much like a geometry proof.
- New **Guided Student Notes** are an amazing resource for instructors to help their students become better note-takers. They contain in-class examples provided in the margin of the text along with additional examples not found in the book to emphasize the given topic so that students spend less time copying down information and more time engaging within the classroom.
- **In-Class Examples** give instructors an additional tool that exactly mirrors the corresponding examples in the book for classroom use.

Mastering Concepts:

- **You Try** problems follow almost every example in the text to provide students the opportunity to immediately apply their knowledge of the concept being presented.
- **Putting It All Together** sections allow the students to synthesize important concepts presented in the chapter sooner rather than later, which helps in their overall mastery of the material.
- **Connect Math hosted by ALEKS** is the combination of an online homework manager with an artificial-intelligent, diagnostic assessment. It allows students to identify their strengths and weaknesses and to take the necessary steps to master those concepts. Instructors will have a platform that was designed through a comprehensive market development process involving full-time and adjunct math faculty to better meet their needs.

Connecting Knowledge

Examples The **examples** in each section begin with an arithmetic equation that mirrors the algebraic equation for the concept being presented. This positioning allows students to apply their knowledge of arithmetic to the algebraic problem, making the concept more easily understandable.

1. Add and Subtract Rational Expressions with a Common Denominator

Let's first look at fractions and rational expressions with common denominators.

Example 1

Add or subtract.

a) $\frac{8}{11} - \frac{5}{11}$ b) $\frac{2x}{4x-9} + \frac{5x+3}{4x-9}$

Solution

a) Since the fractions have the same denominator, subtract the terms in the numerator and keep the common denominator.

$$\frac{8}{11} - \frac{5}{11} = \frac{8-5}{11} = \frac{3}{11} \quad \text{Subtract terms in the numerator.}$$

b) Since the rational expressions have the same denominator, add the terms in the numerator and keep the common denominator.

$$\frac{2x}{4x-9} + \frac{5x+3}{4x-9} = \frac{2x+(5x+3)}{4x-9} \quad \text{Add terms in the numerator.}$$
$$= \frac{7x+3}{4x-9} \quad \text{Combine like terms.}$$

"Messersmith does a great job of addressing students' abilities with the examples and explanations provided, and the thoroughness with which the topic is addressed is excellent." Tina Evans, Shelton State Community College

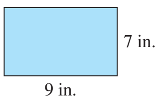
"The author is straightforward, using language that is accessible to students of all levels of ability. The author does an excellent job explaining difficult concepts and working from easier to more difficult problems." Lisa Christman, University of Central Arkansas

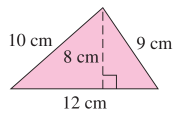
Geometry Review A **geometry review** in Chapter 1 Section 1.3, provides the material necessary for students to revis it the geometry concepts they will need later in the course. Reviewing the geometry early, rather than in an appendix or not at all, removes a common stumbling block for students. The book also includes geometry applications, where appropriate, throughout.

"The Geometry Review is excellent for this level." Abraham Biggs, Broward College-South

Example 3

Find the perimeter and area of each figure.

a) 

b) 

Solution

a) This figure is a rectangle.

Perimeter: $P = 2l + 2w$
$P = 2(9 \text{ in.}) + 2(7 \text{ in.})$
$P = 18 \text{ in.} + 14 \text{ in.}$
$P = 32 \text{ in.}$

Area: $A = lw$
$A = (9 \text{ in.})(7 \text{ in.})$
$A = 63 \text{ in}^2$ or 63 square inches

Worksheets Supplemental **worksheets** for *every* section are available online through Connect. They fall into three categories: worksheets to strengthen basic skills, worksheets to help teach new concepts, and worksheets to tie concepts together. These worksheets provide a quick, engaging way for students to work on key concepts. They save instructors from having to create their own supplemental material, and they address potential stumbling blocks. They are also a great resource for standardizing instruction across a mathematics department.

Worksheet 2F Name: ____________

Messersmith–Beginning & Intermediate Algebra

1)	5^2 ____	16)	11^2 ____
2)	2^4 ____	17)	$(-5)^3$ ____
3)	9^0 ____	18)	2^4 ____
4)	3^3 ____	19)	1^9 ____
5)	-11^2 ____	20)	10^2 ____
6)	8^2 ____	21)	-2^2 ____
7)	2^6 ____	22)	3^2 ____
8)	7^2 ____	23)	4^3 ____
9)	3^4 ____	24)	-1^8 ____
10)	5^0 ____	25)	2^3 ____
11)	6^2 ____	26)	13^2 ____
12)	-9^2 ____	27)	-3^4 ____
13)	5^3 ____	28)	$(-2)^4$ ____
14)	$(-4)^2$ ____	29)	12^0 ____
15)	2^5 ____	30)	5^3 ____

"I really like it and many topics are covered the way I would teach them in my classroom."
Pamela Harden,
Tennessee Tech University

Worksheet 3C Name: ____________

Messersmith–Beginning & Intermediate Algebra

Find 2 numbers that . . .

MULTIPLY TO	*and* ADD TO	ANSWER
−27	−6	−9 and 3
72	18	
24	−11	
−4	3	
10	−7	
121	22	
−54	−3	
54	29	
16	−10	
30	17	
9	−6	
−8	−2	
21	10	
60	−19	
56	15	
−28	3	
−72	−6	
100	25	
−40	6	
11	−12	
20	12	
−35	−2	
77	18	
108	21	
−3	−2	

"Messersmith has a very simple and clear approach to each objective. Messersmith tends to think where students have most difficulties and provide examples and explanation on those areas." Avi Kar Abraham Baldwin, Agricultural College

Presenting Concepts in Bite-Size Pieces

Chapter Organization The **chapter organization** is designed to present the context in "bite-size" pieces, focusing not only on the mathematical concepts but also on the "why" behind those concepts. By breaking down the sections into manageable chunks, the author has identified the core places where students traditionally struggle.

"The material is presented in a very understandable manner, in that it approaches all topics in bite-sized pieces and explains each step thoroughly as it proceeds through the examples."
Lee Ann Spahr,
Durham Technical Community College

CHAPTER 10

Radicals and Rational Exponents

Algebra at Work: Forensics

Forensic scientists use mathematics in many ways to help them analyze evidence and solve crimes. To help him reconstruct an accident scene, Keith can use this formula containing a radical to estimate the minimum speed of a vehicle when the accident occurred:

$$S = \sqrt{30fd}$$

where f = the drag factor, based on the type of road surface
d = the length of the skid, in feet
S = the speed of the vehicle, in miles per hour

Keith is investigating an accident in a residential neighborhood where the speed limit is 25 mph. The car involved in the accident left skid marks 60 ft

In-Class Examples To give instructors additional material to use in the classroom, a matching ***In-Class Example*** is provided in the margin of the Annotated Instructor's Edition for every example in the book. The more examples a student reviews, the better chance he or she will have to understand the related concept.

Example 11

Write an equation and solve.

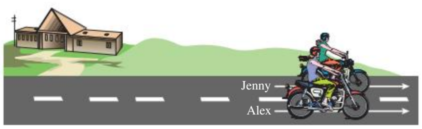
Alex and Jenny are taking a cross-country road trip on their motorcycles. Jenny leaves a rest area first traveling at 60 mph. Alex leaves 30 minutes later, traveling on the same highway, at 70 mph. How long will it take Alex to catch Jenny?

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must determine how long it takes Alex to catch Jenny.

We will use a picture to help us see what is happening in this problem.



Since both girls leave the same rest area and travel on the same highway, when Alex catches Jenny they have driven the *same* distance.

"I like that the teacher's edition gives in-class examples to use so the teacher doesn't have to spend prep time looking for good examples or using potential homework/exam questions for in-class examples."
Judith Atkinson,
University of Alaska–Fairbanks

Fill-It-In **Fill-It-In exercises** take a student through the process of working a problem step-by-step so that students either have to provide the reason for each mathematical step or fill in a mathematical step when the reason is given. These types of exercises are unique to this text and appear throughout.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

39) $64^{-1/2} = \left(\underline{\quad}\right)^{1/2}$ The reciprocal of 64 is ___.

$= \sqrt{\dfrac{1}{64}}$ ______________________

$= \underline{\quad}$ Simplify.

40) $\left(\dfrac{1}{1000}\right)^{-1/3} = (\underline{\quad})^{1/3}$ The reciprocal of $\dfrac{1}{1000}$ is ___.

$= \sqrt[3]{1000}$ ______________________

$= \underline{\quad}$ Simplify.

"I love how your problems are set up throughout each section and how they build upon the problems before them. It has great coverage of all types as well." Keith Pachlhofer, University of Central Arkansas

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

39) $64^{-1/2} = \left(\dfrac{1}{64}\right)^{1/2}$ The reciprocal of 64 is $\dfrac{1}{64}$.

$= \sqrt{\dfrac{1}{64}}$ The denominator of the fractional exponent is the index of the radical.

$= \dfrac{1}{8}$ Simplify.

40) $\left(\dfrac{1}{1000}\right)^{-1/3} = (1000)^{1/3}$ The reciprocal of $\dfrac{1}{1000}$ is 1000.

$= \sqrt[3]{1000}$ The denominator of the fractional exponent is the index of the radical.

$= 10$ Simplify.

"I would describe this text as being student centered, one that offers readability for the student in both explanations and definitions, nice worked examples, and a wide variety of practice exercises to enhance the students', understanding of the concepts being discussed." Kim Cain, Miami University–Hamilton

Guided Student Notes Name: ____________________

Messersmith–Beginning & Intermediate Algebra

1.1 Review of Fractions

Definition of Fraction

What part of the figure is shaded?

1) 

Definition of Lowest Terms

Factors of a Number

2) Find all factors of 18

3) Find all factors of 54

Prime Factorization **Composite Numbers** **Prime Numbers**

Guided Student Notes Name: ____________________

Messersmith–Beginning & Intermediate Algebra

2.1 Basic Rules of Exponents Product Rule and Power Rule

Base

Exponent

Identify the base and the exponent in each expression and evaluate

1) 3^4

2) $(-3)^4$

3) -3^4

4) -5^2

5) $(-5)^3$

6) $2(5)^2$

7) $4a(-3)^2$

8) $-(2)^4$

Product Rule

Find each product

9) $5^2 \cdot 5$

10) $y^4 y^9$

11) $(-2x)^2 \cdot (-2x)^3$

12) $d \cdot d^7 \cdot d^4$

Power Rule

Simplify using the power rule

13) $(4^6)^3$

14) $(m^2)^7$

15) $(q^8)^7$

16) $(df^2)^3$

Guided Student Notes New **Guided Student Notes** are an amazing resource for instructors to help their students become better note-takers. They contain in-class examples provided in the margin of the text along with additional examples not found in the book to emphasize the given topic so that students spend less time copying down information and more time engaging within the classroom. A note will be available for each section of the text.

"This text is easier to read than most texts, using questions to prompt the student's thinking. In general, it breaks the concepts down into smaller bite-size pieces, with exercises that build progressively from just-in-time review problems to more difficult exercises."
Cindy Bond, Butler Community College

"This author has written one of the best books for this level in the past 15 years that I have been teaching. It is one of the top three books I have seen in my teaching career."
Edward Koslowska, Southwest Texas Junior College–Uvalde

Mastering Concepts

You Try After almost every example, there is a ***You Try*** problem that mirrors the example. This provides students the opportunity to practice a problem similar to what the instructor has presented before moving on to the next concept.

"Comparing Martin-Gay to Messersmith is like comparing Tin to Gold." Mark Pavitch, California State University–Los Angeles LA

"Sherri Messersmith weaves well-chosen examples with frequently occurring, student-directed ***You Try*** problems to reinforce concepts in a clear and concise manner." Jon Becker, Indiana University Northwest–Gary

Example 9

Perform the operation and simplify.

a) $\frac{3}{11} + \frac{5}{11}$ b) $\frac{17}{30} - \frac{13}{30}$

Solution

a) $\frac{3}{11} + \frac{5}{11} = \frac{3+5}{11}$ Add the numerators and keep the denominator the same.

$= \frac{8}{11}$

b) $\frac{17}{30} - \frac{13}{30} = \frac{17-13}{30}$ Subtract the numerators and keep the denominator the same.

$= \frac{4}{30}$ This is not in lowest terms, so reduce.

$= \frac{2}{15}$ Simplify.

You Try 9

Perform the operation and simplify.

a) $\frac{5}{9} + \frac{2}{9}$ b) $\frac{19}{20} - \frac{7}{20}$

Putting It All Together Several chapter include a ***Putting It All Together*** section, in keeping with the author's philosophy of breaking sections into manageable chunks to increase student comprehension. These sections help students synthesize key concepts before moving on to the rest of the chapter.

Putting It All Together

Objective

1. Combine the Rules of Exponents

1. Combine the Rules of Exponents

Let's review all the rules for simplifying exponential expressions and then see how we can combine the rules to simplify expressions.

Summary Rules of Exponents

In the rules stated here, a and b are any real numbers and m and n are positive integers.

Product rule	$a^m \cdot a^n = a^{m+n}$
Basic power rule	$(a^m)^n = a^{mn}$
Power rule for a product	$(ab)^n = a^n b^n$
Power rule for a quotient	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \quad (b \neq 0)$
Quotient rule	$\frac{a^m}{a^n} = a^{m-n}, \quad (a \neq 0)$

Changing from negative to positive exponents, where $a \neq 0$, $b \neq 0$, and m and n are any integers:

$$\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m} \qquad \left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$$

In the following definitions, $a \neq 0$, and n is any integer.

Zero as an exponent	$a^0 = 1$
Negative number as an exponent	$a^{-n} = \frac{1}{a^n}$

"I like the Putting It All Together midchapter summary. Any opportunity to spiral back around to previous concepts gives students the opportunity to make important connections with these properties rather than just leaving students with the impression that each section of the chapter is independent." Steve Felzer, Lenoir Community College

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Connect Math Hosted by ALEKS Corporation is an exciting, new assignment and assessment platform combining the strengths of McGraw-Hill Higher Education and ALEKS Corporation. Connect Math Hosted by ALEKS is the first platform on the market to combine an artificially-intelligent, diagnostic assessment with an intuitive ehomework platform designed to meet your needs.

Connect Math Hosted by ALEKS Corporation is the culmination of a one-of-a-kind market development process involving math full-time and adjunct Math faculty at every step of the process. This process enables us to provide you with a solution that best meets your needs.

Connect Math Hosted by ALEKS Corporation is built by Math educators for Math educators!

1 *Your students want a well-organized homepage where key information is easily viewable.*

Modern Student Homepage

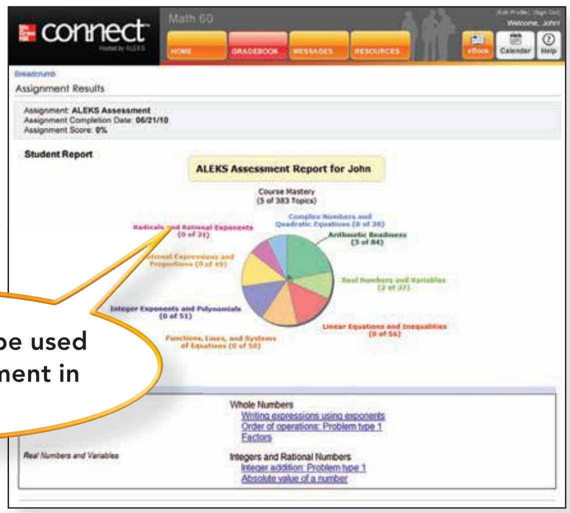
- This homepage provides a dashboard for students to immediately view their assignments, grades, and announcements for their course. (Assignments include HW, quizzes, and tests.)
- Students can access their assignments through the course Calendar to stay up-to-date and organized for their class.

Modern, intuitive, and simple interface.

2 *You want a way to identify the strengths and weaknesses of your class at the beginning of the term rather than after the first exam.*

Integrated ALEKS® Assessment

- This artificially-intelligent (AI), diagnostic assessment identifies precisely what a student knows and is ready to learn next.
- Detailed assessment reports provide instructors with specific information about where students are struggling most.
- This AI-driven assessment is the only one of its kind in an online homework platform.



Recommended to be used as the first assignment in any course.

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Built by Math Educators for Math Educators

3 *Your students want an assignment page that is easy to use and includes lots of extra help resources.*

Efficient Assignment Navigation

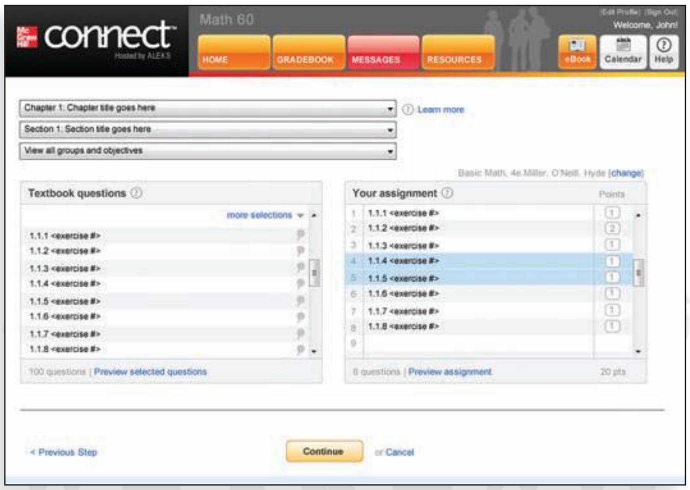
- Students have access to immediate feedback and help while working through assignments.
- Students have direct access to a media-rich eBook for easy referencing.
- Students can view detailed, step-by-step solutions written by instructors who teach the course, providing a unique solution to each and every exercise.

Students can easily monitor and track their progress on a given assignment.

4 *You want a more intuitive and efficient assignment creation process because of your busy schedule.*

Assignment Creation Process

- Instructors can select textbook-specific questions organized by chapter, section, and objective.
- Drag-and-drop functionality makes creating an assignment quick and easy.
- Instructors can preview their assignments for efficient editing.



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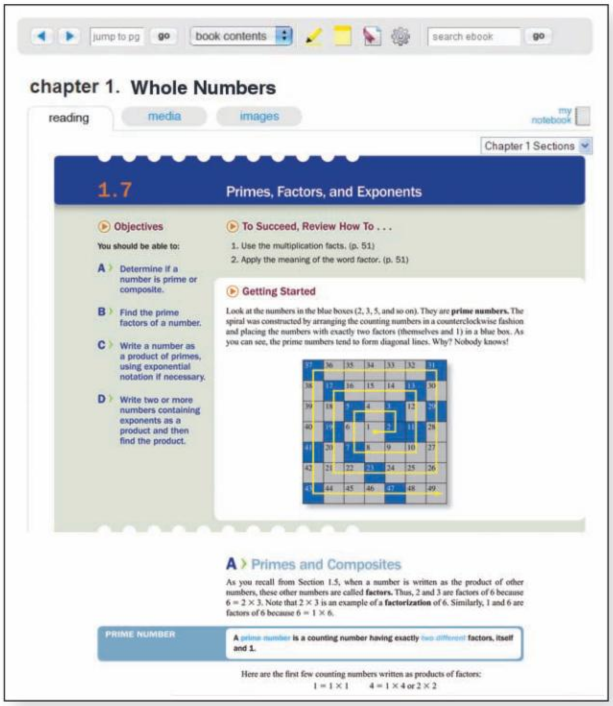
Integrated Media-Rich eBook

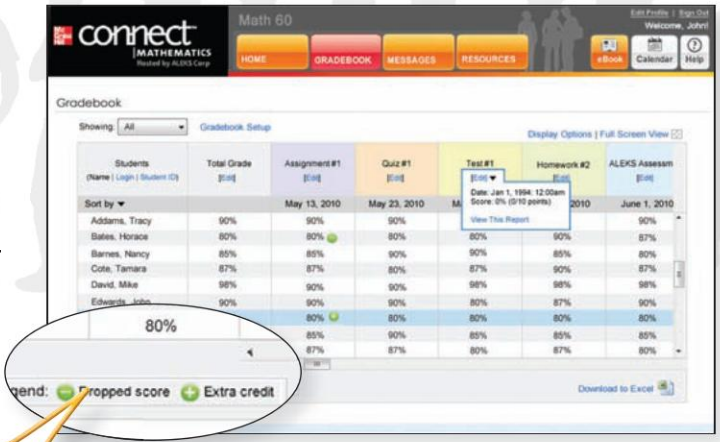
- A Web-optimized eBook is seamlessly integrated within ConnectPlus Math Hosted by ALEKS Corp for ease of use.
- Students can access videos, images, and other media in context within each chapter or subject area to enhance their learning experience.
- Students can highlight, take notes, or even access shared instructor highlights/notes to learn the course material.
- The integrated eBook provides students with a cost-saving alternative to traditional textbooks.



6 *You want a flexible gradebook that is easy to use.*

Flexible Instructor Gradebook

- Based on instructor feedback, Connect Math Hosted by ALEKS Corp's straightforward design creates an intuitive, visually pleasing grade management environment.
- Assignment types are color-coded for easy viewing.
- The gradebook allows instructors the flexibility to import and export additional grades.



Instructors have the ability to drop grades as well as assign extra credit.

Built by Math Educators for Math Educators

7 ***You want algorithmic content that was developed by math faculty to ensure the content is pedagogically sound and accurate.***

Digital Content Development Story

The development of McGraw-Hill's Connect Math Hosted by ALEKS Corp. content involved collaboration between McGraw-Hill, experienced instructors, and ALEKS, a company known for its high-quality digital content. The result of this process, outlined below, is accurate content created with your students in mind. It is available in a simple-to-use interface with all the functionality tools needed to manage your course.

1. McGraw-Hill selected experienced instructors to work as Digital Contributors.
2. The Digital Contributors selected the textbook exercises to be included in the algorithmic content to ensure appropriate coverage of the textbook content.
3. The Digital Contributors created detailed, stepped-out solutions for use in the Guided Solution and Show Me features.
4. The Digital Contributors provided detailed instructions for authoring the algorithm specific to each exercise to maintain the original intent and integrity of each unique exercise.
5. Each algorithm was reviewed by the Contributor, went through a detailed quality control process by ALEKS Corporation, and was copyedited prior to being posted live.

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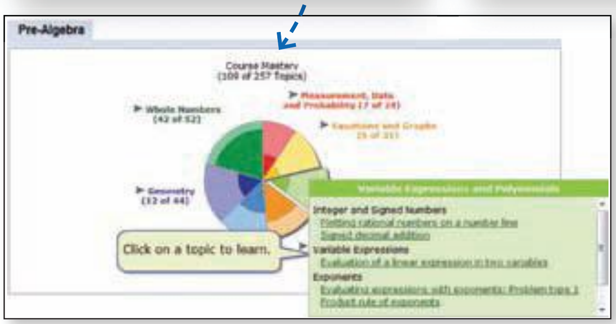

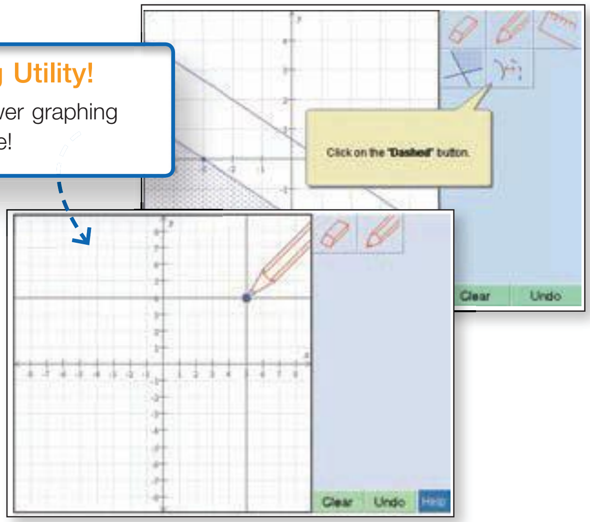

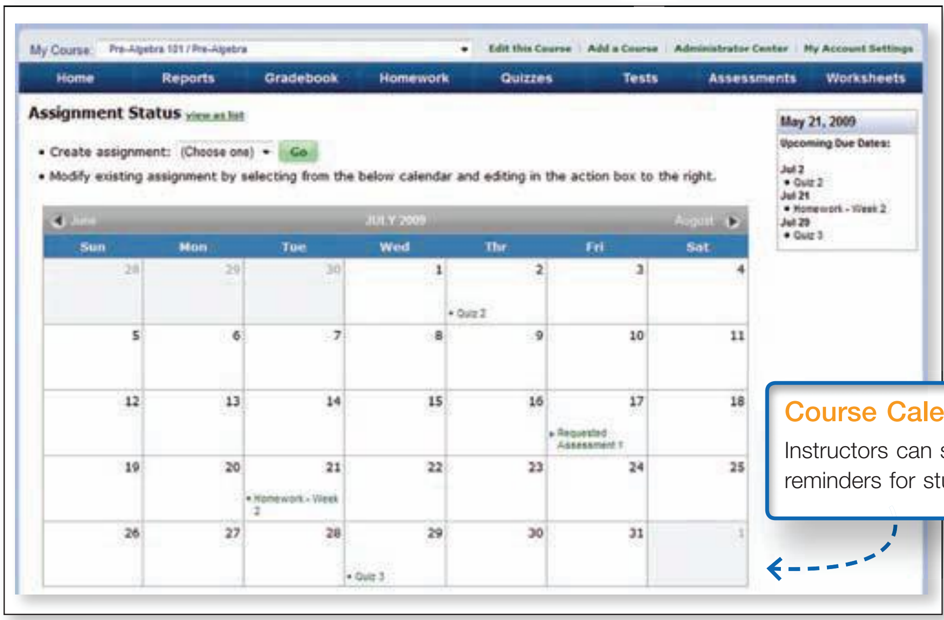

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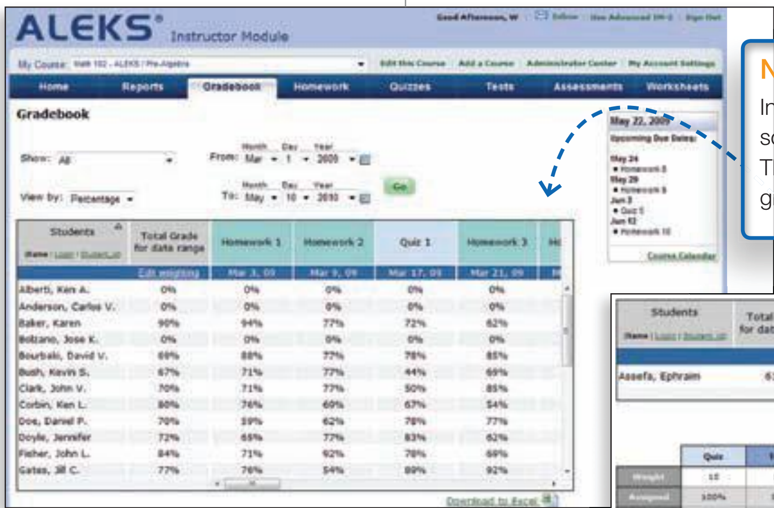

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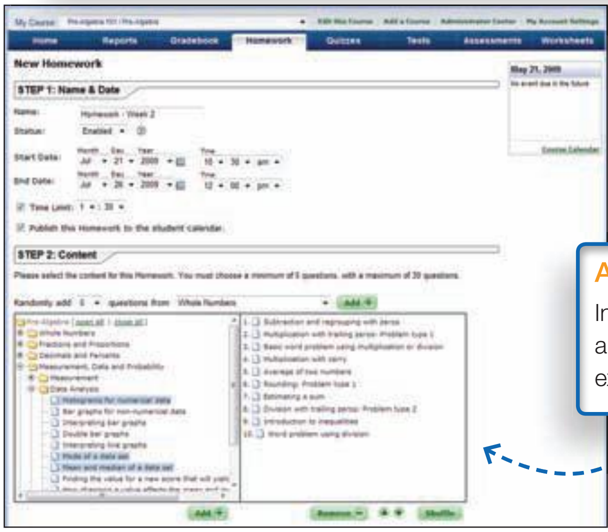

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Content Changes for the Third Edition of *Beginning and Intermediate Algebra*

360° Development Process

McGraw-Hill's 360° Development Process is an ongoing, never ending, market-oriented approach to building accurate and innovative print and digital products. It is dedicated to continual large scale and incremental improvement driven by multiple customer feedback loops and checkpoints. This is initiated during the early planning stages of our new products, and intensifies during the development and production stages, then begins again upon publication, in anticipation of the next edition.

A key principle in the development of any mathematics text is its ability to adapt to teaching specifications in a universal way. The only way to do so is by contacting those universal voices—and learning from their suggestions. We are confident that our book has the most current content the industry has to offer, thus pushing our desire for accuracy to the highest standard possible. In order to accomplish this, we have moved through an arduous road to production. Extensive and open-minded advice is critical in the production of a superior text.

Changes Throughout the Entire Book

1. Enriched exercise sets
   - More conceptual questions
   - More application questions
   - More rigorous questions
   - New **Fill-It-In exercises** that are not found in any other textbook. They take students through harder, multistep problems and help reinforce math vocabulary.
   - New **Putting It All Together** sections
   - Revised Summary, Review Exercises, Chapter Test, and Cumulative Review in every chapter
2. 17 newly created Using Technology and 6 newly created Be Careful pedagogical features

Chapter 1

Section 1.7 has been rewritten to contain Simplifying Expressions from the former Section 2.1.

27 new Examples
28 new In-Class Examples
28 new You Try exercises

Chapter 2

Rewritten summary of the rules of exponents
Rewritten explanation on writing a number in scientific notation

5 new Examples
2 new In-Class Examples
5 new You Try exercises

Chapter 3

All application examples have been reformatted for clearer presentation.
New and improved section material in Sections 3.3–3.6

55 new Examples
55 new In-Class Examples
56 new You Try exercises

New procedures on How to Solve a Linear Equation, How to Eliminate Decimals, and Steps for Solving Applied Problems

Chapter 4

Parallel and perpendicular lines have been absorbed into the section on slope and the section on writing equations of lines.
The topic of functions is now introduced in a single section.

23 new Examples
21 new In-Class Examples
17 new You Try exercises

New and improved explanations on Introduction to Linear Equations in Two Variables and Graphing a Linear Equation of the Form Ax 1 By = 0
New procedure on Writing an Equation of a Line Given Its Slope and y-Intercept

Chapter 5

Application examples have been reformatted for clearer presentation.

19 new Examples
20 new In-Class Examples
17 new You Try exercises

Revised and updated procedure on Solving a System by Graphing
New procedure on Solving an Applied Problem Using a System of Equations

Chapter 6

32 new Examples
32 new In-Class Examples
18 new You Try exercises

New summary of Dividing and Multiplying Polynomials

Chapter 7

33 new Examples
33 new In-Class Examples
33 new You Try exercises

New section material on Applications of Quadratic Equations
New steps for solving Applied Problems

Chapter 8

40 new Examples
40 new In-Class Examples
36 new You Try exercises

New section material on simplifying complex fractions Revised methods for simplifying a complex fraction

Chapter 9

4 new Examples
4 new In-Class Examples
5 new You Try exercises

New section material on Matrices
New procedure on How to Solve a System of Equations Using Gaussian Elimination

Chapter 10

Adding, Subtracting and Multiplying Radicals are now all introduced in the same section.
The topic of complex numbers is now introduced in Chapter 10 before the discussion on Quadratic Equations.

22 new Examples
21 new In-Class Examples
21 new You Try exercises

New section material on Solving Radical Equations, nth roots, Finding the Square Root of a Negative Number, Simplifying the Power of i, and Rationalizing a Numerator

Chapter 11

The sections on the square root property and completing the square have been combined into a single section.
Application examples have been reformatted for clearer presentation.

3 new Examples
3 new In-Class Examples
3 new You Try exercises

Chapter 12

New section material on Finding the Composition of Functions

Chapter 13

New material on Graphing a Natural Logarithmic Function
4 new Examples
1 new In-Class Example
4 new You Try exercises

Chapter 14

Reformatted examples in Section 14.4
New material on the Midpoint Formula and Graphing Other Square Root Functions
2 new Examples
2 new In-Class Examples
2 new You Try exercises

Brand New Videos!

Larry Perez from Saddleback College, has created 27 new exercise videos for the Messersmith *Beginning and Intermediate Algebra* series. The exercise videos are in a student-friendly, easy-to-follow format. They allow a student to attend a virtual lecture by Professor Larry Perez and walked step-by-step through an exercise problem from the textbook.

Acknowledgments and Reviewers

The development of this textbook series would never have been possible without the creative ideas and feedback offered by many reviewers. We are especially thankful to the following instructors for their careful review of the manuscript.

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Sherri Messersmith

Sherri Messersmith
College of DuPage

Supplements for the Student

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Connect Math hosted by ALEKS is an exciting, new assignment and assessment ehomework platform. Starting with an easily viewable, intuitive interface, students will be able to access key information, complete homework assignments, and utilize an integrated, media–rich eBook.

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Student Solution Manual The student's solution manual provides comprehensive, worked-out solutions to the odd-numbered exercises in the section exercises, summary exercises, self-test, and the cumulative review. The steps shown in the solutions match the style of solved examples in the textbook.

Online Videos In the online exercise videos, the author, Sherri Messersmith, works through selected exercises using the solution methodology employed in her text. Each video is available online as part of Connect and is indicated by an icon next to a corresponding exercise in the text. Other supplemental videos include eProfessor videos, which are animations based on examples in the book, and Connect2Developmental Mathematics videos, which use 3D animations and lectures to teach algebra concepts by placing them in a real-world setting. The videos are closed-captioned for the hearing impaired, are subtitled in Spanish, and meet the Americans with Disabilities Act Standards for Accessible Design.

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Instructor's Testing and Resource Online Provides wealth of resources for the instructor. Among the supplements is a computerized test bank utilizing Brownstone Diploma algorithm-based testing software to create customized exams quickly. This user-friendly program enables instructors to search for questions by topic, format, or difficulty level; to edit existing questions, or to add new ones; and to scramble questions and answer keys for multiple versions of a single test. Hundreds of text-specific, open-ended, and multiple-choice questions are included in the question bank. Sample chapter tests are also provided. CD available upon request.

Annotated Instructor's Edition In the Annotated Instructor's Edition (AIE), answers to exercises and tests appear adjacent to each exercise set, in a color used only for annotations. Also found in the AIE are icons with the practice exercises that serve to guide instructors in their preparation of homework assignments and lessons.

Instructor's Solution Manual The instructor's solution manual provides comprehensive, worked-out solutions to all exercises in the section exercises, summary exercises, self-test, and the cumulative review. The steps shown in the solutions match the style of solved examples in the textbook.

Worksheets The very popular author-created worksheets that accompany the text provide instructors with additional exercises to assist with overcoming potential stumbling blocks in student knowledge and to help students see the connection among multiple mathematical concepts. The worksheets fall into three categories: Worksheets to Strengthen Basic Skills, Worksheets to Help Teach New Concepts, Worksheets to Tie Concepts Together.

Guided Student Notes Guided Student Notes are an amazing resource for instructors to help their students become better note-takers. They are similar to "fill-in-the-blank" notes where certain topics or definitions are prompted and spaces are left for the students to write down what the instructor says or writes on the board. The Guided Student Notes contain in-class examples provided in the margin of the text along with additional examples not found in the book to emphasize the given topic. This allows students to spend less time copying down from the board and more time thinking and learning about the solutions and concepts. The notes are specific to the Messersmith textbook and offer ready-made lesson plans for teachers to either "print and go" or require students to print and bring to class.

Powerpoints These powerpoints will present key concepts and definitions with fully editable slides that follow the textbook. Project in class or post to a website in an online course.

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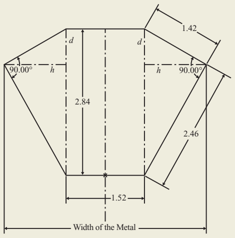

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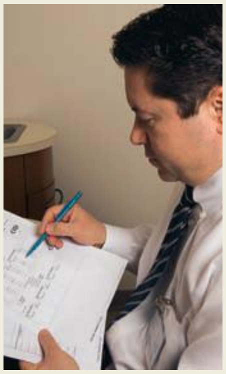

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APPENDIX:

Beginning Algebra Review A-1

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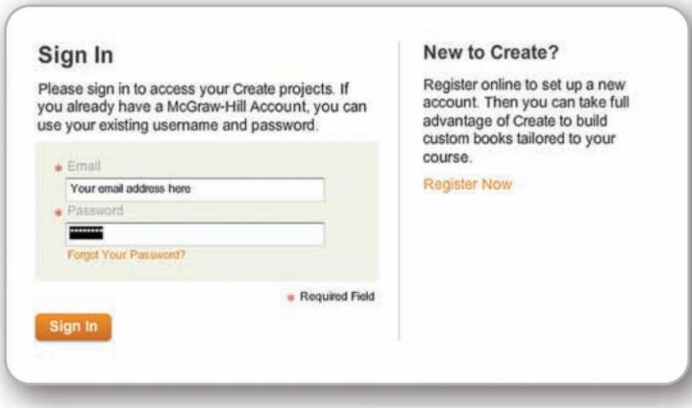

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CHAPTER 1

The Real Number System and Geometry

Algebra at Work: Landscape Architecture

Jill is a landscape architect and uses multiplication, division, and geometry formulas on a daily basis. Here is an example of the type of landscaping she designs. When Jill is asked to create the landscape for a new house, her first job is to draw the plans.

The ground in front of the house will be dug out into shapes that include rectangles and circles, shrubs and flowers will be planted, and mulch will cover the ground. To determine the volume of mulch that will be needed, Jill must use the formulas for the area of a rectangle and a circle and then multiply by the depth of the mulch. She will calculate the total cost of this landscaping job only after determining the cost of the plants, the mulch, and the labor.

It is important that her numbers are accurate. Her company and her clients must have an accurate estimate of the cost of the job. If the estimate is too high, the customer might choose another, less expensive landscaper to do the job. If the estimate is too low, either the client will have to pay more money at the end or the company will not earn as much profit on the job.

In this chapter, we will review formulas from geometry as well as some concepts from arithmetic.

Section 1.1 Review of Fractions

Objectives

1. Understand What a Fraction Represents
2. Write Fractions in Lowest Terms
3. Multiply and Divide Fractions
4. Add and Subtract Fractions

Why review fractions and arithmetic skills? Because the manipulations done in arithmetic and with fractions are precisely the same skills needed to learn algebra.

Let's begin by defining some numbers used in arithmetic:

Natural numbers: 1, 2, 3, 4, 5, . . .

Whole numbers: 0, 1, 2, 3, 4, 5, . . .

Natural numbers are often thought of as the counting numbers. **Whole numbers** consist of the natural numbers and zero.

Natural and whole numbers are used to represent complete quantities. To represent a part of a quantity, we can use a fraction.

1. Understand What a Fraction Represents

What is a fraction?

Definition

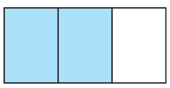
A **fraction** is a number in the form $\frac{a}{b}$ where $b \neq 0$, a is called the **numerator**, and b is the **denominator**.

Note

1) A fraction describes a part of a whole quantity.
2) $\frac{a}{b}$ means $a \div b$.

Example 1

What part of the figure is shaded?



Solution

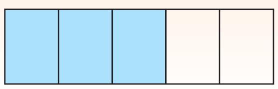
The whole figure is divided into three equal parts. Two of the parts are shaded. Therefore, the part of the figure that is shaded is $\frac{2}{3}$.

$2 \longrightarrow$ Number of shaded parts

$3 \longrightarrow$ Total number of equal parts in the figure

You Try 1

What part of the figure is shaded?



2. Write Fractions in Lowest Terms

A fraction is in **lowest terms** when the numerator and denominator have no common factors except 1. Before discussing how to write a fraction in lowest terms, we need to know about factors.

Consider the number 12.

$$\underset{\text{Product}}{\underset{\uparrow}{12}} = \underset{\text{Factor}}{\underset{\uparrow}{3}} \cdot \underset{\text{Factor}}{\underset{\uparrow}{4}}$$

3 and *4* are *factors* of 12. (When we use the term **factors**, we mean natural numbers.) Multiplying *3* and *4* results in 12. *12* is the **product**.

Does 12 have any other factors?

Example 2

Find all factors of 12.

Solution

$12 = 3 \cdot 4$	Factors are 3 and 4.
$12 = 2 \cdot 6$	Factors are 2 and 6.
$12 = 1 \cdot 12$	Factors are 1 and 12.

These are all of the ways to write *12* as the product of two factors. The factors of 12 are 1, 2, 3, 4, 6, and 12. ■

You Try 2

Find all factors of 30.

We can also write 12 as a product of *prime numbers.*

Definition

A **prime number** is a natural number whose only factors are 1 and itself. (The factors are natural numbers.)

Example 3

Is 7 a prime number?

Solution

Yes. The only way to write 7 as a product of natural numbers is $1 \cdot 7$. ■

You Try 3

Is 19 a prime number?

Definition

A **composite number** is a natural number with factors other than 1 and itself. Therefore, if a natural number is not prime, it is composite.

Note

The number 1 is neither prime nor composite.

You Try 4

a) What are the first six prime numbers?

b) What are the first six composite numbers?

To perform various operations in arithmetic and algebra, it is helpful to write a number as the product of its **prime factors**. This is called finding the **prime factorization** of a number. We can use a **factor tree** to help us find the prime factorization of a number.

Example 4

Write 12 as the product of its prime factors.

Solution

Use a factor tree.

```
  12
  / \
 3 · 4          Think of any two natural numbers that multiply to 12.
    / \
   2 · 2        4 is not prime, so break it down into the product of two factors, 2 × 2.
```

When a factor is a prime number, circle it, and that part of the factor tree is complete. When all of the numbers at the end of the tree are primes, you have found the *prime factorization* of the number.

Therefore, $12 = 2 \cdot 2 \cdot 3$. Write the prime factorization from the smallest factor to the largest.

Example 5

Write 120 as the product of its prime factors.

Solution

```
      120
     /   \
   10  ·  12        Think of any two natural numbers that multiply to 120.
   / \   / \
  2 · 5 2 · 6       10 and 12 are not prime, so write them as the product of two factors.
            / \     Circle the primes.
           2 · 3    6 is not prime, so write it as the product of two factors. The factors are
                    primes. Circle them.
```

Prime factorization: $120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$.

You Try 5

Use a factor tree to write each number as the product of its prime factors.

a) 20 b) 36 c) 90

Let's return to writing a fraction in lowest terms.

Example 6

Write each fraction in lowest terms.

a) $\dfrac{4}{6}$ b) $\dfrac{48}{42}$

Solution

a) $\dfrac{4}{6}$ There are two ways to approach this problem:

Method 1

Write 4 and 6 as the product of their primes, and divide out common factors.

$$\frac{4}{6} = \frac{2 \cdot 2}{2 \cdot 3} \quad \text{Write 4 and 6 as the product of their prime factors.}$$

$$= \frac{\overset{1}{\cancel{2}} \cdot 2}{\underset{1}{\cancel{2}} \cdot 3} \quad \text{Divide out common factor.}$$

$$= \frac{2}{3} \quad \text{Since 2 and 3 have no common factors other than 1, the fraction is in lowest terms.}$$

Method 2

Divide 4 and 6 by a common factor.

$\dfrac{4}{6} = \dfrac{4 \div 2}{6 \div 2} = \dfrac{2}{3}$ $\dfrac{4}{6}$ and $\dfrac{2}{3}$ are **equivalent fractions** since $\dfrac{4}{6}$ simplifies to $\dfrac{2}{3}$.

b) $\dfrac{48}{42}$ is an **improper fraction**. A fraction is *improper* if its numerator is greater than or equal to its denominator. We will use two methods to express this fraction in lowest terms.

Method 1

Using a factor tree to get the prime factorizations of 48 and 42 and then dividing out common factors, we have

$$\frac{48}{42} = \frac{\overset{1}{\cancel{2}} \cdot 2 \cdot 2 \cdot 2 \cdot \overset{1}{\cancel{3}}}{\underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{3}} \cdot 7} = \frac{2 \cdot 2 \cdot 2}{7} = \frac{8}{7} \text{ or } 1\frac{1}{7}$$

The answer may be expressed as an improper fraction, $\dfrac{8}{7}$, or as a **mixed number**, $1\dfrac{1}{7}$, as long as each is in lowest terms.

Method 2

48 and 42 are each divisible by 6, so we can divide each by 6.

$$\frac{48}{42} = \frac{48 \div 6}{42 \div 6} = \frac{8}{7} \text{ or } 1\frac{1}{7}$$

You Try 6

Write each fraction in lowest terms.

a) $\dfrac{8}{14}$ b) $\dfrac{63}{36}$

3. Multiply and Divide Fractions

Procedure Multiplying Fractions

To multiply fractions, $\frac{a}{b} \cdot \frac{c}{d}$, we multiply the numerators and multiply the denominators. That is,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \text{ if } b \neq 0 \text{ and } d \neq 0.$$

Example 7

Multiply. Write each answer in lowest terms.

a) $\frac{3}{8} \cdot \frac{7}{4}$ b) $\frac{10}{21} \cdot \frac{21}{25}$ c) $4\frac{2}{5} \cdot 1\frac{7}{8}$

Solution

a) $\frac{3}{8} \cdot \frac{7}{4} = \frac{3 \cdot 7}{8 \cdot 4}$ Multiply numerators; multiply denominators.

$= \frac{21}{32}$ 21 and 32 contain no common factors, so $\frac{21}{32}$ is in lowest terms.

b) $\frac{10}{21} \cdot \frac{21}{25}$

If we follow the procedure in the previous example, we get

$$\frac{10}{21} \cdot \frac{21}{25} = \frac{10 \cdot 21}{21 \cdot 25}$$

$$= \frac{210}{525}$$ $\frac{210}{525}$ is not in lowest terms.

We must reduce $\frac{210}{525}$ to lowest terms:

$\frac{210 \div 5}{525 \div 5} = \frac{42}{105}$ $\frac{42}{105}$ is not in lowest terms. Each number is divisible by 3.

$= \frac{42 \div 3}{105 \div 3} = \frac{14}{35}$ 14 and 35 have a common factor of 7.

$= \frac{14 \div 7}{35 \div 7} = \frac{2}{5}$ $\frac{2}{5}$ is in lowest terms.

Therefore, $\frac{10}{21} \cdot \frac{21}{25} = \frac{2}{5}$.

However, we can take out the common factors before we multiply to avoid all of the reducing in the steps above.

5 is the greatest common factor of 10 and 25. Divide 10 and 25 by **5**.

$$\frac{\overset{2}{\cancel{10}}}{\underset{1}{\cancel{21}}} \times \frac{\overset{1}{\cancel{21}}}{\underset{5}{\cancel{25}}} = \frac{2}{1} \times \frac{1}{5} = \frac{2 \times 1}{1 \times 5} = \boxed{\frac{2}{5}}$$

21 is the greatest common factor of 21 and 21. Divide each 21 by **21**.

Note

Usually, it is easier to remove the common factors before multiplying rather than after finding the product.

c) $4\frac{2}{5} \cdot 1\frac{7}{8}$

Before multiplying mixed numbers, we must change them to improper fractions. Recall that $4\frac{2}{5}$ is the same as $4 + \frac{2}{5}$. Here is one way to rewrite $4\frac{2}{5}$ as an improper fraction:

1) Multiply the denominator and the whole number: $5 \cdot 4 = 20$.
2) Add the numerator: $20 + 2 = 22$.
3) Put the sum over the denominator: $\frac{22}{5}$

To summarize, $4\frac{2}{5} = \frac{(5 \cdot 4) + 2}{5} = \frac{20 + 2}{5} = \frac{22}{5}$.

Then, $1\frac{7}{8} = \frac{(8 \cdot 1) + 7}{8} = \frac{8 + 7}{8} = \frac{15}{8}$.

$$4\frac{2}{5} \cdot 1\frac{7}{8} = \frac{22}{5} \cdot \frac{15}{8}$$

5 and 15 each divide by **5**.

$$= \frac{\overset{11}{\cancel{22}}}{\underset{1}{\cancel{5}}} \cdot \frac{\overset{3}{\cancel{15}}}{\underset{4}{\cancel{8}}}$$

8 and 22 each divide by **2**.

$$= \frac{11}{1} \cdot \frac{3}{4}$$

$$= \frac{33}{4} \text{ or } 8\frac{1}{4}$$

Express the result as an improper fraction or as a mixed number.

You Try 7

Multiply. Write the answer in lowest terms.

a) $\frac{1}{5} \cdot \frac{4}{9}$ b) $\frac{8}{25} \cdot \frac{15}{32}$ c) $3\frac{3}{4} \cdot 2\frac{2}{3}$

Dividing Fractions

To divide fractions, we must define a reciprocal.

Definition

The **reciprocal** of a number, $\frac{a}{b}$, is $\frac{b}{a}$ since $\frac{a}{b} \cdot \frac{b}{a} = 1$. That is, a nonzero number times its reciprocal equals 1.

For example, the reciprocal of $\frac{5}{9}$ is $\frac{9}{5}$ since $\frac{\overset{1}{\cancel{5}}}{\underset{1}{\cancel{9}}} \cdot \frac{\overset{1}{\cancel{9}}}{\underset{1}{\cancel{5}}} = \frac{1}{1} = 1$.

Definition

Division of fractions: Let a, b, c, and d represent numbers so that b, c, and d do not equal zero. Then,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

Note

To perform division involving fractions, multiply the first fraction by the reciprocal of the second.

Example 8

Divide. Write the answer in lowest terms.

a) $\dfrac{3}{8} \div \dfrac{10}{11}$ b) $\dfrac{3}{2} \div 9$ c) $5\dfrac{1}{4} \div 1\dfrac{1}{13}$

Solution

a) $\dfrac{3}{8} \div \dfrac{10}{11} = \dfrac{3}{8} \cdot \dfrac{11}{10}$ Multiply $\frac{3}{8}$ by the reciprocal of $\frac{10}{11}$.

$= \dfrac{33}{80}$ Multiply.

b) $\dfrac{3}{2} \div 9 = \dfrac{3}{2} \cdot \dfrac{1}{9}$ The reciprocal of 9 is $\frac{1}{9}$.

$= \dfrac{\overset{1}{\cancel{3}}}{2} \cdot \dfrac{1}{\underset{3}{\cancel{9}}}$ Divide out a common factor of 3.

$= \dfrac{1}{6}$ Multiply.

c) $5\dfrac{1}{4} \div 1\dfrac{1}{13} = \dfrac{21}{4} \div \dfrac{14}{13}$ Change the mixed numbers to improper fractions.

$= \dfrac{21}{4} \cdot \dfrac{13}{14}$ Multiply $\frac{21}{4}$ by the reciprocal of $\frac{14}{13}$.

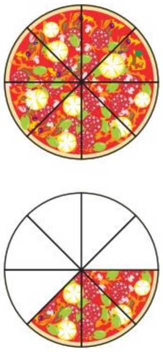
$= \dfrac{\overset{3}{\cancel{21}}}{4} \cdot \dfrac{13}{\underset{2}{\cancel{14}}}$ Divide out a common factor of 7.

$= \dfrac{39}{8}$ or $4\dfrac{7}{8}$ Express the answer as an improper fraction or mixed number.

You Try 8

Divide. Write the answer in lowest terms.

a) $\dfrac{2}{7} \div \dfrac{3}{5}$ b) $\dfrac{3}{10} \div \dfrac{9}{16}$ c) $9\dfrac{1}{6} \div 5$



4. Add and Subtract Fractions

The pizza on top is cut into eight equal slices. If you eat two pieces and your friend eats three pieces, what fraction of the pizza was eaten?

Five out of the eight pieces were eaten. As a fraction, we can say that you and your friend ate $\frac{5}{8}$ of the pizza.

Let's set up this problem as the sum of two fractions.

$$\begin{array}{ccccc} \text{Fraction you ate} & + & \text{Fraction your friend ate} & = & \text{Fraction of the pizza eaten} \\ \frac{2}{8} & + & \frac{3}{8} & = & \frac{5}{8} \end{array}$$

To add $\frac{2}{8} + \frac{3}{8}$, we added the numerators and kept the denominator the same. Notice that these fractions have the same denominator.

Definition

Let a, b, and c be numbers such that $c \neq 0$.

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \quad \text{and} \quad \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

To add or subtract fractions, the denominators must be the same. (This is called a **common denominator**.) Then, add (or subtract) the numerators and keep the same denominator.

Example 9

Perform the operation and simplify.

a) $\frac{3}{11} + \frac{5}{11}$ b) $\frac{17}{30} - \frac{13}{30}$

Solution

a) $\frac{3}{11} + \frac{5}{11} = \frac{3+5}{11}$ Add the numerators and keep the denominator the same.

$= \frac{8}{11}$

b) $\frac{17}{30} - \frac{13}{30} = \frac{17-13}{30}$ Subtract the numerators and keep the denominator the same.

$= \frac{4}{30}$ This is not in lowest terms, so reduce.

$= \frac{2}{15}$ Simplify.

You Try 9

Perform the operation and simplify.

a) $\frac{5}{9} + \frac{2}{9}$ b) $\frac{19}{20} - \frac{7}{20}$

When adding or subtracting mixed numbers, either work with them as mixed numbers or change them to improper fractions first.

Example 10

Add $2\frac{4}{15} + 1\frac{7}{15}$.

Solution

Method 1

To add these numbers while keeping them in mixed number form, add the whole number parts and add the fractional parts.

$$2\frac{4}{15} + 1\frac{7}{15} = (2 + 1) + \left(\frac{4}{15} + \frac{7}{15}\right) = 3\frac{11}{15}$$

Method 2

Change each mixed number to an improper fraction, then add.

$$2\frac{4}{15} + 1\frac{7}{15} = \frac{34}{15} + \frac{22}{15} = \frac{34 + 22}{15} = \frac{56}{15} \text{ or } 3\frac{11}{15}$$

You Try 10

Add $4\frac{3}{7} + 5\frac{1}{7}$.

The examples given so far contain common denominators. How do we add or subtract fractions that do not have common denominators? We find the least common denominator for the fractions and rewrite each fraction with this denominator.

The **least common denominator (LCD)** of two fractions is the least common multiple of the numbers in the denominators.

Example 11

Find the LCD for $\frac{3}{4}$ and $\frac{1}{6}$.

Solution

Method 1

List some multiples of 4 and 6.

4: 4, 8, $\boxed{12}$, 16, 20, *24*, . . .

6: 6, $\boxed{12}$, 18, *24*, 30, . . .

Although 24 is a multiple of 6 and of 4, the *least* common multiple, and therefore the least common denominator, is 12.

Method 2

We can also use the prime factorizations of 4 and 6 to find the LCD.

To find the LCD:

1) Find the prime factorization of each number.
2) The least common denominator will include each different factor appearing in the factorizations.
3) If a factor appears more than once in any prime factorization, use it in the LCD the *maximum number of times* it appears in any single factorization. Multiply the factors.

$$4 = 2 \cdot 2$$
$$6 = 2 \cdot 3$$

The least common multiple of 4 and 6 is

$$\underbrace{2 \cdot 2}_{\substack{\text{2 appears at}\\ \text{most twice in}\\ \text{any single}\\ \text{factorization.}}} \cdot \underbrace{3}_{\substack{\text{3 appears}\\ \text{once in a}\\ \text{factorization.}}} = 12$$

The LCD of $\frac{3}{4}$ and $\frac{1}{6}$ is 12.

You Try 11

Find the LCD for $\frac{5}{6}$ and $\frac{4}{9}$.

To add or subtract fractions with unlike denominators, begin by identifying the least common denominator. Then, we must rewrite each fraction with this LCD. This will not change the value of the fraction; we will obtain an *equivalent* fraction.

Example 12

Rewrite $\frac{3}{4}$ with a denominator of 12.

Solution

We want to find a fraction that is equivalent to $\frac{3}{4}$ so that $\frac{3}{4} = \frac{?}{12}$.

To obtain the new denominator of 12, the "old" denominator, 4, must be multiplied by 3. But, if the denominator is multiplied by 3, the numerator must be multiplied by 3 as well. When we multiply $\frac{3}{4}$ by $\frac{3}{3}$, we have multiplied by 1 since $\frac{3}{3} = 1$. This is why the fractions are equivalent.

$$\frac{3}{4} \cdot \frac{3}{3} = \frac{9}{12} \qquad \text{So, } \frac{3}{4} = \frac{9}{12}.$$

Procedure Adding or Subtracting Fractions with Unlike Denominators

To add or subtract fractions with unlike denominators:

1) Determine, and write down, the least common denominator (LCD).
2) Rewrite each fraction with the LCD.
3) Add or subtract.
4) Express the answer in lowest terms.

You Try 12

Rewrite $\frac{5}{6}$ with a denominator of 42.

Example 13

Add or subtract.

a) $\frac{2}{9}+\frac{1}{6}$ b) $6\frac{7}{8}-3\frac{1}{2}$

Solution

a) $\frac{2}{9}+\frac{1}{6}$ LCD $=18$ Identify the least common denominator.

$\frac{2}{9}\cdot\frac{2}{2}=\frac{4}{18}$ $\frac{1}{6}\cdot\frac{3}{3}=\frac{3}{18}$ Rewrite each fraction with a denominator of 18.

$$\frac{2}{9}+\frac{1}{6}=\frac{4}{18}+\frac{3}{18}=\frac{7}{18}$$

b) $6\frac{7}{8}-3\frac{1}{2}$

Method 1

Keep the numbers in mixed number form. Subtract the whole number parts and subtract the fractional parts. Get a common denominator for the fractional parts.

LCD $=8$ Identify the least common denominator.

$6\frac{7}{8}$: $\frac{7}{8}$ has the LCD of 8.

$3\frac{1}{2}$: $\frac{1}{2}\cdot\frac{4}{4}=\frac{4}{8}$.   So, $3\frac{1}{2}=3\frac{4}{8}$.   Rewrite $\frac{1}{2}$ with a denominator of 8.

$$6\frac{7}{8}-3\frac{1}{2}=6\frac{7}{8}-3\frac{4}{8}$$

$$=3\frac{3}{8}$$

Subtract whole number parts and subtract fractional parts.

Method 2

Rewrite each mixed number as an improper fraction, get a common denominator, then subtract.

$$6\frac{7}{8} - 3\frac{1}{2} = \frac{55}{8} - \frac{7}{2} \qquad \text{LCD} = 8 \qquad \frac{55}{8} \text{ already has a denominator of 8.}$$

$$\frac{7}{2} \cdot \frac{4}{4} = \frac{28}{8} \qquad \text{Rewrite } \frac{7}{2} \text{ with a denominator of 8.}$$

$$6\frac{7}{8} - 3\frac{1}{2} = \frac{55}{8} - \frac{7}{2} = \frac{55}{8} - \frac{28}{8} = \frac{27}{8} \text{ or } 3\frac{3}{8}$$

You Try 13

Perform the operations and simplify.

a) $\frac{11}{12} - \frac{5}{8}$ b) $\frac{1}{3} + \frac{5}{6} + \frac{3}{4}$ c) $4\frac{2}{5} + 1\frac{7}{15}$

Answers to You Try Exercises

1) $\frac{3}{5}$ 2) 1, 2, 3, 5, 6, 10, 15, 30 3) yes 4) a) 2, 3, 5, 7, 11, 13 b) 4, 6, 8, 9, 10, 12

5) a) $2 \cdot 2 \cdot 5$ b) $2 \cdot 2 \cdot 3 \cdot 3$ c) $2 \cdot 3 \cdot 3 \cdot 5$ 6) a) $\frac{4}{7}$ b) $\frac{7}{4}$ or $1\frac{3}{4}$ 7) a) $\frac{4}{45}$ b) $\frac{3}{20}$ c) 10

8) a) $\frac{10}{21}$ b) $\frac{8}{15}$ c) $\frac{11}{6}$ or $1\frac{5}{6}$ 9) a) $\frac{7}{9}$ b) $\frac{3}{5}$ 10) $9\frac{4}{7}$ 11) 18 12) $\frac{35}{42}$

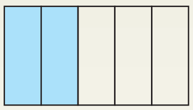
13) a) $\frac{7}{24}$ b) $\frac{23}{12}$ or $1\frac{11}{12}$ c) $\frac{88}{15}$ or $5\frac{13}{15}$

1.1 Exercises

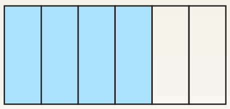
Objective 1: Understand What a Fraction Represents

1) What fraction of each figure is shaded? If the fraction is not in lowest terms, reduce it.

a)



b)



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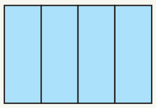
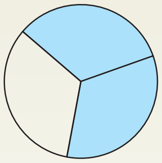
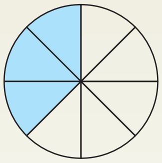


2) What fraction of each figure is *not* shaded? If the fraction is not in lowest terms, reduce it.

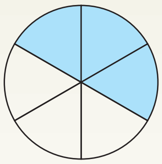
a)



b)



c)



3) Draw a rectangle divided into 8 equal parts. Shade in $\frac{4}{8}$ of the rectangle. Write another fraction to represent how much of the rectangle is shaded.

4) Draw a rectangle divided into 6 equal parts. Shade in $\frac{2}{6}$ of the rectangle. Write another fraction to represent how much of the rectangle is shaded.

Objective 2: Write Fractions in Lowest Terms

5) Find all factors of each number.

a) 18

b) 40

c) 23

6) Find all factors of each number.

a) 20

b) 17

c) 60

7) Identify each number as prime or composite.

a) 27

b) 34

c) 11

8) Identify each number as prime or composite.

a) 2

b) 57

c) 90

9) Is 3072 prime or composite? Explain your answer.

10) Is 4185 prime or composite? Explain your answer.

11) Use a factor tree to find the prime factorization of each number.

a) 18 b) 54

c) 42 d) 150

12) Explain, in words, how to use a factor tree to find the prime factorization of 72.

13) Write each fraction in lowest terms.

a) $\frac{9}{12}$ b) $\frac{54}{72}$

c) $\frac{84}{35}$ d) $\frac{120}{280}$

14) Write each fraction in lowest terms.

a) $\frac{21}{35}$ b) $\frac{48}{80}$

c) $\frac{125}{500}$ d) $\frac{900}{450}$

Objective 3: Multiply and Divide Fractions

15) Multiply. Write the answer in lowest terms.

a) $\frac{2}{7} \cdot \frac{3}{5}$ b) $\frac{15}{26} \cdot \frac{4}{9}$

c) $\frac{1}{2} \cdot \frac{14}{15}$ VIDEO d) $\frac{42}{55} \cdot \frac{22}{35}$

e) $4 \cdot \frac{1}{8}$ f) $6\frac{1}{8} \cdot \frac{2}{7}$

16) Multiply. Write the answer in lowest terms.

a) $\frac{1}{6} \cdot \frac{5}{9}$ b) $\frac{9}{20} \cdot \frac{6}{7}$

c) $\frac{12}{25} \cdot \frac{25}{36}$ d) $\frac{30}{49} \cdot \frac{21}{100}$

e) $\frac{7}{15} \cdot 10$ f) $7\frac{5}{7} \cdot 1\frac{5}{9}$

17) When Elizabeth multiplies $5\frac{1}{2} \cdot 2\frac{1}{3}$, she gets $10\frac{1}{6}$. What was her mistake? What is the correct answer?

18) Explain how to multiply mixed numbers.

19) Divide. Write the answer in lowest terms.

a) $\frac{1}{42} \div \frac{2}{7}$ b) $\frac{3}{11} \div \frac{4}{5}$

c) $\frac{18}{35} \div \frac{9}{10}$ d) $\frac{14}{15} \div \frac{2}{15}$

VIDEO e) $6\frac{2}{5} \div 1\frac{13}{15}$ f) $\frac{4}{7} \div 8$

20) Explain how to divide mixed numbers.

Objective 4: Add and Subtract Fractions

21) Find the least common multiple of 10 and 15.

22) Find the least common multiple of 12 and 9.

23) Find the least common denominator for each group of fractions.

a) $\frac{9}{10}, \frac{11}{30}$ b) $\frac{7}{8}, \frac{5}{12}$

c) $\frac{4}{9}, \frac{1}{6}, \frac{3}{4}$

24) Find the least common denominator for each group of fractions.

a) $\frac{3}{14}, \frac{2}{7}$ b) $\frac{17}{25}, \frac{3}{10}$

c) $\frac{29}{30}, \frac{3}{4}, \frac{9}{20}$

25) Add or subtract. Write the answer in lowest terms.

a) $\frac{6}{11} + \frac{2}{11}$ b) $\frac{19}{20} - \frac{7}{20}$

c) $\frac{4}{25} + \frac{2}{25} + \frac{9}{25}$ d) $\frac{2}{9} + \frac{1}{6}$

e) $\frac{3}{5} + \frac{11}{30}$ f) $\frac{13}{18} - \frac{2}{3}$

g) $\frac{4}{7} + \frac{5}{9}$ VIDEO h) $\frac{5}{6} - \frac{1}{4}$

i) $\frac{3}{10} + \frac{7}{20} + \frac{3}{4}$ j) $\frac{1}{6} + \frac{2}{9} + \frac{10}{27}$

26) Add or subtract. Write the answer in lowest terms.

a) $\frac{8}{9} - \frac{5}{9}$ b) $\frac{14}{15} - \frac{2}{15}$

c) $\frac{11}{36} + \frac{13}{36}$ d) $\frac{16}{45} + \frac{8}{45} + \frac{11}{45}$

e) $\frac{15}{16} - \frac{3}{4}$ f) $\frac{1}{8} + \frac{1}{6}$

g) $\frac{5}{8} - \frac{2}{9}$ h) $\frac{23}{30} - \frac{19}{90}$

i) $\frac{1}{6} + \frac{1}{4} + \frac{2}{3}$ j) $\frac{3}{10} + \frac{2}{5} + \frac{4}{15}$

27) Add or subtract. Write the answer in lowest terms.

a) $8\frac{5}{11} + 6\frac{2}{11}$ b) $2\frac{1}{10} + 9\frac{3}{10}$

c) $7\frac{11}{12} - 1\frac{5}{12}$ d) $3\frac{1}{5} + 2\frac{1}{4}$

e) $5\frac{2}{3} - 4\frac{4}{15}$ VIDEO f) $9\frac{5}{8} - 5\frac{3}{10}$

g) $4\frac{3}{7} + 6\frac{3}{4}$ h) $7\frac{13}{20} + \frac{4}{5}$

28) Add or subtract. Write the answer in lowest terms.

a) $3\frac{2}{7} + 1\frac{3}{7}$ b) $8\frac{5}{16} + 7\frac{3}{16}$

c) $5\frac{13}{20} - 3\frac{5}{20}$ d) $10\frac{8}{9} - 2\frac{1}{3}$

e) $1\frac{5}{12} + 2\frac{3}{8}$ f) $4\frac{1}{9} + 7\frac{2}{5}$

g) $1\frac{5}{6} + 4\frac{11}{18}$ h) $3\frac{7}{8} + 4\frac{2}{5}$

Mixed Exercises: Objectives 3 and 4

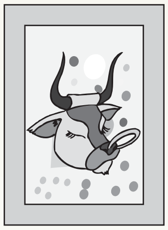
29) For Valentine's Day, Alex wants to sew teddy bears for her friends. Each bear requires $1\frac{2}{3}$ yd of fabric. If she has 7 yd of material, how many bears can Alex make? How much fabric will be left over?

30) A chocolate chip cookie recipe that makes 24 cookies uses $\frac{3}{4}$ cup of brown sugar. If Raphael wants to make 48 cookies, how much brown sugar does he need?

31) Nine weeks into the 2009 Major League Baseball season, Nyjer Morgan of the Pittsburgh Pirates had been up to bat 175 times. He got a hit $\frac{2}{7}$ of the time. How many hits did Nyjer have?

32) When all children are present, Ms. Yamoto has 30 children in her fifth-grade class. One day during flu season, $\frac{3}{5}$ of them were absent. How many children were absent on this day?

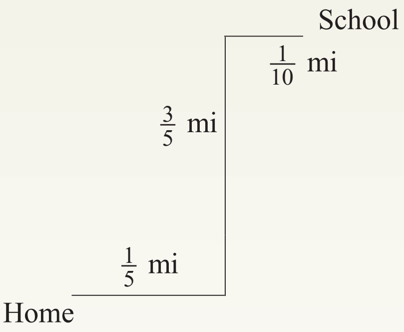
33) Mr. Burnett plans to have a picture measuring $18\frac{3}{8}''$ by $12\frac{1}{4}''$ custom framed. The frame he chose is $2\frac{1}{8}''$ wide. What will be the new length and width of the picture plus the frame?



34) Andre is building a table in his workshop. For the legs, he bought wood that is 30 in. long. If the legs are to be $26\frac{3}{4}$ in. tall, how many inches must he cut off to get the desired height?

35) When Rosa opens the kitchen cabinet, she finds three partially filled bags of flour. One contains $\frac{2}{3}$ cup, another contains $1\frac{1}{4}$ cups, and the third contains $1\frac{1}{2}$ cups. How much flour does she have all together?

36) Tamika takes the same route to school every day. (See the figure.) How far does she walk to school?



37) The gas tank of Jenny's car holds $11\frac{3}{5}$ gal, while Scott's car holds $16\frac{3}{4}$ gal. How much more gasoline does Scott's car hold?

38) Mr. Johnston is building a brick wall along his driveway. He estimates that one row of brick plus mortar will be $4\frac{1}{4}$ in. high. How many rows will he need to construct a wall that is 34 in. high?

39) For homework, Bill's math teacher assigned 42 problems. He finished $\frac{5}{6}$ of them. How many problems did he do?

40) Clarice's parents tell her that she must deposit $\frac{1}{3}$ of the money she earns from babysitting into her savings account, but she can keep the rest. If she earns \$117 in 1 week during the summer, how much does she deposit, and how much does she keep?

41) A welder must construct a beam with a total length of $32\frac{7}{8}$ in. If he has already joined a $14\frac{1}{6}$-in. beam with a $10\frac{3}{4}$-in. beam, find the length of a third beam needed to reach the total length.

42) Telephia, a market research company, surveyed 1500 teenage cell phone owners. The company learned that $\frac{2}{3}$ of them use cell phone data services. How many teenagers surveyed use cell phone data services? (*American Demographics,* May 2004, Vol. 26, Issue 4, p. 10)

43) A study conducted in 2000 indicated that about $\frac{3}{5}$ of the full-time college students surveyed had consumed alcohol sometime during the 30 days preceding the survey. If 400 students were surveyed, how many of them drank alcohol within the 30 days before the survey? (*Alcohol Research & Health,* The Journal of the Nat'l Institute on Alcohol Abuse & Alcoholism, Vol. 27, No. 1, 2003)

Section 1.2 Exponents and Order of Operations

Objectives

1. Use Exponents
2. Use the Order of Operations

1. Use Exponents

In Section 1.1, we discussed the prime factorization of a number. Let's find the prime factorization of 8.

```
      8
     / \
    4 · 2
   / \
  2 · 2
```

$$8 = 2 \cdot 2 \cdot 2$$

We can write $2 \cdot 2 \cdot 2$ another way, by using an *exponent*.

$$2 \cdot 2 \cdot 2 = \underset{\text{base}}{2}^{3 \leftarrow \text{exponent (or power)}}$$

2 is the *base*. 2 is a *factor* that appears three times. 3 is the *exponent* or *power*. An **exponent** represents repeated multiplication. We read 2^3 as "2 to the third power" or "2 cubed." 2^3 is called an **exponential expression**.

Example 1

Rewrite each product in exponential form.

a) $9 \cdot 9 \cdot 9 \cdot 9$ b) $7 \cdot 7$

Solution

a) $9 \cdot 9 \cdot 9 \cdot 9 = 9^4$ — 9 is the base. It appears as a factor 4 times. So, 4 is the exponent.

b) $7 \cdot 7 = 7^2$ — 7 is the base. 2 is the exponent.
This is read as "7 squared."

You Try 1

Rewrite each product in exponential form.

a) $8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$ b) $\frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2}$

We can also evaluate an exponential expression.

Example 2

Evaluate.

a) 2^5 b) 5^3 c) $\left(\frac{4}{7}\right)^2$ d) 8^1 e) 1^4

Solution

a) $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$ — 2 appears as a factor 5 times.

b) $5^3 = 5 \cdot 5 \cdot 5 = 125$ — 5 appears as a factor 3 times.

c) $\left(\frac{4}{7}\right)^2 = \frac{4}{7} \cdot \frac{4}{7} = \frac{16}{49}$ — $\frac{4}{7}$ appears as a factor 2 times.

d) $8^1 = 8$ — 8 is a factor only once.

e) $1^4 = 1 \cdot 1 \cdot 1 \cdot 1 = 1$ — 1 appears as a factor 4 times.

Note

1 raised to any natural number power is 1 since 1 multiplied by itself equals 1.

You Try 2

Evaluate.

a) 3^4 b) 8^2 c) $\left(\frac{3}{4}\right)^3$

It is generally agreed that there are some skills in arithmetic that everyone should have in order to be able to acquire other math skills. Knowing the basic multiplication facts, for example, is essential for learning how to add, subtract, multiply, and divide fractions as well as how to perform many other operations in arithmetic and algebra. Similarly, memorizing powers of certain bases is necessary for learning how to apply the rules of exponents (Chapter 2) and for working with radicals (Chapter 10). Therefore, the powers listed here must be memorized in order to be successful in the previously mentioned, as well as other, topics. Throughout this book, it is assumed that students know these powers:

Powers to Memorize

$2^1 = 2$	$3^1 = 3$	$4^1 = 4$	$5^1 = 5$	$6^1 = 6$	$8^1 = 8$	$10^1 = 10$
$2^2 = 4$	$3^2 = 9$	$4^2 = 16$	$5^2 = 25$	$6^2 = 36$	$8^2 = 64$	$10^2 = 100$
$2^3 = 8$	$3^3 = 27$	$4^3 = 64$	$5^3 = 125$			$10^3 = 1000$
$2^4 = 16$	$3^4 = 81$					
$2^5 = 32$				$7^1 = 7$	$9^1 = 9$	$11^1 = 11$
$2^6 = 64$				$7^2 = 49$	$9^2 = 81$	$11^2 = 121$
						$12^1 = 12$
						$12^2 = 144$
						$13^1 = 13$
						$13^2 = 169$

(Hint: Making flashcards might help you learn these facts.)

2. Use the Order of Operations

We will begin this topic with a problem for the student:

You Try 3

Evaluate $40 - 24 \div 8 + (5 - 3)^2$.

What answer did you get? 41? or 6? or 33? Or, did you get another result?

Most likely you obtained one of the three answers just given. Only one is correct, however. If we do not have rules to guide us in evaluating expressions, it is easy to get the incorrect answer.

Therefore, here are the rules we follow. This is called the **order of operations**.

Procedure The Order of Operations

Simplify expressions in the following order:

1) If parentheses or other grouping symbols appear in an expression, simplify what is in these grouping symbols first.
2) Simplify expressions with exponents.
3) Perform multiplication and division from left to right.
4) Perform addition and subtraction from left to right.

Think about the "You Try" problem. Did you evaluate it using the order of operations? Let's look at that expression:

Example 3

Evaluate $40 - 24 \div 8 + (5 - 3)^2$.

Solution

$40 - 24 \div 8 + (5 - 3)^2$	First, perform the operation in the parentheses.
$40 - 24 \div 8 + 2^2$	Exponents are done before division, addition, and subtraction.
$40 - 24 \div 8 + 4$	Perform division before addition and subtraction.
$40 - 3 + 4$	When an expression contains only addition and subtraction,
$37 + 4$	perform the operations starting at the left and moving to the right.
41	

You Try 4

Evaluate: $12 \cdot 3 - (2 + 1)^2 \div 9$.

A good way to remember the order of operations is to remember the sentence, "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally" (**P**arentheses, **E**xponents, **M**ultiplication and **D**ivision from left to right, **A**ddition and **S**ubtraction from left to right). Don't forget that multiplication and division are at the same "level" in the process of performing operations and that addition and subtraction are at the same "level."

Example 4

Evaluate.

a) $9 + 20 - 5 \cdot 3$

b) $5(8 - 2) + 3^2$

c) $4[3 + (10 \div 2)] - 11$

d) $\dfrac{(9 - 6)^3 \cdot 2}{26 - 4 \cdot 5}$

Solution

a)

$9 + 20 - 5 \cdot 3 = 9 + 20 - 15$	Perform multiplication before addition and subtraction.
$= 29 - 15$	When an expression contains only addition and subtraction, work from left to right.
$= 14$	Subtract.

b)

$5(8 - 2) + 3^2 = 5(6) + 3^2$	Parentheses
$= 5(6) + 9$	Exponent
$= 30 + 9$	Multiply.
$= 39$	Add.

c) $4[3 + (10 \div 2)] - 11$

This expression contains two sets of grouping symbols: **brackets** [ ] and **parentheses** ( ). Perform the operation in the **innermost** grouping symbol first which is the parentheses in this case.

$4[3 + (10 \div 2)] - 11 = 4[3 + 5] - 11$	Innermost grouping symbol
$= 4[8] - 11$	Brackets
$= 32 - 11$	Perform multiplication before subtraction.
$= 21$	Add.

d) $\dfrac{(9-6)^3 \cdot 2}{26 - 4 \cdot 5}$

The fraction bar in this expression acts as a grouping symbol. Therefore, simplify the numerator, simplify the denominator, then simplify the resulting fraction, if possible.

$$\frac{(9-6)^3 \cdot 2}{26 - 4 \cdot 5} = \frac{3^3 \cdot 2}{26 - 20} \quad \begin{array}{l}\text{Parentheses}\\ \text{Multiply.}\end{array}$$

$$= \frac{27 \cdot 2}{6} \quad \begin{array}{l}\text{Exponent}\\ \text{Subtract.}\end{array}$$

$$= \frac{54}{6} \quad \text{Multiply.}$$

$$= 9$$

You Try 5

Evaluate:

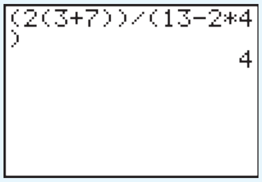
a) $35 - 2 \cdot 6 + 1$ b) $3 \cdot 12 - (7-4)^3 \div 9$

c) $9 + 2[23 - 4(1+2)]$ d) $\dfrac{11^2 - 7 \cdot 3}{20(9-4)}$

Using Technology

We can use a graphing calculator to check our answer when we evaluate an expression by hand. The order of operations is built into the calculator. For example, evaluate the expression $\dfrac{2(3+7)}{13 - 2 \cdot 4}$.

To evaluate the expression using a graphing calculator, enter the following on the home screen: (2(3+7))/(13−2*4) and then press ENTER. The result is 4, as shown on the screen.



Notice that it is important to enclose the numerator and denominator in parentheses since the fraction bar acts as both a division and a grouping symbol.

Evaluate each expression by hand, and then verify your answer using a graphing calculator.

1) $45 - 3 \cdot 2 + 7$ 2) $24 \div \dfrac{6}{7} - 5 \cdot 4$ 3) $5 + 2(9-6)^2$

4) $3 + 2[37 - (4+1)^2 - 2 \cdot 6]$ 5) $\dfrac{5(7-3)}{50 - 3^2 \cdot 4}$ 6) $\dfrac{25 - (1+3)^2}{6 + 14 \div 2 - 8}$

Answers to You Try Exercises

1) a) 8^5 b) $\left(\dfrac{3}{2}\right)^4$ 2) a) 81 b) 64 c) $\dfrac{27}{64}$ 3) 41 4) 35 5) a) 24 b) 33 c) 31 d) 1

Answers to Technology Exercises

1) 46 2) 8 3) 23 4) 3 5) $\dfrac{10}{7}$ or $1\dfrac{3}{7}$ 6) $\dfrac{9}{5}$ or $1\dfrac{4}{5}$

1.2 Exercises

Objective 1: Use Exponents

1) Identify the base and the exponent.
   a) 6^4
   b) 2^3
   c) $\left(\frac{9}{8}\right)^5$

2) Identify the base and the exponent.
   a) 5^1
   b) 1^8
   c) $\left(\frac{3}{7}\right)^2$

3) Write in exponential form.
   a) $9 \cdot 9 \cdot 9 \cdot 9$
   b) $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
   c) $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4}$

4) Explain, in words, why $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = 7^5$.

5) Evaluate.
   a) 8^2
   b) 11^2
   c) 2^4
   d) 5^3
   e) 3^4
   f) 12^2
   g) 1^2
   h) $\left(\frac{3}{10}\right)^2$
   i) $\left(\frac{1}{2}\right)^6$
   j) $(0.3)^2$

6) Evaluate.
   a) 9^2
   b) 13^2
   c) 3^3
   d) 2^5
   e) 4^3
   f) 1^4
   g) 6^2
   h) $\left(\frac{7}{5}\right)^2$
   i) $\left(\frac{2}{3}\right)^4$
   j) $(0.02)^2$

7) Evaluate $(0.5)^2$ two different ways.

8) Explain why $1^{200} = 1$.

Objective 2: Use the Order of Operations

9) In your own words, summarize the order of operations.

Evaluate.

10) $20 + 12 - 5$

11) $17 - 2 + 4$

12) $51 - 18 + 2 - 11$

13) $48 \div 2 + 14$

14) $15 \cdot 2 - 1$

15) $20 - 3 \cdot 2 + 9$

16) $28 + 21 \div 7 - 4$

17) $8 + 12 \cdot \frac{3}{4}$

18) $27 \div \frac{9}{5} - 1$

19) $\frac{2}{5} \cdot \frac{1}{8} + \frac{2}{3} \cdot \frac{9}{10}$

20) $\frac{4}{9} \cdot \frac{5}{6} - \frac{1}{6} \cdot \frac{2}{3}$

21) $2 \cdot \frac{3}{4} - \left(\frac{2}{3}\right)^2$

22) $\left(\frac{3}{2}\right)^2 - \left(\frac{5}{4}\right)^2$

23) $25 - 11 \cdot 2 + 1$

24) $2 + 16 + 14 \div 2$

25) $39 - 3(9 - 7)^3$

26) $1 + 2(7 - 1)^2$

27) $60 \div 15 + 5 \cdot 3$

28) $27 \div (10 - 7)^2 + 8 \cdot 3$

29) $7[45 \div (19 - 10)] + 2$

30) $6[3 + (14 - 12)^3] - 10$

31) $1 + 2[(3 + 2)^3 \div (11 - 6)^2]$

32) $(4 + 7)^2 - 3[5(6 + 2) - 4^2]$

33) $\frac{4(7 - 2)^2}{12^2 - 8 \cdot 3}$

34) $\frac{(8 + 4)^2 - 2^6}{7 \cdot 8 - 6 \cdot 9}$

35) $\frac{4(9 - 6)^3}{2^2 + 3 \cdot 8}$

36) $\frac{7 + 3(10 - 8)^4}{6 + 10 \div 2 + 11}$

Section 1.3 Geometry Review

Objectives

1. Identify Angles and Parallel and Perpendicular Lines
2. Identify Triangles
3. Use Area, Perimeter, and Circumference Formulas
4. Use Volume Formulas

Thousands of years ago, the Egyptians collected taxes based on how much land a person owned. They developed measuring techniques to accomplish such a task. Later the Greeks formalized the process of measurements such as this into a branch of mathematics we call geometry. "Geometry" comes from the Greek words for "earth measurement." In this section, we will review some basic geometric concepts that we will need in the study of algebra.

Let's begin by looking at angles. An angle can be measured in **degrees**. For example, 45° is read as "45 degrees."

1. Identify Angles and Parallel and Perpendicular Lines

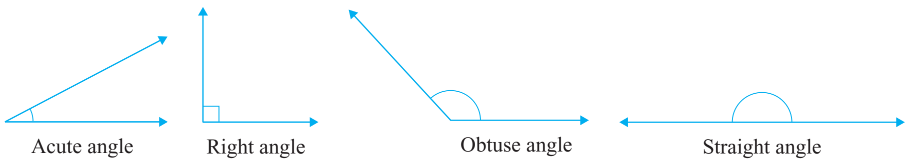
Angles

An **acute angle** is an angle whose measure is greater than 0° and less than 90°.

A **right angle** is an angle whose measure is 90°, indicated by the ⊾ symbol.

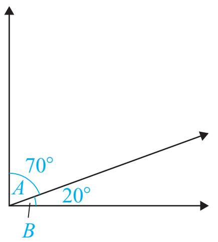
An **obtuse angle** is an angle whose measure is greater than 90° and less than 180°.

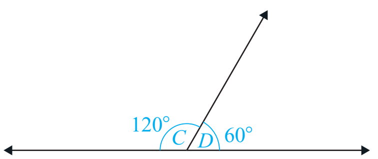
A **straight angle** is an angle whose measure is 180°.



Two angles are **complementary** if their measures add to 90°.

Two angles are **supplementary** if their measures add to 180°.





A and B are **complementary angles** since $m\angle A + m\angle B = 70° + 20° = 90°$.

C and D are **supplementary angles** since $m\angle C + m\angle D = 120° + 60° = 180°$.

Note

The measure of angle A is denoted by $m\angle A$.

Example 1

$m\angle A = 41°$. Find its complement.

Solution

$$\text{Complement} = 90° - 41° = 49°$$

Since the sum of two complementary angles is 90°, if one angle measures 41°, its complement has a measure of $90° - 41° = 49°$. ■

You Try 1

$m\angle A = 62°$. Find its supplement.

Next, we will explore some relationships between lines and angles.

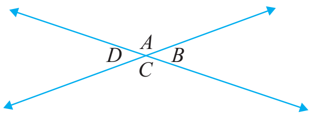


Figure 1.1

Vertical Angles

When two lines intersect, four angles are formed (see Figure 1.1). The pair of opposite angles are called **vertical angles**. Angles A and C are *vertical angles,* and angles B and D are *vertical angles. The measures of vertical angles are equal.* Therefore, $m\angle A = m\angle C$ and $m\angle B = m\angle D$.

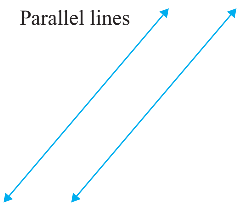


Figure 1.2

Parallel and Perpendicular Lines

Parallel lines are lines in the same plane that do not intersect (Figure 1.2). **Perpendicular lines** are lines that intersect at right angles (Figure 1.3).

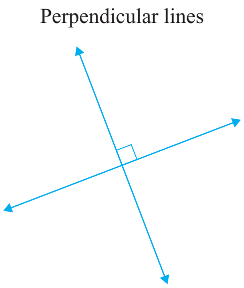
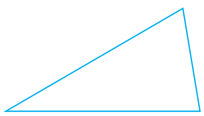
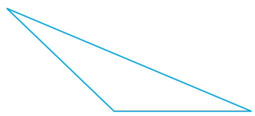


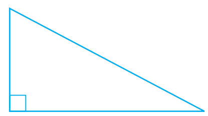
Figure 1.3

2. Identify Triangles

We can classify triangles by their angles and by their sides.


Acute triangle

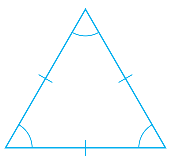

Obtuse triangle

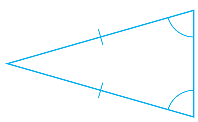

Right triangle

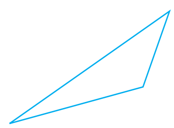
An **acute triangle** is one in which all three angles are acute.
An **obtuse triangle** contains one obtuse angle.
A **right triangle** contains one right angle.

Property

The sum of the measures of the angles of any triangle is 180°.


Equilateral triangle


Isosceles triangle


Scalene triangle

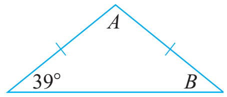
If a triangle has three sides of equal length, it is an **equilateral triangle**. (Each angle measure of an equilateral triangle is 60°.)

If a triangle has two sides of equal length, it is an **isosceles triangle**. (The angles opposite the equal sides have the same measure.)

If a triangle has no sides of equal length, it is a **scalene triangle**. (No angles have the same measure.)

Example 2

Find the measures of angles A and B in this isosceles triangle.



Solution

The single hash marks on the two sides of the triangle mean that those sides are of equal length.

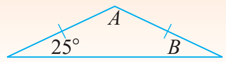
$$m\angle B = 39° \qquad \text{Angle measures opposite sides of equal length are the same.}$$
$$39° + m\angle B = 39° + 39° = 78°.$$

We have found that the sum of two of the angles is 78°. Since all of the angle measures add up to 180°,

$$m\angle A = 180° - 78° = 102°$$

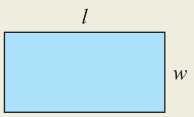
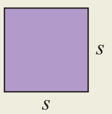
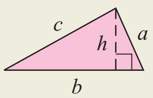
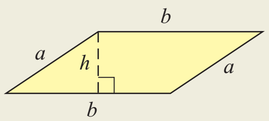
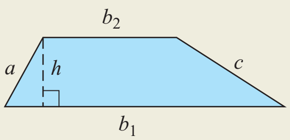
You Try 2

Find the measures of angles A and B in this isosceles triangle.



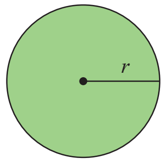
3. Use Area, Perimeter, and Circumference Formulas

The **perimeter** of a figure is the distance around the figure, while the **area** of a figure is the number of square units enclosed within the figure. For some familiar shapes, we have the following formulas:

Figure		Perimeter	Area
Rectangle:		$P = 2l + 2w$	$A = lw$
Square:		$P = 4s$	$A = s^2$
Triangle: h = height		$P = a + b + c$	$A = \frac{1}{2}bh$
Parallelogram: h = height		$P = 2a + 2b$	$A = bh$
Trapezoid: h = height		$P = a + c + b_1 + b_2$	$A = \frac{1}{2}h(b_1 + b_2)$

The perimeter of a circle is called the **circumference**. The **radius**, r, is the distance from the center of the circle to a point on the circle. A line segment that passes through the center of the circle and has its endpoints on the circle is called a **diameter**.

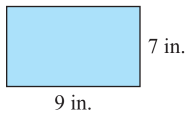
Pi, π, is the ratio of the circumference of any circle to its diameter. $\pi \approx 3.14159265\ldots$, but we will use 3.14 as an approximation for π. The symbol $\approx$ is read as "approximately equal to."

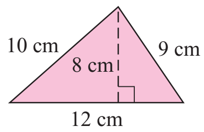


Circumference	**Area**
$C = 2\pi r$	$A = \pi r^2$

Example 3

Find the perimeter and area of each figure.

a) 

b) 

Solution

a) This figure is a rectangle.

Perimeter: $P = 2l + 2w$
$P = 2(9 \text{ in.}) + 2(7 \text{ in.})$
$P = 18 \text{ in.} + 14 \text{ in.}$
$P = 32 \text{ in.}$

Area: $A = lw$
$A = (9 \text{ in.})(7 \text{ in.})$
$A = 63 \text{ in}^2$ or 63 square inches

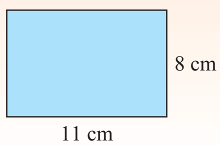
b) This figure is a triangle.

Perimeter: $P = a + b + c$
$P = 9 \text{ cm} + 12 \text{ cm} + 10 \text{ cm}$
$P = 31 \text{ cm}$

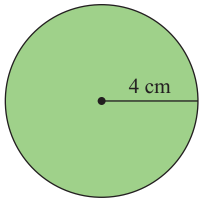
Area: $A = \dfrac{1}{2}bh$
$A = \dfrac{1}{2}(12 \text{ cm})(8 \text{ cm})$
$A = 48 \text{ cm}^2$ or 48 square centimeters

You Try 3

Find the perimeter and area of the figure.



Example 4



Find (a) the circumference and (b) the area of the circle shown at left. Give an exact answer for each and give an approximation using 3.14 for π.

Solution

a) The formula for the circumference of a circle is $C = 2\pi r$. The radius of the given circle is 4 cm. Replace r with 4 cm.

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi(4\text{ cm}) && \text{Replace } r \text{ with 4 cm.} \\ &= 8\pi\text{ cm} && \text{Multiply.} \end{aligned}$$

Leaving the answer in terms of π gives us the exact circumference of the circle, 8π cm.
To find an approximation for the circumference, substitute 3.14 for π and simplify.

$$\begin{aligned} C &= 8\pi\text{ cm} \\ &\approx 8(3.14)\text{ cm} = 25.12\text{ cm} \end{aligned}$$

b) The formula for the area of a circle is $A = \pi r^2$. Replace r with 4 cm.

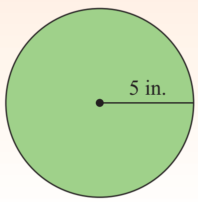
$$\begin{aligned} A &= \pi r^2 \\ &= \pi(4\text{ cm})^2 && \text{Replace } r \text{ with 4 cm.} \\ &= 16\pi\text{ cm}^2 && 4^2 = 16 \end{aligned}$$

Leaving the answer in terms of π gives us the exact area of the circle, $16\pi\text{ cm}^2$.
To find an approximation for the area, substitute 3.14 for π and simplify.

$$\begin{aligned} A &= 16\pi\text{ cm}^2 \\ &\approx 16(3.14)\text{ cm}^2 \\ &= 50.24\text{ cm}^2 \end{aligned}$$

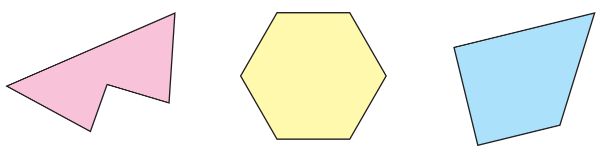
You Try 4

Find (a) the circumference and (b) the area of the circle. Give an exact answer for each and give an approximation using 3.14 for π.



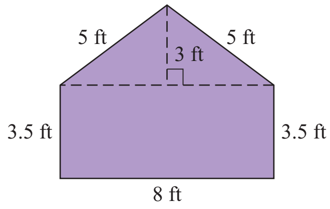
A **polygon** is a closed figure consisting of three or more line segments. (See the figure.) We can extend our knowledge of perimeter and area to determine the area and perimeter of a polygon.

Polygons:



Example 5

Find the perimeter and area of the figure shown here.

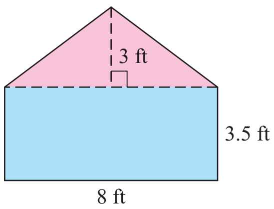


Solution

Perimeter: The perimeter is the distance around the figure.

$$P = 5 \text{ ft} + 5 \text{ ft} + 3.5 \text{ ft} + 8 \text{ ft} + 3.5 \text{ ft}$$
$$P = 25 \text{ ft}$$

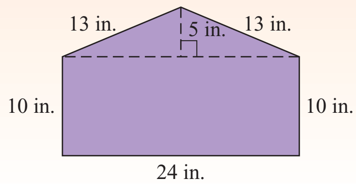
Area: To find the area of this figure, think of it as two regions: a triangle and a rectangle.



$$\begin{aligned}\text{Total area} &= \text{Area of triangle} + \text{Area of rectangle}\\ &= \frac{1}{2}bh + lw\\ &= \frac{1}{2}(8 \text{ ft})(3 \text{ ft}) + (8 \text{ ft})(3.5 \text{ ft})\\ &= 12 \text{ ft}^2 + 28 \text{ ft}^2\\ &= 40 \text{ ft}^2\end{aligned}$$

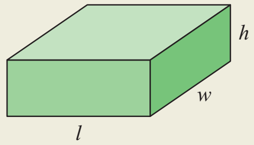
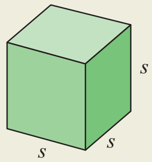
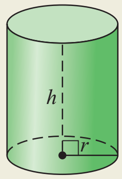
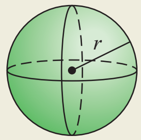
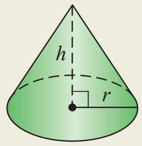
You Try 5

Find the perimeter and area of the figure.



4. Use Volume Formulas

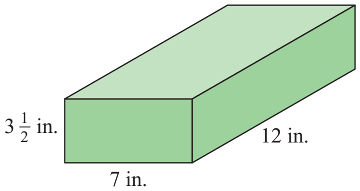
The **volume** of a three-dimensional object is the amount of space occupied by the object. Volume is measured in cubic units such as cubic inches (in^3), cubic centimeters (cm^3), cubic feet (ft^3), and so on. Volume also describes the amount of a substance that can be enclosed within a three-dimensional object. Therefore, volume can also be measured in quarts, liters, gallons, and so on. In the figures, l = length, w = width, h = height, s = length of a side, and r = radius.

Volumes of Three-Dimensional Figures		
Rectangular solid		$V = lwh$
Cube		$V = s^3$
Right circular cylinder		$V = \pi r^2 h$
Sphere		$V = \frac{4}{3}\pi r^3$
Right circular cone		$V = \frac{1}{3}\pi r^2 h$

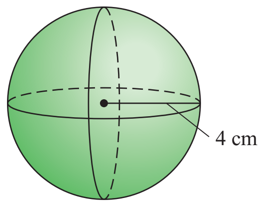
Example 6

Find the volume of each. In (b) give the answer in terms of π.

a)



b)



Solution

a) $V = lwh$ — Volume of a rectangular solid

$= (12 \text{ in.})(7 \text{ in.})\left(3\frac{1}{2} \text{ in.}\right)$ — Substitute values.

$= (12 \text{ in.})(7 \text{ in.})\left(\frac{7}{2} \text{ in.}\right)$ — Change to an improper fraction.

$= \left(84 \cdot \frac{7}{2}\right) \text{in}^3$ — Multiply.

$= 294 \text{ in}^3$ or 294 cubic inches

b) $V = \frac{4}{3}\pi r^3$ Volume of a sphere

$= \frac{4}{3}\pi(4 \text{ cm})^3$ Replace r with 4 cm.

$= \frac{4}{3}\pi(64 \text{ cm}^3)$ $4^3 = 64$

$= \frac{256}{3}\pi \text{ cm}^3$ Multiply.

You Try 6

Find the volume of each figure. In (b) give the answer in terms of π.

a) A box with length = 3 ft, width = 2 ft, and height = 1.5 ft

b) A sphere with radius = 3 in.

Example 7

Application

A large truck has a fuel tank in the shape of a right circular cylinder. Its radius is 1 ft, and it is 4 ft long.

a) How many cubic feet of diesel fuel will the tank hold? (Use 3.14 for π.)

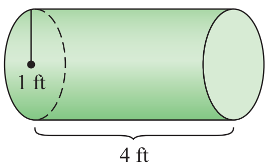
b) How many gallons will it hold? Round to the nearest gallon. ($1 \text{ ft}^3 \approx 7.48$ gal)

c) If diesel fuel costs \$1.75 per gallon, how much will it cost to fill the tank?

Solution

a) We're asked to determine how much fuel the tank will hold. We must find the *volume* of the tank.

$$\begin{aligned} \text{Volume of a cylinder} &= \pi r^2 h \\ &\approx (3.14)(1 \text{ ft})^2(4 \text{ ft}) \\ &= 12.56 \text{ ft}^3 \end{aligned}$$



The tank will hold 12.56 ft^3 of diesel fuel.

b) We must convert 12.56 ft^3 to gallons. Since $1 \text{ ft}^3 \approx 7.48$ gal, we can change units by multiplying:

$$12.56 \cancel{\text{ft}^3} \cdot \left(\frac{7.48 \text{ gal}}{1 \cancel{\text{ft}^3}}\right) = 93.9488 \text{ gal}$$
$$\approx 94 \text{ gal}$$

We can divide out units in fractions the same way we can divide out common factors.

The tank will hold approximately 94 gal.

c) Diesel fuel costs \$1.75 per gallon. We can figure out the total cost of the fuel the same way we did in (b).

\$1.75 *per* gallon
↓

$$94 \cancel{\text{gal}} \cdot \left(\frac{\$1.75}{\cancel{\text{gal}}}\right) = \$164.50$$ Divide out the units of gallons.

It will cost about \$164.50 to fill the tank.

You Try 7

A large truck has a fuel tank in the shape of a right circular cylinder. Its radius is 1 ft, and it is 3 ft long.

a) How many cubic feet of diesel fuel will the tank hold? (Use 3.14 for π).

b) How many gallons of fuel will it hold? Round to the nearest gallon. (1 $\text{ft}^3 \approx 7.48$ gal)

c) If diesel fuel costs \$1.75 per gallon, how much will it cost to fill the tank?

Answers to You Try Exercises

1) 118° 2) $m\angle A = 130°$; $m\angle B = 25°$ 3) $P = 38$ cm; $A = 88\text{ cm}^2$ 4) a) $C = 10\pi$ in.; $C \approx 31.4$ in. b) $A = 25\pi\text{ in}^2$; $A \approx 78.5\text{ in}^2$ 5) $P = 70$ in.; $A = 300\text{ in}^2$ 6) a) 9 ft^3 b) $36\pi\text{ in}^3$ 7) a) 9.42 ft^3 b) 70 gal c) \$122.50

1.3 Exercises

Objective 1: Identify Angles and Parallel and Perpendicular Lines

1) An angle whose measure is between 0° and 90° is a(n) _________ angle.

2) An angle whose measure is 90° is a(n) _________ angle.

3) An angle whose measure is 180° is a(n) _________ angle.

4) An angle whose measure is between 90° and 180° is a(n) _________ angle.

5) If the sum of two angles is 180°, the angles are _________. If the sum of two angles is 90°, the angles are _________.

6) If two angles are supplementary, can both of them be obtuse? Explain.

Find the complement of each angle.

7) 59°

8) 84°

9) 12°

10) 40°

Find the supplement of each angle.

11) 143°

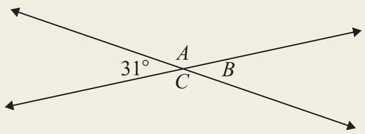
12) 62°

13) 38°

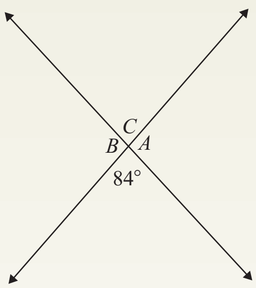
14) 155°

Find the measure of the missing angles.

15)



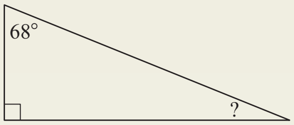
16)

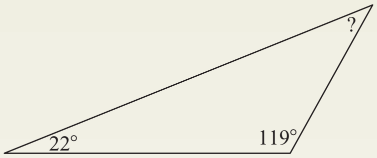


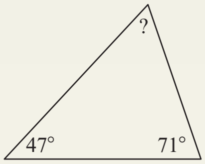
Objective 2: Identify Triangles

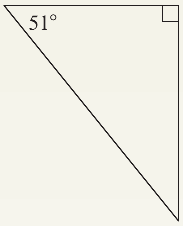
17) The sum of the angles in a triangle is _________ degrees.

Find the missing angle and classify each triangle as acute, obtuse, or right.

18) 

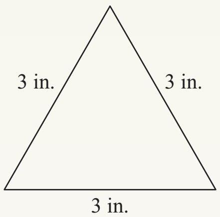
19) 

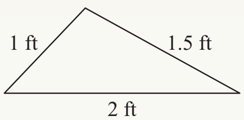
20) 

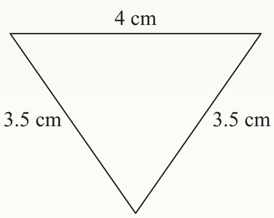
21) 

22) Can a triangle contain more than one obtuse angle? Explain.

Classify each triangle as equilateral, isosceles, or scalene.

23) 

24) 

25) 

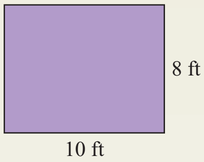
26) What can you say about the measures of the angles in an equilateral triangle?

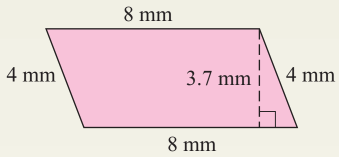
27) True or False: A right triangle can also be isosceles.

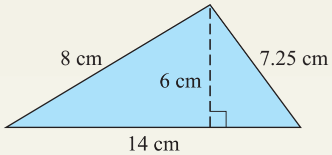
28) True or False: If a triangle has two sides of equal length, then the angles opposite these sides are equal.

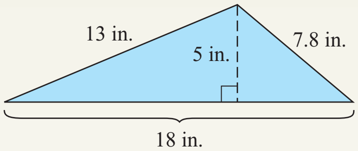
Objective 3: Use Area, Perimeter, and Circumference Formulas

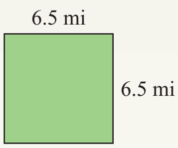
Find the area and perimeter of each figure. Include the correct units.

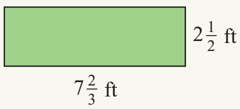
29) 

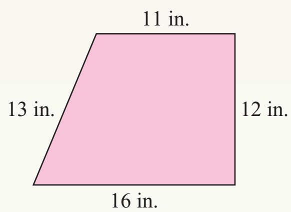
30) 

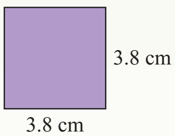
31) 

32) 

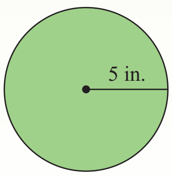
33) 

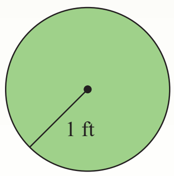
34) 

35) 

36) 

For 37–40, find (a) the area and (b) the circumference of the circle. Give an exact answer for each and give an approximation using 3.14 for π. Include the correct units.

37) 

38) 

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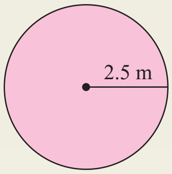

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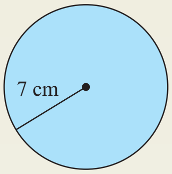
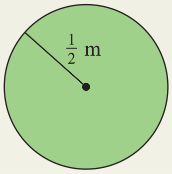


For 41–44, find the exact area and circumference of the circle in terms of π. Include the correct units.

41)



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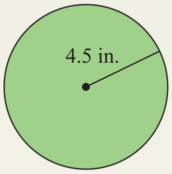

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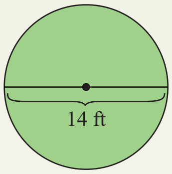

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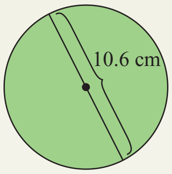
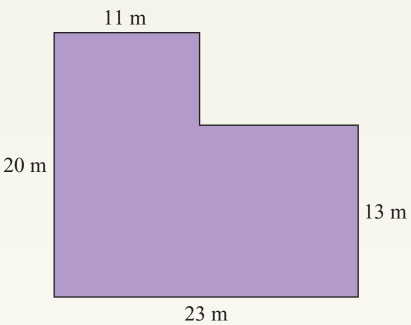


Find the area and perimeter of each figure. Include the correct units.

45)



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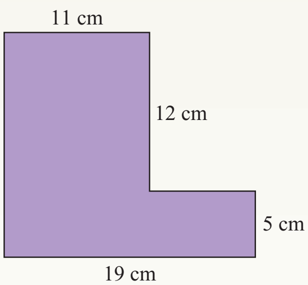

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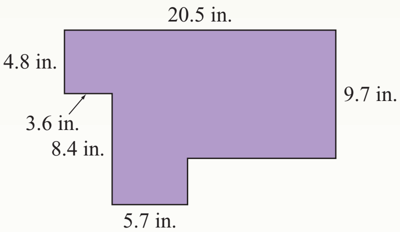

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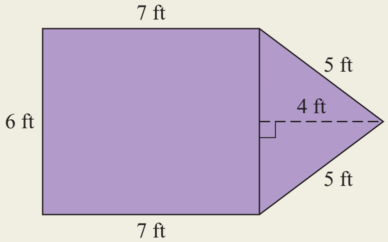

Find the area of the shaded region. Use 3.14 for π. Include the correct units.

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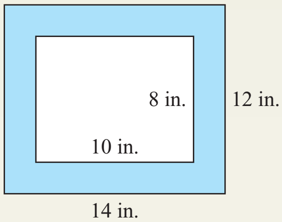

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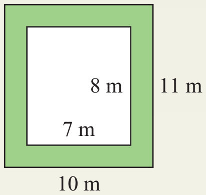

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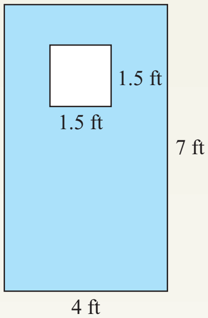

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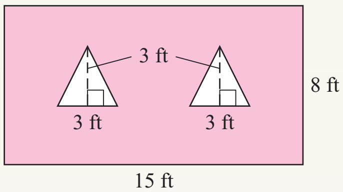

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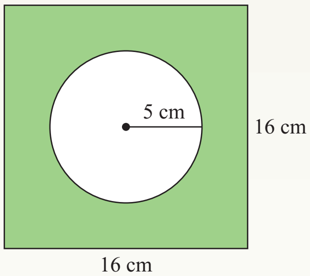

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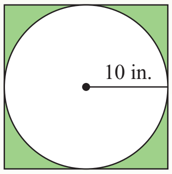
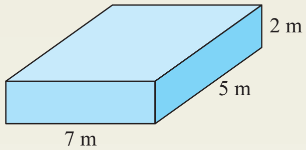


Objective 4: Use Volume Formulas

Find the volume of each figure. Where appropriate, give the answer in terms of π. Include the correct units.

VIDEO 55)



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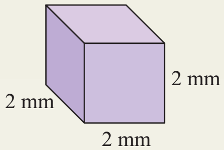

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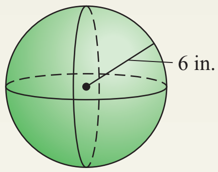

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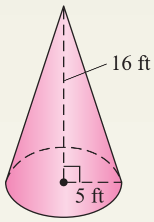

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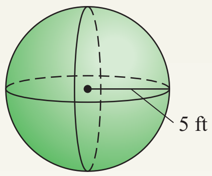

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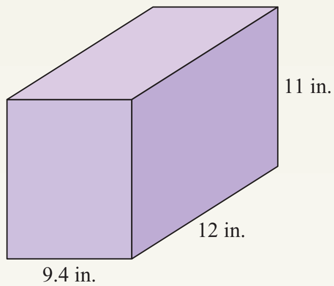

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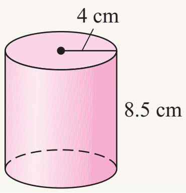

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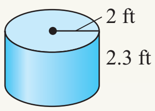


Mixed Exercises: Objectives 3 and 4

Applications of Perimeter, Area, and Volume: Use 3.14 for π and include the correct units.

63) To lower her energy costs, Yun would like to replace her rectangular storefront window with low-emissivity (low-e) glass that costs $20.00/ft^2. The window measures 9 ft by 6.5 ft, and she can spend at most $900.

a) How much glass does she need?

b) Can she afford the low-e glass for this window?

64) An insulated rectangular cooler is 15" long, 10" wide, and 13.6" high. What is the volume of the cooler?

65) A fermentation tank at a winery is in the shape of a right circular cylinder. The diameter of the tank is 6 ft, and it is 8 ft tall.

a) How many cubic feet of wine will the tank hold?

b) How many gallons of wine will the tank hold? Round to the nearest gallon. (1 ft^3 $\approx$ 7.48 gallons)

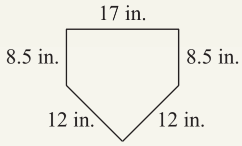
66) Yessenia wants a custom-made area rug measuring 5 ft by 8 ft. She has budgeted $500. She likes the Alhambra carpet sample that costs $9.80/ft^2 and the Sahara pattern that costs $12.20/ft^2. Can she afford either of these fabrics to make the area rug, or does she have to choose the cheaper one to remain within her budget? Support your answer by determining how much it would cost to have the rug made in each pattern.

67) The lazy Susan on a table in a Chinese restaurant has a 10-inch radius. (A lazy Susan is a rotating tray used to serve food.)

a) What is the perimeter of the lazy Susan?

b) What is its area?

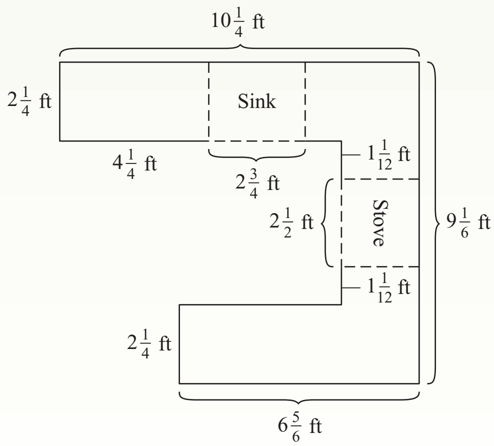
68) Find the perimeter of home plate given the dimensions below.



69) A rectangular reflecting pool is 30 ft long, 19 ft wide and 1.5 ft deep. How many gallons of water will this pool hold? (1 ft^3 $\approx$ 7.48 gallons)

70) Ralph wants to childproof his house now that his daughter has learned to walk. The round, glass-top coffee table in his living room has a diameter of 36 inches. How much soft padding does Ralph need to cover the edges around the table?

71) Nadia is remodeling her kitchen and her favorite granite countertop costs $80.00/ft^2, including installation. The layout of the countertop is shown below, where the counter has a uniform width of $2\frac{1}{4}$ ft. If she can spend at most $2500.00, can she afford her first-choice granite?



72) A container of lip balm is in the shape of a right circular cylinder with a radius of 1.5 cm and a height of 2 cm. How much lip balm will the container hold?

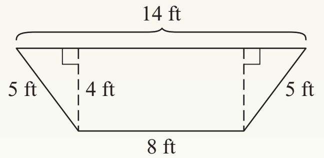
73) The radius of a women's basketball is approximately 4.6 in. Find its circumference to the nearest tenth of an inch.

74) The chamber of a rectangular laboratory water bath measures $6'' \times 11\frac{3}{4}'' \times 5\frac{1}{2}''$.

a) How many cubic inches of water will the water bath hold?

b) How many liters of water will the water bath hold? ($1 \text{ in}^3 \approx 0.016$ liter)

75) A town's public works department will install a flower garden in the shape of a trapezoid. It will be enclosed by decorative fencing that costs \$23.50/ft.



a) Find the area of the garden.

b) Find the cost of the fence.

76) Jaden is making decorations for the bulletin board in his fifth-grade classroom. Each equilateral triangle has a height of 15.6 inches and sides of length 18 inches.

a) Find the area of each triangle.

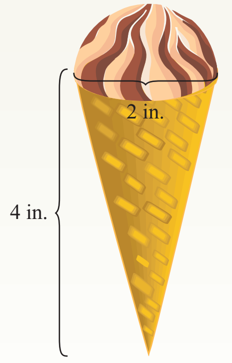
b) Find the perimeter of each triangle.

77) The top of a counter-height pub table is in the shape of an equilateral triangle. Each side has a length of 18 inches, and the height of the triangle is 15.6 inches. What is the area of the table top?

78) The dimensions of Riyad's home office are $10' \times 12'$. He plans to install laminated hardwood flooring that costs \$2.69/ft^2. How much will the flooring cost?

79) Salt used to melt road ice in winter is piled in the shape of a right circular cone. The radius of the base is 12 ft, and the pile is 8 ft high. Find the volume of salt in the pile.

80) Find the volume of the ice cream pictured below. Assume that the right circular cone is completely filled and that the scoop on top is half of a sphere.



Section 1.4 Sets of Numbers and Absolute Value

Objectives
1. Identify and Graph Numbers on a Number Line
2. Compare Numbers Using Inequality Symbols
3. Find the Additive Inverse and Absolute Value of a Number

1. Identify and Graph Numbers on a Number Line

In Section 1.1, we defined the following sets of numbers:

Natural numbers: $\{1, 2, 3, 4, \ldots\}$
Whole numbers: $\{0, 1, 2, 3, 4, \ldots\}$

We will begin this section by discussing other sets of numbers.

On a **number line**, positive numbers are to the right of zero and negative numbers are to the left of zero.

Definition

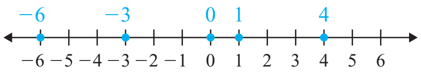
The set of **integers** includes the set of natural numbers, their negatives, and zero. The set of *integers* is $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Example 1

Graph each number on a number line.

$$4, 1, -6, 0, -3$$

Solution



4 and 1 are to the right of zero since they are positive.
-3 is three units to the left of zero, and -6 is six units to the left of zero. ■

You Try 1

Graph each number on a number line. $2, -4, 5, -1, -2$

Positive and negative numbers are also called **signed numbers**.

Example 2

Given the set of numbers $\left\{4, -7, 0, \frac{3}{4}, -6, 10, -3\right\}$, list the

a) whole numbers b) natural numbers c) integers

Solution

a) whole numbers: 0, 4, 10
b) natural numbers: 4, 10
c) integers: $-7, -6, -3, 0, 4, 10$ ■

You Try 2

Given the set of numbers $\left\{-1, 5, \frac{2}{7}, 8, -\frac{4}{5}, 0, -12\right\}$, list the

a) whole numbers b) natural numbers c) integers

Notice in Example 2 that $\frac{3}{4}$ did not belong to any of these sets. That is because the whole numbers, natural numbers, and integers do not contain any fractional parts. $\frac{3}{4}$ is a *rational number*.

Definition

A **rational number** is any number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

That is, a rational number is any number that can be written as a fraction where the numerator and denominator are integers and the denominator does not equal zero.

Rational numbers include much more than numbers like $\frac{3}{4}$, which are already in fractional form.

Example 3

Explain why each of the following numbers is rational.

a) 7 b) 0.8 c) -5

d) $6\frac{1}{4}$ e) $0.\overline{3}$ f) $\sqrt{4}$

Solution

Rational Number	Reason
7	7 can be written as $\frac{7}{1}$.
0.8	0.8 can be written as $\frac{8}{10}$.
-5	-5 can be written as $\frac{-5}{1}$.
$6\frac{1}{4}$	$6\frac{1}{4}$ can be written as $\frac{25}{4}$.
$0.\overline{3}$	$0.\overline{3}$ can be written as $\frac{1}{3}$.
$\sqrt{4}$	$\sqrt{4} = 2$ and $2 = \frac{2}{1}$.

$\sqrt{4}$ is read as "the square root of 4." This means, "What number times itself equals 4?" That number is 2.

You Try 3

Explain why each of the following numbers is rational.

a) 12 b) 0.7 c) -8 d) $2\frac{3}{4}$ e) $0.\overline{6}$ f) $\sqrt{100}$

To summarize, the set of rational numbers includes

1) Integers, whole numbers, and natural numbers.
2) Repeating decimals.
3) Terminating decimals.
4) Fractions and mixed numbers.

The set of rational numbers does *not* include nonrepeating, nonterminating decimals. These decimals cannot be written as the quotient of two integers. Numbers such as these are called *irrational numbers.*

Definition

The set of numbers that cannot be written as the quotient of two integers is called the set of **irrational numbers**. Written in decimal form, an *irrational number* is a nonrepeating, nonterminating decimal.

Example 4

Explain why each of the following numbers is irrational.

a) 0.8271316… b) π c) $\sqrt{3}$

Solution

Irrational Number	Reason
0.827136…	It is a nonrepeating, nonterminating decimal.
π	$\pi \approx 3.14159265\ldots$ It is a nonrepeating, nonterminating decimal.
$\sqrt{3}$	3 is not a perfect square, and the decimal equivalent of the square root of a nonperfect square is a nonrepeating, nonterminating decimal. Here, $\sqrt{3} \approx 1.73205\ldots$.

You Try 4

Explain why each of the following numbers is irrational.

a) 2.41895… b) $\sqrt{2}$

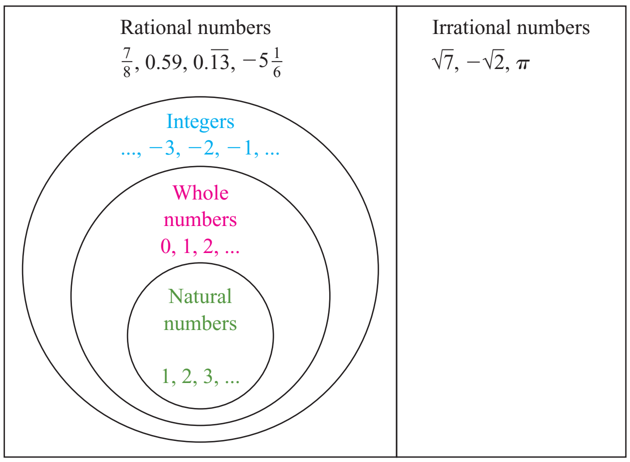
If we put together the sets of numbers we have discussed up to this point, we get the *real numbers.*

Definition

The set of **real numbers** consists of the rational and irrational numbers.

We summarize the information next with examples of the different sets of numbers:

Real Numbers



From the figure we can see, for example, that all whole numbers $\{0, 1, 2, 3, \ldots\}$ are integers, but not all integers are whole numbers (-3, for example).

Example 5

Given the set of numbers $\left\{-16, 3.82, 0, 29, 0.\overline{7}, -\frac{11}{12}, \sqrt{10}, 5.302981\ldots\right\}$, list the

a) integers b) natural numbers c) whole numbers
d) rational numbers e) irrational numbers f) real numbers

Solution

a) integers: $-16, 0, 29$
b) natural numbers: 29
c) whole numbers: 0, 29
d) rational numbers: $-16, 3.82, 0, 29, 0.\overline{7}, -\frac{11}{12}$ Each of these numbers can be written as the quotient of two integers.
e) irrational numbers: $\sqrt{10}, 5.302981\ldots$
f) real numbers: All of the numbers in this set are real.
$\left\{-16, 3.82, 0, 29, 0.\overline{7}, -\frac{11}{12}, \sqrt{10}, 5.302981\ldots\right\}$ ■

You Try 5

Given the set of numbers $\left\{\frac{9}{8}, \sqrt{14}, 34, -41, 6.5, 0.\overline{21}, 0, 7.412835\ldots\right\}$, list the

a) whole numbers b) integers
c) rational numbers d) irrational numbers

2. Compare Numbers Using Inequality Symbols

Let's review the inequality symbols.

$<$ less than	$\le$ less than or equal to
$>$ greater than	$\ge$ greater than or equal to
$\ne$ not equal to	$\approx$ approximately equal to

We use these symbols to compare numbers as in $5 > 2$, $6 \le 17$, $4 \ne 9$, and so on.

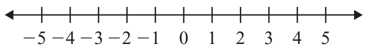
How do we compare negative numbers?

Note

As we move to the *left* on the number line, the numbers get smaller. As we move to the *right* on the number line, the numbers get larger.

Example 6

Insert $>$ or $<$ to make the statement true. Look at the number line, if necessary.



a) 4 ___ 2 b) -3 ___ 1 c) -2 ___ -5 d) -4 ___ -1

Solution

a) $4 \ \underline{>}\ 2$ 4 is to the right of 2.

b) $-3 \ \underline{<}\ 1$ -3 is to the left of 1.

c) $-2 \ \underline{>}\ -5$ -2 is to the right of -5.

d) $-4 \ \underline{<}\ -1$ -4 is to the left of -1.

You Try 6

Insert $>$ or $<$ to make the statement true.

a) 7 ___ 3 b) -5 ___ -1 c) -6 ___ -14

Application of Signed Numbers

Example 7

Use a signed number to represent the change in each situation.

a) During the recession, the number of employees at a manufacturing company decreased by 850.

b) From October 2008 to March 2009, the number of Facebook users increased by over 23,000,000. (www.insidefacebook.com)

Solution

a) -850 The negative number represents a decrease in the number of employees.

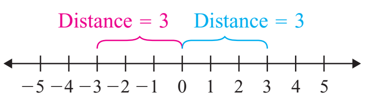
b) $23{,}000{,}000$ The positive number represents an increase in the number of Facebook users.

You Try 7

Use a signed number to represent the change.

After getting off the highway, Huda decreased his car's speed by 25 mph.

3. Find the Additive Inverse and Absolute Value of a Number



Notice that both -3 and 3 are a distance of 3 units from 0 but are on opposite sides of 0. We say that 3 and -3 are *additive inverses.*

Definition

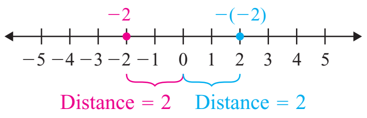
Two numbers are **additive inverses** if they are the same distance from 0 on the number line but on the opposite side of 0. Therefore, if a is any real number, then $-a$ is its additive inverse.

Furthermore, $-(-a) = a$. We can see this on the number line.

Example 8

Find $-(-2)$.

Solution



So, beginning with -2, the number on the opposite side of zero and 2 units away from zero is 2.

$-(-2) = 2$ ■

You Try 8

Find $-(-13)$.

We can explain "distance from zero" in another way: *absolute value.* The absolute value of a number is the distance between that number and 0 on the number line. It just describes the distance, *not* what side of zero the number is on. Therefore, the absolute value of a number is always positive or zero.

Definition

If a is any real number, then the **absolute value of a**, denoted by $|a|$, is

i) a if $a \geq 0$

ii) $-a$ if $a < 0$

Remember, $|a|$ is never negative.

Example 9

Evaluate each.

a) $|6|$ b) $|-5|$ c) $|0|$ d) $-|12|$ e) $|14 - 5|$

Solution

a) $|6| = 6$ 6 is 6 units from 0.

b) $|-5| = 5$ -5 is 5 units from 0.

c) $|0| = 0$

d) $-|12| = -12$ First, evaluate $|12|$: $|12| = 12$. Then, apply the negative symbol to get -12.

e) $|14 - 5| = |9|$ The absolute value symbols work like parentheses. First, evaluate what is inside: $14 - 5 = 9$.

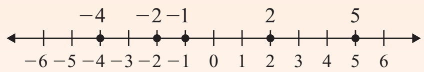
$= 9$ Find the absolute value. ■

You Try 9

Evaluate each.

a) $|19|$ b) $|-8|$ c) $-|7|$ d) $|20 - 9|$

Answers to You Try Exercises

1) 

2) a) 0, 5, 8 b) 5, 8 c) $-12, -1, 0, 5, 8$

3) a) $12 = \frac{12}{1}$ b) $0.7 = \frac{7}{10}$ c) $-8 = \frac{-8}{1}$ d) $2\frac{3}{4} = \frac{11}{4}$ e) $0.\overline{6} = \frac{2}{3}$ f) $\sqrt{100} = 10$ and $10 = \frac{10}{1}$

4) a) It is a nonrepeating, nonterminating decimal. b) 2 is not a perfect square, so the decimal equivalent of $\sqrt{2}$ is a nonrepeating, nonterminating decimal. 5) a) 34, 0 b) 34, -41, 0 c) $\frac{9}{8}$, 34, -41, 6.5, $0.\overline{21}$, 0 d) $\sqrt{14}$, 7.412835 . . . 6) a) > b) < c) > 7) -25

8) 13 9) a) 19 b) 8 c) -7 d) 11

1.4 Exercises

Objective 1: Identify and Graph Numbers on a Number Line

1) In your own words, explain the difference between the set of rational numbers and the set of irrational numbers. Give two examples of each type of number.

2) In your own words, explain the difference between the set of whole numbers and the set of natural numbers. Give two examples of each type of number.

In Exercises 3 and 4, given each set of numbers, list the

a) natural numbers b) whole numbers

c) integers d) rational numbers

e) irrational numbers f) real numbers

3) $\left\{17, 3.8, \frac{4}{5}, 0, \sqrt{10}, -25, 6.\overline{7}, -2\frac{1}{8}, 9.721983\ldots\right\}$

4) $\left\{-6, \sqrt{23}, 21, 5.\overline{62}, 0.4, 3\frac{2}{9}, 0, -\frac{7}{8}, 2.074816\ldots\right\}$

Determine whether each statement is true or false.

5) Every whole number is a real number.

6) Every real number is an integer.

7) Every rational number is a whole number.

8) Every whole number is an integer.

9) Every natural number is a whole number.

10) Every integer is a rational number.

Graph the numbers on a number line. Label each.

11) $5, -2, \frac{3}{2}, -3\frac{1}{2}, 0$

12) $-4, 3, \frac{7}{8}, 4\frac{1}{3}, -2\frac{1}{4}$

VIDEO 13) $-6.8, -\frac{3}{8}, 0.2, 1\frac{8}{9}, -4\frac{1}{3}$

14) $-3.25, \frac{2}{3}, 2, -1\frac{3}{8}, 4.1$

Objective 3: Find the Additive Inverse and Absolute Value of a Number

15) What does the absolute value of a number represent?

16) If a is a real number and if $|a|$ is not a positive number, then what is the value of a?

Find the additive inverse of each.

17) 8 18) 6

19) -15 20) -1

21) $-\frac{3}{4}$ 22) 4.7

Evaluate.

23) $|-10|$ 24) $|9|$

25) $\left|\frac{9}{4}\right|$ 26) $\left|-\frac{5}{6}\right|$

27) $-|-14|$

28) $-|27|$

29) $|17 - 4|$

30) $-|10 - 6|$

31) $-\left|-4\frac{1}{7}\right|$

32) $|-9.6|$

Write each group of numbers from smallest to largest.

VIDEO 33) $7, -2, 3.8, -10, 0, \frac{9}{10}$

34) $2.6, 2.06, -1, -5\frac{3}{8}, 3, \frac{7}{4}$

35) $7\frac{5}{6}, -5, -6.5, -6.51, 7\frac{1}{3}, 2$

36) $-\frac{3}{4}, 0, -0.5, 4, -1, \frac{15}{2}$

Mixed Exercises: Objectives 2 and 3

Decide whether each statement is true or false.

37) $16 \geq -11$

38) $-19 < -18$

39) $\frac{7}{11} \leq \frac{5}{9}$

40) $-1.7 \geq -1.6$

41) $-|-28| = 28$

42) $-|13| = -13$

VIDEO 43) $-5\frac{3}{10} < -5\frac{3}{4}$

44) $\frac{3}{2} \leq \frac{3}{4}$

Use a signed number to represent the change in each situation.

45) In 2007, Alex Rodriguez of the New York Yankees had 156 RBIs (runs batted in) while in 2008 he had 103 RBIs. That was a decrease of 53 RBIs. (http://newyork.yankees.mlb.com)

46) In 2006, Madonna's *Confessions* tour grossed about \$194 million. Her *Sticky and Sweet* tour grossed about \$230 million in 2008, an increase of \$36 million compared to the *Confessions* tour. (www.billboard.com)

47) In January 2009, an estimated 2.6 million people visited the Twitter website. In February 2009, there were about 4 million visitors to the site. This is an increase of 1.4 million people. (www.techcrunch.com)

48) According to the *Statistical Abstract of the United States,* the population of Louisiana decreased by about 58,000 from April 1, 2000 to July 1, 2008. (www.census.gov)

49) From 2006 to 2007, the number of new housing starts decreased by about 419,000. (www.census.gov)

50) Research done by the U.S. Department of Agriculture has found that the per capita consumption of bottled water increased by 2.1 gallons from 2005 to 2006. (www.census.gov)

Section 1.5 Addition and Subtraction of Real Numbers

Objectives

1. Add Integers Using a Number Line
2. Add Real Numbers with the Same Sign
3. Add Real Numbers with Different Signs
4. Subtract Real Numbers
5. Solve Applied Problems
6. Apply the Order of Operations to Real Numbers
7. Translate English Expressions to Mathematical Expressions

In Section 1.4, we defined real numbers. In this section, we will discuss adding and subtracting real numbers.

1. Add Integers Using a Number Line

Let's use a number line to add numbers.

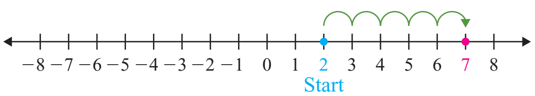
Example 1

Use a number line to add each pair of numbers.

a) $2 + 5$ b) $-1 + (-4)$ c) $2 + (-5)$ d) $-8 + 12$

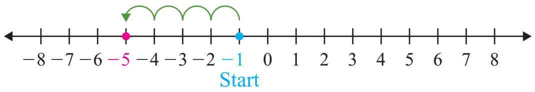
Solution

a) $2 + 5$: Start at 2 and move 5 units to the right.



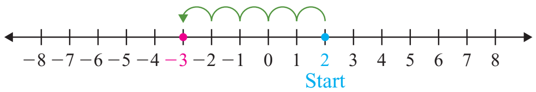
$2 + 5 = 7$

b) $-1 + (-4)$: Start at -1 and move 4 units to the left. (Move to the left when adding a negative.)



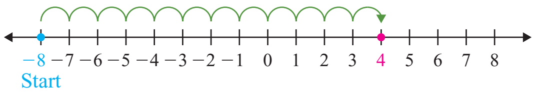
$$-1 + (-4) = -5$$

c) $2 + (-5)$: Start at 2 and move 5 units to the left.



$$2 + (-5) = -3$$

d) $-8 + 12$: Start at -8 and move 12 units to the right.



$$-8 + 12 = 4$$

You Try 1

Use a number line to add each pair of numbers.

a) $1 + 3$ b) $-3 + (-2)$ c) $8 + (-6)$ d) $-10 + 7$

2. Add Real Numbers with the Same Sign

We found that

$$2 + 5 = 7, \qquad -1 + (-4) = -5, \qquad 2 + (-5) = -3, \qquad -8 + 12 = 4.$$

Notice that when we add two numbers with the same sign, the result has the same sign as the numbers being added.

Procedure Adding Numbers with the Same Sign

To add numbers with the same sign, find the absolute value of each number and add them. The sum will have the same sign as the numbers being added.

Apply this rule to $-1 + (-4)$.

The result will be negative

$$-1 + (-4) = -\underbrace{(|-1| + |-4|)}_{\text{Add the absolute value of each number.}} = -(1 + 4) = -5$$

Example 2

Add.

a) $-5 + (-4)$ b) $-23 + (-41)$

Solution

a) $-5 + (-4) = -(|-5| + |-4|) = -(5 + 4) = -9$

b) $-23 + (-41) = -(|-23| + |-41|) = -(23 + 41) = -64$

You Try 2

Add.

a) $-6 + (-10)$ b) $-38 + (-56)$

3. Add Real Numbers with Different Signs

In Example 1, we found that $2 + (-5) = -3$ and $-8 + 12 = 4$.

Procedure Adding Numbers with Different Signs

To add two numbers with different signs, find the absolute value of each number. Subtract the smaller absolute value from the larger. The sum will have the sign of the number with the larger absolute value.

Let's apply this to $2 + (-5)$ and $-8 + 12$.

$2 + (-5)$: $|2| = 2$ $|-5| = 5$
Since $2 < 5$, subtract $5 - 2$ to get 3. Since $|-5| > |2|$, the sum will be negative.
$2 + (-5) = -3$

$-8 + 12$: $|-8| = 8$ $|12| = 12$
Subtract $12 - 8$ to get 4. Since $|12| > |-8|$, the sum will be positive.
$-8 + 12 = 4$

Example 3

Add.

a) $-17 + 5$ b) $9.8 + (-6.3)$ c) $\frac{1}{5} + \left(-\frac{2}{3}\right)$ d) $-8 + 8$

Solution

a) $-17 + 5 = -12$ The sum will be negative since the number with the larger absolute value, $|-17|$, is negative.

b) $9.8 + (-6.3) = 3.5$ The sum will be positive since the number with the larger absolute value, $|9.8|$, is positive.

c) $\frac{1}{5} + \left(-\frac{2}{3}\right) = \frac{3}{15} + \left(-\frac{10}{15}\right)$ Get a common denominator.

$= -\frac{7}{15}$ The sum will be negative since the number with the larger absolute value, $\left|-\frac{10}{15}\right|$, is negative.

d) $-8 + 8 = 0$

Note

The sum of a number and its additive inverse is always 0. That is, if a is a real number, then $a + (-a) = 0$. Notice in part d) of Example 3 that -8 and 8 are additive inverses.

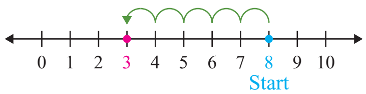
You Try 3

Add.

a) $20 + (-19)$ b) $-14 + (-2)$ c) $-\frac{3}{7} + \frac{1}{4}$ d) $7.2 + (-7.2)$

4. Subtract Real Numbers

We can use the additive inverse to subtract numbers. Let's start with a basic subtraction problem and use a number line to find $8 - 5$.



Start at 8. Then to subtract 5, move 5 units to the left to get 3.

$$8 - 5 = 3$$

We use the same procedure to find $8 + (-5)$. This leads us to a definition of subtraction:

> **Definition**
>
> If a and b are real numbers, then $a - b = a + (-b)$.

The definition tells us that to subtract $a - b$,

1) Change subtraction to addition.
2) Find the additive inverse of b.
3) Add a and the additive inverse of b.

Example 4

Subtract.

a) $4 - 9$ b) $-10 - 8$ c) $24 - 11$ d) $6 - (-25)$

Solution

a) $4 - 9 = 4 + (-9) = -5$

Change to addition Additive inverse of 9

b) $-10 - 8 = -10 + (-8) = -18$

Change to addition Additive inverse of 8

c) $24 - 11 = 24 + (-11)$
$= 13$

d) $6 - (-25) = 6 + 25 = 31$

Change to addition Additive inverse of -25

You Try 4

Subtract.

a) $2 - 14$ b) $-9 - 13$ c) $31 - 14$ d) $23 - (-34)$

In part d) of Example 4, $6 - (-25)$ changed to $6 + 25$. This illustrates that *subtracting a negative number is equivalent to adding a positive number.* Therefore, $-7 - (-15) = -7 + 15 = 8$.

5. Solve Applied Problems

We can use signed numbers to solve real-life problems.

Example 5

According to the National Weather Service, the coldest temperature ever recorded in Wyoming was $-66°F$ on February 9, 1944. The record high was $115°F$ on August 8, 1983. What is the difference between these two temperatures? (www.ncdc.noaa.gov)

Solution

$$\begin{aligned} \text{Difference} &= \text{Highest temperature} - \text{Lowest temperature} \\ &= 115 - (-66) \\ &= 115 + 66 \\ &= 181 \end{aligned}$$

The difference between the temperatures is 181°F. ■

You Try 5

The best score in a golf tournament was -16, and the worst score was $+9$. What is the difference between these two scores?

6. Apply the Order of Operations to Real Numbers

We discussed the order of operations in Section 1.2. Let's explore it further with the real numbers.

Example 6

Simplify.

a) $(10 - 18) + (-4 + 6)$ b) $-13 - (-21 + 5)$
c) $|-31 - 4| - 7|9 - 4|$

Solution

a) $(10 - 18) + (-4 + 6) = -8 + 2$ First, perform the operations in parentheses.
$= -6$ Add.

b) $-13 - (-21 + 5) = -13 - (-16)$ First, perform the operations in parentheses.
$= -13 + 16$ Change to addition.
$= 3$ Add.

c) $|-31 - 4| - 7|9 - 4| = |-31 + (-4)| - 7|9 - 4|$
$= |-35| - 7|5|$ Perform the operations in the absolute values.
$= 35 - 7(5)$ Evaluate the absolute values.
$= 35 - 35$
$= 0$ ■

You Try 6

Simplify.

a) $[12 + (-5)] - [-16 + (-8)]$ b) $-\frac{4}{9} + \left(\frac{1}{6} - \frac{2}{3}\right)$ c) $-|7 - 15| - |4 - 2|$

7. Translate English Expressions to Mathematical Expressions

Knowing how to translate from English expressions to mathematical expressions is a skill students need to learn algebra. Here, we will discuss how to "translate" from English to mathematics.

Let's look at some key words and phrases you may encounter.

English Expression	Mathematical Operation
sum, more than, increased by	addition
difference between, less than, decreased by	subtraction

Here are some examples:

Example 7

Write a mathematical expression for each and simplify.

a) 9 more than -2
b) 10 less than 41
c) -8 decreased by 17
d) the sum of 13 and -4
e) 8 less than the sum of -11 and -3

Solution

a) 9 more than -2

9 more than a quantity means we *add 9* to the quantity, in this case, -2.

$$-2 + 9 = 7$$

b) 10 less than 41

10 less than a quantity means we *subtract 10 from* that quantity, in this case, 41.

$$41 - 10 = 31$$

c) -8 decreased by 17

If -8 is being *decreased by 17*, then we subtract 17 *from* -8.

$$\begin{aligned} -8 - 17 &= -8 + (-17) \\ &= -25 \end{aligned}$$

d) the sum of 13 and -4

Sum means add. $13 + (-4) = 9$

e) 8 less than the sum of -11 and -3.

8 less than means we are subtracting 8 *from* something. From what? From the *sum of -11 and -3.*

Sum means add, so we must find the sum of -11 and -3 and subtract 8 from it.

$$\begin{aligned} [-11 + (-3)] - 8 &= -14 - 8 && \text{First, perform the operation in the brackets.} \\ &= -14 + (-8) && \text{Change to addition.} \\ &= -22 && \text{Add.} \end{aligned}$$

You Try 7

Write a mathematical expression for each and simplify.

a) -14 increased by 6
b) 27 less than 15
c) The sum of 23 and -7 decreased by 5

Answers to You Try Exercises

1) a) 4 b) -5 c) 2 d) -3 2) a) -16 b) -94 3) a) 1 b) -16 c) $-\frac{5}{28}$ d) 0

4) a) -12 b) -22 c) 17 d) 57 5) 25 6) a) 31 b) $-\frac{17}{18}$ c) -10

7) a) $-14 + 6; -8$ b) $15 - 27; -12$ c) $[23 + (-7)] - 5; 11$

1.5 Exercises

Mixed Exercises: Objectives 1–4 and 6

1) Explain, in your own words, how to subtract two negative numbers.

2) Explain, in your own words, how to add two negative numbers.

3) Explain, in your own words, how to add a positive and a negative number.

Use a number line to represent each sum or difference.

4) $-8 + 5$
5) $6 - 11$
6) $-1 - 5$
7) $-2 + (-7)$
8) $10 + (-6)$

Add or subtract as indicated.

9) $8 + (-15)$
10) $-12 + (-6)$
11) $-3 - 11$
12) $-7 + 13$
13) $-31 + 54$
14) $19 - (-14)$
15) $-26 - (-15)$
16) $-20 - (-30)$
17) $-352 - 498$
18) $217 + (-521)$
19) $-\frac{7}{12} + \frac{3}{4}$
20) $\frac{3}{10} - \frac{11}{15}$
21) $-\frac{1}{6} - \frac{7}{8}$
22) $\frac{2}{9} - \left(-\frac{2}{5}\right)$
23) $-\frac{4}{9} - \left(-\frac{4}{15}\right)$
24) $-\frac{1}{8} + \left(-\frac{3}{4}\right)$
25) $19.4 + (-16.7)$
26) $-31.3 - (-19.82)$
27) $-25.8 - (-16.57)$
28) $7.3 - 21.9$
29) $9 - (5 - 11)$
30) $-2 + (3 - 8)$
31) $-1 + (-6 - 4)$
32) $14 - (-10 - 2)$
33) $(-3 - 1) - (-8 + 6)$
34) $[14 + (-9)] + (1 - 8)$
35) $-16 + 4 + 3 - 10$
36) $8 - 28 + 3 - 7$
37) $5 - (-30) - 14 + 2$
38) $-17 - (-9) + 1 - 10$
39) $\frac{4}{9} - \left(\frac{2}{3} + \frac{5}{6}\right)$
40) $-\frac{1}{2} + \left(\frac{3}{5} - \frac{3}{10}\right)$
41) $\left(\frac{1}{8} - \frac{1}{2}\right) + \left(\frac{3}{4} - \frac{1}{6}\right)$
42) $\frac{11}{12} - \left(\frac{3}{8} - \frac{2}{3}\right)$
43) $(2.7 + 3.8) - (1.4 - 6.9)$
44) $-9.7 - (-5.5 + 1.1)$
45) $|7 - 11| + |6 + (-13)|$
46) $|8 - (-1)| - |3 + 12|$
47) $-|-2 - (-3)| - 2|-5 + 8|$
48) $|-6 + 7| + 5|-20 - (-11)|$

Determine whether each statement is true or false. For any real numbers a and b,

49) $|a + b| = |a| + |b|$
50) $|a - b| = |b - a|$
51) $|a + b| = a + b$
52) $|a| + |b| = a + b$
53) $-b - (-b) = 0$
54) $a + (-a) = 0$

Objective 5: Solve Applied Problems

Applications of Signed Numbers: Write an expression for each and simplify. Answer the question with a complete sentence.

55) Tiger Woods won his first Masters championship in 1997 at age 21 with a score of -18. When he won the championship in 2005, his score was 6 strokes higher. What was Tiger's score when he won the Masters in 2005? (www.masters.com)

56) In 1999, the U.S. National Park System recorded 287.1 million visits while in 2007 there were 275.6 million visits. What was the difference in the number of visits from 1999 to 2007? (www.nationalparkstraveler.com)

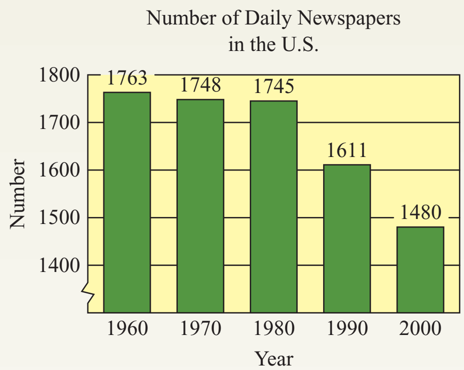
57) In 2006, China's carbon emissions were 6,110,000 thousand metric tons and the carbon emissions of the United States totaled 5,790,000 thousand metric tons. By how much did China's carbon emissions exceed those of the United States? (www.pbl.nl)

58) The budget of the Cincinnati Public Schools was \$428,554,470 in the 2006–2007 school year. This was \$22,430 less than the previous school year. What was the budget in the 2005–2006 school year? (www.cps-k12.org)

59) From 2007 to 2008, the number of flights going through O'Hare Airport in Chicago decreased by 45,407. There were 881,566 flights in 2008. How many flights went through O'Hare in 2007? (www.ohare.com)

60) The lowest temperature ever recorded in Minneapolis was $-41°F$ while the highest temperature on record was 149° greater than that. What was the warmest temperature ever recorded in Minneapolis? (www.weather.com)

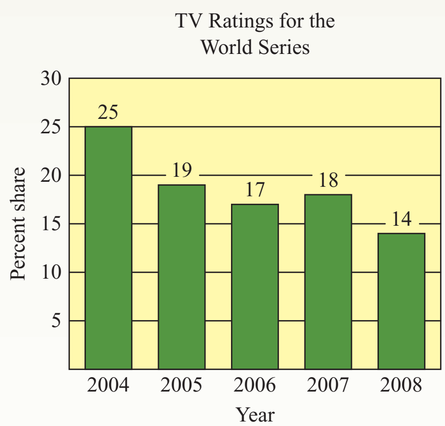
61) The bar graph shows the total number of daily newspapers in the United States in various years. Use a signed number to represent the change in the number of dailies over the given years. (www.naa.org)



a) 1970–1980 b) 1980–1990

c) 1990–2000 d) 1960–2000

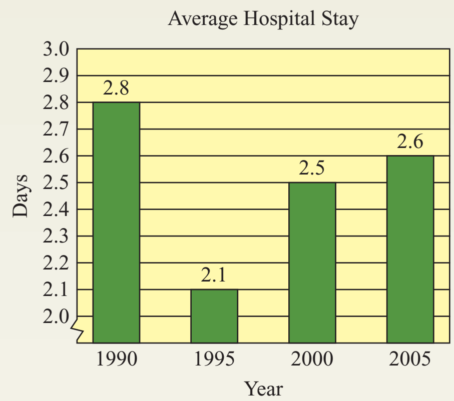
62) The bar graph shows the TV ratings for the World Series over a 5-year period. Each ratings number represents the percentage of people watching TV at the time of the World Series who were tuned into the games. Use a signed number to represent the change in ratings over the given years. (www.baseball-almanac.com)



a) 2004–2005 b) 2006–2007

c) 2007–2008 d) 2004–2008

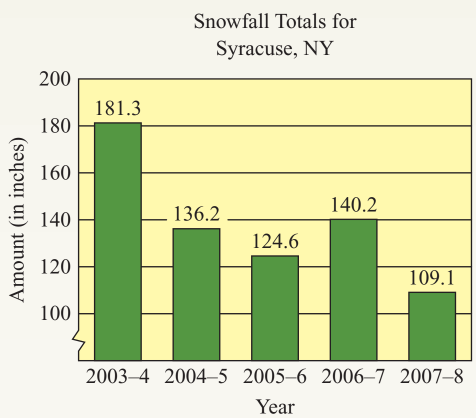
63) The bar graph shows the average number of days a woman was in the hospital for childbirth. Use a signed number to represent the change in hospitalization time over the given years. (www.cdc.gov)



a) 1990–1995 b) 1995–2000

c) 2000–2005 d) 1990–2005

64) The bar graph shows snowfall totals for different seasons in Syracuse, NY. Use a signed number to represent the difference in snowfall totals over different years. (www.erh.noaa.gov)



a) 2003–4 to 2004–5 b) 2005–6 to 2006–7

c) 2006–7 to 2007–8 d) 2003–4 to 2007–8

Objective 7: Translate English Expressions to Mathematical Expressions

Write a mathematical expression for each and simplify.

65) 7 more than 5

66) 3 more than 11

VIDEO 67) 16 less than 10

68) 15 less than 4

69) -8 less than 9

70) -25 less than -19

71) The sum of -21 and 13

72) The sum of -7 and 20

73) -20 increased by 30

74) -37 increased by 22

75) 23 decreased by 19

76) 8 decreased by 18

77) 18 less than the sum of -5 and 11

78) 35 less than the sum of -17 and 3

Section 1.6 Multiplication and Division of Real Numbers

Objectives

1. Multiply Real Numbers
2. Evaluate Exponential Expressions
3. Divide Real Numbers
4. Apply the Order of Operations
5. Translate English Expressions to Mathematical Expressions

1. Multiply Real Numbers

What is the meaning of $4 \cdot 5$? It is repeated addition.

$$4 \cdot 5 = 5 + 5 + 5 + 5 = 20$$

So, what is the meaning of $4 \cdot (-5)$? It, too, represents repeated addition.

$$4 \cdot (-5) = -5 + (-5) + (-5) + (-5) = -20$$

Let's make a table of some products:

$\times$	5	4	3	2	1	0	-1	-2	-3	-4	-5
4	20	16	12	8	4	0	-4	-8	-12	-16	-20

$4 \cdot 3 = 12$

The bottom row represents the product of 4 and the number above it ($4 \cdot 3 = 12$). Notice that as the numbers in the first row decrease by 1, the numbers in the bottom row decrease by 4. Therefore, once we get to $4 \cdot (-1)$, the product is negative. From the table we can see that,

Note

The product of a positive number and a negative number is negative.

Example 1

Multiply.

a) $-6 \cdot 9$ b) $\frac{3}{8} \cdot (-12)$ c) $-5 \cdot 0$

Solution

a) $-6 \cdot 9 = -54$

b) $\frac{3}{8} \cdot (-12) = \frac{3}{\underset{2}{\cancel{8}}} \cdot \left(-\frac{\overset{3}{\cancel{12}}}{1}\right) = -\frac{9}{2}$

c) $-5 \cdot 0 = 0$ The product of zero and any real number is zero.

You Try 1

Multiply.

a) $-7 \cdot 3$ b) $\frac{8}{15} \cdot (-10)$

What is the sign of the product of two negative numbers? Again, we'll make a table.

$\times$	3	2	1	0	-1	-2	-3
-4	-12	-8	-4	0	4	8	12

As we decrease the numbers in the top row by 1, the numbers in the bottom row *increase* by 4. When we reach $-4 \cdot (-1)$, our product is a positive number, 4. The table illustrates that,

Note

The product of two negative numbers is positive.

We can summarize our findings this way:

Procedure Multiplying Real Numbers

1) The product of two positive numbers is positive.
2) The product of two negative numbers is positive.
3) The product of a positive number and a negative number is negative.
4) The product of any real number and zero is zero.

Example 2

Multiply.

a) $-8 \cdot (-5)$ b) $-1.5 \cdot 6$

c) $-\frac{3}{8} \cdot \left(-\frac{4}{5}\right)$ d) $-5 \cdot (-2) \cdot (-3)$

Solution

a) $-8 \cdot (-5) = 40$ The product of two negative numbers is positive.

b) $-1.5 \cdot 6 = -9$ The product of a negative number and a positive number is negative.

c) $-\frac{3}{8} \cdot \left(-\frac{4}{5}\right) = -\frac{3}{\cancel{8}_2} \cdot \left(-\frac{\cancel{4}^1}{5}\right)$

$= \frac{3}{10}$ The product of two negatives is positive.

d) $\underbrace{-5 \cdot (-2)}_{10} \cdot (-3) = 10 \cdot (-3)$ Order of operations—multiply from left to right.

$= -30$

You Try 2

Multiply.

a) $-6 \cdot 7$ b) $-\frac{8}{9} \cdot \frac{3}{4}$ c) $-4 \cdot (-1) \cdot (-5) \cdot (-2)$

Note

It is helpful to know that

1) An **even number** of negative factors in a product gives a positive result.

$-3 \cdot 1 \cdot (-2) \cdot (-1) \cdot (-4) = 24$ Four negative factors

2) An **odd number** of negative factors in a product gives a negative result.

$5 \cdot (-3) \cdot (-1) \cdot (-2) \cdot (3) = -90$ Three negative factors

2. Evaluate Exponential Expressions

In Section 1.2 we discussed exponential expressions. Recall that exponential notation is a shorthand way to represent repeated multiplication:

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$$

Now we will discuss exponents and negative numbers. Consider a base of -2 raised to different powers. (The -2 is in parentheses to indicate that it is the base.)

$$\begin{aligned}
(-2)^1 &= -2 \\
(-2)^2 &= -2 \cdot (-2) = 4 \\
(-2)^3 &= -2 \cdot (-2) \cdot (-2) = -8 \\
(-2)^4 &= -2 \cdot (-2) \cdot (-2) \cdot (-2) = 16 \\
(-2)^5 &= -2 \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2) = -32 \\
(-2)^6 &= -2 \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2) = 64
\end{aligned}$$

Do you notice that

1) -2 raised to an *odd* power gives a negative result?

and

2) -2 raised to an *even* power gives a positive result?

This will always be true.

Note

A negative number raised to an *odd* power will give a *negative* result. A negative number raised to an *even* power will give a *positive* result.

Example 3

Evaluate.

a) $(-6)^2$ b) $(-10)^3$

Solution

a) $(-6)^2 = 36$ b) $(-10)^3 = -1000$

You Try 3

Evaluate.

a) $(-9)^2$ b) $(-5)^3$

How do $(-2)^4$ and -2^4 differ? Let's identify their bases and evaluate each.

$(-2)^4$: Base $= -2$ $(-2)^4 = 16$

-2^4: Since there are no parentheses,

-2^4 is equivalent to $-1 \cdot 2^4$. Therefore, the base is 2.

$$\begin{aligned}
-2^4 &= -1 \cdot 2^4 \\
&= -1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \\
&= -1 \cdot 16 \\
&= -16
\end{aligned}$$

So, $(-2)^4 = 16$ and $-2^4 = -16$.

BE CAREFUL

When working with exponential expressions, be able to identify the base.

Example 4

Evaluate.

a) $(-5)^3$ b) -9^2 c) $\left(-\frac{1}{7}\right)^2$

Solution

a) $(-5)^3$: Base $= -5$ $(-5)^3 = -5 \cdot (-5) \cdot (-5) = -125$

b) -9^2: Base $= 9$

$$\begin{aligned} -9^2 &= -1 \cdot 9^2 \\ &= -1 \cdot 9 \cdot 9 \\ &= -81 \end{aligned}$$

c) $\left(-\frac{1}{7}\right)^2$: Base $= -\frac{1}{7}$ $\left(-\frac{1}{7}\right)^2 = -\frac{1}{7} \cdot \left(-\frac{1}{7}\right) = \frac{1}{49}$

You Try 4

Evaluate.

a) -3^4 b) $(-11)^2$ c) -8^2 d) $-\left(-\frac{2}{3}\right)^3$

3. Divide Real Numbers

Here are the rules for dividing signed numbers:

Procedure Dividing Signed Numbers

1) The quotient of two positive numbers is a positive number.
2) The quotient of two negative numbers is a positive number.
3) The quotient of a positive and a negative number is a negative number.

Example 5

Divide.

a) $-36 \div 9$ b) $-\frac{1}{10} \div \left(-\frac{3}{5}\right)$ c) $\frac{-8}{-1}$ d) $\frac{-24}{42}$

Solution

a) $-36 \div 9 = -4$

b) $-\frac{1}{10} \div \left(-\frac{3}{5}\right) = -\frac{1}{10} \cdot \left(-\frac{5}{3}\right)$ When dividing by a fraction, multiply by the reciprocal.

$$= -\frac{1}{\cancel{10}_{2}} \cdot \left(-\frac{\cancel{5}^{1}}{3}\right) = \frac{1}{6}$$

c) $\dfrac{-8}{-1} = 8$ The quotient of two negative numbers is positive, and $\frac{8}{1}$ simplifies to 8.

d) $\dfrac{-24}{42} = -\dfrac{24}{42}$ The quotient of a negative number and a positive number is negative, so reduce $\frac{24}{42}$.

$= -\dfrac{4}{7}$ 24 and 42 each divide by 6.

It is important to note here in part d) that there are three ways to write the answer: $-\dfrac{4}{7}, \dfrac{-4}{7}$, or $\dfrac{4}{-7}$. These are equivalent. However, we usually write the negative sign in front of the entire fraction as in $-\dfrac{4}{7}$.

You Try 5

Divide.

a) $-\dfrac{8}{5} \div \left(-\dfrac{6}{5}\right)$ b) $\dfrac{-30}{-10}$ c) $\dfrac{21}{-56}$

4. Apply the Order of Operations

Example 6

Simplify.

a) $-24 \div 12 - 2^2$ b) $-5(-3) - 4(2 - 3)$

Solution

a) $-24 \div 12 - 2^2 = -24 \div 12 - 4$ Simplify exponent first.
$= -2 - 4$ Perform division before subtraction.
$= -6$

b) $-5(-3) - 4(2 - 3) = -5(-3) - 4(-1)$ Simplify the difference in parentheses.
$= 15 - (-4)$ Find the products.
$= 15 + 4$
$= 19$

You Try 6

Simplify.

a) $-13 - 4(-5 + 2)$ b) $(-10)^2 + 2[8 - 5(4)]$

5. Translate English Expressions to Mathematical Expressions

Here are some words and phrases you may encounter and how they would translate to mathematical expressions:

English Expression	Mathematical Operation
times, product of	multiplication
divided by, quotient of	division

Example 7

Write a mathematical expression for each and simplify.

a) The quotient of -56 and 7
b) The product of 4 and the sum of 15 and -6
c) Twice the difference of -10 and -3
d) Half of the sum of -8 and 3

Solution

a) The quotient of -56 and 7:
Quotient means division with -56 in the numerator and 7 in the denominator.
The expression is $\dfrac{-56}{7} = -8$.

b) The product of 4 and the sum of 15 and -6:
The *sum of 15 and* -6 means we must add the two numbers. *Product* means multiply.

$$\underbrace{4[\overbrace{15 + (-6)}^{\text{Sum of 15 and } -6}]}_{\text{Product of 4 and the sum}} = 4(9) = 36$$

c) Twice the difference of -10 and -3:
The *difference of* -10 *and* -3 will be in parentheses with -3 being subtracted from -10. *Twice* means "two times."

$$\begin{aligned} 2[-10 - (-3)] &= 2(-10 + 3) \\ &= 2(-7) \\ &= -14 \end{aligned}$$

d) Half of the sum of -8 and 3:
The *sum of* -8 *and 3* means that we will add the two numbers. They will be in parentheses. *Half of* means multiply by $\dfrac{1}{2}$.

$$\frac{1}{2}(-8 + 3) = \frac{1}{2}(-5) = -\frac{5}{2}$$

You Try 7

Write a mathematical expression for each and simplify.

a) 12 less than the product of -7 and 4
b) Twice the sum of 19 and -11
c) The sum of -41 and -23, divided by the square of -2

Answers to You Try Exercises

1) a) -21 b) $-\dfrac{16}{3}$ 2) a) -42 b) $-\dfrac{2}{3}$ c) 40 3) a) 81 b) -125

4) a) -81 b) 121 c) -64 d) $\dfrac{8}{27}$ 5) a) $\dfrac{4}{3}$ b) 3 c) $-\dfrac{3}{8}$ 6) a) -1 b) 76

7) a) $(-7) \cdot 4 - 12; -40$ b) $2[19 + (-11)]; 16$ c) $\dfrac{-41 + (-23)}{(-2)^2}; -16$

1.6 Exercises

Objective 1: Multiply Real Numbers

Fill in the blank with *positive* or *negative*.

1) The product of a positive number and a negative number is ________.
2) The product of two negative numbers is ________.

Multiply.

3) $-8 \cdot 7$
4) $4 \cdot (-9)$
5) $-15 \cdot (-3)$
6) $-23 \cdot (-48)$
7) $-4 \cdot 3 \cdot (-7)$
8) $-5 \cdot (-1) \cdot (-11)$
9) $\frac{4}{33} \cdot \left(-\frac{11}{10}\right)$
10) $-\frac{14}{27} \cdot \left(-\frac{15}{28}\right)$
11) $(-0.5)(-2.8)$
12) $(-6.1)(5.7)$
13) $-9 \cdot (-5) \cdot (-1) \cdot (-3)$
14) $-1 \cdot (-6) \cdot (4) \cdot (-2) \cdot (3)$
15) $\frac{3}{10} \cdot (-7) \cdot (8) \cdot (-1) \cdot (-5)$
16) $-\frac{5}{6} \cdot (-4) \cdot 0 \cdot 3$

Objective 2: Evaluate Exponential Expressions

17) For what values of k is k^5 a negative quantity?
18) For what values of k is k^5 a positive quantity?
19) For what values of k is $-k^2$ a negative quantity?
20) Explain the difference between how you would evaluate $(-8)^2$ and -8^2. Then, evaluate each.

Evaluate.

21) $(-6)^2$
22) -6^2
23) -5^3
24) $(-2)^4$
25) $(-3)^2$
26) $(-1)^5$
27) -7^2
28) -4^3
29) -2^5
30) $(-12)^2$

Objective 3: Divide Real Numbers

Fill in the blank with *positive* or *negative*.

31) The quotient of two negative numbers is __________.
32) The quotient of a negative number and a positive number is ________.

Divide.

33) $-50 \div (-5)$
34) $-84 \div 12$
35) $\frac{64}{-16}$
36) $\frac{-54}{-9}$
37) $\frac{-2.4}{0.3}$
38) $\frac{16}{-0.5}$
39) $-\frac{12}{13} \div \left(-\frac{6}{5}\right)$
40) $20 \div \left(-\frac{15}{7}\right)$
41) $-\frac{0}{7}$
42) $\frac{0}{-6}$
43) $\frac{270}{-180}$
44) $\frac{-64}{-320}$

Objective 4: Apply the Order of Operations

Use the order of operations to simplify.

45) $7 + 8(-5)$
46) $-40 \div 2 - 10$
47) $(9 - 14)^2 - (-3)(6)$
48) $-23 - 6^2 \div 4$
49) $10 - 2(1 - 4)^3 \div 9$
50) $-7(4) + (-8 + 6)^4 + 5$
51) $\left(-\frac{3}{4}\right)(8) - 2[7 - (-3)(-6)]$
52) $-2^5 - (-3)(4) + 5[(-9 + 30) \div 7]$
53) $\frac{-46 - 3(-12)}{(-5)(-2)(-4)}$
54) $\frac{(8)(-6) + 10 - 7}{(-5 + 1)^2 - 12 + 5}$

Objective 5: Translate English Expressions to Mathematical Expressions

Write a mathematical expression for each and simplify.

55) The product of -12 and 6
56) The quotient of -80 and -4
57) 9 more than the product of -7 and -5
58) The product of -10 and 2 increased by 11
59) The quotient of 63 and -9 increased by 7
60) 8 more than the quotient of 54 and -6
61) 19 less than the product of -4 and -8
62) The product of -16 and -3 decreased by 20
63) The quotient of -100 and 4 decreased by the sum of -7 and 2
64) The quotient of -35 and 5 increased by the product of -11 and -2

65) Twice the sum of 18 and -31

66) Twice the difference of -5 and -15

67) Two-thirds of -27

68) Half of -30

69) The product of 12 and -5 increased by half of 36

70) One third of -18 decreased by half the sum of -21 and -5

Section 1.7 Algebraic Expressions and Properties of Real Numbers

Objectives

1. Identify the Terms and Coefficients in an Expression
2. Evaluate Algebraic Expressions
3. Identify Like Terms
4. Use the Commutative Properties
5. Use the Associative Properties
6. Use the Identity and Inverse Properties
7. Use the Distributive Property
8. Combine Like Terms
9. Translate English Expressions to Mathematical Expressions

1. Identify the Terms and Coefficients in an Expression

Here is an algebraic expression:

$$8x^3 - 5x^2 + \frac{2}{7}x + 4$$

x is the *variable*. A **variable** is a symbol, usually a letter, used to represent an unknown number. The *terms* of this algebraic expression are $8x^3$, $-5x^2$, $\frac{2}{7}x$, and 4. A **term** is a number or a variable or a product or quotient of numbers and variables. *4* is the **constant** or **constant term**. The value of a constant does not change. Each term has a **coefficient**.

Term	Coefficient
$8x^3$	8
$-5x^2$	-5
$\frac{2}{7}x$	$\frac{2}{7}$
4	4

Definition

An **algebraic expression** is a collection of numbers, variables, and grouping symbols connected by operation symbols such as $+$, $-$, $\times$, and $\div$.

Examples of expressions:

$$3c + 4, \qquad 9(p^2 - 7p - 2), \qquad -4a^2b^2 + 5ab - 8a + 1.$$

Example 1

List the terms and coefficients of $4x^2y + 7xy - x + \frac{y}{9} - 12$.

Solution

Term	Coefficient	
$4x^2y$	4	
$7xy$	7	
$-x$	-1	The minus sign indicates a negative coefficient.
$\frac{y}{9}$	$\frac{1}{9}$	$\frac{y}{9}$ can be rewritten as $\frac{1}{9}y$.
-12	-12	-12 is also called the "constant."

You Try 1

List the terms and coefficients of $-15r^3 + r^2 - 4r + 8$.

Next, we will use our knowledge of operations with real numbers to evaluate algebraic expressions.

2. Evaluate Algebraic Expressions

We can **evaluate** an algebraic expression by substituting a value for a variable and simplifying. The value of an algebraic expression changes depending on the value that is substituted.

Example 2

Evaluate $3x - 8$ when a) $x = 5$ and b) $x = -4$.

Solution

a) $3x - 8$ when $x = 5$	Substitute 5 for x.
$= 3(5) - 8$	Use parentheses when substituting a value for a variable.
$= 15 - 8$	Multiply.
$= 7$	Subtract.
b) $3x - 8$ when $x = -4$	Substitute -4 for x.
$= 3(-4) - 8$	Use parentheses when substituting a value for a variable.
$= -12 - 8$	Multiply.
$= -20$	

You Try 2

Evaluate $6x + 5$ when $x = -2$.

Example 3

Evaluate $2a^2 - 7ab + 9$ when $a = -2$ and $b = 3$.

Solution

$2a^2 - 7ab + 9$ when $a = -2$ and $b = 3$	Substitute -2 for a and 3 for b.
$= 2(-2)^2 - 7(-2)(3) + 9$	Use parentheses when substituting a value for a variable.
$= 2(4) - 7(-6) + 9$	Evaluate exponent; multiply.
$= 8 - (-42) + 9$	Multiply.
$= 8 + 42 + 9$	
$= 59$	

You Try 3

Evaluate $d^2 + 3cd - 10c - 1$ when $c = \frac{1}{2}$ and $d = -4$.

In algebra, it is important to be able to identify *like terms*.

3. Identify Like Terms

In the expression $15a + 11a - 8a + 3a$, there are four **terms**: $15a$, $11a$, $-8a$, $3a$. In fact, they are **like terms**. *Like terms contain the same variables with the same exponents.*

Example 4

Determine whether the following groups of terms are like terms.

a) $4y^2, -9y^2, \frac{2}{3}y^2$ b) $-5x^6, 0.8x^9, 3x^4$

c) $6a^2b^3, a^2b^3, -\frac{5}{8}a^2b^3$ d) $9c, 4d$

Solution

a) $4y^2, -9y^2, \frac{2}{3}y^2$
Yes. Each contains the variable y with an exponent of 2. They are y^2-terms.

b) $-5x^6, 0.8x^9, 3x^4$
No. Although each contains the variable x, the exponents are not the same.

c) $6a^2b^3, a^2b^3, -\frac{5}{8}a^2b^3$
Yes. Each contains a^2 and b^3.

d) $9c, 4d$
No. The terms contain different variables. ■

You Try 4

Determine whether the following groups of terms are like terms.

a) $2k^2, -9k^2, \frac{1}{5}k^2$ b) $-xy^2, 8xy^2, 7xy^2$ c) $3r^3s^2, -10r^2s^3$

After we discuss the properties of real numbers, we will use them to help us combine like terms.

Properties of Real Numbers

Like the order of operations, the properties of real numbers guide us in our work with numbers and variables. We begin with the commutative properties of real numbers.

True or false?

1) $7 + 3 = 3 + 7$ *True*: $7 + 3 = 10$ and $3 + 7 = 10$

2) $8 - 2 = 2 - 8$ *False*: $8 - 2 = 6$ but $2 - 8 = -6$

3) $(-6)(5) = (5)(-6)$ *True*: $(-6)(5) = -30$ and $(5)(-6) = -30$

4. Use the Commutative Properties

In 1) we see that adding 7 and 3 in any order still equals 10. The third equation shows that multiplying $(-6)(5)$ and $(5)(-6)$ both equal -30. But, 2) illustrates that changing the order in which numbers are subtracted does *not* necessarily give the same result: $8 - 2 \neq 2 - 8$. Therefore, subtraction is **not commutative**, while the addition and multiplication of real numbers **is commutative**. This gives us our first property of real numbers:

Property Commutative Properties

If a and b are real numbers, then

1) $a + b = b + a$ Commutative property of addition
2) $ab = ba$ Commutative property of multiplication

We have already shown that subtraction is not commutative. Is division commutative? No. For example,

$$20 \div 4 \stackrel{?}{=} 4 \div 20$$
$$5 \neq \frac{1}{5}$$

Example 5

Use the commutative property to rewrite each expression.

a) $12 + 5$ b) $k \cdot 3$

Solution

a) $12 + 5 = 5 + 12$ b) $k \cdot 3 = 3 \cdot k$ or $3k$

You Try 5

Use the commutative property to rewrite each expression.

a) $1 + 16$ b) $n \cdot 6$

5. Use the Associative Properties

Another important property involves the use of grouping symbols. Let's determine whether these two statements are true:

$$\begin{aligned} (9 + 4) + 2 &\stackrel{?}{=} 9 + (4 + 2) \\ 13 + 2 &\stackrel{?}{=} 9 + 6 \\ 15 &= 15 \\ &\text{TRUE} \end{aligned} \qquad \text{and} \qquad \begin{aligned} (2 \cdot 3)4 &\stackrel{?}{=} 2(3 \cdot 4) \\ (6)4 &\stackrel{?}{=} 2(12) \\ 24 &= 24 \\ &\text{TRUE} \end{aligned}$$

We can generalize and say that when adding or multiplying real numbers, the way in which we group them to evaluate them will not affect the result. Notice that the *order* in which the numbers are written does not change.

Property Associative Properties

If a, b, and c are real numbers, then

1) $(a + b) + c = a + (b + c)$ Associative property of addition
2) $(ab)c = a(bc)$ Associative property of multiplication

Sometimes, applying the associative property can simplify calculations.

Example 6

Apply the associative property to simplify $\left(7 \cdot \frac{2}{5}\right)5$.

Solution

By the associative property, $\left(7 \cdot \frac{2}{5}\right)5 = 7 \cdot \left(\frac{2}{\cancel{5}} \cdot \overset{1}{\cancel{5}}\right)$

$$= 7 \cdot 2$$
$$= 14$$

You Try 6

Apply the associative property to simplify $\left(9 \cdot \frac{4}{3}\right)3$.

Example 7

Use the associative property to simplify each expression.

a) $-6 + (10 + y)$ b) $\left(-\frac{3}{11} \cdot \frac{8}{5}\right)\frac{5}{8}$

Solution

a) $-6 + (10 + y) = (-6 + 10) + y$
$$= 4 + y$$

b) $\left(-\frac{3}{11} \cdot \frac{8}{5}\right)\frac{5}{8} = -\frac{3}{11}\left(\frac{8}{5} \cdot \frac{5}{8}\right)$

$$= -\frac{3}{11}(1)$$ A number times its reciprocal equals 1.

$$= -\frac{3}{11}$$

You Try 7

Use the associative property to simplify each expression.

a) $(k + 3) + 9$ b) $\left(-\frac{9}{7} \cdot \frac{8}{5}\right)\frac{5}{8}$

The identity properties of addition and multiplication are also ones we need to know.

6. Use the Identity and Inverse Properties

For addition we know that, for example,

$$5 + 0 = 5, \qquad 0 + \frac{2}{3} = \frac{2}{3}, \qquad -14 + 0 = -14.$$

When zero is added to a number, the value of the number is unchanged. *Zero* is the **identity element for addition** (also called the **additive identity**).

What is the identity element for multiplication?

$$-4(1) = -4 \qquad 1(3.82) = 3.82 \qquad \frac{9}{2}(1) = \frac{9}{2}$$

When a number is multiplied by 1, the value of the number is unchanged. *One* is the **identity element for multiplication** (also called the **multiplicative identity**).

Property Identity Properties

If a is a real number, then

1) $a + 0 = 0 + a = a$ Identity property of addition

2) $a \cdot 1 = 1 \cdot a = a$ Identity property of multiplication

The next properties we will discuss give us the additive and multiplicative identities as results. In Section 1.4, we introduced an **additive inverse**.

Number	Additive Inverse
3	-3
-11	11
$-\frac{7}{9}$	$\frac{7}{9}$

Let's add each number and its additive inverse:

$$3 + (-3) = 0, \qquad -11 + 11 = 0, \qquad -\frac{7}{9} + \frac{7}{9} = 0.$$

Note

The sum of a number and its additive inverse is zero (the identity element for addition).

Given a number such as $\frac{3}{5}$, we know that its **reciprocal** (or **multiplicative inverse**) is $\frac{5}{3}$. We have also established the fact that the product of a number and its reciprocal is 1 as in

$$\frac{3}{5} \cdot \frac{5}{3} = 1$$

Therefore, multiplying a number b by its reciprocal (multiplicative inverse) $\frac{1}{b}$ gives us the identity element for multiplication, 1. That is,

$$b \cdot \frac{1}{b} = \frac{1}{b} \cdot b = 1$$

Property Inverse Properties

If a is any real number and b is a real number not equal to 0, then

1) $a + (-a) = -a + a = 0$ Inverse property of addition

2) $b \cdot \frac{1}{b} = \frac{1}{b} \cdot b = 1$ Inverse property of multiplication

Example 8

Which property is illustrated by each statement?

a) $0 + 12 = 12$ b) $-9.4 + 9.4 = 0$

c) $\frac{1}{7} \cdot 7 = 1$ d) $2(1) = 2$

Solution

a) $0 + 12 = 12$ Identity property of addition

b) $-9.4 + 9.4 = 0$ Inverse property of addition

c) $\frac{1}{7} \cdot 7 = 1$ Inverse property of multiplication

d) $2(1) = 2$ Identity property of multiplication

You Try 8

Which property is illustrated by each statement?

a) $5 \cdot \frac{1}{5} = 1$ b) $-26 + 26 = 0$ c) $2.7(1) = 2.7$ d) $-4 + 0 = -4$

7. Use the Distributive Property

The last property we will discuss is the **distributive property**. It involves both multiplication and addition or multiplication and subtraction.

Property Distributive Properties

If a, b, and c are real numbers, then

1) $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$

2) $a(b - c) = ab - ac$ and $(b - c)a = ba - ca$

Example 9

Evaluate using the distributive property.

a) $3(2 + 8)$ b) $-6(7 - 8)$ c) $-(6 + 3)$

Solution

a) $$\begin{aligned} 3(2 + 8) &= 3 \cdot 2 + 3 \cdot 8 \\ &= 6 + 24 \\ &= 30 \end{aligned}$$ Apply distributive property.

Note: We would get the same result if we would apply the order of operations:

$$\begin{aligned} 3(2 + 8) &= 3(10) \\ &= 30 \end{aligned}$$

b) $-6(7-8) = -6 \cdot 7 - (-6)(8)$ Apply distributive property.
$= -42 - (-48)$
$= -42 + 48$
$= 6$

c) $-(6+3) = -1(6+3)$
$= -1 \cdot 6 + (-1)(3)$ Apply distributive property.
$= -6 + (-3)$
$= -9$

A negative sign in front of parentheses is the same as multiplying by -1.

You Try 9

Evaluate using the distributive property.

a) $2(11-5)$ b) $-5(3-7)$ c) $-(4+9)$

The distributive property can be applied when there are more than two terms in parentheses and when there are variables.

Example 10

Use the distributive property to rewrite each expression. Simplify if possible.

a) $-2(3+8-5)$ b) $7(x+4)$ c) $-(-5c+4d-6)$

Solution

a) $-2(3+8-5) = -2 \cdot 3 + (-2)(8) - (-2)(5)$ Apply distributive property.
$= -6 + (-16) - (-10)$ Multiply.
$= -6 + (-16) + 10$
$= -12$

b) $7(x+4) = 7x + 7 \cdot 4$ Apply distributive property.
$= 7x + 28$

c) $-(-5c+4d-6) = -1(-5c+4d-6)$
$= -1(-5c) + (-1)(4d) - (-1)(6)$ Apply distributive property.
$= 5c + (-4d) - (-6)$ Multiply.
$= 5c - 4d + 6$

You Try 10

Use the distributive property to rewrite each expression. Simplify if possible.

a) $6(a+2)$ b) $5(2x-7y-4z)$ c) $-(-r+4s-9)$

The properties stated previously are summarized next.

Summary Properties of Real Numbers

If a, b, and c are real numbers, then

Commutative Properties:	$a + b = b + a$ and $ab = ba$
Associative Properties:	$(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$
Identity Properties:	$a + 0 = 0 + a = a$
	$a \cdot 1 = 1 \cdot a = a$
Inverse Properties:	$a + (-a) = -a + a = 0$
	$b \cdot \frac{1}{b} = \frac{1}{b} \cdot b = 1\ (b \neq 0)$
Distributive Properties:	$a(b + c) = ab + ac$ and $(b + c)a = ba + ca$
	$a(b - c) = ab - ac$ and $(b - c)a = ba - ca$

8. Combine Like Terms

To simplify an expression like $15a + 11a - 8a + 3a$, we combine like terms using the distributive property.

$$\begin{aligned} 15a + 11a - 8a + 3a &= (15 + 11 - 8 + 3)a && \text{Distributive property} \\ &= (26 - 8 + 3)a && \text{Order of operations} \\ &= (18 + 3)a && \text{Order of operations} \\ &= 21a \end{aligned}$$

We can add and subtract only those terms that are like terms.

Example 11

Combine like terms.

a) $-9k + 2k$ b) $n + 8 - 4n + 3$ c) $\frac{3}{5}t^2 + \frac{1}{4}t^2$

d) $10x^2 + 6x - 2x^2 + 5x$

Solution

a) We can use the distributive property to combine like terms.

$$-9k + 2k = (-9 + 2)k = -7k$$

Notice that using the distributive property to combine like terms is the same as combining the coefficients of the terms and leaving the variable and its exponent the same.

b) $$\begin{aligned} n + 8 - 4n + 3 &= n - 4n + 8 + 3 && \text{Rewrite like terms together.} \\ &= -3n + 11 && \text{Remember, } n \text{ is the same as } 1n. \end{aligned}$$

c) $$\begin{aligned} \frac{3}{5}t^2 + \frac{1}{4}t^2 &= \frac{12}{20}t^2 + \frac{5}{20}t^2 && \text{Get a common denominator.} \\ &= \frac{17}{20}t^2 \end{aligned}$$

d) $$\begin{aligned} 10x^2 + 6x - 2x^2 + 5x &= 10x^2 - 2x^2 + 6x + 5x && \text{Rewrite like terms together.} \\ &= 8x^2 + 11x \end{aligned}$$

$8x^2 + 11x$ cannot be simplified more because the terms are *not* like terms. ■

You Try 11

Combine like terms.

a) $6z + 5z$ b) $q - 9 - 4q + 11$ c) $\frac{5}{6}c^2 - \frac{2}{3}c^2$

d) $2y^2 + 8y + y^2 - 3y$

If an expression contains parentheses, we use the distributive property to clear the parentheses, and then combine like terms.

Example 12

Combine like terms.

a) $5(2c + 3) - 3c + 4$ b) $3(2n + 1) - (6n - 11)$

c) $\frac{3}{8}(8 - 4p) + \frac{5}{6}(2p - 6)$

Solution

a) $5(2c + 3) - 3c + 4 = 10c + 15 - 3c + 4$ Distributive property

$= 10c - 3c + 15 + 4$ Rewrite like terms together.

$= 7c + 19$

b) $3(2n + 1) - (6n - 11) = 3(2n + 1) - 1(6n - 11)$ Remember, $-(6n - 11)$ is the same as $-1(6n - 11)$.

$= 6n + 3 - 6n + 11$ Distributive property

$= 6n - 6n + 3 + 11$ Rewrite like terms together.

$= 0n + 14$ $0n = 0$

$= 14$

c) $\frac{3}{8}(8 - 4p) + \frac{5}{6}(2p - 6) = \frac{3}{8}(8) - \frac{3}{8}(4p) + \frac{5}{6}(2p) - \frac{5}{6}(6)$ Distributive property

$= 3 - \frac{3}{2}p + \frac{5}{3}p - 5$ Multiply.

$= -\frac{3}{2}p + \frac{5}{3}p + 3 - 5$ Rewrite like terms together.

$= -\frac{9}{6}p + \frac{10}{6}p + 3 - 5$ Get a common denominator.

$= \frac{1}{6}p - 2$ Combine like terms.

You Try 12

Combine like terms.

a) $9d^2 - 7 + 2d^2 + 3$ b) $10 - 3(2k + 5) + k - 6$

9. Translate English Expressions to Mathematical Expressions

Translating from English to a mathematical expression is a skill that is necessary to solve applied problems. We will practice writing mathematical expressions.

Read the phrase carefully, choose a variable to represent the unknown quantity, then translate the phrase to a mathematical expression.

Example 13

Write a mathematical expression for each and simplify. Define the unknown with a variable.

a) Seven more than twice a number
b) The sum of a number and four times the same number

Solution

a) Seven more than twice a number
   i) **Define the unknown.** This means that you should clearly state on your paper what the variable represents.

$$\text{Let } x = \text{the number.}$$

   ii) **Slowly, break down the phrase.** How do you write an expression for "seven more than" something?

$$+\ 7$$

   iii) **What does "twice a number" mean?** It means two times the number. Since our number is represented by x, "twice a number" is $2x$.
   iv) **Put the information together:**

Seven more than twice a number

$$2x \quad + \quad 7$$

The expression is $2x + 7$.

b) The sum of a number and four times the same number
   i) **Define the unknown.**

$$\text{Let } y = \text{the number.}$$

   ii) **Slowly, break down the phrase.** What does *sum* mean? **Add.** So, we have to add a number and four times the same number:

Number + 4(Number)

   iii) Since y represents the number, *four times the number* is $4y$.
   iv) Therefore, to translate from English to a mathematical expression, we know that we must add the number, y, to four times the number, $4y$. Our expression is $y + 4y$. It simplifies to $5y$.

You Try 13

Write a mathematical expression for each and simplify. Let x equal the unknown number.

a) Five less than twice a number
b) The sum of a number and two times the same number

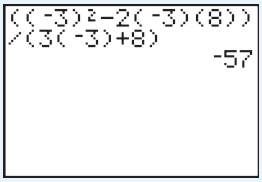
Using Technology

A graphing calculator can be used to evaluate an algebraic expression. This is especially valuable when evaluating expressions for several values of the given variables.

We will evaluate the expression $\dfrac{x^2 - 2xy}{3x + y}$ when $x = -3$ and $y = 8$.

Method 1

Substitute the values for the variables and evaluate the arithmetic expression on the home screen. Each value substituted for a variable should be enclosed in parentheses to guarantee a correct answer. For example $(-3)^2$ gives the result 9, whereas -3^2 gives the result -9. Be careful to press the negative key [(−)] when entering a negative sign and the minus key [−] when entering the minus operator.



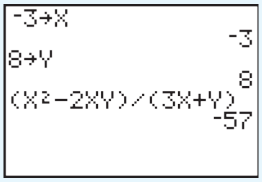
Method 2

Store the given values in the variables and evaluate the algebraic expression on the home screen.

To store -3 in the variable x, press [(−)] [3] [STO>] [X, T, θ, n] [ENTER].

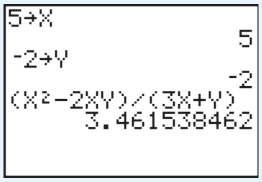
To store 8 in the variable y, press [8] [STO>] [ALPHA] [1] [ENTER].

Enter $\dfrac{x^2 - 2xy}{3x + y}$ on the home screen.

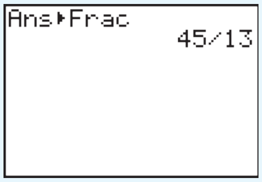


The advantage of Method 2 is that we can easily store two different values in x and y. For example, store 5 in x and -2 in y. It is not necessary to enter the expression again because the calculator can recall previous entries.

Press [2nd] [ENTER] three times; then press [ENTER].



To convert this decimal to a fraction, press [MATH] [ENTER] [ENTER].



Evaluate each expression when $x = -5$ and $y = 2$.

1. $3y - 4x$
2. $2xy - 5y$
3. $y^3 - 2x^2$
4. $\dfrac{x - y}{4x}$
5. $\dfrac{2x + 5y}{x - y}$
6. $\dfrac{x - y^2}{2x}$

Answers to You Try Exercises

1)

Term	Coeff.
$-15r^3$	-15
r^2	1
$-4r$	-4
8	8

2) -7 3) 4 4) a) yes b) yes c) no

5) a) $16 + 1$ b) $6n$ 6) 36 7) a) $k + 12$ b) $-\dfrac{9}{7}$ or $-1\dfrac{2}{7}$ 8) a) inverse property of multiplication b) inverse property of addition c) identity property of multiplication d) identity property of addition

9) a) 12 b) 20 c) -13 10) a) $6a + 12$ b) $10x - 35y - 20z$ c) $r - 4s + 9$ 11) a) $11z$ b) $-3q + 2$ c) $\dfrac{1}{6}c^2$ d) $3y^2 + 5y$

12) a) $11d^2 - 4$ b) $-5k - 11$

13) a) $2x - 5$ b) $x + 2x$; $3x$

Answers to Technology Exercises

1. 26 2. -30 3. -42 4. $\frac{7}{20}$ 5. 0 6. $\frac{9}{10}$

1.7 Exercises

Objective 1: Identify the Terms and Coefficients in an Expression

For each expression, list the terms and their coefficients. Also, identify the constant.

1) $7p^2 - 6p + 4$

2) $-8z + \frac{5}{6}$

3) $x^2y^2 + 2xy - y + 11$

4) $w^3 - w^2 + 9w - 5$

VIDEO 5) $-2g^5 + \frac{g^4}{5} + 3.8g^2 + g - 1$

6) $121c^2 - d^2$

Objective 2: Evaluate Algebraic Expressions

7) Evaluate $4c + 3$ when

a) $c = 2$ b) $c = -5$

8) Evaluate $8m - 5$ when

a) $m = 3$ b) $m = -1$

Evaluate each expression when $x = 3$, $y = -5$, and $z = -2$.

9) $x + 4y$

10) $3z - y$

11) $z^2 - xy - 19$

12) $x^2 + 4yz$

13) $\frac{x^3}{2y + 1}$

14) $\frac{z^3}{x^2 - 1}$

15) $\frac{z^2 - y^2}{2y - 4(x + z)}$

16) $\frac{10 + 3(y + 2z)}{x^3 - z^4}$

Objective 3: Identify Like Terms

17) Are $9k$ and $9k^2$ *like* terms? Why or why not?

18) Are $\frac{3}{4}n$ and $8n$ *like* terms? Why or why not?

19) Are a^3b and $-7a^3b$ *like* terms? Why or why not?

20) Write three *like* terms that are x^2-terms.

Mixed Exercises: Objectives 4–7

21) What is the identity element for multiplication?

22) What is the identity element for addition?

23) What is the additive inverse of 5?

24) What is the multiplicative inverse of 8?

Which property of real numbers is illustrated by each example? Choose from the commutative, associative, identity, inverse, or distributive property.

25) $9(2 + 8) = 9 \cdot 2 + 9 \cdot 8$

26) $(-16 + 7) + 3 = -16 + (7 + 3)$

27) $14 \cdot 1 = 14$

28) $\left(\frac{9}{2}\right)\left(\frac{2}{9}\right) = 1$

29) $-10 + 18 = 18 + (-10)$

30) $4 \cdot 6 - 4 \cdot 1 = 4(6 - 1)$

31) $5(2 \cdot 3) = (5 \cdot 2) \cdot 3$

32) $11 \cdot 7 = 7 \cdot 11$

Rewrite each expression using the indicated property.

33) $p + 19$; commutative

34) $5(m + n)$; distributive

35) $8 + (1 + 9)$; associative

36) $-2c + 0$; identity

37) $3(k - 7)$; distributive

38) $10 + 9x$; commutative

39) $y + 0$; identity

40) $\left(4 \cdot \frac{2}{7}\right) \cdot 7$; associative

41) Is $2a - 7$ equivalent to $7 - 2a$? Why or why not?

42) Is $6 + t$ equivalent to $t + 6$? Why or why not?

Rewrite each expression using the distributive property. Simplify if possible.

43) $2(1 + 9)$

44) $3(9 + 4)$

45) $-2(5 + 7)$

46) $-5(3 + 7)$

47) $4(8 - 3)$

48) $-6(5 - 11)$

49) $-(10 - 4)$

50) $-(3 + 9)$

51) $8(y + 3)$

52) $4(k + 11)$

53) $-10(z + 6)$

54) $-7(m + 5)$

55) $-3(x - 4y - 6)$

56) $6(2a - 5b + 1)$

57) $-(-8c + 9d - 14)$

58) $-(x - 10y - 4z)$

Objective 8: Combine Like Terms

Combine like terms and simplify.

59) $10p + 9 + 14p - 2$

60) $11 - k^2 + 12k^2 - 3 + 6k^2$

61) $-18y^2 - 2y^2 + 19 + y^2 - 2 + 13$

62) $-7x - 3x - 1 + 9x + 6 - 2x$

63) $\frac{4}{9} + 3r - \frac{2}{3} + \frac{1}{5}r$

64) $6a - \frac{3}{8}a + 2 + \frac{1}{4} - \frac{3}{4}a$

65) $2(3w + 5) + w$

66) $-8d^2 + 6(d^2 - 3) + 7$

67) $9 - 4(3 - x) - 4x + 3$

68) $m + 11 + 3(2m - 5) + 1$

69) $3g - (8g + 3) + 5$

70) $-6 + 4(10b - 11) - 8(5b + 2)$

71) $-5(t - 2) - (10 - 2t)$

72) $11 + 8(3u - 4) - 2(u + 6) + 9$

73) $3[2(5x + 7) - 11] + 4(7 - x)$

74) $22 - [6 + 5(2w - 3)] - (7w + 16)$

75) $\frac{4}{5}(2z + 10) - \frac{1}{2}(z + 3)$

76) $\frac{2}{3}(6c - 7) + \frac{5}{12}(2c + 5)$

77) $1 + \frac{3}{4}(10t - 3) + \frac{5}{8}\left(t + \frac{1}{10}\right)$

78) $\frac{7}{15} - \frac{9}{10}(2y + 1) - \frac{2}{5}(4y - 3)$

79) $2.5(x - 4) - 1.2(3x + 8)$

80) $9.4 - 3.8\,(2a + 5) + 0.6 + 1.9a$

Objective 9: Translate English Expressions to Mathematical Expressions

Write a mathematical expression for each phrase, and combine like terms if possible. Let x represent the unknown quantity.

81) Eighteen more than a number

82) Eleven more than a number

83) Six subtracted from a number

84) Eight subtracted from a number

85) Three less than a number

86) Fourteen less than a number

87) The sum of twelve and twice a number

88) Five added to the sum of a number and six

89) Seven less than the sum of three and twice a number

90) Two more than the sum of a number and nine

91) The sum of a number and fifteen decreased by five

92) The sum of -8 and twice a number increased by three

Chapter 1: Summary

1.1 Review of Fractions

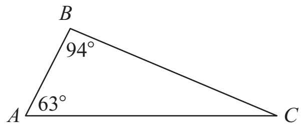
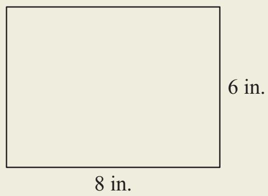
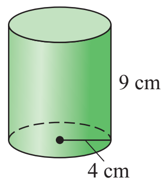
Definition/Procedure	Example
Reducing Fractions A fraction is in **lowest terms** when the numerator and denominator have no common factors other than 1. **(p. 2)**	Write $\frac{36}{48}$ in lowest terms. Divide 36 and 48 by a common factor, 12. Since $36 \div 12 = 3$ and $48 \div 12 = 4$, $\frac{36}{48} = \frac{3}{4}$.
Multiplying Fractions To multiply fractions, multiply the numerators and multiply the denominators. Common factors can be divided out either before or after multiplying. **(p. 6)**	Multiply $\frac{21}{45} \cdot \frac{9}{14}$. $\frac{\overset{3}{\cancel{21}}}{\underset{5}{\cancel{45}}} \cdot \frac{\overset{1}{\cancel{9}}}{\underset{2}{\cancel{14}}}$ 9 and 45 each divide by 9. 21 and 14 each divide by 7. $= \frac{3}{5} \cdot \frac{1}{2} = \frac{3}{10}$
Dividing Fractions To divide fractions, multiply the first fraction by the reciprocal of the second. **(p. 7)**	Divide $\frac{7}{5} \div \frac{4}{3}$. $\frac{7}{5} \div \frac{4}{3} = \frac{7}{5} \cdot \frac{3}{4} = \frac{21}{20}$ or $1\frac{1}{20}$
Adding and Subtracting Fractions To add or subtract fractions, 1) Identify the least common denominator (LCD). 2) Write each fraction as an equivalent fraction using the LCD. 3) Add or subtract. 4) Express the answer in lowest terms. **(p. 12)**	Add $\frac{5}{11} + \frac{2}{11}$. $\frac{5}{11} + \frac{2}{11} = \frac{7}{11}$ Subtract $\frac{8}{9} - \frac{3}{4}$. $\frac{8}{9} - \frac{3}{4} = \frac{32}{36} - \frac{27}{36} = \frac{5}{36}$

1.2 Exponents and Order of Operations

Definition/Procedure	Example
Exponents An **exponent** represents repeated multiplication. **(p. 17)**	Write $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$ in exponential form. $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 9^5$ Evaluate 2^4. $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$
Order of Operations **P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition, **S**ubtraction **(p. 18)**	Evaluate $8 + (5 - 1)^2 - 6 \cdot 3$. $8 + (5 - 1)^2 - 6 \cdot 3$ $= 8 + 4^2 - 6 \cdot 3$ Parentheses $= 8 + 16 - 6 \cdot 3$ Exponents $= 8 + 16 - 18$ Multiply. $= 24 - 18$ Add. $= 6$ Subtract.

1.3 Geometry Review

Definition/Procedure	Example
Important Angles The definitions for an acute angle, an obtuse angle, and a right angle can be found on **p. 22.** Two angles are **complementary** if the sum of their angles is 90°. Two angles are **supplementary** if the sum of their angles is 180°. **(p. 22)**	The measure of an angle is 73°. Find the measure of its complement and its supplement. The measure of its complement is 17° since $90° - 73° = 17°$. The measure of its supplement is 107° since $180° - 73° = 107°$.

Definition/Procedure	Example
Triangle Properties The sum of the measures of the angles of any triangle is 180°. An **equilateral triangle** has three sides of equal length. Each angle measures 60°. An **isosceles triangle** has two sides of equal length. The angles opposite the sides have the same measure. A **scalene triangle** has no sides of equal length. No angles have the same measure. **(p. 23)**	Find the measure of $\angle C$.  $m\angle A + m\angle B = 63° + 94° = 157°$ $m\angle C = 180° - 157° = 23°$
Perimeter and Area The formulas for the perimeter and area of a rectangle, square, triangle, parallelogram, and trapezoid can be found on **p. 24.**	Find the area and perimeter of this rectangle.  Area = (Length)(Width) = (8 in.)(6 in.) = 48 in^2 Perimeter = 2(Length) + 2(Width) = 2(8 in.) + 2(6 in.) = 16 in. + 12 in. = 28 in.
Volume The formulas for the volume of a rectangular solid, cube, right circular cylinder, sphere, and right circular cone can be found on **p. 28.**	Find the volume of the cylinder pictured here.  Give an exact answer and give an approximation using 3.14 for π. $V = \pi r^2 h = \pi(4 \text{ cm})^2(9 \text{ cm}) = \pi(16 \text{ cm}^2)(9 \text{ cm}) = 144\pi \text{ cm}^3$ $V = 144\pi \text{ cm}^3 \approx 144(3.14) \text{ cm}^3 = 452.16 \text{ cm}^3$

1.4 Sets of Numbers and Absolute Value

Definition/Procedure	Example
Natural numbers: {1, 2, 3, 4, . . .} **Whole numbers:** {0, 1, 2, 3, 4, . . .} **Integers:** {. . . , −3, −2, −1, 0, 1, 2, 3, . . .}	
A **rational number** is any number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. **(p. 35)**	The following numbers are rational: $-3, 10, \frac{5}{8}, 7.4, 2.\overline{3}$
An **irrational number** cannot be written as the quotient of two integers. **(p. 36)**	The following numbers are irrational: $\sqrt{6}$, 9.2731. . .
The set of **real numbers** includes the rational and irrational numbers. **(p. 37)**	Any number that can be represented on the number line is a real number.
The **additive inverse** of a is $-a$. **(p. 39)**	The additive inverse of 4 is −4.
Absolute Value $\|a\|$ is the distance of a from zero. **(p. 40)**	$\|-6\| = 6$

Definition/Procedure	Example
1.5 Addition and Subtraction of Real Numbers	
Adding Real Numbers To add numbers with the **same sign**, add the absolute value of each number. The sum will have the same sign as the numbers being added. **(p. 43)**	$-3 + (-9) = -12$
To add two numbers with **different signs**, subtract the smaller absolute value from the larger. The sum will have the sign of the number with the larger absolute value. **(p. 44)**	$-20 + 15 = -5$
Subtracting Real Numbers To subtract $a - b$, change subtraction to addition and add the additive inverse of b: $a - b = a + (-b)$. **(p. 45)**	$2 - 11 = 2 + (-11) = -9$ $-17 - (-7) = -17 + 7 = -10$
1.6 Multiplication and Division of Real Numbers	
Multiplying Real Numbers The product of two real numbers with the *same* sign is positive.	$8 \cdot 3 = 24$ $-7 \cdot (-8) = 56$
The product of a positive number and a negative number is *negative*.	$-2 \cdot 5 = -10$ $9 \cdot (-1) = -9$
An *even number* of negative factors in a product gives a *positive* result.	$\underbrace{(-1)(-6)(-3)(2)(-4)}_{\text{4 negative factors}} = 144$
An *odd number* of negative factors in a product gives a *negative* result. **(p. 51)**	$\underbrace{(5)(-2)(-3)(1)(-1)}_{\text{3 negative factors}} = -30$
Evaluating Exponential Expressions (p. 52)	Evaluate $(-3)^4$. The base is -3. $(-3)^4 = (-3)(-3)(-3)(-3) = 81$ Evaluate -3^4. The base is 3. $-3^4 = -1 \cdot 3^4 = -1 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = -81$
Dividing real numbers The quotient of two numbers with the *same* sign is positive. The quotient of two numbers with *different* signs is negative. **(p. 53)**	$\frac{40}{2} = 20$ $-18 \div (-3) = 6$ $\frac{-56}{8} = -7$ $48 \div (-4) = -12$
1.7 Algebraic Expressions and Properties of Real Numbers	
An **algebraic expression** is a collection of numbers, variables, and grouping symbols connected by operation symbols such as $+, -, \times$, and $\div$. **(p. 57)**	$4y^2 - 7y + \frac{3}{5}$
Important terms Variable Constant Term Coefficient	
We can evaluate expressions for different values of the variables. **(p. 58)**	Evaluate $2xy - 5y + 1$ when $x = -3$ and $y = 4$. Substitute -3 for x and 4 for y and simplify. $2xy - 5y + 1 = 2(-3)(4) - 5(4) + 1$ $= -24 - 20 + 1$ $= -24 + (-20) + 1$ $= -43$

Definition/Procedure	Example
Like Terms **Like terms** contain the same variables with the same exponents. **(p. 59)**	In the group of terms $5k^2, -8k, -4k^2, \frac{1}{3}k$, $5k^2$ and $-4k^2$ are like terms and $-8k$ and $\frac{1}{3}k$ are like terms.
Properties of Real Numbers If *a*, *b*, and *c* are real numbers, then the following properties hold.	
Commutative Properties: $a + b = b + a$ $ab = ba$	$10 + 3 = 3 + 10$ $(-6)(5) = (5)(-6)$
Associative Properties: $(a + b) + c = a + (b + c)$ $(ab)c = a(bc)$	$(9 + 4) + 2 = 9 + (4 + 2)$ $(5 \cdot 2)8 = 5 \cdot (2 \cdot 8)$
Identity Properties: $a + 0 = 0 + a = a$ $a \cdot 1 = 1 \cdot a = a$	$7 + 0 = 7$ $\qquad \frac{2}{3} \cdot 1 = \frac{2}{3}$
Inverse Properties: $a + (-a) = -a + a = 0$ $b \cdot \frac{1}{b} = \frac{1}{b} \cdot b = 1$	$11 + (-11) = 0$ $\qquad 5 \cdot \frac{1}{5} = 1$
Distributive Properties: $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ $a(b - c) = ab - ac$ and $(b - c)a = ba - ca$ **(p. 65)**	$6(5 + 8) = 6 \cdot 5 + 6 \cdot 8$ $= 30 + 48$ $= 78$ $9(w - 2) = 9w - 9 \cdot 2$ $= 9w - 18$
Combining Like Terms We can simplify expressions by combining like terms. **(p. 65)**	Combine like terms and simplify. $4n^2 - 3n + 1 - 2(6n^2 - 5n + 7)$ $= 4n^2 - 3n + 1 - 12n^2 + 10n - 14$ Distributive property $= -8n^2 + 7n - 13$ Combine like terms.
Writing Mathematical Expressions (p. 67)	Write a mathematical expression for the following: *Sixteen more than twice a number* Let $x =$ the number. $\underbrace{\text{Sixteen more than}}_{+16}$ $\quad \underbrace{\text{twice a number}}_{2x}$ $2x + 16$

Chapter 1: Review Exercises

(1.1)

1) **Find all factors of each number.**

a) 16 b) 37

2) **Find the prime factorization of each number.**

a) 28 b) 66

3) **Write each fraction in lowest terms.**

a) $\frac{12}{30}$ b) $\frac{414}{702}$

Perform the indicated operation. Write the answer in lowest terms.

4) $\frac{4}{11} \cdot \frac{3}{5}$

5) $\frac{45}{64} \cdot \frac{32}{75}$

6) $\frac{5}{8} \div \frac{3}{10}$

7) $35 \div \frac{7}{8}$

8) $4\frac{2}{3} \cdot 1\frac{1}{8}$

9) $\frac{30}{49} \div 2\frac{6}{7}$

10) $\frac{2}{9} + \frac{4}{9}$

11) $\frac{2}{3} + \frac{1}{4}$

12) $\frac{9}{40} + \frac{7}{16}$

13) $\frac{1}{5} + \frac{1}{3} + \frac{1}{6}$

14) $\frac{21}{25} - \frac{11}{25}$

15) $\frac{5}{8} - \frac{2}{7}$

16) $3\frac{2}{9} + 5\frac{3}{8}$

17) $9\frac{3}{8} - 2\frac{5}{6}$

18) A pattern for a skirt calls for $1\frac{7}{8}$ yd of fabric. If Mary Kate wants to make one skirt for herself and one for her twin, how much fabric will she need?

(1.2) Evaluate.

19) 3^4

20) 2^6

21) $\left(\frac{3}{4}\right)^3$

22) $(0.6)^2$

23) $13 - 7 + 4$

24) $8 \cdot 3 + 20 \div 4$

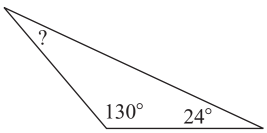
25) $\frac{12 - 56 \div 8}{(1 + 5)^2 - 2^4}$

(1.3)

26) The complement of 51° is ________.

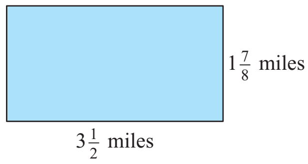
27) The supplement of 78° is ________.

28) Is this triangle acute, obtuse, or right? Find the missing angle.

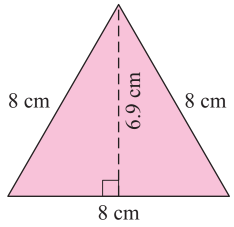


Find the area and perimeter of each figure. Include the correct units.

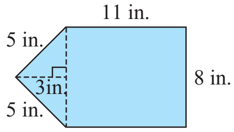
29)



30)



31)



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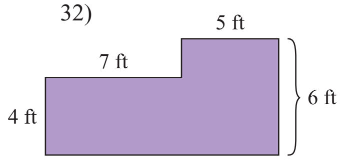
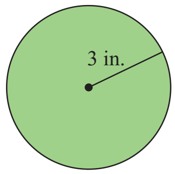


Find a) the area and b) the circumference of each circle. Give an exact answer for each and give an approximation using 3.14 for π. Include the correct units.

33)



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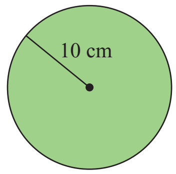
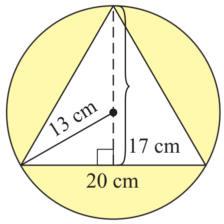


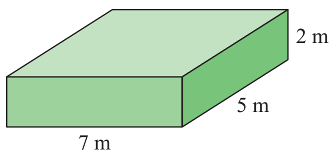
Find the area of the shaded region. Use 3.14 for π. Include the correct units.

35)



Find the volume of each figure. Where appropriate, give the answer in terms of π. Include the correct units.

36)



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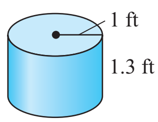

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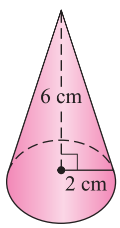

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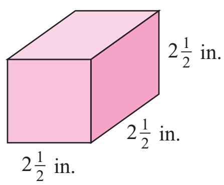


40) The radius of a basketball is approximately 4.7 inches. Find its circumference to the nearest tenth of an inch.

(1.4)

41) Given this set of numbers,
$\left\{\frac{7}{15}, -16, 0, 3.\overline{2}, 8.5, \sqrt{31}, 4, 6.01832\ldots\right\}$
list the

a) integers

b) rational numbers

c) natural numbers

d) whole numbers

e) irrational numbers

42) Graph and label these numbers on a number line.

$$-3.5, 4, \frac{9}{10}, 2\frac{1}{3}, -\frac{3}{4}, -5$$

43) Evaluate.

a) $|-18|$ b) $-|7|$

(1.5) Add or subtract as indicated.

44) $-38 + 13$

45) $-21 - (-40)$

46) $-1.9 + 2.3$

47) $\frac{5}{12} - \frac{5}{8}$

48) The lowest temperature on record in the country of Greenland is -87°F. The coldest temperature ever reached on the African continent was in Morocco and is 76° higher than Greenland's record low. What is the lowest temperature ever recorded in Africa?
(www.ncdc.noaa.gov)

(1.6) Multiply or divide as indicated.

49) $\left(-\frac{3}{2}\right)(8)$

50) $(-4.9)(-3.6)$

51) $(-4)(3)(-2)(-1)(-3)$

52) $\left(-\frac{2}{3}\right)(-5)(2)(-6)$

53) $-108 \div 9$

54) $\frac{56}{-84}$

55) $-3\frac{1}{8} \div \left(-\frac{5}{6}\right)$

56) $-\frac{9}{10} \div 12$

Evaluate.

57) -6^2

58) $(-6)^2$

59) $(-2)^6$

60) -1^{10}

61) 3^3

62) $(-5)^3$

Use the order of operations to simplify.

63) $56 \div (-7) - 1$

64) $15 - (2 - 5)^3$

65) $-11 + 4 \cdot 3 + (-8 + 6)^5$

66) $\frac{1 + 6(7 - 3)}{2[3 - 2(8 - 1)] - 3}$

Write a mathematical expression for each and simplify.

67) The quotient of -120 and -3

68) Twice the sum of 22 and -10

69) 15 less than the product of -4 and 7

70) 11 more than half of -18

(1.7)

71) List the terms and coefficients of

$$5z^4 - 8z^3 + \frac{3}{5}z^2 - z + 14.$$

72) Evaluate $9x - 4y$ when $x = -3$ and $y = 7$.

73) Evaluate $\frac{2a + b}{a^3 - b^2}$ when $a = -3$ and $b = 5$.

Which property of real numbers is illustrated by each example? Choose from the commutative, associative, identity, inverse, or distributive property.

74) $12 + (5 + 3) = (12 + 5) + 3$

75) $\left(\frac{2}{5}\right)\left(\frac{5}{2}\right) = 1$

76) $0 + 19 = 19$

77) $-4(7 + 2) = -4(7) + (-4)(2)$

78) $8 \cdot 3 = 3 \cdot 8$

Rewrite each expression using the distributive property. Simplify if possible.

79) $7(3 - 9)$

80) $(10 + 4)5$

81) $-(15 - 3)$

82) $-6(9p - 4q + 1)$

Combine like terms and simplify.

83) $9m - 14 + 3m + 4$

84) $-5c + d - 2c + 8d$

85) $15y^2 + 8y - 4 + 2y^2 - 11y + 1$

86) $7t + 10 - 3(2t + 3)$

87) $\frac{3}{2}(5n - 4) + \frac{1}{4}(n + 6)$

88) $1.4(a + 5) - (a + 2)$

Chapter 1: Test

1) Find the prime factorization of 210.

2) Write in lowest terms:

a) $\dfrac{45}{72}$ b) $\dfrac{420}{560}$

Perform the indicated operations. Write all answers in lowest terms.

3) $\dfrac{7}{16} \cdot \dfrac{10}{21}$

4) $\dfrac{5}{12} + \dfrac{2}{9}$

5) $10\dfrac{2}{3} - 3\dfrac{1}{4}$

6) $\dfrac{4}{9} \div 12$

7) $\dfrac{3}{5} - \dfrac{17}{20}$

8) $-31 - (-14)$

9) $16 + 8 \div 2$

10) $\dfrac{1}{8} \cdot \left(-\dfrac{2}{3}\right)$

11) $-15 \cdot (-4)$

12) $-9.5 + 5.8$

13) $23 - 6[-4 + (9 - 11)^4]$

14) $\dfrac{7 \cdot 2 - 4}{48 \div 3 - 8^0}$

15) An extreme sports athlete has reached an altitude of 14,693 ft while ice climbing and has dived to a depth of 518 ft below sea level. What is the difference between these two elevations?

16) Evaluate.

a) 5^3

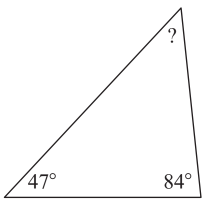
b) -2^4

c) $|-43|$

d) $-|18 - 40| - 3|9 - 4|$

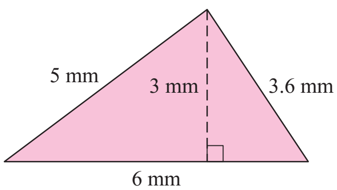
17) The supplement of 31° is ______.

18) Find the missing angle, and classify the triangle as acute, obtuse, or right.

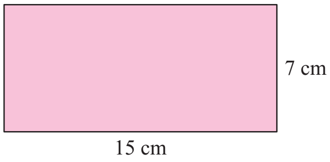


19) Find the area and perimeter of each figure. Include the correct units.

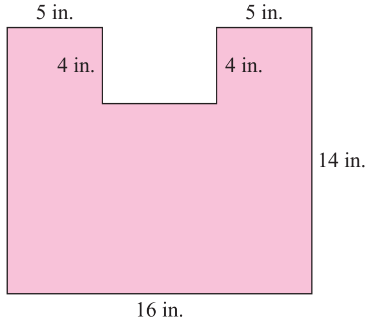
a)



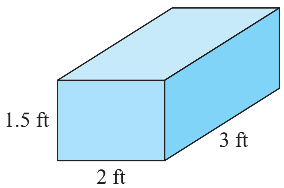
b)



c)



20) Find the volume of this figure:



21) The radius of the pitcher's mound on a major-league baseball diamond is 9 ft.

a) Find the exact area of the pitcher's mound.

b) Find the approximate area of the pitcher's mound using 3.14 for π.

22) Given this set of numbers,

$\{3\frac{1}{5}, 22, -7, \sqrt{43}, 0, 6.2, 1.\overline{5}, 8.0934\ldots\}$ list the

a) whole numbers

b) natural numbers

c) irrational numbers

d) integers

e) rational numbers

23) Graph the numbers on a number line. Label each.

$4, -5, \dfrac{2}{3}, -3\dfrac{1}{2}, -\dfrac{5}{6}, 2.2$

24) Write a mathematical expression for each and simplify.

a) The sum of -4 and 27

b) The product of 5 and -6 subtracted from 17

25) List the terms and coefficients of

$$4p^3 - p^2 + \frac{1}{3}p - 10.$$

26) Evaluate $\dfrac{x^2 - y^2}{6y + x}$ when $x = 3$ and $y = -4$.

27) Which property of real numbers is illustrated by each example? Choose from the commutative, associative, identity, inverse, or distributive property.

a) $9 \cdot 5 = 5 \cdot 9$

b) $16 + (4 + 7) = (16 + 4) + 7$

c) $\left(\dfrac{10}{3}\right)\left(\dfrac{3}{10}\right) = 1$

d) $8(1 - 4) = 8 \cdot 1 - 8 \cdot 4$

28) Rewrite each expression using the distributive property. Simplify if possible.

a) $-4(2 + 7)$

b) $3(8m - 3n + 11)$

29) Combine like terms and simplify.

a) $-8k^2 + 3k - 5 + 2k^2 + k - 9$

b) $\dfrac{4}{3}(6c - 5) - \dfrac{1}{2}(4c + 3)$

30) Write a mathematical expression for "nine less than twice a number." Let x represent the number.

CHAPTER 2

The Rules of Exponents

Algebra at Work: Custom Motorcycle Shop

The people who build custom motorcycles use a lot of mathematics to do their jobs. Mark is building a chopper frame and needs to make the supports for the axle. He has to punch holes in the plates that will be welded to the frame.

Mark has to punch holes with a diameter of 1 in. in mild steel that is $\frac{3}{8}$ in. thick. The press punches two holes at a time. To determine how much power is needed to do this job, he uses a formula containing an exponent, $P = \frac{t^2dN}{3.78}$. After substituting the numbers into the expression, he calculates that the power needed to punch these holes is 0.07 hp.

In this chapter, we will learn more about working with expressions containing exponents.

Section 2.1A The Product Rule and Power Rules

Objectives

1. Evaluate Exponential Expressions
2. Use the Product Rule for Exponents
3. Use the Power Rule $(a^m)^n = a^{mn}$
4. Use the Power Rule $(ab)^n = a^n b^n$
5. Use the Power Rule $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, Where $b \neq 0$

1. Evaluate Exponential Expressions

Recall from Chapter 1 that exponential notation is used as a shorthand way to represent a multiplication problem.

For example, $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$ can be written as 3^5.

Definition

An **exponential expression** of the form a^n, where a is any real number and n is a positive integer, is equivalent to $\underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors of } a}$. We say that a is the **base** and n is the **exponent.**

We can also evaluate an exponential expression.

Example 1

Identify the base and the exponent in each expression and evaluate.

a) 2^4 b) $(-2)^4$ c) -2^4

Solution

a) 2^4 — 2 is the base, 4 is the exponent. Therefore, $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$.

b) $(-2)^4$ — -2 is the base, 4 is the exponent. Therefore, $(-2)^4 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) = 16$.

c) -2^4 — It may be very tempting to say that the base is -2. However, there are no parentheses in this expression. Therefore, 2 is the base, and 4 is the exponent. To evaluate,

$$\begin{aligned} -2^4 &= -1 \cdot 2^4 = -1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \\ &= -16 \end{aligned}$$

BE CAREFUL

The expressions $(-a)^n$ and $-a^n$ are not always equivalent:

$$(-a)^n = \underbrace{(-a) \cdot (-a) \cdot (-a) \cdot \ldots \cdot (-a)}_{n \text{ factors of } -a}$$

$$-a^n = -1 \cdot \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors of } a}$$

You Try 1

Identify the base and exponent in each expression and evaluate.

a) 5^3 b) -8^2 c) $\left(-\frac{2}{3}\right)^3$

2. Use the Product Rule for Exponents

Is there a rule to help us *multiply* exponential expressions? Let's rewrite each of the following products as a single power of the base using what we already know:

$$1)\quad 2^3 \cdot 2^2 = \overbrace{2 \cdot 2 \cdot 2}^{\text{3 factors of 2}} \cdot \overbrace{2 \cdot 2}^{\text{2 factors of 2}} = \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{\text{5 factors of 2}} = 2^5$$

$$2)\quad 5^4 \cdot 5^3 = \overbrace{5 \cdot 5 \cdot 5 \cdot 5}^{\text{4 factors of 5}} \cdot \overbrace{5 \cdot 5 \cdot 5}^{\text{3 factors of 5}} = \underbrace{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}_{\text{7 factors of 5}} = 5^7$$

Let's summarize: $2^3 \cdot 2^2 = 2^5$, $\quad 5^4 \cdot 5^3 = 5^7$

Do you notice a pattern? *When you multiply expressions with the same base, keep the same base and add the exponents.* This is called the **product rule** for exponents.

Property Product Rule

Let a be any real number and let m and n be positive integers. Then,

$$a^m \cdot a^n = a^{m+n}$$

Example 2

Find each product.

a) $2^2 \cdot 2^4$ b) $x^9 \cdot x^6$ c) $5c^3 \cdot 7c^9$ d) $(-k)^8 \cdot (-k) \cdot (-k)^{11}$

Solution

a) $2^2 \cdot 2^4 = 2^{2+4} = 2^6 = 64$ — Since the bases are the same, add the exponents.

b) $x^9 \cdot x^6 = x^{9+6} = x^{15}$

c) $5c^3 \cdot 7c^9 = (5 \cdot 7)(c^3 \cdot c^9) = 35c^{12}$ — Associative and commutative properties

d) $(-k)^8 \cdot (-k) \cdot (-k)^{11} = (-k)^{8+1+11} = (-k)^{20}$ — Product rule

You Try 2

Find each product.

a) $3 \cdot 3^2$ b) $y^{10} \cdot y^4$ c) $-6m^5 \cdot 9m^{11}$ d) $h^4 \cdot h^6 \cdot h^4$ e) $(-3)^2 \cdot (-3)^2$

BE CAREFUL

Can the product rule be applied to $4^3 \cdot 5^2$? **No!** The bases are not the same, so we cannot add the exponents. To evaluate $4^3 \cdot 5^2$, we would evaluate $4^3 = 64$ and $5^2 = 25$, then multiply:

$$4^3 \cdot 5^2 = 64 \cdot 25 = 1600$$

3. Use the Power Rule $(a^m)^n = a^{mn}$

What does $(2^2)^3$ mean?

We can rewrite $(2^2)^3$ first as $2^2 \cdot 2^2 \cdot 2^2$.

$$\begin{aligned} 2^2 \cdot 2^2 \cdot 2^2 &= 2^{2+2+2} && \text{Use the product rule for exponents.} \\ &= 2^6 && \text{Add the exponents.} \\ &= 64 && \text{Simplify.} \end{aligned}$$

Notice that $(2^2)^3 = 2^{2+2+2}$, or $2^{2 \cdot 3}$. This leads us to the basic power rule for exponents: *When you raise a power to another power, keep the base and multiply the exponents.*

Property Basic Power Rule

Let a be any real number and let m and n be positive integers. Then,

$$(a^m)^n = a^{mn}$$

Example 3

Simplify using the power rule.

a) $(3^8)^4$ b) $(n^3)^7$ c) $((-f)^4)^3$

Solution

a) $(3^8)^4 = 3^{8 \cdot 4} = 3^{32}$ b) $(n^3)^7 = n^{3 \cdot 7} = n^{21}$ c) $((-f)^4)^3 = (-f)^{4 \cdot 3} = (-f)^{12}$ ■

You Try 3

Simplify using the power rule.

a) $(5^4)^3$ b) $(j^6)^5$ c) $((-2)^3)^2$

4. Use the Power Rule $(ab)^n = a^n b^n$

We can use another power rule to simplify an expression such as $(5c)^3$. We can rewrite and simplify $(5c)^3$ as $5c \cdot 5c \cdot 5c = 5 \cdot 5 \cdot 5 \cdot c \cdot c \cdot c = 5^3c^3 = 125c^3$. *To raise a product to a power, raise each factor to that power.*

Property Power Rule for a Product

Let a and b be real numbers and let n be a positive integer. Then,

$$(ab)^n = a^n b^n$$

BE CAREFUL

Notice that $(ab)^n = a^n b^n$ is different from $(a + b)^n$. $(a + b)^n \neq a^n + b^n$. We will study this in Chapter 6.

Example 4

Simplify each expression.

a) $(9y)^2$ b) $\left(\frac{1}{4}t\right)^3$ c) $(5c^2)^3$ d) $3(6ab)^2$

Solution

a) $(9y)^2 = 9^2y^2 = 81y^2$

b) $\left(\frac{1}{4}t\right)^3 = \left(\frac{1}{4}\right)^3 \cdot t^3 = \frac{1}{64}t^3$

c) $(5c^2)^3 = 5^3 \cdot (c^2)^3 = 125c^{2 \cdot 3} = 125c^6$

d) $3(6ab)^2 = 3[6^2 \cdot (a)^2 \cdot (b)^2]$ The 3 is not in parentheses; therefore, it will not be squared.

$= 3(36a^2b^2)$

$= 108a^2b^2$ ■

You Try 4

Simplify.

a) $(k^4)^7$ b) $(2k^{10}m^3)^6$ c) $(-r^2s^8)^3$ d) $-4(3tu)^2$

5. Use the Power Rule $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, Where $b \neq 0$

Another power rule allows us to simplify an expression like $\left(\frac{2}{x}\right)^4$. We can rewrite and simplify $\left(\frac{2}{x}\right)^4$ as $\frac{2}{x} \cdot \frac{2}{x} \cdot \frac{2}{x} \cdot \frac{2}{x} = \frac{2 \cdot 2 \cdot 2 \cdot 2}{x \cdot x \cdot x \cdot x} = \frac{2^4}{x^4} = \frac{16}{x^4}$. *To raise a quotient to a power, raise both the numerator and denominator to that power.*

Property Power Rule for a Quotient

Let a and b be real numbers and let n be a positive integer. Then,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \text{ where } b \neq 0$$

Example 5

Simplify using the power rule for quotients.

a) $\left(\frac{3}{8}\right)^2$ b) $\left(\frac{5}{x}\right)^3$ c) $\left(\frac{t}{u}\right)^9$

Solution

a) $\left(\frac{3}{8}\right)^2 = \frac{3^2}{8^2} = \frac{9}{64}$ b) $\left(\frac{5}{x}\right)^3 = \frac{5^3}{x^3} = \frac{125}{x^3}$ c) $\left(\frac{t}{u}\right)^9 = \frac{t^9}{u^9}$

You Try 5

Simplify using the power rule for quotients.

a) $\left(\frac{5}{12}\right)^2$ b) $\left(\frac{2}{d}\right)^5$ c) $\left(\frac{u}{v}\right)^6$

Let's summarize the rules of exponents we have learned in this section:

Summary The Product and Power Rules of Exponents

In the rules below, a and b are any real numbers and m and n are positive integers.

Rule		Example
Product rule	$a^m \cdot a^n = a^{m+n}$	$p^4 \cdot p^{11} = p^{4+11} = p^{15}$
Basic power rule	$(a^m)^n = a^{mn}$	$(c^8)^3 = c^{8 \cdot 3} = c^{24}$
Power rule for a product	$(ab)^n = a^n b^n$	$(3z)^4 = 3^4 \cdot z^4 = 81z^4$
Power rule for a quotient	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, (b \neq 0)$	$\left(\frac{w}{2}\right)^4 = \frac{w^4}{2^4} = \frac{w^4}{16}$

Answers to You Try Exercises

1) a) base: 5; exponent: 3; $5^3 = 125$ b) base: 8; exponent: 2; $-8^2 = -64$ c) base: $-\frac{2}{3}$; exponent: 3; $\left(-\frac{2}{3}\right)^3 = -\frac{8}{27}$ 2) a) 27 b) y^{14} c) $-54m^{16}$ d) h^{14} e) 81 3) a) 5^{12} b) j^{30} c) 64 4) a) k^{28} b) $64k^{60}m^{18}$ c) $-r^6s^{24}$ d) $-36t^2u^2$ 5) a) $\frac{25}{144}$ b) $\frac{32}{d^5}$ c) $\frac{u^6}{v^6}$

2.1A Exercises

Objective 1: Evaluate Exponential Expressions

Rewrite each expression using exponents.

1) $9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$

2) $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

3) $\left(\frac{1}{7}\right)\left(\frac{1}{7}\right)\left(\frac{1}{7}\right)\left(\frac{1}{7}\right)$

4) $(0.8)(0.8)(0.8)$

5) $(-5)(-5)(-5)(-5)(-5)(-5)(-5)$

6) $(-c)(-c)(-c)(-c)(-c)$

7) $(-3y)(-3y)(-3y)(-3y)(-3y)(-3y)(-3y)(-3y)$

8) $\left(-\frac{5}{4}t\right)\left(-\frac{5}{4}t\right)\left(-\frac{5}{4}t\right)\left(-\frac{5}{4}t\right)$

Identify the base and the exponent in each.

9) 6^8

10) 9^4

11) $(0.05)^7$

12) $(0.3)^{10}$

13) $(-8)^5$

14) $(-7)^6$

15) $(9x)^8$

16) $(13k)^3$

17) $(-11a)^2$

18) $(-2w)^9$

19) $5p^4$

20) $-3m^5$

21) $-\frac{3}{8}y^2$

22) $\frac{5}{9}t^7$

VIDEO 23) Evaluate $(3 + 4)^2$ and $3^2 + 4^2$. Are they equivalent? Why or why not?

24) Evaluate $(7 - 3)^2$ and $7^2 - 3^2$. Are they equivalent? Why or why not?

25) For any values of a and b, does $(a + b)^2 = a^2 + b^2$? Why or why not?

26) Does $-2^4 = (-2)^4$? Why or why not?

27) Are $3t^4$ and $(3t)^4$ equivalent? Why or why not?

28) Is there any value of a for which $(-a)^2 = -a^2$? Support your answer with an example.

Evaluate.

29) 2^5

30) 9^2

31) $(11)^2$

32) 4^3

33) $(-2)^4$

34) $(-5)^3$

35) -3^4

36) -6^2

37) -2^3

38) -8^2

39) $\left(\frac{1}{5}\right)^3$

40) $\left(\frac{3}{2}\right)^4$

Objective 2: Use the Product Rule for Exponents

Evaluate the expression using the product rule, where applicable.

41) $2^2 \cdot 2^3$

42) $5^2 \cdot 5$

43) $3^2 \cdot 3^2$

44) $2^3 \cdot 2^3$

45) $5^2 \cdot 2^3$

46) $4^3 \cdot 3^2$

47) $\left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^2$

48) $\left(\frac{4}{3}\right) \cdot \left(\frac{4}{3}\right)^2$

Simplify the expression using the product rule. Leave your answer in exponential form.

49) $8^3 \cdot 8^9$

50) $6^4 \cdot 6^3$

51) $5^2 \cdot 5^4 \cdot 5^5$

52) $12^4 \cdot 12 \cdot 12^2$

53) $(-7)^2 \cdot (-7)^3 \cdot (-7)^3$

54) $(-3)^5 \cdot (-3) \cdot (-3)^6$

55) $b^2 \cdot b^4$

56) $x^4 \cdot x^3$

VIDEO 57) $k \cdot k^2 \cdot k^3$

58) $n^6 \cdot n^5 \cdot n^2$

59) $8y^3 \cdot y^2$

60) $10c^8 \cdot c^2 \cdot c$

61) $(9m^4)(6m^{11})$

62) $(-10p^8)(-3p)$

63) $(-6r)(7r^4)$

64) $(8h^5)(-5h^2)$

65) $(-7t^6)(t^3)(-4t^7)$

66) $(3k^2)(-4k^5)(2k^4)$

67) $\left(\frac{5}{3}x^2\right)(12x)(-2x^3)$

68) $\left(\frac{7}{10}y^9\right)(-2y^4)(3y^2)$

69) $\left(\frac{8}{21}b\right)(-6b^8)\left(-\frac{7}{2}b^6\right)$

70) $(12c^3)\left(\frac{14}{15}c^2\right)\left(\frac{5}{7}c^6\right)$

Mixed Exercises: Objectives 3–5

Simplify the expression using one of the power rules.

71) $(y^3)^4$

72) $(x^5)^8$

73) $(w^{11})^7$

74) $(a^3)^2$

75) $(3^3)^2$

76) $(2^2)^2$

77) $((-5)^3)^2$

78) $((-4)^5)^3$

79) $\left(\frac{1}{3}\right)^4$

80) $\left(\frac{5}{2}\right)^3$

81) $\left(\frac{6}{a}\right)^2$

82) $\left(\frac{v}{4}\right)^3$

83) $\left(\frac{m}{n}\right)^5$

84) $\left(\frac{t}{u}\right)^{12}$

85) $(10y)^4$

86) $(7w)^2$

87) $(-3p)^4$

88) $(2m)^5$

89) $(-4ab)^3$

90) $(-2cd)^4$

91) $6(xy)^3$

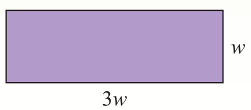
92) $-8(mn)^5$

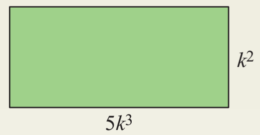
93) $-9(tu)^4$

94) $2(ab)^6$

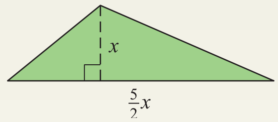
Mixed Exercises: Objectives 2–5

95) Find the area and perimeter of each rectangle.

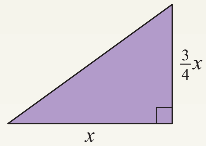
a) 

b) 

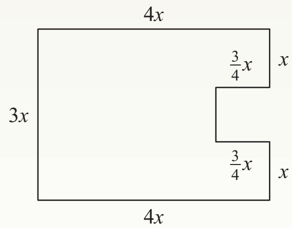
96) Find the area.



97) Find the area.



98) The shape and dimensions of the Millers' family room are given below. They will have wall-to-wall carpeting installed, and the carpet they have chosen costs $\$2.50/\text{ft}^2$.



a) Write an expression for the amount of carpet they will need. (Include the correct units.)

b) Write an expression for the cost of carpeting the family room. (Include the correct units.)

Section 2.1B Combining the Rules

Objective

1. **Combine the Product Rule and Power Rules of Exponents**

1. Combine the Product Rule and Power Rules of Exponents

Now that we have learned the product rule and the power rules for exponents, let's think about how to combine the rules.

If we were asked to evaluate $2^3 \cdot 3^2$, we would follow the order of operations. What would be the first step?

$$\begin{aligned} 2^3 \cdot 3^2 &= 8 \cdot 9 && \text{Evaluate exponents.} \\ &= 72 && \text{Multiply.} \end{aligned}$$

When we combine the rules of exponents, we follow the order of operations.

Example 1

Simplify.

a) $(2c)^3(3c^8)^2$ b) $2(5k^4m^3)^3$ c) $\dfrac{(6t^5)^2}{(2u^4)^3}$

Solution

a) $(2c)^3(3c^8)^2$

Because evaluating exponents comes before multiplying in the order of operations, *evaluate the exponents first.*

$$\begin{aligned}(2c)^3(3c^8)^2 &= (2^3c^3)(3^2)(c^8)^2 && \text{Use the power rule.}\\ &= (8c^3)(9c^{16}) && \text{Use the power rule and evaluate exponents.}\\ &= 72c^{19} && \text{Product rule}\end{aligned}$$

b) $2(5k^4m^3)^3$

Which operation should be performed first, multiplying $2 \cdot 5$ or simplifying $(5k^4m^3)^3$? In the order of operations, we evaluate exponents before multiplying, so *we will begin by simplifying* $(5k^4m^3)^3$.

$$\begin{aligned}2(5k^4m^3)^3 &= 2 \cdot (5)^3(k^4)^3(m^3)^3 && \text{Order of operations and power rule}\\ &= 2 \cdot 125k^{12}m^9 && \text{Power rule}\\ &= 250k^{12}m^9 && \text{Multiply.}\end{aligned}$$

c) $\dfrac{(6t^5)^2}{(2u^4)^3}$

What comes first in the order of operations, dividing or evaluating exponents? *Evaluating exponents*.

$$\begin{aligned}\frac{(6t^5)^2}{(2u^4)^3} &= \frac{36t^{10}}{8u^{12}} && \text{Power rule}\\ &= \frac{\overset{9}{\cancel{36}}t^{10}}{\underset{2}{\cancel{8}}u^{12}} && \text{Divide out the common factor of 4.}\\ &= \frac{9t^{10}}{2u^{12}}\end{aligned}$$

BE CAREFUL

When simplifying the expression in Example 1c, $\dfrac{(6t^5)^2}{(2u^4)^3}$, it may be tempting to simplify before applying the product rule, like this:

$$\frac{(\overset{3}{\cancel{6}}t^5)^2}{(\underset{1}{\cancel{2}}u^4)^3} \neq \frac{(3t^5)^2}{(u^4)^3} = \frac{9t^{10}}{u^{12}} \quad \textbf{Wrong!}$$

You can see, however, that because we did not follow the rules for the order of operations, we did not get the correct answer.

You Try 1

Simplify.

a) $-4(2a^9b^6)^4$ b) $(7x^{10}y)^2(-x^4y^5)^4$ c) $\dfrac{10(m^2n^3)^5}{(5p^4)^2}$ d) $\left(\dfrac{1}{6}w^7\right)^2(3w^{11})^3$

Answers to You Try Exercises

1) a) $-64a^{36}b^{24}$ b) $49x^{36}y^{22}$ c) $\dfrac{2m^{10}n^{15}}{5p^8}$ d) $\dfrac{3}{4}w^{47}$

2.1B Exercises

Objective 1: Combine the Product Rule and Power Rules of Exponents

1) When evaluating expressions involving exponents, always keep in mind the order of ____________.

2) The first step in evaluating $(9-3)^2$ is ____________.

Simplify.

VIDEO 3) $(k^9)^2(k^3)^2$

4) $(d^5)^3(d^2)^4$

5) $(5z^4)^2(2z^6)^3$

6) $(3r)^2(6r^8)^2$

7) $6ab(-a^{10}b^2)^3$

8) $-5pq^4(-p^4q)^4$

9) $(9+2)^2$

10) $(8-5)^3$

11) $(-4t^6u^2)^3(u^4)^5$

12) $(-m^2)^6(-2m^9)^4$

VIDEO 13) $8(6k^7l^2)^2$

14) $5(-7c^4d)^2$

15) $\left(\dfrac{3}{g^5}\right)^3\left(\dfrac{1}{6}\right)^2$

16) $\left(-\dfrac{2}{5}z^5\right)^3(10z)^2$

17) $\left(\dfrac{7}{8}n^2\right)^2(-4n^9)^2$

18) $\left(\dfrac{2}{3}d^8\right)^4\left(\dfrac{9}{2}d^3\right)^2$

19) $h^4(10h^3)^2(-3h^9)^2$

20) $-v^6(-2v^5)^5(-v^4)^3$

21) $3w^{11}(7w^2)^2(-w^6)^5$

22) $5z^3(-4z)^2(2z^3)^2$

23) $\dfrac{(12x^3)^2}{(10y^5)^2}$

24) $\dfrac{(-3a^4)^3}{(6b)^2}$

VIDEO 25) $\dfrac{(4d^9)^2}{(-2c^5)^6}$

26) $\dfrac{(-5m^7)^3}{(5n^{12})^2}$

27) $\dfrac{8(a^4b^7)^9}{(6c)^2}$

28) $\dfrac{(3x^5)^3}{21(yz^2)^6}$

29) $\dfrac{r^4(r^5)^7}{2t(11t^2)^2}$

30) $\dfrac{k^5(k^2)^3}{7m^{10}(2m^3)^2}$

31) $\left(\dfrac{4}{9}x^3y\right)^2\left(\dfrac{3}{2}x^6y^4\right)^3$

32) $(6s^8t^3)^2\left(-\dfrac{10}{3}st^4\right)^2$

33) $\left(-\dfrac{2}{5}c^9d^2\right)^3\left(\dfrac{5}{4}cd^6\right)^2$

34) $-\dfrac{11}{12}\left(\dfrac{3}{2}m^3n^{10}\right)^2$

35) $\left(\dfrac{5x^5y^2}{z^4}\right)^3$

36) $\left(-\dfrac{7a^4b}{8c^6}\right)^2$

37) $\left(-\dfrac{3t^4u^9}{2v^7}\right)^4$

38) $\left(\dfrac{2pr^8}{q^{11}}\right)^5$

39) $\left(\dfrac{12w^5}{4x^3y^6}\right)^2$

40) $\left(\dfrac{10b^3c^5}{15a}\right)^2$

VIDEO 41) The length of a side of a square is $5l^2$ units.

a) Write an expression for its perimeter.

b) Write an expression for its area.

42) The width of a rectangle is $2w$ units, and the length of the rectangle is $7w$ units.

a) Write an expression for its area.

b) Write an expression for its perimeter.

43) The length of a rectangle is x units, and the width of the rectangle is $\dfrac{3}{8}x$ units.

a) Write an expression for its area.

b) Write an expression for its perimeter.

44) The width of a rectangle is $4y^3$ units, and the length of the rectangle is $\dfrac{13}{2}y^3$ units.

a) Write an expression for its perimeter.

b) Write an expression for its area.

Section 2.2A Real-Number Bases

Objectives

1. **Use 0 as an Exponent**
2. **Use Negative Integers as Exponents**

Thus far, we have defined an exponential expression such as 2^3. The exponent of 3 indicates that $2^3 = 2 \cdot 2 \cdot 2$ (3 factors of 2) so that $2^3 = 2 \cdot 2 \cdot 2 = 8$. Is it possible to have an exponent of zero or a negative exponent? If so, what do they mean?

1. Use 0 as an Exponent

Definition

Zero as an Exponent: If $a \neq 0$, then $a^0 = 1$.

How can this be possible? Let's look at an example involving the product rule to help us understand why $a^0 = 1$.

Let's evaluate $2^0 \cdot 2^3$. Using the product rule, we get:

$$2^0 \cdot 2^3 = 2^{0+3} = 2^3 = 8$$

But we know that $2^3 = 8$. Therefore, if $2^0 \cdot 2^3 = 8$, then $2^0 = 1$. This is one way to understand that $a^0 = 1$.

Example 1

Evaluate each expression.

a) 5^0 b) -8^0 c) $(-7)^0$ d) $-3(2^0)$

Solution

a) $5^0 = 1$ b) $-8^0 = -1 \cdot 8^0 = -1 \cdot 1 = -1$

c) $(-7)^0 = 1$ d) $-3(2^0) = -3(1) = -3$

You Try 1

Evaluate.

a) 9^0 b) -2^0 c) $(-5)^0$ d) $3^0(-2)$

2. Use Negative Integers as Exponents

So far we have worked with exponents that are zero or positive. What does a negative exponent mean?

Let's use the product rule to find $2^3 \cdot 2^{-3}$.

$$2^3 \cdot 2^{-3} = 2^{3+(-3)} = 2^0 = 1.$$

Remember that a number multiplied by its reciprocal is 1, and here we have that a quantity, 2^3, times another quantity, 2^{-3}, is 1. Therefore, 2^3 and 2^{-3} are reciprocals!

This leads to the definition of a negative exponent.

Definition

Negative Exponent: If n is any integer and a and b are not equal to zero, then

$$a^{-n} = \left(\frac{1}{a}\right)^n = \frac{1}{a^n} \quad \text{and} \quad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n.$$

Therefore, to rewrite an expression of the form a^{-n} with a positive exponent, *take the reciprocal of the base and make the exponent positive.*

Example 2

Evaluate each expression.

a) 2^{-3} b) $\left(\frac{3}{2}\right)^{-4}$ c) $\left(\frac{1}{5}\right)^{-3}$ d) $(-7)^{-2}$

Solution

a) 2^{-3}: The reciprocal of 2 is $\frac{1}{2}$, so $2^{-3} = \left(\frac{1}{2}\right)^{3} = \frac{1^3}{2^3} = \frac{1}{8}$.

Above we found that $2^3 \cdot 2^{-3} = 1$ using the product rule, but now we can evaluate the product using the definition of a negative exponent.

$$2^3 \cdot 2^{-3} = 8 \cdot \left(\frac{1}{2}\right)^{3} = 8 \cdot \frac{1}{8} = 1$$

b) $\left(\frac{3}{2}\right)^{-4}$: The reciprocal of $\frac{3}{2}$ is $\frac{2}{3}$, so $\left(\frac{3}{2}\right)^{-4} = \left(\frac{2}{3}\right)^{4} = \frac{2^4}{3^4} = \frac{16}{81}$.

BE CAREFUL

Notice that a negative exponent does not make the answer negative!

c) $\left(\frac{1}{5}\right)^{-3}$: The reciprocal of $\frac{1}{5}$ is 5, so $\left(\frac{1}{5}\right)^{-3} = 5^3 = 125$.

d) $(-7)^{-2}$: The reciprocal of -7 is $-\frac{1}{7}$, so

$$(-7)^{-2} = \left(-\frac{1}{7}\right)^{2} = \left(-1 \cdot \frac{1}{7}\right)^{2} = (-1)^2\left(\frac{1}{7}\right)^{2} = 1 \cdot \frac{1^2}{7^2} = \frac{1}{49}$$

You Try 2

Evaluate.

a) $(10)^{-2}$ b) $\left(\frac{1}{4}\right)^{-2}$ c) $\left(\frac{2}{3}\right)^{-3}$ d) -5^{-3}

Answers to You Try Exercises

1) a) 1 b) -1 c) 1 d) -2 2) a) $\frac{1}{100}$ b) 16 c) $\frac{27}{8}$ d) $-\frac{1}{125}$

2.2A Exercises

Mixed Exercises: Objectives 1 and 2

1) True or False: Raising a positive base to a negative exponent will give a negative result. (Example: 2^{-4})

2) True or False: $8^0 = 1$.

3) True or False: The reciprocal of 4 is $\frac{1}{4}$.

4) True or False: $3^{-2} - 2^{-2} = 1^{-2}$.

Evaluate.

5) 2^0

6) $(-4)^0$

7) -5^0

8) -1^0

9) 0^8

10) $-(-9)^0$

11) $(5)^0 + (-5)^0$

12) $\left(\frac{4}{7}\right)^0 - \left(\frac{7}{4}\right)^0$

13) 6^{-2}

14) 9^{-2}

15) 2^{-4}

16) 11^{-2}

17) 5^{-3}

18) 2^{-5}

19) $\left(\frac{1}{8}\right)^{-2}$

20) $\left(\frac{1}{10}\right)^{-3}$

21) $\left(\frac{1}{2}\right)^{-5}$

22) $\left(\frac{1}{4}\right)^{-2}$

23) $\left(\frac{4}{3}\right)^{-3}$

24) $\left(\frac{2}{5}\right)^{-3}$

25) $\left(\frac{9}{7}\right)^{-2}$

26) $\left(\frac{10}{3}\right)^{-2}$

27) $\left(-\frac{1}{4}\right)^{-3}$

28) $\left(-\frac{1}{12}\right)^{-2}$

29) $\left(-\frac{3}{8}\right)^{-2}$

30) $\left(-\frac{5}{2}\right)^{-3}$

31) -2^{-6}

32) -4^{-3}

33) -1^{-5}

34) -9^{-2}

35) $2^{-3} - 4^{-2}$

36) $5^{-2} + 2^{-2}$

37) $2^{-2} + 3^{-2}$

38) $4^{-1} - 6^{-2}$

39) $-9^{-2} + 3^{-3} + (-7)^0$

40) $6^0 - 9^{-1} + 4^0 + 3^{-2}$

Section 2.2B Variable Bases

Objectives

1. **Use 0 as an Exponent**
2. **Rewrite an Exponential Expression with Positive Exponents**

1. Use 0 as an Exponent

We can apply 0 as an exponent to bases containing variables.

Example 1

Evaluate each expression. Assume the variable does not equal zero.

a) t^0 b) $(-k)^0$ c) $-(11p)^0$

Solution

a) $t^0 = 1$ b) $(-k)^0 = 1$

c) $-(11p)^0 = -1 \cdot (11p)^0 = -1 \cdot 1 = -1$

You Try 1

Evaluate. Assume the variable does not equal zero.

a) p^0 b) $(10x)^0$ c) $-(7s)^0$

2. Rewrite an Exponential Expression with Positive Exponents

Next, let's apply the definition of a negative exponent to bases containing variables. As in Example 1, we will assume the variable does not equal zero since having zero in the denominator of a fraction will make the fraction undefined.

Recall that $2^{-4} = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$. That is, to rewrite the expression with a positive exponent, we take the reciprocal of the base.

What is the reciprocal of x? The reciprocal is $\frac{1}{x}$.

Example 2

Rewrite the expression with positive exponents. Assume the variable does not equal zero.

a) x^{-6} b) $\left(\frac{2}{n}\right)^{-6}$ c) $3a^{-2}$

Solution

a) $x^{-6} = \left(\frac{1}{x}\right)^6 = \frac{1^6}{x^6} = \frac{1}{x^6}$

b) $\left(\frac{2}{n}\right)^{-6} = \left(\frac{n}{2}\right)^6$ The reciprocal of $\frac{2}{n}$ is $\frac{n}{2}$.

$= \frac{n^6}{2^6} = \frac{n^6}{64}$

c) $3a^{-2} = 3 \cdot \left(\frac{1}{a}\right)^2$ Remember, the base is a, *not* $3a$, since there are no parentheses. Therefore, the exponent of -2 applies only to a.

$= 3 \cdot \frac{1}{a^2} = \frac{3}{a^2}$

You Try 2

Rewrite the expression with positive exponents. Assume the variable does not equal zero.

a) m^{-4} b) $\left(\frac{1}{z}\right)^{-7}$ c) $-2y^{-3}$

How could we rewrite $\frac{x^{-2}}{y^{-2}}$ with only positive exponents? One way would be to apply the power rule for exponents: $\frac{x^{-2}}{y^{-2}} = \left(\frac{x}{y}\right)^{-2} = \left(\frac{y}{x}\right)^2 = \frac{y^2}{x^2}$

Let's do the same for $\frac{a^{-5}}{b^{-5}}$: $\frac{a^{-5}}{b^{-5}} = \left(\frac{a}{b}\right)^{-5} = \left(\frac{b}{a}\right)^5 = \frac{b^5}{a^5}$

Notice that to rewrite the original expression with only positive exponents, the terms with the negative exponents "switch" their positions in the fraction. We can generalize this way:

Definition

If m and n are any integers and a and b are real numbers not equal to zero, then

$$\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$$

Example 3

Rewrite the expression with positive exponents. Assume the variables do not equal zero.

a) $\dfrac{c^{-8}}{d^{-3}}$ b) $\dfrac{5p^{-6}}{q^7}$ c) $t^{-2}u^{-1}$

d) $\dfrac{2xy^{-3}}{3z^{-2}}$ e) $\left(\dfrac{ab}{4c}\right)^{-3}$

Solution

a) $\dfrac{c^{-8}}{d^{-3}} = \dfrac{d^3}{c^8}$ To make the exponents positive, "switch" the positions of the terms in the fraction.

b) $\dfrac{5p^{-6}}{q^7} = \dfrac{5}{p^6q^7}$ Since the exponent on q is positive, we do not change its position in the expression.

c) $t^{-2}u^{-1} = \dfrac{t^{-2}u^{-1}}{1}$

$= \dfrac{1}{t^2u^1} = \dfrac{1}{t^2u}$ Move $t^{-2}u^{-1}$ to the denominator to write with positive exponents.

d) $\dfrac{2xy^{-3}}{3z^{-2}} = \dfrac{2xz^2}{3y^3}$ To make the exponents positive, "switch" the positions of the factors with negative exponents in the fraction.

e) $\left(\dfrac{ab}{4c}\right)^{-3} = \left(\dfrac{4c}{ab}\right)^3$ To make the exponent positive, use the reciprocal of the base.

$= \dfrac{4^3c^3}{a^3b^3}$ Power rule

$= \dfrac{64c^3}{a^3b^3}$ Simplify.

You Try 3

Rewrite the expression with positive exponents. Assume the variables do not equal zero.

a) $\dfrac{n^{-6}}{y^{-2}}$ b) $\dfrac{z^{-9}}{3k^{-4}}$ c) $8x^{-5}y$ d) $\dfrac{8d^{-4}}{6m^2n^{-1}}$ e) $\left(\dfrac{3x^2}{y}\right)^{-2}$

Answers to You Try Exercises

1) a) 1 b) 1 c) -1 2) a) $\dfrac{1}{m^4}$ b) z^7 c) $-\dfrac{2}{y^3}$ 3) a) $\dfrac{y^2}{n^6}$ b) $\dfrac{k^4}{3z^9}$ c) $\dfrac{8y}{x^5}$

d) $\dfrac{4n}{3m^2d^4}$ e) $\dfrac{y^2}{9x^4}$

2.2B Exercises

Objective 1: Use 0 as an Exponent

1) Identify the base in each expression.

a) w^0 b) $-3n^{-5}$

c) $(2p)^{-3}$ d) $4c^0$

2) True or False: $6^0 - 4^0 = (6-4)^0$

Evaluate. Assume the variables do not equal zero.

3) r^0

4) $(5m)^0$

5) $-2k^0$

6) $-z^0$

7) $x^0 + (2x)^0$

8) $\left(\frac{7}{8}\right)^0 - \left(\frac{3}{5}\right)^0$

Objective 2: Rewrite an Exponential Expression with Positive Exponents

Rewrite each expression with only positive exponents. Assume the variables do not equal zero.

9) d^{-3}

10) y^{-7}

11) p^{-1}

12) a^{-5}

VIDEO 13) $\dfrac{a^{-10}}{b^{-3}}$

14) $\dfrac{h^{-2}}{k^{-1}}$

15) $\dfrac{y^{-8}}{x^{-5}}$

16) $\dfrac{v^{-2}}{w^{-7}}$

17) $\dfrac{t^5}{8u^{-3}}$

18) $\dfrac{9x^{-4}}{y^5}$

19) $5m^6n^{-2}$

20) $\frac{1}{9}a^{-4}b^3$

21) $\dfrac{2}{t^{-11}u^{-5}}$

22) $\dfrac{7r}{2t^{-9}u^2}$

VIDEO 23) $\dfrac{8a^6b^{-1}}{5c^{-10}d}$

24) $\dfrac{17k^{-8}h^5}{20m^{-7}n^{-2}}$

25) $\dfrac{2z^4}{x^{-7}y^{-6}}$

26) $\dfrac{1}{a^{-2}b^{-2}c^{-1}}$

VIDEO 27) $\left(\dfrac{a}{6}\right)^{-2}$

28) $\left(\dfrac{3}{y}\right)^{-4}$

29) $\left(\dfrac{2n}{q}\right)^{-5}$

30) $\left(\dfrac{w}{5v}\right)^{-3}$

VIDEO 31) $\left(\dfrac{12b}{cd}\right)^{-2}$

32) $\left(\dfrac{2tu}{v}\right)^{-6}$

33) $-9k^{-2}$

34) $3g^{-5}$

35) $3t^{-3}$

36) $8h^{-4}$

37) $-m^{-9}$

38) $-d^{-5}$

39) $\left(\dfrac{1}{z}\right)^{-10}$

40) $\left(\dfrac{1}{k}\right)^{-6}$

41) $\left(\dfrac{1}{j}\right)^{-1}$

42) $\left(\dfrac{1}{c}\right)^{-7}$

43) $5\left(\dfrac{1}{n}\right)^{-2}$

44) $7\left(\dfrac{1}{t}\right)^{-8}$

VIDEO 45) $c\left(\dfrac{1}{d}\right)^{-3}$

46) $x^2\left(\dfrac{1}{y}\right)^{-2}$

Section 2.3 The Quotient Rule

Objective

1. Use the Quotient Rule for Exponents

1. Use the Quotient Rule for Exponents

In this section, we will discuss how to simplify the quotient of two exponential expressions with the same base. Let's begin by simplifying $\frac{8^6}{8^4}$. One way to simplify this expression is to write the numerator and denominator without exponents:

$$\frac{8^6}{8^4} = \frac{\cancel{8}\cdot\cancel{8}\cdot\cancel{8}\cdot\cancel{8}\cdot 8\cdot 8}{\cancel{8}\cdot\cancel{8}\cdot\cancel{8}\cdot\cancel{8}} \qquad \text{Divide out common factors.}$$
$$= 8\cdot 8 = 8^2 = 64$$

Therefore,

$$\frac{8^6}{8^4} = 8^2 = 64$$

Do you notice a relationship between the exponents in the original expression and the exponent we get when we simplify?

$$\frac{8^6}{8^4} = 8^{6-4} = 8^2 = 64$$

That's right. We *subtracted* the exponents.

Property Quotient Rule for Exponents

If m and n are any integers and $a \neq 0$, then

$$\frac{a^m}{a^n} = a^{m-n}$$

Notice that the base in the numerator and denominator is a. *To divide expressions with the same base, keep the base and subtract the denominator's exponent from the numerator's exponent.*

Example 1

Simplify. Assume the variables do not equal zero.

a) $\frac{2^9}{2^3}$ b) $\frac{t^{10}}{t^4}$ c) $\frac{3}{3^{-2}}$ d) $\frac{n^5}{n^7}$ e) $\frac{3^2}{2^4}$

Solution

a) $\frac{2^9}{2^3} = 2^{9-3} = 2^6 = 64$ Since the bases are the same, subtract the exponents.

b) $\frac{t^{10}}{t^4} = t^{10-4} = t^6$ Since the bases are the same, subtract the exponents.

c) $\frac{3}{3^{-2}} = \frac{3^1}{3^{-2}} = 3^{1-(-2)}$ Since the bases are the same, subtract the exponents.

$= 3^3 = 27$ Be careful when subtracting the negative exponent!

d) $\frac{n^5}{n^7} = n^{5-7} = n^{-2}$ Same base; subtract the exponents.

$= \left(\frac{1}{n}\right)^2 = \frac{1}{n^2}$ Write with a positive exponent.

e) $\frac{3^2}{2^4} = \frac{9}{16}$ Since the bases are not the same, we cannot apply the quotient rule. Evaluate the numerator and denominator separately.

You Try 1

Simplify. Assume the variables do not equal zero.

a) $\frac{5^7}{5^4}$ b) $\frac{c^4}{c^{-1}}$ c) $\frac{k^2}{k^{10}}$ d) $\frac{2^3}{2^7}$

We can apply the quotient rule to expressions containing more than one variable. Here are more examples:

Example 2

Simplify. Assume the variables do not equal zero.

a) $\dfrac{x^8y^7}{x^3y^4}$ b) $\dfrac{12a^{-5}b^{10}}{8a^{-3}b^2}$

Solution

a) $\dfrac{x^8y^7}{x^3y^4} = x^{8-3}y^{7-4}$ Subtract the exponents.

$= x^5y^3$

b) $\dfrac{12a^{-5}b^{10}}{8a^{-3}b^2}$ We will reduce $\frac{12}{8}$ in addition to applying the quotient rule.

$\dfrac{\overset{3}{\cancel{12}}a^{-5}b^{10}}{\underset{2}{\cancel{8}}a^{-3}b^2} = \dfrac{3}{2}a^{-5-(-3)}b^{10-2}$ Subtract the exponents.

$= \dfrac{3}{2}a^{-5+3}b^8 = \dfrac{3}{2}a^{-2}b^8 = \dfrac{3b^8}{2a^2}$

You Try 2

Simplify. Assume the variables do not equal zero.

a) $\dfrac{r^4s^{10}}{rs^3}$ b) $\dfrac{30m^6n^{-8}}{42m^4n^{-3}}$

Answers to You Try Exercises

1) a) 125 b) c^5 c) $\dfrac{1}{k^8}$ d) $\dfrac{1}{16}$ 2) a) r^3s^7 b) $\dfrac{5m^2}{7n^5}$

2.3 Exercises

Objective 1: Use the Quotient Rule for Exponents

State what is wrong with the following steps and then simplify correctly.

1) $\dfrac{a^5}{a^3} = a^{3-5} = a^{-2} = \dfrac{1}{a^2}$

2) $\dfrac{4^3}{2^6} = \left(\dfrac{4}{2}\right)^{3-6} = 2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

Simplify using the quotient rule. Assume the variables do not equal zero.

3) $\dfrac{d^{10}}{d^5}$

4) $\dfrac{z^{11}}{z^7}$

5) $\dfrac{m^9}{m^5}$

6) $\dfrac{a^6}{a}$

7) $\dfrac{8t^{15}}{t^8}$

8) $\dfrac{4k^4}{k^2}$

9) $\dfrac{6^{12}}{6^{10}}$

10) $\dfrac{4^4}{4}$

11) $\dfrac{3^{12}}{3^8}$

12) $\dfrac{2^7}{2^4}$

13) $\dfrac{2^5}{2^9}$

14) $\dfrac{9^5}{9^7}$

VIDEO 15) $\dfrac{5^6}{5^9}$

16) $\dfrac{8^4}{8^6}$

17) $\dfrac{10d^4}{d^2}$

18) $\dfrac{3x^6}{x^2}$

19) $\dfrac{20c^{11}}{30c^6}$

20) $\dfrac{35t^7}{56t^2}$

21) $\dfrac{y^3}{y^8}$

22) $\dfrac{m^4}{m^{10}}$

VIDEO 23) $\dfrac{x^{-3}}{x^6}$

24) $\dfrac{u^{-20}}{u^{-9}}$

25) $\dfrac{t^{-6}}{t^{-3}}$

26) $\dfrac{y^8}{y^{15}}$

27) $\dfrac{a^{-1}}{a^9}$

28) $\dfrac{m^{-9}}{m^{-3}}$

29) $\dfrac{t^4}{t}$

30) $\dfrac{c^7}{c^{-1}}$

31) $\dfrac{15w^2}{w^{10}}$

32) $\dfrac{-7p^3}{p^{12}}$

VIDEO 33) $\dfrac{-6k}{k^4}$

34) $\dfrac{21h^3}{h^7}$

35) $\dfrac{a^4b^9}{ab^2}$

36) $\dfrac{p^5q^7}{p^2q^3}$

37) $\dfrac{10k^{-2}l^{-6}}{15k^{-5}l^2}$

38) $\dfrac{28tu^{-2}}{14t^5u^{-9}}$

39) $\dfrac{300x^7y^3}{30x^{12}y^8}$

40) $\dfrac{63a^{-3}b^2}{7a^7b^8}$

VIDEO 41) $\dfrac{6v^{-1}w}{54v^2w^{-5}}$

42) $\dfrac{3a^2b^{-11}}{18a^{-10}b^6}$

43) $\dfrac{3c^5d^{-2}}{8cd^{-3}}$

44) $\dfrac{9x^{-5}y^2}{4x^{-2}y^6}$

45) $\dfrac{(x+y)^9}{(x+y)^2}$

46) $\dfrac{(a+b)^9}{(a+b)^4}$

47) $\dfrac{(c+d)^{-5}}{(c+d)^{-11}}$

48) $\dfrac{(a+2b)^{-3}}{(a+2b)^{-4}}$

Putting It All Together

Objective

1. Combine the Rules of Exponents

1. Combine the Rules of Exponents

Let's review all the rules for simplifying exponential expressions and then see how we can combine the rules to simplify expressions.

Summary Rules of Exponents

In the rules stated here, a and b are any real numbers and m and n are positive integers.

Product rule	$a^m \cdot a^n = a^{m+n}$
Basic power rule	$(a^m)^n = a^{mn}$
Power rule for a product	$(ab)^n = a^n b^n$
Power rule for a quotient	$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad (b \neq 0)$
Quotient rule	$\dfrac{a^m}{a^n} = a^{m-n}, \quad (a \neq 0)$

Changing from negative to positive exponents, where $a \neq 0$, $b \neq 0$, and m and n are any integers:

$$\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m} \qquad \left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$$

In the following definitions, $a \neq 0$, and n is any integer.

Zero as an exponent	$a^0 = 1$
Negative number as an exponent	$a^{-n} = \dfrac{1}{a^n}$

Example 1

Simplify using the rules of exponents. Assume all variables represent nonzero real numbers.

a) $(2t^{-6})^3(3t^2)^2$ b) $\left(\dfrac{7c^{10}d^7}{c^4d^2}\right)^2$ c) $\dfrac{w^{-3}\cdot w^4}{w^6}$ d) $\left(\dfrac{12a^{-2}b^9}{30ab^{-2}}\right)^{-3}$

Solution

a) $(2t^{-6})^3\,(3t^2)^2$ We must follow the order of operations. Therefore, evaluate the exponents first.

$$
\begin{aligned}
(2t^{-6})^3\cdot(3t^2)^2 &= 2^3t^{(-6)(3)}\cdot 3^2t^{(2)(2)} && \text{Apply the power rule.}\\
&= 8t^{-18}\cdot 9t^4 && \text{Simplify.}\\
&= 72t^{-18+4} && \text{Multiply } 8\cdot 9 \text{ and add the exponents.}\\
&= 72t^{-14}\\
&= \frac{72}{t^{14}} && \text{Write the answer using a positive exponent.}
\end{aligned}
$$

b) $\left(\dfrac{7c^{10}d^7}{c^4d^2}\right)^2$ How can we begin this problem? We can use the quotient rule to simplify the expression before squaring it.

$$
\begin{aligned}
\left(\frac{7c^{10}d^7}{c^4d^2}\right)^2 &= (7c^{10-4}d^{7-2})^2 && \text{Apply the quotient rule in the parentheses.}\\
&= (7c^6d^5)^2 && \text{Simplify.}\\
&= 7^2c^{(6)(2)}d^{(5)(2)} && \text{Apply the power rule.}\\
&= 49c^{12}d^{10}
\end{aligned}
$$

c) $\dfrac{w^{-3}\cdot w^4}{w^6}$ Let's begin by simplifying the numerator:

$$
\begin{aligned}
\frac{w^{-3}\cdot w^4}{w^6} &= \frac{w^{-3+4}}{w^6} && \text{Add the exponents in the numerator.}\\
&= \frac{w^1}{w^6}
\end{aligned}
$$

Now, we can apply the quotient rule:

$$
\begin{aligned}
&= w^{1-6} = w^{-5} && \text{Subtract the exponents.}\\
&= \frac{1}{w^5} && \text{Write the answer using a positive exponent.}
\end{aligned}
$$

d) $\left(\dfrac{12a^{-2}b^9}{30ab^{-2}}\right)^{-3}$ Eliminate the negative exponent *outside* the parentheses by taking the reciprocal of the base. Notice that we have *not* eliminated the negatives on the exponents *inside* the parentheses.

$$
\left(\frac{12a^{-2}b^9}{30ab^{-2}}\right)^{-3} = \left(\frac{30ab^{-2}}{12a^{-2}b^9}\right)^3
$$

We could apply the exponent of 3 to the quantity inside the parentheses, but we could also reduce $\frac{30}{12}$ first and apply the quotient rule before cubing the quantity.

$$
\begin{aligned}
\left(\frac{30\,ab^{-2}}{12a^{-2}\,b^{9}}\right)^{3} &= \left(\frac{5}{2}a^{1-(-2)}b^{-2-9}\right)^{3} && \text{Reduce } \tfrac{30}{12} \text{ and subtract the exponents.}\\
&= \left(\frac{5}{2}a^{3}b^{-11}\right)^{3} \\
&= \frac{125}{8}a^{9}b^{-33} && \text{Apply the power rule.}\\
&= \frac{125a^{9}}{8b^{33}} && \text{Write the answer using positive exponents.}
\end{aligned}
$$

You Try 1

Simplify using the rules of exponents.

a) $\left(\frac{m^{12}n^{3}}{m^{4}n}\right)^{4}$ b) $(-p^{-5})^{4}(6p^{7})^{2}$ c) $\left(\frac{9x^{4}y^{-5}}{54x^{3}y}\right)^{-2}$

It is possible for variables to appear in exponents. The same rules apply.

Example 2

Simplify using the rules of exponents. Assume that the variables represent nonzero integers. Write your final answer so that the exponents have positive coefficients.

a) $c^{4x} \cdot c^{2x}$ b) $\frac{x^{5y}}{x^{9y}}$

Solution

a) $c^{4x} \cdot c^{2x} = c^{4x+2x} = c^{6x}$ The bases are the same, so apply the product rule. Add the exponents.

b)

$$
\begin{aligned}
\frac{x^{5y}}{x^{9y}} &= x^{5y-9y} && \text{The bases are the same, so apply the quotient rule. Subtract the exponents.}\\
&= x^{-4y} \\
&= \frac{1}{x^{4y}} && \text{Write the answer with a positive coefficient in the exponent.}
\end{aligned}
$$

You Try 2

Simplify using the rules of exponents. Assume that the variables represent nonzero integers. Write your final answer so that the exponents have positive coefficients.

a) $8^{2k} \cdot 8^{k} \cdot 8^{10k}$ b) $(w^{3})^{-2p}$

Answers to You Try Exercises

1) a) $m^{32}n^{8}$ b) $\frac{36}{p^{6}}$ c) $\frac{36y^{12}}{x^{2}}$ 2) a) 8^{13k} b) $\frac{1}{w^{6p}}$

Putting It All Together
Summary Exercises

Objective 1: Combine the Rules of Exponents

Use the rules of exponents to evaluate.

1) $\left(\frac{2}{3}\right)^4$

2) $(2^2)^3$

3) $\frac{3^9}{3^5 \cdot 3^4}$

4) $\frac{(-5)^6 \cdot (-5)^2}{(-5)^5}$

VIDEO 5) $\left(\frac{10}{3}\right)^{-2}$

6) $\left(\frac{3}{7}\right)^{-2}$

7) $(9-6)^2$

8) $(3-8)^3$

9) 10^{-2}

10) 2^{-3}

11) $\frac{2^7}{2^{12}}$

12) $\frac{3^{19}}{3^{15}}$

13) $\left(-\frac{5}{3}\right)^{-7} \cdot \left(-\frac{5}{3}\right)^4$

14) $\left(\frac{1}{8}\right)^{-2}$

15) $3^{-2} - 12^{-1}$

16) $2^{-2} + 3^{-2}$

Simplify. Assume all variables represent nonzero real numbers. The final answer should not contain negative exponents.

17) $-10(-3g^4)^3$

18) $7(2d^3)^3$

19) $\frac{33s}{s^{12}}$

20) $\frac{c^{-7}}{c^{-2}}$

21) $\left(\frac{2xy^4}{3x^{-9}y^{-2}}\right)^4$

22) $\left(\frac{a^6b^5}{10a^3}\right)^3$

23) $\left(\frac{9m^8}{n^3}\right)^{-2}$

24) $\left(\frac{3s^{-6}}{r^2}\right)^{-4}$

25) $(-b^5)^3$

26) $(h^{11})^8$

27) $(-3m^5n^2)^3$

28) $(13a^6b)^2$

29) $\left(-\frac{9}{4}z^5\right)\left(\frac{8}{3}z^{-2}\right)$

30) $(15w^3)\left(-\frac{3}{5}w^6\right)$

31) $\left(\frac{s^7}{t^3}\right)^{-6}$

32) $\frac{m^{-3}}{n^{14}}$

33) $(-ab^3c^5)^2\left(\frac{a^4}{bc}\right)^3$

34) $\frac{(4v^3)^2}{(6v^8)^2}$

VIDEO 35) $\left(\frac{48u^{-7}v^2}{36u^3v^{-5}}\right)^{-3}$

36) $\left(\frac{xy^5}{9x^{-2}y}\right)^{-2}$

37) $\left(\frac{-3t^4u}{t^2u^{-4}}\right)^3$

38) $\left(\frac{k^7m^7}{12k^{-1}m^6}\right)^2$

39) $(h^{-3})^6$

40) $(-d^4)^{-5}$

41) $\left(\frac{h}{2}\right)^4$

42) $13f^{-2}$

43) $-7c^4(-2c^2)^3$

44) $5p^3(4p^6)^2$

VIDEO 45) $(12a^7)^{-1}(6a)^2$

46) $(9r^2s^2)^{-1}$

47) $\left(\frac{9}{20}r^4\right)(4r^{-3})\left(\frac{2}{33}r^9\right)$

48) $\left(\frac{f^8 \cdot f^{-3}}{f^2 \cdot f^9}\right)^6$

49) $\frac{(a^2b^{-5}c)^{-3}}{(a^4b^{-3}c)^{-2}}$

50) $\frac{(x^{-1}y^7z^4)^3}{(x^4yz^{-5})^{-3}}$

51) $\frac{(2mn^{-2})^3(5m^2n^{-3})^{-1}}{(3m^{-3}n^3)^{-2}}$

52) $\frac{(4s^3t^{-1})^2(5s^2t^{-3})^{-2}}{(4s^3t^{-1})^3}$

53) $\left(\frac{4n^{-3}m}{n^8m^2}\right)^0$

54) $\left(\frac{7qr^4}{37r^{-19}}\right)^0$

55) $\left(\frac{49c^4d^8}{21c^4d^5}\right)^{-2}$

56) $\frac{(2x^4y)^{-2}}{(5xy^3)^2}$

Simplify. Assume that the variables represent nonzero integers. Write your final answer so that the exponents have positive coefficients.

57) $(p^{2c})^6$

58) $(5d^{4t})^2$

59) $y^m \cdot y^{3m}$

60) $x^{-5c} \cdot x^{9c}$

61) $t^{5b} \cdot t^{-8b}$

62) $a^{-4y} \cdot a^{-3y}$

63) $\frac{25c^{2x}}{40c^{9x}}$

64) $-\frac{3y^{-10a}}{8y^{-2a}}$

Section 2.4 Scientific Notation

Objectives

1. Multiply a Number by a Power of Ten
2. Understand Scientific Notation
3. Write a Number in Scientific Notation
4. Perform Operations with Numbers in Scientific Notation

The distance from the Earth to the Sun is approximately 150,000,000 km. A single rhinovirus (cause of the common cold) measures 0.00002 mm across. Performing operations on very large or very small numbers like these can be difficult. This is why scientists and economists, for example, often work with such numbers in a shorthand form called *scientific notation*. Writing numbers in scientific notation together with applying rules of exponents can simplify calculations with very large and very small numbers.

1. Multiply a Number by a Power of Ten

Before discussing scientific notation further, we need to understand some principles behind the notation. Let's look at multiplying numbers by positive powers of 10.

Example 1

Multiply.

a) 3.4×10^1 b) 0.0857×10^3 c) 97×10^2

Solution

a) $3.4 \times 10^1 = 3.4 \times 10 = 34$

b) $0.0857 \times 10^3 = 0.0857 \times 1000 = 85.7$

c) $97 \times 10^2 = 97 \times 100 = 9700$

Notice that when we multiply each of these numbers by a positive power of 10, the result is *larger* than the original number. In fact, the exponent determines how many places to the *right* the decimal point is moved.

$$3.40 \times 10^1 = 3.4 \times 10^1 = 34 \qquad 0.0857 \times 10^3 = 85.7$$

1 place to the right 3 places to the right

$$97 \times 10^2 = 97.00 \times 10^2 = 9700$$

2 places to the right

You Try 1

Multiply by moving the decimal point the appropriate number of places.

a) 6.2×10^2 b) 5.31×10^5 c) 0.000122×10^4

What happens to a number when we multiply by a *negative* power of 10?

Example 2

Multiply.

a) 41×10^{-2} b) 367×10^{-4} c) 5.9×10^{-1}

Solution

a) $41 \times 10^{-2} = 41 \times \dfrac{1}{100} = \dfrac{41}{100} = 0.41$

b) $367 \times 10^{-4} = 367 \times \dfrac{1}{10{,}000} = \dfrac{367}{10{,}000} = 0.0367$

c) $5.9 \times 10^{-1} = 5.9 \times \dfrac{1}{10} = \dfrac{5.9}{10} = 0.59$

Is there a pattern? When we multiply each of these numbers by a negative power of 10, the result is *smaller* than the original number. The exponent determines how many places to the *left* the decimal point is moved:

$$41 \times 10^{-2} = \underbrace{41.}_{\text{2 places to the left}} \times 10^{-2} = 0.41 \qquad 367 \times 10^{-4} = \underbrace{0367.}_{\text{4 places to the left}} \times 10^{-4} = 0.0367$$

$$5.9 \times 10^{-1} = \underbrace{5.9}_{\text{1 place to the left}} \times 10^{-1} = 0.59$$

You Try 2

Multiply.

a) 83×10^{-2} b) 45×10^{-3}

It is important to understand the previous concepts to understand how to use scientific notation.

2. Understand Scientific Notation

Definition

A number is in **scientific notation** if it is written in the form $a \times 10^n$ where $1 \le |a| < 10$ and n is an integer.

Multiplying $|a|$ by a *positive* power of 10 will result in a number that is *larger* than $|a|$. Multiplying $|a|$ by a *negative* power of 10 will result in a number that is *smaller* than $|a|$. The double inequality $1 \le |a| < 10$ means that a is a number that has *one* nonzero digit to the left of the decimal point.

Here are some examples of numbers written in scientific notation: 3.82×10^{-5}, 1.2×10^3, and 7×10^{-2}.

The following numbers are *not* in scientific notation:

51.94×10^4	0.61×10^{-3}	300×10^6
↑ 2 digits to left of decimal point	↑ Zero is to left of decimal point	↑ 3 digits to left of decimal point

Now let's convert a number written in scientific notation to a number without exponents.

Example 3

Rewrite without exponents.

a) 5.923×10^4 b) 7.4×10^{-3} c) 1.8875×10^3

Solution

a) $5.923 \times 10^4 \rightarrow \underbrace{5.9230}_{\text{4 places to the right}} = 59{,}230$

Remember, multiplying by a positive power of 10 will make the result *larger* than 5.923.

b) $7.4 \times 10^{-3} \rightarrow \underbrace{007.4}_{\text{3 places to the left}} = 0.0074$

Multiplying by a negative power of 10 will make the result *smaller* than 7.4.

c) $1.8875 \times 10^3 \rightarrow \underbrace{1.8875}_{\text{3 places to the right}} = 1887.5$

You Try 3

Rewrite without exponents.

a) 3.05×10^4 b) 8.3×10^{-5} c) 6.91853×10^3

3. Write a Number in Scientific Notation

We will write the number 48,000 in scientific notation.

To write the number 48,000 in scientific notation, first locate its decimal point.

48,000.

Next, determine where the decimal point will be when the number is in scientific notation:

48,000.

Decimal point will be here.

Therefore, $48{,}000 = 4.8 \times 10^n$, where n is an integer. Will n be positive or negative? We can see that 4.8 must be multiplied by a *positive* power of 10 to make it larger, 48,000.

48000.

Decimal point will be here. Decimal point starts here.

Now we count four places between the original and the final decimal place locations.

48000.

1 2 3 4

We use the number of spaces, 4, as the exponent of 10.

$$48{,}000 = 4.8 \times 10^4$$

Example 4

Write each number in scientific notation.

Solution

a) The distance from the Earth to the Sun is approximately 150,000,000 km.

150,000,000. 150,000,000. Move decimal point eight places.

Decimal point will be here. Decimal point is here.

$150{,}000{,}000 \text{ km} = 1.5 \times 10^8 \text{ km}$

b) A single rhinovirus measures 0.00002 mm across.

0.00002 mm $0.00002 \text{ mm} = 2 \times 10^{-5} \text{ mm}$

Decimal point will be here.

Summary How to Write a Number in Scientific Notation

1) Locate the decimal point in the original number.
2) Determine where the decimal point will be when converting to scientific notation. Remember, there will be *one* nonzero digit to the left of the decimal point.
3) Count how many places you must move the decimal point to take it from its original place to its position for scientific notation.
4) If the absolute value of the resulting number is *smaller* than the absolute value of the original number, you will multiply the result by a *positive* power of 10. Example: $350.9 = 3.509 \times 10^2$.

   If the absolute value of the resulting number is *larger* than the absolute value of the original number, you will multiply the result by a *negative* power of 10. Example: $0.0000068 = 6.8 \times 10^{-6}$.

You Try 4

Write each number in scientific notation.

a) The gross domestic product of the United States in 2008 was approximately \$14,264,600,000,000.

b) The diameter of a human hair is approximately 0.001 in.

4. Perform Operations with Numbers in Scientific Notation

We use the rules of exponents to perform operations with numbers in scientific notation.

Example 5

Perform the operations and simplify.

a) $(-2 \times 10^3)(4 \times 10^2)$ b) $\dfrac{3 \times 10^3}{4 \times 10^5}$

Solution

a) $$\begin{aligned}(-2 \times 10^3)(4 \times 10^2) &= (-2 \times 4)(10^3 \times 10^2) && \text{Commutative property}\\ &= -8 \times 10^5 && \text{Add the exponents.}\\ &= -800{,}000\end{aligned}$$

b) $$\begin{aligned}\frac{3 \times 10^3}{4 \times 10^5} &= \frac{3}{4} \times \frac{10^3}{10^5}\\ &= 0.75 \times 10^{-2} && \text{Write } \tfrac{3}{4} \text{ in decimal form.}\\ &= 7.5 \times 10^{-3} && \text{Use scientific notation.}\end{aligned}$$

or 0.0075

You Try 5

Perform the operations and simplify.

a) $(2.6 \times 10^2)(5 \times 10^4)$ b) $\dfrac{7.2 \times 10^{-9}}{6 \times 10^{-5}}$

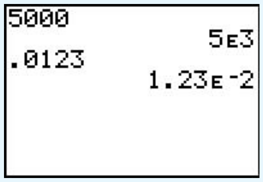
Using Technology

We can use a graphing calculator to convert a very large or very small number to scientific notation, or to convert a number in scientific notation to a number written without an exponent. Suppose we are given a very large number such as 35,000,000,000. If you enter any number with more than 10 digits on the home screen on your calculator and press $\boxed{\text{ENTER}}$, the number will automatically be displayed in scientific notation as shown on the screen below. A small number with more than two zeros to the right of the decimal point (such as .000123) will automatically be displayed in scientific notation as shown below.

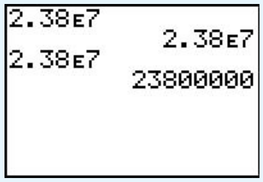
The E shown in the screen refers to a power of 10, so 3.5 E 10 is the number 3.5×10^{10} in scientific notation. 1.23 E-4 is the number 1.23×10^{-4} in scientific notation.

```
35000000000
         3.5E10
.000123
        1.23E-4
```

If a large number has 10 or fewer digits, or if a small number has fewer than three zeros to the right of the decimal point, then the number will not automatically be displayed in scientific notation. To display the number using scientific notation, press MODE, select SCI, and press ENTER. When you return to the home screen, all numbers will be displayed in scientific notation as shown below.



A number written in scientific notation can be entered directly into your calculator. For example, the number 2.38×10^7 can be entered directly on the home screen by typing 2.38 followed by 2nd , 7 ENTER as shown here. If you wish to display this number without an exponent, change the mode back to NORMAL and enter the number on the home screen as shown.



Write each number without an exponent, using a graphing calculator.

1. 3.4×10^5 2. 9.3×10^7 3. 1.38×10^{-3}

Write each number in scientific notation, using a graphing calculator.

4. 186,000 5. 5280 6. 0.0469

Answers to You Try Exercises

1) a) 620 b) 531,000 c) 1.22 2) a) 0.83 b) 0.045 3) a) 30,500 b) 0.000083 c) 6918.53 4) a) 1.42646×10^{13} dollars b) 1.0×10^{-3} in. 5) a) 13,000,000 b) 0.00012

Answers to Technology Exercises

1) 314,000 2) 93,000,000 3) .00138 4) 1.86×10^5 5) 5.28×10^3 6) 4.69×10^{-2}

2.4 Exercises

Mixed Exercises: Objectives 1 and 2

Determine whether each number is in scientific notation.

1) 7.23×10^5 2) 24.0×10^{-3}

3) 0.16×10^{-4} 4) -2.8×10^4

5) -37×10^{-2} 6) 0.9×10^{-1}

7) -5×10^6 8) 7.5×2^{-10}

9) Explain, in your own words, how to determine whether a number is expressed in scientific notation.

10) Explain, in your own words, how to write 4.1×10^{-3} without an exponent.

11) Explain, in your own words, how to write -7.26×10^4 without an exponent.

Multiply.

12) 980.2×10^4 13) 71.765×10^2

14) 0.1502×10^8 15) 40.6×10^{-3}

16) 0.0674×10^{-1} 17) $1,200,006 \times 10^{-7}$

Objective 1: Multiply a Number by a Power of Ten

Write each number without an exponent.

18) 1.92×10^6 VIDEO 19) -6.8×10^{-5}

20) 2.03449×10^3 21) -5.26×10^4

22) -7×10^{-4}

23) 8×10^{-6}

24) -9.5×10^{-3}

25) 6.021967×10^{5}

26) 6×10^{4}

27) 3×10^{6}

28) -9.815×10^{-2}

29) -7.44×10^{-4}

30) 4.1×10^{-6}

Write the following quantities without an exponent.

31) About 2.4428×10^{7} Americans played golf at least 2 times in a year. Write this number using scientific notation. (Statistical Abstract of the U.S., www.census.gov)

32) In 2009, the social network website Facebook claimed that over 2.20×10^{8} photos were uploaded each week. (www.facebook.com)

33) The radius of one hydrogen atom is about 2.5×10^{-11} meters.

34) The length of a household ant is 2.54×10^{-3} meters.

Objective 3: Write a Number in Scientific Notation

Write each number in scientific notation.

35) 2110.5

36) 38.25

37) 0.000096

38) 0.00418

39) −7,000,000

40) 62,000

41) 3400

42) −145,000

43) 0.0008

44) −0.00000022

45) −0.076

46) 990

47) 6000

48) −500,000

Write each number in scientific notation.

49) The total weight of the Golden Gate Bridge is 380,800,000 kg. (www.goldengatebridge.com)

50) A typical hard drive may hold approximately 160,000,000,000 bytes of data.

51) The diameter of an atom is about 0.00000001 cm.

52) The oxygen-hydrogen bond length in a water molecule is 0.000000001 mm.

Objective 4: Perform Operations with Numbers in Scientific Notation

Perform the operation as indicated. Write the final answer without an exponent.

53) $\dfrac{6 \times 10^{9}}{2 \times 10^{5}}$

54) $(7 \times 10^{2})(2 \times 10^{4})$

55) $(2.3 \times 10^{3})(3 \times 10^{2})$

56) $\dfrac{8 \times 10^{7}}{4 \times 10^{4}}$

57) $\dfrac{8.4 \times 10^{12}}{-7 \times 10^{9}}$

58) $\dfrac{-4.5 \times 10^{-6}}{-1.5 \times 10^{-8}}$

59) $(-1.5 \times 10^{-8})(4 \times 10^{6})$

60) $(-3 \times 10^{-2})(-2.6 \times 10^{-3})$

61) $\dfrac{-3 \times 10^{5}}{6 \times 10^{8}}$

62) $\dfrac{2 \times 10^{1}}{5 \times 10^{4}}$

63) $(9.75 \times 10^{4}) + (6.25 \times 10^{4})$

64) $(4.7 \times 10^{-3}) + (8.8 \times 10^{-3})$

65) $(3.19 \times 10^{-5}) + (9.2 \times 10^{-5})$

66) $(2 \times 10^{2}) + (9.7 \times 10^{2})$

For each problem, express each number in scientific notation, then solve the problem.

67) Humans shed about 1.44×10^{7} particles of skin every day. How many particles would be shed in a year? (Assume 365 days in a year.)

68) Scientists send a lunar probe to land on the moon and send back data. How long will it take for pictures to reach the Earth if the distance between the Earth and the moon is 360,000 km and if the speed of light is 3×10^{5} km/sec?

69) In Wisconsin in 2001, approximately 1,300,000 cows produced 2.21×10^{10} lb of milk. On average, how much milk did each cow produce? (www.nass.usda.gov)

70) The average snail can move 1.81×10^{-3} mi in 5 hours. What is its rate of speed in miles per hour?

71) A photo printer delivers approximately 1.1×10^{6} droplets of ink per square inch. How many droplets of ink would a 4 in. $\times$ 6 in. photo contain?

72) In 2007, 3,500,000,000,000 prescription drug orders were filled in the United States. If the average price of each prescription was roughly \$65.00, how much did the United States pay for prescription drugs last year? (National Conference of State Legislatures, www.ncsl.org)

73) In 2006, American households spent a total of about 7.3×10^{11} dollars on food. If there were roughly 120,000,000 households in 2006, how much money did the average household spend on food? (Round to the closest dollar.) (www.census.gov)

74) Find the population density of Australia if the estimated population in 2009 was about 22,000,000 people and the country encompasses about 2,900,000 sq mi. (Australian Bureau of Statistics, www.abs.gov.au)

75) When one of the U.S. space shuttles enters orbit, it travels at about 7800 m/sec. How far does it travel in 2 days? (Hint: Change days to seconds, and write all numbers in scientific notation before doing the computations.) (hypertextbook.com)

76) According to Nielsen Media Research, over 92,000,000 people watched Super Bowl XLIII in 2009 between the Pittsburgh Steelers and the Arizona Cardinals. The California Avocado Commission estimates that about 736,000,000 ounces of avocados were eaten during that Super Bowl, mostly in the form of guacamole. On average, how many ounces of avocados did each viewer eat during the Super Bowl?

77) In 2007, the United States produced about 6×10^{9} metric tons of carbon emissions. The U.S. population that year was about 300 million. Find the amount of carbon emissions produced per person that year. (www.eia.doe.gov, U.S. Census Bureau)

Chapter 2: Summary

Definition/Procedure	Example
2.1A The Product Rule and Power Rules	
Exponential Expression: $a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors of } a}$ a is the **base**, n is the exponent. **(p. 80)**	$5^4 = 5 \cdot 5 \cdot 5 \cdot 5$ 5 is the **base**, 4 is the exponent.
Product Rule: $a^m \cdot a^n = a^{m+n}$ **(p. 81)**	$x^8 \cdot x^2 = x^{10}$
Basic Power Rule: $(a^m)^n = a^{mn}$ **(p. 82)**	$(t^3)^5 = t^{15}$
Power Rule for a Product: $(ab)^n = a^nb^n$ **(p. 82)**	$(2c)^4 = 2^4c^4 = 16c^4$
Power Rule for a Quotient: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, where $b \neq 0$. **(p. 83)**	$\left(\frac{w}{5}\right)^3 = \frac{w^3}{5^3} = \frac{w^3}{125}$
2.1B Combining the Rules	
Remember to follow the order of operations. **(p. 85)**	Simplify $(3y^4)^2(2y^9)^3$. $= 9y^8 \cdot 8y^{27}$ Exponents come before multiplication. $= 72y^{35}$ Use the product rule and multiply coefficients.
2.2A Real-Number Bases	
Zero Exponent: If $a \neq 0$, then $a^0 = 1$. **(p. 88)**	$(-9)^0 = 1$
Negative Exponent: For $a \neq 0$, $a^{-n} = \left(\frac{1}{a}\right)^n = \frac{1}{a^n}$. **(p. 88)**	Evaluate. $\left(\frac{5}{2}\right)^{-3} = \left(\frac{2}{5}\right)^3 = \frac{2^3}{5^3} = \frac{8}{125}$
2.2B Variable Bases	
If $a \neq 0$ and $b \neq 0$, then $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$. **(p. 91)**	Rewrite p^{-10} with a positive exponent (assume $p \neq 0$). $p^{-10} = \left(\frac{1}{p}\right)^{10} = \frac{1}{p^{10}}$
If $a \neq 0$ and $b \neq 0$, then $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$. **(p. 92)**	Rewrite each expression with positive exponents. Assume the variables represent nonzero real numbers. a) $\frac{x^{-3}}{y^{-7}} = \frac{y^7}{x^3}$ b) $\frac{14m^{-6}}{n^{-1}} = \frac{14n}{m^6}$

Definition/Procedure	Example
2.3 The Quotient Rule	
Quotient Rule: If $a \neq 0$, then $\frac{a^m}{a^n} = a^{m-n}$. **(p. 94)**	Simplify. $\frac{4^9}{4^6} = 4^{9-6} = 4^3 = 64$
Putting It All Together	
Combine the Rules of Exponents (p. 96)	Simplify. $\left(\frac{a^4}{2a^7}\right)^{-5} = \left(\frac{2a^7}{a^4}\right)^5 = (2a^3)^5 = 32a^{15}$
2.4 Scientific Notation	
Scientific Notation A number is in **scientific notation** if it is written in the form $a \times 10^n$, where $1 \leq \lvert a \rvert < 10$ and n is an integer. That is, a is a number that has one nonzero digit to the left of the decimal point. **(p. 101)**	Write in scientific notation. a) $78{,}000 \rightarrow 78{,}000 \rightarrow 7.8 \times 10^4$ b) $0.00293 \rightarrow 0.00293 \rightarrow 2.93 \times 10^{-3}$
Converting from Scientific Notation (p. 101)	Write without exponents. a) $5 \times 10^{-4} \rightarrow 0005. \rightarrow 0.0005$ b) $1.7 \times 10^6 = 1.700000 \rightarrow 1{,}700{,}000$
Performing Operations (p. 103)	Multiply $(4 \times 10^2)(2 \times 10^4)$. $= (4 \times 2)(10^2 \times 10^4)$ $= 8 \times 10^6$ $= 8{,}000{,}000$

Chapter 2: Review Exercises

(2.1 A)

1) Write in exponential form.
   a) $8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8$
   b) $(-7)(-7)(-7)(-7)$

2) Identify the base and the exponent.
   a) -6^5 b) $(4t)^3$
   c) $4t^3$ d) $-4t^3$

3) Use the rules of exponents to simplify.
   a) $2^3 \cdot 2^2$ b) $\left(\frac{1}{3}\right)^2 \cdot \left(\frac{1}{3}\right)$
   c) $(7^3)^4$ d) $(k^5)^6$

4) Use the rules of exponents to simplify.
   a) $(3^2)^2$ b) $8^3 \cdot 8^7$
   c) $(m^4)^9$ d) $p^9 \cdot p^7$

5) Simplify using the rules of exponents.
   a) $(5y)^3$ b) $(-7m^4)(2m^{12})$
   c) $\left(\frac{a}{b}\right)^6$ d) $6(xy)^2$
   e) $\left(\frac{10}{9}c^4\right)(2c)\left(\frac{15}{4}c^3\right)$

6) Simplify using the rules of exponents.
   a) $\left(\frac{x}{y}\right)^{10}$ b) $(-2z)^5$
   c) $(6t^7)\left(-\frac{5}{8}t^5\right)\left(\frac{2}{3}t^2\right)$
   d) $-3(ab)^4$ e) $(10j^6)(4j)$

(2.1 B)

7) Simplify using the rules of exponents.
   a) $(z^5)^2(z^3)^4$ b) $-2(3c^5d^8)^2$
   c) $(9-4)^3$ d) $\dfrac{(10t^3)^2}{(2u^7)^3}$

8) Simplify using the rules of exponents.
   a) $\left(\dfrac{-20d^4c}{5b^3}\right)^3$ b) $(-2y^8z)^3(3yz^2)^2$
   c) $\dfrac{x^7 \cdot (x^2)^5}{(2y^3)^4}$ d) $(6-8)^2$

(2.2 A)

9) Evaluate.
   a) 8^0 b) -3^0
   c) 9^{-1} d) $3^{-2} - 2^{-2}$
   e) $\left(\frac{4}{5}\right)^{-3}$

10) Evaluate.
   a) $(-12)^0$ b) $5^0 + 4^0$
   c) -6^{-2} d) 2^{-4}
   e) $\left(\frac{10}{3}\right)^{-2}$

(2.2 B)

11) Rewrite the expression with positive exponents. Assume the variables do not equal zero.
   a) v^{-9} b) $\left(\frac{9}{c}\right)^{-2}$
   c) $\left(\frac{1}{y}\right)^{-8}$ d) $-7k^{-9}$
   e) $\dfrac{19z^{-4}}{a^{-1}}$ f) $20m^{-6}n^5$
   g) $\left(\frac{2j}{k}\right)^{-5}$

12) Rewrite the expression with positive exponents. Assume the variables do not equal zero.
   a) $\left(\frac{1}{x}\right)^{-5}$ b) $3p^{-4}$
   c) $a^{-8}b^{-3}$ d) $\dfrac{12k^{-3}r^5}{16mn^{-6}}$
   e) $\dfrac{c^{-1}d^{-1}}{15}$ f) $\left(-\dfrac{m}{4n}\right)^{-3}$
   g) $\dfrac{10b^4}{a^{-9}}$

(2.3)
In Exercises 13–16, assume the variables represent nonzero real numbers. The answers should not contain negative exponents.

13) Simplify using the rules of exponents.
   a) $\dfrac{3^8}{3^6}$ b) $\dfrac{r^{11}}{r^3}$
   c) $\dfrac{48t^{-2}}{32t^3}$ d) $\dfrac{21xy^2}{35x^{-6}y^3}$

14) Simplify using the rules of exponents.

a) $\frac{2^9}{2^{15}}$ b) $\frac{d^4}{d^{-10}}$

c) $\frac{m^{-5}n^3}{mn^8}$ d) $\frac{100a^8b^{-1}}{25a^7b^{-4}}$

15) Simplify by applying one or more of the rules of exponents.

a) $(-3s^4t^5)^4$ b) $\frac{(2a^6)^5}{(4a^7)^2}$

c) $\left(\frac{z^4}{y^3}\right)^{-6}$ d) $(-x^3y)^5(6x^{-2}y^3)^2$

e) $\left(\frac{cd^{-4}}{c^8d^{-9}}\right)^5$ f) $\left(\frac{14m^5n^5}{7m^4n}\right)^3$

g) $\left(\frac{3k^{-1}t}{5k^{-7}t^4}\right)^{-3}$ h) $\left(\frac{40}{21}x^{10}\right)(3x^{-12})\left(\frac{49}{20}x^2\right)$

16) Simplify by applying one or more of the rules of exponents.

a) $\left(\frac{4}{3}\right)^8\left(\frac{4}{3}\right)^{-2}\left(\frac{4}{3}\right)^{-3}$ b) $\left(\frac{k^{10}}{k^4}\right)^3$

c) $\left(\frac{x^{-4}y^{11}}{xy^2}\right)^{-2}$ d) $(-9z^5)^{-2}$

e) $\left(\frac{g^2 \cdot g^{-1}}{g^{-7}}\right)^{-4}$ f) $(12p^{-3})\left(\frac{10}{3}p^5\right)\left(\frac{1}{4}p^2\right)^2$

g) $\left(\frac{30u^2v^{-3}}{40u^7v^{-7}}\right)^{-2}$ h) $-5(3h^4k^9)^2$

17) Simplify. Assume that the variables represent nonzero integers. Write your final answer so that the exponents have positive coefficients.

a) $y^{3k} \cdot y^{7k}$ b) $(x^{5p})^2$

c) $\frac{z^{12c}}{z^{5c}}$ d) $\frac{t^{6d}}{t^{11d}}$

(2.4)
Write each number without an exponent.

18) 9.38×10^5 19) -4.185×10^2

20) 9×10^3 21) 6.7×10^{-4}

22) 1.05×10^{-6} 23) 2×10^4

24) 8.8×10^{-2}

Write each number in scientific notation.

25) 0.0000575 26) 36,940

27) 32,000,000 28) 0.0000004

29) 178,000 30) 66

31) 0.0009315

Write the number without exponents.

32) Before 2010, golfer Tiger Woods earned over 7×10^7 dollars per year in product endorsements. (www.forbes.com)

Perform the operation as indicated. Write the final answer without an exponent.

33) $\frac{8 \times 10^6}{2 \times 10^{13}}$ 34) $\frac{-1 \times 10^9}{5 \times 10^{12}}$

35) $(9 \times 10^{-8})(4 \times 10^7)$ 36) $(5 \times 10^3)(3.8 \times 10^{-8})$

37) $\frac{-3 \times 10^{10}}{-4 \times 10^6}$ 38) $(-4.2 \times 10^2)(3.1 \times 10^3)$

For each problem, write each of the numbers in scientific notation, then solve the problem. Write the answer without exponents.

39) Eight porcupines have a total of about 2.4×10^5 quills on their bodies. How many quills would one porcupine have?

40) In 2002, Nebraska had approximately 4.6×10^7 acres of farmland and about 50,000 farms. What was the average size of a Nebraska farm in 2002? (www.nass.usda.gov)

41) One molecule of water has a mass of 2.99×10^{-23} g. Find the mass of 100,000,000 molecules.

42) At the end of 2008, the number of SMS text messages sent in one month in the United States was 110.4 billion. If 270.3 million people used SMS text messaging, about how many did each person send that month? (Round to the nearest whole number.) (www.ctia.org/advocacy/research/index.cfm/AID/10323)

43) When the polls closed on the west coast on November 4, 2008, and Barack Obama was declared the new president, there were about 143,000 visits per second to news websites. If the visits continued at that rate for 3 minutes, how many visits did the news websites receive during that time? (www.xconomy.com)

Chapter 2: Test

Write in exponential form.

1) $(-3)(-3)(-3)$

2) $x \cdot x \cdot x \cdot x \cdot x$

Use the rules of exponents to simplify.

3) $5^2 \cdot 5$

4) $\left(\frac{1}{x}\right)^5 \cdot \left(\frac{1}{x}\right)^2$

5) $(8^3)^{12}$

6) $p^7 \cdot p^{-2}$

Evaluate.

7) 3^4

8) 8^0

9) 2^{-5}

10) $4^{-2} + 2^{-3}$

11) $\left(-\frac{3}{4}\right)^3$

12) $\left(\frac{10}{7}\right)^{-2}$

Simplify using the rules of exponents. Assume all variables represent nonzero real numbers. The final answer should not contain negative exponents.

13) $(5n^6)^3$

14) $(-3p^4)(10p^8)$

15) $\frac{m^{10}}{m^4}$

16) $\frac{a^9b}{a^5b^7}$

17) $\left(\frac{-12t^{-6}u^8}{4t^5u^{-1}}\right)^{-3}$

18) $(2y^{-4})^6\left(\frac{1}{2}y^5\right)^3$

19) $\left(\frac{(9x^2y^{-2})^3}{4xy}\right)^0$

20) $\frac{(2m+n)^3}{(2m+n)^2}$

21) $\frac{12a^4b^{-3}}{20c^{-2}d^3}$

22) $\left(\frac{y^{-7} \cdot y^3}{y^5}\right)^{-2}$

23) Simplify $t^{10k} \cdot t^{3k}$. Assume that the variables represent nonzero integers.

24) Rewrite 7.283×10^5 without exponents.

25) Write 0.000165 in scientific notation.

26) Divide. Write the answer without exponents. $\frac{-7.5 \times 10^{12}}{1.5 \times 10^8}$

27) Write the number without an exponent: In 2002, the population of Texas was about 2.18×10^7. (U.S. Census Bureau)

28) An electron is a subatomic particle with a mass of 9.1×10^{-28} g. What is the mass of 2,000,000,000 electrons? Write the answer without exponents.

Cumulative Review: Chapters 1–2

1) Write $\dfrac{90}{150}$ in lowest terms.

Perform the indicated operations. Write the answer in lowest terms.

2) $\dfrac{2}{15} + \dfrac{1}{10} + \dfrac{7}{20}$

3) $\dfrac{4}{15} \div \dfrac{20}{21}$

4) $-144 \div (-12)$

5) $-26 + 5 - 7$

6) -9^2

7) $(-1)^5$

8) $(5 + 1)^2 - 2[17 + 5(10 - 14)]$

9) Glen Crest High School is building a new football field. The dimensions of a regulation-size field are $53\frac{1}{3}$ yd by 120 yd. (There are 10 yd of end zone on each end.) The sod for the field will cost $\$1.80/\text{yd}^2$.

   a) Find the perimeter of the field.

   b) How much will it cost to sod the field?

10) Evaluate $2p^2 - 11q$ when $p = 3$ and $q = -4$.

11) State the formula for the volume of a sphere.

12) Given this set of numbers $\left\{3, -4, -2.1\overline{3}, \sqrt{11}, 2\frac{2}{3}\right\}$, list the

   a) integers

   b) irrational numbers

   c) natural numbers

   d) rational numbers

   e) whole numbers

13) Evaluate $4x^3 + 2x - 3$ when $x = 4$.

14) Rewrite $\frac{3}{4}(6m - 20n + 7)$ using the distributive property.

15) Combine like terms and simplify: $5(t^2 + 7t - 3) - 2(4t^2 - t + 5)$

16) Let x represent the unknown quantity, and write a mathematical expression for "thirteen less than half of a number."

Simplify using the rules of exponents. The answer should not contain negative exponents. Assume the variables represent nonzero real numbers.

17) $4^3 \cdot 4^7$

18) $\left(\dfrac{x}{y}\right)^{-3}$

19) $\left(\dfrac{32x^3}{8x^{-2}}\right)^{-1}$

20) $-(3rt^{-3})^4$

21) $(4z^3)(-7z^5)$

22) $\dfrac{n^2}{n^9}$

23) $(-2a^{-6}b)^5$

24) Write 0.000729 in scientific notation.

25) Perform the indicated operation. Write the final answer without an exponent. $(6.2 \times 10^5)(9.4 \times 10^{-2})$

CHAPTER 3

Linear Equations and Inequalities

Algebra at Work: Landscape Architecture

A landscape architect must have excellent problem-solving skills.

Matthew is designing the driveway, patio, and walkway for this new home. The village has a building code which states that, at most, 70% of the lot can be covered with an impervious surface such as the house, driveway, patio, and walkway leading up to the front door. So, he cannot design just anything.

To begin, Matthew must determine the area of the land and find 70% of that number to determine how much land can be covered with these hard surfaces. He must subtract the area covered by the house to determine how much land he has left for the driveway, patio, and walkway. Using his design experience and problem-solving skills, he must come up with a plan for building the driveway, patio, and walkway that will not only please his client but will meet building codes as well.

In this chapter, we will learn different strategies for solving many different types of problems.

Section 3.1 Solving Linear Equations Part I

Objectives

1. Define a Linear Equation in One Variable
2. Use the Addition and Subtraction Properties of Equality
3. Use the Multiplication and Division Properties of Equality
4. Solve Equations of the Form $ax + b = c$
5. Combine Like Terms on One Side of the Equal Sign, Then Solve

1. Define a Linear Equation in One Variable

What is an equation? It is a mathematical statement that two expressions are equal. $5 + 1 = 6$ is an equation.

Note

An equation contains an "=" sign and an expression does not.

$5x + 4 = 7$ is an *equation*.
$5x + 4x$ is an *expression*.

We can **solve** equations, and we can **simplify** expressions.

There are many different types of algebraic equations, and in Sections 3.1 and 3.2 we will learn how to solve *linear* equations. Here are some examples of linear equations in one variable:

$$p - 1 = 4, \qquad 3x + 5 = 17, \qquad 8(n + 1) - 7 = 2n + 3, \qquad -\frac{5}{6}y + \frac{1}{3} = y - 2$$

Definition

A **linear equation in one variable** is an equation that can be written in the form $ax + b = 0$, where a and b are real numbers and $a \neq 0$.

The exponent of the variable, x, in a linear equation is 1. For this reason, linear equations are also known as first-degree equations. Equations like $k^2 - 13k + 36 = 0$ and $\sqrt{w - 3} = 2$ are not linear equations and are presented later in the text.

To **solve an equation** means to find the value or values of the variable that make the equation true. For example, the solution of the equation $p - 1 = 4$ is $p = 5$ since substituting 5 for the variable makes the equation true.

$$\begin{aligned} p - 1 &= 4 \\ 5 - 1 &= 4 \quad \text{True} \end{aligned}$$

Usually, we use set notation to list all the solutions of an equation. The **solution set** of an equation is the set of all numbers that make the equation true. Therefore, $\{5\}$ is the solution set of the equation $p - 1 = 4$. We also say that 5 *satisfies* the equation $p - 1 = 4$.

2. Use the Addition and Subtraction Properties of Equality

Begin with the true statement $8 = 8$. What happens if we add the same number, say 2, to each side? Is the statement still true? Yes!

$$\begin{aligned} 8 &= 8 \\ 8 + 2 &= 8 + 2 \\ 10 &= 10 \quad \text{True} \end{aligned}$$

Will a statement remain true if we *subtract* the same number from each side? Let's begin with the true statement $5 = 5$ and subtract 9 from each side:

$$\begin{aligned} 5 &= 5 \\ 5 - 9 &= 5 - 9 \\ -4 &= -4 \quad \text{True} \end{aligned}$$

When we subtracted 9 from each side of the equation, the new statement was true.

$8 = 8$ and $8 + 2 = 8 + 2$ are *equivalent equations*.
$5 = 5$ and $5 - 9 = 5 - 9$ are *equivalent equations* as well.

We can apply these principles to equations containing variables. This will help us solve equations.

Property Addition and Subtraction Properties of Equality

Let a, b, and c be expressions representing real numbers. Then,

1) If $a = b$, then $a + c = b + c$ Addition property of equality
2) If $a = b$, then $a - c = b - c$ Subtraction property of equality

Example 1

Solve each equation, and check the solution.

a) $x - 8 = 3$ b) $w + 2.3 = 9.8$ c) $-5 = m + 4$

Solution

Remember, to solve the equation means to find the value of the variable that makes the statement true. To do this, we want to get the variable by itself. We call this **isolating the variable**.

a) On the left side of the equal sign, the 8 is being **subtracted from** the x. To isolate x, we perform the "opposite" operation—that is, we **add 8** to each side.

$$\begin{aligned} x - 8 &= 3 \\ x - 8 + 8 &= 3 + 8 \qquad \text{Add 8 to each side.} \\ x &= 11 \end{aligned}$$

Check: Substitute 11 for x in the original equation.

$$\begin{aligned} x - 8 &= 3 \\ 11 - 8 &= 3 \\ 3 &= 3 \quad ✓ \end{aligned}$$

The solution set is $\{11\}$.

b) Here, 2.3 is being **added to** w. To get the w by itself, **subtract 2.3** from each side.

$$\begin{aligned} w + 2.3 &= 9.8 \\ w + 2.3 - 2.3 &= 9.8 - 2.3 \qquad \text{Subtract 2.3 from each side.} \\ w &= 7.5 \end{aligned}$$

Check: Substitute 7.5 for w in the original equation.

$$\begin{aligned} w + 2.3 &= 9.8 \\ 7.5 + 2.3 &= 9.8 \\ 9.8 &= 9.8 \quad ✓ \end{aligned}$$

The solution set is $\{7.5\}$.

c) $-5 = m + 4$

BE CAREFUL

The variable does not always appear on the left-hand side of the equation. The 4 is being **added to** the m, so we will subtract **4** from each side.

$$\begin{aligned} -5 &= m + 4 \\ -5 - 4 &= m + 4 - 4 \qquad \text{Subtract 4 from each side.} \\ -9 &= m \end{aligned}$$

Check:

$$-5 = m + 4$$
$$-5 = -9 + 4$$
$$-5 = -5 \quad ✓$$

The solution set is $\{-9\}$.

You Try 1

Solve each equation, and check the solution.

a) $b - 8 = 9$ b) $t + 7.2 = 3.8$ c) $-6 = r - 11$

3. Use the Multiplication and Division Properties of Equality

We have just seen that we can add a quantity to each side of an equation or subtract a quantity from each side of an equation, and we will obtain an equivalent equation. It is also true that if we multiply both sides of an equation by the same nonzero number or divide both sides of an equation by the same nonzero number, then we will obtain an equivalent equation.

Property Multiplication and Division Properties of Equality

Let a, b, and c be expressions representing real numbers where $c \neq 0$. Then,

1) If $a = b$, then $ac = bc$ Multiplication property of equality

2) If $a = b$, then $\dfrac{a}{c} = \dfrac{b}{c}$ Division property of equality

Example 2

Solve each equation.

a) $4k = -2.4$ b) $-m = 19$ c) $\dfrac{x}{3} = 5$ d) $\dfrac{3}{8}y = 12$

Solution

The goal is to isolate the variable—that is, get the variable on a side by itself.

a) On the left-hand side of the equation, the k is being **multiplied** by 4. So, we will perform the "opposite" operation and **divide** each side by 4.

$$4k = -2.4$$
$$\frac{4k}{4} = \frac{-2.4}{4} \qquad \text{Divide each side by 4.}$$
$$k = -0.6$$

Check: Substitute -0.6 for k in the original equation.

$$4k = -2.4$$
$$4(-0.6) = -2.4$$
$$-2.4 = -2.4 \quad ✓$$

The solution set is $\{-0.6\}$.

b) The negative sign in front of the m in $-m = 19$ tells us that the coefficient of m is -1. Since m is being **multiplied** by -1, we will **divide** each side by -1.

$$-m = 19$$

$$\frac{-1m}{-1} = \frac{19}{-1} \qquad \text{Divide both sides by } -1.$$

$$m = -19$$

The check is left to the student. The solution set is $\{-19\}$.

c) The x in $\frac{x}{3} = 5$ is being **divided** by 3. Therefore, we will **multiply** each side by 3.

$$\frac{x}{3} = 5$$

$$3 \cdot \frac{x}{3} = 3 \cdot 5 \qquad \text{Multiply both sides by 3.}$$

$$1x = 15 \qquad \text{Simplify.}$$

$$x = 15$$

The check is left to the student. The solution set is $\{15\}$.

d) On the left-hand side of $\frac{3}{8}y = 12$, the y is being **multiplied** by $\frac{3}{8}$. So, we could divide each side by $\frac{3}{8}$. However, recall that dividing a quantity by a fraction is the same as **multiplying by the reciprocal** of the fraction. Therefore, we will multiply each side by the reciprocal of $\frac{3}{8}$.

$$\frac{3}{8}y = 12$$

$$\frac{8}{3} \cdot \frac{3}{8}y = \frac{8}{3} \cdot 12 \qquad \text{The reciprocal of } \frac{3}{8} \text{ is } \frac{8}{3}\text{. Multiply both sides by } \frac{8}{3}.$$

$$1y = \frac{8}{\cancel{3}_{1}} \cdot \cancel{12}^{4} \qquad \text{Perform the multiplication.}$$

$$y = 32 \qquad \text{Simplify.}$$

The check is left to the student. The solution set is $\{32\}$. ■

You Try 2

Solve each equation.

a) $-8w = 9.6$ b) $-y = -7$ c) $\frac{n}{2} = 12$ d) $\frac{5}{9}c = 20$

4. Solve Equations of the Form $ax + b = c$

So far we have not combined the properties of addition, subtraction, multiplication, and division to solve an equation. But that is exactly what we must do to solve equations like $3p + 7 = 31$ and $4x + 9 - 6x + 2 = 17$.

Example 3

Solve $3p + 7 = 31$.

Solution

In this equation, there is a number, 7, being **added** to the term containing the variable, and the variable is being multiplied by a number, 3. **In general, we first eliminate the number being added to or subtracted from the variable.** Then, we eliminate the coefficient.

$$\begin{aligned}
3p + 7 &= 31 \\
3p + 7 - 7 &= 31 - 7 && \text{Subtract 7 from each side.} \\
3p &= 24 && \text{Combine like terms.} \\
\frac{3p}{3} &= \frac{24}{3} && \text{Divide by 3.} \\
p &= 8
\end{aligned}$$

$$\begin{aligned}
\text{Check:} \quad 3p + 7 &= 31 \\
3(8) + 7 &= 31 \\
24 + 7 &= 31 \\
31 &= 31 \quad ✓
\end{aligned}$$

The solution set is $\{8\}$.

You Try 3

Solve $2n + 9 = 15$.

Example 4

Solve $-\frac{6}{5}c - 1 = 13$.

Solution

On the left-hand side, the c is being multiplied by $-\frac{6}{5}$, and 1 is being subtracted from the c-term. To solve the equation, begin by eliminating the number being subtracted from the c-term.

$$\begin{aligned}
-\frac{6}{5}c - 1 &= 13 \\
-\frac{6}{5}c - 1 + 1 &= 13 + 1 && \text{Add 1 to each side.} \\
-\frac{6}{5}c &= 14 && \text{Combine like terms.} \\
-\frac{5}{6} \cdot \left(-\frac{6}{5}c\right) &= -\frac{5}{6} \cdot 14 && \text{Multiply each side by the reciprocal of } -\frac{6}{5}. \\
1c &= -\frac{5}{\cancel{6}_{3}} \cdot \cancel{14}^{7} && \text{Simplify. 14 and 6 each divide by 2.} \\
c &= -\frac{35}{3}
\end{aligned}$$

The check is left to the student. The solution set is $\left\{-\frac{35}{3}\right\}$.

You Try 4

Solve $-\frac{4}{9}z + 3 = -7$.

Example 5

Solve $-8.85 = 2.1y - 5.49$.

Solution

The variable is on the right-hand side of the equation. First, we will add 5.49 to each side, then we will divide by 2.1.

$$\begin{aligned} -8.85 &= 2.1y - 5.49 && \\ -8.85 + 5.49 &= 2.1y - 5.49 + 5.49 && \text{Add 5.49 to each side.} \\ -3.36 &= 2.1y && \text{Combine like terms.} \\ \frac{-3.36}{2.1} &= \frac{2.1y}{2.1} && \text{Divide each side by 2.1.} \\ -1.6 &= y && \text{Simplify.} \end{aligned}$$

Verify that -1.6 is the solution. The solution set is $\{-1.6\}$.

You Try 5

Solve $6.7 = -0.4t - 5.3$.

5. Combine Like Terms on One Side of the Equal Sign, Then Solve

The equations we have solved so far have contained only one term with a variable. So how do we solve an equation like $4x + 9 - 6x + 2 = 17$? *We begin by combining like terms,* then continue as we did in the previous examples.

Example 6

Solve $4x + 9 - 6x + 2 = 17$, and check the solution.

Solution

$$\begin{aligned} 4x + 9 - 6x + 2 &= 17 && \\ -2x + 11 &= 17 && \text{Combine like terms.} \\ -2x + 11 - 11 &= 17 - 11 && \text{Subtract 11 from each side.} \\ -2x &= 6 && \text{Combine like terms.} \\ \frac{-2x}{-2} &= \frac{6}{-2} && \text{Divide by } -2. \\ x &= -3 && \text{Simplify.} \end{aligned}$$

$$\begin{aligned} \text{Check: } 4x + 9 - 6x + 2 &= 17 && \\ 4(-3) + 9 - 6(-3) + 2 &= 17 && \text{Substitute } -3 \text{ for each } x. \\ -12 + 9 + 18 + 2 &= 17 && \\ 17 &= 17 \ \checkmark && \end{aligned}$$

The solution set is $\{-3\}$.

You Try 6

Solve $15 - 7u - 6 + 2u = -1$, and check the solution.

In the next two examples, we will see that sometimes the first step in solving an equation is to use the distributive property to clear parentheses.

Example 7

Solve $2(1 - 3h) - 5(2h + 3) = -21$.

Solution

$$\begin{aligned}
2(1 - 3h) - 5(2h + 3) &= -21 \\
2 - 6h - 10h - 15 &= -21 && \text{Distribute.} \\
-16h - 13 &= -21 && \text{Combine like terms.} \\
-16h - 13 + 13 &= -21 + 13 && \text{Add 13 to each side.} \\
-16h &= -8 && \text{Combine like terms.} \\
\frac{-16h}{-16} &= \frac{-8}{-16} && \text{Divide by } -16. \\
h &= \frac{1}{2} && \text{Simplify.}
\end{aligned}$$

The check is left to the student. The solution set is $\left\{\frac{1}{2}\right\}$.

You Try 7

Solve $-3(4y - 3) + 4(y + 1) = 15$.

Example 8

Solve $\frac{1}{2}(3b + 8) + \frac{3}{4} = -\frac{1}{2}$.

Solution

$$\begin{aligned}
\frac{1}{2}(3b + 8) + \frac{3}{4} &= -\frac{1}{2} \\
\frac{3}{2}b + 4 + \frac{3}{4} &= -\frac{1}{2} && \text{Distribute.} \\
\frac{3}{2}b + \frac{16}{4} + \frac{3}{4} &= -\frac{1}{2} && \text{Rewrite 4 as } \frac{16}{4}. \\
\frac{3}{2}b + \frac{19}{4} &= -\frac{1}{2} && \text{Combine like terms.} \\
\frac{3}{2}b + \frac{19}{4} - \frac{19}{4} &= -\frac{1}{2} - \frac{19}{4} && \text{Subtract } \frac{19}{4} \text{ from each side.} \\
\frac{3}{2}b &= -\frac{21}{4} && \text{Get a common denominator and subtract.} \\
\frac{2}{3} \cdot \frac{3}{2}b &= \frac{2}{3} \cdot \left(-\frac{21}{4}\right) && \text{Multiply both sides by the reciprocal of } \frac{3}{2}. \\
b &= \frac{\cancel{2}^{1}}{\cancel{3}_{1}} \cdot \left(-\frac{\cancel{21}^{7}}{\cancel{4}_{2}}\right) && \text{Perform the multiplication.} \\
b &= -\frac{7}{2}
\end{aligned}$$

The check is left to the student. The solution set is $\left\{-\frac{7}{2}\right\}$.

You Try 8

Solve $\frac{1}{3}(2m - 1) + \frac{5}{9} = \frac{4}{3}$.

In Section 3.2, we will learn another way to solve equations containing several fractions like the one in Example 8.

Answers to You Try Exercises

1) a) $\{17\}$ b) $\{-3.4\}$ c) $\{5\}$ 2) a) $\{-1.2\}$ b) $\{7\}$ c) $\{24\}$ d) $\{36\}$ 3) $\{3\}$

4) $\left\{\frac{45}{2}\right\}$ 5) $\{-30\}$ 6) $\{2\}$ 7) $\left\{-\frac{1}{4}\right\}$ 8) $\left\{\frac{5}{3}\right\}$

3.1 Exercises

Objective 1: Define a Linear Equation in One Variable

Identify each as an expression or an equation.

1) $7t - 2 = 11$

2) $\frac{3}{4}k + 5(k - 6) = 2$

3) $8 - 10p + 4p + 5$

4) $9(2z - 7) + 3z$

5) Can we solve $3(c + 2) + 5(2c - 5)$? Why or why not?

6) Can we solve $3(c + 2) + 5(2c - 5) = -6$? Why or why not?

7) Which of the following are linear equations in one variable?

a) $y^2 + 8y + 15 = 0$

b) $\frac{1}{2}w - 5(3w + 1) = 6$

c) $8m - 7 + 2m + 1$

d) $0.3z + 0.2 = 1.5$

8) Explain how to check the solution of an equation.

Determine whether the given value is a solution to the equation.

9) $a - 4 = -9;\ a = 5$

10) $2d + 1 = 13;\ d = -6$

11) $-8p = 12;\ p = -\frac{3}{2}$

12) $15 = 21m;\ m = \frac{5}{7}$

13) $10 - 2(3y - 1) + y = -8;\ y = 4$

14) $-2w + 9 + w - 11 = -1;\ w = -3$

Objective 2: Use the Addition and Subtraction Properties of Equality

Solve each equation, and check the solution.

15) $r - 6 = 11$

16) $c + 2 = -5$

17) $b + 10 = 4$

18) $x - 3 = 9$

19) $-16 = k - 12$

20) $8 = t + 1$

21) $a + \frac{5}{8} = \frac{1}{2}$

22) $w - \frac{3}{4} = -\frac{1}{6}$

23) $13.1 = v + 7.2$

24) $-8.3 = m - 5.6$

25) Write an equation that can be solved with the subtraction property of equality and that has a solution set of $\{-9\}$.

26) Write an equation that can be solved with the addition property of equality and that has a solution set of $\{5\}$.

Objective 3: Use the Multiplication and Division Properties of Equality

Solve each equation, and check the solution.

27) $3y = 30$

28) $2n = 8$

29) $-5z = 35$

30) $-8b = -24$

31) $-56 = -7v$

32) $-54 = 6m$

33) $\frac{a}{4} = 12$

34) $\frac{w}{5} = 4$

35) $-6 = \frac{k}{8}$

36) $30 = -\frac{x}{2}$

37) $\frac{2}{3}g = -10$

38) $\frac{7}{4}r = 42$

39) $-\frac{5}{3}d = -30$

40) $-\frac{1}{8}h = 3$

41) $\frac{11}{15} = \frac{1}{3}y$

42) $-\frac{5}{6} = -\frac{4}{9}x$

43) $0.5q = 6$

44) $0.3t = 3$

45) $-w = -7$

46) $-p = \frac{6}{7}$

47) $-12d = 0$

48) $4f = 0$

Objective 4: Solve Equations of the Form $ax + b = c$

Solve each equation, and check the solution.

49) $3x - 7 = 17$

50) $2y - 5 = 3$

51) $7c + 4 = 18$

52) $5g + 19 = 4$

VIDEO 53) $8d - 15 = -15$

54) $-3r + 8 = 8$

55) $-11 = 5t - 9$

56) $4 = 7j - 8$

57) $10 = 3 - 7y$

58) $-6 = 9 - 3p$

59) $\frac{4}{9}w - 11 = 1$

60) $\frac{5}{3}a + 6 = 41$

61) $\frac{10}{7}m + 3 = 1$

62) $\frac{9}{10}x - 4 = 11$

63) $\frac{1}{2}d + 7 = 12$

64) $\frac{1}{4}y - 3 = -9$

65) $2 - \frac{5}{6}t = -2$

66) $5 - \frac{3}{4}h = -1$

67) $-\frac{1}{6}z + \frac{1}{2} = \frac{3}{4}$

68) $\frac{3}{5} - \frac{2}{3}k = 1$

69) $5 - 0.4p = 2.6$

70) $1.8 = 1.2n - 7.8$

71) $4.3a + 1.98 = -14.36$

72) $14.74 = -20.6 - 5.7u$

Objective 5: Combine Like Terms on One Side of the Equal Sign, Then Solve

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

Solve each equation.

73) $3x + 7 + 5x + 4 = 27$

$8x + 11 = 27$	________
________	Subtraction property of equality
$8x = 16$	________
________	Division property of equality
________	Simplify.

The solution set is ___

74) $5 - 2(3k + 1) + 2k = 23$

________	Distribute.
________	Combine like terms.
$-4k + 3 - 3 = 23 - 3$	________
$-4k = 20$	________
$\frac{-4k}{-4} = \frac{20}{-4}$	________
________	Simplify.

The solution set is ___.

Solve each equation, and check the solution.

75) $10v + 9 - 2v + 16 = 1$

76) $-8g - 7 + 6g + 1 = 20$

77) $5 - 3m + 9m + 10 - 7m = -4$

78) $t - 12 - 13t - 5 + 2t = -7$

VIDEO 79) $5 = -12p + 7 + 4p - 12$

80) $12 = 9y + 11 - 3y - 7$

81) $2(5x + 3) - 3x + 4 = -11$

82) $6(2c - 1) + 3 - 7c = 42$

83) $-12 = 7(2a - 3) - (8a - 9)$

84) $20 = 5r - 3 + 2(9 - 3r)$

85) $\frac{1}{3}(3w + 4) - \frac{2}{3} = -\frac{1}{3}$

86) $\frac{3}{4}(2r - 5) + \frac{1}{2} = \frac{5}{4}$

87) $\frac{1}{2}(c - 2) + \frac{1}{4}(2c + 1) = \frac{5}{4}$

88) $\frac{2}{3}(m + 3) - \frac{4}{15}(3m + 7) = \frac{4}{5}$

89) $\frac{4}{3}(t + 1) - \frac{1}{6}(4t - 3) = 2$

90) $\frac{1}{4}(3x - 2) - \frac{1}{2}(x - 1) = -\frac{1}{7}$

Section 3.2 Solving Linear Equations Part II

Objectives

1. Solve Equations Containing Variables on Both Sides of the Equal Sign
2. Solve Equations Containing Fractions or Decimals
3. Solve Equations with No Solution or an Infinite Number of Solutions
4. Use the Five Steps for Solving Applied Problems

In Section 3.1, we learned how to solve equations like $n - 3 = 18$, $17 = \frac{4}{5}t + 9$, and $2(x + 5) - 9x = -11$.

Did you notice that all of these equations contained variables on only one side of the equal sign? In this section, we will discuss how to solve equations containing variables on both sides of the equal sign. We can use the following steps to solve most linear equations.

Procedure How to Solve a Linear Equation

Step 1: **Clear parentheses** and **combine like terms** on each side of the equation.

Step 2: **Get the variable on one side of the equal sign and the constant on the other side of the equal sign** (isolate the variable) using the addition or subtraction property of equality.

Step 3: **Solve for the variable** using the multiplication or division property of equality.

Step 4: **Check the solution** in the original equation.

1. Solve Equations Containing Variables on Both Sides of the Equal Sign

To solve an equation such as $3y - 11 + 7y = 6y + 9$, our goal is the same as it was when we solved equations in the previous section: get the variables on one side of the equal sign and the constants on the other side. Let's use the steps.

Example 1

Solve $3y - 11 + 7y = 6y + 9$.

Solution

Step 1: Combine like terms on the left side of the equal sign.

$$\begin{aligned} 3y - 11 + 7y &= 6y + 9 \\ 10y - 11 &= 6y + 9 && \text{Combine like terms.} \end{aligned}$$

Step 2: Isolate the variable using the addition and subtraction properties of equality. Combine like terms so that there is a single variable term on one side of the equation and a constant on the other side.

$$\begin{aligned} 10y - 6y - 11 &= 6y - 6y + 9 && \text{Subtract } 6y \text{ from each side.} \\ 4y - 11 &= 9 && \text{Combine like terms.} \\ 4y - 11 + 11 &= 9 + 11 && \text{Add 11 to each side.} \\ 4y &= 20 && \text{Combine like terms.} \end{aligned}$$

Step 3: Solve for y using the division property of equality.

$$\begin{aligned} \frac{4y}{4} &= \frac{20}{4} && \text{Divide each side by 4.} \\ y &= 5 && \text{Simplify.} \end{aligned}$$

Step 4: Check:

$$\begin{aligned} 3y - 11 + 7y &= 6y + 9 \\ 3(5) - 11 + 7(5) &= 6(5) + 9 \\ 15 - 11 + 35 &= 30 + 9 \\ 39 &= 39 \checkmark \end{aligned}$$

The solution set is $\{5\}$. ■

You Try 1

Solve $-3k + 4 = 8k - 15 - 6k - 11$.

Example 2

Solve $9t + 4 - (7t - 2) = t + 6(t + 1)$.

Solution

Step 1: Clear the parentheses and combine like terms.

$$\begin{aligned} 9t + 4 - (7t - 2) &= t + 6(t + 1) \\ 9t + 4 - 7t + 2 &= t + 6t + 6 && \text{Distribute.} \\ 2t + 6 &= 7t + 6 && \text{Combine like terms.} \end{aligned}$$

Step 2: Isolate the variable.

$$\begin{aligned} 2t - 7t + 6 &= 7t - 7t + 6 && \text{Subtract } 7t \text{ from each side.} \\ -5t + 6 &= 6 && \text{Combine like terms.} \\ -5t + 6 - 6 &= 6 - 6 && \text{Subtract 6 from each side.} \\ -5t &= 0 && \text{Combine like terms.} \end{aligned}$$

Step 3: Solve for t using the division property of equality.

$$\begin{aligned} \frac{-5t}{-5} &= \frac{0}{-5} && \text{Divide each side by } -5. \\ t &= 0 && \text{Simplify.} \end{aligned}$$

Step 4: Check:

$$\begin{aligned} 9t + 4 - (7t - 2) &= t + 6(t + 1) \\ 9(0) + 4 - [7(0) - 2] &= 0 + 6[(0) + 1] \\ 0 + 4 - (0 - 2) &= 0 + 6(1) \\ 4 - (-2) &= 0 + 6 \\ 6 &= 6 \quad ✓ \end{aligned}$$

The solution set is $\{0\}$.

You Try 2

Solve $5 + 3(a + 4) = 7a - (9 - 10a) + 4$.

2. Solve Equations Containing Fractions or Decimals

To solve $\frac{1}{2}(3b + 8) + \frac{3}{4} = -\frac{1}{2}$ in Section 3.1, we began by using the distributive property to clear the parentheses, and we worked with the fractions throughout the solving process. But, there is another way we can solve equations containing several fractions. Before applying the steps for solving a linear equation, we can eliminate the fractions from the equation.

Procedure Eliminating Fractions from an Equation

To eliminate the fractions, determine the least common denominator (LCD) for all the fractions in the equation. Then, multiply both sides of the equation by the LCD.

Let's solve the equation above using this new approach.

Example 3

Solve $\frac{1}{2}(3b + 8) + \frac{3}{4} = -\frac{1}{2}$.

Solution

The least common denominator of all the fractions in the equation is 4. Multiply both sides of the equation by 4 to eliminate the fractions.

$$4\left[\frac{1}{2}(3b + 8) + \frac{3}{4}\right] = 4\left(-\frac{1}{2}\right)$$

Step 1: Distribute the 4, clear the parentheses, and combine like terms.

$$\begin{aligned} 4 \cdot \frac{1}{2}(3b + 8) + 4 \cdot \frac{3}{4} &= -2 && \text{Distribute.} \\ 2(3b + 8) + 3 &= -2 && \text{Multiply.} \\ 6b + 16 + 3 &= -2 && \text{Distribute.} \\ 6b + 19 &= -2 && \text{Combine like terms.} \end{aligned}$$

Step 2: Isolate the variable.

$$\begin{aligned} 6b + 19 - 19 &= -2 - 19 && \text{Subtract 19 from each side.} \\ 6b &= -21 && \text{Combine like terms.} \end{aligned}$$

Step 3: Solve for b using the division property of equality.

$$\begin{aligned} \frac{6b}{6} &= \frac{-21}{6} && \text{Divide each side by 6.} \\ b &= -\frac{7}{2} && \text{Simplify.} \end{aligned}$$

Step 4: The check is left to the student. The solution set is $\left\{-\frac{7}{2}\right\}$. This is the same as the result we obtained in Section 3.1, Example 8.

You Try 3

Solve $\frac{1}{6}x + \frac{5}{4} = \frac{1}{2}x - \frac{5}{12}$.

Just as we can eliminate the fractions from an equation to make it easier to solve, we can eliminate decimals from an equation before applying the four-step equation-solving process.

Procedure Eliminating Decimals from an Equation

To eliminate the decimals from an equation, multiply both sides of the equation by the smallest power of 10 that will eliminate all decimals from the problem.

Example 4

Solve $0.05a + 0.2(a + 3) = 0.1$

Solution

We want to eliminate the decimals. The number containing a decimal place farthest to the right is 0.05. The 5 is in the hundredths place. Therefore, multiply both sides of the equation by 100 to eliminate all decimals in the equation.

$$100[0.05a + 0.2(a + 3)] = 100(0.1)$$

Step 1: Distribute the 100, clear the parentheses, and combine like terms.

$$\begin{aligned} 100[0.05a + 0.2(a + 3)] &= 100(0.1) & \\ 100 \cdot (0.05a) + 100[0.2(a + 3)] &= 10 & \text{Distribute.} \\ 5a + 20(a + 3) &= 10 & \text{Multiply.} \\ 5a + 20a + 60 &= 10 & \text{Distribute.} \\ 25a + 60 &= 10 & \text{Combine like terms.} \end{aligned}$$

Step 2: Isolate the variable.

$$\begin{aligned} 25a + 60 - 60 &= 10 - 60 & \text{Subtract 60 from each side.} \\ 25a &= -50 & \text{Combine like terms.} \end{aligned}$$

Step 3: Solve for a using the division property of equality.

$$\begin{aligned} \frac{25a}{25} &= \frac{-50}{25} & \text{Divide each side by 25.} \\ a &= -2 & \text{Simplify.} \end{aligned}$$

Step 4: The check is left to the student. The solution set is $\{-2\}$.

You Try 4

Solve $0.08k - 0.2(k + 5) = -0.1$.

3. Solve Equations with No Solution or an Infinite Number of Solutions

Does every equation have a solution? Consider the next example.

Example 5

Solve $9h + 2 = 6h + 3(h - 5)$.

Solution

$$\begin{aligned} 9h + 2 &= 6h + 3(h - 5) & \\ 9h + 2 &= 6h + 3h - 15 & \text{Distribute.} \\ 9h + 2 &= 9h - 15 & \text{Combine like terms.} \\ 9h - 9h + 2 &= 9h - 9h - 15 & \text{Subtract } 9h \text{ from both sides.} \\ 2 &= -15 & \text{False} \end{aligned}$$

Notice that the variable has "dropped out." Is $2 = -15$ a true statement? No! This means that there is no value for h that will make the statement true. Therefore, the equation has *no solution*. We can say that the solution set is the **empty set**, or **null set**, and it is denoted by $\varnothing$.

We have seen that a linear equation may have one solution or no solution. There is a third possibility—a linear equation may have an infinite number of solutions.

Example 6

Solve $p - 3p + 8 = 8 - 2p$.

Solution

$$\begin{aligned} p - 3p + 8 &= 8 - 2p & \\ -2p + 8 &= 8 - 2p & \text{Combine like terms.} \\ -2p + 2p + 8 &= 8 - 2p + 2p & \text{Add } 2p \text{ to each side.} \\ 8 &= 8 & \text{True} \end{aligned}$$

Here, the variable has "dropped out," and we are left with an equation, $8 = 8$, that is true. This means that any real number we substitute for p will make the original equation true. Therefore, this equation has an *infinite number of solutions*. The solution set is **{all real numbers}**. ■

Summary Outcomes When Solving Linear Equations

There are three possible outcomes when solving a linear equation. The equation may have

1) **one solution.** Solution set: {a real number}. An equation that is true for some values and not for others is called a **conditional equation**.

or

2) **no solution.** In this case, the variable will drop out, and there will be a false statement such as $2 = -15$. Solution set: $\varnothing$. An equation that has no solution is called a **contradiction**.

or

3) **an infinite number of solutions.** In this case, the variable will drop out, and there will be a true statement such as $8 = 8$. Solution set: {all real numbers}. An equation that has all real numbers as its solution set is called an **identity**.

You Try 5

Solve.

a) $6 + 5x - 4 = 3x + 2(1 + x)$ b) $3x - 4x + 9 = 5 - x$

4. Use the Five Steps for Solving Applied Problems

Mathematical equations can be used to describe many situations in the real world. To do this, we must learn how to translate information presented in English into an algebraic equation. We will begin slowly and work our way up to more challenging problems. Yes, it may be difficult at first, but with patience and persistence, you can do it!

Although no single method will work for solving all applied problems, the following approach is suggested to help in the problem-solving process.

Procedure Steps for Solving Applied Problems

Step 1: **Read** the problem carefully, more than once if necessary, until you understand it. Draw a picture, if applicable. Identify what you are being asked to find.

Step 2: **Choose a variable** to represent an unknown quantity. If there are any other unknowns, define them in terms of the variable.

Step 3: **Translate** the problem from English into an equation using the chosen variable. Some suggestions for doing so are:

- Restate the problem in your own words.
- Read and think of the problem in "small parts."
- Make a chart to separate these "small parts" of the problem to help you translate into mathematical terms.
- Write an equation in English, then translate it into an algebraic equation.

Step 4: **Solve** the equation.

Step 5: **Check** the answer in the original problem, and **interpret** the solution as it relates to the problem. Be sure your answer makes sense in the context of the problem.

Example 7

Write the following statement as an equation, and find the number.

Nine more than twice a number is fifteen. Find the number.

Solution

Step 1: **Read** the problem carefully. We must find an unknown number.

Step 2: **Choose a variable** to represent the unknown.

Let $x =$ the number.

Step 3: **Translate** the information that appears in English into an algebraic equation by rereading the problem slowly and "in parts."

Statement:	Nine more than	twice a number	is	fifteen
Meaning:	Add 9 to	2 times the unknown	equals	15
	↓	↓	↓	↓
Equation:	$9 +$	$2x$	$=$	15

The equation is $9 + 2x = 15$.

Step 4: **Solve** the equation.

$$
\begin{aligned}
9 + 2x &= 15 \\
9 - 9 + 2x &= 15 - 9 && \text{Subtract 9 from each side.} \\
2x &= 6 && \text{Combine like terms.} \\
x &= 3 && \text{Divide each side by 2.}
\end{aligned}
$$

Step 5: **Check** the answer. Does the answer make sense? Nine more than twice three is $9 + 2(3) = 15$. The answer is correct. The number is 3.

You Try 6

Write the following statement as an equation, and find the number.

Three more than twice a number is twenty-nine.

Sometimes, dealing with subtraction in an application can be confusing. So, let's look at an arithmetic problem first.

Example 8

What is two less than seven?

Solution

To solve this problem, do we subtract $7 - 2$ or $2 - 7$? "Two less than seven" is written as $7 - 2$, and $7 - 2 = 5$. Five is two less than seven. To get the correct answer, the 2 is *subtracted from* the 7.

Keep this in mind as you read the next problem.

Example 9

Write the following statement as an equation, and find the number.

Five less than three times a number is the same as the number increased by seven. Find the number.

Solution

Step 1: **Read** the problem carefully. We must find an unknown number.

Step 2: **Choose a variable** to represent the unknown.

Let $x =$ the number.

Step 3: **Translate** the information that appears in English into an algebraic equation by rereading the problem slowly and "in parts."

Statement:	Five less than	three times a number	is the same as	the number	increased by 7
Meaning:	Subtract 5 from	3 times the unknown	equals	the unknown	add 7
Equation:	$3x$	-5	$=$	x	$+\ 7$

The equation is $3x - 5 = x + 7$.

Step 4: **Solve** the equation.

$$
\begin{aligned}
3x - 5 &= x + 7 \\
3x - x - 5 &= x - x + 7 && \text{Subtract } x \text{ from each side.} \\
2x - 5 &= 7 && \text{Combine like terms.} \\
2x - 5 + 5 &= 7 + 5 && \text{Add 5 to each side.} \\
2x &= 12 && \text{Combine like terms.} \\
x &= 6 && \text{Divide each side by 2.}
\end{aligned}
$$

Step 5: **Check** the answer. Does it make sense? Five less than three times 6 is $3(6) - 5 = 13$. The number increased by seven is $6 + 7 = 13$. The answer is correct. The number is 6.

You Try 7

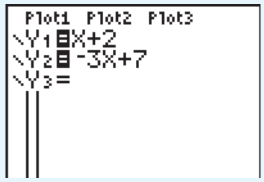
Write the following statement as an equation, and find the number.

Three less than five times a number is the same as the number increased by thirteen.

Using Technology

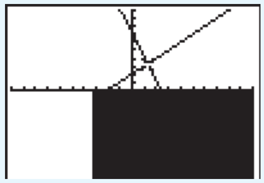
We can use a graphing calculator to solve a linear equation in one variable. First, enter the left side of the equation in Y_1 and the right side of the equation in Y_2. Then graph the equations. The x-coordinate of the point of intersection is the solution to the equation.

We will solve $x + 2 = -3x + 7$ algebraically and by using a graphing calculator, and then compare the results. First, use algebra to solve $x + 2 = -3x + 7$. You should get $\frac{5}{4}$.

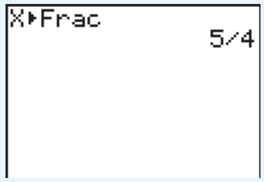


Next, use a graphing calculator to solve $x + 2 = -3x + 7$.

1. Enter $x + 2$ in Y_1 by pressing Y= and entering $x + 2$ to the right of $\backslash Y_1 =$. Then Press ENTER.
2. Enter $-3x + 7$ in Y_2 by pressing the Y= key and entering $-3x + 7$ to the right of $\backslash Y_2 =$. Press ENTER.
3. Press ZOOM and select 6:ZStandard to graph the equations.
4. To find the intersection point, press 2nd TRACE and select 5:intersect. Press ENTER three times. The x-coordinate of the intersection point is shown on the left side of the screen, and is stored in the variable x.



5. Return to the home screen by pressing 2nd MODE. Press X,T, Θ, n ENTER to display the solution. Since the result in this case is a decimal value, we can convert it to a fraction by pressing X,T, Θ, n MATH, selecting Frac, then pressing ENTER.



The calculator then gives us the solution set $\left\{\frac{5}{4}\right\}$.

Solve each equation algebraically; then verify your answer using a graphing calculator.

1. $x + 6 = -2x - 3$
2. $2x + 3 = -x - 4$
3. $\frac{5}{6}x + \frac{1}{2} = \frac{1}{6}x - \frac{3}{4}$
4. $0.3x - 1 = -0.2x - 5$
5. $3x - 7 = -x + 5$
6. $6x - 7 = 5$

Answers to You Try Exercises

1) $\{6\}$ 2) $\left\{\frac{11}{7}\right\}$ 3) $\{5\}$ 4) $\{-7.5\}$ 5) a) {all real numbers} b) $\varnothing$
6) $3 + 2x = 29$; 13 7) $5x - 3 = x + 13$; 4

Answers to Technology Exercises

1) $\{-3\}$ 2) $\left\{-\frac{7}{3}\right\}$ 3) $\left\{-\frac{15}{8}\right\}$ 4) $\{-8\}$ 5) $\{3\}$ 6) $\{2\}$

3.2 Exercises

Objective 1: Solve Equations Containing Variables on Both Sides of the Equal Sign

1) Explain, in your own words, the steps for solving a linear equation.

2) What is the first step for solving $8n + 3 + 2n - 9 = 13 - 5n + 11$? Do not solve the equation.

Solve each equation.

3) $2y + 7 = 5y - 2$

4) $8n - 21 = 3n - 1$

5) $6 - 7p = 2p + 33$

6) $z + 19 = 5 - z$

7) $-8x + 6 - 2x + 11 = 3 + 3x - 7x$

8) $10 - 13a + 2a - 16 = -5 + 7a + 11$

9) $18 - h + 5h - 11 = 9h + 19 - 3h$

10) $4m - 1 - 6m + 7 = 11m + 3 - 10m$

11) $4(2t + 5) - 7 = 5(t + 5)$

12) $3(2m + 10) = 6(m + 4) - 8m$

13) $2(1 - 8c) = 5 - 3(6c + 1) + 4c$

14) $13u + 6 - 5(2u - 3) = 1 + 4(u + 5)$

15) $2(6d + 5) = 16 - (7d - 4) + 11d$

16) $-3(4r + 9) + 2(3r + 8) = r - (9r - 5)$

Objective 2: Solve Equations Containing Fractions or Decimals

17) If an equation contains fractions, what is the first step you can perform to make it easier to solve?

18) If an equation contains decimals, what is the first step you can perform to make it easier to solve?

19) How can you eliminate the fractions from the equation $\frac{3}{8}x - \frac{1}{2} = \frac{1}{8}x + \frac{3}{4}$?

20) How can you eliminate the decimals from the equation $0.02n + 0.1(n - 3) = 0.06$?

Solve each equation.

21) $\frac{3}{8}x - \frac{1}{2} = \frac{1}{8}x + \frac{3}{4}$

22) $\frac{3}{4}n + \frac{1}{2} = \frac{1}{2}n + \frac{1}{4}$

23) $\frac{2}{3}d - 1 = \frac{1}{5}d + \frac{2}{5}$

24) $\frac{1}{5}c + \frac{2}{7} = 2 - \frac{1}{7}c$

VIDEO 25) $\frac{1}{3} + \frac{1}{9}(k + 5) - \frac{k}{4} = 2$

26) $\frac{1}{2} = \frac{2}{9}(3x - 2) - \frac{x}{9} - \frac{x}{6}$

27) $\frac{3}{4}(y + 7) + \frac{1}{2}(3y - 5) = \frac{9}{4}(2y - 1)$

28) $\frac{5}{8}(2w + 3) + \frac{5}{4}w = \frac{3}{4}(4w + 1)$

29) $\frac{2}{3}(3h - 5) + 1 = \frac{3}{2}(h - 2) + \frac{1}{6}h$

30) $\frac{1}{2}(4r + 1) - r = \frac{2}{5}(2r - 3) + \frac{3}{2}$

31) $0.06d + 0.13 = 0.31$

32) $0.09x - 0.14 = 0.4$

33) $0.04n - 0.05(n + 2) = 0.1$

34) $0.07t + 0.02(3t + 8) = -0.1$

35) $0.35a - 0.1a = 0.03(5a + 4)$

36) $0.12(5q - 1) - q = 0.15(7 - 2q)$

37) $27 = 0.04y + 0.03(y + 200)$

38) $98 = 0.06r + 0.1(r - 300)$

39) $0.2(12) + 0.08z = 0.12(z + 12)$

40) $0.1x + 0.15(8 - x) = 0.125(8)$

Objective 3: Solve Equations with No Solution or an Infinite Number of Solutions

41) How do you know that an equation has no solution.

42) How do you know that the solution set of an equation is {all real numbers}?

Determine whether each of the following equations has a solution set of {all real numbers} or has no solution, $\varnothing$.

VIDEO 43) $-9r + 4r - 11 + 2 = 3r + 7 - 8r + 9$

44) $3(4b - 7) + 8 = 6(2b + 5)$

45) $j - 15j + 8 = -3(4j - 3) - 2j - 1$

46) $n - 16 + 10n + 4 = 2(7n - 6) - 3n$

VIDEO 47) $8(3t + 4) = 10t - 3 + 7(2t + 5)$

48) $2(9z - 1) + 7 = 10z - 14 + 8z + 2$

49) $\frac{5}{6}k - \frac{2}{3} = \frac{1}{6}(5k - 4) + \frac{1}{2}$

50) $0.4y + 0.3(20 - y) = 0.1y + 6$

Mixed Exercises: Objectives 1–3

Solve each equation.

51) $7(2q + 3) = 6 + 3(q + 5)$

52) $\frac{5}{3}w + \frac{2}{5} = w - \frac{7}{3}$

53) $0.16h + 0.4(2000) = 0.22(2000 + h)$

54) $5x + 12 = -11x + 8(2x - 9)$

55) $t + 18 = 3(5 - t) + 4t + 3$

56) $9 - (7p - 2) + 2p = 4(p + 3) + 5$

57) $\frac{1}{2}(2r + 9) = 1 + \frac{1}{3}(r + 12)$

58) $0.3m + 0.18(5000 - m) = 0.21(5000)$

59) $2d + 7 = -4d + 3(2d - 5)$

60) $\frac{4}{9} + \frac{2}{3}(c - 1) + \frac{5}{9}c = \frac{2}{9}(5c + 3)$

Objective 4: Use the Five Steps for Solving Applied Problems

61) What are the five steps for solving applied problems?

62) If you are solving an applied problem in which you have to find the length of a side of a rectangle, would a solution of -12 be reasonable? Explain your answer.

Write each statement as an equation, and find the number.

63) Twelve more than a number is five.

64) Fifteen more than a number is nineteen.

65) Nine less than a number is twelve.

66) Fourteen less than a number is three.

67) Five more than twice a number is seventeen.

68) Seven more than twice a number is twenty-three.

69) Eighteen more than twice a number is eight.

70) Eleven more than twice a number is thirteen.

VIDEO 71) Three times a number decreased by eight is forty.

72) Four times a number decreased by five is forty-three.

73) Three-fourths of a number is thirty-three.

74) Two-thirds of a number is twenty-six.

75) Nine less than half a number is three.

76) Two less than one-fourth of a number is three.

77) Six more than a number is eight.

78) Fifteen more than a number is nine.

79) Three less than twice a number is the same as the number increased by eight.

80) Twelve less than five times a number is the same as the number increased by sixteen.

81) Ten more than one-third of a number is the same as the number decreased by two.

82) A number decreased by nine is the same as seven more than half the number.

83) If forty-five is subtracted from a number, the result is the same as the number divided by four.

84) If twenty-four is subtracted from a number, the result is the same as the number divided by nine.

85) If two-thirds of a number is added to the number, the result is twenty-five.

86) If three-eighths of a number is added to twice the number, the result is thirty-eight.

87) When a number is decreased by twice the number, the result is thirteen.

88) When three times a number is subtracted from the number, the result is ten.

Section 3.3 Applications of Linear Equations

Objectives

1. Solve Problems Involving General Quantities
2. Solve Problems Involving Lengths
3. Solve Consecutive Integer Problems

In the previous section, we learned the five steps for solving applied problems and used this procedure to solve problems involving unknown numbers. Now we will apply this problem-solving technique to other types of applications.

1. Solve Problems Involving General Quantities

Example 1

Write an equation and solve.

Swimmers Michael Phelps and Natalie Coughlin both competed in the 2004 Olympics in Athens and in the 2008 Olympics in Beijing, where they won a total of 27 medals. Phelps won five more medals than Coughlin. How many Olympic medals has each athlete won? (http://swimming.teamusa.org)

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the number of medals each Olympian won.

Step 2: **Choose a variable** to represent an unknown, and define the other unknown in terms of this variable.

In the statement "Phelps won five more medals than Coughlin," the number of medals that Michael Phelps won is expressed *in terms of* the number medals won by Natalie Coughlin. Therefore, let

$x =$ the number of medals Coughlin won

Define the other unknown (the number of medals that Michael Phelps won) in terms of x. The statement "Phelps won two more medals than Coughlin" means

number of Coughlin's medals $+ 5 =$ number of Phelps' medals

$x + 5 =$ number of Phelps' medals

Step 3: **Translate** the information that appears in English into an algebraic equation. One approach is to restate the problem in your own words.

Since these two athletes won a total of 27 medals, we can think of the situation in this problem as:

The number of medals Coughlin won plus the number of medals Phelps won is 27.

Let's write this as an equation.

Statement:	Number of medals Coughlin won	plus	Number of medals Phelps won	is	27
	↓	↓	↓	↓	↓
Equation:	x	$+$	$(x + 5)$	$=$	27

The equation is $x + (x + 5) = 27$.

Step 4: **Solve** the equation.

$$x + (x + 5) = 27$$
$$2x + 5 = 27$$
$$2x + 5 - 5 = 27 - 5 \qquad \text{Subtract 5 from each side.}$$
$$2x = 22 \qquad \text{Combine like terms.}$$
$$\frac{2x}{2} = \frac{22}{2} \qquad \text{Divide each side by 2.}$$
$$x = 11 \qquad \text{Simplify.}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

Since x represents the number of medals that Natalie Coughlin won, she won 11 medals.

The expression $x + 5$ represents the number of medals Michael Phelps won, so he won $x + 5 = 11 + 5 = 16$ medals.

The answer makes sense because the total number of medals they won was $11 + 16 = 27$. ■

You Try 1

Write an equation and solve.

An employee at a cellular phone store is doing inventory. The store has 23 more conventional cell phones in stock than smart phones. If the store has a total of 73 phones, how many of each type of phone is in stock?

Example 2

Write an equation and solve.

Nick has half as many songs on his iPod as Mariah. Together they have a total of 4887 songs. How many songs does each of them have?

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the number of songs on Nick's iPod and the number on Mariah's iPod.

Step 2: **Choose a variable** to represent an unknown, and define the other unknown in terms of this variable.

In the sentence "Nick has half as many songs on his iPod as Mariah," the number of songs Nick has is expressed *in terms of* the number of songs Mariah has. Therefore, let

$$x = \text{the number of songs on Mariah's iPod}$$

Define the other unknown in terms of x.

$$\frac{1}{2}x = \text{the number of songs on Nick's iPod}$$

Step 3: **Translate** the information that appears in English into an algebraic equation. One approach is to restate the problem in your own words.

Since Mariah and Nick have a total of 4887 songs, we can think of the situation in this problem as:

The number of Mariah's songs plus the number of Nick's songs equals 4887.

Let's write this as an equation.

Statement:	Number of Mariah's songs	plus	Number of Nick's songs	equals	4887
	↓	↓	↓	↓	↓
Equation:	x	$+$	$\frac{1}{2}x$	$=$	4887

The equation is $x + \frac{1}{2}x = 4887$.

Step 4: **Solve** the equation.

$$\begin{aligned} x + \frac{1}{2}x &= 4887 \\ \frac{3}{2}x &= 4887 && \text{Combine like terms.} \\ \frac{2}{3} \cdot \frac{3}{2}x &= \frac{2}{3} \cdot 4887 && \text{Multiply by the reciprocal of } \tfrac{3}{2}. \\ x &= 3258 && \text{Multiply.} \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

Mariah has 3258 songs on her iPod.

The expression $\frac{1}{2}x$ represents the number of songs on Nick's iPod, so there are $\frac{1}{2}(3258) = 1629$ songs on Nick's iPod.

The answer makes sense because the total number of songs on their iPods is $3258 + 1629 = 4887$ songs. ■

You Try 2

Write an equation and solve.

Terrance and Janay are in college. Terrance has earned twice as many credits as Janay. How many credits does each student have if together they have earned 51 semester hours?

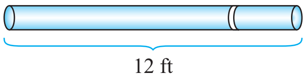
2. Solve Problems Involving Lengths

Example 3

Write an equation and solve.

A plumber has a section of PVC pipe that is 12 ft long. He needs to cut it into two pieces so that one piece is 2 ft shorter than the other. How long will each piece be?

Solution



Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the length of each of two pieces of pipe.

A picture will be very helpful in this problem.

Step 2: **Choose a variable** to represent an unknown, and define the other unknown in terms of this variable.

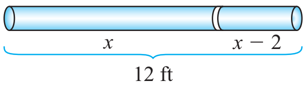
One piece of pipe must be 2 ft shorter than the other piece. Therefore, let

$x =$ the length of one piece

Define the other unknown in terms of x.

$x - 2 =$ the length of the second piece

Step 3: **Translate** the information that appears in English into an algebraic equation. Let's label the picture with the expressions representing the unknowns and then restate the problem in our own words.



From the picture we can see that the

length of one piece plus the length of the second piece equals 12 ft.

Let's write this as an equation.

Statement:	Length of one piece	plus	Length of second piece	equals	12 ft
	↓	↓	↓	↓	↓
Equation:	x	$+$	$(x - 2)$	$=$	12

The equation is $x + (x - 2) = 12$.

Step 4: **Solve** the equation.

$$
\begin{aligned}
x + (x - 2) &= 12 \\
2x - 2 &= 12 \\
2x - 2 + 2 &= 12 + 2 && \text{Add 2 to each side.} \\
2x &= 14 && \text{Combine like terms.} \\
\frac{2x}{2} &= \frac{14}{2} && \text{Divide each side by 2.} \\
x &= 7 && \text{Simplify.}
\end{aligned}
$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

One piece of pipe is 7 ft long.

The expression $x - 2$ represents the length of the other piece of pipe, so the length of the other piece is $x - 2 = 7 - 2 = 5$ ft.

The answer makes sense because the length of the original pipe was 7 ft + 5 ft = 12 ft. ■

You Try 3

Write the following statement as an equation, and find the number.

An electrician has a 20-ft wire. He needs to cut the wire so that one piece is 4 ft shorter than the other. What will be the length of each piece?

3. Solve Consecutive Integer Problems

Consecutive means one after the other, in order. In this section, we will look at consecutive integers, consecutive even integers, and consecutive odd integers.

Consecutive integers differ by 1. Look at the consecutive integers 5, 6, 7, and 8. If $x = 5$, then $x + 1 = 6$, $x + 2 = 7$, and $x + 3 = 8$. Therefore, to define the unknowns for consecutive integers, let

$$\begin{aligned} x &= \text{first integer} \\ x + 1 &= \text{second integer} \\ x + 2 &= \text{third integer} \\ x + 3 &= \text{fourth integer} \end{aligned}$$

and so on.

Example 4

The sum of three consecutive integers is 87. Find the integers.

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find three consecutive integers with a sum of 87.

Step 2: **Choose a variable** to represent an unknown, and define the other unknowns in terms of this variable.

There are three unknowns. We will let x represent the first consecutive integer and then define the other unknowns in terms of x.

$x =$ the first integer

Define the other unknowns in terms of x.

$x + 1 =$ the second integer $\qquad$ $x + 2 =$ the third integer

Step 3: **Translate** the information that appears in English into an algebraic equation. What does the original statement mean?

"The sum of three consecutive integers is 87" means that when the three numbers are *added,* the sum is 87.

Statement:	The sum of three consecutive integers						is	87
Meaning:	The first integer	+	The second integer	+	The third integer		equals	87
	↓		↓		↓		↓	↓
Equation:	x	+	$(x + 1)$	+	$(x + 2)$		$=$	87

The equation is $x + (x + 1) + (x + 2) = 87$.

Step 4: **Solve** the equation.

$$\begin{aligned} x + (x + 1) + (x + 2) &= 87 \\ 3x + 3 &= 87 \\ 3x + 3 - 3 &= 87 - 3 && \text{Subtract 3 from each side.} \\ 3x &= 84 && \text{Combine like terms.} \\ \frac{3x}{3} &= \frac{84}{3} && \text{Divide each side by 3.} \\ x &= 28 && \text{Simplify.} \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

The first integer is 28. The second integer is 29 since $x + 1 = 28 + 1 = 29$, and the third integer is 30 since $x + 2 = 28 + 2 = 30$.

The answer makes sense because their sum is $28 + 29 + 30 = 87$. ■

You Try 4

The sum of three consecutive integers is 162. Find the integers.

Next, let's look at **consecutive even integers,** which are even numbers that differ by 2, such as -10, -8, -6, and -4. If x is the first even integer, we have

-10	-8	-6	-4
x	$x + 2$	$x + 4$	$x + 6$

Therefore, to define the unknowns for consecutive even integers, let

$$\begin{aligned} x &= \text{the first even integer} \\ x + 2 &= \text{the second even integer} \\ x + 4 &= \text{the third even integer} \\ x + 6 &= \text{the fourth even integer} \end{aligned}$$

and so on.

Will the expressions for **consecutive odd integers** be any different? No! When we count by consecutive odds, we are still counting by 2's. Look at the numbers 9, 11, 13, and 15 for example. If x is the first odd integer, we have

9	11	13	15
x	$x + 2$	$x + 4$	$x + 6$

To define the unknowns for consecutive odd integers, let

$$\begin{aligned} x &= \text{the first odd integer} \\ x + 2 &= \text{the second odd integer} \\ x + 4 &= \text{the third odd integer} \\ x + 6 &= \text{the fourth odd integer} \end{aligned}$$

Example 5

The sum of two consecutive odd integers is 19 more than five times the larger integer. Find the integers.

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find two consecutive odd integers.

Step 2: **Choose a variable** to represent an unknown, and define the other unknown in terms of this variable.

There are two unknowns. We will let x represent the first consecutive odd integer and then define the other unknown in terms of x.

$$\begin{aligned} x &= \text{the first odd integer} \\ x + 2 &= \text{the second odd integer} \end{aligned}$$

Step 3: **Translate** the information that appears in English into an algebraic equation. Read the problem slowly and carefully, breaking it into small parts.

Statement:	The sum of two consecutive odd integers			is	19 more than	five times the larger integer
Meaning:	The first odd integer	+	The second odd integer	equals	Add 19 to	5 times the larger integer
	↓		↓	↓	↓	↓
Equation:	x	+	$(x + 2)$	$=$	$19 +$	$5(x + 2)$

The equation is $x + (x + 2) = 19 + 5(x + 2)$.

Step 4: **Solve** the equation.

$$\begin{aligned} x + (x + 2) &= 19 + 5(x + 2) & \\ 2x + 2 &= 19 + 5x + 10 & &\text{Combine like terms; distribute.} \\ 2x + 2 &= 5x + 29 & &\text{Combine like terms.} \\ 2x + 2 - 2 &= 5x + 29 - 2 & &\text{Subtract 2 from each side.} \\ 2x &= 5x + 27 & &\text{Combine like terms.} \\ 2x - 5x &= 5x - 5x + 27 & &\text{Subtract } 5x \text{ from each side.} \\ -3x &= 27 & &\text{Combine like terms.} \\ \frac{-3x}{-3} &= \frac{27}{-3} & &\text{Divide each side by } -3. \\ x &= -9 & &\text{Simplify.} \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

The first odd integer is -9. The second integer is -7 since $x + 2 = -9 + 2 = -7$.

Check these numbers in the original statement of the problem. The sum of -9 and -7 is -16. Then, 19 more than five times the larger integer is $19 + 5(-7) = 19 + (-35) = -16$. The numbers are -9 and -7. ■

You Try 5

The sum of two consecutive even integers is 16 less than three times the larger number. Find the integers.

Answers to You Try Exercises

1) Smart phones: 25; conventional phones: 48 2) Janay: 17 hours; Terrance: 34 hours
3) 8 ft and 12 ft 4) 53, 54, 55 5) 12 and 14

3.3 Exercises

Objective 1: Solve Problems Involving General Quantities

1) During the month of June, a car dealership sold 14 more compact cars than SUVs. Write an expression for the number of compact cars sold if c SUVs were sold.

2) During a Little League game, the Tigers scored 3 more runs than the Eagles. Write an expression for the number of runs the Tigers scored if the Eagles scored r runs.

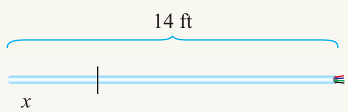
3) A restaurant had 37 fewer customers on a Wednesday night than on a Thursday night. If there were c customers on Thursday, write an expression for the number of customers on Wednesday.

4) After a storm rolled through Omaha, the temperature dropped 15 degrees. If the temperature before the storm was t degrees, write an expression for the temperature after the storm.

5) Due to the increased use of e-mail to send documents, the shipping expenses of a small business in 2010 were half of what they were in 2000. Write an expression for the cost of shipping in 2010 if the cost in 2000 was s dollars.

6) A coffee shop serves three times as many cups of regular coffee as decaffeinated coffee. If the shop serves d cups of decaffeinated coffee, write an expression for the number of cups of regular coffee it sells.

7) An electrician cuts a 14-foot wire into two pieces. If one is x feet long, how long is the other piece?



8) Ralph worked for a total of 8.5 hours one day, some at his office and some at home. If he worked h hours in his office, write an expression for the number of hours he worked at home.

9) If you are asked to find the number of children in a class, why would 26.5 not be a reasonable answer?

10) If you are asked to find the length of a piece of wire, why would -7 not be a reasonable answer?

11) If you are asked to find consecutive odd integers, why would -10 not be a reasonable answer?

12) If you are asked to find the number of workers at an ice cream shop, why would $5\frac{1}{4}$ not be a reasonable answer?

Solve using the five-step method. See Examples 1 and 2.

13) The wettest April (greatest rainfall amount) for Albuquerque, NM, was recorded in 1905. The amount was 1.2 inches more than the amount recorded for the second-wettest April, in 2004. If the total rainfall for these two months was 7.2 inches, how much rain fell in April of each year? (www.srh.noaa.gov)

14) Bo-Lin applied to three more colleges than his sister Liling. Together they applied to 13 schools. To how many colleges did each apply?

15) Miguel Indurain of Spain won the Tour de France two fewer times than Lance Armstrong. They won a total of 12 titles. How many times did each cyclist win this race? (www.letour.fr)

16) In 2009, an Apple MacBook weighed 11 lb less than the Apple Macintosh Portable did in 1989. Find the weight of each computer if they weighed a total of 21 lb. (http://oldcomputers.net; www.apple.com)

17) A 12-oz cup of regular coffee at Starbucks has 13 times the amount of caffeine found in the same-sized serving of decaffeinated coffee. Together they contain 280 mg of caffeine. How much caffeine is in each type of coffee? (www.starbucks.com)

18) A farmer plants soybeans and corn on his 540 acres of land. He plants twice as many acres with soybeans as with corn. How many acres are planted with each crop?

19) In the sophomore class at Dixon High School, the number of students taking French is two-thirds of the number

taking Spanish. How many students are studying each language if the total number of students in French and Spanish is 310?

20) A serving of salsa contains one-sixth of the number of calories of the same-sized serving of guacamole. Find the number of calories in each snack if they contain a total of 175 calories.

Objective 2: Solve Problems Involving Lengths

Solve using the five-step method. See Example 3.

VIDEO 21) A plumber has a 36-in. pipe. He must cut it into two pieces so that one piece is 14 inches longer than the other. How long is each piece?

22) A 40-in. board is to be cut into two pieces so that one piece is 8 inches shorter than the other. Find the length of each piece.

23) Trisha has a 28.5-inch piece of wire to make a necklace and a bracelet. She has to cut the wire so that the piece for the necklace will be twice as long as the piece for the bracelet. Find the length of each piece.

24) Ethan has a 20-ft piece of rope that he will cut into two pieces. One piece will be one-fourth the length of the other piece. Find the length of each piece of rope.

25) Derek orders a 6-ft sub sandwich for himself and two friends. Cory wants his piece to be 2 feet longer than Tamara's piece, and Tamara wants half as much as Derek. Find the length of each person's sub.

26) A 24-ft pipe must be cut into three pieces. The longest piece will be twice as long as the shortest piece, and the medium-sized piece will be 4 feet longer than the shortest piece. Find the length of each piece of pipe.

Objective 3: Solve Consecutive Integer Problems

Solve using the five-step method. See Examples 4 and 5.

27) The sum of three consecutive integers is 126. Find the integers.

28) The sum of two consecutive integers is 171. Find the integers.

29) Find two consecutive even integers such that twice the smaller is 16 more than the larger.

30) Find two consecutive odd integers such that the smaller one is 12 more than one-third the larger.

VIDEO 31) Find three consecutive odd integers such that their sum is five more than four times the largest integer.

32) Find three consecutive even integers such that their sum is 12 less than twice the smallest.

33) Two consecutive page numbers in a book add up to 215. Find the page numbers.

34) The addresses on the west side of Hampton Street are consecutive even numbers. Two consecutive house numbers add up to 7446. Find the addresses of these two houses.

Mixed Exercises: Objectives 1–3

Solve using the five-step method.

35) In a fishing derby, Jimmy caught six more trout than his sister Kelly. How many fish did each person catch if they caught a total of 20 fish?

36) Five times the sum of two consecutive integers is two more than three times the larger integer. Find the integers.

37) A 16-ft steel beam is to be cut into two pieces so that one piece is 1 foot longer than twice the other. Find the length of each piece.

38) A plumber has a 9-ft piece of copper pipe that has to be cut into three pieces. The longest piece will be 4 ft longer than the shortest piece. The medium-sized piece will be three times the length of the shortest. Find the length of each piece of pipe.

39) The attendance at the 2008 Lollapalooza Festival was about 15,000 more than three times the attendance at Bonnaroo that year. The total number of people attending those festivals was about 295,000. How many people went to each event? (www.chicagotribune.com, www.ilmc.com)

40) A cookie recipe uses twice as much flour as sugar. If the total amount of these ingredients is $2\frac{1}{4}$ cups, how much sugar and how much flour are in these cookies?

41) The sum of three consecutive page numbers in a book is 174. What are the page numbers?

42) At a ribbon-cutting ceremony, the mayor cuts a 12-ft ribbon into two pieces so that the length of one piece is 2 ft shorter than the other. Find the length of each piece.

43) During season 7 of *The Biggest Loser,* Tara lost 15 lb more than Helen. The amount of weight that Mike lost was 73 lb less than twice Helen's weight loss. They lost a combined 502 lb. How much weight did each contestant lose? (www.msnbc.msn.com)

44) Find three consecutive odd integers such that three times the middle number is 23 more than the sum of the other two.

45) A builder is installing hardwood floors. He has to cut a 72-in piece into three separate pieces so that the smallest piece is one-third the length of the longest piece, and the third piece is 12 inches shorter than the longest. How long is each piece?

46) In 2008, there were 395 fewer cases of tuberculosis in the United States than in 2007. If the total number of TB cases in those two years was 26,191. How many people tested positive for TB in 2007 and in 2008? (www.cdc.gov)

47) In 2008, Lil Wayne's "The Carter III" sold 0.73 million more copies than Coldplay's "Viva La Vida. . . ." Taylor Swift came in third place with her "Fearless" album, selling 0.04 million fewer copies than Coldplay. The three artists sold a total of 7.14 million albums. How many albums did each artist sell? (www.billboard.com)

48) Workers cutting down a large tree have a rope that is 33 ft long. They need to cut it into two pieces so that one piece is half the length of the other piece. How long is each piece of rope?

VIDEO 49) One-sixth of the smallest of three consecutive even integers is three less than one-tenth the sum of the other even integers. Find the integers.

50) Caedon's mom is a math teacher, and when he asks her on which pages he can find the magazine article on LeBron James, she says, "The article is on three consecutive pages so that 62 less than four times the last page number is the same as the sum of all the page numbers." On what page does the LeBron James article begin?

Section 3.4 Applications Involving Percentages

Objectives

1. Solve Problems Involving Percent Change
2. Solve Problems Involving Simple Interest
3. Solve Mixture Problems

Problems involving percents are everywhere—at the mall, in a bank, in a laboratory, just to name a few places. In this section, we begin learning how to solve different types of applications involving percents.

Before trying to solve a percent problem using algebra, let's look at an arithmetic problem we might see in a store. Relating an algebra problem to an arithmetic problem can make it easier to solve an application that requires the use of algebra.

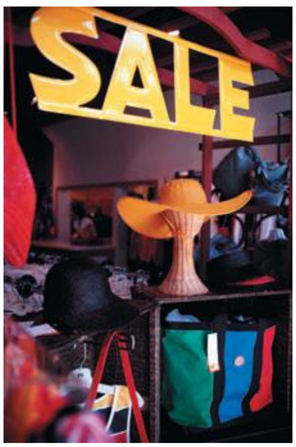
1. Solve Problems Involving Percent Change

Example 1

A hat that normally sells for \$60.00 is marked down 40%. What is the sale price?

Solution

Concentrate on the **procedure** used to obtain the answer. This is the same procedure we will use to solve algebra problems with percent increase and percent decrease.



$$\text{Sale price} = \text{Original price} - \text{Amount of discount}$$

How much is the discount? It is 40% of \$60.00.

Change the percent to a decimal. The amount of the discount is calculated by multiplying:

$$\begin{aligned}\text{Amount of discount} &= (\text{Rate of discount})(\text{Original price})\\ \text{Amount of discount} &= (0.40) \cdot (\$60.00) = \$24.00\end{aligned}$$

$$\begin{aligned}\text{Sale price} &= \text{Original price} - \text{Amount of discount}\\ &= \$60.00 - (0.40)(\$60.00)\\ &= \$60.00 - \$24.00\\ &= \$36.00\end{aligned}$$

The sale price is \$36.00. ■

You Try 1

A pair of running shoes that normally sells for \$80.00 is marked down 30%. What is the sale price?

Next, let's solve an algebra problem involving percent change.

Example 2

The sale price of a video game is \$48.75 after a 25% discount. What was the original price of the game?

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the original price of the video game.

Step 2: **Choose a variable** to represent the unknown.

x = the original price of the video game

Step 3: **Translate** the information that appears in English into an algebraic equation. One way to figure out how to write an algebraic equation is to relate this problem to the arithmetic problem in Example 1. To find the sale price of the hat in Example 1, we found that

Sale price = Original price − Amount of discount

where we found the amount of the discount by multiplying the rate of the discount by the original price. We will write an algebraic equation using the same procedure.

Statement:	Sale price	=	Original price	−	Amount of discount
	↓	↓	↓	↓	↓
Equation:	48.75	=	x	−	$0.25x$

$$0.25x = \left(\text{Rate of discount}\right) \cdot \left(\text{Original price}\right)$$

The equation is $48.75 = x - 0.25x$.

Step 4: **Solve** the equation.

$$\begin{aligned} 48.75 &= x - 0.25x & \\ 48.75 &= 0.75x & &\text{Combine like terms.} \\ \frac{48.75}{0.75} &= \frac{0.75x}{0.75} & &\text{Divide each side by 0.75.} \\ x &= 65 & &\text{Simplify.} \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

The original price of the video game was \$65.00.

The answer makes sense because the amount of the discount is (0.25)(\$65.00) = \$16.25, which makes the sale price \$65.00 − \$16.25 = \$48.75. ■

You Try 2

A circular saw is on sale for \$120.00 after a 20% discount. What was the original price of the saw?

2. Solve Problems Involving Simple Interest

When customers invest their money in bank accounts, their accounts earn interest. There are different ways to calculate the amount of interest earned from an investment, and in this section we will discuss *simple interest*. **Simple interest** calculations are based on the initial amount of money deposited in an account. This is known as the **principal.**

The formula used to calculate simple interest is $I = PRT$, where

$$
\begin{aligned}
I &= \text{interest (simple) earned} \\
P &= \text{principal (initial amount invested)} \\
R &= \text{annual interest rate (expressed as a decimal)} \\
T &= \text{amount of time the money is invested (in years)}
\end{aligned}
$$

We will begin with two arithmetic problems. The procedures used will help you understand more clearly how we arrive at the algebraic equation in Example 5.

Example 3

If \$600 is invested for 1 year in an account earning 4% simple interest, how much interest will be earned?

Solution

We are given that $P = \$600$, $R = 0.04$, $T = 1$. We need to find I.

$$
\begin{aligned}
I &= PRT \\
I &= (600)(0.04)(1) \\
I &= 24
\end{aligned}
$$

The interest earned will be \$24. ■

You Try 3

If \$1400 is invested for 1 year in an account earning 3% simple interest, how much interest will be earned?

Example 4

Gavin invests \$1000 in an account earning 6% interest and \$7000 in an account earning 3% interest. After 1 year, how much interest will he have earned?

Solution

Gavin will earn interest from two accounts. Therefore,

$$\text{Total interest earned} = \text{Interest from 6\% account} + \text{Interest from 3\% account}$$

$$
\begin{aligned}
\text{Total interest earned} &= \overset{P}{(1000)}\overset{R}{(0.06)}\overset{T}{(1)} + \overset{P}{(7000)}\overset{R}{(0.03)}\overset{T}{(1)} \\
&= \quad 60 \quad + \quad 210 \\
&= \$270
\end{aligned}
$$

Gavin will earn a total of \$270 in interest from the two accounts. ■

You Try 4

Taryn invests \$2500 in an account earning 4% interest and \$6000 in an account earning 5.5% interest. After 1 year, how much interest will she have earned?

Note

When money is invested for 1 year, $T = 1$. Therefore, the formula $I = PRT$ can be written as $I = PR$.

In the next example, we will use the same procedure for solving an algebraic problem that we used for solving the arithmetic problems in Examples 3 and 4.

Example 5

Samira had \$8000 to invest. She invested some of it in a savings account that paid 4% simple interest and the rest in a certificate of deposit that paid 6% simple interest. In 1 year, she earned a total of \$360 in interest. How much did Samira invest in each account?

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the amounts Samira invested in the 4% account and in the 6% account.

Step 2: **Choose a variable** to represent an unknown, and define the other unknown in terms of this variable.

Let x = amount Samira invested in the 4% account.

How do we write an expression, in terms of x, for the amount invested in the 6% account?

$$\underset{\substack{\text{Total invested} \\ \downarrow}}{8000} \quad - \quad \underset{\substack{\text{Amount invested in 4\% account} \\ \downarrow}}{x} \quad = \text{Amount invested in the 6\% account}$$

We define the unknowns as:

x = amount Samira invested in the 4% account
$8000 - x$ = amount Samira invested in the 6% account

Step 3: **Translate** the information that appears in English into an algebraic equation. Use the "English equation" we used in Example 4. Remember, since $T = 1$, we can compute the interest using $I = PR$.

Total interest earned = Interest from 4% account + Interest from 6% account

$$360 = \overset{P}{(x)}\overset{R}{(0.04)} + \overset{P}{(8000 - x)}\overset{R}{(0.06)}$$

The equation is $360 = 0.04x + 0.06(8000 - x)$.

We can also get the equation by organizing the information in a table:

Amount Invested, in Dollars *P*	Interest Rate *R*	Interest Earned After 1 Year *I*
x	0.04	$0.04x$
$8000 - x$	0.06	$0.06(8000 - x)$

Total interest earned = Interest from 4% account + Interest from 6% account

$$360 = 0.04x + 0.06(8000 - x)$$

The equation is $360 = 0.04x + 0.06(8000 - x)$.

Either way of organizing the information will lead us to the correct equation.

Step 4: **Solve** the equation. Begin by multiplying both sides of the equation by 100 to eliminate the decimals.

$$\begin{aligned} 360 &= 0.04x + 0.06(8000 - x) && \\ 100(360) &= 100[0.04x + 0.06(8000 - x)] && \\ 36{,}000 &= 4x + 6(8000 - x) && \text{Multiply by 100.} \\ 36{,}000 &= 4x + 48{,}000 - 6x && \text{Distribute.} \\ 36{,}000 &= -2x + 48{,}000 && \text{Combine like terms.} \\ -12{,}000 &= -2x && \text{Subtract 48,000.} \\ 6000 &= x && \text{Divide by } -2. \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

Samira invested \$6000 at 4% interest. The amount invested at 6% is $\$8000 - x$ or $\$8000 - \$6000 = \$2000$.

Check:

$$\begin{aligned} \text{Total interest earned} &= \text{Interest from 4\% account} + \text{Interest from 6\% account} \\ 360 &= 6000(0.04) + 2000(0.06) \\ &= 240 + 120 \\ &= 360 \end{aligned}$$

You Try 5

Jeff inherited \$10,000 from his grandfather. He invested part of it at 3% simple interest and the rest at 5% simple interest. Jeff earned a total of \$440 in interest after 1 year. How much did he deposit in each account?

3. Solve Mixture Problems

Percents can also be used to solve mixture problems. Let's look at an arithmetic example before solving a problem using algebra.

Example 6

The state of Illinois mixes ethanol (made from corn) in its gasoline to reduce pollution. If a customer purchases 12 gallons of gasoline and it has a 10% ethanol content, how many gallons of ethanol are in the 12 gallons of gasoline?

Solution

Write an equation in English first:

English:	Percent of ethanol in the gasoline (as a decimal)	times	Gallons of gasoline	=	Gallons of pure ethanol in the gasoline
	↓	↓	↓	↓	↓
Equation:	0.10	·	12	=	1.2

The equation is $0.10(12) = 1.2$.

We can also organize the information in a table:

Percent of Ethanol in the Gasoline (as a decimal)	Gallons of Gasoline	Gallons of Pure Ethanol in the Gasoline
0.10	12	$0.10(12) = 1.2$

Either way, we find that there are 1.2 gallons of ethanol in 12 gallons of gasoline. ■

We will use this same idea to help us solve the next mixture problem.

Example 7

A chemist needs to make 24 liters (L) of an 8% acid solution. She will make it from some 6% acid solution and some 12% acid solution that is in the storeroom. How much of the 6% solution and the 12% solution should she use?

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the amount of 6% acid solution and the amount of 12% acid solution she should use.

Step 2: **Choose a variable** to represent an unknown, and define the other unknown in terms of this variable. Let

x = the number of liters of 6% acid solution needed

Define the other unknown (the amount of 12% acid solution needed) in terms of x. Since she wants to make a total of 24 L of acid solution,

$24 - x$ = the number of liters of 12% acid solution needed

Step 3: **Translate** the information that appears in English into an algebraic equation.

Let's begin by arranging the information in a table. *Remember, to obtain the expression in the last column, multiply the percent of acid in the solution by the number of liters of solution to get the number of liters of acid in the solution.*

	Percent of Acid in Solution (as a decimal)	Liters of Solution	Liters of Acid in Solution
Mix these	0.06	x	$0.06x$
	0.12	$24 - x$	$0.12(24 - x)$
to make →	0.08	24	$0.08(24)$

Now, write an equation in English. Since we make the 8% solution by mixing the 6% and 12% solutions,

English:	Liters of acid in 6% solution	plus	Liters of acid in 12% solution	equals	Liters of acid in 8% solution
	↓	↓	↓	↓	↓
Equation:	$0.06x$	$+$	$0.12(24 - x)$	$=$	$0.08(24)$

The equation is $0.06x + 0.12(24 - x) = 0.08(24)$.

Step 4: **Solve** the equation.

$$\begin{aligned} 0.06x + 0.12(24 - x) &= 0.08(24) \\ 100[0.06x + 0.12(24 - x)] &= 100[0.08(24)] && \text{Multiply by 100 to eliminate decimals.} \\ 6x + 12(24 - x) &= 8(24) \\ 6x + 288 - 12x &= 192 && \text{Distribute.} \\ -6x + 288 &= 192 && \text{Combine like terms.} \\ -6x &= -96 && \text{Subtract 288 from each side.} \\ x &= 16 && \text{Divide by } -6. \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

The chemist needs 16 L of the 6% solution.

Find the other unknown, the amount of 12% solution needed.

$$24 - x = 24 - 16 = 8\text{ L of 12\% solution.}$$

Check:

$$\begin{array}{ccccc} \text{Acid in 6\% solution} & + & \text{Acid in 12\% solution} & = & \text{Acid in 8\% solution} \\ 0.06(16) & + & 0.12(8) & = & 0.08(24) \\ 0.96 & + & 0.96 & = & 1.92 \\ & & 1.92 & = & 1.92 \end{array}$$

■

You Try 6

Write an equation and solve.

How many milliliters (mL) of a 10% alcohol solution and how many milliliters of a 20% alcohol solution must be mixed to obtain 30 mL of a 16% alcohol solution?

Answers to You Try Exercises

1) \$56.00 2) \$150.00 3) \$42 4) \$430 5) \$3000 at 3% and \$7000 at 5%

6) 12 mL of the 10% solution and 18 mL of the 20% solution

3.4 Exercises

Objective 1: Solve Problems Involving Percent Change

Find the sale price of each item.

1) A USB thumb drive that regularly sells for \$50.00 is marked down 15%.

2) A surfboard that retails for \$525.00 is on sale at 20% off.

3) A sign reads, "Take 30% off the original price of all Blu-ray Disc movies." The original price on the movie you want to buy is \$29.50.

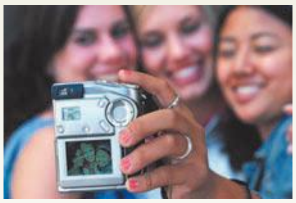
4) The \$100.00 basketball shoes Frank wants are now on sale at 20% off.

5) At the end of the summer, the bathing suit that sold for \$49.00 is marked down 60%.

6) An advertisement states that a flat-screen TV that regularly sells for \$899.00 is being discounted 25%.

Solve using the five-step method. See Example 2.

VIDEO 7) A digital camera is on sale for \$119 after a 15% discount. What was the original price of the camera?



8) Candace paid $21.76 for a hardcover book that was marked down 15%. What was the original selling price of the book?

9) In March, a store discounted all of its calendars by 75%. If Bruno paid $4.40 for a calendar, what was its original price?

10) An appliance store advertises 20% off all of its dishwashers. Mr. Petrenko paid $479.20 for the dishwasher. Find its original price.

11) The sale price of a coffeemaker is $40.08. This is 40% off the original price. What was the original price of the coffeemaker?

12) Katrina paid $25.50 for a box fan that was marked down 15%. Find the original retail price of the box fan.

13) In 2009, there were about 1224 acres of farmland in Crane County. This is 32% less than the number of acres of farmland in 2000. Calculate the number of acres of farmland in Crane County in 2000.

14) One hundred forty countries participated in the 1984 Summer Olympics in Los Angeles. This was 75% more than the number of countries that took part in the Summer Olympics in Moscow 4 years earlier. How many countries participated in the 1980 Olympics in Moscow? (www.mapsofworld.com)

15) In 2006, there were 12,440 Starbucks stores worldwide. This is approximately 1126% more stores than 10 years earlier. How many Starbucks stores were there in 1996? (Round to the nearest whole number.) (www.starbucks.com)

16) McDonald's total revenue in 2003 was $17.1 billion. This is a 28.5% increase over the 1999 revenue. What was McDonald's revenue in 1999? (Round to the tenths place.) (www.mcdonalds.com)

17) From 2001 to 2003, the number of employees at Kmart's corporate headquarters decreased by approximately 34%. If 2900 people worked at the headquarters in 2003, how many worked there in 2001? (Round to the hundreds place.) (www.detnews.com)

18) Jet Fi's salary this year is 14% higher than it was 3 years ago. If he earns $37,050 this year, what did he earn 3 years ago?

Objective 2: Solve Problems Involving Simple Interest

Solve.

19) Kristi invests $300 in an account for 1 year earning 3% simple interest. How much interest was earned from this account?

20) Last year, Mr. Doubtfire deposited $14,000 into an account earning 8.5% simple interest for 1 year. How much interest was earned?

21) Jake Thurmstrom invested $6500 in an account earning 7% simple interest. How much money will be in the account 1 year later?

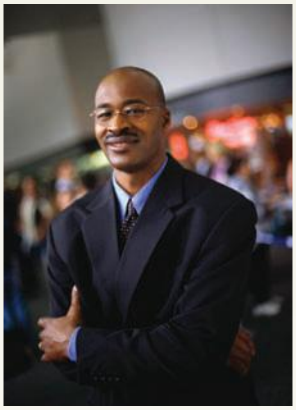
22) If $4000 is deposited into an account for 1 year earning 5.5% simple interest, how much money will be in the account after 1 year?

23) Rachel Rays has a total of $4500 to invest for 1 year. She deposits $3000 into an account earning 6.5% annual simple interest and the rest into an account earning 8% annual simple interest. How much interest did Rachel earn?

24) Bob Farker plans to invest a total of $11,000 for 1 year. Into the account earning 5.2% simple interest he will deposit $6000, and into an account earning 7% simple interest he will deposit the rest. How much interest will Bob earn?

Solve using the five-step method. See Example 5.

VIDEO 25) Amir Sadat receives a $15,000 signing bonus upon accepting his new job. He plans to invest some of it at 6% annual simple interest and the rest at 7% annual simple interest. If he will earn $960 in interest after 1 year, how much will Amir invest in each account?



26) Angelica invested part of her $15,000 inheritance in an account earning 5% simple interest and the rest in an account earning 4% simple interest. How much did Angelica invest in each account if she earned $680 in total interest after 1 year?

27) Barney's money earned $204 in interest after 1 year. He invested some of his money in an account earning 6% simple interest and $450 more than that amount in an account earning 5% simple interest. Find the amount Barney invested in each account.

28) Saori Yamachi invested some money in an account earning 7.4% simple interest and three times that amount in an account earning 12% simple interest. She earned $1085 in interest after 1 year. How much did Saori invest in each account?

29) Last year, Taz invested a total of $7500 in two accounts earning simple interest. Some of it he invested at 9.5%, and the rest he invested at 6.5%. How much did he invest in each account if he earned a total of $577.50 in interest last year?

30) Luke has $3000 to invest. He deposits a portion of it into an account earning 4% simple interest and the rest at 6.5% simple interest. After 1 year, he has earned $170 in interest. How much did Luke deposit into each account?

Objective 3: Solve Mixture Problems

Solve.

31) How many ounces of alcohol are in 50 oz of a 6% alcohol solution?

32) How many milliliters of acid are in 50 mL of a 5% acid solution?

33) Seventy-five milliliters of a 10% acid solution are mixed with 30 mL of a 2.5% acid solution. How much acid is in the mixture?

34) Fifty ounces of a 9% alcohol solution are mixed with 60 ounces of a 7% alcohol solution. How much alcohol is in the mixture?

Solve using the five-step method. See Example 7.

VIDEO 35) How many ounces of a 4% acid solution and how many ounces of a 10% acid solution must be mixed to make 24 oz of a 6% acid solution?

36) How many milliliters of a 17% alcohol solution must be added to 40 mL of a 3% alcohol solution to make a 12% alcohol solution?

37) How many liters of a 25% antifreeze solution must be mixed with 4 liters of a 60% antifreeze solution to make a mixture that is 45% antifreeze?

38) How many milliliters of an 8% hydrogen peroxide solution and how many milliliters of a 2% hydrogen peroxide solution should be mixed to get 300 mL of a 4% hydrogen peroxide solution?

39) All-Mixed-Up Nut Shop sells a mix consisting of cashews and pistachios. How many pounds of cashews, which sell for $7.00 per pound, should be mixed with 4 pounds of pistachios, which sell for $4.00 per pound, to get a mix worth $5.00 per pound?

40) Creative Coffees blends its coffees for customers. How much of the Aromatic coffee, which sells for $8.00 per pound, and how much of the Hazelnut coffee, which sells for $9.00 per pound, should be mixed to make 3 pounds of the Smooth blend to be sold at $8.75 per pound?

41) An alloy that is 50% silver is mixed with 500 g of a 5% silver alloy. How much of the 50% alloy must be used to obtain an alloy that is 20% silver?

42) A pharmacist needs to make 20 cubic centimeters (cc) of a 0.05% steroid solution to treat allergic rhinitis. How much of a 0.08% solution and a 0.03% solution should she use?

43) How much pure acid must be added to 6 gallons of a 4% acid solution to make a 20% acid solution?

44) How many milliliters of pure alcohol and how many milliliters of a 4% alcohol solution must be combined to make 480 milliliters of an 8% alcohol solution?

Mixed Exercises: Objectives 1–3

Solve using the five-step method.

45) In her gift shop, Cheryl sells all stuffed animals for 60% more than what she paid her supplier. If one of these toys sells for $14.00 in her shop, what did it cost Cheryl?

46) Aaron has $7500 to invest. He will invest some of it in a long-term IRA paying 4% simple interest and the rest in a short-term CD earning 2.5% simple interest. After 1 year, Aaron's investments have earned $225 in interest. How much did Aaron invest in each account?

47) In Johnson County, 8330 people were collecting unemployment benefits in September 2010. This is 2% less than the number collecting the benefits in September 2009. How many people in Johnson County were getting unemployment benefits in September 2009?

48) Andre bought a new car for $15,225. This is 13% less than the car's sticker price. What was the sticker price of the car?

49) Erica invests some money in three different accounts. She puts some of it in a CD earning 3% simple interest and twice as much in an IRA paying 4% simple interest. She also decides to invest $1000 more than what she's invested in the CD into a mutual fund earning 5% simple interest. Determine how much money Erica invested in each account if she earned $370 in interest after 1 year.

50) Gil marks up the prices of his fishing poles by 55%. Determine what Gil paid his supplier for his best-selling fishing pole if Gil charges his customers $124.

51) Find the original price of a desk lamp if it costs $25.60 after a 20% discount.

52) It is estimated that in 2003 the number of Internet users in Slovakia was 40% more than the number of users in Kenya. If Slovakia had 700,000 Internet users in 2003, how many people used the Internet in Kenya that year? (*2003 CIA World Factbook,* www.theodora.com)

VIDEO 53) Zoe's current salary is $40,144. This is 4% higher than last year's salary. What was Zoe's salary last year?

54) Jackson earns \$284 in interest from 1-year investments. He invested some money in an account earning 6% simple interest, and he deposited \$1500 more than that amount into an account paying 5% simple interest. How much did Jackson invest in each account?

55) How many ounces of a 9% alcohol solution and how many ounces of a 17% alcohol solution must be mixed to get 12 ounces of a 15% alcohol solution?

56) How many milliliters of a 4% acid solution and how many milliliters of a 10% acid solution must be mixed to obtain 54 mL of a 6% acid solution?

57) How many pounds of peanuts that sell for \$1.80 per pound should be mixed with cashews that sell for \$4.50 per pound so that a 10-pound mixture is obtained that will sell for \$2.61 per pound?

58) Sally invested \$4000 in two accounts, some of it at 3% simple interest and the rest in an account earning 5% simple interest. How much did she invest in each account if she earned \$144 in interest after 1 year?

59) Diego inherited \$20,000 and put some of it into an account earning 4% simple interest and the rest into an account earning 7% simple interest. He earned a total of \$1130 in interest after a year. How much did he deposit into each account?

60) How much pure acid and how many liters of a 10% acid solution should be mixed to get 12 liters of a 40% acid solution?

61) How many ounces of pure orange juice and how many ounces of a citrus fruit drink containing 5% fruit juice should be mixed to get 76 ounces of a fruit drink that is 25% fruit juice?

62) A store owner plans to make 10 pounds of a candy mix worth \$1.92/lb. How many pounds of gummi bears worth \$2.40/lb and how many pounds of jelly beans worth \$1.60/lb must be combined to make the candy mix?

63) The number of plastic surgery procedures performed in the United States in 2003 was 293% more than the number performed in 1997. If approximately 8,253,000 cosmetic procedures were performed in 2003, how many took place in 1997?
(American Society for Aesthetic Plastic Surgery)

Section 3.5 Geometry Applications and Solving Formulas

Objectives

1. Substitute Values into a Formula, and Find the Unknown Variable
2. Solve Problems Using Formulas from Geometry
3. Solve Problems Involving Angle Measures
4. Solve a Formula for a Specific Variable

A **formula** is a rule containing variables and mathematical symbols to state relationships between certain quantities.

Some examples of formulas we have used already are

$$P = 2l + 2w \qquad A = \frac{1}{2}bh \qquad C = 2\pi r \qquad I = PRT$$

In this section we will solve problems using *formulas,* and then we will learn how to solve a formula for a specific variable.

1. Substitute Values into a Formula, and Find the Unknown Variable

Example 1

The formula for the area of a triangle is $A = \frac{1}{2}bh$. If $A = 30$ when $b = 8$, find h.

Solution

The only unknown variable is h since we are given the values of A and b. Substitute $A = 30$ and $b = 8$ into the formula, and solve for h.

$$A = \frac{1}{2}bh$$

$$30 = \frac{1}{2}(8)h \qquad \text{Substitute the given values.}$$

Since h is the only remaining variable in the equation, we can solve for it.

$$30 = 4h \qquad \text{Multiply.}$$

$$\frac{30}{4} = \frac{4h}{4} \qquad \text{Divide by 4.}$$

$$\frac{15}{2} = h \qquad \text{Simplify.}$$

You Try 1

The area of a trapezoid is $A = \frac{1}{2}h(b_1 + b_2)$. If $A = 21$ when $b_1 = 10$ and $b_2 = 4$, find h.

2. Solve Problems Using Formulas from Geometry

Next we will solve applied problems using concepts and formulas from geometry. Unlike in Example 1, you will not be given a formula. You will need to know the geometry formulas that we reviewed in Section 1.3. They are also found at the end of the book.

Example 2

A soccer field is in the shape of a rectangle and has an area of 9000 yd². Its length is 120 yd. What is the width of the field?

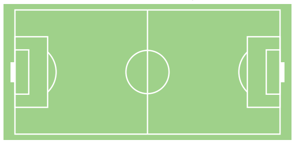
Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the length of the soccer field.

A picture will be very helpful in this problem.

Area = 9000 yd²



w

120 yd

Step 2: **Choose a variable** to represent the unknown.

$w =$ the width of the soccer field

Label the picture with the length, 120 yd, and the width, w.

Step 3: **Translate** the information that appears in English into an algebraic equation. We will use a known geometry formula. How do we know which formula to use? List the information we are given and what we want to find:

The field is in the shape of a rectangle; its area $= 9000$ yd² and its length $= 120$ yd. We must find the width. Which formula involves the area, length, and width of a rectangle?

$$A = lw$$

Substitute the known values into the formula for the area of a rectangle, and solve for w.

$$\begin{aligned} A &= lw \\ 9000 &= 120w \qquad \text{Substitute the known values.} \end{aligned}$$

Step 4: **Solve** the equation.

$$\begin{aligned} 9000 &= 120w \\ \frac{9000}{120} &= \frac{120w}{120} \qquad \text{Divide by 120.} \\ 75 &= w \qquad \text{Simplify.} \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

If $w = 75$ yd, then $l \cdot w = 120 \text{ yd} \cdot 75 \text{ yd} = 9000 \text{ yd}^2$. Therefore, the width of the soccer field is 75 yd. ■

Note

Remember to include the correct units in your answer!

You Try 2

Write an equation and solve.

The area of a rectangular room is 270 ft^2. Find the length of the room if the width is 15 ft.

Example 3

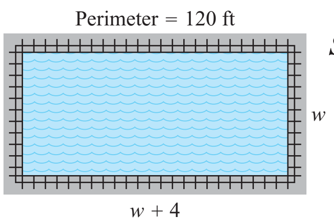
Stewart wants to put a rectangular safety fence around his backyard pool. He calculates that he will need 120 feet of fencing and that the length will be 4 feet longer than the width. Find the dimensions of the safety fence.

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the length and width of the safety fence.

Draw a picture.



Step 2: **Choose a variable** to represent an unknown, and define the other unknown in terms of this variable.

The length is 4 feet longer than the width. Therefore, let

w = the width of the safety fence

Define the other unknown in terms of w.

$w + 4$ = the length of the safety fence

Label the picture with the expressions for the width and length.

Step 3: **Translate** the information that appears in English into an algebraic equation.

Use a known geometry formula. What does the 120 ft of fencing represent? *Since the fencing will go around the pool, the 120 ft represents the perimeter of the rectangular safety fence*. We need to use a formula that involves the length, width, and perimeter of a rectangle. The formula we will use is

$$P = 2l + 2w$$

Substitute the known values and expressions into the formula.

$$P = 2l + 2w$$
$$120 = 2(w + 4) + 2w \qquad \text{Substitute.}$$

Step 4: **Solve** the equation.

$$\begin{aligned} 120 &= 2(w + 4) + 2w \\ 120 &= 2w + 8 + 2w && \text{Distribute.} \\ 120 &= 4w + 8 && \text{Combine like terms.} \\ 120 - 8 &= 4w + 8 - 8 && \text{Subtract 8 from each side.} \\ 112 &= 4w && \text{Combine like terms.} \\ \frac{112}{4} &= \frac{4w}{4} && \text{Divide each side by 4.} \\ 28 &= w && \text{Simplify.} \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

The width of the safety fence is 28 ft. The length is $w + 4 = 28 + 4 = 32$ ft.

The answer makes sense because the perimeter of the fence is $2(32\text{ft}) + 2(28\text{ ft}) = 64\text{ ft} + 56\text{ ft} = 120\text{ ft}$.

You Try 3

Write an equation and solve.

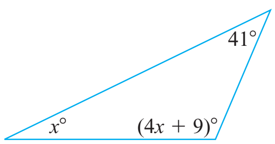
Marina wants to make a rectangular dog run in her backyard. It will take 46 feet of fencing to enclose it, and the length will be 1 foot less than three times the width. Find the dimensions of the dog run.

3. Solve Problems Involving Angle Measures

Recall from Section 1.3 that the sum of the angle measures in a triangle is 180°. We will use this fact in our next example.

Example 4

Find the missing angle measures.



Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

Find the missing angle measures.

Step 2: The unknowns are already defined. We must find x, the measure of one angle, and then $4x + 9$, the measure of the other angle.

Step 3: **Translate** the information into an algebraic equation. Since the sum of the angles in a triangle is 180°, we can write

English:	Measure of one angle	plus	Measure of second angle	plus	Measure of third angle	is	180°
	↓	↓	↓	↓	↓	↓	↓
Equation:	x	$+$	41	$+$	$4x + 9$	$=$	180

The equation is $x + 41 + (4x + 9) = 180$.

Step 4: **Solve** the equation.

$$\begin{aligned} x + 41 + (4x + 9) &= 180 \\ 5x + 50 &= 180 && \text{Combine like terms.} \\ 5x + 50 - 50 &= 180 - 50 && \text{Subtract 50 from each side.} \\ 5x &= 130 && \text{Combine like terms.} \\ \frac{5x}{5} &= \frac{130}{5} && \text{Divide each side by 5.} \\ x &= 26 && \text{Simplify.} \end{aligned}$$

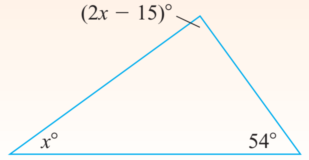
Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

One angle, x, has a measure of 26°. The other unknown angle measure is $4x + 9 = 4(26) + 9 = 113°$.

The answer makes sense because the sum of the angle measures is $26° + 41° + 113° = 180°$.

You Try 4

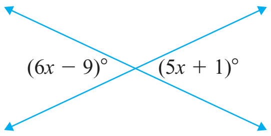
Find the missing angle measures.



Let's look at another type of problem involving angle measures.

Example 5

Find the measure of each indicated angle.



Solution

The indicated angles are *vertical angles*, and **vertical angles** have the same measure. (See Section 1.3.) Since their measures are the same, set $6x - 9 = 5x + 1$ and solve for x.

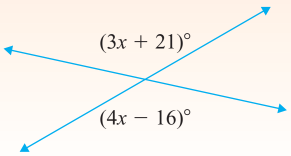
$$\begin{aligned} 6x - 9 &= 5x + 1 \\ 6x - 9 + 9 &= 5x + 1 + 9 && \text{Add 9 to each side.} \\ 6x &= 5x + 10 && \text{Combine like terms.} \\ 6x - 5x &= 5x - 5x + 10 && \text{Subtract } 5x \text{ from each side.} \\ x &= 10 && \text{Combine like terms.} \end{aligned}$$

Be careful! Although $x = 10$, the angle measure is *not* 10. To find the angle measures, substitute $x = 10$ into the expressions for the angles.

The measure of the angle on the left is $6x - 9 = 6(10) - 9 = 51°$. The other angle measure is also 51° since these are vertical angles. We can verify this by substituting 10 into the expression for the other angle, $5x + 1$: $5x + 1 = 5(10) + 1 = 51°$.

You Try 5

Find the measure of each indicated angle.



In Section 1.3, we learned that two angles are **complementary** if the sum of their angles is 90°, and two angles are **supplementary** if the sum of their angles is 180°.

For example, if the measure of $\angle A$ is 71°, then

a) the measure of its complement is $90° - 71° = 19°$.
b) the measure of its supplement is $180° - 71° = 109°$.

Now let's say the measure of an angle is x. Using the same reasoning as above,

a) the measure of its complement is $90 - x$.
b) the measure of its supplement is $180 - x$.

We will use these ideas to solve the problem in Example 6.

Example 6

The supplement of an angle is 34° more than twice the complement of the angle. Find the measure of the angle.

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the measure of the angle.

Step 2: **Choose a variable** to represent an unknown, and define the other unknowns in terms of this variable.

This problem has three unknowns: the measures of the angle, its complement, and its supplement. Choose a variable to represent the original angle, then define the other unknowns in terms of this variable.

x = the measure of the angle

Define the other unknowns in terms of x.

$90 - x$ = the measure of the complement
$180 - x$ = the measure of the supplement

Step 3: **Translate** the information that appears in English into an algebraic equation.

Statement:	The supplement of an angle	is	34° more than	twice the complement of the angle.
	↓	↓	↓	↓
Equation:	$180 - x$	$=$	$34 +$	$2(90 - x)$

The equation is $180 - x = 34 + 2(90 - x)$.

Step 4: **Solve** the equation.

$$\begin{aligned}
180 - x &= 34 + 2(90 - x) & & \\
180 - x &= 34 + 180 - 2x & & \text{Distribute.} \\
180 - x &= 214 - 2x & & \text{Combine like terms.} \\
180 - 180 - x &= 214 - 180 - 2x & & \text{Subtract 180 from each side.} \\
-x &= 34 - 2x & & \text{Combine like terms.} \\
-x + 2x &= 34 - 2x + 2x & & \text{Add } 2x \text{ to each side.} \\
x &= 34 & & \text{Simplify.}
\end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

The measure of the angle is 34°.

To check the answer, we first need to find its complement and supplement. The complement is $90° - 34° = 56°$, and its supplement is $180° - 34° = 146°$. Now we can check these values in the original statement: The supplement is 146°. Thirty-four degrees more than twice the complement is $34° + 2(56°) = 34° + 112° = 146°$. ■

You Try 6

Write an equation and solve.

Twice the complement of an angle is 18° less than the supplement of the angle. Find the measure of the angle.

4. Solve a Formula for a Specific Variable

The formula $P = 2l + 2w$ allows us to find the perimeter of a rectangle when we know its length (l) and width (w). But what if we were solving problems where we repeatedly needed to find the value of w? Then, we could rewrite $P = 2l + 2w$ so that it is solved for w:

$$w = \frac{P - 2l}{2}$$

Doing this means that we have *solved the formula* $P = 2l + 2w$ *for the specific variable* w.

Solving a formula for a specific variable may seem confusing at first because the formula contains more than one letter. Keep in mind that we will solve for a specific variable the same way we have been solving equations up to this point.

We'll start by solving $3x + 4 = 19$ step-by-step for x and then applying the same procedure to solving $ax + b = c$ for x.

Example 7

Solve $3x + 4 = 19$ and $ax + b = c$ for x.

Solution

Look at these equations carefully, and notice that they have the same form. Read the parts of the solution in numerical order.

Part 1 Solve $3x + 4 = 19$.

Don't quickly run through the solution of this equation. **The emphasis here is on the steps used to solve the equation and why we use those steps!**

$$3\boxed{x} + 4 = 19$$

We are solving for x. We'll put a box around it. What is the first step? "Get rid of" what is being added to the $3x$; that is, "get rid of" the 4 on the left. Subtract 4 from each side.

$$3\boxed{x} + 4 - 4 = 19 - 4$$

Combine like terms.

$$3\boxed{x} = 15$$

Part 2 Solve $ax + b = c$ for x.

Since we are solving for x, we'll put a box around it.

$$a\boxed{x} + b = c$$

The goal is to get the x on a side by itself. What do we do first? As in part 1, "get rid of" what is being added to the ax term; that is, "get rid of" the b on the left. Since b is being added to ax, we will subtract it from each side. (We are performing the same steps as in part 1!)

$$a\boxed{x} + b - b = c - b$$

Combine like terms.

$$a\boxed{x} = c - b$$

We cannot combine the terms on the right, so it remains $c - b$.

Part 3 We left off needing to solve $3\boxed{x} = 15$ for x. We need to eliminate the 3 on the left. Since x is being multiplied by 3, we will **divide** each side by 3.

$$\frac{3\boxed{x}}{3} = \frac{15}{3}$$

Simplify.

$$x = 5$$

Part 4 Now, we have to solve $a\boxed{x} = c - b$ for x. We need to eliminate the a on the left. Since x is being multiplied by a, we will **divide** each side by a.

$$\frac{a\boxed{x}}{a} = \frac{c - b}{a}$$

These are the same steps used in part 3!

Simplify.

$$\frac{a\boxed{x}}{a} = \frac{c - b}{a}$$

$$x = \frac{c - b}{a} \text{ or } \frac{c}{a} - \frac{b}{a}$$

Note

To obtain the result $x = \frac{c}{a} - \frac{b}{a}$, we distributed the a in the denominator to each term in the numerator. Either form of the answer is correct.

When you are solving a formula for a specific variable, think about the steps you use to solve an equation in one variable.

You Try 7

Solve $rt - n = k$ for t.

Example 8

$U = \frac{1}{2}LI^2$ is a formula used in physics. Solve this equation for L.

Solution

$$U = \frac{1}{2}\boxed{L}I^2 \qquad \text{Solve for } L\text{. Put it in a box.}$$

$$2U = 2 \cdot \frac{1}{2}\boxed{L}I^2 \qquad \text{Multiply by 2 to eliminate the fraction.}$$

$$\frac{2U}{I^2} = \frac{\boxed{L}I^2}{I^2} \qquad \text{Divide each side by } I^2.$$

$$\frac{2U}{I^2} = L \qquad \text{Simplify.}$$

Example 9

$A = \frac{1}{2}h(b_1 + b_2)$ is the formula for the area of a trapezoid. Solve it for b_1.

Solution

There are two ways to solve this for b_1.

Method 1: We will put b_1 in a box to remind us that this is what we must solve for. In Method 1, we will start by eliminating the fraction.

$$2A = 2 \cdot \frac{1}{2}h(\boxed{b_1} + b_2) \qquad \text{Multiply each side by 2.}$$

$$2A = h(\boxed{b_1} + b_2) \qquad \text{Simplify.}$$

$$\frac{2A}{h} = \frac{h(\boxed{b_1} + b_2)}{h} \qquad \text{Divide each side by } h.$$

$$\frac{2A}{h} = \boxed{b_1} + b_2$$

$$\frac{2A}{h} - b_2 = \boxed{b_1} + b_2 - b_2 \qquad \text{Subtract } b_2 \text{ from each side.}$$

$$\frac{2A}{h} - b_2 = b_1 \qquad \text{Simplify.}$$

Method 2: Another way to solve $A = \frac{1}{2}h(b_1 + b_2)$ for b_1 is to begin by distributing $\frac{1}{2}h$ on the right.

$$A = \frac{1}{2}h\boxed{b_1} + \frac{1}{2}hb_2 \qquad \text{Distribute.}$$

$$2A = 2\left(\frac{1}{2}h\boxed{b_1} + \frac{1}{2}hb_2\right) \qquad \text{Multiply by 2 to eliminate the fractions.}$$

$$2A = h\boxed{b_1} + hb_2 \qquad \text{Distribute.}$$

$$2A - hb_2 = h\boxed{b_1} + hb_2 - hb_2 \qquad \text{Subtract } hb_2 \text{ from each side.}$$

$$2A - hb_2 = h\boxed{b_1} \qquad \text{Simplify.}$$

$$\frac{2A - hb_2}{h} = \frac{h\boxed{b_1}}{h} \qquad \text{Divide by } h.$$

$$\frac{2A - hb_2}{h} = b_1 \qquad \text{Simplify.}$$

Therefore, b_1 can be written as $b_1 = \frac{2A}{h} - \frac{hb_2}{h}$ or $b_1 = \frac{2A}{h} - b_2$. These two forms are equivalent.

You Try 8

Solve for the indicated variable.

a) $t = \frac{qr}{s}$ for q b) $R = t(k - c)$ for c

Answers to You Try Exercises

1) 3 2) 18 ft 3) 6 ft $\times$ 17 ft 4) 47°, 79° 5) 132°, 132° 6) 18°

7) $t = \frac{k + n}{r}$ 8) a) $q = \frac{st}{r}$ b) $c = \frac{kt - R}{t}$ or $c = k - \frac{R}{t}$

3.5 Exercises

Objective 1: Substitute Values into a Formula, and Find the Unknown Variable

1) If you are using the formula $A = \frac{1}{2}bh$, is it reasonable to get an answer of $h = -6$? Explain your answer.

2) If you are finding the area of a rectangle and the lengths of the sides are given in inches, the area of the rectangle would be expressed in which unit?

3) If you are asked to find the volume of a sphere and the radius is given in centimeters, the volume would be expressed in which unit?

4) If you are asked to find the perimeter of a football field and the length and width are given in yards, the perimeter of the field would be expressed in which unit?

Substitute the given values into the formula and solve for the remaining variable.

5) $A = lw$; If $A = 44$ when $l = 16$, find w.

6) $A = \frac{1}{2}bh$; If $A = 21$ when $h = 14$, find b.

7) $I = PRT$; If $I = 240$ when $R = 0.04$ and $T = 2$, find P.

8) $I = PRT$; If $I = 600$ when $P = 2500$ and $T = 4$, find R.

9) $d = rt$
(Distance formula: *distance* = *rate* · *time*);
If $d = 150$ when $r = 60$, find t.

10) $d = rt$
(Distance formula: *distance* = *rate* · *time*);
If $r = 36$ and $t = 0.75$, find d.

11) $C = 2\pi r$; If $r = 4.6$, find C.

12) $C = 2\pi r$; If $C = 15\pi$, find r.

13) $P = 2l + 2w$; If $P = 11$ when $w = \frac{3}{2}$, find l.

14) $P = s_1 + s_2 + s_3$ (Perimeter of a triangle); If $P = 11.6$ when $s_2 = 2.7$ and $s_3 = 3.8$, find s_1.

15) $V = lwh$; If $V = 52$ when $l = 6.5$ and $h = 2$, find w.

16) $V = \frac{1}{3}Ah$ (Volume of a pyramid); If $V = 16$ when $A = 24$, find h.

17) $V = \frac{1}{3}\pi r^2 h$; If $V = 48\pi$ when $r = 4$, find h.

18) $V = \frac{1}{3}\pi r^2 h$; If $V = 50\pi$ when $r = 5$, find h.

19) $S = 2\pi r^2 + 2\pi rh$ (Surface area of a right circular cylinder); If $S = 154\pi$ when $r = 7$, find h.

20) $S = 2\pi r^2 + 2\pi rh$; If $S = 132\pi$ when $r = 6$, find h.

21) $A = \frac{1}{2}h(b_1 + b_2)$; If $A = 136$ when $b_1 = 7$ and $h = 16$, find b_2.

22) $A = \frac{1}{2}h(b_1 + b_2)$; If $A = 1.5$ when $b_1 = 3$ and $b_2 = 1$, find h.

Objective 2: Solve Problems Using Formulas from Geometry

Use a known formula to solve. See Example 2.

23) The area of a tennis court is 2808 ft^2. Find the length of the court if it is 36 ft wide.

24) A rectangular tabletop has an area of 13.5 ft^2. What is the width of the table if it is 4.5 ft long?

25) A rectangular flower box holds 1232 in^3 of soil. Find the height of the box if it is 22 in. long and 7 in. wide.

26) A rectangular storage box is 2.5 ft wide, 4 ft long, and 1.5 ft high. What is the storage capacity of the box?

27) The center circle on a soccer field has a radius of 10 yd. What is the area of the center circle? Use 3.14 for π.

28) The face of the clock on Big Ben in London has a radius of 11.5 feet. What is the area of this circular clock face? Use 3.14 for π.
(www.bigben.freeservers.com)

29) Abbas drove 134 miles on the highway in 2 hours. What was his average speed?

30) If Reza drove 108 miles at 72 mph, without stopping, for how long did she drive?

31) A stainless steel garbage can is in the shape of a right circular cylinder. If its radius is 6 inches and its volume is 864π in^3, what is the height of the can?

32) A coffee can in the shape of a right circular cylinder has a volume of 50π in^3. Find the height of the can if its diameter is 5 inches.

33) A flag is in the shape of a triangle and has an area of 6 ft^2. Find the length of the base if its height is 4 ft.

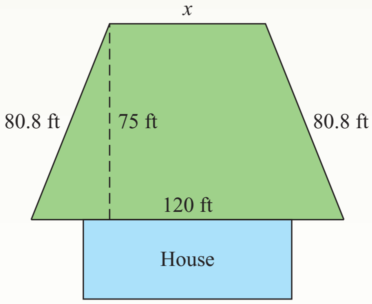
34) A championship banner hanging from the rafters of a stadium is in the shape of a triangle and has an area of 20 ft^2. How long is the banner if its base is 5 ft?

35) Leilani invested \$1500 in a bank account for 2 years and earned \$75 in interest. What interest rate did she receive?

36) The backyard of a house is in the shape of a trapezoid as pictured here. If the area of the yard is 6750 ft^2:

a) Find the length of the missing side, x.

b) How much fencing would be needed to completely enclose the plot?

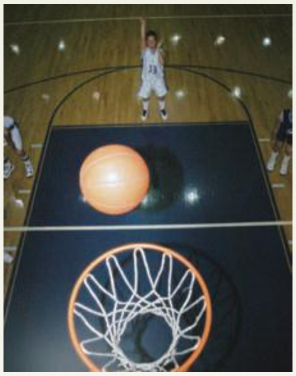


Use a known formula to solve. See Example 3.

37) Vivian is making a rectangular wooden picture frame that will have a width that is 10 in. shorter than its length. If she will use 92 in. of wood, what are the dimensions of the frame?

38) A construction crew is making repairs next to a school, so they have to enclose the rectangular area with a fence. They determine that they will need 176 ft of fencing for the work area, which is 22 ft longer than it is wide. Find the dimensions of the fenced area.

VIDEO 39) The "lane" on a basketball court is a rectangle that has a perimeter of 62 ft. Find the dimensions of the "lane" given that its length is 5 ft less than twice the width.



40) A rectangular whiteboard in a classroom is twice as long as it is high. Its perimeter is 24 ft. What are the dimensions of the whiteboard?

41) One base of a trapezoid is 2 in. longer than three times the other base. Find the lengths of the bases if the trapezoid is 5 in. high and has an area of 25 in^2.

42) A caution flag on the side of a road is shaped like a trapezoid. One base of the trapezoid is 1 ft shorter than the other base. Find the lengths of the bases if the trapezoid is 4 ft high and has an area of 10 ft^2.

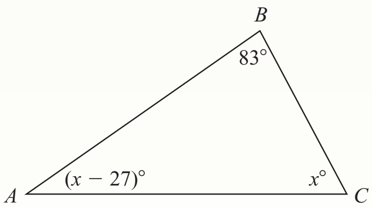
43) A triangular sign in a store window has a perimeter of 5.5 ft. Two of the sides of the triangle are the same length while the third side is 1 foot longer than those sides. Find the lengths of the sides of the sign.

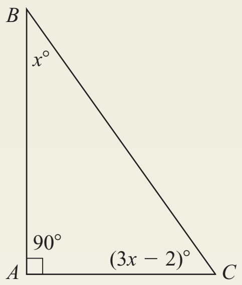
44) A triangle has a perimeter of 31 in. The longest side is 1 in. less than twice the shortest side, and the third side is 4 in. longer than the shortest side. Find the lengths of the sides.

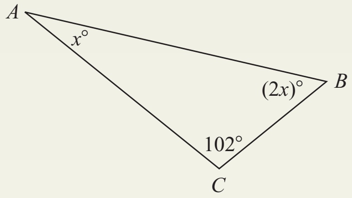
Objective 3: Solve Problems Involving Angle Measures

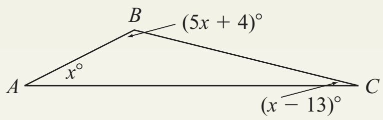
Find the missing angle measures.

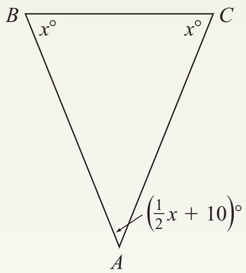
45)

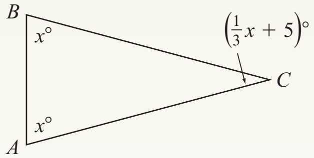


46) 

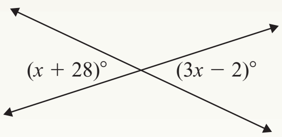
VIDEO 47) 

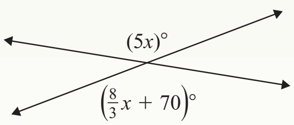
48) 

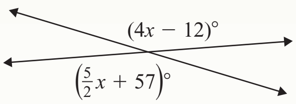
49) 

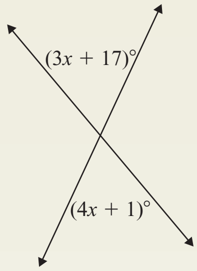
50) 

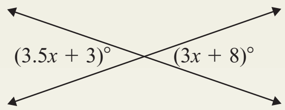
Find the measure of each indicated angle.

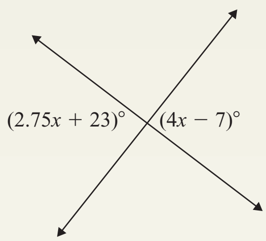
51) 

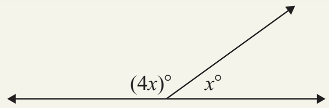
52) 

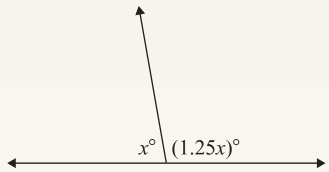
53) 

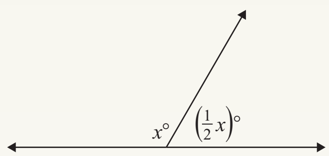
54) 

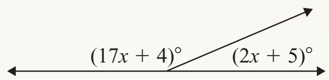
55) 

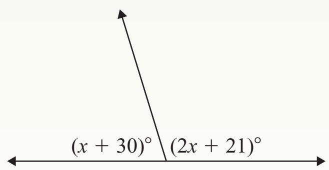
56) 

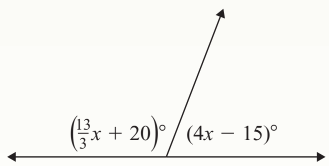
57) 

58) 

59) 

60) 

61) 

62) 

63) If x = the measure of an angle, write an expression for its supplement.

64) If x = the measure of an angle, write an expression for its complement.

Write an equation and solve.

65) The supplement of an angle is 63° more than twice the measure of its complement. Find the measure of the angle.

66) Twice the complement of an angle is 49° less than its supplement. Find the measure of the angle.

67) Six times an angle is 12° less than its supplement. Find the measure of the angle.

68) An angle is 1° less than 12 times its complement. Find the measure of the angle.

VIDEO 69) Four times the complement of an angle is 40° less than twice the angle's supplement. Find the angle, its complement, and its supplement.

70) Twice the supplement of an angle is 30° more than eight times its complement. Find the angle, its complement, and its supplement.

71) The sum of an angle and half its supplement is seven times its complement. Find the measure of the angle.

72) The sum of an angle and three times its complement is 62° more than its supplement. Find the measure of the angle.

73) The sum of four times an angle and twice its complement is 270°. Find the angle.

74) The sum of twice an angle and half its supplement is 192°. Find the angle.

Objective 4: Solve a Formula for a Specific Variable

75) Solve for x.

a) $x + 16 = 37$ b) $x + h = y$

c) $x + r = c$

76) Solve for t.

a) $t - 8 = 17$ b) $t - p = z$

c) $t - k = n$

77) Solve for c.

a) $8c = 56$ b) $ac = d$

c) $mc = v$

78) Solve for k.

a) $9k = 54$ b) $nk = t$

c) $wk = h$

79) Solve for a.

a) $\dfrac{a}{4} = 11$ b) $\dfrac{a}{y} = r$

c) $\dfrac{a}{w} = d$

80) Solve for d.

a) $\dfrac{d}{6} = 3$ b) $\dfrac{d}{t} = q$

c) $\dfrac{d}{x} = a$

VIDEO 81) Solve for d.

a) $8d - 7 = 17$ b) $kd - a = z$

82) Solve for w.

a) $5w + 18 = 3$ b) $pw + r = \pi$

83) Solve for h.

a) $9h + 23 = 17$ b) $qh + v = n$

84) Solve for b.

a) $12b - 5 = 17$ b) $mb - c = a$

Solve each formula for the indicated variable.

85) $F = ma$ for m (Physics)

86) $C = 2\pi r$ for r

87) $n = \dfrac{c}{v}$ for c (Physics)

88) $f = \dfrac{R}{2}$ for R (Physics)

89) $E = \sigma T^4$ for σ (Meteorology)

90) $p = \rho g y$ for ρ (Geology)

91) $V = \dfrac{1}{3}\pi r^2 h$ for h

92) $d = rt$ for r

93) $R = \dfrac{E}{I}$ for E (Electricity)

94) $A = \dfrac{1}{2}bh$ for b

95) $I = PRT$ for R

96) $I = PRT$ for P

97) $P = 2l + 2w$ for l

98) $A = P + PRT$ for T (Finance)

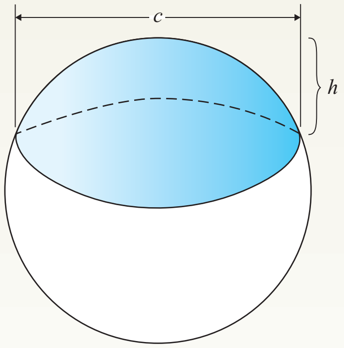
99) $H = \dfrac{D^2 N}{2.5}$ for N (Auto mechanics)

100) $V = \frac{AH}{3}$ for A (Geometry)

101) $A = \frac{1}{2}h(b_1 + b_2)$ for b_2

102) $A = \pi(R^2 - r^2)$ for r^2 (Geometry)

For Exercises 103 and 104, refer to the figure below.



The surface area, S, of the spherical segment shown in the figure is given by $S = \frac{\pi}{4}(4h^2 + c^2)$, where h is the height of the segment and c is the diameter of the segment's base.

103) Solve the formula for h^2.

104) Solve the formula for c^2.

VIDEO 105) The perimeter, P, of a rectangle is $P = 2l + 2w$, where l = length and w = width.

a) Solve $P = 2l + 2w$ for w.

b) Find the width of the rectangle with perimeter 28 cm and length 11 cm.

106) The area, A, of a triangle is $A = \frac{1}{2}bh$, where b = length of the base and h = height.

a) Solve $A = \frac{1}{2}bh$ for h.

b) Find the height of the triangle that has an area of 39 cm^2 and a base of length 13 cm.

107) The formula $C = \frac{5}{9}(F - 32)$ can be used to convert from degrees Fahrenheit, F, to degrees Celsius, C.

a) Solve this formula for F.

b) The average high temperature in Paris, France, in May is 20°C. Use the result in part a) to find the equivalent temperature in degrees Fahrenheit. (www.bbc.co.uk)

108) The average low temperature in Buenos Aires, Argentina, in June is 5°C. Use the result in Exercise 107 a) to find the equivalent temperature in degrees Fahrenheit. (www.bbc.co.uk)

Section 3.6 Applications of Linear Equations to Proportions, Money Problems, and $d = rt$

Objectives

1. Use Ratios
2. Solve a Proportion
3. Solve Problems Involving Denominations of Money
4. Solve Problems Involving Distance, Rate, and Time

1. Use Ratios

We hear about *ratios* and use them in many ways in everyday life. For example, if a survey on cell phone use revealed that 80 teenagers prefer texting their friends while 25 prefer calling their friends, we could write the ratio of teens who prefer texting to teens who prefer calling as

$$\frac{\text{Number who prefer texting}}{\text{Number who prefer calling}} = \frac{80}{25} = \frac{16}{5}$$

Here is a formal definition of a ratio:

Definition

A **ratio** is a quotient of two quantities. The ratio of the number x to the number y, where $y \neq 0$, can be written as $\frac{x}{y}$, x to y, or $x:y$.

A percent is actually a ratio. For example, we can think of 39% as $\frac{39}{100}$ or as the ratio of 39 to 100.

Example 1

Write the ratio of 4 feet to 2 yards.

Solution

Write each quantity with the same units. Let's change yards to feet. Since there are 3 feet in 1 yard,

$$2 \text{ yards} = 2 \cdot 3 \text{ feet} = 6 \text{ feet}$$

Then the ratio of 4 feet to 2 yards is

$$\frac{4 \text{ feet}}{2 \text{ yds}} = \frac{4 \text{ feet}}{6 \text{ feet}} = \frac{4}{6} = \frac{2}{3}$$

You Try 1

Write the ratio of 3 feet to 24 inches.

We can use ratios to help us figure out which item in a store gives us the most value for our money. To do this, we will determine the *unit price* of each item. The **unit price** is the ratio of the price of the item to the amount of the item.

Example 2

A store sells Haagen-Dazs vanilla ice cream in three different sizes. The sizes and prices are listed here. Which size is the best buy?

Size	Price
4 oz	\$1.00
14 oz	\$3.49
28 oz	\$7.39

Solution

For each carton of ice cream, we must find the unit price, or how much the ice cream costs per ounce. We will find the unit price by dividing.

$$\text{Unit price} = \frac{\text{Price of ice cream}}{\text{Number of ounces in the container}} = \text{Cost per ounce}$$

Size	Unit Price
4 oz	$\frac{\$1.00}{4 \text{ oz}} = \$0.250$ per oz
14 oz	$\frac{\$3.49}{14 \text{ oz}} = \$0.249$ per oz
28 oz	$\frac{\$7.39}{28 \text{ oz}} = \$0.264$ per oz

We round the answers to the thousandths place because, as you can see, there is not much difference in the unit price. Since the 14-oz carton of ice cream has the smallest unit price, it is the best buy.

You Try 2

A store sells Gatorade fruit punch in three different sizes. A 20-oz bottle costs \$1.00, a 32-oz bottle sells for \$1.89, and the price of a 128-oz bottle is \$5.49. Which size is the best buy, and what is its unit price?

2. Solve a Proportion

We have learned that a ratio is a way to compare two quantities. If two ratios are equivalent, like $\frac{4}{6}$ and $\frac{2}{3}$, we can set them equal to make a *proportion*.

Definition

A **proportion** is a statement that two ratios are equal.

How can we be certain that a proportion is true? We can find the **cross products.** If the cross products are equal, then the proportion is true. If the cross products are not equal, then the proportion is false.

Property

Cross Products If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$ provided that $b \neq 0$ and $d \neq 0$.

We will see later in the book that finding the cross products is the same as multiplying both sides of the equation by the least common denominator of the fractions.

Example 3

Determine whether each proportion is true or false.

a) $\frac{5}{7} = \frac{15}{21}$ b) $\frac{2}{9} = \frac{7}{36}$

Solution

a) Find the cross products.

Multiply.

$$\frac{5}{7} = \frac{15}{21}$$

Multiply.

$$5 \cdot 21 = 7 \cdot 15$$
$$105 = 105 \quad \text{True}$$

The cross products are equal, so the proportion is true.

b) Find the cross products.

Multiply.

$$\frac{2}{9} = \frac{7}{36}$$

Multiply.

$$2 \cdot 36 = 9 \cdot 7$$
$$72 = 63 \quad \text{False}$$

The cross products are not equal, so the proportion is false. ■

You Try 3

Determine whether each proportion is true or false.

a) $\frac{4}{9} = \frac{24}{56}$ b) $\frac{3}{8} = \frac{12}{32}$

We can use cross products to solve equations.

Example 4

Solve each proportion.

a) $\frac{16}{24} = \frac{x}{3}$ b) $\frac{k+2}{2} = \frac{k-4}{5}$

Solution

Find the cross products.

a) $\frac{16}{24} = \frac{x}{3}$ Multiply. Multiply.

b) $\frac{k+2}{2} = \frac{k-4}{5}$ Multiply. Multiply.

a)

$16 \cdot 3 = 24 \cdot x$ Set the cross products equal.
$48 = 24x$ Multiply.
$2 = x$ Divide by 24.

The solution set is $\{2\}$.

b)

$5(k + 2) = 2(k - 4)$ Set the cross products equal.
$5k + 10 = 2k - 8$ Distribute.
$3k + 10 = -8$ Subtract $2k$.
$3k = -18$ Subtract 10.
$k = -6$ Divide by 3.

The solution set is $\{-6\}$. ■

You Try 4

Solve each proportion.

a) $\frac{2}{3} = \frac{w}{27}$ b) $\frac{b-6}{12} = \frac{b+2}{20}$

Proportions are often used to solve real-world problems. When we solve problems by setting up a proportion, we must be sure that the numerators contain the same quantities and the denominators contain the same quantities.

Example 5

Write an equation and solve.

Cailen is an artist, and she wants to make turquoise paint by mixing the green and blue paints that she already has. To make turquoise, she will have to mix 4 parts of green with 3 parts of blue. If she uses 6 oz of green paint, how much blue paint should she use?

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the amount of blue paint needed.

Step 2: **Choose a variable** to represent the unknown.

x = the number of ounces of blue paint

Step 3: **Translate** the information that appears in English into an algebraic equation. Write a proportion. We will write our ratios in the form of

$\dfrac{\text{Amount of green paint}}{\text{Amount of blue paint}}$ so that the numerators contain the same quantities and the denominators contain the same quantities.

$$\begin{array}{r} \text{Amount of green paint} \rightarrow \\ \text{Amount of blue paint} \rightarrow \end{array} \frac{4}{3} = \frac{6}{x} \begin{array}{l} \leftarrow \text{Amount of green paint} \\ \leftarrow \text{Amount of blue paint} \end{array}$$

The equation is $\dfrac{4}{3} = \dfrac{6}{x}$.

Step 4: **Solve** the equation.

Multiply.

$$\frac{4}{3} = \frac{6}{x}$$

Multiply.

$$\begin{aligned} 4x &= 6 \cdot 3 && \text{Set the cross products equal.} \\ 4x &= 18 && \text{Multiply.} \\ x &= 4.5 && \text{Divide by 4.} \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

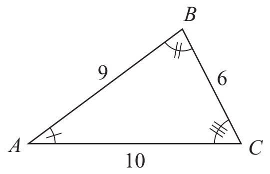
Cailen should mix 4.5 oz of blue paint with the 6 oz of green paint to make the turquoise paint she needs. The check is left to the student. ■

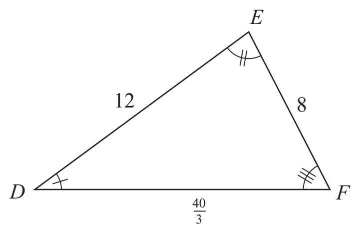
You Try 5

Write an equation and solve.

If 3 lb of coffee costs \$21.60, how much would 5 lb of the same coffee cost?

Another application of proportions is for solving similar triangles.





$$m\angle A = m\angle D, \quad m\angle B = m\angle E, \quad \text{and} \quad m\angle C = m\angle F$$

We say that ΔABC and ΔDEF are *similar triangles*. Two triangles are **similar** if they have the same shape, the corresponding angles have the same measure, and the corresponding sides are proportional.

The ratio of each of the corresponding sides is $\dfrac{3}{4}$:

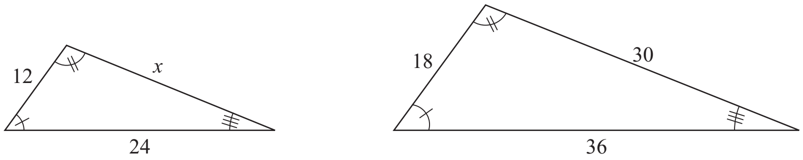
$$\frac{9}{12} = \frac{3}{4}; \quad \frac{6}{8} = \frac{3}{4}; \quad \frac{10}{\frac{40}{3}} = 10 \cdot \frac{3}{40} = \frac{3}{4}.$$

We can use a proportion to find the length of an unknown side in two similar triangles.

Example 6

Given the following similar triangles, find x.

Solution



$$\frac{12}{18} = \frac{x}{30}$$ Set the ratios of two corresponding sides equal to each other. (Set up a proportion.)

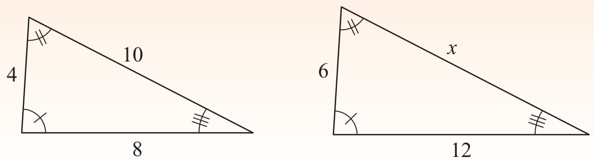
$$12 \cdot 30 = 18 \cdot x$$ Solve the proportion.

$$360 = 18x$$ Multiply.

$$20 = x$$ Divide by 18.

You Try 6

Given the following similar triangles, find x.



3. Solve Problems Involving Denominations of Money

Many applications in algebra involve the number of coins or bills and their values. We will begin this topic with two arithmetic examples; then we will solve an algebraic problem.

Example 7

Determine the amount of money you have in cents *and* in dollars if you have

a) 9 nickels b) 2 quarters c) 9 nickels and 2 quarters

Solution

You may be able to figure out these answers quickly and easily, but what is important here is to understand the *procedure* that is used to do this arithmetic problem so that you can apply the same procedure to algebra. So, read this carefully!

a) and b): Let's begin with part a), finding the value of 9 nickels.

Value in Cents

$$5 \cdot 9 = 45¢$$

Value of a nickel
Number of nickels
Value of 9 nickels

Value in Dollars

$$0.05 \cdot 9 = \$0.45$$

Value of a nickel
Number of nickels
Value of 9 nickels

Here's how we find the value of 2 quarters:

Value in Cents

$$25 \cdot 2 = 50¢$$

Value of a quarter
Number of quarters
Value of 2 quarters

Value in Dollars

$$0.25 \cdot 2 = \$0.50$$

Value of a quarter
Number of quarters
Value of 2 quarters

A table can help us organize the information, so let's put both part a) and part b) in a table so that we can see a pattern.

Value of the Coins **(in cents)**

	Value of the Coin	Number of Coins	Total Value of the Coins
Nickels	5	9	$5 \cdot 9 = 45$
Quarters	25	2	$25 \cdot 2 = 50$

Value of the Coins **(in dollars)**

	Value of the Coin	Number of Coins	Total Value of the Coins
Nickels	0.05	9	$0.05 \cdot 9 = 0.45$
Quarters	0.25	2	$0.25 \cdot 2 = 0.50$

In each case, notice that we find the total value of the coins by multiplying:

Value of the coin $\cdot$ Number of coins $=$ Value of all of the coins

c) Now let's write an equation in English to find the total value of the 9 nickels and 2 quarters.

English:	Value of 9 nickels	plus	Value of 2 quarters	equals	Total value of all the coins
	↓	↓	↓	↓	↓
Cents:	$5(9)$	$+$	$25(2)$		
	45	$+$	50	$=$	$95¢$
Dollars:	$0.05(9)$	$+$	$0.25(2)$		
	0.45	$+$	0.50	$=$	$\$0.95$

We will use the same procedure that we just used to solve these arithmetic problems to write algebraic expressions to represent the value of a collection of coins.

Example 8

Write expressions for the amount of money you have in cents *and* in dollars if you have

a) n nickels b) q quarters c) n nickels and q quarters

Solution

a) and b): Let's use tables just like we did in Example 7. We will put parts a) and b) in the same table.

Value of the Coins **(in cents)**

	Value of the Coin	Number of Coins	Total Value of the Coins
Nickels	5	n	$5 \cdot n = 5n$
Quarters	25	q	$25 \cdot q = 25q$

Value of the Coins **(in dollars)**

	Value of the Coin	Number of Coins	Total Value of the Coins
Nickels	0.05	n	$0.05 \cdot n = 0.05n$
Quarters	0.25	q	$0.25 \cdot q = 0.25q$

If you have n nickels, then the expression for the amount of money in cents is $5n$. The amount of money in dollars is $0.05n$.

If you have q quarters, then the expression for the amount of money in cents is $25q$. The amount of money in dollars is $0.25q$.

c) Write an equation in English to find the total value of n nickels and q quarters. It is based on the same idea that we used in Example 7.

English:	Value of n nickels	plus	Value of q quarters	equals	Total value of all the coins
	↓	↓	↓	↓	↓
Equation in cents:	$5n$	$+$	$25q$	$=$	$5n + 25q$
Equation in dollars:	$0.05n$	$+$	$0.25q$	$=$	$0.05n + 0.25q$

The expression in cents is $5n + 25q$. The expression in dollars is $0.05n + 0.25q$. ■

You Try 7

Determine the amount of money you have in cents and in dollars if you have

a) 8 dimes b) 67 pennies c) 8 dimes and 67 pennies
d) d dimes e) p pennies f) d dimes and p pennies

Now we are ready to solve an algebraic application involving denominations of money.

Example 9

Write an equation and solve.

At the end of the day, Annah counts the money in the cash register at the bakery where she works. There are twice as many dimes as nickels, and they are worth a total of \$5.25. How many dimes and nickels are in the cash register?

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the number of dimes and nickels in the cash register.

Step 2: **Choose a variable** to represent an unknown, and define the other unknown in terms of this variable.

In the statement "there are twice as many dimes as nickels," the number of dimes is expressed *in terms of* the number of nickels. Therefore, let

n = the number of nickels

Define the other unknown (the number of dimes) in terms of n:

$2n$ = the number of dimes

Step 3: **Translate** the information that appears in English into an algebraic equation.

Let's begin by making a table to write an expression for the value of the nickels and the value of the dimes. We will write the expression in terms of dollars because the total value of the coins, \$5.25, is given in dollars.

	Value of the Coin	Number of Coins	Total Value of the Coins
Nickels	0.05	n	$0.05n$
Dimes	0.10	$2n$	$0.10 \cdot (2n)$

Write an equation in English and substitute the expressions we found in the table and the total value of the coins to get an algebraic equation.

English:	Value of the nickels	plus	Value of the dimes	equals	Total value of the coins
	↓	↓	↓	↓	↓
Equation:	$0.05n$	$+$	$0.10(2n)$	$=$	5.25

Step 4: **Solve** the equation.

$$\begin{aligned} 0.05n + 0.10(2n) &= 5.25 \\ 100[0.05n + 0.10(2n)] &= 100(5.25) && \text{Multiply by 100 to eliminate the decimals.} \\ 5n + 10(2n) &= 525 && \text{Distribute.} \\ 5n + 20n &= 525 && \text{Multiply.} \\ 25n &= 525 && \text{Combine like terms.} \\ \frac{25n}{25} &= \frac{525}{25} && \text{Divide each side by 25.} \\ n &= 21 && \text{Simplify.} \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

There were 21 nickels in the cash register and $2(21) = 42$ dimes in the register.

Check: The value of the nickels is $\$0.05(21) = \$1.05$, and the value of the dimes is $\$0.10(42) = \$4.20$. Their total value is $\$1.05 + \$4.20 = \$5.25$. ■

You Try 8

Write an equation and solve.

A collection of coins consists of pennies and quarters. There are three times as many pennies as quarters, and the coins are worth $8.40. How many of each type of coin is in the collection?

4. Solve Problems Involving Distance, Rate, and Time

An important mathematical relationship is one involving distance, rate, and time. These quantities are related by the formula

$$\textbf{Distance} = \textbf{Rate} \times \textbf{Time}$$

and is also written as $d = rt$. We use this formula often in mathematics and in everyday life. Let's use the formula $d = rt$ to answer the following question: If you drive on a highway at a rate of 65 mph for 3 hours, how far will you drive? Using $d = rt$, we get

$$\begin{aligned} d &= rt \\ d &= (65 \text{ mph}) \cdot (3 \text{ hr}) && \text{Substitute the values.} \\ d &= 195 \text{ mi} \end{aligned}$$

Notice that the rate is in miles per *hour*, and the time is in *hours*. That is, the units are consistent, and they must always be consistent to correctly solve a problem like this. If the time had been expressed in minutes, we would have had to convert minutes to hours.

Next we will use the relationship $d = rt$ to solve two algebraic applications.

Example 10

Write an equation and solve.

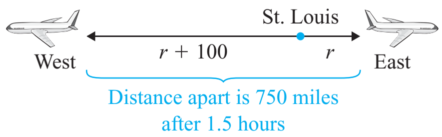
Two planes leave St. Louis, one flying east and the other flying west. The westbound plane travels 100 mph faster than the eastbound plane, and after 1.5 hours they are 750 miles apart. Find the speed of each plane.

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the speed of the eastbound and westbound planes.

We will draw a picture to help us see what is happening in this problem.



Step 2: **Choose a variable** to represent an unknown, and define the other unknown in terms of this variable.

The westbound plane is traveling 100 mph faster than the eastbound plane, so let

$$r = \text{the rate of the eastbound plane}$$
$$r + 100 = \text{the rate of the westbound plane}$$

Label the picture.

Step 3: **Translate** the information that appears in English into an algebraic equation.

Let's make a table using the equation $d = rt$. Fill in the time, 1.5 hr, and the rates first, then multiply those together to fill in the values for the distance.

	d	r	t
Eastbound	$1.5r$	r	1.5
Westbound	$1.5(r + 100)$	$r + 100$	1.5

We will write an equation in English to help us write an algebraic equation. The picture shows that

English:	Distance of westbound plane	plus	Distance of eastbound plane	equals	Distance between the planes after 1.5 hours
	↓	↓	↓	↓	↓
Equation:	$1.5(r + 100)$	$+$	$1.5r$	$=$	750

The expressions for the distances in the equation come from the table.

The equation is $1.5(r + 100) + 1.5r = 750$.

Step 4: **Solve** the equation.

$$\begin{aligned} 1.5(r + 100) + 1.5r &= 750 \\ 10[1.5(r + 100) + 1.5r] &= 10(750) && \text{Multiply by 10 to eliminate the decimals.} \\ 15(r + 100) + 15r &= 7500 && \text{Distribute.} \\ 15r + 1500 + 15r &= 7500 && \text{Distribute.} \\ 30r + 1500 &= 7500 && \text{Combine like terms.} \\ 30r &= 6000 && \text{Subtract 1500.} \\ \frac{30r}{30} &= \frac{6000}{30} && \text{Divide each side by 30.} \\ x &= 200 && \text{Simplify.} \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

The speed of the eastbound plane is 200 mph, and the speed of the westbound plane is $200 + 100 = 300$ mph.

Check to see that $1.5(200) + 1.5(300) = 300 + 450 = 750$ miles. ■

You Try 9

Write an equation and solve.

Two drivers leave Albany, Oregon, on Interstate 5. Dhaval heads south traveling 4 mph faster than Pradeep, who is driving north. After $\frac{1}{2}$ hr, they are 62 miles apart. How fast is each man driving?

Example 11

Write an equation and solve.

Alex and Jenny are taking a cross-country road trip on their motorcycles. Jenny leaves a rest area first traveling at 60 mph. Alex leaves 30 minutes later, traveling on the same highway, at 70 mph. How long will it take Alex to catch Jenny?

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must determine how long it takes Alex to catch Jenny.

We will use a picture to help us see what is happening in this problem.

Since both girls leave the same rest area and travel on the same highway, when Alex catches Jenny they have driven the *same* distance.

Step 2: **Choose a variable** to represent an unknown, and define the other unknown in terms of this variable.

Alex's time is in terms of Jenny's time, so let

t = the number of hours Jenny has been riding when Alex catches her

Alex leaves 30 minutes ($\frac{1}{2}$ hour) after Jenny, so Alex travels $\frac{1}{2}$ hour *less than* Jenny.

$$t - \frac{1}{2} = \text{the number of hours it takes Alex to catch Jenny}$$

Label the picture.

Step 3: **Translate** the information that appears in English into an algebraic equation.

Let's make a table using the equation $d = rt$. Fill in the time and the rates first; then multiply those together to fill in the values for the distance.

	d	**r**	**t**
Jenny	$60t$	60	t
Alex	$70(t - \frac{1}{2})$	70	$t - \frac{1}{2}$

We will write an equation in English to help us write an algebraic equation. The picture shows that

English: Jenny's distance | is the same as | Alex's distance

Equation: $$60t = 70\left(t - \frac{1}{2}\right)$$

The expressions for the distances come from the table.

The equation is $60t = 70\left(t - \frac{1}{2}\right)$.

Step 4: **Solve** the equation.

$$\begin{aligned} 60t &= 70\left(t - \frac{1}{2}\right) & \\ 60t &= 70t - 35 & \text{Distribute.} \\ -10t &= -35 & \text{Subtract } 70t. \\ \frac{-10t}{-10} &= \frac{-35}{-10} & \text{Divide each side by } -10. \\ t &= 3.5 & \text{Simplify.} \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

Remember, Jenny's time is t. Alex's time is $t - \frac{1}{2} = 3\frac{1}{2} - \frac{1}{2} = 3$ hr.

It took Alex 3 hr to catch Jenny.

Check to see that Jenny travels 60 mph · (3.5 hr) = 210 miles, and Alex travels 70 mph · (3 hr) = 210 miles. The girls traveled the same distance. ■

You try 10

Write an equation and solve.

Brad leaves home driving 40 mph. Angelina leaves the house 30 minutes later driving the same route at 50 mph. How long will it take Angelina to catch Brad?

Answers to You Try Exercises

1) $\frac{3}{2}$ 2) 128-oz bottle; \$0.043/oz 3) a) false b) true 4) a) {18} b) {18}

5) \$36.00 6) 15 7) a) 80¢; \$0.80 b) 67¢; \$0.67 c) 147¢; \$1.47 d) $10d$ cents; $0.10d$ dollars e) p cents; $0.01p$ dollars f) $10d + p$ cents; $0.10d + 0.01p$ dollars 8) 30 quarters, 90 pennies

9) Dhaval: 64 mph, Pradeep: 60 mph 10) 2 hr

3.6 Exercises

Objective 1: Use Ratios

1) Write three ratios that are equivalent to $\frac{3}{4}$.

2) Is 0.65 equivalent to the ratio 13 to 20? Explain.

3) Is a percent a type of ratio? Explain.

4) Write 57% as a ratio.

Write as a ratio in lowest terms.

5) 16 girls to 12 boys

6) 9 managers to 90 employees

7) 4 coaches to 50 team members

8) 30 blue marbles to 18 red marbles

9) 20 feet to 80 feet

10) 7 minutes to 4 minutes

11) 2 feet to 36 inches

12) 30 minutes to 3 hours

13) 18 hours to 2 days

14) 20 inches to 3 yards

A store sells the same product in different sizes. Determine which size is the best buy based on the unit price of each item.

15) Batteries

Number	Price
8	\$ 6.29
16	\$12.99

16) Cat litter

Size	Price
30 lb	\$ 8.48
50 lb	\$12.98

17) Mayonnaise

Size	Price
8 oz	\$2.69
15 oz	\$3.59
48 oz	\$8.49

18) Applesauce

Size	Price
16 oz	\$1.69
24 oz	\$2.29
48 oz	\$3.39

19) Cereal

Size	Price
11 oz	\$4.49
16 oz	\$5.15
24 oz	\$6.29

20) Shampoo

Size	Price
14 oz	\$3.19
25 oz	\$5.29
32 oz	\$6.99

Objective 2: Solve a Proportion

21) What is the difference between a ratio and a proportion?

22) In the proportion $\frac{a}{b} = \frac{c}{d}$, can $b = 0$? Explain.

Determine whether each proportion is true or false.

23) $\frac{4}{7} = \frac{20}{35}$

24) $\frac{54}{64} = \frac{7}{8}$

25) $\frac{72}{54} = \frac{8}{7}$

26) $\frac{120}{140} = \frac{30}{35}$

27) $\frac{8}{10} = \frac{\frac{2}{5}}{\frac{1}{2}}$

28) $\frac{3}{4} = \frac{\frac{1}{2}}{\frac{2}{3}}$

Solve each proportion.

29) $\frac{8}{36} = \frac{c}{9}$

30) $\frac{n}{3} = \frac{20}{15}$

31) $\frac{w}{15} = \frac{32}{12}$

32) $\frac{8}{14} = \frac{d}{21}$

33) $\frac{40}{24} = \frac{30}{a}$

34) $\frac{10}{x} = \frac{12}{54}$

35) $\frac{2}{k} = \frac{9}{12}$

36) $\frac{15}{27} = \frac{m}{6}$

37) $\frac{3z + 10}{14} = \frac{2}{7}$

38) $\frac{8t - 9}{20} = \frac{3}{4}$

39) $\dfrac{r+7}{9} = \dfrac{r-5}{3}$

40) $\dfrac{b+6}{5} = \dfrac{b+10}{15}$

41) $\dfrac{3h+15}{16} = \dfrac{2h+5}{4}$

42) $\dfrac{a+7}{8} = \dfrac{4a-11}{6}$

43) $\dfrac{4m-1}{6} = \dfrac{6m}{10}$

44) $\dfrac{9w+8}{10} = \dfrac{5-3w}{12}$

Set up a proportion and solve.

45) If 4 containers of yogurt cost \$2.36, find the cost of 6 containers of yogurt.

46) Find the cost of 3 scarves if 2 scarves cost \$29.00.

47) A marinade for chicken uses 2 parts of lime juice for every 3 parts of orange juice. If the marinade uses $\frac{1}{3}$ cup of lime juice, how much orange juice should be used?

48) The ratio of salt to baking soda in a cookie recipe is 0.75 to 1. If a recipe calls for $1\frac{1}{2}$ teaspoons of salt, how much baking soda is in the cookie dough?

VIDEO 49) A 12-oz serving of Mountain Dew contains 55 mg of caffeine. How much caffeine is in an 18-oz serving of Mountain Dew?
(www.energyfiend.com)

50) An 8-oz serving of Red Bull energy drink contains about 80 mg of caffeine. Approximately how much caffeine is in 12 oz of Red Bull?
(www.energyfiend.com)

51) Approximately 9 out of 10 smokers began smoking before the age of 21. In a group of 400 smokers, about how many of them started before they reached their 21st birthday?
(www.lungusa.org)

52) Ridgemont High School administrators estimate that 2 out of 3 members of its student body attended the homecoming football game. If there are 1941 students in the school, how many went to the game?

53) At the end of a week, Ernest put 20 lb of yard waste and some kitchen scraps on the compost pile. If the ratio of yard waste to kitchen scraps was 5 to 2, how many pounds of kitchen scraps did he put on the pile?

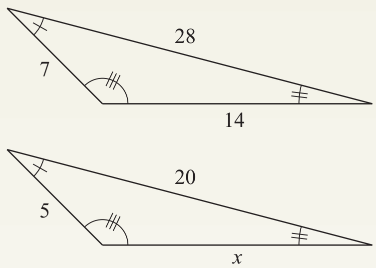
54) On a map of the United States, 1 inch represents 120 miles. If two cities are 3.5 inches apart on the map, what is the actual distance between the two cities?

55) On July 4, 2009, the exchange rate was such that \$20.00 (American) was worth 14.30 Euros. How many Euros could you get for \$50.00?
(www.xe.com)

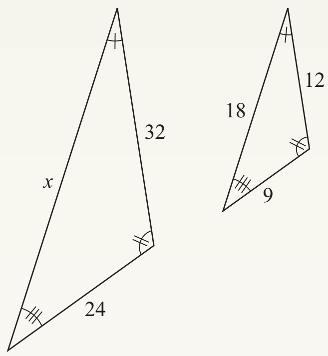
56) On July 4, 2009, the exchange rate was such that 100 British pounds were worth \$163.29 (American). How many dollars could you get for 280 British pounds?
(www.xe.com)

Given the following similar triangles, find x.

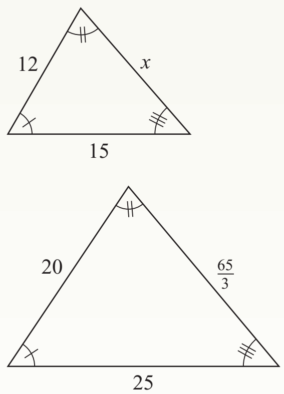
VIDEO 57)



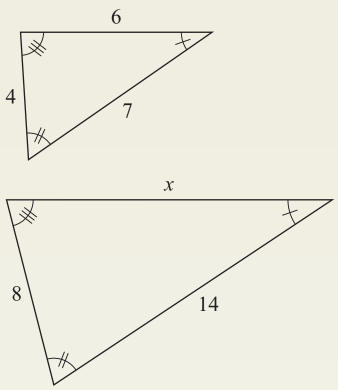
58)



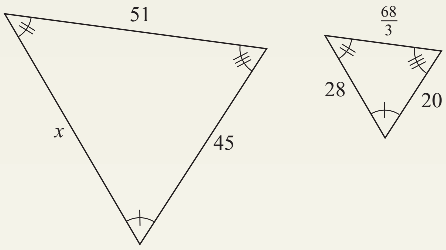
59)



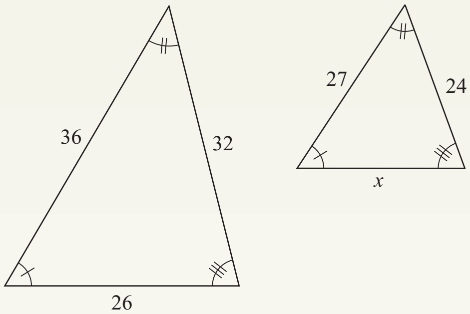
60)



61)



62)



Objective 3: Solve Problems Involving Denominations of Money

For Exercises 63–68, determine the amount of money a) in dollars and b) in cents given the following quantities.

63) 7 dimes

64) 17 nickels

65) 422 pennies

66) 14 quarters

67) 9 nickels and 7 quarters

68) 73 pennies and 14 dimes

For Exercises 69–74, write an expression that represents the amount of money in a) dollars and b) cents given the following quantities.

69) q quarters

70) p pennies

71) d dimes

72) n nickels

73) p pennies and n nickels

74) q quarters and d dimes

Solve using the five-step method.

75) Turtle and Vince combine their coins to find they have all dimes and quarters. They have 8 more quarters than dimes, and the coins are worth a total of \$5.15. How many dimes and quarters do they have?

76) Johnny saves all of his nickels and dimes in a jar. One day he counted them and found that there were 131 coins worth \$9.20. How many pennies and how many nickels were in the jar?

77) Eric has been saving his paper route money. He has \$73.00 consisting of \$5 bills and \$1 bills. If he has a total of 29 bills, how many \$5 bills and how many \$1 bills does he have?

78) A bank employee is servicing the automated teller machine after a busy Friday night. She finds the machine contains only \$20 bills and \$10 bills and that there are twice as many \$20 bills remaining as there are \$10 bills. If there is a total of \$600.00 left in the machine, how many of the bills are twenties, and how many are tens?

79) The community pool charges \$9.00 for adults and \$7.00 for children. The total revenue for a particular cloudy day is \$437.00. Determine the number of adults and the number of children who went to the pool that day if twice as many children paid for admission as adults.

80) At the convenience store, Sandeep buys 12 more 44¢ stamps than 28¢ stamps. If he spends \$11.04 on the stamps, how many of each type did he buy?

81) Carlos attended two concerts with his friends at the American Airlines Arena in Miami. He bought five tickets to see Marc Anthony and two tickets to the Santana concert for \$563. If the Santana ticket cost \$19.50 less than the Marc Anthony ticket, find the cost of a ticket to each concert. (www.pollstaronline.com)

82) Both of the pop groups Train and Maroon 5 played at the House of Blues in North Myrtle Beach, South Carolina, in 2003. If Train tickets cost \$14.50 more than Maroon 5 tickets and four Maroon 5 tickets and four Train tickets would have cost \$114, find the cost of a ticket to each concert. (www.pollstaronline.com)

Objective 4: Solve Problems Involving Distance, Rate, and Time

83) If you use the formula $d = rt$ to find the distance traveled by a car when its rate is given in miles per hour and its time traveled is given in hours, what would be the units of its distance?

84) If you use the formula $d = rt$ to find the distance traveled by a car when its rate is given in miles per hour and its time traveled is given in minutes, what must you do before you substitute the rate and time into the formula?

Solve using the five-step method.

85) Two planes leave San Francisco, one flying north and the other flying south. The southbound plane travels 50 mph faster than the northbound plane, and after 2 hours they are 900 miles apart. Find the speed of each plane.

86) Two cars leave Indianapolis, one driving east and the other driving west. The eastbound car travels 8 mph slower than the westbound car, and after 3 hours they are 414 miles apart. Find the speed of each car.

87) When Lance and Danica pass each other on their bikes going in opposite directions, Lance is riding at 22 mph, and Danica is pedaling at 18 mph. If they continue at those speeds, after how long will they be 200 miles apart?

88) A car and a truck leave the same location, the car headed east and the truck headed west. The truck's speed is 10 mph less than the speed of the car. After 3 hours, the car and truck are 330 miles apart. Find the speed of each vehicle.

89) Ahmad and Davood leave the same location traveling the same route, but Davood leaves 20 minutes after Ahmad. If Ahmad drives 30 mph and Davood drives 36 mph, how long will it take Davood to catch Ahmad?

90) Nayeli and Elena leave the gym to go to work traveling the same route, but Nayeli leaves 10 minutes after Elena. If Elena drives 60 mph and Nayeli drives 72 mph, how long will it take Nayeli to catch Elena?

VIDEO 91) A truck and a car leave the same intersection traveling in the same direction. The truck is traveling at 35 mph, and the car is traveling at 45 mph. In how many minutes will they be 6 miles apart?

92) Greg is traveling north on a road while Peter is traveling south on the same road. They pass by each other at noon, Greg driving 30 mph and Peter driving 40 mph. At what time will they be 105 miles apart?

93) Nick and Scott leave opposite ends of a bike trail 13 miles apart and travel toward each other. Scott is traveling 2 mph slower than Nick. Find each of their speeds if they meet after 30 minutes.

94) At 3:00 P.M., a truck and a car leave the same intersection traveling in the same direction. The truck is traveling at 30 mph, and the car is traveling at 42 mph. At what time will they be 9 miles apart?

95) A passenger train and a freight train leave cities 400 miles apart and travel toward each other. The passenger train is traveling 20 mph faster than the freight train. Find the speed of each train if they pass each other after 5 hours.

96) A freight train passes the Old Towne train station at 11:00 A.M. going 30 mph. Ten minutes later, a passenger train, headed in the same direction on an adjacent track, passes the same station at 45 mph. At what time will the passenger train catch the freight train?

Mixed Exercises: Objectives 2–4

Solve using the five-step method.

97) If the exchange rate between the American dollar and the Japanese yen is such that $4.00 = 442 yen, how many yen could be exchanged for $70.00? (moneycentral.msn.com)

98) A collection of coins contains 73 coins, all nickels and quarters. If the value of the coins is $14.05, determine the number of each type of coin in the collection.

99) Sherri is riding her bike at 10 mph when Bill passes her going in the opposite direction at 14 mph. How long will it take before the distance between them is 6 miles?

100) The ratio of sugar to flour in a brownie recipe is 1 to 2. If the recipe used 3 cups of flour, how much sugar is used?

101) At the end of her shift, a cashier has a total of $6.70 in dimes and quarters. There are 11 more dimes than quarters. How many of each of these coins does she have?

102) Paloma leaves Mateo's house traveling 30 mph. Mateo leaves 15 minutes later, trying to catch up to Paloma, going 40 mph. If they drive along the same route, how long will it take Mateo to catch Paloma?

103) A jet flying at an altitude of 30,000 ft passes over a small plane flying at 15,000 ft headed in the same direction. The jet is flying twice as fast as the small plane, and 45 minutes later they are 150 miles apart. Find the speed of each plane.

104) Tickets for a high school play cost $3.00 each for children and $5.00 each for adults. The revenue from one performance was $663, and 145 tickets were sold. How many adult tickets and how many children's tickets were sold?

105) A survey of teenage girls found that 3 out of 5 of them earned money by babysitting. If 400 girls were surveyed, how many of them were babysitters?

106) A car and a tour bus leave the same location and travel in opposite directions. The car's speed is 12 mph more than the speed of the bus. If they are 270 miles apart after $2\frac{1}{2}$ hours, how fast is each vehicle traveling?

Section 3.7 Solving Linear Inequalities in One Variable

Objectives

1. Use Graphs and Set and Interval Notations
2. Solve Inequalities Using the Addition and Subtraction Properties of Inequality
3. Solve Inequalities Using the Multiplication Property of Inequality
4. Solve Inequalities Using a Combination of the Properties
5. Solve Three-Part Inequalities
6. Solve Applications Involving Linear Inequalities

Recall the inequality symbols

$<$ "is less than" $\leq$ "is less than or equal to"

$>$ "is greater than" $\geq$ "is greater than or equal to"

We will use the symbols to form *linear inequalities in one variable*.

Some examples of linear inequalities in one variable are $2x + 11 \leq 19$ and $y > -4$.

Definition

A **linear inequality in one variable** can be written in the form $ax + b < c$, $ax + b \leq c$, $ax + b > c$, or $ax + b \geq c$, where a, b, and c are real numbers and $a \neq 0$.

The solution to a linear inequality is a set of numbers that can be represented in one of three ways:

1) On a graph
2) In *set notation*
3) In *interval notation*

In this section, we will learn how to solve linear inequalities in one variable and how to represent the solution in each of those three ways.

Graphing an Inequality and Using the Notations

1. Use Graphs and Set and Interval Notations

Example 1

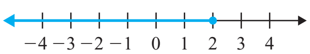
Graph each inequality and express the solution in set notation and interval notation.

a) $x \leq 2$ b) $k > -3$

Solution

a) $x \leq 2$

When we graph $x \leq 2$, we are finding the solution set of $x \leq 2$. What value(s) of x will make the inequality true? The largest solution is 2. Also, any number *less than* 2 will make $x \leq 2$ true. We represent this **on the number line** as follows:



The graph illustrates that the solution is the set of all numbers less than and including 2.

Notice that the dot on 2 is shaded. This tells us that 2 is included in the solution set. The shading to the left of 2 indicates that *any* real number (not just integers) in this region is a solution.

We can write the solution set in **set notation** this way: $\{x | x \leq 2\}$. This means

$\{x$ — The set of all values of x

$|$ — such that

$x \leq 2\}$ — x is less than or equal to 2.

In **interval notation** we write

$$(-\infty, 2]$$

$-\infty$ is not a number. x gets infinitely more negative without bound. Use a "(" instead of a bracket.

The bracket indicates the 2 is included in the interval.

Note

The variable does not appear anywhere in interval notation.

b) $k > -3$

We will plot -3 as an *open circle* on the number line because the symbol is "$>$" and *not* "$\geq$." The inequality $k > -3$ means that we must find the set of all numbers, k, *greater than* (but *not* equal to) -3. Shade to the right of -3.

The graph illustrates that the solution is the set of all numbers greater than -3 but not including -3.

We can write the solution set in *set notation* this way: $\{k \mid k > -3\}$

In *interval notation* we write

$$(-3, \infty)$$

The "(" indicates that -3 is the lower bound of the interval but that it is not included.

∞ is not a number. k gets increasingly bigger without bound. Use a "(" instead of a bracket.

Hints for using interval notation:

1) The variable never appears in interval notation.
2) A number *included* in the solution set gets a bracket: $x \leq -2 \rightarrow (-\infty, -2]$
3) A number *not included* in the solution set gets a parenthesis: $k > -3 \rightarrow (-3, \infty)$
4) The symbols $-\infty$ and ∞ *always* get parentheses.
5) The smaller number is always placed to the left. The larger number is placed to the right.
6) Even if we are not asked to graph the solution set, the graph may be helpful in writing the interval notation correctly.

You Try 1

Graph each inequality and express the solution in interval notation.

a) $z \geq -1$ b) $n < 4$

2. Solve Inequalities Using the Addition and Subtraction Properties of Inequality

The addition and subtraction properties of equality help us to solve equations. Similar properties hold for inequalities as well.

Property Addition and Subtraction Properties of Inequality

Let a, b, and c be real numbers. Then,

1) $a < b$ and $a + c < b + c$ are equivalent

and

2) $a < b$ and $a - c < b - c$ are equivalent.

Adding the same number to both sides of an inequality or subtracting the same number from both sides of an inequality will not change the solution.

Note

The above properties hold for any of the inequality symbols.

Example 2

Solve $n - 9 \geq -8$. Graph the solution set and write the answer in interval and set notations.

Solution

$$n - 9 \geq -8$$
$$n - 9 + 9 \geq -8 + 9 \qquad \text{Add 9 to each side.}$$
$$n \geq 1$$

The solution set in interval notation is $[1, \infty)$.
In set notation, we write $\{n \mid n \geq 1\}$.

You Try 2

Solve $q - 5 \geq -3$. Graph the solution set and write the answer in interval and set notations.

3. Solve Inequalities Using the Multiplication Property of Inequality

Let's see how multiplication works in inequalities.

Begin with an inequality we know is true and multiply both sides by a *positive* number.

$$2 < 5 \qquad \text{True}$$
$$3(2) < 3(5) \qquad \text{Multiply by 3.}$$
$$6 < 15 \qquad \text{True}$$

Begin again with $2 < 5$ and multiply both sides by a *negative* number.

$$2 < 5 \qquad \text{True}$$
$$-3(2) < -3(5) \qquad \text{Multiply by } -3.$$
$$-6 < -15 \qquad \text{False}$$

To make $-6 < -15$ into a *true* statement, we must *reverse the direction of the inequality symbol.*

$$-6 > -15 \qquad \text{True}$$

If you begin with a true inequality and *divide* by a positive number or by a negative number, the results will be the same as above since division can be defined in terms of multiplication. This leads us to the multiplication property of inequality.

Property Multiplication Property of Inequality

Let a, b, and c be real numbers.

1) If c is a *positive* number, then $a < b$ and $ac < bc$ are equivalent inequalities and have the same solutions.

2) If c is a *negative* number, then $a < b$ and $ac > bc$ are equivalent inequalities and have the same solutions.

It is also true that if $c > 0$ and $a < b$, then $\frac{a}{c} < \frac{b}{c}$. If $c < 0$ and $a < b$, then $\frac{a}{c} > \frac{b}{c}$.

For the most part, the procedures used to solve linear inequalities are the same as those for solving linear equations **except** *when you multiply or divide an inequality by a negative number, you must reverse the direction of the inequality symbol.*

Example 3

Solve each inequality. Graph the solution set and write the answer in interval and set notations.

a) $-6t \le 12$ b) $6t \le -12$

Solution

a) $-6t \le 12$

First, divide each side by -6. *Since we are dividing by a negative number, we must remember to reverse the direction of the inequality symbol.*

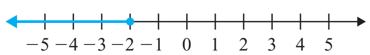
$$-6t \le 12$$
$$\frac{-6t}{-6} \ge \frac{12}{-6} \quad \text{Divide by } -6 \text{, so reverse the inequality symbol.}$$
$$t \ge -2$$

Interval notation: $[-2, \infty)$ Set notation: $\{t \mid t \ge -2\}$

b) $6t \le -12$

First, divide by 6. Since we are dividing by a *positive* number, the inequality symbol remains the same.

$$6t \le -12$$
$$\frac{6t}{6} \le \frac{-12}{6} \quad \text{Divide by 6. Do } \textit{not} \text{ reverse the inequality symbol.}$$
$$t \le -2$$



Interval notation: $(-\infty, -2]$ Set notation: $\{t \mid t \le -2\}$ ■

You Try 3

Solve $-\frac{1}{2}t < 3$. Graph the solution set and write the answer in interval and set notations.

4. Solve Inequalities Using a Combination of the Properties

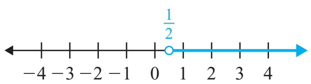
Often it is necessary to combine the properties to solve an inequality.

Example 4

Solve $3(1 - 4a) + 15 < 2(2a + 5)$. Graph the solution set and write the answer in interval and set notations.

Solution

$$
\begin{aligned}
3(1 - 4a) + 15 &< 2(2a + 5) && \\
3 - 12a + 15 &< 4a + 10 && \text{Distribute.} \\
18 - 12a &< 4a + 10 && \text{Combine like terms.} \\
18 - 12a - 4a &< 4a - 4a + 10 && \text{Subtract } 4a \text{ from each side.} \\
18 - 16a &< 10 && \\
18 - 18 - 16a &< 10 - 18 && \text{Subtract 18 from each side.} \\
-16a &< -8 && \\
\frac{-16a}{-16} &> \frac{-8}{-16} && \text{Divide both sides by } -16. \text{ Reverse the inequality symbol.} \\
a &> \frac{1}{2} && \text{Simplify.}
\end{aligned}
$$



The solution set in interval notation is $\left(\frac{1}{2}, \infty\right)$.

In set notation, we write $\left\{a \,\middle|\, a > \frac{1}{2}\right\}$.

You Try 4

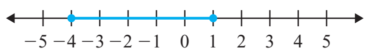
Solve $5(b + 2) - 3 > 4(2b - 1)$. Graph the solution set and write the answer in interval and set notations.

5. Solve Three-Part Inequalities

A **three-part inequality** states that one number is between two other numbers. Some examples are $5 < 8 < 12$, $-4 \le x \le 1$, and $0 < r + 2 < 5$. They are also called **compound inequalities** because they contain more than one inequality symbol.

The inequality $-4 \le x \le 1$ means that x is *between* -4 and 1, and -4 and 1 are included in the interval.

On a number line, the inequality would be represented as



Notice that the **lower bound** of the interval on the number line is -4 (including -4), and the **upper bound** is 1 (including 1). Therefore, we can write the interval notation as

$$[-4, 1]$$

The endpoint, -4, is included in the interval, so use a bracket.

The endpoint, 1, is included in the interval, so use a bracket.

The set notation to represent $-4 \le x \le 1$ is $\{x \mid -4 \le x \le 1\}$.

Next, we will solve the inequality $0 < r + 2 < 5$. To solve this type of compound inequality, you must remember that *whatever operation you perform on one part of the inequality must be performed on all three parts.* All properties of inequalities apply.

Example 5

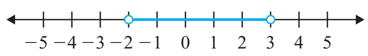
Solve $0 < r + 2 < 5$. Graph the solution set, and write the answer in interval notation.

Solution

$$0 < r + 2 < 5$$

$$0 - 2 < r + 2 - 2 < 5 - 2 \qquad \text{To get the } r \text{ by itself, subtract 2 from each part of the inequality.}$$

$$-2 < r < 3$$



The solution set is $(-2, 3)$. ■

Note

Use parentheses here since -2 and 3 are not included in the solution set.

You Try 5

Solve $-1 \le 5w + 4 \le 14$. Graph the solution set, and write the answer in interval notation.

We can eliminate fractions in an inequality by multiplying by the LCD of all of the fractions.

Example 6

Solve $-\frac{5}{6} < \frac{1}{4}p + \frac{5}{12} \le \frac{4}{3}$. Graph the solution set, and write the answer in interval notation.

Solution

The LCD of the fractions is 12. Multiply by 12 to eliminate the fractions.

$$-\frac{5}{6} < \frac{1}{4}p + \frac{5}{12} \le \frac{4}{3}$$

$$12\left(-\frac{5}{6}\right) < 12\left(\frac{1}{4}p + \frac{5}{12}\right) \le 12\left(\frac{4}{3}\right) \qquad \text{Multiply all parts of the inequality by 12.}$$

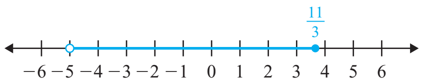
$$-10 < 3p + 5 \le 16$$

$$-10 - 5 < 3p + 5 - 5 \le 16 - 5 \qquad \text{Subtract 5 from each part.}$$

$$-15 < 3p \le 11 \qquad \text{Combine like terms.}$$

$$-\frac{15}{3} < \frac{3p}{3} \le \frac{11}{3} \qquad \text{Divide each part by 3.}$$

$$-5 < p \le \frac{11}{3} \qquad \text{Simplify.}$$



The solution set is $\left(-5, \frac{11}{3}\right]$. ■

You Try 6

Solve $-\frac{3}{2} \le \frac{1}{4}x - \frac{3}{2} < \frac{1}{8}$. Graph the solution set, and write the answer in interval notation.

Remember, if we multiply or divide an inequality by a negative number, we reverse the direction of the inequality symbol. When solving a compound inequality like these, reverse *both* symbols.

Example 7

Solve $3 < -2m + 7 < 13$. Graph the solution set, and write the answer in interval notation.

Solution

$$3 < -2m + 7 < 13$$

$$3 - 7 < -2m + 7 - 7 < 13 - 7 \qquad \text{Subtract 7 from each part.}$$

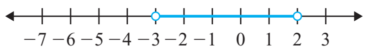
$$-4 < -2m < 6$$

$$\frac{-4}{-2} > \frac{-2m}{-2} > \frac{6}{-2} \qquad \text{Divide by } -2 \text{ and reverse the direction of the inequality symbols.}$$

$$2 > m > -3 \qquad \text{Simplify.}$$

Think carefully about what $2 > m > -3$ means. It means "m is less than 2 *and* m is greater than -3." This is especially important to understand when writing the correct interval notation.

The graph of the solution set is



Even though our result is $2 > m > -3$, -3 is actually the lower bound of the solution set and 2 is the upper bound. The inequality $2 > m > -3$ can also be written as $-3 < m < 2$.

Interval notation: $(-3, 2)$.

Lower bound on the left ↑ ↑ Upper bound on the right ■

You Try 7

Solve $-1 < -4p + 11 < 15$. Graph the solution set, and write the answer in interval notation.

6. Solve Applications Involving Linear Inequalities

Certain phrases in applied problems indicate the use of inequality symbols:

at least: $\ge$	no less than: $\ge$
at most: $\le$	no more than: $\le$

There are others. Next, we will look at an example of a problem involving the use of an inequality symbol. We will use the same steps that were used to solve applications involving equations.

Example 8

Keisha is planning a baby shower for her sister. The restaurant charges \$450 for the first 25 people plus \$15 for each additional guest. If Keisha can spend at most \$700, find the greatest number of people who can attend the shower.

Solution

Step 1: **Read** the problem carefully. We must find the greatest number of people who can attend the shower.

Step 2: **Choose a variable** to represent the unknown quantity. We know that the first 25 people will cost \$450, but we do not know how many *additional* guests Keisha can afford to invite.

$x =$ number of people **over** the first 25 who attend the shower

Step 3: **Translate** from English to an algebraic inequality.

English:	Cost of first 25 people	$+$	Cost of additional guests		is at most		\$700
	$\downarrow$		$\downarrow$		$\downarrow$		$\downarrow$
Inequality:	450	$+$	$15x$		$\leq$		700

The inequality is $450 + 15x \leq 700$.

Step 4: **Solve** the inequality.

$$450 + 15x \leq 700$$
$$15x \leq 250 \qquad \text{Subtract 450.}$$
$$x \leq 16.\overline{6} \qquad \text{Divide by 15.}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

The result was $x \leq 16.\overline{6}$, where x represents the number of additional people who can attend the baby shower. Since it is not possible to have $16.\overline{6}$ people, and $x \leq 16.\overline{6}$, in order to stay within budget, Keisha can afford to pay for at most 16 additional guests *over* the initial 25.

Therefore, the greatest number of people who can attend the shower is

$$\begin{array}{ccccc} \text{The first 25} & + & \text{Additional} & = & \text{Total} \\ \downarrow & & \downarrow & & \\ 25 & + & 16 & = & 41 \end{array}$$

At most, 41 people can attend the baby shower.

Does the answer make sense?

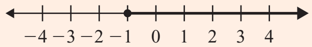
Total cost of shower $= \$450 + \$15(16) = \$450 + \$240 = \$690$

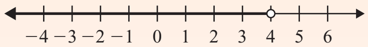
We can see that one more guest (at a cost of \$15) would put Keisha over budget. ■

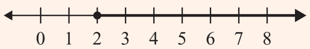
You Try 8

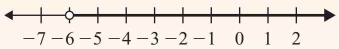
Tristan's basic mobile phone plan gives him 500 minutes of calling per month for \$40.00. Each additional minute costs \$0.25. If he can spend at most \$55.00 per month on his phone bill, find the greatest number of minutes Tristan can talk each month.

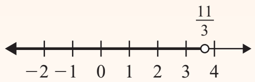
Answers to You Try Exercises

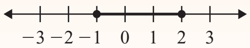
1) a) $[-1, \infty)$ 

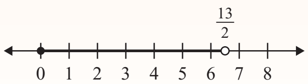
b) $(-\infty, 4)$ 

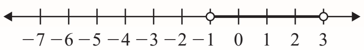
2) interval: $[2, \infty)$, set: $\{q \mid q \geq 2\}$ 

3) interval: $(-6, \infty)$, set: $\{t \mid t > -6\}$ 

4) interval: $\left(-\infty, \frac{11}{3}\right)$, set: $\left\{b \,\middle|\, b < \frac{11}{3}\right\}$ 

5) $[-1, 2]$ 

6) $\left[0, \frac{13}{2}\right)$ 

7) $(-1, 3)$ 

8) 560

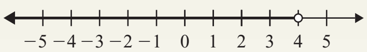
3.7 Exercises

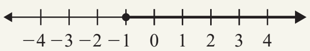
Objective 1: Use Graphs and Set and Interval Notations

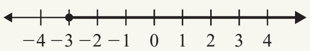
1) When do you use brackets when writing a solution set in interval notation?

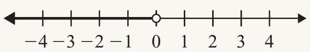
2) When do you use parentheses when writing a solution set in interval notation?

Write each set of numbers in interval notation.

3) 

4) 

5) 

6) 

Graph the inequality. Express the solution in a) set notation and b) interval notation.

7) $k \leq 2$

8) $y \geq 3$

9) $c < \frac{5}{2}$

10) $n > -\frac{11}{3}$

11) $a \geq -4$

12) $x < -1$

Mixed Exercises: Objectives 2 and 3

13) When solving an inequality, when do you change the direction of the inequality symbol?

14) What is solution set of $-4x \leq 12$?

a) $(-\infty, -3]$

b) $[-3, \infty)$

c) $(-\infty, 3]$

d) $[3, \infty)$

Solve each inequality. Graph the solution set and write the answer in a) set notation and b) interval notation.

15) $k + 9 \geq 7$

16) $t - 3 \leq 2$

17) $c - 10 \leq -6$

18) $x + 12 \geq 8$

19) $-3 + d < -4$

20) $1 + k > 1$

21) $16 < z + 11$

22) $-5 > p - 7$

23) $5m > 15$

24) $10r > 40$

25) $12x < -21$

26) $6y < -22$

27) $-4b \leq 32$

28) $-7b \geq 21$

29) $-24a < -40$

30) $-12n > -36$

31) $\frac{1}{3}k \geq -5$

32) $\frac{1}{2}w < -3$

33) $-\frac{3}{8}c < -3$

34) $-\frac{7}{2}d \geq 35$

Objective 4: Solve Inequalities Using a Combination of the Properties

Solve each inequality. Graph the solution set and write the answer in interval notation.

35) $4p - 11 \leq 17$

36) $6y + 5 > -13$

37) $9 - 2w \leq 11$

38) $17 - 7x \geq 20$

39) $-\frac{3}{4}m + 10 > 1$

40) $\frac{1}{2}k - 3 < -2$

41) $3c + 10 > 5c + 13$

42) $a + 2 < 2a + 3$

43) $3(n + 1) - 16 \leq 2(6 - n)$

44) $-6 - (t + 8) \leq 2(11 - 3t) + 4$

45) $\frac{8}{3}(2k + 1) > \frac{1}{6}k + \frac{8}{3}$

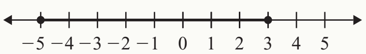
46) $\frac{11}{6} + \frac{3}{2}(d - 2) > \frac{2}{3}(d + 5) + \frac{1}{2}d$

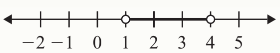
47) $0.05x + 0.09(40 - x) > 0.07(40)$

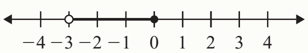
48) $0.02c + 0.1(30) < 0.08(30 + c)$

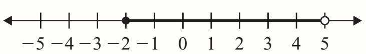
Objective 5: Solve Three–Part Inequalities

Write each set of numbers in interval notation.

49) 

50) 

51) 

52) 

Graph the inequality. Express the solution in a) set notation and b) interval notation.

53) $-4 < y < 0$

54) $1 \leq t \leq 4$

55) $-3 \leq k \leq 2$

56) $-2 < p < 1$

57) $\frac{1}{2} < n \leq 3$

58) $-2 \leq a < 3$

Solve each inequality. Graph the solution set and write the answer in interval notation.

59) $-11 \leq b - 8 \leq -7$

60) $4 < k + 9 < 10$

61) $-10 < 2a < 7$

62) $-5 \leq 5m \leq -2$

63) $-5 \leq 4x - 13 \leq 7$

64) $-4 < 2y - 7 < -1$

65) $-17 < \frac{3}{2}c - 5 < 1$

66) $2 \leq \frac{1}{2}n + 3 \leq 5$

67) $-6 \leq 4c - 13 < -1$

68) $-4 \leq 3w - 1 \leq 3$

69) $4 \leq \frac{k + 11}{4} \leq 5$

70) $0 < \frac{5t + 2}{3} < \frac{7}{3}$

71) $-7 \leq 8 - 5y < 3$

72) $-9 < 7 - 4m < 9$

73) $2 < 10 - p \leq 5$

74) $6 \leq 4 - 3b < 10$

Objective 6: Solve Applications Involving Linear Inequalities

Write an inequality for each problem and solve.

75) Leslie is planning a party for her daughter at Princess Party Palace. The cost of a party is \$180 for the first 10 children plus \$16.00 for each additional child. If Leslie can spend at most \$300, find the greatest number of children who can attend the party.

76) Big-City Parking Garage charges \$36.00 for the first 4 hours plus \$3.00 for each additional half-hour. Eduardo has \$50.00 for parking. For how long can Eduardo park his car in this garage?

77) Heinrich is planning an Oktoberfest party at the House of Bratwurst. It costs \$150.00 to rent a tent plus \$11.50 per person for food. If Heinrich can spend at most \$450.00, find the greatest number of people he can invite to the party.

78) A marketing company plans to hold a meeting in a large conference room at a hotel. The cost of renting the room is \$500, and the hotel will provide snacks and beverages for an additional \$8.00 per person. If the company has budgeted \$1000.00 for the room and refreshments, find the greatest number of people who can attend the meeting.

79) A taxi in a large city charges \$2.50 plus \$0.40 for every $\frac{1}{5}$ of a mile. How many miles can you go if you have \$14.50?

80) A taxi in a small city charges \$2.00 plus \$0.30 for every $\frac{1}{4}$ of a mile. How many miles can you go if you have \$14.00?

VIDEO 81) Melinda's first two test grades in Psychology were 87 and 94. What does she need to make on the third test to maintain an average of at least 90?

82) Eliana's first three test scores in Algebra were 92, 85, and 96. What does she need to make on the fourth test to maintain an average of at least 90?

Section 3.8 Solving Compound Inequalities

Objectives

1. Find the Intersection and Union of Two Sets
2. Solve Compound Inequalities Containing the Word *And*
3. Solve Compound Inequalities Containing the Word *Or*
4. Solve Special Compound Inequalities

Compound inequalities like $-8 \le 3x + 4 \le 13$ were introduced in Section 3.7. In this section, we will discuss how to solve compound inequalities like the following two:

$$t \le \frac{1}{2} \text{ or } t \ge 3 \quad \text{and} \quad 2z + 9 < 5 \text{ and } z - 1 > 6$$

First, we must talk about set notation and operations.

1. Find the Intersection and Union of Two Sets

Example 1

Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{3, 5, 7, 9, 11\}$.

The **intersection** of sets A and B is the set of numbers that are elements of A **and** of B. The *intersection* of A and B is denoted by $A \cap B$.

$$A \cap B = \{3, 5\} \text{ since 3 and 5 are found in both } A \text{ and } B.$$

The **union** of sets A and B is the set of numbers that are elements of A **or** of B. The *union* of A and B is denoted by $A \cup B$. The set $A \cup B$ consists of the elements in A or in B or in both.

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 9, 11\}$$ ■

Note

Although the elements 3 and 5 appear both in set A and in set B, we do not write them twice in the set $A \cup B$.

You Try 1

Let $A = \{2, 4, 6, 8, 10\}$ and $B = \{1, 2, 5, 6, 9, 10\}$. Find $A \cap B$ and $A \cup B$.

Note

The word "*and*" indicates *intersection*, while the word "*or*" indicates *union*. This same principle holds when solving compound inequalities involving "*and*" or "*or*."

Example 2

The following table of selected National Basketball Association (NBA) teams contains the number of times they have appeared in the play-offs as well as the number of NBA championships they have won through the 2007–2008 season. (www.basketball-reference.com)

Team	Play-off Appearances	Championships
Boston Celtics	46	17
Chicago Bulls	27	6
Cleveland Cavaliers	16	0
Detroit Pistons	31	3
Los Angeles Lakers	44	9
New York Knicks	38	2

List the elements of the sets:

a) The set of teams with more than 20 play-off appearances and more than 5 championships
b) The set of teams with fewer than 30 play-off appearances or more than 5 championships

Solution

a) Since the two conditions in this statement are connected by *and*, we must find the team or teams that satisfy *both* conditions. The set of teams is

{Boston Celtics, Chicago Bulls, Los Angeles Lakers}

b) Since the two conditions in this statement are connected by *or*, we must find the team or teams that satisfy the first condition, *or* the second condition, *or* both. The set of teams is

{Boston Celtics, Chicago Bulls, Cleveland Cavaliers, Los Angeles Lakers} ■

You Try 2

Use the table in Example 2 and list the elements of the sets:

a) The set of teams with less than 40 play-off appearances and at least one championship
b) The set of teams with more than 30 play-off appearances or no championships

2. Solve Compound Inequalities Containing the Word *And*

Example 3

Solve the compound inequality $c + 5 \geq 3$ and $8c \leq 32$. Graph the solution set, and write the answer in interval notation.

Solution

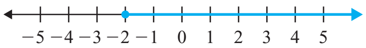
Step 1: Identify the inequality as "*and*" or "*or*" and understand what that means. These two inequalities are connected by "*and*." That means the solution set will consist of the values of c that make *both* inequalities true. The solution set will be the *intersection* of the solution sets of $c + 5 \geq 3$ and $8c \leq 32$.

Step 2: Solve each inequality separately.

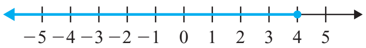
$$c + 5 \geq 3 \quad \text{and} \quad 8c \leq 32$$
$$c \geq -2 \quad \text{and} \quad c \leq 4$$

Step 3: Graph the solution set to each inequality on its own number line even if the problem does not require you to graph the solution set. This will help you visualize the solution set of the compound inequality.

$c \geq -2$:



$c \leq 4$:



Step 4: Look at the number lines and think about where the solution set for the compound inequality would be graphed.

Since this is an "*and*" inequality, the solution set of $c + 5 \geq 3$ and $8c \leq 32$ consists of the numbers that are solutions to *both* inequalities. We can visualize it this way: If we take the number line above representing $c \geq -2$ and place it on top of the number line representing $c \leq 4$, what shaded areas would overlap (intersect)?

$c \geq -2$ and $c \leq 4$:

They intersect between -2 and 4, *including* those endpoints.

Step 5: Write the answer in interval notation.

The final number line illustrates that the solution to $c + 5 \geq 3$ and $8c \leq 32$ is $[-2, 4]$. The graph of the solution set is the final number line above in Step 4.

Here are the steps to follow when solving a compound inequality.

Procedure Steps for Solving a Compound Inequality

1) Identify the inequality as "*and*" or "*or*" and understand what that means.
2) Solve each inequality separately.
3) Graph the solution set to each inequality on its own number line even if the problem does not explicitly tell you to graph the solution set. This will help you visualize the solution to the compound inequality.
4) Use the separate number lines to graph the solution set of the compound inequality.
   a) If it is an "*and*" inequality, the solution set consists of the regions on the separate number lines that would *overlap* (intersect) if one number line was placed on top of the other.
   b) If it is an "*or*" inequality, the solution set consists of the *total* (union) of what would be shaded if you took the separate number lines and put one on top of the other.
5) Use the graph of the solution set to write the answer in interval notation.

You Try 3

Solve the compound inequality and $y - 2 \leq 1$ and $7y > -28$. Graph the solution set, and write the answer in interval notation.

Example 4

Solve the compound inequality $7y + 2 > 37$ and $5 - \frac{1}{3}y < 6$. Write the solution set in interval notation.

Solution

Step 1: This is an "*and*" inequality. The solution set will be the *intersection* of the solution sets of the separate inequalities $7y + 2 > 37$ and $5 - \frac{1}{3}y < 6$.

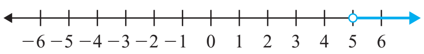
Step 2: We must solve each inequality separately.

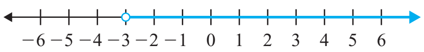
$$7y + 2 > 37 \quad \text{and} \quad 5 - \frac{1}{3}y < 6$$

$$7y > 35 \quad \text{and} \quad -\frac{1}{3}y < 1 \qquad \text{Multiply both sides by } -3.$$

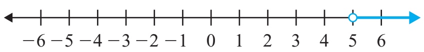
$$y > 5 \quad \text{and} \quad y > -3 \qquad \text{Reverse the direction of the inequality symbol.}$$

Step 3: Graph the solution sets separately so that it is easier to find their intersection.

$y > 5$: 

$y > -3$: 

Step 4: If we were to put these number lines on top of each other, where would they intersect?

$y > 5$ and $y > -3$: 

Step 5: The solution, shown in the shaded region above, is $(5, \infty)$.

You Try 4

Solve each compound inequality and write the answer in interval notation.

a) $4x - 3 > 1$ and $x + 6 < 13$ b) $-\frac{4}{5}m > -8$ and $2m + 5 \leq 12$

3. Solve Compound Inequalities Containing the Word *Or*

Recall that the word "*or*" indicates the union of two sets.

Example 5

Solve the compound inequality $6p + 5 \leq -1$ or $p - 3 \geq 1$. Write the answer in interval notation.

Solution

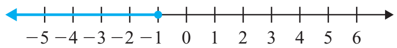
Step 1: These two inequalities are joined by "*or*." Therefore, the solution set will consist of the values of p that are in the solution set of $6p + 5 \leq -1$ *or* in the solution set of $p - 3 \geq 1$ *or* in *both* solution sets.

Step 2: Solve each inequality separately.

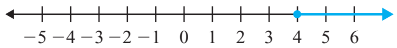
$$\begin{aligned} 6p + 5 &\le -1 \quad \text{or} \quad p - 3 \ge 1 \\ 6p &\le -6 \\ p &\le -1 \quad \text{or} \quad p \ge 4 \end{aligned}$$

Step 3: Graph the solution sets separately so that it is easier to find the *union* of the sets.

$p \le -1$:

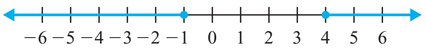


$p \ge 4$:



Step 4: The solution set of the compound inequality $6p + 5 \le -1$ or $p - 3 \ge 1$ consists of the numbers that are solutions to the first inequality *or* the second inequality *or* both. We can visualize it this way: If we put the number lines on top of each other, the solution set of the compound inequality is the **total** (union) of what is shaded.

$p \le -1$ or $p \ge 4$:



Step 5: The solution, shown above, is $(-\infty, -1] \cup [4, \infty)$.

↑

Use the *union* symbol for "or."

You Try 5

Solve $t + 8 \ge 14$ or $\frac{3}{2}t < 6$ and write the solution in interval notation.

4. Solve Special Compound Inequalities

Example 6

Solve each compound inequality and write the answer in interval notation.

a) $k - 5 < -2$ or $4k + 9 > 6$ b) $\frac{1}{2}w \ge 3$ and $1 - w \ge 0$

Solution

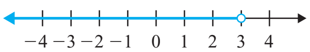
a) $k - 5 < -2$ or $4k + 9 > 6$

Step 1: The solution to this "*or*" inequality is the *union* of the solution sets of $k - 5 < -2$ and $4k + 9 > 6$.

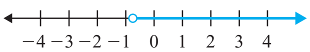
Step 2: Solve each inequality separately.

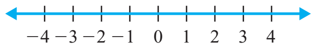
$$\begin{aligned} k - 5 < -2 \quad &\text{or} \quad 4k + 9 > 6 \\ & \qquad\quad 4k > -3 \\ k < 3 \quad &\text{or} \quad\quad k > -\tfrac{3}{4} \end{aligned}$$

Step 3: $k < 3$:



$k > -\frac{3}{4}$:



Step 4: $k < 3$ or $k > -\frac{3}{4}$: 

If the number lines in step 3 were placed on top of each other, the *total* (union) of what would be shaded is the entire number line. This represents all real numbers.

Step 5: The solution set is $(-\infty, \infty)$.

b) $\dfrac{1}{2}w \geq 3$ and $1 - w \geq 0$

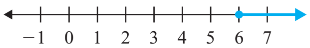
Step 1: The solution to this "*and*" inequality is the *intersection* of the solution sets of $\frac{1}{2}w \geq 3$ and $1 - w \geq 0$.

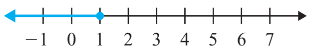
Step 2: Solve each inequality separately.

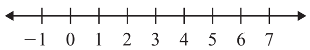
$$\frac{1}{2}w \geq 3 \quad \text{and} \quad 1 - w \geq 0$$

$$1 \geq w \qquad \text{Add } w.$$

$$\text{Multiply by 2.} \qquad w \geq 6 \quad \text{and} \quad w \leq 1 \qquad \text{Rewrite } 1 \geq w \text{ as } w \leq 1.$$

Step 3: $w \geq 6$: 

$w \leq 1$: 

Step 4: $w \geq 6$ and $w \leq 1$: 

If the number lines in step 3 were placed on top of each other, the shaded regions would *not* intersect. Therefore, the solution set is the empty set, $\varnothing$.

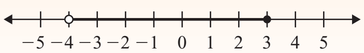
Step 5: The solution set is $\varnothing$. (There is no solution.) ■

You Try 6

Solve the compound inequalities and write the solution in interval notation.

a) $-3w \leq w - 6$ and $5w < 4$ b) $9z - 8 \leq -8$ or $z + 7 \geq 2$

Answers to You Try Exercises

1) $A \cap B = \{2, 6, 10\}$, $A \cup B = \{1, 2, 4, 5, 6, 8, 9, 10\}$ 2) a) {Chicago Bulls, Detroit Pistons, New York Knicks} b) {Boston Celtics, Cleveland Cavaliers, Detroit Pistons, Los Angeles Lakers, New York Knicks} 3) 

$(-4, 3]$ 4) a) $(1, 7)$ b) $\left(-\infty, \dfrac{7}{2}\right]$
5) $(-\infty, 4) \cup [6, \infty)$ 6) a) $\varnothing$ b) $(-\infty, \infty)$

3.8 Exercises

Objective 1: Find the Intersection and Union of Two Sets

1) Given sets A and B, explain how to find $A \cap B$.

2) Given sets X and Y, explain how to find $X \cup Y$.

Given sets $A = \{2, 4, 6, 8, 10\}$, $B = \{1, 3, 5\}$, $X = \{8, 10, 12, 14\}$, and $Y = \{5, 6, 7, 8, 9\}$ find

3) $X \cap Y$

4) $A \cap X$

5) $A \cup Y$

6) $B \cup Y$

7) $X \cap B$

8) $B \cap A$

9) $A \cup B$

10) $X \cup Y$

The following table lists the net worth (in billions of dollars) of some of the wealthiest women in the world for the years 2004 and 2008. (www.forbes.com)

Name	Net Worth in 2004	Net Worth in 2008
Liliane Bettencourt	17.2	22.9
Abigail Johnson	12.0	15.0
J. K. Rowling	1.0	1.0
Alice Walton	18.0	19.0
Oprah Winfrey	1.3	2.5

List the elements of the sets:

11) The set of women with a net worth more than \$15 billion in 2004 and in 2008

12) The set of women with a net worth more than \$10 billion in 2004 and less than \$20 billion in 2008

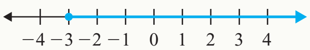
13) The set of women with a net worth less than \$2 billion in 2004 or more than \$20 billion in 2008

14) The set of women with a net worth more than \$15 billion in 2004 or more than \$2 billion in 2008

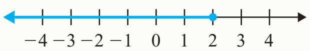
Objective 2: Solve Compound Inequalities Containing the Word *And*

Each number line represents the solution set of an inequality. Graph the *intersection* of the solution sets and write the intersection in interval notation.

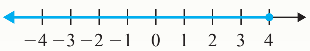
15) $x \geq -3$:



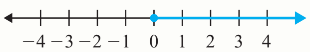
$x \leq 2$:



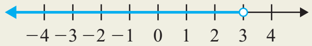
16) $n \leq 4$:



$n \geq 0$:



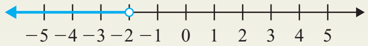
17) $t < 3$:



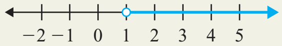
$t > -1$:

18) $y > -4$:

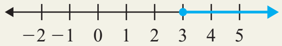
$y < -2$:



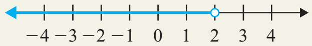
19) $c > 1$:



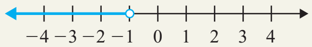
$c \geq 3$:



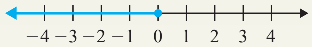
20) $p < 2$:



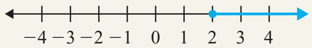
$p < -1$:



21) $z \leq 0$:

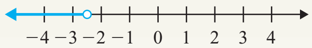


$z \geq 2$:



22) $g \geq -1$:

$g < -\frac{5}{2}$:



Solve each compound inequality. Graph the solution set, and write the answer in interval notation.

23) $a \leq 5$ and $a \geq 2$

24) $k > -3$ and $k < 4$

25) $b - 7 > -9$ and $8b < 24$

26) $3x \leq 1$ and $x + 11 \geq 4$

27) $5w + 9 \leq 29$ and $\frac{1}{3}w - 8 > -9$

28) $4y - 11 > -7$ and $\frac{3}{2}y + 5 \leq 14$

29) $2m + 15 \geq 19$ and $m + 6 < 5$

30) $d - 1 > 8$ and $3d - 12 < 4$

31) $r - 10 > -10$ and $3r - 1 > 8$

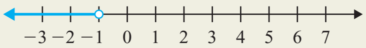
32) $2t - 3 \leq 6$ and $5t + 12 \leq 17$

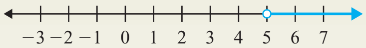
33) $9 - n \leq 13$ and $n - 8 \leq -7$

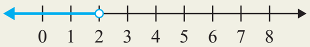
34) $c + 5 \geq 6$ and $10 - 3c \geq -5$

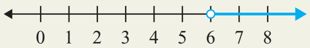
Objective 3: Solve Compound Inequalities Containing the Word *Or*

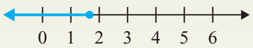
Each number line represents the solution set of an inequality. Graph the *union* of the solution sets and write the union in interval notation.

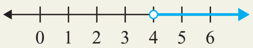
VIDEO 35) $p < -1$:


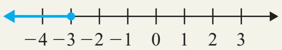
$p > 5$:


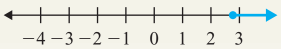
36) $z < 2$:


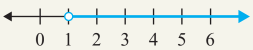
$z > 6$:


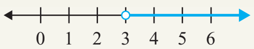
37) $a \le \frac{5}{3}$:


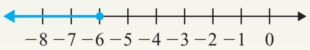
$a > 4$:


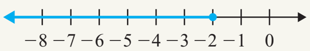
38) $v \le -3$:


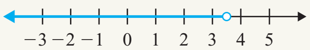
$v \ge \frac{11}{4}$:


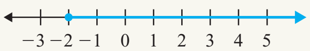
39) $y > 1$:


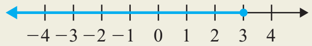
$y > 3$:


40) $x \le -6$:


$x \le -2$:


41) $c < \frac{7}{2}$:


$c \ge -2$:


42) $q \le 3$:


$q > -2.7$:


Mixed Exercises: Objectives 3 and 4

Solve each compound inequality. Graph the solution set, and write the answer in interval notation.

43) $z < -1$ or $z > 3$

44) $x \le -4$ or $x \ge 0$

VIDEO 45) $6m \le 21$ or $m - 5 > 1$

46) $a + 9 > 7$ or $8a \le -44$

47) $3t + 4 > -11$ or $t + 19 > 17$

48) $5y + 8 \le 13$ or $2y \le -6$

49) $-2v - 5 \le 1$ or $\frac{7}{3}v < -14$

50) $k - 11 < -4$ or $-\frac{2}{9}k \le -2$

VIDEO 51) $c + 3 \ge 6$ or $\frac{4}{5}c \le 10$

52) $\frac{8}{3}g \ge -12$ or $2g + 1 \le 7$

53) $7 - 6n \ge 19$ or $n + 14 < 11$

54) $d - 4 > -7$ or $-6d \le 2$

Chapter 3: Summary

Definition/Procedure	Example
3.1 Solving Linear Equations Part I	
The Addition and Subtraction Properties of Equality 1) If $a = b$, then $a + c = b + c$. 2) If $a = b$, then $a - c = b - c$. **(p. 115)**	Solve $3 + b = 20$ $3 - 3 + b = 20 - 3$ Subtract 3 from each side. $b = 17$ The solution set is $\{17\}$.
The Multiplication and Division Properties of Equality 1) If $a = b$, then $ac = bc$. 2) If $a = b$, then $\frac{a}{c} = \frac{b}{c}$ $(c \neq 0)$. **(p. 116)**	Solve $\frac{3}{5}m = 6$ $\frac{5}{3} \cdot \frac{3}{5}m = \frac{5}{3} \cdot 6$ Multiply each side by $\frac{5}{3}$. $m = 10$ The solution set is $\{10\}$.
Sometimes it is necessary to *combine like terms* as the first step in solving a linear equation. **(p. 119)**	Solve $11w - 2 - 4w + 3 = -6$ $7w + 1 = -6$ Combine like terms. $7w + 1 - 1 = -6 - 1$ Subtract 1 from each side. $7w = -7$ $\frac{7w}{7} = \frac{-7}{7}$ Divide by 7. $w = -1$ The solution set is $\{-1\}$.
3.2 Solving Linear Equations Part II	
How to Solve a Linear Equation ***Step 1:*** **Clear parentheses** and **combine like terms** on each side of the equation. ***Step 2:*** **Get the variable on one side of the equal sign and the constant on the other side of the equal sign** (isolate the variable) using the addition or subtraction property of equality. ***Step 3:*** **Solve for the variable** using the multiplication or division property of equality. ***Step 4:*** **Check the solution** in the original equation. **(p. 123)**	Solve $2(c + 2) + 11 = 5c + 9$. $2c + 4 + 11 = 5c + 9$ Distribute. $2c + 15 = 5c + 9$ Combine like terms. $2c - 5c + 15 = 5c - 5c + 9$ Get variable terms on one side. $-3c + 15 = 9$ $-3c = -6$ Get constants on one side. $\frac{-3c}{-3} = \frac{-6}{-3}$ Division property of equality $c = 2$ The solution set is $\{2\}$.
Solve Equations Containing Fractions or Decimals To eliminate the fractions, determine the least common denominator (LCD) for all of the fractions in the equation. Then, multiply both sides of the equation by the LCD. **(p. 124)** To eliminate the decimals from an equation, multiply both sides of the equation by the smallest power of 10 that will eliminate all decimals from the problem. **(p. 125)**	$\frac{3}{4}y - 3 = \frac{1}{4}y - \frac{2}{3}$ LCD = 12 $12\left(\frac{3}{4}y - 3\right) = 12\left(\frac{1}{4}y - \frac{2}{3}\right)$ Multiply each side of the equation by 12. $9y - 36 = 3y - 8$ Distribute. $9y - 3y - 36 = 3y - 3y - 8$ Get the y terms on one side. $6y - 36 = -8$ $6y - 36 + 36 = -8 + 36$ Get the constants on the other side. $6y = 28$ $\frac{6y}{6} = \frac{28}{6}$ Divide each side by 6. $y = \frac{28}{6} = \frac{14}{3}$ Reduce.

Definition/Procedure	Example
Steps for Solving Applied Problems 1) **Read** and reread the problem. Draw a picture, if applicable. 2) **Choose a variable** to represent an unknown. Define other unknown quantities in terms of the variable. 3) **Translate** from English to math. 4) **Solve** the equation. 5) **Check** the answer in the original problem, and **interpret** the solution as it relates to the problem. **(p. 127)**	Nine less than twice a number is the same as the number plus thirteen. 1) Read the problem carefully, then read it again. 2) Choose a variable to represent the unknown. $x =$ the number 3) "Nine less than twice a number is the same as the number plus thirteen" means $2x - 9 = x + 13$. 4) Solve the equation. $2x - 9 = x + 13$ $2x - 9 + 9 = x + 13 + 9$ $2x = x + 22$ $x = 22$ The number is 22.

3.3 Applications of Linear Equations

The "**Steps for Solving Applied Problems**" can be used to solve problems involving general quantities, lengths, and consecutive integers. **(p. 132)**	The sum of three consecutive even integers is 72. Find the integers. 1) Read the problem carefully, then read it again. 2) Define the unknowns. $x =$ the first even integer $x + 2 =$ the second even integer $x + 4 =$ the third even integer 3) "The sum of three consecutive even integers is 72" means First even + Second even + Third even = 72 $x + (x + 2) + (x + 4) = 72$ *Equation:* $x + (x + 2) + (x + 4) = 72$ 4) Solve $x + (x + 2) + (x + 4) = 72$ $3x + 6 = 72$ $3x + 6 - 6 = 72 - 6$ $3x = 66$ $\frac{3x}{3} = \frac{66}{3}$ $x = 22$ 5) Find the values of all the unknowns. $x = 22, \quad x + 2 = 24, \quad x + 4 = 26$ The numbers are 22, 24, and 26.

3.4 Applications Involving Percentages

The "**Steps for Solving Applied Problems**" can be used to solve applications involving percent increase/decrease, interest earned on investments, and mixture problems. **(p. 142)**	A Lady Gaga poster is on sale for \$7.65 after a 15% discount. Find the original price. 1) Read the problem carefully, then read it again. 2) Choose a variable to represent the unknown. $x =$ the original price of the poster 3) Write an equation in English, then translate it to math. Original price of the poster − Amount of the discount = Sale price of the poster $x - 0.15x = 7.65$ *Equation*: $x - 0.15x = 7.65$ 4) Solve $x - 0.15x = 7.65$ $0.85x = 7.65$ $\frac{0.85x}{0.85} = \frac{7.65}{0.85}$ $x = 9.00$ 5) The original price of the Lady Gaga poster was \$9.00.

Definition/Procedure	Example

3.5 Geometry Applications and Solving Formulas

Definition/Procedure: Formulas from geometry can be used to solve applications. **(p. 151)**

Example: A rectangular bulletin board has an area of 180 in^2. It is 12 in. wide. Find its length.

Use $A = lw$. Formula for the area of a rectangle

$A = 180$ in^2, $w = 12$ in. Find l.

$$\begin{aligned} A &= lw \\ 180 &= l(12) \quad \text{Substitute values into } A = lw. \\ \frac{180}{12} &= \frac{l(12)}{12} \\ 15 &= l \end{aligned}$$

The length is 15 inches.

Definition/Procedure: To solve a formula for a specific variable, think about the steps involved in solving a linear equation in one variable. **(p. 157)**

Example: Solve $C = kr - w$ for k.

$$\begin{aligned} C + w &= \boxed{k}r - w + w \quad \text{Add } w \text{ to each side.} \\ C + w &= \boxed{k}r \\ \frac{C + w}{r} &= \frac{\boxed{k}r}{r} \quad \text{Divide each side by } r. \\ \frac{C + w}{r} &= k \end{aligned}$$

3.6 Applications of Linear Equations to Proportions, Money Problems, and $d = rt$

Definition/Procedure: A **proportion** is a statement that two ratios are equivalent.

We can use the same principles for solving equations involving similar triangles, money, and distance = rate · time. **(p. 166)**

Example: If Geri can watch 4 movies in 3 weeks, how long will it take her to watch 7 movies?

1) Read the problem carefully twice.
2) Choose a variable to represent the unknown.
   x = number of weeks to watch 7 movies
3) Set up a proportion. $\dfrac{4 \text{ movies}}{3 \text{ weeks}} = \dfrac{7 \text{ movies}}{x \text{ weeks}}$

Equation: $\dfrac{4}{3} = \dfrac{7}{x}$

4) Solve $\dfrac{4}{3} = \dfrac{7}{x}$

$$\begin{aligned} 4x &= 3(7) \quad \text{Set cross products equal.} \\ \frac{4x}{4} &= \frac{21}{4} \\ x &= \frac{21}{4} = 5\frac{1}{4} \end{aligned}$$

5) It will take Geri $5\frac{1}{4}$ weeks to watch 7 movies.

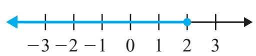
3.7 Solving Linear Inequalities in One Variable

Definition/Procedure: We solve linear inequalities in very much the same way we solve linear equations *except that when we multiply or divide by a negative number, we must reverse the direction of the inequality symbol.*

We can graph the solution set, write the solution in set notation, or write the solution in interval notation. **(p. 180)**

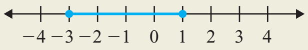
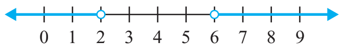
Example: Solve $x - 9 \le -7$. Graph the solution set and write the answer in both set notation and interval notation.

$$\begin{aligned} x - 9 &\le -7 \\ x - 9 + 9 &\le -7 + 9 \\ x &\le 2 \end{aligned}$$



$\{x \mid x \le 2\}$ Set notation

$(-\infty, 2]$ Interval notation

Definition/Procedure	Example
3.8 Solving Compound Inequalities	
The solution set of a compound inequality joined by "**and**" is the **intersection** of the solution sets of the individual inequalities. **(p. 191)**	Solve the compound inequality $5x - 2 \geq -17$ and $x + 8 \leq 9$. $5x - 2 \geq -17$ and $x + 8 \leq 9$ $5x \geq -15$ $x \geq -3$ and $x \leq 1$  Solution in interval notation: $[-3, 1]$
The solution set of a compound inequality joined by "**or**" is the **union** of the solution sets of the individual inequalities. **(p. 193)**	Solve the compound inequality $x - 3 < -1$ or $7x > 42$. $x - 3 < -1$ or $7x > 42$ $x < 2$ or $x > 6$  Solution in interval notation: $(-\infty, 2) \cup (6, \infty)$

Chapter 3: Review Exercises

Sections 3.1–3.2

Determine whether the given value is a solution to the equation.

1) $\frac{3}{2}k - 5 = 1; \quad k = -4$

2) $5 - 2(3p + 1) = 9p - 2; \quad p = \frac{1}{3}$

3) How do you know that an equation has no solution?

4) What can you do to make it easier to solve an equation with fractions?

Solve each equation.

5) $h + 14 = -5$

6) $w - 9 = 16$

7) $-7g = 56$

8) $-0.78 = -0.6t$

9) $4 = \frac{c}{9}$

10) $-\frac{10}{3}y = 16$

11) $23 = 4m - 7$

12) $\frac{1}{6}v - 7 = -3$

13) $4c + 9 + 2(c - 12) = 15$

14) $\frac{5}{9}x + \frac{1}{6} = -\frac{3}{2}$

15) $2z + 11 = 8z + 15$

16) $8 - 5(2y - 3) = 14 - 9y$

17) $k + 3(2k - 5) = 4(k - 2) - 7$

18) $10 - 7b = 4 - 5(2b + 9) + 3b$

19) $0.18a + 0.1(20 - a) = 0.14(20)$

20) $16 = -\frac{12}{5}d$

21) $3(r + 4) - r = 2(r + 6)$

22) $\frac{1}{2}(n - 5) - 1 = \frac{2}{3}(n - 6)$

Write each statement as an equation, and find the number.

23) Nine less than twice a number is twenty-five.

24) One more than two-thirds of a number is the same as the number decreased by three.

Section 3.3

Solve using the five-step method.

25) Kendrick received 24 fewer e-mails on Friday than he did on Thursday. If he received a total of 126 e-mails on those two days, how many did he get on each day?

26) The number of Michael Jackson solo albums sold the week after his death was 42.2 times the number sold the previous week. If a total of 432,000 albums were sold during those two weeks, how many albums were sold the week after his death? (http://abcnews.go.com)

27) A plumber has a 36-inch pipe that he has to cut into two pieces so that one piece is 8 in. longer than the other. Find the length of each piece.

28) The sum of three consecutive integers is 249. Find the integers.

Section 3.4

Solve using the five-step method.

29) Today's typical hip implant weighs about 50% less than it did 20 years ago. If an implant weighs about 3 lb today, how much did it weigh 20 years ago?

30) By mid-February of 2009, the number of out-of-state applicants to the University of Colorado had decreased by about 19% compared to the same time the previous year. If the school received about 11,500 out-of-state applications in 2009, how many did it receive in 2008? Round the answer to the nearest hundred. (www.dailycamera.com)

31) Jose had \$6000 to invest. He put some of it into a savings account earning 2% simple interest and the rest into an account earning 4% simple interest. If he earned \$210 of interest in 1 year, how much did he invest in each account?

32) How many milliliters of a 10% hydrogen peroxide solution and how many milliliters of a 2% hydrogen peroxide solution should be mixed to obtain 500 mL of a 4% hydrogen peroxide solution?

Section 3.5

Substitute the given values into the formula and solve for the remaining variable.

33) $P = 2l + 2w$; If $P = 32$ when $l = 9$, find w.

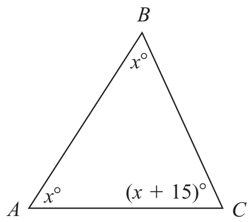
34) $V = \frac{1}{3}\pi r^2 h$; If $V = 60\pi$ when $r = 6$, find h.

Use a known formula to solve.

35) The base of a triangle measures 12 in. If the area of the triangle is 42 in^2, find the height.

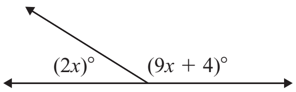
36) The Statue of Liberty holds a tablet in her left hand that is inscribed with the date, in Roman numerals, that the Declaration of Independence was signed. The length of this rectangular tablet is 120 in. more than the width, and the perimeter of the tablet is 892 in. What are the dimensions of the tablet? (www.nps.gov)

37) Find the missing angle measures.

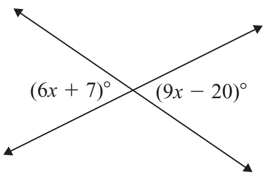


Find the measure of each indicated angle.

38)



39)



Solve using the five-step method.

40) The sum of the supplement of an angle and twice the angle is 10° more than four times the measure of its complement. Find the measure of the angle.

Solve for the indicated variable.

41) $p - n = z$ for p

42) $r = ct + a$ for t

43) $A = \frac{1}{2}bh$ for b

44) $M = \frac{1}{4}k(d + D)$ for D

Section 3.6

45) Can 15% be written as a ratio? Explain.

46) What is the difference between a ratio and a proportion?

47) Write the ratio of 12 girls to 15 boys in lowest terms.

48) A store sells olive oil in three different sizes. Which size is the best buy, and what is its unit price?

Size	Price
17 oz	\$ 8.69
25 oz	\$11.79
101 oz	\$46.99

Solve each proportion.

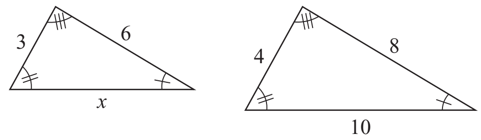
49) $\frac{x}{15} = \frac{8}{10}$

50) $\frac{2c + 3}{6} = \frac{c - 4}{2}$

Set up a proportion and solve.

51) The 2007 Youth Risk Behavior Survey found that about 9 out of 20 high school students drank some amount of alcohol in the 30 days preceding the survey. If a high school has 2500 students, how many would be expected to have used alcohol within a 30-day period? (www.cdc.gov)

52) Given these two similar triangles, find x.



Solve using the five-step method.

53) At the end of his shift, Bruno had \$340 worth of tips, all in \$10 and \$20 bills. If he had two more \$20 bills than \$10 bills, how many of each bill did Bruno have?

54) At Ralph's grocery store, green peppers cost \$0.88 each and red peppers cost \$0.95 each. Chung-Hee buys twice as many green peppers as red peppers and spends \$5.42. How many green peppers and how many red peppers did he buy?

55) Jared and Meg leave opposite ends of a hiking trail 11 miles apart and travel toward each other. Jared is jogging 1 mph slower than Meg. Find each of their speeds if they meet after an hour.

56) Ceyda jogs past the library at 9:00 A.M. going 4 mph. Twenty minutes later, Turgut runs past the library at 6 mph following the same trail. At what time will Turgut catch up to Ceyda?

Section 3.7

Solve each inequality. Graph the solution set, and write the answer in interval notation.

57) $w + 8 > 5$

58) $-6k \le 15$

59) $5x - 2 \le 18$

60) $3(3c + 8) - 7 > 2(7c + 1) - 5$

61) $-19 \le 7p + 9 \le 2$

62) $-3 < \frac{3}{4}a - 6 \le 0$

63) $\frac{1}{2} < \frac{1 - 4t}{6} < \frac{3}{2}$

64) **Write an inequality and solve.** Gia's scores on her first three History tests were 94, 88, and 91. What does she need to make on her fourth test to have an average of at least 90?

Section 3.8

The following table lists the number of hybrid vehicles sold in the United States by certain manufacturers in June and July of 2008. (www.hybridcars.com)

Manufacturer	Number Sold in June	Number Sold in July
Toyota	16,330	18,801
Honda	2,717	3,443
Ford	1,910	1,265
Lexus	1,476	1,562
Nissan	1,333	715

List the elements of the set.

65) The set of manufacturers who sold more than 3000 hybrid vehicles in each of June and July

66) The set of manufacturers who sold more than 5000 hybrid vehicles in June or fewer than 1500 hybrids in July

Use the follwing sets for Exercises 67 and 68:
$A = \{10, 20, 30, 40, 50\}$, $B = \{20, 25, 30, 35\}$

67) Find $A \cup B$.

68) Find $A \cap B$.

Solve each compound inequality. Graph the solution set and write the answer in interval notation.

69) $a + 6 \le 9$ and $7a - 2 \ge 5$

70) $3r - 1 > 5$ or $-2r \ge 8$

71) $8 - y < 9$ or $\frac{1}{10}y > \frac{3}{5}$

72) $x + 12 \le 9$ and $0.2x \ge 3$

Mixed Exercises: Solving Equations and Applications

Solve each equation.

73) $-8k + 13 = -7$

74) $-7 - 4(3w - 2) = 1 - 9w$

75) $29 = -\frac{4}{7}m + 5$

76) $\frac{c}{20} = \frac{18}{12}$

77) $10p + 11 = 5(2p + 3) - 1$

78) $0.14a + 0.06(36 - a) = 0.12(36)$

79) $\frac{2x + 9}{5} = \frac{x + 1}{2}$

80) $14 = 8 - h$

81) $\frac{5}{6} - \frac{3}{4}(r + 2) = \frac{1}{2}r + \frac{7}{12}$

82) $\frac{1}{4}d + \frac{9}{4} = 1 + \frac{1}{4}(d + 5)$

Solve using the five-step method.

83) How many ounces of a 5% alcohol solution must be mixed with 60 oz of a 17% alcohol solution to obtain a 9% alcohol solution?

84) A library offers free tutoring after school for children in grades 1–5. The number of students who attended on Friday was half the number who attended on Thursday. How many students came for tutoring each day if the total number of students served on both days was 42?

85) The sum of two consecutive odd integers is 21 less than three times the larger integer. Find the numbers.

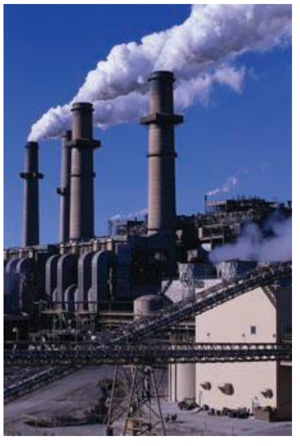
86) Blair and Serena emptied their piggy banks before heading to the new candy store. They have 45 coins in nickels and quarters, for a total of \$8.65. How many nickels and quarters did they have?

87) The perimeter of a triangle is 35 cm. One side is 3 cm longer than the shortest side, and the longest side is twice as long as the shortest. How long is each side of the triangle?

88) Yvette and Celeste leave the same location on their bikes and head in opposite directions. Yvette travels at 10 mph, and Celeste travels at 12 mph. How long will it take before they are 33 miles apart?

89) A 2008 poll revealed that 9 out of 25 residents of Quebec, Canada, wanted to secede from the rest of the country. If 1000 people were surveyed, how many said they would like to see Quebec separate from the rest of Canada? (www.bloomberg.com)

90) If a certain environmental bill is passed by Congress, the United States would have to reduce greenhouse gas emissions by 17% from 2005 levels by the year 2020. The University of New Hampshire has been at the forefront of reducing emissions, and if this bill is passed they would be required to have a greenhouse gas emission level of about 56,440 MTCDE (metric tons carbon dioxide equivalents) by 2020. Find their approximate emission level in 2005. (www.sustainableunh.unh.edu, www.knoxnews.com)



Chapter 3: Test

Solve each equation.

1) $-18y = 14$

2) $16 = 7 - a$

3) $\frac{8}{3}n - 11 = 5$

4) $3c - 2 = 8c + 13$

5) $\frac{1}{2} - \frac{1}{6}(x - 5) = \frac{1}{3}(x + 1) + \frac{2}{3}$

6) $7(3k + 4) = 11k + 8 + 10k - 20$

7) $\frac{9 - w}{4} = \frac{3w + 1}{2}$

8) What is the difference between a ratio and a proportion?

Solve using the five-step method.

9) The sum of three consecutive even integers is 114. Find the numbers.

10) How many milliliters of a 20% acid solution should be mixed with 50 mL of an 8% acid solution to obtain an 18% acid solution?

11) Ray buys 14 gallons of gas and pays \$40.60. His wife, Debra, goes to the same gas station later that day and buys 11 gallons of the same gasoline. How much did she spend?

12) The tray table on the back of an airplane seat is in the shape of a rectangle. It is 5 in. longer than it is wide and has a perimeter of 50 in. Find the dimensions.

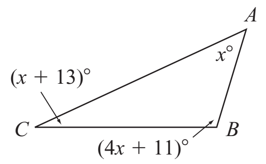
13) Two cars leave the same location at the same time, one traveling east and the other going west. The westbound car is driving 6 mph faster than the eastbound car. After 2.5 hr they are 345 miles apart. What is the speed of each car?

Solve for the indicated variable.

14) $B = \frac{an}{4}$ for a

15) $S = 2\pi r^2 + 2\pi rh$ for h

16) Find the measure of each indicated angle.



Solve. Graph the solution set, and write the answer in interval notation.

17) $6m + 19 \le 7$

18) $1 - 2(3x - 5) < 2x + 5$

19) $-\frac{5}{6} < \frac{4c - 1}{6} \le \frac{3}{2}$

20) Write an inequality and solve.

Anton has grades of 87 and 76 on his first two Biology tests. What does he need on the third test to keep an average of at least 80?

Solve each compound inequality. Write the answer in interval notation.

21) $3n + 5 > 12$ or $\frac{1}{4}n < -2$

22) $y - 8 \le -5$ and $2y \ge 0$

23) $6 - p < 10$ or $p - 7 < 2$

Cumulative Review: Chapters 1–3

Perform the operations and simplify.

1) $\frac{3}{8} - \frac{5}{6}$

2) $\frac{5}{8} \cdot 12$

3) $26 - 14 \div 2 + 5 \cdot 7$

4) $-82 + 15 + 10(1 - 3)$

5) $-39 - |7 - 15|$

6) Find the area of a triangle with a base of length 9 cm and height of 6 cm.

Given the set of numbers $\left\{\frac{3}{4}, -5, \sqrt{11}, 2.5, 0, 9, 0.\overline{4}\right\}$ identify

7) the integers

8) the rational numbers

9) the whole numbers

10) Which property is illustrated by $6(5 + 2) = 6 \cdot 5 + 6 \cdot 2$?

11) Does the commutative property apply to the subtraction of real numbers? Explain.

12) Combine like terms.

$11y^2 - 14y + 6y^2 + y - 5y$

Simplify. The answer should not contain any negative exponents.

13) $\frac{35r^{16}}{28r^4}$

14) $(-2m^5)^3 (3m^9)^2$

15) $(-12z^{10})\left(\frac{3}{8}z^{-16}\right)$

16) $\left(\frac{10c^{12}d^2}{5c^9d^{-3}}\right)^{-2}$

17) Write 0.00000895 in scientific notation.

Solve.

18) $8t - 17 = 10t + 6$

19) $\frac{3}{2}n + 14 = 20$

20) $3(7w - 5) - w = -7 + 4(5w - 2)$

21) $\frac{x + 3}{10} = \frac{2x - 1}{4}$

22) $-\frac{1}{2}c + \frac{1}{5}(2c - 3) = \frac{3}{10}(2c + 1) - \frac{3}{4}c$

Solve using the five-step method.

23) Stu and Phil were racing from Las Vegas back to Napa Valley. Stu can travel 140 miles by train in the time it takes Phil to travel 120 miles by car. What are the speeds of the train and the car if the train is traveling 10 mph faster than the car?

Solve. Write the answer in interval notation.

24) $7k + 4 \geq 9k + 16$

25) $8x \leq -24 \quad \text{or} \quad 4x - 5 \geq 6$

CHAPTER 4

Linear Equations in Two Variables

Algebra at Work: Landscape Architecture

We will take a final look at how mathematics is used in landscape architecture.

A landscape architect uses slope in many different ways. David explains that one important application of slope is in designing driveways after a new house has been built. Towns often have building codes that restrict the slope or steepness of a driveway. In this case, the rise of the land is the difference in height between the top and the bottom of the driveway. The run is the linear horizontal distance between those two points. By finding $\frac{rise}{run}$, a landscape architect knows how to design the driveway so that it meets the town's building code. This is especially important in cold weather climates, where if a driveway is too steep, a car will too easily slide into the street. If it doesn't meet the code, the driveway may have to be removed and rebuilt, or coils that radiate heat might have to be installed under the driveway to melt the snow in the wintertime. Either way, a mistake in calculating slope could cost the landscape company or the client a lot of extra money.

In Chapter 4, we will learn about slope, its meaning, and different ways to use it.

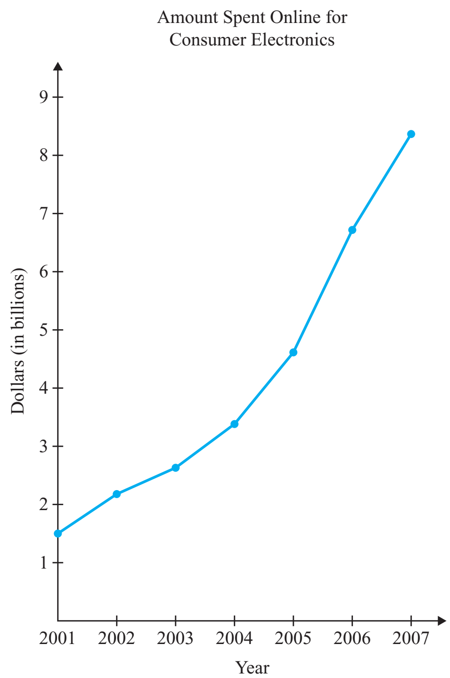
Section 4.1 Introduction to Linear Equations in Two Variables

Objectives

1. Define a Linear Equation in Two Variables
2. Decide Whether an Ordered Pair Is a Solution of a Given Equation
3. Complete Ordered Pairs for a Given Equation
4. Plot Ordered Pairs
5. Solve Applied Problems Involving Ordered Pairs

Graphs are everywhere—online, in newspapers, and in books. The accompanying graph shows how many billions of dollars consumers spent shopping online for consumer electronics during the years 2001–2007.

We can get different types of information from this graph. For example, in the year 2001, consumers spent about \$1.5 billion on electronics, while in 2007, they spent about \$8.4 billion on electronics. The graph also illustrates a general trend in online shopping: More and more people are buying their consumer electronics online.



Source: www.census.gov

1. Define a Linear Equation in Two Variables

Later in this section, we will see that graphs like this one are based on the *Cartesian coordinate system*, also known as the *rectangular coordinate system,* which gives us a way to graphically represent the relationship between two quantities. We will also learn about different ways to represent relationships between two quantities, like year and online spending, when we learn about *linear equations in two variables.* Let's begin with a definition.

> **Definition**
>
> A **linear equation in two variables** can be written in the form $Ax + By = C$, where A, B, and C are real numbers and where both A and B do not equal zero.

Some examples of linear equations in two variables are

$$5x - 2y = 11 \qquad y = \frac{3}{4}x + 1 \qquad -3a + b = 2 \qquad y = x \qquad x = -3$$

(We can write $x = -3$ as $x + 0y = -3$; therefore it is a linear equation in two variables.)

2. Decide Whether an Ordered Pair Is a Solution of a Given Equation

A solution to a linear equation in two variables is written as an *ordered pair* so that when the values are substituted for the appropriate variables, we obtain a true statement.

Example 1

Determine whether each ordered pair is a solution of $5x - 2y = 11$.

a) $(1, -3)$ b) $\left(\frac{3}{5}, 4\right)$

Solution

a) Solutions to the equation $5x - 2y = 11$ are written in the form (x, y) where (x, y) is called an *ordered pair*. Therefore, the ordered pair $(1, -3)$ means that $x = 1$ and $y = -3$.

$$(\underset{x\text{-coordinate}}{1}, \underset{y\text{-coordinate}}{-3})$$

To determine whether $(1, -3)$ is a solution of $5x - 2y = 11$, we substitute 1 for x and -3 for y. Remember to put these values in parentheses.

$$\begin{aligned} 5x - 2y &= 11 \\ 5(1) - 2(-3) &= 11 && \text{Substitute } x = 1 \text{ and } y = -3. \\ 5 + 6 &= 11 && \text{Multiply.} \\ 11 &= 11 && \text{True} \end{aligned}$$

Since substituting $x = 1$ and $y = -3$ into the equation gives the true statement $11 = 11$, $(1, -3)$ *is a solution* of $5x - 2y = 11$. We say that $(1, -3)$ *satisfies* $5x - 2y = 11$.

b) The ordered pair $\left(\frac{3}{5}, 4\right)$ tells us that $x = \frac{3}{5}$ and $y = 4$.

$$\begin{aligned} 5x - 2y &= 11 \\ 5\left(\frac{3}{5}\right) - 2(4) &= 11 && \text{Substitute } \tfrac{3}{5} \text{ for } x \text{ and 4 for } y. \\ 3 - 8 &= 11 && \text{Multiply.} \\ -5 &= 11 && \text{False} \end{aligned}$$

Since substituting $\left(\frac{3}{5}, 4\right)$ into the equation gives the false statement $-5 = 11$, the ordered pair is *not* a solution to the equation. ■

You Try 1

Determine whether each ordered pair is a solution of the equation $y = -\frac{3}{4}x + 5$.

a) $(12, -4)$ b) $(0, 7)$ c) $(-8, 11)$

If the variables in the equation are not x and y, then the variables in the ordered pairs are written in alphabetical order. For example, solutions to $-3a + b = 2$ are ordered pairs of the form (a, b).

3. Complete Ordered Pairs for a Given Equation

Often, we are given the value of one variable in an equation and we can find the value of the other variable that makes the equation true.

Example 2

Complete the ordered pair $(-3, \quad)$ for $y = 2x + 10$.

Solution

To complete the ordered pair $(-3, \quad)$, we must find the value of y from $y = 2x + 10$ when $x = -3$.

$$\begin{aligned} y &= 2x + 10 \\ y &= 2(-3) + 10 && \text{Substitute } -3 \text{ for } x. \\ y &= -6 + 10 \\ y &= 4 \end{aligned}$$

When $x = -3$, $y = 4$. The ordered pair is $(-3, 4)$. ■

You Try 2

Complete the ordered pair $(5, \quad)$ for $y = 3x - 7$.

If we want to complete more than one ordered pair for a particular equation, we can organize the information in a **table of values**.

Example 3

Complete the table of values for each equation and write the information as ordered pairs.

a) $-x + 3y = 8$

x	y
1	
	-4
	$\frac{2}{3}$

b) $y = 2$

x	y
7	
-5	
0	

Solution

a) $-x + 3y = 8$

x	y
1	
	-4
	$\frac{2}{3}$

The first ordered pair is (1,), and we must find y.

$$\begin{aligned} -x + 3y &= 8 \\ -(1) + 3y &= 8 && \text{Substitute 1 for } x. \\ -1 + 3y &= 8 \\ 3y &= 9 && \text{Add 1 to each side.} \\ y &= 3 && \text{Divide by 3.} \end{aligned}$$

The ordered pair is (1, 3).

The second ordered pair is (, -4), and we must find x.

$$\begin{aligned} -x + 3y &= 8 \\ -x + 3(-4) &= 8 && \text{Substitute } -4 \text{ for } y. \\ -x + (-12) &= 8 && \text{Multiply.} \\ -x &= 20 && \text{Add 12 to each side.} \\ x &= -20 && \text{Divide by } -1. \end{aligned}$$

The ordered pair is $(-20, -4)$.

The third ordered pair is $\left(\ , \frac{2}{3}\right)$, and we must find x.

$$\begin{aligned} -x + 3y &= 8 \\ -x + 3\left(\frac{2}{3}\right) &= 8 && \text{Substitute } \frac{2}{3} \text{ for } y. \\ -x + 2 &= 8 && \text{Multiply.} \\ -x &= 6 && \text{Subtract 2 from each side.} \\ x &= -6 && \text{Divide by } -1. \end{aligned}$$

The ordered pair is $\left(-6, \frac{2}{3}\right)$.

As you complete each ordered pair, fill in the table of values. The completed table will look like this:

x	y
1	3
-20	-4
-6	$\frac{2}{3}$

b) $y = 2$

x	y
7	
−5	
0	

The first ordered pair is (7,), and we must find y. The equation $y = 2$ means that *no matter the value of* x, y *always equals 2*. Therefore, when $x = 7$, $y = 2$.

The ordered pair is (7, 2).

Since $y = 2$ for every value of x, we can complete the table of values as follows:

The ordered pairs are (7, 2), (−5, 2), and (0, 2).

x	y
7	2
−5	2
0	2

You Try 3

Complete the table of values for each equation and write the information as ordered pairs.

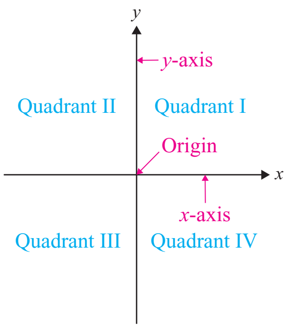
a) $x - 2y = 9$

x	y
5	
12	
	−7
	$\frac{5}{2}$

b) $x = -3$

x	y
	1
	3
	−8

4. Plot Ordered Pairs



When we completed the table of values for the last two equations, we were finding solutions to each linear equation in two variables.

How can we represent the solutions graphically? We will use the **Cartesian coordinate system**, also known as the **rectangular coordinate system**, to graph the ordered pairs, (x, y).

In the Cartesian coordinate system, we have a horizontal number line, called the x-axis, and a vertical number line, called the y-axis.

The x-axis and y-axis in the Cartesian coordinate system determine a flat surface called a **plane**. The axes divide this plane into four **quadrants**, as shown in the figure. The point at which the x-axis and y-axis intersect is called the **origin**. The arrow at one end of the x-axis and one end of the y-axis indicates the positive direction on each axis.

Ordered pairs can be represented by **points** in the plane. Therefore, to graph the ordered pair (4, 2) we *plot the point* (4, 2). We will do this in Example 4.

Example 4

Plot the point (4, 2).

Solution

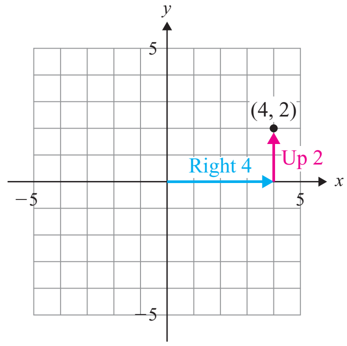
Since $x = 4$, we say that the *x-coordinate* of the point is 4. Likewise, the *y-coordinate* is 2.

The *origin* has coordinates (0, 0). The **coordinates** of a point tell us how far from the origin, in the x-direction and y-direction, the point is located. So, the coordinates of the point (4, 2) tell us that to locate the point we do the following:

(4, 2)

First, from the origin, move 4 units to the right along the x-axis.

Then, from the current position, move 2 units up, parallel to the y-axis.



Example 5

Plot the points.

a) $(-2, 5)$ b) $(1, -4)$ c) $\left(\frac{5}{2}, 3\right)$

d) $(-5, -2)$ e) $(0, 1)$ f) $(-4, 0)$

Solution

The points are plotted on the following graph.

a) $(-2, 5)$

First
From the origin, move left 2 units on the x-axis.

Then
From the current position, move 5 units up, parallel to the y-axis.

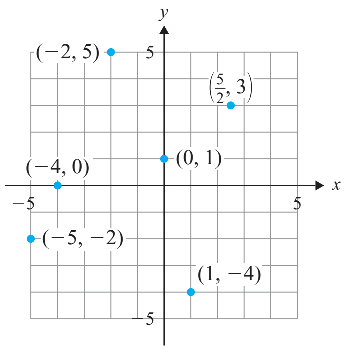
b) $(1, -4)$

First
From the origin, move right 1 unit on the x-axis.

Then
From the current position, move 4 units down, parallel to the y-axis.

c) $\left(\frac{5}{2}, 3\right)$

Think of $\frac{5}{2}$ as $2\frac{1}{2}$. From the origin, move right $2\frac{1}{2}$ units, then up 3 units.



d) $(-5, -2)$ From the origin, move left 5 units, then down 2 units.

e) $(0, 1)$ The x-coordinate of 0 means that we don't move in the x-direction (horizontally). From the origin, move up 1 on the y-axis.

f) $(-4, 0)$ From the origin, move left 4 units. Since the y-coordinate is zero, we do not move in the y-direction (vertically).

You Try 4

Plot the points.

a) (3, 1) b) $(-2, 4)$ c) $(0, -5)$ d) (2, 0) e) $(-4, -3)$ f) $\left(1, \frac{7}{2}\right)$

Note

The coordinate system should always be labeled to indicate how many units each mark represents.

We can graph sets of ordered pairs for a linear equation in two variables.

Example 6

Complete the table of values for $2x - y = 5$, then plot the points.

x	y
0	
1	
	3

Solution

The first ordered pair is (0,), and we must find y.

$$\begin{aligned} 2x - y &= 5 \\ 2(0) - y &= 5 && \text{Substitute 0 for } x. \\ 0 - y &= 5 \\ -y &= 5 \\ y &= -5 && \text{Divide by } -1. \end{aligned}$$

The ordered pair is $(0, -5)$.

The second ordered pair is (1,), and we must find y.

$$\begin{aligned} 2x - y &= 5 \\ 2(1) - y &= 5 && \text{Substitute 1 for } x. \\ 2 - y &= 5 \\ -y &= 3 && \text{Subtract 2 from each side.} \\ y &= -3 && \text{Divide by } -1. \end{aligned}$$

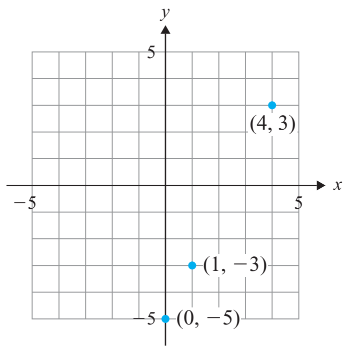
The ordered pair is $(1, -3)$.

The third ordered pair is (, 3), and we must find x.

$$\begin{aligned} 2x - y &= 5 \\ 2x - (3) &= 5 && \text{Substitute 3 for } y. \\ 2x &= 8 && \text{Add 3 to each side.} \\ x &= 4 && \text{Divide by 2.} \end{aligned}$$

The ordered pair is $(4, 3)$.

Each of the points $(0, -5)$, $(1, -3)$, and $(4, 3)$ satisfies the equation $2x - y = 5$.



You Try 5

Complete the table of values for $3x + y = 1$, then plot the points.

x	y
0	
−1	
	−5

5. Solve Applied Problems Involving Ordered Pairs

Next, we will look at an application of ordered pairs.

Example 7

The length of an 18-year-old female's hair is measured to be 250 millimeters (mm) (almost 10 in.). The length of her hair after x days can be approximated by

$$y = 0.30x + 250$$

where y is the length of her hair in millimeters.

a) Find the length of her hair (i) 10 days, (ii) 60 days, and (iii) 90 days after the initial measurement and write the results as ordered pairs.

b) Graph the ordered pairs.

c) How long would it take for her hair to reach a length of 274 mm (almost 11 in.)?

Solution

a) The problem states that in the equation $y = 0.30x + 250$,

$x =$ number of days after the hair was measured
$y =$ length of the hair (in millimeters)

We must determine the length of her hair after 10 days, 60 days, and 90 days. We can organize the information in a table of values.

x	y
10	
60	
90	

i) $x = 10$:

$$\begin{aligned} y &= 0.30x + 250 \\ y &= 0.30(10) + 250 && \text{Substitute 10 for } x. \\ y &= 3 + 250 && \text{Multiply.} \\ y &= 253 \end{aligned}$$

After 10 days, her hair is 253 mm long. We can write this as the ordered pair (10, 253).

ii) $x = 60$:

$$\begin{aligned} y &= 0.30x + 250 \\ y &= 0.30(60) + 250 && \text{Substitute 60 for } x. \\ y &= 18 + 250 && \text{Multiply.} \\ y &= 268 \end{aligned}$$

After 60 days, her hair is 268 mm long. We can write this as the ordered pair (60, 268).

iii) $x = 90$:

$$\begin{aligned} y &= 0.30x + 250 \\ y &= 0.30(90) + 250 && \text{Substitute 90 for } x. \\ y &= 27 + 250 && \text{Multiply.} \\ y &= 277 \end{aligned}$$

After 90 days, her hair is 277 mm long. We can write this as the ordered pair (90, 277).

We can complete the table of values:

x	y
10	253
60	268
90	277

The ordered pairs are (10, 253), (60, 268), and (90, 277).

b) Graph the ordered pairs.

The x-axis represents the number of days after the hair was measured. Since it does not make sense to talk about a negative number of days, we will not continue the x-axis in the negative direction.

The y-axis represents the length of the female's hair. Likewise, a negative number does not make sense in this situation, so we will not continue the y-axis in the negative direction.

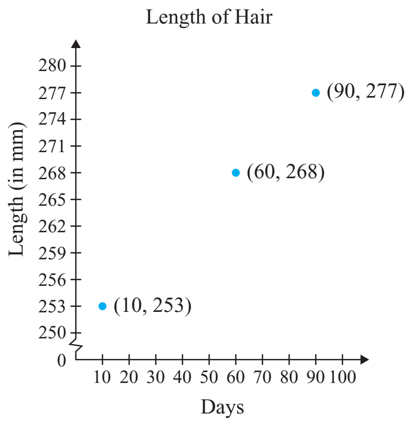
The scales on the x-axis and y-axis are different. This is because the size of the numbers they represent are quite different.

Here are the ordered pairs we must graph: (10, 253), (60, 268), and (90, 277).

The x-values are 10, 60, and 90, so we will let each mark in the x-direction represent 10 units.

The y-values are 253, 268, and 277. While the numbers are rather large, they do not actually differ by much. We will begin labeling the y-axis at 250, but each mark in the y-direction will represent 3 units. Because there is a large jump in values from 0 to 250 on the y-axis, we indicate this with "$\wr$" on the axis between the 0 and 250.

Notice also that we have labeled both axes. The ordered pairs are plotted on the following graph.



c) We must determine how many days it would take for the hair to grow to a length of 274 mm.

The length, 274 mm, is the y-value. We must find the value of x that corresponds to $y = 274$ since x represents the number of days.

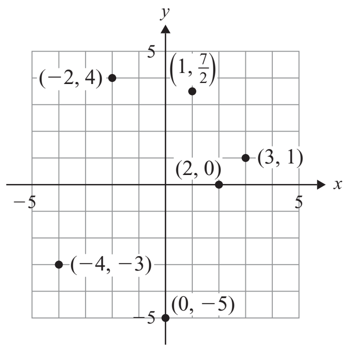
The equation relating x and y is $y = 0.30x + 250$. We will substitute 274 for y and solve for x.

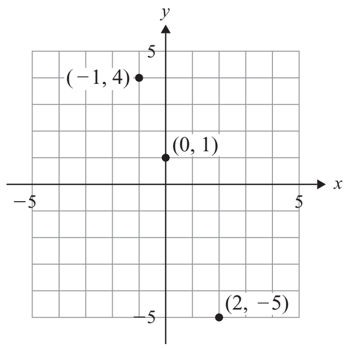
$$\begin{aligned} y &= 0.30x + 250 \\ 274 &= 0.30x + 250 \\ 24 &= 0.30x \\ 80 &= x \end{aligned}$$

It will take 80 days for her hair to grow to a length of 274 mm.

Answers to You Try Exercises

1) a) yes b) no c) yes 2) (5, 8) 3) a) $(5, -2), (12, \frac{3}{2}), (-5, -7), (14, \frac{5}{2})$ b) $(-3, 1), (-3, 3), (-3, -8)$ 4) a) b) c) d) e) f) 5) $(0, 1), (-1, 4), (2, -5)$

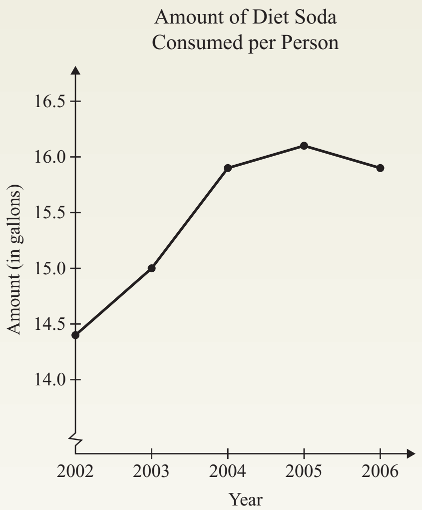




4.1 Exercises

Mixed Exercises: Objectives 1 and 2

The graph shows the number of gallons of diet soda consumed per person for the years 2002–2006. (U.S. Dept of Agriculture)



1) How many gallons of diet soda were consumed per person in 2005?

2) During which year was the consumption level about 15.0 gallons per person?

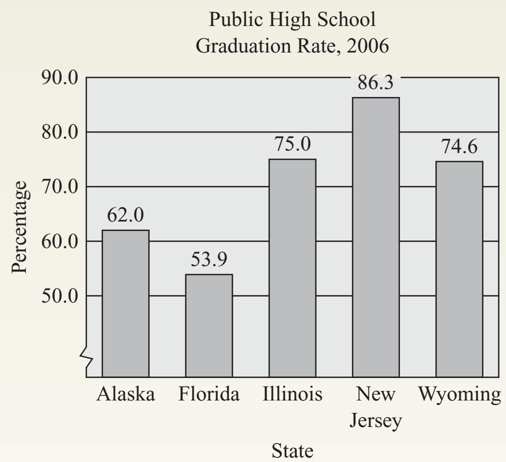
3) During which two years was consumption the same, and how much diet soda was consumed each of these years?

4) During which year did people drink the least amount of diet soda?

5) What was the general consumption trend from 2002 to 2005?

6) Compare the consumption level in 2002 with that in 2006.

The bar graph shows the public high school graduation rate in certain states in 2006. (www.higheredinfo.org)



7) Which state had the highest graduation rate, and what percentage of its public high school students graduated?

8) Which states graduated between 70% and 80% of its students?

9) How does the graduation rate of Florida compare with that of New Jersey?

10) Which state had a graduation rate of about 62%?

11) Explain the difference between a linear equation in one variable and a linear equation in two variables. Give an example of each.

12) True or False: $3x + 6y^2 = -1$ is a linear equation in two variables.

Determine whether each ordered pair is a solution of the given equation.

13) $2x + 5y = 1$; $(-2, 1)$

14) $2x + 7y = -4$; $(2, -5)$

15) $-3x - 2y = -15$; $(7, -3)$

16) $y = 5x - 6$; $(3, 9)$

17) $y = -\frac{3}{2}x - 7$; $(8, 5)$

18) $5y = \frac{2}{3}x + 1$; $(6, 1)$

19) $y = -7$; $(9, -7)$

20) $x = 8$; $(-10, 8)$

Objective 3: Complete Ordered Pairs for a Given Equation

Complete the ordered pair for each equation.

21) $y = 3x - 7$; $(4, \quad)$

22) $y = -2x + 3$; $(6, \quad)$

23) $2x - 15y = 13$; $\left(\quad , -\frac{4}{3}\right)$

24) $-x + 10y = 8$; $\left(\quad , \frac{2}{5}\right)$

25) $x = 5$; $(\quad , -200)$

26) $y = -10$; $(12, \quad)$

Complete the table of values for each equation.

27) $y = 2x - 4$

x	y
0	
1	
-1	
-2	

28) $y = -5x + 1$

x	y
0	
1	
2	
-1	

29) $y = 4x$

x	y
0	
$\frac{1}{2}$	
	12
	-20

30) $y = 9x - 8$

x	y
0	
$-\frac{1}{3}$	
	-17
	1

31) $5x + 4y = -8$

x	y
0	
	0
1	
$-\frac{12}{5}$	

32) $2x - y = 12$

x	y
	0
0	
	-2
$\frac{5}{2}$	

33 $y = -2$

x	y
0	
-3	
8	
17	

34) $x = 20$

x	y
	0
	3
	-4
	-9

35) Explain, in words, how to complete the table of values for $x = -13$.

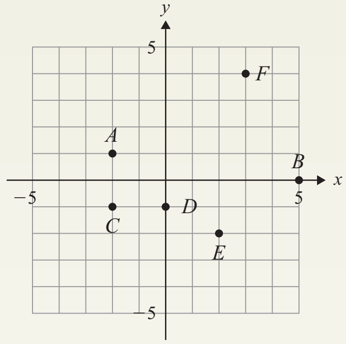
x	y
	0
	2
	-1

36. Explain, in words, how to complete the ordered pair $(\quad , -3)$ for $y = -x - 2$.

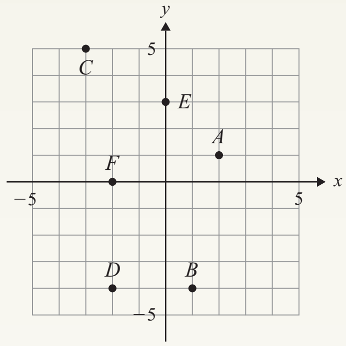
Objective 4: Plot Ordered Pairs

Name each point with an ordered pair, and identify the quadrant in which each point lies.

37)



38)



Graph each ordered pair and explain how you plotted the points.

39) $(2, 4)$

40) $(4, 1)$

41) $(-3, -5)$

42) $(-2, 1)$

Graph each ordered pair.

43) $(-6, 1)$

44) $(-2, -3)$

45) $(0, -1)$

46) $(4, -5)$

47) $(0, 4)$

48) $(-5, 0)$

49) $(-2, 0)$

50) $(0, -1)$

51) $\left(-2, \frac{3}{2}\right)$

52) $\left(\frac{4}{3}, 3\right)$

53) $\left(3, -\frac{1}{4}\right)$

54) $\left(-2, -\frac{9}{4}\right)$

55) $\left(0, -\frac{11}{5}\right)$

56) $\left(-\frac{9}{2}, -\frac{2}{3}\right)$

Mixed Exercises: Objectives 3 and 4

Complete the table of values for each equation, and plot the points.

57) $y = -4x + 3$

x	y
0	
	0
2	
	7

58) $y = -3x + 4$

x	y
0	
	0
	−2
−1	

59) $y = x$

x	y
0	
−1	
	3
	−5

60) $y = -2x$

x	y
	0
	4
2	
−3	

61) $3x + 4y = 12$

x	y
0	
	0
1	
	6

62) $2x - 3y = 6$

x	y
0	
	0
	2
2	

63) $y + 1 = 0$

x	y
0	
1	
−3	
−1	

64) $x = 3$

x	y
	0
	−2
	3
	1

65) $y = \frac{1}{4}x + 2$

x	y
0	
−2	
4	
−1	

66) $y = -\frac{5}{2}x + 3$

x	y
0	
4	
2	
1	

VIDEO 67) For $y = \frac{2}{3}x - 7$,

a) find y when $x = 3$, $x = 6$, and $x = -3$. Write the results as ordered pairs.

b) find y when $x = 1$, $x = 5$, and $x = -2$. Write the results as ordered pairs.

c) why is it easier to find the y-values in part a) than in part b)?

68) Which ordered pair is a solution to every linear equation of the form $y = mx$, where m is a real number?

Fill in the blank with *positive, negative,* or *zero.*

69) The x-coordinate of every point in quadrant III is ________.

70) The y-coordinate of every point in quadrant I is ________.

71) The x-coordinate of every point in quadrant II is ________.

72) The y-coordinate of every point in quadrant II is ________.

73) The x-coordinate of every point in quadrant I is ________.

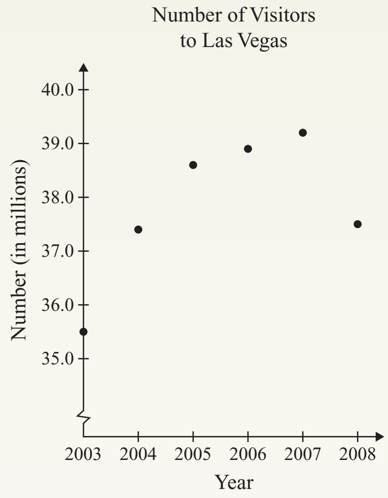
74) The y-coordinate of every point in quadrant IV is ________.

75) The x-coordinate of every point on the y-axis is ________.

76) The y-coordinate of every point on the x-axis is ________.

Objective 5: Solve Applied Problems Involving Ordered Pairs

77) The graph shows the number of people who visited Las Vegas from 2003 to 2008. (www.lvcva.com)



a) If a point on the graph is represented by the ordered pair (x, y), then what do x and y represent?

b) What does the ordered pair (2004, 37.4) represent in the context of this problem?

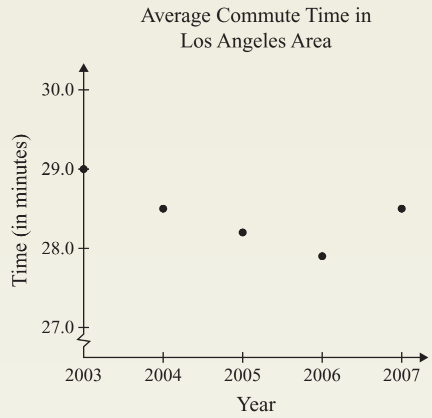
c) Approximately how many people went to Las Vegas in 2006?

d) In which year were there approximately 38.6 million visitors?

e) Approximately how many more people visited Las Vegas in 2008 than in 2003?

f) Represent the following with an ordered pair: During which year did Las Vegas have the most visitors, and how many visitors were there?

78) The graph shows the average amount of time people spent commuting to work in the Los Angeles metropolitan area from 2003 to 2007. (American Community Survey, U.S. Census)



a) If a point on the graph is represented by the ordered pair (x, y), then what do x and y represent?

b) What does the ordered pair (2004, 28.5) represent in the context of this problem?

c) Which year during this time period had the shortest commute? What was the approximate commute time?

d) When was the average commute 28.5 minutes?

e) Write an ordered pair to represent when the average commute time was 28.2 minutes.

79) The percentage of deadly highway crashes involving alcohol is given in the table. (www.bts.gov)

Year	Percentage
1985	52.9
1990	50.6
1995	42.4
2000	41.4
2005	40.0

a) Write the information as ordered pairs (x, y) where x represents the year and y represents the percentage of accidents involving alcohol.

b) Label a coordinate system, choose an appropriate scale, and graph the ordered pairs.

c) Explain the meaning of the ordered pair (2000, 41.4) in the context of the problem.

80) The average annual salary of a social worker is given in the table. (www.bts.gov)

Year	Salary
2005	\$42,720
2006	\$44,950
2007	\$47,170
2008	\$48,180

a) Write the information as ordered pairs (x, y) where x represents the year and y represents the average annual salary.

b) Label a coordinate system, choose an appropriate scale, and graph the ordered pairs.

c) Explain the meaning of the ordered pair (2007, 47,170) in the context of the problem.

81) The amount of sales tax paid by consumers in Seattle in 2009 is given by $y = 0.095x$, where x is the price of an item in dollars and y is the amount of tax to be paid.

x	y
100.00	
140.00	
210.72	
250.00	

a) Complete the table of values, and write the information as ordered pairs.

b) Label a coordinate system, choose an appropriate scale, and graph the ordered pairs.

c) Explain the meaning of the ordered pair (140.00, 13.30) in the context of the problem.

d) How much tax would a customer pay if the cost of an item was \$210.72?

e) Look at the graph. Is there a pattern indicated by the points?

f) If a customer paid \$19.00 in sales tax, what was the cost of the item purchased?

82) Kyle is driving from Atlanta to Oklahoma City. His distance from Atlanta, y (in miles), is given by $y = 66x$, where x represents the number of hours driven.

a) Complete the table of values, and write the information as ordered pairs.

x	y
1	
1.5	
2	
4.5	

b) Label a coordinate system, choose an appropriate scale, and graph the ordered pairs.

c) Explain the meaning of the ordered pair (4.5, 297) in the context of the problem.

d) Look at the graph. Is there a pattern indicated by the points?

e) What does the 66 in $y = 66x$ represent?

f) How many hours of driving time will it take for Kyle to get to Oklahoma City if the distance between Atlanta and Oklahoma City is about 860 miles?

Section 4.2 Graphing by Plotting Points and Finding Intercepts

Objectives

1. Graph a Linear Equation by Plotting Points
2. Graph a Linear Equation in Two Variables by Finding the Intercepts
3. Graph a Linear Equation of the Form $Ax + By = 0$
4. Graph Linear Equations of the Forms $x = c$ and $y = d$
5. Model Data with a Linear Equation

In Example 3 of Section 4.1 we found that the ordered pairs $(1, 3)$, $(-20, -4)$, and $\left(-6, \frac{2}{3}\right)$ are three solutions to the equation $-x + 3y = 8$. But how many solutions does the equation have? It has an infinite number of solutions. Every linear equation in two variables has an infinite number of solutions because we can choose any real number for one of the variables and we will get another real number for the other variable.

Property Solutions of Linear Equations in Two Variables

Every linear equation in two variables has an infinite number of solutions, and the solutions are ordered pairs.

How can we represent all of the solutions to a linear equation in two variables? We can represent them with a graph, and that graph is a line.

Property The Graph of a Linear Equation in Two Variables

The graph of a linear equation in two variables, $Ax + By = C$, is a straight line. Each point on the line is a solution to the equation.

1. Graph a Linear Equation by Plotting Points

Example 1

Complete the table of values and graph $4x - y = 5$.

x	y
2	3
0	−5
1	
	0

Solution

When $x = 1$, we get

$$\begin{aligned} 4x - y &= 5 \\ 4(1) - y &= 5 && \text{Substitute 1 for } x. \\ 4 - y &= 5 \\ -y &= 1 \\ y &= -1 && \text{Solve for } y. \end{aligned}$$

When $y = 0$, we get

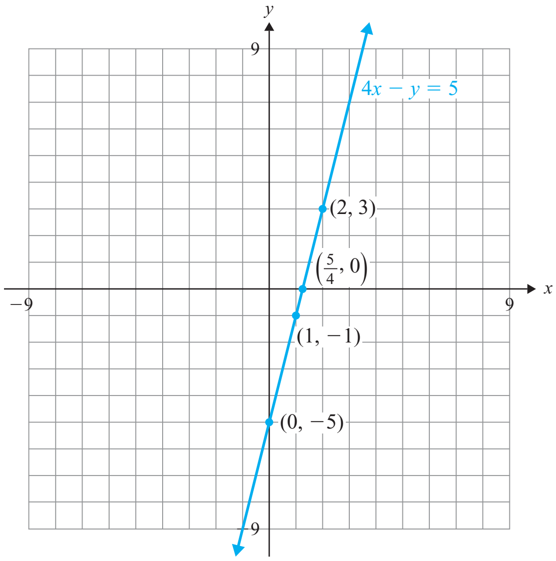
$$\begin{aligned} 4x - y &= 5 \\ 4x - (0) &= 5 && \text{Substitute 0 for } y. \\ 4x &= 5 \\ x &= \frac{5}{4} && \text{Solve for } x. \end{aligned}$$

The completed table of values is

x	y
2	3
0	−5
1	−1
$\frac{5}{4}$	0

This gives us the ordered pairs (2, 3), (0, −5), (1, −1), and $\left(\frac{5}{4}, 0\right)$. Each is a solution to the equation $4x - y = 5$.

Plot the points. They lie on a straight line. We draw the line through these points to get the graph.



The line represents all solutions to the equation $4x - y = 5$. Every point on the line is a solution to the equation. The arrows on the ends of the line indicate that the line extends indefinitely in each direction. Although it is true that we need to find only two points to graph a line, it is best to plot at least three as a check.

You Try 1

Complete the table of values and graph $x - 2y = 3$.

x	y
1	−1
3	0
0	
	2
5	

Example 2

Graph $-x + 2y = 4$.

Solution

We will find three ordered pairs that satisfy the equation. Let's complete a table of values for $x = 0$, $x = 2$, and $x = -4$.

$x = 0$:
$$\begin{aligned} -x + 2y &= 4 \\ -(0) + 2y &= 4 \\ 2y &= 4 \\ y &= 2 \end{aligned}$$

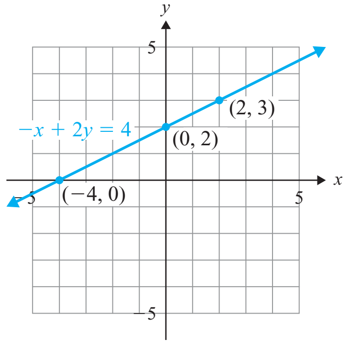
$x = 2$:
$$\begin{aligned} -x + 2y &= 4 \\ -(2) + 2y &= 4 \\ -2 + 2y &= 4 \\ 2y &= 6 \\ y &= 3 \end{aligned}$$

$x = -4$:
$$\begin{aligned} -x + 2y &= 4 \\ -(-4) + 2y &= 4 \\ 4 + 2y &= 4 \\ 2y &= 0 \\ y &= 0 \end{aligned}$$

We get the table of values

x	y
0	2
2	3
−4	0

Plot the points (0, 2), (−4, 0), and (2, 3), and draw the line through them.



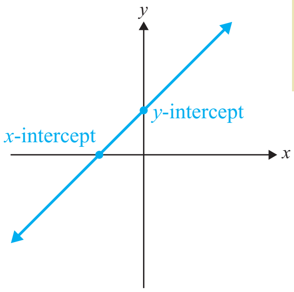
You Try 2

Graph each line.

a) $3x + 2y = 6$ b) $y = 4x - 8$

2. Graph a Linear Equation in Two Variables by Finding the Intercepts

In Example 2, the line crosses the x-axis at −4 and crosses the y-axis at 2. These points are called **intercepts**.

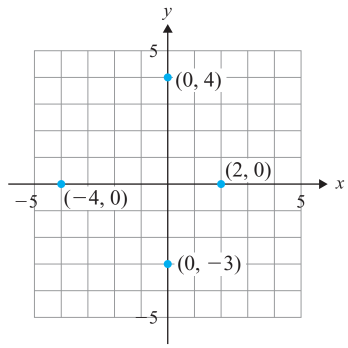


Definitions

The **x-intercept** of the graph of an equation is the point where the graph intersects the x-axis.

The **y-intercept** of the graph of an equation is the point where the graph intersects the y-axis.

What is the y-coordinate of any point on the x-axis? It is zero. Likewise, the x-coordinate of any point on the y-axis is zero.



Therefore,

Procedure Finding Intercepts

To find the *x-intercept* of the graph of an equation, let $y = 0$ and solve for x.

To find the *y-intercept* of the graph of an equation, let $x = 0$ and solve for y.

Finding intercepts is very helpful for graphing linear equations in two variables.

Example 3

Graph $y = -\frac{1}{3}x + 1$ by finding the intercepts and one other point.

Solution

We will begin by finding the intercepts.

x-intercept: Let $y = 0$, and solve for x.

$$0 = -\frac{1}{3}x + 1$$
$$-1 = -\frac{1}{3}x$$
$$3 = x \quad \text{Multiply both sides by } -3 \text{ to solve for } x.$$

The x-intercept is $(3, 0)$.

y-intercept: Let $x = 0$, and solve for y.

$$y = -\frac{1}{3}(0) + 1$$
$$y = 0 + 1$$
$$y = 1$$

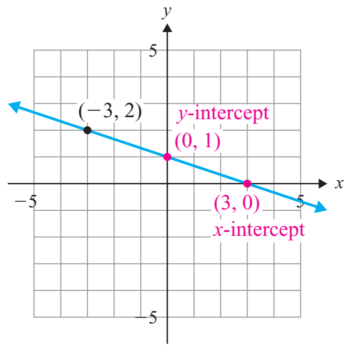
The y-intercept is $(0, 1)$.

We must find another point. Let's look closely at the equation $y = -\frac{1}{3}x + 1$. The coefficient of x is $-\frac{1}{3}$. If we choose a value for x that is a multiple of 3 (the denominator of the fraction), then $-\frac{1}{3}x$ will not be a fraction.

Let $x = -3$.

$$y = -\frac{1}{3}x + 1$$
$$y = -\frac{1}{3}(-3) + 1$$
$$y = 1 + 1$$
$$y = 2$$

The third point is $(-3, 2)$.



Plot the points, and draw the line through them. See the graph above.

You Try 3

Graph $y = \frac{4}{3}x - 2$ by finding the intercepts and one other point.

3. Graph a Linear Equation of the Form $Ax + By = 0$

Sometimes the x- and y-intercepts are the same point.

Example 4

Graph $-2x + y = 0$.

Solution

If we begin by finding the x-intercept, let $y = 0$ and solve for x.

$$\begin{aligned} -2x + y &= 0 \\ -2x + (0) &= 0 \\ -2x &= 0 \\ x &= 0 \end{aligned}$$

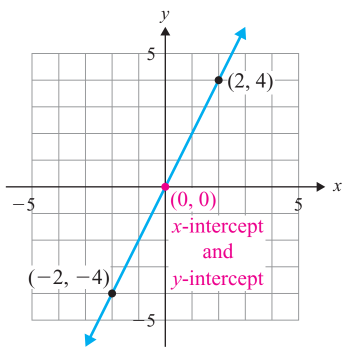
The x-intercept is (0, 0). But this is the same as the y-intercept since we find the y-intercept by substituting 0 for x and solving for y. Therefore, *the x- and y-intercepts are the same point.*

Instead of the intercepts giving us two points on the graph of $-2x + y = 0$, we have only one. We will find two other points on the line.

$x = 2$:
$$\begin{aligned} -2x + y &= 0 \\ -2(2) + y &= 0 \\ -4 + y &= 0 \\ y &= 4 \end{aligned}$$

$x = -2$:
$$\begin{aligned} -2x + y &= 0 \\ -2(-2) + y &= 0 \\ 4 + y &= 0 \\ y &= -4 \end{aligned}$$

The ordered pairs (2, 4) and $(-2, -4)$ are also solutions to the equation. Plot all three points on the graph and draw the line through them.



Property The Graph of $Ax + By = 0$

If A and B are nonzero real numbers, then the graph of $Ax + By = 0$ is a line passing through the origin, (0, 0).

You Try 4

Graph $x - y = 0$.

4. Graph Linear Equations of the Forms $x = c$ and $y = d$

In Section 4.1 we said that an equation like $x = -2$ is a linear equation in two variables since it can be written in the form $x + 0y = -2$. The same is true for $y = 3$. It can be written as $0x + y = 3$. Let's see how we can graph these equations.

Example 5

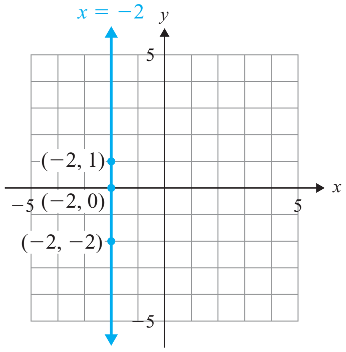
Graph $x = -2$.

Solution

The equation $x = -2$ means that *no matter the value of y, x always equals* -2. We can make a table of values where we choose any value for y, but x is always -2.

Plot the points, and draw a line through them. The graph of $x = -2$ is a *vertical line*.

x	y
-2	0
-2	1
-2	-2



We can generalize the result as follows:

Property The Graph of $x = c$

If c is a constant, then the graph of $x = c$ is a *vertical line* going through the point $(c, 0)$.

You Try 5

Graph $x = 2$.

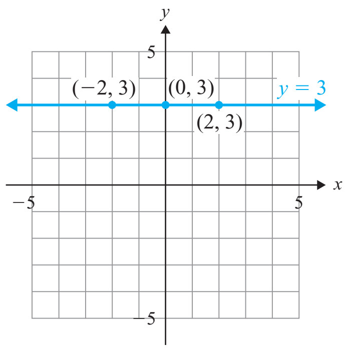
Example 6

Graph $y = 3$.

Solution

The equation $y = 3$ means that *no matter the value of x, y always equals* 3. Make a table of values where we choose any value for x, but y is always 3.

x	y
0	3
2	3
-2	3



Plot the points, and draw a line through them. The graph of $y = 3$ is a **horizontal line.**

We can generalize the result as follows:

Property The Graph of $y = d$

If d is a constant, then the graph of $y = d$ is a *horizontal line* going through the point $(0, d)$.

You Try 6

Graph $y = -4$.

5. Model Data with a Linear Equation

Linear equations are often used to model (or describe mathematically) real-world data. We can use these equations to learn what has happened in the past or predict what will happen in the future.

Example 7

The average annual cost of college tuition and fees at private, 4-year institutions can be modeled by

$$y = 907x + 12{,}803$$

where x is the number of years after 1996 and y is the average tuition and fees, in dollars. (Source: The College Board)

a) Find the y-intercept of the graph of this equation and explain its meaning.

b) Find the approximate cost of tuition and fees in 2000 and 2006. Write the information as ordered pairs.

c) Graph $y = 907x + 12{,}803$.

d) Use the graph to approximate the average cost of tuition and fees in 2005. Is this the same result as when you use the equation to estimate the average cost?

Solution

a) To find the y-intercept, let $x = 0$.

$$\begin{aligned} y &= 907(0) + 12{,}803 \\ y &= 12{,}803 \end{aligned}$$

The y-intercept is $(0, 12{,}803)$. What does this represent?

The problem states that x is the number of years *after* 1996. Therefore, $x = 0$ represents zero years after 1996, which is the year 1996.

The y-intercept $(0, 12{,}803)$ tells us that in 1996 the average cost of tuition and fees at a private 4-year institution was \$12,803.

b) The approximate cost of tuition and fees in

2000: First, realize that $x \neq 2000$. x is the number of years *after* 1996. Since 2000 is 4 years after 1996, $x = 4$. Let $x = 4$ in $y = 907x + 12{,}803$ and find y.

$$\begin{aligned} y &= 907(4) + 12{,}803 \\ y &= 3628 + 12{,}803 \\ y &= 16{,}431 \end{aligned}$$

In 2000, the approximate cost of college tuition and fees at these schools was \$16,431. We can write this information as the ordered pair $(4, 16{,}431)$.

2006: Begin by finding x. 2006 is 10 years after 1996, so $x = 10$.

$$\begin{aligned} y &= 907(10) + 12{,}803 \\ y &= 9070 + 12{,}803 \\ y &= 21{,}873 \end{aligned}$$

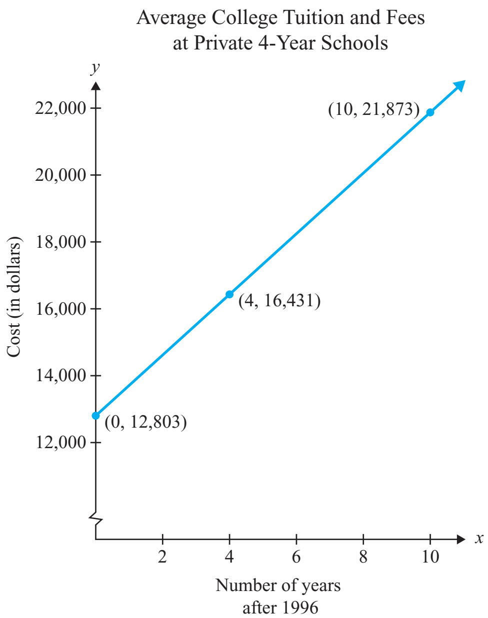
In 2006, the approximate cost of college tuition and fees at private 4-year schools was \$21,873.

The ordered pair $(10, 21{,}873)$ can be written from this information.

c) We will plot the points $(0, 12{,}803)$, $(4, 16{,}431)$, and $(10, 21{,}873)$. Label the axes, and choose an appropriate scale for each.

The x-coordinates of the ordered pairs range from 0 to 10, so we will let each mark in the x-direction represent 2 units.

The y-coordinates of the ordered pairs range from 12,803 to 21,873. We will let each mark in the y-direction represent 2000 units.



d) Using the graph to estimate the cost of tuition and fees in 2005, we locate $x = 9$ on the x-axis since 2005 is 9 years after 1996. When $x = 9$, we move straight up the graph to $y \approx 21{,}000$. Our approximation from the graph is \$21,000.

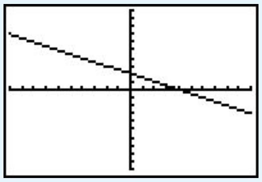
If we use the equation and let $x = 9$, we get

$$y = 907x + 12{,}803$$
$$y = 907(9) + 12{,}803$$
$$y = 8163 + 12{,}803$$
$$y = 20{,}966$$

From the equation we find that the cost of college tuition and fees at private 4-year schools was about \$20,966. The numbers are not exactly the same, but they are close.

Using Technology

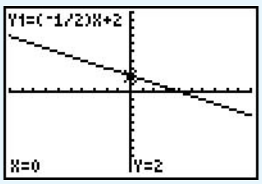
A graphing calculator can be used to graph an equation and to verify information that we find using algebra. We will graph the equation $y = -\frac{1}{2}x + 2$ and then find the intercepts both algebraically and using the calculator.

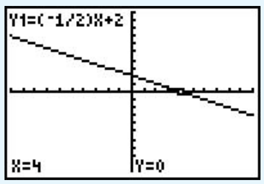


First, enter the equation into the calculator. Press $\boxed{\text{ZOOM}}$ and select 6:Zstandard to graph the equation.

1. Find, algebraically, the y-intercept of the graph of $y = -\frac{1}{2}x + 2$. Is it consistent with the graph of the equation?
2. Find, algebraically, the x-intercept of the graph of $y = -\frac{1}{2}x + 2$. Is it consistent with the graph of the equation?

Now let's verify the intercepts using the graphing calculator. To find the y-intercept, press $\boxed{\text{TRACE}}$ after displaying the graph. The cursor is automatically placed at the center x-value on the screen, which is at the point (0, 2) as shown next on the left. To find the x-intercept, press $\boxed{\text{TRACE}}$, type 4, and press $\boxed{\text{ENTER}}$. The calculator displays (4, 0) as shown next on the right. This is consistent with the intercepts found in 1 and 2, using algebra.



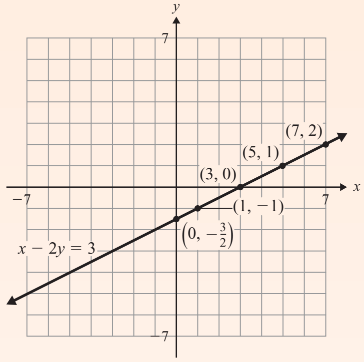


Use algebra to find the x- and y-intercepts of the graph of each equation. Then, use the graphing calculator to verify your results.

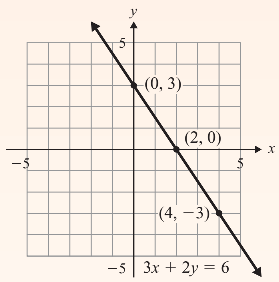
1. $y = 2x - 4$
2. $y = x + 3$
3. $y = -x + 5$
4. $2x - 5y = 10$
5. $3x + 4y = 24$
6. $3x - 7y = 21$

Answers to You Try Exercises

1) $(0, -\frac{3}{2})$, (7, 2), (5, 1)



2) a)



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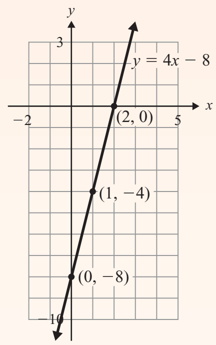
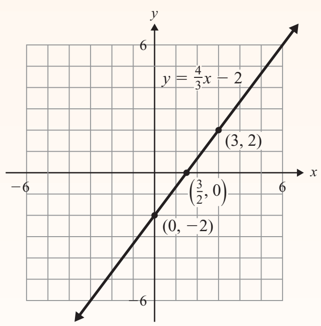
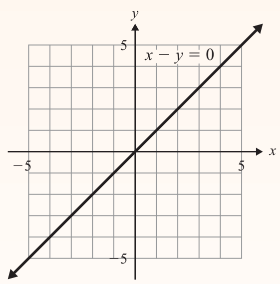


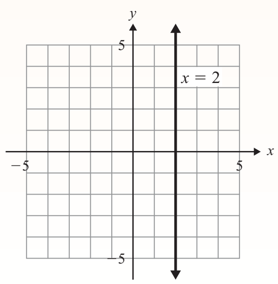
3)



4)



5)



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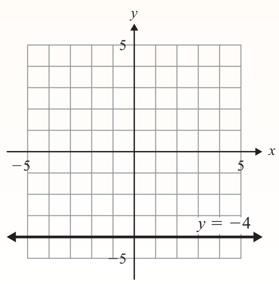


Answers to Technology Exercises

1. (2, 0), (0, −4)
2. (−3, 0), (0, 3)
3. (5, 0), (0, 5)
4. (5, 0), (0, −2)
5. (8, 0), (0, 6)
6. (7, 0), (0, −3)

4.2 Exercises

Objective 1: Graph a Linear Equation by Plotting Points

1) The graph of a linear equation in two variables is a ________.

2) Every linear equation in two variables has how many solutions?

Complete the table of values and graph each equation.

3) $y = -2x + 4$

x	y
0	
−1	
2	
3	

4) $y = 3x - 2$

x	y
0	
1	
2	
−1	

5) $y = \frac{3}{2}x + 7$

x	y
0	
2	
−2	
−4	

6) $y = -\frac{5}{3}x + 3$

x	y
0	
−3	
3	
6	

7) $2x = 3 - y$

x	y
	0
0	
$\frac{1}{2}$	
	5

8) $-x + 5y = 10$

x	y
0	
	0
	4
−3	

9) $x = -\frac{4}{9}$

x	y
	5
	0
	−1
	−2

10) $y + 5 = 0$

x	y
0	
−3	
−1	
2	

Mixed Exercises: Objectives 1–4

11) What is the y-intercept of the graph of an equation? How do you find it?

12) What is the x-intercept of the graph of an equation? How do you find it?

Graph each equation by finding the intercepts and at least one other point.

13) $y = x - 1$

14) $y = -x + 3$

VIDEO 15) $3x - 4y = 12$

16) $2x - 7y = 14$

17) $x = -\frac{4}{3}y - 2$

18) $x = \frac{5}{4}y - 5$

19) $2x - y = 8$

20) $3x + y = -6$

VIDEO 21) $y = -x$

22) $y = 3x$

23) $4x - 3y = 0$

24) $6y - 5x = 0$

VIDEO 25) $x = 5$

26) $y = -4$

27) $y = 0$

28) $x = 0$

29) $x - \frac{4}{3} = 0$

30) $y + 1 = 0$

31) $4x - y = 9$

32) $x + 3y = -5$

33) Which ordered pair is a solution to every linear equation of the form $Ax + By = 0$?

34) True or False: The graph of $Ax + By = 0$ will always pass through the origin.

Objective 5: Model Data with a Linear Equation

35) The cost of downloading popular songs from iTunes is given by $y = 1.29x$, where x represents the number of songs downloaded and y represents the cost, in dollars.

a) Make a table of values using $x = 0, 4, 7,$ and 12, and write the information as ordered pairs.

b) Explain the meaning of each ordered pair in the context of the problem.

c) Graph the equation. Use an appropriate scale.

d) How many songs could you download for \$11.61?

36) The force, y, measured in newtons (N), required to stretch a particular spring x meters is given by $y = 100x$.

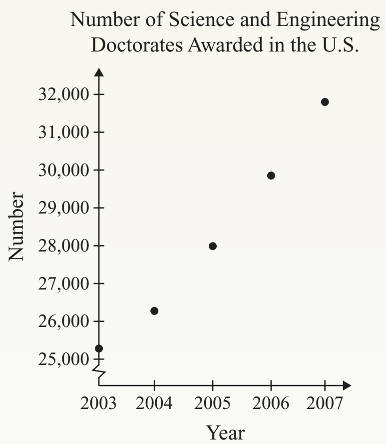
a) Make a table of values using $x = 0, 0.5, 1.0,$ and 1.5, and write the information as ordered pairs.

b) Explain the meaning of each ordered pair in the context of the problem.

c) Graph the equation. Use an appropriate scale.

d) If the spring was pulled with a force of 80 N, how far did it stretch?

37) The number of doctorate degrees awarded in science and engineering in the United States from 2003 to 2007 can be modeled by $y = 1662x + 24{,}916$, where x represents the number of years after 2003, and y represents the number of doctorate degrees awarded. The actual data are graphed here. (www.nsf.gov)



a) From the graph, estimate the number of science and engineering doctorates awarded in 2004 and 2007.

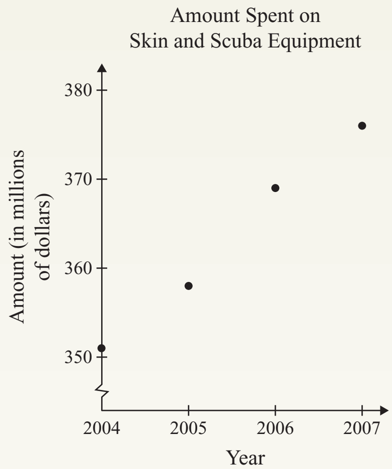
b) Determine the number of degrees awarded during the same years using the equation. Are the numbers close?

c) Graph the line that models the data given on the original graph.

d) What is the y-intercept of the graph of this equation, and what does it represent? How close is it to the actual point plotted on the given graph?

e) If the trend continues, how many science and engineering doctorates will be awarded in 2012? Use the equation.

38) The amount of money Americans spent on skin and scuba diving equipment from 2004 to 2007 can be modeled by $y = 8.6x + 350.6$, where x represents the number of years after 2004, and y represents the amount spent on equipment in millions of dollars. The actual data are graphed here. (www.census.gov)



a) From the graph, estimate the amount spent in 2005 and 2006.

b) Determine the amount of money spent during the same years using the equation. Are the numbers close?

c) Graph the line that models the data given on the original graph.

d) What is the y-intercept of the graph of this equation, and what does it represent? How close is it to the actual point plotted on the given graph?

e) If the trend continues, how much will be spent on skin and scuba gear in 2014? Use the equation.

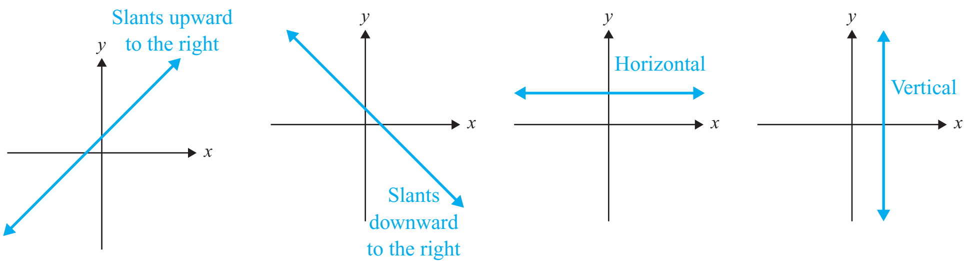
Section 4.3 The Slope of a Line

Objectives

1. Understand the Concept of Slope
2. Find the Slope of a Line Given Two Points on the Line
3. Use Slope to Solve Applied Problems
4. Find the Slope of Horizontal and Vertical Lines
5. Use Slope and One Point on a Line to Graph the Line

1. Understand the Concept of Slope

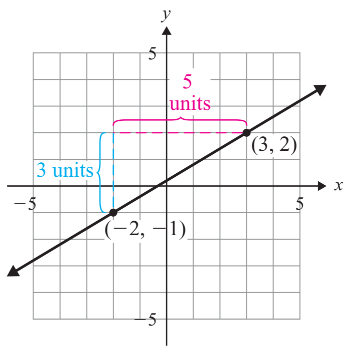
In Section 4.2, we learned to graph lines by plotting points. You may have noticed that some lines are steeper than others. Their "slants" are different, too.

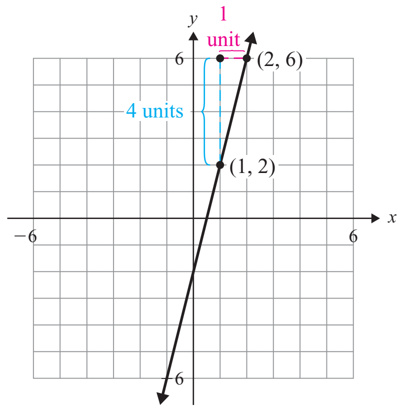


We can describe the steepness of a line with its *slope*.

> **Property** Slope of a Line
>
> The **slope** of a line measures its steepness. It is the ratio of the vertical change in y to the horizontal change in x. Slope is denoted by m.

We can also think of slope as a rate of change. *Slope* is the rate of change between two points. More specifically, it describes the rate of change in y to the change in x.





$$\text{Slope} = \frac{3}{5} \begin{array}{l} \leftarrow \text{vertical change} \\ \leftarrow \text{horizontal change} \end{array} \qquad \text{Slope} = 4 \text{ or } \frac{4}{1} \begin{array}{l} \leftarrow \text{vertical change} \\ \leftarrow \text{horizontal change} \end{array}$$

For example, in the graph on the left, the line changes 3 units vertically for every 5 units it changes horizontally. Its slope is $\frac{3}{5}$. The line on the right changes 4 units vertically for every 1 unit of horizontal change. It has a slope of $\frac{4}{1}$ or 4.

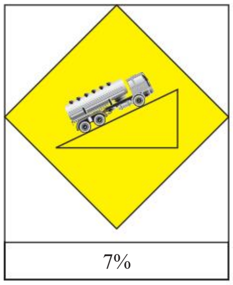
Notice that the line with slope 4 is steeper than the line that has a slope of $\frac{3}{5}$.

> **Note**
>
> As the magnitude of the slope gets larger, the line gets steeper.

Here is an application of slope.

Example 1

A sign along a highway through the Rocky Mountains is shown on the left. What does it mean?



Solution

Percent means "out of 100." Therefore, we can write 7% as $\frac{7}{100}$. We can interpret $\frac{7}{100}$ as the ratio of the vertical change in the road to horizontal change in the road.

$$\text{The slope of the road is } \frac{7}{100}. \quad \begin{array}{l} \longleftarrow \text{Vertical change} \\ \longleftarrow \text{Horizontal change} \end{array}$$

The highway rises 7 ft for every 100 horizontal feet. ■

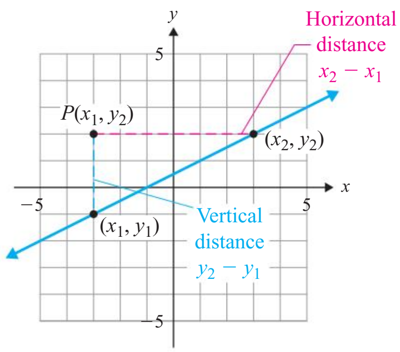
You Try 1

The slope of a conveyer belt is $\frac{5}{12}$, where the dimensions of the ramp are in inches.

What does this mean?

2. Find the Slope of a Line Given Two Points on the Line

Here is line L. The points (x_1, y_1) and (x_2, y_2) are two points on line L. *We will find the ratio of the vertical change in y to the horizontal change in x between the points* (x_1, y_1) *and* (x_2, y_2).



To get from (x_1, y_1) to (x_2, y_2), we move *vertically* to point P then *horizontally* to (x_2, y_2). The x-coordinate of point P is x_1, and the y-coordinate of P is y_2.

When we moved *vertically* from (x_1, y_1) to point $P(x_1, y_2)$, how far did we go? We moved a vertical distance $y_2 - y_1$.

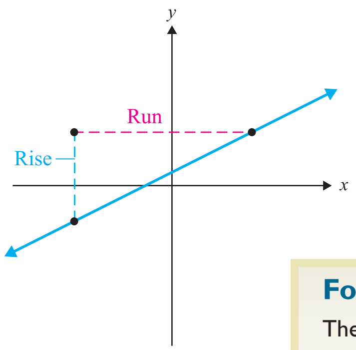
Note

The vertical change is $y_2 - y_1$ and is called the **rise**.

Then we moved *horizontally* from point $P(x_1, y_2)$ to (x_2, y_2). How far did we go? We moved a horizontal distance $x_2 - x_1$.

Note

The horizontal change is $x_2 - x_1$ and is called the **run**.



We said that the slope of a line is the ratio of the vertical change (rise) to the horizontal change (run). Therefore,

Formula The Slope of a Line

The **slope,** m, of a line containing the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{\text{Vertical change}}{\text{Horizontal change}} = \frac{y_2 - y_1}{x_2 - x_1}$$

We can also think of slope as:

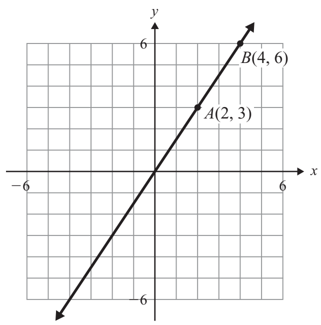
$$\frac{\text{Rise}}{\text{Run}} \quad \text{or} \quad \frac{\text{Change in } y}{\text{Change in } x}.$$

Let's look at some different ways to determine the slope of a line.

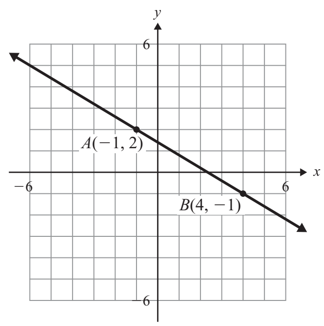
Example 2

Determine the slope of each line.

a)



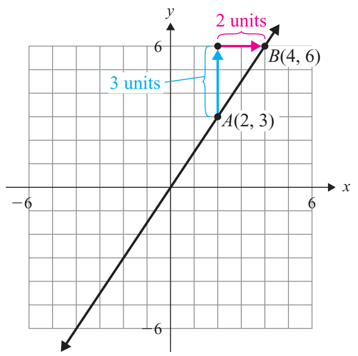
b)



Solution

a) We will find the slope in two ways.

i) First, we will find the vertical change and the horizontal change by counting these changes as we go from A to B.



Vertical change (change in y) from A to B: 3 units

Horizontal change (change in x) from A to B: 2 units

$$\text{Slope} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{3}{2} \quad \text{or} \quad m = \frac{3}{2}$$

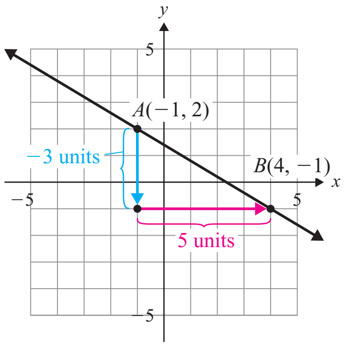
ii) We can also find the slope using the formula.

Let $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (4, 6)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 3}{4 - 2} = \frac{3}{2}.$$

You can see that we get the same result either way we find the slope.

b) i) First, find the slope by counting the vertical change and horizontal change as we go from A to B.



Vertical change (change in y) from A to B: -3 units

Horizontal change (change in x) from A to B: 5 units

$$\text{Slope} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{-3}{5} = -\frac{3}{5}$$

$$\text{or} \quad m = -\frac{3}{5}$$

ii) We can also find the slope using the formula.

Let $(x_1, y_1) = (-1, 2)$ and $(x_2, y_2) = (4, -1)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 2}{4 - (-1)} = \frac{-3}{5} = -\frac{3}{5}.$$

Again, we obtain the same result using either method for finding the slope. ■

Note

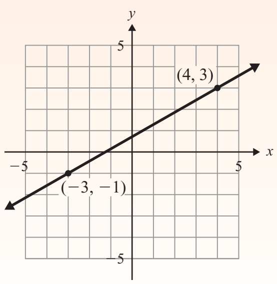
The slope of $-\frac{3}{5}$ can be thought of as $\frac{-3}{5}$, $\frac{3}{-5}$, or $-\frac{3}{5}$.

You Try 2

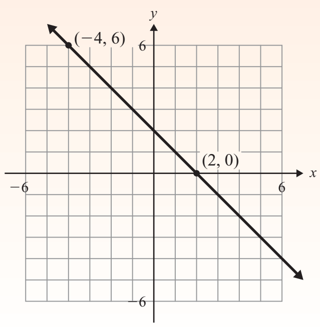
Determine the slope of each line by

a) counting the vertical change and horizontal change. b) using the formula.

a)



b)



Notice that in Example 2a, the line has a positive slope and slants upward from left to right. As the value of x increases, the value of y increases as well. The line in Example 2b has a negative slope and slants downward from left to right. Notice, in this case, that as the line goes from left to right, the value of x increases while the value of y decreases. We can summarize these results with the following general statements.

Property Positive and Negative Slopes

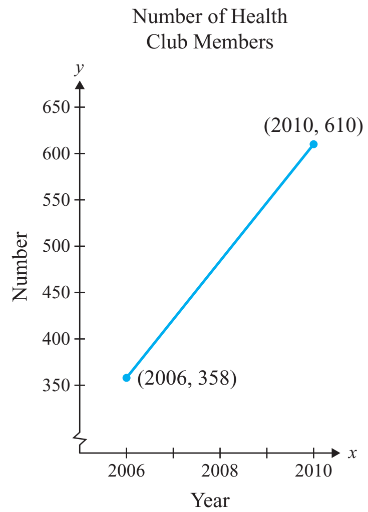
A line with a **positive slope** slants upward from left to right. As the value of x increases, the value of y increases as well.

A line with a **negative slope** slants downward from left to right. As the value of x increases, the value of y decreases.

3. Use Slope to Solve Applied Problems

Example 3

The graph models the number of members of a certain health club from 2006 to 2010.



a) How many members did the club have in 2006? in 2010?

b) What does the sign of the slope of the line segment mean in the context of the problem?

c) Find the slope of the line segment, and explain what it means in the context of the problem.

Solution

a) We can determine the number of members by reading the graph. In 2006, there were 358 members, and in 2010 there were 610 members.

b) The positive slope tells us that from 2006 to 2010 the number of members was increasing.

c) Let $(x_1, y_1) = (2006, 358)$ and $(x_2, y_2) = (2010, 610)$.

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{610 - 358}{2010 - 2006} = \frac{252}{4} = 63$$

The slope of the line is 63. Therefore, the number of members of the health club between 2006 and 2010 increased by 63 per year.

4. Find the Slope of Horizontal and Vertical Lines

Example 4

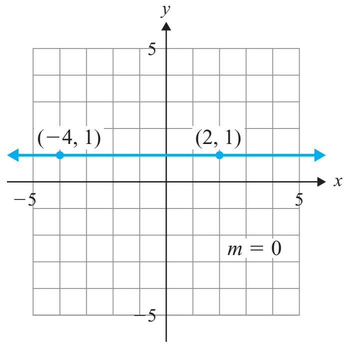
Find the slope of the line containing each pair of points.

a) $(-4, 1)$ and $(2, 1)$ b) $(2, 4)$ and $(2, -3)$

Solution

a) Let $(x_1, y_1) = (-4, 1)$ and $(x_2, y_2) = (2, 1)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 1}{2 - (-4)} = \frac{0}{6} = 0$$

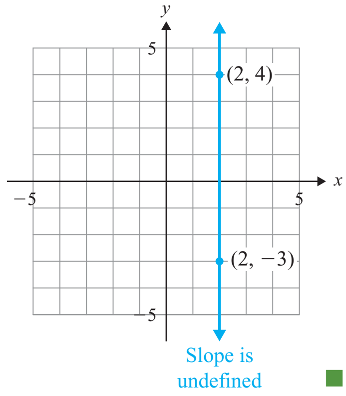


If we plot the points, we see that they lie on a horizontal line. Each point on the line has a y-coordinate of 1, so $y_2 - y_1$ *always* equals zero.

The slope of every horizontal line is zero.

b) Let $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (2, -3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 4}{2 - 2} = \frac{-7}{0} \quad \text{undefined}$$



We say that the slope is undefined. Plotting these points gives us a vertical line. Each point on the line has an x-coordinate of 2, so $x_2 - x_1$ *always* equals zero.

The slope of every vertical line is undefined.

You Try 3

Find the slope of the line containing each pair of points.

a) $(4, 9)$ and $(-3, 9)$ b) $(-7, 2)$ and $(-7, 0)$

Property Slopes of Horizontal and Vertical Lines

The slope of a horizontal line, $y = d$, is **zero**. The slope of a vertical line, $x = c$, is **undefined**. (c and d are constants.)

5. Use Slope and One Point on a Line to Graph the Line

We have seen how we can find the slope of a line given two points on the line. Now, we will see how we can use the slope and *one* point on the line to graph the line.

Example 5

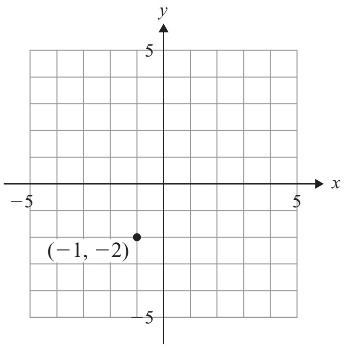
Graph the line containing the point

a) $(-1, -2)$ with a slope of $\frac{3}{2}$.

b) $(0, 1)$ with a slope of -3.

Solution

a) Plot the point.

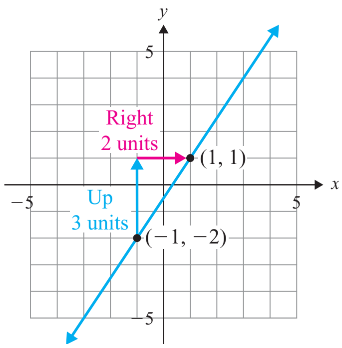


Use the slope to find another point on the line.

$$m = \frac{3}{2} = \frac{\text{Change in } y}{\text{Change in } x}$$

To get from the point $(-1, -2)$ to another point on the line, move up 3 units in the y-direction and right 2 units in the x-direction.

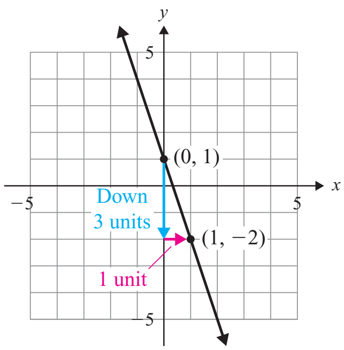
Plot this point, and draw a line through the two points.



b) Plot the point $(0, 1)$.

What does the slope, $m = -3$, mean?

$$m = -3 = \frac{-3}{1} = \frac{\text{Change in } y}{\text{Change in } x}$$



To get from $(0, 1)$ to another point on the line, we will move *down* 3 units in the y-direction and *right* 1 unit in the x-direction. We end up at $(1, -2)$.

Plot this point, and draw a line through $(0, 1)$ and $(1, -2)$.

In part b), we could have written $m = -3$ as $m = \frac{3}{-1}$. This would have given us a different point on the same line. ■

You Try 4

Graph the line containing the point

a) $(-2, 1)$ with a slope of $-\frac{3}{2}$.

b) $(0, -3)$ with a slope of 2.

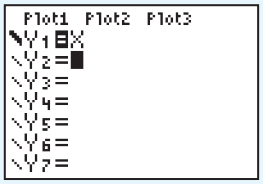
c) $(3, 2)$ with an undefined slope.

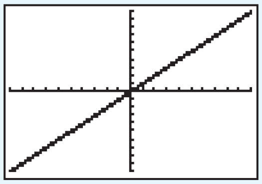
Using Technology

When we look at the graph of a linear equation, we should be able to estimate its slope. Use the equation $y = x$ as a guideline.

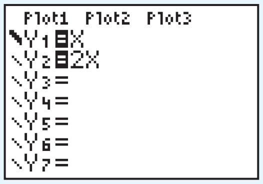
Step 1: Graph the equation $y = x$.

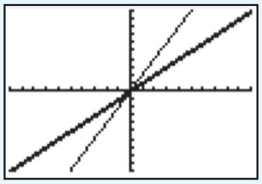
We can make the graph a thick line (so we can tell it apart from the others) by moving the arrow all the way to the left and pressing enter:





Step 2: Keeping this equation, graph the equation $y = 2x$:

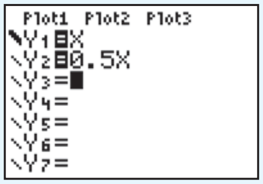


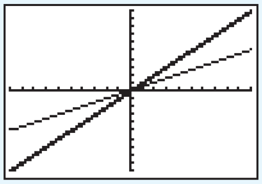


a. Is the new graph steeper or flatter than the graph of $y = x$?

b. Make a guess as to whether $y = 3x$ will be steeper or flatter than $y = x$. Test your guess by graphing $y = 3x$.

Step 3: Clear the equation $y = 2x$ and graph the equation $y = 0.5x$:

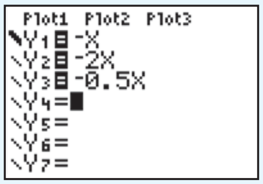


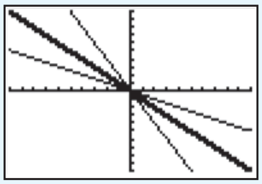


a. Is the new graph steeper or flatter than the graph of $y = x$?

b. Make a guess as to whether $y = 0.65x$ will be steeper or flatter than $y = x$. Test your guess by graphing $y = 0.65x$.

Step 4: Test similar situations, except with negative slopes: $y = -x$





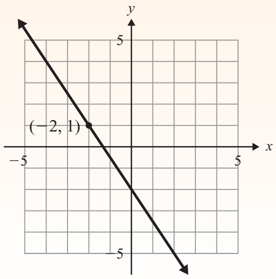
Did you notice that we have the same relationship, except in the opposite direction? That is, $y = 2x$ is steeper than $y = x$ in the positive direction, and $y = -2x$ is steeper than $y = -x$, but in the negative direction. And $y = 0.5x$ is flatter than $y = x$ in the positive direction, and $y = -0.5x$ is flatter than $y = -x$, but in the negative direction.

Answers to You Try Exercises

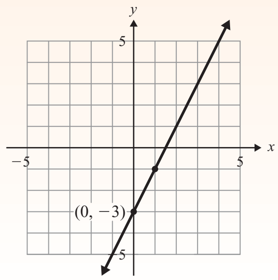
1) The belt rises 5 in. for every 12 horizontal inches. 2) a) $m = \frac{4}{7}$ b) $m = -1$

3) a) $m = 0$ b) undefined

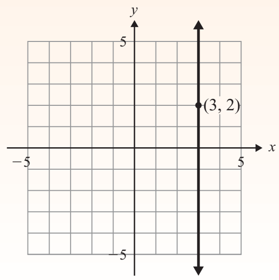
4) a)



b)



c)



4.3 Exercises

Objective 1: Understand the Concept of Slope

1) Explain the meaning of slope.

2) Describe the slant of a line with a negative slope.

3) Describe the slant of a line with a positive slope.

4) The slope of a horizontal line is ________.

5) The slope of a vertical line is ________.

6) If a line contains the points (x_1, y_1) and (x_2, y_2), write the formula for the slope of the line.

Mixed Exercises: Objectives 2 and 4

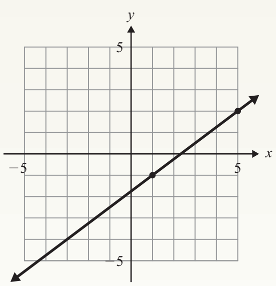
Determine the slope of each line by

a) counting the vertical change and the horizontal change as you move from one point to the other on the line;

and

b) using the slope formula. (See Example 2.)

VIDEO 7)



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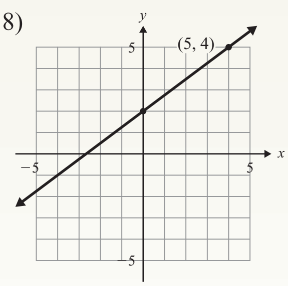

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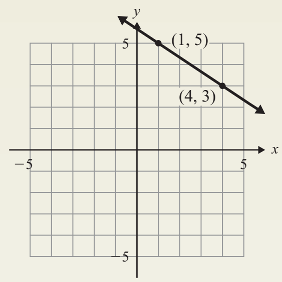

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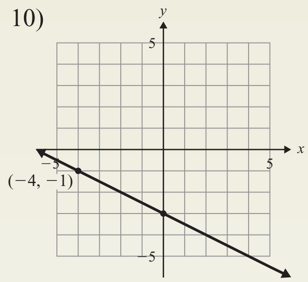

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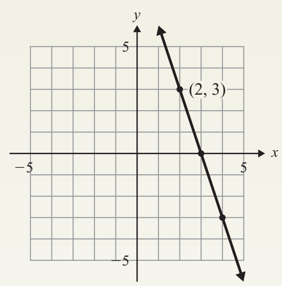

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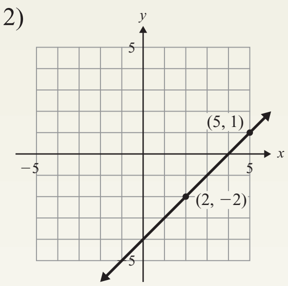

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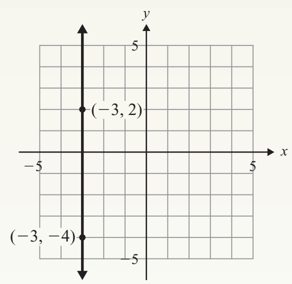

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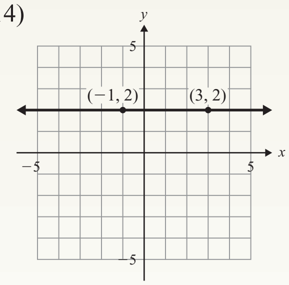


15) Graph a line with a positive slope and a negative y-intercept.

16) Graph a line with a negative slope and a positive x-intercept.

Use the slope formula to find the slope of the line containing each pair of points.

17) $(2, 1)$ and $(0, -3)$

18) $(0, 3)$ and $(9, 6)$

19) $(2, -6)$ and $(-1, 6)$

20) $(-3, 9)$ and $(2, 4)$

21) $(-4, 3)$ and $(1, -8)$

22) $(2, 0)$ and $(-5, 4)$

23) $(-2, -2)$ and $(-2, 7)$

24) $(0, -6)$ and $(-9, -6)$

VIDEO 25) $(3, 5)$ and $(-1, 5)$

26) $(1, 3)$ and $(1, -1)$

27) $\left(\frac{2}{3}, \frac{5}{2}\right)$ and $\left(-\frac{1}{2}, 2\right)$

28) $\left(-\frac{1}{5}, \frac{3}{4}\right)$ and $\left(\frac{1}{3}, -\frac{3}{5}\right)$

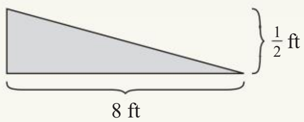
29) $(3.5, -1.4)$ and $(7.5, 1.6)$

30) $(-1.7, 10.2)$ and $(0.8, -0.8)$

Objective 3: Use Slope to Solve Applied Problems

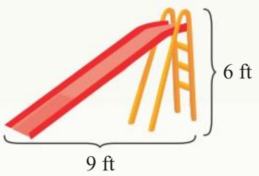
31) The longest run at Ski Dubai, an indoor ski resort in the Middle East, has a vertical drop of about 60 m with a horizontal distance of about 395 m. What is the slope of this ski run? (www.skidxb.com)

32) The federal government requires that all wheelchair ramps in new buildings have a maximum slope of $\frac{1}{12}$. Does the following ramp meet this requirement? Give a reason for your answer. (www.access-board.gov)

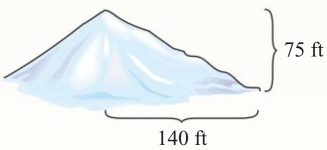


Use the following information for Exercises 33 and 34. To minimize accidents, the Park District Risk Management Agency recommends that playground slides and sledding hills have a maximum slope of about 0.577. (Illinois Parks and Recreation)

33) Does this slide meet the agency's recommendations?



34) Does this sledding hill meet the agency's recommendations?



35) In Granby, Colorado, the first 50 ft of a driveway cannot have a slope of more than 5%. If the first 50 ft of a driveway rises 0.75 ft for every 20 ft of horizontal distance, does this driveway meet the building code? (http://co.grand.co.us)

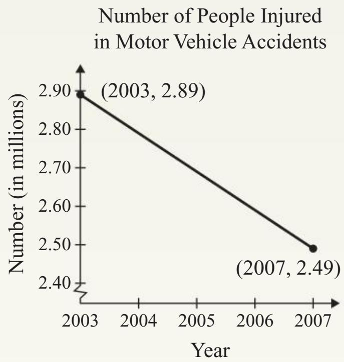
Use the following information for Exercises 36–38. The steepness (slope) of a roof on a house in a certain town cannot exceed $\frac{7}{12}$, also known as a 7:12 *pitch*. The first number refers to the rise of the roof. The second number refers to how far over you must go (the run) to attain that rise.

36) Find the slope of a roof with a 12:20 pitch.

37) Find the slope of a roof with a 12:26 pitch.

38) Does the slope in Exercise 36 meet the town's building code? Give a reason for your answer.

39) The graph shows the approximate number of people in the United States injured in a motor vehicle accident from 2003 to 2007. (www.bts.gov)



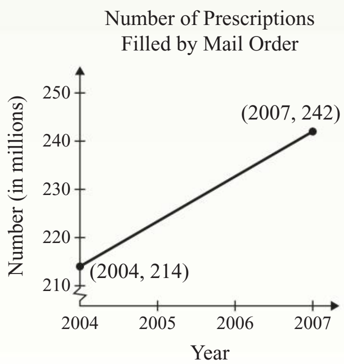
a) Approximately how many people were injured in 2003? in 2005?

b) Without computing the slope, determine whether it is positive or negative.

c) What does the sign of the slope mean in the context of the problem?

d) Find the slope of the line segment, and explain what it means in the context of the problem.

40) The graph shows the approximate number of prescriptions filled by mail order from 2004 to 2007. (www.census.gov)



a) Approximately how many prescriptions were filled by mail order in 2004? in 2007?

b) Without computing the slope, determine whether it is positive or negative.

c) What does the sign of the slope mean in the context of the problem?

d) Find the slope of the line segment, and explain what it means in the context of the problem.

Objective 5: Use Slope and One Point on a Line to Graph the Line

Graph the line containing the given point and with the given slope.

41) $(2, 1); m = \frac{3}{4}$

42) $(1, 2); m = \frac{1}{3}$

43) $(-2, -3); m = \frac{1}{4}$

44) $(-5, 0); m = \frac{2}{5}$

45) $(1, 2); m = -\frac{3}{4}$

46) $(1, -3); m = -\frac{2}{5}$

47) $(-1, -3); m = 3$

48) $(0, -2); m = -2$

49) $(6, 2); m = -4$

50) $(4, 3); m = -5$

51) $(3, -4); m = -1$

52) $(-1, -2); m = 0$

53) $(-2, 3); m = 0$

54) $(2, 0)$; slope is undefined.

55) $(-1, -4)$; slope is undefined.

56) $(0, 0); m = 1$

57) $(0, 0); m = -1$

Section 4.4 The Slope-Intercept Form of a Line

Objectives

1. Define the Slope-Intercept Form of a Line
2. Graph a Line Expressed in Slope-Intercept Form
3. Rewrite an Equation in Slope-Intercept Form and Graph the Line
4. Use Slope to Determine Whether Two Lines Are Parallel or Perpendicular

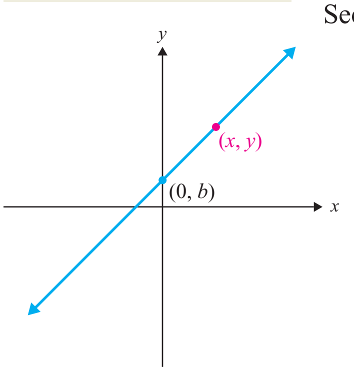
In Section 4.1, we learned that a linear equation in two variables can be written in the form $Ax + By = C$ (this is called **standard form**), where A, B, and C are real numbers and where both A and B do not equal zero. Equations of lines can take other forms, too, and we will look at one of those forms in this section.

1. Define the Slope-Intercept Form of a Line

We know that if (x_1, y_1) and (x_2, y_2) are points on a line, then the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Recall that to find the y-intercept of a line, we let $x = 0$ and solve for y. Let one of the points on a line be the y-intercept $(0, b)$, where b is a number. Let another point on the line be (x, y).



See the graph on the left.

Substitute the points $(0, b)$ and (x, y) into the slope formula:

Subtract y-coordinates.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y - b}{x - 0} = \frac{y - b}{x}$$

Subtract x-coordinates.

Solve $m = \dfrac{y - b}{x}$ for y.

$$mx = \frac{y-b}{x} \cdot x \qquad \text{Multiply by } x \text{ to eliminate the fraction.}$$
$$mx = y - b$$
$$mx + b = y - b + b \qquad \text{Add } b \text{ to each side to solve for } y.$$
$$mx + b = y$$
OR
$$y = mx + b \qquad \text{Slope-intercept form}$$

Definition

The **slope-intercept form of a line** is $y = mx + b$, where m is the slope and $(0, b)$ is the y-intercept.

When an equation is in the form $y = mx + b$, we can quickly recognize the y-intercept and slope to graph the line.

2. Graph a Line Expressed in Slope-Intercept Form

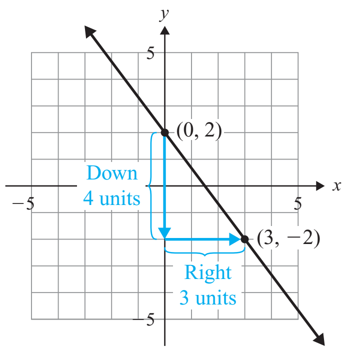
Example 1

Graph each equation.

a) $y = -\dfrac{4}{3}x + 2$ b) $y = 4x - 3$ c) $y = \dfrac{1}{2}x$

Solution

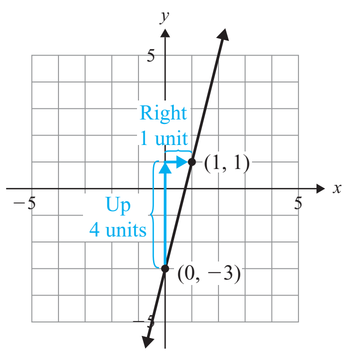
Notice that each equation is in slope-intercept form, $y = mx + b$, where m is the slope and $(0, b)$ is the y-intercept.



a) Graph $y = -\dfrac{4}{3}x + 2$.

Slope $= -\dfrac{4}{3}$, y-intercept is $(0, 2)$.

Graph the line by first plotting the y-intercept and then using the slope to locate another point on the line.



b) Graph $y = 4x - 3$.

Slope $= 4$, y-intercept is $(0, -3)$.

Plot the y-intercept first, then use the slope to locate another point on the line. Since the slope is 4, think of it as $\dfrac{4}{1}$. $\begin{array}{l} \leftarrow \text{Change in } y \\ \leftarrow \text{Change in } x \end{array}$

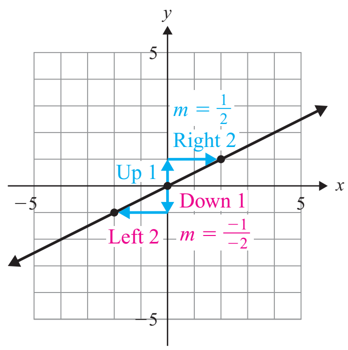
c) The equation $y = \frac{1}{2}x$ is the same as $y = \frac{1}{2}x + 0$.

Identify the slope and y-intercept.

$$\text{Slope} = \frac{1}{2}, \quad y\text{-intercept is } (0, 0).$$

Plot the y-intercept, then use the slope to locate another point on the line.

$\frac{1}{2}$ is equivalent to $\frac{-1}{-2}$, so we can use $\frac{-1}{-2}$ as the slope to locate yet another point on the line.



You Try 1

Graph each line using the slope and y-intercept.

a) $y = \frac{1}{4}x + 1$ b) $y = x - 3$ c) $y = -2x$

3. Rewrite an Equation in Slope-Intercept Form and Graph the Line

Lines are not always written in slope-intercept form. They may be written in *standard form* (like $7x + 4y = 12$) or in another form such as $2x = 2y + 10$. We can put equations like these into slope-intercept form by solving the equation for y.

Example 2

Put each equation into slope-intercept form, and graph.

a) $7x + 4y = 12$ b) $2x = 2y + 10$

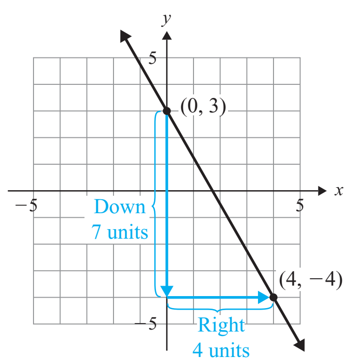
Solution

a) The slope-intercept form of a line is $y = mx + b$. We must solve the equation for y.

$$7x + 4y = 12$$
$$4y = -7x + 12 \quad \text{Add } -7x \text{ to each side.}$$
$$y = \frac{-7}{4}x + 3 \quad \text{Divide each side by 4.}$$

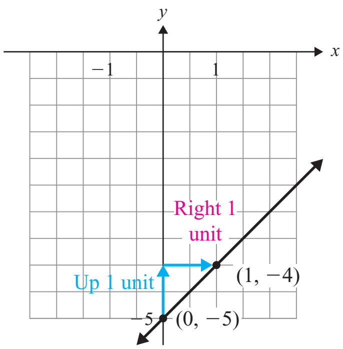
$$\text{Slope} = -\frac{7}{4} \text{ or } \frac{-7}{4}; \quad y\text{-intercept is } (0, 3).$$

$\left(\text{We could also have thought of the slope as } \frac{7}{-4}.\right)$



b) We must solve $2x = 2y + 10$ for y.

$$\begin{aligned} 2x &= 2y + 10 \\ 2x - 10 &= 2y && \text{Subtract 10 from each side.} \\ x - 5 &= y && \text{Divide each side by 2.} \end{aligned}$$



The slope-intercept form is $y = x - 5$. We can also think of this as $y = 1x - 5$.

$$\text{slope} = 1, \quad y\text{-intercept is } (0, -5).$$

We will think of the slope as $\dfrac{1}{1}$. $\begin{matrix} \leftarrow \text{Change in } y \\ \leftarrow \text{Change in } x \end{matrix}$

$\left(\text{We could also think of it as } \dfrac{-1}{-1}.\right)$

You Try 2

Put each equation into slope-intercept form, and graph.

a) $10x - 5y = -5$ b) $2x = -3 - 3y$

Summary Different Methods for Graphing a Line Given Its Equation

We have learned that we can use different methods for graphing lines. Given the equation of a line we can

1) Make a table of values, plot the points, and draw the line through the points.
2) Find the x-intercept by letting $y = 0$ and solving for x, and find the y-intercept by letting $x = 0$ and solving for y. Plot the points, then draw the line through the points.
3) Put the equation into slope-intercept form, $y = mx + b$, identify the slope and y-intercept, then graph the line.

4. Use Slope to Determine Whether Two Lines Are Parallel or Perpendicular

Recall that two lines in a plane are **parallel** if they do not intersect. If we are given the equations of two lines, how can we determine whether they are parallel?

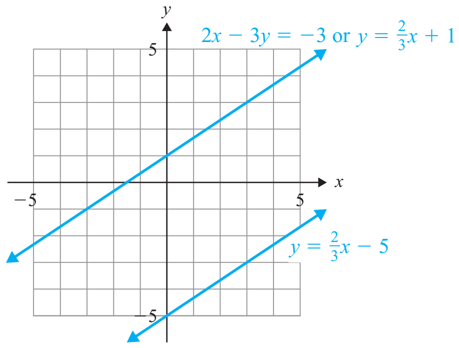
Here are the equations of two lines:

$$2x - 3y = -3 \qquad y = \frac{2}{3}x - 5$$

We will graph each line. To graph the first line, we write it in slope-intercept form.

$$\begin{aligned} -3y &= -2x - 3 && \text{Add } -2x \text{ to each side.} \\ y &= \frac{-2}{-3}x - \frac{3}{-3} && \text{Divide by } -3. \\ y &= \frac{2}{3}x + 1 && \text{Simplify.} \end{aligned}$$

The slope-intercept form of the first line is $y = \frac{2}{3}x + 1$, and the second line is already in slope-intercept form, $y = \frac{2}{3}x - 5$. Now, graph each line.



These lines are parallel. Their slopes are the same, but they have different y-intercepts. (If the y-intercepts were the same, they would be the same line.) This is how we determine whether two (nonvertical) lines are parallel. They have the same slope, but different y-intercepts.

Property Parallel Lines

Parallel lines have the same slope. (If two lines are vertical, they are parallel. However, their slopes are undefined.)

Example 3

Determine whether each pair of lines is parallel.

a) $2x + 8y = 12$
$x + 4y = -20$

b) $y = -5x + 2$
$5x - y = 7$

Solution

a) To determine whether the lines are parallel, we must find the slope of each line. If the slopes are the same, but the y-intercepts are different, the lines are parallel.

Write each equation in slope-intercept form.

$$\begin{aligned} 2x + 8y &= 12 \\ 8y &= -2x + 12 \\ y &= -\frac{2}{8}x + \frac{12}{8} \\ y &= -\frac{1}{4}x + \frac{3}{2} \end{aligned} \qquad \begin{aligned} x + 4y &= -20 \\ 4y &= -x - 20 \\ y &= -\frac{x}{4} - \frac{20}{4} \\ y &= -\frac{1}{4}x - 5 \end{aligned}$$

Each line has a slope of $-\frac{1}{4}$. Their y-intercepts are different. Therefore, $2x + 8y = 12$ and $x + 4y = -20$ are parallel lines.

b) Again, we must find the slope of each line. $y = -5x + 2$ is already in slope-intercept form. Its slope is -5.

Write $5x - y = 7$ in slope-intercept form.

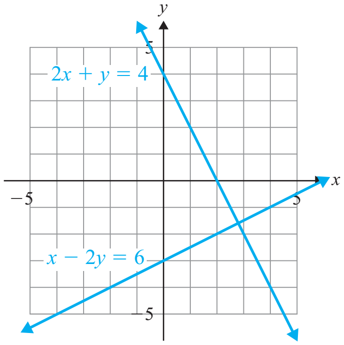
$$-y = -5x + 7 \qquad \text{Add } -5x \text{ to each side.}$$
$$y = 5x - 7 \qquad \text{Divide each side by } -1.$$

The slope of $y = -5x + 2$ is -5. The slope of $5x - y = 7$ is 5. The slopes are different; therefore, the lines are not parallel.

The slopes of two lines can tell us about another relationship between the lines. The slopes can tell us whether two lines are *perpendicular*.

Recall that two lines are **perpendicular** if they intersect at 90° angles.

The graphs of two perpendicular lines and their equations are on the left. We will see how their slopes are related.



Find the slope of each line by writing them in slope-intercept form.

Line 1: $2x + y = 4$

$$y = -2x + 4$$

$$m = -2$$

Line 2:

$$\begin{aligned} x - 2y &= 6 \\ -2y &= -x + 6 \\ y &= \frac{-x}{-2} + \frac{6}{-2} \\ y &= \frac{1}{2}x - 3 \\ m &= \frac{1}{2} \end{aligned}$$

How are the slopes related? They are **negative reciprocals**. That is, if the slope of one line is a, then the slope of a line perpendicular to it is $-\frac{1}{a}$.

This is how we determine whether two lines are perpendicular (where neither one is vertical).

Property Perpendicular Lines

Perpendicular lines have slopes that are negative reciprocals of each other.

Example 4

Determine whether each pair of lines is perpendicular.

a) $15x - 12y = -4$
   $4x - 5y = 10$

b) $2x - 9y = -9$
   $9x + 2y = 8$

Solution

a) To determine whether the lines are perpendicular, we must find the slope of each line. If the slopes are negative reciprocals, then the lines are perpendicular.

Write each equation in slope-intercept form.

$$\begin{aligned} 15x - 12y &= -4 \\ -12y &= -15x - 4 \\ y &= \frac{-15}{-12}x - \frac{4}{-12} \\ y &= \frac{5}{4}x + \frac{1}{3} \\ m &= \frac{5}{4} \end{aligned}$$

$$\begin{aligned} 4x - 5y &= 10 \\ -5y &= -4x + 10 \\ y &= \frac{-4}{-5}x + \frac{10}{-5} \\ y &= \frac{4}{5}x - 2 \\ m &= \frac{4}{5} \end{aligned}$$

The slopes are reciprocals, but they are not *negative* reciprocals. Therefore, the lines are *not* perpendicular.

b) Begin by writing each equation in slope-intercept form so that we can find their slopes.

$$\begin{aligned} 2x - 9y &= -9 \\ -9y &= -2x - 9 \\ y &= \frac{-2}{-9}x - \frac{9}{-9} \\ y &= \frac{2}{9}x + 1 \\ m &= \frac{2}{9} \end{aligned} \qquad\qquad \begin{aligned} 9x + 2y &= 8 \\ 2y &= -9x + 8 \\ y &= -\frac{9}{2}x + \frac{8}{2} \\ y &= -\frac{9}{2}x + 4 \\ m &= -\frac{9}{2} \end{aligned}$$

The slopes are negative reciprocals; therefore, the lines are perpendicular. ■

You Try 3

Determine whether each pair of lines is parallel, perpendicular, or neither.

a) $5x - y = -2$
$3x + 15y = -20$

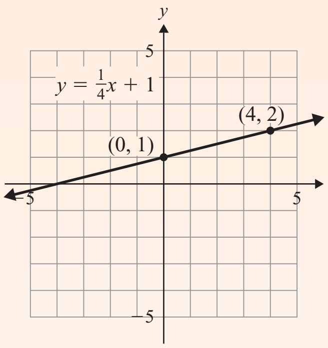
b) $y = \frac{8}{3}x + 9$
$-32x + 12y = 15$

c) $x + 2y = 8$
$2x = 4y + 3$

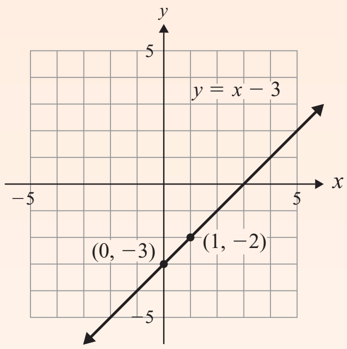
d) $x = 7$
$y = -4$

Answers to You Try Exercises

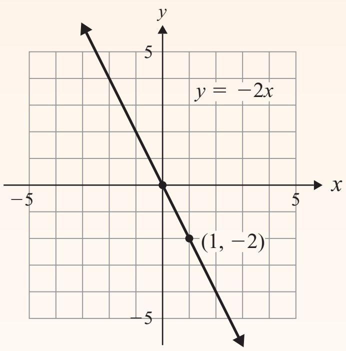
1) a)



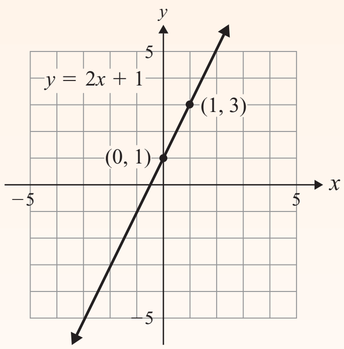
b)



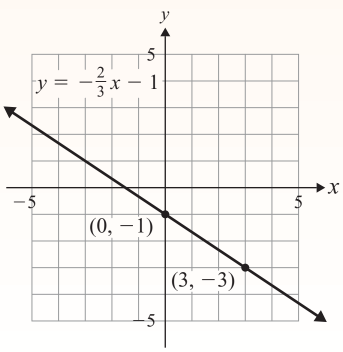
c)



2) a)



b)



3) a) perpendicular
b) parallel
c) neither
d) perpendicular

4.4 Exercises

Mixed Exercises: Objectives 1 and 2

1) The slope-intercept form of a line is $y = mx + b$. What is the slope? What is the y-intercept?

2) How do you put an equation that is in standard form, $Ax + By = C$, into slope-intercept form?

Each of the following equations is in slope-intercept form. Identify the slope and the y-intercept, then graph each line using this information.

VIDEO 3) $y = \frac{2}{5}x - 6$

4) $y = \frac{7}{5}x - 1$

5) $y = -\frac{3}{2}x + 3$

6) $y = -\frac{1}{3}x + 2$

7) $y = \frac{3}{4}x + 2$

8) $y = \frac{2}{3}x + 5$

9) $y = -2x - 3$

10) $y = 3x - 1$

11) $y = 5x$

12) $y = -2x + 5$

13) $y = -\frac{3}{2}x - \frac{7}{2}$

14) $y = \frac{3}{5}x + \frac{3}{4}$

15) $y = 6$

16) $y = -4$

Objective 3: Rewrite an Equation in Slope-Intercept Form and Graph the Line

Put each equation into slope-intercept form, if possible, and graph.

VIDEO 17) $x + 3y = -6$

18) $x + 2y = -8$

19) $4x + 3y = 21$

20) $2x - 5y = 5$

21) $2 = x + 3$

22) $x + 12 = 4$

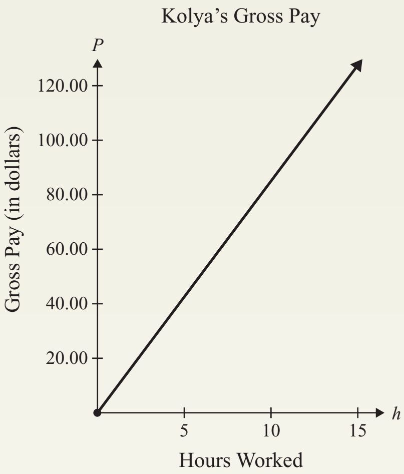
23) $2x = 18 - 3y$

24) $98 = 49y - 28x$

25) $y + 2 = -3$

26) $y + 3 = 3$

27) Kolya has a part-time job, and his gross pay can be described by the equation $P = 8.50h$, where P is his gross pay, in dollars, and h is the number of hours worked.

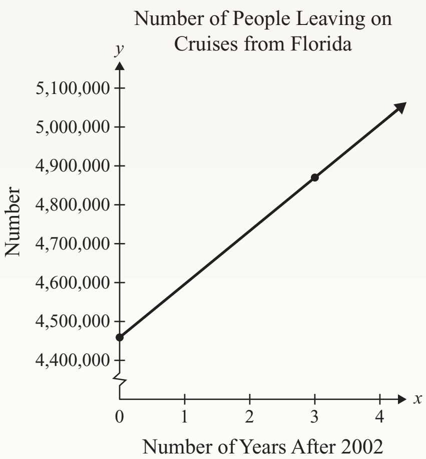


a) What is the P-intercept? What does it mean in the context of the problem?

b) What is the slope? What does it mean in the context of the problem?

c) Use the graph to find Kolya's gross pay when he works 12 hours. Confirm your answer using the equation.

28) The number of people, y, leaving on cruises from Florida from 2002 to 2006 can be approximated by $y = 137{,}000x + 4{,}459{,}000$, where x is the number of years after 2002. (www.census.gov)



a) What is the y-intercept? What does it mean in the context of the problem?

b) What is the slope? What does it mean in the context of the problem?

c) Use the graph to determine how many people left on cruises from Florida in 2005. Confirm your answer using the equation.

29) A Tasmanian devil is a marsupial that lives in Australia. Once a joey leaves its mother's pouch, its weight for the first 8 weeks can be approximated by $y = 2x + 18$, where x represents the number of weeks it has been out of the pouch and y represents its weight, in ounces. (Wikipedia and Animal Planet)

a) What is the y-intercept, and what does it represent?

b) How much does a Tasmanian devil weigh 3 weeks after emerging from the pouch?

c) Explain the meaning of the slope in the context of this problem.

d) How long would it take for a joey to weigh 32 oz?

30) The number of active physicians in Idaho, y, from 2002 to 2006 can be approximated by $y = 74.7x + 2198.8$, where x represents the number of years after 2002. (www.census.gov)

a) What is the y-intercept and what does it represent?

b) How many doctors were practicing in 2006?

c) Explain the meaning of the slope in the context of this problem.

d) If the current trend continues, how many practicing doctors would Idaho have in 2015?

31) On a certain day in 2009, the exchange rate between the American dollar and the Indian rupee was given by $r = 48.2d$, where d represents the number of dollars and r represents the number of rupees.

a) What is the r-intercept and what does it represent?

b) What is the slope? What does it mean in the context of the problem?

c) If Madhura is going to India to visit her family, how many rupees could she get for \$80.00?

d) How many dollars could be exchanged for 2410 rupees?

32) The value of a car, v, in dollars, t years after it is purchased is given by $v = -1800t + 20{,}000$.

a) What is the v-intercept and what does it represent?

b) What is the slope? What does it mean in the context of the problem?

c) What is the car worth after 3 years?

d) When will the car be worth \$11,000?

Write the slope-intercept form for the equation of a line with the given slope and y-intercept.

VIDEO 33) $m = -4$; y-int: $(0, 7)$

34) $m = -7$; y-int: $(0, 4)$

35) $m = \frac{9}{5}$; y-int: $(0, -3)$

36) $m = \frac{7}{4}$; y-int: $(0, -2)$

37) $m = -\frac{5}{2}$; y-int: $(0, -1)$

38) $m = \frac{1}{4}$; y-int: $(0, 7)$

39) $m = 1$; y-int: $(0, 2)$

40) $m = -1$; y-int: $(0, 0)$

41) $m = 0$; y-int: $(0, 0)$

42) $m = 0$; y-int: $(0, -8)$

Objective 4: Use Slope to Determine Whether Two Lines Are Parallel or Perpendicular

43) How do you know whether two lines are perpendicular?

44) How do you know whether two lines are parallel?

Determine whether each pair of lines is parallel, perpendicular, or neither.

45) $y = -x - 5$
$y = x + 8$

46) $y = \frac{3}{4}x + 2$
$y = \frac{3}{4}x - 1$

VIDEO 47) $y = \frac{2}{9}x + 4$
$4x - 18y = 9$

48) $y = \frac{4}{5}x + 2$
$5x + 4y = 12$

49) $3x - y = 4$
$2x - 5y = -9$

50) $-4x + 3y = -5$
$4x - 6y = -3$

51) $-x + y = -21$
$y = 2x + 5$

52) $x + 3y = 7$
$y = 3x$

53) $x + 7y = 4$
$y - 7x = 4$

54) $5y - 3x = 1$
$3x - 5y = -8$

55) $y = -\frac{1}{2}x$
$x + 2y = 4$

56) $-4x + 6y = 5$
$2x - 3y = -12$

57) $x = -1$
$y = 6$

58) $y = 12$
$y = 4$

59) $x = -4.3$
$x = 0$

60) $x = 7$
$y = 0$

Lines L_1 and L_2 contain the given points. Determine whether lines L_1 and L_2 are parallel, perpendicular, or neither.

VIDEO 61) L_1: $(-1, -7), (2, 8)$
L_2: $(10, 2), (0, 4)$

62) L_1: $(0, -3), (-4, -11)$
L_2: $(-2, 0), (3, 10)$

63) L_1: $(1, 10), (3, 8)$
L_2: $(2, 4), (-5, -17)$

64) L_1: $(-1, 4), (2, -8)$
L_2: $(8, 5), (0, 3)$

65) L_1: $(-3, 6), (4, -1)$
L_2: $(-6, -5), (-10, -1)$

66) L_1: $(5, -5), (7, 11)$
L_2: $(-3, 0), (6, 3)$

67) L_1: $(-6, 2), (-6, 1)$
L_2: $(4, 0), (4, -5)$

68) L_1: $(8, 1), (7, 1)$
L_2: $(12, -1), (-2, -1)$

69) L_1: $(7, 2), (7, 5)$
L_2: $(-2, 0), (1, 0)$

70) L_1: $(-6, 4), (-6, -1)$
L_2: $(-1, 10), (-3, 10)$

Section 4.5 Writing an Equation of a Line

Objectives
1. Rewrite an Equation in Standard Form
2. Write an Equation of a Line Given Its Slope and y-Intercept
3. Use the Point-Slope Formula to Write an Equation of a Line Given Its Slope and a Point on the Line
4. Use the Point-Slope Formula to Write an Equation of a Line Given Two Points on the Line
5. Write Equations of Horizontal and Vertical Lines
6. Write an Equation of a Line That is Parallel or Perpendicular to a Given Line
7. Write a Linear Equation to Model Real-World Data

So far in this chapter, we have been graphing lines given their equations. Now we will write an equation of a line when we are given information about it.

1. Rewrite an Equation in Standard Form

In Section 4.4, we practiced writing equations of lines in slope-intercept form. Here we will discuss how to write a line in **standard form,** $Ax + By = C$, **with the additional conditions that A, B, and C are integers and A is positive.**

Example 1

Rewrite each linear equation in standard form.

a) $3x + 8 = -2y$ b) $y = -\frac{3}{4}x + \frac{1}{6}$

Solution

a) In standard form, the x- and y-terms are on the same side of the equation.

$$\begin{aligned} 3x + 8 &= -2y \\ 3x &= -2y - 8 && \text{Subtract 8 from each side.} \\ 3x + 2y &= -8 && \text{Add } 2y \text{ to each side; the equation is now in standard form.} \end{aligned}$$

b) Since an equation $Ax + By = C$ is considered to be in standard form when A, B, and C are integers, the first step in writing $y = -\frac{3}{4}x + \frac{1}{6}$ in standard form is to eliminate the fractions.

$$\begin{aligned} y &= -\frac{3}{4}x + \frac{1}{6} \\ 12 \cdot y &= 12\left(-\frac{3}{4}x + \frac{1}{6}\right) && \text{Multiply both sides of the equation by 12.} \\ 12y &= -9x + 2 \\ 9x + 12y &= 2 && \text{Add } 9x \text{ to each side.} \end{aligned}$$

The standard form is $9x + 12y = 2$. ■

You Try 1

Rewrite each equation in standard form.

a) $5x = 3 + 11y$ b) $y = \frac{1}{3}x - 7$

In the rest of this section, we will learn how to write equations of lines given information about their graphs.

2. Write an Equation of a Line Given Its Slope and *y*-Intercept

Procedure Write an Equation of a Line Given Its Slope and *y*-Intercept

If we are given the slope and *y*-intercept of a line, use $y = mx + b$ and substitute those values into the equation.

Example 2

Find an equation of the line with slope $= -6$ and y-intercept $(0, 5)$.

Solution

Since we are told the slope and y-intercept, use $y = mx + b$.

$$m = -6 \qquad \text{and} \qquad b = 5$$

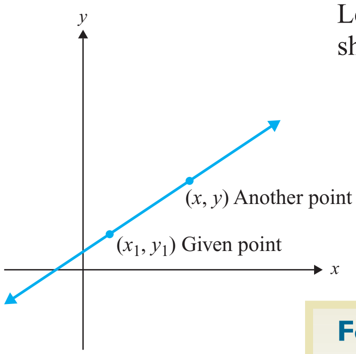
Substitute these values into $y = mx + b$ to get $y = -6x + 5$.

You Try 2

Find an equation of the line with slope $= \frac{5}{8}$ and y-intercept $(0, -9)$.

3. Use the Point-Slope Formula to Write an Equation of a Line Given Its Slope and a Point on the Line

When we are given the slope of a line and a point on that line, we can use another method to find its equation. This method comes from the formula for the slope of a line.



Let (x_1, y_1) be a given point on a line, and let (x, y) be any other point on the same line, as shown in the figure. The slope of that line is

$$m = \frac{y - y_1}{x - x_1} \qquad \text{Definition of slope}$$

$$m(x - x_1) = y - y_1 \qquad \text{Multiply each side by } x - x_1.$$

$$y - y_1 = m(x - x_1) \qquad \text{Rewrite the equation.}$$

We have found the *point-slope form* of the equation of a line.

Formula Point-Slope Form of a Line

The **point-slope form of a line** is

$$y - y_1 = m(x - x_1)$$

where (x_1, y_1) is a point on the line and m is its slope.

Note

If we are given the slope of the line and a point on the line, we can use the point-slope formula to find an equation of the line.

The point-slope formula will help us write an equation of a line. We will not express our final answer in this form. We will write our answer in either slope-intercept form or in standard form.

Example 3

Find an equation of the line containing the point $(-4, 3)$ with slope $= \frac{1}{2}$. Express the answer in slope-intercept form.

Solution

First, ask yourself, *"What kind of information am I given?"* Since the problem tells us the slope of the line and a point on the line, we will use the point-slope formula: $y - y_1 = m(x - x_1)$

Substitute $\frac{1}{2}$ for m. Substitute $(-4, 3)$ for (x_1, y_1).

(Notice we do *not* substitute anything for x and y.)

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{1}{2}(x - (-4)) \qquad \text{Substitute } -4 \text{ for } x_1 \text{ and } 3 \text{ for } y_1\text{; let } m = \frac{1}{2}.$$

$$y - 3 = \frac{1}{2}(x + 4)$$

$$y - 3 = \frac{1}{2}x + 2 \qquad \text{Distribute.}$$

We must write our answer in slope-intercept form, $y = mx + b$, so solve the equation for y.

$$y = \frac{1}{2}x + 5 \qquad \text{Add 3 to each side.}$$

The equation is $y = \frac{1}{2}x + 5$.

You Try 3

Find an equation of the line containing the point $(5, 3)$ with slope $= 2$. Express the answer in slope-intercept form.

Example 4

A line has slope -4 and contains the point $(1, 5)$. Find the standard form for the equation of the line.

Solution

Although we are told to find the *standard form* for the equation of the line, we do not try to immediately "jump" to standard form. First, ask yourself, *"What information am I given?"*

We are given the *slope* and a *point on the line*. Therefore, we will begin by using the point-slope formula. Our *last* step will be to put the equation in standard form.

Use $y - y_1 = m(x - x_1)$. Substitute -4 for m. Substitute $(1, 5)$ for (x_1, y_1).

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 5 &= -4(x - 1) && \text{Substitute 1 for } x_1 \text{ and 5 for } y_1\text{; let } m = -4. \\ y - 5 &= -4x + 4 && \text{Distribute.} \end{aligned}$$

To write the answer in standard form, we must get the x- and y-terms on the same side of the equation so that the coefficient of x is positive.

$$\begin{aligned} 4x + y - 5 &= 4 && \text{Add } 4x \text{ to each side.} \\ 4x + y &= 9 && \text{Add 5 to each side.} \end{aligned}$$

The standard form of the equation is $4x + y = 9$.

You Try 4

A line has slope -8 and contains the point $(-4, 5)$. Find the standard form for the equation of the line.

4. Use the Point-Slope Formula to Write an Equation of a Line Given Two Points on the Line

We are now ready to discuss how to write an equation of a line when we are given two points on a line.

To write an equation of a line given two points on the line,

a) use the points to find the slope of line
then
b) use the slope and *either one* of the points in the point-slope formula.

Example 5

Write an equation of the line containing the points (4, 9) and (2, 6). Express the answer in slope-intercept form.

Solution

We are given two points on the line, so first, we will find the slope.

$$m = \frac{6 - 9}{2 - 4} = \frac{-3}{-2} = \frac{3}{2}$$

We will use the slope and *either one* of the points in the point-slope formula. (Each point will give the same result.) We will use (4, 9).

Substitute $\frac{3}{2}$ for m. Substitute $(4, 9)$ for (x_1, y_1).

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 9 &= \frac{3}{2}(x - 4) && \text{Substitute 4 for } x_1 \text{ and 9 for } y_1\text{; let } m = \frac{3}{2}. \\ y - 9 &= \frac{3}{2}x - 6 && \text{Distribute.} \\ y &= \frac{3}{2}x + 3 && \text{Add 9 to each side to solve for } y. \end{aligned}$$

The equation is $y = \frac{3}{2}x + 3$.

You Try 5

Find the slope-intercept form for the equation of the line containing the points (4, 2) and (1, −5).

5. Write Equations of Horizontal and Vertical Lines

Earlier we learned that the slope of a horizontal line is zero and that it has equation $y = d$, where d is a constant. The slope of a vertical line is undefined, and its equation is $x = c$, where c is a constant.

Formula Equations of Horizontal and Vertical Lines

Equation of a Horizontal Line: The equation of a horizontal line containing the point (c, d) is $y = d$.

Equation of a Vertical Line: The equation of a vertical line containing the point (c, d) is $x = c$.

Example 6

Write an equation of the horizontal line containing the point $(7, -1)$.

Solution

The equation of a horizontal line has the form $y = d$, where d is the y-coordinate of the point. The equation of the line is $y = -1$.

You Try 6

Write an equation of the horizontal line containing the point (3, −8).

Summary Writing Equations of Lines

If you are given

1) **the slope and y-intercept of the line,** use $y = mx + b$ and substitute those values into the equation.
2) **the slope of the line and a point on the line,** use the point-slope formula:

$$y - y_1 = m(x - x_1)$$

Substitute the slope for m and the point you are given for (x_1, y_1). Write your answer in slope-intercept or standard form.

3) **two points on the line,** find the slope of the line and then use the slope and *either one* of the points in the point-slope formula. Write your answer in slope-intercept or standard form.

The equation of a **horizontal line** containing the point (c, d) is $\mathbf{y = d}$.

The equation of a **vertical line** containing the point (c, d) is $\mathbf{x = c}$.

6. Write an Equation of a Line that is Parallel or Perpendicular to a Given Line

In Section 4.4, we learned that parallel lines have the same slope, and perpendicular lines have slopes that are negative reciprocals of each other. We will use this information to write the equation of a line that is parallel or perpendicular to a given line.

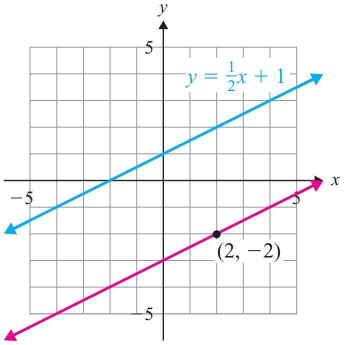
Example 7

A line contains the point $(2, -2)$ and is parallel to the $y = \frac{1}{2}x + 1$. Write the equation of the line in slope-intercept form.

Solution

Let's look at the graph on the left to help us understand what is happening in this example.

We must find the equation of the line in red. It is the line containing the point $(2, -2)$ that is parallel to $y = \frac{1}{2}x + 1$.



The line $y = \frac{1}{2}x + 1$ has $m = \frac{1}{2}$. Therefore, the red line will have $m = \frac{1}{2}$ as well.

We know the slope, $\frac{1}{2}$, and a point on the line, $(2, -2)$, so we use the point-slope formula to find its equation.

Substitute $\frac{1}{2}$ for m. Substitute $(2, -2)$ for (x_1, y_1).

$$y - y_1 = m(x - x_1)$$

$$y - (-2) = \frac{1}{2}(x - 2) \qquad \text{Substitute 2 for } x_1 \text{ and } -2 \text{ for } y_1\text{; let } m = \frac{1}{2}.$$

$$y + 2 = \frac{1}{2}x - 1 \qquad \text{Distribute.}$$

$$y = \frac{1}{2}x - 3 \qquad \text{Subtract 2 from each side.}$$

The equation is $y = \frac{1}{2}x - 3$.

You Try 7

A line contains the point $(-6, 2)$ and is parallel to the line $y = -\frac{3}{2}x + \frac{1}{4}$. Write the equation of the line in slope-intercept form.

Example 8

Find the standard form for the equation of the line that contains the point $(-4, 3)$ and that is perpendicular to $3x - 4y = 8$.

Solution

Begin by finding the slope of $3x - 4y = 8$ by putting it into slope-intercept form.

$$3x - 4y = 8$$

$$-4y = -3x + 8 \qquad \text{Add } -3x \text{ to each side.}$$

$$y = \frac{-3}{-4}x + \frac{8}{-4} \qquad \text{Divide by } -4.$$

$$y = \frac{3}{4}x - 2 \qquad \text{Simplify.}$$

$$m = \frac{3}{4}$$

Then, determine the slope of the line containing $(-4, 3)$ by finding the *negative reciprocal* of the slope of the given line.

$$m_{\text{perpendicular}} = -\frac{4}{3}$$

The line we want has $m = -\frac{4}{3}$ and contains the point $(-4, 3)$. Use the point-slope formula to find an equation of the line.

Substitute $-\frac{4}{3}$ for m. Substitute $(-4, 3)$ for (x_1, y_1).

$$y - y_1 = m(x - x_1)$$
$$y - 3 = -\frac{4}{3}(x - (-4)) \qquad \text{Substitute } -4 \text{ for } x_1 \text{ and } 3 \text{ for } y_1\text{; let } m = -\frac{4}{3}.$$
$$y - 3 = -\frac{4}{3}(x + 4)$$
$$y - 3 = -\frac{4}{3}x - \frac{16}{3} \qquad \text{Distribute.}$$

Since we are asked to write the equation in standard form, eliminate the fractions by multiplying each side by 3.

$$3(y - 3) = 3\left(-\frac{4}{3}x - \frac{16}{3}\right)$$
$$3y - 9 = -4x - 16 \qquad \text{Distribute.}$$
$$3y = -4x - 7 \qquad \text{Add 9 to each side.}$$
$$4x + 3y = -7 \qquad \text{Add } 4x \text{ to each side.}$$

The equation is $4x + 3y = -7$.

You Try 8

Find the equation of the line perpendicular to $5x - y = -6$ containing the point $(10, 0)$. Write the equation in standard form.

7. Write a Linear Equation to Model Real-World Data

Equations of lines are often used to describe real-world situations. We will look at an example in which we must find the equation of a line when we are given some data.

Example 9

Since 2003, vehicle consumption of E85 fuel (ethanol, 85%) has increased by about 8262.4 thousand gallons per year. In 2006, approximately 61,029.4 thousand gallons were used. (Statistical Abstract of the United States)

a) Write a linear equation to model these data. Let x represent the number of years after 2003, and let y represent the amount of E85 fuel (in thousands of gallons) consumed.

b) How much E85 fuel did vehicles use in 2003? in 2005?

Solution

a) The statement "vehicle consumption of E85 fuel … has increased by about 8262.4 thousand gallons per year" tells us the rate of change of fuel use with respect to time. Therefore, this is the *slope.* It will be *positive* since consumption is increasing.

$$\mathbf{m = 8262.4}$$

The statement "In 2006, approximately 61,029.4 thousand gallons were used" gives us a point on the line.

If x = the number of years after 2003, then the year 2006 corresponds to $x = 3$.

If y = number of gallons (in thousands) of E85 consumed, then 61,029.4 thousand gallons corresponds to $y = 61{,}029.4$.

A point on the line is **(3, 61,029.4).**

Now that we know the slope and a point on the line, we can write an equation of the line using the point-slope formula.

Substitute 8262.4 for m. Substitute (3, 61,029.4) for (x_1, y_1).

$$\begin{aligned} y - y_1 &= m(x - x_1) & \\ y - 61{,}029.4 &= 8262.4(x - 3) & &\text{Substitute 3 for } x_1\text{, 61,029.4 for } y_1. \\ y - 61{,}029.4 &= 8262.4x - 24{,}787.2 & &\text{Distribute.} \\ y &= 8262.4x + 36{,}242.2 & &\text{Add 61,029.4 to each side.} \end{aligned}$$

The equation is $y = 8262.4x + 36{,}242.2$.

b) To determine the amount of E85 used in 2003, let $x = 0$ since x = the number of years *after* 2003.

$$\begin{aligned} y &= 8262.4(0) + 36{,}242.2 & &\text{Substitute } x = 0. \\ y &= 36{,}242.2 & & \end{aligned}$$

In 2003, vehicles used about 36,242.2 thousand gallons of E85 fuel. Notice that the equation is in slope-intercept form, $y = mx + b$, and our result is b. That is because when we find the y-intercept we let $x = 0$.

To determine how much E85 fuel was used in 2005, let $x = 2$ since 2005 is 2 years after 2003.

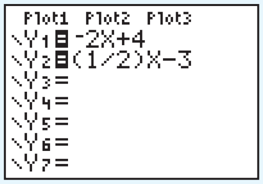
$$\begin{aligned} y &= 8262.4(2) + 36{,}242.2 & &\text{Substitute } x = 2. \\ y &= 16{,}524.8 + 36{,}242.2 & &\text{Multiply.} \\ y &= 52{,}767.0 & & \end{aligned}$$

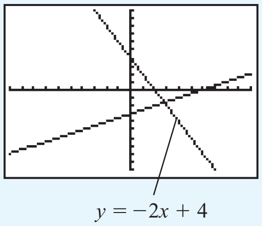
In 2003, vehicles used approximately 52,767.0 thousand gallons of E85. ■

Using Technology

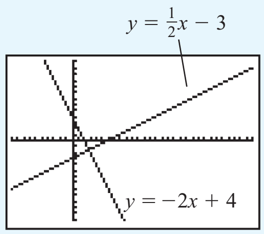
We can use a graphing calculator to explore what we have learned about perpendicular lines.

1. Graph the line $y = -2x + 4$. What is its slope?
2. Find the slope of the line perpendicular to the graph of $y = -2x + 4$.
3. Find the equation of the line perpendicular to $y = -2x + 4$ that passes through the point $(6, 0)$. Express the equation in slope-intercept form.
4. Graph both the original equation and the equation of the perpendicular line:





5. Do the lines above appear to the perpendicular?
6. Press ZOOM and choose 5:Zsquare.
7. Do the graphs look perpendicular now? Because the viewing window on a graphing calculator is a rectangle, *squaring* the window will give a more accurate picture of the graphs of the equations.



Answers to You Try Exercises

1) a) $5x - 11y = 3$ b) $x - 3y = 21$ 2) $y = \frac{5}{8}x - 9$ 3) $y = 2x - 7$ 4) $8x + y = -27$

5) $y = \frac{7}{3}x - \frac{22}{3}$ 6) $y = -8$ 7) $y = -\frac{3}{2}x - 7$ 8) $x + 5y = 10$

Answers to Technology Exercises

1) -2 2) $\frac{1}{2}$ 3) $y = \frac{1}{2}x - 3$ 5) No, because they do not meet at 90° angles.

7) Yes, because they meet at 90° angles.

4.5 Exercises

Objective 1: Rewrite an Equation in Standard Form

Rewrite each equation in standard form.

1) $y = -2x - 4$
2) $y = 3x + 5$
3) $x = y + 1$
4) $x = -4y - 9$
5) $y = \frac{4}{5}x + 1$
6) $y = \frac{2}{3}x - 6$
7) $y = -\frac{1}{3}x - \frac{5}{4}$
8) $y = -\frac{1}{4}x + \frac{2}{5}$

Objective 2: Write an Equation of a Line Given Its Slope and y-Intercept

9) Explain how to find an equation of a line when you are given the slope and y-intercept of the line.

Find an equation of the line with the given slope and y-intercept. Express your answer in the indicated form.

10) $m = -3$, y-int: $(0, 3)$; slope-intercept form

VIDEO 11) $m = -7$, y-int: $(0, 2)$; slope-intercept form

12) $m = 1$, y-int: $(0, -4)$; standard form

13) $m = -4$, y-int: $(0, 6)$; standard form

14) $m = -\frac{2}{5}$, y-int: $(0, -4)$; standard form

15) $m = \frac{2}{7}$, y-int: $(0, -3)$; standard form

16) $m = 1$, y-int: $(0, 0)$; slope-intercept form

17) $m = -1$, y-int: $(0, 0)$; slope-intercept form

18) $m = \frac{5}{9}$, y-int: $\left(0, -\frac{1}{3}\right)$; standard form

Objective 3: Use the Point-Slope Formula to Write an Equation of a Line Given Its Slope and a Point on the Line

19) a) If (x_1, y_1) is a point on a line with slope m, then the point-slope formula is ________.

b) Explain how to find an equation of a line when you are given the slope and a point on the line.

Find an equation of the line containing the given point with the given slope. Express your answer in the indicated form.

20) $(2, 3)$, $m = 4$; slope-intercept form

21) $(5, 7)$, $m = 1$; slope-intercept form

22) $(-2, 5)$, $m = -3$; slope-intercept form

23) $(4, -1)$, $m = -5$; slope-intercept form

24) $(-1, -2)$, $m = 2$; standard form

VIDEO 25) $(-2, -1)$, $m = 4$; standard form

26) $(9, 3)$, $m = -\frac{1}{3}$; standard form

27) $(-5, 8)$, $m = \frac{2}{5}$; standard form

28) $(-2, -3)$, $m = \frac{1}{8}$; slope-intercept form

29) $(5, 1)$, $m = -\frac{5}{4}$; slope-intercept form

30) $(4, 0)$, $m = -\frac{3}{16}$; standard form

31) $(-3, 0)$, $m = \frac{5}{6}$; standard form

32) $\left(\frac{1}{4}, -1\right)$, $m = 3$; slope-intercept form

Objective 4: Use the Point-Slope Formula to Write an Equation of a Line Given Two Points on the Line

33) Explain how to find an equation of a line when you are given two points on the line.

Find an equation of the line containing the two given points. Express your answer in the indicated form.

34) $(-2, 1)$ and $(8, 11)$; slope-intercept form

35) $(-1, 7)$ and $(3, -5)$; slope-intercept form

36) $(6, 8)$ and $(-1, -4)$; slope-intercept form

37) $(4, 5)$ and $(7, 11)$; slope-intercept form

38) $(2, -1)$ and $(5, 1)$; standard form

VIDEO 39) $(-2, 4)$ and $(1, 3)$; slope-intercept form

40) $(-1, 10)$ and $(3, -2)$; standard form

41) $(-5, 1)$ and $(4, -2)$; standard form

42) $(4.2, 1.3)$ and $(-3.4, -17.7)$; slope-intercept form

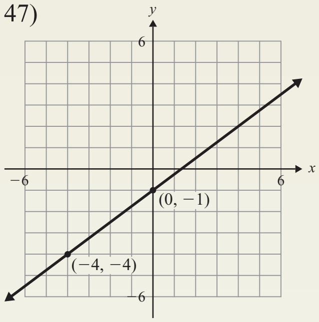
43) $(-3, -11)$ and $(3, -1)$; standard form

44) $(-6, 0)$ and $(3, -1)$; standard form

45) $(-2.3, 8.3)$ and $(5.1, -13.9)$; slope-intercept form

46) $(-7, -4)$ and $(14, 2)$; standard form

Write the slope-intercept form of the equation of each line, if possible.

47)


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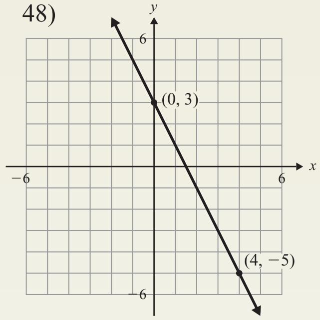
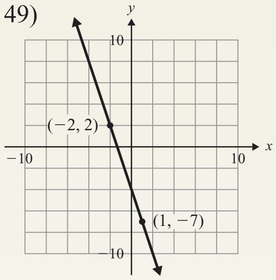


49)


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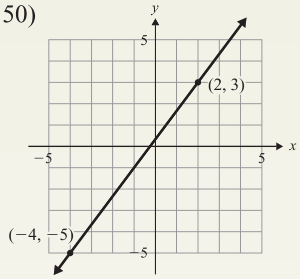
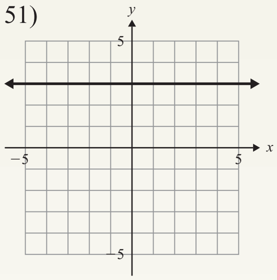
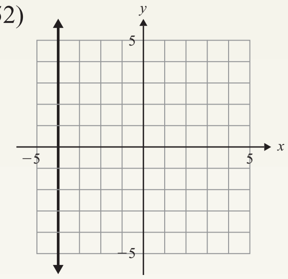


51)


52)


Mixed Exercises: Objectives 2–5

Write the slope-intercept form of the equation of the line, if possible, given the following information.

53) contains $(-4, 7)$ and $(2, -1)$

54) $m = 2$ and contains $(-3, -2)$

55) $m = 1$ and contains $(3, 5)$

56) $m = \frac{7}{5}$ and y-intercept $(0, -4)$

57) y-intercept $(0, 6)$ and $m = 7$

58) contains $(-3, -3)$ and $(1, -7)$

VIDEO 59) vertical line containing $(3, 5)$

60) vertical line containing $\left(\frac{1}{2}, 6\right)$

61) horizontal line containing (2, 3)

62) horizontal line containing (5, −4)

63) $m = -4$ and y-intercept $(0, -4)$

64) $m = -\frac{2}{3}$ and contains $(3, -1)$

65) $m = -3$ and contains $(10, -10)$

66) contains (0, 3) and (5, 0)

67) contains $(-4, -4)$ and $(2, -1)$

68) $m = -1$ and y-intercept $(0, 0)$

Objective 6: Write an Equation of a Line That Is Parallel or Perpendicular to a Given Line

69) What can you say about the equations of two parallel lines?

70) What can you say about the equations of two perpendicular lines?

Write an equation of the line *parallel* to the given line and containing the given point. Write the answer in slope-intercept form or in standard form, as indicated.

VIDEO 71) $y = 4x + 9$; (0, 2); slope-intercept form

72) $y = 8x + 3$; $(0, -3)$; slope-intercept form

73) $y = 4x + 2$; $(-1, -4)$; standard form

74) $y = \frac{2}{3}x - 6$; (6, 6); standard form

75) $x + 2y = 22$; $(-4, 7)$; standard form

76) $3x + 5y = -6$; $(-5, 8)$; standard form

77) $15x - 3y = 1$; $(-2, -12)$; slope-intercept form

78) $x + 6y = 12$; $(-6, 8)$; slope-intercept form

Write an equation of the line *perpendicular* to the given line and containing the given point. Write the answer in slope-intercept form or in standard form, as indicated.

79) $y = -\frac{2}{3}x + 4$; (4, 2); slope-intercept form

80) $y = -\frac{5}{3}x + 10$; (10, 5); slope-intercept form

VIDEO 81) $y = -5x + 1$; (10, 0); standard form

82) $y = \frac{1}{4}x - 9$; $(-1, 7)$; standard form

83) $y = x$; $(4, -9)$; slope-intercept form

84) $x + y = 9$; (4, 4); slope-intercept form

85) $x + 3y = 18$; (4, 2); standard form

86) $12x - 15y = 10$; $(16, -25)$; standard form

Write the slope-intercept form (if possible) of the equation of the line meeting the given conditions.

87) parallel to $3x + y = 8$ containing $(-4, 0)$

88) perpendicular to $x - 5y = -4$ containing (3, 5)

89) perpendicular to $y = x - 2$ containing (2, 9)

90) parallel to $y = 4x - 1$ containing $(-3, -8)$

91) parallel to $y = 1$ containing $(-3, 4)$

92) parallel to $x = -3$ containing $(-7, -5)$

93) perpendicular to $x = 0$ containing (9, 2)

94) perpendicular to $y = 4$ containing $(-4, -5)$

VIDEO 95) perpendicular to $21x - 6y = 2$ containing $(4, -1)$

96) parallel to $-3x + 4y = 8$ containing (9, 4)

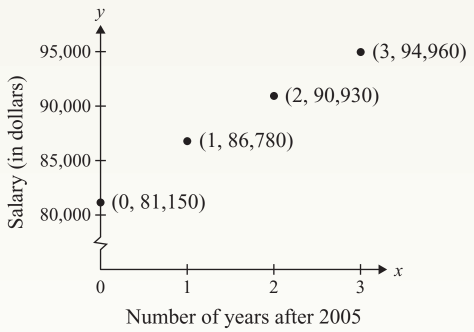
97) parallel to $y = 0$ containing $\left(4, -\frac{3}{2}\right)$

98) perpendicular to $y = \frac{7}{3}$ containing $(-7, 9)$

Objective 7: Write a Linear Equation to Model Real-World Data

99) The graph shows the average annual wage of a mathematician in the United States from 2005 to 2008. x represents the number of years after 2005 so that $x = 0$ represents 2005, $x = 1$ represents 2006, and so on. Let y represent the average annual wage of a mathematician. (www.bls.gov)

Average Annual Salary of a Mathematician

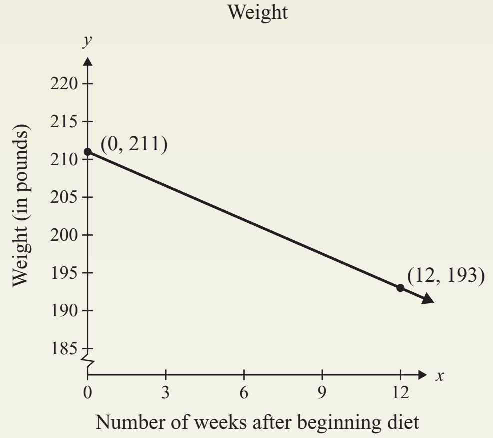


a) Write a linear equation to model these data. Use the data points for 2005 and 2008, and round the slope to the nearest tenth.

b) Explain the meaning of the slope in the context of this problem.

c) If the current trend continues, find the average salary of a mathematician in 2014.

100) The graph shows a dieter's weight over a 12-week period. Let y represent his weight x weeks after beginning his diet.



a) Write a linear equation to model these data. Use the data points for week 0 and week 12.

b) What is the meaning of the slope in the context of this problem?

c) If he keeps losing weight at the same rate, what will he weigh 13 weeks after the started his diet?

101) In 2007, a grocery store chain had an advertising budget of \$500,000 per year. Every year since then its budget has been cut by \$15,000 per year. Let y represent the advertising budget, in dollars, x years after 2007.

a) Write a linear equation to model these data.

b) Explain the meaning of the slope in the context of the problem.

c) What was the advertising budget in 2010?

d) If the current trend continues, in what year will the advertising budget be \$365,000?

102) A temperature of $-10°C$ is equivalent to 14°F, while 15°C is the same as 59°F. Let F represent the temperature on the Fahrenheit scale and C represent the temperature on the Celsius scale.

a) Write a linear equation to convert from degrees Celsius to degrees Fahrenheit. That is, write an equation for F in terms of C.

b) Explain the meaning of the slope in the context of the problem.

c) Convert 24°C to degrees Fahrenheit.

d) Change 95°F to degrees Celsius.

103) A kitten weighs an average of 100 g at birth and should gain about 8 g per day for the first few weeks of life. Let y represent the weight of a kitten, in grams, x days after birth. (http://veterinarymedicine.dvm360.com).

a) Write a linear equation to model these data.

b) Explain the meaning of the slope in the context of the problem.

c) How much would an average kitten weigh 5 days after birth? 2 weeks after birth?

d) How long would it take for a kitten to reach a weight of 284 g?

104) In 2000, Red Delicious apples cost an average of \$0.82 per lb, and in 2007 they cost \$1.12 per lb. Let y represent the cost of a pound of Red Delicious apples x years after 2000. (www.census.gov)

a) Write a linear equation to model these data. Round the slope to the nearest hundredth.

b) Explain the meaning of the slope in the context of the problem.

c) Find the cost of a pound of apples in 2003.

d) When was the average cost about \$1.06/lb?

105) If a woman wears a size 6 on the U.S. shoe size scale, her European size is 38. A U.S. women's size 8.5 corresponds to a European size 42. Let A represent the U.S. women's shoe size, and let E represent that size on the European scale.

a) Write a liner equation that models the European shoe size in terms of the U.S. shoe size.

b) If a woman's U.S. shoe size is 7.5, what is her European shoe size? (Round to the nearest unit.)

106) If a man's foot is 11.5 inches long, his U.S. shoe size is 12.5. A man wears a size 8 if his foot is 10 inches long. Let L represent the length of a man's foot, and let S represent his shoe size.

a) Write a linear equation that describes the relationship between shoe size in terms of the length of a man's foot.

b) If a man's foot is 10.5 inches long, what is his shoe size?

Section 4.6 Introduction to Functions

Objectives

1. Define and Identify Relations, Functions, Domain, and Range
2. Identify Functions and Find Their Domains
3. Use Function Notation
4. Define and Graph a Linear Function
5. Use Linear Functions to Solve Problems

If you are driving on a highway at a constant speed of 60 miles per hour, the distance you travel depends on the amount of time you spend driving.

Driving Time	Distance Traveled
1 hr	60 mi
2 hr	120 mi
2.5 hr	150 mi
3 hr	180 mi

We can express these relationships with the ordered pairs (1, 60), (2, 120), (2.5, 150), and (3, 180), where the first coordinate represents the driving time (in hours), and the second coordinate represents the distance traveled (in miles).

We can also describe this relationship with the equation

$$\underbrace{y}_{\text{Dependent variable}} = 60\underbrace{x}_{\text{Independent variable}}$$

where y is the distance traveled, in miles, and x is the number of hours spent driving.

The distance traveled *depends on* the amount of time spent driving. Therefore, the distance traveled, y, is the **dependent variable**, and the driving time, x, is the **independent variable**.

1. Define and Identify Relations, Functions, Domain, and Range

If we form a set of ordered pairs from the ones listed above,

$$\{(1, 60), (2, 120), (2.5, 150), (3, 180)\},$$

we get a ***relation***.

Definition

A **relation** is any set of ordered pairs.

Definition

The **domain** of a relation is the set of all values of the independent variable (the first coordinates in the set of ordered pairs). The **range** of a relation is the set of all values of the dependent variable (the second coordinates in the set of ordered pairs).

The domain of the last relation is $\{1, 2, 2.5, 3\}$. The range of the relation is $\{60, 120, 150, 180\}$.

The relation $\{(1, 60), (2, 120), (2.5, 150), (3, 180)\}$ is also a *function* because every first coordinate corresponds to *exactly one* second coordinate. A function is a very important concept in mathematics.

Definition

A **function** is a special type of relation. If each element of the domain corresponds to *exactly one* element of the range, then the relation is a function.

Relations and functions can be represented in another way—as a *correspondence* or a *mapping* from one set, the domain, to another, the range.

In this representation, the domain is the set of all values in the first set, and the range is the set of all values in the second set.

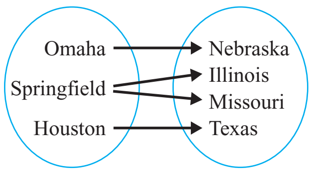
Example 1

Identify the domain and range of each relation, and determine whether each relation is a function.

a) $\{(2, 0), (3, 1), (6, 2), (6, -2)\}$

b) $\{(-2, -6), (0, -5), (1, -\frac{9}{2}), (4, -3), (5, -\frac{5}{2})\}$

c)



Solution

a) The *domain* is the set of first coordinates, $\{2, 3, 6\}$. (We write the 6 in the set only once even though it appears in two ordered pairs.)

The *range* is the set of second coordinates, $\{0, 1, 2, -2\}$.

To determine whether this relation is a function, ask yourself, *"Does every first coordinate correspond to exactly one second coordinate?"* No: one of the first coordinates, 6, corresponds to *two* different second coordinates, 2 and -2. Therefore, this relation is *not* a function.

b) The *domain* is $\{-2, 0, 1, 4, 5\}$.

The *range* is $\{-6, -5, -\frac{9}{2}, -3, -\frac{5}{2}\}$.

Does every first coordinate in this relation correspond to *exactly one* second coordinate? *Yes*, so this relation *is* a function.

c) The *domain* is {Omaha, Springfield, Houston}.
The *range* is {Nebraska, Illinois, Missouri, Texas}.

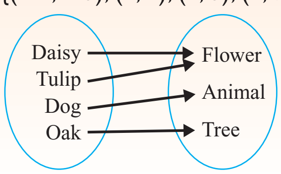
One of the elements in the domain, Springfield, corresponds to *two* elements in the range, Illinois and Missouri. Therefore, this relation is *not* a function. ■

You Try 1

Identify the domain and range of each relation, and determine whether each relation is a function.

a) $\{(-1, -3), (1, 1), (2, 3), (4, 7)\}$ b) $\{(-12, -6), (-12, 6), (-1, \sqrt{3}), (0, 0)\}$

c)

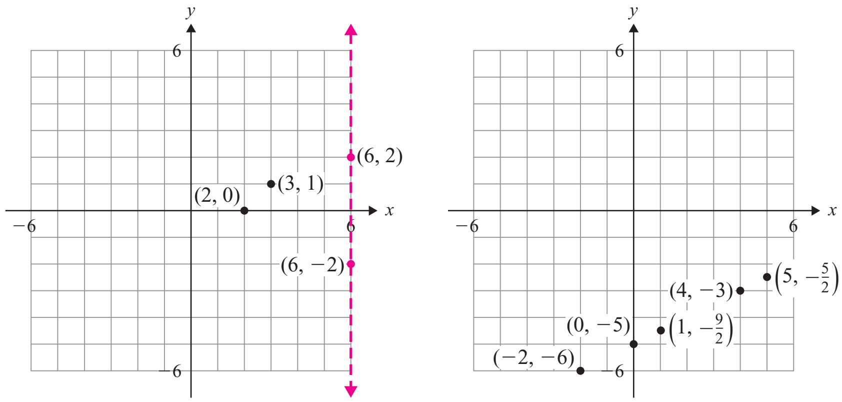


If the ordered pairs of a relation are in the form (x, y), then we can write an alternative definition of a function:

Definition

A relation is a **function** if each x-value corresponds to exactly one y-value.

What does a function look like when it is graphed? Let's examine the graphs of the ordered pairs in the relations of Example 1a and 1b.



Example 1a
not a function

Example 1b
is a function

The relation in Example 1a is *not* a function since the x-value of 6 corresponds to *two different* y-values, 2 and -2. Note that we can draw a vertical line that intersects the graph in more than one point—the line through $(6, 2)$ and $(6, -2)$.

The relation in Example 1b, however, *is* a function—each x-value corresponds to only one y-value. We cannot draw a vertical line through more than one point on this graph.

This leads us to the *vertical line test* for a function.

Definition The Vertical Line Test

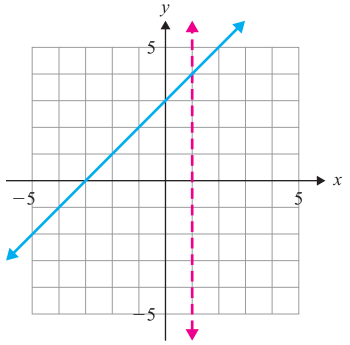
If no vertical line can be drawn through a graph that intersects the graph more than once, then the graph represents a function.

If a vertical line *can* be drawn that intersects the graph more than once, then the graph does *not* represent a function.

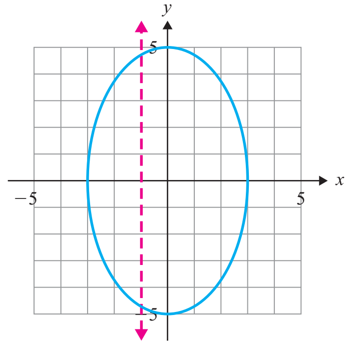
Example 2

Use the vertical line test to determine whether each graph, in blue, represents a function. Identify the domain and range using interval notation.

a)



b)



Solution

a) Any vertical line you can draw through the graph can intersect it only once, so *this graph represents a function.*

The domain of this function is the set of the line's x-values. Since the arrows show that the line continues indefinitely in the x-direction, the *domain* is the set of all real numbers, or $(-\infty, \infty)$.

The range of this function is the set of the line's y-values. Since the arrows show that the line continues indefinitely in the y-direction, the *range* is the set of all real numbers, or $(-\infty, \infty)$.

b) This graph fails the vertical line test because we can draw a vertical line through the graph that intersects it more than once. *This graph does not represent a function.*

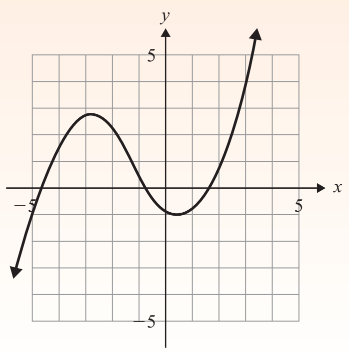
The set of the graph's x-values includes all real numbers from -3 to 3, so the *domain* is $[-3, 3]$.

The set of the graph's y-values includes all real numbers from -5 to 5, so the *range* is $[-5, 5]$.

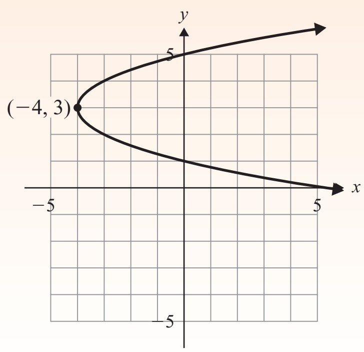
You Try 2

Use the vertical line test to determine whether each relation is also a function. Then, identify the domain (D) and range (R).

a)



b)



2. Identify Functions and Find Their Domains

We can also represent relations and functions with equations. The example given at the beginning of the section illustrates this. The equation $y = 60x$ describes the distance traveled (y, in miles) after x hours of driving at 60 mph.

If $x = 2, y = 60(2) = 120$. If $x = 3, y = 60(3) = 180$, and so on. For *every* value of x that could be substituted into $y = 60x$, there is *exactly one* corresponding value of y. Therefore, $y = 60x$ is a function.

Furthermore, we can say that y *is a function of* x. In the function described by $y = 60x$, the value of y *depends on* the value of x. That is, x is the independent variable and y is the dependent variable.

Definition

If a function describes the relationship between x and y so that x is the independent variable and y is the dependent variable, then we say that **y is a function of x.**

Example 3

Determine whether each relation describes y as a function of x.

a) $y = x + 2$ b) $y^2 = x$

Solution

a) To begin, substitute a few values for x and solve for y to get an idea of what is happening in this relation.

$x = 0$	$x = 3$	$x = -4$
$y = x + 2$	$y = x + 2$	$y = x + 2$
$y = 0 + 2$	$y = 3 + 2$	$y = -4 + 2$
$y = 2$	$y = 5$	$y = -2$

The ordered pairs (0, 2), (3, 5) and $(-4, -2)$ satisfy $y = x + 2$. Each of the values substituted for x has *one* corresponding y-value. Ask yourself, "For *any* value that I substitute for x, how *many* corresponding values of y will there be?" In this case, there will be *exactly one* corresponding value of y. Therefore, $y = x + 2$ *is* a function.

b) Substitute a few values for x and solve for y to get an idea of what is happening in this relation.

$x = 0$	$x = 4$	$x = 9$
$y^2 = x$	$y^2 = x$	$y^2 = x$
$y^2 = 0$	$y^2 = 4$	$y^2 = 9$
$y = 0$	$y = \pm 2$	$y = \pm 3$

The ordered pairs (0, 0), (4, 2), $(4, -2)$, (9, 3), and $(9, -3)$ satisfy $y^2 = x$. Since $2^2 = 4$ and $(-2)^2 = 4$, $x = 4$ corresponds to *two* different y-values, 2 and -2. Likewise, $x = 9$ corresponds to *two* different y-values, 3 and -3. Finding one such example is enough to determine that $y^2 = x$ is *not* a function. ■

You Try 3

Determine whether each relation describes y as a function of x.

a) $y = 3x - 5$ b) $y^2 = x + 1$

We have seen how to determine the domain of a relation written as a set of ordered pairs, as a correspondence (or mapping), and as a graph. Next, we will discuss how to determine the domain of a relation written as an equation.

Sometimes, it is helpful to ask yourself, "Is there any number that *cannot* be substituted for x?"

Example 4

Determine the domain of each relation, and determine whether each relation describes y as a function of x.

a) $y = \dfrac{1}{x}$ b) $y = \dfrac{7}{x - 3}$ c) $y = -2x + 6$

Solution

a) To determine the domain of $y = \dfrac{1}{x}$, ask yourself, "Is there any number that *cannot* be substituted for x?" Yes: x cannot equal zero because a *fraction* is undefined if its denominator equals zero. Any other number can be substituted for x and $y = \dfrac{1}{x}$ will be defined.

The domain contains all real numbers *except* 0. In interval notation, the domain is $(-\infty, 0) \cup (0, \infty)$.

The equation $y = \dfrac{1}{x}$ *is* a function since each value of x in the domain has exactly one corresponding value of y.

b) Ask yourself, "Is there any number that *cannot* be substituted for x in $y = \dfrac{7}{x - 3}$?" Look at the denominator. When will it equal 0?

$$x - 3 = 0 \quad \text{Set the denominator equal to 0.}$$
$$x = 3 \quad \text{Solve.}$$

When $x = 3$, the denominator of $y = \dfrac{7}{x - 3}$ equals zero. The domain contains all real numbers *except* 3. In interval notation, the domain is $(-\infty, 3) \cup (3, \infty)$.

The equation $y = \dfrac{7}{x - 3}$ *is* a function, since each value of x in the domain has exactly one corresponding value of y.

c) Is there any number that *cannot* be substituted for x in $y = -2x + 6$? No. Any real number can be substituted for x, and $y = -2x + 6$ will be defined. The domain consists of all real numbers, or $(-\infty, \infty)$.

Every value substituted for x has exactly one corresponding y-value, so $y = -2x + 6$ *is* a function.

Procedure Finding the Domain of a Relation

If a relation is written as an equation with y in terms of x, then the domain of the relation is the set of all real numbers that can be substituted for the independent variable, x. To determine the domain of a relation, use these tips.

1) Ask yourself, "Is there any number that *cannot* be substituted for x?"
2) If x is in the denominator of a fraction, determine the value of x that will make the denominator equal 0 by setting the denominator equal to zero. Solve for x. This x-value is *not* in the domain.

You Try 4

Determine the domain of each relation, and determine whether each relation describes y as a function of x.

a) $y = x - 9$ b) $y = -x^2 + 6$ c) $y = \frac{4}{x + 1}$

3. Use Function Notation

We know that if a function describes the relationship between x and y so that x is the independent variable and y is the dependent variable, then *y is a function of x*. That is, *the value of y depends on the value of x*. We use a special notation to represent this relationship.

Definition

The **function notation** $y = f(x)$ means that y is a function of x (y depends on x). We read $y = f(x)$ as "y equals f of x."

If y is a function of x, then $f(x)$ can be used in place of y: *$f(x)$ is the same as y.*

For example, $y = x + 3$, a function, can be written as $f(x) = x + 3$. *They mean the same thing*.

Example 5

a) Evaluate $y = x + 3$ for $x = 2$. b) If $f(x) = x + 3$, find $f(2)$.

Solution

a) To evaluate $y = x + 3$ for $x = 2$ means to substitute 2 for x and find the corresponding value of y.

$y = x + 3$
$y = 2 + 3$ Substitute 2 for x.
$y = 5$

When $x = 2$, $y = 5$. We can also say that the ordered pair (2, 5) satisfies $y = x + 3$.

b) To find $f(2)$ (read as "f of 2") means to find the value of the function f when $x = 2$.

$f(x) = x + 3$
$f(2) = 2 + 3$ Substitute 2 for x.
$f(2) = 5$

We can also say that the ordered pair (2, 5) satisfies $f(x) = x + 3$, where the ordered pair represents $(x, f(x))$. ■

Note

Example 5 illustrates that evaluating $y = x + 3$ for $x = 2$ and finding $f(2)$ when $f(x) = x + 3$ are exactly the same thing. Remember, $f(x)$ is another name for y.

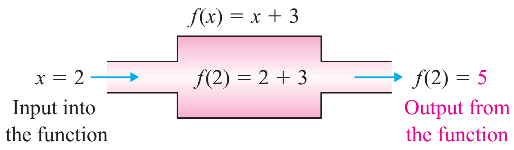
You Try 5

a) Evaluate $y = -2x + 4$ for $x = 1$. b) If $f(x) = -2x + 4$, find $f(1)$.

Different letters can be used to name functions: $g(x)$ is read as "g of x," $h(x)$ is read as "h of x," and so on. Also, the parentheses used in function notation do *not* indicate multiplication.

BE CAREFUL The notation $f(x)$ does not mean f times x.

We can also think of a function as a machine: We put values into it and the function determines the values that come out. We can visualize the function in Example 5b as the figure on the right.



Sometimes, we call evaluating a function for a certain value *finding a function value.*

Example 6

Let $f(x) = 6x - 5$ and $g(x) = x^2 - 8x + 3$. Find the following function values.

a) $f(0)$ b) $g(-1)$

Solution

a) "Find $f(0)$" means find the value of function when $x = 0$. Substitute 0 for x.

$$\begin{aligned} f(x) &= 6x - 5 \\ f(0) &= 6(0) - 5 = 0 - 5 = -5 \\ f(0) &= -5 \end{aligned}$$

The ordered pair $(0, -5)$ satisfies $f(x) = 6x - 5$.

b) To find $g(-1)$, substitute -1 for every x in the function $g(x)$.

$$\begin{aligned} g(x) &= x^2 - 8x + 3 \\ g(-1) &= (-1)^2 - 8(-1) + 3 = 1 + 8 + 3 = 12 \\ g(-1) &= 12 \end{aligned}$$

The ordered pair $(-1, 12)$ satisfies $g(x) = x^2 - 8x + 3$.

You Try 6

Let $f(x) = -4x + 1$ and $h(x) = 2x^2 + 3x - 7$. Find the following function values.

a) $f(5)$ b) $f(-2)$ c) $h(0)$ d) $h(3)$

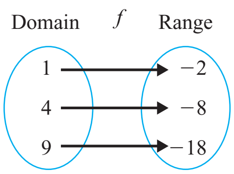
We can also find function values for functions represented by a set of ordered pairs, a correspondence, or a graph.

Example 7

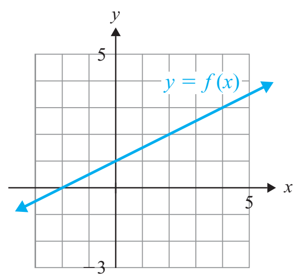
Find $f(4)$ for each function.

a) $f = \{(-2, -11), (0, -5), (3, 4), (4, 7)\}$

b)



c)

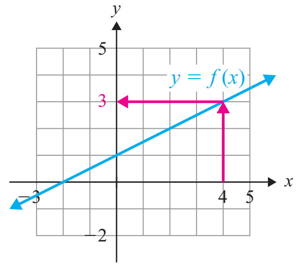


Solution

a) Since this function is expressed as a set of ordered pairs, finding $f(4)$ means finding the y-coordinate of the ordered pair with x-coordinate 4. The ordered pair with x-coordinate 4 is (4, 7), so $f(4) = 7$.

b) In this function, the element 4 in the domain corresponds to the element -8 in the range. Therefore, $f(4) = -8$.

c) To find $f(4)$ from the graph of this function means to find the y-coordinate of the point on the line that has an x-coordinate of 4. Find 4 on the x-axis. Then, go straight up to the graph and move to the left to read the y-coordinate of the point on the graph where x-coordinate is 4. That y-coordinate is 3, so $f(4) = 3$.

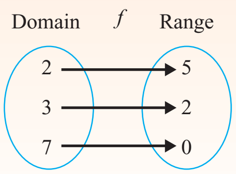


You Try 7

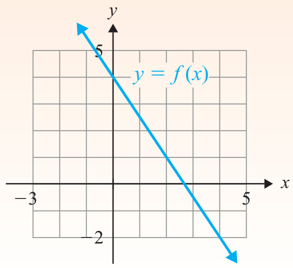
Find $f(2)$ for each function.

a) $f = \{(-5, 8), (-1, 2), (2, -3), (6, -9)\}$

b)



c)



We can also evaluate functions for variables or expressions.

Example 8

Let $h(x) = 5x + 3$. Find each of the following and simplify.

a) $h(c)$ b) $h(t - 4)$

Solution

a) Finding $h(c)$ (read as *h of c*) means to substitute c for x in the function h, and simplify the expression as much as possible.

$$h(x) = 5x + 3$$
$$h(c) = 5c + 3 \qquad \text{Substitute } c \text{ for } x.$$

b) Finding $h(t - 4)$ (read as *h of t minus 4*) means to substitute $t - 4$ for x in function h, and simplify the expression as much as possible. *Since $t - 4$ contains two terms, we must put it in parentheses.*

$$h(x) = 5x + 3$$
$$h(t - 4) = 5(t - 4) + 3 \qquad \text{Substitute } t - 4 \text{ for } x.$$
$$h(t - 4) = 5t - 20 + 3 \qquad \text{Distribute.}$$
$$h(t - 4) = 5t - 17 \qquad \text{Combine like terms.}$$

You Try 8

Let $f(x) = 2x - 7$. Find each of the following and simplify.

a) $f(k)$ b) $f(p + 3)$

4. Define and Graph a Linear Function

We know that a linear equation can have the form $y = mx + b$. A *linear function* has a similar form:

Definition

A **linear function** has the form $\boldsymbol{f(x) = mx + b}$, where m and b are real numbers, m is the *slope* of the line, and $(0, b)$ is the *y-intercept*.

A linear equation is a function except when the line is vertical and has the equation $x = c$. The domain of a linear function is all real numbers.

Example 9

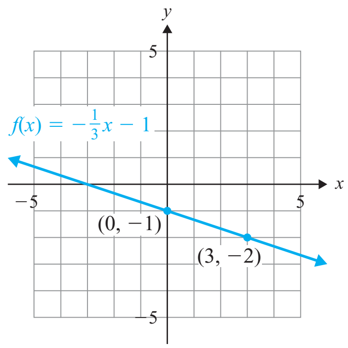
Graph $f(x) = -\frac{1}{3}x - 1$ using the slope and y-intercept. State the domain and range.

Solution

$$f(x) = -\frac{1}{3}x - 1$$

$$\uparrow \qquad \uparrow$$

$$m = -\frac{1}{3} \qquad y\text{-int: } (0, -1)$$



To graph this function, first plot the y-intercept, $(0, -1)$, then use the slope to locate another point on the line.

The domain and range are both $(-\infty, \infty)$.

You Try 9

Graph $f(x) = \frac{3}{4}x - 2$ using the slope and y-intercept. State the domain and range.

5. Use Linear Functions to Solve Problems

The independent variable of a function does not have to be x. When using functions to solve problems, we often choose a more "meaningful" letter to represent a quantity. The same is true for naming the function.

Note

No matter what letter is chosen for the independent variable, the *horizontal axis* represents the values of the *independent variable*, and the *vertical axis* represents the *function values*.

Example 10

A compact disk is read at 44.1 kHz (kilohertz). This means that a CD player scans 44,100 samples of sound per second on a CD to produce the sound that we hear. The function

$$S(t) = 44{,}100t$$

tells us how many samples of sound, $S(t)$, are read in t seconds. (www.mediatechnics.com)

a) How many samples of sound are read in 20 sec?

b) How many samples of sound are read in 1.5 min?

c) How long would it take the CD player to scan 1,764,000 samples of sound?

d) What is the smallest value t could equal in the context of this problem?

e) Graph the function.

Solution

a) To determine how much sound is read in 20 sec, let $t = 20$ and find $S(20)$.

$$\begin{aligned} S(t) &= 44{,}100t \\ S(20) &= 44{,}100(20) && \text{Substitute 20 for } t. \\ S(20) &= 882{,}000 && \text{Multiply.} \end{aligned}$$

The number of samples read is 882,000.

b) To determine how much sound is read in 1.5 min, do we let $t = 1.5$ and find $S(1.5)$? *No*. Recall that t is in *seconds*. Change 1.5 min to seconds: 1.5 min = 90 sec.

$$\begin{aligned} S(t) &= 44{,}100t \\ S(90) &= 44{,}100(90) && \text{Let } t = 90 \text{ and find } S(90). \\ S(90) &= 3{,}969{,}000 \end{aligned}$$

The number of samples read is 3,969,000.

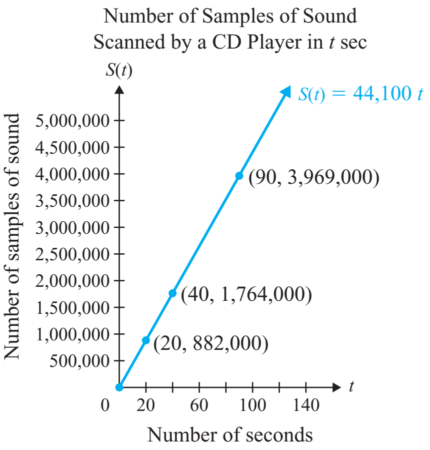
c) Since we are asked to determine *how long* it would take a CD player to scan 1,764,000 samples of sound, we will be solving for t. What do we substitute for $S(t)$? We substitute 1,764,000 for $S(t)$ and find t.

$$\begin{aligned} S(t) &= 44{,}100t \\ 1{,}764{,}000 &= 44{,}100t && \text{Substitute 1,764,000 for } S(t). \\ 40 &= t && \text{Divide by 44,100.} \end{aligned}$$

It will take 40 sec for the CD player to scan 1,764,000 samples of sound.

d) Since t represents the number of seconds a CD plays, the smallest value that makes sense for t is 0.

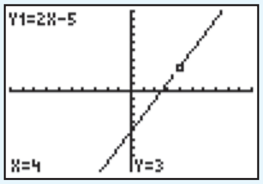
e) The information we obtained in parts a), b), and c) can be written as the ordered pairs (20, 882,000), (90, 3,969,000), and (40, 1,764,000). In addition, when $t = 0$ (from part d), $S(0) = 44{,}100(0) = 0$. (0, 0) is an additional ordered pair on the graph.



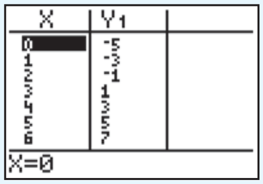
Using Technology

A graphing calculator can be used to represent a function as a graph and also as a table of values.

Consider the function $f(x) = 2x - 5$. To graph the function, press $\boxed{\text{Y=}}$, then type $2x - 5$ to the right of \Y1 =. Press $\boxed{\text{ZOOM}}$ and select 6: ZStandard to graph the equation. We can select a point on the graph. For example, press $\boxed{\text{TRACE}}$, type 4, and press $\boxed{\text{ENTER}}$. The point (4, 3) is displayed on the screen as shown on the right.



The function can also be represented as a table on a graphing calculator. To set up the table, press 2nd $\boxed{\text{WINDOW}}$ and move the cursor after TblStart=. Try entering a number such as 0 to set the starting x-value for the table. Enter 1 after Tbl= to set the increment between x-values as shown on the left below. Press 2nd $\boxed{\text{GRAPH}}$ to display the table as shown below on the right.



The point (4, 3) is represented in the table above as well as on the graph.

Given the function, find the function value on a graph and a table using a graphing calculator.

1. $f(x) = 3x - 4; f(2)$
2. $f(x) = 4x - 1; f(1)$
3. $f(x) = -3x + 7; f(1)$
4. $f(x) = 2x + 5; f(-1)$
5. $f(x) = 2x - 7; f(-1)$
6. $f(x) = -x + 5; f(4)$

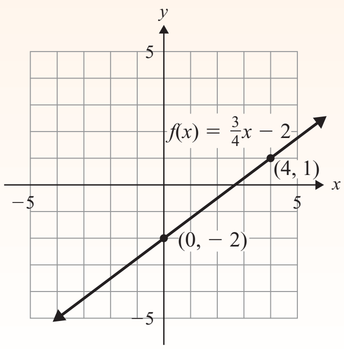
Answers to You Try Exercises

1) a) domain: $\{-1, 1, 2, 4\}$; range: $\{-3, 1, 3, 7\}$; yes b) domain: $\{-12, -1, 0\}$; range: $\{-6, 6, \sqrt{3}, 0\}$; no c) domain: {Daisy, Tulip, Dog, Oak}; range: {Flower, Animal, Tree}; yes

2) a) function; D: $(-\infty, \infty)$; R: $(-\infty, \infty)$ b) not a function; D: $[-4, \infty)$; R: $(-\infty, \infty)$

3) a) yes b) no 4) a) $(-\infty, \infty)$; function b) $(-\infty, \infty)$; function c) $(-\infty, -1) \cup (-1, \infty)$; function 5) a) 2 b) 2 6) a) -19 b) 9 c) -7 d) 20 7) a) -3 b) 5 c) 1

8) a) $f(k) = 2k - 7$ b) $f(p + 3) = 2p - 1$ 9) $m = \frac{3}{4}$, y-int: $(0, -2)$



D: $(-\infty, \infty)$; R: $(-\infty, \infty)$

Answers to Technology Exercises

1. 2
2. 3
3. 4
4. 3
5. -9
6. 1

4.6 Exercises

Objective 1: Define and Identify Relations, Functions, Domain, and Range

1) a) What is a relation?

   b) What is a function?

   c) Give an example of a relation that is also a function.

2) Give an example of a relation that is *not* a function.

Identify the domian and range of each relation, and determine whether each relation is a function.

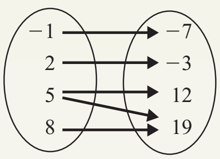
3) $\{(5, 13), (-2, 6), (1, 4), (-8, -3)\}$

4) $\{(0, -3), (1, -4), (1, -2), (16, -5), (16, -1)\}$

5) $\{(9, -1), (25, -3), (1, 1), (9, 5), (25, 7)\}$

6) $\left\{(-4, -2), \left(-3, -\frac{1}{2}\right), \left(-1, -\frac{1}{2}\right), (0, -2)\right\}$

7)



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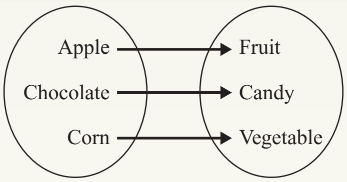

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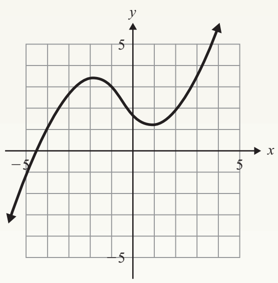

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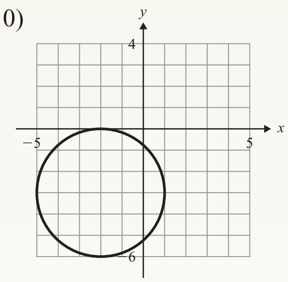

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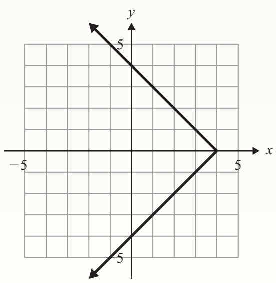

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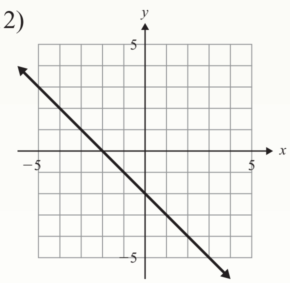

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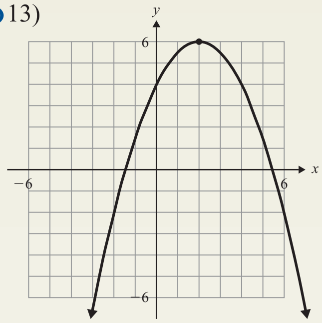

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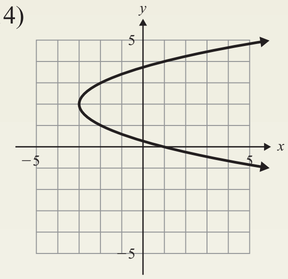


Objective 2: Identify Functions and Find Their Domains

Determine whether each relation describes y as a function of x.

15) $y = x - 9$

16) $y = x + 4$

17) $y = 2x + 7$

18) $y = \frac{2}{3}x + 1$

19) $x = y^4$

20) $x = y^2 - 3$

21) $y^2 = x - 4$

22) $y^2 = x + 9$

Determine the domain of each relation, and determine whether each relation describes y as a function of x.

23) $y = x - 5$

24) $y = 2x + 1$

25) $y = x^3 + 2$

26) $y = -x^3 + 4$

27) $x = y^4$

28) $x = |y|$

29) $y = -\frac{8}{x}$

30) $y = \frac{5}{x}$

31) $y = \frac{9}{x + 4}$

32) $y = \frac{2}{x - 7}$

33) $y = \frac{3}{x - 5}$

34) $y = \frac{1}{x + 10}$

35) $y = \frac{6}{5x - 3}$

36) $y = -\frac{4}{9x + 8}$

37) $y = \frac{15}{3x + 4}$

38) $y = \frac{5}{6x - 1}$

39) $y = -\frac{5}{9 - 3x}$

40) $y = \frac{1}{-6 + 4x}$

41) $y = \frac{x}{12}$

42) $y = \frac{x + 8}{7}$

Objective 3: Use Function Notation

43) Explain what it means when an equation is written in the form $y = f(x)$.

44) Does $y = f(x)$ mean "$y = f$ times x"? Explain.

45) a) Evaluate $y = 5x - 8$ for $x = 3$.

b) If $f(x) = 5x - 8$, find $f(3)$.

46) a) Evaluate $y = -3x - 2$ for $x = -4$.

b) If $f(x) = -3x - 2$, find $f(-4)$.

Let $f(x) = -4x + 7$ and $g(x) = x^2 + 9x - 2$. Find the following function values.

47) $f(5)$ 48) $f(2)$ 49) $f(0)$

50) $f\left(-\frac{3}{2}\right)$ 51) $g(4)$ 52) $g(1)$

53) $g(-1)$ 54) $g(0)$ 55) $g\left(-\frac{1}{2}\right)$

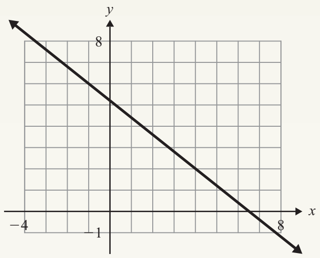
56) $g\left(\frac{1}{3}\right)$ 57) $f(6) - g(6)$ 58) $f(-4) - g(-4)$

For each function f in Exercises 59–64, find $f(-1)$ and $f(4)$.

59) $f = \{(-3, 16), (-1, 10), (0, 7), (1, 4), (4, -5)\}$

60) $f = \left\{(-8, -1), \left(-1, \frac{5}{2}\right), (4, 5), (10, 8)\right\}$

61)



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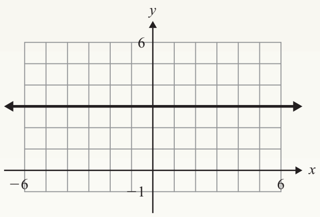

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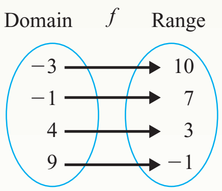

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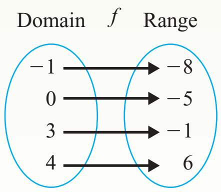


65) $f(x) = -3x - 2$. Find x so that $f(x) = 10$.

66) $f(x) = 5x + 4$. Find x so that $f(x) = 9$.

67) $g(x) = \frac{2}{3}x + 1$. Find x so that $g(x) = 5$.

68) $h(x) = -\frac{1}{2}x - 6$. Find x so that $h(x) = -2$.

Fill It In

In Exercises 69–70, fill in the blanks with either the missing mathematical step or the reason for given step.

69) Let $f(x) = 4x - 5$. Find $f(k + 6)$.

$f(k + 6) = 4(k + 6) - 5$ ____________

____________ Distribute.

____________ Simplify.

70) Let $f(x) = -9x + 2$. Find $f(n - 3)$.

____________ Substitute $n - 3$ for x.

$= -9n + 27 + 2$ ____________

____________ Simplify.

71) $f(x) = -7x + 2$ and $g(x) = x^2 - 5x + 12$. Find each of the following and simplify.

a) $f(c)$ b) $f(t)$

c) $f(a + 4)$ d) $f(z - 9)$

e) $g(k)$ f) $g(m)$

g) $f(x + h)$ h) $f(x + h) - f(x)$

72) $f(x) = 5x + 6$ and $g(x) = x^2 - 3x - 11$. Find each of the following and simplify.

a) $f(n)$ b) $f(p)$

c) $f(w + 8)$ d) $f(r - 7)$

e) $g(b)$ f) $g(s)$

g) $f(x + h)$ h) $f(x + h) - f(x)$

Objective 4: Define and Graph a Linear Function

Graph each function by making a table of values and plotting points.

73) $f(x) = x - 4$ 74) $f(x) = x + 2$

75) $f(x) = \frac{2}{3}x + 2$ 76) $g(x) = -\frac{3}{5}x + 2$

77) $h(x) = -3$ 78) $g(x) = 1$

Graph each function by finding the x- and y-intercepts and one other point.

79) $g(x) = 3x + 3$ 80) $k(x) = -2x + 6$

81) $f(x) = -\frac{1}{2}x + 2$ 82) $f(x) = \frac{1}{3}x + 1$

83) $h(x) = x$ 84) $f(x) = -x$

Graph each function using the slope and y-intercept.

VIDEO 85) $f(x) = -4x - 1$

86) $f(x) = -x + 5$

87) $h(x) = \frac{3}{5}x - 2$

88) $g(x) = -\frac{1}{4}x - 2$

89) $g(x) = 2x + \frac{1}{2}$

90) $g(x) = \frac{1}{2}x - \frac{3}{2}$

Graph each function.

91) $s(t) = -\frac{1}{3}t - 2$

92) $k(d) = d - 1$

93) $A(r) = -3r$

94) $N(t) = 3.5t + 1$

Objective 5: Use Linear Functions to Solve Problems

95) A truck on the highway travels at a constant speed of 54 mph. The distance, D (in miles), that the truck travels after t hr can be defined by the function

$$D(t) = 54t$$

a) How far will the truck travel after 2 hr?

b) How long does it take the truck to travel 135 mi?

c) Graph the function.

96) The velocity of an object, v (in feet per second), of an object during free-fall t sec after being dropped can be defined by the function

$$v(t) = 32t$$

a) Find the velocity of an object 3 sec after being dropped.

b) When will the object be traveling at 256 ft/sec?

c) Graph the function.

97) Jenelle earns \$7.50 per hour at her part-time job. Her total earnings, E (in dollars), for working t hr can be defined by the function

$$E(t) = 7.50t$$

a) Find $E(10)$, and explain what this means in the context of the problem.

b) Find t so that $E(t) = 210$, and explain what this means in the context of the problem.

98) If gasoline costs \$2.50 per gallon, then the cost, C (in dollars), of filling a gas tank with g gal of gas is defined by

$$C(g) = 2.50g$$

a) Find $C(8)$, and explain what this means in the context of the problem.

b) Find g so that $C(g) = 30$, and explain what this means in the context of the problem.

99) A 16× DVD recorder can transfer 21.13 MB (megabytes) of data per second onto a recordable DVD. The function $D(t) = 21.13t$ describes how much data, D (in megabytes), is recorded on a DVD in t sec. (www.osta.org)

a) How much data is recorded in 12 sec?

b) How much data is recorded in 1 min?

c) How long would it take to record 422.6 MB of data?

d) Graph the function.

100) The median hourly wage of an embalmer in Illinois in 2002 was \$17.82. Seth's earnings, E (in dollars), for working t hr in a week can be defined by the function $E(t) = 17.82t$. (www.igpa.uillinois.edu)

a) How much does Seth earn if he works 30 hr?

b) How many hours would Seth have to work to make \$623.70?

c) If Seth can work at most 40 hr per week, what is the domain of this function?

d) Graph the function.

VIDEO 101) Law enforcement agencies use a computerized system called AFIS (Automated Fingerprint Identification System) to identify fingerprints found at crime scenes. One AFIS system can compare 30,000 fingerprints per second. The function

$$F(s) = 30s$$

describes how many fingerprints, $F(s)$ in thousands, are compared in s sec.

a) How many fingerprints can be compared in 2 sec?

b) How long would it take AFIS to search through 105,000 fingerprints?

102) Refer to the function in Exercise 101 to answer the following questions.

a) How many fingerprints can be compared in 3 sec?

b) How long would it take AFIS to search through 45,000 fingerprints?

103) Refer to the function in Example 10 on p. 272 to determine the following.

a) Find $S(50)$, and explain what this means in the context of the problem.

b) Find t so that $S(t) = 2{,}646{,}000$, and explain what this means in the context of the problem.

104) Refer to the function in Exercise 99 to determine the following.

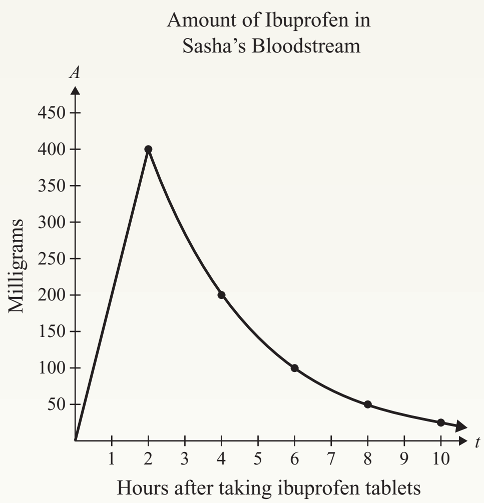
a) Find $D(120)$, and explain what this means in the context of the problem.

b) Find t so that $D(t) = 633.9$, and explain what this means in the context of the problem.

105) The graph shows the amount, A, of ibuprofen in Sasha's bloodstream t hours after she takes two tablets for a headache.

a) How long after taking the tablets will the amount of ibuprofen in her bloodstream be the greatest? How much ibuprofen is in her bloodstream at this time?

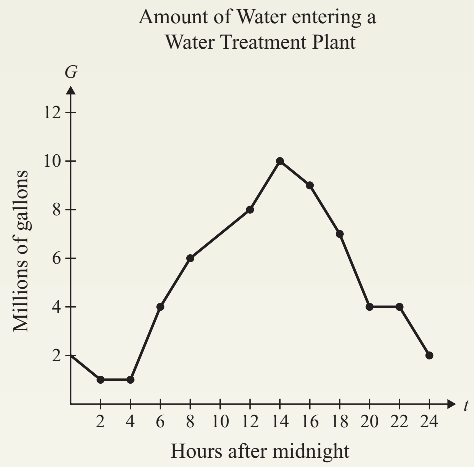
b) When will there be 100 mg of ibuprofen in Sasha's bloodstream?



c) How much of the drug is in her bloodstream after 4 hours?

d) Find $A(8)$, and explain what it means in the context of the problem.

106) The graph shows the number of gallons, G (in millions), of water entering a water treatment plant t hours after midnight on a certain day.



a) Identify the domain and range of this function.

b) How many gallons of water enter the facility at noon? At 10 P.M.?

c) At what time did the most water enter the treatment plant? How much water entered the treatment plant at this time?

d) At what time did the least amount of water enter the treatment plant?

e) Find $G(18)$, and explain what it means in the context of the problem.

Chapter 4: Summary

Definition/Procedure	Example

4.1 Introduction to Linear Equations in Two Variables

A **linear equation in two variables** can be written in the form $Ax + By = C$, where A, B, and C are real numbers and where both A and B do not equal zero.

To determine whether an ordered pair is a solution of an equation, substitute the values for the variables. **(p. 208)**

Is $(4, -1)$ a solution of $3x - 5y = 17$?
Substitute 4 for x and -1 for y.

$$\begin{aligned} 3x - 5y &= 17 \\ 3(4) - 5(-1) &= 17 \\ 12 + 5 &= 17 \\ 17 &= 17 \checkmark \end{aligned}$$

Yes, $(4, -1)$ is a solution.

4.2 Graphing by Plotting Points and Finding Intercepts

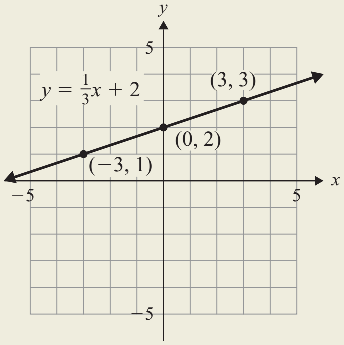
The graph of a linear equation in two variables, $Ax + By = C$, is a straight line. Each point on the line is a solution to the equation.

We can graph the line by plotting the points and drawing the line through them. **(p. 220)**

Graph $y = \frac{1}{3}x + 2$ by plotting points.

Make a table of values. Plot the points, and draw a line through them.

x	y
0	2
3	3
−3	1



The ***x*-intercept** of an equation is the point where the graph intersects the x-axis. To find the *x-intercept* of the graph of an equation, let $y = 0$ and solve for x.

The ***y*-intercept** of an equation is the point where the graph intersects the y-axis. To find the *y-intercept* of the graph of an equation, let $x = 0$ and solve for y. **(p. 222)**

Graph $2x + 5y = -10$ by finding the intercepts and another point on the line.

x-intercept: Let $y = 0$, and solve for x.

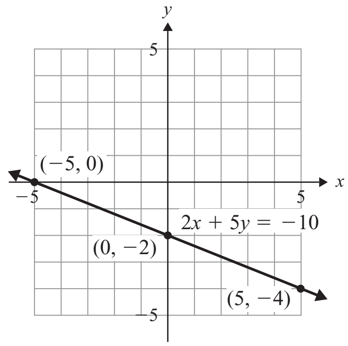
$$\begin{aligned} 2x + 5(0) &= -10 \\ 2x &= -10 \\ x &= -5 \end{aligned}$$

The x-intercept is $(-5, 0)$.

y-intercept: Let $x = 0$, and solve for y.

$$\begin{aligned} 2(0) + 5y &= -10 \\ 5y &= -10 \\ y &= -2 \end{aligned}$$

The y-intercept is $(0, -2)$. Another point on the line is $(5, -4)$. Plot the points, and draw the line through them.

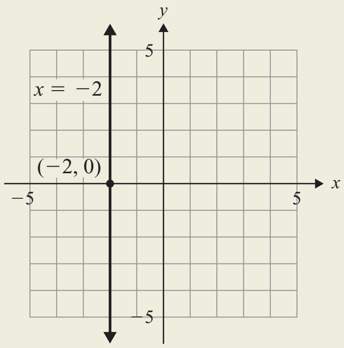


Definition/Procedure	Example

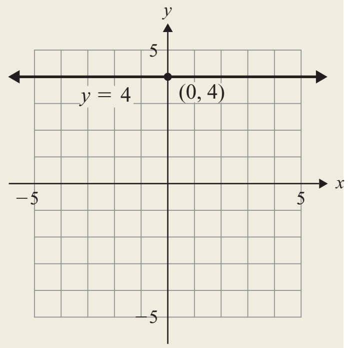
If c is a constant, then the graph of $x = c$ is a *vertical line* going through the point $(c, 0)$.

If d is a constant, then the graph of $y = d$ is a *horizontal line* going through the point $(0, d)$. **(p. 225)**

Graph $x = -2$.



Graph $y = 4$.



4.3 The Slope of a Line

The **slope** of a line is the ratio of the vertical change in y to the horizontal change in x. Slope is denoted by m.

The slope of a line containing the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope of a horizontal line is zero.
The slope of a vertical line is undefined. **(p. 231)**

Find the slope of the line containing the points $(4, -3)$ and $(-1, 5)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-3)}{-1 - 4} = \frac{8}{-5} = -\frac{8}{5}$$

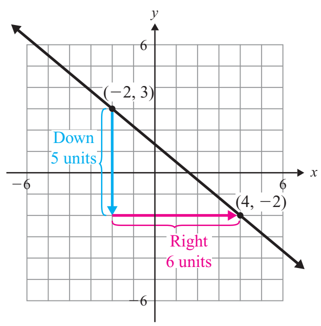
The slope of the line is $-\frac{8}{5}$.

If we know the slope of a line and a point on the line, we can graph the line. **(p. 236)**

Graph the line containing the point $(-2, 3)$ with a slope of $-\frac{5}{6}$.

Start with the point $(-2, 3)$, and use the slope to plot another point on the line.

$$m = \frac{-5}{6} = \frac{\text{Change in } y}{\text{Change in } x}$$



Definition/Procedure	Example

4.4 The Slope-Intercept Form of a Line

The **slope-intercept form of a line** is $y = mx + b$, where m is the slope and $(0, b)$ is the y-intercept.

If a line is written in slope-intercept form, we can use the y-intercept and the slope to graph the line. **(p. 242)**

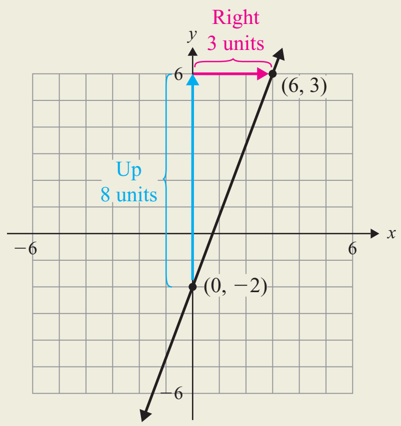
Write the equation in slope-intercept form and graph it.

$$\begin{aligned} 8x - 3y &= 6 \\ -3y &= -8x + 6 \\ y &= \frac{-8}{-3}x + \frac{6}{-3} \\ y &= \frac{8}{3}x - 2 \qquad \text{Slope-intercept form} \end{aligned}$$

$m = \frac{8}{3}$, y-intercept $(0, -2)$

Plot $(0, -2)$, then use the slope to locate another point on the line. We will think of the slope as

$m = \frac{8}{3} = \frac{\text{Change in } y}{\text{Change in } x}$.



Parallel lines have the same slope.

Perpendicular lines have slopes that are negative reciprocals of each other. **(p. 244)**

Determine whether the lines $2x + y = 18$ and $x - 2y = 7$ are parallel, perpendicular, or neither.

Put each line into slope-intercept form to find their slopes.

$$\begin{aligned} 2x + y &= 18 \\ y &= -2x + 18 \\ m &= -2 \end{aligned} \qquad \begin{aligned} x - 2y &= 7 \\ -2y &= -x + 7 \\ y &= \frac{1}{2}x - \frac{7}{2} \\ m &= \frac{1}{2} \end{aligned}$$

The lines are *perpendicular* since their slopes are negative reciprocals of each other.

4.5 Writing an Equation of a Line

To write the equation of a line given its slope and y-intercept, use $y = mx + b$ and substitute those values into the equation. (p. 251)

Find an equation of the line with slope $= 5$ and y-intercept $(0, -3)$.

Use $y = mx + b$. Substitute 5 for m and -3 for b.

$y = 5x - 3$

Definition/Procedure	Example
If (x_1, y_1) is a point on a line and m is the slope of the line, then the equation of the line is given by $y - y_1 = m(x - x_1)$. This is the **point-slope formula.** **If we are given the slope of the line and a point on the line, we can use the point-slope formula to find an equation of the line. (p. 251)**	Find an equation of the line containing the point $(7, -2)$ with slope $= 3$. Express the answer in standard form. Use $y - y_1 = m(x - x_1)$. Substitute 3 for m. Substitute $(7, -2)$ for (x_1, y_1). $y - (-2) = 3(x - 7)$ $y + 2 = 3x - 21$ $-3x + y = -23$ $3x - y = 23$ Standard form
To write an equation of a line given two points on the line, **a) use the points to find the slope of the line** then **b) use the slope and either one of the points in the point-slope formula. (p. 253)**	Find an equation of the line containing the points $(4, 1)$ and $(-4, 5)$. Express the answer in slope-intercept form. $m = \frac{5 - 1}{-4 - 4} = \frac{4}{-8} = -\frac{1}{2}$ We will use $m = -\frac{1}{2}$ and the point $(4, 1)$ in the point-slope formula. $y - y_1 = m(x - x_1)$ Substitute $-\frac{1}{2}$ for m. Substitute $(4, 1)$ for (x_1, y_1). $y - 1 = -\frac{1}{2}(x - 4)$ Substitute. $y - 1 = -\frac{1}{2}x + 2$ Distribute. $y = -\frac{1}{2}x + 3$ Slope-intercept form
The equation of a **horizontal line** containing the point (c, d) is $y = d$. The equation of a **vertical line** containing the point (c, d) is $x = c$. **(p. 254)**	The equation of a horizontal line containing the point $(3, -2)$ is $y = -2$. The equation of a vertical line containing the point $(6, 4)$ is $x = 6$.
To write an equation of the line parallel or perpendicular to a given line, we must first find the slope of the given line. **(p. 254)**	Write an equation of the line parallel to $4x - 5y = 20$ containing the point $(4, -3)$. Express the answer in slope-intercept form. Find the slope of $4x - 5y = 20$. $-5y = -4x + 20$ $y = \frac{4}{5}x - 4$ $m = \frac{4}{5}$ The slope of the parallel line is also $\frac{4}{5}$. Since this line contains $(4, -3)$, use the point-slope formula to write its equation. $y - y_1 = m(x - x_1)$ $y - (-3) = \frac{4}{5}(x - 4)$ Substitute values. $y + 3 = \frac{4}{5}x - \frac{16}{5}$ Distribute. $y = \frac{4}{5}x - \frac{31}{5}$ Slope-intercept form

Definition/Procedure	Example

4.6 Introduction to Functions

A **relation** is any set of ordered pairs. A relation can also be represented as a correspondence or mapping from one set to another. **(p. 262)**

The **domain** of a relation is the set of values of the independent variable (the first coordinates in the set of ordered pairs).

The **range** of a relation is the set of all values of the dependent variable (the second coordinates in the set of ordered pairs). **(p. 263)**

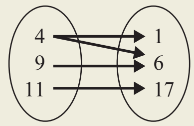
A **function** is a relation in which each element of the domain corresponds to exactly one element of the range.

Alternative definition: A relation is a **function** if each x-value corresponds to one y-value. **(p. 263)**

Relations:

a) $\{(-4, -12), (-1, -3), (3, 9), (5, 15)\}$

b)



In a), the domain is $\{-4, -1, 3, 5\}$, and the range is $\{-12, -3, 9, 15\}$.

In b), the domain is $\{4, 9, 11\}$, and the range is $\{1, 6, 17\}$.

The relation in a) *is* a function.

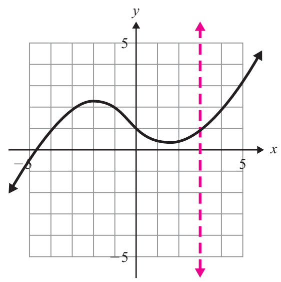
The relation in b) *is not* a function.

The Vertical Line Test

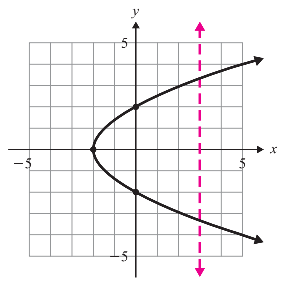
If no vertical line can be drawn through a graph that intersects the graph more than once, then the graph represents a function.

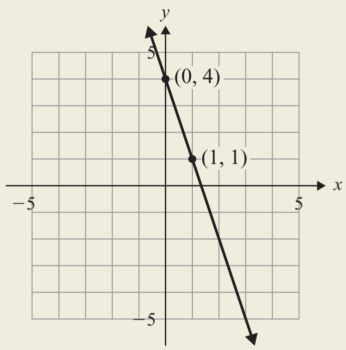
If a vertical line *can* be drawn that intersects the graph more than once, then the graph does not represent a function. **(p. 264)**

This graph represents a function. Any vertical line you can draw through the graph can intersect it only once.



This is not the graph of a function. We can draw a vertical line through the graph that intersects it more than once.



Definition/Procedure	Example
If a relation is written as an equation with y in terms of x, then the domain of the relation is the set of all real numbers that can be substituted for the independent variable x. **To determine the domain of a relation,** use these tips. 1) Ask yourself, "Is there any number that *cannot* be substituted for x?" 2) If x is in the denominator of a fraction, determine what value of x will make the denominator equal 0 by setting the denominator equal to zero. Solve for x. This x-value is *not* in the domain. **(p. 267)**	Determine the domain of $f(x) = \frac{9}{x+8}$. First, determine the value of x that will make the denominator equal zero. $x + 8 = 0$ $x = -8$ When $x = -8$, the denominator of $f(x) = \frac{9}{x+8}$ equals zero. The domain contains all real numbers *except* -8. The domain of the function is $(-\infty, -8) \cup (-8, \infty)$.
If a function describes the relationship between x and y so that x is the independent variable and y is the dependent variable, then y is a function of x. The **function notation $y = f(x)$** is read as "y equals f of x." Finding a function value means evaluating the function for the given value of the variable. **(p. 268)**	If $f(x) = 9x - 4$, find $f(2)$. Substitute 2 for x and evaluate. $f(2) = 9(2) - 4 = 18 - 4 = 14$ Therefore, $f(2) = 14$.
A **linear function** has the form $$\boldsymbol{f(x) = mx + b}$$ where m and b are real numbers, m is the slope of the line, and $(0, b)$ is the y-*intercept*. The domain of a linear function is all real numbers. **(p. 271)**	Graph $f(x) = -3x + 4$ using the slope and y-intercept. The slope is -3 and the y-intercept is $(0, 4)$. Plot the y-intercept and use the slope to locate another point on the line. 

Chapter 4: Review Exercises

(4.1) Determine whether each ordered pair is a solution of the given equation.

1) $5x - y = 13$; $(2, -3)$

2) $2x + 3y = 8$; $(-1, 5)$

3) $y = -\frac{4}{3}x + \frac{7}{3}$; $(4, -3)$

4) $x = 6$; $(6, 2)$

Complete the ordered pair for each equation.

5) $y = -2x + 4$; $(-5, \quad)$

6) $y = \frac{5}{2}x - 3$; $(6, \quad)$

7) $y = -9$; $(7, \quad)$

8) $8x - 7y = -10$; $(\quad, 4)$

Complete the table of values for each equation.

9) $y = x - 14$

x	y
0	
6	
−3	
−8	

10) $3x - 2y = 9$

x	y
	0
0	
2	
	−1

Plot the ordered pairs on the same coordinate system.

11) a) $(4, 0)$ b) $(-2, 3)$
c) $(5, 1)$ d) $(-1, -4)$

12) a) $(0, -3)$ b) $(-4, 4)$
c) $(1, \frac{3}{2})$ d) $(-\frac{1}{3}, -2)$

13) The cost of renting a pick-up for one day is given by $y = 0.5x + 45.00$, where x represents the number of miles driven and y represents the cost, in dollars.

a) Complete the table of values, and write the information as ordered pairs.

x	y
10	
18	
29	
36	

b) Label a coordinate system, choose an appropriate scale, and graph the ordered pairs.

c) Explain the meaning of the ordered pair $(58, 74)$ in the context of the problem.

14) Fill in the blank with positive, negative, or zero.

a) The x-coordinate of every point in quadrant III is _________.

b) The y-coordinate of every point in quadrant II is _______.

(4.2) Complete the table of values and graph each equation.

15) $y = -2x + 4$

x	y
0	
1	
2	
3	

16) $2x + 3y = 6$

x	y
0	
3	
−2	
−3	

Graph each equation by finding the intercepts and at least one other point.

17) $x - 2y = 2$

18) $3x - y = -3$

19) $y = -\frac{1}{2}x + 1$

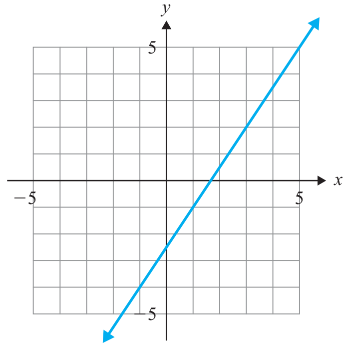
20) $2x + y = 0$

21) $y = 4$

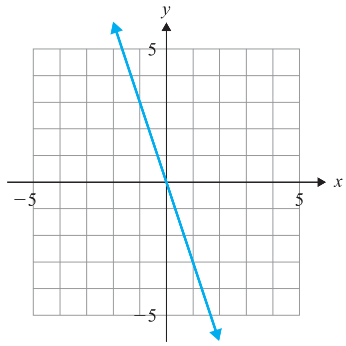
22) $x = -1$

(4.3) Determine the slope of each line.

23)



24)



Use the slope formula to find the slope of the line containing each pair of points.

25) $(5, 8)$ and $(1, -12)$

26) $(-3, 4)$ and $(1, -1)$

27) $(-7, -2)$ and $(2, 4)$

28) $(7, 3)$ and $(15, 1)$

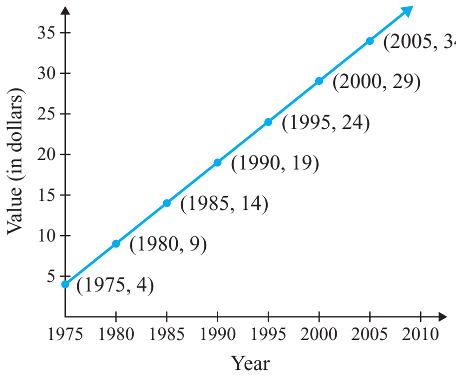
29) $\left(-\frac{1}{4}, 1\right)$ and $\left(\frac{3}{4}, -6\right)$

30) $(3.7, 2.3)$ and $(5.8, 6.5)$

31) $(-2, 5)$ and $(4, 5)$

32) $(-9, 3)$ and $(-9, 2)$

33) Christine collects old record albums. The graph shows the value of an original, autographed copy of one of her albums from 1975.



a) How much did she pay for the album in 1975?

b) Is the slope of the line positive or negative? What does the sign of the slope mean in the context of the problem?

c) Find the slope. What does it mean in the context of the problem?

Graph the line containing the given point and with the given slope.

34) $(3, -4)$; $m = 2$

35) $(-2, 2)$; $m = -3$

36) $(1, 3)$; $m = -\frac{1}{2}$

37) $(-4, 1)$; slope undefined

38) $(-2, -3)$; $m = 0$

(4.4) Identify the slope and *y*-intercept, then graph the line.

39) $y = -x + 5$

40) $y = 4x - 2$

41) $y = \frac{2}{5}x - 6$

42) $y = -\frac{1}{2}x + 5$

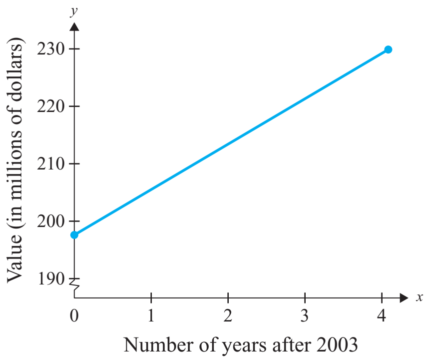
43) $x + 3y = -6$

44) $18 = 6y - 15x$

45) $x + y = 0$

46) $y + 6 = 1$

47) The value of the squash crop in the United States since 2003 can be modeled by $y = 7.9x + 197.6$, where x represents the number of years after 2003, and y represents the value of the crop in millions of dollars. (U.S. Dept. of Agriculture)



a) What is the *y*-intercept? What does it mean in the context of the problem?

b) Has the value of the squash crop been increasing or decreasing since 2003? By how much per year?

c) Use the graph to estimate the value of the squash crop in the United States in 2005. Then use the equation to determine this number.

Determine whether each pair of lines is parallel, perpendicular, or neither.

48) $y = \frac{3}{5}x - 8$
$5x + 3y = 3$

49) $x - 4y = 20$
$-x + 4y = 6$

50) $5x + y = 4$
$2x + 10y = 1$

51) $x = 7$
$y = -3$

52) Write the point-slope formula for the equation of a line with slope m and which contains the point (x_1, y_1).

Write the *slope-intercept form* of the equation of the line, if possible, given the following information.

53) $m = 6$ and contains $(-1, 4)$

54) $m = -5$ and *y*-intercept $(0, -3)$

55) $m = -\frac{3}{4}$ and *y*-intercept $(0, 7)$

56) contains $(-4, 2)$ and $(-2, 5)$

57) contains $(4, 1)$ and $(6, -3)$

58) $m = \frac{2}{3}$ and contains $(5, -2)$

59) horizontal line containing $(3, 7)$

60) vertical line containing $(-5, 1)$

Write the *standard form* of the equation of the line given the following information.

61) contains $(4, 5)$ and $(-1, -10)$

62) $m = -\frac{1}{2}$ and contains $(3, 0)$

63) $m = \frac{5}{2}$ and contains $\left(1, -\frac{3}{2}\right)$

64) contains $(-4, 1)$ and $(4, 3)$

65) $m = -4$ and y-intercept $(0, 0)$

66) $m = -\frac{3}{7}$ and y-intercept $(0, 1)$

67) contains $(6, 1)$ and $(2, 5)$

68) $m = \frac{3}{4}$ and contains $\left(-2, \frac{7}{2}\right)$

69) Mr. Romanski works as an advertising consultant, and his salary has been growing linearly. In 2005 he earned \$62,000, and in 2010 he earned \$79,500. Let y represent Mr. Romanski's salary, in dollars, x years after 2005.

a) Write a linear equation to model these data.

b) Explain the meaning of the slope in the context of the problem.

c) How much did he earn in 2008?

d) If the trend continues, in what year could he expect to earn \$93,500?

Write an equation of the line *parallel* to the given line and containing the given point. Write the answer in slope-intercept form or in standard form, as indicated.

70) $y = 2x + 10$; $(2, -5)$; slope-intercept form

71) $y = -8x + 8$; $(-1, 14)$; slope-intercept form

72) $3x + y = 5$; $(-3, 5)$; standard form

73) $x - 2y = 6$; $(4, 11)$; standard form

74) $3x + 4y = 1$; $(-1, 2)$; slope-intercept form

75) $x + 5y = 10$; $(15, 7)$; slope-intercept form

Write an equation of the line *perpendicular* to the given line and containing the given point. Write the answer in slope-intercept form or in standard form, as indicated.

76) $y = -\frac{1}{5}x + 7$; $(1, 7)$; slope-intercept form

77) $y = -x + 9$; $(3, -9)$; slope-intercept form

78) $4x - 3y = 6$; $(8, -5)$; slope-intercept form

79) $2x + 3y = -3$; $(-4, -4)$; slope-intercept form

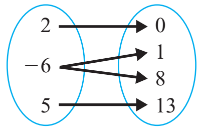
80) $x + 8y = 8$; $(-2, -7)$; standard form

81) Write an equation of the line parallel to $y = 5$ containing $(8, 4)$

82) Write an equation of the line perpendicular to $x = -2$ containing $(4, -3)$.

(4.6) Identify the domain and range of each relation, and determine whether each relation is function.

83) $\{(-3, 1), (5, 3), (5, -3), (12, 4)\}$

84)


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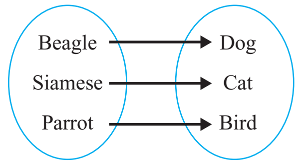

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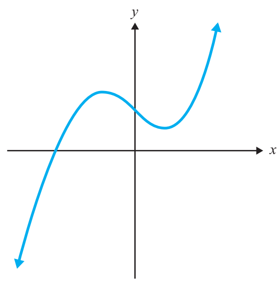

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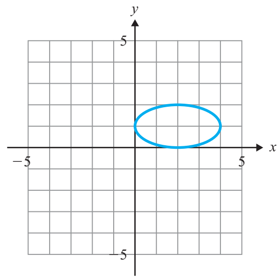


Determine the domain of each relation, and determine whether each relation describes y as a function of x.

88) $y = 4x - 7$

89) $y = \frac{8}{x + 3}$

90) $y = \frac{15}{x}$

91) $y^2 = x$

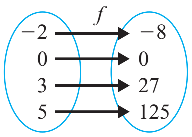
92) $y = x^2 - 6$

93) $y = \frac{5}{7x - 2}$

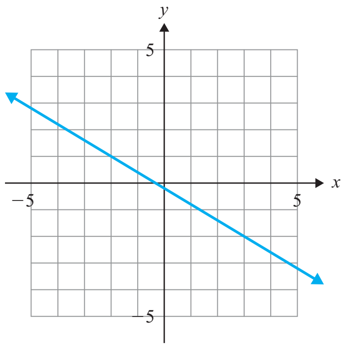
For each function, f, find $f(3)$ and $f(-2)$.

94) $f = \{(-7, -2), (-2, -5), (1, -10), (3, -14)\}$

95)



96)



97) Let $f(x) = 5x - 12$, $g(x) = x^2 + 6x + 5$. Find each of the following and simplify.

a) $f(4)$ b) $f(-3)$

c) $g(3)$ d) $g(0)$

e) $f(a)$ f) $g(t)$

g) $f(k + 8)$ h) $f(c - 2)$

i) $f(x + h)$ j) $f(x + h) - f(x)$

98) $h(x) = -3x + 7$. Find x so that $h(x) = 19$.

99) $f(x) = \frac{3}{2}x + 5$. Find x so that $f(x) = \frac{11}{2}$.

100) Graph $f(x) = -2x + 6$ by making a table of values and plotting points.

101) Graph each function using the slope and y-intercept.

a) $f(x) = \frac{2}{3}x - 1$ b) $f(x) = -3x + 2$

102) Graph $g(x) = \frac{3}{2}x + 3$ by finding the x- and y-intercepts and one other point.

Graph each function.

103) $h(c) = -\frac{5}{2}c + 4$

104) $D(t) = 3t$

105) A USB 2.0 device can transfer data at a rate of 480 MB/sec (megabytes/second). Let $f(t) = 480t$ represent the number of megabytes of data that can be transferred in t sec. (www.usb.org)

a) How many megabytes of a file can be transferred in 2 sec? in 6 sec?

b) How long would it take to transfer a 1200 MB file?

106) A jet travels at a constant speed of 420 mph. The distance D (in miles) that the jet travels after t hr can be defined by the function

$$D(t) = 420t$$

a) Find $D(2)$, and explain what this means in the context of the problem.

b) Find t so that $D(t) = 2100$, and explain what this means in the context of the problem.

Chapter 4: Test

1) Is $(-3, -2)$ a solution of $2x - 7y = 8$?

2) Complete the table of values and graph $y = \frac{3}{2}x - 2$.

x	y
0	
−2	
4	
	1

3) Fill in the blanks with *positive* or *negative.* In quadrant IV, the x-coordinate of every point is ___________ and the y-coordinate is ___________.

4) For $3x - 4y = 6$,

a) find the x-intercept.

b) find the y-intercept.

c) find one other point on the line.

d) graph the line.

5) Graph $y = -3$.

6) Graph $x + y = 0$.

7) Find the slope of the line containing the points

a) $(3, -1)$ and $(-5, 9)$

b) $(8, 6)$ and $(11, 6)$

8) Graph the line containing the point $(-1, 4)$ with slope $-\frac{3}{2}$.

9) Graph the line containing the point $(2, 3)$ with an undefined slope.

10) Put $3x - 2y = 10$ into slope-intercept form. Then, graph the line.

11) Write the slope-intercept form for the equation of the line with slope 7 and y-intercept $(0, -10)$.

12) Write the standard form for the equation of a line with slope $-\frac{1}{3}$ containing the point $(-3, 5)$.

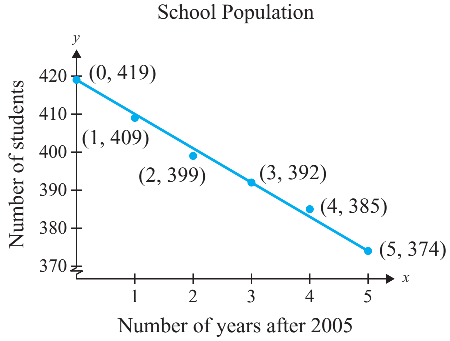
13) Determine whether $4x + 18y = 9$ and $9x - 2y = -6$ are parallel, perpendicular, or neither.

14) Find the slope-intercept form of the equation of the line

a) perpendicular to $y = 2x - 9$ containing $(-6, 10)$.

b) parallel to $3x - 4y = -4$ containing $(11, 8)$.

15) The graph shows the number of children attending a neighborhood school from 2005 to 2010. Let y represent the number of children attending the school x years after 2005.



a) According to the graph, how many children attended this school in 2007?

b) Write a linear equation (in slope-intercept form) to model these data. Use the data points for 2005 and 2010.

c) Use the equation in part b) to determine the number of students attending the school in 2007. How does your answer in part a) compare to the number predicted by the equation?

d) Explain the meaning of the slope in the context of the problem.

e) What is the y-intercept? What does it mean in the context of the problem?

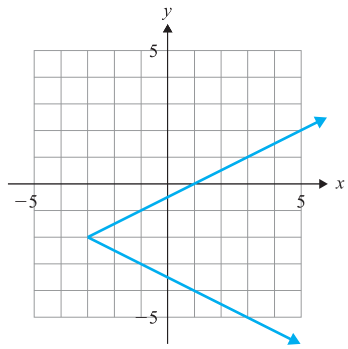
f) If the current trend continues, how many children can be expected to attend this school in 2013?

16) What is a function?

Identify the domain and range of each relation, and determine whether each relation is a function.

17) $\{(-2, -5), (1, -1), (3, 1), (8, 4)\}$

18)



For each function, (a) determine the domain. (b) Is y a function of x?

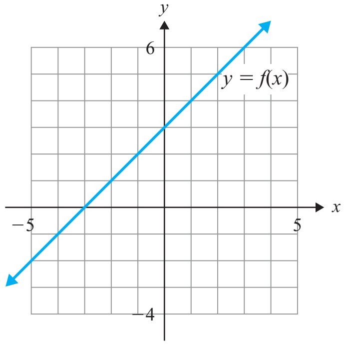
19) $y = \frac{7}{3}x - 5$

20) $y = \frac{8}{2x - 5}$

For each function, f, find $f(2)$.

21) $f = \{(-3, -8), (0, -5), (2, -3), (7, 2)\}$

22)



Let $f(x) = -4x + 2$ and $g(x) = x^2 - 3x + 7$. Find each of the following and simplify.

23) $f(6)$

24) $g(2)$

25) $g(t)$

26) $f(h - 7)$

Graph each function.

27) $h(x) = -\frac{3}{4}x + 5$

28) A USB 1.1 device can transfer data at a rate of 12 MB/sec (megabytes/second). Let $f(t) = 12t$ represent the number of megabytes of data that can be transferred in t sec. (www.usb.org)

a) How many megabytes of a file can be transferred in 3 sec?

b) How long would it take to transfer 132 MB?

Cumulative Review: Chapters 1–4

1) Write $\frac{336}{792}$ in lowest terms.

2) A rectangular picture frame measures 7 in. by 12.5 in. Find the perimeter of the frame.

Evaluate.

3) -3^4

4) $\frac{24}{35} \cdot \frac{49}{60}$

5) $\frac{3}{8} - 2$

6) $4 + 2^6 \div |5 - 13|$

7) Write an expression for "9 less than twice 17" and simplify.

Simplify. The answer should not contain any negative exponents.

8) $(5k^6)(-4k^9)$

9) $\left(\frac{30w^5}{15w^{-3}}\right)^{-4}$

Solve.

10) $-\frac{2}{5}y + 9 = 15$

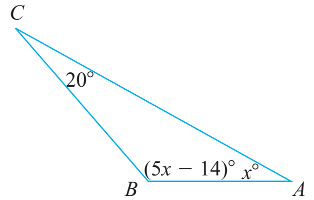
11) $\frac{3}{2}(7c - 5) - 1 = \frac{2}{3}(2c + 1)$

12) $7 + 2(p - 6) = 8(p + 3) - 6p + 1$

13) Solve. Write the solution in interval notation.
$3x + 14 \leq 7x + 4$

14) The Chase family put their house on the market for \$306,000. This is 10% less than what they paid for it 3 years ago. What did they pay for the house?

15) Find the missing angle measures.



16) Lynette's age is 7 yr less than three times her daughter's age. If the sum of their ages is 57, how old is Lynette, and how old is her daughter?

17) Find the slope of the line containing the points $(-7, 8)$ and $(2, 17)$.

18) Graph $4x + y = 5$.

19) Write an equation of the line with slope $-\frac{5}{4}$ containing the point $(-8, 1)$. Express the answer in standard form.

20) Write an equation of the line perpendicular to $y = \frac{1}{3}x + 11$ containing the point $(4, -12)$. Express the answer in slope-intercept form.

21) Determine the domain of $y = \frac{3}{x + 7}$.

Let $f(x) = 8x + 3$. Find each of the following and simplify.

22) $f(-5)$

23) $f(a)$

24) $f(t + 2)$

25) Graph $f(x) = 2$

CHAPTER 5

Solving Systems of Linear Equations

Algebra at Work: Custom Motorcycles

We will take another look at how algebra is used in a custom motorcycle shop.

Tanya took apart a transmission to make repairs when she realized that she had mixed up the gears. She was able to replace the shafts onto the bearings, but she could not remember which gear went on which shaft. Tanya measured the distance (in inches) between the shafts, sketched the layout on a piece of paper, and came up with a system of equations to determine which gear goes on which shaft.

If x = the radius of the gear on the left, y = the radius of the gear on the right, and z = the radius of the gear on the bottom, then the system of equations Tanya must solve to determine where to put each gear is

$$\begin{aligned} x + y &= 2.650 \\ x + z &= 2.275 \\ y + z &= 1.530 \end{aligned}$$

Solving this system, Tanya determines that $x = 1.698$ in., $y = 0.952$ in., and $z = 0.578$ in. Now she knows on which shaft to place each gear.

In this chapter, we will learn how to write and solve systems of two and three equations.

Section 5.1 Solving Systems by Graphing

Objectives

1. Determine Whether an Ordered Pair Is a Solution of a System
2. Solve a Linear System by Graphing
3. Solve a Linear System by Graphing: Special Cases
4. Determine the Number of Solutions of a System Without Graphing

What is a system of linear equations? A **system of linear equations** consists of two or more linear equations with the same variables. In Sections 5.1–5.3, we will learn how to solve systems of two equations in two variables. Some examples of such systems are

$$\begin{aligned} 2x + 5y &= 5 \\ x + 4y &= -1 \end{aligned} \qquad \begin{aligned} y &= \frac{1}{3}x - 8 \\ 5x - 6y &= 10 \end{aligned} \qquad \begin{aligned} -3x + y &= 1 \\ x &= -2 \end{aligned}$$

In the third system, we see that $x = -2$ is written with only one variable. However, we can think of it as an equation in two variables by writing it as $x + 0y = -2$.

It is possible to solve systems of equations containing more than two variables. In Section 5.5, we will learn how to solve systems of linear equations in three variables.

1. Determine Whether an Ordered Pair Is a Solution of a System

We will begin our work with systems of equations by determining whether an ordered pair is a solution of the system.

Definition

A **solution of a system** of two equations in two variables is an ordered pair that is a solution of each equation in the system.

Example 1

Determine whether (2, 3) is a solution of each system of equations.

a) $\begin{aligned} y &= x + 1 \\ x + 2y &= 8 \end{aligned}$ b) $\begin{aligned} 4x - 5y &= -7 \\ 3x + y &= 4 \end{aligned}$

Solution

a) If (2, 3) is a solution of $\begin{aligned} y &= x + 1 \\ x + 2y &= 8 \end{aligned}$ then when we substitute 2 for x and 3 for y, the ordered pair will make each equation true.

$$\begin{aligned} y &= x + 1 \\ 3 &\stackrel{?}{=} 2 + 1 && \text{Substitute.} \\ 3 &= 3 && \text{True} \end{aligned} \qquad \begin{aligned} x + 2y &= 8 \\ 2 + 2(3) &\stackrel{?}{=} 8 && \text{Substitute.} \\ 2 + 6 &\stackrel{?}{=} 8 \\ 8 &= 8 && \text{True} \end{aligned}$$

Since (2, 3) is a solution of each equation, it is a solution of the system.

b) We will substitute 2 for x and 3 for y to see whether (2, 3) satisfies (is a solution of) each equation.

$$\begin{aligned} 4x - 5y &= -7 \\ 4(2) - 5(3) &\stackrel{?}{=} -7 && \text{Substitute.} \\ 8 - 15 &\stackrel{?}{=} -7 \\ -7 &= -7 && \text{True} \end{aligned} \qquad \begin{aligned} 3x + y &= 4 \\ 3(2) + 3 &\stackrel{?}{=} 4 && \text{Substitute.} \\ 6 + 3 &\stackrel{?}{=} 4 \\ 9 &= 4 && \text{False} \end{aligned}$$

Although (2, 3) is a solution of the first equation, it does *not* satisfy $3x + y = 4$. Therefore, (2, 3) is *not* a solution of the system.

You Try 1

Determine whether $(-4, 3)$ is a solution of each system of equations.

a) $3x + 5y = 3$
$-2x - y = -5$

b) $y = \frac{1}{2}x + 5$
$-x + 3y = 13$

If we are given a system and no solution is given, how do we *find* the solution to the system of equations? In this chapter, we will discuss three methods for solving systems of equations:

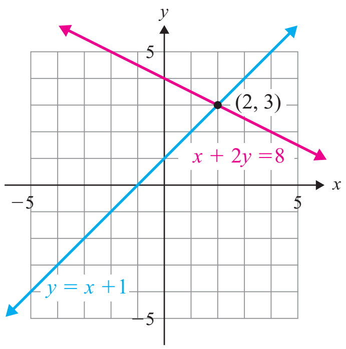
1) Graphing (this section)
2) Substitution (Section 5.2)
3) Elimination (Section 5.3)

Let's begin with the graphing method.

2. Solve a Linear System by Graphing

To **solve a system of equations in two variables** means to find the ordered pair (or pairs) that satisfies each equation in the system.

Recall from Chapter 4 that the graph of a linear equation is a line. This line represents all solutions of the equation.



If two lines intersect at one point, that point of intersection is a solution of each equation.

For example, the graph shows the lines representing the two equations in Example 1(a). The solution to that system is their point of intersection, (2, 3).

Definition

When solving a system of equations by graphing, the point of intersection is the solution of the system. If a system has at least one solution, we say that the system is **consistent.** The equations are **independent** if the system has one solution.

Example 2

Solve the system by graphing.

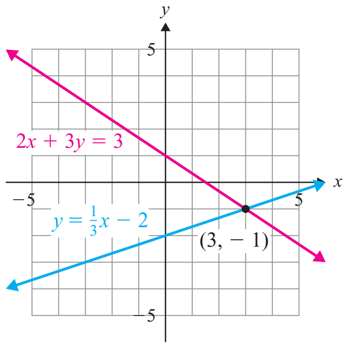
$$y = \frac{1}{3}x - 2$$
$$2x + 3y = 3$$

Solution

Graph each line on the same axes. The first equation is in slope-intercept form, and we see that $m = \frac{1}{3}$ and $b = -2$. Its graph is in blue.

Let's graph $2x + 3y = 3$ by plotting points.

x	y
0	1
-3	3
3	-1



The point of intersection is $(3, -1)$. Therefore, the solution to the system is $(3, -1)$.

This is a consistent system. ■

Note

It is important that you use a straightedge to graph the lines. If the graph is not precise, it will be difficult to correctly locate the point of intersection. Furthermore, if the solution of a system contains numbers that are not integers, it may be impossible to accurately read the point of intersection. This is one reason why solving a system by graphing is not always the best way to find the solution. But it can be a useful method, and it is one that is used to solve problems not only in mathematics, but in areas such as business, economics, and chemistry as well.

You Try 2

Solve the system by graphing.

$$3x + 2y = 2$$
$$y = \frac{1}{2}x - 3$$

3. Solve a Linear System by Graphing: Special Cases

Do two lines *always* intersect? No! Then if we are trying to solve a system of two linear equations by graphing and the graphs do not intersect, what does this tell us about the solution to the system?

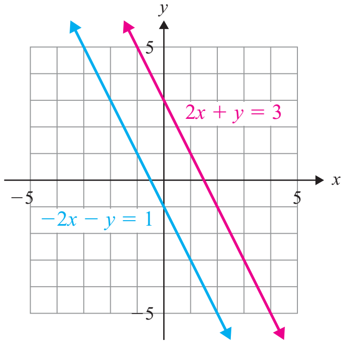
Example 3

Solve the system by graphing.

$$-2x - y = 1$$
$$2x + y = 3$$

Solution

Graph each line on the same axes.



The lines are parallel; they will never intersect. Therefore, there is *no solution* to the system. We write the solution set as $\varnothing$.

Definition

When solving a system of equations by graphing, if the lines are parallel, then the system has **no solution.** We write this as $\varnothing$. Furthermore, a system that has no solution is **inconsistent,** and the equations are **independent.**

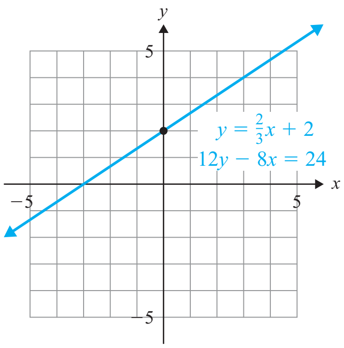
What if the graphs of the equations in a system are the same line?

Example 4

Solve the system by graphing.

$$y = \frac{2}{3}x + 2$$
$$12y - 8x = 24$$

Solution



If we write the second equation in slope-intercept form, we see that it is the same as the first equation. This means that the graph of each equation is the same line. Therefore, each point on the line satisfies each equation. The system has an *infinite number of solutions* of the form $y = \frac{2}{3}x + 2$.

The solution set is $\left\{(x, y) \,\middle|\, y = \frac{2}{3}x + 2\right\}$.

We read this as "the set of all ordered pairs (x, y) such that $y = \frac{2}{3}x + 2$."

We could have used either equation to write the solution set in Example 4. However, we will use either the equation that is written in slope-intercept form or the equation written in standard form with integer coefficients that have no common factor other than 1.

Definition

When solving a system of equations by graphing, if the graph of each equation is the same line, then the system has an **infinite number of solutions.** The system is **consistent,** and the equations are **dependent.**

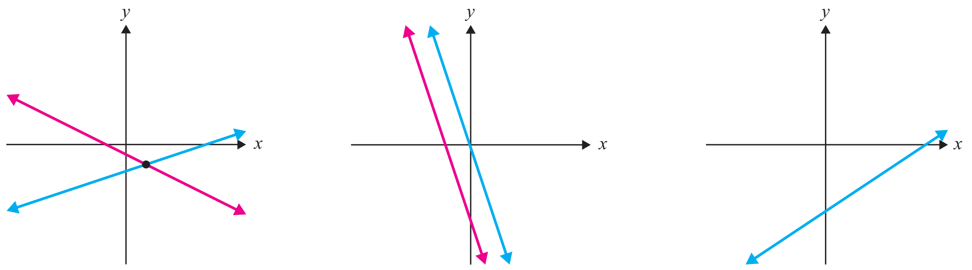
We will summarize what we have learned so far about solving a system of linear equations by graphing:

Procedure Solving a System by Graphing

To solve a system by graphing, graph each line on the same axes.

1) If the lines intersect at a single point, then the point of intersection is the solution of the system. The system is *consistent*, and the equations are *independent*. (See Figure 5.1a.)
2) If the lines are parallel, then the system has *no solution*. We write the solution set as $\varnothing$. The system is *inconsistent*. The equations are *independent*. (See Figure 5.1b.)
3) If the graphs are the same line, then the system has an *infinite number of solutions*. We say that the system is *consistent*, and the equations are *dependent*. (See Figure 5.1c.)

Figure 5.1



a) One solution—the point of intersection Consistent system Independent equations

b) No solution Inconsistent system Independent equations

c) Infinite number of solutions Consistent system Dependent equations

You Try 3

Solve each system by graphing.

a) $-x = y + 4$
$x + y = 1$

b) $2x - 6y = 9$
$-4x + 12y = -18$

4. Determine the Number of Solutions of a System Without Graphing

The graphs of lines can lead us to the solution of a system. We can also determine the number of solutions a system has without graphing.

We saw in Example 4 that if a system has lines with the same slope and the same y-intercept (they are the same line), then the system has an *infinite number of solutions*.

Example 3 shows that if a system contains lines with the same slope and different y-intercepts, then the lines are parallel and the system has *no solution*.

Finally, we learned in Example 2 that if the lines in a system have different slopes, then they will intersect and the system has *one solution*.

Example 5

Without graphing, determine whether each system has no solution, one solution, or an infinite number of solutions.

a) $y = \frac{3}{4}x + 7$
$5x + 8y = -8$

b) $4x - 8y = 10$
$-6x + 12y = -15$

c) $9x + 6y = -13$
$3x + 2y = 4$

Solution

a) The first equation is already in slope-intercept form, so write the second equation in slope-intercept form.

$$\begin{aligned} 5x + 8y &= -8 \\ 8y &= -5x - 8 \\ y &= -\frac{5}{8}x - 1 \end{aligned}$$

The slopes, $\frac{3}{4}$ and $-\frac{5}{8}$, are different; therefore, this system has *one solution.*

b) Write each equation in slope-intercept form.

$$\begin{aligned} 4x - 8y &= 10 \\ -8y &= -4x + 10 \\ y &= \frac{-4}{-8}x + \frac{10}{-8} \\ y &= \frac{1}{2}x - \frac{5}{4} \end{aligned}$$

$$\begin{aligned} -6x + 12y &= -15 \\ 12y &= 6x - 15 \\ y &= \frac{6}{12}x - \frac{15}{12} \\ y &= \frac{1}{2}x - \frac{5}{4} \end{aligned}$$

The equations are the same: they have the same slope and y-intercept. Therefore, this system has an *infinite number of solutions.*

c) Write each equation in slope-intercept form.

$$\begin{aligned} 9x + 6y &= -13 \\ 6y &= -9x - 13 \\ y &= \frac{-9}{6}x - \frac{13}{6} \\ y &= -\frac{3}{2}x - \frac{13}{6} \end{aligned}$$

$$\begin{aligned} 3x + 2y &= 4 \\ 2y &= -3x + 4 \\ y &= \frac{-3}{2}x + \frac{4}{2} \\ y &= -\frac{3}{2}x + 2 \end{aligned}$$

The equations have the same slope but different y-intercepts. If we graphed them, the lines would be parallel. Therefore, this system has *no solution.* ■

You Try 4

Without graphing, determine whether each system has no solution, one solution, or an infinite number of solutions.

a) $-2x = 4y - 8$
$x + 2y = -6$

b) $y = -\frac{5}{6}x + 1$
$10x + 12y = 12$

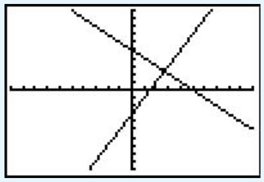
c) $-5x + 3y = 12$
$3x - y = 2$

Using Technology

In this section, we have learned that the solution of a system of equations is the point at which their graphs intersect. We can solve a system by graphing using a graphing calculator. On the calculator, we will solve the following system by graphing:

$$x + y = 5$$
$$y = 2x - 3$$

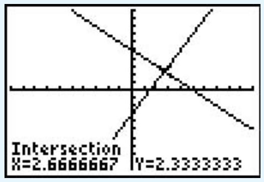
Begin by entering each equation using the $\boxed{\text{Y=}}$ key. Before entering the first equation, we must solve for *y*.



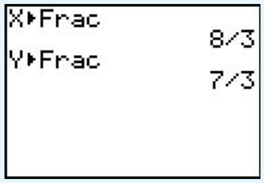
$$x + y = 5$$
$$y = -x + 5$$

Enter $-x + 5$ in Y1 and $2x - 3$ in Y2, press $\boxed{\text{ZOOM}}$, and select 6: ZStandard to graph the equations.

Since the lines intersect, the system has a solution. How can we find that solution? Once you see from the graph that the lines intersect, press 2nd $\boxed{\text{TRACE}}$. Select 5: intersect and then press $\boxed{\text{ENTER}}$ three times. The screen will move the cursor to the point of intersection and display the solution to the system of equations on the bottom of the screen.



To obtain the exact solution to the system of equations, first return to the home screen by pressing 2nd $\boxed{\text{MODE}}$. To display the *x*-coordinate of the solution, press $\boxed{\text{X, T, }\Theta\text{, n}}$ MATH $\boxed{\text{ENTER}}$ $\boxed{\text{ENTER}}$, and to display the *y*-coordinate of the solution, press ALPHA $\boxed{1}$ MATH $\boxed{\text{ENTER}}$ $\boxed{\text{ENTER}}$. The solution to the system is $\left(\frac{8}{3}, \frac{7}{3}\right)$.



Use a graphing calculator to solve each system.

1) $y = x + 4$
   $y = -x + 2$

2) $y = -3x + 7$
   $y = x - 5$

3) $y = -4x - 2$
   $y = x + 5$

4) $5x + y = -1$
   $4x - y = 2$

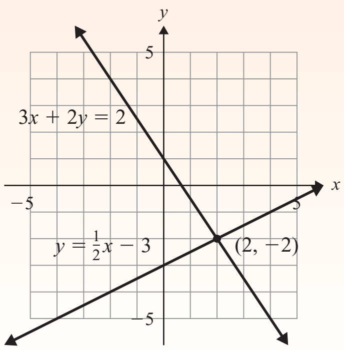
5) $5x + 2y = 7$
   $2x + 4y = 3$

6) $3x + 2y = -2$
   $-x - 3y = -5$

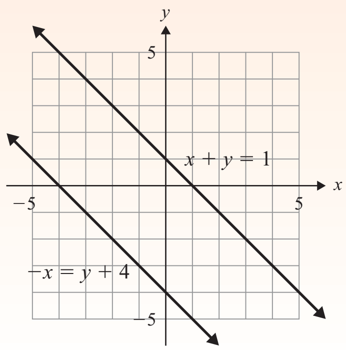
Answers to You Try Exercises

1) a) no b) yes

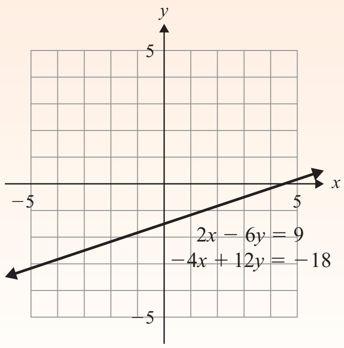
2) $(2, -2)$



3) a) ∅



b) infinite number of solutions of the form $\{(x, y) \mid 2x - 6y = 9\}$



4) a) no solution b) infinite number of solutions c) one solution

Answers to Technology Exercises

1) $(-1, 3)$ 2) $(3, -2)$ 3) $\left(-\frac{7}{5}, \frac{18}{5}\right)$

4) $\left(\frac{1}{9}, -\frac{14}{9}\right)$ 5) $\left(\frac{11}{8}, \frac{1}{16}\right)$ 6) $\left(-\frac{16}{7}, \frac{17}{7}\right)$

5.1 Exercises

Objective 1: Determine Whether an Ordered Pair Is a Solution of a System

Determine whether the ordered pair is a solution of the system of equations.

1) $x + 2y = -6$
$-x - 3y = 13$
$(8, -7)$

2) $y - x = 4$
$x + 3y = 8$
$(-1, 3)$

3) $5x + y = 21$
$2x - 3y = 11$
$(4, 1)$

4) $7x + 2y = 14$
$-5x + 6y = -12$
$(2, 0)$

5) $5y - 4x = -5$
$6x + 2y = -21$
$\left(-\frac{5}{2}, -3\right)$

6) $x = 9y - 7$
$18y = 7x + 4$
$\left(-1, \frac{2}{3}\right)$

7) $y = -x + 11$
$x = 5y - 2$
$(0, 9)$

8) $x = -y$
$y = \frac{5}{8}x - 13$
$(8, -8)$

Mixed Exercises: Objectives 2 and 3

9) If you are solving a system of equations by graphing, how do you know whether the system has no solution?

10) If you are solving a system of equations by graphing, how do you know whether the system has an infinite number of solutions?

Solve each system of equations by graphing. If the system is inconsistent or the equations are dependent, identify this.

VIDEO 11) $y = -\frac{2}{3}x + 3$
$y = x - 2$

12) $y = \frac{1}{2}x + 2$
$y = 2x - 1$

13) $y = x + 1$
$y = -\frac{1}{2}x + 4$

14) $y = -2x + 3$
$y = x - 3$

15) $x + y = -1$
$x - 2y = 14$

16) $2x - 3y = 6$
$x + y = -7$

17) $x - 2y = 7$
$-3x + y = -1$

18) $-x + 2y = 4$
$3x + 4y = -12$

VIDEO 19) $\frac{3}{4}x - y = 0$
$3x - 4y = 20$

20) $y = -x$
$4x + 4y = 2$

21) $y = \frac{1}{3}x - 2$
$4x - 12y = 24$

22) $5x + 5y = 5$
$x + y = 1$

23) $x = 8 - 4y$
$3x + 2y = 4$

24) $x - y = 0$
$7x - 3y = 12$

VIDEO 25) $y = -3x + 1$
$12x + 4y = 4$

26) $2x - y = 1$
$-2x + y = -3$

27) $x + y = 0$
$y = \frac{1}{2}x + 3$

28) $x = -2$
$y = -\frac{5}{2}x - 1$

29) $-3x + y = -4$
$y = -1$

30) $5x + 2y = 6$
$-15x - 6y = -18$

31) $y = \frac{3}{5}x - 6$
$-3x + 5y = 10$

32) $y - x = -2$
$2x + y = -5$

Write a system of equations so that the given ordered pair is a solution of the system.

33) (2, 5)

34) (3, 1)

35) (−4, −3)

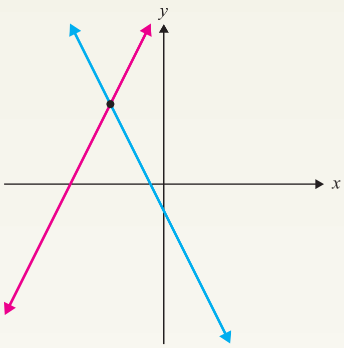
36) (6, −1)

37) $\left(-\frac{1}{3}, 4\right)$

38) $\left(0, \frac{3}{2}\right)$

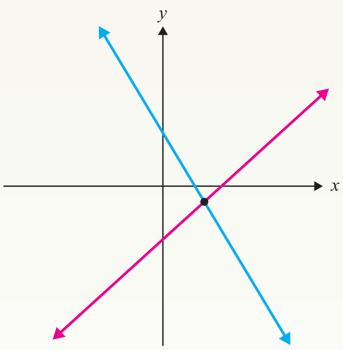
For Exercises 39–42, determine which ordered pair could be a solution to the system of equations that is graphed. Explain why you chose that ordered pair.

39)



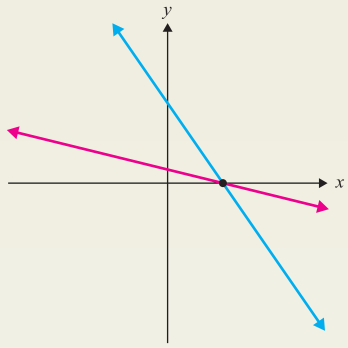
A. (2, −6) C. (−3, 4)
B. (3, 4) D. (−2, −3)

40)



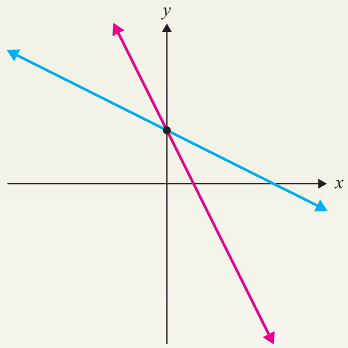
A. $\left(\frac{7}{2}, -\frac{1}{2}\right)$ C. $\left(\frac{9}{4}, \frac{3}{4}\right)$
B. (−4, −1) D. $\left(-\frac{10}{3}, -\frac{2}{3}\right)$

41)



A. (0, 3.8) C. (−2.1, 0)
B. (4.1, 0) D. (0, 5)

42)



A. (4, 0) C. (0, −3)
B. $\left(\frac{1}{3}, 0\right)$ D. (0, 2)

Objective 4: Determine the Number of Solutions of a System Without Graphing

43) How do you determine, *without graphing*, that a system of equations has exactly one solution?

44) How do you determine, *without graphing*, that a system of equations has no solution?

Without graphing, determine whether each system has no solution, one solution, or an infinite number of solutions.

45) $y = 5x - 4$
$y = -3x + 7$

46) $y = \frac{2}{3}x + 9$
$y = \frac{2}{3}x + 1$

47) $y = -\frac{3}{8}x + 1$
$6x + 16y = -9$

48) $y = -\frac{1}{4}x + 3$
$2x + 8y = 24$

49) $-15x + 9y = 27$
$10x - 6y = -18$

50) $7x - y = 6$
$x + y = 13$

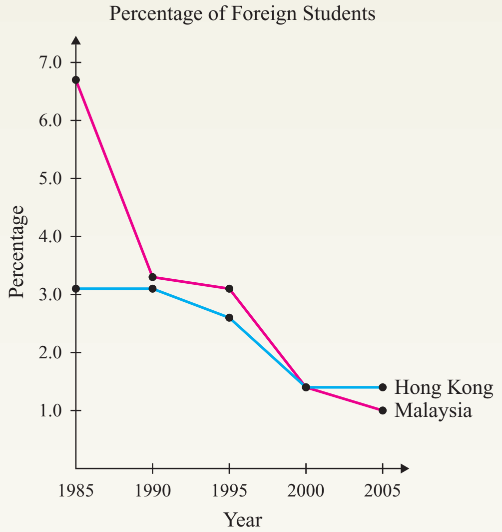
51) $3x + 12y = 9$
$x - 4y = 3$

52) $6x - 4y = -10$
$-21x + 14y = 35$

53) $x = 5$
$x = -1$

54) $y = x$
$y = 2$

55) The graph shows the percentage of foreign students in U.S. institutions of higher learning from Hong Kong and Malaysia from 1980 to 2005. (http://nces.ed.gov)



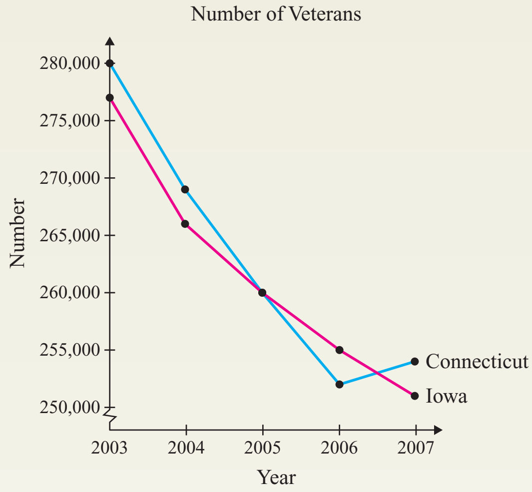
a) When was there a greater percentage of students from Malaysia?

b) Write the point of intersection of the graphs as an ordered pair in the form (year, percentage) and explain its meaning.

c) During which years did the percentage of students from Hong Kong remain the same?

d) During which years did the percentage of students from Malaysia decrease the most? How can this be related to the slope of this line segment?

56) The graph shows the approximate number of veterans living in Connecticut and Iowa from 2003 to 2007. (www.census.gov)



a) In which year were there fewer veterans living in Connecticut than in Iowa? Approximately how many were living in each state?

b) Write the data point for Iowa in 2003 as an ordered pair of the form (year, number) and explain its meaning.

c) Write the point of intersection of the graphs for the year 2005 as an ordered pair in the form (year, number) and explain its meaning.

d) Which line segment on the Connecticut graph has a positive slope? How can this be explained in the context of this problem?

Solve each system using a graphing calculator.

57) $y = -2x + 2$
$y = x - 7$

58) $y = x + 1$
$y = 3x + 3$

59) $x - y = 3$
$x + 4y = 8$

60) $2x + 3y = 3$
$y - x = -4$

61) $4x + 5y = -17$
$3x - 7y = 4.45$

62) $-5x + 6y = 22.8$
$3x - 2y = -5.2$

Section 5.2 Solving Systems by the Substitution Method

Objectives

1. Solve a Linear System by Substitution
2. Solve a System Containing Fractions or Decimals
3. Solve a System by Substitution: Special Cases

In Section 5.1, we learned to solve a system of equations by graphing. This method, however, is not always the *best* way to solve a system. If your graphs are not precise, you may read the solution incorrectly. And, if a solution consists of numbers that are not integers, like $\left(\frac{2}{3}, -\frac{1}{4}\right)$, it may not be possible to accurately identify the point of intersection of the graphs.

1. Solve a Linear System by Substitution

Another way to solve a system of equations is to use the *substitution method.* When we use the **substitution method,** we solve one of the equations for one of the variables in terms of the other. Then we substitute that expression into the other equation. We can do this because solving a system means finding the ordered pair, or pairs, that satisfy *both* equations. *The substitution method is especially good when one of the variables has a coefficient of 1 or −1.*

Example 1

Solve the system using substitution.

$$\begin{aligned} 2x + 3y &= -1 \\ y &= 2x - 3 \end{aligned}$$

Solution

The second equation is already solved for y; it tells us that y *equals* $2x - 3$. Therefore, we can substitute $2x - 3$ for y in the first equation, then solve for x.

$$\begin{aligned} 2x + 3y &= -1 && \text{First equation} \\ 2x + 3(2x - 3) &= -1 && \text{Substitute.} \\ 2x + 6x - 9 &= -1 && \text{Distribute.} \\ 8x - 9 &= -1 \\ 8x &= 8 \\ x &= 1 \end{aligned}$$

We have found that $x = 1$, but we still need to find y. Substitute $x = 1$ into *either* equation, and solve for y. In this case, we will substitute $x = 1$ into the second equation since it is already solved for y.

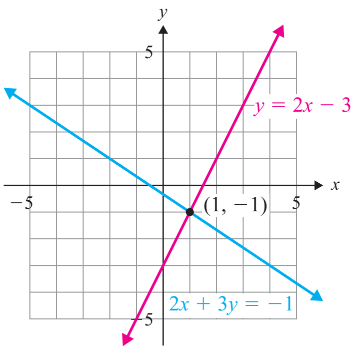
$$\begin{aligned} y &= 2x - 3 && \text{Second equation} \\ y &= 2(1) - 3 && \text{Substitute.} \\ y &= 2 - 3 \\ y &= -1 \end{aligned}$$

Check $x = 1$, $y = -1$ in *both* equations.

$$\begin{aligned} 2x + 3y &= -1 \\ 2(1) + 3(-1) &\stackrel{?}{=} -1 && \text{Substitute.} \\ 2 - 3 &\stackrel{?}{=} -1 \\ -1 &= -1 && \text{True} \end{aligned}$$

$$\begin{aligned} y &= 2x - 3 \\ -1 &\stackrel{?}{=} 2(1) - 3 && \text{Substitute.} \\ -1 &\stackrel{?}{=} 2 - 3 \\ -1 &= -1 && \text{True} \end{aligned}$$

We write the solution as an ordered pair, $(1, -1)$.



If we solve the system in Example 1 by graphing, we can see that the lines intersect at $(1, -1)$, giving us the same solution we obtained using the substitution method.

Let's summarize the steps we use to solve a system by the substitution method:

Procedure Solving a System by Substitution

1) Solve one of the equations for one of the variables. If possible, solve for a variable that has a coefficient of 1 or -1.
2) Substitute the expression found in *step 1* into the *other* equation. The equation you obtain should contain only one variable.
3) Solve the equation you obtained in *step 2*.
4) Substitute the value found in *step 3* into either of the equations to obtain the value of the other variable.
5) Check the values in each of the original equations, and write the solution as an ordered pair.

Example 2

Solve the system by substitution.

$$\begin{aligned} x - 2y &= 7 &\quad (1) \\ 2x + 3y &= -21 &\quad (2) \end{aligned}$$

Solution

We will follow the steps listed above.

1) For which variable should we solve? The x in the first equation is the only variable with a coefficient of 1 or -1. Therefore, we will solve the first equation for x.

$$\begin{aligned} x - 2y &= 7 &\quad \text{First equation (1)} \\ x &= 2y + 7 &\quad \text{Add } 2y. \end{aligned}$$

2) Substitute $2y + 7$ for the x in equation (2).

$$\begin{aligned} 2x + 3y &= -21 &\quad \text{Second equation (2)} \\ 2(2y + 7) + 3y &= -21 &\quad \text{Substitute.} \end{aligned}$$

3) Solve this new equation for y.

$$\begin{aligned} 2(2y + 7) + 3y &= -21 \\ 4y + 14 + 3y &= -21 &\quad \text{Distribute.} \\ 7y + 14 &= -21 \\ 7y &= -35 \\ y &= -5 \end{aligned}$$

4) To determine the value of x, we can substitute -5 for y in either equation. We will use equation (1).

$$\begin{aligned} x - 2y &= 7 && \text{Equation (1)} \\ x - 2(-5) &= 7 && \text{Substitute.} \\ x + 10 &= 7 \\ x &= -3 \end{aligned}$$

5) The check is left to the reader. The solution of the system is $(-3, -5)$. ■

You Try 1

Solve the system by substitution.

$$\begin{aligned} -3x + 4y &= -2 \\ 6x - y &= -3 \end{aligned}$$

If no variable in the system has a coefficient of 1 or -1, solve for any variable.

2. Solve a System Containing Fractions or Decimals

If a system contains an equation with fractions, first multiply the equation by the least common denominator to eliminate the fractions. Likewise, if an equation in the system contains decimals, begin by multiplying the equation by the lowest power of 10 that will eliminate the decimals.

Example 3

Solve the system by substitution.

$$\begin{aligned} \frac{3}{10}x - \frac{1}{5}y &= 1 && (1) \\ -\frac{1}{12}x + \frac{1}{3}y &= \frac{5}{6} && (2) \end{aligned}$$

Solution

Before applying the steps for solving the system, eliminate the fractions in each equation.

$$\begin{aligned} \frac{3}{10}x - \frac{1}{5}y &= 1 \\ 10\left(\frac{3}{10}x - \frac{1}{5}y = 1\right) &= 10 \cdot 1 && \text{Multiply by the LCD: 10.} \\ 3x - 2y &= 10 \quad (3) && \text{Distribute.} \end{aligned}$$

$$\begin{aligned} -\frac{1}{12}x + \frac{1}{3}y &= \frac{5}{6} \\ 12\left(-\frac{1}{12}x + \frac{1}{3}y = \frac{5}{6}\right) &= 12 \cdot \frac{5}{6} && \text{Multiply by the LCD: 12.} \\ -x + 4y &= 10 \quad (4) && \text{Distribute.} \end{aligned}$$

From the original equations, we obtain an equivalent system of equations.

$$\begin{aligned} 3x - 2y &= 10 && (3) \\ -x + 4y &= 10 && (4) \end{aligned}$$

Now, we will work with equations (3) and (4).
Apply the steps:

1) The x in equation (4) has a coefficient of -1. Solve this equation for x.

$$\begin{aligned} -x + 4y &= 10 && \text{Equation (4)} \\ -x &= 10 - 4y && \text{Subtract } 4y. \\ x &= -10 + 4y && \text{Divide by } -1. \end{aligned}$$

2) Substitute $-10 + 4y$ for x in equation (3).

$$\begin{aligned} 3x - 2y &= 10 && (3) \\ 3(-10 + 4y) - 2y &= 10 && \text{Substitute.} \end{aligned}$$

3) Solve the equation above for y.

$$\begin{aligned} 3(-10 + 4y) - 2y &= 10 && \\ -30 + 12y - 2y &= 10 && \text{Distribute.} \\ -30 + 10y &= 10 && \\ 10y &= 40 && \\ y &= 4 && \text{Divide by 10.} \end{aligned}$$

4) Find x by substituting 4 for y in either equation (3) or (4). Let's use equation (4) since it has smaller coefficients.

$$\begin{aligned} -x + 4y &= 10 && (4) \\ -x + 4(4) &= 10 && \text{Substitute.} \\ -x + 16 &= 10 && \\ -x &= -6 && \\ x &= 6 && \text{Divide by } -1. \end{aligned}$$

5) Check $x = 6$ and $y = 4$ in the original equations. The solution of the system is (6, 4).

You Try 2

Solve each system by substitution.

a) $-\frac{1}{6}x + \frac{1}{3}y = \frac{2}{3}$
$\quad \frac{3}{2}x - \frac{5}{2}y = -7$

b) $0.1x + 0.03y = -0.05$
$\quad 0.1x - 0.1y = 0.6$

3. Solve a System by Substitution: Special Cases

We saw in Section 5.1 that a system may have no solution or an infinite number of solutions. If we are solving a system by graphing, we know that a system has no solution if the lines are parallel, and a system has an infinite number of solutions if the graphs are the same line.

When we solve a system by *substitution*, how do we know whether the system is inconsistent or dependent? Read Examples 4 and 5 to find out.

Example 4

Solve the system by substitution.

$$\begin{aligned} 3x + y &= 5 && (1) \\ 12x + 4y &= -7 && (2) \end{aligned}$$

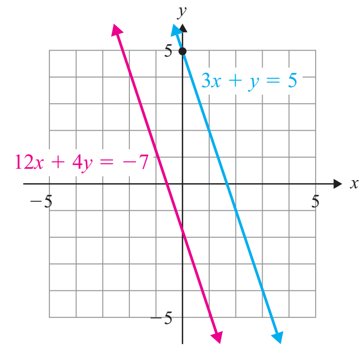
Solution

1) $y = -3x + 5$ Solve equation (1) for y.

2)
$$\begin{aligned} 12x + 4y &= -7 && \text{Substitute } -3x + 5 \text{ for } y \text{ in equation (2).} \\ 12x + 4(-3x + 5) &= -7 && \end{aligned}$$

3)
$$\begin{aligned} 12x + 4(-3x + 5) &= -7 && \text{Solve the resulting equation for } x. \\ 12x - 12x + 20 &= -7 && \text{Distribute.} \\ 20 &= -7 && \text{False} \end{aligned}$$

Since the variables drop out, and we get a false statement, there is no solution to the system. The system is inconsistent, so the solution set is ∅.



The graph of the equations in the system supports our work. The lines are parallel; therefore, the system has no solution. ■

Example 5

Solve the system by substitution.

$$\begin{aligned} 2x - 6y &= 10 \qquad (1) \\ x &= 3y + 5 \qquad (2) \end{aligned}$$

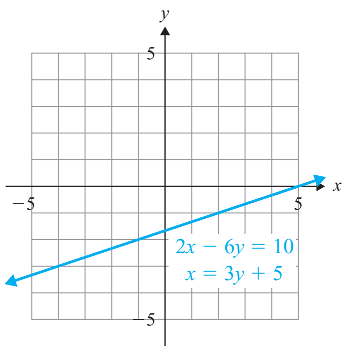
Solution

1) Equation (2) is already solved for x.

2) $$\begin{aligned} 2x - 6y &= 10 \\ 2(3y + 5) - 6y &= 10 \end{aligned}$$ Substitute $3y + 5$ for x in equation (1).

3) $$\begin{aligned} 2(3y + 5) - 6y &= 10 \\ 6y + 10 - 6y &= 10 \\ 10 &= 10 \end{aligned}$$ Solve the equation for y. True

Since the variables drop out and we get a true statement, the system has an infinite number of solutions. The equations are dependent, and the solution set is $\{(x, y) \mid x - 3y = 5\}$.



The graph shows that the equations in the system are the same line, therefore the system has an infinite number of solutions. ■

Note

When you are solving a system of equations and the variables drop out:

1) If you get a *false statement*, like $3 = 5$, then the system has *no solution* and is *inconsistent*.
2) If you get a *true statement*, like $-4 = -4$, then the system has an *infinite number of solutions*. The equations are *dependent*.

You Try 3

Solve each system by substitution.

a) $$\begin{aligned} -20x + 5y &= 3 \\ 4x - y &= -1 \end{aligned}$$

b) $$\begin{aligned} x - 3y &= 5 \\ 4x - 12y &= 20 \end{aligned}$$

Answers to You Try Exercises

1) $\left(-\frac{2}{3}, -1\right)$ 2) a) $(-8, -2)$ b) $(1, -5)$ 3) a) $\varnothing$ b) $\{(x, y) \mid x - 3y = 5\}$

5.2 Exercises

Mixed Exercises: Objectives 1 and 3

1) If you were asked to solve this system by substitution, why would it be easiest to begin by solving for y in the first equation?

$$\begin{aligned} 7x + y &= 1 \\ -2x + 5y &= 9 \end{aligned}$$

2) When is the best time to use substitution to solve a system?

3) When solving a system of linear equations, how do you know whether the system has no solution?

4) When solving a system of linear equations, how do you know whether the system has an infinite number of solutions?

Solve each system by substitution.

5) $y = 4x - 3$
 $5x + y = 15$

6) $y = 3x + 10$
 $-5x + 2y = 14$

7) $x = 7y + 11$
 $4x - 5y = -2$

8) $x = 9 - y$
 $-3x + 4y = 8$

9) $x + 2y = -3$
 $4x + 5y = -6$

10) $x + 4y = 1$
 $5x + 3y = 5$

11) $2y - 7x = -14$
 $4x - y = 7$

12) $-2x - y = 3$
 $3x + 2y = -3$

VIDEO 13) $9y - 18x = 5$
 $2x - y = 3$

14) $2x + 30y = 9$
 $x = 6 - 15y$

15) $x - 2y = 10$
 $3x - 6y = 30$

16) $6x + y = -6$
 $-12x - 2y = 12$

VIDEO 17) $10x + y = -5$
 $-5x + 2y = 10$

18) $2y - x = 4$
 $x + 6y = 8$

19) $x = -\frac{3}{5}y + 7$
 $x + 4y = 24$

20) $y = \frac{3}{2}x - 5$
 $2x - y = 5$

21) $4y = 2x + 4$
 $2y - x = 2$

22) $3x + y = -12$
 $6x = 10 - 2y$

23) $2x + 3y = 6$
 $5x + 2y = -7$

24) $2x - 5y = -4$
 $8x - 9y = 6$

25) $6x - 7y = -4$
 $9x - 2y = 11$

26) $4x + 6y = -13$
 $7x - 4y = -1$

27) $18x + 6y = -66$
 $12x + 4y = -19$

28) $6y - 15x = -12$
 $5x - 2y = 4$

Objective 2: Solve a System Containing Fractions or Decimals

29) If an equation in a system contains fractions, what should you do first to make the system easier to solve?

30) If an equation in a system contains decimals, what should you do first to make the system easier to solve?

Solve each system by substitution.

VIDEO 31) $\frac{1}{4}x - \frac{1}{2}y = 1$
 $\frac{2}{3}x + \frac{1}{6}y = \frac{25}{6}$

32) $\frac{2}{9}x + \frac{2}{9}y = 2$
 $\frac{7}{4}x - \frac{1}{8}y = \frac{3}{4}$

33) $\frac{1}{6}x + \frac{4}{3}y = \frac{13}{3}$
 $\frac{2}{5}x + \frac{3}{2}y = \frac{18}{5}$

34) $\frac{1}{10}x + \frac{1}{2}y = \frac{1}{5}$
 $-\frac{1}{3}x + \frac{1}{2}y = \frac{3}{2}$

35) $\frac{x}{10} - \frac{y}{2} = \frac{13}{10}$
 $\frac{x}{3} + \frac{5}{4}y = -\frac{3}{2}$

36) $-\frac{x}{3} + \frac{y}{2} = \frac{5}{3}$
 $\frac{x}{5} - \frac{4}{5}y = -1$

37) $y - \frac{5}{2}x = -2$
 $\frac{3}{4}x - \frac{3}{10}y = \frac{3}{5}$

38) $-\frac{2}{15}x - \frac{1}{3}y = \frac{2}{3}$
 $\frac{2}{3}x + \frac{5}{3}y = \frac{1}{2}$

39) $\frac{3}{4}x + \frac{1}{2}y = 6$
 $x = 3y + 8$

40) $\frac{5}{3}x - \frac{4}{3}y = -\frac{4}{3}$
 $y = 2x + 4$

VIDEO 41) $0.2x - 0.1y = 0.1$
 $0.01x + 0.04y = 0.23$

42) $0.01x - 0.09y = -0.5$
 $0.02x + 0.05y = 0.38$

43) $0.6x - 0.1y = 1$
 $-0.4x + 0.5y = -1.1$

44) $0.8x + 0.7y = -1.7$
 $0.6x - 0.1y = 0.6$

45) $0.02x + 0.01y = -0.44$
 $-0.1x - 0.2y = 4$

46) $0.3x + 0.1y = 3$
 $0.01x - 0.05y = -0.06$

47) $2.8x + 0.7y = 0.1$
 $0.04x + 0.01y = -0.06$

48) $0.1x - 0.3y = -1.2$
 $1.5y - 0.5x = 6$

Solve by substitution. Begin by combining like terms.

49) $8 + 2(3x - 5) - 7x + 6y = 16$
 $9(y - 2) + 5x - 13y = -4$

50) $3 + 4(2y - 9) + 5x - 2y = 8$
 $3(x + 3) - 4(2y + 1) - 2x = 4$

51) $10(x + 3) - 7(y + 4) = 2(4x - 3y) + 3$
 $10 - 3(2x - 1) + 5y = 3y - 7x - 9$

52) $7x + 3(y - 2) = 7y + 6x - 1$
 $18 + 2(x - y) = 4(x + 2) - 5y$

53) $-(y + 3) = 5(2x + 1) - 7x$
 $x + 12 - 8(y + 2) = 6(2 - y)$

54) $9y - 4(2y + 3) = -2(4x + 1)$
 $16 - 5(2x + 3) = 2(4 - y)$

55) Jamari wants to rent a cargo trailer to move his son into an apartment when he returns to college. A+ Rental charges \$0.60 per mile while Rock Bottom Rental charges \$70 plus \$0.25 per mile. Let $x =$ the number of miles driven and let $y =$ the cost of the rental. The cost of renting a cargo trailer from each company can be expressed with the following equations:

A+ Rental: $y = 0.60x$

Rock Bottom Rental: $y = 0.25x + 70$

a) How much would it cost Jamari to rent a cargo trailer from each company if he will drive a total of 160 miles?

b) How much would it cost Jamari to rent a trailer from each company if he planned to drive 300 miles?

c) Solve the system of equations using the substitution method, and explain the meaning of the solution.

d) Graph the system of equations, and explain when it is cheaper to rent a cargo trailer from A+ and when it is cheaper to rent it from Rock Bottom Rental. When is the cost the same?

56) To rent a pressure washer, Walsh Rentals charges \$16.00 per hour while Discount Company charges \$24.00 plus \$12.00 per hour. Let $x =$ the number of hours, and let $y =$ the cost of the rental. The cost of renting a pressure washer from each company can be expressed with the following equations:

Walsh Rentals: $y = 16.00x$

Discount company: $y = 12.00x + 24$

a) How much would it cost to rent a pressure washer from each company if it would be used for 4 hours?

b) How much would it cost to rent a pressure washer from each company if it would be rented for 9 hours?

c) Solve the system of equations using the substitution method, and explain the meaning of the solution.

d) Graph the system of equations, and explain when it is cheaper to rent a pressure washer from Walsh and when it is cheaper to rent it from Discount. When is the cost the same?

Section 5.3 Solving Systems by the Elimination Method

Objectives

1. Solve a Linear System Using the Elimination Method
2. Solve a Linear System Using the Elimination Method: Special Cases
3. Use the Elimination Method Twice to Solve a Linear System

1. Solve a Linear System Using the Elimination Method

The next technique we will learn for solving a system of equations is the **elimination method.** (This is also called the **addition method.**) It is based on the addition property of equality that says that we can add the *same* quantity to each side of an equation and preserve the equality.

$$\text{If } a = b, \text{ then } a + c = b + c.$$

We can extend this idea by saying that we can add *equal* quantities to each side of an equation and still preserve the equality.

$$\text{If } a = b \text{ and } c = d, \text{ then } a + c = b + d.$$

The object of the elimination method is to add the equations (or multiples of one or both of the equations) so that one variable is eliminated. Then, we can solve for the remaining variable.

Example 1

Solve the system using the elimination method.

$$\begin{aligned} x + y &= 11 \quad (1) \\ x - y &= -5 \quad (2) \end{aligned}$$

Solution

The left side of each equation is equal to the right side of each equation. Therefore, if we add the left sides together and add the right sides together, we can set them equal. We will add these equations vertically. The y-terms are eliminated, enabling us to solve for x.

$$\begin{array}{rll} x + y = 11 & (1) & \\ + \quad \underline{x - y = -5} & (2) & \\ 2x + 0y = 6 & & \text{Add equations (1) and (2).} \\ 2x = 6 & & \text{Simplify.} \\ x = 3 & & \text{Divide by 2.} \end{array}$$

Now we substitute $x = 3$ into either equation to find the value of y. Here, we will use equation (1).

$$\begin{aligned} x + y &= 11 && \text{Equation (1)} \\ 3 + y &= 11 && \text{Substitute 3 for } x. \\ y &= 8 && \text{Subtract 3.} \end{aligned}$$

Check $x = 3$ and $y = 8$ in *both* equations.

$$\begin{aligned} x + y &= 11 \\ 3 + 8 &\stackrel{?}{=} 11 && \text{Substitute.} \\ 11 &= 11 && \text{True} \end{aligned} \qquad \begin{aligned} x - y &= -5 \\ 3 - 8 &\stackrel{?}{=} -5 && \text{Substitute.} \\ -5 &= -5 && \text{True} \end{aligned}$$

The solution is (3, 8).

You Try 1

Solve the system using the elimination method.

$$\begin{aligned} 3x + y &= 10 \\ x - y &= 6 \end{aligned}$$

In Example 1, simply adding the equations eliminated a variable. But what can we do if we *cannot* eliminate a variable just by adding the equations together?

Example 2

Solve the system using the elimination method.

$$\begin{aligned} 2x + 5y &= 5 && (1) \\ x + 4y &= 7 && (2) \end{aligned}$$

Solution

Just adding these equations will *not* eliminate a variable. The multiplication property of equality tells us that multiplying both sides of an equation by the same quantity results in an equivalent equation. If we multiply equation (2) by -2, the coefficient of x will be -2.

$$\begin{aligned} -2(x + 4y) &= -2(7) && \text{Multiply equation (2) by } -2. \\ -2x - 8y &= -14 && \text{New, equivalent equation} \end{aligned}$$

Original System $\longrightarrow$ **Rewrite the System**

$$\begin{aligned} 2x + 5y &= 5 \\ x + 4y &= 7 \end{aligned} \quad \longrightarrow \quad \begin{aligned} 2x + 5y &= 5 \\ -2x - 8y &= -14 \end{aligned}$$

Add the equations in the rewritten system. The x is eliminated.

$$\begin{aligned} 2x + 5y &= 5 \\ + \quad -2x - 8y &= -14 \\ \hline 0x - 3y &= -9 && \text{Add equations.} \\ -3y &= -9 && \text{Simplify.} \\ y &= 3 && \text{Solve for } y. \end{aligned}$$

Substitute $y = 3$ into (1) or (2) to find x. We will use equation (2).

$$\begin{aligned} x + 4y &= 7 && \text{Equation (2)} \\ x + 4(3) &= 7 && \text{Substitute 3 for } y. \\ x + 12 &= 7 \\ x &= -5 \end{aligned}$$

The solution is $(-5, 3)$. Check the solution in equations (1) and (2).

You Try 2

Solve the system using the elimination method.

$$\begin{aligned} 8x - y &= -5 \\ -6x + 2y &= 15 \end{aligned}$$

Next we summarize the steps for solving a system using the elimination method.

Procedure Solving a System of Two Linear Equations by the Elimination Method

1) Write each equation in the form $Ax + By = C$.
2) Determine which variable to eliminate. If necessary, multiply one or both of the equations by a number so that the coefficients of the variable to be eliminated are negatives of one another.
3) Add the equations, and solve for the remaining variable.
4) Substitute the value found in *Step 3* into either of the original equations to find the value of the other variable.
5) Check the solution in each of the original equations.

Example 3

Solve the system using the elimination method.

$$\begin{aligned} 2x &= 9y + 4 & (1) \\ 3x - 7 &= 12y & (2) \end{aligned}$$

Solution

1) **Write each equation in the form $Ax + By = C$.**

$$\begin{aligned} 2x &= 9y + 4 & (1) \\ 2x - 9y &= 4 & \text{Subtract } 9y. \end{aligned}$$

$$\begin{aligned} 3x - 7 &= 12y & (2) \\ 3x - 12y &= 7 & \text{Subtract } 12y \text{ and add } 7. \end{aligned}$$

When we rewrite the equations in the form $Ax + By = C$, we get

$$\begin{aligned} 2x - 9y &= 4 & (3) \\ 3x - 12y &= 7 & (4) \end{aligned}$$

2) **Determine which variable to eliminate from equations (3) and (4).** Often, it is easier to eliminate the variable with the smaller coefficients. Therefore, *we will eliminate x.*

The least common multiple of 2 and 3 (the x-coefficients) is 6. Before we add the equations, one x-coefficient should be 6, and the other should be -6. Multiply equation (3) by 3 and equation (4) by -2.

Rewrite the System

$$\begin{aligned} 3(2x - 9y) &= 3(4) & 3 \text{ times } (3) \\ -2(3x - 12y) &= -2(7) & -2 \text{ times } (4) \end{aligned} \longrightarrow \begin{aligned} 6x - 27y &= 12 \\ -6x + 24y &= -14 \end{aligned}$$

3) **Add the resulting equations to eliminate x. Solve for y.**

$$\begin{aligned} 6x - 27y &= 12 \\ + \quad -6x + 24y &= -14 \\ \hline -3y &= -2 \\ y &= \frac{2}{3} \end{aligned}$$

4) **Substitute $y = \frac{2}{3}$ into equation (1) and solve for x.**

$$\begin{aligned} 2x &= 9y + 4 && \text{Equation (1)} \\ 2x &= 9\left(\frac{2}{3}\right) + 4 && \text{Substitute.} \\ 2x &= 6 + 4 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

5) **Check** to verify that $\left(5, \frac{2}{3}\right)$ satisfies each of the original equations. The solution is $\left(5, \frac{2}{3}\right)$.

You Try 3

Solve the system using the elimination method.

$$\begin{aligned} 5x &= 2y - 14 \\ 4x + 3y &= 21 \end{aligned}$$

2. Solve a Linear System Using the Elimination Method: Special Cases

We have seen in Sections 5.1 and 5.2 that some systems have no solution, and some have an infinite number of solutions. How does the elimination method illustrate these results?

Example 4

Solve the system using the elimination method.

$$\begin{aligned} 4y &= 10x + 3 && (1) \\ 6y - 15x &= -8 && (2) \end{aligned}$$

Solution

1) **Write each equation in the form $Ax + By = C$.**

$$\begin{aligned} 4y &= 10x + 3 \\ 6y - 15x &= -8 \end{aligned} \quad \longrightarrow \quad \begin{aligned} -10x + 4y &= 3 && (3) \\ -15x + 6y &= -8 && (4) \end{aligned}$$

2) **Determine which variable to eliminate from equations (3) and (4).** Eliminate y.

The least common multiple of 4 and 6, the y-coefficients, is 12. One y-coefficient must be 12, and the other must be -12.

Rewrite the System

$$\begin{aligned} -3(-10x + 4y) &= -3(3) \\ 2(-15x + 6y) &= 2(-8) \end{aligned} \quad \longrightarrow \quad \begin{aligned} 30x - 12y &= -9 \\ -30x + 12y &= -16 \end{aligned}$$

3) **Add the equations.**

$$\begin{array}{rr} & 30x - 12y = -9 \\ + & -30x + 12y = -16 \\ \hline & 0 = -25 \end{array} \quad \text{False}$$

The variables drop out, and we get a false statement. Therefore, the system is inconsistent, and the solution set is $\varnothing$.

You Try 4

Solve the system using the elimination method.

$$\begin{aligned} 24x + 6y &= -7 \\ 4y + 3 &= -16x \end{aligned}$$

Example 5

Solve the system using the elimination method.

$$\begin{aligned} 12x - 18y &= 9 & (1) \\ y &= \frac{2}{3}x - \frac{1}{2} & (2) \end{aligned}$$

Solution

1) **Write equation (2) in the form $Ax + By = C$.**

$$\begin{aligned} y &= \frac{2}{3}x - \frac{1}{2} & & \text{Equation (2)} \\ 6y &= 6\left(\frac{2}{3}x - \frac{1}{2}\right) & & \text{Multiply by 6 to eliminate fractions.} \\ 6y &= 4x - 3 & & \\ -4x + 6y &= -3 & (3) \quad & \text{Rewrite as } Ax + By = C. \end{aligned}$$

We can rewrite $y = \frac{2}{3}x - \frac{1}{2}$ as $-4x + 6y = -3$, equation (3).

2) **Determine which variable to eliminate from equations (1) and (3).**

$$\begin{aligned} 12x - 18y &= 9 & (1) \\ -4x + 6y &= -3 & (3) \end{aligned}$$

Eliminate x. Multiply equation (3) by 3.

$$\begin{aligned} 12x - 18y &= 9 & & (1) \\ -12x + 18y &= -9 & & \text{3 times (3)} \end{aligned}$$

3) **Add the equations.**

$$\begin{array}{rr} & 12x - 18y = 9 \\ + & -12x + 18y = -9 \\ \hline & 0 = 0 \end{array} \quad \text{True}$$

The variables drop out, and we get a true statement. The equations are dependent, so there are an infinite number of solutions. The solution set is $\left\{(x, y) \,\middle|\, y = \frac{2}{3}x - \frac{1}{2}\right\}$. ■

You Try 5

Solve the system using the elimination method.

$$\begin{aligned} -6x + 8y &= 4 \\ 3x - 4y &= -2 \end{aligned}$$

3. Use the Elimination Method Twice to Solve a Linear System

Sometimes, applying the elimination method *twice* is the best strategy.

Example 6

Solve using the elimination method.

$$\begin{aligned} 5x - 6y &= 2 && (1) \\ 9x + 4y &= -3 && (2) \end{aligned}$$

Solution

Each equation is written in the form $Ax + By = C$, so we begin with step 2.

2) We will eliminate y from equations (1) and (2).

Rewrite the System

$$\begin{aligned} 2(5x - 6y) &= 2(2) \\ 3(9x + 4y) &= 3(-3) \end{aligned} \longrightarrow \begin{aligned} 10x - 12y &= 4 \\ 27x + 12y &= -9 \end{aligned}$$

3) Add the resulting equations to eliminate y. Solve for x.

$$\begin{array}{rr} & 10x - 12y = 4 \\ + & 27x + 12y = -9 \\ \hline & 37x = -5 \end{array}$$

$$x = -\frac{5}{37} \qquad \text{Solve for } x.$$

Normally, we would substitute $x = -\frac{5}{37}$ into equation (1) or equation (2) and solve for y. This time, however, working with a number like $-\frac{5}{37}$ would be difficult, so *we will use the elimination method a second time.*

Go back to the original equations, (1) and (2), and use the elimination method again but eliminate the other variable, x. Then, solve for y.

Eliminate x from
$$\begin{aligned} 5x - 6y &= 2 && (1) \\ 9x + 4y &= -3 && (2) \end{aligned}$$

Rewrite the System

$$\begin{aligned} -9(5x - 6y) &= -9(2) \\ 5(9x + 4y) &= 5(-3) \end{aligned} \longrightarrow \begin{aligned} -45x + 54y &= -18 \\ 45x + 20y &= -15 \end{aligned}$$

Add the equations
$$\begin{array}{rr} & -45x + 54y = -18 \\ + & 45x + 20y = -15 \\ \hline & 74y = -33 \end{array}$$

$$y = -\frac{33}{74} \qquad \text{Solve for } y.$$

Check to verify that the solution is $\left(-\frac{5}{37}, -\frac{33}{74}\right)$.

You Try 6

Solve using the elimination method.

$$\begin{aligned} -9x + 2y &= -3 \\ 2x - 5y &= 4 \end{aligned}$$

Answers to You Try Exercises

1) $(4, -2)$ 2) $\left(\frac{1}{2}, 9\right)$ 3) $(0, 7)$ 4) $\varnothing$

5) Infinite number of solutions of the form $\{(x, y)|3x - 4y = -2\}$ 6) $\left(\frac{7}{41}, -\frac{30}{41}\right)$

5.3 Exercises

Mixed Exercises: Objectives 1 and 2

1) What is the first step you would use to solve this system by elimination if you wanted to eliminate y?

$$\begin{aligned} 5x + y &= 2 \\ 3x - y &= 6 \end{aligned}$$

2) What is the first step you would use to solve this system by elimination if you wanted to eliminate x?

$$\begin{aligned} 4x - 3y &= 14 \\ 8x - 11y &= 18 \end{aligned}$$

Solve each system using the elimination method.

3) $x - y = -3$; $2x + y = 18$

4) $x + 3y = 1$; $-x + 5y = -9$

5) $-x + 2y = 2$; $x - 7y = 8$

6) $4x - y = -15$; $3x + y = -6$

7) $x + 4y = 1$; $3x - 4y = -29$

8) $5x - 4y = -10$; $-5x + 7y = 25$

9) $-8x + 5y = -16$; $4x - 7y = 8$

10) $7x + 6y = 3$; $3x + 2y = -1$

11) $4x + 15y = 13$; $3x + 5y = 16$

12) $12x + 7y = 7$; $-3x + 8y = 8$

VIDEO 13) $9x - 7y = -14$; $4x + 3y = 6$

14) $5x - 2y = -6$; $4x + 5y = -18$

15) $-9x + 2y = -4$; $6x - 3y = 11$

16) $12x - 2y = 3$; $8x - 5y = -9$

17) $9x - y = 2$; $18x - 2y = 4$

18) $-4x + 7y = 13$; $12x - 21y = -5$

19) $x = 12 - 4y$; $2x - 7 = 9y$

20) $5x + 3y = -11$; $y = 6x + 4$

21) $4y = 9 - 3x$; $5x - 16 = -6y$

22) $8x = 6y - 1$; $10y - 6 = -4x$

23) $2x - 9 = 8y$; $20y - 5x = 6$

24) $3x + 2y = 6$; $4y = 12 - 6x$

25) $6x - 11y = -1$; $-7x + 13y = 2$

26) $10x - 4y = 7$; $12x - 3y = -15$

27) $9x + 6y = -2$; $-6x - 4y = 11$

28) $4x - 9y = -3$; $36y - 16x = 12$

29) What is the first step in solving this system by the elimination method? DO NOT SOLVE.

$$\begin{aligned} \frac{x}{4} + \frac{y}{2} &= -1 \\ \frac{3}{8}x + \frac{5}{3}y &= -\frac{7}{12} \end{aligned}$$

30) What is the first step in solving this system by the elimination method? DO NOT SOLVE.

$$\begin{aligned} 0.1x + 2y &= -0.8 \\ 0.03x + 0.10y &= 0.26 \end{aligned}$$

Solve each system by elimination.

31) $\frac{4}{5}x - \frac{1}{2}y = -\frac{3}{2}$; $2x - \frac{1}{4}y = \frac{1}{4}$

32) $\frac{1}{3}x - \frac{4}{5}y = \frac{13}{15}$; $\frac{1}{6}x - \frac{3}{4}y = -\frac{1}{2}$

33) $\frac{5}{4}x - \frac{1}{2}y = \frac{7}{8}$; $\frac{2}{5}x - \frac{1}{10}y = -\frac{1}{2}$

34) $\frac{1}{2}x - \frac{11}{8}y = -1$; $-\frac{2}{5}x + \frac{3}{10}y = \frac{4}{5}$

VIDEO 35) $\frac{x}{4} + \frac{y}{2} = -1$; $\frac{3}{8}x + \frac{5}{3}y = -\frac{7}{12}$

36) $\frac{x}{12} - \frac{y}{6} = \frac{2}{3}$; $\frac{x}{4} + \frac{y}{3} = 2$

37) $\frac{x}{12} - \frac{y}{8} = \frac{7}{8}$; $y = \frac{2}{3}x - 7$

38) $\frac{5}{3}x + \frac{1}{3}y = \frac{2}{3}$; $\frac{3}{4}x + \frac{3}{20}y = -\frac{5}{4}$

39) $-\frac{1}{2}x + \frac{5}{4}y = \frac{3}{4}$; $\frac{2}{5}x - \frac{1}{2}y = -\frac{1}{10}$

40) $y = 2 - 4x$; $\frac{1}{3}x - \frac{3}{8}y = \frac{5}{8}$

41) $0.08x + 0.07y = -0.84$; $0.32x - 0.06y = -2$

42) $0.06x + 0.05y = 0.58$; $0.18x - 0.13y = 1.18$

VIDEO 43) $0.1x + 2y = -0.8$; $0.03x + 0.10y = 0.26$

44) $0.6x - 0.1y = 0.5$; $0.1x - 0.03y = -0.01$

45) $-0.4x + 0.2y = 0.1$
$0.6x - 0.3y = 1.5$

46) $x - 0.5y = 0.2$
$-0.3x + 0.15y = -0.06$

47) $0.04x + 0.03y = 0.16$
$0.6x + 0.2y = 1.15$

48) $-0.5x + 0.8y = 0.3$
$0.03x + 0.1y = -0.24$

49) $17x - 16(y + 1) = 4(x - y)$
$19 - 10(x + 2) = -4(x + 6) - y + 2$

50) $28 - 4(y + 1) = 3(x - y) + 4$
$-5(x + 4) - y + 3 = 28 - 5(2x + 5)$

51) $5 - 3y = 6(3x + 4) - 8(x + 2)$
$6x - 2(5y + 2) = -7(2y - 1) - 4$

52) $5(y + 3) = 6(x + 1) + 6x$
$7 - 3(2 - 3x) - y = 2(3y + 8) - 5$

53) $6(x - 3) + x - 4y = 1 + 2(x - 9)$
$4(2y - 3) + 10x = 5(x + 1) - 4$

54) $8y + 2(4x + 5) - 5x = 7y - 11$
$11y - 3(x + 2) = 16 + 2(3y - 4) - x$

Objective 3: Use the Elimination Method Twice to Solve a Linear System

Solve each system using the elimination method twice.

VIDEO 55) $4x + 5y = -6$
$3x + 8y = 15$

56) $8x - 4y = -21$
$-5x + 6y = 12$

57) $4x + 9y = 7$
$6x + 11y = -14$

58) $10x + 3y = 18$
$9x - 4y = 5$

Find k so that the given ordered pair is a solution of the given system.

59) $x + ky = 17;\quad (5, 4)$
$2x - 3y = -2$

60) $kx + y = -13;\quad (-1, -8)$
$9x - 2y = 7$

61) $3x + 4y = -9;\quad (-7, 3)$
$kx - 5y = 41$

62) $4x + 3y = -7;\quad (2, -5)$
$3x + ky = 16$

63) Given the following system of equations,

$$x - y = 5$$
$$x - y = c$$

find c so that the system has

a) an infinite number of solutions.

b) no solution.

64) Given the following system of equations,

$$2x + y = 9$$
$$2x + y = c$$

find c so that the system has

a) an infinite number of solutions.

b) no solution.

65) Given the following system of equations,

$$9x + 12y = -15$$
$$ax + 4y = -5$$

find a so that the system has

a) an infinite number of solutions.

b) exactly one solution.

66) Given the following system of equations,

$$-2x + 7y = 3$$
$$4x + by = -6$$

find b so that the system has

a) an infinite number of solutions.

b) exactly one solution.

Extension

Let a, b, and c represent nonzero constants. Solve each system for x and y.

67) $-5x + 4by = 6$
$5x + 3by = 8$

68) $ax - 6y = 4$
$-ax + 9y = 2$

69) $3ax + by = 4$
$ax - by = -5$

70) $2ax + by = c$
$ax + 3by = 4c$

Putting It All Together

Objective

1. Choose the Best Method for Solving a System of Linear Equations

1. Choose the Best Method for Solving a System of Linear Equations

We have learned three methods for solving systems of linear equations:

1) Graphing 2) Substitution 3) Elimination

How do we know which method is best for a particular system? We will answer this question by looking at a few examples, and then we will summarize our findings.

First, solving a system by graphing is the least desirable of the methods. The point of intersection can be difficult to read, especially if one of the numbers is a fraction. But, the graphing method is important in certain situations and is one you should know.

Example 1

Decide which method to use to solve each system, substitution or elimination, and explain why this method was chosen. Then, solve the system.

a) $\begin{aligned} -5x + 2y &= -8 \\ x &= 4y + 16 \end{aligned}$ b) $\begin{aligned} 4x + y &= 10 \\ -3x - 8y &= 7 \end{aligned}$ c) $\begin{aligned} 4x - 5y &= -3 \\ 6x + 8y &= 11 \end{aligned}$

Solution

a) $\begin{aligned} -5x + 2y &= -8 \\ x &= 4y + 16 \end{aligned}$

The second equation in this system is solved for x, *and* there are no fractions in this equation. *Solve this system by substitution.*

$$\begin{aligned} -5x + 2y &= -8 && \text{First equation} \\ -5(4y + 16) + 2y &= -8 && \text{Substitute } 4y + 16 \text{ for } x. \\ -20y - 80 + 2y &= -8 && \text{Distribute.} \\ -18y - 80 &= -8 && \text{Combine like terms.} \\ -18y &= 72 && \text{Add 80.} \\ y &= -4 && \text{Solve for } y. \end{aligned}$$

Substitute $y = -4$ into $x = 4y + 16$:

$$\begin{aligned} x &= 4(-4) + 16 && \text{Substitute.} \\ x &= 0 && \text{Solve for } x. \end{aligned}$$

The solution is $(0, -4)$. The check is left to the student.

b) $\begin{aligned} 4x + y &= 10 \\ -3x - 8y &= 7 \end{aligned}$

In the first equation, y has a coefficient of 1, so we can easily solve for y and *substitute* the expression into the second equation. (Solving for another variable would result in having fractions in the equation.) *Or*, since each equation is in the form $Ax + By = C$, elimination would work well too. *Either substitution or elimination would be a good choice to solve this system*. The student should choose whichever method he or she prefers. Here, we use substitution.

$$\begin{aligned} y &= -4x + 10 && \text{Solve the first equation for } y. \\ -3x - 8y &= 7 && \text{Second equation} \\ -3x - 8(-4x + 10) &= 7 && \text{Substitute } -4x + 10 \text{ for } y. \\ -3x + 32x - 80 &= 7 && \text{Distribute.} \\ 29x &= 87 && \text{Simplify.} \\ x &= 3 && \text{Solve for } x. \end{aligned}$$

Substitute $x = 3$ into $y = -4x + 10$.

$$\begin{aligned} y &= -4(3) + 10 && \text{Substitute.} \\ y &= -2 && \text{Solve for } y. \end{aligned}$$

The check is left to the student. The solution is $(3, -2)$.

c) $\begin{aligned} 4x - 5y &= -3 \\ 6x + 8y &= 11 \end{aligned}$

None of the variables has a coefficient of 1 or -1. Therefore, we do *not* want to use substitution because we would have to work with fractions in the equation. *To solve a system like this, where none of the coefficients are 1 or -1, use the elimination method.*

Eliminate x.

Rewrite the System

$$-3(4x - 5y) = -3(-3) \longrightarrow -12x + 15y = 9$$
$$2(6x + 8y) = 2(11) \qquad\qquad + \;\; 12x + 16y = 22$$
$$31y = 31$$
$$y = 1 \qquad \text{Solve for } y.$$

Substitute $y = 1$ into $4x - 5y = -3$.

$$
\begin{aligned}
4x - 5(1) &= -3 && \text{Substitute.}\\
4x - 5 &= -3 && \text{Multiply.}\\
4x &= 2 && \text{Add 5.}\\
x &= \frac{2}{4} = \frac{1}{2} && \text{Solve for } x.
\end{aligned}
$$

The check is left to the student. The solution is $\left(\frac{1}{2}, 1\right)$.

Procedure Choosing Between Substitution and the Elimination Method to Solve a System

1) If at least one equation is solved for a variable and contains no fractions, *use substitution*.

$$-5x + 2y = -8$$
$$x = 4y + 16 \qquad \text{Example 1(a)}$$

2) If a variable has a coefficient of 1 or -1, you can solve for that variable and *use substitution*.

$$4x + y = 10$$
$$-3x - 8y = 7 \qquad \text{Example 1(b)}$$

Or, leave each equation in the form $Ax + By = C$ *and use elimination*. Either approach is good and is a matter of personal preference.

3) If no variable has a coefficient of 1 or -1, *use elimination.*

$$4x - 5y = -3$$
$$6x + 8y = 11 \qquad \text{Example 1(c)}$$

Remember, if an equation contains fractions or decimals, begin by eliminating them. Then, decide which method to use following the guidelines listed here.

You Try 1

Decide which method to use to solve each system, substitution or elimination, and explain why this method was chosen. Then, solve the system.

a) $9x - 7y = -9$, $2x + 9y = -2$ b) $9x - 2y = 0$, $x = y - 7$ c) $4x + y = 13$, $-3x - 2y = 4$

Answers to You Try Exercises

1) a) Elimination; $(-1, 0)$ b) Substitution; $(2, 9)$ c) Substitution or elimination; $(6, -11)$

Putting It All Together Summary Exercises

Objective 1: Choose the Best Method for Solving a System of Linear Equations

Decide which method to use to solve each system, substitution or addition, and explain why this method was chosen. Then solve the system.

1) $8x - 5y = 10$
$2x - 3y = -8$

2) $x = 2y - 7$
$8x - 3y = 9$

3) $12x - 5y = 18$
$8x + y = -1$

4) $11x + 10y = -4$
$9x - 5y = 2$

5) $y - 4x = -11$
$x = y + 8$

6) $4x - 5y = 4$
$y = \frac{3}{4}x - \frac{1}{2}$

Solve each system using either the substitution or elimination method.

7) $4x + 5y = 24$
$x - 3y = 6$

8) $6y - 5x = 22$
$-9x - 8y = 2$

9) $6x + 15y = -1$
$9x = 10y - 8$

10) $x + 2y = 9$
$7x - y = 3$

11) $10x + 4y = 7$
$15x + 6y = -2$

12) $y = -6x + 5$
$12x + 2y = 10$

13) $10x + 9y = 4$
$x = -\frac{1}{2}$

14) $6x - 4y = 11$
$\frac{3}{2}x + \frac{1}{4}y = \frac{7}{8}$

15) $7y - 2x = 13$
$3x - 2y = 6$

16) $y = 6$
$12x + y = 8$

17) $\frac{2}{5}x + \frac{4}{5}y = -2$
$\frac{1}{6}x + \frac{1}{6}y = \frac{1}{3}$

18) $5x + 4y = 14$
$y = -\frac{8}{5}x + 7$

19) $-0.3x + 0.1y = 0.4$
$0.01x + 0.05y = 0.2$

20) $0.01x - 0.06y = 0.03$
$0.4x + 0.3y = -1.5$

21) $-6x + 2y = -10$
$21x - 7y = 35$

22) $\frac{5}{3}x + \frac{4}{3}y = \frac{2}{3}$
$10x + 8y = -5$

23) $2 = 5y - 8x$
$y = \frac{3}{2}x - \frac{1}{2}$

24) $\frac{5}{6}x - \frac{3}{4}y = \frac{2}{3}$
$\frac{1}{3}x + 2y = \frac{10}{3}$

25) $2x - 3y = -8$
$7x + 10y = 4$

26) $6x = 9 - 13y$
$4x + 3y = -2$

27) $6(2x - 3) = y + 4(x - 3)$
$5(3x + 4) + 4y = 11 - 3y + 27x$

28) $3 - 5(x - 4) = 2(1 - 4y) + 2$
$2(x + 10) + y + 1 = 3x + 5(y + 6) - 17$

29) $2y - 2(3x + 4) = -5(y - 2) - 17$
$4(2x + 3) = 10 + 5(y + 1)$

30) $x - y + 23 = 2y + 3(2x + 7)$
$9y - 8 + 4(x + 2) = 2(4x - 1) - 3x + 10y$

31) $y = -4x$
$10x + 2y = -5$

32) $x = \frac{2}{3}y$
$9x - 5y = -6$

Solve each system by graphing.

33) $y = \frac{1}{2}x + 1$
$x + y = 4$

34) $x + y = -3$
$y = 3x + 1$

35) $x + y = 0$
$x - 2y = -12$

36) $2y - x = 6$
$y = 2x$

37) $2x - 3y = 3$
$y = \frac{2}{3}x + 1$

38) $y = -\frac{5}{2}x - 3$
$10x + 4y = -12$

Solve each system using a graphing calculator.

39) $8x - 6y = -7$
$4x - 16y = 3$

40) $4x + 3y = -9$
$2x + y = 2$

Section 5.4 Applications of Systems of Two Equations

Objectives

1. Solve Problems Involving General Quantities
2. Solve Geometry Problems
3. Solve Problems Involving Cost
4. Solve Mixture Problems
5. Solve Distance, Rate, and Time Problems

In Section 3.2, we introduced the five-step method for solving applied problems. Here, we modify the method for problems with *two* unknowns and *two* equations.

Procedure Solving an Applied Problem Using a System of Equations

Step 1: **Read** the problem carefully, more than once if necessary. Draw a picture, if applicable. Identify what you are being asked to find.

Step 2: **Choose variables** to represent the unknown quantities. Label any pictures with the variables.

Step 3: **Write a system of equations using two variables.** It may be helpful to begin by writing the equations in words.

Step 4: **Solve** the system.

Step 5: **Check** the answer in the original problem, and **interpret** the solution as it relates to the problem. Be sure your answer makes sense in the context of the problem.

1. Solve Problems Involving General Quantities

Example 1

Write a system of equations and solve.

Pink Floyd's album, *The Dark Side of the Moon,* spent more weeks on the Billboard 200 chart for top-selling albums than any other album in history. It was on the chart 251 more weeks than the second-place album, *Johnny's Greatest Hits,* by Johnny Mathis. If they were on the charts for a total of 1231 weeks, how many weeks did each album spend on the Billboard 200 chart? (www.billboard.com)

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the number of weeks each album was on the chart.

Step 2: **Choose variables** to represent the unknown quantities.

x = the number of weeks *The Dark Side of the Moon* was on the Billboard 200 chart

y = the number of weeks *Johnny's Greatest Hits* was on the Billboard 200 chart

Step 3: **Write a system of equations using two variables.** First, let's think of the equations in English. Then we will translate them into algebraic equations.

To get one equation, use the information that says these two albums were on the Billboard 200 chart for a total of 1231 weeks. Write an equation in words, then translate it into an algebraic equation.

English:	Number of weeks Pink Floyd was on Billboard 200 chart	plus	Number of weeks Johnny Mathis was on Billboard 200 chart	equals	Total weeks
	↓	↓	↓	↓	↓
Equation:	x	$+$	y	$=$	1231

The first equation is $x + y = 1231$.

To get the second equation, use the information that says the Pink Floyd album was on the chart 251 weeks more than the Johnny Mathis album.

English:	Number of weeks Pink Floyd was on Billboard 200 chart	was	251 weeks more than		the Johnny Mathis album.
	↓	↓	↓		↓
	x	$=$	251	$+$	y

The second equation is $x = 251 + y$.

The system of equations is
$$\begin{aligned} x + y &= 1231 \\ x &= 251 + y. \end{aligned}$$

Step 4: **Solve** the system.

$$\begin{aligned}
x + y &= 1231 \\
(251 + y) + y &= 1231 && \text{Substitute.} \\
251 + 2y &= 1231 && \text{Combine like terms.} \\
2y &= 980 && \text{Subtract 251.} \\
\frac{2y}{2} &= \frac{980}{2} && \text{Divide each side by 2.} \\
y &= 490 && \text{Simplify}
\end{aligned}$$

Find x by substituting $y = 490$ into $x = 251 + y$.

$$x = 251 + 490 = 741$$

The solution to the system is (741, 490).

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

The Dark Side of the Moon was on the Billboard 200 for 741 weeks, and *Johnny's Greatest Hits* was on the chart for 490 weeks.

They were on the chart for a total of $741 + 490 = 1231$ weeks, and the Pink Floyd album was on there $741 - 490 = 251$ weeks longer than the other album.

You Try 1

Write a system of equations and solve.

In 2007, Carson City, NV, had about 33,000 fewer citizens than Elmira, NY. Find the population of each city if together they had approximately 143,000 residents. (www.census.gov)

Next we will see how we can use two variables and a system of equations to solve geometry problems.

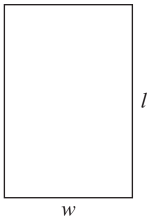
2. Solve Geometry Problems

Example 2

Write a system of equations and solve.

A builder installed a rectangular window in a new house and needs 182 in. of trim to go around it on the inside of the house. Find the dimensions of the window if the width is 23 in. less than the length.

Solution



Step 1: **Read** the problem carefully, and identify what we are being asked to find. Draw a picture.

We must find the length and width of the window.

Step 2: **Choose variables** to represent the unknown quantities.

$w =$ the width of the window
$l =$ the length of the window

Label the picture with the variables.

Step 3: **Write a system of equations using two variables.**

To get one equation, we know that the width is 23 in. less than the length. We can write the equation $w = l - 23$.

If it takes 182 in. of trim to go around the window, this is the *perimeter* of the rectangular window. Use the equation for the perimeter of a rectangle.

$$2l + 2w = 182$$

The system of equations is

$$\begin{aligned} w &= l - 23 \\ 2l + 2w &= 182. \end{aligned}$$

Step 4: **Solve** the system.

$$\begin{aligned} 2l + 2w &= 182 && \\ 2l + 2(l - 23) &= 182 && \text{Substitute.} \\ 2l + 2l - 46 &= 182 && \text{Distribute.} \\ 4l - 46 &= 182 && \text{Combine like terms.} \\ 4l &= 228 && \text{Add 46.} \\ \frac{4l}{4} &= \frac{228}{4} && \text{Divide each side by 4.} \\ l &= 57 && \text{Simplify.} \end{aligned}$$

Find w by substituting $l = 57$ into $w = l - 23$.

$$w = 57 - 23 = 34$$

The solution to the system is (57, 34). (The ordered pair is written as (l, w), in alphabetical order.)

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

The length of the window is 57 in., and the width is 34 in. The check is left to the student.

You Try 2

Write a system of equations and solve.

The top of a desk is twice as long as it is wide. If the perimeter of the desk is 162 in., find its dimensions.

3. Solve Problems Involving Cost

Example 3

Write a system of equations and solve.

Ari buys two mezzanine tickets to a Broadway play and four tickets to the top of the Empire State Building for \$352. Lloyd spends \$609 on four mezzanine tickets and three tickets to the top of the Empire State Building. Find the cost of a ticket to each attraction.

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the cost of a ticket to a Broadway play and to the top of the Empire State Building.

Step 2: **Choose variables** to represent the unknown quantities.

x = the cost of a ticket to a Broadway play

y = the cost of a ticket to the Empire State Building

Step 3: **Write a system of equations using two variables.** First, let's think of the equations in English. Then we will translate them into algebraic equations.

First, use the information about Ari's purchase.

English:	Cost of two play tickets	plus	Cost of four Empire State tickets	equals	Ari's total cost
	↓	↓	↓	↓	↓
Equation:	$2x$	$+$	$4y$	$=$	352
	Number of tickets ↗ ↖ Cost of each ticket		Number of tickets ↗ ↖ Cost of each ticket		

One equation is $2x + 4y = 352$.

Next, use the information about Lloyd's purchase.

English:	Cost of four play tickets	plus	Cost of three Empire State tickets	equals	Lloyd's total cost
	↓	↓	↓	↓	↓
Equation:	$4x$	$+$	$3y$	$=$	609
	Number of tickets ↗ ↖ Cost of each ticket		Number of tickets ↗ ↖ Cost of each ticket		

The other equation is $4x + 3y = 609$.

The system of equations is
$$\begin{aligned} 2x + 4y &= 352 \\ 4x + 3y &= 609. \end{aligned}$$

Step 4: **Solve** the system.

Use the elimination method. Multiply the first equation by -2 to eliminate x.

$$\begin{array}{rr} & -4x - 8y = -704 \\ + & 4x + 3y = 609 \\ \hline & -5y = -95 \\ & y = 19 \end{array}$$

Add the equations.

Find x. We will substitute $y = 19$ into $2x + 4y = 352$.

$$\begin{aligned} 2x + 4(19) &= 352 \qquad \text{Substitute.} \\ 2x + 76 &= 352 \\ 2x &= 276 \\ x &= 138 \end{aligned}$$

The solution to the system is (138, 19).

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

A Broadway play ticket costs \$138.00, and a ticket to the top of the Empire State Building costs \$19.00.

The check is left to the student.

You Try 3

Write a system of equations and solve.

Torie bought three scarves and a belt for \$105 while Liz bought two scarves and two belts for \$98. Find the cost of a scarf and a belt.

In Chapter 3, we learned how to solve mixture problems by writing an equation in one variable. Now we will learn how to solve the same type of problem using two variables and a system of equations.

4. Solve Mixture Problems

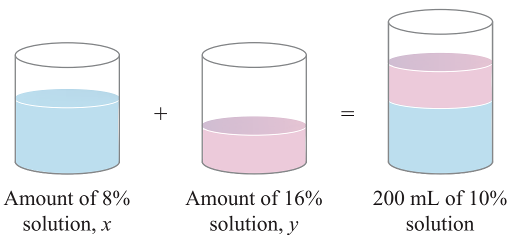
Example 4

A pharmacist needs to make 200 mL of a 10% hydrogen peroxide solution. She will make it from some 8% hydrogen peroxide solution and some 16% hydrogen peroxide solution that are in the storeroom. How much of the 8% solution and 16% solution should she use?

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find. Draw a picture.

We must find the amount of 8% solution and 16% solution she should use.



Step 2: **Choose variables** to represent the unknown quantities.

x = amount of 8% solution needed

y = amount of 16% solution needed

Step 3: **Write a system of equations using two variables.**

Let's begin by arranging the information in a table. Remember, to obtain the expression in the last column, multiply the percent of hydrogen peroxide in the solution by the amount of solution to get the amount of pure hydrogen peroxide in the solution.

	Percent of Hydrogen Peroxide in Solution (as a decimal)	Amount of Solution	Amount of Pure Hydrogen Peroxide in Solution
Mix these	0.08	x	$0.08x$
	0.16	y	$0.16y$
to make →	0.10	200	0.10(200)

To get one equation, use the information in the second column. It tells us that

English: Amount of 8% solution | plus | Amount of 16% solution | equals | Amount of 10% solution

Equation: x | $+$ | y | $=$ | 200

The equation is $x + y = 200$.

To get the second equation, use the information in the third column. It tells us that

English: Amount of pure hydrogen peroxide in the 8% solution | plus | Amount of pure hydrogen peroxide in the 16% solution | equals | Amount of pure hydrogen peroxide in the 10% solution

Equation: $0.08x$ | $+$ | $0.16y$ | $=$ | $0.10(200)$

The equation is $0.08x + 0.16y = 0.10(200)$.

The system of equations is
$$\begin{aligned} x + y &= 200 \\ 0.08x + 0.16y &= 0.10(200). \end{aligned}$$

Step 4: **Solve** the system.

Multiply the second equation by 100 to eliminate the decimals. Our system becomes

$$\begin{aligned} x + y &= 200 \\ 8x + 16y &= 2000 \end{aligned}$$

Use the elimination method. Multiply the first equation by -8 to eliminate x.

$$\begin{array}{rrcr} & -8x - 8y & = & -1600 \\ + & 8x + 16y & = & 2000 \\ \hline & 8y & = & 400 \\ & y & = & 50 \end{array}$$

Find x. Substitute $y = 50$ into $x + y = 200$.

$$\begin{aligned} x + 50 &= 200 \\ x &= 150 \end{aligned}$$

The solution to the system is (150, 50).

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

The pharmacist needs 150 mL of the 8% solution and 50 mL of the 16% solution. Check the answers in the original problem to verify that they are correct. ■

You Try 4

Write an equation and solve.

How many milliliters of a 5% acid solution and how many milliliters of a 17% acid solution must be mixed to obtain 60 mL of a 13% acid solution?

5. Solve Distance, Rate, and Time Problems

Example 5

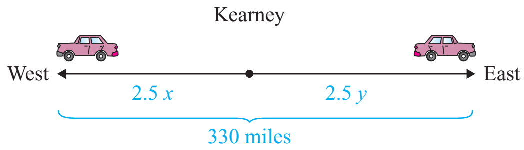
Write an equation and solve.

Two cars leave Kearney, Nebraska, one driving east and the other heading west. The eastbound car travels 4 mph faster than the westbound car, and after 2.5 hours they are 330 miles apart. Find the speed of each car.

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the speed of the eastbound and westbound cars. We will draw a picture to help us see what is happening in this problem. After 2.5 hours, the position of the cars looks like this:



Step 2: **Choose variables** to represent the unknown quantities.

$x =$ the speed of the westbound car

$y =$ the speed of the eastbound car

Step 3: **Write a system of equations using two variables.**

Let's make a table using the equation $d = rt$. Fill in the time, 2.5 hr, and the rates first, then multiply those together to fill in the values for the distance.

	d	r	t
Westbound	$2.5x$	x	2.5
Eastbound	$2.5y$	y	2.5

Label the picture with the expressions for the distances.

To get one equation, look at the picture and think about the distance between the cars after 2.5 hr.

English:	Distance traveled by westbound car	plus	Distance traveled by eastbound car	equals	Distance between them
	↓	↓	↓	↓	↓
Equation:	$2.5x$	$+$	$2.5y$	$=$	330

The equation is $2.5x + 2.5y = 330$.

To get the second equation, use the information that says the eastbound car goes 4 mph faster than the westbound car.

English:	Speed of eastbound car	is	4 mph faster than	Speed of westbound car
	↓	↓	↓	↓
Equation:	y	$=$	$4 +$	x

The equation is $y = 4 + x$.

The system of equations is
$$\begin{aligned} 2.5x + 2.5y &= 330 \\ y &= 4 + x. \end{aligned}$$

Step 4: **Solve** the system.

Use substitution.

$$\begin{aligned} 2.5x + 2.5y &= 330 & & \\ 2.5x + 2.5(4 + x) &= 330 & & \text{Substitute } 4 + x \text{ for } y. \\ 2.5x + 10 + 2.5x &= 330 & & \text{Distribute.} \\ 5x + 10 &= 330 & & \text{Combine like terms.} \\ 5x &= 320 & & \\ x &= 64 & & \end{aligned}$$

Find y by substituting $x = 64$ into $y = 4 + x$.

$$y = 4 + 64 = 68$$

The solution to the system is (64, 68).

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

The speed of the westbound car is 64 mph, and the speed of the eastbound car is 68 mph.

Check.

$$\underbrace{2.5(64)}_{\text{Distance of westbound car}} + \underbrace{2.5(68)}_{\text{Distance of eastbound car}} = 160 + 170 = 330 \text{ mi}$$

You Try 5

Write an equation and solve.

Two planes leave the same airport, one headed north and the other headed south. The northbound plane goes 100 mph slower than the southbound plane. Find each of their speeds if they are 1240 miles apart after 2 hours.

Answers to You Try Exercises

1) Carson City: 55,000; Elmira: 88,000 2) width: 27 in.; length: 54 in.
3) scarf: \$28; belt: \$21 4) 20 mL of 5% solution; 40 mL of 17% solution
5) northbound plane: 260 mph; southbound plane: 360 mph

5.4 Exercises

Objective 1: Solve Problems Involving General Quantities

Write a system of equations and solve.

1) The sum of two numbers is 87, and one number is eleven more than the other. Find the numbers.

2) One number is half another number. The sum of the two numbers is 141. Find the numbers.

3) Through the summer of 2009, *The Dark Knight* and *Transformers: Revenge of the Fallen* earned more money on their opening days than any other movies. *The Dark Knight* grossed \$6.6 million more than *Transformers.* Together, they brought in \$127.8 million. How much did each film earn on opening day? (http://hollywoodinsider.ew.com)

4) In the 1976–1977 season, Kareem Abdul-Jabbar led all players in blocked shots. He blocked 50 more shots than Bill Walton, who finished in second place. How many shots did each man block if they rejected a total of 472 shots? (www.nba.com)

5) Through 2009, Beyonce had been nominated for five more BET Awards than T.I. Determine how many nominations each performer received if they got a total of 27 nominations. (http://en.wikipedia.org)

6) From 1965 to 2000, twice as many people immigrated to the United States from The Philippines as from Vietnam. The total number of immigrants from these two countries was 2,100,000. How many people came to the United States from each country? (www.ellisisland.org)

7) According to a U.S. Census Bureau survey in 2006, about half as many people in the United States spoke Urdu at home as spoke Polish. If a total of about 975,000 people spoke these languages in their homes, how many spoke Urdu and how many spoke Polish? (www.census.gov)

8) During one week, a hardware store sold 27 fewer "regular" incandescent lightbulbs than energy-efficient compact fluorescent light (CFL) bulbs. How many of each type of bulb was sold if the store sold a total of 79 of these two types of lightbulbs?

9) On April 12, 1961, Yuri Gagarin of the Soviet Union became the first person in space when he piloted the Vostok 1 mission. The next month, Alan B. Shepard became the first American in space in the Freedom 7 space capsule. The two of them spent a total of about 123 minutes in space, with Gagarin logging 93 more minutes than Shepard. How long did each man spend in space? (www-pao.ksc.nasa.gov, www.enchantedlearning.com)

10) Mr. Monet has 85 students in his Art History lecture. For their assignment on impressionists, one-fourth as many students chose to recreate an impressionist painting as chose to write a paper. How many students will be painting, and how many will be writing papers?

Objective 2: Solve Geometry Problems

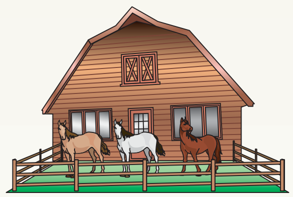
VIDEO 11) Find the dimensions of a rectangular door that has a perimeter of 220 in. if the width is 50 in. less than the height of the door.

12) The length of a rectangle is 3.5 in. more than its width. If the perimeter is 23 in., what are the dimensions of the rectangle?

13) An iPod Touch is rectangular in shape and has a perimeter of 343.6 mm. Find its length and width given that it is 48.2 mm longer than it is wide.

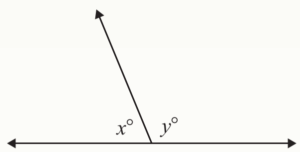
14) Eliza needs 332 in. of a decorative border to sew around a rectangular quilt she just made. Its width is 26 in. less than its length. Find the dimensions of the quilt.

15) A rectangular horse corral is bordered on one side by a barn as pictured here. The length of the corral is 1.5 times the width. If 119 ft of fencing was used to make the corral, what are its dimensions?

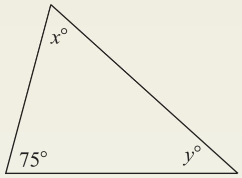


16) The length of a rectangular mirror is twice its width. Find the dimensions of the mirror if its perimeter is 246 cm.

17) Find the measures of angles x and y if the measure of angle x is three-fifths the measure of angle y and if the angles are related according to the figure.



18) Find the measures of angles x and y if the measure of angle y is two-thirds the measure of angle x and if the angles are related according to the figure.



Objective 3: Solve Problems Involving Cost

19) Kenny and Kyle are huge Colorado Avalanche fans. Kenny buys a T-shirt and two souvenir hockey pucks for \$36.00, and Kyle spends \$64.00 on two T-shirts and three pucks. Find the price of a T-shirt and the price of a puck.

20) Bruce Springsteen and Jimmy Buffett each played in Chicago in 2009. Four Springsteen tickets and four Buffett tickets cost \$908.00, while three Springsteen tickets and two Buffett tickets cost \$552.00. Find the cost of a ticket to each concert. (www.ticketmaster.com)

21) Angela and Andy watch *The Office* every week with their friends and decide to buy them some gifts. Angela buys three Dwight bobbleheads and four star mugs for \$105.00, while Andy spends \$74.00 on two bobbleheads and three mugs. Find the cost of each item. (www.nbcuniversalstore.com)

22) Manny and Hiroki buy tickets in advance to some Los Angeles Dodgers games. Manny buys three left-field pavilion seats and six club seats for \$423.00. Hiroki spends \$413.00 on eight left-field pavilion seats and five club seats. Find the cost of each type of ticket. (www.dodgers.com)

23) Carol orders five White Castle hamburgers and a small order of french fries for \$4.44, and Momar orders four hamburgers and two small fries for \$5.22. Find the cost of a hamburger and the cost of a small order of french fries at White Castle. (White Castle menu)

24) Phuong buys New Jersey lottery tickets every Friday. One day she spent \$17.00 on four Gold Strike tickets and three Super Cashout tickets. The next Friday, she bought three Gold Strike tickets and six Super Cashout tickets for \$24.00. How much did she pay for each type of lottery ticket?

25) Lakeisha is selling wrapping paper products for a school fund-raiser. Her mom buys four rolls of wrapping paper and three packages of gift bags for \$52.00. Her grandmother spends \$29.00 on three rolls of wrapping paper and one package of gift bags. Find the cost of a roll of wrapping paper and a package of gift bags.

26) Alberto is selling popcorn to raise money for his Cub Scout den. His dad spends \$86.00 on two tins of popcorn and three tins of caramel corn. His neighbor buys two tins of popcorn and one tin of caramel corn for \$48.00. How much does each type of treat cost?

Objective 4: Solve Mixture Problems

VIDEO 27) How many ounces of a 9% alcohol solution and how many ounces of a 17% alcohol solution must be mixed to obtain 12 oz of a 15% alcohol solution?

28) How many milliliters of a 15% acid solution and how many milliliters of a 3% acid solution must be mixed to get 45 mL of a 7% alcohol solution?

29) How many liters of pure acid and how many liters of a 25% acid solution should be mixed to get 10 L of a 40% acid solution?

30) How many ounces of pure cranberry juice and how many ounces of a citrus fruit drink containing 10% fruit juice should be mixed to get 120 oz of a fruit drink that is 25% fruit juice?

31) How many ounces of Asian Treasure tea that sells for \$7.50/oz should be mixed with Pearadise tea that sells for \$5.00/oz so that a 60-oz mixture is obtained that will sell for \$6.00/oz?

32) How many pounds of peanuts that sell for \$1.80 per pound should be mixed with cashews that sell for \$4.50 per pound so that a 10-pound mixture is obtained that will sell for \$2.61 per pound?

33) During a late-night visit to Taco Bell, Giovanni orders three Crunchy Tacos and a chicken chalupa supreme. His order contains 1640 mg of sodium. Jurgis orders two Crunchy Tacos and two chicken chalupa supremes, and his order contains 1960 mg of sodium. How much sodium is in each item? (www.tacobell.com)

34) Five White Castle hamburgers and one small order of french fries contain 1010 calories. Four hamburgers and two orders of fries contain 1180 calories. Determine how many calories are in a White Castle hamburger and in a small order of french fries. (www.whitecastle.com)

35) Mahmud invested \$6000 in two accounts, some of it at 2% simple interest, the rest in an account earning 4% simple interest. How much did he invest in each account if he earned \$190 in interest after 1 year?

36) Marijke inherited \$15,000 and puts some of it into an account earning 5% simple interest and the rest in an account earning 4% simple interest. She earns a total of \$660 in interest after 1 year. How much did she deposit into each account?

37) Oscar purchased 16 stamps. He bought some \$0.44 stamps and some \$0.28 stamps and spent \$6.40. How many of each type of stamp did he buy?

38) Kelly saves all of her dimes and nickels in a jar on her desk. When she counts her money, she finds that she has 133 coins worth a total of \$10.45. How many dimes and how many nickels does she have?

Objective 5: Solve Distance, Rate, and Time Problems

39) Michael and Jan leave the same location but head in opposite directions on their bikes. Michael rides 1 mph faster than Jan, and after 3 hr they are 51 miles apart. How fast was each of them riding?

40) A passenger train and a freight train leave cities 400 miles apart and travel toward each other. The passenger train is traveling 16 mph faster than the freight train. Find the speed of each train if they pass each other after 5 hours.

41) A small plane leaves an airport and heads south, while a jet takes off at the same time heading north. The speed of the small plane is 160 mph less than the speed of the jet. If they are 1280 miles apart after 2 hours, find the speeds of both planes.

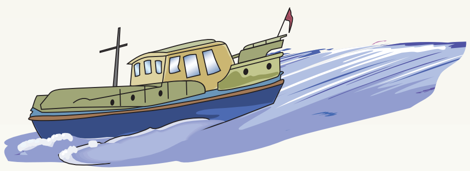
42) Tyreese and Justine start jogging toward each other from opposite ends of a trail 6.5 miles apart. They meet after 30 minutes. Find their speeds if Tyreese jogs 3 mph faster than Justine.

43) Pam and Jim leave opposite ends of a bike trail 9 miles apart and travel toward each other. Pam is traveling 2 mph slower than Jim. Find each of their speeds if they meet after 30 minutes.

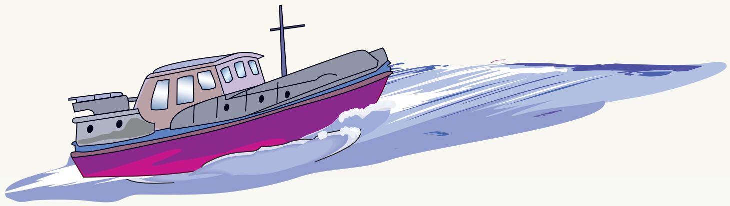
44) Stanley and Phyllis leave the office and travel in opposite directions. Stanley drives 6 mph slower than Phyllis, and after 1 hr they are 76 miles apart. How fast was each person driving?

Other types of distance, rate, and time problems involve a boat traveling upstream and downstream, and a plane traveling with and against the wind. To solve problems like these, we will still use a table to help us organize our information, but we must understand what is happening in such problems.

Let's think about the case of a boat traveling upstream and downstream.



Downstream



Upstream

Let x = the speed of the boat in still water and let y = the speed of the current.

When the boat is going *downstream* (*with* the current), the boat is being pushed along by the current so that

The speed of the boat going downstream	=	The speed of the boat in still water	plus	The speed of the current
↓	↓	↓	↓	↓
The speed of the boat going downstream	=	x	+	y

When the boat is going *upstream* (*against* the current), the boat is being slowed down by the current so that

The speed of the boat going upstream	=	The speed of the boat in still water	minus	The speed of the current
↓	↓	↓	↓	↓
The speed of the boat going upstream	=	x	−	y

Use this idea to solve Exercises 45–50.

45) It takes 2 hours for a boat to travel 14 miles downstream. The boat can travel 10 miles upstream in the same amount of time. Find the speed of the boat in still water and the speed of the current. (Hint: Use the information in the following table, and write a system of equations.)

	d	*r*	*t*
Downstream	14	$x + y$	2
Upstream	10	$x - y$	2

46) A boat can travel 15 miles downstream in 0.75 hours. It takes the same amount of time for the boat to travel 9 miles upstream. Find the speed of the boat in still water and the speed of the current. (Hint: Use the information in the following table, and write a system of equations.)

	d	*r*	*t*
Downstream	15	$x + y$	0.75
Upstream	9	$x - y$	0.75

47) It takes 5 hours for a boat to travel 80 miles downstream. The boat can travel the same distance back upstream in 8 hours. Find the speed of the boat in still water and the speed of the current.

48) A boat can travel 12 miles downstream in 1.5 hours. It takes 3 hours for the boat to travel back to the same spot going upstream. Find the speed of the boat in still water and the speed of the current.

49) A jet can travel 1000 miles against the wind in 2.5 hours. Going with the wind, the jet could travel 1250 miles in the same amount of time. Find the speed of the jet in still air and speed of the wind.

50) It takes 2 hours for a small plane to travel 390 miles with the wind. Going against the wind, the plane can travel 330 miles in the same amount of time. Find the speed of the plane in still air and the speed of the wind.

Section 5.5 Solving Systems of Three Equations and Applications

Objectives

1. Understand Systems of Three Equations in Three Variables
2. Solve Systems of Linear Equations in Three Variables
3. Solve Special Systems in Three Variables
4. Solve a System with Missing Terms
5. Solve Applied Problems

In this section, we will learn how to solve a system of linear equations in *three* variables.

1. Understand Systems of Three Equations in Three Variables

Definition

A **linear equation in three variables** is an equation of the form $Ax + By + Cz = D$ where A, B, and C are not all zero and where A, B, C, and D are real numbers. Solutions to this type of an equation are **ordered triples** of the form (x, y, z).

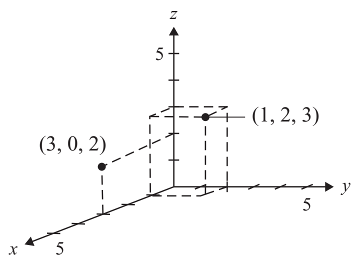
An example of a linear equation in three variables is

$$2x - y + 3z = 12.$$

This equation has infinitely many solutions. Here are a few:

$$\begin{array}{lll} (5, 1, 1) & \text{since} & 2(5) - (1) + 3(1) = 12 \\ (3, 0, 2) & \text{since} & 2(3) - (0) + 3(2) = 12 \\ (6, -3, -1) & \text{since} & 2(6) - (-3) + 3(-1) = 12 \end{array}$$

Ordered triples, like (1, 2, 3) and (3, 0, 2), are graphed on a three-dimensional coordinate system, as shown to the right. Notice that the ordered triples are *points*.



The graph of a linear equation in three variables is a *plane*.

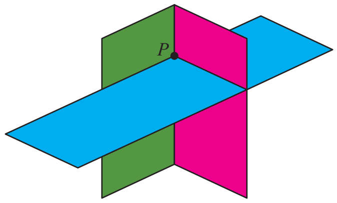
A **solution to a system of linear equations in three variables** is an *ordered triple* that satisfies each equation in the system. Like systems of linear equations in two variables, systems of linear equations in *three* variables can have *one* solution, *no* solution, or *infinitely many* solutions.

Here is an example of a system of linear equations in three variables:

$$\begin{aligned} x + 4y + 2z &= 10 \\ 3x - y + z &= 6 \\ 2x + 3y - z &= -4 \end{aligned}$$

In Section 5.1, we solved systems of linear equations in *two* variables by graphing. Since the graph of an equation like $x + 4y + 2z = 10$ is a *plane*, however, solving a system in three variables by graphing would not be practical. But let's look at the graphs of systems of linear equations in three variables that have one solution, no solution, or an infinite number of solutions.

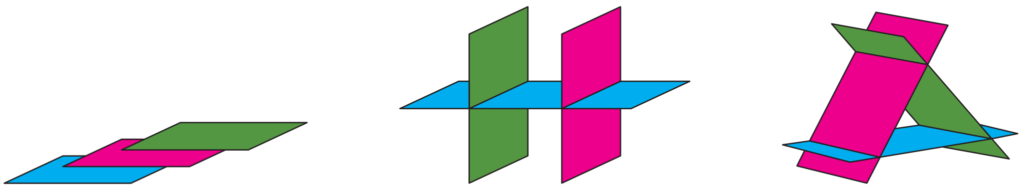
One solution:



Intersection is at point P.

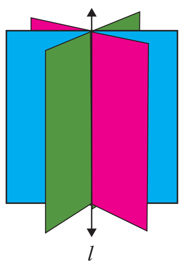
All three planes intersect at one point; this is the solution of the system.

No solution:

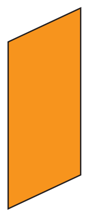


None of the planes may intersect or *two* of the planes may intersect, but if there is no solution to the system, *all three planes* do not have a common point of intersection.

Infinite number of solutions:



Intersection is the set of points on line l.



Intersection is the set of points on the plane.

The three planes may intersect so that they have a line or a plane in common. The solution to the system is the infinite set of points on the line or the plane, respectively.

2. Solve Systems of Linear Equations in Three Variables

First we will learn how to solve a system in which each equation has three variables.

Procedure Solving a System of Linear Equations in Three Variables

1) **Label** the equations (1), (2), and (3).
2) **Choose a variable to eliminate. Eliminate** this variable from *two* sets of *two* equations using the elimination method. You will obtain two equations containing the same two variables. Label one of these new equations $\boxed{A}$ and the other $\boxed{B}$.
3) Use the elimination method to **eliminate a variable from equations $\boxed{A}$ and $\boxed{B}$.** You have now found the value of one variable.
4) **Find the value of another variable** by substituting the value found in *Step 3* into equation $\boxed{A}$ or $\boxed{B}$ and solving for the second variable.
5) **Find the value of the third variable** by substituting the values of the two variables found in *Steps 3 and 4* into equation (1), (2), or (3).
6) **Check** the solution in each of the original equations, and **write the solution as an ordered triple.**

Example 1

Solve

$$\begin{array}{rl} (1) & x + 2y - 2z = 3 \\ (2) & 2x + y + 3z = 1 \\ (3) & x - 2y + z = -10 \end{array}$$

Solution

Steps 1) and 2) We have already **labeled** the equations. We'll **choose** to eliminate the variable y from *two* sets of *two* equations:

a) Add equations (1) and (3) to eliminate y. Label the resulting equation $\boxed{A}$.

$$\begin{array}{rrl} (1) & & x + 2y - 2z = 3 \\ (3) & + & x - 2y + z = -10 \\ \hline \boxed{A} & & 2x \qquad\; - z = -7 \end{array}$$

b) Multiply equation (2) by 2 and add it to equation (3) to eliminate y. Label the resulting equation $\boxed{B}$.

$$\begin{array}{rrl} 2 \times (2) & & 4x + 2y + 6z = 2 \\ (3) & + & x - 2y + z = -10 \\ \hline \boxed{B} & & 5x \qquad\; + 7z = -8 \end{array}$$

Note

Equations $\boxed{A}$ and $\boxed{B}$ contain only *two* variables and they are the same variables, x and z.

3) Use the elimination method to **eliminate a variable from equations $\boxed{A}$ and $\boxed{B}$.** We will eliminate z from $\boxed{A}$ and $\boxed{B}$. Multiply $\boxed{A}$ by 7 and add it to $\boxed{B}$.

$$\begin{array}{rrl} 7 \times \boxed{A} & & 14x - 7z = -49 \\ \boxed{B} & + & 5x + 7z = -8 \\ \hline & & 19x \qquad = -57 \\ & & \boxed{x = -3} \end{array}$$

Divide by 19.

4) **Find the value of another variable** by substituting $x = -3$ into equation $\boxed{A}$ or $\boxed{B}$. We will use $\boxed{A}$ since it has smaller coefficients.

$$\begin{aligned} \boxed{A} \quad 2x - z &= -7 \\ 2(-3) - z &= -7 && \text{Substitute } -3 \text{ for } x. \\ -6 - z &= -7 && \text{Multiply.} \\ \boxed{z = 1} & && \text{Add 6 and divide by } -1. \end{aligned}$$

5) **Find the value of the third variable** by substituting $x = -3$ and $z = 1$ into equation (1), (2), or (3). We will use equation (1).

$$\begin{aligned} (1) \quad x + 2y - 2z &= 3 \\ -3 + 2y - 2(1) &= 3 && \text{Substitute } -3 \text{ for } x \text{ and } 1 \text{ for } z. \\ -3 + 2y - 2 &= 3 && \text{Multiply.} \\ 2y - 5 &= 3 && \text{Combine like terms.} \\ \boxed{y = 4} & && \text{Add 5 and divide by 2.} \end{aligned}$$

6) **Check** the solution, $(-3, 4, 1)$, in each of the original equations, and **write the solution**.

$$\begin{aligned} (1) \quad x + 2y - 2z &= 3 \\ -3 + 2(4) - 2(1) &\stackrel{?}{=} 3 \\ -3 + 8 - 2 &\stackrel{?}{=} 3 \\ 3 &= 3 \quad \checkmark \text{ True} \end{aligned}$$

$$\begin{aligned} (2) \quad 2x + y + 3z &= 1 \\ 2(-3) + 4 + 3(1) &\stackrel{?}{=} 1 \\ -6 + 4 + 3 &\stackrel{?}{=} 1 \\ 1 &= 1 \quad \checkmark \text{ True} \end{aligned}$$

$$\begin{aligned} (3) \quad x - 2y + z &= -10 \\ -3 - 2(4) + 1 &\stackrel{?}{=} -10 \\ -3 - 8 + 1 &\stackrel{?}{=} -10 \\ -10 &= -10 \quad \checkmark \text{ True} \end{aligned}$$

The solution is $(-3, 4, 1)$.

You Try 1

Solve
$$\begin{aligned} x + 2y + 3z &= -11 \\ 3x - y + z &= 0 \\ -2x + 3y - z &= 4 \end{aligned}$$

3. Solve Special Systems in Three Variables

Some systems in three variables have no solution and some have an infinite number of solutions.

Example 2

Solve
$$\begin{aligned} (1) \quad -3x + 2y - z &= 5 \\ (2) \quad x + 4y + z &= -4 \\ (3) \quad 9x - 6y + 3z &= -2 \end{aligned}$$

Solution

Steps 1) and 2) We have already *labeled* the equations. The *variable we choose to eliminate* is z, the easiest.

a) Add equations (1) and (2) to eliminate z. Label the resulting equation $\boxed{A}$.

$$\begin{aligned} (1) \quad -3x + 2y - z &= 5 \\ (2) \quad + \quad x + 4y + z &= -4 \\ \hline \boxed{A} \quad -2x + 6y \quad &= 1 \end{aligned}$$

b) Multiply equation 1 by 3 and add it to equation 3 to eliminate z. Label the resulting equation $\boxed{B}$.

$$1 \quad -3x + 2y - z = 5 \longrightarrow \begin{array}{r r} 3 \times 1 & -9x + 6y - 3z = 15 \\ 3 & +\ \ 9x - 6y + 3z = -2 \\ \hline \boxed{B} & 0 = 13 \quad \text{False} \end{array}$$

Since the variables are eliminated and we get the false statement $0 = 13$, equations 1 and 3 have no ordered triple that satisfies each equation.

The system is inconsistent, so there is no solution. The solution set is $\varnothing$. ■

Note

If the variables are eliminated and you get a false statement, there is *no solution* to the system. The system is inconsistent, so the solution set is $\varnothing$.

Example 3

Solve

$$\begin{aligned} 1 \quad -4x - 2y + 8z &= -12 \\ 2 \quad 2x + y - 4z &= 6 \\ 3 \quad 6x + 3y - 12z &= 18 \end{aligned}$$

Solution

Steps 1) and 2) We label the equations and choose a variable, y, to eliminate.

a) Multiply equation 2 by 2 and add it to equation 1. Label the resulting equation $\boxed{A}$.

$$\begin{array}{r r} 2 \times 2 & 4x + 2y - 8z = \ \ 12 \\ 1 & +\ -4x - 2y + 8z = -12 \\ \hline \boxed{A} & 0 = 0 \quad \text{True} \end{array}$$

The variables were eliminated and we obtained the true statement $0 = 0$. This is because equation 1 is a multiple of equation 2.

Notice, also, that equation 3 is a multiple of equation 2.

The equations in this system are dependent. There are an infinite number of solutions and we write the solution set as $\{(x, y, z) | 2x + y - 4z = 6\}$. The equations all have the same graph. ■

You Try 2

Solve each system of equations.

a)
$$\begin{aligned} 8x + 20y - 4z &= -16 \\ -6x - 15y + 3z &= 12 \\ 2x + 5y - z &= -4 \end{aligned}$$

b)
$$\begin{aligned} x + 4y - 3z &= 2 \\ 2x - 5y + 2z &= -8 \\ -3x - 12y + 9z &= 7 \end{aligned}$$

4. Solve a System with Missing Terms

Example 4

Solve
$$\begin{aligned} (1)\quad 5x - 2y &= 6 \\ (2)\quad y + 2z &= 1 \\ (3)\quad 3x - 4z &= -8 \end{aligned}$$

Solution

First, notice that while this *is* a system of three equations in three variables, none of the equations contains three variables. Furthermore, each equation is "missing" a different variable.

Note

We will use many of the *ideas* outlined in the steps for solving a system of three equations, but we will use *substitution* rather than the elimination method.

1) Label the equations (1), (2), and (3).

2) The goal of step 2 is to obtain two equations that contain the same two variables. We will modify this step from the way it was outlined on p. 332.

   In order to obtain *two* equations with the same *two* variables, we will use *substitution*.

   Since y in equation (2) is the only variable in the system with a coefficient of 1, we will solve equation (2) for y.

$$\begin{aligned} (2)\quad y + 2z &= 1 \\ y &= 1 - 2z && \text{Subtract } 2z. \end{aligned}$$

   Substitute $y = 1 - 2z$ into equation (1) to obtain an equation containing the variables x and z. Simplify. Label the resulting equation $\boxed{A}$.

$$\begin{aligned} (1)\quad 5x - 2y &= 6 \\ 5x - 2(1 - 2z) &= 6 && \text{Substitute } 1 - 2z \text{ for } y. \\ 5x - 2 + 4z &= 6 && \text{Distribute.} \\ \boxed{A}\quad 5x + 4z &= 8 && \text{Add 2.} \end{aligned}$$

3) The goal of step 3 is to solve for one of the variables. Equations $\boxed{A}$ and (3) contain only x and z.

   We will eliminate z from $\boxed{A}$ and (3). Add the two equations to eliminate z, then solve for x.

$$\begin{aligned} \boxed{A}\quad 5x + 4z &= 8 \\ (3)\quad + \; 3x - 4z &= -8 \\ \hline 8x \qquad &= 0 \\ \boxed{x = 0} & && \text{Divide by 8.} \end{aligned}$$

4) Find the value of another variable by substituting $x = 0$ into either $\boxed{A}$, (1), or (3).

$$\begin{aligned} \boxed{A}\quad 5x + 4z &= 8 \\ 5(0) + 4z &= 8 && \text{Substitute 0 for } x. \\ 4z &= 8 \\ \boxed{z = 2} & && \text{Divide by 4.} \end{aligned}$$

5) Find the value of the third variable by substituting $x = 0$ into ① or $z = 2$ into ②.

$$
\begin{aligned}
① \quad 5x - 2y &= 6 \\
5(0) - 2y &= 6 && \text{Substitute 0 for } x. \\
-2y &= 6 \\
\boxed{y = -3} & && \text{Divide by } -2.
\end{aligned}
$$

6) Check the solution $(0, -3, 2)$ in each of the original equations. The check is left to the student. The solution is $(0, -3, 2)$. ■

You Try 3

Solve
$$
\begin{aligned}
x + 2y &= 8 \\
2y + 3z &= 1 \\
3x - z &= -3
\end{aligned}
$$

5. Solve Applied Problems

To solve applications involving a system of three equations in three variables, we will extend the method used for two equations in two variables.

Example 5

Write a system of equations and solve.

The top three gold-producing nations in 2002 were South Africa, the United States, and Australia. Together, these three countries produced 37% of the gold during that year. Australia's share was 2% less than that of the United States, while South Africa's percentage was 1.5 times Australia's percentage of world gold production. Determine what percentage of the world's gold supply was produced by each country in 2002.
(Market Share Reporter—2005, Vol. 1, "Mine Product" http://www.gold.org/value/market/supply-demand/min_production.html from World Gold Council)

Solution

Step 1: **Read** the problem carefully. We must determine the percentage of the world's gold produced by South Africa, the United States, and Australia in 2002.

Step 2: **Choose variables** to represent the unknown quantities.

x = percentage of world's gold supply produced by South Africa

y = percentage of world's gold supply produced by the United States

z = percentage of world's gold supply produced by Australia

Step 3: **Write a system of equations using the variables.**

To write one equation, we will use the information that says *together* the three countries produced 37% of the gold.

$$x + y + z = 37 \qquad \text{Equation (1)}$$

To write a second equation, we will use the information that says Australia's share was 2% less than that of the United States.

$$z = y - 2 \qquad \text{Equation (2)}$$

To write the third equation, we will use the statement that says South Africa's percentage was 1.5 times Australia's percentage.

$$x = 1.5z \qquad \text{Equation (3)}$$

The system is

$$\begin{aligned} (1)\quad x + y + z &= 37 \\ (2)\quad z &= y - 2 \\ (3)\quad x &= 1.5z \end{aligned}$$

Step 4: **Solve** the system. Since two of the equations contain only two variables, we will modify our steps to solve the system.

Our plan is to rewrite equation (1) in terms of a single variable, z, and solve for z.

Solve equation (2) for y.

$$\begin{aligned} (2)\quad z &= y - 2 \\ z + 2 &= y && \text{Solve for } y. \end{aligned}$$

Now rewrite equation (1) using the value for y from equation (2) and the value for x from equation (3).

$$\begin{aligned} (1)\quad x + y + z &= 37 && \text{Equation (1)} \\ (1.5z) + (z + 2) + z &= 37 && \text{Substitute } 1.5z \text{ for } x \text{ and } z + 2 \text{ for } y. \\ 3.5z + 2 &= 37 && \text{Combine like terms.} \\ 3.5z &= 35 && \text{Subtract 2.} \\ \boxed{z = 10} & && \text{Divide by 3.5.} \end{aligned}$$

To solve for x, we substitute $z = 10$ into equation (3).

$$\begin{aligned} (3)\quad x &= 1.5z \\ x &= 1.5(10) && \text{Substitute 10 for } z. \\ \boxed{x = 15} & && \text{Multiply.} \end{aligned}$$

To solve for y, we substitute $z = 10$ into equation (2).

$$\begin{aligned} (2)\quad z &= y - 2 \\ 10 &= y - 2 && \text{Substitute 10 for } z. \\ \boxed{12 = y} & && \text{Solve for } y. \end{aligned}$$

The solution of the system is (15, 12, 10).

Step 5: **Check** and **interpret** the solution.

In 2002, South Africa produced 15% of the world's gold, the United States produced 12%, and Australia produced 10%. The check is left to the student. ■

You Try 4

Write a system of equations and solve.

Amelia, Bella, and Carmen are sisters. Bella is 5 yr older than Carmen, and Amelia's age is 5 yr less than twice Carmen's age. The sum of their ages is 48. How old is each girl?

Answers to You Try Exercises

1) $(2, 1, -5)$ 2) a) $\{(x, y, z) | 2x + 5y - z = -4\}$ b) $\varnothing$ 3) $(-2, 5, -3)$

4) Amelia: 19; Bella: 17; Carmen: 12

5.5 Exercises

Objective 1: Understand Systems of Three Equations in Three Variables

Determine whether the ordered triple is a solution of the system.

1) $\begin{aligned} 4x + 3y - 7z &= -6 \\ x - 2y + 5z &= -3 \\ -x + y + 2z &= 7 \end{aligned}$
$(-2, 3, 1)$

2) $\begin{aligned} 3x + y + 2z &= 2 \\ -2x - y + z &= 5 \\ x + 2y - z &= -11 \end{aligned}$
$(1, -5, 2)$

3) $\begin{aligned} -x + y - 2z &= 2 \\ 3x - y + 5z &= 4 \\ 2x + 3y - z &= 7 \end{aligned}$
$(0, 6, 2)$

4) $\begin{aligned} 6x - y + 4z &= 4 \\ -2x + y - z &= 5 \\ 2x - 3y + z &= 2 \end{aligned}$
$\left(-\frac{1}{2}, -3, 1\right)$

5) Write a system of equations in x, y, and z so that the ordered triple $(4, -1, 2)$ is a solution of the system.

6) Find the value of c so that $(6, 0, 5)$ is a solution of the system $\begin{aligned} 2x - 5y - 3z &= -3. \\ -x + y + 2z &= 4 \\ -2x + 3y + cz &= 8 \end{aligned}$

Objective 2: Solve Systems of Linear Equations in Three Variables

Solve each system. See Example 1.

VIDEO 7) $\begin{aligned} x + 3y + z &= 3 \\ 4x - 2y + 3z &= 7 \\ -2x + y - z &= -1 \end{aligned}$

8) $\begin{aligned} x - y + 2z &= -7 \\ -3x - 2y + z &= -10 \\ 5x + 4y + 3z &= 4 \end{aligned}$

9) $\begin{aligned} 5x + 3y - z &= -2 \\ -2x + 3y + 2z &= 3 \\ x + 6y + z &= -1 \end{aligned}$

10) $\begin{aligned} -2x - 2y + 3z &= 2 \\ 3x + 3y - 5z &= -3 \\ -x + y - z &= 9 \end{aligned}$

11) $\begin{aligned} 3a + 5b - 3c &= -4 \\ a - 3b + c &= 6 \\ -4a + 6b + 2c &= -6 \end{aligned}$

12) $\begin{aligned} a - 4b + 2c &= -7 \\ 3a - 8b + c &= 7 \\ 6a - 12b + 3c &= 12 \end{aligned}$

Objective 3: Solve Special Systems in Three Variables

Solve each system. Identify any systems that are inconsistent or that have dependent equations. See Examples 2 and 3.

13) $\begin{aligned} a - 5b + c &= -4 \\ 3a + 2b - 4c &= -3 \\ 6a + 4b - 8c &= 9 \end{aligned}$

14) $\begin{aligned} -a + 2b - 12c &= 8 \\ -6a + 2b - 8c &= -3 \\ 3a - b + 4c &= 4 \end{aligned}$

15) $\begin{aligned} -15x - 3y + 9z &= 3 \\ 5x + y - 3z &= -1 \\ 10x + 2y - 6z &= -2 \end{aligned}$

16) $\begin{aligned} -4x + 10y - 16z &= -6 \\ -6x + 15y - 24z &= -9 \\ 2x - 5y + 8z &= 3 \end{aligned}$

17) $\begin{aligned} -3a + 12b - 9c &= -3 \\ 5a - 20b + 15c &= 5 \\ -a + 4b - 3c &= -1 \end{aligned}$

18) $\begin{aligned} 3x - 12y + 6z &= 4 \\ -x + 4y - 2z &= 7 \\ 5x + 3y + z &= -2 \end{aligned}$

Objective 4: Solve a System with Missing Terms

Solve each system. See Example 4.

19) $\begin{aligned} 5x - 2y + z &= -5 \\ x - y - 2z &= 7 \\ 4y + 3z &= 5 \end{aligned}$

20) $\begin{aligned} -x + z &= 9 \\ -2x + 4y - z &= 4 \\ 7x + 2y + 3z &= -1 \end{aligned}$

21) $\begin{aligned} a + 15b &= 5 \\ 4a + 10b + c &= -6 \\ -2a - 5b - 2c &= -3 \end{aligned}$

22) $\begin{aligned} 2x - 6y - 3z &= 4 \\ -3y + 2z &= -6 \\ -x + 3y + z &= -1 \end{aligned}$

23) $\begin{aligned} x + 2y + 3z &= 4 \\ -3x + y &= -7 \\ 4y + 3z &= -10 \end{aligned}$

24) $\begin{aligned} -3a + 5b + c &= -4 \\ a + 5b &= 3 \\ 4a - 3c &= -11 \end{aligned}$

25) $\begin{aligned} -5x + z &= -3 \\ 4x - y &= -1 \\ 3y - 7z &= 1 \end{aligned}$

26) $\begin{aligned} a + b &= 1 \\ a - 5c &= 2 \\ b + 2c &= -4 \end{aligned}$

VIDEO 27) $\begin{aligned} 4a + 2b &= -11 \\ -8a - 3c &= -7 \\ b + 2c &= 1 \end{aligned}$

28) $\begin{aligned} 3x + 4y &= -6 \\ -x + 3z &= 1 \\ 2y + 3z &= -1 \end{aligned}$

Mixed Exercises: Objectives 2–4

Solve each system. Identify any systems that are inconsistent or that have dependent equations.

29) $\begin{aligned} 6x + 3y - 3z &= -1 \\ 10x + 5y - 5z &= 4 \\ x - 3y + 4z &= 6 \end{aligned}$

30) $$\begin{aligned} 2x + 3y - z &= 0 \\ x - 4y - 2z &= -5 \\ -4x + 5y + 3z &= -4 \end{aligned}$$

31) $$\begin{aligned} 7x + 8y - z &= 16 \\ -\frac{1}{2}x - 2y + \frac{3}{2}z &= 1 \\ \frac{4}{3}x + 4y - 3z &= -\frac{2}{3} \end{aligned}$$

32) $$\begin{aligned} 3a + b - 2c &= -3 \\ 9a + 3b - 6c &= -9 \\ -6a - 2b + 4c &= 6 \end{aligned}$$

33) $$\begin{aligned} 2a - 3b &= -4 \\ 3b - c &= 8 \\ -5a + 4c &= -4 \end{aligned}$$

34) $$\begin{aligned} 5x + y - 2z &= -2 \\ -\frac{1}{2}x - \frac{3}{4}y + 2z &= \frac{5}{4} \\ x - 6z &= 3 \end{aligned}$$

35) $$\begin{aligned} -4x + 6y + 3z &= 3 \\ -\frac{2}{3}x + y + \frac{1}{2}z &= \frac{1}{2} \\ 12x - 18y - 9z &= -9 \end{aligned}$$

36) $$\begin{aligned} x - \frac{5}{2}y + \frac{1}{2}z &= \frac{5}{4} \\ x + 3y - z &= 4 \\ -6x + 15y - 3z &= -1 \end{aligned}$$

37) $$\begin{aligned} a + b + 9c &= -3 \\ -5a - 2b + 3c &= 10 \\ 4a + 3b + 6c &= -15 \end{aligned}$$

38) $$\begin{aligned} 2x + 3y &= 2 \\ -3x + 4z &= 0 \\ y - 5z &= -17 \end{aligned}$$

39) $$\begin{aligned} x + 5z &= 10 \\ 4y + z &= -2 \\ 3x - 2y &= 2 \end{aligned}$$

40) $$\begin{aligned} a + 3b - 8c &= 2 \\ -2a - 5b + 4c &= -1 \\ 4a + b + 16c &= -4 \end{aligned}$$

VIDEO 41) $$\begin{aligned} 2x - y + 4z &= -1 \\ x + 3y + z &= -5 \\ -3x + 2y &= 7 \end{aligned}$$

42) $$\begin{aligned} -2a + 3b &= 3 \\ a + 5c &= -1 \\ b - 2c &= -5 \end{aligned}$$

43) Given the following two equations, write a third equation to obtain a system of three equations in x, y, and z so that the system has no solution.

$$\begin{aligned} x + 3y - 2z &= -9 \\ 2x - 5y + z &= 1 \end{aligned}$$

44) Given the following two equations, write a third equation to obtain a system of three equations in x, y, and z so that the system has an infinite number of solutions.

$$\begin{aligned} 9x - 12y + 3z &= 21 \\ -3x + 4y - z &= -7 \end{aligned}$$

Objective 5: Solve Applied Problems

Write a system of equations and solve.

45) Moe buys two hot dogs, two orders of fries, and a large soda for \$9.00. Larry buys two hot dogs, one order of fries, and two large sodas for \$9.50, and Curly spends \$11.00 on three hot dogs, two orders of fries, and a large soda. Find the price of a hot dog, an order of fries, and a large soda.

46) A movie theater charges \$9.00 for an adult's ticket, \$7.00 for a ticket for seniors 60 and over, and \$6.00 for a child's ticket. For a particular movie, the theater sold a total of 290 tickets, which brought in \$2400. The number of seniors' tickets sold was twice the number of children's tickets sold. Determine the number of adults', seniors', and children's tickets sold.

47) A Chocolate Chip Peanut Crunch Clif Bar contains 4 fewer grams of protein than a Chocolate Peanut Butter Balance Bar Plus. A Chocolate Peanut Butter Protein Plus PowerBar contains 9 more grams of protein than the Balance Bar Plus. All three bars contain a total of 50 g of protein. How many grams of protein are in each type of bar? (www.clifbar.com, www.balance.com, www.powerbar.com)

48) A 1-tablespoon serving size of Hellman's Real Mayonnaise has 55 more calories than the same serving size of Hellman's Light Mayonnaise. Miracle Whip and Hellman's Light have the same number of calories in a 1-tablespoon serving size. If the three spreads have a total of 160 calories in one serving, determine the number of calories in one serving of each. (product labels)

49) The three NBA teams with the highest revenues in 2002–2003 were the New York Knicks, the Los Angeles Lakers, and the Chicago Bulls. Their revenues totaled \$428 million. The Lakers took in \$30 million more than the Bulls, and the Knicks took in \$11 million more than the Lakers. Determine the revenue of each team during the 2002–2003 season. (*Forbes,* Feb. 16, 2004, p. 66)

50) The best-selling paper towel brands in 2002 were Bounty, Brawny, and Scott. Together they accounted for 59% of the market. Bounty's market share was 25% more than Brawny's, and Scott's market share was 2% less than Brawny's. What percentage of the market did each brand hold in 2002? (*USA Today,* Oct. 23, 2003, p. 3B from Information Resources, Inc.)

51) Ticket prices to a Cubs game at Wrigley Field vary depending on whether they are on a value date, a regular date, or a prime date. At the beginning of the 2008 season, Bill, Corrinne, and Jason bought tickets in the bleachers for several games. Bill spent \$367 on four value dates, four regular dates, and three prime dates. Corrinne bought tickets for four value dates, three regular dates, and two prime dates for \$286. Jason spent \$219 on three value dates, three regular dates, and one prime date. How much did it cost to sit in the bleachers at Wrigley Field on a value date, regular date, and prime date in 2008? (http://chicago.cubs.mlb.com)

52) To see the Boston Red Sox play at Fenway Park in 2009, two field box seats, three infield grandstand seats, and five bleacher seats cost \$530. The cost of four field box seats, two infield grandstand seats, and three bleacher seats was \$678. The total cost of buying one of each type of ticket was \$201. What was the cost of each type of ticket during the 2009 season? (http://boston.redsox.mlb.com)

53) The measure of the largest angle of a triangle is twice the middle angle. The smallest angle measures 28° less than the middle angle. Find the measures of the angles of the triangle. (Hint: Recall that the sum of the measures of the angles of a triangle is 180°.)

54) The measure of the smallest angle of a triangle is one-third the measure of the largest angle. The middle angle measures 30° less than the largest angle. Find the measures of the angles of the triangle. (Hint: Recall that the sum of the measures of the angles of a triangle is 180°.)

55) The smallest angle of a triangle measures 44° less than the largest angle. The sum of the two smaller angles is 20° more than the measure of the largest angle. Find the measures of the angles of the triangle.

56) The sum of the measures of the two smaller angles of a triangle is 40° less than the largest angle. The measure of the largest angle is twice the measure of the middle angle. Find the measures of the angles of the triangle.

57) The perimeter of a triangle is 29 cm. The longest side is 5 cm longer than the shortest side, and the sum of the two smaller sides is 5 cm more than the longest side. Find the lengths of the sides of the triangle.

58) The shortest side of a triangle is half the length of the longest side. The sum of the two smaller sides is 2 in. more than the longest side. Find the lengths of the sides if the perimeter is 58 in.

Extension

Extend the concepts of this section to solve each system. Write the solution in the form (a, b, c, d)

59) $$\begin{aligned} a - 2b - c + d &= 0 \\ -a + 2b + 3c + d &= 6 \\ 2a + b + c - d &= 8 \\ a - b + 2c + d &= 7 \end{aligned}$$

60) $$\begin{aligned} -a + 4b + 3c - d &= 4 \\ 2a + b - 3c + d &= -6 \\ a + b + c + d &= 0 \\ a - b + 2c - d &= -1 \end{aligned}$$

61) $$\begin{aligned} 3a + 4b + c - d &= -7 \\ -3a - 2b - c + d &= 1 \\ a + 2b + 3c - 2d &= 5 \\ 2a + b + c - d &= 2 \end{aligned}$$

62) $$\begin{aligned} 3a - 4b + c + d &= 12 \\ -3a + 2b - c + 3d &= -4 \\ a - 2b + 2c - d &= 2 \\ -a + 4b + c + d &= 8 \end{aligned}$$

Chapter 5: Summary

Definition/Procedure	Example

5.1 Solving Systems by Graphing

A **system of linear equations** consists of two or more linear equations with the same variables. A **solution of a system** of two equations in two variables is an ordered pair that is a solution of each equation in the system. **(p. 292)**

Determine whether (4, 2) is a solution of the system

$$\begin{aligned} x + 2y &= 8 \\ -3x + 4y &= -4 \end{aligned}$$

$$\begin{aligned} x + 2y &= 8 \\ 4 + 2(2) &\stackrel{?}{=} 8 && \text{Substitute.} \\ 4 + 4 &\stackrel{?}{=} 8 \\ 8 &= 8 && \text{TRUE} \end{aligned}$$

$$\begin{aligned} -3x + 4y &= -4 \\ -3(4) + 4(2) &\stackrel{?}{=} -4 && \text{Substitute.} \\ -12 + 8 &\stackrel{?}{=} -4 \\ -4 &= -4 && \text{TRUE} \end{aligned}$$

Since (4, 2) is a solution of each equation in the system, **yes,** it is a solution of the system.

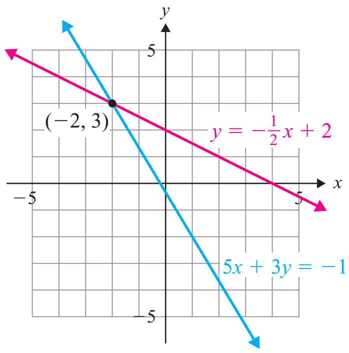
To **solve a system by graphing,** graph each line on the same axes.

a) If the lines intersect at a single point, then this point is the solution of the system. The system is **consistent.**

b) If the lines are parallel, then the system has **no solution.** We write the solution set as $\varnothing$. The system is **inconsistent.**

c) If the graphs are the same line, then the system has an **infinite number of solutions.** The system is **dependent. (p. 293)**

Solve by graphing.
$$\begin{aligned} y &= -\frac{1}{2}x + 2 \\ 5x + 3y &= -1 \end{aligned}$$



The solution of the system is (−2, 3).

The system is **consistent.**

5.2 Solving Systems by the Substitution Method

Steps for Solving a System by Substitution

1) Solve one of the equations for one of the variables. If possible, solve for a variable that has a coefficient of 1 or −1.

2) Substitute the expression in *step 1* into the *other* equation. The equation you obtain should contain only one variable.

3) Solve the equation in *step 2*.

4) Substitute the value found in *step 3* into either of the equations to obtain the value of the other variable.

5) Check the values in the original equations. **(p. 303)**

Solve by substitution.
$$\begin{aligned} 7x - 3y &= 8 \\ x + 2y &= -11 \end{aligned}$$

1) Solve for x in the second equation since its coefficient is 1.

$$x = -2y - 11$$

2) Substitute $-2y - 11$ for the x in the first equation.

$$7(-2y - 11) - 3y = 8$$

3) Solve the equation above for y.

$$\begin{aligned} 7(-2y - 11) - 3y &= 8 \\ -14y - 77 - 3y &= 8 && \text{Distribute.} \\ -17y - 77 &= 8 && \text{Combine like terms.} \\ -17y &= 85 && \text{Add 77.} \\ y &= -5 && \text{Divide by } -17. \end{aligned}$$

4) Substitute −5 into the equation in step 1 to find x.

$$\begin{aligned} x &= -2(-5) - 11 && \text{Substitute } -5 \text{ for } y. \\ x &= 10 - 11 && \text{Multiply.} \\ x &= -1 \end{aligned}$$

5) The solution is (−1, −5). Verify this by substituting (−1, −5) into each of the original equations.

Definition/Procedure	Example
If the variables drop out and a false equation is obtained, the system has **no solution. The system is inconsistent, and the solution set is $\varnothing$. (p. 305)**	Solve by substitution. $\begin{aligned} 2x - 8y &= 9 \\ x &= 4y + 2 \end{aligned}$ 1) The second equation is solved for x. 2) Substitute $4y + 2$ for x in the first equation. $2(4y + 2) - 8y = 9$ 3) Solve the above equation for y. $\begin{aligned} 2(4y + 2) - 8y &= 9 \\ 8y + 4 - 8y &= 9 \quad \text{Distribute.} \\ 4 &= 9 \quad \text{FALSE} \end{aligned}$ 4) The system has no solution. The solution set is $\varnothing$.
If the variables drop out and a true equation is obtained, the system has an **infinite number of solutions. The equations are dependent. (p. 306)**	Solve by substitution. $\begin{aligned} y &= x - 3 \\ 3x - 3y &= 9 \end{aligned}$ 1) The first equation is solved for y. 2) Substitute $x - 3$ for y in the second equation. $3x - 3(x - 3) = 9$ 3) Solve the above equation for x. $\begin{aligned} 3x - 3(x - 3) &= 9 \\ 3x - 3x + 9 &= 9 \quad \text{Distribute.} \\ 9 &= 9 \quad \text{TRUE} \end{aligned}$ 4) The system has an infinite number of solutions of the form $\{(x, y)\|y = x - 3\}$.

5.3 Solving Systems by the Elimination Method

Definition/Procedure	Example
Steps for Solving a System of Two Linear Equations by the Elimination Method 1) Write each equation in the form $Ax + By = C$. 2) Determine which variable to eliminate. If necessary, multiply one or both of the equations by a number so that the coefficients of the variable to be eliminated are negatives of one another. 3) Add the equations, and solve for the remaining variable. 4) Substitute the value found in *Step 3* into either of the original equations to find the value of the other variable. 5) Check the solution in each of the original equations. **(p. 310)**	Solve using the elimination method. $\begin{aligned} 4x + 5y &= -7 \\ -5x - 6y &= 8 \end{aligned}$ Eliminate x. Multiply the first equation by 5, and multiply the second equation by 4 to rewrite the system with equivalent equations. Rewrite the system $\begin{aligned} 5(4x + 5y) &= 5(-7) \rightarrow 20x + 25y = -35 \\ 4(-5x - 6y) &= 4(8) \rightarrow -20x - 24y = 32 \end{aligned}$ Add the equations: $\begin{array}{r} 20x + 25y = -35 \\ + \quad -20x - 24y = 32 \\ \hline y = -3 \end{array}$ Substitute $y = -3$ into either of the original equations and solve for x. $\begin{aligned} 4x + 5y &= -7 \\ 4x + 5(-3) &= -7 \\ 4x - 15 &= -7 \\ 4x &= 8 \\ x &= 2 \end{aligned}$ The solution is $(2, -3)$. Verify this by substituting $(2, -3)$ into each of the original equations.

Definition/Procedure	Example
5.4 Applications of Systems of Two Equations	
Use the **Five Steps for Solving Applied Problems** outlined in the section to solve an applied problem. 1) **Read** the problem carefully. Draw a picture, if applicable. 2) **Choose variables** to represent the unknown quantities. If applicable, label the picture with the variables. 3) **Write a system of equations using two variables.** It may be helpful to begin by writing an equation in words. 4) **Solve the system.** 5) **Check** the answer in the original problem, and **interpret** the solution as it relates to the problem. **(p. 319)**	Amana spent \$40.20 at a second-hand movie and music store when she purchased some DVDs and CDs. Each DVD cost \$6.30, and each CD cost \$2.50. How many DVDs and CDs did she buy if she purchased 10 items all together? 1) Read the problem carefully. 2) Choose the variables. x = number of DVDs she bought y = number of CDs she bought 3) One equation involves the *cost* of the items: Cost DVDs + Cost CDs = Total Cost $6.30x + 2.50y = 40.20$ The second equation involves the number of items: Number of DVDs + Number of CDs = Total number of items $x + y = 10$ The system is $6.30x + 2.50y = 40.20$. $x + y = 10$ 4) Multiply by 10 to eliminate the decimals in the first equation, and then solve the system using substitution. $10(6.30x + 2.50y) = 10(40.20)$ Eliminate decimals. $63x + 25y = 402$ Solve the system $63x + 25y = 402$, $x + y = 10$ to determine that the solution is (4, 6). 5) **Amana bought 4 DVDs and 6 CDs.** Verify the solution.
5.5 Solving Systems of Three Equations and Applications	
A **linear equation in three variables** is an equation of the form $Ax + By + Cz = D$, where A, B, and C are not all zero and where A, B, C, and D are real numbers. Solutions of this type of equation are **ordered triples** of the form (x, y, z). **(p. 330)**	$5x + 3y + 9z = -2$ One solution of this equation is $(-1, -2, 1)$ since substituting the values for x, y, and z satisfies the equation. $5x + 3y + 9z = -2$ $5(-1) + 3(-2) + 9(1) = -2$ $-5 - 6 + 9 = -2$ $-2 = -2$ TRUE
Solving a System of Linear Equations in Three Variables 1) **Label** the equations (1), (2), and (3). 2) **Choose a variable to eliminate. Eliminate** this variable from *two* sets of *two* equations using the elimination method. You will obtain two equations containing the same two variables. Label one of these new equations $\boxed{A}$ and the other $\boxed{B}$. 3) **Use the elimination method to eliminate a variable from equations $\boxed{A}$ and $\boxed{B}$.** You have now found the value of one variable.	Solve (1) $x + 2y + 3z = 5$ (2) $4x - 2y - z = -1$ (3) $-3x + y + 4z = -12$ 1) **Label** the equations (1), (2), and (3). 2) We will eliminate y from *two* sets of *two* equations. a) *Equations (1) and (2)*: Add the equations to eliminate y. Label the resulting equation $\boxed{A}$. (1) $x + 2y + 3z = 5$ (2) $+\ 4x - 2y - z = -1$ $\boxed{A}$ $5x + 2z = 4$

Definition/Procedure	Example
4) **Find the value of another variable** by substituting the value found in *Step 3* into either equation $\boxed{A}$ or $\boxed{B}$ and solving for the second variable. 5) **Find the value of the third variable** by substituting the values of the two variables found in *Steps 3 and 4* into equation 1, 2, or 3. 6) **Check** the solution in each of the original equations, and **write the solution as an ordered triple. (p. 332)**	b) *Equations* 2 *and* 3: To eliminate y, multiply equation 3 by 2 and add it to equation 2. Label the resulting equation $\boxed{B}$. $2 \times 3 \quad -6x + 2y + 8z = -24$ $2 \quad + \underline{4x - 2y - z = -1}$ $\boxed{B} \quad -2x + 7z = -25$ 3) Eliminate x from $\boxed{A}$ and $\boxed{B}$. Multiply $\boxed{A}$ by 2 and $\boxed{B}$ by 5. Add the resulting equations. $2 \times \boxed{A} \quad 10x + 4z = 8$ $5 \times \boxed{B} \quad + \underline{-10x + 35z = -125}$ $39z = -117$ $\boxed{z = -3}$ 4) Substitute $z = -3$ into either $\boxed{A}$ or $\boxed{B}$. Substitute $z = -3$ into $\boxed{A}$ and solve for x. $\boxed{A} \quad 5x + 2z = 4$ $5x + 2(-3) = 4$ Substitute -3 for z. $5x - 6 = 4$ Multiply. $5x = 10$ Add 6. $\boxed{x = 2}$ Divide by 5. 5) Substitute $x = 2$ and $z = -3$ into either equation 1, 2, or 3. Substitute $x = 2$ and $z = -3$ into 1 to solve for y. 1 $\quad x + 2y + 3z = 5$ $2 + 2y + 3(-3) = 5$ Substitute 2 for x and -3 for z. $2 + 2y - 9 = 5$ Multiply. $2y - 7 = 5$ Combine like terms. $2y = 12$ Add 7. $\boxed{y = 6}$ Divide by 2. 6) The solution is $(2, 6, -3)$. The check is left to the student.

Chapter 5: Review Exercises

(5.1) Determine whether the ordered pair is a solution of the system of equations.

1) $x - 5y = 13$
$2x + 7y = 20$
$(-4, -5)$

2) $8x + 3y = 16$
$10x - 6y = 7$
$\left(\frac{3}{2}, \frac{4}{3}\right)$

3) If you are solving a system of equations by graphing, how do you know whether the system has no solution?

Solve each system by graphing.

4) $y = \frac{1}{2}x + 1$
$x + y = 4$

5) $x - 3y = 9$
$-x + 3y = 6$

6) $6x - 3y = 12$
$-2x + y = -4$

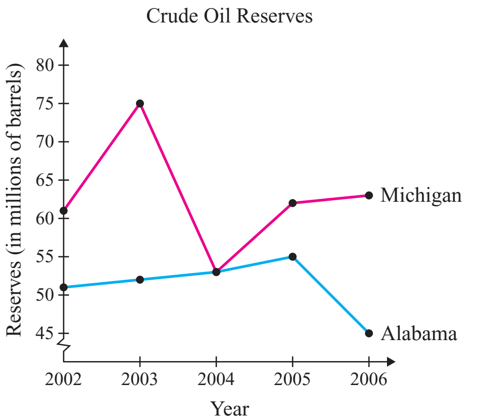
7) $-x + 2y = 1$
$2x + 3y = -9$

Without graphing, determine whether each system has no solution, one solution, or an infinite number of solutions.

8) $8x + 9y = -2$
$x - 4y = 1$

9) $y = -\frac{5}{2}x + 3$
$5x + 2y = 6$

10) The graph shows the number of millions of barrels of crude oil reserves in Alabama and Michigan from 2002 to 2006. (www.census.gov)



a) In 2006, approximately how much more crude oil did Michigan have in reserve than Alabama?

b) Write the point of intersection as an ordered pair in the form (year, reserves) and explain its meaning.

c) Which line segment has the most negative slope? How can this be explained in the context of the problem.

(5.2) Solve each system by substitution.

11) $9x - 2y = 8$
$y = 2x + 1$

12) $y = -6x + 5$
$12x + 2y = 10$

13) $-x + 8y = 19$
$4x - 3y = 11$

14) $-12x + 7y = 9$
$8x - y = -6$

(5.3) Solve each system using the elimination method.

15) $x - 7y = 3$
$-x + 5y = -1$

16) $5x + 4y = -23$
$3x - 8y = -19$

17) $-10x + 4y = -8$
$5x - 2y = 4$

18) $7x - 4y = 13$
$6x - 5y = 8$

Solve each system using the elimination method twice.

19) $2x + 9y = -6$
$5x + y = 3$

20) $7x - 4y = 10$
$6x + 3y = 8$

(5.2–5.3)

21) When is the best time to use substitution to solve a system?

22) If an equation in a system contains fractions, what should you do first to make the system easier to solve?

Solve each system by either the substitution or elimination method.

23) $6x + y = -8$
$9x + 7y = -1$

24) $4y - 5x = -23$
$2x + 3y = -23$

25) $\frac{1}{3}x - \frac{2}{9}y = -\frac{2}{3}$
$\frac{5}{12}x + \frac{1}{3}y = 1$

26) $0.02x - 0.01y = 0.13$
$-0.1x + 0.4y = 1.8$

27) $6(2x - 3) = y + 4(x - 3)$
$5(3x + 4) + 4y = 11 - 3y + 27x$

28) $x - 3y = 36$
$y = \frac{5}{3}x$

29) $\frac{3}{4}x - \frac{5}{4}y = \frac{7}{8}$
$4 - 2(x + 5) - y = 3(1 - 2y) + x$

30) $y = -\frac{9}{7}x + \frac{6}{7}$
$18x + 14y = 12$

(5.4) Write a system of equations and solve.

31) One day in the school cafeteria, the number of children who bought white milk was twice the number who bought chocolate milk. How many cartons of each type of milk were sold if the cafeteria sold a total of 141 cartons of milk?

32) How many ounces of a 7% acid solution and how many ounces of a 23% acid solution must be mixed to obtain 20 oz of a 17% acid solution?

33) Edwin and Camille leave from opposite ends of a jogging trail 7 miles apart and travel toward each other. Edwin jogs 2 mph faster than Camille, and they meet after half an hour. How fast does each of them jog?

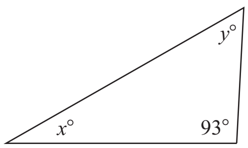
34) At a movie theater concession stand, three candy bars and two small sodas cost \$14.00. Four candy bars and three small sodas cost \$19.50. Find the cost of a candy bar and the cost of a small soda.

35) The width of a rectangle is 5 cm less than the length. Find the dimensions of the rectangle if the perimeter is 38 cm.

36) Two planes leave the same airport and travel in opposite directions. The northbound plane flies 40 mph slower than the southbound plane. After 1.5 hours they are 1320 miles apart. Find the speed of each plane.

37) Shawanna saves her quarters and dimes in a piggy bank. When she opens it, she has 63 coins worth a total of \$11.55. How many of each type of coin does she have?

38) Find the measure of angles x and y if the measure of angle x is half the measure of angle y.



39) At a ski shop, two packs of hand warmers and one pair of socks cost \$27.50. Five packs of hand warmers and three pairs of socks cost \$78.00. Find the cost of a pack of hand warmers and a pair of socks.

40) A 7 P.M. spinning class has 9 more members than a 10 A.M. spinning class. The two classes have a total of 71 students. How many are in each class?

(5.5) Determine whether the ordered triple is a solution of the system.

41) $\begin{aligned} x - 6y + 4z &= 13 \\ 5x + y + 7z &= 8 \\ 2x + 3y - z &= -5 \end{aligned}$
$(-3, -2, 1)$

42) $\begin{aligned} -4x + y + 2z &= 1 \\ x - 3y - 4z &= 3 \\ -x + 2y + z &= -7 \end{aligned}$
$(0, -5, 3)$

Solve each system using one of the methods of Section 5.5. Identify any inconsistent systems or dependent equations.

43) $\begin{aligned} 2x - 5y - 2z &= 3 \\ x + 2y + z &= 5 \\ -3x - y + 2z &= 0 \end{aligned}$

44) $\begin{aligned} x - 2y + 2z &= 6 \\ x + 4y - z &= 0 \\ 5x + 3y + z &= -3 \end{aligned}$

45) $\begin{aligned} 5a - b + 2c &= -6 \\ -2a - 3b + 4c &= -2 \\ a + 6b - 2c &= 10 \end{aligned}$

46) $\begin{aligned} 2x + 3y - 15z &= 5 \\ -3x - y + 5z &= 3 \\ -x + 6y - 10z &= 12 \end{aligned}$

47) $\begin{aligned} 4x - 9y + 8z &= 2 \\ x + 3y &= 5 \\ 6y + 10z &= -1 \end{aligned}$

48) $\begin{aligned} -a + 5b - 2c &= -3 \\ 3a + 2c &= -3 \\ 2a + 10b &= -2 \end{aligned}$

49) $\begin{aligned} x + 3y - z &= 0 \\ 11x - 4y + 3z &= 8 \\ 5x + 15y - 5z &= 1 \end{aligned}$

50) $\begin{aligned} 4x + 2y + z &= 0 \\ 8x + 4y + 2z &= 0 \\ 16x + 8y + 4z &= 0 \end{aligned}$

51) $\begin{aligned} 12a - 8b + 4c &= 8 \\ 3a - 2b + c &= 2 \\ -6a + 4b - 2c &= -4 \end{aligned}$

52) $\begin{aligned} 3x - 12y - 6z &= -8 \\ x + y - z &= 5 \\ -4x + 16y + 8z &= 10 \end{aligned}$

53) $\begin{aligned} 5y + 2z &= 6 \\ -x + 2y &= -1 \\ 4x - z &= 1 \end{aligned}$

54) $\begin{aligned} 2a - b &= 4 \\ 3b + c &= 8 \\ -3a + 2c &= -5 \end{aligned}$

55) $\begin{aligned} 8x + z &= 7 \\ 3y + 2z &= -4 \\ 4x - y &= 5 \end{aligned}$

56) $\begin{aligned} 6y - z &= -2 \\ x + 3y &= 1 \\ -3x + 2z &= 8 \end{aligned}$

Write a system of equations and solve.

57) One serving (8 fl oz) of Powerade has 17 mg more sodium than one serving of Propel. One serving of Gatorade has 58 mg more sodium than the same serving size of Powerade. Together the three drinks have 197 mg of sodium. How much sodium is in one serving of each drink? (product labels)

58) In 2003, the top highway truck tire makers were Goodyear, Michelin, and Bridgestone. Together, they held 53% of the market. Goodyear's market share was 3% more than Bridgestone's, and Michelin's share was 1% less than Goodyear's. What percent of this tire market did each company hold in 2003? (*Market Share Reporter*, Vol. I, 2005, p. 361: from: *Tire Business*, Feb. 2, 2004, p. 9)

59) One Friday, Serena, Blair, and Chuck were busy texting their friends. Together, they sent a total of 140 text messages. Blair sent 15 more texts than Serena while Chuck sent half as many as Serena. How many texts did each person send that day?

60) Digital downloading of albums has been on the rise. The Recording Industry Association of America reports that in 2005 there were 14 million fewer downloads than in 2006, and in 2007 there were 14.9 million more downloads than the previous year. During all three years, 83.7 million albums were downloaded. How many albums were downloaded in 2005, 2006, and 2007? (www.riaa.com)

61) A family of six people goes to an ice cream store every Sunday after dinner. One week, they order two ice cream cones, three shakes, and one sundae for \$13.50. The next week, they get three cones, one shake, and two sundaes for \$13.00. The week after that, they spend \$11.50 on one shake, one sundae, and four ice cream cones. Find the price of an ice cream cone, a shake, and a sundae.

62) An outdoor music theater sells three types of seats—reserved, behind-the-stage, and lawn seats. Two reserved, three behind-the-stage, and four lawn seats cost \$360. Four reserved, two behind-the-stage, and five lawn seats cost \$470. One of each type of seat would total \$130. Determine the cost of a reserved seat, a behind-the-stage seat, and a lawn seat.

63) The measure of the smallest angle of a triangle is one-third the measure of the middle angle. The measure of the largest angle is 70° more than the measure of the smallest angle. Find the measures of the angles of the triangle.

64) The perimeter of a triangle is 40 in. The longest side is twice the length of the shortest side, and the sum of the two smaller sides is four inches longer than the longest side. Find the lengths of the sides of the triangles.

Chapter 5: Test

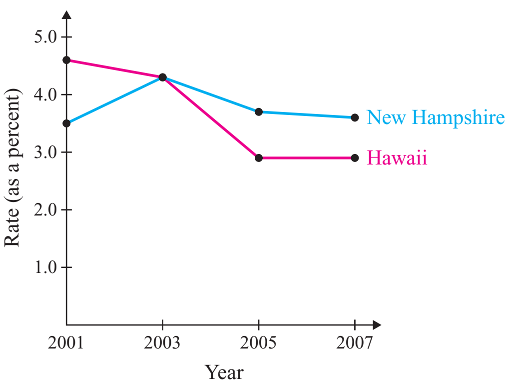
1) Determine whether $\left(-\frac{2}{3}, 4\right)$ is a solution of the system
$$9x + 5y = 14$$
$$-6x - y = 0.$$

Solve each system by graphing.

2) $y = -x + 2$
$3x - 4y = 20$

3) $3y - 6x = 6$
$2x - y = 1$

4) The graph shows the unemployment rate in the civilian labor force in Hawaii and New Hampshire in various years from 2001 to 2007. (www.bls.gov)



a) When were more people unemployed in Hawaii? Approximately what percent of the state's population was unemployed at that time?

b) Write the point of intersection of the graphs as an ordered pair in the form (year, percentage) and explain its meaning.

c) Which line segment has the most negative slope? How can this be explained in the context of the problem?

Solve each system by substitution.

5) $3x - 10y = -10$
$x + 8y = -9$

6) $y = \frac{1}{2}x - 3$
$4x - 8y = 24$

Solve each system by the elimination method.

7) $2x + 5y = 11$
$7x - 5y = 16$

8) $3x + 4y = 24$
$7x - 3y = -18$

9) $-6x + 9y = 14$
$4x - 6y = 5$

Solve each system using any method.

10) $11x - y = -14$
$-9x + 7y = -38$

11) $\frac{5}{8}x + \frac{1}{4}y = \frac{1}{4}$
$\frac{1}{3}x + \frac{1}{2}y = -\frac{4}{3}$

12) $7 - 4(2x + 3) = x + 7 - y$
$5(x - y) + 20 = 8(2 - x) + x - 12$

13) $-x + 4y + 3z = 6$
$3x - 2y + 6z = -18$
$x + y + 2z = -1$

14) Write a system of equations in two variables that has $(5, -1)$ as its only solution.

Write a system of equations and solve.

15) The area of Yellowstone National Park is about 1.1 million fewer acres than the area of Death Valley National Park. If they cover a total of 5.5 million acres, how big is each park? (www.nps.gov)

16) The Mahmood and Kuchar families take their kids to an amusement park. The Mahmoods buy one adult ticket and two children's tickets for \$85.00. The Kuchars spend \$150.00 on two adult and three children's tickets. How much did they pay for each type of ticket?

17) The width of a rectangle is half its length. Find the dimensions of the rectangle if the perimeter is 114 cm.

18) How many milliliters of a 12% alcohol solution and how many milliliters of a 30% alcohol solution must be mixed to obtain 72 mL of a 20% alcohol solution?

19) Rory and Lorelei leave Stars Hollow, Connecticut, and travel in opposite directions. Rory drives 4 mph faster than Lorelei, and after 1.5 hr they are 120 miles apart. How fast was each driving?

20) The measure of the smallest angle of a triangle is 9° less than the measure of the middle angle. The largest angle is 30° more than the sum of the two smaller angles. Find the measures of the angles of the triangle.

Cumulative Review: Chapters 1–5

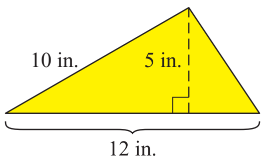
Perform the operations and simplify.

1) $\dfrac{7}{15} + \dfrac{9}{10}$

2) $4\dfrac{4}{5} \div \dfrac{9}{20}$

3) $3(5 - 7)^3 + 18 \div 6 - 8$

4) Find the area of the triangle.



5) Simplify $-3(4x^2 + 5x - 1)$.

Simplify. The answer should not contain any negative exponents.

6) $(2p^4)^5$

7) $9x^2 \cdot 7x^{-6}$

8) $\dfrac{36m^{-7}n^5}{24m^3n}$

9) Write 0.0007319 in scientific notation.

10) Solve $0.04(3p - 2) - 0.02p = 0.1(p + 3)$.

11) Solve $11 - 3(2k - 1) = 2(6 - k)$.

12) Solve. Write the answer in interval notation.

$$-5 < 4v - 9 < 15$$

13) Write an equation and solve.

During the first week of the "Cash for Clunkers" program, the average increase in gas mileage for the new car purchased versus the car traded in was 61%. If the average gas mileage of the new cars was 25.4 miles per gallon, what was the average gas mileage of the cars traded in? Round the answer to the nearest tenth. (www.yahoo.com)

14) The area, A of a trapezoid is $A = \dfrac{1}{2}h(b_1 + b_2)$, where h = height of the trapezoid, b_1 = length of one base of the trapezoid, and b_2 = length of the second base of the trapezoid.

a) Solve the equation for h.

b) Find the height of the trapezoid that has an area of 39 cm^2 and bases of length 8 cm and 5 cm.

15) Graph $2x - 3y = 9$.

16) Find the x- and y-intercepts of the graph of $x - 8y = 16$.

17) Write the slope-intercept form of the equation of the line containing $(3, 2)$ and $(-9, -1)$

18) Determine whether the lines are parallel, perpendicular, or neither. $10x + 18y = 9$
$9x - 5y = 17$

Solve each system of equations.

19) $9x - 3y = 6$
$3x - 2y = -8$

20) $3(2x - 1) - (y + 10) = 2(2x - 3) - 2y$
$3x + 13 = 4x - 5(y - 3)$

21) $x + 2y = 4$
$-3x - 6y = 6$

22) $-\dfrac{1}{4}x - \dfrac{3}{4}y = \dfrac{1}{6}$
$\dfrac{1}{2}x + \dfrac{3}{2}y = -\dfrac{1}{3}$

23) $4a - 3b = -5$
$-a + 5c = 2$
$2b + c = -2$

Write a system of equations and solve.

24) Dhaval used twice as many 6-foot boards as 4-foot boards when he made a treehouse for his children. If he used a total of 48 boards, how many of each size did he use?

25) Through 2008, Juanes had won 3 more Latin Grammy Awards than Alejandro Sanz while Sanz had won twice as many as Shakira. Together, these three performers had won 38 Latin Grammy Awards. How many had each person won? (www.grammy.com/latin)

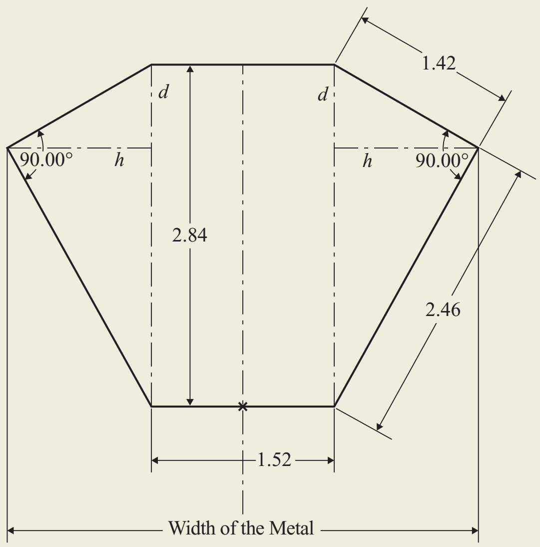
CHAPTER 6

Polynomials

Algebra at Work: Custom Motorcycles

This is a final example of how algebra is used to build motorcycles in a custom chopper shop.

The support bracket for the fender of a custom motorcycle must be fabricated. To save money, Jim's boss told him to use a piece of scrap metal and not a new piece. So, he has to figure out how big a piece of scrap metal he needs to be able to cut out the shape needed to make the fender.



Jim drew the sketch on the left that showed the shape and dimension of the piece of metal to be cut so that it could be bent into the correct shape and size for the fender. He knows that the height of the piece must be 2.84 in.

To determine the width of the piece of metal that he needs, Jim analyzes the sketch and writes the equation

$$[(1.42)^2 - d^2] + (2.84 - d)^2 = (2.46)^2$$

In order to solve this equation, he must know how to square the binomial $(2.84 - d)^2$, something we will learn in this chapter. When he solves the equation, Jim determines that $d \approx 0.71$ in. He uses this value of d to find that the width of the piece of metal that he must use to cut the correct shape for the fender is 3.98 in.

We will learn how to square binomials and perform other operations with polynomials in this chapter.

Section 6.1 Review of the Rules of Exponents

Objective

1. Review the Rules of Exponents

In Chapter 2, we learned the rules of exponents. We will review them here to prepare us for the topics in the rest of this chapter—adding, subtracting, multiplying, and dividing polynomials.

1. Review of the Rules of Exponents

Rules of Exponents

For real numbers a and b and integers m and n, the following rules apply:

Summary The Rules of Exponents

Rule	Example
Product Rule: $a^m \cdot a^n = a^{m+n}$	$y^6 \cdot y^9 = y^{6+9} = y^{15}$
Basic Power Rule: $(a^m)^n = a^{mn}$	$(k^4)^7 = k^{28}$
Power Rule for a Product: $(ab)^n = a^n b^n$	$(9t)^2 = 9^2t^2 = 81t^2$
Power Rule for a Quotient: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, where $b \neq 0$.	$\left(\frac{2}{r}\right)^5 = \frac{2^5}{r^5} = \frac{32}{r^5}$
Zero Exponent: If $a \neq 0$, then $a^0 = 1$.	$(7)^0 = 1$
Negative Exponent:	
For $a \neq 0$, $a^{-n} = \left(\frac{1}{a}\right)^n = \frac{1}{a^n}$.	$\left(\frac{3}{4}\right)^{-3} = \left(\frac{4}{3}\right)^3 = \frac{4^3}{3^3} = \frac{64}{27}$
If $a \neq 0$ and $b \neq 0$, then $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$;	$\frac{x^{-6}}{y^{-3}} = \frac{y^3}{x^6}$
also, $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$.	$\left(\frac{8c}{d}\right)^{-2} = \left(\frac{d}{8c}\right)^2 = \frac{d^2}{64c^2}$
Quotient Rule: If $a \neq 0$, then $\frac{a^m}{a^n} = a^{m-n}$.	$\frac{2^9}{2^4} = 2^{9-4} = 2^5 = 32$

When we use the rules of exponents to simplify an expression, we must remember to use the order of operations.

Example 1

Simplify. Assume all variables represent nonzero real numbers. The answer should contain only positive exponents.

a) $(7k^{10})(-2k)$ b) $\dfrac{(-4)^5 \cdot (-4)^2}{(-4)^4}$ c) $\dfrac{10x^5y^{-3}}{2x^2y^5}$

d) $\left(\dfrac{c^2}{2d^4}\right)^{-5}$ e) $4(3p^5q)^2$

Solution

a) $(7k^{10})(-2k) = -14k^{10+1}$ Multiply coefficients and add the exponents.

$= -14k^{11}$ Simplify.

b) $\dfrac{(-4)^5 \cdot (-4)^2}{(-4)^4} = \dfrac{(-4)^{5+2}}{(-4)^4} = \dfrac{(-4)^7}{(-4)^4}$ Product rule—the bases in the numerator are the same, so add the exponents.

$= (-4)^{7-4}$ Quotient rule

$= (-4)^3$

$= -64$ Evaluate.

c) $\dfrac{10x^5y^{-3}}{2x^2y^5} = 5x^{5-2}y^{-3-5}$ Divide coefficients and subtract the exponents.

$= 5x^3y^{-8}$ Simplify.

$= \dfrac{5x^3}{y^8}$ Write the answer with only positive exponents.

d) $\left(\dfrac{c^2}{2d^4}\right)^{-5} = \left(\dfrac{2d^4}{c^2}\right)^5$ Take the reciprocal of the base and make the exponent positive.

$= \dfrac{2^5d^{20}}{c^{10}}$ Power rule

$= \dfrac{32d^{20}}{c^{10}}$ Simplify.

e) In this expression, a quantity is raised to a power and that quantity is being multiplied by 4. Remember what the order of operations says: Perform exponents before multiplication.

$4(3p^5q)^2 = 4(9p^{10}q^2)$ Apply the power rule *before* multiplying factors.

$= 36p^{10}q^2$ Multiply.

You Try 1

Simplify. Assume all variables represent nonzero real numbers. The answer should contain only positive exponents.

a) $(-6u^2)(-4u^3)$ b) $\dfrac{8^3 \cdot 8^4}{8^5}$ c) $\dfrac{8n^9}{12n^5}$

d) $(3y^{-9})^2(2y^7)$ e) $\left(\dfrac{3a^3b^{-4}}{2ab^6}\right)^{-4}$

Answers to You Try Exercises

1) a) $24u^5$ b) 64 c) $\dfrac{2n^4}{3}$ d) $\dfrac{18}{y^{11}}$ e) $\dfrac{16b^{40}}{81a^8}$

6.1 Exercises

Objective 1: Review the Rules of Exponents

State which exponent rule must be used to simplify each exercise. Then simplify.

1) $\dfrac{k^{10}}{k^4}$

2) $p^5 \cdot p^2$

3) $(2h)^4$

4) $\left(\dfrac{5}{w}\right)^3$

Evaluate using the rules of exponents.

5) $2^2 \cdot 2^4$

6) $(-3)^2 \cdot (-3)$

7) $\dfrac{(-4)^8}{(-4)^5}$

8) $\dfrac{2^{10}}{2^6}$

9) 6^{-1}

10) $(12)^{-2}$

11) $\left(\dfrac{1}{9}\right)^{-2}$

12) $\left(-\dfrac{1}{5}\right)^{-3}$

13) $\left(\dfrac{3}{2}\right)^{-4}$

14) $\left(\dfrac{7}{9}\right)^{-2}$

15) $6^0 + \left(-\dfrac{1}{2}\right)^{-5}$

16) $\left(\dfrac{1}{4}\right)^{-2} + \left(\dfrac{1}{4}\right)^0$

17) $\dfrac{8^5}{8^7}$

18) $\dfrac{2^7}{2^{12}}$

Simplify. Assume all variables represent nonzero real numbers. The answer should not contain negative exponents.

19) $t^5 \cdot t^8$

20) $n^{10} \cdot n^6$

21) $(-8c^4)(2c^5)$

22) $(3w^9)(-7w)$

23) $(z^6)^4$

24) $(y^3)^2$

25) $(5p^{10})^3$

26) $(-6m^4)^2$

27) $\left(-\dfrac{2}{3}a^7b\right)^3$

28) $\left(\dfrac{7}{10}r^2s^5\right)^2$

29) $\dfrac{f^{11}}{f^7}$

30) $\dfrac{u^9}{u^8}$

31) $\dfrac{35v^9}{5v^8}$

32) $\dfrac{36k^8}{12k^5}$

33) $\dfrac{9d^{10}}{54d^6}$

34) $\dfrac{7m^4}{56m^2}$

35) $\dfrac{x^3}{x^9}$

36) $\dfrac{v^2}{v^5}$

37) $\dfrac{m^2}{m^3}$

38) $\dfrac{t^3}{t^3}$

39) $\dfrac{45k^{-2}}{30k^2}$

40) $\dfrac{22n^{-9}}{55n^{-3}}$

41) $5(2m^4n^7)^2$

42) $2(-3a^8b)^3$

43) $(6y^2)(2y^3)^2$

44) $(-c^4)(5c^9)^3$

45) $\left(\dfrac{7a^4}{b^{-1}}\right)^{-2}$

46) $\left(\dfrac{3t^{-3}}{2u}\right)^{-4}$

47) $\dfrac{a^{-12}b^7}{a^{-9}b^2}$

48) $\dfrac{mn^{-4}}{m^9n^7}$

49) $\dfrac{(x^2y^{-3})^4}{x^5y^8}$

50) $\dfrac{10r^{-6}t}{(4r^{-5}t^4)^3}$

51) $\dfrac{12a^6bc^{-9}}{(3a^2b^{-7}c^4)^2}$

52) $\dfrac{(-7k^2m^{-3}n^{-1})^2}{14km^{-2}n^2}$

53) $(xy^{-3})^{-5}$

54) $-(s^{-6}t^2)^{-4}$

55) $\left(\dfrac{a^2b}{4c^2}\right)^{-3}$

56) $\left(\dfrac{2s^3}{rt^4}\right)^{-5}$

57) $\left(\dfrac{7h^{-1}k^9}{21h^{-5}k^5}\right)^{-2}$

58) $\left(\dfrac{24m^8n^{-3}}{16mn}\right)^{-3}$

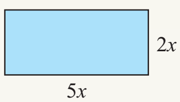
59) $\left(\dfrac{15cd^{-4}}{5c^3d^{-10}}\right)^{-3}$

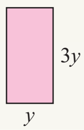
60) $\left(\dfrac{10x^{-5}y}{20x^5y^{-3}}\right)^{-2}$

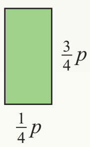
61) $\dfrac{(2u^{-5}v^2w^4)^{-5}}{(u^6v^{-7}w^{-10})^2}$

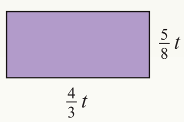
62) $\dfrac{(a^{-10}b^{-5}c^2)^4}{6(a^9bc^{-4})^{-2}}$

Write expressions for the area and perimeter for each rectangle.

63) 

64) 

65) 

66) 

Simplify. Assume that the variables represent nonzero integers.

67) $k^{4a} \cdot k^{2a}$

68) $r^{9y} \cdot r^y$

69) $(g^{2x})^4$

70) $(t^{5c})^3$

71) $\dfrac{x^{7b}}{x^{4b}}$

72) $\dfrac{m^{10u}}{m^{3u}}$

73) $(2r^{6m})^{-3}$

74) $(5a^{-2x})^{-2}$

Section 6.2 Addition and Subtraction of Polynomials

Objectives

1. Learn the Vocabulary Associated with Polynomials
2. Evaluate Polynomials
3. Add Polynomials
4. Subtract Polynomials
5. Add and Subtract Polynomials in More Than One Variable
6. Define and Evaluate a Polynomial Function

1. Learn the Vocabulary Associated with Polynomials

In Section 1.7, we defined an *algebraic expression* as a collection of numbers, variables, and grouping symbols connected by operation symbols such as $+$, $-$, $\times$, and $\div$.

An example of an algebraic expression is

$$5x^3 + \frac{7}{4}x^2 - x + 9$$

The *terms* of this algebraic expression are $5x^3$, $\frac{7}{4}x^2$, $-x$, and 9. A *term* is a number or a variable or a product or quotient of numbers and variables.

The expression $5x^3 + \frac{7}{4}x^2 - x + 9$ is also a *polynomial*.

Definition

A **polynomial in *x*** is the sum of a finite number of terms of the form ax^n, where n is a whole number and a is a real number. (The exponents must be whole numbers.)

Let's look more closely at the polynomial $5x^3 + \frac{7}{4}x^2 - x + 9$.

1) The polynomial is written **in descending powers of *x*** since the powers of x decrease from left to right. Generally, we write polynomials in descending powers of the variable.
2) Recall that the term without a variable is called a **constant**. The constant is 9. The **degree of a term** equals the exponent on its variable. (If a term has more than one variable, the degree equals the *sum* of the exponents on the variables.) We will list each term, its coefficient, and its degree.

Term	Coefficient	Degree
$5x^3$	5	3
$\frac{7}{4}x^2$	$\frac{7}{4}$	2
$-x$	-1	1
9	9	$0\ (9 = 9x^0)$

3) The **degree of the polynomial** equals the highest degree of any nonzero term. The degree of $5x^3 + \frac{7}{4}x^2 - x + 9$ is 3. Or, we say that this is a **third-degree polynomial**.

Example 1

Decide whether each expression *is* or *is not* a polynomial. If it is a polynomial, identify each term and the degree of each term. Then, find the degree of the polynomial.

a) $-8p^4 + 5.7p^3 - 9p^2 - 13$ b) $4c^2 - \frac{2}{5}c + 3 + \frac{6}{c^2}$

c) $a^3b^3 + 4a^3b^2 - ab + 1$ d) $7n^6$

Solution

a) The expression $-8p^4 + 5.7p^3 - 9p^2 - 13$ is a polynomial in p. Its terms have whole-number exponents and real coefficients. The term with the highest degree is $8p^4$, so the degree of the polynomial is 4.

Term	Degree
$-8p^4$	4
$5.7p^3$	3
$-9p^2$	2
-13	0

b) The expression $4c^2 - \frac{2}{5}c + 3 + \frac{6}{c^2}$ is *not* a polynomial because one of its terms has a variable in the denominator. $\left(\frac{6}{c^2} = 6c^{-2}\text{; the exponent } -2 \text{ is not a whole number.}\right)$

c) The expression $a^3b^3 + 4a^3b^2 - ab + 1$ *is* a polynomial because the variables have whole-number exponents and the coefficients are real numbers. Since this is a polynomial in two variables, we find the degree of each term by adding the exponents. The first term, a^3b^3, has the highest degree, 6, so the polynomial has degree 6.

Term	Degree
a^3b^3	6
$4a^3b^2$	5
$-ab$	2
1	0

Add the exponents to get the degree.

d) The expression $7n^6$ *is* a polynomial even though it has only one term. The degree of the term is 6, and that is the degree of the polynomial as well.

You Try 1

Decide whether each expression *is* or *is not* a polynomial. If it is a polynomial, identify each term and the degree of each term. Then, find the degree of the polynomial.

a) $d^4 + 7d^3 + \frac{3}{d}$ b) $k^3 - k^2 - 3.8k + 10$

c) $5x^2y^2 + \frac{1}{2}xy - 6x - 1$ d) $2r + 3r^{1/2} - 7$

The polynomial in Example 1d) is $7n^6$ and has one term. We call $7n^6$ a *monomial*. A **monomial** is a polynomial that consists of one term ("mono" means one). Some other examples of monomials are

$$y^2, \quad -4t^5, \quad x, \quad m^2n^2, \quad \text{and} \quad -3$$

A **binomial** is a polynomial that consists of exactly two terms ("bi" means two). Some examples are

$$w + 2, \quad 4z^2 - 11, \quad a^4 - b^4, \quad \text{and} \quad -8c^3d^2 + 3cd$$

A **trinomial** is a polynomial that consists of exactly three terms (“tri” means three). Here are some examples:

$$x^2 - 3x - 40, \qquad 2q^4 - 18q^2 + 10q, \qquad \text{and} \qquad 6a^4 + 29a^2b + 28b^2$$

In Section 1.7, we saw that expressions have different values depending on the value of the variable(s). The same is true for polynomials.

2. Evaluate Polynomials

Example 2

Evaluate the trinomial $n^2 - 7n + 4$ when

a) $n = 3$ and b) $n = -2$

Solution

a) Substitute 3 for n in $n^2 - 7n + 4$. Remember to put 3 in parentheses.

$$\begin{aligned} n^2 - 7n + 4 &= (3)^2 - 7(3) + 4 && \text{Substitute.} \\ &= 9 - 21 + 4 \\ &= -8 && \text{Add.} \end{aligned}$$

b) Substitute -2 for n in $n^2 - 7n + 4$. Put -2 in parentheses.

$$\begin{aligned} n^2 - 7n + 4 &= (-2)^2 - 7(-2) + 4 && \text{Substitute.} \\ &= 4 + 14 + 4 \\ &= 22 && \text{Add.} \end{aligned}$$

You Try 2

Evaluate $t^2 - 9t - 6$ when

a) $t = 5$ b) $t = -4$

Adding and Subtracting Polynomials

Recall in Section 1.7 we said that **like terms** contain the same variables with the same exponents. We add or subtract like terms by adding or subtracting the coefficients and leaving the variable(s) and exponent(s) the same. We use the same idea for adding and subtracting polynomials.

3. Add Polynomials

Procedure Adding Polynomials

To add polynomials, add like terms.

We can set up an addition problem vertically or horizontally.

Example 3

Add $(m^3 - 9m^2 + 5m - 4) + (2m^3 + 3m^2 - 1)$.

Solution

We will add these vertically. Line up like terms in columns and add.

$$\begin{array}{rrrrr} & m^3 - 9m^2 & + 5m & - 4 \\ + & 2m^3 + 3m^2 & & - 1 \\ \hline & 3m^3 - 6m^2 & + 5m & - 5 \end{array}$$

You Try 3

Add $(6b^3 - 11b^2 + 3b + 3) + (-2b^3 - 6b^2 + b - 8)$.

Example 4

Add $10k^2 + 2k - 1$ and $5k^2 + 7k + 9$.

Solution

Let's add these horizontally. Put the polynomials in parentheses since each contains more than one term. Use the associative and commutative properties to rewrite like terms together.

$$\begin{aligned} (10k^2 + 2k - 1) + (5k^2 + 7k + 9) &= (10k^2 + 5k^2) + (2k + 7k) + (-1 + 9) \\ &= 15k^2 + 9k + 8 \qquad \text{Combine like terms.} \end{aligned}$$

4. Subtract Polynomials

To subtract two polynomials such as $(8x + 3) - (5x - 4)$ we will be using the distributive property to clear the parentheses in the second polynomial.

Example 5

Subtract $(8x + 3) - (5x - 4)$.

Solution

$$\begin{aligned} (8x + 3) - (5x - 4) &= (8x + 3) - 1(5x - 4) \\ &= (8x + 3) + (-1)(5x - 4) && \text{Change } -1 \text{ to } +(-1). \\ &= (8x + 3) + (-5x + 4) && \text{Distribute.} \\ &= 3x + 7 && \text{Combine like terms.} \end{aligned}$$

In Example 5, notice that we changed the sign of each term in the second polynomial and then added it to the first.

Procedure Subtracting Polynomials

To subtract two polynomials, change the sign of each term in the second polynomial. Then, add the polynomials.

Let's see how we use this rule to subtract polynomials both horizontally and vertically.

Example 6

Subtract $(-6w^3 - w^2 + 10w + 1) - (2w^3 - 4w^2 + 9w - 5)$ vertically.

Solution

To subtract vertically, line up like terms in columns.

$$\begin{array}{rrrrr} & -6w^3 & -\ \ w^2 & +\ 10w & +\ 1 \\ - & \underline{(2w^3} & \underline{-\ 4w^2} & \underline{+\ \ 9w} & \underline{-\ 5)} \end{array}$$

Change the signs in the second polynomial and add the polynomials.

$$\begin{array}{rrrrr} & -6w^3 & -\ \ w^2 & +\ 10w & +\ 1 \\ + & \underline{-2w^3} & \underline{+\ 4w^2} & \underline{-\ \ 9w} & \underline{+\ 5} \\ & -8w^3 & +\ 3w^2 & +\ \ \ w & +\ 6 \end{array}$$

You Try 4

Subtract $(-7h^2 - 8h + 1) - (-3h^2 + h - 4)$.

5. Add and Subtract Polynomials in More Than One Variable

To add and subtract polynomials in more than one variable, remember that like terms contain the same variables with the same exponents.

Example 7

Perform the indicated operation.

a) $(a^2b^2 + 2a^2b - 13ab - 4) + (9a^2b^2 - 5a^2b - ab + 17)$

b) $(6tu - t + 2u + 5) - (4tu + 8t - 2)$

Solution

a) $(a^2b^2 + 2a^2b - 13ab - 4) + (9a^2b^2 - 5a^2b - ab + 17)$
$= 10a^2b^2 - 3a^2b - 14ab + 13$ Combine like terms.

b) $(6tu - t + 2u + 5) - (4tu + 8t - 2) = (6tu - t + 2u + 5) - 4tu - 8t + 2$
$= 2tu - 9t + 2u + 7$ Combine like terms.

You Try 5

Perform the indicated operation.

a) $(-12x^2y^2 + xy - 6y + 1) - (-4x^2y^2 - 10xy + 3y + 6)$

b) $(3.6m^3n^2 + 8.1mn - 10n) + (8.5m^3n^2 - 11.2mn + 4.3)$

6. Define and Evaluate a Polynomial Function

Look at the polynomial $3x^2 - x + 5$. If we substitute 2 for x, the *only* value of the expression is 15:

$$\begin{aligned} 3(2)^2 - (2) + 5 &= 3(4) - 2 + 5 \\ &= 12 - 2 + 5 \\ &= 15 \end{aligned}$$

For any value we substitute for x in a polynomial like $3x^2 - x + 5$, there will be *only one value* of the expression. Since each value substituted for the variable produces *only one value* of the expression, we can use function notation to represent a polynomial like $3x^2 - x + 5$.

Note

$f(x) = 3x^2 - x + 5$ is a **polynomial function** since $3x^2 - x + 5$ is a polynomial.

Therefore, finding $f(2)$ when $f(x) = 3x^2 - x + 5$ is the same as evaluating $3x^2 - x + 5$ when $x = 2$.

Example 8

If $f(x) = x^3 - 2x^2 + 4x + 19$, find $f(-3)$.

Solution

Substitute -3 for x.

$$f(x) = x^3 - 2x^2 + 4x + 19$$
$$f(-3) = (-3)^3 - 2(-3)^2 + 4(-3) + 19 \qquad \text{Substitute } -3 \text{ for } x.$$
$$f(-3) = -27 - 2(9) - 12 + 19$$
$$f(-3) = -27 - 18 - 12 + 19$$
$$f(-3) = -38$$

You Try 6

If $h(t) = 5t^4 + 8t^3 - t^2 + 7$, find $h(-1)$.

Answers to You Try Exercises

1) a) not a polynomial b) polynomial of degree 3 c) polynomial of degree 4

b)

Term	Degree
k^3	3
$-k^2$	2
$-3.8k$	1
10	0

c)

Term	Degree
$5x^2y^2$	4
$\frac{1}{2}xy$	2
$-6x$	1
-1	0

d) not a polynomial 2) a) -26 b) 46 3) $4b^3 - 17b^2 + 4b - 5$ 4) $4h^2 - 9h + 5$
5) a) $-8x^2y^2 + 11xy - 9y - 5$ b) $12.1m^3n^2 - 3.1mn - 10n + 4.3$ 6) 3

6.2 Exercises

Objective 1: Learn the Vocabulary Associated with Polynomials

Is the given expression a polynomial? Why or why not?

1) $-2p^2 - 5p + 6$

2) $8r^3 + 7r^2 - t + \frac{4}{5}$

3) $c^3 + 5c^2 + 4c^{-1} - 8$

4) $9a^5$

5) $f^{3/4} + 6f^{2/3} + 1$

6) $7y - 1 + \frac{3}{y}$

Determine whether each is a monomial, a binomial, or a trinomial.

7) $4x - 1$

8) $-5q^2$

9) $m^2n^2 - mn + 13$

10) $11c^2 + 3c$

11) 8

12) $k^5 + 2k^3 + 8k$

13) How do you determine the degree of a polynomial in one variable?

14) Write a third-degree polynomial in one variable.

15) How do you determine the degree of a term in a polynomial in more than one variable?

16) Write a fourth-degree monomial in x and y.

For each polynomial, identify each term in the polynomial, the coefficient and degree of each term, and the degree of the polynomial.

17) $3y^4 + 7y^3 - 2y + 8$

18) $6a^2 + 2a - 11$

19) $-4x^2y^3 - x^2y^2 + \frac{2}{3}xy + 5y$

20) $3c^2d^2 + 0.7c^2d + cd - 1$

Objective 2: Evaluate Polynomials

Evaluate each polynomial when a) $r = 3$ and b) $r = -1$.

21) $2r^2 - 7r + 4$

22) $2r^3 + 5r - 6$

Evaluate each polynomial when $x = 5$ and $y = -2$.

23) $9x + 4y$

24) $-2x + 3y + 16$

25) $x^2y^2 - 5xy + 2y$

26) $-2xy^2 + 7xy + 12y - 6$

27) $\frac{1}{2}xy - 4x - y$

28) $x^2 - y^2$

29) Bob will make a new gravel road from the highway to his house. The cost of building the road, y (in dollars), includes the cost of the gravel and is given by $y = 60x + 380$, where x is the number of hours he rents the equipment needed to complete the job.

a) Evaluate the binomial when $x = 5$, and explain what it means in the context of the problem.

b) If he keeps the equipment for 9 hours, how much will it cost to build the road?

c) If it cost \$860.00 to build the road, for how long did Bob rent the equipment?

30) An object is thrown upward so that its height, y (in feet), x seconds after being thrown is given by $y = -16x^2 + 48x + 64$.

a) Evaluate the polynomial when $x = 2$, and explain what it means in the context of the problem.

b) What is the height of the object 3 seconds after it is thrown?

c) Evaluate the polynomial when $x = 4$, and explain what it means in the context of the problem.

Objective 3: Add Polynomials

Add like terms.

31) $-6z + 8z + 11z$

32) $m^2 + 7m^2 - 14m^2$

33) $5c^2 + 9c - 16c^2 + c - 3c$

34) $-4y^3 + 3y^5 + 17y^5 + 6y^3 - 5y^5$

35) $6.7t^2 - 9.1t^6 - 2.5t^2 + 4.8t^6$

36) $\frac{5}{4}w^3 + \frac{3}{8}w^4 - \frac{2}{3}w^4 - \frac{5}{6}w^3$

37) $7a^2b^2 + 4ab^2 - 16ab^2 - a^2b^2 + 5ab^2$

38) $x^5y^2 - 14xy + 6xy + 5x^5y^2 + 8xy$

Add the polynomials.

39) $\begin{array}{r} 5n - 8 \\ + \underline{4n + 3} \end{array}$

40) $\begin{array}{r} 9d + 14 \\ + \underline{2d + 5} \end{array}$

41) $\begin{array}{r} -7a^3 + 11a \\ + \underline{2a^3 - 4a} \end{array}$

42) $\begin{array}{r} -h^4 + 6h^2 \\ + \underline{5h^4 - 3h^2} \end{array}$

43) $\begin{array}{r} 9r^2 + 16r + 2 \\ + \underline{3r^2 - 10r + 9} \end{array}$

44) $\begin{array}{r} m^2 - 3m - 8 \\ + \underline{2m^2 + 7m + 1} \end{array}$

45) $\begin{array}{r} b^2 - 8b - 14 \\ + \underline{3b^2 + 8b + 11} \end{array}$

46) $\begin{array}{r} 8g^2 + g + 5 \\ + \underline{5g^2 - 6g - 5} \end{array}$

47) $\begin{array}{r} \frac{5}{6}w^4 - \frac{2}{3}w^2 \qquad + \frac{1}{2} \\ + \underline{\frac{4}{9}w^4 + \frac{1}{6}w^2 - \frac{3}{8}w - 2} \end{array}$

48) $\begin{array}{r} -1.7p^3 - 2p^2 + 3.8p - 6 \\ + \underline{6.2p^3 \qquad - 1.2p + 14} \end{array}$

49) $(6m^2 - 5m + 10) + (-4m^2 + 8m + 9)$

50) $(3t^4 - 2t^2 + 11) + (t^4 + t^2 - 7)$

51) $\left(-2c^4 - \frac{7}{10}c^3 + \frac{3}{4}c - \frac{2}{9}\right) + \left(12c^4 + \frac{1}{2}c^3 - c + 3\right)$

52) $\left(\frac{7}{4}y^3 - \frac{3}{8}\right) + \left(\frac{5}{6}y^3 + \frac{7}{6}y^2 - \frac{9}{16}\right)$

53) $(2.7d^3 + 5.6d^2 - 7d + 3.1) + (-1.5d^3 + 2.1d^2 - 4.3d - 2.5)$

54) $(0.2t^4 - 3.2t + 4.1) + (-2.7t^4 + 0.8t^3 - 6.4t + 3.9)$

Objective 4: Subtract Polynomials

Subtract the polynomials.

55) $\begin{array}{r} 15w + 7 \\ - \underline{3w + 11} \end{array}$

56) $\begin{array}{r} 12a - 8 \\ - \underline{2a + 9} \end{array}$

57) $\begin{array}{r} y - 6 \\ - \underline{2y - 8} \end{array}$

58) $\begin{array}{r} 6p + 1 \\ - \underline{9p - 17} \end{array}$

59) $\begin{array}{r} 3b^2 - 8b + 12 \\ - \underline{5b^2 + 2b - 7} \end{array}$

60) $\begin{array}{r} -7d^2 + 15d + 6 \\ - \underline{8d^2 + 3d - 9} \end{array}$

61) $$\begin{array}{r} f^4 - 6f^3 + 5f^2 - 8f + 13 \\ - \;\; \underline{-3f^4 + 8f^3 - \;\; f^2 \qquad\;\; + \;\; 4} \end{array}$$

62) $$\begin{array}{r} 11x^4 + x^3 - 9x^2 + 2x - 4 \\ - \;\; \underline{-3x^4 + x^3 \qquad\qquad - \;\; x + 1} \end{array}$$

63) $$\begin{array}{r} 10.7r^2 + 1.2r + \;\; 9 \\ - \;\; \underline{4.9r^2 - 5.3r - 2.8} \end{array}$$

64) $$\begin{array}{r} -\frac{11}{10}m^3 + \frac{1}{2}m - \frac{5}{8} \\ - \;\; \underline{\frac{2}{5}m^3 + \frac{1}{7}m - \frac{5}{6}} \end{array}$$

65) $(j^2 + 16j) - (-6j^2 + 7j + 5)$

66) $(-3p^2 + p + 4) - (4p^2 + p + 1)$

67) $(17s^5 - 12s^2) - (9s^5 + 4s^4 - 8s^2 - 1)$

68) $(-5d^4 - 8d^2 + d + 3) - (-3d^4 + 17d^3 - 6d^2 - 20)$

69) $\left(-\frac{3}{8}r^2 + \frac{2}{9}r + \frac{1}{3}\right) - \left(-\frac{7}{16}r^2 - \frac{5}{9}r + \frac{7}{6}\right)$

70) $(3.8t^5 + 7.5t - 9.6) - (-1.5t^5 + 2.9t^2 - 1.1t + 3.4)$

71) Explain, in your own words, how to subtract two polynomials.

72) Do you prefer adding and subtracting polynomials vertically or horizontally? Why?

73) Will the sum of two trinomials always be a trinomial? Why or why not? Give an example.

74) Write a third-degree polynomial in x that does not contain a second-degree term.

Mixed Exercises: Objectives 3 and 4

Perform the indicated operations.

VIDEO 75) $(8a^4 - 9a^2 + 17) - (15a^4 + 3a^2 + 3)$

76) $(-x + 15) + (-5x - 12)$

77) $(-11n^2 - 8n + 21) + (4n^2 + 15n - 3) + (7n^2 - 10)$

78) $(-15a^3 + 8) - (-7a^3 + 3a + 5) + (10a^3 - a + 17)$

79) $(w^3 + 5w^2 + 3) - (6w^3 - 2w^2 + w + 12) + (9w^3 + 7)$

80) $(3r + 2) - (r^2 + 5r - 1) - (-9r^3 - r + 6)$

81) $\left(y^3 - \frac{3}{4}y^2 - 5y + \frac{3}{7}\right) + \left(\frac{1}{3}y^3 - y^2 + 8y - \frac{1}{2}\right)$

82) $\left(\frac{3}{5}c^4 + c^3 - \frac{3}{2}c^2 + 1\right)$
$+ \left(c^4 - 6c^3 - \frac{1}{4}c^2 + 6c - 1\right)$

83) $(3m^3 - 5m^2 + m + 12) - [(7m^3 + 4m^2 - m + 11)$
$+ (-5m^3 - 2m^2 + 6m + 8)]$

84) $(j^2 - 13j - 9) - [(-7j^2 + 10j - 2) + (4j^2 - 11j - 6)]$

Perform the indicated operations.

85) Find the sum of $p^2 - 7$ and $8p^2 + 2p - 1$.

86) Add $12n - 15$ to $5n + 4$.

87) Subtract $z^6 - 8z^2 + 13$ from $6z^6 + z^2 + 9$.

88) Subtract $-7x^2 + 8x + 2$ from $2x^2 + x$.

89) Subtract the sum of $6p^2 + 1$ and $3p^2 - 8p + 4$ from $2p^2 + p + 5$.

90) Subtract $17g^3 + 2g - 10$ from the sum of $5g^3 + g^2 + g$ and $3g^3 - 2g - 7$.

Objective 5: Add and Subtract Polynomials in More Than One Variable

Each of the polynomials is a polynomial in two variables. Perform the indicated operations.

91) $(5w + 17z) - (w + 3z)$

92) $(-4g - 7h) + (9g + h)$

93) $(ac + 8a + 6c) + (-6ac + 4a - c)$

94) $(11rt - 6r + 2) - (10rt - 7r + 12t + 2)$

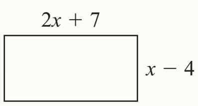
95) $(-6u^2v^2 + 11uv + 14)$
$- (-10u^2v^2 - 20uv + 18)$

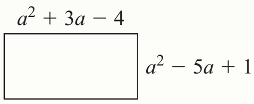
96) $(-7j^2k^2 + 9j^2k - 23jk^2 + 13)$
$+ (10j^2k^2 + 5j^2k - 17)$

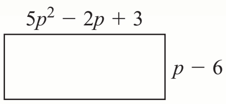
97) $(12x^3y^2 - 5x^2y^2 + 9x^2y - 17) + (5x^3y^2 + x^2y - 1)$
$- (6x^2y^2 + 10x^2y + 2)$

98) $(r^3s^2 + 2r^2s^2 + 10) - (7r^3s^2 + 18r^2s^2 - 9)$
$+ (11r^3s^2 - 3r^2s^2 - 4)$

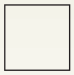
Find the polynomial that represents the perimeter of each rectangle.

99)


100)


101)


102) $\frac{2}{3}m + 4$



Objective 6: Define and Evaluate a Polynomial Function

VIDEO 103) If $f(x) = 5x^2 + 7x - 8$, find

a) $f(-3)$ b) $f(1)$

104) If $h(a) = -a^2 - 4a + 11$, find

a) $h(4)$ b) $h(-5)$

105) If $P(t) = t^3 - 2t^2 + 5t + 8$, find

a) $P(3)$ b) $P(0)$

106) If $G(c) = 4c^4 + c^2 - 3c - 5$, find

a) $G(0)$ b) $G(-1)$

107) $H(z) = -3z + 11$. Find z so that $H(z) = 13$.

108) $f(x) = \frac{1}{4}x + 7$. Find x so that $f(x) = 9$.

109) $r(k) = \frac{3}{5}k - 4$. Find k so that $r(k) = 14$.

110) $Q(a) = 4a - 3$. Find a so that $Q(a) = -9$.

Section 6.3 Multiplication of Polynomials

Objectives

1. Multiply a Monomial and a Polynomial
2. Multiply Two Polynomials
3. Multiply Two Binomials Using FOIL
4. Find the Product of More Than Two Polynomials
5. Find the Product of Binomials of the Form $(a + b)(a - b)$
6. Square a Binomial
7. Find Higher Powers of a Binomial

We have already learned that when multiplying two monomials, we multiply the coefficients and add the exponents of the same bases:

$$4c^5 \cdot 3c^6 = 12c^{11} \qquad -3x^2y^4 \cdot 7xy^3 = -21x^3y^7$$

In this section, we will discuss how to multiply other types of polynomials.

1. Multiply a Monomial and a Polynomial

To multiply a monomial and a polynomial, we use the distributive property.

Example 1

Multiply $2k^2(6k^2 + 5k - 3)$.

Solution

$$\begin{aligned} 2k^2(6k^2 + 5k - 3) &= (2k^2)(6k^2) + (2k^2)(5k) + (2k^2)(-3) && \text{Distribute.} \\ &= 12k^4 + 10k^3 - 6k^2 && \text{Multiply.} \end{aligned}$$

You Try 1

Multiply $5z^4(4z^3 - 7z^2 - z + 8)$.

2. Multiply Two Polynomials

To multiply two polynomials, we use the distributive property repeatedly. For example, to multiply $(2x - 3)(x^2 + 7x + 4)$, we multiply each term in the second polynomial by $(2x - 3)$.

$$(2x - 3)(x^2 + 7x + 4) = (2x - 3)(x^2) + (2x - 3)(7x) + (2x - 3)(4) \qquad \text{Distribute.}$$

Next, we distribute again.

$$\begin{aligned} &(2x - 3)(x^2) + (2x - 3)(7x) + (2x - 3)(4) \\ &= (2x)(x^2) - (3)(x^2) + (2x)(7x) - (3)(7x) + (2x)(4) - (3)(4) \\ &= 2x^3 - 3x^2 + 14x^2 - 21x + 8x - 12 && \text{Multiply.} \\ &= 2x^3 + 11x^2 - 13x - 12 && \text{Combine like terms.} \end{aligned}$$

This process of repeated distribution leads us to the following rule.

Procedure Multiplying Polynomials

To multiply two polynomials, multiply each term in the second polynomial by each term in the first polynomial. Then combine like terms. The answer should be written in descending powers.

Let's use this rule to multiply the polynomials in Example 2.

Example 2

Multiply $(n^2 + 5)(2n^3 + n - 9)$.

Solution

Multiply each term in the second polynomial by each term in the first.

$$\begin{aligned} &(n^2 + 5)(2n^3 + n - 9) \\ &\quad = (n^2)(2n^3) + (n^2)(n) + (n^2)(-9) + (5)(2n^3) + (5)(n) + (5)(-9) && \text{Distribute.} \\ &\quad = 2n^5 + n^3 - 9n^2 + 10n^3 + 5n - 45 && \text{Multiply.} \\ &\quad = 2n^5 + 11n^3 - 9n^2 + 5n - 45 && \text{Combine like terms.} \end{aligned}$$

Polynomials can be multiplied vertically as well. The process is very similar to the way we multiply whole numbers, so let's review a multiplication problem here.

$$\begin{array}{rl} 271 & \\ \times\quad 53 & \\ \hline 813 & \text{Multiply the 271 by 3.} \\ 13\,55 & \text{Multiply the 271 by 5.} \\ \hline 14{,}363 & \text{Add.} \end{array}$$

In the next example, we will find a product of polynomials by multiplying vertically.

Example 3

Multiply vertically. $(a^3 - 4a^2 + 5a - 1)(3a + 7)$

Solution

Set up the multiplication problem like you would for whole numbers:

$$\begin{array}{rrrrrl} & a^3 & - 4a^2 & + 5a & - 1 & \\ \times & & & 3a & + 7 & \\ \hline & 7a^3 & - 28a^2 & + 35a & - 7 & \text{Multiply each term in } a^3 - 4a^2 + 5a - 1 \text{ by 7.} \\ 3a^4 & - 12a^3 & + 15a^2 & - 3a & & \text{Multiply each term in } a^3 - 4a^2 + 5a - 1 \text{ by } 3a. \\ \hline 3a^4 & - 5a^3 & - 13a^2 & + 32a & - 7 & \text{Line up like terms in the same column. Add.} \end{array}$$

You Try 2

Multiply.

a) $(9x + 5)(2x^2 - x - 3)$

b) $\left(t^2 - \frac{2}{3}t - 4\right)(4t^2 + 6t - 5)$

3. Multiply Two Binomials Using FOIL

Multiplying two binomials is one of the most common types of polynomial multiplication used in algebra. A method called **FOIL** is one that is often used to multiply two binomials, and it comes from using the distributive property.

Let's use the distributive property to multiply $(x + 6)(x + 4)$.

$$\begin{aligned}(x + 6)(x + 4) &= (x + 6)(x) + (x + 6)(4) && \text{Distribute.}\\ &= x(x) + 6(x) + x(4) + 6(4) && \text{Distribute.}\\ &= x^2 + 6x + 4x + 24 && \text{Multiply.}\\ &= x^2 + 10x + 24 && \text{Combine like terms.}\end{aligned}$$

To be sure that each term in the first binomial has been multiplied by each term in the second binomial, we can use FOIL. FOIL stands for **F**irst **O**uter **I**nner **L**ast. Let's see how we can apply FOIL to the example above:

First, Last, Inner, Outer

$$\begin{aligned}(x + 6)(x + 4) = (x + 6)(x + 4) &= \overset{F}{x \cdot x} + \overset{O}{x \cdot 4} + \overset{I}{6 \cdot x} + \overset{L}{6 \cdot 4}\\ &= x^2 + 4x + 6x + 24 && \text{Multiply.}\\ &= x^2 + 10x + 24 && \text{Combine like terms.}\end{aligned}$$

You can see that we get the same result.

Example 4

Use FOIL to multiply the binomials.

a) $(p + 5)(p - 2)$ b) $(4r - 3)(r - 1)$

c) $(a + 4b)(a - 3b)$ d) $(2x + 9)(3y + 5)$

Solution

First, Last, Inner, Outer

$$\begin{aligned}\text{a)}\ (p + 5)(p - 2) = (p + 5)(p - 2) &= \overset{F}{p(p)} + \overset{O}{p(-2)} + \overset{I}{5(p)} + \overset{L}{5(-2)} && \text{Use FOIL.}\\ &= p^2 - 2p + 5p - 10 && \text{Multiply.}\\ &= p^2 + 3p - 10 && \text{Combine like terms.}\end{aligned}$$

Notice that the middle terms are like terms, so we can combine them.

First, Last, Inner, Outer

$$\begin{aligned}\text{b)}\ (4r - 3)(r - 1) = (4r - 3)(r - 1) &= \overset{F}{4r(r)} + \overset{O}{4r(-1)} - \overset{I}{3(r)} - \overset{L}{3(-1)} && \text{Use FOIL.}\\ &= 4r^2 - 4r - 3r + 3 && \text{Multiply.}\\ &= 4r^2 - 7r + 3 && \text{Combine like terms.}\end{aligned}$$

The middle terms are like terms, so we can combine them.

$$\begin{aligned}\text{c)}\ (a + 4b)(a - 3b) &= \overset{F}{a(a)} + \overset{O}{a(-3b)} + \overset{I}{4b(a)} + \overset{L}{4b(-3b)} && \text{Use FOIL.}\\ &= a^2 - 3ab + 4ab - 12b^2 && \text{Multiply.}\\ &= a^2 + ab - 12b^2 && \text{Combine like terms.}\end{aligned}$$

As in parts a) and b), we combined the middle terms.

$$\begin{aligned}\text{d)}\ (2x + 9)(3y + 5) &= \overset{F}{2x(3y)} + \overset{O}{2x(5)} + \overset{I}{9(3y)} + \overset{L}{9(5)} && \text{Use FOIL.}\\ &= 6xy + 10x + 27y + 45 && \text{Multiply.}\end{aligned}$$

In this case the middle terms were not like terms, so we could not combine them. ■

You Try 3

Use FOIL to multiply the binomials.

a) $(n + 8)(n + 5)$ b) $(3k + 7)(k - 4)$
c) $(x - 2y)(x - 6y)$ d) $(5c - 8)(2d + 1)$

With practice, you should get to the point where you can find the product of two binomials "in your head." Remember that, as in the case of parts a)–c) in Example 4, it is often possible to combine the middle terms.

4. Find the Product of More Than Two Polynomials

Sometimes we must find the product of more than two polynomials.

Example 5

Multiply $3t^2(5t + 7)(t - 2)$.

Solution

We can approach this problem a couple of ways.

Method 1

Begin by multiplying the binomials, *then* multiply by the monomial.

$$\begin{aligned} 3t^2(5t + 7)(t - 2) &= 3t^2(5t^2 - 10t + 7t - 14) && \text{Use FOIL to multiply the binomials.} \\ &= 3t^2(5t^2 - 3t - 14) && \text{Combine like terms.} \\ &= 15t^4 - 9t^3 - 42t^2 && \text{Distribute.} \end{aligned}$$

Method 2

Begin by multiplying $3t^2$ and $(5t + 7)$, then multiply *that* product by $(t - 2)$.

$$\begin{aligned} 3t^2(5t + 7)(t - 2) &= (15t^3 + 21t^2)(t - 2) && \text{Multiply } 3t^2 \text{ and } (5t + 7). \\ &= 15t^4 - 30t^3 + 21t^3 - 42t^2 && \text{Use FOIL to multiply.} \\ &= 15t^4 - 9t^3 - 42t^2 && \text{Combine like terms.} \end{aligned}$$

The result is the same. These may be multiplied by whichever method you prefer. ■

You Try 4

Multiply $-7m^3(m - 1)(2m - 3)$.

There are special types of binomial products that come up often in algebra. We will look at these next.

5. Find the Product of Binomials of the Form $(a + b)(a - b)$

Let's find the product $(y + 6)(y - 6)$. Using FOIL, we get

$$\begin{aligned} (y + 6)(y - 6) &= y^2 - 6y + 6y - 36 \\ &= y^2 - 36 \end{aligned}$$

Notice that the middle terms, the y-terms, drop out. In the result, $y^2 - 36$, the first term (y^2) is the square of y and the last term (36) is the square of 6. The resulting polynomial is a *difference of squares*. This pattern always holds when multiplying two binomials of the form $(a + b)(a - b)$.

Procedure The Product of the Sum and Difference of Two Terms

$$(a + b)(a - b) = a^2 - b^2$$

Example 6

Multiply.

a) $(z + 9)(z - 9)$ b) $(2 + c)(2 - c)$

c) $(5n - 8)(5n + 8)$ d) $\left(\frac{3}{4}t + u\right)\left(\frac{3}{4}t - u\right)$

Solution

a) The product $(z + 9)(z - 9)$ is in the form $(a + b)(a - b)$, where $a = z$ and $b = 9$. Use the rule that says $(a + b)(a - b) = a^2 - b^2$.

$$\begin{aligned}(z + 9)(z - 9) &= z^2 - 9^2 \\ &= z^2 - 81\end{aligned}$$

b) $$\begin{aligned}(2 + c)(2 - c) &= 2^2 - c^2 && a = 2 \text{ and } b = c \\ &= 4 - c^2\end{aligned}$$

Be very careful on a problem like this. The answer is $4 - c^2$, NOT $c^2 - 4$; subtraction is not commutative.

c) $$\begin{aligned}(5n - 8)(5n + 8) &= (5n + 8)(5n - 8) && \text{Commutative property} \\ &= (5n)^2 - 8^2 && a = 5n \text{ and } b = 8\text{; put } 5n \text{ in parentheses.} \\ &= 25n^2 - 64\end{aligned}$$

d) $$\begin{aligned}\left(\frac{3}{4}t + u\right)\left(\frac{3}{4}t - u\right) &= \left(\frac{3}{4}t\right)^2 - u^2 && a = \frac{3}{4}t \text{ and } b = u\text{; put } \frac{3}{4}t \text{ in parentheses.} \\ &= \frac{9}{16}t^2 - u^2\end{aligned}$$

You Try 5

Multiply.

a) $(k + 7)(k - 7)$ b) $(3c + 4)(3c - 4)$

c) $(8 - p)(8 + p)$ d) $\left(\frac{5}{2}m + n\right)\left(\frac{5}{2}m - n\right)$

6. Square a Binomial

Another type of special binomial product is a **binomial square** such as $(x + 5)^2$. $(x + 5)^2$ means $(x + 5)(x + 5)$. Therefore, we can use FOIL to multiply.

$$\begin{aligned}(x + 5)^2 = (x + 5)(x + 5) &= x^2 + 5x + 5x + 25 && \text{Use FOIL.} \\ &= x^2 + 10x + 25 && \text{Note that } 10x = 2(x)(5).\end{aligned}$$

Let's square the binomial $(y - 3)$.

$$\begin{aligned}(y - 3)^2 = (y - 3)(y - 3) &= y^2 - 3y - 3y + 9 && \text{Use FOIL.} \\ &= y^2 - 6y + 9 && \text{Note that } -6y = 2(y)(-3).\end{aligned}$$

In each case, notice that the outer and inner products are the same. When we add those terms, we see that the middle term of the result is *twice* the product of the terms in each binomial.

In the expansion of $(x + 5)^2$, $10x$ is $2(x)(5)$.
In the expansion of $(y - 3)^2$, $-6y$ is $2(y)(-3)$.

The *first* term in the result is the square of the *first* term in the binomial, and the *last* term in the result is the square of the *last* term in the binomial. We can express these relationships with the following formulas:

Procedure The Square of a Binomial

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$

We can think of the formulas in words as:

To square a binomial, you square the first term, square the second term, then multiply 2 times the first term times the second term and add.

Finding the products $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$ is also called *expanding* the binomial squares $(a + b)^2$ and $(a - b)^2$.

$$(a + b)^2 \neq a^2 + b^2 \quad \text{and} \quad (a - b)^2 \neq a^2 - b^2.$$

Example 7

Expand.

a) $(d + 7)^2$ b) $(m - 9)^2$ c) $(2x - 5y)^2$ d) $\left(\frac{1}{3}t + 4\right)^2$

Solution

a) $$(d + 7)^2 = \underset{\substack{\uparrow\\ \text{Square the}\\ \text{first term}}}{d^2} + \underset{\substack{\uparrow\\ \text{Two times}\\ \text{first term}\\ \text{times second}\\ \text{term}}}{2(d)(7)} + \underset{\substack{\uparrow\\ \text{Square the}\\ \text{second term}}}{(7)^2} \qquad a = d,\ b = 7$$

$$= d^2 + 14d + 49$$

Notice, $(d + 7)^2 \neq d^2 + 49$. Do not "distribute" the power of 2 to each term in the binomial!

b) $$(m - 9)^2 = \underset{\substack{\uparrow\\ \text{Square the}\\ \text{first term}}}{m^2} \underset{\substack{\uparrow\\ \text{Two times}\\ \text{first term}\\ \text{times second}\\ \text{term}}}{-2(m)(9)} + \underset{\substack{\uparrow\\ \text{Square the}\\ \text{second term}}}{(9)^2} \qquad a = m,\ b = 9$$

$$= m^2 - 18m + 81$$

c) $(2x - 5y)^2 = (2x)^2 - 2(2x)(5y) + (5y)^2 \qquad a = 2x,\ b = 5y$
$= 4x^2 - 20xy + 25y^2$

d) $\left(\frac{1}{3}t + 4\right)^2 = \left(\frac{1}{3}t\right)^2 + 2\left(\frac{1}{3}t\right)(4) + (4)^2 \qquad a = \frac{1}{3}t,\ b = 4$
$= \frac{1}{9}t^2 + \frac{8}{3}t + 16$

You Try 6

Expand.

a) $(r + 10)^2$ b) $(h - 1)^2$ c) $(2p + 3q)^2$ d) $\left(\frac{3}{4}y - 5\right)^2$

7. Find Higher Powers of a Binomial

To find higher powers of binomials, we use techniques we have already discussed.

Example 8

Expand.

a) $(n + 2)^3$ b) $(3v - 2)^4$

Solution

a) Just as $x^2 \cdot x = x^3$, it is true that $(n + 2)^2 \cdot (n + 2) = (n + 2)^3$. So we can think of $(n + 2)^3$ as $(n + 2)^2(n + 2)$.

$$\begin{aligned} (n + 2)^3 &= (n + 2)^2(n + 2) \\ &= (n^2 + 4n + 4)(n + 2) && \text{Square the binomial.} \\ &= n^3 + 2n^2 + 4n^2 + 8n + 4n + 8 && \text{Multiply.} \\ &= n^3 + 6n^2 + 12n + 8 && \text{Combine like terms.} \end{aligned}$$

b) Since we can write $x^4 = x^2 \cdot x^2$, we can write $(3v - 2)^4 = (3v - 2)^2 \cdot (3v - 2)^2$.

$$\begin{aligned} (3v - 2)^4 &= (3v - 2)^2 \cdot (3v - 2)^2 \\ &= (9v^2 - 12v + 4)(9v^2 - 12v + 4) && \text{Square each binomial.} \\ &= 81v^4 - 108v^3 + 36v^2 - 108v^3 + 144v^2 - 48v && \text{Multiply.} \\ &\quad + 36v^2 - 48v + 16 \\ &= 81v^4 - 216v^3 + 216v^2 - 96v + 16 && \text{Combine like terms.} \end{aligned}$$

You Try 7

Expand. a) $(k - 3)^3$ b) $(2h + 1)^4$

Answers to You Try Exercises

1) $20z^7 - 35z^6 - 5z^5 + 40z^4$ 2) a) $18x^3 + x^2 - 32x - 15$ b) $4t^4 + \frac{10}{3}t^3 - 25t^2 - \frac{62}{3}t + 20$

3) a) $n^2 + 13n + 40$ b) $3k^2 - 5k - 28$ c) $x^2 - 8xy + 12y^2$ d) $10cd + 5c - 16d - 8$

4) $-14m^5 + 35m^4 - 21m^3$ 5) a) $k^2 - 49$ b) $9c^2 - 16$ c) $64 - p^2$ d) $\frac{25}{4}m^2 - n^2$

6) a) $r^2 + 20r + 100$ b) $h^2 - 2h + 1$ c) $4p^2 + 12pq + 9q^2$ d) $\frac{9}{16}y^2 - \frac{15}{2}y + 25$

7) a) $k^3 - 9k^2 + 27k - 27$ b) $16h^4 + 32h^3 + 24h^2 + 8h + 1$

6.3 Exercises

Objective 1: Multiply a Monomial and a Polynomial

1) Explain how to multiply a monomial and a binomial.
2) Explain how to multiply two binomials.

Multiply.

3) $(3m^5)(8m^3)$
4) $(2k^6)(7k^3)$
5) $(-8c)(4c^5)$
6) $\left(-\frac{2}{9}z^3\right)\left(\frac{3}{4}z^9\right)$

Multiply.

7) $5a(2a - 7)$
8) $3y(10y - 1)$
9) $-6c(7c + 2)$
10) $-15d(11d - 2)$
11) $6v^3(v^2 - 4v - 2)$
12) $8f^5(f^2 - 3f - 6)$
13) $-9b^2(4b^3 - 2b^2 - 6b - 9)$
14) $-4h^7(5h^6 + 4h^3 + 11h - 3)$
15) $3a^2b(ab^2 + 6ab - 13b + 7)$
16) $4x^6y^2(-5x^2y + 11xy^2 - xy + 2y - 1)$
17) $-\frac{3}{5}k^4(15k^2 + 20k - 3)$
18) $\frac{3}{4}t^5(12t^3 - 20t^2 + 9)$

Objective 2: Multiply Two Polynomials

Multiply.

19) $(c + 4)(6c^2 - 13c + 7)$
20) $(d + 8)(7d^2 + 3d - 9)$
21) $(f - 5)(3f^2 + 2f - 4)$
22) $(k - 2)(9k^2 - 4k - 12)$
23) $(4x^3 - x^2 + 6x + 2)(2x - 5)$
24) $(3m^3 + 3m^2 - 4m - 9)(4m - 7)$
25) $\left(\frac{1}{3}y^2 + 4\right)(12y^2 + 7y - 9)$
26) $\left(\frac{3}{5}q^2 - 1\right)(10q^2 - 7q + 20)$
27) $(s^2 - s + 2)(s^2 + 4s - 3)$
28) $(t^2 + 4t + 1)(2t^2 - t - 5)$
29) $(4h^2 - h + 2)(-6h^3 + 5h^2 - 9h)$
30) $(n^4 + 8n^2 - 5)(n^2 - 3n - 4)$

Multiply both horizontally and vertically. Which method do you prefer and why?

31) $(3y - 2)(5y^2 - 4y + 3)$
32) $(2p^2 + p - 4)(5p + 3)$

Objective 3: Multiply Two Binomials Using FOIL

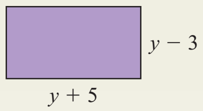
33) What do the letters in the word FOIL represent?
34) Can FOIL be used to expand $(x + 8)^2$? Explain your answer.

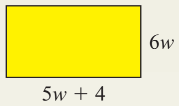
Use FOIL to multiply.

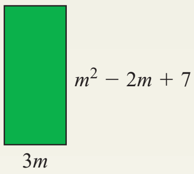
35) $(w + 5)(w + 7)$
36) $(u + 5)(u + 3)$
37) $(r - 3)(r + 9)$
38) $(w - 12)(w - 4)$
39) $(y - 7)(y - 1)$
40) $(g + 4)(g - 8)$
41) $(3p + 7)(p - 2)$
42) $(5u + 1)(u + 7)$
43) $(7n + 4)(3n + 1)$
44) $(4y - 3)(7y + 6)$
45) $(5 - 4w)(3 - w)$
46) $(2 - 3r)(4 - 5r)$
47) $(4a - 5b)(3a + 4b)$
48) $(3c + 2d)(c - 5d)$
49) $(6x + 7y)(8x + 3y)$
50) $(0.5p - 0.3q)(0.7p - 0.4q)$
51) $\left(v + \frac{1}{3}\right)\left(v + \frac{3}{4}\right)$
52) $\left(t + \frac{5}{2}\right)\left(t + \frac{6}{5}\right)$
53) $\left(\frac{1}{2}a + 5b\right)\left(\frac{2}{3}a - b\right)$

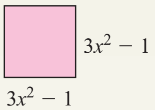
54) $\left(\frac{3}{4}x - y\right)\left(\frac{1}{3}x + 4y\right)$

Write an expression for a) the perimeter of each figure and b) the area of each figure.

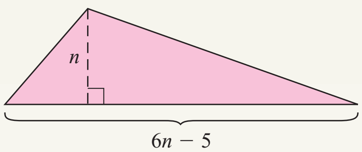
55) 

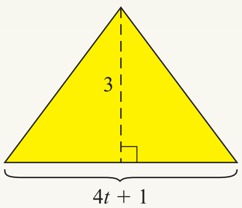
56) 

57) 

58) 

Express the area of each triangle as a polynomial.

59) 

60) 

Objective 4: Find the Product of More Than Two Polynomials

61) To find the product $3(c + 5)(c - 1)$, Parth begins by multiplying $3(c + 5)$ and then he multiplies that result by $(c - 1)$. Yolanda begins by multiplying $(c + 5)(c - 1)$ and multiplies that result by 3. Who is right?

62) Find the product $(2y + 3)(y - 5)(y - 4)$

a) by first multiplying $(2y + 3)(y - 5)$ and then multiplying that result by $(y - 4)$.

b) by first multiplying $(y - 5)(y - 4)$ and then multiplying that result by $(2y + 3)$.

c) What do you notice about the results?

Multiply.

63) $2(n + 3)(4n - 5)$

64) $-13(3p - 1)(p + 4)$

65) $-5z^2(z - 8)(z - 2)$

66) $11r^2(2r + 7)(-r + 1)$

67) $(c + 3)(c + 4)(c - 1)$

68) $(2t + 3)(t + 1)(t + 4)$

69) $(3x - 1)(x - 2)(x - 6)$

70) $(2m + 7)(m + 1)(m + 5)$

71) $8p\left(\frac{1}{4}p^2 + 3\right)(p^2 + 5)$

72) $10c\left(\frac{3}{5}c^2 - \frac{1}{2}\right)(c^2 + 1)$

Mixed Exercises: Objectives 5 and 6

Find the following special products.

73) $(y + 5)(y - 5)$

74) $(w + 2)(w - 2)$

75) $(a - 7)(a + 7)$

76) $(f - 11)(f + 11)$

77) $(3 - p)(3 + p)$

78) $(12 + d)(12 - d)$

79) $\left(u + \frac{1}{5}\right)\left(u - \frac{1}{5}\right)$

80) $\left(g - \frac{1}{4}\right)\left(g + \frac{1}{4}\right)$

81) $\left(\frac{2}{3} - k\right)\left(\frac{2}{3} + k\right)$

82) $\left(\frac{7}{4} + c\right)\left(\frac{7}{4} - c\right)$

83) $(2r + 7)(2r - 7)$

84) $(4h - 3)(4h + 3)$

85) $-(8j - k)(8j + k)$

86) $-(3m + 5n)(3m - 5n)$

87) $(d + 4)^2$

88) $(g + 2)^2$

89) $(n - 13)^2$

90) $(b - 3)^2$

91) $(h - 0.6)^2$

92) $(q - 1.5)^2$

93) $(3u + 1)^2$

94) $(5n + 4)^2$

VIDEO 95) $(2d - 5)^2$

96) $(4p - 3)^2$

97) $(3c + 2d)^2$

98) $(2a - 5b)^2$

99) $\left(\frac{3}{2}k + 8m\right)^2$

100) $\left(\frac{4}{5}x - 3y\right)^2$

101) $[(2a + b) + 3]^2$

102) $[(3c - d) + 5]^2$

103) $[(f - 3g) + 4][(f - 3g) - 4]$

104) $[(5j + 2k) + 1][(5j + 2k) - 1]$

105) Does $3(r + 2)^2 = (3r + 6)^2$? Why or why not?

106) Explain how to find the product $4(a - 5)^2$, then find the product.

Find each product.

107) $7(y + 2)^2$

108) $3(m + 4)^2$

109) $4c(c + 3)^2$

110) $-3a(a + 1)^2$

Objective 7: Find Higher Powers of a Binomial

Expand.

VIDEO 111) $(r + 5)^3$

112) $(u + 3)^3$

113) $(g - 4)^3$

114) $(w - 5)^3$

115) $(2a - 1)^3$

116) $(3x + 4)^3$

117) $(h + 3)^4$

118) $(y + 5)^4$

119) $(5t - 2)^4$

120) $(4c - 1)^4$

121) Does $(x + 2)^2 = x^2 + 4$? Why or why not?

122) Does $(y - 1)^3 = y^3 - 1$? Why or why not?

Mixed Exercises: Objectives 1–7

Find each product.

123) $(c - 12)(c + 7)$

124) $(11t^4)(-2t^6)$

125) $4(6 - 5a)(2a - 1)$

126) $(3y^2 - 8z)(3y^2 + 8z)$

127) $(2k - 9)(5k^2 + 4k - 1)$

128) $(m + 12)^2$

129) $\left(\frac{1}{6} - h\right)\left(\frac{1}{6} + h\right)$

130) $3(7p^3 + 4p^2 + 2) - (5p^3 - 18p - 11)$

131) $(3c + 1)^3$

132) $(4w - 5)(2w - 3)$

133) $\left(\frac{3}{8}p^7\right)\left(\frac{3}{4}p^4\right)$

134) $xy(2x - y)(x - 3y)(x - 2y)$

135) $(a^2 + 7b^2)^2$

136) $(r - 6)^3$

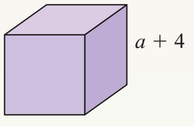
137) $-5z(z - 3)^2$

138) $(2n^2 + 9n - 3)(4n^2 - n - 8)$

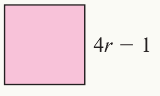
139) $[(x - 4y) + 5][(x - 4y) - 5]$

140) $[(3a + 2) - b]^2$

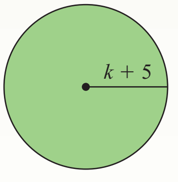
141) Express the volume of the cube as a polynomial.



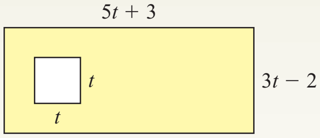
142) Express the area of the square as a polynomial.

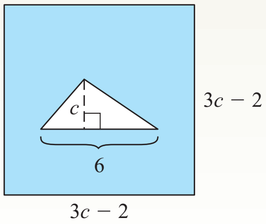


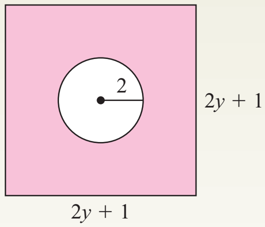
143) Express the area of the circle as a polynomial. Leave π in your answer.



Express the area of the shaded region as a polynomial. Leave π in your answer where appropriate.

144)


145)


146)


Section 6.4 Division of Polynomials

Objectives

1. **Divide a Polynomial by a Monomial**
2. **Divide a Polynomial by a Polynomial**
3. **Divide a Polynomial by a Binomial Using Synthetic Division**

The last operation with polynomials we need to discuss is division of polynomials. We will consider this in two parts:

dividing a polynomial by a monomial and

dividing a polynomial by a polynomial.

Let's begin with dividing a polynomial by a monomial.

1. Divide a Polynomial by a Monomial

The procedure for dividing a polynomial by a monomial is based on the procedure for adding or subtracting fractions.

To add $\frac{2}{9} + \frac{5}{9}$, we add the numerators and keep the denominator.

$$\frac{2}{9} + \frac{5}{9} = \frac{2+5}{9}$$
$$= \frac{7}{9}$$

Reversing the process above, we can write $\frac{7}{9} = \frac{2+5}{9} = \frac{2}{9} + \frac{5}{9}$.

We can generalize this result and say that

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \quad (c \neq 0)$$

This leads us to the following rule.

Procedure Dividing a Polynomial by a Monomial

To divide a polynomial by a monomial, divide *each term* in the polynomial by the monomial and simplify.

Example 1

Divide.

a) $\dfrac{24x^2 - 8x + 20}{4}$ b) $\dfrac{12c^3 + 54c^2 + 6c}{6c}$

Solution

a) First, note that the polynomial is being divided by a *monomial.* Divide each term in the numerator by the monomial 4.

$$\frac{24x^2 - 8x + 20}{4} = \frac{24x^2}{4} - \frac{8x}{4} + \frac{20}{4}$$
$$= 6x^2 - 2x + 5$$

Let's label the components of our division problem the same way as when we divide with integers.

$$\begin{array}{r} \text{Dividend} \longrightarrow \\ \text{Divisor} \longrightarrow \end{array} \frac{24x^2 - 8x + 20}{4} = 6x^2 - 2x + 5 \longleftarrow \text{Quotient}$$

We can check our answer by multiplying the quotient by the divisor. The answer should be the dividend.

Check: $4(6x^2 - 2x + 5) = 24x^2 - 8x + 20$ ✓ The quotient is correct.

b) $\dfrac{12c^3 + 54c^2 + 6c}{6c} = \dfrac{12c^3}{6c} + \dfrac{54c^2}{6c} + \dfrac{6c}{6c}$ Divide each term in numerator by $6c$.

$= 2c^2 + 9c + 1$ Apply the quotient rule for exponents.

BE CAREFUL

Students will often incorrectly "cancel out" $\dfrac{6c}{6c}$ and get nothing. But $\dfrac{6c}{6c} = 1$ since a quantity divided by itself equals one.

Check: $6c(2c^2 + 9c + 1) = 12c^3 + 54c^2 + 6c$ ✓ The quotient is correct. ■

Note

In Example 1b), c cannot equal zero because then the denominator of $\dfrac{12c^3 + 54c^2 + 6c}{6c}$ would equal zero. Remember, a fraction is undefined when its denominator equals zero!

You Try 1

Divide $\dfrac{35n^5 + 20n^4 - 5n^2}{5n^2}$.

Example 2

Divide $(6a - 7 - 36a^2 + 27a^3) \div (9a^2)$.

Solution

This is another example of a polynomial divided by a monomial. Notice, however, the terms in the numerator are not written in descending powers. Rewrite them in descending powers before dividing.

$$\begin{aligned}\frac{6a - 7 - 36a^2 + 27a^3}{9a^2} &= \frac{27a^3 - 36a^2 + 6a - 7}{9a^2} \\ &= \frac{27a^3}{9a^2} - \frac{36a^2}{9a^2} + \frac{6a}{9a^2} - \frac{7}{9a^2} \\ &= 3a - 4 + \frac{2}{3a} - \frac{7}{9a^2} \qquad \text{Apply quotient rule and simplify.}\end{aligned}$$

The quotient is *not* a polynomial since a and a^2 appear in denominators. The quotient of polynomials is not necessarily a polynomial.

You Try 2

Divide $(30z^2 + 3 - 50z^3 + 18z) \div (10z^2)$.

2. Divide a Polynomial by a Polynomial

When dividing a polynomial by a polynomial containing two or more terms, we use long division of polynomials. This method is similar to long division of whole numbers, so let's look at a long division problem and compare the procedure with polynomial long division.

Example 3

Divide 854 by 3.

Solution

$$\begin{array}{r} 2 \\ 3\overline{)854} \\ -6\downarrow \\ \hline 25 \end{array}$$

1) How many times does 3 divide into 8 evenly? 2
2) Multiply $2 \times 3 = 6$.
3) Subtract $8 - 6 = 2$.
4) Bring down the 5.

Start the process again.

$$\begin{array}{r} 28 \\ 3\overline{)854} \\ -6 \\ \hline 25 \\ -24\downarrow \\ \hline 14 \end{array}$$

1) How many times does 3 divide into 25 evenly? 8
2) Multiply $8 \times 3 = 24$.
3) Subtract $25 - 24 = 1$.
4) Bring down the 4.

Do the procedure again.

$$\begin{array}{r} 284 \\ 3\overline{)\ 854} \\ \underline{-6} \\ 25 \\ \underline{-24} \\ 14 \\ \underline{-12} \\ 2 \end{array}$$

1) How many times does 3 divide into 14 evenly? 4
2) Multiply $4 \times 3 = 12$.
3) Subtract $14 - 12 = 2$.
4) There are no more numbers to bring down, so the remainder is 2.

Write the result.

$$854 \div 3 = 284\frac{2}{3} \begin{array}{l} \leftarrow \text{Remainder} \\ \leftarrow \text{Divisor} \end{array}$$

Check: $(3 \times 284) + 2 = 852 + 2 = 854$ ✓ ■

You Try 3

Divide 638 by 5.

Next we will divide two polynomials using a long division process similar to that of Example 3.

Example 4

Divide $\dfrac{5x^2 + 13x + 6}{x + 2}$.

Solution

First, notice that we are dividing by a polynomial containing more than one term. That tells us to use long division of polynomials.

We will work with the x in $x + 2$ like we worked with the 3 in Example 3.

$$\begin{array}{r} 5x\phantom{{}+6} \\ x + 2\overline{)\ 5x^2 + 13x + 6} \\ \underline{-(5x^2 + 10x)}\ \downarrow \\ 3x + 6 \end{array}$$

1) By what do we multiply x to get $5x^2$? $5x$
   Line up terms in the quotient according to exponents, so write $5x$ above $13x$.
2) Multiply $5x$ by $(x + 2)$: $5x(x + 2) = 5x^2 + 10x$.
3) Subtract $(5x^2 + 13x) - (5x^2 + 10x) = 3x$.
4) Bring down the $+6$.

Start the process again. Remember, work with the x in $x + 2$ like we worked with the 3 in Example 3.

$$\begin{array}{r} 5x + 3 \\ x + 2\overline{)\ 5x^2 + 13x + 6} \\ \underline{-(5x^2 + 10x)}\phantom{{}+6} \\ 3x + 6 \\ \underline{-(3x + 6)} \\ 0 \end{array}$$

1) By what do we multiply x to get $3x$? 3
   Write $+3$ above $+6$.
2) Multiply 3 by $(x + 2)$. $3(x + 2) = 3x + 6$
3) Subtract $(3x + 6) - (3x + 6) = 0$.
4) There are no more terms. The remainder is 0.

Write the result.

$$\frac{5x^2 + 13x + 6}{x + 2} = 5x + 3$$

Check: $(x + 2)(5x + 3) = 5x^2 + 3x + 10x + 6 = 5x^2 + 13x + 6$ ✓

You Try 4

Divide.

a) $\dfrac{r^2 + 11r + 28}{r + 4}$ b) $\dfrac{3k^2 + 17k + 10}{k + 5}$

Next, we will look at a division problem with a remainder.

Example 5

Divide $\dfrac{-28n + 15n^3 + 41 - 17n^2}{3n - 4}$.

Solution

When we write our long division problem, the polynomial in the numerator must be rewritten so that the exponents are in descending order. Then, perform the long division.

$$\begin{array}{r}
5n^2 \phantom{{}- 17n^2 - 28n + 41} \\
3n - 4\overline{)\,15n^3 - 17n^2 - 28n + 41} \\
\underline{-(15n^3 - 20n^2)} \quad \downarrow \\
3n^2 - 28n \phantom{{}+ 41}
\end{array}$$

1) By what do we multiply $3n$ to get $15n^3$? $5n^2$
2) Multiply $5n^2(3n - 4) = 15n^3 - 20n^2$
3) Subtract. $(15n^3 - 17n^2) - (15n^3 - 20n^2)$
$= 15n^3 - 17n^2 - 15n^3 + 20n^2$
$= 3n^2$
4) Bring down the $-28n$.

Repeat the process.

$$\begin{array}{r}
5n^2 + n \phantom{{}- 28n + 41} \\
3n - 4\overline{)\,15n^3 - 17n^2 - 28n + 41} \\
\underline{-(15n^3 - 20n^2)} \phantom{{}- 28n + 41} \\
3n^2 - 28n \phantom{{}+ 41} \\
\underline{-(3n^2 - 4n)} \phantom{{}+ 41} \\
-24n + 41
\end{array}$$

1) By what do we multiply $3n$ to get $3n^2$? n
2) Multiply $n(3n - 4) = 3n^2 - 4n$.
3) Subtract. $(3n^2 - 28n) - (3n^2 - 4n)$
$= 3n^2 - 28n - 3n^2 + 4n$
$= -24n$
4) Bring down the $+41$.

Continue.

$$\begin{array}{r}
5n^2 + n - 8 \phantom{{}+ 41} \\
3n - 4\overline{)\,15n^3 - 17n^2 - 28n + 41} \\
\underline{-(15n^3 - 20n^2)} \phantom{{}- 28n + 41} \\
3n^2 - 28n \phantom{{}+ 41} \\
\underline{-(3n^2 - 4n)} \phantom{{}+ 41} \\
-24n + 41 \\
\underline{-(-24n + 32)} \\
9
\end{array}$$

1) By what do we multiply $3n$ to get $-24n$? -8
2) Multiply $-8(3n - 4) = -24n + 32$.
3) Subtract. $(-24n + 41) - (-24n + 32) = 9$

We are done with the long division process. How do we know that? Since the degree of 9 (degree zero) is less than the degree of $3n - 4$ (degree one), we cannot divide anymore. *The remainder is 9.*

$$\frac{15n^3 - 17n^2 - 28n + 41}{3n - 4} = 5n^2 + n - 8 + \frac{9}{3n - 4}$$

Check: $(3n - 4)(5n^2 + n - 8) + 9 = 15n^3 + 3n^2 - 24n - 20n^2 - 4n + 32 + 9$
$= 15n^3 - 17n^2 - 28n + 41$ ✓

You Try 5

Divide $-34a^2 + 57 + 8a^3 - 21a$ by $2a - 9$.

In Example 5, we saw that we must write our polynomials in descending order. We have to watch out for something else as well—missing terms. If a polynomial is missing one or more terms, we put them into the polynomial with coefficients of zero.

Example 6

Divide.

a) $x^3 + 64$ by $x + 4$ b) $t^4 + 3t^3 + 6t^2 + 11t + 5$ by $t^2 + 2$

Solution

a) The polynomial $x^3 + 64$ is missing the x^2-term and the x-term. We will insert these terms into the polynomial by giving them coefficients of zero.

$$x^3 + 64 = x^3 + 0x^2 + 0x + 64$$

Divide.

$$\begin{array}{r}
x^2 - 4x + 16 \\
x + 4 \overline{) x^3 + 0x^2 + 0x + 64} \\
\underline{-(x^3 + 4x^2)} \qquad\qquad \\
-4x^2 + 0x \qquad \\
\underline{-(-4x^2 - 16x)} \qquad \\
16x + 64 \\
\underline{-(16x + 64)} \\
0
\end{array}$$

$$(x^3 + 64) \div (x + 4) = x^2 - 4x + 16$$

Check: $(x + 4)(x^2 - 4x + 16) = x^3 - 4x^2 + 16x + 4x^2 - 16x + 64$
$= x^3 + 64$ ✓

b) In this case, the divisor, $t^2 + 2$, is missing a t-term. Rewrite it as $t^2 + 0t + 2$ and divide.

$$\begin{array}{rl}
 & t^2 + 3t + 4 \\
t^2 + 0t + 2 & \overline{)\, t^4 + 3t^3 + 6t^2 + 11t + 5} \\
 & \underline{-(t^4 + 0t^3 + 2t^2)} \\
 & \qquad 3t^3 + 4t^2 + 11t \\
 & \qquad \underline{-(3t^3 + 0t^2 + 6t)} \\
 & \qquad\qquad 4t^2 + 5t + 5 \\
 & \qquad\qquad \underline{-(4t^2 + 0t + 8)} \\
 & \qquad\qquad\qquad 5t - 3 \quad \leftarrow \text{Remainder}
\end{array}$$

We are done with the long division process because the degree of $5t - 3$ (degree one) is less than the degree of the divisor, $t^2 + 0t + 2$ (degree two).

Write the answer as $t^2 + 3t + 4 + \dfrac{5t - 3}{t^2 + 2}$. The check is left to the student. ■

You Try 6

Divide.

a) $\dfrac{4m^3 + 17m^2 - 38}{m + 3}$

b) $\dfrac{p^4 + 6p^3 + 3p^2 + 10p + 1}{p^2 + 1}$

3. Divide a Polynomial by a Binomial Using Synthetic Division

When we divide a polynomial by a binomial of the form $x - c$, another method called **synthetic division** can be used. Synthetic division uses only the numerical coefficients of the variables to find the quotient.

Consider the division problem $(3x^3 - 5x^2 - 6x + 13) \div (x - 2)$. On the left, we will illustrate the long division process as we have already presented it. On the right, we will show the process using only the coefficients of the variables.

$$\begin{array}{rl}
 & 3x^2 + x - 4 \\
x - 2 & \overline{)\, 3x^3 - 5x^2 - 6x + 13} \\
 & \underline{-(3x^3 - 6x^2)} \\
 & \qquad x^2 - 6x \\
 & \qquad \underline{-(x^2 - 2x)} \\
 & \qquad\qquad -4x + 13 \\
 & \qquad\qquad \underline{-(-4x + 8)} \\
 & \qquad\qquad\qquad 5
\end{array}
\qquad\qquad
\begin{array}{rl}
 & 3 + 1 - 4 \\
1 - 2 & \overline{)\, 3 - 5 - 6 + 13} \\
 & \underline{-(3 - 6)} \\
 & \qquad 1 - 6 \\
 & \qquad \underline{-(1 - 2)} \\
 & \qquad\qquad -4 + 13 \\
 & \qquad\qquad \underline{-(-4 + 8)} \\
 & \qquad\qquad\qquad 5
\end{array}$$

Note

As long as we keep the like terms lined up in the correct columns, the variables do not affect the numerical coefficients of the quotient. This process of using only the numerical coefficients to divide a polynomial by a binomial of the form $x - c$ is called *synthetic division*. Using synthetic division is often quicker than using the traditional long division process.

We will present the steps for performing synthetic division by looking at the previous example again.

Example 7

Use synthetic division to divide $(3x^3 - 5x^2 - 6x + 13)$ by $(x - 2)$.

Solution

Remember, in order to be able to use synthetic division, the divisor must be in the form $x - c$. $x - 2$ is in the form $x - c$, and $c = 2$.

Procedure How to Perform Synthetic Division

Step 1: Write the dividend in descending powers of x. If a term of any degree is missing, insert the term into the polynomial with a coefficient of 0.
The dividend in the example is $3x^3 - 5x^2 - 6x + 13$. *It is written in descending order, and no terms are missing.*

Step 2: Write the value of c in an open box. Next to it, on the right, write the coefficients of the terms of the dividend. Skip a line and draw a horizontal line under the coefficients. Bring down the first coefficient.

In this example, $c = 2$.

$$\begin{array}{r|rrrr} 2 & 3 & -5 & -6 & 13 \\ & & & & \\ \hline & 3 & & & \end{array}$$

Step 3: Multiply the number in the box by the coefficient under the horizontal line. (Here, that is $2 \cdot 3 = 6$.) Write the product under the next coefficient. (Write the 6 under the -5.) Then, *add* the numbers in the second column. (Here, we get 1.)

$$\begin{array}{r|rrrr} 2 & 3 & -5 & -6 & 13 \\ & & 6 & & \\ \hline & 3 & 1 & & \end{array} \quad 2 \cdot 3 = 6; -5 + 6 = 1$$

Step 4: Multiply the number in the box by the number under the horizontal line in the second column. (Here, that is $2 \cdot 1 = 2$.) Write the product under the next coefficient. (Write the 2 under the -6.) Then, *add* the numbers in the third column. (Here, we get -4.)

$$\begin{array}{r|rrrr} 2 & 3 & -5 & -6 & 13 \\ & & 6 & 2 & \\ \hline & 3 & 1 & -4 & \end{array} \quad 2 \cdot 1 = 2; -6 + 2 = -4$$

Step 5: Repeat the procedure of step 4 with subsequent columns until there is a number in each column in the row under the horizontal line.

$$\begin{array}{r|rrrr} 2 & 3 & -5 & -6 & 13 \\ & & 6 & 2 & -8 \\ \hline & 3 & 1 & -4 & 5 \end{array} \quad 2 \cdot (-4) = -8; 13 + (-8) = 5$$

The numbers in the last row represent the quotient and the remainder. The last number is the remainder. The numbers before it are the coefficients of the quotient. *The degree of the quotient is one less than the degree of the dividend.*

In our example, the dividend is a *third-degree* polynomial. Therefore, the quotient is a *second-degree* polynomial.

Since the 3 in the first row is the coefficient of x^3 in the dividend, the 3 in the last row is the coefficient of x^2 in the quotient, and so on.

$$
\begin{array}{r|rrrrl}
\multicolumn{5}{c}{\text{Dividend}} \\
\multicolumn{5}{c}{\overbrace{3x^3 - 5x^2 - 6x + 13}} \\
2 & 3 & -5 & -6 & 13 & \\
 & & 6 & 2 & -8 & \\
\hline
 & 3 & 1 & -4 & 5 & \longrightarrow \text{Remainder} \\
\multicolumn{5}{c}{\underbrace{3x^2 + 1x - 4}} \\
\multicolumn{5}{c}{\text{Quotient}}
\end{array}
$$

$$(3x^3 - 5x^2 - 6x + 13) \div (x - 2) = 3x^2 + x - 4 + \frac{5}{x - 2}.$$ ■

You Try 7

Use synthetic division to divide $(2x^3 + x^2 - 16x - 7) \div (x - 3)$.

Note

If the divisor was $x + 2$, then we would write it in the form $x - c$ as $x - (-2)$. So, $c = -2$.

BE CAREFUL

Synthetic division can be used only when dividing a polynomial by a binomial of the form $x - c$. If the divisor is not in the form $x - c$, use long division.

Synthetic division can be used to find $(4x^2 - 19x + 16) \div (x - 4)$ because $x - 4$ is in the form $x - c$.

Synthetic division *cannot* be used to find $(4x^2 - 19x + 16) \div (2x - 3)$ because $2x - 3$ is *not* in the form $x - c$. Use long division.

Summary Dividing Polynomials

Remember, when asked to divide two polynomials, first identify which type of division problem it is.

1) To divide a *polynomial* by a *monomial,* divide *each term* in the polynomial by the monomial and simplify.

$$\text{Monomial} \longrightarrow \quad \frac{56k^3 + 24k^2 - 8k + 2}{8k} = \frac{56k^3}{8k} + \frac{24k^2}{8k} - \frac{8k}{8k} + \frac{2}{8k}$$
$$= 7k^2 + 3k - 1 + \frac{1}{4k}$$

2) To divide a *polynomial* by a *polynomial* containing two or more terms, use *long division*.

$$\frac{15x^3 + 34x^2 - 11x - 2}{5x - 2}$$

Binomial ⟶ $5x - 2$

$$\begin{array}{r}
3x^2 + 8x + 1 \\
5x - 2\overline{\big)\,15x^3 + 34x^2 - 11x - 2} \\
\underline{-(15x^3 - 6x^2)} \\
40x^2 - 11x \\
\underline{-(40x^2 - 16x)} \\
5x - 2 \\
\underline{-(5x - 2)} \\
0
\end{array}$$

$$\frac{15x^3 + 34x^2 - 11x - 2}{5x - 2} = 3x^2 + 8x + 1$$

3) To divide a *polynomial* by a *binomial of the form* $x - c$, use long division or synthetic division.

Answers to You Try Exercises

1) $7n^3 + 4n^2 - 1$ 2) $-5z + 3 + \frac{9}{5z} + \frac{3}{10z^2}$ 3) $127\frac{3}{5}$ 4) a) $r + 7$ b) $3k + 2$

5) $4a^2 + a - 6 + \frac{3}{2a - 9}$ 6) a) $4m^2 + 5m - 15 + \frac{7}{m + 3}$ b) $p^2 + 6p + 2 + \frac{4p - 1}{p^2 + 2}$

7) $2x^2 + 7x + 5 + \frac{8}{x - 3}$

6.4 Exercises

Objective 1: Divide a Polynomial by a Monomial

1) Label the dividend, divisor, and quotient of $\frac{6c^3 + 15c^2 - 9c}{3c} = 2c^2 + 5c - 3$.

2) How would you check the answer to the division problem in Exercise 1?

3) Explain, in your own words, how to divide a polynomial by a monomial.

4) Without dividing, determine what the degree of the quotient will be when performing the division $\frac{48y^5 - 16y^4 + 5y^3 - 32y^2}{16y^2}$.

Divide.

5) $\frac{49p^4 + 21p^3 + 28p^2}{7}$

6) $\frac{10m^3 + 45m^2 + 30m}{5}$

7) $\frac{12w^3 - 40w^2 - 36w}{4w}$

8) $\frac{3a^5 - 27a^4 + 12a^3}{3a}$

9) $\frac{22z^6 + 14z^5 - 38z^3 + 2z}{2z}$

10) $\frac{48u^7 - 18u^4 - 90u^2 + 6u}{6u}$

11) $\frac{9h^8 + 54h^6 - 108h^3}{9h^2}$

12) $\frac{72x^9 - 24x^7 - 56x^4}{8x^2}$

13) $\frac{36r^7 - 12r^4 + 6}{12r}$

14) $\frac{20t^6 + 130t^2 + 2}{10t}$

15) $\dfrac{8d^6 - 12d^5 + 18d^4}{2d^4}$

16) $\dfrac{21p^4 - 15p^3 + 6p^2}{3p^2}$

17) $\dfrac{28k^7 + 8k^5 - 44k^4 - 36k^2}{4k^2}$

18) $\dfrac{42n^7 + 14n^6 - 49n^4 + 63n^3}{7n^3}$

VIDEO 19) $(35d^5 - 7d^2) \div (-7d^2)$

20) $(-30h^7 - 8h^5 + 2h^3) \div (-2h^3)$

21) $\dfrac{10w^5 + 12w^3 - 6w^2 + 2w}{6w^2}$

22) $\dfrac{-48r^5 + 14r^3 - 4r^2 + 10}{4r}$

23) $(12k^8 - 4k^6 - 15k^5 - 3k^4 + 1) \div (2k^5)$

24) $(56m^6 + 4m^5 - 21m^2) \div (7m^3)$

Divide.

25) $\dfrac{48p^5q^3 + 60p^4q^2 - 54p^3q + 18p^2q}{6p^2q}$

26) $(-45x^5y^4 - 27x^4y^5 + 9x^3y^5 + 63x^3y^3) \div (-9x^3y^2)$

27) $\dfrac{14s^6t^6 - 28s^5t^4 - s^3t^3 + 21st}{7s^2t}$

28) $(4a^5b^4 - 32a^4b^4 - 48a^3b^4 + a^2b^3) \div (4ab^2)$

29) Chandra divides $40p^3 - 10p^2 + 5p$ by $5p$ and gets a quotient of $8p^2 - 2p$. Is this correct? Why or why not?

30) Ryan divides $\dfrac{20y^2 + 15y}{15y}$ and gets a quotient of $20y^2$. What was his mistake? What is the correct answer?

Objective 2: Divide a Polynomial by a Polynomial

31) Label the dividend, divisor, and quotient of

$$\begin{array}{r} 4w^2 - 2w - 7 \\ 3w + 1\overline{)12w^3 - 2w^2 - 23w - 7} \end{array}.$$

32) When do you use long division to divide polynomials?

33) If a polynomial of degree 3 is divided by a binomial of degree 1, determine the degree of the quotient.

34) How would you check the answer to the division problem in Exercise 31?

Divide.

35) $6\overline{)949}$

36) $4\overline{)857}$

37) $9\overline{)3937}$

38) $8\overline{)4189}$

VIDEO 39) $\dfrac{g^2 + 9g + 20}{g + 5}$

40) $\dfrac{m^2 + 8m + 15}{m + 3}$

41) $\dfrac{a^2 + 13a + 42}{a + 7}$

42) $\dfrac{w^2 + 5w + 4}{w + 1}$

43) $\dfrac{k^2 - k - 30}{k + 5}$

44) $\dfrac{c^2 - 13c + 36}{c - 4}$

45) $\dfrac{6h^3 + 7h^2 - 17h - 4}{3h - 4}$

46) $\dfrac{20f^3 - 23f^2 + 41f - 14}{5f - 2}$

47) $(p + 23p^2 - 1 + 12p^3) \div (4p + 1)$

48) $(16y^2 + 6 + 15y^3 + 13y) \div (3y + 2)$

49) $(7m^2 - 16m - 41) \div (m - 4)$

50) $(2t^2 + 5t + 8) \div (t + 7)$

51) $\dfrac{24a + 20a^3 - 12 - 43a^2}{5a - 2}$

52) $\dfrac{17v^3 + 33v - 18 + 9v^4 - 56v^2}{9v - 1}$

53) $\dfrac{n^3 + 27}{n + 3}$

54) $\dfrac{d^3 - 8}{d - 2}$

55) $(8r^3 + 6r^2 - 25) \div (4r - 5)$

56) $(12c^3 + 23c^2 + 61) \div (3c + 8)$

57) $\dfrac{12x^3 - 17x + 4}{2x + 3}$

58) $\dfrac{16h^3 - 106h + 15}{2h - 5}$

59) $\dfrac{k^4 + k^3 + 9k^2 + 4k + 20}{k^2 + 4}$

60) $\dfrac{a^4 + 7a^3 + 6a^2 + 21a + 9}{a^2 + 3}$

61) $\dfrac{15t^4 - 40t^3 - 33t^2 + 10t + 2}{5t^2 - 1}$

62) $\dfrac{18v^4 - 15v^3 - 18v^2 + 13v - 10}{3v^2 - 4}$

63) Is the quotient of two polynomials always a polynomial? Explain.

64) Write a division problem that has a divisor of $2c - 1$ and a quotient of $8c - 5$.

Objective 3: Divide a Polynomial by a Binomial Using Synthetic Division

65) Explain when synthetic division may be used to divide polynomials.

66) Can synthetic division be used to divide $\dfrac{x^3 - 15x^2 + 8x + 12}{x^2 - 2}$? Why or why not?

Use synthetic division to divide the polynomials.

67) $(t^2 + 5t - 36) \div (t - 4)$

68) $(m^2 - 2m - 24) \div (m - 6)$

69) $\dfrac{5n^2 + 21n + 20}{n + 3}$

70) $\dfrac{6k^2 + 4k - 19}{k + 2}$

71) $(2y^3 + 7y^2 - 10y + 21) \div (y + 5)$

72) $(4p - 3 - 10p^2 + 3p^3) \div (p - 3)$

73) $(10c^2 + 3c + 2c^3 - 20) \div (c + 4)$

74) $(2 + 5x^4 - 8x + 7x^3 - x^2) \div (x + 1)$

75) $(-4w^3 + w - 8 + w^4 + 7w^2) \div (w - 2)$

76) $\dfrac{r^3 - 3r^2 + 4}{r - 2}$

77) $\dfrac{m^4 - 81}{m - 3}$

78) $(n^5 - 29n^2 - 2n) \div (n - 3)$

79) $(2x^3 + 7x^2 - 16x + 6) \div \left(x - \dfrac{1}{2}\right)$

80) $(3t^3 - 25t^2 + 14t - 2) \div \left(t - \dfrac{1}{3}\right)$

Mixed Exercises: Objectives 1–3

Divide.

81) $\dfrac{50a^4b^4 + 30a^4b^3 - a^2b^2 + 2ab}{10a^2b^2}$

82) $\dfrac{12n^2 - 37n + 16}{4n - 3}$

83) $\dfrac{-15f^4 - 22f^2 + 5 + 49f + 36f^3}{5f - 2}$

84) $(8p^2 + 4p - 32p^3 + 36p^4) \div (-4p)$

85) $\dfrac{8t^2 - 19t - 4}{t - 3}$

86) $\dfrac{-27x^3y^3 + 9x^2y^3 + 36xy + 72}{9x^2y}$

87) $(64p^3 - 27) \div (4p - 3)$

88) $(7g^3 + 26g^2 - 47g - 10) \div (g + 5)$

89) $(11x^2 + x^3 - 21 + 6x^4 + 3x) \div (x^2 + 3)$

90) $\dfrac{125t^3 + 8}{5t + 2}$

91) $\dfrac{10h^4 - 6h^3 - 49h^2 + 27h + 19}{2h^2 - 9}$

92) $\dfrac{m^4 - 16}{m^2 + 4}$

93) $\dfrac{j^4 - 1}{j^2 - 1}$

94) $\dfrac{45t^4 - 81t^2 - 27t^3 + 8t^6}{-9t^3}$

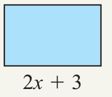
95) $\dfrac{9q^2 + 42q^4 - 9 + 6q - 8q^3}{3q^2}$

96) $\left(x^2 + \dfrac{17}{2}x - 15\right) \div (2x - 3)$

97) $\dfrac{21p^4 - 29p^3 - 15p^2 + 28p + 16}{7p^2 + 2p - 4}$

98) $\dfrac{8c^4 + 26c^3 + 29c^2 + 14c + 3}{2c^2 + 5c + 3}$

For each rectangle, find a polynomial that represents the missing side.

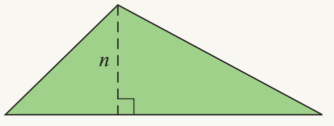
99) 

Find the width if the area is given by $6x^2 + 23x + 21$ sq units.

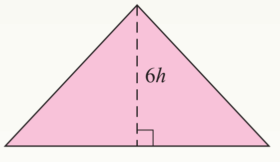
100) 

Find the width if the area is given by $20k^3 + 8k^5$ sq units.

101) Find the base of the triangle if the area is given by $15n^3 - 18n^2 + 6n$ sq units.



102) Find the base of the triangle if the area is given by $8h^3 + 7h^2 + 2h$ sq units.



103) If Joelle travels $(3x^3 + 5x^2 - 26x + 8)$ miles in $(x + 4)$ hours, find an expression for her rate.

104) If Lorenzo spent $(4a^3 - 11a^2 + 3a - 18)$ dollars on chocolates that cost $(a - 3)$ dollars per pound, find an expression for the number of pounds of chocolates he purchased.

Chapter 6: Summary

Definition/Procedure	Example
6.1 Review of the Rules of Exponents For real numbers a and b and integers m and n, the following rules apply:	
Product rule: $a^m \cdot a^n = a^{m+n}$ **(p. 352)**	$p^3 \cdot p^5 = p^{3+5} = p^8$
Power rules: a) $(a^m)^n = a^{mn}$ b) $(ab)^n = a^n b^n$ c) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $(b \neq 0)$ **(p. 352)**	a) $(c^4)^3 = c^{4 \cdot 3} = c^{12}$ b) $(2g)^5 = 2^5 g^5 = 32g^5$ c) $\left(\frac{t}{4}\right)^3 = \frac{t^3}{4^3} = \frac{t^3}{64}$
Zero exponent: $a^0 = 1$ if $a \neq 0$ **(p. 352)**	$9^0 = 1$
Negative exponent: a) $a^{-n} = \left(\frac{1}{a}\right)^n = \frac{1}{a^n}$ $(a \neq 0)$ b) $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$ $(a \neq 0, b \neq 0)$ **(p. 352)**	a) $6^{-2} = \left(\frac{1}{6}\right)^2 = \frac{1}{6^2} = \frac{1}{36}$ b) $\frac{a^{-5}}{b^{-3}} = \frac{b^3}{a^5}$
Quotient rule: a) $\frac{a^m}{a^n} = a^{m-n}$ $(a \neq 0)$ **(p. 352)**	$\frac{k^{14}}{k^4} = k^{14-4} = k^{10}$
6.2 Addition and Subtraction of Polynomials	
A **polynomial in *x*** is the sum of a finite number of terms of the form ax^n, where n is a whole number and a is a real number. (The exponents must be whole numbers.) The **degree of a term** equals the exponent on its variable. If a term has more than one variable, the degree equals the *sum* of the exponents on the variables. The **degree of the polynomial** equals the highest degree of any nonzero term. **(p. 355)**	Identify each term in the polynomial, the coefficient and degree of each term, and the degree of the polynomial. $3m^4n^2 - m^3n^2 + 2m^2n^3 + mn - 5n$ **Term** / **Coeff.** / **Degree** $3m^4n^2$ / 3 / 6 $-m^3n^2$ / -1 / 5 $2m^2n^3$ / 2 / 5 mn / 1 / 2 $-5n$ / -5 / 1 The degree of the polynomial is 6.
To **add polynomials,** add like terms. Polynomials may be added horizontally or vertically. **(p. 357)**	Add the polynomials. $(4q^2 + 2q - 12) + (-5q^2 + 3q + 8)$ $= (4q^2 + (-5q^2)) + (2q + 3q) + (-12 + 8) = -q^2 + 5q - 4$
To **subtract two polynomials,** change the sign of each term in the second polynomial. Then add the polynomials. **(p. 358)**	Subtract $(4t^3 - 7t^2 + 4t + 4) - (12t^3 - 8t^2 + 3t + 9)$ $= (4t^3 - 7t^2 + 4t + 4) + (-12t^3 + 8t^2 - 3t - 9)$ $= -8t^3 + t^2 + t - 5$
$f(x) = 2x^2 + 9x - 5$ is an example of a **polynomial function** because $2x^2 + 9x - 5$ is a polynomial and each real number that is substituted for x produces only one value for the expression. Finding $f(3)$ is the same as evaluating $2x^2 + 9x - 5$ when $x = 3$. **(p. 359)**	If $f(x) = 2x^2 + 9x - 5$, find $f(3)$. $f(3) = 2(3)^2 + 9(3) - 5$ $f(3) = 2(9) + 27 - 5$ $f(3) = 18 + 27 - 5$ $f(3) = 40$

Definition/Procedure	Example

6.3 Multiplication of Polynomials

Definition/Procedure	Example
When multiplying a **monomial** and a **polynomial,** use the distributive property. **(p. 363)**	Multiply. $5y^3(-2y^2 + 8y - 3)$ $= (5y^3)(-2y^2) + (5y^3)(8y) + (5y^3)(-3)$ $= -10y^5 + 40y^4 - 15y^3$
To **multiply two polynomials,** multiply each term in the second polynomial by each term in the first polynomial. Then combine like terms. **(p. 364)**	Multiply. $(5p + 2)(p^2 - 3p + 6)$ $= (5p)(p^2) + (5p)(-3p) + (5p)(6)$ $+ (2)(p^2) + (2)(-3p) + (2)(6)$ $= 5p^3 - 15p^2 + 30p + 2p^2 - 6p + 12$ $= 5p^3 - 13p^2 + 24p + 12$
Multiplying Two Binomials We can use FOIL to multiply two binomials. **FOIL** stands for **F**irst, **O**uter, **I**nner, **L**ast. Then add like terms. **(p. 365)**	Use FOIL to multiply $(4a - 5)(a + 3)$. First Last $(4a - 5)(a + 3)$ Inner Outer $(4a - 5)(a + 3) = 4a^2 + 7a - 15$
Special Products a) $(a + b)(a - b) = a^2 - b^2$ b) $(a + b)^2 = a^2 + 2ab + b^2$ c) $(a - b)^2 = a^2 - 2ab + b^2$ **(pp. 367–368)**	a) Multiply. $(c + 9)(c - 9) = c^2 - 9^2 = c^2 - 81$ b) Expand. $(x + 4)^2 = x^2 + 2(x)(4) + 4^2$ $= x^2 + 8x + 16$ c) Expand. $(6v - 7)^2 = (6v)^2 - 2(6v)(7) + 7^2$ $= 36v^2 - 84v + 49$

6.4 Division of Polynomials

Definition/Procedure	Example
To **divide a polynomial by a monomial,** divide *each term* in the polynomial by the monomial and simplify. **(p. 373)**	Divide. $\dfrac{22s^4 + 6s^3 - 7s^2 + 3s - 8}{4s^2}$ $= \dfrac{22s^4}{4s^2} + \dfrac{6s^3}{4s^2} - \dfrac{7s^2}{4s^2} + \dfrac{3s}{4s^2} - \dfrac{8}{4s^2}$ $= \dfrac{11s^2}{2} + \dfrac{3s}{2} - \dfrac{7}{4} + \dfrac{3}{4s} - \dfrac{2}{s^2}$

Definition/Procedure	Example
To **divide a polynomial by another polynomial** containing two or more terms, use *long division*. **(p. 375)**	Divide. $\dfrac{10w^3 + 2w^2 + 13w + 18}{5w + 6}$ $$\begin{array}{r} 2w^2 - 2w + 5 \\ 5w+6 \overline{\smash{)}\, 10w^3 + 2w^2 + 13w + 18} \\ \underline{-(10w^3 + 12w^2)} \downarrow \quad \\ -10w^2 + 13w \quad \\ \underline{-(-10w^2 - 12w)} \downarrow \\ 25w + 18 \\ \underline{-(25w + 30)} \\ -12 \end{array} \rightarrow \text{Remainder}$$ $\dfrac{10w^3 + 2w^2 + 13w + 18}{5w + 6} = 2w^2 - 2w + 5 - \dfrac{12}{5w + 6}$
We can use **synthetic division** to divide a polynomial by a binomial of the form $x - c$. **(p. 379)**	Use synthetic division to divide $(4x^3 - 17x^2 + 17x - 6) \div (x - 3)$. $$\begin{array}{r\|rrrr} 3 & 4 & -17 & 17 & -6 \\ & & 12 & -15 & 6 \\ \hline & 4 & -5 & 2 & 0 \end{array} \rightarrow \text{Remainder}$$ $\underbrace{4x^2 - 5x + 2}_{\text{Quotient}}$ The degree of the quotient is one less than the degree of the dividend. $(4x^3 - 17x^2 + 17x - 6) \div (x - 3) = 4x^2 - 5x + 2$

Chapter 6: Review Exercises

(6.1)
Evaluate using the rules of exponents.

1) $\dfrac{2^{11}}{2^6}$

2) 6^{-2}

3) $\left(\dfrac{2}{5}\right)^{-3}$

4) $-3^0 + 8^0$

Simplify. Assume all variables represent nonzero real numbers. The answer should not contain negative exponents.

5) $(p^7)^4$

6) $(5m^3)(-3m^8)$

7) $\dfrac{60t^9}{12t^3}$

8) $(-3a^5)^4$

9) $(-7c)(6c^8)$

10) $\dfrac{4p^{13}}{32p^9}$

11) $\dfrac{k^7}{k^{12}}$

12) $\dfrac{f^{-3}}{f^7}$

13) $(-2r^2s)^3(6r^{-9}s)$

14) $\dfrac{a^4b^{-2}}{a^7b^{-5}}$

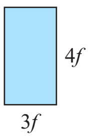
15) $\left(\dfrac{2xy^{-8}}{3x^{-2}y^6}\right)^{-2}$

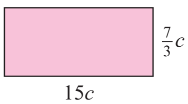
16) $(8p^{-7}q^3)(5p^{-2}q)^2$

17) $\dfrac{m^{-1}n^8}{mn^{14}}$

18) $\dfrac{(4b^{-3}c^4d)^2}{(2b^5cd^{-2})^{-3}}$

Write expressions for the area and perimeter of each rectangle.

19) 

20) 

Simplify. Assume that the variables represent nonzero integers. Write the final answer so that the exponents have positive coefficients.

21) $y^{4a} \cdot y^{3a}$

22) $d^{5n} \cdot d^n$

23) $\dfrac{r^{11x}}{r^{2x}}$

24) $\dfrac{g^{13h}}{g^{5h}}$

(6.2)
Identify each term in the polynomial, the coefficient and degree of each term, and the degree of the polynomial.

25) $7s^3 - 9s^2 + s + 6$

26) $a^2b^3 + 7ab^2 + 2ab + 9b$

27) Evaluate $2r^2 - 8r - 11$ for $r = -3$.

28) Evaluate $p^3q^2 + 4pq^2 - pq - 2q + 9$ for $p = -1$ and $q = 4$.

29) If $h(x) = 5x^2 - 3x - 6$, find

a) $h(-2)$

b) $h(0)$

30) $f(t) = \dfrac{2}{5}t + 4$. Find t so that $f(t) = \dfrac{4}{5}$.

Add or subtract as indicated.

31) $(6c^2 + 2c - 8) - (8c^2 + c - 13)$

32) $(-2m^2 - m + 11) + (6m^2 - 12m + 1)$

33)
$$\begin{array}{r} 6.7j^3 - 1.4j^2 + j - 5.3 \\ \underline{+3.1j^3 + 5.7j^2 + 2.4j + 4.8} \end{array}$$

34)
$$\begin{array}{r} -4.2p^3 + 12.5p^2 - 7.2p + 6.1 \\ -\ \underline{1.3p^3 - 3.3p^2 + 2.5p + 4.3} \end{array}$$

35) $\left(\dfrac{3}{5}k^2 + \dfrac{1}{2}k + 4\right) - \left(\dfrac{1}{10}k^2 + \dfrac{3}{2}k - 2\right)$

36) $\left(\dfrac{2}{7}u^2 - \dfrac{5}{8}u + \dfrac{4}{3}\right) + \left(\dfrac{3}{7}u^2 + \dfrac{3}{8}u - \dfrac{11}{12}\right)$

37) Subtract $4x^2y^2 - 7x^2y + xy + 5$ from $x^2y^2 + 2x^2y - 4xy + 11$.

38) Find the sum of $3c^3d^3 - 7c^2d^2 - c^2d + 8d + 1$ and $14c^3d^3 + 3c^2d^2 - 12cd - 2d - 6$.

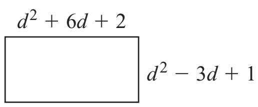
39) Find the sum of $6m + 2n - 17$ and $-3m + 2n + 14$.

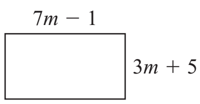
40) Subtract $-h^4 + 8j^4 - 2$ from $12h^4 - 3j^4 + 19$.

41) Subtract $2x^2 + 3x + 18$ from the sum of $7x - 16$ and $8x^2 - 15x + 6$.

42) Find the sum of $7xy + 2x - 3y - 11$ and $-3xy + 5y + 1$ and subtract it from $-5xy - 9x + y + 4$.

Find the polynomial that represents the perimeter of each rectangle.

43) 

44) 

(6.3)
Multiply.

45) $3r(8r - 13)$

46) $-5m^2(7m^2 - 4m + 8)$

47) $(4w + 3)(-8w^3 - 2w + 1)$

48) $\left(2t^2 - \frac{1}{3}\right)(-9t^2 + 7t - 12)$

49) $(y - 3)(y - 9)$

50) $(f - 5)(f - 8)$

51) $(3n - 4)(2n - 7)$

52) $(3p + 4)(3p + 1)$

53) $-(a - 13)(a + 10)$

54) $-(5d + 2)(6d + 5)$

55) $6pq^2(7p^3q^2 + 11p^2q^2 - pq + 4)$

56) $9x^3y^4(-6x^2y + 2xy^2 + 8x - 1)$

57) $(2x - 9y)(2x + y)$

58) $(7r + 3s)(r - s)$

59) $(x^2 + 5x - 12)(10x^4 - 3x^2 + 6)$

60) $(3m^2 - 4m + 2)(m^2 + m - 5)$

61) $4f^2(2f - 7)(f - 6)$

62) $-3(5u - 11)(u + 4)$

63) $(z + 3)(z + 1)(z + 4)$

64) $(p + 2)(p + 5)(p + 4)$

65) $\left(\frac{2}{7}d + 3\right)\left(\frac{1}{2}d - 8\right)$

66) $\left(\frac{3}{10}t - 6\right)\left(\frac{2}{9}t - 5\right)$

Expand.

67) $(c + 4)^2$

68) $(x - 12)^2$

69) $(4p - 3)^2$

70) $(9 - 2y)^2$

71) $(x - 3)^3$

72) $(p + 4)^3$

73) $[(m - 3) + n]^2$

Find the special products.

74) $(z + 7)(z - 7)$

75) $(p - 13)(p + 13)$

76) $\left(\frac{1}{4}n - 5\right)\left(\frac{1}{4}n + 5\right)$

77) $\left(\frac{9}{2} + \frac{5}{6}x\right)\left(\frac{9}{2} - \frac{5}{6}x\right)$

78) $(0.9 - r^2)(0.9 + r^2)$

79) $\left(3a - \frac{1}{2}b\right)\left(3a + \frac{1}{2}b\right)$

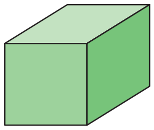
80) $-4(2d - 7)^2$

81) $3u(u + 4)^2$

82) $[(2p + 5) + q][(2p + 5) - q]$

83) Write an expression for the a) area and b) perimeter of the rectangle.

84) Express the volume of the cube as a polynomial.



(6.4)

Divide.

85) $\dfrac{12t^6 - 30t^5 - 15t^4}{3t^4}$

86) $\dfrac{42p^4 + 12p^3 - 18p^2 + 6p}{-6p}$

87) $\dfrac{w^2 + 9w + 20}{w + 4}$

88) $\dfrac{a^2 - 2a - 24}{a - 6}$

89) $\dfrac{8r^3 + 22r^2 - r - 15}{2r + 5}$

90) $\dfrac{-36h^3 + 99h^2 + 4h + 1}{12h - 1}$

91) $\dfrac{14t^4 + 28t^3 - 21t^2 + 20t}{14t^3}$

92) $\dfrac{48w^4 - 30w^3 + 24w^2 + 3w}{6w^2}$

93) $(14v + 8v^2 - 3) \div (4v + 9)$

94) $(-8 + 12r^2 - 19r) \div (3r - 1)$

95) $\dfrac{6v^4 - 14v^3 + 25v^2 - 21v + 24}{2v^2 + 3}$

96) $\dfrac{8t^4 + 20t^3 - 30t^2 - 65t + 13}{4t^2 - 13}$

97) $\dfrac{c^3 - 8}{c - 2}$

98) $\dfrac{g^3 + 64}{g + 4}$

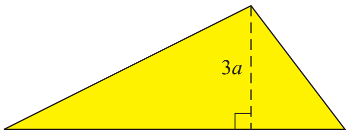
99) $\dfrac{-4 + 13k + 18k^3}{3k + 2}$

100) $\dfrac{10 + 12m^3 - 34m^2}{6m + 1}$

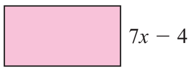
101) $(20x^4y^4 - 48x^2y^4 - 12xy^2 + 15x) \div (-12xy^2)$

102) $(3u^4 - 31u^3 - 13u^2 + 76u - 30) \div (u^2 - 11u + 5)$

103) Find the base of the triangle if the area is given by $12a^2 + 3a$ sq units.



104) Find the length of the rectangle if the area is given by $28x^3 - 51x^2 + 34x - 8$.



Mixed Exercises

Perform the operations and simplify. Assume all variables represent nonzero real numbers. The answer should not contain negative exponents.

105) $$\begin{array}{rrrrr} & 18c^3 + & 7c^2 & - 11c & + 2 \\ + & 2c^3 - & 19c^2 & & - 1 \\ \hline \end{array}$$

106) $\dfrac{15a - 11 + 14a^2}{7a - 3}$

107) $(12 - 7w)(12 + 7w)$

108) $(5p - 9)(2p^2 - 4p - 7)$

109) $5(-2r^7t^9)^3$

110) $(7k^2 + k - 9) - (-4k^2 + 8k - 3)$

111) $(39a^6b^6 + 21a^4b^5 - 5a^3b^4 + a^2b) \div (3a^3b^3)$

112) $\dfrac{(6x^{-4}y^5)^{-2}}{(3xy^{-2})^{-4}}$

113) $(h - 5)^3$

114) $\left(\dfrac{1}{8}m - \dfrac{2}{3}n\right)^2$

115) $\dfrac{-23c + 41 + 2c^3}{c + 4}$

116) $-7d^3(5d^2 + 12d - 8)$

117) $\left(\dfrac{5}{y^4}\right)^{-3}$

118) $(27q^3 + 8) \div (3q + 2)$

119) $(6p^4 + 11p^3 - 20p^2 - 17p + 20) \div (3p^2 + p - 4)$

120) $\left(\dfrac{3b^{-2}c}{a^5}\right)^{-3}\left(\dfrac{4a^{-2}b}{c^4}\right)\left(\dfrac{2ab^3}{c^2}\right)$

Chapter 6: Test

Evaluate.

1) $\left(\frac{3}{4}\right)^{-3}$

2) $\frac{5^7}{5^3}$

Simplify. Assume all variables represent nonzero real numbers. The answer should not contain negative exponents.

3) $(8p^3)(-4p^6)$

4) $(2t^3)^5$

5) $\frac{g^{11}h^{-4}}{g^7h^6}$

6) $\left(\frac{54ab^7}{90a^4b^{-2}}\right)^{-2}$

7) Given the polynomial $6n^3 + 6n^2 - n - 7$,

a) what is the coefficient of n?

b) what is the degree of the polynomial?

8) What is the degree of the polynomial $6a^4b^5 + 11a^4b^3 - 2a^3b + 5ab^2 - 3$?

9) Evaluate $-2r^2 + 7s$ when $r = -4$ and $s = 5$.

Perform the indicated operation(s).

10) $4h^3(6h^2 - 3h + 1)$

11) $(7a^3b^2 + 9a^2b^2 - 4ab + 8) + (5a^3b^2 - 12a^2b^2 + ab + 1)$

12) Subtract $6y^2 - 5y + 13$ from $15y^2 - 8y + 6$.

13) $3(-c^3 + 3c - 6) - 4(2c^3 + 3c^2 + 7c - 1)$

14) $(u - 5)(u - 9)$

15) $(4g + 3)(2g + 1)$

16) $\left(v + \frac{2}{5}\right)\left(v - \frac{2}{5}\right)$

17) $(3x - 7y)(2x + y)$

18) $(5 - 6n)(2n^2 + 3n - 8)$

19) $2y(y + 6)^2$

Expand.

20) $(3m - 4)^2$

21) $\left(\frac{4}{3}x + y\right)^2$

22) $(t - 2)^3$

Divide.

23) $\frac{w^2 + 9w + 18}{w + 6}$

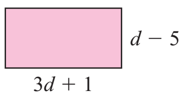
24) $\frac{24m^6 - 40m^5 + 8m^4 - 6m^3}{8m^4}$

25) $(22p - 50 + 18p^3 - 45p^2) \div (3p - 7)$

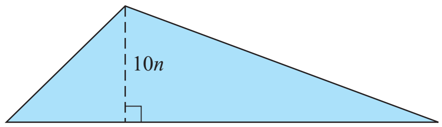
26) $\frac{y^3 - 27}{y - 3}$

27) $(2r^4 + 3r^3 + 6r^2 + 15r - 20) \div (r^2 + 5)$

28) Write an expression for a) the area and b) the perimeter of the rectangle.



29) Write an expression for the base of the triangle if the area is given by $20n^2 + 15n$ sq units.



Cumulative Review: Chapters 1–6

1) Given the set of numbers

$$\left\{\frac{3}{8}, -15, 2.1, \sqrt{17}, 41, 0.\overline{52}, 0, 9.32087326\ldots\right\}$$

list the

a) whole numbers

b) integers

c) rational numbers

2) Evaluate $-3^4 + 2 \cdot 9 \div (-3)$.

3) Divide $3\frac{1}{8} \div 1\frac{7}{24}$.

Simplify. The answers should not contain negative exponents.

4) $-8(2a^7)^2$

5) $c^{10} \cdot c^7$

6) $\left(\dfrac{4p^{-12}}{p^{-5}}\right)^3$

Solve.

7) $-\dfrac{18}{7}m - 9 = 21$

8) $5(u + 3) + 2u = 1 + 7(u - 2)$

9) Solve $5y - 16 \geq 8y - 1$. Write the answer in interval notation.

10) *Write an equation in one variable and solve*. How many milliliters of a 15% alcohol solution and how many milliliters of an 8% alcohol solution must be mixed to obtain 70 mL of a 12% alcohol solution?

11) Find the x- and y- intercepts of $3x - 8y = 24$ and sketch a graph of the equation.

12) Graph $y = -4$.

13) Write an equation of the line containing the points $(-4, 7)$ and $(2, -11)$. Express the answer in standard form.

14) Write an equation of the line perpendicular to $4x - y = 1$ containing the point $(8, 1)$. Express the answer in slope-intercept form.

15) Solve this system using the elimination method.

$$\begin{aligned} 3x - 4y &= -17 \\ x + 2y &= -4 \end{aligned}$$

16) *Write a system of two equations in two variables and solve.* The length of a rectangle is 1 cm less than three times its width. The perimeter of the rectangle is 94 cm. What are the dimensions of the figure?

Perform the indicated operations.

17) $(6q^2 + 7q - 1) - 4(2q^2 - 5q + 8) + 3(-9q - 4)$

18) $(n - 7)(n + 8)$

19) $(3a - 11)(3a + 11)$

20) $\dfrac{12a^4b^4 - 18a^3b + 60ab + 6b}{12a^3b^2}$

21) $(5p^3 - 14p^2 - 10p + 5) \div (p - 3)$

22) $(4n^2 - 9)(3n^2 + n - 2)$

23) $5c(c - 4)^2$

24) $\dfrac{8z^3 + 1}{2z + 1}$

25) Given $g(x) = -3x^2 + 2x + 9$, find $g(-2)$.

CHAPTER 7

Factoring Polynomials

Algebra at Work: Ophthalmology

Mark is an ophthalmologist, a doctor specializing in the treatment of diseases of the eye. He says that he could not do his job without a background in mathematics. While formulas are very important in his work, he says that the thinking skills learned in math courses are the same kinds of thinking skills he uses to treat his patients on a daily basis.

As a physician, Mark follows a very logical, analytical progression to form an accurate diagnosis and treatment plan. He examines a patient, performs tests, and then analyzes the results to form a diagnosis. Next, he must think of different ways to solve the problem and decide on the treatment plan that is best for that patient. He says that the skills he learned in his mathematics courses prepared him for the kind of problem solving he must do every day as an ophthalmologist. Factoring requires the kinds of skills that are so important to Mark in his job—the ability to think through and solve a problem in an analytical and logical manner.

In this chapter, we will learn different techniques for factoring polynomials.

Section 7.1 The Greatest Common Factor and Factoring by Grouping

Objectives

1. Find the GCF of a Group of Monomials
2. Factoring vs. Multiplying Polynomials
3. Factor Out the Greatest Common Monomial Factor
4. Factor Out the Greatest Common Binomial Factor
5. Factor by Grouping

In Section 1.1, we discussed writing a number as the product of factors:

$$\underset{\text{Product}}{18} = \underset{\text{Factor}}{3} \cdot \underset{\text{Factor}}{6}$$

To **factor** an integer is to write it as the product of two or more integers. Therefore, 18 can also be factored in other ways:

$$18 = 1 \cdot 18 \qquad 18 = 2 \cdot 9 \qquad 18 = -1 \cdot (-18)$$
$$18 = -2 \cdot (-9) \qquad 18 = -3 \cdot (-6) \qquad 18 = 2 \cdot 3 \cdot 3$$

The last **factorization**, $2 \cdot 3 \cdot 3$ or $2 \cdot 3^2$, is called the **prime factorization** of 18 since all of the factors are prime numbers. (See Section 1.1.) The factors of 18 are 1, 2, 3, 6, 9, 18, −1, −2, −3, −6, −9, and −18.

We can also write the factors as ±1, ±2, ±3, ±6, ±9, and ±18. (Read ±1 as "plus or minus 1.")

In this chapter, we will learn how to factor polynomials, a skill that is used in many ways throughout algebra.

1. Find the GCF of a Group of Monomials

Definition

The **greatest common factor (GCF)** of a group of two or more integers is the *largest* common factor of the numbers in the group.

For example, if we want to find the GCF of 18 and 24, we can list their positive factors.

18: 1, 2, 3, 6, 9, 18

24: 1, 2, 3, 4, 6, 8, 12, 24

The greatest common factor of 18 and 24 is 6. We can also use prime factors.

We begin our study of factoring polynomials by discussing how to find the greatest common factor of a group of monomials.

Example 1

Find the greatest common factor of x^4 and x^6.

Solution

We can write x^4 as $1 \cdot x^4$, and we can write x^6 as $x^4 \cdot x^2$. The largest power of x that is a factor of both x^4 and x^6 is x^4. Therefore, the GCF is x^4.

In Example 1, notice that the power of 4 in the GCF is the smallest of the powers when comparing x^4 and x^6. This will always be true.

Note

The exponent on the variable in the GCF will be the *smallest* exponent appearing on the variable in the group of terms.

You Try 1

Find the greatest common factor of y^5 and y^8.

Example 2

Find the greatest common factor for each group of terms.

a) $24n^5, 8n^9, 16n^3$ b) $-15x^{10}y, 25x^6y^8$

c) $49a^4b^5, 21a^3, 35a^2b^4$

Solution

a) The GCF of the coefficients, 24, 8, and 16, is 8. The smallest exponent on n is 3, so n^3 is part of the GCF.

The GCF is $8n^3$.

b) The GCF of the coefficients, -15 and 25, is 5. The smallest exponent on x is 6, so x^6 is part of the GCF. The smallest exponent on y is 1, so y is part of the GCF.

The GCF is $5x^6y$.

c) The GCF of the coefficients is 7. The smallest exponent on a is 2, so a^2 is part of the GCF. There is no b in the term $21a^3$, so there will be no b in the GCF.

The GCF is $7a^2$. ■

You Try 2

Find the greatest common factor for each group of terms.

a) $18w^6, 45w^{10}, 27w^5$ b) $-14hk^3, 18h^4k^2$ c) $54c^5d^5, 66c^8d^3, 24c^2$

Factoring Out the Greatest Common Factor

Earlier we said that to **factor an integer** is to write it as the product of two or more integers.

To **factor a polynomial** is to write it as a product of two or more polynomials.

Throughout this chapter, we will study different factoring techniques. We will begin by discussing how to factor out the greatest common factor.

2. Factoring vs. Multiplying Polynomials

Factoring a polynomial is the opposite of multiplying polynomials. Let's see how factoring and multiplying are related.

Example 3

a) Multiply $3y(y + 4)$. b) Factor out the GCF from $3y^2 + 12y$.

Solution

a) Use the distributive property to multiply.

$$\begin{aligned} 3y(y + 4) &= (3y)y + (3y)(4) \\ &= 3y^2 + 12y \end{aligned}$$

b) Use the distributive property to factor out the greatest common factor from $3y^2 + 12y$.

First, identify the GCF of $3y^2$ and $12y$. The GCF is $3y$.

Then, rewrite each term as a product of two factors with one factor being $3y$.

$$3y^2 = (3y)(y) \text{ and } 12y = (3y)(4)$$

$$\begin{aligned} 3y^2 + 12y &= (3y)(y) + (3y)(4) \\ &= 3y(y + 4) \qquad \text{Distributive property} \end{aligned}$$

When we factor $3y^2 + 2y$, we get $3y(y + 4)$. We can check our result by multiplying.

$$3y(y + 4) = 3y^2 + 12y$$

Procedure Steps for Factoring Out the Greatest Common Factor

1) Identify the GCF of all of the terms of the polynomial.
2) Rewrite each term as the product of the GCF and another factor.
3) Use the distributive property to factor out the GCF from the terms of the polynomial.
4) Check the answer by multiplying the factors. The result should be the original polynomial.

3. Factor Out the Greatest Common Monomial Factor

Example 4

Factor out the greatest common factor.

a) $28p^5 + 12p^4 + 4p^3$ b) $w^8 - 7w^6$

c) $6a^5b^3 + 30a^5b^2 - 54a^4b^2 - 6a^3b$

Solution

a) Identify the GCF of all of the terms: GCF $= 4p^3$.

$$28p^5 + 12p^4 + 4p^3 = (4p^3)(7p^2) + (4p^3)(3p) + (4p^3)(1)$$ Rewrite each term using the GCF as one of the factors.

$$= 4p^3(7p^2 + 3p + 1)$$ Distributive property

Check: $4p^3(7p^2 + 3p + 1) = 28p^5 + 12p^4 + 4p^3$ ✓

b) The GCF of all of the terms is w^6.

$$w^8 - 7w^6 = (w^6)(w^2) - (w^6)(7)$$ Rewrite each term using the GCF as one of the factors.

$$= w^6(w^2 - 7)$$ Distributive property

Check: $w^6(w^2 - 7) = w^8 - 7w^6$ ✓

c) The GCF of all of the terms is $6a^3b$.

$$6a^5b^3 + 30a^5b^2 - 54a^4b^2 - 6a^3b$$

$$= (6a^3b)(a^2b^2) + (6a^3b)(5a^2b) - (6a^3b)(9ab) - (6a^3b)(1)$$ Rewrite using the GCF.

$$= 6a^3b(a^2b^2 + 5a^2b - 9ab - 1)$$ Distributive property

Check: $6a^3b(a^2b^2 + 5a^2b - 9ab - 1) = 6a^5b^3 + 30a^5b^2 - 54a^4b^2 - 6a^3b$ ✓

You Try 3

Factor out the greatest common factor.

a) $3u^6 + 36u^5 + 15u^4$ b) $z^5 - 9z^2$ c) $45r^4t^3 + 36r^4t^2 + 18r^3t^2 - 9r^2t$

Sometimes we need to take out a negative factor.

Example 5

Factor out $-2k$ from $-6k^4 + 10k^3 - 8k^2 + 2k$.

Solution

$-6k^4 + 10k^3 - 8k^2 + 2k$

$= (-2k)(3k^3) + (-2k)(-5k^2) + (-2k)(4k) + (-2k)(-1)$ Rewrite using $-2k$ as one of the factors.

$= -2k[3k^3 + (-5k^2) + 4k + (-1)]$ Distributive property

$= -2k(3k^3 - 5k^2 + 4k - 1)$ Rewrite $+(-5k^2)$ as $-5k^2$ and $+(-1)$ as -1.

Check: $-2k(3k^3 - 5k^2 + 4k - 1) = -6k^4 + 10k^3 - 8k^2 + 2k$ ✓

BE CAREFUL

When taking out a negative factor, be very careful with the signs!

You Try 4

Factor out $-y^2$ from $-y^4 + 10y^3 - 8y^2$.

4. Factor Out the Greatest Common Binomial Factor

Until now, all of the GCFs have been monomials. Sometimes, however, the greatest common factor is a *binomial*.

Example 6

Factor out the greatest common factor.

a) $a(b + 5) + 8(b + 5)$ b) $c^2(c + 9) - 2(c + 9)$ c) $x(y + 3) - (y + 3)$

Solution

a) In the polynomial $\underbrace{a(b + 5)}_{\text{Term}} + \underbrace{8(b + 5)}_{\text{Term}}$, $a(b + 5)$ is a term and $8(b + 5)$ is a term. What do these terms have in common? $b + 5$

The GCF of $a(b + 5)$ and $8(b + 5)$ is $(b + 5)$. Use the distributive property to factor out $b + 5$.

$a(b + 5) + 8(b + 5) = (b + 5)(a + 8)$ Distributive property

Check: $(b + 5)(a + 8) = (b + 5)a + (b + 5)8$ Distribute.
$= a(b + 5) + 8(b + 5)$ Commutative property

b) The GCF of $c^2(c + 9) - 2(c + 9)$ is $c + 9$.

$c^2(c + 9) - 2(c + 9) = (c + 9)(c^2 - 2)$ Distributive property

Check: $(c + 9)(c^2 - 2) = (c + 9)c^2 + (c + 9)(-2)$ Distributive property
$= c^2(c + 9) - 2(c + 9)$ ✓ Commutative property

c) Begin by writing $x(y + 3) - (y + 3)$ as $x(y + 3) - 1(y + 3)$. The GCF is $y + 3$.

$x(y + 3) - 1(y + 3) = (y + 3)(x - 1)$ Distributive property

The check is left to the student.

BE CAREFUL

It is important to write -1 in front of $(y + 3)$. Otherwise, the following mistake is often made:

$$x(y + 3) - (y + 3) = (y + 3)x \quad \text{This is incorrect!}$$

The correct factor is $x - 1$, *not* x.

You Try 5

Factor out the GCF.

a) $c(d - 8) + 2(d - 8)$ b) $k(k^2 + 15) - 7(k^2 + 15)$ c) $u(v + 2) - (v + 2)$

Taking out a binomial factor leads us to our next method of factoring—factoring by grouping.

5. Factor by Grouping

When we are asked to factor a polynomial containing four terms, we often try to **factor by grouping**.

Example 7

Factor by grouping.

a) $rt + 7r + 2t + 14$ b) $3xz - 4yz + 18x - 24y$

c) $n^3 + 8n^2 - 5n - 40$

Solution

a) Begin by grouping terms together so that each group has a common factor.

$$\underbrace{rt + 7r}_{\downarrow} + \underbrace{2t + 14}_{\downarrow}$$

Factor out r to get $r(t + 7)$.

$$= r(t + 7) + 2(t + 7)$$

Factor out 2 to get $2(t + 7)$.

$$= (t + 7)(r + 2)$$

Factor out $(t + 7)$.

Check: $(t + 7)(r + 2) = rt + 7r + 2t + 14$ ✓

b) Group terms together so that each group has a common factor.

$$\underbrace{3xz - 4yz}_{\downarrow} + \underbrace{18x - 24y}_{\downarrow}$$

Factor out z to get $z(3x - 4y)$.

$$= z(3x - 4y) + 6(3x - 4y)$$

Factor out 6 to get $6(3x - 4y)$.

$$= (3x - 4y)(z + 6)$$

Factor out $(3x - 4y)$.

Check: $(3x - 4y)(z + 6) = 3xz - 4yz + 18x - 24y$ ✓

c) Group terms together so that each group has a common factor.

$$\underbrace{n^3 + 8n^2}_{\downarrow} \underbrace{-5n - 40}_{\downarrow}$$

Factor out n^2 to get $n^2(n + 8)$.

$$= n^2(n + 8) - 5(n + 8)$$

Factor out -5 to get $-5(n + 8)$.

$$= (n + 8)(n^2 - 5)$$

Factor out $(n + 8)$.

We must factor out –5, *not* 5, from the second group so that the binomial factors for both groups are the same! [If we had factored out 5, then the factorization of the second group would have been $5(-n - 8)$.]

Check: $(n + 8)(n^2 - 5) = n^3 + 8n^2 - 5n - 40$ ✓

You Try 6

Factor by grouping.

a) $xy + 4x + 10y + 40$ b) $5pr - 8qr + 10p - 16q$ c) $w^3 + 9w^2 - 6w - 54$

Sometimes we have to rearrange the terms before we can factor.

Example 8

Factor completely. $12c^2 - 2d + 3c - 8cd$

Solution

Group terms together so that each group has a common factor.

$$\underbrace{12c^2 - 2d}_{\text{Factor out 2 to get } 2(6c^2 - d).} + \underbrace{3c - 8cd}_{\text{Factor out } c \text{ to get } c(3 - 8d).}$$

$$= 2(6c^2 - d) + c(3 - 8d)$$

These groups do not have common factors! Let's rearrange the terms in the original polynomial and group the terms differently.

$$\underbrace{12c^2 + 3c}_{\text{Factor out } 3c \text{ to get } 3c(4c + 1).} - \underbrace{8cd - 2d}_{\text{Factor out } -2d \text{ to get } -2d(4c + 1).}$$

$$= 3c(4c + 1) - 2d(4c + 1)$$

$$= (4c + 1)(3c - 2d) \quad \text{Factor out } (4c + 1).$$

Check: $(4c + 1)(3c - 2d) = 12c^2 - 2d + 3c - 8cd$ ✓

Note

Often, there is more than one way that the terms can be rearranged so that the polynomial can be factored by grouping.

You Try 7

Factor completely. $3k^2 - 48m + 8k - 18km$

Often, we have to combine the two factoring techniques we have learned here. That is, we begin by factoring out the GCF and then we factor by grouping. Let's summarize how to factor a polynomial by grouping and then look at another example.

Procedure Steps for Factoring by Grouping

1) Before trying to factor by grouping, look at each term in the polynomial and ask yourself, "*Can I factor out a GCF first?*" If so, factor out the GCF from all of the terms.
2) Make two groups of two terms so that each group has a common factor.
3) Take out the common factor in each group of terms.
4) Factor out the common binomial factor using the distributive property.
5) Check the answer by multiplying the factors.

Example 9

Factor completely. $7h^4 + 7h^3 - 42h^2 - 42h$

Solution

Notice that this polynomial has four terms. This is a clue for us to try factoring by grouping. *However*, look at the polynomial carefully and ask yourself, "*Can I factor out a GCF?*" Yes! *Therefore, the first step in factoring this polynomial is to factor out* $7h$.

$$7h^4 + 7h^3 - 42h^2 - 42h = 7h(h^3 + h^2 - 6h - 6)$$ Factor out the GCF, $7h$.

The polynomial in parentheses has 4 terms. Try to factor it by grouping.

$$7h(\underbrace{h^3 + h^2} - \underbrace{6h - 6})$$

$$= 7h[h^2(h + 1) - 6(h + 1)]$$ Take out the common factor in each group.

$$= 7h(h + 1)(h^2 - 6)$$ Factor out $(h + 1)$ using the distributive property.

Check: $7h(h + 1)(h^2 - 6) = 7h(h^3 + h^2 - 6h - 6)$

$= 7h^4 + 7h^3 - 42h^2 - 42h$ ✓

You Try 8

Factor completely. $12t^3 + 12t^2 - 3t^2u - 3tu$

Remember, seeing a polynomial with four terms is a clue to try factoring by grouping. Not all polynomials will factor this way, however. We will learn other techniques later, and some polynomials must be factored using methods learned in later courses.

Answers to You Try Exercises

1) y^5 2) a) $9w^5$ b) $2hk^2$ c) $6c^2$ 3) a) $3u^4(u^2 + 12u + 5)$ b) $z^2(z^3 - 9)$ c) $9r^2t(5r^2t^2 + 4r^2t + 2rt - 1)$ 4) $-y^2(y^2 - 10y + 8)$ 5) a) $(d - 8)(c + 2)$ b) $(k^2 + 15)(k - 7)$ c) $(v + 2)(u - 1)$ 6) a) $(y + 4)(x + 10)$ b) $(5p - 8q)(r + 2)$ c) $(w + 9)(w^2 - 6)$ 7) $(3k + 8)(k - 6m)$ 8) $3t(t + 1)(4t - u)$

7.1 Exercises

Objective 1: Find the GCF of a Group of Monomials

Find the greatest common factor of each group of terms.

1) $28, 21c$
2) $9t, 36$
3) $18p^3, 12p^2$
4) $32z^5, 56z^3$
5) $12n^6, 28n^{10}, 36n^7$
6) $63b^4, 45b^7, 27b$
7) $35a^3b^2, 15a^2b$
8) $10x^5y^4, 2x^4y^4$
9) $21r^3s^6, 63r^3s^2, -42r^4s^5$
10) $-60p^2q^2, 36pq^5, 96p^3q^3$
11) $a^2b^2, 3ab^2, 6a^2b$
12) $n^3m^4, -n^3m^4, -n^4$
13) $c(k-9), 5(k-9)$
14) $a^2(h+8), b^2(h+8)$
15) Explain how to find the GCF of a group of terms.
16) What does it mean to factor a polynomial?

Objective 2: Factoring vs. Multiplying Polynomials

Determine whether each expression is written in factored form.

17) $5p(p+9)$
18) $8h^2 - 24h$
19) $18w^2 + 30w$
20) $-3z^2(2z+7)$
21) $a^2b^2(-4ab)$
22) $c^3d - (2c+d)$

Objective 3: Factor Out the Greatest Common Monomial Factor

Factor out the greatest common factor. Be sure to check your answer.

23) $2w + 10$
24) $3y + 18$
25) $18z^2 - 9$
26) $14h - 12h^2$
27) $100m^3 - 30m$
28) $t^5 - t^4$
29) $r^9 + r^2$
30) $\frac{1}{2}a^2 + \frac{3}{2}a$
31) $\frac{1}{5}y^2 + \frac{4}{5}y$
32) $9a^3 + 2b^2$
33) $s^7 - 4t^3$
34) $14u^7 + 63u^6 - 42u^5$
35) $10n^5 - 5n^4 + 40n^3$
36) $3d^8 - 33d^7 - 24d^6 + 3d^5$
37) $40p^6 + 40p^5 - 8p^4 + 8p^3$
38) $44m^3n^3 - 24mn^4$
39) $63a^3b^3 - 36a^3b^2 + 9a^2b$
40) $8p^4q^3 + 8p^3q^3 - 72p^2q^2$
41) Factor out -6 from $-30n - 42$.
42) Factor out $-c$ from $-9c^3 + 2c^2 - c$.
43) Factor out $-4w^3$ from $-12w^5 - 16w^3$.
44) Factor out $-m$ from $-6m^3 - 3m^2 + m$
45) Factor out -1 from $-k + 3$.
46) Factor out -1 from $-p - 10$.

Objective 4: Factor Out the Greatest Common Binomial Factor

Factor out the common binomial factor.

47) $u(t-5) + 6(t-5)$
48) $c(b+9) + 2(b+9)$
49) $y(6x+1) - z(6x+1)$
50) $s(4r-3) - t(4r-3)$
51) $p(q+12) + (q+12)$
52) $8x(y-2) + (y-2)$
53) $5h^2(9k+8) - (9k+8)$
54) $3a(4b+1) - (4b+1)$

Objective 5: Factor by Grouping

Factor by grouping.

55) $ab + 2a + 7b + 14$
56) $cd + 8c - 5d - 40$
57) $3rt + 4r - 27t - 36$
58) $5pq + 15p - 6q - 18$
59) $8b^2 + 20bc + 2bc^2 + 5c^3$
60) $4a^3 - 12ab + a^2b - 3b^2$
61) $fg - 7f + 4g - 28$
62) $xy - 8y - 7x + 56$
63) $st - 10s - 6t + 60$
64) $cd + 3c - 11d - 33$
65) $5tu + 6t - 5u - 6$
66) $qr + 3q - r - 3$
67) $36g^4 + 3gh - 96g^3h - 8h^2$
68) $40j^3 + 72jk - 55j^2k - 99k^2$
69) Explain, in your own words, how to factor by grouping.
70) What should be the first step in factoring $6ab + 24a + 18b + 54$?

Factor completely. You may need to begin by factoring out the GCF first or by rearranging terms.

Fill It In

Fill in the blanks with either the missing mathematical step or the reason for the given step.

71) $4xy + 12x + 20y + 60$

$4xy + 12x + 20y + 60$

$=$ ________________ Factor out the GCF.

$= 4[x(y + 3) + 5(y + 3)]$ ________________

$=$ ________________ Take out the binomial factor.

72) $2m^2n - 4m^2 - 18mn + 36m$

$2m^2n - 4m^2 - 18mn + 36m$

$=$ ________________ Factor out the GCF.

$= 2m[m(n - 2) - 9(n - 2)]$ ________________

$=$ ________________ Take out the binomial factor.

73) $3cd + 6c + 21d + 42$

74) $5xy + 15x - 5y - 15$

75) $2p^2q - 10p^2 - 8pq + 40p$

76) $3uv^2 - 24uv + 3v^2 - 24v$

77) $10st + 5s - 12t - 6$

78) $8pq + 12p + 10q + 15$

79) $3a^3 - 21a^2b - 2ab + 14b^2$

80) $8c^3 + 32c^2d + cd + 4d^2$

81) $8u^2v^2 + 16u^2v + 10uv^2 + 20uv$

82) $10x^2y^2 - 5x^2y - 60xy^2 + 30xy$

Mixed Exercises: Objectives 1–5

Factor completely.

83) $3mn + 21m + 10n + 70$

84) $4yz + 7z - 20y - 35$

85) $16b - 24$

86) $2yz^3 + 14yz^2 + 3z^3 + 21z^2$

87) $cd + 6c - 4d - 24$

88) $5x^3 - 30x^2y^2 + xy - 6y^3$

89) $6a^4b + 12a^4 - 8a^3b - 16a^3$

90) $6x^2 + 48x^3$

VIDEO 91) $7cd + 12 + 28c + 3d$

92) $2uv + 12u - 7v - 42$

93) $dg - d + g - 1$

94) $2ab - 10a - 12b + 60$

95) $x^4y^2 + 12x^3y^3$

96) $8u^2 - 16uv^2 + 3uv - 6v^3$

97) $4mn + 8m + 12n + 24$

98) $5c^2 - 20$

99) Factor out -2 from $-6p^2 - 20p + 2$.

100) Factor out $-5g$ from $-5g^3 + 50g^2 - 25g$.

Section 7.2 Factoring Trinomials of the Form $x^2 + bx + c$

Objectives

1. Practice Arithmetic Skills Needed for Factoring Trinomials
2. Factor a Trinomial of the Form $x^2 + bx + c$
3. More on Factoring a Trinomial of the Form $x^2 + bx + c$
4. Factor a Trinomial Containing Two Variables

One of the factoring problems encountered most often in algebra is the factoring of trinomials. In this section, we will discuss how to factor a trinomial of the form $x^2 + bx + c$; notice that the coefficient of the squared term is 1.

We will begin with arithmetic skills we need to be able to factor a trinomial of the form $x^2 + bx + c$.

1. Practice Arithmetic Skills Needed for Factoring Trinomials

Example 1

Find two integers whose

a) product is 15 and sum is 8.
b) product is 24 and sum is -10.
c) product is -28 and sum is 3.

Solution

a) If the product of two numbers is *positive* 15 and the sum of the numbers is *positive* 8, *then the two numbers will be positive*. (The product of two positive numbers is positive, and their sum is positive as well.)

First, list the pairs of *positive* integers whose product is 15—the *factors* of 15. Then, find the *sum* of those factors.

Factors of 15	Sum of the Factors
$1 \cdot 15$	$1 + 15 = 16$
$3 \cdot 5$	$3 + 5 = 8$

The product of 3 and 5 is 15, and their sum is 8.

b) If the product of two numbers is *positive* 24 and the sum of those numbers is *negative* 10, *then the two numbers will be negative*. (The product of two negative numbers is positive, while the sum of two negative numbers is negative.)

First, list the pairs of negative numbers that are factors of 24. Then, find the sum of those factors. You can stop making your list when you find the pair that works.

Factors of 24	Sum of the Factors
$-1 \cdot (-24)$	$-1 + (-24) = -25$
$-2 \cdot (-12)$	$-2 + (-12) = -14$
$-3 \cdot (-8)$	$-3 + (-8) = -11$
$-4 \cdot (-6)$	$-4 + (-6) = -10$

The product of -4 and -6 is 24, and their sum is -10.

c) If two numbers have a product of *negative* 28 and their sum is *positive* 3, *one number must be positive and one number must be negative*. (The product of a positive number and a negative number is negative, while the sum of the numbers can be either positive *or* negative.)

First, list pairs of factors of -28. Then, find the sum of those factors.

Factors of -28	Sum of the Factors
$-1 \cdot 28$	$-1 + 28 = 27$
$1 \cdot (-28)$	$1 + (-28) = -27$
$-4 \cdot 7$	$-4 + 7 = 3$

The product of -4 and 7 is -28 and their sum is 3. ■

You Try 1

Find two integers whose

a) product is 21 and sum is 10.

b) product is -18 and sum is -3.

c) product is 20 and sum is -12.

Note

You should try to get to the point where you can come up with the correct numbers *in your head* without making a list.

2. Factor a Trinomial of the Form $x^2 + bx + c$

In Section 7.1, we said that the process of factoring is the opposite of multiplying. Let's see how this will help us understand how to factor a trinomial of the form $x^2 + bx + c$.

Multiply $(x + 4)(x + 5)$ using FOIL.

$$\begin{aligned}(x + 4)(x + 5) &= x^2 + 5x + 4x + 4 \cdot 5 && \text{Multiply using FOIL.}\\ &= x^2 + (5 + 4)x + 20 && \text{Use the distributive property and multiply } 4 \cdot 5.\\ &= x^2 + 9x + 20\end{aligned}$$

$$(x + 4)(x + 5) = x^2 + \underbrace{9x}_{\text{The sum of 4 and 5 is 9.}} + 20 \longleftarrow \text{The product of 4 and 5 is 20.}$$

So, if we were asked to *factor* $x^2 + 9x + 20$, we need to think of two integers whose *product* is 20 and whose *sum* is 9. Those numbers are 4 and 5. The *factored form* of $x^2 + 9x + 20$ is $(x + 4)(x + 5)$.

Procedure Factoring a Polynomial of the Form $x^2 + bx + c$

To factor a polynomial of the form $x^2 + bx + c$, find two integers m and n whose product is c and whose sum is b. Then, $x^2 + bx + c = (x + m)(x + n)$.

1) If b and c are positive, then both m and n must be positive.

2) If c is positive and b is negative, then both m and n must be negative.

3) If c is negative, then one integer, m, must be positive and the other integer, n, must be negative.

You can check the answer by multiplying the binomials. The result should be the original polynomial.

Example 2

Factor, if possible.

a) $x^2 + 7x + 12$ b) $p^2 - 9p + 14$ c) $w^2 + w - 30$

d) $a^2 - 3a - 54$ e) $c^2 - 6c + 9$ f) $y^2 + 11y + 35$

Solution

a) To factor $x^2 + 7x + 12$, we must find two integers whose *product* is 12 and whose *sum* is 7. Both integers will be positive.

Factors of 12	Sum of the Factors
$1 \cdot 12$	$1 + 12 = 13$
$2 \cdot 6$	$2 + 6 = 8$
$3 \cdot 4$	$3 + 4 = 7$

The numbers are 3 and 4: $x^2 + 7x + 12 = (x + 3)(x + 4)$

Check: $(x + 3)(x + 4) = x^2 + 4x + 3x + 12 = x^2 + 7x + 12$ ✓

Note

The order in which the factors are written does not matter. In this example, $(x + 3)(x + 4) = (x + 4)(x + 3)$.

b) To factor $p^2 - 9p + 14$, find two integers whose *product* is 14 and whose *sum* is -9. Since 14 is positive and the coefficient of p is a negative number, -9, both integers will be negative.

Factors of 14	Sum of the Factors
$-1 \cdot (-14)$	$-1 + (-14) = -15$
$-2 \cdot (-7)$	$-2 + (-7) = -9$

The numbers are -2 and -7: $p^2 - 9p + 14 = (p - 2)(p - 7)$.

Check: $(p - 2)(p - 7) = p^2 - 7p - 2p + 14 = p^2 - 9p + 14$ ✓

c) $w^2 + w - 30$

The coefficient of w is 1, so we can think of this trinomial as $w^2 + 1w - 30$.

Find two integers whose *product* is -30 and whose *sum* is 1. Since the last term in the trinomial is negative, one of the integers must be positive and the other must be negative.

Try to find these integers mentally. Two numbers with a product of *positive* 30 are 5 and 6. We need a product of -30, so either the 5 is negative or the 6 is negative.

Factors of -30	Sum of the Factors
$-5 \cdot 6$	$-5 + 6 = 1$

The numbers are -5 and 6: $w^2 + w - 30 = (w - 5)(w + 6)$.

Check: $(w - 5)(w + 6) = w^2 + 6w - 5w - 30 = w^2 + w - 30$ ✓

d) To factor $a^2 - 3a - 54$, find two integers whose *product* is -54 and whose *sum* is -3. Since the last term in the trinomial is negative, one of the integers must be positive and the other must be negative.

Find the integers mentally. First, think about two integers whose product is *positive* 54: 1 and 54, 2 and 27, 3 and 18, 6 and 9. One number must be positive and the other negative, however, to get our product of -54, and they must add up to -3.

Factors of −54	Sum of the Factors
$-6 \cdot 9$	$-6 + 9 = 3$
$6 \cdot (-9)$	$6 + (-9) = -3$

The numbers are 6 and -9: $a^2 - 3a - 54 = (a + 6)(a - 9)$.

The check is left to the student.

e) To factor $c^2 - 6c + 9$, notice that the *product*, 9, is positive and the *sum*, -6, is negative, so both integers must be negative. The numbers that multiply to 9 and add to -6 are the same number, -3 and -3: $(-3) \cdot (-3) = 9$ and $-3 + (-3) = -6$.

$$\text{So } c^2 - 6c + 9 = (c - 3)(c - 3) \text{ or } (c - 3)^2.$$

Either form of the factorization is correct.

f) To factor $y^2 + 11y + 35$, find two integers whose *product* is 35 and whose *sum* is 11. We are looking for two positive numbers.

Factors of 35	Sum of the Factors
$1 \cdot 35$	$1 + 35 = 36$
$5 \cdot 7$	$5 + 7 = 12$

There are no such factors! Therefore, $y^2 + 11y + 35$ does not factor using the methods we have learned here. We say that it is **prime**. ■

Note

We say that trinomials like $y^2 + 11y + 35$ are **prime** if they cannot be factored using the method presented here.

In later mathematics courses you may learn how to factor such polynomials using other methods so that they are not considered prime.

You Try 2

Factor, if possible.

a) $m^2 + 11m + 28$ b) $c^2 - 16c + 48$ c) $a^2 - 5a - 21$

d) $r^2 - 4r - 45$ e) $r^2 + 5r - 24$ f) $h^2 + 12h + 36$

3. More on Factoring a Trinomial of the Form $x^2 + bx + c$

Sometimes it is necessary to factor out the GCF before applying this method for factoring trinomials.

Note

From this point on, the *first* step in factoring *any* polynomial should be to ask yourself, *"Can I factor out a greatest common factor?"*

Since some polynomials can be factored more than once, after performing one factorization, ask yourself, *"Can I factor again?"* If so, factor again. If not, you know that the polynomial has been completely factored.

Example 3

Factor completely. $4n^3 - 12n^2 - 40n$

Solution

Ask yourself, *"Can I factor out a GCF?"* Yes. The GCF is $4n$.

$$4n^3 - 12n^2 - 40n = 4n(n^2 - 3n - 10)$$

Look at the trinomial and ask yourself, *"Can I factor again?"* Yes. The integers whose product is -10 and whose sum is -3 are -5 and 2. Therefore,

$$\begin{aligned} 4n^3 - 12n^2 - 40n &= 4n(n^2 - 3n - 10) \\ &= 4n(n - 5)(n + 2) \end{aligned}$$

We cannot factor again.

$$\begin{aligned} \textit{Check}:\ 4n(n - 5)(n + 2) &= 4n(n^2 + 2n - 5n - 10) \\ &= 4n(n^2 - 3n - 10) \\ &= 4n^3 - 12n^2 - 40n \quad \checkmark \end{aligned}$$

You Try 3

Factor completely.

a) $7p^4 + 42p^3 + 56p^2$ b) $3a^2b - 33ab + 90b$

4. Factor a Trinomial Containing Two Variables

If a trinomial contains two variables and we cannot take out a GCF, the trinomial may still be factored according to the method outlined in this section.

Example 4

Factor completely. $x^2 + 12xy + 32y^2$

Solution

Ask yourself, *"Can I factor out a GCF?"* No. Notice that the first term is x^2. Let's rewrite the trinomial as

$$x^2 + 12yx + 32y^2$$

so that we can think of $12y$ as the coefficient of x. Find two expressions whose product is $32y^2$ and whose sum is $12y$. They are $4y$ and $8y$ since $4y \cdot 8y = 32y^2$ and $4y + 8y = 12y$.

$$x^2 + 12xy + 32y^2 = (x + 4y)(x + 8y)$$

We cannot factor $(x + 4y)(x + 8y)$ any more, so this is the complete factorization. The check is left to the student.

You Try 4

Factor completely.

a) $m^2 + 10mn + 16n^2$ b) $5a^3 + 40a^2b - 45ab^2$

Answers to You Try Exercises

1) a) 3, 7 b) $-6, 3$ c) $-2, -10$ 2) a) $(m + 4)(m + 7)$ b) $(c - 12)(c - 4)$ c) prime d) $(r - 9)(r + 5)$ e) $(r + 8)(r - 3)$ f) $(h + 6)(h + 6)$ or $(h + 6)^2$ 3) a) $7p^2(p + 4)(p + 2)$ b) $3b(a - 5)(a - 6)$ 4) a) $(m + 2n)(m + 8n)$ b) $5a(a - b)(a + 9b)$

7.2 Exercises

Objective 1: Practice Arithmetic Skills Needed for Factoring Trinomials

1) Find two integers whose

	PRODUCT IS	and whose SUM IS	ANSWER
a)	10	7	
b)	-56	-1	
c)	-5	4	
d)	36	-13	

2) Find two integers whose

	PRODUCT IS	and whose SUM IS	ANSWER
a)	42	-13	
b)	-14	13	
c)	54	15	
d)	-21	-4	

Objective 2: Factor a Trinomial of the Form $x^2 + bx + c$

3) If $x^2 + bx + c$ factors to $(x + m)(x + n)$ and if c is positive and b is negative, what do you know about the signs of m and n?

4) If $x^2 + bx + c$ factors to $(x + m)(x + n)$ and if b and c are positive, what do you know about the signs of m and n?

5) When asked to factor a polynomial, what is the first question you should ask yourself?

6) What does it mean to say that a polynomial is prime?

7) After factoring a polynomial, what should you ask yourself to be sure that the polynomial is completely factored?

8) How do you check the factorization of a polynomial?

Complete the factorization.

9) $n^2 + 7n + 10 = (n + 5)(\quad)$

10) $p^2 + 11p + 28 = (p + 4)(\quad)$

11) $c^2 - 16c + 60 = (c - 6)(\quad)$

12) $t^2 - 12t + 27 = (t - 9)(\quad)$

13) $x^2 + x - 12 = (x - 3)(\quad)$

14) $r^2 - 8r - 9 = (r + 1)(\quad)$

Factor completely, if possible. Check your answer.

VIDEO 15) $g^2 + 8g + 12$

16) $p^2 + 9p + 14$

17) $y^2 + 10y + 16$

18) $a^2 + 11a + 30$

19) $w^2 - 17w + 72$

20) $d^2 - 14d + 33$

21) $b^2 - 3b - 4$

22) $t^2 + 2t - 48$

23) $z^2 + 6z - 11$

24) $x^2 - 7x - 15$

25) $c^2 - 13c + 36$
26) $h^2 - 13h + 12$
27) $m^2 + 4m - 60$
28) $v^2 - 4v - 45$
29) $r^2 - 4r - 96$
30) $a^2 - 21a + 110$
31) $q^2 + 12q + 42$
32) $d^2 - 15d + 32$
33) $x^2 + 16x + 64$
34) $c^2 - 10c + 25$
35) $n^2 - 2n + 1$
36) $w^2 + 20w + 100$
37) $24 + 14d + d^2$
38) $10 + 7k + k^2$
39) $-56 + 12a + a^2$
40) $63 + 21w + w^2$

Objective 3: More on Factoring a Trinomial of the Form $x^2 + bx + c$

Factor completely, if possible. Check your answer.

41) $2k^2 - 22k + 48$
42) $6v^2 + 54v + 120$
43) $50h + 35h^2 + 5h^3$
44) $3d^3 - 33d^2 - 36d$
45) $r^4 + r^3 - 132r^2$
46) $2n^4 - 40n^3 + 200n^2$
47) $7q^3 - 49q^2 - 42q$
48) $8b^4 + 24b^3 + 16b^2$
49) $3z^4 + 24z^3 + 48z^2$
50) $-36w + 6w^2 + 2w^3$
51) $xy^3 - 2xy^2 - 63xy$
52) $2c^3d - 14c^2d - 24cd$

Factor completely by first taking out -1 and then by factoring the trinomial, if possible. Check your answer.

53) $-m^2 - 12m - 35$
54) $-x^2 - 15x - 36$
55) $-c^2 - 3c + 28$
56) $-t^2 + 2t + 48$
57) $-z^2 + 13z - 30$
58) $-n^2 + 16n - 55$
59) $-p^2 + p + 56$
60) $-w^2 - 2w + 3$

Objective 4: Factor a Trinomial Containing Two Variables

Factor completely. Check your answer.

61) $x^2 + 7xy + 12y^2$
62) $a^2 + 11ab + 18b^2$
63) $c^2 - 7cd - 8d^2$
64) $p^2 + 6pq - 72q^2$
65) $u^2 - 14uv + 45v^2$
66) $h^2 - 8hk + 7k^2$
67) $m^2 + 4mn - 21n^2$
68) $a^2 - 6ab - 40b^2$
69) $a^2 + 24ab + 144b^2$
70) $g^2 + 6gh + 5h^2$

Determine whether each polynomial is factored completely. If it is not, explain why and factor it completely.

71) $3x^2 + 21x + 30 = (3x + 6)(x + 5)$
72) $6a^2 + 24a - 72 = 6(a + 6)(a - 2)$
73) $n^4 - 3n^3 - 108n^2 = n^2(n - 12)(n + 9)$
74) $9y^3 - 45y^2 + 54y = (y - 2)(9y^2 - 27y)$

Mixed Exercises: Objectives 2–4

Factor completely. Begin by asking yourself, *"Can I factor out a GCF?"*

75) $2x^2 + 16x + 30$
76) $3c^2 + 21c + 18$
77) $n^2 - 6n + 8$
78) $a^2 + a - 6$
79) $m^2 + 7mn - 44n^2$
80) $a^2 + 10ab + 24b^2$
81) $h^2 - 10h + 32$
82) $z^2 + 9z + 36$
83) $4q^3 - 28q^2 + 48q$
84) $3w^3 - 9w^2 - 120w$
85) $-k^2 - 18k - 81$
86) $-y^2 + 8y - 16$
87) $4h^5 + 32h^4 + 28h^3$
88) $3r^4 - 6r^3 - 45r^2$
89) $k^2 + 21k + 108$
90) $j^2 - 14j - 15$
91) $p^3q - 17p^2q^2 + 70pq^3$
92) $u^3v^2 - 2u^2v^3 - 15uv^4$
93) $a^2 + 9ab + 24b^2$
94) $m^2 - 8mn - 35n^2$
95) $x^2 - 13xy + 12y^2$
96) $p^2 - 3pq - 40q^2$
97) $5v^5 + 55v^4 - 45v^3$
98) $6t^4 + 42t^3 + 48t^2$
99) $6x^3y^2 - 48x^2y^2 - 54xy^2$
100) $2c^2d^4 - 18c^2d^3 + 28c^2d^2$
101) $36 - 13z + z^2$
102) $121 + 22w + w^2$
103) $a^2b^2 + 13ab + 42$
104) $h^2k^2 + 8hk - 20$
105) $(x + y)z^2 + 7(x + y)z - 30(x + y)$
106) $(m + n)k^2 + 17(m + n)k + 66(m + n)$
107) $(a - b)c^2 - 11(a - b)c + 28(a - b)$
108) $(r - t)u^2 - 4(r - t)u - 45(r - t)$
109) $(p + q)r^2 + 24r(p + q)r + 144(p + q)$
110) $(a + b)d^2 - 8(a + b)d + 16(a + b)$

Section 7.3 Factoring Trinomials of the Form $ax^2 + bx + c$ ($a \neq 1$)

Objectives

1. **Factor $ax^2 + bx + c$ ($a \neq 1$) by Grouping**
2. **Factor $ax^2 + bx + c$ ($a \neq 1$) by Trial and Error**

In the previous section, we learned that we could factor $2x^2 + 10x + 8$ by first taking out the GCF of 2 and then factoring the trinomial.

$$2x^2 + 10x + 8 = 2(x^2 + 5x + 4) = 2(x + 4)(x + 1)$$

In this section, we will learn how to factor a trinomial like $2x^2 + 11x + 15$ where we *cannot* factor out the leading coefficient of 2.

1. Factor $ax^2 + bx + c$ ($a \neq 1$) by Grouping

To factor $2x^2 + \overset{\text{Sum is 11.}}{11x} + 15$, first find the product of 2 and 15. Then, find two integers

Product: $2 \cdot 15 = 30$

whose *product* is 30 and whose *sum* is 11. The numbers are 6 and 5. Rewrite the middle term, $11x$, as $6x + 5x$, then factor by grouping.

$$2x^2 + 11x + 15 = \underbrace{2x^2 + 6x} + \underbrace{5x + 15} \quad \text{Take out the common factor from each group.}$$
$$= 2x(x + 3) + 5(x + 3) \quad \text{Factor out } (x + 3).$$
$$= (x + 3)(2x + 5)$$

Check: $(x + 3)(2x + 5) = 2x^2 + 5x + 6x + 15 = 2x^2 + 11x + 15$ ✓

Example 1

Factor completely.

a) $8k^2 + 14k + 3$ b) $6c^2 - 17c + 12$ c) $7x^2 - 34xy - 5y^2$

Solution

a) Since we cannot factor out a GCF (the GCF = 1), we begin with a new method.

Sum is 14.

$8k^2 + 14k + 3$

Product: $8 \cdot 3 = 24$

Think of two integers whose *product* is 24 and whose *sum* is 14. The integers are 2 and 12. Rewrite the middle term, $14k$, as $2k + 12k$. Factor by grouping.

$$8k^2 + 14k + 3 = \underbrace{8k^2 + 2k} + \underbrace{12k + 3}$$
$$= 2k(4k + 1) + 3(4k + 1) \quad \text{Take out the common factor from each group.}$$
$$= (4k + 1)(2k + 3) \quad \text{Factor out } (4k + 1).$$

Check by multiplying: $(4k + 1)(2k + 3) = 8k^2 + 14k + 3$ ✓

b) *Sum* is -17.

$6c^2 - 17c + 12$

Product: $6 \cdot 12 = 72$

Think of two integers whose *product* is 72 and whose *sum* is -17. (Both numbers will be negative.) The integers are -9 and -8. Rewrite the middle term, $-17c$, as $-9c - 8c$. Factor by grouping.

$$6c^2 - 17c + 12 = \underbrace{6c^2 - 9c} - \underbrace{8c + 12}$$
$$= 3c(2c - 3) - 4(2c - 3) \quad \text{Take out the common factor from each group.}$$
$$= (2c - 3)(3c - 4) \quad \text{Factor out } (2c - 3).$$

Check: $(2c - 3)(3c - 4) = 6c^2 - 17c + 12$ ✓

c) Sum is -34.

$$\underbrace{7x^2 \overset{\downarrow}{-\ 34xy} - 5y^2}_{\text{Product: } 7 \cdot (-5) = -35}$$

The integers whose *product* is -35 and whose *sum* is -34 are -35 and 1. Rewrite the middle term, $-34xy$, as $-35xy + xy$. Factor by grouping.

$$\begin{aligned} 7x^2 - 34xy - 5y^2 &= \underbrace{7x^2 - 35xy} + \underbrace{xy - 5y^2} \\ &= 7x(x - 5y) + y(x - 5y) && \text{Take out the common factor from each group.} \\ &= (x - 5y)(7x + y) && \text{Factor out } (x - 5y). \end{aligned}$$

Check: $(x - 5y)(7x + y) = 7x^2 - 34xy - 5y^2$ ✓

You Try 1

Factor completely.

a) $4p^2 + 16p + 15$ b) $10y^2 - 13y + 4$ c) $5a^2 - 29ab - 6b^2$

Example 2

Factor completely. $12n^2 + 64n - 48$

Solution

It is tempting to jump right in and multiply $12 \cdot (-48) = -576$ and try to think of two integers with a product of -576 and a sum of 64. However, first ask yourself, *"Can I factor out a GCF?"* Yes! We can factor out 4.

$$12n^2 + 64n - 48 = 4(\underbrace{3n^2 + 16n - 12}_{\text{Product: } 3 \cdot (-12) = -36}) \quad \text{Factor out 4.}$$

Now factor $3n^2 + 16n - 12$ by finding two integers whose *product* is -36 and whose *sum* is 16. The numbers are 18 and -2.

$$\begin{aligned} &= 4(\underbrace{3n^2 + 18n} \underbrace{- 2n - 12}) \\ &= 4[3n(n + 6) - 2(n + 6)] && \text{Take out the common factor from each group.} \\ &= 4(n + 6)(3n - 2) && \text{Factor out } (n + 6). \end{aligned}$$

Check by multiplying: $4(n + 6)(3n - 2) = 4(3n^2 + 16n - 12)$
$= 12n^2 + 64n - 48$ ✓

You Try 2

Factor completely.

a) $24h^2 - 54h - 15$ b) $20d^3 + 38d^2 + 12d$

2. Factor $ax^2 + bx + c$ $(a \neq 1)$ by Trial and Error

At the beginning of this section, we factored $2x^2 + 11x + 15$ by grouping. Now we will factor it by trial and error, which is just reversing the process of FOIL.

Example 3

Factor $2x^2 + 11x + 15$ completely.

Solution

Can we factor out a GCF? No. So try to factor $2x^2 + 11x + 15$ as the product of two binomials. Notice that all terms are positive, so all factors will be positive.

Begin with the squared term, $2x^2$. Which two expressions with integer coefficients can we multiply to get $2x^2$? $2x$ and x. Put these in the binomials.

$$2x^2 + 11x + 15 = (2x \quad\ )(x \quad\ ) \qquad 2x \cdot x = 2x^2$$

Next, look at the last term, 15. What are the pairs of positive integers that multiply to 15? They are 15 and 1 as well as 5 and 3.

Try these numbers as the last terms of the binomials. The middle term, $11x$, comes from finding the sum of the products of the outer terms and inner terms.

First Try

$$2x^2 + 11x + 15 \stackrel{?}{=} (2x + 15)(x + 1) \qquad \text{Incorrect!}$$

$$\begin{array}{r} 15x \\ +\ \ 2x \\ \hline 17x \end{array}$$

These must both be $11x$. $\longrightarrow$ $17x$

Switch the 15 and the 1

$$2x^2 + 11x + 15 \stackrel{?}{=} (2x + 1)(x + 15) \qquad \text{Incorrect!}$$

$$\begin{array}{r} 1x \\ +\ 30x \\ \hline 31x \end{array}$$

These must both be $11x$. $\longrightarrow$ $31x$

Try using 5 and 3. $\quad 2x^2 + 11x + 15 \stackrel{?}{=} (2x + 5)(x + 3) \qquad$ Correct!

$$\begin{array}{r} 5x \\ +\ \ 6x \\ \hline 11x \end{array}$$

These must both be $11x$. $\longrightarrow$ $11x$

Therefore, $2x^2 + 11x + 15 = (2x + 5)(x + 3)$. Check by multiplying. ■

Example 4

Factor $3t^2 - 29t + 18$ completely.

Solution

Can we factor out a GCF? No. To get a product of $3t^2$, we will use $3t$ and t.

$$3t^2 - 29t + 18 = (3t \quad\ )(t \quad\ ) \qquad 3t \cdot t = 3t^2$$

Since the last term is positive and the middle term is negative, we want pairs of negative integers that multiply to 18. The pairs are -1 and -18, -2 and -9, and -3 and -6. Try these numbers as the last terms of the binomials. The middle term, $-29t$, comes from finding the sum of the products of the outer terms and inner terms.

$$3t^2 - 29t + 18 \stackrel{?}{=} (3t - 1)(t - 18) \qquad \text{Incorrect!}$$

$$\begin{array}{r} -t \\ +\ (-54t) \\ \hline -55t \end{array}$$

These must both be $-29t$. $\longrightarrow$ $-55t$

Switch the -1 and the -18: $3t^2 - 29t + 18 \stackrel{?}{=} (3t - 18)(t - 1)$

Without multiplying, we know that this choice is incorrect. How? In the factor $(3t - 18)$, a 3 can be factored out to get $3(t - 6)$. But, we concluded that we could not factor out a GCF from the original polynomial, $3t^2 - 29t + 18$. Therefore, it will not be possible to take out a common factor from one of the binomial factors.

Note

If you cannot factor out a GCF from the original polynomial, then you cannot take out a factor from one of the binomial factors either.

Try using -2 and -9. $3t^2 - 29t + 18 = (3t - 2)(t - 9)$ Correct !

$$\begin{array}{r} -2t \\ +\ (-27t) \\ \hline -29t \end{array}$$

These must both be $-29t$.

So, $3t^2 - 29t + 18 = (3t - 2)(t - 9)$. Check by multiplying. ■

You Try 3

Factor completely.

a) $2k^2 + 17k + 8$ b) $6z^2 - 23z + 20$

Example 5

Factor completely.

a) $16a^2 + 62a - 8$ b) $-2c^2 + 3c + 20$

Solution

a) Ask yourself, *"Can I take out a common factor?"* Yes!

$$16a^2 + 62a - 8 = 2(8a^2 + 31a - 4)$$

Now, try to factor $8a^2 + 31a - 4$. To get a product of $8a^2$, we can try either $8a$ *and* a or $4a$ *and* $2a$. Let's start by trying $8a$ and a.

$$8a^2 + 31a - 4 = (8a \quad\ )(a \quad\ )$$

List pairs of integers that multiply to -4: 4 and -1, -4 and 1, 2 and -2.

Try 4 and -1. Do not put 4 in the same binomial as $8a$ since then it would be possible to factor out 2. But, 2 does not factor out of $8a^2 + 31a - 4$. Put the 4 in the same binomial as a.

$$8a^2 + 31a - 4 \stackrel{?}{=} (8a - 1)(a + 4)$$

$$\begin{array}{r} -a \\ +\ 32a \\ \hline 31a \end{array}$$

Correct

Don't forget that the very first step was to factor out a 2. Therefore,

$$16a^2 + 62a - 8 = 2(8a^2 + 31a - 4) = 2(8a - 1)(a + 4)$$

Check by multiplying.

b) Since the coefficient of the squared term is negative, begin by factoring out -1. (There is no other common factor except 1.)

$$-2c^2 + 3c + 20 = -1(2c^2 - 3c - 20)$$

Try to factor $2c^2 - 3c - 20$. To get a product of $2c^2$, we will use $2c$ and c in the binomials.

$$2c^2 - 3c - 20 = (2c \quad)(c \quad)$$

We need pairs of integers whose product is -20. They are 1 and -20, -1 and 20, 2 and -10, -2 and 10, 4 and -5, -4 and 5.

Do *not* start with 1 and -20 or -1 and 20 because the middle term, $-3c$, is not very large. Using 1 and -20 or -1 and 20 would likely result in a larger middle term.

Think about 2 and -10 *and* -2 and 10. *These will not work because if we put any of these numbers in the factor containing 2c, then it will be possible to factor out 2.*

Try 4 and -5. Do not put 4 in the same binomial as $2c$ since then it would be possible to factor out 2.

$$2c^2 - 3c - 20 \stackrel{?}{=} (2c - 5)(c + 4)$$

$$\begin{array}{rl} -5c & \\ +\ 8c & \\ \hline 3c & \text{This must equal } -3c. \quad \text{Incorrect} \end{array}$$

Only the sign of the sum is incorrect. *Change the signs in the binomials to get the correct sum.*

$$2c^2 - 3c - 20 \stackrel{?}{=} (2c + 5)(c - 4)$$

$$\begin{array}{rl} 5c & \\ +\ (-8c) & \\ \hline -3c & \text{Correct} \end{array}$$

Remember that we factored out -1 to begin the problem.

$$-2c^2 + 3c + 20 = -1(2c^2 - 3c - 20) = -(2c + 5)(c - 4)$$

Check by multiplying. ■

You Try 4

Factor completely.

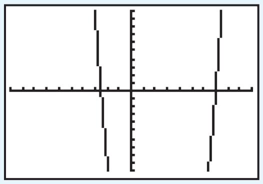
a) $10y^2 - 58y + 40$ b) $-4n^2 - 5n + 6$

We have seen two methods for factoring $ax^2 + bx + c$ $(a \neq 1)$: factoring by grouping and factoring by trial and error. In either case, remember to begin by taking out a common factor from all terms whenever possible.

Using Technology

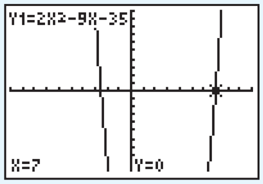
We found some ways to narrow down the possibilities when factoring $ax^2 + bx + c$ $(a \neq 1)$ using the trial and error method.

We can also use a graphing calculator to help with the process. Consider the trinomial $2x^2 - 9x - 35$. Enter the trinomial into Y_1 and press $\boxed{\text{ZOOM}}$; then enter 6 to display the graph in the standard viewing window.



Look on the graph for the x-intercept (if any) that appears to be an integer. It appears that 7 is an x-intercept.

To check whether 7 is an x-intercept, press $\boxed{\text{TRACE}}$ then $\boxed{7}$ and press $\boxed{\text{ENTER}}$. As shown on the graph, when $x = 7$, $y = 0$, so 7 is an x-intercept.



When an x-intercept is an integer, then x minus that x-intercept is a factor of the trinomial. In this case, $x - 7$ is a factor of $2x^2 - 9x - 35$. We can then complete the factoring as $(2x + 5)(x - 7)$, since we must multiply -7 by 5 to obtain -35.

Find an x-intercept using a graphing calculator and factor the trinomial.

1) $3x^2 + 11x - 4$ 2) $2x^2 + x - 15$ 3) $5x^2 + 6x - 8$
4) $2x^2 - 5x + 3$ 5) $4x^2 - 3x - 10$ 6) $14x^2 - x - 4$

Answers to You Try Exercises

1) a) $(2p + 5)(2p + 3)$ b) $(5y - 4)(2y - 1)$ c) $(5a + b)(a - 6b)$ 2) a) $3(2h - 5)(4h + 1)$ b) $2d(5d + 2)(2d + 3)$ 3) a) $(2k + 1)(k + 8)$ b) $(3z - 4)(2z - 5)$ 4) a) $2(5y - 4)(y - 5)$ b) $-(4n - 3)(n + 2)$

Answers to Technology Exercises

1) $(3x - 1)(x + 4)$ 2) $(2x - 5)(x + 3)$ 3) $(x + 2)(5x - 4)$ 4) $(x - 1)(2x - 3)$
5) $(x - 2)(4x + 5)$ 6) $(2x + 1)(7x - 4)$

7.3 Exercises

Objective 1: Factor $ax^2 + bx + c$ $(a \neq 1)$ by Grouping

1) Find two integers whose

	PRODUCT IS	and whose SUM IS	ANSWER
a)	-50	5	
b)	27	-28	
c)	12	8	
d)	-72	-6	

2) Find two integers whose

	PRODUCT IS	and whose SUM IS	ANSWER
a)	18	19	
b)	-132	1	
c)	-30	-13	
d)	63	-16	

Factor by grouping.

3) $3c^2 + 12c + 8c + 32$

4) $5y^2 + 15y + 2y + 6$

5) $6k^2 - 6k - 7k + 7$

6) $4r^2 - 4r + 9r - 9$

7) $6x^2 - 27xy + 8xy - 36y^2$

8) $14p^2 - 8pq - 7pq + 4q^2$

9) When asked to factor a polynomial, what is the first question you should ask yourself?

10) After factoring a polynomial, what should you ask yourself to be sure that the polynomial is factored completely?

11) Find the polynomial that factors to $(4k + 9)(k + 2)$.

12) Find the polynomial that factors to $(6m - 5)(2m - 3)$.

Complete the factorization.

13) $5t^2 + 13t + 6 = (5t + 3)(\qquad)$

14) $4z^2 + 29z + 30 = (4z + 5)(\qquad)$

15) $6a^2 - 11a - 10 = (2a - 5)(\qquad)$

16) $15c^2 - 23c + 4 = (3c - 4)(\qquad)$

17) $12x^2 - 25xy + 7y^2 = (4x - 7y)(\qquad)$

18) $12r^2 - 52rt - 9t^2 = (6r + t)(\qquad)$

Factor by grouping. See Example 1.

19) $2h^2 + 13h + 15$

20) $3z^2 + 13z + 14$

21) $7y^2 - 11y + 4$

22) $5a^2 - 21a + 18$

23) $5b^2 + 9b - 18$

24) $11m^2 - 18m - 8$

25) $6p^2 + p - 2$

26) $8c^2 - 22c + 5$

27) $4t^2 + 16t + 15$

28) $10k^2 + 23k + 12$

29) $9x^2 - 13xy + 4y^2$

30) $6a^2 + ab - 5b^2$

Objective 2: Factor $ax^2 + bx + c$ ($a \neq 1$) by Trial and Error

31) How do we know that $(2x - 4)$ cannot be a factor of $2x^2 + 13x - 24$?

32) How do we know that $(3p + 2)$ cannot be a factor of $6p^2 - 25p + 14$?

Factor by trial and error. See Examples 3 and 4.

33) $2r^2 + 9r + 10$

34) $3q^2 + 10q + 8$

VIDEO 35) $3u^2 - 23u + 30$

36) $7m^2 - 15m + 8$

37) $7a^2 + 31a - 20$

38) $5x^2 - 11x - 36$

39) $6y^2 + 23y + 10$

40) $8u^2 + 18u + 7$

41) $9w^2 + 20w - 21$

42) $10h^2 - 59h - 6$

43) $8c^2 - 42c + 27$

44) $15v^2 - 16v + 4$

45) $4k^2 + 40k + 99$

46) $4n^2 - 41n + 10$

47) $20b^2 - 32b - 45$

48) $14g^2 + 27g - 20$

49) $2r^2 + 13rt - 24t^2$

50) $3c^2 - 17cd - 6d^2$

51) $6a^2 - 25ab + 4b^2$

52) $6x^2 + 31xy + 18y^2$

Mixed Exercises: Objectives 1 and 2

53) Factor $4z^2 + 5z - 6$ using each method. Do you get the same answer? Which method do you prefer? Why?

54) Factor $10a^2 + 27a + 18$ using each method. Do you get the same answer? Which method do you prefer? Why?

Factor completely.

55) $3p^2 - 16p - 12$

56) $2t^2 - 19t + 24$

57) $4k^2 + 15k + 9$

58) $12x^3 + 15x^2 - 18x$

59) $30w^3 + 76w^2 + 14w$

60) $12d^2 - 28d - 5$

61) $21r^2 - 90r + 24$

62) $45q^2 + 57q + 18$

63) $6y^2 - 10y + 3$

64) $9z^2 + 14z + 8$

65) $42b^2 + 11b - 3$

66) $13u^2 + 17u - 18$

67) $7x^2 - 17xy + 6y^2$

68) $5a^2 + 23ab + 12b^2$

VIDEO 69) $2d^2 + 2d - 40$

70) $6c^2 + 42c + 72$

71) $30r^4t^2 + 23r^3t^2 + 3r^2t^2$

72) $8m^2n^3 + 4m^2n^2 - 60m^2n$

73) $9k^2 - 42k + 49$

74) $25p^2 + 20p + 4$

75) $2m^2(n + 9) - 5m(n + 9) - 7(n + 9)$

76) $3s^2(t - 8)^2 + 19s(t - 8)^2 + 20(t - 8)^2$

77) $6v^2(u + 4)^2 + 23v(u + 4)^2 + 20(u + 4)^2$

78) $8c^2(d - 7)^3 + 69c(d - 7)^3 - 27(d - 7)^3$

79) $15b^2(2a - 1)^4 - 28b(2a - 1)^4 + 12(2a - 1)^4$

80) $12x^2(3y + 2)^3 - 28x(3y + 2)^3 + 15(3y + 2)^3$

Factor completely by first taking out a negative common factor. See Example 5.

81) $-n^2 - 8n + 48$

82) $-c^2 - 16c - 63$

83) $-7a^2 + 4a + 3$

84) $-3p^2 + 14p - 16$

VIDEO 85) $-10z^2 + 19z - 6$

86) $-16k^3 + 48k^2 - 36k$

87) $-20m^3 - 120m^2 - 135m$

88) $-3z^3 - 15z^2 + 198z$

89) $-6a^3b + 11a^2b^2 + 2ab^3$

90) $-35u^4 - 203u^3v - 140u^2v^2$

Section 7.4 Factoring Special Trinomials and Binomials

Objectives

1. Factor a Perfect Square Trinomial
2. Factor the Difference of Two Squares
3. Factor the Sum and Difference of Two Cubes

1. Factor a Perfect Square Trinomial

Recall that we can square a binomial using the formulas

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$

For example, $(x + 3)^2 = x^2 + 2x(3) + 3^2 = x^2 + 6x + 9$.

Since factoring a polynomial means writing the polynomial as a product of its factors, $x^2 + 6x + 9$ factors to $(x + 3)^2$.

The expression $x^2 + 6x + 9$ is a *perfect square trinomial*. A **perfect square trinomial** is a trinomial that results from squaring a binomial.

We can use the factoring method presented in Section 7.2 to factor a perfect square trinomial, or we can learn to recognize the special pattern that appears in these trinomials.

How are the terms of $x^2 + 6x + 9$ and $(x + 3)^2$ related?

x^2 is the square of x, the first term in the binomial.

9 is the square of 3, the last term in the binomial.

We get the term $6x$ by doing the following:

$$6x = \underset{\text{Two times}}{2} \cdot \underset{\text{First term in binomial}}{x} \cdot \underset{\text{Last term in binomial}}{3}$$

This follows directly from how we found $(x + 3)^2$ using the formula.

Formula Factoring a Perfect Square Trinomial

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$

Note

In order for a trinomial to be a perfect square, two of its terms must be perfect squares.

Example 1

Factor completely. $k^2 + 18k + 81$

Solution

We cannot take out a common factor, so let's see whether this follows the pattern of a perfect square trinomial.

$$k^2 + 18k + 81$$

What do you square to get k^2? k — $(k)^2$

$(9)^2$ — What do you square to get 81? 9

Does the middle term equal $2 \cdot k \cdot 9$? Yes.

$$2 \cdot k \cdot 9 = 18k$$

Therefore, $k^2 + 18k + 81 = (k + 9)^2$. Check by multiplying. ■

Example 2

Factor completely.

a) $c^2 - 16c + 64$ b) $4t^3 + 32t^2 + 64t$

c) $9w^2 + 12w + 4$ d) $4c^2 + 20c + 9$

Solution

a) We cannot take out a common factor. However, since the middle term is negative and the first and last terms are positive, the sign in the binomial will be a minus (−) sign. Does this fit the pattern of a perfect square trinomial?

$$c^2 - 16c + 64$$

$$\downarrow \qquad\qquad \downarrow$$

What do you square to get c^2? c $(c)^2$ $(8)^2$ What do you square to get 64? 8

Does the middle term equal $2 \cdot c \cdot 8$? Yes. $2 \cdot c \cdot 8 = 16c$.

Notice that $c^2 - 16c + 64$ fits the pattern of $a^2 - 2ab + b^2 = (a - b)^2$ with $a = c$ and $b = 8$.

Therefore, $c^2 - 16c + 64 = (c - 8)^2$. Check by multiplying.

b) From $4t^3 + 32t^2 + 64t$ we *can* begin by taking out the GCF of $4t$.

$$4t^3 + 32t^2 + 64t = 4t(t^2 + 8t + 16)$$

$$\downarrow \qquad\qquad \downarrow$$

What do you square to get t^2? t $(t)^2$ $(4)^2$ What do you square to get 16? 4

Does the middle term of the trinomial in parentheses equal $2 \cdot t \cdot 4$? Yes. $2 \cdot t \cdot 4 = 8t$.

$$4t^3 + 32t^2 + 64t = 4t(t^2 + 8t + 16) = 4t(t + 4)^2$$

Check by multiplying.

c) We cannot take out a common factor. Since the first and last terms of $9w^2 + 12w + 4$ are perfect squares, let's see whether this is a perfect square trinomial.

$$9w^2 + 12w + 4$$

$$\downarrow \qquad\qquad \downarrow$$

What do you square to get $9w^2$? $3w$ $(3w)^2$ $(2)^2$ What do you square to get 4? 2

Does the middle term equal $2 \cdot 3w \cdot 2$? Yes. $2 \cdot 3w \cdot 2 = 12w$.

Therefore, $9w^2 + 12w + 4 = (3w + 2)^2$. Check by multiplying.

d) We cannot take out a common factor. The first and last terms of $4c^2 + 20c + 9$ are perfect squares. Is this a perfect square trinomial?

$$4c^2 + 20c + 9$$

$$\downarrow \qquad\qquad \downarrow$$

What do you square to get $4c^2$? $2c$ $(2c)^2$ $(3)^2$ What do you square to get 9? 3

Does the middle term equal $2 \cdot 2c \cdot 3$? No! $2 \cdot 2c \cdot 3 = 12c$.

This is *not* a perfect square trinomial. Applying a method from Section 7.3, we find that the trinomial does factor, however.

$4c^2 + 20c + 9 = (2c + 9)(2c + 1)$. Check by multiplying. ■

You Try 1

Factor completely.

a) $g^2 + 14g + 49$ b) $6y^3 - 36y^2 + 54y$

c) $25v^2 - 10v + 1$ d) $9b^2 + 15b + 4$

2. Factor the Difference of Two Squares

Another common factoring problem is a **difference of two squares**. Some examples of these types of binomials are

$$c^2 - 36 \qquad 49x^2 - 25y^2 \qquad 64 - t^2 \qquad h^4 - 16$$

Notice that in each binomial, the terms are being *subtracted*, and each term is a perfect square.

In Section 6.3, multiplication of polynomials, we saw that

$$(a + b)(a - b) = a^2 - b^2$$

If we reverse the procedure, we get the factorization of the difference of two squares.

Formula Factoring the Difference of Two Squares

$$a^2 - b^2 = (a + b)(a - b)$$

Don't forget that we can check all factorizations by multiplying.

Example 3

Factor completely.

a) $c^2 - 36$ b) $49x^2 - 25y^2$ c) $t^2 - \frac{4}{9}$ d) $k^2 + 81$

Solution

a) First, notice that $c^2 - 36$ is the difference of two terms *and* those terms are perfect squares. We can use the formula $a^2 - b^2 = (a + b)(a - b)$.

Identify a and b.

$$\begin{array}{cc} c^2 & - \ 36 \\ \downarrow & \downarrow \\ (c)^2 & (6)^2 \end{array}$$

What do you square to get c^2? c

What do you square to get 36? 6

Then, $a = c$ and $b = 6$. Therefore, $c^2 - 36 = (c + 6)(c - 6)$.

b) Look carefully at $49x^2 - 25y^2$. Each term *is* a perfect square, and they are being subtracted.

Identify a and b.

$$\begin{array}{cc} 49x^2 & - \ 25y^2 \\ \downarrow & \downarrow \\ (7x)^2 & (5y)^2 \end{array}$$

What do you square to get $49x^2$? $7x$

What do you square to get $25y^2$? $5y$

Then, $a = 7x$ and $b = 5y$. So, $49x^2 - 25y^2 = (7x + 5y)(7x - 5y)$.

c) Each term in $t^2 - \frac{4}{9}$ is a perfect square, and they are being subtracted.

$$t^2 - \frac{4}{9}$$
$$\downarrow \quad \downarrow$$

What do you square to get t^2? t $\quad (t)^2 \left(\frac{2}{3}\right)^2 \quad$ What do you square to get $\frac{4}{9}$? $\frac{2}{3}$

So, $a = t$ and $b = \frac{2}{3}$. Therefore, $t^2 - \frac{4}{9} = \left(t + \frac{2}{3}\right)\left(t - \frac{2}{3}\right)$.

d) Each term in $k^2 + 81$ is a perfect square, but the expression is the *sum* of two squares. This polynomial does not factor.

$$k^2 + 81 \neq (k + 9)(k - 9) \text{ since } (k + 9)(k - 9) = k^2 - 81.$$
$$k^2 + 81 \neq (k + 9)(k + 9) \text{ since } (k + 9)(k + 9) = k^2 + 18k + 81.$$

So, $k^2 + 81$ is prime.

Note

If the sum of two squares does not contain a common factor, then it cannot be factored.

You Try 2

Factor completely.

a) $m^2 - 100$ b) $4c^2 - 81d^2$ c) $h^2 - \frac{64}{25}$ d) $p^2 + 49$

Remember that sometimes we can factor out a GCF first. And, after factoring once, ask yourself, *"Can I factor again?"*

Example 4

Factor completely.

a) $300p - 3p^3$ b) $7w^2 + 28$ c) $x^4 - 81$

Solution

a) Ask yourself, *"Can I take out a common factor?"* Yes. Factor out $3p$.

$$300p - 3p^3 = 3p(100 - p^2)$$

Now ask yourself, *"Can I factor again?"* Yes. $100 - p^2$ is the difference of two squares. Identify a and b.

$$100 - p^2$$
$$\downarrow \quad \downarrow$$
$$(10)^2 \quad (p)^2$$

So, $a = 10$ and $b = p$. $100 - p^2 = (10 + p)(10 - p)$.

Therefore, $300p - 3p^3 = 3p(10 + p)(10 - p)$.

BE CAREFUL

$(10 + p)(10 - p)$ is *not* the same as $(p + 10)(p - 10)$ because subtraction is not commutative. While $10 + p = p + 10$, $10 - p$ *does not equal* $p - 10$. You must write the terms in the correct order. Another way to see that they are not equivalent is to multiply $(p + 10)(p - 10)$. $(p + 10)(p - 10) = p^2 - 100$. This is not the same as $100 - p^2$.

b) Look at $7w^2 + 28$. Ask yourself, *"Can I take out a common factor?"* Yes. Factor out 7.

$$7(w^2 + 4)$$

"Can I factor again?" No, because $w^2 + 4$ is the *sum* of two squares.

Therefore, $7w^2 + 28 = 7(w^2 + 4)$.

c) The terms in $x^4 - 81$ have no common factors, but they are perfect squares. Identify a and b.

$$\begin{array}{ccc} x^4 & - & 81 \\ \downarrow & & \downarrow \\ (x^2)^2 & & (9)^2 \end{array}$$

What do you square to get x^4? x^2

What do you square to get 81? 9

So, $a = x^2$ and $b = 9$. $x^4 - 81 = (x^2 + 9)(x^2 - 9)$.

Can we factor again?

$x^2 + 9$ is the *sum* of two squares. It will not factor.

$x^2 - 9$ is the difference of two squares, so it *will* factor.

$$\begin{array}{ccc} x^2 & - & 9 \\ \downarrow & & \downarrow \\ (x)^2 & & (3)^2 \end{array} \qquad x^2 - 9 = (x + 3)(x - 3)$$

$a = x$ and $b = 3$

Therefore,

$$\begin{aligned} x^4 - 81 &= (x^2 + 9)(x^2 - 9) \\ &= (x^2 + 9)(x + 3)(x - 3) \end{aligned}$$

You Try 3

Factor completely.

a) $125d - 5d^3$ b) $3r^2 + 48$ c) $z^4 - 1$

3. Factor the Sum and Difference of Two Cubes

Before we give the formulas for factoring the sum and difference of two cubes, let's look at two products.

$$\begin{aligned} (a + b)(a^2 - ab + b^2) &= a(a^2 - ab + b^2) + b(a^2 - ab + b^2) && \text{Distributive property} \\ &= a^3 - \cancel{a^2b} + \cancel{ab^2} + \cancel{a^2b} - \cancel{ab^2} + b^3 && \text{Distribute.} \\ &= a^3 + b^3 && \text{Combine like terms.} \end{aligned}$$

So, $(a + b)(a^2 - ab + b^2) = a^3 + b^3$, the sum of two cubes.

Let's look at another product:

$$\begin{aligned}(a - b)(a^2 + ab + b^2) &= a(a^2 + ab + b^2) - b(a^2 + ab + b^2) && \text{Distributive property}\\ &= a^3 + \cancel{a^2b} + \cancel{ab^2} - \cancel{a^2b} - \cancel{ab^2} - b^3 && \text{Distribute.}\\ &= a^3 - b^3 && \text{Combine like terms.}\end{aligned}$$

So, $(a - b)(a^2 + ab + b^2) = a^3 - b^3$, the difference of two cubes.

The formulas for factoring the sum and difference of two cubes, then, are as follows:

Formula Factoring the Sum and Difference of Two Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Note

Notice that each factorization is the product of a binomial and a trinomial. To factor the sum and difference of two cubes

Step 1: Identify a and b.

Step 2: Place them in the binomial factor and write the trinomial based on a and b.

Step 3: Simplify.

Example 5

Factor completely.

a) $k^3 + 27$ b) $n^3 - 125$ c) $64c^3 + 125d^3$

Solution

a) Use steps 1–3 to factor.

Step 1: Identify a and b.

$$k^3 + 27$$
$$\downarrow \qquad \downarrow$$

What do you cube to get k^3? k $(k)^3$ $(3)^3$ What do you cube to get 27? 3

So, $a = k$ and $b = 3$.

Step 2: Remember, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

Write the binomial factor, then write the trinomial.

Same sign Square a. Product of a and b Square b.

$$k^3 + 27 = (k + 3)[(k)^2 - (k)(3) + (3)^2]$$

Opposite sign

Step 3: Simplify: $k^3 + 27 = (k + 3)(k^2 - 3k + 9)$.

b) ***Step 1:*** Identify a and b.

$$n^3 - 125$$

What do you cube to get n^3? n $\quad (n)^3 \quad (5)^3 \quad$ What do you cube to get 125? 5

So, $a = n$ and $b = 5$.

Step 2: Write the binomial factor, then write the trinomial. Remember, $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

Same sign $\qquad$ Square a. $\qquad$ Product of a and b $\qquad$ Square b.

$$n^3 - 125 = (n - 5)[(n)^2 + \underbrace{(n)(5)}_{\text{Product of } a \text{ and } b} + (5)^2]$$

Opposite sign

Step 3: Simplify: $\quad n^3 - 125 = (n - 5)(n^2 + 5n + 25)$.

c) $64c^3 + 125d^3$

Step 1: Identify a and b.

$$64c^3 + 125d^3$$

What do you cube to get $64c^3$? $4c$ $\quad (4c)^3 \quad (5d)^3 \quad$ What do you cube to get $125d^3$? $5d$

So, $a = 4c$ and $b = 5d$.

Step 2: Write the binomial factor, then write the trinomial. Remember, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

Same sign $\qquad$ Square a. $\qquad$ Product of a and b $\qquad$ Square b.

$$64c^3 + 125d^3 = (4c + 5d)[(4c)^2 - \underbrace{(4c)(5d)}_{\text{Product of } a \text{ and } b} + (5d)^2]$$

Opposite sign

Step 3: Simplify: $\quad 64c^3 + 125d^3 = (4c + 5d)(16c^2 - 20cd + 25d^2)$. ■

You Try 4

Factor completely.

a) $m^3 + 1000$ $\qquad$ b) $h^3 - 1$ $\qquad$ c) $27p^3 - 64q^3$

Just as in the other factoring problems we've studied so far, the first step in factoring *any* polynomial should be to ask ourselves, *"Can I factor out a GCF?"*

Example 6

Factor $5z^3 - 40$ completely.

Solution

"Can I factor out a GCF?" Yes. The GCF is 5.

$$5z^3 - 40 = 5(z^3 - 8)$$

Factor $z^3 - 8$. Use $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

$$\begin{aligned} z^3 \quad - \quad 8 \quad &= (z - 2)[(z)^2 + (z)(2) + (2)^2] \\ \downarrow \qquad \downarrow \quad & \\ (z)^3 - (2)^3 &= (z - 2)(z^2 + 2z + 4) \end{aligned}$$

Therefore,
$$\begin{aligned} 5z^3 - 40 &= 5(z^3 - 8) \\ &= 5(z - 2)(z^2 + 2z + 4) \end{aligned}$$

You Try 5

Factor completely.

a) $2t^3 - 54$ b) $2a^7 + 128ab^3$

As always, the first thing you should do when factoring is ask yourself, *"Can I factor out a GCF?"* and the last thing you should do is ask yourself, *"Can I factor again?"* Now we will summarize the factoring methods discussed in this section.

Summary Special Factoring Rules

Perfect square trinomials: $a^2 + 2ab + b^2 = (a + b)^2$
$a^2 - 2ab + b^2 = (a - b)^2$

Difference of two squares: $a^2 - b^2 = (a + b)(a - b)$

Sum of two cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Difference of two cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Answers to You Try Exercises

1) a) $(g + 7)^2$ b) $6y(y - 3)^2$ c) $(5v - 1)^2$ d) $(3b + 4)(3b + 1)$ 2) a) $(m + 10)(m - 10)$ b) $(2c + 9d)(2c - 9d)$ c) $\left(h + \frac{8}{5}\right)\left(h - \frac{8}{5}\right)$ d) prime

3) a) $5d(5 + d)(5 - d)$ b) $3(r^2 + 16)$ c) $(z^2 + 1)(z + 1)(z - 1)$

4) a) $(m + 10)(m^2 - 10m + 100)$ b) $(h - 1)(h^2 + h + 1)$ c) $(3p - 4q)(9p^2 + 12pq + 16q^2)$

5) a) $2(t - 3)(t^2 + 3t + 9)$ b) $2a(a^2 + 4b)(a^4 - 4a^2b + 16b^2)$

7.4 Exercises

Objective 1: Factor a Perfect Square Trinomial

1) Find the following.

a) 7^2 b) 9^2

c) 6^2 d) 10^2

e) 5^2 f) 4^2

g) 11^2 h) $\left(\frac{1}{3}\right)^2$

i) $\left(\frac{3}{8}\right)^2$

2) What is perfect square trinomial?

3) Fill in the blank with a term that has a positive coefficient.

a) $(\_\_)^2 = c^4$ b) $(\_\_)^2 = 9r^2$

c) $(\_\_)^2 = 81p^2$ d) $(\_\_)^2 = 36m^4$

e) $(\_\_)^2 = \frac{1}{4}$ f) $(\_\_)^2 = \frac{144}{25}$

4) If x^n is a perfect square, then n is divisible by what number?

5) What perfect square trinomial factors to $(y + 6)^2$?

6) What perfect square trinomial factors to $(3k - 8)^2$?

7) Why isn't $4a^2 - 10a + 9$ a perfect square trinomial?

8) Why isn't $x^2 + 5x + 12$ a perfect square trinomial?

Factor completely.

9) $h^2 + 10h + 25$
10) $q^2 + 8q + 16$
11) $b^2 - 14b + 49$
12) $t^2 - 24t + 144$
13) $4w^2 + 4w + 1$
14) $25m^2 + 20m + 4$
VIDEO 15) $9k^2 - 24k + 16$
16) $16a^2 - 56a + 49$
17) $c^2 + c + \frac{1}{4}$
18) $h^2 + \frac{1}{3}h + \frac{1}{36}$
19) $k^2 - \frac{14}{5}k + \frac{49}{25}$
20) $p^2 - \frac{4}{3}h + \frac{4}{9}$
21) $a^2 + 8ab + 16b^2$
22) $4x^2 - 12xy + 9y^2$
23) $25m^2 - 30mn + 9n^2$
24) $49p^2 + 14pq + q^2$
VIDEO 25) $4f^2 + 24f + 36$
26) $8r^2 - 16r + 8$
27) $5a^4 + 30a^3 + 45a^2$
28) $3k^3 - 42k^2 + 147k$
29) $-16y^2 - 80y - 100$
30) $-81n^2 + 54n - 9$
31) $75h^3 - 6h^2 + 12h$
32) $98b^5 + 42b^4 + 18b^3$

Objective 2: Factor the Difference of Two Squares

33) What binomial factors to

a) $(x + 9)(x - 9)$?
b) $(9 + x)(9 - x)$?

34) What binomial factors to

a) $(y - 10)(y + 10)$?
b) $(10 - y)(10 + y)$?

Complete the factorization.

35) $w^2 - 64 = (w + 8)(\quad)$
36) $t^2 - 1 = (t - 1)(\quad)$
37) $121 - p^2 = (11 + p)(\quad)$
38) $9h^2 - 49 = (3h + 7)(\quad)$
39) $64c^2 - 25b^2 = (8c + 5b)(\quad)$
40) Does $n^2 + 9 = (n + 3)^2$? Explain.

Factor completely.

41) $k^2 - 4$
42) $z^2 - 100$
43) $c^2 - 25$
44) $y^2 - 81$
45) $w^2 + 49$
46) $b^2 + 64$
47) $x^2 - \frac{1}{9}$
48) $p^2 - \frac{1}{4}$
49) $a^2 - \frac{4}{49}$
50) $t^2 - \frac{121}{64}$
51) $144 - v^2$
52) $36 - r^2$
53) $1 - h^2$
54) $169 - d^2$
55) $\frac{36}{25} - b^2$
56) $\frac{9}{100} - q^2$
VIDEO 57) $100m^2 - 49$
58) $25a^2 - 121$
59) $169k^2 - 1$
60) $36p^2 - 1$
61) $4y^2 + 49$
62) $9d^2 + 25$
63) $\frac{1}{9}t^2 - \frac{25}{4}$
64) $\frac{16}{9}x^2 - \frac{1}{49}$
65) $u^4 - 100$
66) $a^4 - 4$
67) $36c^2 - d^4$
68) $25y^2 - 144z^4$
VIDEO 69) $r^4 - 1$
70) $h^4 - 16$
71) $r^4 - 81t^4$
72) $y^4 - x^4$
73) $5u^2 - 45$
74) $3k^2 - 300$
75) $2n^2 - 288$
76) $11p^2 - 11$
77) $12z^4 - 75z^2$
78) $45b^5 - 245b^3$

Objective 3: Factor the Sum and Difference of Two Cubes

79) Find the following.

a) 4^3
b) 1^3
c) 10^3
d) 3^3
e) 5^3
f) 2^3

80) If x^n is a perfect cube, then n is divisible by what number?

81) Fill in the blank.

a) $(\_\_)^3 = m^3$
b) $(\_\_)^3 = 27t^3$
c) $(\_\_)^3 = 8b^3$
d) $(\_\_)^3 = h^6$

82) If x^n is a perfect square *and* a perfect cube, then n is divisible by what number?

Complete the factorization.

83) $y^3 + 8 = (y + 2)(\quad)$
84) $p^3 - 1000 = (p - 10)(\quad)$

Factor completely.

85) $t^3 + 64$
86) $d^3 - 125$
87) $z^3 - 1$
88) $r^3 + 27$
VIDEO 89) $27m^3 - 125$
90) $64c^3 + 1$
91) $125y^3 - 8$
92) $27a^3 + 64$
93) $1000c^3 - d^3$
94) $125v^3 + w^3$
95) $8j^3 + 27k^3$
96) $125m^3 - 27n^3$
97) $64x^3 + 125y^3$
98) $27a^3 - 1000b^3$
VIDEO 99) $6c^3 + 48$
100) $9k^3 - 9$
101) $7v^3 - 7000w^3$
102) $216a^3 + 64b^3$
103) $h^6 - 64$
104) $p^6 - 1$

Extend the concepts of this section to factor completely.

105) $(d + 4)^2 - (d - 3)^2$
106) $(w - 9)^2 - (w + 2)^2$
107) $(3k + 1)^2 - (k + 5)^2$
108) $(2m - 3)^2 - (m - 1)^2$
109) $(r - 2)^3 + 27$
110) $(x + 7)^3 + 8$
111) $(c + 4)^3 - 125$
112) $(p - 3)^3 - 1$

Putting It All Together

Objective

1. Learn Strategies for Factoring a Given Polynomial

1. Learn Strategies for Factoring a Given Polynomial

In this chapter, we have discussed several different types of factoring problems:

1) Factoring out a GCF (Section 7.1)
2) Factoring by grouping (Section 7.1)
3) Factoring a trinomial of the form $x^2 + bx + c$ (Section 7.2)
4) Factoring a trinomial of the form $ax^2 + bx + c$ (Section 7.3)
5) Factoring a perfect square trinomial (Section 7.4)
6) Factoring the difference of two squares (Section 7.4)
7) Factoring the sum and difference of two cubes (Section 7.4)

We have practiced the factoring methods separately in each section, but how do we know which factoring method to use given many different types of polynomials together? We will discuss some strategies in this section. First, recall the steps for factoring *any* polynomial:

Summary To Factor a Polynomial

1) *Always* begin by asking yourself, *"Can I factor out a GCF?"* If so, factor it out.
2) Look at the expression to decide whether it will factor further. Apply the appropriate method to factor. If there are
   a) *two terms*, see whether it is a difference of two squares or the sum or difference of two cubes as in Section 7.4.
   b) *three terms*, see whether it can be factored using the methods of Section 7.2 or Section 7.3 *or* determine whether it is a perfect square trinomial (Section 7.4).
   c) *four terms*, see whether it can be factored by grouping as in Section 7.1.
3) After factoring, *always* look carefully at the result and ask yourself, *"Can I factor it again?"* If so, factor again.

Next, we will discuss how to decide which factoring method should be used to factor a particular polynomial.

Example 1

Factor completely.

a) $8x^2 - 50y^2$ b) $t^2 - t - 56$ c) $a^2b - 9b + 4a^2 - 36$
d) $k^2 - 12k + 36$ e) $15p^2 + 51p + 18$ f) $27k^3 + 8$
g) $c^2 + 4$

Solution

a) *"Can I factor out a GCF?"* is the first thing you should ask yourself. Yes. Factor out 2.

$$8x^2 - 50y^2 = 2(4x^2 - 25y^2)$$

Ask yourself, *"Can I factor again?"* Examine $4x^2 - 25y^2$. It has two terms that are being subtracted and each term is a perfect square. $4x^2 - 25y^2$ is the difference of squares.

$$\underset{\substack{\downarrow \\ (2x)^2}}{4x^2} - \underset{\substack{\downarrow \\ (5y)^2}}{25y^2} = (2x + 5y)(2x - 5y)$$

$$8x^2 - 50y^2 = 2(4x^2 - 25y^2) = 2(2x + 5y)(2x - 5y)$$

"Can I factor again?" No. It is completely factored.

b) Look at $t^2 - t - 56$. *"Can I factor out a GCF?"* No. Think of two numbers whose *product* is -56 and *sum* is -1. The numbers are -8 and 7.

$$t^2 - t - 56 = (t - 8)(t + 7)$$

"Can I factor again?" No. It is completely factored.

c) We have to factor $a^2b - 9b + 4a^2 - 36$. *"Can I factor out a GCF?"* No. Notice that this polynomial has *four terms*. When a polynomial has *four terms*, think about *factoring by grouping*.

$$\underbrace{a^2b - 9b}_{\downarrow} + \underbrace{4a^2 - 36}_{\downarrow}$$

$$= b(a^2 - 9) + 4(a^2 - 9) \quad \text{Take out the common factor from each pair of terms.}$$
$$= (a^2 - 9)(b + 4) \quad \text{Factor out } (a^2 - 9) \text{ using the distributive property.}$$

Examine $(a^2 - 9)(b + 4)$ and ask yourself, *"Can I factor again?"* Yes! $(a^2 - 9)$ is the difference of two squares. Factor again.

$$(a^2 - 9)(b + 4) = (a + 3)(a - 3)(b + 4)$$

"Can I factor again?" No. So, $a^2b - 9b + 4a^2 - 36 = (a + 3)(a - 3)(b + 4)$.

Note

Seeing four terms is a clue to try factoring by grouping.

d) We cannot take out a GCF from $k^2 - 12k + 36$. It is a trinomial, and notice that the first and last terms are perfect squares. *Is this a perfect square trinomial?*

$$\underbrace{k^2}_{(k)^2} - 12k + \underbrace{36}_{(6)^2}$$

Does the middle term equal $2 \cdot k \cdot 6$? Yes. $2 \cdot k \cdot 6 = 12k$

Use $a^2 - 2ab + b^2 = (a - b)^2$ with $a = k$ and $b = 6$.

Then, $k^2 - 12k + 36 = (k - 6)^2$.

"Can I factor again?" No. It is completely factored.

e) It is tempting to jump right in and try to factor $15p^2 + 51p + 18$ as the product of two binomials, but ask yourself, *"Can I take out a GCF?"* Yes! Factor out 3.

$$15p^2 + 51p + 18 = 3(5p^2 + 17p + 6)$$

"Can I factor again?" Yes.

$$3(5p^2 + 17p + 6) = 3(5p + 2)(p + 3)$$

"Can I factor again?" No. So, $15p^2 + 51p + 18 = 3(5p + 2)(p + 3)$.

f) We cannot take out a GCF from $27k^3 + 8$. Notice that $27k^3 + 8$ has two terms, so think about squares and cubes. Neither term is a perfect square *and* the positive terms are being added, so this *cannot* be the difference of squares.

Is each term a perfect cube? Yes! $27k^3 + 8$ is the sum of two cubes. We will factor $27k^3 + 8$ using $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ with $a = 3k$ and $b = 2$.

$$\underbrace{27k^3}_{(3k)^3} + \underbrace{8}_{(2)^3} = (3k + 2)[(3k)^2 - (3k)(2) + (2)^2]$$
$$= (3k + 2)(9k^2 - 6k + 4)$$

"Can I factor again?" No. It is completely factored.

g) Look at $c^2 + 4$ and ask yourself, *"Can I factor out a GCF?"* No.

The binomial $c^2 + 4$ is the sum of two squares, so it does not factor. This polynomial is prime. ■

You Try 1

Factor completely.

a) $6a^2 + 27a - 54$ b) $5h^3 - h^2 + 15h - 3$ c) $d^2 - 11d + 24$

d) $8 - 8t^4$ e) $1000x^3 - y^3$ f) $4m^3 + 9$ g) $w^2 + 22w + 121$

Answers to You Try Exercises

1) a) $3(2a - 3)(a + 6)$ b) $(h^2 + 3)(5h - 1)$ c) $(d - 3)(d - 8)$ d) $8(1 + t^2)(1 + t)(1 - t)$ e) $(10x - y)(100x^2 + 10xy + y^2)$ f) prime g) $(w + 11)^2$

Putting It All Together Summary Exercises

Objective 1: Learn Strategies for Factoring a Given Polynomial

Factor completely.

1) $c^2 + 15c + 56$
2) $r^2 - 100$
3) $uv + 6u + 9v + 54$
4) $5t^2 - 36t - 32$
5) $2p^2 - 13p + 21$
6) $h^2 - 22h + 121$
7) $9v^5 + 90v^4 - 54v^3$
8) $m^2 + 6mn - 40n^2$
9) $24q^3 + 52q^2 - 32q$
10) $5k^3 - 40$
11) $g^3 + 125$
12) $xy - x - 9y + 9$
13) $144 - w^2$
14) $z^2 - 11z + 42$
15) $9r^2 + 12rt + 4t^2$
16) $40b - 35$
17) $7n^4 - 63n^3 - 70n^2$
18) $4x^2 + 4x - 15$
19) $9h^2 + 25$
20) $4abc - 24ab + 12ac - 72a$
21) $40x^3 - 135$
22) $49c^2 + 56c + 16$
23) $m^2 - \dfrac{1}{100}$
24) $p^2 + 10p + 14$
25) $20x^2y + 6 - 24x^2 - 5y$
26) $100a^5b - 36ab^3$
27) $p^2 + 17pq + 30q^2$
28) $8k^3 + 64$
29) $t^2 - 2t - 16$
30) $12g^4h^3 + 54g^3h + 30g^2h$
31) $50n^2 - 40n + 8$
32) $8a^2 - a - 9$
33) $36r^2 + 57rs + 21s^2$
34) $t^2 - \dfrac{81}{169}$
35) $81x^4 - y^4$
36) $v^2 - 23v + 132$
37) $2a^2 - 10a - 72$
38) $p^2q - q - 6p^2 + 6$
39) $h^2 - \dfrac{2}{5}h + \dfrac{1}{25}$
40) $m^3 - 64$
41) $16uv + 24u - 10v - 15$
42) $-27r^3 + 144r^2 - 180r$
43) $8b^2 - 14b - 15$
44) $12b^2 + 36b + 27$
45) $8y^4z^3 - 28y^3z^3 - 40y^3z^2 + 4y^2z^2$
46) $49 - p^2$
47) $2a^2 - 7a + 8$
48) $6h^3k + 54h^2k^2 + 48hk^3$
49) $16u^2 + 40uv + 25v^2$
50) $b^4 - 16$
51) $24k^2 + 31k - 15$
52) $r^2 + 81$
53) $5s^3 - 320t^3$
54) $36w^6 - 84w^4 + 12w^3$
55) $ab - a - b + 1$
56) $d^2 + 16d + 64$
57) $7h^2 - 7$
58) $9p^2 - 18pq + 8p^2$
59) $6m^2 - 60m + 150$
60) $100x^4 + 49y^2$
61) $121z^2 - 169$
62) $64a^3 - 125b^3$
63) $-12r^2 - 75r - 18$
64) $9c^2 + 54c + 72$
65) $n^3 + 1$
66) $16t^2 + 8t + 1$
67) $81u^4 - v^4$

68) $14v^3 + 12u^2 + 28uv^2 + 6uv$

69) $13h^2 + 15h + 2$

70) $2g^3 - 2g^2 - 112g$

71) $5t^7 - 8t^4$

72) $m^2 - \dfrac{144}{25}$

73) $d^2 - 7d - 30$

74) $25k^2 - 60k + 36$

75) $z^2 + 144$

76) $54w^3 + 16$

77) $r^2 + 2r + 1$

78) $b^2 - 19b + 84$

79) $49n^2 - 100$

80) $9y^4 - 81y^2$

Extend the concepts of Sections 7.1–7.4 to factor completely.

VIDEO 81) $(2z + 1)y^2 + 6(2z + 1)y - 55(2z + 1)$

82) $(2k - 7)h^2 - 4(2k - 7)h - 45(2k - 7)$

83) $(t - 3)^2 + 3(t - 3) - 4$

84) $(v + 8)^2 - 14(v + 8) + 48$

85) $(z + 7)^2 - 11(z + 7) + 28$

86) $(3n - 1)^2 - (3n - 1) - 72$

87) $(a + b)^2 - (a - b)^2$

88) $(x + y)^2 - (x + 3y)^2$

89) $(5p - 2q)^2 - (2p + q)^2$

90) $(4s + t)^2 - (3s - 2t)^2$

91) $(r + 2)^3 + 27$

92) $(d - 5)^3 + 8$

93) $(k - 7)^3 - 1$

94) $(2w + 3)^3 - 125$

95) $a^2 - 8a + 16 - b^2$

96) $x^2 + 6x + 9 - y^2$

97) $s^2 + 18s + 81 - t^2$

98) $m^2 - 2m + 1 - n^2$

Section 7.5 Solving Quadratic Equations by Factoring

Objectives

1. Solve a Quadratic Equation of the Form $ab = 0$
2. Solve Quadratic Equations by Factoring
3. Solve Higher-Degree Equations by Factoring

Earlier, we learned that a *linear equation in one variable* is an equation that can be written in the form $ax + b = 0$, where a and b are real numbers and $a \neq 0$.

In this section, we will learn how to solve *quadratic equations*.

Definition

A **quadratic equation** can be written in the form $ax^2 + bx + c = 0$ where a, b, and c are real numbers and $a \neq 0$.

We say that a quadratic equation written in the form $ax^2 + bx + c = 0$ is in **standard form**. But quadratic equations can be written in other forms, too.

Some examples of quadratic equations are

$$x^2 + 13x + 36 = 0, \quad 5n(n - 3) = 0, \quad \text{and} \quad (z + 4)(z - 7) = -10.$$

Quadratic equations are also called *second-degree equations* because the highest power on the variable is 2.

There are many different ways to solve quadratic equations. In this section, we will learn how to solve them by factoring; other methods will be discussed later in this book.

Solving a quadratic equation by factoring is based on the *zero product rule:* If the product of two quantities is zero, then one or both of the quantities is zero.

For example, if $5y = 0$, then $y = 0$. If $p \cdot 4 = 0$, then $p = 0$. If $ab = 0$, then either $a = 0$, $b = 0$, or *both* a and b equal zero.

Definition

Zero product rule: If $ab = 0$, then $a = 0$ or $b = 0$.

We will use this idea to solve quadratic equations by factoring.

1. Solve a Quadratic Equation of the Form $ab = 0$

Example 1

Solve. a) $p(p + 4) = 0$ b) $(3x + 1)(x - 7) = 0$

Solution

a) The zero product rule says that at least one of the factors on the left must equal zero in order for the *product* to equal zero.

$$p(p + 4) = 0$$

$$p = 0 \quad \text{or} \quad p + 4 = 0 \qquad \text{Set each factor equal to 0.}$$

$$p = -4 \qquad \text{Solve.}$$

Check the solutions in the original equation:

If $p = 0$:
$0(0 + 4) \stackrel{?}{=} 0$
$0(4) = 0$ ✓

If $p = -4$:
$-4(-4 + 4) \stackrel{?}{=} 0$
$-4(0) = 0$ ✓

The solution set is $\{-4, 0\}$.

Note

It is important to remember that the factor p gives us the solution 0.

b) At least one of the factors on the left must equal zero for the *product* to equal zero.

$$(3x + 1)(x - 7) = 0$$

$$3x + 1 = 0 \quad \text{or} \quad x - 7 = 0 \qquad \text{Set each factor equal to 0.}$$

$$3x = -1$$

$$x = -\frac{1}{3} \qquad x = 7 \qquad \text{Solve each equation.}$$

Check in the original equation:

If $x = -\frac{1}{3}$:

$$\left[3\left(-\frac{1}{3}\right) + 1\right]\left(-\frac{1}{3} - 7\right) \stackrel{?}{=} 0$$

$$(-1 + 1)\left(-\frac{22}{3}\right) \stackrel{?}{=} 0$$

$$0\left(-\frac{22}{3}\right) = 0 \quad ✓$$

If $x = 7$:

$$[3(7) + 1](7 - 7) \stackrel{?}{=} 0$$

$$22(0) = 0 \quad ✓$$

The solution set is $\left\{-\frac{1}{3}, 7\right\}$.

You Try 1

Solve

a) $k(k + 2) = 0$ b) $(2r - 3)(r + 6) = 0$

2. Solve Quadratic Equations by Factoring

If the equation is in standard form, $ax^2 + bx + c = 0$, begin by factoring the expression.

Example 2

Solve $y^2 - 6y - 16 = 0$.

Solution

$$\begin{aligned} y^2 - 6y - 16 &= 0 \\ (y-8)(y+2) &= 0 && \text{Factor.} \\ \swarrow \qquad\qquad & \searrow \\ y - 8 = 0 \quad \text{or} \quad & y + 2 = 0 && \text{Set each factor equal to zero.} \\ y = 8 \quad \text{or} \quad & y = -2 && \text{Solve.} \end{aligned}$$

Check in the original equation:

If $y = 8$:
$$(8)^2 - 6(8) - 16 \stackrel{?}{=} 0$$
$$64 - 48 - 16 = 0 \quad \checkmark$$

If $y = -2$:
$$(-2)^2 - 6(-2) - 16 \stackrel{?}{=} 0$$
$$4 + 12 - 16 = 0 \quad \checkmark$$

The solution set is $\{-2, 8\}$.

Here are the steps to use to solve a quadratic equation by factoring:

Procedure Solving a Quadratic Equation by Factoring

1) Write the equation in the form $ax^2 + bx + c = 0$ (standard form) so that all terms are on one side of the equal sign and zero is on the other side.
2) Factor the expression.
3) Set each factor equal to zero, and solve for the variable. (Use the zero product rule.)
4) Check the answer(s).

You Try 2

Solve $r^2 + 7r + 6 = 0$.

Example 3

Solve each equation by factoring.

a) $2t^2 + 7t = 15$ b) $9v^2 = -54v$ c) $h^2 = -5(2h + 5)$
d) $4(a^2 + 3) - 11a = 7a(a - 3) + 20$ e) $(w - 4)(w - 5) = 2$

Solution

a) Begin by writing $2t^2 + 7t = 15$ in standard form, $at^2 + bt + c = 0$.

$$\begin{aligned} 2t^2 + 7t - 15 &= 0 && \text{Standard form} \\ (2t-3)(t+5) &= 0 && \text{Factor.} \\ \swarrow \qquad\qquad & \searrow \\ 2t - 3 = 0 \quad \text{or} \quad & t + 5 = 0 && \text{Set each factor equal to zero.} \\ 2t = 3 \qquad & \\ t = \frac{3}{2} \quad \text{or} \quad & t = -5 && \text{Solve.} \end{aligned}$$

Check in the original equation:

If $t = \frac{3}{2}$:

$$\begin{aligned} 2\left(\frac{3}{2}\right)^2 + 7\left(\frac{3}{2}\right) &\stackrel{?}{=} 15 \\ 2\left(\frac{9}{4}\right) + \frac{21}{2} &\stackrel{?}{=} 15 \\ \frac{9}{2} + \frac{21}{2} &\stackrel{?}{=} 15 \\ \frac{30}{2} &= 15 \quad \checkmark \end{aligned}$$

If $t = -5$:

$$\begin{aligned} 2(-5)^2 + 7(-5) &\stackrel{?}{=} 15 \\ 2(25) - 35 &\stackrel{?}{=} 15 \\ 50 - 35 &= 15 \quad \checkmark \end{aligned}$$

The solution set is $\left\{-5, \frac{3}{2}\right\}$.

b) Write $9v^2 = -54v$ in standard form.

$$\begin{aligned} 9v^2 + 54v &= 0 && \text{Standard form} \\ 9v(v + 6) &= 0 && \text{Factor.} \end{aligned}$$

$$\begin{aligned} 9v = 0 \quad &\text{or} \quad v + 6 = 0 && \text{Set each factor equal to zero.} \\ v = 0 \quad &\text{or} \quad v = -6 && \text{Solve.} \end{aligned}$$

Check. The solution set is $\{-6, 0\}$.

Note

Since both terms in $9v^2 = -54v$ are divisible by 9, we could have started part b) by dividing by 9:

$$\begin{aligned} \frac{9v^2}{9} &= \frac{-54v}{9} && \text{Divide by 9.} \\ v^2 &= -6v \\ v^2 + 6v &= 0 && \text{Write in standard form.} \\ v(v + 6) &= 0 && \text{Factor.} \end{aligned}$$

$$\begin{aligned} v = 0 \quad \text{or} \quad v + 6 &= 0 && \text{Set each factor equal to zero.} \\ d &= -6 && \text{Solve.} \end{aligned}$$

The solution set is $\{-6, 0\}$. We get the same result.

BE CAREFUL

We cannot divide by v even though each term contains a factor of v. Doing so would eliminate the solution of zero. *In general, we can divide an equation by a nonzero real number but we cannot divide an equation by a variable because we may eliminate a solution, and we may be dividing by zero.*

c) To solve $h^2 = -5(2h + 5)$, begin by writing the equation in standard form.

$$\begin{aligned} h^2 &= -10h - 25 && \text{Distribute.} \\ h^2 + 10h + 25 &= 0 && \text{Write in standard form.} \\ (h + 5)^2 &= 0 && \text{Factor.} \end{aligned}$$

Since $(h + 5)^2 = 0$ means $(h + 5)(h + 5) = 0$, setting each factor equal to zero will result in the same value for h.

$$\begin{aligned} h + 5 &= 0 && \text{Set } h + 5 = 0. \\ h &= -5 && \text{Solve.} \end{aligned}$$

Check. The solution set is $\{-5\}$.

d) We will have to perform several steps to write the equation in standard form.

$$4(a^2 + 3) - 11a = 7a(a - 3) + 20$$
$$4a^2 + 12 - 11a = 7a^2 - 21a + 20 \qquad \text{Distribute.}$$

Move the terms on the left side of the equation to the right side so that the coefficient of a^2 is positive.

$$0 = 3a^2 - 10a + 8 \qquad \text{Write in standard form.}$$
$$0 = (3a - 4)(a - 2) \qquad \text{Factor.}$$
$$\swarrow \qquad \searrow$$
$$3a - 4 = 0 \quad \text{or} \quad a - 2 = 0 \qquad \text{Set each factor equal to zero.}$$
$$3a = 4$$
$$a = \frac{4}{3} \quad \text{or} \quad a = 2 \qquad \text{Solve.}$$

Check. The solution set is $\left\{\frac{4}{3}, 2\right\}$.

BE CAREFUL

e) It is tempting to solve $(w - 4)(w - 5) = 2$ like this:

$$(w - 4)(w - 5) = 2$$
$$\swarrow \qquad \searrow$$
$$w - 4 = 2 \quad \text{or} \quad w - 5 = 2 \qquad \text{This is incorrect!}$$

One side of the equation must equal zero in order to use the zero product rule. Begin by multiplying on the left.

$$(w - 4)(w - 5) = 2$$
$$w^2 - 9w + 20 = 2 \qquad \text{Multiply using FOIL.}$$
$$w^2 - 9w + 18 = 0 \qquad \text{Standard form}$$
$$(w - 6)(w - 3) = 0 \qquad \text{Factor.}$$
$$\swarrow \qquad \searrow$$
$$w - 6 = 0 \quad \text{or} \quad w - 3 = 0 \qquad \text{Set each factor equal to zero.}$$
$$w = 6 \quad \text{or} \quad w = 3 \qquad \text{Solve.}$$

The check is left to the student. The solution set is $\{3, 6\}$. ■

You Try 3

Solve.

a) $5c^2 = 6c + 8$ b) $3q^2 = -18q$ c) $6n(n + 2) = -7(n - 1)$

d) $(m - 5)(m - 10) = 6$ e) $z(z + 3) = 40$

3. Solve Higher-Degree Equations by Factoring

Sometimes, equations that are not quadratics can be solved by factoring as well.

Example 4

Solve each equation.

a) $(4x - 1)(x^2 - 8x - 20) = 0$ b) $12n^3 - 108n = 0$

Solution

a) This is *not* a quadratic equation because if we multiplied the factors on the left we would get $4x^3 - 33x^2 - 72x + 20 = 0$. This is a *cubic* equation because the degree of the polynomial on the left is 3.

The original equation is the product of two factors so we can use the zero product rule.

$$
\begin{aligned}
(4x - 1)(x^2 - 8x - 20) &= 0 \\
(4x - 1)(x - 10)(x + 2) &= 0 && \text{Factor.}
\end{aligned}
$$

$$
\begin{array}{ccccc|l}
4x - 1 = 0 & \text{or} & x - 10 = 0 & \text{or} & x + 2 = 0 & \text{Set each factor equal to zero.} \\
4x = 1 & & & & & \\
x = \dfrac{1}{4} & \text{or} & x = 10 & \text{or} & x = -2 & \text{Solve.}
\end{array}
$$

The check is left to the student. The solution set is $\left\{-2, \frac{1}{4}, 10\right\}$.

b) The GCF of the terms in the equation $12n^3 - 108n = 0$ is $12n$. Remember, however, that *we can divide an equation by a constant but we cannot divide an equation by a variable.* Dividing by a variable may eliminate a solution and may mean we are dividing by zero. So let's begin by dividing each term by 12.

$$
\begin{aligned}
\frac{12n^3}{12} - \frac{108n}{12} &= \frac{0}{12} && \text{Divide by 12.} \\
n^3 - 9n &= 0 && \text{Simplify.} \\
n(n^2 - 9) &= 0 && \text{Factor out } n. \\
n(n + 3)(n - 3) &= 0 && \text{Factor } n^2 - 9. \\
n = 0 \quad \text{or} \quad n + 3 &= 0 \quad \text{or} \quad n - 3 = 0 && \text{Set each factor equal to zero.} \\
n &= -3 \qquad\qquad n = 3 && \text{Solve.}
\end{aligned}
$$

Check. The solution set is $\{0, -3, 3\}$. ■

You Try 4

Solve.

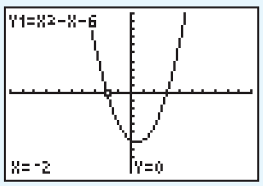
a) $(5y + 3)(y^2 - 10y + 21) = 0$ b) $8k^3 - 32k = 0$

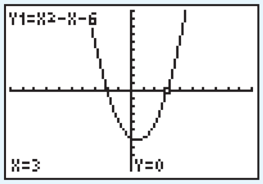
In this section, it was possible to solve all of the equations by factoring. Below we show the relationship between solving a quadratic equation by factoring and solving it using a graphing calculator. In Chapter 11, we will learn other methods for solving quadratic equations.

Using Technology

Solve $x^2 - x - 6 = 0$ using a graphing calculator.

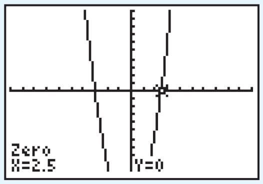
Recall from Chapter 4 that to find the x-intercepts of the graph of an equation, we let $y = 0$ and solve the equation for x. If we let $y = x^2 - x - 6$, then solving $x^2 - x - 6 = 0$ is the same as finding the x-intercepts of the graph of $y = x^2 - x - 6$. The x-intercepts are also called zeros of the equation since they are the values of x that make $y = 0$. Enter $y = x^2 - x - 6$ into the calculator and display the graph using the standard viewing window. We obtain a graph called a *parabola,* and we can see that it has two x-intercepts.



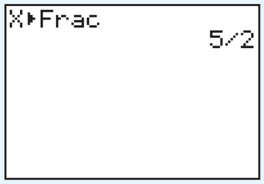


Since the scale for each tick mark on the graph is 1, it appears that the x-intercepts are -2 and 3. To verify this, press TRACE, type -2, and press ENTER as shown on the first screen. Since $x = -2$ and $y = 0$, $x = -2$ is an x-intercept. While still in "Trace" mode, type 3 and press ENTER as shown. Since $x = 3$ and $y = 0$, $x = 3$ is an x-intercept.

Sometimes an x-intercept is not an integer. Solve $2x^2 + x - 15 = 0$ using a graphing calculator.



Enter $2x^2 + x - 15$ into the calculator and press GRAPH. The x-intercept on the right side of the graph is between two tick marks, so it is not an integer. To find this x-intercept press 2nd TRACE and select 2: zero. Now move the cursor to the left of one of the intercepts and press ENTER, then move the cursor again, so that it is the right of the same intercept and press ENTER. Press ENTER one more time, and the calculator will reveal the intercept and, therefore, one solution to the equation as shown on the third screen.



Press 2nd MODE to return to the home screen. Press X,T,Θ,n MATH ENTER ENTER to display the x-intercept in fraction form: $x = \frac{5}{2}$, as shown on the final screen. Since the other x-intercept appears to be -3, press TRACE -3 ENTER to reveal that $x = -3$ and $y = 0$.

Solve using a graphing calculator.

1) $x^2 - 5x - 6 = 0$
2) $2x^2 - 9x - 5 = 0$
3) $x^2 + 4x - 21 = 0$
4) $5x^2 - 12x + 4 = 0$
5) $x^2 + 2x - 35 = 0$
6) $2x^2 - 11x + 12 = 0$

Answers to You Try Exercises

1) a) $\{-2, 0\}$ b) $\left\{-6, \frac{3}{2}\right\}$ 2) $\{-6, -1\}$ 3) a) $\left\{-\frac{4}{5}, 2\right\}$ b) $\{-6, 0\}$ c) $\left\{-\frac{7}{2}, \frac{1}{3}\right\}$ d) $\{4, 11\}$ e) $\{-8, 5\}$ 4 a) $\left\{-\frac{3}{5}, 3, 7\right\}$ b) $\{0, -2, 2\}$

Answers to Technology Exercises

1) $\{-1, 6\}$ 2) $\left\{-\frac{1}{2}, 5\right\}$ 3) $\{-7, 3\}$ 4) $\left\{\frac{2}{5}, 2\right\}$ 5) $\{5, -7\}$ 6) $\left\{4, \frac{3}{2}\right\}$

7.5 Exercises

Objective 1: Solve a Quadratic Equation of the Form $ab = 0$

1) What is the standard form of an equation that is quadratic in x?

2) A quadratic equation is also called a ________-degree equation.

3) Identify the following equations as linear or quadratic.

a) $5x^2 + 3x - 7 = 0$

b) $6(p + 1) = 0$

c) $(n + 4)(n - 9) = 8$

d) $2w + 3(w - 5) = 4w + 9$

4) Which of the following equations are quadratic?

a) $t^3 - 6t^2 - 4t + 24 = 0$

b) $2(y^2 - 7) + 3y = 6y + 1$

c) $3a(a - 11) = 0$

d) $(c + 4)(2c^2 - 5c - 3) = 0$

5) Explain the zero product rule.

6) When Stephanie solves $m(m - 8) = 0$, she gets a solution set of $\{8\}$. Is this correct? Why or why not?

Solve each equation.

7) $(z + 11)(z - 4) = 0$
8) $(b + 1)(b + 7) = 0$
9) $(2r - 3)(r - 10) = 0$
10) $(5k - 4)(k + 9) = 0$
11) $d(d - 12) = 0$
12) $6w(w + 2) = 0$
13) $(3x + 5)^2 = 0$
14) $(c - 14)^2 = 0$
15) $(9h + 2)(2h + 1) = 0$
16) $(6q - 5)(2q - 3) = 0$
17) $\left(m + \frac{1}{4}\right)\left(m - \frac{2}{5}\right) = 0$
18) $\left(v + \frac{7}{3}\right)\left(v + \frac{4}{3}\right) = 0$
19) $n(n - 4.6) = 0$
20) $g(g + 0.7) = 0$

Objective 2: Solve Quadratic Equations by Factoring

21) Can we solve $(k - 4)(k - 8) = 5$ by setting each factor equal to 5 like this: $k - 4 = 5$ or $k - 8 = 5$? Why or why not?

22) State two ways you could begin to solve $3x^2 + 18x + 24 = 0$.

Solve each equation.

23) $p^2 + 8p + 12 = 0$
24) $c^2 + 3c - 28 = 0$
25) $t^2 - t - 110 = 0$
26) $w^2 - 17w + 72 = 0$
27) $3a^2 - 10a + 8 = 0$
28) $2y^2 + 7y + 5 = 0$
29) $12z^2 + z - 6 = 0$
30) $8b^2 - 18b - 5 = 0$
31) $r^2 = 60 - 7r$
32) $h^2 + 20 = 12h$
VIDEO 33) $d^2 - 15d = -54$
34) $h^2 + 17h = -66$
35) $x^2 - 64 = 0$
36) $n^2 - 144 = 0$
37) $49 = 100u^2$
38) $81 = 4a^2$
39) $22k = -10k^2 - 12$
40) $4m - 48 = -24m^2$
41) $v^2 = 4v$
42) $x^2 = x$
43) $(z + 3)(z + 1) = 15$
44) $(c - 10)(c - 1) = -14$
45) $t(19 - t) = 84$
46) $48 = w(w - 2)$
47) $6k(k + 4) + 3 = 5(k^2 - 12) + 8k$
48) $7b(b + 1) + 15 = 6(b^2 + 2) + 11b$
49) $3(n^2 - 15) + 4n = 4n(n - 3) + 19$
50) $8(p^2 - 6) + 9p = 3p(3p + 7) - 13$
VIDEO 51) $\frac{1}{2}(m + 1)^2 = -\frac{3}{4}m(m + 5) - \frac{5}{2}$
52) $\frac{1}{8}(2y - 3)^2 + \frac{1}{8}y = \frac{1}{8}(y - 5)^2 - \frac{3}{4}$

Objective 3: Solve Higher-Degree Equations by Factoring

53) To solve $5t^3 - 20t = 0$, Julio begins by dividing the equation by $5t$ to get $t^2 - 4 = 0$. Is this correct? Why or why not?

54) What are two possible first steps for solving $5t^3 - 20t = 0$?

The following equations are not quadratic but can be solved by factoring and applying the zero product rule. Solve each equation.

55) $7w(8w - 9)(w + 6) = 0$
56) $-5q(4q - 7)(q + 3) = 0$
57) $(6m + 7)(m^2 - 5m + 6) = 0$
58) $(9c - 2)(c^2 + 9c + 8) = 0$
59) $49h = h^3$
60) $r^3 = 36r$
VIDEO 61) $5w^2 + 36w = w^3$
62) $10p^2 - 25p = p^3$
63) $60a = 44a^2 - 8a^3$
64) $6z^3 + 16z = -50z^2$
65) $162b^3 - 8b = 0$
66) $75x = 27x^3$

Mixed Exercises: Objectives 1–3

Solve each equation.

67) $-63 = 4y(y - 8)$
68) $-84 = g(g + 19)$
69) $\frac{1}{2}d(2 - d) - \frac{3}{2} = \frac{2}{5}d(d + 1) - \frac{7}{5}$
70) $(9p - 2)(p^2 - 10p - 11) = 0$
71) $a^2 - a = 30$
72) $45k + 27 = 18k^2$
73) $48t = 3t^3$
74) $\frac{1}{2}c(c + 3) - \frac{5}{4} = \frac{5}{8}(c^2 + 6) - \frac{1}{4}c$
75) $104r + 36 = 12r^2$
76) $3t(t - 5) + 14 = 5 - t(t + 3)$
77) $w^2 - 121 = 0$
78) $h^2 + 15h + 54 = 0$
79) $(2n - 5)(n^2 - 6n + 9) = 0$
80) $36b^2 + 60b = 0$

Extend the concepts of this section to solve each of the following equations.

81) $(2d - 5)^2 - (d + 6)^2 = 0$
82) $(3x + 8)^2 - (x + 4)^2 = 0$
83) $(11z - 4)^2 - (2z + 5)^2 = 0$
84) $(8g - 7)^2 - (g - 6)^2 = 0$
85) $2p^2(p - 4) + 9p(p - 4) + 9(p - 4) = 0$

86) $4a^2(a+5) - 23a(a+5) - 6(a+5) = 0$

87) $10c^2(2c-7) + 7(2c-7) = 37c(2c-7)$

88) $14y^2(5y-8) = 4(5y-8) - y(5y-8)$

89) $h^3 + 8h^2 - h - 8 = 0$

90) $r^3 - 7r^2 - 4r + 28 = 0$

Find the indicated values for the following polynomial functions.

91) $f(x) = x^2 + 10x + 16$. Find x so that $f(x) = 0$.

92) $g(t) = t^2 - 9t - 36$. Find t so that $g(t) = 0$.

VIDEO 93) $g(a) = 2a^2 - 13a + 24$. Find a so that $g(a) = 4$.

94) $h(c) = 2c^2 - 11c + 11$. Find c so that $h(c) = -3$.

95) $P(a) = a^2 - 12$. Find a so that $P(a) = 13$.

96) $Q(x) = 4x^2 + 19$. Find x so that $Q(x) = 20$.

97) $h(t) = 3t^3 - 21t^2 + 18t$. Find t so that $h(t) = 0$.

98) $F(x) = 2x^3 - 72x^2$. Find x so that $F(x) = 0$.

Section 7.6 Applications of Quadratic Equations

Objectives

1. Solve Problems Involving Geometry
2. Solve Problems Involving Consecutive Integers
3. Solve Problems Using the Pythagorean Theorem
4. Solve Applied Problems Using Given Quadratic Equations

In Chapters 3 and 5, we learned how to solve applied problems involving linear equations. In this section, we will learn how to solve applications involving quadratic equations. Let's begin by reviewing the five steps for solving applied problems.

Procedure Steps for Solving Applied Problems

Step 1: **Read** the problem carefully, more than once if necessary, until you understand it. Draw a picture, if applicable. Identify what you are being asked to find.

Step 2: **Choose a variable** to represent an unknown quantity. If there are any other unknowns, define them in terms of the variable.

Step 3: **Translate** the problem from English into an equation using the chosen variable.

Step 4: **Solve** the equation.

Step 5: **Check** the answer in the original problem, and **interpret** the solution as it relates to the problem. Be sure your answer makes sense in the context of the problem.

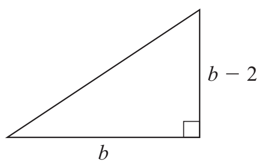
1. Solve Problems Involving Geometry

Example 1

A builder must cut a piece of tile into a right triangle. The tile will have an area of 40 in^2, and the height will be 2 in. shorter than the base. Find the base and height.

Solution

Step 1: **Read** the problem carefully. Draw a picture.



Step 2: **Choose a variable** to represent the unknown.

Let $\quad b$ = the base

$b - 2$ = the height

Step 3: **Translate** the information that appears in English into an algebraic equation. We are given the area of a triangular-shaped tile, so let's use the formula for the area of a triangle and substitute the expressions above for the base and the height and 40 for the area.

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height})$$

$$40 = \frac{1}{2}(b)(b-2) \qquad \text{Substitute Area} = 40\text{, base} = b\text{, height} = b - 2.$$

Step 4: **Solve** the equation. Eliminate the fraction first.

$$\begin{aligned} 80 &= (b)(b-2) && \text{Multiply by 2.} \\ 80 &= b^2 - 2b && \text{Distribute.} \\ 0 &= b^2 - 2b - 80 && \text{Write the equation in standard form.} \\ 0 &= (b-10)(b+8) && \text{Factor.} \end{aligned}$$

$$\begin{aligned} b - 10 &= 0 \quad \text{or} \quad b + 8 = 0 && \text{Set each factor equal to zero.} \\ b &= 10 \quad \text{or} \quad\quad\; b = -8 && \text{Solve.} \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem. Since b represents the length of the base of the triangle, it cannot be a negative number. So, $b = -8$ cannot be a solution. Therefore, the length of the base is 10 in., which will make the height $10 - 2 = 8$ in. The area, then, is $\frac{1}{2}(10)(8) = 40 \text{ in}^2$.

You try 1

The height of a triangle is 3 cm more than its base. Find the height and base if its area is 35 cm^2.

2. Solve Problems Involving Consecutive Integers

In Chapter 3, we solved problems involving consecutive integers. Some applications involving consecutive integers lead to quadratic equations.

Example 2

Twice the sum of three consecutive odd integers is five less than the product of the two larger integers. Find the numbers.

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find three consecutive odd integers.

Step 2: **Choose a variable** to represent an unknown, and define the other unknowns in terms of the variable.

$$\begin{aligned} x &= \text{the first odd integer} \\ x + 2 &= \text{the second odd integer} \\ x + 4 &= \text{the third odd integer} \end{aligned}$$

Step 3: **Translate** the information that appears in English into an algebraic equation. Read the problem slowly and carefully, breaking it into small parts.

Statement:	Twice the sum of three consecutive odd integers	is	5 less than	the product of the two larger integers.
	$\downarrow$	$\downarrow$		
Equation:	$2[x + (x + 2) + (x + 4)]$	$=$	$(x + 2)(x + 4)$	-5

Step 4: **Solve** the equation.

$$\begin{aligned} 2[x + (x + 2) + (x + 4)] &= (x + 2)(x + 4) - 5 & \\ 2(3x + 6) &= x^2 + 6x + 8 - 5 & \text{Combine like terms; distribute.} \\ 6x + 12 &= x^2 + 6x + 3 & \text{Combine like terms; distribute.} \\ 0 &= x^2 - 9 & \text{Write in standard form.} \\ 0 &= (x + 3)(x - 3) & \text{Factor.} \end{aligned}$$

$$\begin{aligned} x + 3 &= 0 \quad \text{or} \quad x - 3 = 0 & \text{Set each factor equal to zero.} \\ x &= -3 \qquad\qquad\;\; x = 3 & \text{Solve.} \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

We get two sets of solutions. If $x = -3$, then the other odd integers are -1 and 1. If $x = 3$, the other odd integers are 5 and 7.

Check these numbers in the original statement of the problem.

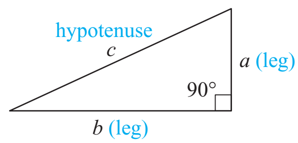
$$\begin{aligned} 2[-3 + (-1) + 1] &= (-1)(1) - 5 \\ 2(-3) &= -1 - 5 \\ -6 &= -6 \end{aligned} \qquad \begin{aligned} 2(3 + 5 + 7) &= (5)(7) - 5 \\ 2(15) &= 35 - 5 \\ 30 &= 30 \end{aligned}$$

You Try 2

Find three consecutive odd integers such that the product of the smaller two is 15 more than four times the sum of the three integers.

3. Solve Problems Using the Pythagorean Theorem

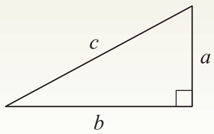
Recall that a **right triangle** contains a 90° angle. We can label it this way.



The side opposite the 90° angle is the longest side of the triangle and is called the **hypotenuse**. The other two sides are called the **legs**. The Pythagorean theorem states a relationship between the lengths of the sides of a right triangle. This is a very important relationship in mathematics and is used in many different ways.

Definition Pythagorean Theorem

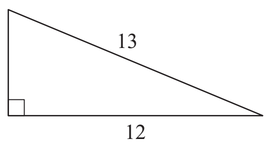
Given a right triangle with legs of length a and b and hypotenuse of length c, the **Pythagorean theorem** states that $a^2 + b^2 = c^2$ [or $(\text{leg})^2 + (\text{leg})^2 = (\text{hypotenuse})^2$].



The Pythagorean theorem is true *only* for right triangles.

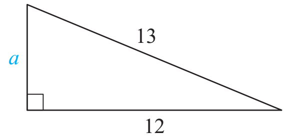
Example 3

Find the length of the missing side.



Solution

Since this is a right triangle, we can use the Pythagorean theorem to find the length of the side. Let a represent its length, and label the triangle.



The length of the hypotenuse is 13, so $c = 13$. a and 12 are legs. Let $b = 12$.

$$\begin{aligned} a^2 + b^2 &= c^2 && \text{Pythagorean theorem} \\ a^2 + (12)^2 &= (13)^2 && \text{Substitute values.} \\ a^2 + 144 &= 169 && \\ a^2 - 25 &= 0 && \text{Write the equation in standard form.} \\ (a+5)(a-5) &= 0 && \text{Factor.} \end{aligned}$$

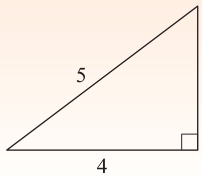
$$\begin{aligned} a + 5 = 0 \quad &\text{or} \quad a - 5 = 0 && \text{Set each factor equal to 0.} \\ a = -5 \quad &\text{or} \quad a = 5 && \text{Solve.} \end{aligned}$$

$a = -5$ does not make sense as an answer because the length of a side of a triangle cannot be negative. Therefore, $a = 5$.

Check: $5^2 + (12)^2 \stackrel{?}{=} (13)^2$
$25 + 144 = 169$ ✓

You Try 3

Find the length of the missing side.



Example 4

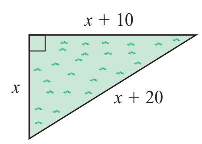
A community garden sits on a corner lot and is in the shape of a right triangle. One side is 10 ft longer than the shortest side, while the longest side is 20 ft longer than the shortest side. Find the lengths of the sides of the garden.

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find. Draw a picture.

We must find the lengths of the sides of the garden.

Step 2: **Choose a variable** to represent an unknown, and define the other unknowns in terms of this variable. Draw and label the picture.



x = length of the shortest side (a leg)
$x + 10$ = length of the second side (a leg)
$x + 20$ = length of the third side (hypotenuse)

Step 3: **Translate** the information that appears in English into an algebraic equation. We will use the Pythagorean theorem.

$$\begin{aligned} a^2 + b^2 &= c^2 && \text{Pythagorean theorem} \\ x^2 + (x+10)^2 &= (x+20)^2 && \text{Substitute.} \end{aligned}$$

Step 4: **Solve** the equation.

$$
\begin{aligned}
x^2 + (x+10)^2 &= (x+20)^2 \\
x^2 + x^2 + 20x + 100 &= x^2 + 40x + 400 && \text{Multiply using FOIL.} \\
2x^2 + 20x + 100 &= x^2 + 40x + 400 \\
x^2 - 20x - 300 &= 0 && \text{Write in standard form.} \\
(x-30)(x+10) &= 0 && \text{Factor.}
\end{aligned}
$$

$$
\begin{aligned}
x - 30 = 0 \quad &\text{or} \quad x + 10 = 0 && \text{Set each factor equal to 0.} \\
x = 30 \quad &\text{or} \quad x = -10 && \text{Solve.}
\end{aligned}
$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

The length of the shortest side, x, cannot be a negative number, so x cannot equal -10. Therefore, the length of the shortest side must be 30 ft.

The length of the second side $= x + 10$, so $30 + 10 = 40$ ft.

The length of the longest side $= x + 20$, so $30 + 20 = 50$ ft.

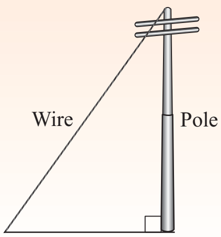
Do these lengths satisfy the Pythagorean theorem? Yes.

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
(30)^2 + (40)^2 &\stackrel{?}{=} (50)^2 \\
900 + 1600 &= 2500 \quad \checkmark
\end{aligned}
$$

Therefore, the lengths of the sides are 30 ft, 40 ft, and 50 ft. ■

You Try 4

A wire is attached to the top of a pole. The pole is 2 ft shorter than the wire, and the distance from the wire on the ground to the bottom of the pole is 9 ft less than the length of the wire. Find the length of the wire and the height of the pole.



4. Solve Applied Problems Using Given Quadratic Equations

Let's see how to use a quadratic equation to model a real-life situation.

Example 5

An object is launched from a platform with an initial velocity of 32 ft/sec. The height h (in feet) of the object t seconds after it is released is given by the quadratic equation

$$h = -16t^2 + 32t + 20.$$

a) What is the initial height of the ball?

b) How long does it take the ball to reach a height of 32 feet?

c) How long does it take for the ball to hit the ground?

Solution

a) Since t represents the number of seconds after the ball is thrown, $t = 0$ at the time of release.

Let $t = 0$ and solve for h.

$$\begin{aligned} h &= -16(0)^2 + 32(0) + 20 && \text{Substitute 0 for } t. \\ h &= 0 + 0 + 20 \\ h &= 20 \end{aligned}$$

The initial height of the ball is 20 ft.

b) We must find the *time* it takes for the ball to reach a height of 32 feet. Find t when $h = 32$.

$$\begin{aligned} h &= -16t^2 + 32t + 20 \\ 32 &= -16t^2 + 32t + 20 && \text{Substitute 32 for } h. \\ 0 &= -16t^2 + 32t - 12 && \text{Write in standard form.} \\ 0 &= 4t^2 - 8t + 3 && \text{Divide by } -4. \\ 0 &= (2t - 1)(2t - 3) && \text{Factor.} \end{aligned}$$

$$\begin{aligned} 2t - 1 &= 0 \quad \text{or} \quad & 2t - 3 &= 0 && \text{Set each factor equal to 0.} \\ 2t &= 1 & 2t &= 3 \\ t &= \frac{1}{2} \quad \text{or} & t &= \frac{3}{2} && \text{Solve.} \end{aligned}$$

How can two answers be possible? After $\frac{1}{2}$ sec, the ball is 32 feet above the ground *on its way up*, and after $\frac{3}{2}$ sec, the ball is 32 feet above the ground *on its way down*.

The ball reaches a height of 32 feet after $\frac{1}{2}$ sec and after $\frac{3}{2}$ sec.

c) When the ball hits the ground, how high off the ground is it? *It is 0 feet high.*

Find t when $h = 0$.

$$\begin{aligned} h &= -16t^2 + 32t + 20 \\ 0 &= -16t^2 + 32t + 20 && \text{Substitute 0 for } h. \\ 0 &= 4t^2 - 8t - 5 && \text{Divide by } -4. \\ 0 &= (2t + 1)(2t - 5) && \text{Factor.} \end{aligned}$$

$$\begin{aligned} 2t + 1 &= 0 \quad \text{or} \quad & 2t - 5 &= 0 && \text{Set each factor equal to 0.} \\ 2t &= -1 & 2t &= 5 \\ t &= -\frac{1}{2} \quad \text{or} & t &= \frac{5}{2} && \text{Solve.} \end{aligned}$$

Since t represents time, t cannot equal $-\frac{1}{2}$. Therefore, $t = \frac{5}{2}$.

The ball will hit the ground after $\frac{5}{2}$ sec (or 2.5 sec). ■

Note

In Example 5, the equation can also be written using function notation $h(t) = -16t^2 + 32t + 20$ since the expression $-16t^2 + 32t + 20$ is a polynomial. Furthermore, $h(t) = -16t^2 + 32t + 20$ is a *quadratic function,* and we say that the height, h, is a function of the time, t. We will study quadratic functions in more detail in Chapter 10.

You Try 5

An object is thrown upward from a building. The height h of the object (in feet) t sec after the object is released is given by the quadratic equation

$$h = -16t^2 + 36t + 36.$$

a) What is the initial height of the object?

b) How long does it take the object to reach a height of 44 ft?

c) How long does it take for the object to hit the ground?

Answers to You Try Exercises

1) base = 7 cm; height = 10 cm 2) 13, 15, 17 or $-3, -1, 1$ 3) 3

4) length of wire = 17 ft; height of pole = 15 ft

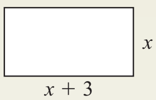
5) a) 36 ft b) 0.25 sec and 2 sec c) 3 sec

7.6 Exercises

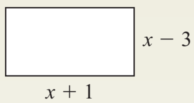
Objective 1: Solve Problems Involving Geometry

Find the length and width of each rectangle.

1) Area = 28 in^2

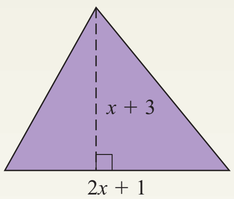


2) Area = 96 cm^2

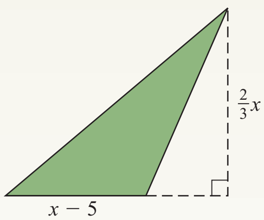


Find the base and height of each triangle.

3) Area = 44 cm^2

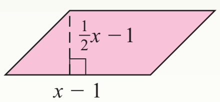


4) Area = 12 ft^2

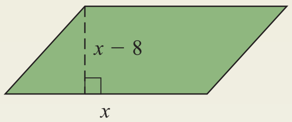


Find the base and height of each parallelogram.

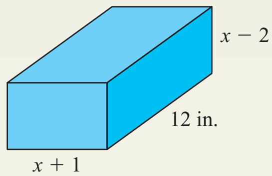
5) Area = 36 in^2



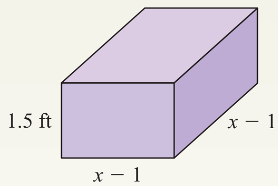
6) Area = 240 mm^2



7) The volume of the box is 648 in^3. Find its height and width.



8) The volume of the box is 6 ft^3. Find its width and length.



Write an equation and solve.

9) A rectangular sign is twice as long as it is wide. If its area is 8 ft^2, what are its length and width?

10) An ad in a magazine is in the shape of a rectangle and occupies 88 in^2. The length is three inches longer than the width. Find the dimensions of the ad.

11) The top of a kitchen island is a piece of granite that has an area of 15 ft^2. It is 3.5 ft longer than it is wide. Find the dimensions of the surface.

12) To install an exhaust fan, a builder cuts a rectangular hole in the ceiling so that the width is 3 in. less than the length. The area of the hole is 180 in^2. Find the length and width of the hole.

13) A rectangular make-up case is 3 in. high and has a volume of 90 in^3. The width is 1 in. less than the length. Find the length and width of the case.

14) An artist's sketchbox is 4 in. high and shaped like a rectangular solid. The width is three-fourths as long as the length. Find the length and width of the box if its volume is 768 in^3.

15) The height of a triangle is 1 cm more than its base. Find the height and base if its area is 21 cm^2.

16) The area of a triangle is 24 in^2. Find the height and base if its height is one-third the length of the base.

Objective 2: Solve Problems Involving Consecutive Integers

Write an equation and solve.

17) The product of two consecutive integers is 13 less than five times their sum. Find the integers.

18) The product of two consecutive integers is 10 less than four times their sum. Find the integers.

19) Find three consecutive even integers such that the product of the two smaller numbers is the same as twice the sum of all three integers.

20) Find three consecutive even integers such that five times the sum of the smallest and largest integers is the same as the square of the middle number.

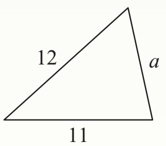
21) Find three consecutive odd integers such that the product of the two larger numbers is 18 more than three times the sum of all three numbers.

22) Find three consecutive odd integers such that the square of the largest integer is 9 more than six times the sum of the two smaller numbers.

Objective 3: Solve Problems Using the Pythagorean Theorem

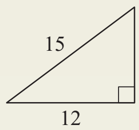
23) In your own words, explain the Pythagorean theorem.

24) Can the Pythagorean theorem be used to find a in this triangle? Why or why not?



Use the Pythagorean theorem to find the length of the missing side.

VIDEO 25)



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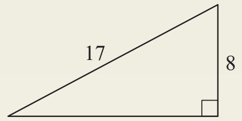

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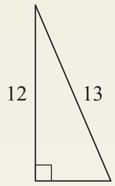

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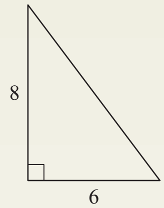

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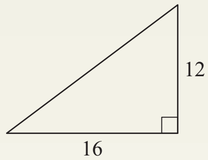

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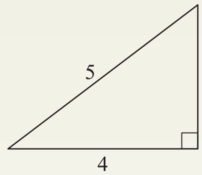
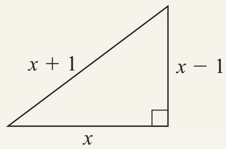


Find the lengths of the sides of each right triangle.

31)



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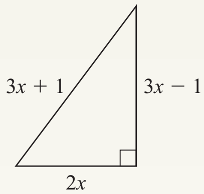

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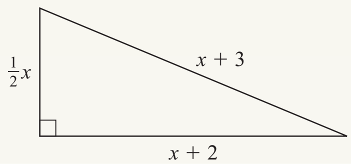

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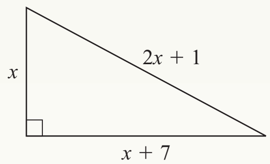
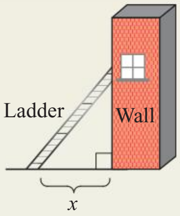


Write an equation and solve.

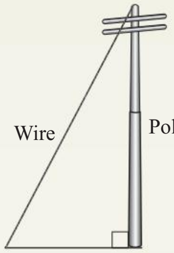
35) The longer leg of a right triangle is 2 cm more than the shorter leg. The length of the hypotenuse is 4 cm more than the shorter leg. Find the length of the hypotenuse.

36) The hypotenuse of a right triangle is 1 in. longer than the longer leg. The shorter leg measures 7 in. less than the longer leg. Find the measure of the longer leg of the triangle.

VIDEO 37) A 13-ft ladder is leaning against a wall. The distance from the top of the ladder to the bottom of the wall is 7 ft more than the distance from the bottom of the ladder to the wall. Find the distance from the bottom of the ladder to the wall.



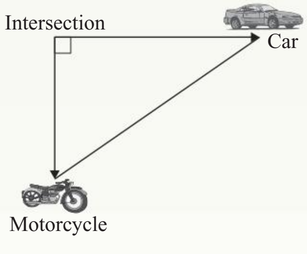
38) A wire is attached to the top of a pole. The wire is 4 ft longer than the pole, and the distance from the wire on the ground to the bottom of the pole is 4 ft less than the height of the pole. Find the length of the wire and the height of the pole.



Write an equation and solve.

39) Lance and Alberto leave the same location with Lance heading due north and Alberto riding due east. When Alberto has ridden 4 miles, the distance between him and Lance is 2 miles more than Lance's distance from the starting point. Find the distance between Lance and Alberto.

40) A car heads east from an intersection while a motorcycle travels south. After 20 minutes, the car is 2 miles farther from the intersection than the motorcycle. The distance between the two vehicles is 4 miles more than the motorcycle's distance from the intersection. What is the distance between the car and the motorcycle?



Objective 4: Solve Applied Problems Using Given Quadratic Equations

Solve.

VIDEO 41) A rock is dropped from a cliff and into the ocean. The height h (in feet) of the rock after t sec is given by $h = -16t^2 + 144$.

a) What is the initial height of the rock?

b) When is the rock 80 ft above the water?

c) How long does it take the rock to hit the water?

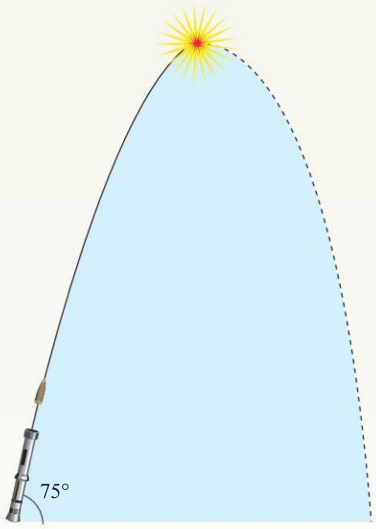
42) A Little League baseball player throws a ball upward. The height h of the ball (in feet) t seconds after the ball is released is given by $h = -16t^2 + 30t + 4$.

a) What is the initial height of the ball?

b) When is the ball 18 feet above the ground?

c) How long does it take for the ball to hit the ground?

Organizers of fireworks shows use quadratic and linear equations to help them design their programs. *Shells* contain the chemicals that produce the bursts we see in the sky. At a fireworks show, the shells are shot from *mortars* and when the chemicals inside the shells ignite, they explode, producing the brilliant bursts we see in the night sky.



43) At a fireworks show, a 3-in. shell is shot from a mortar at an angle of 75°. The height, y (in feet), of the shell t sec after being shot from the mortar is given by the quadratic equation

$$y = -16t^2 + 144t$$

and the horizontal distance of the shell from the mortar, x (in feet), is given by the linear equation

$$x = 39t.$$

(http://library.thinkquest.org/15384/physics/physics.html)

a) How high is the shell after 3 sec?

b) What is the shell's horizontal distance from the mortar after 3 sec?

c) The maximum height is reached when the shell explodes. How high is the shell when it bursts after 4.5 sec?

d) What is the shell's horizontal distance from its launching point when it explodes? (Round to the nearest foot.)

44) When a 10-in. shell is shot from a mortar at an angle of 75°, the height, y (in feet), of the shell t sec after being shot from the mortar is given by

$$y = -16t^2 + 264t$$

and the horizontal distance of the shell from the mortar, x (in feet), is given by

$$x = 71t.$$

a) How high is the shell after 3 sec?

b) Find the shell's horizontal distance from the mortar after 3 sec.

c) The shell explodes after 8.25 sec. What is its height when it bursts?

d) What is the shell's horizontal distance from its launching point when it explodes? (Round to the nearest foot.)

e) Compare your answers to 43 a) and 44 a). What is the difference in their heights after 3 sec?

f) Compare your answers to 43 c) and 44 c). What is the difference in the shells' heights when they burst?

g) Use the information from Exercises 43 and 44. Assuming that the technicians timed the firings of the 3-in. shell and the 10-in. shell so that they exploded at the same time, how far apart would their respective mortars need to be so that the 10-in. shell would burst directly above the 3-in. shell?

45) An object is launched upward with an initial velocity of 96 ft/sec. The height h (in feet) of the object after t seconds is given by $h(t) = -16t^2 + 96t$.

a) From what height is the object launched?

b) When does the object reach a height of 128 ft?

c) How high is the object after 3 sec?

d) When does the object hit the ground?

46) An object is launched upward with an initial velocity of 128 ft/sec. The height h (in feet) of the object after t seconds is given by $h(t) = -16t^2 + 128t$.

a) From what height is the object launched?

b) Find the height of the object after 2 sec.

c) When does the object hit the ground?

47) The equation $R(p) = -9p^2 + 324p$ describes the relationship between the price of a ticket, p, in dollars, and the revenue, R, in dollars, from ticket sales at a music club. That is, the revenue is a function of price.

a) Determine the club's revenue from ticket sales if the price of a ticket is \$15.

b) Determine the club's revenue from ticket sales if the price of a ticket is \$20.

c) If the club is expecting its revenue from ticket sales to be \$2916, how much should it charge for each ticket?

48) The equation $R(p) = -7p^2 + 700p$ describes the revenue, R, from ticket sales, in dollars, as a function of the price, p, in dollars, of a ticket to a fund-raising dinner. That is, the revenue is a function of price.

a) Determine the revenue if ticket price is \$40.

b) Determine the revenue if the ticket price is \$70.

c) If the goal of the organizers is to have ticket revenue of \$17,500, how much should it charge for each ticket?

Chapter 7: Summary

Definition/Procedure	Example

7.1 The Greatest Common Factor and Factoring by Grouping

To **factor a polynomial** is to write it as a product of two or more polynomials.

To factor out a greatest common factor (GCF):

1) Identify the GCF of all of the terms of the polynomial.
2) Rewrite each term as the product of the GCF and another factor.
3) Use the distributive property to factor out the GCF from the terms of the polynomial.
4) Check the answer by multiplying the factors. **(p. 396)**

Factor out the greatest common factor.

$$6k^6 - 27k^5 + 15k^4$$

The GCF is $3k^4$.

$$\begin{aligned} 6k^6 - 27k^5 + 15k^4 &= (3k^4)(2k^2) - (3k^4)(9k) + (3k^4)(5) \\ &= 3k^4(2k^2 - 9k + 5) \end{aligned}$$

Check: $3k^4(2k^2 - 9k + 5) = 6k^6 - 27k^5 + 15k^4$ ✓

The first step in factoring any polynomial is to ask yourself, "*Can I factor out a GCF?*"

The last step in factoring any polynomial is to ask yourself, "*Can I factor again?*"

Try to **factor by grouping** when you are asked to factor a polynomial containing four terms.

1) Make two groups of two terms so that each group has a common factor.
2) Take out the common factor from each group of terms.
3) Factor out the common factor using the distributive property.
4) Check the answer by multiplying the factors. **(p. 400)**

Factor completely. $10xy + 5y - 8x - 4$

Since the four terms have a GCF of 1, we will not factor out a GCF. Begin by grouping two terms together so that each group has a common factor.

$$\underbrace{10xy + 5y}_{\downarrow} \; - \; \underbrace{8x - 4}_{\downarrow}$$

$$= 5y(2x + 1) - 4(2x + 1) \quad \text{Take out the common factor.}$$
$$= (2x + 1)(5y - 4) \quad \text{Factor out } (2x + 1).$$

Check: $(2x + 1)(5y - 4) = 10xy + 5y - 8x - 4$ ✓

7.2 Factoring Trinomials of the Form $x^2 + bx + c$

Factoring $x^2 + bx + c$

If $x^2 + bx + c = (x + m)(x + n)$, then

1) if b and c are positive, then both m and n must be positive.
2) if c is positive and b is negative, then both m and n must be negative.
3) if c is negative, then one integer, m, must be positive and the other integer, n, must be negative. **(p. 404)**

Factor completely.

a) $t^2 + 9t + 20$

Think of two numbers whose **product** is 20 and whose **sum** is 9. **4 and 5** Then,

$$t^2 + 9t + 20 = (t + 4)(t + 5)$$

b) $3s^3 - 33s^2 + 54s$

Begin by factoring out the GCF of $3s$.

$$3s^3 - 33s^2 + 54s = 3s(s^2 - 11s + 18) = 3s(s - 2)(s - 9)$$

7.3 Factoring Trinomials of the Form $ax^2 + bx + c$ ($a \neq 1$)

Factoring $ax^2 + bx + c$ by **grouping. (p. 410)**

Factor completely. $5t^2 + 18t - 8$

$$\overset{\text{Sum is 18.}}{\downarrow}$$
$$\underbrace{5t^2 + 18t - 8}_{\text{Product: } 5 \cdot (-8) = -40}$$

Think of two integers whose **product** is -40 and whose **sum** is 18. **20 and -2**

Definition/Procedure	Example
	Factor by grouping. $5t^2 + 18t - 8 = \underbrace{5t^2 + 20t}_{\text{Group}} \underbrace{- 2t - 8}_{\text{Group}}$ Write $18t$ as $20t - 2t$. $= 5t(t + 4) - 2(t + 4)$ $= (t + 4)(5t - 2)$
Factoring $ax^2 + bx + c$ by **trial and error.** When approaching a problem in this way, we must keep in mind that we are reversing the FOIL process. **(p. 411)**	Factor completely. $4x^2 - 16x + 15$ $4x^2 - 16x + 15 = (2x - 3)(2x - 5)$ $4x^2$, $15x$ $-6x$ $+ -10x$ $-16x$ $4x^2 - 16x + 15 = (2x - 3)(2x - 5)$

7.4 Factoring Special Trinomials and Binomials

A **perfect square trinomial** is a trinomial that results from squaring a binomial. **Factoring a Perfect Square Trinomial** $a^2 + 2ab + b^2 = (a + b)^2$ $a^2 - 2ab + b^2 = (a - b)^2$ **(p. 417)**	Factor completely. a) $g^2 + 22g + 121 = (g + 11)^2$ $a = g \quad b = 11$ b) $16d^2 - 8d + 1 = (4d - 1)^2$ $a = 4d \quad b = 1$
Factoring the Difference of Two Squares $a^2 - b^2 = (a + b)(a - b)$ **(p. 419)**	Factor completely. $w^2 - 64 = (w + 8)(w - 8)$ $\downarrow \quad \downarrow$ $(w)^2 \quad (8)^2 \quad a = w, b = 8$
Factoring the Sum and Difference of Two Cubes $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ **(p. 422)**	Factor completely. $r^3 + 64 = (r + 4)[(r)^2 - (r)(4) + (4)^2]$ $\downarrow \quad \downarrow$ $(r)^3 \quad (4)^3 \quad a = r, b = 4$ $r^3 + 64 = (r + 4)(r^2 - 4r + 16)$

7.5 Solving Quadratic Equations by Factoring

A **quadratic equation** can be written in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. **(p. 429)**	Some examples of quadratic equations are $5x^2 + 9 = 0, \quad y^2 = 4y + 21, \quad 4(p - 2)^2 = 8 - 7p$
To solve a quadratic equation by factoring, use the **zero product rule:** If $ab = 0$, then $a = 0$ or $b = 0$. **(p. 429)**	Solve $(y + 7)(y - 4) = 0$ $\swarrow \quad \searrow$ $y + 7 = 0$ or $y - 4 = 0$ Set each factor equal to zero. $y = -7$ or $y = 4$ Solve. The solution set is $\{-7, 4\}$.

Definition/Procedure	Example

Steps for **Solving a Quadratic Equation by Factoring**

1) Write the equation in the form $ax^2 + bx + c = 0$.
2) Factor the expression.
3) Set each factor equal to zero, and solve for the variable.
4) Check the answer(s). **(p. 431)**

Solve $5p^2 - 11 = 3p^2 + 3p - 9$.

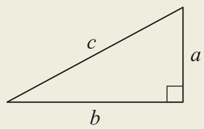
$$
\begin{aligned}
2p^2 - 3p - 2 &= 0 \quad \text{Standard form} \\
(2p + 1)(p - 2) &= 0 \quad \text{Factor.}
\end{aligned}
$$

$$
\begin{aligned}
2p + 1 = 0 \quad &\text{or} \quad p - 2 = 0 \\
2p = -1 \quad & \\
p = -\frac{1}{2} \quad &\text{or} \quad p = 2
\end{aligned}
$$

The solution set is $\left\{-\frac{1}{2}, 2\right\}$. Check the answers.

7.6 Applications of Quadratic Equations

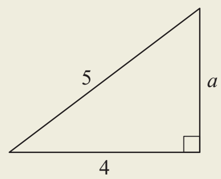
Pythagorean Theorem
Given a right triangle with legs of length a and b and hypotenuse of length c,



the Pythagorean theorem states that

$$a^2 + b^2 = c^2. \text{ (p. 439)}$$

Find the length of side a.



Let $b = 4$ and $c = 5$ in $\boldsymbol{a^2 + b^2 = c^2}$.

$$
\begin{aligned}
a^2 + (4)^2 &= (5)^2 \\
a^2 + 16 &= 25 \\
a^2 - 9 &= 0 \\
(a + 3)(a - 3) &= 0
\end{aligned}
$$

$$
\begin{aligned}
a + 3 = 0 \quad &\text{or} \quad a - 3 = 0 \\
a = 3 \quad &\text{or} \quad a = -3
\end{aligned}
$$

Reject -3 as a solution since the length of a side cannot be negative.
Therefore, $a = 3$.

Chapter 7: Review Exercises

(7.1) Find the greatest common factor of each group of terms.

1) 40, 56
2) $36y, 12y^2, 54y^2$
3) $15h^4, 45h^5, 20h^3$
4) $4c^4d^3, 20c^4d^2, 28c^2d$

Factor out the greatest common factor.

5) $63t + 45$
6) $21w^5 - 56w$
7) $2p^6 - 20p^5 + 2p^4$
8) $18a^3b^3 - 3a^2b^3 - 24ab^3$
9) $n(m + 8) - 5(m + 8)$
10) $x(9y - 4) + w(9y - 4)$
11) Factor out $-5r$ from $-15r^3 - 40r^2 + 5r$.
12) Factor out -1 from $-z^2 + 9z - 4$.

Factor by grouping.

13) $ab + 2a + 9b + 18$
14) $cd - 3c + 8d - 24$
15) $4xy - 28y - 3x + 21$
16) $hk^2 + 6h - k^2 - 6$

(7.2) Factor completely.

17) $q^2 + 10q + 24$
18) $t^2 - 12t + 27$
19) $z^2 - 6z - 72$
20) $h^2 + 6h - 7$
21) $m^2 - 13mn + 30n^2$
22) $a^2 + 11ab + 30b^2$
23) $4v^2 - 24v - 64$
24) $7c^2 - 7c - 84$
25) $9w^4 + 9w^3 - 18w^2$
26) $5x^3y - 25x^2y^2 + 20xy^3$

(7.3) Factor completely.

27) $3r^2 - 23r + 14$
28) $5k^2 + 11k + 6$
29) $4p^2 - 8p - 5$
30) $8d^2 + 29d - 12$
31) $12c^2 + 38c + 20$
32) $21n^2 - 54n + 24$
33) $10x^2 + 39xy - 27y^2$
34) $6g^2(h - 8)^2 + 23g(h - 8)^2 + 20(h - 8)^2$

(7.4) Factor completely.

35) $w^2 - 49$
36) $121 - p^2$
37) $64t^2 - 25u^2$
38) $y^4 - 81$
39) $4b^2 + 9$
40) $12c^2 - 48d^2$
41) $64x - 4x^3$
42) $\dfrac{25}{9} - h^2$
43) $r^2 + 12r + 36$
44) $9z^2 - 24z + 16$
45) $20k^2 - 60k + 45$
46) $25a^2 + 20ab + 4b^2$
47) $v^3 - 27$
48) $8m^3 + 125$
49) $125x^3 + 64y^3$
50) $81p^4 - 3pq^3$

(7.1–7.4) Mixed Exercises
Factor completely.

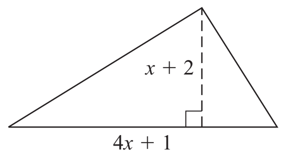
51) $10z^2 - 7z - 12$
52) $4c^2 + 24c + 36$
53) $9k^4 - 16k^2$
54) $14m^5 + 63m^4 + 21m^3$
55) $d^2 - 17d + 60$
56) $\dfrac{4}{25}t^2 - \dfrac{1}{9}u^2$
57) $3a^2b + a^2 - 12b - 4$
58) $h^2 + 100$
59) $48p^3 - 6q^3$
60) $t^4 + t$
61) $(x + 4)^2 - (y - 5)^2$
62) $8mn - 8m + 56n - 56$
63) $25c^2 - 20c + 4$
64) $12v^2 + 32v + 5$

(7.5) Solve each equation.

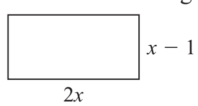
65) $y(3y + 7) = 0$
66) $(2n - 3)^2 = 0$
67) $2k^2 + 18 = 13k$
68) $3t^2 - 75 = 0$
69) $h^2 + 17h + 72 = 0$
70) $21 = 8p^2 - 2p$
71) $121 = 81r^2$
72) $12c = -c^2$
73) $3m^2 - 120 = 18m$
74) $x(16 - x) = 63$
75) $(w + 3)(w + 8) = -6$
76) $(2a + 3)^2 + 3a = a(a - 10) + 1$
77) $(5z + 4)(3z^2 - 7z + 4) = 0$
78) $18 = 9b^2 + 9b$
79) $3v + (v - 3)^2 = 5(v^2 - 4v + 1) + 8$
80) $6d^3 + 45d = 33d^2$
81) $45p^3 - 20p = 0$
82) $(r + 6)^2 - (4r - 1)^2 = 0$

(7.6)

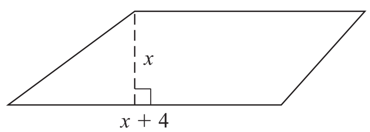
83) Find the base and height if the area of the triangle is 18 cm^2.



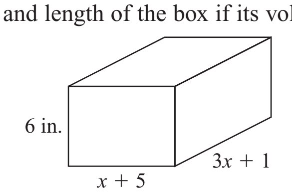
84) Find the length and width of the rectangle if its area is 60 in^2.



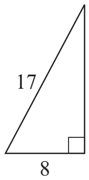
85) Find the base and height of the parallelogram if its area is 12 ft^2.



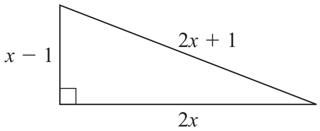
86) Find the height and length of the box if its volume is 480 in^3.



87) Use the Pythagorean theorem to find the length of the missing side.



88) Find the length of the hypotenuse.



Write an equation and solve.

89) A rectangular mirror has an area of 10 ft^2, and it is 1.5 ft longer than it is wide. Find the dimensions of the mirror.

90) The base of a triangular banner is 1 ft less than its height. If the area is 3 ft^2, find the base and height.

91) The sum of three consecutive integers is one less than the square of the smallest number. Find the integers.

92) The product of two consecutive odd integers is 18 more than the square of the smaller number. Find the integers.

93) Desmond and Marcus leave an intersection with Desmond jogging north and Marcus jogging west. When Marcus is 1 mile farther from the intersection than Desmond, the distance between them is 2 miles more than Desmond's distance from the intersection. How far is Desmond from the intersection?

94) An object is thrown upward with an initial velocity of 68 ft/sec. The height h (in feet) of the object t seconds after it is thrown is given by

$$h = -16t^2 + 68t + 60.$$

a) How long does it take for the object to reach a height of 120 ft?

b) What is the initial height of the object?

c) What is the height of the object after 2 seconds?

d) How long does it take the object to hit the ground?

Chapter 7: Test

1) What is the first thing you should do when you are asked to factor a polynomial?

Factor completely.

2) $h^2 - 14h + 48$

3) $36 - v^2$

4) $7p^2 + 6p - 16$

5) $20a^3b^4 + 36a^2b^3 + 4ab^2$

6) $k^2 + 81$

7) $64t^3 - 27u^3$

8) $4z^3 + 28z^2 + 48z$

9) $36r^2 - 60r + 25$

10) $n^3 + 7n^2 - 4n - 28$

11) $x^2 - 3xy - 18y^2$

12) $81c^4 - d^4$

13) $p^2(q - 4)^2 + 17p(q - 4)^2 + 30(q - 4)^2$

14) $32w^2 + 28w - 30$

15) $k^8 + k^5$

Solve each equation.

16) $t^2 + 5t - 36 = 0$

17) $144r = r^3$

18) $49a^2 = 16$

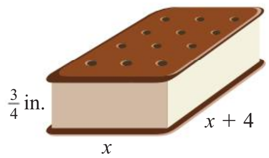
19) $(y - 7)(y - 5) = 3$

20) $x - 2x(x + 4) = (2x + 3)^2 - 1$

21) $20k^2 - 52k = 24$

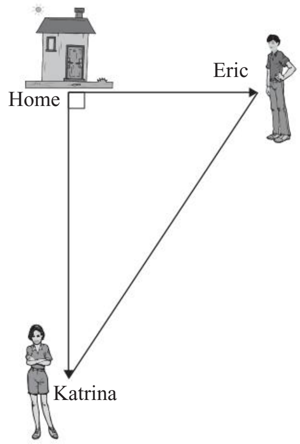
Write an equation and solve.

22) Find the length and width of a rectangular ice cream sandwich if its volume is 9 in^3.



23) Find three consecutive odd integers such that the sum of the three numbers is 110 less than the product of the larger two integers.

24) Eric and Katrina leave home to go to work. Eric drives due east while his wife drives due south. At 8:30 A.M., Katrina is 3 miles farther from home than Eric, and the distance between them is 6 miles more than Eric's distance from home. Find Eric's distance from his house.



25) A rectangular cheerleading mat has an area of 252 ft^2. It is 7 times longer than it is wide. Find the dimensions of the mat.

26) An object is launched upward from the ground with an initial velocity of 200 ft/sec. The height h (in feet) of the object after t seconds is given by

$$h = -16t^2 + 200t.$$

a) Find the height of the object after 1 second.

b) Find the height of the object after 4 seconds.

c) When is the object 400 feet above the ground?

d) How long does it take for the object to hit the ground?

Cumulative Review: Chapters 1–7

Perform the indicated operation(s) and simplify.

1) $\frac{2}{9} - \frac{5}{6} + \frac{1}{3}$

2) $\frac{35}{48} \cdot \frac{32}{63}$

Simplify. The answer should not contain any negative exponents.

3) $2(-3p^4q)^2$

4) $\frac{28a^8b^4}{40a^{-3}b^9}$

5) Write 0.0000839 in scientific notation.

6) Solve $\frac{3}{8}t + \frac{1}{2} = \frac{1}{4}(4t + 1) - \frac{3}{4}t$

7) Solve for b_2. $A = \frac{1}{2}h(b_1 + b_2)$

8) Solve $5 - \frac{2}{3}k \le 13$.

9) Graph $y = -\frac{1}{4}x + 2$.

10) Write the equation of the line parallel to $5x - 3y = 7$ containing the point $(-5, -7)$. Express the answer in slope-intercept form.

11) Use any method to solve this system of equations.

$$\begin{aligned} 4 + 2(1 - 3x) + 7y &= 3x - 5y \\ 10y + 9 &= 3(2x + 5) + 2y \end{aligned}$$

12) Write a system of equations and solve.
Tickets to a high school football game cost \$6.00 for an adult and \$2.00 for a student. The revenue from the sale of 465 tickets was \$1410. Find the number of adult tickets and number of student tickets sold.

Multiply and simplify.

13) $(4w - 7)(2w + 3)$

14) $(3n + 4)^2$

15) $(6z - 5)(2z^2 + 7z - 3)$

16) Add $(11v^2 - 16v + 4) + (2v^2 + 3v - 9)$.

Divide.

17) $\frac{16x^3 - 57x + 14}{4x - 7}$

18) $\frac{24m^2 - 8m + 4}{8m}$

Factor completely.

19) $6c^2 + 15c - 54$

20) $r^2 + 16$

21) $xy^2 + 4y^2 - x - 4$

22) $\frac{1}{4} - b^2$

23) $h^3 + 125$

Solve.

24) $12 = 13t - 3t^2$

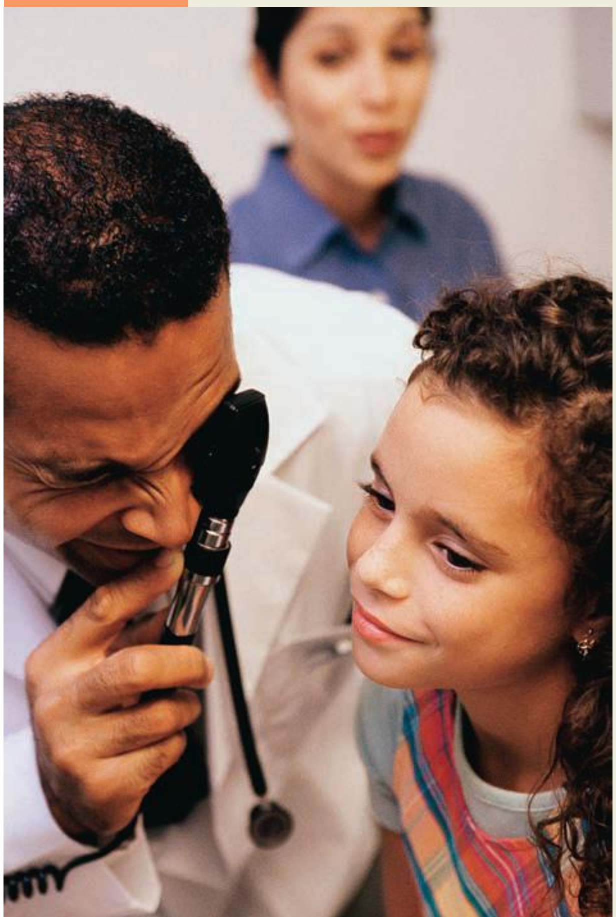
25) $24n^3 = 54n$

CHAPTER 8

Rational Expressions

Algebra at Work: Ophthalmology

At the beginning of Chapter 7, we saw how an ophthalmologist, a doctor specializing in diseases of the eye, uses mathematics every day to treat his patients. Here we will see another example of how math is used in this branch of medicine.



Some formulas in optics involve rational expressions. If Calvin determines that one of his patients needs glasses, he would use the following formula to figure out the proper prescription:

$$P = \frac{1}{f}$$

where f is the focal length, in meters, and P is the power of the lens, in diopters.

While computers now aid in these calculations, physicians believe that it is still important to double-check the calculations by hand.

In this chapter, we will learn how to perform operations with rational expressions and how to solve equations, like the one above, for a specific variable.

Section 8.1 Simplifying Rational Expressions

Objectives

1. Evaluate a Rational Expression
2. Find the Values of the Variable That Make a Rational Expression Undefined or Equal to Zero
3. Write a Rational Expression in Lowest Terms
4. Simplify a Rational Expression of the Form $\frac{a-b}{b-a}$
5. Write Equivalent Forms of a Rational Expression
6. Determine the Domain of a Rational Function

1. Evaluate a Rational Expression

In Section 1.4, we defined a **rational number** as the quotient of two integers where the denominator does not equal zero. Some examples of rational numbers are

$$\frac{7}{8}, \quad -\frac{2}{5}, \quad 18\left(\text{since } 18 = \frac{18}{1}\right)$$

We can define a rational expression in a similar way. A rational expression is a quotient of two polynomials provided that the denominator does not equal zero. We state the definition formally next.

Definition

A **rational expression** is an expression of the form $\frac{P}{Q}$, where P and Q are polynomials and where $Q \neq 0$.

Some examples of rational expressions are

$$\frac{4k^3}{7}, \quad \frac{2n-1}{n+6}, \quad \frac{5}{t^2-3t-28}, \quad -\frac{9x+2y}{x^2+y^2}$$

We can *evaluate* rational expressions for given values of the variables as long as the values do not make any denominators equal zero.

Example 1

Evaluate $\frac{x^2-9}{x+1}$ (if possible) for each value of x.

a) $x = 7$ b) $x = 3$ c) $x = -1$

Solution

a) $\frac{x^2-9}{x+1} = \frac{(7)^2-9}{(7)+1}$ Substitute 7 for x.

$$= \frac{49-9}{7+1} = \frac{40}{8} = 5$$

b) $\frac{x^2-9}{x+1} = \frac{(3)^2-9}{(3)+1}$ Substitute 3 for x.

$$= \frac{9-9}{3+1} = \frac{0}{4} = 0$$

c) $\frac{x^2-9}{x+1} = \frac{(-1)^2-9}{(-1)+1}$ Substitute -1 for x.

$$= \frac{1-9}{0} = \frac{-8}{0}$$

Undefined

Remember, a fraction is **undefined** when its denominator equals zero. Therefore, we say that $\frac{x^2-9}{x+1}$ is *undefined* when $x = -1$ since this value of x makes the denominator equal zero. So, x *cannot equal* -1 in this expression. ■

You Try 1

Evaluate $\dfrac{k^2 - 1}{k + 11}$ (if possible) for each value of k.

a) $k = 9$ b) $k = -2$ c) $k = -11$ d) $k = 1$

2. Find the Values of the Variable That Make a Rational Expression Undefined or Equal to Zero

Parts b) and c) in Example 1 remind us about two important aspects of fractions and rational expressions.

Note

1) A fraction (rational expression) *equals zero* when its numerator equals zero.

2) A fraction (rational expression) is *undefined* when its denominator equals zero.

Example 2

For each rational expression, for what values of the variable

i) does the expression equal zero?

ii) is the expression undefined?

a) $\dfrac{m + 8}{m - 3}$ b) $\dfrac{2z}{z^2 - 5z - 36}$ c) $\dfrac{9c^2 - 49}{6}$ d) $\dfrac{4}{2w + 1}$

Solution

a) i) $\dfrac{m + 8}{m - 3} = 0$ when its *numerator* equals zero. Set the numerator equal to zero, and solve for m.

$$m + 8 = 0$$
$$m = -8$$

Therefore, $\dfrac{m + 8}{m - 3} = 0$ when $m = -8$.

ii) $\dfrac{m + 8}{m - 3}$ is *undefined* when its *denominator* equals zero. Set the denominator equal to zero, and solve for m.

$$m - 3 = 0$$
$$m = 3$$

$\dfrac{m + 8}{m - 3}$ is *undefined* when $m = 3$. This means that any real number *except* 3 can be substituted for m in this expression.

b) i) $\dfrac{2z}{z^2 - 5z - 36} = 0$ when its *numerator* equals zero. Set the numerator equal to zero, and solve for z.

$$2z = 0$$
$$z = \frac{0}{2} = 0$$

So, $\dfrac{2z}{z^2 - 5z - 36} = 0$ when $z = 0$.

ii) $\dfrac{2z}{z^2 - 5z - 36}$ is *undefined* when its *denominator* equals zero. Set the denominator equal to zero, and solve for z.

$$\begin{aligned} z^2 - 5z - 36 &= 0 \\ (z+4)(z-9) &= 0 && \text{Factor.} \\ z + 4 = 0 \quad &\text{or} \quad z - 9 = 0 && \text{Set each factor equal to 0.} \\ z = -4 \quad &\text{or} \quad z = 9 && \text{Solve.} \end{aligned}$$

$\dfrac{2z}{z^2 - 5z - 36}$ is undefined when $z = -4$ or $z = 9$. All real numbers *except* -4 and 9 can be substituted for z in this expression.

c) i) To determine the values of c that make $\dfrac{9c^2 - 49}{6} = 0$, set $9c^2 - 49 = 0$ and solve.

$$\begin{aligned} 9c^2 - 49 &= 0 \\ (3c+7)(3c-7) &= 0 && \text{Factor.} \\ 3c + 7 = 0 \quad &\text{or} \quad 3c - 7 = 0 \\ 3c = -7 \quad & \quad\quad 3c = 7 && \text{Set each factor equal to 0.} \\ c = -\frac{7}{3} \quad &\text{or} \quad c = \frac{7}{3} && \text{Solve.} \end{aligned}$$

So, $\dfrac{9c^2 - 49}{6} = 0$ when $c = -\dfrac{7}{3}$ or $c = \dfrac{7}{3}$.

ii) $\dfrac{9c^2 - 49}{6}$ is *undefined* when the denominator equals zero. However, the denominator is 6 and $6 \neq 0$. Therefore, there *is no value of* c that makes $\dfrac{9c^2 - 49}{6}$ undefined. *We say that* $\dfrac{9c^2 - 49}{6}$ *is defined for all real numbers.*

d) i) $\dfrac{4}{2w + 1} = 0$ when the numerator equals zero. The numerator is 4, and $4 \neq 0$. Therefore, $\dfrac{4}{2w + 1}$ will *never* equal zero.

ii) $\dfrac{4}{2w + 1}$ is *undefined* when $2w + 1 = 0$. Solve for w.

$$\begin{aligned} 2w + 1 &= 0 \\ 2w &= -1 \\ w &= -\frac{1}{2} \end{aligned}$$

So, $\dfrac{4}{2w + 1}$ is undefined when $w = -\dfrac{1}{2}$. So, $w \neq -\dfrac{1}{2}$ in the expression.

You Try 2

For each rational expression, determine which values of the variable

i) make the expression equal zero.

ii) make the expression undefined.

a) $\dfrac{v - 6}{v + 11}$ b) $\dfrac{9w}{w^2 - 12w + 20}$ c) $\dfrac{x^2 - 25}{8}$ d) $\dfrac{1}{5q + 4}$

All of the operations that can be performed with fractions can also be done with rational expressions. We begin our study of these operations with rational expressions by learning how to write a rational expression in lowest terms.

3. Write a Rational Expression in Lowest Terms

One way to think about writing a fraction such as $\frac{8}{12}$ in lowest terms is

$$\frac{8}{12} = \frac{2 \cdot 4}{3 \cdot 4} = \frac{2}{3} \cdot \frac{4}{4} = \frac{2}{3} \cdot 1 = \frac{2}{3}$$

Since $\frac{4}{4} = 1$, we can also think of reducing $\frac{8}{12}$ as $\frac{8}{12} = \frac{2 \cdot \cancel{4}}{3 \cdot \cancel{4}} = \frac{2}{3}$.

To write $\frac{8}{12}$ in lowest terms we can *factor* the numerator and denominator, then *divide* the numerator and denominator by the common factor, 4. This is the approach we use to write a rational expression in lowest terms.

Definition Fundamental Property of Rational Expressions

If P, Q, and C are polynomials such that $Q \neq 0$ and $C \neq 0$, then

$$\frac{PC}{QC} = \frac{P}{Q}$$

This property mirrors the example above since

$$\frac{PC}{QC} = \frac{P}{Q} \cdot \frac{C}{C} = \frac{P}{Q} \cdot 1 = \frac{P}{Q}$$

Or, we can also think of the reducing procedure as dividing the numerator and denominator by the common factor, C.

$$\frac{P\cancel{C}}{Q\cancel{C}} = \frac{P}{Q}$$

Procedure Writing a Rational Expression in Lowest Terms

1) Completely **factor** the numerator and denominator.
2) **Divide** the numerator and denominator by the greatest common factor.

Example 3

Write each rational expression in lowest terms.

a) $\frac{21r^{10}}{3r^4}$ b) $\frac{8a + 40}{3a + 15}$ c) $\frac{5n^2 - 20}{n^2 + 5n + 6}$

Solution

a) We can simplify $\frac{21r^{10}}{3r^4}$ using the quotient rule presented in Chapter 2.

$$\frac{21r^{10}}{3r^4} = 7r^6$$

Divide 21 by 3 and use the quotient rule: $\frac{r^{10}}{r^4} = r^{10-4} = r^6$.

b) $\dfrac{8a + 40}{3a + 15} = \dfrac{8\cancel{(a + 5)}}{3\cancel{(a + 5)}}$ Factor.

$= \dfrac{8}{3}$ Divide out the common factor, $a + 5$.

c) $\dfrac{5n^2 - 20}{n^2 + 5n + 6} = \dfrac{5(n^2 - 4)}{(n + 2)(n + 3)}$ Factor.

$= \dfrac{5\cancel{(n + 2)}(n - 2)}{\cancel{(n + 2)}(n + 3)}$ Factor completely.

$= \dfrac{5(n - 2)}{n + 3}$ Divide out the common factor, $n + 2$. ■

BE CAREFUL

Notice that we divided by *factors* not *terms*.

$\dfrac{\cancel{x + 5}}{2\cancel{(x + 5)}} = \dfrac{1}{2}$

Divide by the *factor* $x + 5$.

$\dfrac{\cancel{x}}{\cancel{x} + 5} \neq \dfrac{1}{5}$

We cannot divide by x because the x in the denominator is a *term* in a sum.

You Try 3

Write each rational expression in lowest terms.

a) $\dfrac{12b^8}{18b^3}$ b) $\dfrac{2h + 8}{7h + 28}$ c) $\dfrac{y^3 - 8}{9y^4 - 9y^3 - 18y^2}$

4. Simplify a Rational Expression of the Form $\dfrac{a - b}{b - a}$

Do you think that $\dfrac{x - 4}{4 - x}$ is in lowest terms? Let's look at it more closely to understand the answer.

$\dfrac{x - 4}{4 - x} = \dfrac{x - 4}{-1(-4 + x)}$ Factor -1 out of the denominator.

$= \dfrac{1\cancel{(x - 4)}}{-1\cancel{(x - 4)}}$ Rewrite $-4 + x$ as $x - 4$.

$= -1$ Divide out the common factor, $x - 4$.

$\dfrac{x - 4}{4 - x} = -1$

We can generalize this result as

Note

1) $b - a = -1(a - b)$ and 2) $\dfrac{a - b}{b - a} = -1$

The terms in the numerator and denominator in 2) differ only in sign. The rational expression simplifies to -1.

Example 4

Write each rational expression in lowest terms.

a) $\dfrac{7-t}{t-7}$ b) $\dfrac{4h^2-25}{5-2h}$ c) $\dfrac{2x^3-12x^2+18x}{9-x^2}$

Solution

a) $\dfrac{7-t}{t-7} = -1$ since $\dfrac{7-t}{t-7} = \dfrac{-1\cancel{(t-7)}}{\cancel{(t-7)}} = -1.$

b) $\dfrac{4h^2-25}{5-2h} = \dfrac{(2h+5)\overset{-1}{\cancel{(2h-5)}}}{\cancel{5-2h}}$ Factor. $\dfrac{2h-5}{5-2h} = -1$

$= -1(2h+5)$

$= -2h-5$ Distribute.

c) $\dfrac{2x^3-12x^2+18x}{9-x^2} = \dfrac{2x(x^2-6x+9)}{(3+x)(3-x)}$ Factor $2x$ out of the numerator.

$= \dfrac{2x(x-3)^2}{(3+x)(3-x)}$ Factor the trinomial.

$= \dfrac{2x\overset{-1}{\cancel{(x-3)}}(x-3)}{(3+x)\cancel{(3-x)}}$ $\dfrac{x-3}{3-x} = -1$

$= -\dfrac{2x(x-3)}{3+x}$ The negative sign can be written in front of the fraction. ■

You Try 4

Write each rational expression in lowest terms.

a) $\dfrac{10-m}{m-10}$ b) $\dfrac{8a-4b}{6b-12a}$ c) $\dfrac{100-4k^2}{k^2-8k+15}$

5. Write Equivalent Forms of a Rational Expression

The answer to Example 4c) can be written in several different ways. You should be able to recognize equivalent forms of rational expressions because there isn't always just one way to write the correct answer.

Example 5

Write the answer to Example 4c), $-\dfrac{2x(x-3)}{3+x}$, in three different ways.

Solution

The negative sign in front of a fraction can be applied to the numerator or to the denominator. $\left(\text{For example, } -\dfrac{4}{9} = \dfrac{-4}{9} = \dfrac{4}{-9}.\right)$ Applying this concept to rational expressions can result in expressions that look quite different but that are, actually, equivalent.

i) Apply the negative sign to the denominator.

$$-\frac{2x(x-3)}{3+x} = \frac{2x(x-3)}{-1(3+x)}$$

$$= \frac{2x(x-3)}{-3-x} \qquad \text{Distribute.}$$

ii) Apply the negative sign to the numerator.

$$-\frac{2x(x-3)}{3+x} = \frac{-2x(x-3)}{3+x}$$

iii) Apply the negative sign to the numerator, but distribute the -1.

$$-\frac{2x(x-3)}{3+x} = \frac{(2x)(-1)(x-3)}{x+3}$$

$$= \frac{2x(-x+3)}{3+x} \qquad \text{Distribute.}$$

$$= \frac{2x(3-x)}{3+x} \qquad \text{Rewrite } -x+3 \text{ as } 3-x.$$

Therefore, $\frac{2x(x-3)}{-3-x}$, $\frac{-2x(x-3)}{3+x}$, and $\frac{2x(3-x)}{3+x}$ are *all* equivalent forms of $-\frac{2x(x-3)}{3+x}$.

Keep this idea of equivalent forms of rational expressions in mind when checking your answers against the answers in the back of the book. Sometimes students believe their answer is wrong because it "looks different" when, in fact, it is an *equivalent form* of the given answer!

You Try 5

Find three equivalent forms of $\frac{-(1-t)}{5t-8}$.

6. Determine the Domain of a Rational Function

We can combine what we have learned about rational expressions with what we learned about functions in Section 4.6. $f(x) = \frac{x+3}{x-8}$ is an example of a **rational function** since $\frac{x+3}{x-8}$ is a rational expression and since each value that can be substituted for x will produce *only one* value for the expression.

Recall from Chapter 4 that the domain of a function $f(x)$ is the set of all real numbers that can be substituted for x. Since a rational expression is undefined when its denominator equals zero, we define the domain of a rational function as follows.

Definition

The **domain of a rational function** consists of all real numbers except the values of the variable that make the denominator equal zero.

Therefore, to determine the domain of a rational function we set the denominator equal to zero and solve for the variable. Any value that makes the denominator equal to zero is *not* in the domain of the function.

To determine the domain of a rational function, sometimes it is helpful to ask yourself, "Is there any number that *cannot* be substituted for the variable?"

Example 6

Determine the domain of each rational function.

a) $f(x) = \dfrac{x + 3}{x - 8}$ b) $g(c) = \dfrac{6}{c^2 + 3c - 4}$ c) $h(n) = \dfrac{4n^2 - 9}{7}$

Solution

a) To determine the domain of $f(x) = \dfrac{x + 3}{x - 8}$, ask yourself, "Is there any number that *cannot* be substituted for x?" Yes. $f(x)$ is *undefined* when the denominator equals zero. Set the denominator equal to zero, and solve for x.

$$x - 8 = 0 \quad \text{Set the denominator} = 0.$$
$$x = 8 \quad \text{Solve.}$$

When $x = 8$, the denominator of $f(x) = \dfrac{x + 3}{x - 8}$ equals zero. The domain contains all real numbers *except* 8. Write the domain in interval notation as $(-\infty, 8) \cup (8, \infty)$.

b) To determine the domain of $g(c) = \dfrac{6}{c^2 + 3c - 4}$, ask yourself, "Is there any number that *cannot* be substituted for c?" Yes. $g(c)$ is *undefined* when its *denominator* equals zero. Set the denominator equal to zero and solve for c.

$$c^2 + 3c - 4 = 0 \quad \text{Set the denominator} = 0.$$
$$(c + 4)(c - 1) = 0 \quad \text{Factor.}$$
$$c + 4 = 0 \quad \text{or} \quad c - 1 = 0 \quad \text{Set each factor equal to 0.}$$
$$c = -4 \quad \text{or} \quad c = 1 \quad \text{Solve.}$$

When $c = -4$ or $c = 1$, the denominator of $g(c) = \dfrac{6}{c^2 + 3c - 4}$ equals zero. The domain contains all real numbers *except* -4 and 1. Write the domain in interval notation as $(-\infty, -4) \cup (-4, 1) \cup (1, \infty)$.

c) Ask yourself, "Is there any number that *cannot* be substituted for n?" No! Looking at the denominator, we see that the number 7 will never equal zero. Therefore, there is *no value of n* that makes $h(n) = \dfrac{4n^2 - 9}{7}$ undefined. Any real number may be substituted for n and the function will be defined.

The domain of the function is the set of all real numbers. We can write the domain in interval notation as $(-\infty, \infty)$. ■

You Try 6

Determine the domain of each rational function.

a) $h(t) = \dfrac{9}{t + 5}$ b) $f(x) = \dfrac{2x - 3}{x^2 - 8x + 12}$ c) $g(a) = \dfrac{a + 4}{10}$

Answers to You Try Exercises

1) a) 4 b) $\frac{1}{3}$ c) undefined d) 0 2) a) i) 6 ii) -11 b) i) 0 ii) 2, 10 c) i) -5, 5 ii) defined for all real numbers d) i) never equals zero ii) $-\frac{4}{5}$

3) a) $\frac{2b^5}{3}$ b) $\frac{2}{7}$ c) $\frac{y^2+2y+4}{9y^2(y+1)}$ 4) a) -1 b) $-\frac{2}{3}$ c) $-\frac{4(5+k)}{k-3}$

5) Some possibilities are $\frac{t-1}{5t-8}, -\frac{1-t}{5t-8}, \frac{1-t}{8-5t}$. 6) a) $(-\infty,-5)\cup(-5,\infty)$ b) $(-\infty,2)\cup(2,6)\cup(6,\infty)$ c) $(-\infty,\infty)$

8.1 Exercises

Objective 1: Evaluate a Rational Expression

1) When is a fraction or a rational expression undefined?

2) When does a fraction or a rational expression equal 0?

Evaluate (if possible) for a) $x = 3$ and b) $x = -2$.

3) $\frac{2x-1}{5x+2}$

4) $\frac{3(x^2+1)}{x^2+2x+1}$

Evaluate (if possible) for a) $z = 1$ and b) $z = -3$.

5) $\frac{(4z)^2}{z^2-z-12}$

6) $\frac{3(z^2-9)}{z^2+8}$

7) $\frac{15+5z}{16-z^2}$

8) $\frac{4z-3}{z^2+6z-7}$

Objective 2: Find the Values of the Variable That Make a Rational Expression Undefined or Equal to Zero

9) How do you determine the values of the variable for which a rational expression is undefined?

10) If $x^2 + 9$ is the numerator of a rational expression, can that expression equal zero? Give a reason.

Determine the value(s) of the variable for which

a) the expression equals zero.

b) the expression is undefined.

11) $\frac{m+4}{3m}$

12) $\frac{-y}{y+3}$

13) $\frac{2w-7}{4w+1}$

14) $\frac{3x+13}{2x+13}$

15) $\frac{11v-v^2}{5v-9}$

16) $-\frac{r+5}{r^2-100}$

17) $\frac{8}{p}$

18) $\frac{22}{m-1}$

19) $-\frac{7k}{k^2+9k+20}$

20) $\frac{a-9}{a^2+8a-9}$

21) $\frac{c+20}{2c^2+3c-9}$

22) $\frac{4}{3f^2-13f+10}$

23) $\frac{g^2+9g+18}{9g}$

24) $\frac{6m-11}{10}$

VIDEO 25) $\frac{4y}{y^2+9}$

26) $\frac{q^2+49}{7}$

Objective 3: Write a Rational Expression in Lowest Terms

Write each rational expression in lowest terms.

27) $\frac{7x(x-11)}{3(x-11)}$

28) $\frac{24(g+3)}{6(g+3)(g-5)}$

29) $\frac{24g^2}{56g^4}$

30) $\frac{99d^7}{9d^3}$

31) $\frac{4d-20}{5d-25}$

32) $\frac{12c-3}{8c-2}$

33) $\frac{-14h-56}{6h+24}$

34) $\frac{-15v^2+12}{40v^2-32}$

35) $\frac{39u^2+26}{30u^2+20}$

36) $\frac{3q+15}{-7q-35}$

37) $\frac{g^2-g-56}{g+7}$

38) $\frac{b^2+9b+20}{b^2+b-12}$

39) $\frac{t-5}{t^2-25}$

40) $\frac{r+9}{r^2+7r-18}$

41) $\frac{3c^2+28c+32}{c^2+10c+16}$

42) $\frac{3k^2-36k+96}{k-8}$

43) $\frac{q^2-25}{2q^2-7q-15}$

44) $\frac{6p^2+11p-10}{9p^2-4}$

45) $\dfrac{w^3 + 125}{5w^2 - 25w + 125}$ 46) $\dfrac{6w^3 - 48}{w^2 + 2w + 4}$

47) $\dfrac{9c^2 - 27c + 81}{c^3 + 27}$ 48) $\dfrac{4x + 20}{x^3 + 125}$

49) $\dfrac{4u^2 - 20u + 4uv - 20v}{13u + 13v}$ 50) $\dfrac{ab + 3a - 6b - 18}{b^2 - 9}$

51) $\dfrac{m^2 - n^2}{m^3 - n^3}$ 52) $\dfrac{a^3 + b^3}{a^2 - b^2}$

Objective 4: Simplify a Rational Expression of the Form $\dfrac{a - b}{b - a}$

53) Any rational expression of the form $\dfrac{a - b}{b - a}$ $(a \neq b)$ reduces to what?

54) Does $\dfrac{h + 4}{h - 4} = -1$?

Write each rational expression in lowest terms.

55) $\dfrac{8 - q}{q - 8}$ 56) $\dfrac{m - 15}{15 - m}$

57) $\dfrac{m^2 - 121}{11 - m}$ 58) $\dfrac{k - 9}{162 - 2k^2}$

59) $\dfrac{36 - 42x}{7x^2 + 8x - 12}$ 60) $\dfrac{a^2 - 6a - 27}{9 - a}$

61) $\dfrac{16 - 4b^2}{b - 2}$ 62) $\dfrac{45 - 9v}{v^2 - 25}$

63) $\dfrac{y^3 - 3y^2 + 2y - 6}{21 - 7y}$ 64) $\dfrac{8t^3 - 27}{9 - 4t^2}$

Mixed Exercises: Objectives 1–4

Write each rational expression in lowest terms.

65) $\dfrac{18c + 45}{12c^2 + 18c - 30}$ 66) $\dfrac{36n^3}{42n^9}$

67) $\dfrac{r^3 - t^3}{t^2 - r^2}$ 68) $\dfrac{k^2 - 16k + 64}{k^3 - 8k^2 + 9k - 72}$

69) $\dfrac{b^2 + 6b - 72}{4b^2 + 52b + 48}$ 70) $\dfrac{5p^2 - 13p + 6}{32 - 8p^2}$

71) $\dfrac{28h^4 - 56h^3 + 7h}{7h}$ 72) $\dfrac{z^2 + 5z - 36}{64 - z^3}$

73) $\dfrac{14 - 6w}{12w^3 - 28w^2}$

74) $\dfrac{54d^6 + 6d^5 - 42d^3 - 18d^2}{6d^2}$

75) $\dfrac{-5v - 10}{v^3 - v^2 - 4v + 4}$ 76) $\dfrac{38x^2 + 38}{-12x^2 - 12}$

Objective 5: Write Equivalent Forms of a Rational Expression

Find three equivalent forms of each rational expression.

77) $-\dfrac{u + 7}{u - 2}$ 78) $-\dfrac{8y - 1}{2y + 5}$

79) $-\dfrac{9 - 5t}{2t - 3}$ 80) $\dfrac{w - 6}{-4w + 7}$

81) $\dfrac{-12m}{m^2 - 3}$ 82) $\dfrac{-9x - 11}{18 - x}$

Reduce to lowest terms

a) using long division.

b) using the methods of this section.

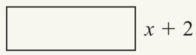
83) $\dfrac{4y^2 - 11y + 6}{y - 2}$ 84) $\dfrac{2x^2 + x - 28}{x + 4}$

85) $\dfrac{8a^3 + 125}{2a + 5}$ 86) $\dfrac{27t^3 - 8}{3t - 2}$

Recall that the area of a rectangle is $A = lw$, where $w =$ width and $l =$ length. Solving for the width, we get $w = \dfrac{A}{l}$ and solving for the length gives us $l = \dfrac{A}{w}$.

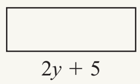
Find the missing side in each rectangle.

87) Area $= 3x^2 + 8x + 4$



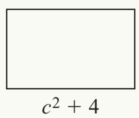
Find the length.

88) Area $= 2y^2 - 3y - 20$



Find the width.

89) Area $= 2c^3 + 4c^2 + 8c + 16$



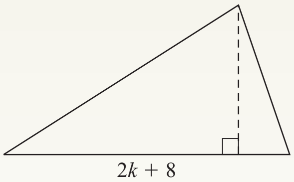
Find the width.

90) Area $= 3n^3 - 12n^2 - n + 4$

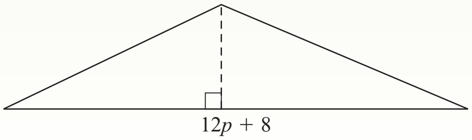
Find the length.

Recall that the area of a triangle is $A = \frac{1}{2}bh$, where b = length of the base and h = height. Solving for the height, we get $h = \frac{2A}{b}$. Find the height of each triangle.

91) Area $= 3k^2 + 13k + 4$



92) Area $= 6p^2 + 52p + 32$



Objective 6: Determine the Domain of a Rational Function

Determine the domain of each rational function.

93) $f(p) = \dfrac{1}{p - 7}$

94) $h(z) = \dfrac{z + 8}{z + 3}$

95) $k(r) = \dfrac{r}{5r + 2}$

96) $f(a) = \dfrac{6a}{7 - 2a}$

97) $g(t) = \dfrac{3t - 4}{t^2 - 9t + 8}$

98) $r(c) = \dfrac{c + 9}{c^2 - c - 42}$

99) $h(w) = \dfrac{w + 7}{w^2 - 81}$

100) $k(t) = \dfrac{t}{t^2 - 14t + 33}$

101) $A(c) = \dfrac{8}{c^2 + 6}$

102) $C(n) = \dfrac{3n + 1}{2}$

103) Write your own example of a rational function, $f(x)$, that has a domain of $(-\infty, -8) \cup (-8, \infty)$.

104) Write your own example of a rational function, $g(x)$, that has a domain of $(-\infty, -5) \cup (-5, 6) \cup (6, \infty)$.

Section 8.2 Multiplying and Dividing Rational Expressions

Objectives

1. Multiply Fractions
2. Multiply Rational Expressions
3. Divide Rational Expressions

We multiply rational expressions the same way we multiply rational numbers. Multiply numerators, multiply denominators, and simplify.

Procedure Multiplying Rational Expressions

If $\frac{P}{Q}$ and $\frac{R}{T}$ are rational expressions, then $\frac{P}{Q} \cdot \frac{R}{T} = \frac{PR}{QT}$.

To multiply two rational expressions, multiply their numerators, multiply their denominators, and simplify.

Let's begin by reviewing how we multiply two fractions.

1. Multiply Fractions

Example 1

Multiply $\frac{9}{16} \cdot \frac{8}{15}$.

Solution

We can multiply numerators, multiply denominators, then simplify by dividing out common factors, *or* we can divide out the common factors before multiplying.

$$\frac{9}{16} \cdot \frac{8}{15} = \frac{\cancel{3} \cdot 3}{2 \cdot \cancel{8}} \cdot \frac{\cancel{8}}{\cancel{3} \cdot 5} \qquad \text{Factor and divide out common factors.}$$

$$= \frac{3}{2 \cdot 5} \qquad \text{Multiply.}$$

$$= \frac{3}{10} \qquad \text{Simplify.}$$

You Try 1

Multiply $\frac{8}{35} \cdot \frac{5}{12}$.

Multiplying rational expressions works the same way.

2. Multiply Rational Expressions

Procedure Multiplying Rational Expressions

1) Factor. 2) Reduce and multiply.

All products must be written in lowest terms.

Example 2

Multiply.

a) $\frac{18x^3}{y^4} \cdot \frac{y^7}{9x^8}$ b) $\frac{9c+45}{6c^{10}} \cdot \frac{c^6}{c^2-25}$

c) $\frac{2n^2-11n-6}{n^2-2n-24} \cdot \frac{n^2+8n+16}{2n^2+n}$

Solution

a) $\frac{18x^3}{y^4} \cdot \frac{y^7}{9x^8} = \frac{\overset{2}{\cancel{18}}\cancel{x^3}}{\cancel{y^4}} \cdot \frac{\cancel{y^4} \cdot y^3}{\cancel{9}\cancel{x^3} \cdot x^5}$ Factor and reduce.

$= \frac{2y^3}{x^5}$ Multiply.

b) $\frac{9c+45}{6c^{10}} \cdot \frac{c^6}{c^2-25} = \frac{\overset{3}{\cancel{9}}\cancel{(c+5)}}{\underset{2}{\cancel{6}}\cancel{c^6} \cdot c^4} \cdot \frac{\cancel{c^6}}{\cancel{(c+5)}(c-5)}$ Factor and reduce.

$= \frac{3}{2c^4(c-5)}$ Multiply.

c) $\frac{2n^2-11n-6}{n^2-2n-24} \cdot \frac{n^2+8n+16}{2n^2+n} = \frac{\cancel{(2n+1)}\cancel{(n-6)}}{\cancel{(n+4)}\cancel{(n-6)}} \cdot \frac{\overset{(n+4)}{\cancel{(n+4)^2}}}{n\cancel{(2n+1)}}$ Factor and reduce.

$= \frac{n+4}{n}$ Multiply.

You Try 2

Multiply.

a) $\frac{n^7}{20m^9} \cdot \frac{10m^5}{n^2}$ b) $\frac{d^2}{d^2-4} \cdot \frac{4d-8}{12d^5}$ c) $\frac{h^2+10h+25}{3h^2-4h} \cdot \frac{3h^2+5h-12}{h^2+8h+15}$

3. Divide Rational Expressions

When we divide rational numbers, we multiply by a reciprocal. For example,

$\frac{7}{4} \div \frac{3}{8} = \frac{7}{\underset{1}{\cancel{4}}} \cdot \frac{\overset{2}{\cancel{8}}}{3} = \frac{14}{3}$. We divide rational expressions the same way.

To divide rational expressions, we multiply the first rational expression by the reciprocal of the second rational expression.

Procedure Dividing Rational Expressions

If $\frac{P}{Q}$ and $\frac{R}{T}$ are rational expressions with Q, R, and T not equal to zero, then

$$\frac{P}{Q} \div \frac{R}{T} = \frac{P}{Q} \cdot \frac{T}{R} = \frac{PT}{QR}.$$

Multiply the first rational expression by the reciprocal of the second rational expression.

Example 3

Divide.

a) $\dfrac{15a^7}{b^3} \div \dfrac{3a^4}{b^9}$ b) $\dfrac{t^2 - 16t + 63}{t^2} \div (t-7)^2$

c) $\dfrac{x^2 - 9}{x^2 + 3x - 10} \div \dfrac{24 - 8x}{x^2 + 9x + 20}$

Solution

a) $\dfrac{15a^7}{b^3} \div \dfrac{3a^4}{b^9} = \dfrac{\overset{5a^3}{\cancel{15a^7}}}{\cancel{b^3}} \cdot \dfrac{\overset{b^6}{\cancel{b^9}}}{\cancel{3}\,\cancel{a^4}}$ Multiply by the reciprocal and reduce.

$= 5a^3b^6$ Multiply.

Notice that we used the *quotient rule* for exponents to reduce:

$$\frac{a^7}{a^4} = a^3, \quad \frac{b^9}{b^3} = b^6$$

b) $\dfrac{t^2 - 16t + 63}{t^2} \div (t-7)^2 = \dfrac{(t-9)\cancel{(t-7)}}{t^2} \cdot \dfrac{1}{\underset{(t-7)}{\cancel{(t-7)^2}}}$ Since $(t-7)^2$ can be written as $\dfrac{(t-7)^2}{1}$, its reciprocal is $\dfrac{1}{(t-7)^2}$.

$= \dfrac{t-9}{t^2(t-7)}$ Reduce and multiply.

c) $\dfrac{x^2 - 9}{x^2 + 3x - 10} \div \dfrac{24 - 8x}{x^2 + 9x + 20} = \dfrac{x^2 - 9}{x^2 + 3x - 10} \cdot \dfrac{x^2 + 9x + 20}{24 - 8x}$ Multiply by the reciprocal.

$= \dfrac{(x+3)\overset{-1}{\cancel{(x-3)}}}{\cancel{(x+5)}(x-2)} \cdot \dfrac{(x+4)\cancel{(x+5)}}{8\cancel{(3-x)}}$ Factor; $\dfrac{x-3}{3-x} = -1$.

$= -\dfrac{(x+3)(x+4)}{8(x-2)}$ Reduce and multiply.

You Try 3

Divide.

a) $\dfrac{k^3}{12h^7} \div \dfrac{k^4}{16h^2}$ b) $\dfrac{w^2 + 4w - 45}{w} \div (w+9)^2$

c) $\dfrac{2m^2 - m - 15}{1 - m^2} \div \dfrac{m^2 - 10m + 21}{12m - 12}$

Remember that a fraction, itself, represents division. That is, $\frac{30}{5} = 30 \div 5 = 6$. We can write division problems involving fractions and rational expressions in a similar way.

Example 4

Divide.

a) $\dfrac{\frac{8}{35}}{\frac{16}{45}}$ b) $\dfrac{\frac{3w+2}{5}}{\frac{9w^2-4}{10}}$

Solution

a) $\dfrac{\frac{8}{35}}{\frac{16}{45}}$ means $\dfrac{8}{35} \div \dfrac{16}{45}$. Then,

$$\frac{8}{35} \div \frac{16}{45} = \frac{8}{35} \cdot \frac{45}{16} \qquad \text{Multiply by the reciprocal.}$$

$$= \frac{\overset{1}{\cancel{8}}}{\underset{7}{\cancel{35}}} \cdot \frac{\overset{9}{\cancel{45}}}{\underset{2}{\cancel{16}}} \qquad \text{Divide 8 and 16 by 8. Divide 45 and 35 by 5.}$$

$$= \frac{9}{14} \qquad \text{Multiply.}$$

b) $\dfrac{\frac{3w+2}{5}}{\frac{9w^2-4}{10}}$ means $\dfrac{3w+2}{5} \div \dfrac{9w^2-4}{10}$. Then,

$$\frac{3w+2}{5} \div \frac{9w^2-4}{10} = \frac{3w+2}{5} \cdot \frac{10}{9w^2-4} \qquad \text{Multiply by the reciprocal.}$$

$$= \frac{\cancel{3w+2}}{\underset{1}{\cancel{5}}} \cdot \frac{\overset{2}{\cancel{10}}}{\cancel{(3w+2)}(3w-2)} \qquad \text{Factor and reduce.}$$

$$= \frac{2}{3w-2} \qquad \text{Multiply.}$$

You Try 4

Divide.

a) $\dfrac{\frac{4}{45}}{\frac{20}{27}}$ b) $\dfrac{\frac{25u^2-9}{24}}{\frac{5u+3}{16}}$

Answers to You Try Exercises

1) $\dfrac{2}{21}$ 2) a) $\dfrac{n^5}{2m^4}$ b) $\dfrac{1}{3d^3(d+2)}$ c) $\dfrac{h+5}{h}$

3) a) $\dfrac{4}{3h^5k}$ b) $\dfrac{w-5}{w(w+9)}$ c) $-\dfrac{12(2m+5)}{(m+1)(m-7)}$ 4) a) $\dfrac{3}{25}$ b) $\dfrac{2(5u-3)}{3}$

8.2 Exercises

Objective 1: Multiply Fractions

Multiply.

1) $\dfrac{5}{6} \cdot \dfrac{7}{9}$

2) $\dfrac{4}{11} \cdot \dfrac{2}{3}$

3) $\dfrac{6}{15} \cdot \dfrac{25}{42}$

4) $\dfrac{12}{21} \cdot \dfrac{7}{4}$

Objective 2: Multiply Rational Expressions

Multiply.

5) $\dfrac{16b^5}{3} \cdot \dfrac{4}{36b}$

6) $\dfrac{26}{25r^3} \cdot \dfrac{15r^6}{2}$

7) $\dfrac{21s^4}{15t^2} \cdot \dfrac{5t^4}{42s^{10}}$

8) $\dfrac{15u^4}{14v^2} \cdot \dfrac{7v^7}{20u^8}$

9) $\dfrac{9c^4}{42c} \cdot \dfrac{35}{3c^3}$

10) $-\dfrac{10}{8x^7} \cdot \dfrac{24x^9}{9x}$

11) $\dfrac{5t^2}{(3t-2)^2} \cdot \dfrac{3t-2}{10t^3}$

12) $\dfrac{11(z+5)^5}{6(z-4)} \cdot \dfrac{3}{22(z+5)}$

13) $\dfrac{4u-5}{9u^3} \cdot \dfrac{6u^5}{(4u-5)^3}$

14) $\dfrac{5k+6}{2k^3} \cdot \dfrac{12k^5}{(5k+6)^4}$

15) $\dfrac{6}{n+5} \cdot \dfrac{n^2+8n+15}{n+3}$

16) $\dfrac{9p^2-1}{12} \cdot \dfrac{9}{9p+3}$

17) $\dfrac{18y-12}{4y^2} \cdot \dfrac{y^2-4y-5}{3y^2+y-2}$

18) $\dfrac{12v-3}{8v+12} \cdot \dfrac{2v^2-5v-12}{3v-12}$

19) $(c-6) \cdot \dfrac{5}{c^2-6c}$

20) $(r^2+r-2) \cdot \dfrac{18r^2}{3r^2+6r}$

21) $\dfrac{7x}{11-x} \cdot (x^2-121)$

22) $\dfrac{4b}{2b^2-3b-5} \cdot (b+1)^2$

Objective 3: Divide Rational Expressions

Divide.

23) $\dfrac{20}{9} \div \dfrac{10}{27}$

24) $\dfrac{4}{5} \div \dfrac{12}{7}$

25) $42 \div \dfrac{9}{2}$

26) $\dfrac{18}{7} \div 6$

Divide.

27) $\dfrac{12}{5m^5} \div \dfrac{21}{8m^{12}}$

28) $\dfrac{12k^3}{35} \div \dfrac{42k^6}{25}$

29) $-\dfrac{50g}{7h^3} \div \dfrac{15g^4}{14h}$

30) $-\dfrac{c^{12}}{b} \div \dfrac{c^2}{6b}$

31) $\dfrac{2(k-2)}{21k^6} \div \dfrac{(k-2)^2}{28}$

32) $\dfrac{18}{(x+4)^3} \div \dfrac{36(x-7)}{x+4}$

33) $\dfrac{16q^5}{p+7} \div \dfrac{2q^4}{(p+7)^2}$

34) $\dfrac{(2a-5)^5}{32a^5} \div \dfrac{2a-5}{8a^3}$

35) $\dfrac{q+8}{q} \div \dfrac{q^2+q-56}{5}$

36) $\dfrac{4y^2-25}{10} \div \dfrac{18y-45}{18}$

37) $\dfrac{z^2+18z+80}{2z+1} \div (z+8)^2$

38) $\dfrac{6w^2-30w}{7} \div (w-5)^2$

39) $\dfrac{36a-12}{16} \div (9a^2-1)$

40) $\dfrac{h^2-21h+108}{4h} \div (144-h^2)$

41) $\dfrac{7n^2-14n}{8n} \div \dfrac{n^2+4n-12}{4n+24}$

42) $\dfrac{4j+24}{9} \div \dfrac{j^2-36}{9j-54}$

43) $\dfrac{4c-9}{2c^2-8c} \div \dfrac{12c-27}{c^2-3c-4}$

44) $\dfrac{p+13}{p+3} \div \dfrac{p^3+13p^2}{p^2-5p-24}$

45) Explain how to multiply rational expressions.

46) Explain how to divide rational expressions.

47) Find the missing polynomial in the denominator of $\dfrac{9h+45}{h^4} \cdot \dfrac{h^3}{\underline{\qquad\qquad}} = \dfrac{9}{h(h-2)}$.

48) Find the missing monomial in the numerator of $\dfrac{\underline{\qquad\qquad}}{m^2-81} \cdot \dfrac{3m-27}{2m^2} = \dfrac{15m^3}{m+9}$.

49) Find the missing binomial in the numerator of $\dfrac{4z^2-49}{z^2-3z-40} \div \dfrac{\underline{\qquad\qquad}}{z+5} = \dfrac{2z+7}{8-z}$.

50) In the division problem $\dfrac{12}{x} \div \dfrac{3y}{2}$, can $y = 0$? Explain your answer.

Divide.

51) $\dfrac{\frac{25}{42}}{\frac{8}{21}}$

52) $\dfrac{\frac{9}{35}}{\frac{4}{15}}$

53) $\dfrac{\frac{5}{24}}{\frac{15}{4}}$

54) $\dfrac{\frac{4}{3}}{\frac{2}{9}}$

55) $\dfrac{\frac{3d+7}{24}}{\frac{3d+7}{6}}$

56) $\dfrac{\frac{8s-7}{4}}{\frac{8s-7}{16}}$

VIDEO 57) $\dfrac{\frac{16r+24}{r^3}}{\frac{12r+18}{r}}$

58) $\dfrac{\frac{44m-33}{3m^2}}{\frac{8m-6}{m}}$

59) $\dfrac{\frac{a^2-25}{3a^{11}}}{\frac{4a+20}{a^3}}$

60) $\dfrac{\frac{4z-8}{z^8}}{\frac{z^2-4}{z^6}}$

61) $\dfrac{\frac{16x^2-25}{x^7}}{\frac{36x-45}{6x^3}}$

62) $-\dfrac{\frac{16a^2}{3a^2+2a}}{\frac{12}{9a^2-4}}$

Mixed Exercises: Objectives 2 and 3

63) $\dfrac{c^2+c-30}{9c+9} \cdot \dfrac{c^2+2c+1}{c^2-25}$

64) $\dfrac{d^2+3d-54}{d-12} \cdot \dfrac{d^2-10d-24}{7d+63}$

65) $\dfrac{3x+2}{9x^2-4} \div \dfrac{4x}{15x^2-7x-2}$

66) $\dfrac{b^2-10b+25}{8b-40} \div \dfrac{2b^2-5b-25}{2b+5}$

67) $\dfrac{3k^2-12k}{12k^2-30k-72} \cdot (2k+3)^2$

68) $\dfrac{4a^3}{a^2+a-72} \cdot (a^2-a-56)$

69) $\dfrac{7t^6}{t^2-4} \div \dfrac{14t^2}{3t^2-7t+2}$

70) $\dfrac{4n^2-1}{10n^3} \div \dfrac{2n^2-7n-4}{6n^5}$

71) $\dfrac{4h^3}{h^2-64} \cdot \dfrac{8h-h^2}{12}$

72) $\dfrac{c^2-36}{c+6} \div \dfrac{30-5c}{c-9}$

73) $\dfrac{54x^8}{22x^3y^2} \div \dfrac{36xy^5}{11x^2y}$

74) $\dfrac{28cd^9}{2c^3d} \cdot \dfrac{5d^2}{84c^{10}d^2}$

75) $\dfrac{r^3+8}{r+2} \cdot \dfrac{7}{3r^2-6r+12}$

76) $\dfrac{2t^2-6t+18}{5t-5} \cdot \dfrac{t^2-9}{t^3+27}$

VIDEO 77) $\dfrac{a^2-4a}{6a+54} \cdot \dfrac{a^2+13a+36}{16-a^2}$

78) $\dfrac{64-u^2}{40-5u} \div \dfrac{u^2+10u+16}{2u+3}$

79) $\dfrac{2a^2}{a^2+a-20} \cdot \dfrac{a^3+5a^2+4a+20}{2a^2+8}$

80) $\dfrac{18x^4}{x^3+3x^2-9x-27} \cdot \dfrac{6x^2+19x+3}{18x^2+3x}$

81) $\dfrac{30}{4y^2-4x^2} \div \dfrac{10x^2+10xy+10y^2}{x^3-y^3}$

82) $\dfrac{a^2-b^2}{a^3+b^3} \div \dfrac{8b-8a}{9}$

83) $\dfrac{3m^2+8m+4}{4} \div (12m+8)$

84) $\dfrac{w^2-17w+72}{3w} \div (w-8)$

Perform the operations and simplify.

85) $\dfrac{4j^2-21j+5}{j^3} \div \left(\dfrac{3j+2}{j^3-j^2} \cdot \dfrac{j^2-6j+5}{j}\right)$

86) $\dfrac{2a}{a^2+18a+81} \div \left(\dfrac{a^2+3a-4}{a^2+9a+20} \cdot \dfrac{a^2+5a}{a^2+8a-9}\right)$

87) $\dfrac{x}{3x^2-15x+75} \div \left(\dfrac{4x+20}{x+9} \cdot \dfrac{x^2-81}{x^3+125}\right)$

88) $\dfrac{t^3-8}{t-2} \div \left(\dfrac{3t+11}{5t+15} \cdot \dfrac{t^2+2t+4}{3t^2+11t}\right)$

89) If the area of a rectangle is $\dfrac{3x}{2y^2}$ and the width is $\dfrac{y}{8x^4}$, what is the length of the rectangle?

90) If the area of a triangle is $\dfrac{2n}{n^2-4n+3}$ and the height is $\dfrac{n+3}{n-1}$, what is the length of the base of the triangle?

Section 8.3 Finding the Least Common Denominator

Objectives

1. Find the Least Common Denominator for a Group of Rational Expressions
2. Rewrite Rational Expressions Using Their LCD

1. Find the Least Common Denominator for a Group of Rational Expressions

Recall that to add or subtract fractions, they must have a common denominator. Similarly, rational expressions must have common denominators in order to be added or subtracted. In this section, we will discuss how to find the least common denominator (LCD) of rational expressions.

We begin by looking at the fractions $\frac{3}{8}$ and $\frac{5}{12}$. By inspection, we can see that the LCD = 24. But, *why* is that true? Let's write each of the denominators, 8 and 12, as the product of their prime factors:

$$8 = 2 \cdot 2 \cdot 2 = 2^3$$
$$12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$$

The LCD will contain each factor the *greatest* number of times it appears in any single factorization.

The LCD will contain 2^3 because 2 appears *three* times in the factorization of 8.

The LCD will contain 3 because 3 appears *one* time in the factorization of 12.

The LCD, then, is the product of the factors we have identified.

$$\text{LCD of } \frac{3}{8} \text{ and } \frac{5}{12} = 2^3 \cdot 3 = 8 \cdot 3 = 24$$

This is the same result as the one we obtained just by inspecting the two denominators.

We use the same procedure to find the least common denominator of rational expressions.

Procedure Finding the Least Common Denominator (LCD)

Step 1: Factor the denominators.

Step 2: The LCD will contain each unique factor the *greatest* number of times it appears in any single factorization.

Step 3: The LCD is the *product* of the factors identified in step 2.

Example 1

Find the LCD of each pair of rational expressions.

a) $\frac{17}{24}, \frac{5}{36}$ b) $\frac{1}{12n}, \frac{10}{21n}$ c) $\frac{8}{49c^3}, \frac{13}{14c^2}$

Solution

a) Follow the steps for finding the least common denominator.

Step 1: Factor the denominators.

$$24 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3$$
$$36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$$

Step 2: The LCD will contain each unique factor the *greatest* number of times it appears in any factorization. *The LCD will contain* 2^3 *and* 3^2.

Step 3: The LCD is the *product* of the factors in step 2.

$$\text{LCD} = 2^3 \cdot 3^2 = 8 \cdot 9 = 72$$

b) ***Step 1:*** Factoring the denominators of $\frac{1}{12n}$ and $\frac{10}{21n}$ gives us

$$12n = 2 \cdot 2 \cdot 3 \cdot n = 2^2 \cdot 3 \cdot n$$
$$21n = 3 \cdot 7 \cdot n$$

Step 2: The LCD will contain each unique factor the *greatest* number of times it appears in any factorization. *It will contain 2^2, 3, 7, and n.*

Step 3: The LCD is the *product* of the factors in step 2.

$$\text{LCD} = 2^2 \cdot 3 \cdot 7 \cdot n = 84n.$$

c) ***Step 1:*** Factoring the denominators of $\frac{8}{49c^3}$ and $\frac{13}{14c^2}$ gives us

$$49c^3 = 7 \cdot 7 \cdot c^3 = 7^2 \cdot c^3$$
$$14c^2 = 2 \cdot 7 \cdot c^2$$

Step 2: The LCD will contain each unique factor the *greatest* number of times it appears in any factorization. *It will contain 2, 7^2, and c^3.*

Step 3: The LCD is the *product* of the factors in step 2.

$$\text{LCD} = 2 \cdot 7^2 \cdot c^3 = 98c^3$$

You Try 1

Find the LCD of each pair of rational expressions.

a) $\frac{14}{15}, \frac{11}{18}$ b) $\frac{3}{14}, \frac{7}{10}$ c) $\frac{20}{27h^2}, \frac{1}{6h^4}$

Example 2

Find the LCD of each group of rational expressions.

a) $\frac{4}{k}, \frac{6}{k+3}$ b) $\frac{7}{a-6}, \frac{2a}{a^2+3a-54}$ c) $\frac{3p}{p^2+4p+4}, \frac{1}{5p^2+10p}$

Solution

a) The denominators of $\frac{4}{k}$, and $\frac{6}{k+3}$ are already in simplest form. It is important to recognize that *k and $k+3$ are different factors.*

The LCD will be the product of k and $k+3$: LCD $= k(k+3)$.

Usually, we leave the LCD in this form; we do not distribute.

b) ***Step 1:*** Factor the denominators of $\frac{7}{a-6}$ and $\frac{2a}{a^2+3a-54}$.

$a - 6$ cannot be factored. $a^2 + 3a - 54 = (a-6)(a+9)$

Step 2: The LCD will contain each unique factor the *greatest* number of times it appears in any factorization. *It will contain $a - 6$ and $a + 9$.*

Step 3: The LCD is the *product* of the factors identified in step 2.

$$\text{LCD} = (a-6)(a+9)$$

c) ***Step 1:*** Factor the denominators of $\dfrac{3p}{p^2+4p+4}$ and $\dfrac{1}{5p^2+10p}$.

$$p^2+4p+4=(p+2)^2 \quad 5p^2+10p=5p(p+2)$$

Step 2: The unique factors are 5, p, and $p+2$, with $p+2$ *appearing at most twice. The factors we will use in the LCD are 5, p, and* $(p+2)^2$.

Step 3: $\text{LCD}=5p(p+2)^2$ ■

You Try 2

Find the LCD of each group of rational expressions.

a) $\dfrac{6}{w}, \dfrac{9w}{w+1}$ b) $\dfrac{1}{r+8}, \dfrac{5r}{r^2+r-56}$ c) $\dfrac{4b}{b^2-18b+81}, \dfrac{3}{8b^2-72b}$

At first glance it may appear that the least common denominator of $\dfrac{9}{x-7}$ and $\dfrac{5}{7-x}$ is $(x-7)(7-x)$. This is *not* the case. Recall from Section 8.1 that $a-b=-1(b-a)$. We will use this idea to find the LCD of $\dfrac{9}{x-7}$ and $\dfrac{5}{7-x}$.

Example 3

Find the LCD of $\dfrac{9}{x-7}$ and $\dfrac{5}{7-x}$.

Solution

Since $7-x=-(x-7)$, we can rewrite $\dfrac{5}{7-x}$ as $\dfrac{5}{-(x-7)}=-\dfrac{5}{x-7}$.

Therefore, we can now think of our task as finding the LCD of $\dfrac{9}{x-7}$ and $-\dfrac{5}{x-7}$.

The least common denominator is $x-7$. ■

You Try 3

Find the LCD of $\dfrac{2}{k-5}$ and $\dfrac{13}{5-k}$.

2. Rewrite Rational Expressions Using Their LCD

As we know from our previous work with fractions, after *determining* the least common denominator, we must *rewrite* those fractions as equivalent fractions with the LCD so that they can be added or subtracted.

Example 4

Identify the LCD of $\frac{1}{6}$ and $\frac{8}{9}$, and rewrite each as an equivalent fraction with the LCD as its denominator.

Solution

The LCD of $\frac{1}{6}$ and $\frac{8}{9}$ is 18. We must rewrite each fraction with a denominator of 18.

$\frac{1}{6}$: By what number should we multiply 6 to get 18? 3

$$\frac{1}{6} \cdot \frac{3}{3} = \frac{3}{18}$$ Multiply the numerator *and* denominator by 3 to obtain an equivalent fraction.

$\frac{8}{9}$: By what number should we multiply 9 to get 18? 2

$$\frac{8}{9} \cdot \frac{2}{2} = \frac{16}{18}$$ Multiply the numerator *and* denominator by 2 to obtain an equivalent fraction.

Therefore, $\frac{1}{6} = \frac{3}{18}$ and $\frac{8}{9} = \frac{16}{18}$.

You Try 4

Identify the LCD of $\frac{7}{12}$ and $\frac{2}{9}$, and rewrite each as an equivalent fraction with the LCD as its denominator.

The procedure for rewriting rational expressions as equivalent expressions with the least common denominator is very similar to the process used in Example 4.

Procedure Writing Rational Expressions as Equivalent Expressions with the Least Common Denominator

Step 1: Identify and write down the LCD.

Step 2: Look at each rational expression (with its denominator in factored form) and compare its denominator with the LCD. Ask yourself, "What factors are missing?"

Step 3: Multiply the numerator and denominator by the "missing" factors to obtain an equivalent rational expression with the desired LCD.

Note

Use the distributive property to multiply the terms in the numerator, but leave the denominator as the product of factors. (We will see why this is done in Section 8.4.)

Example 5

Identify the LCD of each pair of rational expressions, and rewrite each as an equivalent expression with the LCD as its denominator.

a) $\dfrac{5}{12z}, \dfrac{4}{9z^3}$ b) $\dfrac{m}{m-4}, \dfrac{3}{m+8}$ c) $\dfrac{2}{3x^2-6x}, \dfrac{9x}{x^2-7x+10}$

d) $\dfrac{t}{t-3}, \dfrac{8}{3-t}$

Solution

a) ***Step 1:*** Identify and write down the LCD of $\dfrac{5}{12z}$ and $\dfrac{4}{9z^3}$: LCD $= 36z^3$.

Step 2: Compare the denominators of $\dfrac{5}{12z}$ and $\dfrac{4}{9z^3}$ to the LCD and ask yourself, "What's missing?"

$\dfrac{5}{12z}$: *12z is "missing" the factors 3 and z^2.* | $\dfrac{4}{9z^3}$: *$9z^3$ is "missing" the factor 4.*

Step 3: Multiply the numerator and denominator by $3z^2$.

$$\frac{5}{12z}\cdot\frac{3z^2}{3z^2}=\frac{15z^2}{36z^3}$$

Multiply the numerator and denominator by 4.

$$\frac{4}{9z^3}\cdot\frac{4}{4}=\frac{16}{36z^3}$$

$$\frac{5}{12z}=\frac{15z^2}{36z^3} \quad\text{and}\quad \frac{4}{9z^3}=\frac{16}{36z^3}$$

b) ***Step 1:*** Identify and write down the LCD of $\dfrac{m}{m-4}$ and $\dfrac{3}{m+8}$: LCD $= (m-4)(m+8)$.

Step 2: Compare the denominators of $\dfrac{m}{m-4}$ and $\dfrac{3}{m+8}$ to the LCD and ask yourself, "What's missing?"

$\dfrac{m}{m-4}$: *$m-4$ is "missing" the factor $m+8$.* | $\dfrac{3}{m+8}$: *$m+8$ is "missing" the factor $m-4$.*

Step 3: Multiply the numerator and denominator by $m+8$.

$$\frac{m}{m-4}\cdot\frac{m+8}{m+8}=\frac{m(m+8)}{(m-4)(m+8)}=\frac{m^2+8m}{(m-4)(m+8)}$$

Multiply the numerator and denominator by $m-4$.

$$\frac{3}{m+8}\cdot\frac{m-4}{m-4}=\frac{3(m-4)}{(m+8)(m-4)}=\frac{3m-12}{(m-4)(m+8)}$$

Notice that we multiplied the factors in the numerator but left the denominator in factored form.

$$\frac{m}{m-4}=\frac{m^2+8m}{(m-4)(m+8)} \quad\text{and}\quad \frac{3}{m+8}=\frac{3m-12}{(m-4)(m+8)}$$

c) ***Step 1:*** Identify and write down the LCD of $\dfrac{2}{3x^2 - 6x}$ and $\dfrac{9x}{x^2 - 7x + 10}$.

First, we must factor the denominators.

$$\frac{2}{3x^2 - 6x} = \frac{2}{3x(x - 2)}, \quad \frac{9x}{x^2 - 7x + 10} = \frac{9x}{(x - 2)(x - 5)}$$

We will work with the factored forms of the expressions.

$$\text{LCD} = 3x(x - 2)(x - 5)$$

Step 2: Compare the denominators of $\dfrac{2}{3x(x - 2)}$ and $\dfrac{9x}{(x - 2)(x - 5)}$ to the LCD and ask yourself, "What's missing?"

$\dfrac{2}{3x(x - 2)}$: $3x(x - 2)$ *is "missing" the factor* $x - 5$.

$\dfrac{9x}{(x - 2)(x - 5)}$: $(x - 2)(x - 5)$ *is "missing"* $3x$.

Step 3: Multiply the numerator and denominator by $x - 5$.

$$\frac{2}{3x(x - 2)} \cdot \frac{x - 5}{x - 5} = \frac{2(x - 5)}{3x(x - 2)(x - 5)} = \frac{2x - 10}{3x(x - 2)(x - 5)}$$

Multiply the numerator and denominator by $3x$.

$$\frac{9x}{(x - 2)(x - 5)} \cdot \frac{3x}{3x} = \frac{27x^2}{3x(x - 2)(x - 5)}$$

$$\frac{2}{3x^2 - 6x} = \frac{2x - 10}{3x(x - 2)(x - 5)} \quad \text{and} \quad \frac{9x}{x^2 - 7x + 10} = \frac{27x^2}{3x(x - 2)(x - 5)}$$

d) To find the LCD of $\dfrac{t}{t - 3}$ and $\dfrac{8}{3 - t}$, recall that $3 - t$ can be rewritten as $-(t - 3)$. So,

$$\frac{8}{3 - t} = \frac{8}{-(t - 3)} = -\frac{8}{t - 3}$$

Therefore, the LCD of $\dfrac{t}{t - 3}$ and $-\dfrac{8}{t - 3}$ is $t - 3$.

$\dfrac{t}{t - 3}$ already has the LCD, and $\dfrac{8}{3 - t} = -\dfrac{8}{t - 3}$. ■

You Try 5

Identify the least common denominator of each pair of rational expressions, and rewrite each as an equivalent expression with the LCD as its denominator.

a) $\dfrac{3}{10a^6}, \dfrac{7}{8a^5}$

b) $\dfrac{6}{n + 10}, \dfrac{n}{2n - 3}$

c) $\dfrac{v - 9}{v^2 + 15v + 44}, \dfrac{8}{5v^2 + 55v}$

d) $\dfrac{c}{10 - c}, \dfrac{5}{c - 10}$

Answers to You Try Exercises

1) a) 90 b) 70 c) $54h^4$ 2) a) $w(w+1)$ b) $(r+8)(r-7)$ c) $8b(b-9)^2$

3) $k-5$ 4) LCD $=36$; $\frac{7}{12}=\frac{21}{36}, \frac{2}{9}=\frac{8}{36}$ 5) a) LCD $=40a^6$; $\frac{3}{10a^6}=\frac{12}{40a^6}, \frac{7}{8a^5}=\frac{35a}{40a^6}$

b) LCD $=(2n-3)(n+10)$; $\frac{6}{n+10}=\frac{12n-18}{(2n-3)(n+10)}, \frac{n}{2n-3}=\frac{n^2+10n}{(2n-3)(n+10)}$

c) LCD $=5v(v+4)(v+11)$; $\frac{v-9}{v^2+15v+44}=\frac{5v^2-45v}{5v(v+4)(v+11)}, \frac{8}{5v^2+55v}=\frac{8v+32}{5v(v+4)(v+11)}$

d) LCD $=c-10$; $\frac{c}{10-c}=-\frac{c}{c-10}, \frac{5}{c-10}=\frac{5}{c-10}$

8.3 Exercises

Objective 1: Find the LCD for a Group of Rational Expressions

Find the LCD of each group of fractions.

1) $\frac{7}{12}, \frac{2}{15}$

2) $\frac{3}{8}, \frac{9}{7}$

3) $\frac{27}{40}, \frac{11}{10}, \frac{5}{12}$

4) $\frac{19}{8}, \frac{1}{12}, \frac{3}{32}$

5) $\frac{3}{n^7}, \frac{5}{n^{11}}$

6) $\frac{4}{c^2}, \frac{8}{c^3}$

7) $\frac{13}{14r^4}, \frac{3}{4r^7}$

8) $\frac{11}{6p^4}, \frac{3}{10p^9}$

9) $-\frac{5}{6z^5}, \frac{7}{36z^2}$

10) $\frac{5}{24w^5}, -\frac{1}{4w^{10}}$

11) $\frac{7}{10m}, \frac{9}{22m^4}$

12) $-\frac{3}{2k^2}, \frac{5}{14k^5}$

13) $\frac{4}{24x^3y^2}, \frac{11}{6x^3y}$

14) $\frac{3}{10a^4b^2}, \frac{8}{15ab^4}$

15) $\frac{4}{11}, \frac{8}{z-3}$

16) $\frac{3}{n+8}, \frac{1}{5}$

VIDEO 17) $\frac{10}{w}, \frac{6}{2w+1}$

18) $\frac{1}{y}, -\frac{6}{6y+1}$

19) What is the first step for finding the LCD of $\frac{9}{8t-10}$ and $\frac{3t}{20t-25}$?

20) Is $(h-9)(9-h)$ the LCD of $\frac{2h}{h-9}$ and $\frac{4}{9-h}$? Explain your answer.

Find the LCD of each group of fractions.

21) $\frac{8}{5c-5}, \frac{9}{2c-2}$

22) $\frac{5}{7k+14}, \frac{11}{4k+8}$

23) $\frac{2}{9p^4-6p^3}, \frac{3}{3p^6-2p^5}$

24) $\frac{21}{6a^2-8a}, \frac{13}{18a^3-24a^2}$

25) $\frac{4m}{m-7}, \frac{2}{m-3}$

26) $\frac{5}{r+9}, \frac{7}{r-1}$

27) $\frac{11}{z^2+11z+24}, \frac{7z}{z^2+5z-24}$

28) $\frac{7x}{x^2-12x+35}, \frac{x}{x^2-x-20}$

29) $\frac{14t}{t^2-3t-18}, -\frac{6}{t^2-36}, \frac{t}{t^2+9t+18}$

30) $\frac{6w}{w^2-10w+16}, \frac{3}{w^2-7w-8}, \frac{4w}{w^2-w-2}$

31) $\frac{6}{a-8}, \frac{7}{8-a}$

32) $\frac{6}{b-3}, \frac{5}{3-b}$

33) $\frac{12}{y-x}, \frac{5y}{x-y}$

34) $\frac{u}{v-u}, \frac{8}{u-v}$

Objective 2: Rewrite Rational Expressions Using Their LCD

35) Explain, in your own words, how to rewrite $\frac{4}{x+9}$ as an equivalent rational expression with a denominator of $(x+9)(x-3)$.

36) Explain, in your own words, how to rewrite $\frac{7}{5-m}$ as an equivalent rational expression with a denominator of $m-5$.

Rewrite each rational expression with the indicated denominator.

37) $\frac{7}{12} = \frac{}{48}$

38) $\frac{5}{7} = \frac{}{42}$

39) $\frac{8}{z} = \frac{}{9z}$

40) $\frac{-6}{b} = \frac{}{4b}$

41) $\frac{3}{8k} = \frac{}{56k^4}$

42) $\frac{5}{3p^4} = \frac{}{9p^6}$

43) $\frac{6}{5t^5u^2} = \frac{}{10t^7u^5}$

44) $\frac{13}{6cd^2} = \frac{}{24c^3d^3}$

45) $\frac{7}{3r+4} = \frac{}{r(3r+4)}$

46) $\frac{8}{m-8} = \frac{}{m(m-8)}$

47) $\frac{v}{4(v-3)} = \frac{}{16v^5(v-3)}$

48) $\frac{a}{5(2a+7)} = \frac{}{15a(2a+7)}$

49) $\frac{9x}{x+6} = \frac{}{(x+6)(x-5)}$

50) $\frac{5b}{b+3} = \frac{}{(b+3)(b+7)}$

51) $\frac{z-3}{2z-5} = \frac{}{(2z-5)(z+8)}$

52) $\frac{w+2}{4w-1} = \frac{}{(4w-1)(w-4)}$

53) $\frac{5}{3-p} = \frac{}{p-3}$

54) $\frac{10}{10-n} = \frac{}{n-10}$

55) $-\frac{8c}{6c-7} = \frac{}{7-6c}$

56) $-\frac{g}{3g-2} = \frac{}{2-3g}$

Identify the least common denominator of each pair of rational expressions, and rewrite each as an equivalent rational expression with the LCD as its denominator.

57) $\frac{8}{15}, \frac{1}{6}$

58) $\frac{3}{8}, \frac{5}{12}$

59) $\frac{4}{u}, \frac{8}{u^3}$

60) $\frac{9}{d^5}, \frac{7}{d^2}$

VIDEO 61) $\frac{9}{8n^6}, \frac{2}{3n^2}$

62) $\frac{5}{8a}, \frac{7}{10a^5}$

63) $\frac{6}{4a^3b^5}, \frac{6}{a^4b}$

64) $\frac{3}{x^3y}, \frac{6}{5xy^5}$

65) $\frac{r}{5}, \frac{2}{r-4}$

66) $\frac{t}{5t-1}, \frac{8}{7}$

67) $\frac{3}{d}, \frac{7}{d-9}$

68) $\frac{5}{c}, \frac{4}{c+2}$

69) $\frac{m}{m+7}, \frac{3}{m}$

70) $\frac{z}{z-4}, \frac{5}{z}$

71) $\frac{a}{30a-15}, \frac{1}{12a-6}$

72) $\frac{7}{24x-16}, \frac{x}{18x-12}$

73) $\frac{8}{k-9}, \frac{5k}{k+3}$

74) $\frac{6}{h+1}, \frac{11h}{h+7}$

75) $\frac{3}{a+2}, \frac{2a}{3a+4}$

76) $\frac{b}{6b-5}, \frac{8}{b-9}$

VIDEO 77) $\frac{9y}{y^2-y-42}, \frac{3}{2y^2+12y}$

78) $\frac{12q}{q^2-6q-16}, \frac{4}{2q^2-16q}$

79) $\frac{c}{c^2+9c+18}, \frac{11}{c^2+12c+36}$

80) $\frac{z}{z^2-8z+16}, \frac{9z}{z^2+4z-32}$

VIDEO 81) $\frac{11}{g-3}, \frac{4}{9-g^2}$

82) $\frac{6}{g-9}, \frac{1}{81-g^2}$

83) $\frac{4}{3x-4}, \frac{7x}{16-9x^2}$

84) $\frac{12}{5k-2}, \frac{4k}{4-25k^2}$

85) $\frac{2}{z^2+3z}, \frac{6}{3z^2+9z}, \frac{8}{z^2+6z+9}$

86) $\frac{4}{w^2-4w}, \frac{6}{7w^2-28w}, \frac{11}{w^2-8w+16}$

87) $\frac{t}{t^2-13t+30}, \frac{6}{t-10}, \frac{7}{t^2-9}$

88) $-\frac{2}{a+2}, \frac{a}{a^2-4}, \frac{15}{a^2-3a+2}$

89) $-\frac{9}{h^3+8}, \frac{2h}{5h^2-10h+20}$

90) $\frac{5x}{x^3-y^3}, \frac{7}{8x-8y}$

Section 8.4 Adding and Subtracting Rational Expressions

Objectives

1. Add and Subtract Rational Expressions with a Common Denominator
2. Add and Subtract Rational Expressions with Different Denominators
3. Add and Subtract Rational Expressions with Special Denominators

We know that in order to add or subtract fractions, they must have a common denominator. The same is true for rational expressions.

1. Add and Subtract Rational Expressions with a Common Denominator

Let's first look at fractions and rational expressions with common denominators.

Example 1

Add or subtract.

a) $\dfrac{8}{11} - \dfrac{5}{11}$ b) $\dfrac{2x}{4x-9} + \dfrac{5x+3}{4x-9}$

Solution

a) Since the fractions have the same denominator, subtract the terms in the numerator and keep the common denominator.

$$\frac{8}{11} - \frac{5}{11} = \frac{8-5}{11} = \frac{3}{11} \qquad \text{Subtract terms in the numerator.}$$

b) Since the rational expressions have the same denominator, add the terms in the numerator and keep the common denominator.

$$\frac{2x}{4x-9} + \frac{5x+3}{4x-9} = \frac{2x+(5x+3)}{4x-9} \qquad \text{Add terms in the numerator.}$$

$$= \frac{7x+3}{4x-9} \qquad \text{Combine like terms.}$$

We can generalize the procedure for adding and subtracting rational expressions that have a common denominator as follows.

Procedure Adding and Subtracting Rational Expressions

If $\dfrac{P}{Q}$ and $\dfrac{R}{Q}$ are rational expressions with $Q \neq 0$, then

1) $\dfrac{P}{Q} + \dfrac{R}{Q} = \dfrac{P+R}{Q}$ and 2) $\dfrac{P}{Q} - \dfrac{R}{Q} = \dfrac{P-R}{Q}$

You Try 1

Add or subtract.

a) $\dfrac{11}{12} - \dfrac{7}{12}$ b) $\dfrac{6h}{5h-2} + \dfrac{3h+8}{5h-2}$

All answers to a sum or difference of rational expressions should be in lowest terms. Sometimes it is necessary to simplify our result to lowest terms by factoring the numerator and dividing the numerator and denominator by the greatest common factor.

Example 2

Add or subtract.

a) $\dfrac{4}{15k} + \dfrac{8}{15k}$ b) $\dfrac{c^2 - 3}{c(c+4)} - \dfrac{5 - 2c}{c(c+4)}$

Solution

a) $\dfrac{4}{15k} + \dfrac{8}{15k} = \dfrac{4+8}{15k} = \dfrac{12}{15k}$ Add terms in the numerator.

$= \dfrac{4}{5k}$ Reduce to lowest terms.

b) $\dfrac{c^2 - 3}{c(c+4)} - \dfrac{5 - 2c}{c(c+4)} = \dfrac{(c^2 - 3) - (5 - 2c)}{c(c+4)}$ Subtract terms in the numerator.

$= \dfrac{c^2 - 3 - 5 + 2c}{c(c+4)}$ Distribute.

$= \dfrac{c^2 + 2c - 8}{c(c+4)}$ Combine like terms.

$= \dfrac{\cancel{(c+4)}(c-2)}{c\cancel{(c+4)}}$ Factor the numerator.

$= \dfrac{c-2}{c}$ Reduce to lowest terms.

You Try 2

Add or subtract.

a) $\dfrac{19}{32w} + \dfrac{9}{32w}$ b) $\dfrac{m^2 - 5}{m(m+6)} - \dfrac{3m + 49}{m(m+6)}$

BE CAREFUL

After combining like terms in the numerator, ask yourself, *"Can I factor the numerator?"* If so, factor it. Sometimes, the expression can be reduced by dividing the numerator and denominator by the greatest common factor.

2. Add and Subtract Rational Expressions with Different Denominators

If we are asked to add or subtract rational expressions with different denominators, we must begin by rewriting each expression with the least common denominator. Then, add or subtract. Simplify the result.

Using the procedure studied in Section 8.3, here are the steps to follow to add or subtract rational expressions with different denominators.

Procedure Steps for Adding and Subtracting Rational Expressions with Different Denominators

1) Factor the denominators.
2) Write down the LCD.
3) Rewrite each rational expression as an equivalent rational expression with the LCD.
4) Add or subtract the numerators and keep the common denominator in factored form.
5) After combining like terms in the numerator ask yourself, *"Can I factor it?"* If so, factor.
6) Reduce the rational expression, if possible.

Example 3

Add or subtract.

a) $\dfrac{t+6}{4}+\dfrac{t-8}{12}$ b) $\dfrac{3}{10a}-\dfrac{7}{8a^2}$ c) $\dfrac{7n-30}{n^2-36}+\dfrac{n}{n+6}$

Solution

a) The LCD is 12. $\dfrac{t-8}{12}$ already has the LCD.

Rewrite $\dfrac{t+6}{4}$ with the LCD: $\dfrac{t+6}{4}\cdot\dfrac{3}{3}=\dfrac{3(t+6)}{12}$

$$\begin{aligned}\frac{t+6}{4}+\frac{t-8}{12}&=\frac{3(t+6)}{12}+\frac{t-8}{12} && \text{Write each expression with the LCD.}\\ &=\frac{3(t+6)+(t-8)}{12} && \text{Add the numerators.}\\ &=\frac{3t+18+t-8}{12} && \text{Distribute.}\\ &=\frac{4t+10}{12} && \text{Combine like terms.}\end{aligned}$$

Ask yourself, *"Can I factor the numerator?"* Yes.

$$\begin{aligned}&=\frac{\cancel{2}(2t+5)}{\underset{6}{\cancel{12}}} && \text{Factor.}\\ &=\frac{2t+5}{6} && \text{Reduce.}\end{aligned}$$

b) The LCD of $\dfrac{3}{10a}$ and $\dfrac{7}{8a^2}$ is $40a^2$. Rewrite each expression with the LCD.

$$\frac{3}{10a}\cdot\frac{4a}{4a}=\frac{12a}{40a^2} \quad\text{and}\quad \frac{7}{8a^2}\cdot\frac{5}{5}=\frac{35}{40a^2}$$

$$\begin{aligned}\frac{3}{10a}-\frac{7}{8a^2}&=\frac{12a}{40a^2}-\frac{35}{40a^2} && \text{Write each expression with the LCD.}\\ &=\frac{12a-35}{40a^2} && \text{Subtract the numerators.}\end{aligned}$$

"Can I factor the numerator?" No. The expression is in simplest form since the numerator and denominator have no common factors.

c) First, factor the denominator of $\dfrac{7n - 30}{n^2 - 36}$ to get $\dfrac{7n - 30}{(n + 6)(n - 6)}$.

The LCD of $\dfrac{7n - 30}{(n + 6)(n - 6)}$ and $\dfrac{n}{n + 6}$ is $(n + 6)(n - 6)$.

Rewrite $\dfrac{n}{n + 6}$ with the LCD: $\dfrac{n}{n + 6} \cdot \dfrac{n - 6}{n - 6} = \dfrac{n(n - 6)}{(n + 6)(n - 6)}$

$$
\begin{aligned}
\frac{7n - 30}{n^2 - 36} + \frac{n}{n + 6} &= \frac{7n - 30}{(n + 6)(n - 6)} + \frac{n}{n + 6} && \text{Factor the denominator.} \\
&= \frac{7n - 30}{(n + 6)(n - 6)} + \frac{n(n - 6)}{(n + 6)(n - 6)} && \text{Write each expression with the LCD.} \\
&= \frac{7n - 30 + n(n - 6)}{(n + 6)(n - 6)} && \text{Add the numerators.} \\
&= \frac{7n - 30 + n^2 - 6n}{(n + 6)(n - 6)} && \text{Distribute.} \\
&= \frac{n^2 + n - 30}{(n + 6)(n - 6)} && \text{Combine like terms.}
\end{aligned}
$$

Ask yourself, *"Can I factor the numerator?"* Yes.

$$
\begin{aligned}
&= \frac{\cancel{(n + 6)}(n - 5)}{\cancel{(n + 6)}(n - 6)} && \text{Factor.} \\
&= \frac{n - 5}{n - 6} && \text{Reduce.}
\end{aligned}
$$

■

You Try 3

Add or subtract.

a) $\dfrac{t + 4}{5} + \dfrac{2t - 7}{15}$ b) $\dfrac{7}{12v} - \dfrac{9}{16v^2}$ c) $\dfrac{15h - 8}{h^2 - 64} + \dfrac{h}{h + 8}$

Example 4

Subtract $\dfrac{6r}{r^2 + 10r + 16} - \dfrac{3r + 4}{r^2 + 3r - 40}$.

Solution

Factor the denominators, then write down the LCD.

$$\frac{6r}{r^2 + 10r + 16} = \frac{6r}{(r + 8)(r + 2)}, \qquad \frac{3r + 4}{r^2 + 3r - 40} = \frac{3r + 4}{(r + 8)(r - 5)}$$

Rewrite each expression with the LCD, $(r + 8)(r + 2)(r - 5)$.

$$\frac{6r}{(r+8)(r+2)} \cdot \frac{r-5}{r-5} = \frac{6r(r-5)}{(r+8)(r+2)(r-5)}$$

$$\frac{3r+4}{(r+8)(r-5)} \cdot \frac{r+2}{r+2} = \frac{(3r+4)(r+2)}{(r+8)(r+2)(r-5)}$$

$$\frac{6r}{r^2+10r+16} - \frac{3r+4}{r^2+3r-40}$$

$$= \frac{6r}{(r+8)(r+2)} - \frac{3r+4}{(r+8)(r-5)} \qquad \text{Factor denominators.}$$

$$= \frac{6r(r-5)}{(r+8)(r+2)(r-5)} - \frac{(3r+4)(r+2)}{(r+8)(r+2)(r-5)} \qquad \text{Write each expression with the LCD.}$$

$$= \frac{6r(r-5) - (3r+4)(r+2)}{(r+8)(r+2)(r-5)} \qquad \text{Subtract the numerators.}$$

$$= \frac{6r^2 - 30r - (3r^2 + 10r + 8)}{(r+8)(r+2)(r-5)} \qquad \text{Distribute. You must use parentheses.}$$

$$= \frac{6r^2 - 30r - 3r^2 - 10r - 8}{(r+8)(r+2)(r-5)} \qquad \text{Distribute.}$$

$$= \frac{3r^2 - 40r - 8}{(r+8)(r+2)(r-5)} \qquad \text{Combine like terms.}$$

Ask yourself, *"Can I factor the numerator?"* No. The expression is in simplest form since the numerator and denominator have no common factors. ■

BE CAREFUL

In Example 4, when you move from

$$\frac{6r(r-5) - (3r+4)(r+2)}{(r+8)(r+2)(r-5)} \quad \text{to} \quad \frac{6r^2 - 30r - (3r^2 + 10r + 8)}{(r+8)(r+2)(r-5)}$$

you *must* use parentheses since the entire quantity $3r^2 + 10r + 8$ is being subtracted from $6r^2 - 30r$.

You Try 4

Subtract $\dfrac{4z}{z^2 + 10z + 21} - \dfrac{3z + 5}{z^2 - z - 12}$.

3. Add and Subtract Rational Expressions with Special Denominators

Example 5

Add or subtract.

a) $\dfrac{z}{z-9} - \dfrac{8}{9-z}$ b) $\dfrac{4}{7-w} + \dfrac{10}{w^2-49}$

Solution

a) Recall that $a - b = -(b - a)$. The least common denominator of $\dfrac{z}{z-9}$ and $\dfrac{8}{9-z}$ is $z - 9$ or $9 - z$. We will use LCD $= z - 9$.

Rewrite $\dfrac{8}{9-z}$ with the LCD:

$$\frac{8}{9-z} = \frac{8}{-(z-9)} = -\frac{8}{z-9}$$

$$\begin{aligned}
\frac{z}{z-9} - \frac{8}{9-z} &= \frac{z}{z-9} - \left(-\frac{8}{z-9}\right) && \text{Write each expression with the LCD.}\\
&= \frac{z}{z-9} + \frac{8}{z-9} && \text{Distribute.}\\
&= \frac{z+8}{z-9} && \text{Add the numerators.}
\end{aligned}$$

b) Factor the denominator of $\dfrac{10}{w^2-49}$: $\dfrac{10}{w^2-49} = \dfrac{10}{(w+7)(w-7)}$

Rewrite $\dfrac{4}{7-w}$ with a denominator of $w-7$:

$$\frac{4}{7-w} = \frac{4}{-(w-7)} = -\frac{4}{w-7}$$

Now we must find the LCD of $\dfrac{10}{(w+7)(w-7)}$ and $-\dfrac{4}{w-7}$.

$$\text{LCD} = (w+7)(w-7)$$

Rewrite $-\dfrac{4}{w-7}$ with the LCD.

$$-\frac{4}{w-7} \cdot \frac{w+7}{w+7} = -\frac{4(w+7)}{(w+7)(w-7)} = \frac{-4(w+7)}{(w+7)(w-7)}$$

$$\begin{aligned}
\frac{4}{7-w} + \frac{10}{w^2-49} &= -\frac{4}{w-7} + \frac{10}{(w+7)(w-7)}\\
&= \frac{-4(w+7)}{(w+7)(w-7)} + \frac{10}{(w+7)(w-7)} && \text{Write each expression with the LCD.}\\
&= \frac{-4(w+7)+10}{(w+7)(w-7)} && \text{Add the numerators.}\\
&= \frac{-4w-28+10}{(w+7)(w-7)} && \text{Distribute.}\\
&= \frac{-4w-18}{(w+7)(w-7)} && \text{Combine like terms.}
\end{aligned}$$

Ask yourself, *"Can I factor the numerator?"* Yes.

$$= \frac{-2(2w+9)}{(w+7)(w-7)} \qquad \text{Factor.}$$

Although the numerator factors, the numerator and denominator do not contain any common factors. The result, $\dfrac{-2(2w+9)}{(w+7)(w-7)}$, is in simplest form. ■

You Try 5

Add or subtract.

a) $\dfrac{n}{n-12} - \dfrac{7}{12-n}$ b) $\dfrac{15}{4-t} + \dfrac{20}{t^2-16}$

Answers to You Try Exercises

1) a) $\frac{1}{3}$ b) $\frac{9h+8}{5h-2}$ c) $\frac{h-1}{h-8}$ 2) a) $\frac{7}{8w}$ b) $\frac{m-9}{m}$ 3) a) $\frac{t+1}{3}$ b) $\frac{28v-27}{48v^2}$ 4) $\frac{z^2-42z-35}{(z+7)(z+3)(z-4)}$ 5) a) $\frac{n+7}{n-12}$ b) $\frac{-5(3t+8)}{(t+4)(t-4)}$

8.4 Exercises

Objective 1: Add and Subtract Rational Expressions with a Common Denominator

Add or subtract.

1) $\frac{5}{16}+\frac{9}{16}$

2) $\frac{5}{7}-\frac{3}{7}$

3) $\frac{11}{14}-\frac{3}{14}$

4) $\frac{1}{10}+\frac{9}{10}$

5) $\frac{5}{p}-\frac{23}{p}$

6) $\frac{6}{a}+\frac{3}{a}$

7) $\frac{7}{3c}+\frac{8}{3c}$

8) $\frac{10}{3k^2}-\frac{2}{3k^2}$

9) $\frac{6}{z-1}+\frac{z}{z-1}$

10) $\frac{4n}{n+9}-\frac{6}{n+9}$

VIDEO 11) $\frac{8}{x+4}+\frac{2x}{x+4}$

12) $\frac{5m}{m+7}+\frac{35}{m+7}$

13) $\frac{25t+17}{t(4t+3)}-\frac{5t+2}{t(4t+3)}$

14) $\frac{9w-20}{w(2w-5)}-\frac{20-7w}{w(2w-5)}$

15) $\frac{d^2+15}{(d+5)(d+2)}+\frac{8d-3}{(d+5)(d+2)}$

16) $\frac{2r+15}{(r-5)(r+4)}+\frac{r^2-10r}{(r-5)(r+4)}$

Objective 2: Add and Subtract Rational Expressions with Different Denominators

17) For $\frac{4}{9b^2}$ and $\frac{5}{6b^4}$,

a) find the LCD.

b) explain, in your own words, how to rewrite each expression with the LCD.

c) Rewrite each expression with the LCD.

18) For $\frac{8}{x-3}$ and $\frac{2}{x}$,

a) find the LCD.

b) explain, in your own words, how to rewrite each expression with the LCD.

c) Rewrite each expression with the LCD.

Add or subtract.

19) $\frac{3}{8}+\frac{2}{5}$

20) $\frac{5}{12}-\frac{1}{8}$

21) $\frac{4t}{3}+\frac{3}{2}$

22) $\frac{14x}{15}-\frac{5x}{6}$

23) $\frac{10}{3h^3}+\frac{2}{5h}$

24) $\frac{5}{8u}-\frac{2}{3u^2}$

25) $\frac{3}{2f^2}-\frac{7}{f}$

26) $\frac{8}{5a}+\frac{2}{5a^2}$

27) $\frac{13}{y+3}+\frac{3}{y}$

28) $\frac{3}{k}+\frac{11}{k+9}$

VIDEO 29) $\frac{15}{d-8}-\frac{4}{d}$

30) $\frac{14}{r-5}-\frac{3}{r}$

31) $\frac{9}{c-4}+\frac{6}{c+8}$

32) $\frac{2}{z+5}+\frac{1}{z+2}$

33) $\frac{m}{3m+5}-\frac{2}{m-10}$

34) $\frac{x}{x+4}-\frac{3}{2x+1}$

35) $\frac{8u+2}{u^2-1}+\frac{3u}{u+1}$

36) $\frac{t}{t+7}+\frac{9t-35}{t^2-49}$

37) $\dfrac{7g}{g^2 + 10g + 16} + \dfrac{3}{g^2 - 64}$

38) $\dfrac{b}{b^2 - 25} + \dfrac{8}{b^2 - 3b - 40}$

39) $\dfrac{5a}{a^2 - 6a - 27} - \dfrac{2a + 1}{a^2 + 2a - 3}$

40) $\dfrac{3c}{c^2 + 4c - 12} - \dfrac{2c - 5}{c^2 + 2c - 24}$

41) $\dfrac{2x}{x^2 + x - 20} - \dfrac{4}{x^2 + 2x - 15}$

42) $\dfrac{3m}{m^2 + 10m + 24} - \dfrac{2}{m^2 + 3m - 4}$

43) $\dfrac{4b + 1}{3b - 12} + \dfrac{5b}{b^2 - b - 12}$

44) $\dfrac{k + 12}{2k - 18} + \dfrac{3k}{k^2 - 12k + 27}$

Objective 3: Add and Subtract Rational Expressions with Special Denominators

45) Is $(x - 6)(6 - x)$ the LCD for $\dfrac{9}{x - 6} + \dfrac{4}{6 - x}$? Why or why not?

46) When Lamar adds $\dfrac{u}{7 - 2u} + \dfrac{5}{2u - 7}$, he gets $\dfrac{u - 5}{7 - 2u}$, but when he checks his answer in the back of the book, it says that the answer is $\dfrac{5 - u}{2u - 7}$. Which is the correct answer?

Add or subtract.

47) $\dfrac{16}{q - 4} + \dfrac{10}{4 - q}$

48) $\dfrac{8}{z - 9} + \dfrac{4}{9 - z}$

49) $\dfrac{11}{f - 7} - \dfrac{15}{7 - f}$

50) $\dfrac{5}{a - b} - \dfrac{4}{b - a}$

51) $\dfrac{7}{x - 4} + \dfrac{x - 1}{4 - x}$

52) $\dfrac{10}{m - 5} + \dfrac{m + 21}{5 - m}$

53) $\dfrac{8}{3 - a} + \dfrac{a + 5}{a - 3}$

54) $\dfrac{9}{6 - n} + \dfrac{n + 3}{n - 6}$

55) $\dfrac{3}{2u - 3v} - \dfrac{6u}{3v - 2u}$

56) $\dfrac{3c}{11b - 5c} - \dfrac{9}{5c - 11b}$

57) $\dfrac{8}{x^2 - 9} + \dfrac{2}{3 - x}$

58) $\dfrac{4}{8 - y} + \dfrac{12}{y^2 - 64}$

59) $\dfrac{a}{4a^2 - 9} - \dfrac{4}{3 - 2a}$

60) $\dfrac{3b}{9b^2 - 25} - \dfrac{3}{5 - 3b}$

Mixed Exercises: Objectives 2 and 3

Perform the indicated operations.

61) $\dfrac{5}{a^2 - 2a} + \dfrac{8}{a} - \dfrac{10a}{a - 2}$

62) $\dfrac{3}{j^2 + 6j} + \dfrac{2j}{j + 6} - \dfrac{2}{3j}$

63) $\dfrac{3b - 1}{b^2 + 8b} + \dfrac{b}{3b^2 + 25b + 8} + \dfrac{2}{3b^2 + b}$

64) $\dfrac{2k + 7}{k^2 - 4k} + \dfrac{9k}{2k^2 - 15k + 28} + \dfrac{15}{2k^2 - 7k}$

65) $\dfrac{c}{c^2 - 8c + 16} - \dfrac{5}{c^2 - c - 12}$

66) $\dfrac{n}{n^2 + 11n + 30} - \dfrac{6}{n^2 + 10n + 25}$

67) $\dfrac{9}{4a + 4b} + \dfrac{8}{a - b} - \dfrac{6a}{a^2 - b^2}$

68) $\dfrac{1}{x + y} + \dfrac{x}{x^2 - y^2} - \dfrac{10}{5x - 5y}$

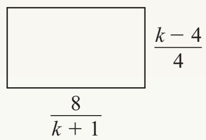
69) $\dfrac{2v + 1}{6v^2 - 29v - 5} - \dfrac{v - 2}{3v^2 - 13v - 10}$

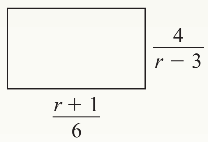
70) $\dfrac{n + 2}{4n^2 + 11n - 3} - \dfrac{n - 3}{2n^2 + 7n + 3}$

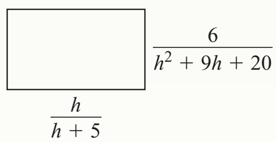
71) $\dfrac{g - 5}{5g^2 - 30g} + \dfrac{g}{2g^2 - 17g + 30} - \dfrac{6}{2g^2 - 5g}$

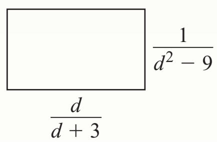
72) $\dfrac{y + 6}{y^2 - 4y} + \dfrac{y}{2y^2 - 13y + 20} - \dfrac{1}{2y^2 - 5y}$

For each rectangle, find a rational expression in simplest form to represent its a) area and b) perimeter.

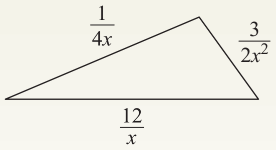
73) 

74) 

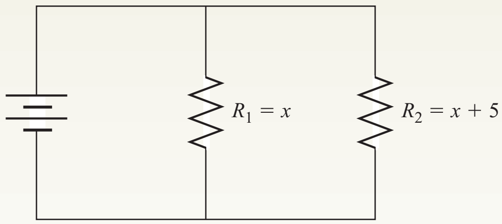
75) 

76) 

77) Find a rational expression in simplest form to represent the perimeter of the triangle.



78) The total resistance of a set of resistors in parallel in an electrical circuit can be found using the formula $\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$, where $R_1 =$ the resistance in resistor 1, $R_2 =$ the resistance in resistor 2, and $R_T =$ the total resistance in the circuit. (Resistance is measured in ohms.) For the given circuit,



a) find the sum $\frac{1}{x} + \frac{1}{x+5}$.

b) find an expression for the total resistance, R_T, in the circuit.

c) if $R_1 = 10$ ohms, what is the total resistance in the circuit?

Putting It All Together

Objective

1. Review the Concepts Presented in Sections 8.1–8.4

1. Review the Concepts Presented in Sections 8.1–8.4

In Section 8.1, we defined a rational expression, and we evaluated expressions. We also discussed how to write a rational expression in lowest terms.

Example 1

Write in lowest terms: $\frac{5n^2 - 45n}{n^2 - 11n + 18}$

Solution

$$\frac{5n^2 - 45n}{n^2 - 11n + 18} = \frac{5n\cancel{(n-9)}}{(n-2)\cancel{(n-9)}} \quad \text{Factor.}$$

$$= \frac{5n}{n-2} \quad \text{Divide by } n - 9.$$

Recall that a rational expression *equals zero* when its *numerator equals zero*. A rational expression is *undefined* when its *denominator equals zero*.

Example 2

Determine the values of c for which $\frac{c+8}{c^2 - 25}$

a) equals zero. b) is undefined.

Solution

a) $\frac{c+8}{c^2-25}$ equals zero when its numerator equals zero.
Let $c + 8 = 0$, and solve for c.

$$c + 8 = 0$$
$$c = -8$$

$\frac{c+8}{c^2-25}$ equals zero when $c = -8$.

b) $\dfrac{c+8}{c^2-25}$ is undefined when its denominator equals zero.
Let $c^2 - 25 = 0$, and solve for c.

$$\begin{aligned} c^2 - 25 &= 0 \\ (c+5)(c-5) &= 0 && \text{Factor.} \\ c+5 = 0 \quad &\text{or} \quad c-5=0 && \text{Set each factor equal to zero.} \\ c = -5 \quad &\text{or} \quad c = 5 && \text{Solve.} \end{aligned}$$

$\dfrac{c+8}{c^2-25}$ is undefined when $c = 5$ or $c = -5$. So, $c \neq 5$ and $c \neq -5$ in the expression.

In Sections 8.2–8.4, we learned how to multiply, divide, add, and subtract rational expressions. Now we will practice these operations together so that we will learn to recognize the techniques needed to perform these operations.

Example 3

Divide $\dfrac{t^2 - 3t - 28}{16t^2 - 81} \div \dfrac{t^2 - 7t}{54 - 24t}$.

Solution

Do we need a common denominator to divide? *No.* A common denominator is needed to add or subtract but not to multiply or divide.

To divide, multiply the first rational expression by the reciprocal of the second expression, then factor, reduce, and multiply.

$$\begin{aligned} \frac{t^2 - 3t - 28}{16t^2 - 81} \div \frac{t^2 - 7t}{54 - 24t} &= \frac{t^2 - 3t - 28}{16t^2 - 81} \cdot \frac{54 - 24t}{t^2 - 7t} && \text{Multiply by the reciprocal.} \\ &= \frac{(t+4)\cancel{(t-7)}}{(4t+9)\cancel{(4t-9)}} \cdot \frac{6\overset{-1}{\cancel{(9-4t)}}}{t\cancel{(t-7)}} && \text{Factor and reduce.} \\ &= -\frac{6(t+4)}{t(4t+9)} && \text{Multiply.} \end{aligned}$$

Recall that $\dfrac{9-4t}{4t-9} = -1$.

Example 4

Add $\dfrac{x}{x+2} + \dfrac{4}{3x-1}$.

Solution

To add or subtract rational expressions, we need a common denominator. We do not need to factor these denominators, so we are ready to identify the LCD.

$$\text{LCD} = (x+2)(3x-1)$$

Rewrite each expression with the LCD.

$$\frac{x}{x+2}\cdot\frac{3x-1}{3x-1}=\frac{x(3x-1)}{(x+2)(3x-1)},\quad \frac{4}{3x-1}\cdot\frac{x+2}{x+2}=\frac{4(x+2)}{(x+2)(3x-1)}$$

$$\frac{x}{x+2}+\frac{4}{3x-1}=\frac{x(3x-1)}{(x+2)(3x-1)}+\frac{4(x+2)}{(x+2)(3x-1)}$$ Write each expression with the LCD.

$$=\frac{x(3x-1)+4(x+2)}{(x+2)(3x-1)}$$ Add the numerators.

$$=\frac{3x^2-x+4x+8}{(x+2)(3x-1)}$$ Distribute.

$$=\frac{3x^2+3x+8}{(x+2)(3x-1)}$$ Combine like terms.

Although this numerator will not factor, remember that sometimes it *is* possible to factor the numerator and simplify the result. ■

You Try 1

a) Write in lowest terms: $\dfrac{3k^2-14k+16}{4-k^2}$ b) Subtract $\dfrac{b}{b+10}-\dfrac{3}{b}$.

c) Multiply $\dfrac{r^3+9r^2-r-9}{12r+12}\cdot\dfrac{5}{r^2+3r-54}$.

d) Determine the values of w for which $\dfrac{6w-1}{2w^2+16w}$ i) equals zero. ii) is undefined.

Answers to You Try Exercises

1) a) $\dfrac{8-3k}{k+2}$ b) $\dfrac{b^2-3b-30}{b(b+10)}$ c) $\dfrac{5(r-1)}{12(r-6)}$ d) i) $\dfrac{1}{6}$ ii) $-8, 0$

Putting It All Together Summary Exercises

Objective 1: Review the Concepts Presented in Sections 8.1–8.4

Evaluate, if possible, for a) $x=-3$ and b) $x=2$.

1) $\dfrac{x+3}{3x+4}$

2) $\dfrac{x}{x-2}$

3) $\dfrac{5x-3}{x^2+10x+21}$

4) $-\dfrac{x^2}{x^2-12}$

Determine the values of the variable for which

a) the expression is undefined.

b) the expression equals zero.

5) $-\dfrac{5w}{w^2-36}$

6) $\dfrac{m-4}{2m^2+11m+15}$

VIDEO 7) $\dfrac{3-5b}{b^2+2b-8}$

8) $\dfrac{5k-8}{64-k^2}$

9) $\dfrac{12}{5r}$

10) $\dfrac{t-15}{t^2+4}$

Write each rational expression in lowest terms.

11) $\dfrac{12w^{16}}{3w^5}$

12) $\dfrac{42n^3}{18n^8}$

13) $\dfrac{m^2+6m-27}{2m^2+2m-24}$

14) $\dfrac{2j+20}{2j^2+10j-100}$

VIDEO 15) $\dfrac{12-15n}{5n^2+6n-8}$

16) $\dfrac{-x-y}{xy+y^2+5x+5y}$

Perform the operations and simplify.

17) $\dfrac{4c^2 + 4c - 24}{c + 3} \div \dfrac{3c - 6}{8}$

18) $\dfrac{6}{f + 11} - \dfrac{2}{f}$

19) $\dfrac{4j}{j^2 - 81} + \dfrac{3}{j^2 - 3j - 54}$

20) $\dfrac{27a^4}{8b} \cdot \dfrac{40b^2}{81a^2}$

21) $\dfrac{12y^7}{4z^6} \cdot \dfrac{8z^4}{72y^6}$

22) $\dfrac{3}{q^2 - q - 20} + \dfrac{8q}{q^2 + 11q + 28}$

23) $\dfrac{x}{2x^2 - 7x - 4} - \dfrac{x + 3}{4x^2 + 4x + 1}$

24) $\dfrac{n - 4}{4n - 44} \cdot \dfrac{121 - n^2}{n + 11}$

25) $\dfrac{16 - m^2}{m + 4} \div \dfrac{8m - 32}{m + 7}$

26) $\dfrac{16}{r - 7} + \dfrac{4}{7 - r}$

27) $\dfrac{3xy - 24x - 5y + 40}{y^2 - 64} \div \dfrac{27x^3 - 125}{9x}$

28) $\dfrac{\dfrac{10d}{d + 11}}{\dfrac{5d^7}{3d + 33}}$

29) $\dfrac{9}{d + 3} + \dfrac{8}{d^2}$

30) $\dfrac{3a^2 - 6a + 12}{5a - 10} \cdot \dfrac{a^2 - 4}{a^3 + 8}$

31) $\dfrac{\dfrac{9k^2 - 1}{14k}}{\dfrac{3k - 1}{21k^4}}$

32) $\dfrac{13}{4z} - \dfrac{1}{3z}$

33) $\dfrac{2w}{25 - w^2} + \dfrac{w - 3}{w^2 - 12w + 35}$

34) $\dfrac{12a^4}{10a - 30} \div \dfrac{4a}{a^3 - 3a^2 + 5a - 15}$

35) $\dfrac{10}{x - 8} + \dfrac{4}{x + 3}$

36) $\dfrac{1}{4y} + \dfrac{8}{6y^4}$

37) $\dfrac{2h^2 + 11h + 5}{8} \div (2h + 1)^2$

38) $\dfrac{b^2 - 15b + 36}{b^2 - 8b - 48} \cdot (b + 4)^2$

39) $\dfrac{3m}{7m - 4n} - \dfrac{20n}{4n - 7m}$

40) $\dfrac{10d}{8c - 10d} + \dfrac{8c}{10d - 8c}$

41) $\dfrac{2p + 3}{p^2 + 7p} - \dfrac{4p}{p^2 - p - 56} + \dfrac{5}{p^2 - 8p}$

42) $\dfrac{6u + 1}{3u^2 - 2u} - \dfrac{u}{3u^2 + u - 2} + \dfrac{10}{u^2 + u}$

43) $\dfrac{6t + 6}{3t^2 - 24t} \cdot (t^2 - 7t - 8)$

44) $\dfrac{3r^2 + r - 14}{5r^3 - 10r^2} \div (9r^2 - 49)$

45) $\dfrac{\dfrac{3c^3}{8c + 40}}{\dfrac{9c}{c + 5}}$

46) $\dfrac{\dfrac{6v - 30}{4}}{\dfrac{v - 5}{3}}$

47) $\dfrac{f - 8}{f - 4} - \dfrac{4}{4 - f}$

48) $\dfrac{12p}{4p^2 + 11p + 6} - \dfrac{5}{p^2 - 4p - 12}$

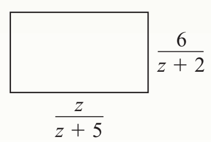
49) $\left(\dfrac{3m}{3m - 1} - \dfrac{4}{m + 4}\right) \cdot \dfrac{9m^2 - 1}{21m^2 + 28}$

50) $\left(\dfrac{2c}{c + 8} + \dfrac{4}{c - 2}\right) \div \dfrac{6}{4c + 32}$

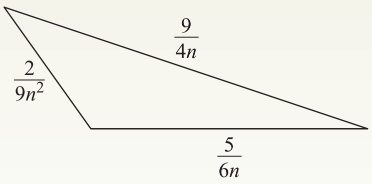
51) $\dfrac{3}{k^2 + 3k} - \dfrac{4}{3k} + \dfrac{1}{k + 3}$

52) $\dfrac{3}{w^2 - w} + \dfrac{4}{5w} - \dfrac{3}{w - 1}$

53) Find a rational expression in simplest form to represent the a) area and b) perimeter of the rectangle.



54) Find a rational expression in simplest form to represent the perimeter of the triangle.



Determine the domain of each rational function.

55) $f(t) = \dfrac{t + 8}{4t^2 - 1}$

56) $h(a) = \dfrac{3a}{a + 7}$

57) $g(x) = \dfrac{4x - 5}{3}$

58) $k(x) = \dfrac{2}{x^2 - 10x}$

59) $P(n) = \dfrac{7}{4n}$

60) $F(c) = \dfrac{c + 1}{c^2 + 3c - 54}$

Section 8.5 Simplifying Complex Fractions

Objectives

1. **Simplify a Complex Fraction with One Term in the Numerator and One Term in the Denominator**
2. **Simplify a Complex Fraction with More Than One Term in the Numerator and/or Denominator by Rewriting It as a Division Problem**
3. **Simplify a Complex Fraction with More Than One Term in the Numerator and/or Denominator by Multiplying by the LCD**

In algebra, we sometimes encounter fractions that contain fractions in their numerators, denominators, or both. These are called **complex fractions.** Some examples of complex fractions are

$$\frac{\frac{3}{7}}{\frac{9}{2}}, \quad \frac{\frac{1}{8} + \frac{5}{6}}{2 - \frac{2}{3}}, \quad \frac{\frac{4}{xy^2}}{\frac{1}{x} - \frac{1}{y}}, \quad \frac{\frac{5a - 15}{4}}{\frac{a - 3}{a}}$$

Definition

A **complex fraction** is a rational expression that contains one or more fractions in its numerator, its denominator, or both.

A complex fraction is not considered to be an expression in simplest form. In this section, we will learn how to simplify two different types of complex fractions:

1) Complex fractions with *one term* in the numerator and *one term* in the denominator
2) Complex fractions that have *more than one term* in their numerators, their denominators or both.

1. Simplify a Complex Fraction with One Term in the Numerator and One Term in the Denominator

We studied these expressions in Section 8.2 when we learned how to divide fractions. We will look at another example.

Example 1

Simplify. $\dfrac{\frac{5a - 15}{4}}{\frac{a - 3}{a}}$

This is the numerator.

This is the main fraction bar.

This is the denominator.

Solution

This complex fraction contains one term in the numerator and one term in the denominator. To simplify, rewrite as a division problem, multiply by the reciprocal, and simplify.

$$\frac{\frac{5a-15}{4}}{\frac{a-3}{a}} = \frac{5a-15}{4} \div \frac{a-3}{a} \qquad \text{Rewrite as a division problem.}$$

$$= \frac{5a-15}{4} \cdot \frac{a}{a-3} = \frac{5(a-3)}{4} \cdot \frac{a}{a-3} = \frac{5\cancel{(a-3)}}{4} \cdot \frac{a}{\cancel{a-3}} = \frac{5a}{4}$$

You Try 1

Simplify. $\dfrac{\frac{9}{z^2-64}}{\frac{3z+3}{z+8}}$

Let's summarize how to simplify this first type of complex fraction.

Procedure Simplify a Complex Fraction Containing One Term in the Numerator and One Term in the Denominator

To simplify a complex fraction containing one term in the numerator and one term in the denominator:

1) Rewrite the complex fraction as a division problem.
2) Perform the division by multiplying the first fraction by the reciprocal of the second (that is, multiply the numerator of the complex fraction by the reciprocal of the denominator).

2. Simplify a Complex Fraction with More Than One Term in the Numerator and/or Denominator by Rewriting It as a Division Problem

When a complex fraction has more than one term in the numerator and/or the denominator, we can use one of two methods to simplify.

Procedure Simplify a Complex Fraction Using Method I

1) Combine the terms in the numerator and combine the terms in the denominator so that each contains only one fraction.
2) Rewrite as a division problem.
3) Perform the division by multiplying the first fraction by the reciprocal of the second.

Example 2

Simplify.

a) $\dfrac{\frac{1}{4}+\frac{2}{3}}{2-\frac{1}{2}}$ b) $\dfrac{\frac{5}{a^2b}}{\frac{a}{b}+\frac{1}{a}}$

Solution

a) The numerator, $\frac{1}{4} + \frac{2}{3}$, contains two terms; the denominator, $2 - \frac{1}{2}$, contains two terms. We will add the terms in the numerator and subtract the terms in the denominator so that the numerator and denominator will each contain one fraction.

$$\frac{\frac{1}{4} + \frac{2}{3}}{2 - \frac{1}{2}} = \frac{\frac{3}{12} + \frac{8}{12}}{\frac{4}{2} - \frac{1}{2}} = \frac{\frac{11}{12}}{\frac{3}{2}}$$

Add the fractions in the numerator.
Subtract the fractions in the denominator.

Rewrite as a division problem, multiply by the reciprocal, and simplify.

$$\frac{11}{12} \div \frac{3}{2} = \frac{11}{\underset{6}{\cancel{12}}} \cdot \frac{\overset{1}{\cancel{2}}}{3} = \frac{11}{18}$$

b) The numerator, $\frac{5}{a^2b}$, contains one term; the denominator, $\frac{a}{b} + \frac{1}{a}$, contains two terms.

We will add the terms in the denominator so that it, like the numerator, will contain only one term. The LCD of the expressions in the denominator is ab.

$$\frac{\frac{5}{a^2b}}{\frac{a}{b} + \frac{1}{a}} = \frac{\frac{5}{a^2b}}{\frac{a}{b} \cdot \frac{a}{a} + \frac{1}{a} \cdot \frac{b}{b}} = \frac{\frac{5}{a^2b}}{\frac{a^2}{ab} + \frac{b}{ab}} = \frac{\frac{5}{a^2b}}{\frac{a^2 + b}{ab}}$$

Rewrite as a division problem, multiply by the reciprocal, and simplify.

$$\frac{5}{a^2b} \div \frac{a^2 + b}{ab} = \frac{5}{\underset{a}{\cancel{a^2}\cancel{b}}} \cdot \frac{\cancel{a}\cancel{b}}{a^2 + b} = \frac{5}{a(a^2 + b)}$$

You Try 2

Simplify.

a) $\dfrac{\frac{5}{8} - \frac{1}{2}}{1 + \frac{1}{4}}$ b) $\dfrac{\frac{6}{r^2t^2}}{\frac{2}{r} - \frac{r}{t}}$

3. Simplify a Complex Fraction with More than One Term in the Numerator and/or Denominator by Multiplying by the LCD

Another method we can use to simplify complex fractions involves multiplying the numerator and denominator of the complex fraction by the LCD of *all* of the fractions in the expression.

Procedure Simplify a Complex Fraction Using Method 2

1) Identify and write down the LCD of *all* of the fractions in the complex fraction.
2) Multiply the numerator and denominator of the complex fraction by the LCD.
3) Simplify.

We will simplify the complex fractions we simplified in Example 2 using method 2.

Example 3

Simplify using method 2.

a) $\dfrac{\frac{1}{4}+\frac{2}{3}}{2-\frac{1}{2}}$ b) $\dfrac{\frac{5}{a^2b}}{\frac{a}{b}+\frac{1}{a}}$

Solution

a) List all of the fractions in the complex fraction: $\frac{1}{4}, \frac{2}{3}, \frac{1}{2}$. Write down their LCD: LCD $= 12$.

Multiply the numerator and denominator of the complex fraction by the LCD, 12, then simplify.

$$\frac{12\left(\frac{1}{4}+\frac{2}{3}\right)}{12\left(2-\frac{1}{2}\right)} \quad \text{We are multiplying the expression by } \frac{12}{12}, \text{ which equals 1.}$$

$$= \frac{12 \cdot \frac{1}{4} + 12 \cdot \frac{2}{3}}{12 \cdot 2 - 12 \cdot \frac{1}{2}} \quad \text{Distribute.}$$

$$= \frac{3+8}{24-6} = \frac{11}{18} \quad \text{Simplify.}$$

This is the same result we obtained in Example 2 using method 1.

Note

In the denominator, we multiplied the 2 by 12 even though 2 is not a fraction. Remember, *all* terms, not just the fractions, must be multiplied by the LCD.

b) List all of the fractions in the complex fraction: $\frac{5}{a^2b}, \frac{a}{b}, \frac{1}{a}$. Write down their LCD: LCD $= a^2b$.

Multiply the numerator and denominator of the complex fraction by the LCD, a^2b, then simplify.

$$\frac{a^2b\left(\frac{5}{a^2b}\right)}{a^2b\left(\frac{a}{b}+\frac{1}{a}\right)} \quad \text{We are multiplying the expression by } \frac{a^2b}{a^2b}, \text{ which equals 1.}$$

$$= \frac{a^2b \cdot \frac{5}{a^2b}}{a^2b \cdot \frac{a}{b} + a^2b \cdot \frac{1}{a}} \quad \text{Distribute.}$$

$$= \frac{5}{a^3+ab} = \frac{5}{a(a^2+b)} \quad \text{Simplify.}$$

If the numerator and denominator factor, factor them. Sometimes, you can divide by a common factor to simplify.

Notice that the result is the same as what was obtained in Example 2 using method 1. ■

You Try 3

Simplify using method 2.

a) $\dfrac{\dfrac{5}{8}-\dfrac{1}{2}}{1+\dfrac{1}{4}}$ b) $\dfrac{\dfrac{6}{r^2t^2}}{\dfrac{2}{r}-\dfrac{r}{t}}$

You should be familiar with both methods for simplifying complex fractions containing two terms in the numerator or denominator. After a lot of practice, you will be able to decide which method is better for a particular problem.

Example 4

Determine which method to use to simplify each complex fraction, then simplify.

a) $\dfrac{\dfrac{4}{x}+\dfrac{1}{x+3}}{\dfrac{2}{x+3}-\dfrac{1}{x}}$ b) $\dfrac{\dfrac{n^2-1}{7n+28}}{\dfrac{6n-6}{n^2-16}}$

Solution

a) This complex fraction contains two terms in the numerator and two terms in the denominator. Let's use method 2: multiply the numerator and denominator by the LCD of all of the fractions.

List all of the fractions in the complex fraction: $\dfrac{4}{x}, \dfrac{1}{x+3}, \dfrac{2}{x+3}, \dfrac{1}{x}$. Write down their LCD: LCD $= x(x+3)$.

Multiply the numerator and denominator of the complex fraction by the LCD, $x(x+3)$, then simplify.

$$\frac{x(x+3)\left(\dfrac{4}{x}+\dfrac{1}{x+3}\right)}{x(x+3)\left(\dfrac{2}{x+3}-\dfrac{1}{x}\right)} = \frac{\cancel{x}(x+3)\cdot\dfrac{4}{\cancel{x}}+x\cancel{(x+3)}\cdot\dfrac{1}{\cancel{x+3}}}{x\cancel{(x+3)}\cdot\dfrac{2}{\cancel{x+3}}-\cancel{x}(x+3)\cdot\dfrac{1}{\cancel{x}}}$$

Multiply the numerator and denominator by $x(x+3)$ and distribute.

$$= \frac{4(x+3)+x}{2x-(x+3)}$$

Multiply.

$$= \frac{4x+12+x}{2x-x-3} = \frac{5x+12}{x-3}$$

Distribute and combine like terms.

b) The complex fraction $\dfrac{\dfrac{n^2-1}{7n+28}}{\dfrac{6n-6}{n^2-16}}$ contains one term in the numerator, $\dfrac{n^2-1}{7n+28}$, and one term in the denominator, $\dfrac{6n-6}{n^2-16}$. To simplify, rewrite as a division problem, multiply by the reciprocal, and simplify.

$$\frac{\dfrac{n^2-1}{7n+28}}{\dfrac{6n-6}{n^2-16}} = \frac{n^2-1}{7n+28} \div \frac{6n-6}{n^2-16}$$ Rewrite as a division problem.

$$= \frac{n^2-1}{7n+28} \cdot \frac{n^2-16}{6n-6}$$ Multiply by the reciprocal.

$$= \frac{(n+1)\cancel{(n-1)}}{7\cancel{(n+4)}} \cdot \frac{\cancel{(n+4)}(n-4)}{6\cancel{(n-1)}}$$ Factor and reduce.

$$= \frac{(n+1)(n-4)}{42}$$ Multiply.

You Try 4

Determine which method to use to simplify each complex fraction, then simplify.

a) $\dfrac{\dfrac{8}{k} - \dfrac{1}{k+5}}{\dfrac{3}{k+5} + \dfrac{5}{k}}$ b) $\dfrac{\dfrac{c^2-9}{8c-56}}{\dfrac{2c+6}{c^2-49}}$

Answers to You Try Exercises

1) $\dfrac{3}{(z-8)(z+1)}$ 2) a) $\dfrac{1}{10}$ b) $\dfrac{6}{rt(2t-r^2)}$ 3) a) $\dfrac{1}{10}$ b) $\dfrac{6}{rt(2t-r^2)}$

4) a) $\dfrac{7k+40}{8k+25}$ b) $\dfrac{(c-3)(c+7)}{16}$

8.5 Exercises

Objective 1: Simplify a Complex Fraction with One Term in the Numerator and One Term in the Denominator

1) Explain, in your own words, two ways to simplify $\dfrac{\dfrac{2}{9}}{\dfrac{5}{18}}$. Then simplify it both ways. Which method do you prefer and why?

2) Explain, in your own words, two ways to simplify $\dfrac{\dfrac{3}{2} - \dfrac{1}{5}}{\dfrac{1}{10} + \dfrac{3}{5}}$. Then simplify it both ways. Which method do you prefer and why?

Simplify completely.

3) $\dfrac{\frac{5}{9}}{\frac{7}{4}}$

4) $\dfrac{\frac{3}{10}}{\frac{5}{6}}$

5) $\dfrac{\frac{u^4}{v^2}}{\frac{u^3}{v}}$

6) $\dfrac{\frac{a^3}{b}}{\frac{a}{b^3}}$

7) $\dfrac{\frac{x^4}{y}}{\frac{x^2}{y^2}}$

8) $\dfrac{\frac{s^3}{t^3}}{\frac{s^4}{t}}$

9) $\dfrac{\frac{14m^5n^4}{9}}{\frac{35mn^6}{3}}$

10) $\dfrac{\frac{11b^4c^2}{4}}{\frac{55bc}{12}}$

11) $\dfrac{\frac{m-7}{m}}{\frac{m-7}{18}}$

12) $\dfrac{\frac{t-4}{9}}{\frac{t-4}{t^2}}$

13) $\dfrac{\frac{g^2-36}{20}}{\frac{g+6}{60}}$

14) $\dfrac{\frac{6}{y^2-49}}{\frac{8}{y+7}}$

15) $\dfrac{\frac{d^3}{16d-24}}{\frac{d}{40d-60}}$

16) $\dfrac{\frac{45w-63}{w^5}}{\frac{30w-42}{w^2}}$

17) $\dfrac{\frac{c^2-7c-8}{11c}}{\frac{c+1}{c}}$

18) $\dfrac{\frac{5x}{x-3}}{\frac{5}{x^2+4x-21}}$

Objective 2: Simplify a Complex Fraction with More Than One Term in the Numerator and/or Denominator by Rewriting It as a Division Problem

Simplify using method 1.

19) $\dfrac{\frac{7}{9}-\frac{2}{3}}{3+\frac{1}{9}}$

20) $\dfrac{\frac{1}{2}+\frac{3}{4}}{\frac{2}{3}+\frac{3}{2}}$

21) $\dfrac{\frac{r}{s}-4}{\frac{3}{s}+\frac{1}{r}}$

22) $\dfrac{\frac{4}{c}-c^2}{1+\frac{8}{c}}$

23) $\dfrac{\frac{8}{r^2t}}{\frac{3}{r}-\frac{r}{t}}$

24) $\dfrac{\frac{9}{h^2k^3}}{\frac{6}{hk}-\frac{24}{k^2}}$

25) $\dfrac{\frac{5}{w-1}+\frac{3}{w+4}}{\frac{6}{w+4}+\frac{4}{w-1}}$

26) $\dfrac{\frac{5}{z-2}-\frac{2}{z+3}}{\frac{4}{z-2}+\frac{1}{z+3}}$

Objective 3: Simplify a Complex Fraction with More Than One Term in the Numerator and/or Denominator by Multiplying by the LCD

Simplify the complex fractions in Exercises 19–26 using method 2. Think about which method you prefer. (You will discuss your preference in Exercises 35 and 36.)

27) Rework Exercise 19.

28) Rework Exercise 20.

29) Rework Exercise 21.

30) Rework Exercise 22.

31) Rework Exercise 23.

32) Rework Exercise 24.

33) Rework Exercise 25.

34) Rework Exercise 26.

35) In Exercises 19–34, which types of complex fractions did you prefer to simplify using method 1? Why?

36) In Exercises 19–34, which types of complex fractions did you prefer to simplify using method 2? Why?

Mixed Exercises: Objectives 1–3

Simplify completely using any method.

37) $\dfrac{\frac{a-4}{12}}{\frac{a-4}{a}}$

38) $\dfrac{\frac{z^2+1}{5}}{z+\frac{1}{z}}$

39) $\dfrac{\frac{3}{n}-\frac{4}{n-2}}{\frac{1}{n-2}+\frac{5}{n}}$

40) $\dfrac{\frac{1}{6}-\frac{5}{4}}{\frac{3}{5}+\frac{1}{3}}$

41) $\dfrac{\frac{6}{w}-w}{1+\frac{6}{w}}$

42) $\dfrac{\frac{6t+48}{t}}{\frac{9t+72}{7}}$

43) $\dfrac{\frac{6}{5}}{\frac{9}{15}}$

44) $\dfrac{\frac{8}{k+7} + \frac{1}{k}}{\frac{9}{k} + \frac{2}{k+7}}$

45) $\dfrac{1 - \frac{4}{t+5}}{\frac{4}{t^2-25} + \frac{t}{t-5}}$

46) $\dfrac{\frac{c^2}{d} + \frac{2}{c^2 d}}{\frac{d}{c} - \frac{c}{d}}$

47) $\dfrac{\frac{9}{x} - \frac{9}{y}}{\frac{2}{x^2} - \frac{2}{y^2}}$

48) $\dfrac{\frac{m^2}{n^2}}{\frac{m^5}{n}}$

49) $\dfrac{\frac{24c-60}{5}}{\frac{8c-20}{c^2}}$

50) $\dfrac{\frac{3}{x^2y^2}}{\frac{x}{y} + \frac{1}{x}}$

51) $\dfrac{\frac{4}{9} + \frac{2}{5}}{\frac{1}{5} - \frac{2}{3}}$

52) $\dfrac{1 + \frac{4}{t-3}}{\frac{t}{t-3} + \frac{2}{t^2-9}}$

53) $\dfrac{\frac{1}{10}}{\frac{7}{8}}$

54) $\dfrac{\frac{r^2-6}{40}}{r - \frac{6}{r}}$

55) $\dfrac{\frac{2}{uv^2}}{\frac{6}{v} - \frac{4v}{u}}$

56) $\dfrac{\frac{y^2-9}{3y+15}}{\frac{7y-21}{y^2-25}}$

57) $\dfrac{1 + \frac{b}{a-b}}{\frac{b}{a^2-b^2} + \frac{1}{a+b}}$

58) $\dfrac{\frac{c}{c+2} + \frac{1}{c^2-4}}{1 - \frac{3}{c+2}}$

59) $\dfrac{\frac{x^2-x-42}{2x-14}}{\frac{x^2-36}{8x+16}}$

60) $\dfrac{3b + \frac{1}{b}}{b - \frac{13}{b}}$

61) $\dfrac{\frac{y^4}{z^3}}{\frac{y^6}{z^4}}$

62) $\dfrac{\frac{k+6}{k}}{\frac{k+6}{5}}$

63) $\dfrac{7 - \frac{8}{m}}{\frac{7m-8}{11}}$

64) $\dfrac{\frac{7}{a} - \frac{7}{b}}{\frac{1}{a^2} - \frac{1}{b^2}}$

65) $\dfrac{\frac{1}{h^2-4} + \frac{2}{h+2}}{h - \frac{3}{2}}$

66) $\dfrac{\frac{w^2+10w+25}{25-w^2}}{\frac{w^3+125}{4w-20}}$

67) $\dfrac{\frac{6}{v+3} - \frac{4}{v-1}}{\frac{2}{v-1} + \frac{1}{v+2}}$

68) $\dfrac{\frac{5}{r+2} + \frac{7}{2r-3}}{\frac{1}{r-3} + \frac{3}{2r-3}}$

Section 8.6 Solving Rational Equations

Objectives

1. **Differentiate Between Rational Expressions and Rational Equations**
2. **Solve Rational Equations**
3. **Solve a Proportion**
4. **Solve an Equation for a Specific Variable**

A **rational equation** is an equation that contains a rational expression. Some examples of rational equations are

$$\frac{1}{2}a + \frac{7}{10} = \frac{3}{5}a - 4, \qquad \frac{8}{p+7} - \frac{p}{p-10} = 2, \qquad \frac{3n}{n^2+10n+16} + \frac{5}{n+8} = \frac{1}{n+2}$$

1. Differentiate Between Rational Expressions and Rational Equations

In Chapter 3, we solved rational equations like the first one above, and we learned how to add and subtract rational expressions in Section 8.4. Let's summarize the difference between the two because this is often a point of confusion for students.

Summary Expressions Versus Equations

1) *The sum or difference of rational expressions does* not *contain an = sign.* To add or subtract, rewrite each expression with the LCD, and *keep the denominator* while performing the operations.

2) *An equation contains an = sign.* To solve an equation containing rational expressions, *multiply* the equation by the LCD of all fractions to *eliminate* the denominators, then solve.

Example 1

Determine whether each is an equation or is a sum or difference of expressions. Then, solve the equation or find the sum or difference.

a) $\dfrac{c-5}{6} + \dfrac{c}{8} = \dfrac{3}{2}$ b) $\dfrac{c-5}{6} + \dfrac{c}{8}$

Solution

a) This is an *equation* because it contains an = sign. We will *solve* for c using the method we learned in Chapter 3: eliminate the denominators by multiplying by the LCD of all of the expressions. LCD = 24

$$24\left(\frac{c-5}{6} + \frac{c}{8}\right) = 24 \cdot \frac{3}{2} \quad \text{Multiply by LCD of 24 to eliminate the denominators.}$$

$$\begin{aligned} 4(c-5) + 3c &= 36 && \text{Distribute and eliminate denominators.} \\ 4c - 20 + 3c &= 36 && \text{Distribute.} \\ 7c - 20 &= 36 && \text{Combine like terms.} \\ 7c &= 56 \\ c &= 8 \end{aligned}$$

Check to verify that the solution set is $\{8\}$.

b) $\dfrac{c-5}{6} + \dfrac{c}{8}$ is *not* an equation to be solved because it does *not* contain an = sign. *It is a sum of rational expressions.* Rewrite each expression with the LCD, then subtract, *keeping the denominators* while performing the operations.

LCD = 24

$$\frac{(c-5)}{6} \cdot \frac{4}{4} = \frac{4(c-5)}{24} \qquad \frac{c}{8} \cdot \frac{3}{3} = \frac{3c}{24}$$

$$\begin{aligned} \frac{c-5}{6} + \frac{c}{8} &= \frac{4(c-5)}{24} + \frac{3c}{24} && \text{Rewrite each expression with a denominator of 24.} \\ &= \frac{4(c-5) + 3c}{24} && \text{Add the numerators.} \\ &= \frac{4c - 20 + 3c}{24} && \text{Distribute.} \\ &= \frac{7c - 20}{24} && \text{Combine like terms.} \end{aligned}$$

You Try 1

Determine whether each is an equation or is a sum or difference of expressions. Then solve the equation or find the sum or difference.

a) $\dfrac{m+1}{6} - \dfrac{m}{2}$ b) $\dfrac{m+1}{6} - \dfrac{m}{2} = \dfrac{5}{6}$

2. Solve Rational Equations

Let's list the steps we use to solve a rational equation. Then we will look at more examples.

Procedure How to Solve a Rational Equation

1) If possible, factor all denominators.
2) Write down the LCD of all of the expressions.
3) Multiply both sides of the equation by the LCD to *eliminate* the denominators.
4) Solve the equation.
5) Check the solution(s) in the original equation. If a proposed solution makes a denominator equal 0, then it is rejected as a solution.

Example 2

Solve $\frac{t}{16} + \frac{2}{t} = \frac{3}{4}$.

Solution

Since this is an equation, we will eliminate the denominators by multiplying the equation by the LCD of all of the expressions. LCD $= 16t$

$$16t\left(\frac{t}{16} + \frac{2}{t}\right) = 16t\left(\frac{3}{4}\right)$$ Multiply both sides of the equation by the LCD, $16t$.

$$\cancel{16}t\left(\frac{t}{\cancel{16}}\right) + 16\cancel{t}\left(\frac{2}{\cancel{t}}\right) = \overset{4}{\cancel{16}}t\left(\frac{3}{\cancel{4}}\right)$$ Distribute and divide out common factors.

$$t^2 + 32 = 12t$$

$$t^2 - 12t + 32 = 0$$ Subtract $12t$.

$$(t - 8)(t - 4) = 0$$ Factor.

$$t - 8 = 0 \quad \text{or} \quad t - 4 = 0$$

$$t = 8 \quad \text{or} \quad t = 4$$

Check:

$t = 8$

$$\frac{t}{16} + \frac{2}{t} \stackrel{?}{=} \frac{3}{4}$$

$$\frac{8}{16} + \frac{2}{8} \stackrel{?}{=} \frac{3}{4}$$

$$\frac{2}{4} + \frac{1}{4} = \frac{3}{4} \checkmark$$

$t = 4$

$$\frac{t}{16} + \frac{2}{t} \stackrel{?}{=} \frac{3}{4}$$

$$\frac{4}{16} + \frac{2}{4} \stackrel{?}{=} \frac{3}{4}$$

$$\frac{1}{4} + \frac{2}{4} = \frac{3}{4} \checkmark$$

The solution set is $\{4, 8\}$.

You Try 2

Solve $\frac{d}{3} + \frac{4}{d} = \frac{13}{3}$.

It is *very* important to check the proposed solution. Sometimes, what appears to be a solution actually is not.

Example 3

Solve $2 - \dfrac{9}{k+9} = \dfrac{k}{k+9}$.

Solution

Since this is an equation, we will eliminate the denominators by multiplying the equation by the LCD of all of the expressions. LCD $= k + 9$

$$(k+9)\left(2 - \frac{9}{k+9}\right) = (k+9)\left(\frac{k}{k+9}\right) \qquad \text{Multiply both sides of the equation by the LCD, } k+9.$$

$$(k+9)2 - \cancel{(k+9)} \cdot \frac{9}{\cancel{k+9}} = \cancel{(k+9)}\left(\frac{k}{\cancel{k+9}}\right) \qquad \text{Distribute and divide out common factors.}$$

$$2k + 18 - 9 = k \qquad \text{Multiply.}$$
$$2k + 9 = k$$
$$9 = -k \qquad \text{Subtract } 2k.$$
$$-9 = k \qquad \text{Divide by } -1.$$

Check: $$2 - \frac{9}{(-9)+9} \stackrel{?}{=} \frac{-9}{(-9)+9} \qquad \text{Substitute } -9 \text{ for } k \text{ in the original equation.}$$
$$2 - \frac{9}{0} = \frac{-9}{0}$$

Since $k = -9$ makes the denominator equal zero, -9 cannot be a solution to the equation. Therefore, this equation has no solution. The solution set is $\varnothing$. ■

BE CAREFUL

Always check what *appears* to be the solution or solutions to an equation containing rational expressions. If one of these values makes a denominator zero, then it *cannot* be a solution to the equation.

You Try 3

Solve $\dfrac{3m}{m-4} - 1 = \dfrac{12}{m-4}$.

Example 4

Solve $\dfrac{1}{4} - \dfrac{1}{a+2} = \dfrac{a+18}{4a^2-16}$.

Solution

This is an equation. Eliminate the denominators by multiplying by the LCD. Begin by factoring the denominator of $\dfrac{a+18}{4a^2-16}$.

$$\frac{1}{4} - \frac{1}{a+2} = \frac{a+18}{4(a+2)(a-2)} \qquad \text{Factor the denominator.}$$

$$\text{LCD} = 4(a+2)(a-2) \qquad \text{Write down the LCD of all of the expressions.}$$

$$4(a+2)(a-2)\left(\frac{1}{4} - \frac{1}{a+2}\right) = 4(a+2)(a-2)\left(\frac{a+18}{4(a+2)(a-2)}\right) \qquad \text{Multiply both sides of the equation by the LCD.}$$

$$\cancel{4}(a+2)(a-2)\left(\frac{1}{\cancel{4}}\right) - 4\cancel{(a+2)}(a-2)\left(\frac{1}{\cancel{a+2}}\right) = \cancel{4(a+2)(a-2)}\left(\frac{a+18}{\cancel{4(a+2)(a-2)}}\right)$$

Distribute and divide out common factors.

$$\begin{aligned}
(a+2)(a-2)-4(a-2) &= a+18 && \text{Multiply.}\\
a^2-4-4a+8 &= a+18 && \text{Distribute.}\\
a^2-4a+4 &= a+18 && \text{Combine like terms.}\\
a^2-5a-14 &= 0 && \text{Subtract } a \text{ and subtract 18.}\\
(a-7)(a+2) &= 0 && \text{Factor.}\\
a-7=0 \quad \text{or} \quad a+2 &= 0 && \text{Set each factor equal to zero.}\\
a=7 \quad \text{or} \quad a &= -2 && \text{Solve.}
\end{aligned}$$

Look at the factored form of the equation. If $a = 7$, no denominator will equal zero. *If* $a = -2$, *however, two of the denominators will equal zero. Therefore, we must reject* $a = -2$ *as a solution.* Check only $a = 7$.

$$\begin{aligned}
\textit{Check}: \frac{1}{4}-\frac{1}{7+2} &\stackrel{?}{=} \frac{7+18}{4(7)^2-16} && \text{Substitute } a=7 \text{ into the original equation.}\\
\frac{1}{4}-\frac{1}{9} &\stackrel{?}{=} \frac{25}{180} && \text{Simplify.}\\
\frac{9}{36}-\frac{4}{36} &\stackrel{?}{=} \frac{5}{36} && \text{Get a common denominator and reduce } \frac{25}{180}.\\
\frac{5}{36} &= \frac{5}{36} \quad \checkmark && \text{Subtract.}
\end{aligned}$$

The solution set is $\{7\}$.

The previous problem is a good example of why it is necessary to check all "solutions" to equations containing rational expressions.

You Try 4

Solve $\dfrac{1}{3}-\dfrac{1}{z+2}=\dfrac{z+14}{3z^2-12}$.

Example 5

Solve $\dfrac{11}{6h^2+48h+90}=\dfrac{h}{3h+15}+\dfrac{1}{2h+6}$.

Solution

Since this is an equation, we will eliminate the denominators by multiplying by the LCD. Begin by factoring all denominators, then identify the LCD.

$$\frac{11}{6(h+5)(h+3)}=\frac{h}{3(h+5)}+\frac{1}{2(h+3)} \qquad \text{LCD} = 6(h+5)(h+3)$$

$$6(h+5)(h+3)\left(\frac{11}{6(h+5)(h+3)}\right)=6(h+5)(h+3)\left(\frac{h}{3(h+5)}+\frac{1}{2(h+3)}\right) \qquad \text{Multiply by the LCD.}$$

$$\cancel{6(h+5)(h+3)}\left(\frac{11}{\cancel{6(h+5)(h+3)}}\right)=\overset{2}{\cancel{6}}\cancel{(h+5)}(h+3)\left(\frac{h}{\cancel{3(h+5)}}\right)+\overset{3}{\cancel{6}}(h+5)\cancel{(h+3)}\left(\frac{1}{\cancel{2(h+3)}}\right) \qquad \text{Distribute.}$$

$$\begin{aligned}
11 &= 2h(h+3)+3(h+5) && \text{Multiply.}\\
11 &= 2h^2+6h+3h+15 && \text{Distribute.}\\
11 &= 2h^2+9h+15 && \text{Combine like terms.}\\
0 &= 2h^2+9h+4 && \text{Subtract 11.}\\
0 &= (2h+1)(h+4) && \text{Factor.}
\end{aligned}$$

$$2h+1=0 \quad \text{or} \quad h+4=0$$

$$h=-\frac{1}{2} \quad \text{or} \quad h=-4 \qquad \text{Solve.}$$

You can see from the factored form of the equation that neither $h = -\frac{1}{2}$ nor $h = -4$ will make a denominator zero. Check the values in the original equation to verify that the solution set is $\left\{-4, -\frac{1}{2}\right\}$. ■

You Try 5

Solve $\dfrac{5}{6n^2 + 18n + 12} = \dfrac{n}{2n + 2} + \dfrac{1}{3n + 6}$.

3. Solve a Proportion

Example 6

Solve $\dfrac{18}{r + 7} = \dfrac{6}{r - 1}$.

Solution

This rational equation is also a *proportion*. A **proportion** is a statement that two ratios are equal. We can solve this proportion as we have solved the other equations in this section, by multiplying both sides of the equation by the LCD. Or, recall from Section 3.6 that *we can solve a proportion by setting the cross products equal to each other*.

$$\frac{18}{r + 7} = \frac{6}{r - 1}$$

Multiply. Multiply.

$$\begin{aligned} 18(r - 1) &= 6(r + 7) && \text{Set the cross products equal to each other.} \\ 18r - 18 &= 6r + 42 && \text{Distribute.} \\ 12r &= 60 \\ r &= 5 && \text{Solve.} \end{aligned}$$

The proposed solution, $r = 5$, does *not* make a denominator equal zero. Check to verify that the solution set is $\{5\}$. ■

You Try 6

Solve $\dfrac{7}{d + 3} = \dfrac{14}{3d + 5}$.

4. Solve an Equation for a Specific Variable

In Section 3.5, we learned how to solve an equation for a specific variable. For example, to solve $2l + 2w = P$ for w, we do the following:

$$\begin{aligned} 2l + 2\boxed{w} &= P && \text{Put a box around } w\text{, the variable for which we are solving.} \\ 2\boxed{w} &= P - 2l && \text{Subtract } 2l. \\ w &= \frac{P - 2l}{2} && \text{Divide by 2.} \end{aligned}$$

Next we discuss how to solve for a specific variable in a rational expression.

Example 7

Solve $z = \dfrac{n}{d - D}$ for d.

Solution

Note that the equation contains a lowercase d and an uppercase D. These represent different quantities, so be sure to write them correctly. Put d in a box.

Since d is in the denominator of the rational expression, multiply both sides of the equation by $d - D$ to eliminate the denominator.

$$\begin{aligned}
z &= \frac{n}{\boxed{d} - D} && \text{Put } d \text{ in a box.}\\
(\boxed{d} - D)z &= \cancel{(\boxed{d} - D)}\left(\frac{n}{\cancel{\boxed{d} - D}}\right) && \text{Multiply both sides by } d - D \text{ to eliminate the denominator.}\\
\boxed{d}z - Dz &= n && \text{Distribute.}\\
\boxed{d}z &= n + Dz && \text{Add } Dz.\\
d &= \frac{n + Dz}{z} && \text{Divide by } z.
\end{aligned}$$

You Try 7

Solve $v = \dfrac{k}{m + M}$ for m.

Example 8

Solve $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{z}$ for y.

Solution

Put the y in a box. The LCD of all of the fractions is xyz. Multiply both sides of the equation by xyz.

$$\begin{aligned}
\frac{1}{x} + \frac{1}{\boxed{y}} &= \frac{1}{z} && \text{Put } y \text{ in a box.}\\
x\boxed{y}z\left(\frac{1}{x} + \frac{1}{\boxed{y}}\right) &= x\boxed{y}z\left(\frac{1}{z}\right) && \text{Multiply both sides by } xyz \text{ to eliminate the denominator.}\\
\cancel{x}\boxed{y}z \cdot \frac{1}{\cancel{x}} + x\boxed{\cancel{y}}z \cdot \frac{1}{\boxed{\cancel{y}}} &= x\boxed{y}\cancel{z}\left(\frac{1}{\cancel{z}}\right) && \text{Distribute.}\\
\boxed{y}z + xz &= x\boxed{y} && \text{Divide out common factors.}
\end{aligned}$$

Since we are solving for y and there are terms containing y on each side of the equation, we must get yz and xy on one side of the equation and xz on the other side.

$$xz = x\boxed{y} - \boxed{y}z \qquad \text{Subtract } yz \text{ from each side.}$$

To isolate y, we will *factor* y out of each term on the right-hand side of the equation.

$$\begin{aligned}
xz &= \boxed{y}(x - z) && \text{Factor out } y.\\
\frac{xz}{x - z} &= y && \text{Divide by } x - z.
\end{aligned}$$

You Try 8

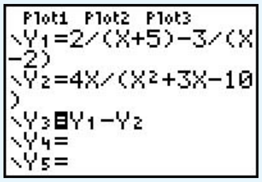
Solve $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ for z.

Using Technology

We can use a graphing calculator to solve a rational equation in one variable. First, enter the left side of the equation in Y_1 and the right side of the equation in Y_2. Then enter $Y_1 - Y_2$ in Y_3. Then graph the equation in Y_3. The zeros or x-intercepts of the graph are the solutions to the equation.

We will solve $\frac{2}{x+5} - \frac{3}{x-2} = \frac{4x}{x^2+3x-10}$ using a graphing calculator.

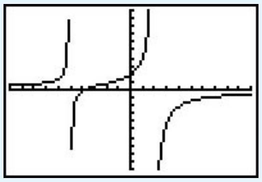
1) Enter $\frac{2}{x+5} - \frac{3}{x-2}$ by entering $2/(x+5) - 3/(x-2)$ in Y_1.



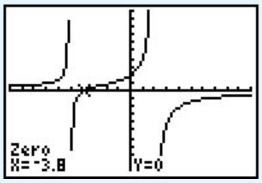
2) Enter $\frac{4x}{x^2+3x-10}$ by entering $4x/(x^2+3x-10)$ in Y_2.

3) Enter $Y_1 - Y_2$ in Y_3 as follows: press VARS, select Y-VARS using the right arrow key, and press ENTER ENTER to select Y_1. Then press –. Press VARS, select Y-VARS using the right arrow key, press ENTER 2 to select Y_2. Then press ENTER.

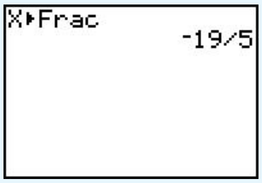
4) Move the cursor onto the = sign just right of \Y_1 and press ENTER to deselect Y_1. Repeat to deselect Y_2. Press GRAPH to graph $Y_1 - Y_2$.



5) Press 2nd TRACE 2:zero, move the cursor to the left of the zero and press ENTER, move the cursor to the right of the zero and press ENTER, and move the cursor close to the zero and press ENTER to display the zero.



6) Press X,T,Θ,n MATH ENTER ENTER to display the zero $x = -\frac{19}{5}$.



If there is more than one zero, repeat steps 5 and 6 above for each zero.

Solve each equation using a graphing calculator.

1) $\frac{2x}{x-3} + \frac{1}{x+5} = \frac{4-2x}{x^2+2x-15}$

2) $\frac{4}{x-3} + \frac{5}{x+3} = \frac{15}{x^2-9}$

3) $\frac{2}{x+2} + \frac{4}{x-5} = \frac{16}{x^2-3x-10}$

4) $\frac{1}{x-7} + \frac{3}{x+4} = \frac{7}{x^2-3x-28}$

5) $\frac{6}{x+1} = \frac{5x+3}{x^2-x-2} - \frac{x}{x-2}$

6) $\frac{4}{x+3} - \frac{x}{x+2} = \frac{3x}{x^2+5x+6}$

Answers to You Try Exercises

1) a) difference; $\frac{1-2m}{6}$ b) equation; $\{-2\}$ 2) $\{1, 12\}$ 3) $\varnothing$ 4) $\{6\}$ 5) $\left\{-3, \frac{1}{3}\right\}$

6) $\{1\}$ 7) $m = \frac{k - vM}{v}$ 8) $z = \frac{xy}{x + y}$

Answers to Technology Exercises

1) $\left\{-7, \frac{1}{2}\right\}$ 2) $\{2\}$ 3) $\{3\}$ 4) $\{6\}$ 5) $\{-5, 3\}$ 6) $\{-4, 2\}$

8.6 Exercises

Objective 1: Differentiate Between Rational Expressions and Rational Equations

1) When solving an equation containing rational expressions, do you keep the LCD throughout the problem or do you eliminate the denominators?

2) When adding or subtracting two rational expressions, do you keep the LCD throughout the problem or do you eliminate the denominators?

Determine whether each is an equation or a sum or difference of expressions. Then solve the equation or find the sum or difference.

3) $\frac{3r+5}{2} - \frac{r}{6}$

4) $\frac{m}{12} + \frac{m-8}{3}$

5) $\frac{3h}{2} + \frac{4}{3} = \frac{2h+3}{3}$

6) $\frac{7f-24}{12} = f + \frac{1}{2}$

7) $\frac{3}{a^2} + \frac{1}{a+11}$

8) $\frac{z}{z-5} - \frac{4}{z}$

9) $\frac{8}{b-11} - 5 = \frac{3}{b-11}$

10) $1 + \frac{2}{c+5} = \frac{11}{c+5}$

Mixed Exercises: Objectives 2 and 3

Values that make the denominators equal zero cannot be solutions of an equation. Find *all* of the values that make the denominators zero and which, therefore, cannot be solutions of each equation. Do NOT solve the equation.

11) $\frac{k+3}{k-2} + 1 = \frac{7}{k}$

12) $\frac{t}{t+12} - \frac{5}{t} = 3$

13) $\frac{8}{p+3} - \frac{6}{p} = \frac{p}{p^2-9}$

14) $\frac{7}{d^2-64} + \frac{6}{d} = \frac{8}{d+8}$

15) $\frac{9h}{h^2-5h-36} + \frac{1}{h+4} = \frac{h+7}{3h-27}$

16) $\frac{v+8}{v^2-8v+12} - \frac{5}{3v-4} = \frac{2v}{v-6}$

Solve each equation.

17) $\frac{a}{3} + \frac{7}{12} = \frac{1}{4}$

18) $\frac{y}{2} - \frac{4}{3} = \frac{1}{6}$

19) $\frac{1}{4}j - j = -4$

20) $\frac{1}{3}h + h = -4$

21) $\frac{8m-5}{24} = \frac{m}{6} - \frac{7}{8}$

22) $\frac{13u-1}{20} = \frac{3u}{5} - 1$

23) $\frac{8}{3x+1} = \frac{2}{x+3}$

24) $\frac{4}{5t+2} = \frac{2}{2t-1}$

25) $\frac{r+1}{2} = \frac{4r+1}{5}$

26) $\frac{w}{3} = \frac{6w-4}{9}$

27) $\frac{23}{z} + 8 = -\frac{25}{z}$

28) $\frac{18}{a} - 2 = \frac{10}{a}$

29) $\frac{5q}{q+1} - 2 = \frac{5}{q+1}$

30) $\frac{n}{n+3} + 5 = \frac{12}{n+3}$

VIDEO 31) $\frac{2}{s+6} + 4 = \frac{2}{s+6}$

32) $\frac{u}{u-5} + 3 = \frac{5}{u-5}$

33) $\frac{3b}{b+7} - 6 = \frac{3}{b+7}$

34) $\frac{c}{c-5} - 5 = \frac{20}{c-5}$

35) $\frac{8}{r} - 1 = \frac{6}{r}$

36) $\frac{11}{g} + 3 = -\frac{10}{g}$

37) $z + \frac{12}{z} = -8$

38) $y - \frac{28}{y} = 3$

39) $\frac{15}{b} = 8 - b$

40) $n = 13 - \frac{12}{n}$

41) $\frac{8}{c+2} - \frac{12}{c-4} = \frac{2}{c+2}$

42) $\frac{2}{m-1} + \frac{1}{m+4} = \frac{4}{m+4}$

43) $\frac{9}{c-8} - \frac{15}{c} = 1$

44) $\frac{6}{r+5} - \frac{2}{r} = -1$

45) $\frac{3}{p-4} + \frac{8}{p+4} = \frac{13}{p^2-16}$

46) $\frac{5}{w-7} - \frac{8}{w+7} = \frac{52}{w^2-49}$

47) $\frac{9}{k+5} - \frac{4}{k+1} = \frac{10}{k^2+6k+5}$

48) $\frac{3}{a+2} + \frac{10}{a^2-6a-16} = \frac{5}{a-8}$

49) $\frac{12}{g^2-9} + \frac{2}{g+3} = \frac{7}{g-3}$

50) $\frac{9}{t+4} + \frac{8}{t^2-16} = \frac{1}{t-4}$

51) $\frac{5}{p-3} - \frac{7}{p^2-7p+12} = \frac{8}{p-4}$

52) $\frac{8}{x^2+2x-15} = \frac{6}{x-3} + \frac{4}{x+5}$

53) $\frac{x^2}{2} = \frac{x^2-6x}{3}$

54) $\frac{k^2}{3} = \frac{k^2+3k}{4}$

55) $\frac{3}{t^2} = \frac{6}{t^2+8t}$

56) $\frac{5}{m^2-36} = \frac{4}{m^2+6m}$

57) $\frac{b+3}{3b-18} - \frac{b+2}{b-6} = \frac{b}{3}$

58) $\frac{3y-2}{y+2} = \frac{y}{4} + \frac{1}{4y+8}$

59) $\frac{4}{n+1} = \frac{10}{n^2-1} - \frac{5}{n-1}$

60) $\frac{2}{c-6} - \frac{24}{c^2-36} = -\frac{3}{c+6}$

61) $-\frac{a}{5} = \frac{3}{a+8}$

62) $\frac{u}{7} = \frac{2}{9-u}$

VIDEO 63) $\frac{8}{p+2} + \frac{p}{p+1} = \frac{5p+2}{p^2+3p+2}$

64) $\frac{6}{x-1} + \frac{x}{x+3} = \frac{2x+28}{x^2+2x-3}$

65) $\frac{-14}{3a^2+15a-18} = \frac{a}{a-1} + \frac{2}{3a+18}$

66) $\frac{3}{2n^2+10n+8} = \frac{n}{2n+2} + \frac{1}{n+1}$

67) $\frac{3}{f+4} = \frac{f}{f+6} - \frac{2}{f^2+10f+24}$

68) $\frac{11}{c+9} = \frac{c}{c-4} - \frac{36-8c}{c^2+5c-36}$

69) $\frac{b}{b^2+b-6} + \frac{3}{b^2+9b+18} = \frac{8}{b^2+4b-12}$

70) $\frac{h}{h^2+2h-8} + \frac{4}{h^2+8h-20} = \frac{4}{h^2+14h+40}$

71) $\frac{r}{r^2+8r+15} - \frac{2}{r^2+r-6} = \frac{2}{r^2+3r-10}$

72) $\frac{5}{t^2+5t-6} - \frac{t}{t^2+10t+24} = \frac{1}{t^2+3t-4}$

73) $\frac{k}{k^2-6k-16} - \frac{12}{5k^2-65k+200} = \frac{28}{5k^2-15k-50}$

74) $\frac{q}{q^2+4q-32} + \frac{2}{q^2-14q+40} = \frac{6}{q^2-2q-80}$

Objective 4: Solve an Equation for a Specific Variable

Solve for the indicated variable.

75) $W = \frac{CA}{m}$ for m

76) $V = \frac{nRT}{P}$ for P

77) $a = \frac{rt}{2b}$ for b

78) $y = \frac{kx}{z}$ for z

VIDEO 79) $B = \frac{t+u}{3x}$ for x

80) $Q = \frac{n-k}{5r}$ for r

81) $d = \frac{t}{z-n}$ for n

82) $z = \frac{a}{b+c}$ for b

83) $h = \frac{3A}{r+s}$ for s

84) $A = \frac{4r}{q-t}$ for t

85) $r = \frac{kx}{y-az}$ for y

86) $w = \frac{na}{kc+b}$ for c

VIDEO 87) $\frac{1}{t} = \frac{1}{r} - \frac{1}{s}$ for r

88) $\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R_3}$ for R_2

89) $\frac{5}{x} = \frac{1}{y} - \frac{4}{z}$ for z

90) $\frac{2}{A} + \frac{1}{C} = \frac{3}{B}$ for C

Section 8.7 Applications of Rational Equations

Objectives

1. Solve Problems Involving Proportions
2. Solve Problems Involving Distance, Rate, and Time
3. Solve Problems Involving Work

We have studied applications of linear and quadratic equations. Now we turn our attention to applications involving equations with rational expressions. We will continue to use the five-step problem-solving method outlined in Section 3.2.

1. Solve Problems Involving Proportions

We first solved application problems involving proportions in Section 3.6. We begin this section with a problem involving a proportion.

Example 1

Write an equation and solve.

At a small business, the ratio of employees who ride their bikes to work to those who drive a car is 4 to 3. The number of people who bike to work is three more than the number who drive. How many people bike to work, and how many drive their cars?

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the number of people who bike to work and the number who drive.

Step 2: **Choose a variable** to represent the unknown, and define the other unknown in terms of this variable.

$$x = \text{the number of people who drive}$$
$$x + 3 = \text{the number of people who bike}$$

Step 3: **Translate** the information that appears in English into an algebraic equation.

Write a proportion. We will write our ratios in the form of $\dfrac{\text{number who bike}}{\text{number who drive}}$ so that the numerators contain the same quantities and the denominators contain the same quantities.

$$\begin{array}{r} \text{Number who bike} \rightarrow \\ \text{Number who drive} \rightarrow \end{array} \frac{4}{3} = \frac{x+3}{x} \begin{array}{l} \leftarrow \text{Number who bike} \\ \leftarrow \text{Number who drive} \end{array}$$

The equation is $\dfrac{4}{3} = \dfrac{x+3}{x}$.

Step 4: **Solve** the equation.

Multiply.

$$\frac{4}{3} = \frac{x+3}{x}$$

Multiply.

$$4x = 3(x+3) \quad \text{Set the cross products equal.}$$
$$4x = 3x + 9 \quad \text{Distribute.}$$
$$x = 9 \quad \text{Subtract } 3x.$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

Therefore, 9 people drive to work and $9 + 3 = 12$ people ride their bikes. The check is left to the student.

You Try 1

Write an equation and solve.

In a classroom of college students, the ratio of students who have Internet access on their cell phones to those who do not is 3 to 5. The number of students who have Internet access is eight less than the number who do not. How many students have Internet access on their phones?

2. Solve Problems Involving Distance, Rate, and Time

In Section 3.6, we solved problems involving distance (d), rate (r), and time (t).

The basic formula is $d = rt$. We can solve this formula for r and then for t to obtain

$$r = \frac{d}{t} \qquad \text{and} \qquad t = \frac{d}{r}$$

The problems in this section involve boats going with and against a current, and planes going with and against the wind. Both situations use the same idea.

Suppose a boat's speed is 18 mph in still water. If that same boat had a 4-mph current pushing *against* it, how fast would it be traveling? (The current will cause the boat to slow down.)

$$\text{Speed } \textit{against} \text{ the current} = 18 \text{ mph} - 4 \text{ mph} = 14 \text{ mph}$$

$$\begin{matrix}\text{Speed } \textit{against} \\ \text{the current}\end{matrix} = \begin{matrix}\text{Speed in} \\ \text{still water}\end{matrix} - \begin{matrix}\text{Speed of} \\ \text{the current}\end{matrix}$$

If the speed of the boat in still water is 18 mph and a 4-mph current is *pushing* the boat, how fast would the boat be traveling *with* the current? (The current will cause the boat to travel faster.)

$$\text{Speed } \textit{with} \text{ the current} = 18 \text{ mph} + 4 \text{ mph} = 22 \text{ mph}$$

$$\begin{matrix}\text{Speed } \textit{with} \\ \text{the current}\end{matrix} = \begin{matrix}\text{Speed in} \\ \text{still water}\end{matrix} + \begin{matrix}\text{Speed of} \\ \text{the current}\end{matrix}$$

A boat traveling *against* the current is said to be traveling *upstream*. A boat traveling *with* the current is said to be traveling *downstream*.

We will use these ideas in Example 2.

Example 2

Write an equation and solve.

A boat can travel 15 mi downstream in the same amount of time it can travel 9 mi upstream. If the speed of the current is 4 mph, what is the speed of the boat in still water?

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

First, we must understand that "15 mi downstream" means *15 mi with the current*, and "9 mi upstream" means *9 miles against the current.*

We must find the speed of the boat in still water.

Step 2: **Choose a variable** to represent the unknown, and define the other unknowns in terms of this variable.

x = the speed of the boat in still water
$x + 4$ = the speed of the boat *with* the current (downstream)
$x - 4$ = the speed of the boat *against* the current (upstream)

Step 3: **Translate** from English into an algebraic equation. Use a table to organize the information.

First, fill in the distances and the rates (or speeds).

	d	**r**	**t**
Downstream	15	$x + 4$	
Upstream	9	$x - 4$	

Next we must write expressions for the times it takes the boat to go downstream and upstream. We know that $d = rt$, so if we solve for t we get $t = \frac{d}{r}$. Substitute the information from the table to get the expressions for the time.

$$\text{Downstream: } t = \frac{d}{r} = \frac{15}{x + 4} \qquad \text{Upstream: } t = \frac{d}{r} = \frac{9}{x - 4}$$

Put these values into the table.

	d	**r**	**t**
Downstream	15	$x + 4$	$\frac{15}{x + 4}$
Upstream	9	$x - 4$	$\frac{9}{x - 4}$

The problem states that it takes the boat the *same amount of time* to travel 15 mi downstream as it does to go 9 mi upstream. We can write an equation in English:

$$\begin{array}{c} \text{Time for boat to go} \\ \text{15 miles downstream} \end{array} = \begin{array}{c} \text{Time for boat to go} \\ \text{9 miles upstream} \end{array}$$

Looking at the table, we can write the algebraic equation using the expressions for time. The equation is $\frac{15}{x + 4} = \frac{9}{x - 4}$.

Step 4: **Solve** the equation.

$$\frac{15}{x + 4} = \frac{9}{x - 4}$$

Multiply.

Multiply.

$$\begin{aligned} 15(x - 4) &= 9(x + 4) && \text{Set the cross products equal.} \\ 15x - 60 &= 9x + 36 && \text{Distribute.} \\ 6x &= 96 \\ x &= 16 && \text{Solve.} \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

The speed of the boat in still water is 16 mph.

Check: The speed of the boat going downstream is $16 + 4 = 20$ mph, so the time to travel downstream is

$$t = \frac{d}{r} = \frac{15}{20} = \frac{3}{4} \text{ hr}$$

The speed of the boat going upstream is $16 - 4 = 12$ mph, so the time to travel upstream is

$$t = \frac{d}{r} = \frac{9}{12} = \frac{3}{4} \text{ hr}$$

So, time upstream = time downstream. ✓ ■

You Try 2

Write an equation and solve.

It takes a boat the same amount of time to travel 12 mi downstream as it does to travel 6 mi upstream. Find the speed of the boat in still water if the speed of the current is 3 mph.

3. Solve Problems Involving Work

Suppose it takes Tara 3 hr to paint a fence. What is the *rate* at which she does the job?

$$\text{rate} = \frac{1 \text{ fence}}{3 \text{ hr}} = \frac{1}{3} \text{ fence/hr}$$

Tara works at a rate of $\frac{1}{3}$ of a fence per hour.

In general, we can say that if it takes t units of time to do a job, then the *rate* at which the job is done is $\frac{1}{t}$ job per unit of time.

This idea of *rate* is what we use to determine how long it can take for 2 or more people or things to do a job.

Let's assume, again, that Tara can paint the fence in 3 hr. At this rate, how much of the job can she do in 2 hr?

$$\begin{aligned}\begin{matrix}\text{Fractional part}\\ \text{of the job done}\end{matrix} &= \begin{matrix}\text{Rate of}\\ \text{work}\end{matrix} \cdot \begin{matrix}\text{Amount of}\\ \text{time worked}\end{matrix}\\ &= \frac{1}{3} \cdot 2\\ &= \frac{2}{3}\end{aligned}$$

She can paint $\frac{2}{3}$ of the fence in 2 hr.

Procedure Solving Work Problems

The basic equation used to solve work problems is:

$$\begin{matrix}\text{Fractional part of a job}\\ \text{done by one person or thing}\end{matrix} + \begin{matrix}\text{Fractional part of a job}\\ \text{done by another person or thing}\end{matrix} = 1 \text{ (whole job)}$$

Example 3

Write an equation and solve.

If Tara can paint the backyard fence in 3 hr but her sister, Grace, could paint the fence in 2 hr, how long would it take them to paint the fence together?

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must determine how long it would take Tara and Grace to paint the fence together.

Step 2: **Choose a variable** to represent the unknown.

t = the number of hours to paint the fence together

Step 3: **Translate** the information that appears in English into an algebraic equation.

Let's write down their rates:

$$\text{Tara's rate} = \frac{1}{3}\text{ fence/hr (since the job takes her 3 hours)}$$

$$\text{Grace's rate} = \frac{1}{2}\text{ fence/hr (since the job takes her 2 hours)}$$

It takes them t hours to paint the room together. Recall that

$$\begin{array}{rcccccl} \text{Fractional part of job done} & = & \text{Rate of work} & \cdot & \text{Amount of time worked} & & \\ \text{Tara's fractional part} & = & \frac{1}{3} & \cdot & t & = & \frac{1}{3}t \\ \text{Grace's fractional part} & = & \frac{1}{2} & \cdot & t & = & \frac{1}{2}t \end{array}$$

The equation we can write comes from

$$\begin{array}{ccccc} \text{Fractional part of the job done by Tara} & + & \text{Fractional part of the job done by Grace} & = & 1\text{ whole job} \\ \frac{1}{3}t & + & \frac{1}{2}t & = & 1 \end{array}$$

The equation is $\frac{1}{3}t + \frac{1}{2}t = 1$.

Step 4: **Solve** the equation.

$$6\left(\frac{1}{3}t + \frac{1}{2}t\right) = 6(1) \quad \text{Multiply by the LCD of 6 to eliminate the fractions.}$$

$$6\left(\frac{1}{3}t\right) + 6\left(\frac{1}{2}t\right) = 6(1) \quad \text{Distribute.}$$

$$2t + 3t = 6 \quad \text{Multiply.}$$

$$5t = 6 \quad \text{Combine like terms.}$$

$$t = \frac{6}{5} \quad \text{Divide by 5.}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

Tara and Grace could paint the fence together in $\frac{6}{5}$ hr or $1\frac{1}{5}$ hr.

Check:

$$\begin{array}{ccccc} \text{Fractional part of the job done by Tara} & + & \text{Fractional part of the job done by Grace} & = & \text{1 whole job} \\ \frac{1}{3}\cdot\left(\frac{6}{5}\right) & + & \frac{1}{2}\cdot\left(\frac{6}{5}\right) & \stackrel{?}{=} & 1 \\ \frac{2}{5} & + & \frac{3}{5} & = & 1 \end{array}$$

You Try 3

Write an equation and solve.

Javier can put up drywall in a house in 6 hr while it would take his coworker, Frank, 8 hr to drywall the same space. How long would it take them to install the drywall if they worked together?

Answers to You Try Exercises

1) 12 2) 9 mph 3) $3\frac{3}{7}$ hr

8.7 Exercises

Objective 1: Solve Problems Involving Proportions

Solve the following proportions.

1) $\frac{8}{15} = \frac{32}{x}$

2) $\frac{9}{12} = \frac{6}{a}$

3) $\frac{4}{7} = \frac{n}{n+9}$

4) $\frac{5}{3} = \frac{c}{c-10}$

Write an equation for each and solve. See Example 1.

5) The scale on a blueprint is 2.5 in. to 10 ft in actual room length. Find the length of a room that is 3 in. long on the blueprint.

6) A survey conducted by the U.S. Centers for Disease Control revealed that, in Michigan, approximately 2 out of 5 adults between the ages of 18 and 24 smoke cigarettes. In a group of 400 Michigan citizens in this age group, how many would be expected to be smokers? (www.cdc.gov)

7) The ratio of employees at a small company who have their paychecks directly deposited into their bank accounts to those who do not is 9 to 2. If the number of people who have direct deposit is 14 more than the number who do not, how many employees do not have direct deposit?

8) In a gluten-free flour mixture, the ratio of potato-starch flour to tapioca flour is 2 to 1. If a mixture contains 3 more cups of potato-starch flour than tapioca flour, how much of each type of flour is in the mixture?

9) At a state university, the ratio of the number of freshmen who graduated in four years to those who took longer was about 2 to 5. If the number of students who graduated in four years was 1200 less than the number who graduated in more than four years, how many students graduated in four years?

10) Francesca makes her own ricotta cheese for her restaurant. The ratio of buttermilk to whole milk in her recipe is 1 to 4. How much of each type of milk will she need if she will use 18 more cups of whole milk than buttermilk?

11) The ancient Greeks believed that the rectangle most pleasing to the eye, the golden rectangle, had sides in which the ratio of its length to its width was approximately 8 to 5. They erected many buildings, including the Parthenon, using this golden ratio. The marble floor of a museum foyer is to be designed as a golden rectangle. If its width is to be 18 feet less than its length, find the length and width of the foyer.

12) The ratio of seniors at Central High School who drive to school to those who take the school bus is 7 to 2. If the number of students who drive is 320 more than the number who take the bus, how many students drive and how many take the bus?

13) A math professor surveyed her class and found that the ratio of students who used the school's tutoring service to those who did not was 3 to 8. The number of students who did not use the tutoring lab was 15 more than the number who did. How many students used the tutoring service and how many did not?

14) An industrial cleaning solution calls for 5 parts water to 2 parts concentrated cleaner. If a worker uses 15 more quarts of water than concentrated cleaner to make a solution,

a) how much concentrated cleaner did she use?

b) how much water did she use?

c) how much solution did she make?

Objective 2: Solve Problems Involving Distance, Rate, and Time

Answer the following questions about rates.

15) If the speed of a boat in still water is 8 mph,

a) what is its speed going *against* a 2-mph current?

b) what is its speed *with* a 2-mph current?

16) If an airplane travels at a constant rate of 350 mph,

a) what is its speed going *into* a 50-mph wind?

b) what is its speed going *with* a 25-mph wind?

17) If an airplane travels at a constant rate of x mph,

a) what is its speed going *with* a 40-mph wind?

b) what is its speed going *against* a 30-mph wind?

18) If the speed of a boat in still water is 11 mph,

a) what is its speed going *against* a current with a rate of x mph?

b) what is its speed going *with* a current with a rate of x mph?

Write an equation for each and solve. See Example 2.

19) A boat can travel 4 mi upstream in the same amount of time it can travel 6 mi downstream. If the speed of the current is 2 mph, what is the speed of the boat in still water?

20) Flying at a constant speed, a plane can travel 800 miles with the wind in the same amount of time it can fly 650 miles against the wind. If the wind blows at 30 mph, what is the speed of the plane?

21) When the wind is blowing at 25 mph, a plane flying at a constant speed can travel 500 miles with the wind in the same amount of time it can fly 400 miles against the wind. Find the speed of the plane.

22) With a current flowing at 3 mph, a boat can travel 9 mi downstream in the same amount of time it can travel 6 mi upstream. What is the speed of the boat in still water?

23) The speed of a boat in still water is 28 mph. The boat can travel 32 mi with the current in the same amount of time it can travel 24 mi against the current. Find the speed of the current.

24) A boat can travel 20 mi downstream in the same amount of time it can travel 12 mi upstream. The speed of the boat in still water is 20 mph. Find the speed of the current.

25) The speed of a plane in still air is 280 mph. Flying against the wind, it can fly 600 mi in the same amount of time it takes to go 800 mi with the wind. What is the speed of the wind?

26) The speed of a boat in still water is 10 mph. If the boat can travel 9 mi downstream in the same amount of time it can travel 6 mi upstream, find the speed of the current.

27) Bill drives 120 miles from his house in San Diego to Los Angeles for a business meeting. Afterward, he drives from LA to Las Vegas, a distance of 240 miles. If he averages the same speed on both legs of the trip and it takes him 2 hours less to go from San Diego to Los Angeles, what is his average driving speed?

28) Rashard drives 80 miles from Detroit to Lansing, and later drives 60 miles more from Lansing to Grand Rapids. The trip from Lansing to Grand Rapids takes him a half hour less than the drive from Detroit to Lansing. Find his average driving speed if it is the same on both parts of the trip.

Objective 3: Solve Problems Involving Work

Answer the following questions about work rate.

29) It takes Midori 3 hr to do her homework. What is her rate?

30) It takes Signe 20 hr to complete her self-portrait for art class. How much of the job does she do in 12 hr?

31) Tomasz can set up his new computer in t hours. What is the rate at which he does this job?

32) It takes Jesse twice as long to edit a chapter in a book as it takes Curtis. If it takes Curtis t hours to edit the chapter, at what rate does Jesse do the job?

Write an equation for each and solve. See Example 3.

33) It takes Rupinderjeet 4 hr to paint a room while the same job takes Sana 5 hr. How long would it take for them to paint the room together?

34) A hot-water faucet can fill a sink in 9 min while it takes the cold-water faucet only 7 min. How long would it take to fill the sink if both faucets were on?

35) Wayne can clean the carpets in his house in 4 hr but it would take his son, Garth, 6 hr to clean them on his own. How long would it take them to clean the carpets together?

36) Janice and Blanca have to type a report on a project they did together. Janice could type it in 40 min, and Blanca could type it in 1 hr. How long would it take them if they worked together?

VIDEO 37) A faucet can fill a tub in 12 minutes. The leaky drain can empty the tub in 30 minutes. If the faucet is on and the drain is leaking, how long would it take to fill the tub?

38) A pipe can fill a pool in 8 hr, and another pipe can empty a pool in 12 hr. If both pipes are accidentally left open, how long would it take to fill the pool?

39) A new machine in a factory can do a job in 5 hr. When it is working together with an older machine, the job can be done in 3 hr. How long would it take the old machine to do the job by itself?

40) It takes Lily 75 minutes to mow the lawn. When she works with her brother, Preston, it takes only 30 minutes. How long would it take Preston to mow the lawn himself?

41) It would take Mei twice as long as Ting to make decorations for a party. If they worked together, they could make the decorations in 40 min. How long would it take Mei to make the decorations by herself?

42) It takes Lemar three times as long as his boss, Emilio, to build a custom shelving unit. Together they can build the unit in 4.5 hr. How long would it take Lemar to build the shelves by himself?

Chapter 8: Summary

Definition/Procedure	Example

8.1 Simplifying Rational Expressions

A **rational expression** is an expression of the form $\frac{P}{Q}$, where P and Q are polynomials and where $Q \neq 0$.

We can *evaluate* rational expressions. **(p. 456)**

Evaluate $\frac{5a - 8}{a + 3}$ for $a = 2$.

$$\frac{5(2) - 8}{2 + 3} = \frac{10 - 8}{5} = \frac{2}{5}$$

How to Determine When a Rational Expression Equals Zero and When It Is Undefined

1) To determine what values of the variable make the expression equal zero, set the numerator equal to zero and solve for the variable.

2) To determine what values of the variable make the expression undefined, set the denominator equal to zero and solve for the variable. **(p. 457)**

For what value(s) of x is $\frac{x - 7}{x + 9}$

a) equal to zero? b) undefined?

a) $\frac{x - 7}{x + 9} = 0$ when $x - 7 = 0$.

$$x - 7 = 0$$
$$x = 7$$

When $x = 7$, the expression equals zero.

b) $\frac{x - 7}{x + 9}$ is undefined when its denominator equals zero.

Solve $x + 9 = 0$.

$$x + 9 = 0$$
$$x = -9$$

When $x = -9$, the expression is undefined.

To Write an Expression in Lowest Terms

1) Completely *factor* the numerator and denominator.

2) *Divide* the numerator and denominator by the greatest common factor. **(p. 459)**

Simplify $\frac{3r^2 - 10r + 8}{2r^2 - 8}$.

$$\frac{3r^2 - 10r + 8}{2r^2 - 8} = \frac{(3r - 4)\cancel{(r - 2)}}{2(r + 2)\cancel{(r - 2)}} = \frac{3r - 4}{2(r + 2)}$$

Simplifying $\frac{a - b}{b - a}$.

A rational expression of the form $\frac{a - b}{b - a}$ will simplify to -1. **(p. 460)**

Simplify $\frac{5 - w}{w^2 - 25}$.

$$\frac{5 - w}{w^2 - 25} = \frac{\overset{-1}{\cancel{5 - w}}}{(w + 5)\cancel{(w - 5)}} = -\frac{1}{w + 5}$$

Rational Functions

The **domain of a rational function** consists of all real numbers except the value(s) of the variable that make the denominator equal zero. **(p. 462)**

Determine the domain of the rational function

$$f(x) = \frac{x - 10}{x + 6}.$$

$x + 6 = 0$ Set the denominator $= 0$.

$x = -6$ Solve.

When $x = -6$, the denominator of $f(x) = \frac{x - 10}{x + 6}$ equals zero.

The domain contains all real numbers except -6. Write the domain in interval notation as $(-\infty, -6) \cup (-6, \infty)$.

Definition/Procedure	Example

8.2 Multiplying and Dividing Rational Expressions

Multiplying Rational Expressions

1) Factor numerators and denominators.
2) Reduce and multiply. **(p. 466)**

Multiply $\dfrac{16v^4}{v^2+10v+21}\cdot\dfrac{3v+21}{4v}$.

$$\frac{16v^4}{v^2+10v+21}\cdot\frac{3v+21}{4v}=\frac{\overset{4}{\cancel{16}}v^3\cdot\cancel{v}}{(v+3)\cancel{(v+7)}}\cdot\frac{3\cancel{(v+7)}}{\cancel{4v}}=\frac{12v^3}{v+3}$$

Dividing Rational Expressions

To **divide** rational expressions, multiply the first expression by the reciprocal of the second. **(p. 467)**

Divide $\dfrac{2x^2+5x}{x+4}\div\dfrac{4x^2-25}{12x-30}$.

$$\frac{2x^2+5x}{x+4}\div\frac{4x^2-25}{12x-30}=\frac{2x^2+5x}{x+4}\cdot\frac{12x-30}{4x^2-25}$$
$$=\frac{x\cancel{(2x+5)}}{x+4}\cdot\frac{6\cancel{(2x-5)}}{\cancel{(2x+5)}\cancel{(2x-5)}}=\frac{6x}{x+4}$$

8.3 Finding the Least Common Denominator

To Find the Least Common Denominator (LCD)

1) Factor the denominators.
2) The LCD will contain each unique factor the greatest number of times it appears in any single factorization.
3) The LCD is the *product* of the factors identified in step 2. **(p. 472)**

Find the LCD of $\dfrac{9b}{b^2+8b}$ and $\dfrac{6}{b^2+16a+64}$.

1) $b^2+8b=b(b+8)$
   $b^2+16a+64=(b+8)^2$
2) The factors we will use in the LCD are b and $(b+8)^2$.
3) LCD $=b(b+8)^2$

8.4 Adding and Subtracting Rational Expressions

Adding and Subtracting Rational Expressions

1) Factor the denominators.
2) Write down the LCD.
3) Rewrite each rational expression as an equivalent expression with the LCD.
4) Add or subtract the numerators and keep the common denominator in factored form.
5) After combining like terms in the numerator, ask yourself, "*Can I factor it?*" If so, factor.
6) Reduce the rational expression, if possible. **(p. 490)**

Add $\dfrac{y}{y+7}+\dfrac{10y-28}{y^2-49}$.

1) Factor the denominator of $\dfrac{10y-28}{y^2-49}$.

$$\frac{10y-28}{y^2-49}=\frac{10y-28}{(y+7)(y-7)}$$

2) The LCD is $(y+7)(y-7)$.

3) Rewrite $\dfrac{y}{y+7}$ with the LCD.

$$\frac{y}{y+7}\cdot\frac{y-7}{y-7}=\frac{y(y-7)}{(y+7)(y-7)}$$

4) $$\frac{y}{y+7}+\frac{10y-28}{y^2-49}=\frac{y(y-7)}{(y+7)(y-7)}+\frac{10y-28}{(y+7)(y-7)}$$
$$=\frac{y(y-7)+10y-28}{(y+7)(y-7)}$$
$$=\frac{y^2-7y+10y-28}{(y+7)(y-7)}$$
$$=\frac{y^2+3y-28}{(y+7)(y-7)}$$

5) $$=\frac{\cancel{(y+7)}(y-4)}{\cancel{(y+7)}(y-7)}$$ Factor.

6) $$=\frac{y-4}{y-7}$$ Reduce.

Definition/Procedure	Example

8.5 Simplifying Complex Fractions

Definition/Procedure	Example
A **complex fraction** is a rational expression that contains one or more fractions in its numerator, its denominator, or both. **(p. 492)**	Some examples of complex fractions are $\dfrac{\frac{9}{16}}{\frac{3}{4}}, \quad \dfrac{\frac{b+3}{2}}{\frac{6b+18}{7}}, \quad \dfrac{\frac{1}{x}-\frac{1}{y}}{1-\frac{x}{y}}$
To simplify a complex fraction containing one term in the numerator and one term in the denominator, 1) Rewrite the complex fraction as a division problem. 2) Perform the division by multiplying the first fraction by the reciprocal of the second. **(p. 492)**	Simplify $\dfrac{\frac{b+3}{2}}{\frac{6b+18}{7}}$. $\dfrac{\frac{b+3}{2}}{\frac{6b+18}{7}} = \dfrac{b+3}{2} \div \dfrac{6b+18}{7}$ $= \dfrac{b+3}{2} \cdot \dfrac{7}{6(b+3)} = \dfrac{\cancel{b+3}}{2} \cdot \dfrac{7}{6(\cancel{b+3})} = \dfrac{7}{12}$
To simplify complex fractions containing more than one term in the numerator and/or the denominator, **Method 1** 1) Combine the terms in the numerator and combine the terms in the denominator so that each contains only one fraction. 2) Rewrite as a division problem. 3) Perform the division. **(p. 493)**	**Method 1** Simplify $\dfrac{\frac{1}{x}-\frac{1}{y}}{1-\frac{x}{y}}$. $\dfrac{\frac{1}{x}-\frac{1}{y}}{1-\frac{x}{y}} = \dfrac{\frac{y}{xy}-\frac{x}{xy}}{\frac{y}{y}-\frac{x}{y}} = \dfrac{\frac{y-x}{xy}}{\frac{y-x}{y}}$ $= \dfrac{y-x}{xy} \div \dfrac{y-x}{y} = \dfrac{\cancel{y-x}}{x\cancel{y}} \cdot \dfrac{\cancel{y}}{\cancel{y-x}} = \dfrac{1}{x}$
Method 2 1) Write down the LCD of *all* of the fractions in the complex fraction. 2) Multiply the numerator and denominator of the complex fraction by the LCD. 3) Simplify. **(p. 494)**	**Method 2** Simplify $\dfrac{\frac{1}{x}-\frac{1}{y}}{1-\frac{x}{y}}$. ***Step 1:*** LCD $= xy$ ***Step 2:*** $\dfrac{xy\left(\frac{1}{x}-\frac{1}{y}\right)}{xy\left(1-\frac{x}{y}\right)}$ $\dfrac{xy\left(\frac{1}{x}-\frac{1}{y}\right)}{xy\left(1-\frac{x}{y}\right)} = \dfrac{xy \cdot \frac{1}{x} - xy \cdot \frac{1}{y}}{xy \cdot 1 - xy \cdot \frac{x}{y}}$ Distribute. ***Step 3:*** $= \dfrac{y-x}{xy-x^2}$ Simplify. $= \dfrac{\cancel{y-x}}{x(\cancel{y-x})} = \dfrac{1}{x}$

Definition/Procedure	Example

8.6 Solving Rational Equations

An **equation** contains an $=$ sign. To solve a rational equation, *multiply* the equation by the LCD to *eliminate* the denominators, then solve. Always check the answer to be sure it does not make a denominator equal zero. **(p. 499)**	Solve $\frac{n}{n+6} + 1 = \frac{18}{n+6}$. This is an equation because it contains an $=$ sign. We must eliminate the denominators. Identify the LCD of all of the expressions in the equation. LCD $= (n + 6)$ Multiply both sides of the equation by $(n + 6)$. $(n+6)\left(\frac{n}{n+6} + 1\right) = (n+6)\left(\frac{18}{n+6}\right)$ $\cancel{(n+6)} \cdot \left(\frac{n}{\cancel{n+6}}\right) + (n+6) \cdot 1 = \cancel{(n+6)} \cdot \frac{18}{\cancel{n+6}}$ $n + n + 6 = 18$ $2n + 6 = 18$ $2n = 12$ $n = 6$ The solution set is $\{6\}$. The check is left to the student.
Solve an Equation for a Specific Variable (p. 504)	Solve $x = \frac{3b}{n+m}$ for n. Since we are solving for n, put it in a box. $x = \frac{3b}{\boxed{n} + m}$ $(\boxed{n} + m)x = \cancel{(\boxed{n} + m)} \cdot \frac{3b}{\cancel{\boxed{n} + m}}$ $(\boxed{n} + m)x = 3b$ $\boxed{n}x + mx = 3b$ $\boxed{n}x = 3b - mx$ $n = \frac{3b - mx}{x}$

8.7 Applications of Rational Equations

Use the five steps for solving word problems outlined in Section 3.2. **(p. 509)**	*Write an equation and solve.* Jeff can wash and wax his car in 3 hours, but it takes his dad only 2 hours to wash and wax the car. How long would it take the two of them to wash and wax together? **Step 1:** Read the problem carefully. **Step 2:** $t =$ number of hours to wash and wax the car together.

Definition/Procedure	Example
	Step 3: Translate from English into an algebraic equation.

Jeff's rate $= \frac{1}{3}$ wash/hr Dad's rate $= \frac{1}{2}$ wash/hr

$$\begin{array}{rcccl} \text{Fractional part} = & \text{Rate} & \cdot & \text{Time} & \\ \text{Jeff's part} = & \frac{1}{3} & \cdot & t & = \frac{1}{3}t \\ \text{Dad's part} = & \frac{1}{2} & \cdot & t & = \frac{1}{2}t \end{array}$$

$$\begin{array}{ccccc} \text{Fractional job by Jeff} & + & \text{Fractional job by his dad} & = & \text{1 whole job} \\ \frac{1}{3}t & + & \frac{1}{2}t & = & 1 \end{array}$$

Equation: $\frac{1}{3}t + \frac{1}{2}t = 1$

Step 4: Solve the equation.

$$\begin{aligned} 6\left(\frac{1}{3}t + \frac{1}{2}t\right) &= 6(1) && \text{Multiply by 6, the LCD.} \\ 6 \cdot \frac{1}{3}t + 6 \cdot \frac{1}{2}t &= 6(1) && \text{Distribute.} \\ 2t + 3t &= 6 && \text{Multiply.} \\ 5t &= 6 \\ t &= \frac{6}{5} \end{aligned}$$

Step 5: Interpret the solution as it relates to the problem.

Jeff and his dad could wash and wax the car together in $\frac{6}{5}$ hours or $1\frac{1}{5}$ hours.

The check is left to the student.

Chapter 8: Review Exercises

(8.1) Evaluate, if possible, for a) $n = 5$ and b) $n = -2$.

1) $\dfrac{n^2 - 3n - 10}{3n + 2}$

2) $\dfrac{3n - 2}{n^2 - 4}$

Determine the value(s) of the variable for which
a) the expression equals zero.
b) the expression is undefined.

3) $\dfrac{2s}{4s + 11}$

4) $\dfrac{k + 3}{k - 4}$

5) $\dfrac{15}{4t^2 - 9}$

6) $\dfrac{2c^2 - 3c - 9}{c^2 - 7c}$

7) $\dfrac{3m^2 - m - 10}{m^2 + 49}$

8) $\dfrac{15 - 5d}{d^2 + 25}$

Write each rational expression in lowest terms.

9) $\dfrac{77k^9}{7k^3}$

10) $\dfrac{54a^3}{9a^{11}}$

11) $\dfrac{r^2 - 14r + 48}{4r^2 - 24r}$

12) $\dfrac{18c - 66}{39c - 143}$

13) $\dfrac{3z - 5}{6z^2 - 7z - 5}$

14) $\dfrac{y^2 + 8y - yz - 8z}{yz - 3y - z^2 + 3z}$

15) $\dfrac{11 - x}{x^2 - 121}$

16) $\dfrac{4t^4 - 32t}{(t - 2)(t^2 + 2t + 4)}$

Find three equivalent forms of each rational expression.

17) $-\dfrac{4n + 1}{5 - 3n}$

18) $-\dfrac{u - 8}{u + 2}$

Find the missing side in each rectangle.

19) Area $= 2b^2 + 13b + 21$

Find the width.

20) Area $= 3x^2 - 8x - 3$

Find the length.

Determine the domain of each rational function.

21) $h(x) = \dfrac{7}{x - 2}$

22) $g(a) = \dfrac{a + 5}{4a + 9}$

23) $f(t) = \dfrac{3t}{t^2 - 64}$

24) $k(x) = \dfrac{x - 12}{x^2 + 25}$

(8.2) Perform the operations and simplify.

25) $\dfrac{64}{45} \cdot \dfrac{27}{56}$

26) $\dfrac{6}{25} \div \dfrac{9}{10}$

27) $\dfrac{t + 6}{4} \cdot \dfrac{2(t + 2)}{(t + 6)^2}$

28) $\dfrac{4m^3}{30n} \div \dfrac{20m^6}{3n^5}$

29) $\dfrac{3x^2 + 11x + 8}{15x + 40} \div \dfrac{9x + 9}{x - 3}$

30) $\dfrac{6w - 1}{6w^2 + 5w - 1} \cdot \dfrac{3w + 3}{12w}$

31) $\dfrac{r^2 - 16r + 63}{2r^3 - 18r^2} \div (r - 7)^2$

32) $(h^2 + 10h + 24) \cdot \dfrac{h}{h^2 + h - 12}$

33) $\dfrac{3p^5}{20q^2} \cdot \dfrac{4q^3}{21p^7}$

34) $\dfrac{25 - a^2}{4a^2 + 12a} \div \dfrac{a^3 - 125}{a^2 + 3a}$

Divide.

35) $\dfrac{\dfrac{3s + 8}{12}}{\dfrac{3s + 8}{4}}$

36) $\dfrac{\dfrac{16m - 8}{m^2}}{\dfrac{12m - 6}{m^4}}$

37) $\dfrac{\dfrac{9}{8}}{\dfrac{15}{4}}$

38) $\dfrac{\dfrac{2r + 10}{r^2}}{\dfrac{r^2 - 25}{4r}}$

(8.3) Find the LCD of each group of fractions.

39) $\dfrac{9}{10}, \dfrac{7}{15}, \dfrac{6}{5}$

40) $\dfrac{3}{9x^2y}, \dfrac{13}{4xy^4}$

41) $\dfrac{3}{k^5}, \dfrac{11}{k^2}$

42) $\dfrac{3}{2m}, \dfrac{4}{m + 4}$

43) $\dfrac{1}{4x + 9}, \dfrac{3x}{x - 7}$

44) $\dfrac{8}{3d^2 - d}, \dfrac{11}{9d - 3}$

45) $\dfrac{w}{w - 5}, \dfrac{11}{5 - w}$

46) $\dfrac{6m}{m^2 - n^2}, \dfrac{n}{n - m}$

47) $\dfrac{3c - 11}{c^2 + 9c + 20}, \dfrac{8c}{c^2 - 2c - 35}$

48) $\dfrac{6}{x^2 + 7x}, \dfrac{1}{2x^2 + 14x}, \dfrac{13}{x^2 + 14x + 49}$

Rewrite each rational expression with the indicated denominator.

49) $\dfrac{3}{5y} = \dfrac{}{20y^3}$

50) $\dfrac{4k}{k - 9} = \dfrac{}{(k - 6)(k - 9)}$

51) $\dfrac{6}{2z + 5} = \dfrac{}{z(2z + 5)}$

52) $\dfrac{n}{9 - n} = \dfrac{}{n - 9}$

53) $\dfrac{t - 3}{3t + 1} = \dfrac{}{(3t + 1)(t + 4)}$

Identify the LCD of each group of fractions, and rewrite each as an equivalent fraction with the LCD as its denominator.

54) $\dfrac{4}{5a^3b}, \dfrac{3}{8ab^5}$

55) $\dfrac{8c}{c^2 + 5c - 24}, \dfrac{5}{c^2 - 6c + 9}$

56) $\dfrac{6}{p + 9}, \dfrac{3}{p}$

57) $\dfrac{7}{2q^2 - 12q}, \dfrac{3q}{36 - q^2}, \dfrac{q - 5}{2q^2 + 12q}$

58) $\dfrac{1}{g-12}, \dfrac{6}{12-g}$

(8.4) Add or subtract.

59) $\dfrac{5}{9c}+\dfrac{7}{9c}$

60) $\dfrac{5}{6z^2}+\dfrac{9}{12z}$

61) $\dfrac{9}{10u^2v^2}-\dfrac{1}{8u^3v}$

62) $\dfrac{3m}{m-4}-\dfrac{1}{m-4}$

63) $\dfrac{n}{3n-5}-\dfrac{4}{n}$

64) $\dfrac{8}{t+2}+\dfrac{8}{t}$

65) $\dfrac{9}{y+2}-\dfrac{5}{y-3}$

66) $\dfrac{7d-3}{d^2+3d-28}+\dfrac{3d}{5d+35}$

67) $\dfrac{k-3}{k^2+14k+49}-\dfrac{2}{k^2+7k}$

68) $\dfrac{8p+3}{2p+2}-\dfrac{6}{p^2-3p-4}$

69) $\dfrac{t+9}{t-18}-\dfrac{11}{18-t}$

70) $\dfrac{1}{12-r}+\dfrac{24}{r^2-144}$

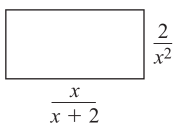
71) $\dfrac{4w}{w^2+11w+24}-\dfrac{3w-1}{2w^2-w-21}$

72) $\dfrac{2a+7}{a^2-6a+9}+\dfrac{6}{a^2+2a-15}$

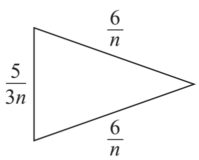
73) $\dfrac{b}{9b^2-4}+\dfrac{b+1}{6b^2-4b}-\dfrac{1}{6b+4}$

74) $\dfrac{d+4}{d^2+3d}+\dfrac{d}{5d^2+12d-9}-\dfrac{8}{5d^2-3d}$

75) Find a rational expression in simplest form to represent the a) area and b) perimeter of the rectangle.



76) Find a rational expression in simplest form to represent the perimeter of the triangle.



(8.5) Simplify completely.

77) $\dfrac{\frac{x}{y}}{\frac{x^3}{y^2}}$

78) $\dfrac{\frac{a}{b}-\frac{2a}{b^2}}{\frac{4}{ab}-\frac{a}{b}}$

79) $\dfrac{p+\frac{4}{p}}{\frac{9}{p}+p}$

80) $\dfrac{\frac{n}{6n+48}}{\frac{n^2}{8n+64}}$

81) $\dfrac{\frac{4}{5}-\frac{2}{3}}{\frac{1}{2}+\frac{1}{6}}$

82) $\dfrac{\frac{4q}{7q+70}}{\frac{q^3}{8q+80}}$

83) $\dfrac{1-\frac{1}{y-8}}{\frac{2}{y+4}+1}$

84) $\dfrac{\frac{10}{21}}{\frac{16}{9}}$

85) $\dfrac{1+\frac{1}{r-t}}{\frac{1}{r^2-t^2}+\frac{1}{r+t}}$

86) $\dfrac{\frac{z}{z+2}+\frac{1}{z^2-4}}{1-\frac{3}{z+2}}$

(8.6) Solve each equation.

87) $\dfrac{5a+4}{15}=\dfrac{a}{5}+\dfrac{4}{5}$

88) $\dfrac{16}{9c-27}+\dfrac{2c-4}{c-3}=\dfrac{c}{9}$

89) $\dfrac{m}{7}=\dfrac{5}{m+2}$

90) $\dfrac{2}{y-7}=\dfrac{8}{y+5}$

91) $\dfrac{r}{r+5}+4=\dfrac{5}{r+5}$

92) $\dfrac{3}{j+9}+\dfrac{j}{j-3}=\dfrac{2j^2+2}{j^2+6j-27}$

93) $\dfrac{5}{t^2+10t+24}+\dfrac{5}{t^2+3t-18}=\dfrac{t}{t^2+t-12}$

94) $p-\dfrac{20}{p}=8$

95) $\dfrac{3}{x+1}=\dfrac{6x}{x^2-1}-\dfrac{4}{x-1}$

96) $\dfrac{9}{4k^2+28k+48}=\dfrac{k}{4k+16}+\dfrac{9}{8k+24}$

Solve for the indicated variable.

97) $R=\dfrac{s+T}{D}$ for D

98) $A=\dfrac{2p}{c}$ for c

99) $w=\dfrac{N}{c-ak}$ for k

100) $n=\dfrac{t}{a+b}$ for a

101) $\dfrac{1}{R_1}+\dfrac{1}{R_2}=\dfrac{1}{R_3}$ for R_1

102) $\dfrac{1}{r}=\dfrac{1}{s}+\dfrac{1}{t}$ for s

(8.7) Write an equation and solve.

103) A boat can travel 8 miles downstream in the same amount of time it can travel 6 miles upstream. If the speed of the boat in still water is 14 mph, what is the speed of the current?

104) The ratio of saturated fat to total fat in a Starbucks tall Caramel Frappuccino is 2 to 3. If there are 4 more grams of total fat in the drink than there are grams of saturated fat, how much total fat is in 2 Caramel Frappuccinos? (Starbucks brochure)

105) Crayton and Flow must put together notebooks for each person attending a conference. Working alone, it would take Crayton 5 hours while it would take Flow 8 hours. How long would it take for them to assemble the notebooks together?

106) An airplane flying at constant speed can fly 350 miles with the wind in the same amount of time it can fly 300 miles against the wind. What is the speed of the plane if the wind blows at 20 mph?

Mixed Exercises

Perform the operation and simplify.

107) $\dfrac{5n}{2n-1} - \dfrac{2n+3}{n+2}$

108) $\dfrac{27w^3}{3w^2+w-4} \cdot \dfrac{2-2w}{15w}$

109) $\dfrac{2a^2+9a+10}{4a-7} \div (2a+5)^2$

110) $\dfrac{5}{8b} + \dfrac{2}{9b^4}$

111) $\dfrac{c^2}{c^2-d^2} + \dfrac{c}{d-c}$

112) $\dfrac{\dfrac{7}{x}+\dfrac{8}{y}}{1-\dfrac{6}{y}}$

Solve.

113) $\dfrac{h}{5} = \dfrac{h-3}{h+1} + \dfrac{12}{5h+5}$

114) $\dfrac{5w}{6} - \dfrac{2}{3} = -\dfrac{1}{6}$

115) $\dfrac{8}{3g^2-7g-6} - \dfrac{8}{3g+2} = -\dfrac{4}{g-3}$

116) $\dfrac{4k}{k+16} = \dfrac{4}{k+1}$

Chapter 8: Test

1) Evaluate, if possible, for $k = -4$.

$$\frac{5k + 8}{k^2 + 16}$$

Determine the values of the variable for which
a) the expression is undefined.
b) the expression equals zero.

2) $\dfrac{2c - 9}{c + 10}$

3) $\dfrac{n^2 + 1}{n^2 - 5n - 36}$

Write each rational expression in lowest terms.

4) $\dfrac{21t^8u^2}{63t^{12}u^5}$

5) $\dfrac{3h^2 - 25h + 8}{27h^3 - 1}$

6) Write three equivalent forms of $\dfrac{7 - m}{4m - 5}$.

7) Identify the LCD of $\dfrac{2z}{z + 6}$ and $\dfrac{9}{z}$.

Perform the operations and simplify.

8) $\dfrac{8}{15r} + \dfrac{2}{15r}$

9) $\dfrac{28a^9}{b^2} \div \dfrac{20a^{15}}{b^3}$

10) $\dfrac{5h}{12} - \dfrac{7h}{9}$

11) $\dfrac{6}{c + 2} + \dfrac{c}{3c + 5}$

12) $\dfrac{k^3 - 9k^2 + 2k - 18}{4k - 24} \cdot \dfrac{k^2 + 3k - 54}{81 - k^2}$

13) $\dfrac{8d^2 + 24d}{20} \div (d + 3)^2$

14) $\dfrac{2t - 5}{t - 7} + \dfrac{t + 9}{7 - t}$

15) $\dfrac{3}{2v^2 - 7v + 6} - \dfrac{v + 4}{v^2 + 7v - 18}$

Simplify completely.

16) $\dfrac{1 - \dfrac{1}{m + 2}}{\dfrac{m}{m + 2} - \dfrac{1}{m}}$

17) $\dfrac{\dfrac{5x + 5y}{x^2y^2}}{\dfrac{20}{xy}}$

Solve each equation.

18) $\dfrac{3r + 1}{2} + \dfrac{1}{10} = \dfrac{6r}{5}$

19) $\dfrac{28}{w^2 - 4} = \dfrac{7}{w - 2} - \dfrac{5}{w + 2}$

20) $\dfrac{3}{x + 8} + \dfrac{x}{x - 4} = \dfrac{7x + 9}{x^2 + 4x - 32}$

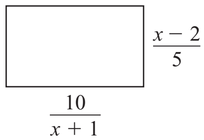
21) Solve for b.

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$$

For the given rectangle, find a rational expression in simplest form to represent

22) its area.

23) its perimeter.



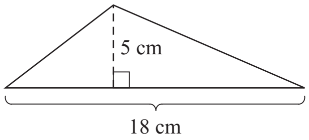
Write an equation for each and solve.

24) Every Sunday night, the equipment at a restaurant must be taken apart and cleaned. Ricardo can do this job twice as fast as Michael. When they work together, they can do the cleaning in 2 hr. How long would it take each man to do the job on his own?

25) A current flows at 4 mph. If a boat can travel 12 mi downstream in the same amount of time it can go 6 mi upstream, find the speed of the boat in still water.

Cumulative Review: Chapters 1–8

1) Find the area of the triangle.



2) Evaluate $72 - 30 \div 6 + 4(3^2 - 10)$

Simplify. The answer should not contain negative exponents.

3) $(2p^3)^5$

4) $(5y^2)^{-3}$

5) Write an equation and solve.

The length of a rectangular garden is 4 ft longer than the width. Find the dimensions of the garden if its perimeter is 28 ft.

Solve each inequality. Write the answer in interval notation.

6) $19 - 8w > 5$

7) $4 \leq \frac{3}{5}t + 4 \leq 13$

8) Find the x- and y-intercepts of $4x - 3y = 6$, and graph the equation.

9) Find the slope of the line containing the points $(4, 1)$ and $(-2, 9)$.

10) Solve the system.

$$\begin{aligned} 5x + 4y &= 5 \\ 7x - 6y &= 36 \end{aligned}$$

Multiply and simplify.

11) $(2n - 3)^2$

12) $(8a + b)(8a - b)$

Divide.

13) $\dfrac{45h^4 - 25h^3 + 15h^2 - 10}{15h^2}$

14) $\dfrac{5k^3 + 18k^2 - 11k - 8}{k + 4}$

Factor completely.

15) $4d^2 + 4d - 15$

16) $3z^4 - 48$

17) $3m^4 - 24m$

18) $rt + 8t - r - 8$

19) Solve $x(x + 16) = x - 36$

20) For what values of a is $\dfrac{7a + 2}{a^2 - 6a}$

a) undefined?

b) equal to zero?

21) Write $\dfrac{3c^2 + 21c - 54}{c^2 + 3c - 54}$ in lowest terms.

Perform the operations and simplify.

22) $\dfrac{10n^2}{n^2 - 8n + 16} \cdot \dfrac{3n^2 - 14n + 8}{10n - 15n^2}$

23) $\dfrac{6}{y + 5} - \dfrac{3}{y}$

24) Simplify $\dfrac{\dfrac{2}{r - 8} + 1}{1 - \dfrac{3}{r - 8}}$.

25) Solve $\dfrac{1}{v - 1} + \dfrac{2}{5v - 3} = \dfrac{37}{5v^2 - 8v + 3}$.

CHAPTER 9

More Equations and Inequalities

Algebra at Work: Finance

When a client comes to a financial planner for help investing money, the adviser may recommend many different types of places to invest the money. Stocks and bonds are just two of the investments the financial planner could suggest.

Theresa has a client who has six different stocks and four different bonds in her investment portfolio. The closing prices of the stocks change each day as do the interest rates the bonds yield. Theresa keeps track of these closing figures each day and then organizes this information weekly so that she knows how much her client's portfolio is worth at any given time.

To make it easier to organize this information, Theresa puts all of the information into matrices (the plural of matrix). Each week she creates a matrix containing information about the closing price of each stock each day of the week. She creates another matrix that contains information about the interest rates each of the bonds yields each day of the week.

When there is a lot of information to organize, financial planners often turn to matrices to help them keep track of all of their information.

In this chapter, we will learn about using augmented matrices to solve systems of equations.

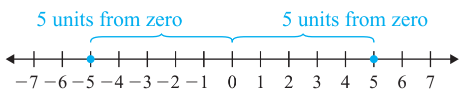
Section 9.1 Solving Absolute Value Equations

Objectives

1. Understand the Meaning of an Absolute Value Equation
2. Solve an Equation of the Form $|ax + b| = k$ for $k > 0$
3. Solve an Equation of the Form $|ax + b| + t = c$
4. Solve an Equation of the Form $|ax + b| = |cx + d|$
5. Solve Special Absolute Value Equations

In Section 1.4, we learned that the absolute value of a number describes its *distance from zero*.

$$|5| = 5 \quad \text{and} \quad |-5| = 5$$



We use this idea of *distance from zero* to solve absolute value equations and inequalities.

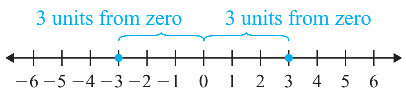
1. Understand the Meaning of an Absolute Value Equation

Example 1

Solve $|x| = 3$.

Solution

Since the equation contains an absolute value, **solve $|x| = 3$** means *"Find the number or numbers whose distance from zero is 3."*



Those numbers are 3 and -3. Each of them is 3 units from zero.

The solution set is $\{-3, 3\}$.

Check: $|3| = 3$, $|-3| = 3$ ✓

You Try 1

Solve $|y| = 8$.

Procedure Solving an Absolute Value Equation

If P represents an expression and k is a positive real number, then to solve $|P| = k$, we rewrite the absolute value equation as the *compound equation*

$$P = k \quad \text{or} \quad P = -k$$

and solve for the variable.

P can represent expressions like x, $3a + 2$, $\frac{1}{4}t - 9$, and so on.

2. Solve an Equation of the Form $|ax + b| = k$ for $k > 0$

Example 2

Solve each equation.

a) $|m + 1| = 5$ b) $|5r - 3| = 13$

Solution

a) Solving $|m + 1| = 5$ means, *"Find the number or numbers that can be substituted for m so that the quantity m + 1 is 5 units from 0."*

$m + 1$ will be 5 units from zero if $m + 1 = 5$ or if $m + 1 = -5$, since both 5 and -5 are 5 units from zero. Therefore, we can solve the equation this way:

$$|m + 1| = 5$$

$$m + 1 = 5 \quad \text{or} \quad m + 1 = -5$$ Set the quantity inside the absolute value equal to 5 and -5.

$$m = 4 \quad \text{or} \quad m = -6$$ Solve.

Check: $m = 4$: $|4 + 1| \stackrel{?}{=} 5$, $|5| = 5$ ✓ | $m = -6$: $|-6 + 1| \stackrel{?}{=} 5$, $|-5| = 5$ ✓

The solution set is $\{-6, 4\}$.

b) Solving $|5r - 3| = 13$ means, "*Find the number or numbers that can be substituted for r so that the quantity 5r − 3 is 13 units from zero.*"

$$|5r - 3| = 13$$

$$5r - 3 = 13 \quad \text{or} \quad 5r - 3 = -13$$ Set the quantity inside the absolute value equal to 13 and -13.

$$5r = 16 \qquad\qquad 5r = -10$$

$$r = \frac{16}{5} \quad \text{or} \quad r = -2$$ Solve.

The check is left to the student. The solution set is $\left\{-2, \frac{16}{5}\right\}$.

You Try 2

Solve each equation.

a) $|c - 4| = 3$ b) $|2k + 1| = 9$

3. Solve an Equation of the Form $|ax + b| + t = c$

Example 3

Solve $\left|\frac{3}{2}t + 7\right| + 5 = 6$.

Solution

Before we rewrite this equation as a compound equation, we must *isolate* the absolute value (get the absolute value on a side by itself).

$$\left|\frac{3}{2}t + 7\right| + 5 = 6$$

$$\left|\frac{3}{2}t + 7\right| = 1 \qquad \text{Subtract 5 to isolate the absolute value.}$$

$$\swarrow \qquad \searrow$$

$$\begin{array}{rclcrcll}
\frac{3}{2}t + 7 &=& 1 & \text{or} & \frac{3}{2}t + 7 &=& -1 & \text{Set the quantity inside the absolute value equal to 1 and } -1. \\
\frac{3}{2}t &=& -6 & & \frac{3}{2}t &=& -8 & \text{Subtract 7.} \\
\frac{2}{3}\cdot\frac{3}{2}t &=& \frac{2}{3}\cdot(-6) & & \frac{2}{3}\cdot\frac{3}{2}t &=& \frac{2}{3}\cdot(-8) & \text{Multiply by } \frac{2}{3} \text{ to solve for } t. \\
t &=& -4 & \text{or} & t &=& -\frac{16}{3} & \text{Solve.}
\end{array}$$

The check is left to the student. The solution set is $\left\{-\frac{16}{3}, -4\right\}$.

You Try 3

Solve $\left|\frac{1}{4}n - 3\right| + 2 = 5$.

4. Solve an Equation of the Form $|ax + b| = |cx + d|$

Another type of absolute value equation involves two absolute values.

Procedure Solving an Absolute Value Equation Containing Two Absolute Values

If P and Q are expressions, then to solve $|P| = |Q|$, we rewrite the absolute value equation as the *compound equation*

$$P = Q \quad \text{or} \quad P = -Q$$

and solve for the variable.

Example 4

Solve $|2w - 3| = |w + 9|$.

Solution

This equation is true when the quantities inside the absolute values are the *same* or when they are *negatives* of each other.

$$|2w - 3| = |w + 9|$$

$$\begin{array}{rcl|rcl}
\multicolumn{3}{c}{\text{The quantities are the same}} & \multicolumn{3}{c}{\text{or the quantities are negatives of each other.}} \\
2w - 3 &=& w + 9 & 2w - 3 &=& -(w + 9) \\
w &=& 12 & 2w - 3 &=& -w - 9 \\
 & & & 3w &=& -6 \\
 & & & w &=& -2
\end{array}$$

Check:

$$w = 12: \quad \begin{aligned} |2(12) - 3| &\stackrel{?}{=} |12 + 9| \\ |24 - 3| &\stackrel{?}{=} |21| \\ |21| &= 21 \quad \checkmark \end{aligned} \qquad\Bigg|\qquad w = -2: \quad \begin{aligned} |2(-2) - 3| &\stackrel{?}{=} |-2 + 9| \\ |-4 - 3| &\stackrel{?}{=} |7| \\ |-7| &= 7 \quad \checkmark \end{aligned}$$

The solution set is $\{-2, 12\}$. ■

BE CAREFUL

In Example 4 and other examples like it, you *must* put parentheses around the expression with the negative as in $-(w + 9)$.

You Try 4

Solve $|c + 7| = |3c - 1|$.

5. Solve Special Absolute Value Equations

It is important to understand the *meaning* of an absolute value equation to understand how to solve special types of absolute value equations.

Example 5

Solve $|4y - 11| = -9$.

Solution

This equation says that the absolute value of the quantity $4y - 11$ equals *negative* 9. Can an absolute value be negative? No! This equation has *no solution*.

The solution set is $\varnothing$. ■

Example 6

Solve $|0.3w + 12| = 0$.

Solution

The absolute value of an expression will be 0 when the expression equals 0. Let $0.3w + 12 = 0$ and solve.

$$\begin{aligned} 0.3w + 12 &= 0 & \\ 0.3w &= -12 & &\text{Subtract 12 from each side.} \\ \frac{0.3w}{0.3} &= \frac{-12}{0.3} & &\text{Divide each side by 0.3.} \\ w &= -40 & &\text{Simplify.} \end{aligned}$$

In this case, there is only one solution to the equation. The solution set is $\{-40\}$. ■

You Try 5

Solve each equation.

a) $|d + 3| = -5$ b) $|0.2p + 7| = 0$

Using Technology

We can use a graphing calculator to solve an equation by entering one side of the equation as Y_1 and the other side as Y_2. Then graph the equations. Remember that absolute value equations like the ones found in this section can have 0, 1, or 2 solutions. *The x-coordinates of their points of intersection are the solutions to the equation.*

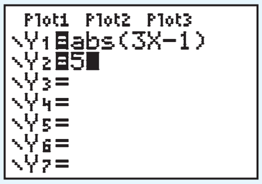
We will solve $|3x - 1| = 5$ algebraically and by using a graphing calculator, and then compare the results.

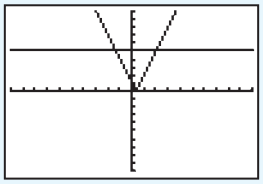
First, use algebra to solve $|3x - 1| = 5$. You should get $\left\{-\frac{4}{3}, 2\right\}$.

Next, use a graphing calculator to solve $|3x - 1| = 5$.

We will enter $|3x - 1|$ as Y_1 and 5 as Y_2. To enter $Y_1 = |3x - 1|$,

1. Press the $\boxed{Y =}$ key, so that the cursor is to the right of \$Y_1 =$.
2. Press $\boxed{\text{MATH}}$ and then press the right arrow, to highlight **NUM**. Also highlighted is 1:abs (which stands for *absolute value*).
3. Press $\boxed{\text{ENTER}}$ and you are now back on the \$Y_1 =$ screen. Enter $3x - 1$ with a closing parenthesis so that you have now entered $Y_1 = \text{abs}(3x - 1)$.
4. Press the down arrow to enter \$Y_2 = 5$.
5. Press $\boxed{\text{GRAPH}}$.



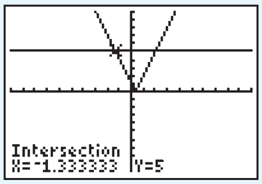


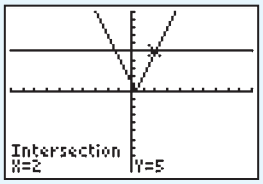
The graphs intersect at two points because there are two solutions to this equation. *Remember that the solutions to the equation are the x-coordinates of the points of intersection.*

To find these x-coordinates we will use the INTERSECT feature introduced in Chapter 5.

To find the left-hand intersection point, press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ and select 5:intersect. Press $\boxed{\text{ENTER}}$. Move the cursor close to the point on the left and press $\boxed{\text{ENTER}}$ three times. You get the result in the screen below on the left.

To find the right-hand intersect point, press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$, select 5:intersect, and press $\boxed{\text{ENTER}}$. Move the cursor close to the point, and press $\boxed{\text{ENTER}}$ three times. You will see the screen that is below on the right.





The screen on the left shows $x = -1.333333$. This is the calculator's approximation of $x = -1.\overline{3}$, the decimal equivalent of $x = -\frac{4}{3}$, one of the solutions found using algebra.

The screen on the right shows $x = 2$ as a solution, the same solution we obtained algebraically.

The calculator gives us a solution set of $\{-1.333333, 2\}$, while the solution set found using algebra is $\left\{-\frac{4}{3}, 2\right\}$.

Solve each equation algebraically, then verify your answer using a graphing calculator.

1) $|x - 1| = 2$
2) $|x + 4| = 6$
3) $|2x + 3| = 3$
4) $|4x - 5| = 1$
5) $|3x + 7| - 6 = -8$
6) $|6 - x| + 3 = 3$

Answers to You Try Exercises

1) $\{-8, 8\}$ 2) a) $\{1, 7\}$ b) $\{-5, 4\}$ 3) $\{0, 24\}$ 4) $\left\{-\frac{3}{2}, 4\right\}$ 5) $\varnothing$ 6) $\{-35\}$

Answers to Technology Exercises

1) $\{-1, 3\}$ 2) $\{-10, 2\}$ 3) $\{-3, 0\}$ 4) $\{1, 1.5\}$ 5) $\varnothing$ 6) $\{6\}$

9.1 Exercises

Objective 1: Understand the Meaning of an Absolute Value Equation

1) In your own-words, explain the *meaning* of the absolute value of a number.

2) Does $|x| = -8$ have a solution? Why or why not?

VIDEO 3) Write an absolute value equation that means *x is 9 units from zero*.

4) Write an absolute value equation that means *y is 6 units from zero*.

Objective 2: Solve an Equation of the Form $|ax + b| = k$ for $k > 0$

Solve.

5) $|q| = 6$

6) $|z| = 7$

7) $|q - 5| = 3$

8) $|a + 2| = 13$

VIDEO 9) $|4t - 5| = 7$

10) $|9x - 8| = 10$

11) $1 = |12c + 5|$

12) $8 = |10 + 7g|$

13) $|1 - 8m| = 9$

14) $|4 - 5k| = 11$

15) $\left|\frac{2}{3}b + 3\right| = 13$

16) $\left|\frac{3}{4}h + 8\right| = 7$

17) $9 = |9 - 1.5d|$

18) $8 = |3 + 0.4w|$

19) $\left|\frac{3}{4}y - 2\right| = \frac{3}{5}$

20) $\left|\frac{3}{2}r + 5\right| = \frac{3}{4}$

21) Write an absolute value equation that has a solution set of $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$.

22) Write an absolute value equation that has a solution set of $\{-1.4, 1.4\}$.

Objective 3: Solve an Equation of the Form $|ax + b| + t = c$

Solve.

23) $|z - 6| + 4 = 20$

24) $|q + 3| - 1 = 14$

VIDEO 25) $|5b + 3| + 6 = 19$

26) $|2a + 5| + 8 = 13$

27) $|6t - 11| + 5 = 10$

28) $|9k - 4| - 11 = 2$

29) $1 = |7 - 8x| - 4$

30) $8 = |5 - 6y| + 2$

31) $-3 = \left|\frac{2}{3}c + 4\right| - 7$

32) $-6 = -14 + |0.8h + 2|$

Objective 4: Solve an Equation of the Form $|ax + b| = |cx + d|$

Solve the following equations containing two absolute values.

VIDEO 33) $|s + 9| = |2s + 5|$

34) $|j - 8| = |4j - 7|$

35) $|3z + 2| = |6 - 5z|$

36) $|1 - 2a| = |10a + 3|$

37) $\left|\frac{3}{2}x - 1\right| = |x|$

38) $|y| = \left|\frac{4}{7}y + 12\right|$

39) $|7c + 10| = |5c + 2|$

40) $|4 - 11r| = |5r + 3|$

41) $\left|\frac{1}{4}t - \frac{5}{2}\right| = \left|5 - \frac{1}{2}t\right|$

42) $\left|k + \frac{1}{6}\right| = \left|\frac{2}{3}k + \frac{1}{2}\right|$

43) $|1.6 - 0.3p| = |0.7p + 0.4|$

44) $|2.9m - 7.2| = |1.9m + 7.2|$

Objective 5: Solve Special Absolute Value Equations

Solve.

45) $|m - 5| = -3$

46) $|2k + 7| = -15$

47) $|10p + 2| = 0$

48) $|4c - 11| = 0$

49) $|w + 14| = 0$

50) $|5h + 7| = -5$

51) $|8n + 11| = -1$

52) $|4p - 3| = 0$

53) $|3m - 1| + 5 = 2$

54) $\left|\frac{5}{4}k + 2\right| + 9 = 7$

Mixed Exercises: Objectives 2–5

Solve.

55) $11 = |7 - v| + 4$

56) $|2q + 9| = 13$

57) $\left|\frac{3}{5}p + 3\right| - 7 = -5$

58) $2.8 = -1.4 + |3 - 0.2y|$

59) $|10h - 3| = 0$

60) $|6z + 1| = |4z + 15|$

61) $|1.8a - 3| = |4.2 - 1.2a|$

62) $-7 = |5w + 8|$

63) $15 + |2k + 1| = 6$

64) $\left|\frac{3}{4} - \frac{5}{6}t\right| + \frac{1}{2} = \frac{7}{6}$

65) $\left|9 - \frac{3}{2}n\right| = 1$

66) $|r - 3| + 8 = 8$

67) $\left|\frac{1}{3}g - 2\right| = \left|\frac{7}{9}g + \frac{1}{6}\right|$

68) $|0.6 + 7y| + 12 = 9$

69) $7.6 = |2.8d + 3.5| + 7.6$

70) $|x + 10| = |5 - 4x|$

Section 9.2 Solving Absolute Value Inequalities

Objectives

1. **Solve Absolute Value Inequalities Containing < or ≤**
2. **Solve Absolute Value Inequalities Containing > or ≥**
3. **Solve Special Cases of Absolute Value Inequalities**
4. **Solve an Applied Problem Using an Absolute Value Inequality**

In Section 9.1, we learned how to solve absolute value equations. In this section, we will learn how to solve **absolute value inequalities**. Some examples of absolute value inequalities are

$$|t| < 6, \qquad |n + 2| \le 5, \qquad |3k - 1| > 11, \qquad \left|5 - \frac{1}{2}y\right| \ge 3.$$

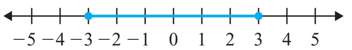
1. Solve Absolute Value Inequalities Containing < or ≤

What does it mean to solve $|x| \le 3$? It means to find the set of all real numbers whose distance from zero is *3 units or less*.

$$3 \text{ is 3 units from 0.}$$

$$-3 \text{ is 3 units from 0.}$$

Any number *between* 3 and -3 is less than 3 units from zero. For example, if $x = 1$, $|1| \le 3$. If $x = -2$, $|-2| \le 3$. We can represent the solution set on a number line as



We can write the solution set in interval notation as $[-3, 3]$.

Procedure Solving $|P| \le k$

Let P be an expression and let k be a positive real number. To solve $|P| \le k$, solve the three-part inequality $-k \le P \le k$. ($<$ may be substituted for $\le$.)

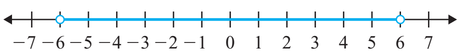
Example 1

Solve $|t| < 6$. Graph the solution set and write the answer in interval notation.

Solution

We must find the set of all real numbers whose distance from zero is less than 6. We can do this by solving the three-part inequality $-6 < t < 6$.

We can represent this on a number line as



We can write the solution set in interval notation as $(-6, 6)$. Any number between -6 and 6 will satisfy the inequality.

You Try 1

Solve $|u| < 2$. Graph the solution set and write the answer in interval notation.

Example 2

Solve each inequality. Graph the solution set and write the answer in interval notation.

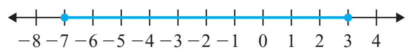
a) $|n + 2| \le 5$ b) $|4 - 5p| < 16$

Solution

a) We must find the set of all real numbers, n, so that $n + 2$ is less than or equal to 5 units from zero. To solve $|n + 2| \le 5$, we must solve the three-part inequality

$$-5 \le n + 2 \le 5$$
$$-7 \le n \le 3 \qquad \text{Subtract 2.}$$

The number line representation is



In interval notation, the solution set is $[-7, 3]$. Any number between -7 and 3 will satisfy the inequality.

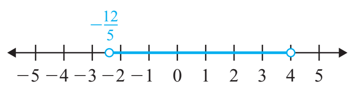
b) Solve the three-part inequality.

$$-16 < 4 - 5p < 16$$
$$-20 < -5p < 12 \qquad \text{Subtract 4.}$$
$$4 > p > -\frac{12}{5} \qquad \text{Divide by } -5 \text{ and change the direction of the inequality symbols.}$$

This inequality means p *is less than 4 and greater than* $-\frac{12}{5}$. We can rewrite it as

$$-\frac{12}{5} < p < 4.$$

The number line representation of the solution set is



In interval notation, we write $\left(-\frac{12}{5}, 4\right)$.

You Try 2

Solve each inequality. Graph the solution set and write the answer in interval notation.

a) $|6k + 5| \le 13$ b) $|9 - 2w| < 3$

2. Solve Absolute Value Inequalities Containing > or ≥

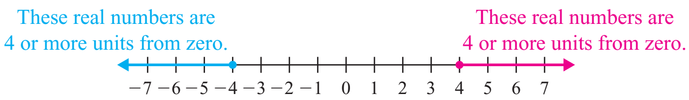
To solve $|x| \geq 4$ means to find the set of all real numbers whose distance from zero is *4 units or more.*

$$4 \text{ is 4 units from 0.}$$

$$-4 \text{ is 4 units from 0.}$$

Any number greater than 4 *or* less than -4 is more than 4 units from zero.

For example, if $x = 6$, $|6| \geq 4$. If $x = -5$, then $|-5| = 5$ and $5 \geq 4$. We can represent the solution set to $|x| \geq 4$ as



The solution set consists of two separate regions, so we can write a compound inequality $x \leq -4$ or $x \geq 4$.

In interval notation, we write $(-\infty, -4] \cup [4, \infty)$.

Procedure Solving $|P| \geq k$

Let P be an expression and let k be a positive, real number. To solve $|P| \geq k$ ($>$ may be substituted for $\geq$), solve the compound inequality $P \geq k$ or $P \leq -k$.

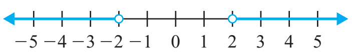
Example 3

Solve $|r| > 2$. Graph the solution set and write the answer in interval notation.

Solution

We must find the set of all real numbers whose distance from zero is greater than 2. The solution is the compound inequality $r > 2$ or $r < -2$.

On the number line, we can represent the solution set as



In interval notation, we write $(-\infty, -2) \cup (2, \infty)$. Any number in the shaded region will satisfy the inequality. For example, to the right of 2, if $r = 3$, then $|3| > 2$. To the left of -2, if $r = -4$, then $|-4| > 2$. ■

You Try 3

Solve $|d| \geq 5$. Graph the solution set and write the answer in interval notation.

Example 4

Solve each inequality. Graph the solution set and write the answer in interval notation.

a) $|3k - 1| > 11$ b) $\left|\frac{1}{2}c + 3\right| + 7 \geq 8$

Solution

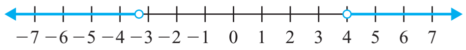
a) To solve $|3k - 1| > 11$ means to find the set of all real numbers, k, so that $3k - 1$ is more than 11 units from zero on the number line. We will solve the compound inequality

$$3k - 1 > 11 \quad \text{or} \quad 3k - 1 < -11$$

$3k - 1$ is more than 11 units away from zero to the *right* of zero. $3k - 1$ is more than 11 units away from zero to the *left* of zero.

$$\begin{aligned} 3k - 1 &> 11 && \text{or} & 3k - 1 &< -11 & & \\ 3k &> 12 && \text{or} & 3k &< -10 & & \text{Add 1.} \\ k &> 4 && \text{or} & k &< -\frac{10}{3} & & \text{Divide by 3.} \end{aligned}$$

On the number line, we get



From the number line, we can write the interval notation $\left(-\infty, -\frac{10}{3}\right) \cup (4, \infty)$.

Any number in the shaded region will satisfy the inequality.

b) Begin by getting the absolute value on a side by itself.

$$\left|\frac{1}{2}c + 3\right| + 7 \geq 8$$

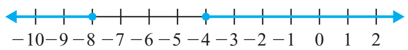
$$\left|\frac{1}{2}c + 3\right| \geq 1 \qquad \text{Subtract 7.}$$

$$\frac{1}{2}c + 3 \geq 1 \quad \text{or} \quad \frac{1}{2}c + 3 \leq -1 \qquad \text{Rewrite as a compound inequality.}$$

$$\frac{1}{2}c \geq -2 \quad \text{or} \quad \frac{1}{2}c \leq -4 \qquad \text{Subtract 3.}$$

$$c \geq -4 \qquad\qquad c \leq -8 \qquad \text{Multiply by 2.}$$

The graph of the solution set is



The interval notation is $(-\infty, -8] \cup [-4, \infty)$. ■

You Try 4

Solve each inequality. Graph the solution set and write the answer in interval notation.

a) $|8q + 9| \geq 7$ b) $\left|\frac{1}{2}k + 2\right| - 1 \geq 1$

Example 5 illustrates why it is important to understand what the absolute value inequality means before trying to solve it.

3. Solve Special Cases of Absolute Value Inequalities

Example 5

Solve each inequality.

a) $|z + 3| < -6$ b) $|2s - 1| \geq 0$ c) $|4d + 7| + 9 \leq 9$

Solution

a) Look carefully at this inequality, $|z + 3| < -6$. It says that the absolute value of a quantity, $z + 3$, is *less than* a negative number. Since the absolute value of a quantity is always zero or positive, this inequality has *no solution*.

The solution set is $\varnothing$.

b) $|2s - 1| \geq 0$ says that the absolute value of a quantity, $2s - 1$, is greater than or equal to zero. An absolute value is *always* greater than or equal to zero, so *any* value of s will make the inequality true.

The solution set consists of all real numbers, which we can write in interval notation as $(-\infty, \infty)$.

c) Begin by isolating the absolute value.

$$
\begin{aligned}
|4d + 7| + 9 &\leq 9 \\
|4d + 7| &\leq 0 \qquad \text{Subtract 9.}
\end{aligned}
$$

The absolute value of a quantity can *never be less than zero* but it *can equal zero*. To solve this, we must solve $4d + 7 = 0$.

$$
\begin{aligned}
4d + 7 &= 0 \\
4d &= -7 \qquad \text{Subtract 7.} \\
d &= -\frac{7}{4} \qquad \text{Divide by 4.}
\end{aligned}
$$

The solution set is $\left\{-\frac{7}{4}\right\}$.

You Try 5

Solve each inequality.

a) $|p + 4| \geq 0$ b) $|5n - 7| < -2$ c) $|6y - 1| + 3 \leq 3$

4. Solve an Applied Problem Using an Absolute Value Inequality

Example 6

On an assembly line, a machine is supposed to fill a can with 19 oz of soup. However, the possibility for error is ± 0.25 oz. Let x represent the range of values for the amount of soup in the can. Write an absolute value inequality to represent the range for the number of ounces of soup in the can, then solve the inequality and explain the meaning of the answer.

Solution

If the *actual* amount of soup in the can is x and there is supposed to be 19 oz in the can, then the error in the amount of soup in the can is $|x - 19|$. If the possible error is ± 0.25 oz, then we can write the inequality

$$|x - 19| \le 0.25$$

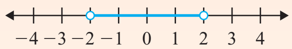
Solve. $\quad -0.25 \le x - 19 \le 0.25$

$18.75 \le x \le 19.25 \quad$ Add 19.

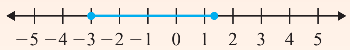
The actual amount of soup in the can is between 18.75 and 19.25 oz.

Answers to You Try Exercises

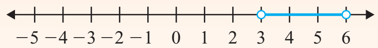
1) $(-2, 2)$



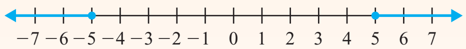
2) a) $\left[-3, \frac{4}{3}\right]$



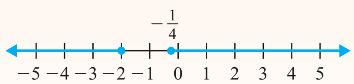
b) $(3, 6)$



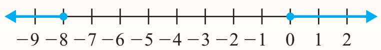
3) $(-\infty, -5] \cup [5, \infty)$



4) a) $(-\infty, -2] \cup \left[-\frac{1}{4}, \infty\right)$



b) $(-\infty, -8] \cup [0, \infty)$



5) a) $(-\infty, \infty)$ b) $\varnothing$ c) $\left\{\frac{1}{6}\right\}$

9.2 Exercises

Graph each inequality on a number line and represent the sets of numbers using interval notation.

1) $-1 \le p \le 5$

2) $7 < t < 11$

3) $y < 2$ or $y > 9$

4) $a \le -8$ or $a \ge \frac{1}{2}$

5) $n \le -\frac{9}{2}$ or $n \ge \frac{3}{5}$

6) $-\frac{1}{4} \le q \le \frac{11}{4}$

7) $4 < b < \frac{17}{3}$

8) $x < -12$ or $x > -9$

Objective 1: Solve Absolute Value Inequalities Containing $<$ or $\le$

Solve each inequality. Graph the solution set and write the answer in interval notation.

9) $|m| \le 7$

10) $|c| < 1$

11) $|3k| < 12$

12) $\left|\frac{5}{4}z\right| \le 30$

13) $|w - 2| < 4$

14) $|k - 6| \le 2$

VIDEO 15) $|3r + 10| \le 4$

16) $|4a + 1| \le 12$

17) $|7 - 6p| \le 3$

18) $|17 - 9d| < 8$

19) $|8c - 3| + 15 < 20$

20) $|2v + 5| + 3 < 14$

21) $\left|\frac{3}{2}h + 6\right| - 2 \le 10$

22) $7 + \left|\frac{8}{3}u - 9\right| < 12$

Objective 2: Solve Absolute Value Inequalities Containing $>$ or $\ge$

Solve each inequality. Graph the solution set and write the answer in interval notation.

23) $|t| \ge 7$

24) $|p| > 3$

25) $|5a| > 2$

26) $|2c| \geq 11$

27) $|d + 10| \geq 4$

28) $|q - 7| > 12$

29) $|4v - 3| \geq 9$

30) $|6a + 19| > 11$

31) $|17 - 6x| > 5$

32) $|1 - 4g| \geq 10$

33) $|2m - 1| + 4 > 5$

34) $|w + 6| - 4 \geq 2$

35) $-3 + \left|\frac{5}{6}n + \frac{1}{2}\right| \geq 1$

36) $\left|\frac{3}{2}y - \frac{5}{4}\right| + 9 \geq 11$

Objective 3: Solve Special Cases of Absolute Value Inequalities

37) $|5q + 11| < 0$

38) $|6t + 16| < 0$

39) $|2x + 7| \leq -12$

40) $|8m - 15| \leq -5$

41) $|8k + 5| \geq 0$

42) $|5b - 6| \geq 0$

43) $|z - 3| \geq -5$

44) $|3r + 10| > -11$

45) Explain why $|3t - 7| < 0$ has no solution.

46) Explain why $|4l + 9| \leq -10$ has no solution.

47) Explain why the solution to $|2x + 1| \geq -3$ is $(-\infty, \infty)$.

48) Explain why the solution to $|7y - 3| \geq 0$ is $(-\infty, \infty)$.

Mixed Exercises: Objectives 1–3

The following exercises contain absolute value equations, linear inequalities, and both types of absolute value inequalities. Solve each. Write the solution set for equations in set notation and use interval notation for inequalities.

49) $|2v + 9| > 3$

50) $\left|\frac{5}{3}a + 2\right| = 8$

51) $3 = |4t + 5|$

52) $|4k + 9| \leq 5$

53) $9 \leq |7 - 8q|$

54) $|2p - 5| - 12 = 11$

55) $2(x - 8) + 10 < 4x$

56) $\frac{1}{2}n + 11 < 8$

57) $|8 - r| \leq 5$

58) $|d + 6| > 7$

59) $|6y + 5| \leq -9$

60) $8 \leq |5v + 2|$

61) $\left|\frac{4}{3}x + 1\right| = \left|\frac{5}{3}x + 8\right|$

62) $|7z - 8| \leq 0$

63) $|3m - 8| - 11 > -3$

64) $|6c - 1| = -14$

65) $|4 - 9t| + 2 = 1$

66) $|5b - 11| - 18 < -10$

67) $-\frac{3}{5} \geq \frac{5}{2}a - \frac{1}{2}$

68) $4 + 3(2r - 5) > 9 - 4r$

69) $|6k + 17| > -4$

70) $|5 - w| \geq 3$

71) $5 \geq |c + 8| - 2$

72) $0 \leq |4a + 1|$

73) $|5h - 8| > 7$

74) $\left|\frac{2}{3}y - 1\right| = \left|\frac{3}{2}y + 4\right|$

75) $\left|\frac{1}{2}d - 4\right| + 7 = 13$

76) $|9q - 8| < 0$

77) $|5j + 3| + 1 \leq 9$

78) $|7 - x| \leq 12$

Objective 4: Solve an Applied Problem Using an Absolute Value Inequality

79) A gallon of milk should contain 128 oz. The possible error in this measurement, however, is ± 0.75 oz. Let a represent the range of values for the amount of milk in the container. Write an absolute value inequality to represent the range for the number of ounces of milk in the container, then solve the inequality and explain the meaning of the answer.

80) Dawn buys a 27-oz box of cereal. The possible error in this amount, however, is ± 0.5 oz. Let c represent the range of values for the amount of cereal in the box. Write an absolute value inequality to represent the range for the number of ounces of cereal in the box, then solve the inequality and explain the meaning of the answer.

81) Emmanuel spent \$38 on a birthday gift for his son. He plans on spending within \$5 of that amount on his daughter's birthday gift. Let b represent the range of values for the amount he will spend on his daughter's gift. Write an absolute value inequality to represent the range for the amount of money Emmanuel will spend on his daughter's birthday gift, then solve the inequality and explain the meaning of the answer.

82) An employee at a home-improvement store is cutting a window shade for a customer. The customer wants the shade to be 32 in. wide. If the machine's possible error in cutting the shade is $\pm\frac{1}{16}$ in., write an absolute value inequality to represent the range for the width of the window shade, and solve the inequality. Explain the meaning of the answer. Let w represent the range of values for the width of the shade.

Section 9.3 Solving Linear and Compound Linear Inequalities in Two Variables

Objectives

1. Define a Linear Inequality in Two Variables
2. Graph a Linear Inequality in Two Variables
3. Graph a Compound Linear Inequality in Two Variables
4. Solve a Linear Programming Problem

In Chapter 3, we learned how to solve linear inequalities in *one variable* such as $2x - 3 \geq 5$. We will begin this section by learning how to graph the solution set of linear inequalities in *two variables*. Then we will learn how to graph the solution set of *systems* of linear inequalities in two variables.

1. Define a Linear Inequality in Two Variables

Definition

A **linear equality in two variables** is an inequality that can be written in the form $Ax + By \geq C$ or $Ax + By \leq C$ where A, B, and C are real numbers and where A and B are not both zero. ($>$ and $<$ may be substituted for $\geq$ and $\leq$.)

Here are some examples of linear inequalities in two variables.

$$5x - 3y \geq 6, \qquad y < \frac{1}{4}x + 3, \qquad x \leq 2, \qquad y > -4$$

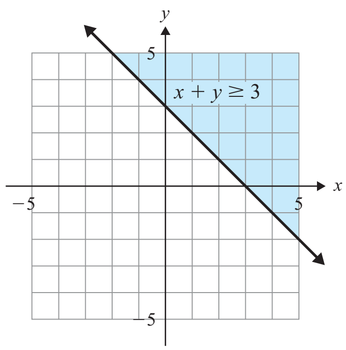
Note

We can call $x \leq 2$ a linear inequality in two variables because we can write it as $x + 0y \leq 2$. Likewise, we can write $y > -4$ as $0x + y > -4$.

The solutions to linear inequalities in two variables, such as $x + y \geq 3$, are *ordered pairs* of the form (x, y) that make the inequality true. We graph a linear inequality in two variables on a rectangular coordinate system.

Example 1

Shown here is the graph of $x + y \geq 3$. Find three points that solve $x + y \geq 3$, and find three points that are not in the solution set.



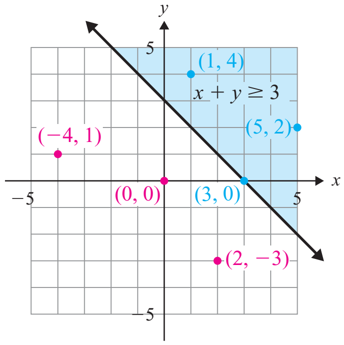
Solution

The solution set of $x + y \geq 3$ consists of all points either on the line or in the shaded region. *Any* point on the line or in the shaded region will make $x + y \geq 3$ true.

Are These Points Solutions?	Check by Substituting into $x + y \geq 3$
$(5, 2)$	$5 + 2 \geq 3$ True
$(1, 4)$	$1 + 4 \geq 3$ True
$(3, 0)$ (on the line)	$3 + 0 \geq 3$ True
$(0, 0)$	$0 + 0 \geq 3$ False
$(-4, 1)$	$-4 + 1 \geq 3$ False
$(2, -3)$	$2 + (-3) \geq 3$ False

The points $(5, 2)$, $(1, 4)$, and $(3, 0)$ are some of the points that satisfy $x + y \geq 3$. There are infinitely many solutions. The points $(0, 0)$, $(-4, 1)$, and $(2, -3)$ are three of the points that do not satisfy $x + y \geq 3$. There are infinitely many points that are not solutions.

Points in the shaded region and on the line are in the solution set.



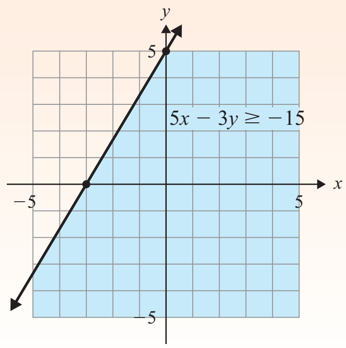
The points in the unshaded region are *not* in the solution set.

Note

If the inequality in Example 1 had been $x + y > 3$, then the line would have been drawn as a *dotted line* and all points on the line would *not* be part of the solution set.

You Try 1

Shown here is the graph of $5x - 3y \geq -15$. Find three points that solve $5x - 3y \geq -15$, and find three points that are not in the solution set.



2. Graph a Linear Inequality in Two Variables

As you saw in the graph in Example 1, the line divides the *plane* into two regions or **half planes.** The line $x + y = 3$ is the **boundary line** between the two half planes. We will use this boundary line to graph a linear inequality in two variables. Notice that the boundary line is written as an equation: it uses an equal sign.

Procedure Graphing a Linear Inequality in Two Variables Using the Test Point Method

1) **Graph the boundary line.** If the inequality contains $\geq$ or $\leq$, make the boundary line *solid*. If the inequality contains $>$ or $<$, make it *dotted*.
2) **Choose a test point not on the line, and shade the appropriate region.** Substitute the test point into the inequality. (If $(0, 0)$ is not on the line, it is an easy point to test in the inequality.)
    a) If it *makes the inequality true,* shade the region *containing* the test point. All points in the shaded region are part of the solution set.
    b) If the test point *does not satisfy the inequality,* shade the region on the *other* side of the line. All points in the shaded region are part of the solution set.

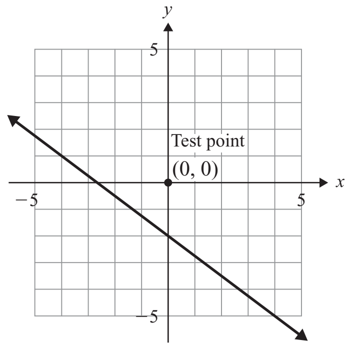
Example 2

Graph $3x + 4y \leq -8$.

Solution

1) Graph the boundary line $3x + 4y = -8$ as a solid line.

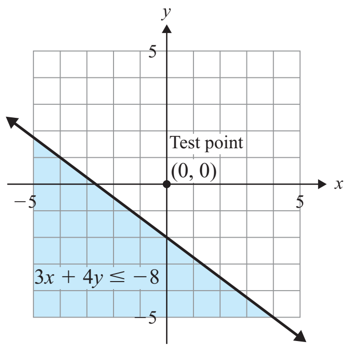
2) Choose a test point not on the line and substitute it into the inequality to determine whether it makes the inequality true.



Test Point	Substitute into $3x + 4y \leq -8$
(0, 0)	$3(0) + 4(0) \leq -8$ $0 \leq -8$ False

Since the test point (0, 0) does *not* satisfy the inequality, we will shade the region that does *not* contain the point (0, 0).

All points on the line and in the shaded region satisfy the inequality $3x + 4y \leq -8$.



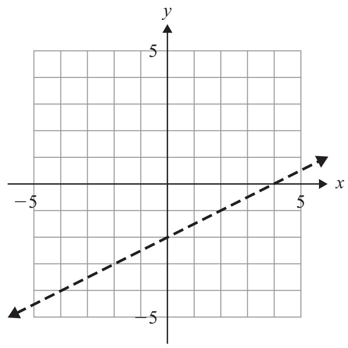
Example 3

Graph $-x + 2y > -4$.

Solution

1) Since the inequality symbol is $>$, graph a *dotted* boundary line, $-x + 2y = -4$. (This means that the points *on* the line are not part of the solution set.)

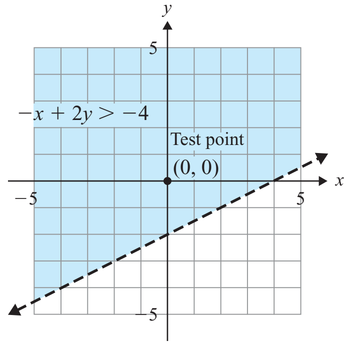
2) Choose a test point not on the line and substitute it into the inequality to determine whether it makes the inequality true.



Test Point	Substitute into $-x + 2y > -4$
(0, 0)	$-(0) + 2(0) > -4$ $0 > -4$ True

Since the test point (0, 0) satisfies the inequality, shade the region containing that point.

All points in the shaded region satisfy the inequality $-x + 2y > -4$.



You Try 2

Graph each inequality.

a) $2x + y \le 4$ b) $x + 4y > 12$

If we write the inequality in *slope-intercept form,* we can decide which region to shade without using test points:

Procedure Using the Slope-Intercept Method to Graph a Linear Inequality in Two Variables

1) If the inequality is in the form $y \ge mx + b$ or $y > mx + b$, shade *above* the line.
2) If the inequality is in the form $y \le mx + b$ or $y < mx + b$, shade *below* the line.

Example 4

Graph each inequality using the slope-intercept method.

a) $y < -\frac{1}{3}x + 5$ b) $2x - y \le -2$

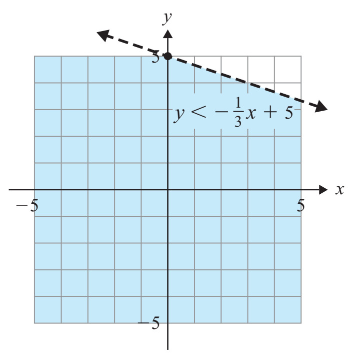
Solution

a) The inequality $y < -\frac{1}{3}x + 5$ is already in slope-intercept form.

Graph the boundary line $y = -\frac{1}{3}x + 5$ as a *dotted line*.

Since $y < -\frac{1}{3}x + 5$ has a *less than* symbol, shade *below* the line.

All points in the shaded region satisfy $y < -\frac{1}{3}x + 5$. We can choose a point such as (0, 0) in the shaded region as a check. Substituting this point into $y < -\frac{1}{3}x + 5$ gives us $0 < -\frac{1}{3} \cdot 0 + 5$, or $0 < 5$, which is true.



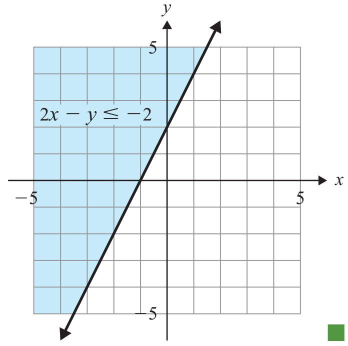
b) Solve $2x - y \leq -2$ for y.

$$\begin{aligned} 2x - y &\leq -2 \\ -y &\leq -2x - 2 && \text{Subtract } 2x. \\ y &\geq 2x + 2 && \text{Divide by } -1 \text{ and change the direction of the inequality symbol.} \end{aligned}$$

Graph $y = 2x + 2$ as a *solid line*.

Since $y \geq 2x + 2$ has a *greater than or equal to* symbol, shade *above* the line.

All points on the line and in the shaded region satisfy $2x - y \leq -2$.



You Try 3

Graph each inequality using the slope-intercept method.

a) $y \geq -\frac{3}{4}x - 6$ b) $5x - 2y > 4$

3. Graph a Compound Linear Inequality in Two Variables

Linear inequalities in two variables are called **compound linear inequalities** if they are connected by the word *and* or *or*.

The solution set of a compound inequality containing *and* is the *intersection* of the solution sets of the inequalities.

The solution set of a compound inequality containing *or* is the *union* of the solution sets of the inequalities.

Procedure Graphing Compound Linear Inequalities in Two Variables

1) Graph each inequality separately on the same axes. Shade lightly.
2) If the inequality contains *and,* the solution set is the *intersection* of the shaded regions. Heavily shade this region.
3) If the inequality contains *or,* the solution set is the *union* (total) of the shaded regions. Heavily shade this region.

Example 5

Graph $x \leq 2$ and $2x + 3y > 3$.

Solution

To graph $x \leq 2$, graph the boundary line $x = 2$ as a solid line. The x-values are *less than* 2 to the *left* of 2, so shade the region to the left of the line $x = 2$. See Figure 1.

Graph $2x + 3y > 3$. Use a dotted boundary line. See Figure 2.

The region shaded blue in Figure 3 is the *intersection* of the first two shaded regions and is the solution set of the compound inequality. The part of the line $x = 2$ that is above the line $2x + 3y = 3$ is included in the solution set.

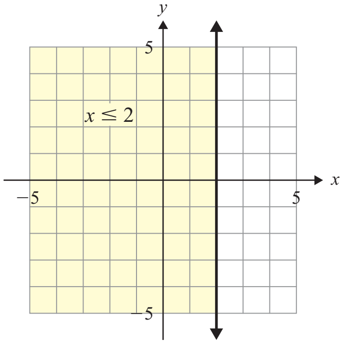


Figure 1

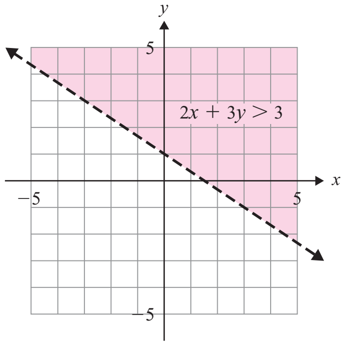


Figure 2

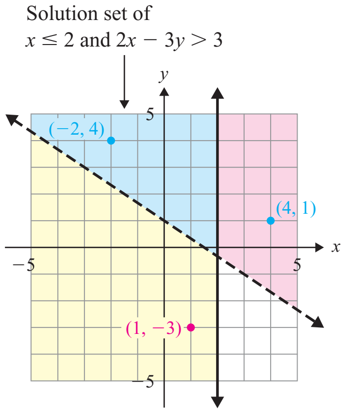


Figure 3

Any point in the solution set must satisfy *both* inequalities, and any point *not* in the solution set will not satisfy *both* inequalities. We check three test points next. (See the graph.)

Test point	**Substitute into $x \le 2$**	**Substitute into $2x + 3y > 3$**	**Solution?**
$(-2, 4)$	$-2 \le 2$ True	$2(-2) + 3(4) > 3$ $8 > 3$ True	Yes
$(4, 1)$	$4 \le 2$ False	$2(4) + 3(1) > 3$ $11 > 3$ True	No
$(1, -3)$	$1 \le 2$ True	$2(1) + 3(-3) > 3$ $-7 > 3$ False	No

Although we show three separate graphs in Example 5, it is customary to graph everything on the same axes, shading lightly at first, then to heavily shade the region that is the graph of the compound inequality.

You Try 4

Graph the compound inequality $y \le 3x - 1$ and $y + 2x \le 4$.

Example 6

Graph $y \le \frac{1}{2}x$ or $2x + y \ge 2$.

Solution

Graph each inequality separately. See Figures 4 and 5.

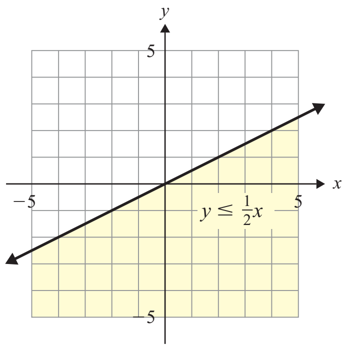


Figure 4

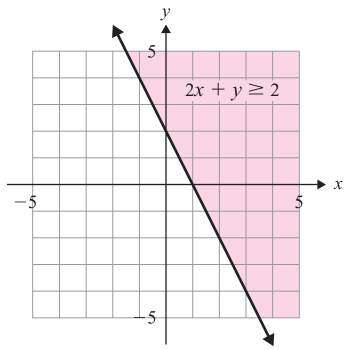


Figure 5

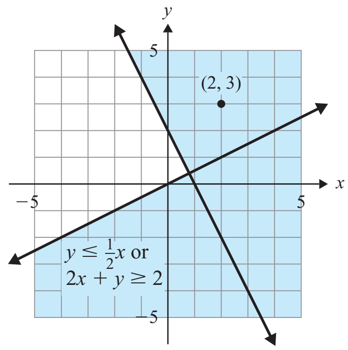


Figure 6

The solution set of the compound inequality will be the *union* (total) of the shaded regions. Any point in the shaded region of Figure 6 will be a solution to the compound inequality $y \le \frac{1}{2}x$ or $2x + y \ge 2$. This means the point must satisfy $y \le \frac{1}{2}x$ *or* $2x + y \ge 2$ *or* both. One point in the shaded region is (2, 3).

Test Point	**Substitute into $y \le \frac{1}{2}x$**	**Substitute into $2x + y \ge 2$**	**Solution?**
(2, 3)	$3 \le \frac{1}{2}(2)$ $3 \le 1$ False	$2(2) + 3 \ge 2$ $7 \ge 2$ True	Yes

Although (2, 3) does not satisfy $y \le \frac{1}{2}x$, it *does* satisfy $2x + y \ge 2$, so it *is* a solution of the compound inequality.

Choose a point in the region that is *not* shaded to verify that it does not satisfy either inequality.

You Try 5

Graph the compound inequality $x \ge -4$ or $x - 3y \le -3$.

4. Solve a Linear Programming Problem

A practical application of linear inequalities in two variables is a process called **linear programming.** Companies use linear programming to determine the best way to use their machinery, employees, and other resources.

A linear programming problem may consist of several inequalities called **constraints.** Constraints describe the conditions that the variables must meet. The graph of the *intersection* of these inequalities is called the **feasible region**—the ordered pairs that are the possible solutions to the problem.

Example 7

During a particular week, a company wants Harvey and Amy to work at most 40 hours between them.

Let x = the number of hours Harvey works
y = the number of hours Amy works

a) Write the linear inequalities that describe the constraints on the number of hours available to work.

b) Graph the feasible region (solution set of the intersection of the inequalities), which describes the possible number of hours each person can work.

c) Find a point in the feasible region and discuss its meaning.

d) Find a point outside the feasible region and discuss its meaning.

Solution

a) Since x and y represent the number of hours worked, x and y cannot be negative. We can write $x \geq 0$ and $y \geq 0$.

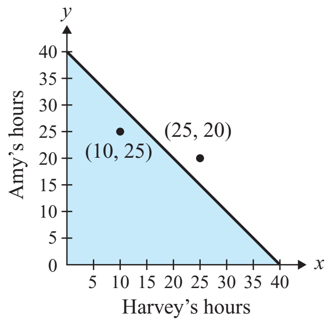
Together they can work at most 40 hours. We can write $x + y \leq 40$.

The inequalities that describe the constraints on the number of hours available are $x \geq 0$ *and* $y \geq 0$ *and* $x + y \leq 40$. We want to find the *intersection* of these inequalities.

b) The graphs of $x \geq 0$ *and* $y \geq 0$ give us the set of points in the first quadrant since x and y are both positive here.

Graph $x + y \leq 40$. This will be the region *below and including* the line $x + y = 40$ in quadrant I.

The feasible region is shown here.



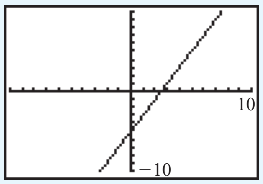
c) One point in the feasible region is (10, 25). It represents Harvey working 10 hours and Amy working 25 hours. It satisfies all three inequalities.

Test Point	Substitute into $x \geq 0$	Substitute into $y \geq 0$	Substitute into $x + y \leq 40$
(10, 25)	$10 \geq 0$ True	$25 \geq 0$ True	$10 + 25 \leq 40$ $35 \leq 40$ True

d) One point outside the feasible region is (25, 20). It represents Harvey working 25 hours and Amy working 20 hours. This is not possible since it does not satisfy the inequality $x + y \leq 40$.

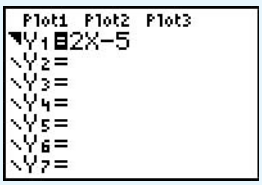
Test Point	Substitute into $x \geq 0$	Substitute into $y \geq 0$	Substitute into $x + y \leq 40$
(25, 20)	$25 \geq 0$ True	$20 \geq 0$ True	$25 + 20 \leq 40$ $45 \leq 40$ False

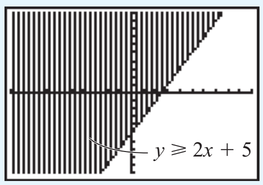
Using Technology



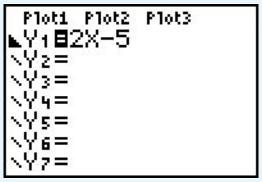
To graph a linear inequality in two variables using a graphing calculator, first solve the inequality for y. Then graph the boundary line found by replacing the inequality symbol with an = symbol. For example, to graph the inequality $2x - y \leq 5$, solve it for y giving $y \geq 2x - 5$. Graph the boundary equation $y = 2x - 5$ using a solid line since the inequality symbol is $\leq$. Press $\boxed{Y =}$, then enter $2x - 5$ in Y_1, press $\boxed{\text{ZOOM}}$, and select 6:ZStandard to graph the equation as shown.

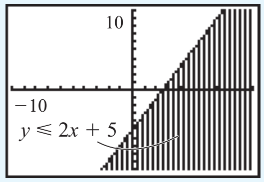
If the inequality symbol is $\leq$, shade below the boundary line. If the inequality symbol is $\geq$, shade above the boundary line. To shade above the line, press $\boxed{Y =}$ and move the cursor to the left of Y_1 using the left arrow key. Press $\boxed{\text{ENTER}}$ twice and then move the cursor to the next line as shown below left. Press $\boxed{\text{GRAPH}}$ to graph the inequality as shown below right.





To shade below the line, press $\boxed{Y=}$ and move the cursor to the left of Y_1 using the left arrow key. Press $\boxed{\text{ENTER}}$ one time and then move the cursor to the next line as shown below left. Press $\boxed{\text{GRAPH}}$ to graph the inequality $y \le 2x - 5$ as shown below right.





Graph the linear inequalities in two variables.

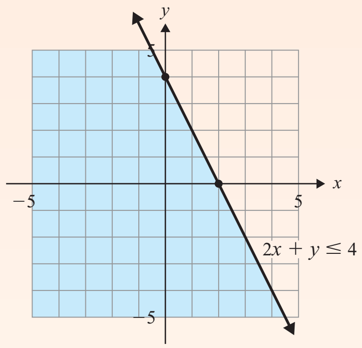
1) $y \le 5x - 2$ 2) $y \ge x - 4$ 3) $x - 2y \le 6$

4) $y - x \ge 5$ 5) $y \le -4x + 1$ 6) $y \ge 3x - 6$

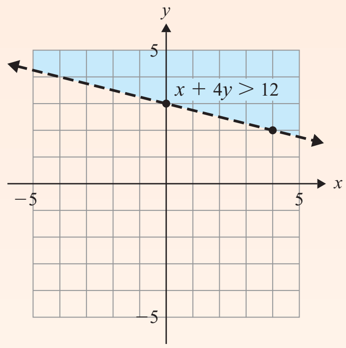
Answers to You Try Exercises

1) Answers may vary.

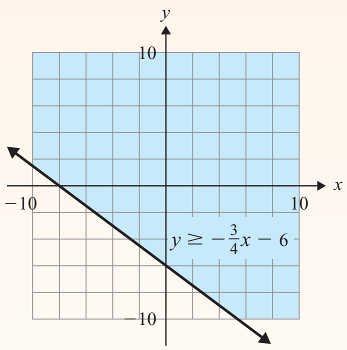
2) a)



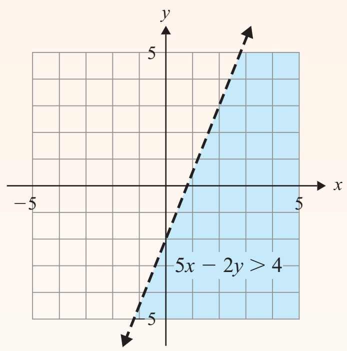
b)



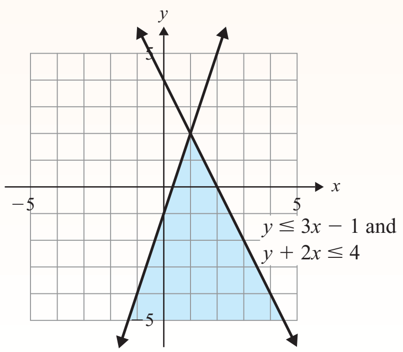
3) a)



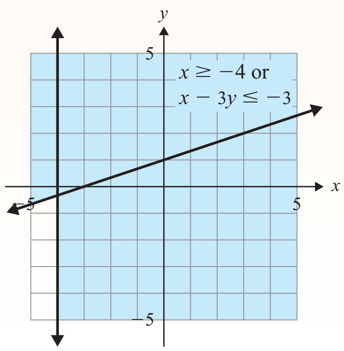
b)



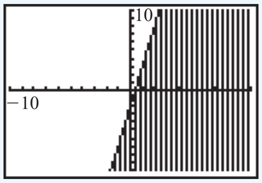
4)

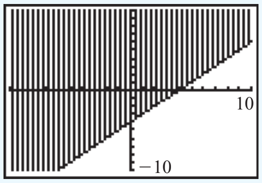


5)



Answers to Technology Exercises

1)


2)


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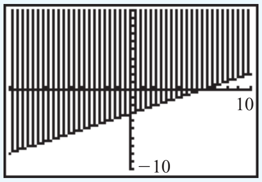

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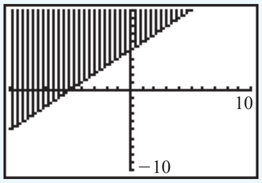

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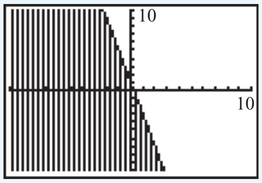

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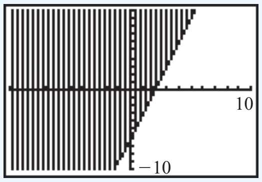

9.3 Exercises

Objective 1: Define a Linear Inequality in Two Variables

The graphs of linear inequalities are given next. For each, find three points that satisfy the inequality and three that are not in the solution set.

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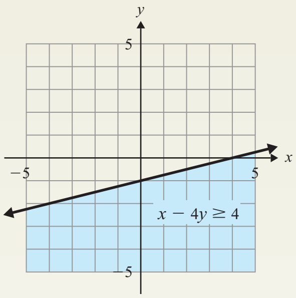

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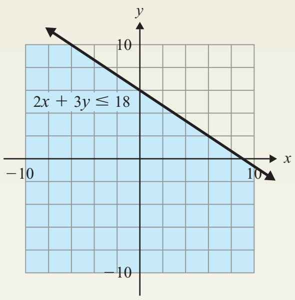

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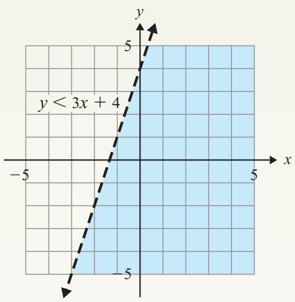

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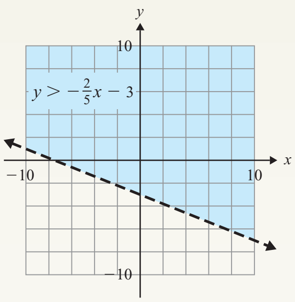

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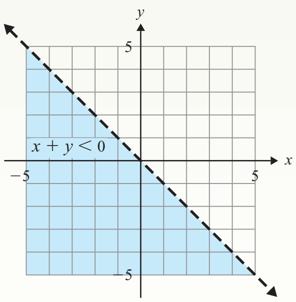

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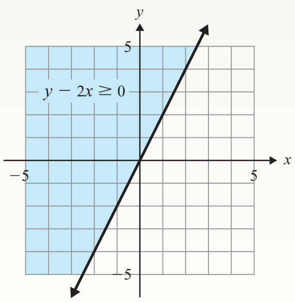


Objective 2: Graph a Linear Inequality in Two Variables

7) Will the boundary line you draw to graph $3x - 4y < 5$ be solid or dotted?

8) Are points on solid boundary lines included in the inequality's solution set?

Graph the inequalities. Use a test point.

9) $2x + y \geq 6$

10) $4x + y \leq 3$

11) $y < x + 2$

12) $y > \frac{1}{2}x - 1$

13) $2x - 7y \leq 14$

14) $4x + 3y < 15$

VIDEO 15) $y < x$

16) $y \geq 3x$

17) $y \geq -5$

18) $x < 1$

19) Should you shade the region above or below the boundary line for the inequality $y \leq 7x + 2$?

20) Should you shade the region above or below the boundary line for the inequality $y > 2x + 4$?

Use the slope-intercept method to graph each inequality.

21) $y \leq 4x - 3$

22) $y \geq \frac{5}{2}x - 8$

23) $y > \frac{2}{5}x - 4$

24) $y < \frac{1}{4}x + 1$

25) $6x + y > 3$

26) $2x + y > -5$

VIDEO 27) $9x - 3y \leq -21$

28) $3x + 5y < -20$

29) $x > 2y$

30) $x - y \leq 0$

31) To graph an inequality like $y \geq \frac{1}{3}x + 2$, would you rather use a test point or use the slope-intercept method? Why?

32) To graph an inequality like $7x + 2y < 10$, would you rather use a test point or the slope-intercept method? Why?

Graph using either a test point or the slope-intercept method.

33) $y > -\frac{3}{4}x + 1$

34) $y \leq \frac{1}{3}x - 6$

35) $5x + 2y < -8$

36) $4x + y < 7$

37) $9x - 3y \leq 21$

38) $5x - 3y \geq -9$

VIDEO 39) $x > 2$

40) $y \leq 4$

41) $3x - 4y > 12$

42) $6x - y \leq 2$

Objective 3: Graph a Compound Linear Inequality in Two Variables

43) Is (3, 5) in the solution set of the compound inequality $x - y \geq -6$ and $2x + y < 7$? Why or why not?

44) Is (3, 5) in the solution set of the compound inequality $x - y \geq -6$ or $2x + y < 7$? Why or why not?

Graph each compound inequality.

VIDEO 45) $x \leq 4$ and $y \geq -\frac{3}{2}x + 3$

46) $y \leq \frac{1}{4}x + 2$ and $y \geq -1$

47) $y < x + 4$ and $y \geq -3$

48) $x < 3$ and $y > \frac{2}{3}x - 1$

49) $2x - 3y < -9$ and $x + 6y < 12$

50) $5x - 3y > 9$ and $2x + 3y \leq 12$

51) $y \leq -x - 1$ or $x \geq 6$

52) $y \leq 2$ or $y \leq \frac{4}{5}x + 2$

53) $y \leq 4$ or $4y - 3x \geq -8$

54) $x + 3y \geq 3$ or $x \geq -2$

55) $y > -\frac{2}{3}x + 1$ or $-2x + 5y \leq 0$

56) $y > x - 4$ or $3x + 2y \geq 12$

57) $x \geq 5$ and $y \leq -3$

58) $x \leq 6$ and $y \geq 1$

59) $y < 4$ or $x \geq -3$

60) $x \geq 2$ or $y \geq -6$

VIDEO 61) $2x + 5y < 15$ or $y \leq \frac{3}{4}x - 1$

62) $y - 2x \leq 1$ and $y \geq -\frac{1}{5}x - 2$

63) $y \geq \frac{2}{3}x - 4$ and $4x + y \leq 3$

64) $y < 5x + 2$ or $x + 4y < 12$

Objective 4: Solve a Linear Programming Problem

65) During the school year, Tazia earns money by babysitting and tutoring. She can work at most 15 hr per week.

Let x = number of hours Tazia babysits
y = number of hours Tazia tutors

a) Write the linear inequalities that describe the constraints on the number of hours Tazia can work per week.

b) Graph the feasible region that describes how her hours can be distributed between babysitting and tutoring.

c) Find three points in the feasible region and discuss their meanings.

d) Find one point outside the feasible region and discuss its meaning.

66) A machine in a factory can be calibrated to fill either large or small bags of potato chips. The machine will run at most 12 hr per day.

Let x = number of hours the machine fills large bags
y = number of hours the machine fills small bags

a) Write the linear inequalities that describe the constraints on the number of hours the machine fills the bags each day.

$$x \geq 0 \text{ and } y \geq 0 \text{ and } x + y \leq 12$$

b) Graph the feasible region that describes how the hours can be distributed between filling the large and small bags of chips.

c) Find three points in the feasible region and discuss their meanings.

d) Find one point outside the feasible region and discuss its meaning.

67) A lawn mower company produces a push mower and a riding mower. Company analysts predict that, for next spring, the company will need to produce at least 150 push mowers and 100 riding mowers per day, but they can produce at most 250 push mowers and 200 riding mowers per day. To satisfy demand, they will have to ship a total of at least 300 mowers per day.

Let p = number of push mowers produced per day
r = number of riding mowers produced per day

a) Write the linear inequalities that describe the constraints on the number of mowers that can be produced per day.

b) Graph the feasible region that describes how production can be distributed between the riding mowers and the push mowers. Let p be the horizontal axis and r be the vertical axis.

c) What does the point (175, 110) represent? Will this level of production meet the needs of the company?

d) Find three points in the feasible region and discuss their meanings.

e) Find one point outside the feasible region and discuss its meaning.

68) A dog food company produces adult dog food and puppy food. The company estimates that for next month, it will need to produce at least 12,000 pounds of adult dog food and 8000 pounds of puppy food per day. The factory can produce at most 18,000 pounds of adult dog food and 14,000 pounds of puppy food per day. The company will need to ship a total of at least 25,000 pounds of dog food per day to its customers.

Let a = pounds of adult dog food produced per day
p = pounds of puppy food produced per day

a) Write the linear inequalities that describe the constraints on the number of pounds of dog food that can be produced per day.

b) Graph the feasible region that describes how production can be distributed between the adult dog food and the puppy food. Let a be the horizontal axis and p be the vertical axis.

c) What does the point (17,000, 9000) represent? Will this level of production meet the needs of the company?

d) Find three points in the feasible region and discuss their meanings.

e) Find one point outside the feasible region and discuss its meaning.

Section 9.4 Solving Systems of Linear Equations Using Matrices

Objectives

1. Learn the Vocabulary Associated with Gaussian Elimination
2. Solve a System Using Gaussian Elimination

We have learned how to solve systems of linear equations by graphing, substitution, and the elimination method. In this section, we will learn how to use *row operations* and *Gaussian elimination* to solve systems of linear equations. We begin by defining some terms.

1. Learn the Vocabulary Associated with Gaussian Elimination

A **matrix** is a rectangular array of numbers. (The plural of *matrix* is *matrices*.) Each number in the matrix is an **element** of the matrix. An example of a matrix is

$$\begin{array}{rc} & \begin{array}{ccc}\text{Column 1} & \text{Column 2} & \text{Column 3}\\ \downarrow & \downarrow & \downarrow\end{array}\\ \begin{array}{r}\text{Row 1}\rightarrow\\ \text{Row 2}\rightarrow\end{array} & \begin{bmatrix}3 & -1 & 4\\ 0 & 2 & -5\end{bmatrix}\end{array}$$

We can represent a system of equations in an *augmented matrix*. An **augmented matrix** has a vertical line to distinguish between different parts of the equation. For example, we can represent the system below with the augmented matrix shown here:

$$\begin{array}{rl} 5x + 4y = 1 & \text{Equation (1)}\\ x - 3y = 6 & \text{Equation (2)}\end{array} \qquad \left[\begin{array}{rr|r}5 & 4 & 1\\ 1 & -3 & 6\end{array}\right] \begin{array}{l}\text{Row 1}\\ \text{Row 2}\end{array}$$

Notice that the vertical line separates the system's coefficients from its constants on the other side of the = sign. The system needs to be in standard form, so the first column in the matrix represents the x-coefficients. The second column represents the y-coefficients, and the column on the right represents the constants.

Gaussian elimination is the process of using row operations on an augmented matrix to solve the corresponding system of linear equations. It is a variation of the elimination method and can be very efficient. Computers often use augmented matrices and row operations to solve systems.

The goal of Gaussian elimination is to obtain a matrix of the form $\left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right]$ or $\left[\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}\right]$ when solving a system of two or three equations, respectively. Notice the 1's along the **diagonal** of the matrix and zeros below the diagonal. We say a matrix is in **row echelon form** when it has 1's along the diagonal and 0's below the diagonal. We get matrices in row echelon form by performing row operations. When we rewrite a matrix that is in row echelon form back into a system, its solution is easy to find.

2. Solve a System Using Gaussian Elimination

The row operations we can perform on augmented matrices are similar to the operations we use to solve a system of equations using the elimination method.

Definition Matrix Row Operations

Performing the following row operations on a matrix produces an equivalent matrix.

1) Interchanging two rows
2) Multiplying every element in a row by a nonzero real number
3) Replacing a row by the sum of it and the multiple of another row

Let's use these operations to solve a system using Gaussian elimination. Notice the similarities between this method and the elimination method.

Example 1

Solve using Gaussian elimination.
$$\begin{aligned} x + 5y &= -1 \\ 2x - y &= 9 \end{aligned}$$

Solution

Begin by writing the system as an augmented matrix. $\left[\begin{array}{cc|c} 1 & 5 & -1 \\ 2 & -1 & 9 \end{array}\right]$

We will use the 1 in Row 1 to make the element below it a zero. If we multiply the 1 by -2 (to get -2) and add it to the 2, we get zero. We must do this operation to the entire row. Denote this as $-2R_1 + R_2 \to R_2$. (Read as, "-2 times Row 1 plus Row 2 makes the new Row 2.") We get a new Row 2.

Use this

to make this 0.

$$\left[\begin{array}{cc|c} 1 & 5 & -1 \\ \boxed{2} & -1 & 9 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 5 & -1 \\ -2(1) + 2 & -2(5) + (-1) & -2(-1) + 9 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & 5 & -1 \\ 0 & -11 & 11 \end{array}\right]$$

Multiply each element of Row 1 by -2 and add it to the corresponding element of Row 2.

Note

We are not *making* a new Row 1, so it stays the same.

We have obtained the first 1 on the diagonal with a 0 below it. Next we need a 1 on the diagonal in Row 2.

This column is in the correct form.

$$\left[\begin{array}{cc|c} 1 & 5 & -1 \\ 0 & \boxed{-11} & 11 \end{array}\right] \xrightarrow{-\frac{1}{11}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 5 & -1 \\ 0 & 1 & -1 \end{array}\right]$$

Make this 1.

Multiply each element of Row 2 by $-\dfrac{1}{11}$ to get a 1 on the diagonal.

We have obtained the final matrix because there are 1's on the diagonal and a 0 below. The matrix is in row echelon form. From this matrix, write a system of equations. The last row gives us the value of y.

$$\left[\begin{array}{cc|c} 1 & 5 & -1 \\ 0 & 1 & -1 \end{array}\right] \quad \begin{aligned} 1x + 5y &= -1 \\ 0x + 1y &= -1 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 5y &= -1 \\ y &= -1 \end{aligned} \quad \begin{aligned} &\text{Equation (1)} \\ &\text{Equation (2)} \end{aligned}$$

$$\begin{aligned} x + 5(-1) &= -1 && \text{Substitute } -1 \text{ for } y \text{ in equation (1).} \\ x - 5 &= -1 && \text{Multiply.} \\ x &= 4 && \text{Add 5.} \end{aligned}$$

The solution is $(4, -1)$. Check by substituting $(4, -1)$ into both equations of the original system.

Here are the steps for using Gaussian elimination to solve a system of any number of equations. Our goal is to obtain a matrix with 1's along the diagonal and 0's below—row echelon form.

Procedure How to Solve a System of Equations Using Gaussian Elimination

Step 1: Write the system as an *augmented matrix.*

Step 2: Use row operations to make the *first entry in column 1* be a 1.

Step 3: Use row operations to make *all entries below the 1 in column 1* be 0's.

Step 4: Use row operations to make the *second entry in column 2* be a 1.

Step 5: Use row operations to make *all entries below the 1 in column 2* be 0's.

Step 6: Continue this procedure until the matrix is in *row echelon form*—1's along the diagonal and 0's below.

Step 7: Write the matrix in step 6 as a *system of equations.*

Step 8: *Solve the system* from step 7. The last equation in the system will give you the value of one of the variables; find the values of the other variables by using substitution.

Step 9: *Check the solution* in each equation of the original system.

You Try 1

Solve the system using Gaussian elimination.

$$\begin{aligned} x - y &= -1 \\ -3x + 5y &= 9 \end{aligned}$$

Next we will solve a system of three equations using Gaussian elimination.

Example 2

Solve using Gaussian elimination.

$$\begin{aligned} 2x + y - z &= -3 \\ x + 2y - 3z &= 1 \\ -x - y + 2z &= 2 \end{aligned}$$

Solution

Step 1: Write the system as an *augmented matrix.*

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & -3 \\ 1 & 2 & -3 & 1 \\ -1 & -1 & 2 & 2 \end{array}\right]$$

Step 2: To make the *first entry in column 1* be a 1, we *could* multiply Row 1 by $\frac{1}{2}$, but this would make the rest of the entries in the first row fractions. Instead, recall that we can interchange two rows. If we interchange Row 1 and Row 2, the first entry in column 1 will be 1.

$R_1 \leftrightarrow R_2$
Interchange Row 1 and Row 2.

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 2 & 1 & -1 & -3 \\ -1 & -1 & 2 & 2 \end{array}\right]$$

Step 3: We want to make *all the entries below the 1 in column 1* be 0's. To obtain a 0 in place of the 2 in column 1, multiply the 1 by -2 (to get -2) and add it to the 2. Perform that same operation on the entire row to obtain the new Row 2.

Use this→ to make→ this zero.

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ \boxed{2} & 1 & -1 & -3 \\ -1 & -1 & 2 & 2 \end{array}\right] \xrightarrow[\text{Row 2 = new Row 2}]{\substack{-2R_1 + R_2 \to R_2 \\ -2 \text{ times Row 1 +}}} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & -3 & 5 & -5 \\ -1 & -1 & 2 & 2 \end{array}\right]$$

To obtain a 0 in place of the -1 in column 1, add the 1 and the -1. Perform that same operation on the entire row to obtain a new Row 3.

Use this→ to make this zero.→

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & -3 & 5 & -5 \\ \boxed{-1} & -1 & 2 & 2 \end{array}\right] \xrightarrow[\text{Row 1 + Row 3 = new Row 3}]{R_1 + R_3 \to R_3} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & -3 & 5 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

Step 4: Next, we want the *second entry in column 2* to be a 1. We *could* multiply Row 2 by $-\frac{1}{3}$ to get the 1, but the other entries would be fractions. Instead, interchanging Row 2 and Row 3 will give us a 1 on the diagonal and keep 0's in column 1. (Sometimes, though, fractions are unavoidable.)

$R_2 \leftrightarrow R_3$
Interchange Rows 2 and 3.

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & -3 & 5 & -5 \end{array}\right]$$

Step 5: We want to make *all the entries below the 1 in column 2* be 0's. To obtain a 0 in place of -3 in column 2, multiply the 1 above it by 3 (to get 3) and add it to -3. Perform that same operation on the entire row to obtain a new Row 3.

Use this to make this zero.

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & \boxed{-3} & 5 & -5 \end{array}\right] \xrightarrow[\text{3 times Row 2 + Row 3 = new Row 3}]{3R_2 + R_3 \to R_3} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

We have completed step 5 because there is only one entry below the 1 in column 2.

Step 6: *Continue this procedure*. The last entry in column 3 needs to be a 1. (This is the last 1 we need along the diagonal.) Multiply Row 3 by $\frac{1}{2}$ to obtain the last 1.

$\frac{1}{2}R_3 \to R_3$
Multiply Row 3 by $\frac{1}{2}$.

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

We are done performing row operations because there are 1's on the diagonal and zeros below.

Step 7: Write the matrix in step 6 as a *system of equations*.

$$\begin{aligned} 1x + 2y - 3z &= 1 \\ 0x + 1y - 1z &= 3 \\ 0x + 0y + 1z &= 2 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y - 3z &= 1 \\ y - z &= 3 \\ z &= 2 \end{aligned}$$

Step 8: *Solve the system* in step 7. The last row tells us that $z = 2$. Substitute $z = 2$ into the equation above it ($y - z = 3$) to get the value of y: $y - 2 = 3$, so $y = 5$.

Substitute $y = 5$ and $z = 2$ into $x + 2y - 3z = 1$ to solve for x.

$$\begin{aligned} x + 2y - 3z &= 1 \\ x + 2(5) - 3(2) &= 1 && \text{Substitute values.} \\ x + 10 - 6 &= 1 && \text{Multiply.} \\ x + 4 &= 1 && \text{Subtract.} \\ x &= -3 \end{aligned}$$

The solution of the system is $(-3, 5, 2)$.

Step 9: *Check the solution* in each equation of the original system. The check is left to the student.

This procedure may seem long and complicated at first, but as you practice and become more comfortable with the steps, you will see that it is actually quite efficient.

You Try 2

Solve the system using Gaussian elimination.

$$\begin{aligned} x + 3y - 2z &= 10 \\ 3x + 2y + z &= 9 \\ -x + 4y - z &= -1 \end{aligned}$$

If we are performing Gaussian elimination and obtain a matrix that produces a false equation as shown, then the system has *no solution*. The system is *inconsistent*.

$$\left[\begin{array}{cc|c} 1 & -6 & 9 \\ 0 & 0 & 8 \end{array}\right] \quad 0x + 0y = 8 \quad \text{False}$$

If, however, we obtain a matrix that produces a row of zeros as shown, then the system has an *infinite number of solutions*. The system is *consistent* with *dependent* equations. We write its solution as we did in previous sections.

$$\left[\begin{array}{cc|c} 1 & 5 & -1 \\ 0 & 0 & 0 \end{array}\right] \quad 0x + 0y = 0 \quad \text{True}$$

Using Technology

In this section, we have learned how to solve a system of three equations using Gaussian elimination. The row operations used to convert an augmented matrix to row echelon form can be performed on a graphing calculator.

Follow the nine-step method given in the text to solve the system using Gaussian elimination:

$$\begin{aligned} x + 2y - 3z &= 1 \\ y - z &= 3 \\ -2y + 4z &= -4 \end{aligned}$$

Step 1: Write the system as an augmented matrix:

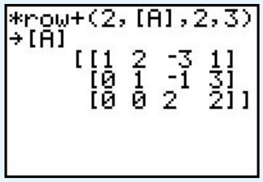
$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & -2 & 4 & -4 \end{array}\right]$$

Store the matrix in matrix [A] using a graphing calculator. Press 2nd x^{-1} to select [A]. Press the right arrow key two times and press ENTER to select EDIT. Press 3 ENTER then 4 ENTER to enter the number of rows and number of columns in the augmented matrix. Enter the coefficients one row at a time as follows: 1 ENTER 2 ENTER (–) 3 ENTER 1 ENTER 0 ENTER 1 ENTER (–) 1 ENTER 3 ENTER 0 ENTER (–) 2 ENTER 4 ENTER (–) 4 ENTER.

Press 2nd MODE to return to the home screen. Press 2nd x^{-1} ENTER ENTER to display matrix [A].

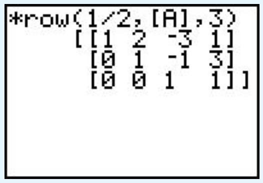
```
[A]
   [[1  2   -3  1 ]
    [0  1   -1  3 ]
    [0  -2  4   -4]]
```

Notice that we can omit steps 2–4 because we already have two 1's on the diagonal and 0's below the first 1.



Step 5: Get the element in row 3, column 2 to be 0. Multiply row 2 by the opposite of the number in row 3, column 2 and add to row 3. The graphing calculator row operation used to multiply a row by a nonzero number and add to another row is ***row+(nonzero number, matrix name, first row, second row).**

In this case, we have *row+(2, [A], 2, 3). To enter this row operation on your calculator, press 2nd x^{-1}, then press the right arrow to access the MATH menu. Scroll down to option F and press ENTER to display *row+(then enter 2 , 2nd x^{-1} ENTER , 2 , 3) as shown. Store the result back in matrix [A] by pressing STO> 2nd x^{-1} ENTER ENTER.



Step 6: To make the last number on the diagonal be 1, multiply row 3 by $\frac{1}{2}$. The graphing calculator row operation used to multiply a row by a nonzero number is ***row(nonzero number, matrix name, row).** In this case, we have *row(1/2, [A], 3). On your calculator, press 2nd x^{-1}, then press the right arrow to access the MATH menu. Scroll down to option E and press ENTER to display *row(then enter 1 ÷ 2 , 2nd x^{-1} ENTER , 3) as shown.

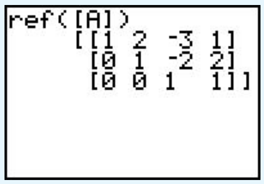
Step 7: Write the matrix from step 6 as:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \begin{aligned} 1x + 2y - 3z &= 1 \\ 0x + 1y - 1z &= 3 \\ 0x + 0y + 1z &= 1 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y - 3z &= 1 \\ y - z &= 3 \\ z &= 1 \end{aligned}$$

Step 8: Solve the system using substitution to obtain the solution $x = -4, y = 4, z = 1$ or $(-4, 4, 1)$.

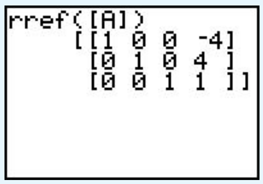
Step 9: Check the solution.

Using Row Echelon Form



The row echelon form shown above is not unique. Another row echelon form can be obtained *in one step* using a graphing calculator. Given the original augmented matrix stored in [A], press 2nd x^{-1}, then press the right arrow, scroll down to option A, and press ENTER to display ref(which stands for row echelon form, and press ENTER. Press 2nd x^{-1} ENTER) ENTER to show the matrix in row echelon form.

Using Reduced Row Echelon Form



The **reduced row echelon form** of an augmented matrix contains 1's on the diagonal and 0's *above* and *below* the 1's. We can find this using row operations as shown in the 9-step process, or directly in one step. Given the original augmented matrix stored in [A], press 2nd x^{-1}, then press the right arrow, scroll down to option B, and press ENTER to display ref(which stands for reduced row-echelon form, and press ENTER. Press 2nd x^{-1} ENTER) ENTER to show the matrix in reduced row echelon form.

Write a system of equations from the matrix that is in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \begin{aligned} 1x + 0y + 0z &= -4 \\ 0x + 1y + 0z &= 4 \\ 0x + 0y + 1z &= 1 \end{aligned} \quad \text{or} \quad \begin{aligned} x &= -4 \\ y &= 4 \\ z &= 1 \end{aligned}$$

Use a graphing calculator to solve each system using Gaussian elimination.

1) $\begin{aligned} x + 2y &= 1 \\ 3x - y &= 17 \end{aligned}$

2) $\begin{aligned} x - 5y &= -3 \\ 2x - 7y &= 3 \end{aligned}$

3) $\begin{aligned} -5x + 2y &= -4 \\ 3x - y &= 8 \end{aligned}$

4) $\begin{aligned} 3x - 5y - 3z &= 6 \\ -x + 3y + 2z &= 1 \\ -2x + 7y + 5z &= 6 \end{aligned}$

5) $\begin{aligned} 3x + 2y + z &= 9 \\ -5x - 2y - z &= -7 \\ 4x + y + z &= 3 \end{aligned}$

6) $\begin{aligned} 2x - y + 2z &= -4 \\ -x + y - 2z &= 7 \\ -3x + y - z &= -1 \end{aligned}$

Answers to You Try Exercises

1) (2, 3) 2) (4, 0, −3)

Answers to Technology Exercises

1) (5, −2) 2) (12, 3) 3) (12, 28) 4) (4, −3, 7) 5) (−1, 5, 2) 6) (3, 6, −2)

9.4 Exercises

Objective 1: Learn the Vocabulary Associated with Gaussian Elimination

Write each system in an augmented matrix.

1) $\begin{aligned} x - 7y &= 15 \\ 4x + 3y &= -1 \end{aligned}$

2) $\begin{aligned} x + 6y &= 4 \\ -5x + y &= -3 \end{aligned}$

3) $\begin{aligned} x + 6y - z &= -2 \\ 3x + y + 4z &= 7 \\ -x - 2y + 3z &= 8 \end{aligned}$

4) $\begin{aligned} x + 2y - 7z &= 3 \\ 3x - 5y &= -1 \\ -x + 2z &= -4 \end{aligned}$

Write a system of linear equations in x and y represented by each augmented matrix.

5) $\left[\begin{array}{cc|c} 3 & 10 & -4 \\ 1 & -2 & 5 \end{array}\right]$

6) $\left[\begin{array}{cc|c} 1 & -1 & 6 \\ -4 & 7 & 2 \end{array}\right]$

7) $\left[\begin{array}{cc|c} 1 & -6 & 8 \\ 0 & 1 & -2 \end{array}\right]$

8) $\left[\begin{array}{cc|c} 1 & 2 & 11 \\ 0 & 1 & 3 \end{array}\right]$

Write a system of linear equations in x, y, and z represented by each augmented matrix.

9) $\left[\begin{array}{ccc|c} 1 & -3 & 2 & 7 \\ 4 & -1 & 3 & 0 \\ -2 & 2 & -3 & -9 \end{array}\right]$

10) $\left[\begin{array}{ccc|c} 1 & 4 & -3 & -5 \\ -1 & 2 & 5 & 8 \\ 6 & -2 & -1 & 3 \end{array}\right]$

11) $\left[\begin{array}{ccc|c} 1 & 5 & 2 & 14 \\ 0 & 1 & -8 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$

12) $\left[\begin{array}{ccc|c} 1 & 4 & -7 & -11 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 6 \end{array}\right]$

Objective 2: Solve a System Using Gaussian Elimination

Solve each system using Gaussian elimination. Identify any inconsistent systems or dependent equations.

13) $\begin{aligned} x + 4y &= -1 \\ 3x + 5y &= 4 \end{aligned}$

14) $\begin{aligned} x - 3y &= 1 \\ -3x + 7y &= 3 \end{aligned}$

15) $\begin{aligned} x - 3y &= 9 \\ -6x + 5y &= 11 \end{aligned}$

16) $\begin{aligned} x + 4y &= -6 \\ 2x + 5y &= 0 \end{aligned}$

17) $\begin{aligned} 4x - 3y &= 6 \\ x + y &= -2 \end{aligned}$

18) $\begin{aligned} -4x + 5y &= -3 \\ x - 8y &= -6 \end{aligned}$

19) $\begin{aligned} x + y - z &= -5 \\ 4x + 5y - 2z &= 0 \\ 8x - 3y + 2z &= -4 \end{aligned}$

20) $\begin{aligned} x - 2y + 2z &= 3 \\ 2x - 3y + z &= 13 \\ -4x - 5y - 6z &= 8 \end{aligned}$

21) $\begin{aligned} x - 3y + 2z &= -1 \\ 3x - 8y + 4z &= 6 \\ -2x - 3y - 6z &= 1 \end{aligned}$

22) $\begin{aligned} x - 2y + z &= -2 \\ 2x - 3y + z &= 3 \\ 3x - 6y + 2z &= 1 \end{aligned}$

23) $\begin{aligned} -4x - 3y + z &= 5 \\ x + y - z &= -7 \\ 6x + 4y + z &= 12 \end{aligned}$

24) $\begin{aligned} 6x - 9y - 2z &= 7 \\ -3x + 4y + z &= -4 \\ x - y - z &= 1 \end{aligned}$

25) $\begin{aligned} x - 3y + z &= -4 \\ 4x + 5y - z &= 0 \\ 2x - 6y + 2z &= 1 \end{aligned}$

26) $\begin{aligned} x - y + 3z &= 1 \\ 5x - 5y + 15z &= 5 \\ -4x + 4y - 12z &= -4 \end{aligned}$

Extension

Extend the concepts of this section to solve these systems using Gaussian elimination.

27) $\begin{aligned} a + b + 3c + d &= -1 \\ -a + c - d &= 7 \\ 2a + 3b + 9c - 2d &= 7 \\ a - 2b + c + 3d &= -11 \end{aligned}$

28) $\begin{aligned} a - 2b - c + 3d &= 15 \\ 2a - 3b + c + 4d &= 22 \\ -a + 4b + 6c + 7d &= -3 \\ 3a + 2b - c - d &= -7 \end{aligned}$

29) $\begin{aligned} w - 3x + 2y - z &= -2 \\ -3w + 8x - 5y + z &= 2 \\ 2w - x + y + 3z &= 7 \\ w - 2x + y + 2z &= 3 \end{aligned}$

30) $\begin{aligned} w + x - 4y + 2z &= -21 \\ 3w + 2x + y - z &= 6 \\ -2w - x - 2y + 6z &= -30 \\ -w + 3x + 4y + z &= 1 \end{aligned}$

Chapter 9: Summary

Definition/Procedure	Example

9.1 Solving Absolute Value Equations

If P represents an expression and k is a positive, real number, then to solve $|P| = k$ we rewrite the absolute value equation as the *compound equation* $P = k$ or $P = -k$ and solve for the variable. **(p. 528)**

Solve $|4a + 10| = 18$.

$$|4a + 10| = 18$$

$$\begin{aligned} 4a + 10 &= 18 \quad \text{or} \quad & 4a + 10 &= -18 \\ 4a &= 8 & 4a &= -28 \\ a &= 2 \quad \text{or} & a &= -7 \end{aligned}$$

Check the solutions in the original equation. The solution set is $\{-7, 2\}$.

9.2 Solving Absolute Value Inequalities

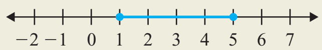
Inequalities Containing $<$ or $\le$

Let P be an expression and let k be a positive, real number. To solve $|P| \le k$, solve the three-part inequality $-k \le P \le k$.

($<$ may be substituted for $\le$.) **(p. 534)**

Solve $|x - 3| \le 2$. Graph the solution set and write the answer in interval notation.

$$\begin{aligned} -2 \le x - 3 &\le 2 \\ 1 \le x &\le 5 \end{aligned}$$



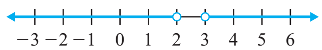
In interval notation, we write $[1, 5]$.

Inequalities Containing $>$ or $\ge$

Let P be an expression and let k be a positive, real number. To solve $|P| \ge k$ ($>$ may be substituted for $\ge$), solve the compound inequality $p \ge k$ or $p \le -k$. **(p. 536)**

Solve $|2n - 5| > 1$. Graph the solution set and write the answer in interval notation.

$$\begin{aligned} 2n - 5 > 1 \quad &\text{or} \quad 2n - 5 < -1 && \text{Solve.} \\ 2n > 6 \quad &\text{or} \quad 2n < 4 && \text{Add 5.} \\ n > 3 \quad &\text{or} \quad n < 2 && \text{Divide by 2.} \end{aligned}$$



In interval notation, we write $(-\infty, 2) \cup (3, \infty)$.

9.3 Solving Linear and Compound Linear Inequalities in Two Variables

A **linear inequality in two variables** is an inequality that can be written in the form $Ax + By \ge C$ or $Ax + By \le C$, where A, B, and C are real numbers and where A and B are not both zero. ($>$ and $<$ may be substituted for $\ge$ and $\le$.) **(p. 541)**

Some examples of linear inequalities in two variables are

$$x + 3y \le 2, \quad y > -\frac{2}{3}x + 5, \quad y \ge -1, \quad x < 4$$

Definition/Procedure	Example

Graphing a Linear Inequality in Two Variables
Using the Test Point Method

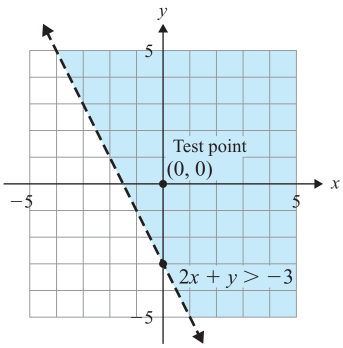
1) *Graph the boundary line.*
   a) If the inequality contains $\geq$ or $\leq$, make the boundary line solid.
   b) If the inequality contains $>$ or $<$, make the boundary line dotted.
2) *Choose a test point not on the line, and shade the appropriate region.* Substitute the test point into the inequality. (If (0, 0) is not on the line, it is an easy point to test in the inequality.) **(p. 542)**
   a) If it *makes the inequality true*, shade the region *containing* the test point. All points in the shaded region are part of the solution set.
   b) If the test point *does not satisfy the inequality*, shade the region on the *other* side of the line. All points in the shaded region are part of the solution set.

Graph $2x + y > -3$.

1) Graph the boundary line as a *dotted* line.
2) Choose a test point not on the line and substitute it into the inequality to determine whether it makes the inequality true.

Test Point	Substitute into $2x + y > -3$
(0, 0)	$2(0) + (0) > -3$ $0 > -3$ True

Since the test point satisfies the inequality, shade the region containing (0, 0).



All points in the shaded region satisfy $2x + y > -3$.

Slope-Intercept method
If the inequality is written in *slope-intercept form*, we can decide which region to shade without using test points:

1) If the inequality is in the form $y \geq mx + b$ or $y > mx + b$, shade *above* the line.
2) If the inequality is in the form $y \leq mx + b$ or $y < mx + b$, shade *below* the line. **(p. 544)**

Graph using the slope-intercept method.

$$-x + 3y \leq 6$$

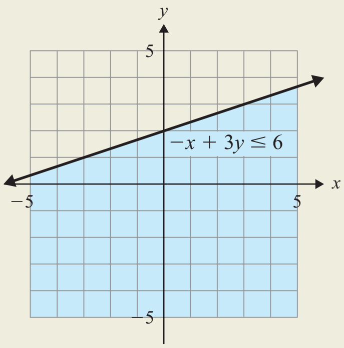
Write the inequality in slope-intercept form by solving

$-x + 3y \leq 6$ for y:

$$-x + 3y \leq 6$$
$$3y \leq x + 6$$
$$y \leq \frac{1}{3}x + 2$$

Graph $y = \frac{1}{3}x + 2$ as a *solid line*.

Since $y \leq \frac{1}{3}x + 2$ has a $\leq$ symbol, shade *below* the line.



All points on the line and in the shaded region satisfy $-x + 3y \leq 6$.

Definition/Procedure	Example

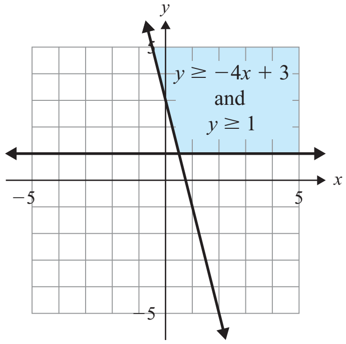
Graphing Compound Linear Inequalities in Two Variables

1) Graph each inequality separately on the same axes. Shade lightly.
2) If the inequality contains *and*, the solution set is the *intersection* of the shaded regions. (Heavily shade this region.)
3) If the inequality contains *or*, the solution set is the *union* (total) of the shaded regions. Heavily shade this region. **(p. 545)**

Graph the compound inequality $y \geq -4x + 3$ and $y \geq 1$.

Since the inequality contains *and*, the solution set is the *intersection* of the shaded regions.

Any point in the shaded area will satisfy *both* inequalities.



9.4 Solving Systems of Linear Equations Using Matrices

An **augmented matrix** contains a vertical line to separate different parts of the matrix. **(p. 552)**

An example of an augmented matrix is $\left[\begin{array}{cc|c} 1 & 4 & -9 \\ 2 & -3 & 8 \end{array}\right]$.

Matrix Row Operations
Performing the following row operations on a matrix produces an equivalent matrix.

1) Interchanging two rows
2) Multiplying every element in a row by a nonzero real number
3) Replacing a row by the sum of it and the multiple of another row **(p. 553)**

Gaussian elimination is the process of performing row operations on a matrix to put it into *row echelon form*.

A matrix is in **row echelon form** when it has 1's along the diagonal and 0's below. **(p. 553)**

$$\left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}\right]$$

Solve using Gaussian elimination.

$$\begin{aligned} x - y &= 5 \\ 2x + 7y &= 1 \end{aligned}$$

Write the system in an augmented matrix. Then, perform row operations to get it into row echelon form.

$$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 2 & 7 & 1 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 9 & -9 \end{array}\right]$$

$$\xrightarrow{\frac{1}{9}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & -1 \end{array}\right]$$

The matrix is in row echelon form since it has 1's on the diagonal and a 0 below.

Now, write a system of equations from the matrix that is in row echelon form.

$$\left[\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & -1 \end{array}\right] \quad \begin{aligned} 1x - 1y &= 5 \\ 0x + 1y &= -1 \end{aligned} \quad \text{or} \quad \begin{aligned} x - y &= 5 \\ y &= -1 \end{aligned}$$

Solving the system, we obtain the solution $(4, -1)$.

Chapter 9: Review Exercises

(9.1) Solve.

1) $|m| = 9$

2) $\left|\frac{1}{2}c\right| = 5$

3) $|7t + 3| = 4$

4) $|4 - 3y| = 12$

5) $|8p + 11| - 7 = -3$

6) $|5k + 3| - 8 = 4$

7) $\left|4 - \frac{5}{3}x\right| = \frac{1}{3}$

8) $\left|\frac{2}{3}w + 6\right| = \frac{5}{2}$

9) $|7r - 6| = |8r + 2|$

10) $|3z - 4| = |5z - 6|$

11) $|2a - 5| = -10$

12) $|5.2h + 6| + 8.3 = 2.3$

13) $|9d + 4| = 0$

14) $|6q - 7| = 0$

15) Write an absolute value equation that means *a is 4 units from zero*.

16) Write an absolute value equation that means *t is 7 units from zero*.

(9.2) Solve each inequality. Graph the solution set and write the answer in interval notation.

17) $|c| \le 3$

18) $|w + 1| < 11$

19) $|4t| > 8$

20) $|2v - 7| \ge 15$

21) $|12r + 5| \ge 7$

22) $|3k - 11| < 4$

23) $|4 - a| < 9$

24) $|2 - 5q| > 6$

25) $|4c + 9| - 8 \le -2$

26) $|3m + 5| + 2 \ge 7$

27) $|5y + 12| - 15 \ge -8$

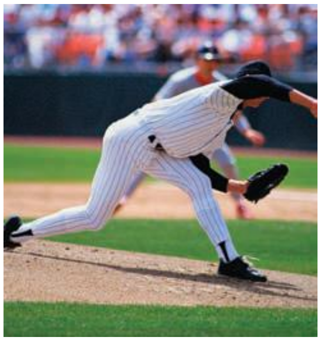
28) $3 + |z - 6| \le 13$

29) $|k + 5| > -3$

30) $|4q - 9| < 0$

31) $|12s + 1| \le 0$

32) A radar gun indicated that a pitcher threw a 93-mph fastball. The radar gun's possible error in measuring the speed of a pitch is ± 1 mph. Write an absolute value inequality to represent the range for the speed of the pitch, and solve the inequality. Explain the meaning of the answer. Let s represent the range of values for the speed of the pitch.



(9.3) Graph each linear inequality in two variables.

33) $y \le -2x + 7$

34) $y \ge -\frac{3}{2}x + 2$

35) $y > -\frac{1}{3}x - 4$

36) $y < \frac{3}{4}x - 5$

37) $-3x + 4y > 12$

38) $5x - 2y \ge 8$

39) $4x - y > -5$

40) $y < x$

41) $x \ge 4$

42) $y \le 3$

Graph each compound inequality.

43) $y \ge \frac{3}{4}x - 4$ and $y \le -5$

44) $y \le -\frac{1}{3}x - 2$ and $x \le 4$

45) $y \le -\frac{1}{2}x + 7$ and $x \le 1$

46) $y \ge -\frac{2}{3}x - 4$ or $x < 1$

47) $y < \frac{5}{4}x - 5$ or $y < -3$

48) $4x - y < -1$ or $y > \frac{1}{2}x + 5$

49) $2x + y \le 3$ or $6x + y > 4$

50) $2x + 5y \le 10$ and $y \ge \frac{1}{3}x + 4$

51) $4x + 2y \ge -6$ and $y \le 2$

52) $3x - 4y < 20$ or $y < -2$

(9.4) Solve each system using Gaussian elimination.

53) $\begin{aligned} x - y &= -11 \\ 2x + 9y &= 0 \end{aligned}$

54) $\begin{aligned} x - 8y &= -13 \\ 4x + 9y &= -11 \end{aligned}$

55) $\begin{aligned} 5x + 3y &= 5 \\ -x + 8y &= -1 \end{aligned}$

56) $\begin{aligned} 3x + 5y &= 5 \\ -4x - 9y &= 5 \end{aligned}$

57) $\begin{aligned} x - 3y - 3z &= -7 \\ 2x - 5y - 3z &= 2 \\ -3x + 5y + 4z &= -1 \end{aligned}$

58) $\begin{aligned} x - 3y + 5z &= 3 \\ 2x - 5y + 6z &= -3 \\ 3x + 2y + 2z &= 3 \end{aligned}$

Chapter 9: Test

Solve.

1) $|4y - 9| = 11$

2) $7 = \left|6 - \frac{3}{8}d\right| - 5$

3) $|3k + 5| = |k - 11|$

4) $\left|\frac{1}{2}n - 1\right| = -8$

5) Write an absolute value equation that means *x is 8 units from zero.*

6) Explain why the solution to $|0.8a + 1.3| \geq 0$ is $(-\infty, \infty)$.

Solve each inequality. Graph the solution set and write the answer in interval notation.

7) $|c| > 4$

8) $|2z - 7| \leq 9$

9) $|4m + 9| - 8 \geq 5$

10) $\left|\frac{2}{3}w - 4\right| + 10 < 4$

11) A scale in a doctor's office has a possible error of ± 0.75 lb. If Thanh's weight is measured as 168 lb, write an absolute value inequality to represent the range for his weight, and solve the inequality. Let w represent the range of values for Thanh's weight. Explain the meaning of the answer.

Graph each inequality.

12) $y \geq 3x + 1$

13) $2x - 5y > 10$

Graph each compound inequality.

14) $-2x + 3y \geq -12$ and $x \leq 3$

15) $y < -x$ or $2x - y > 1$

Solve using Gaussian elimination.

16) $$\begin{aligned} x + 5y &= -4 \\ 3x + 2y &= 14 \end{aligned}$$

17) $$\begin{aligned} -3x + 5y + 8z &= 0 \\ x - 3y + 4z &= 8 \\ 2x - 4y - 3z &= 3 \end{aligned}$$

Cumulative Review: Chapters 1–9

Perform the operations and simplify.

1) $5 \times 6 - 36 \div 3^2$

2) $\frac{5}{12} - \frac{7}{8}$

Evaluate.

3) 3^4

4) 2^5

5) $\left(\frac{1}{8}\right)^2$

6) 4^{-3}

7) Write 0.00000914 in scientific notation.

8) Solve $8 - 3(2y - 5) = 4y + 1$.

9) Solve $3 - \frac{2}{7}n \geq 9$. Write the answer in interval notation.

10) Write an equation and solve.

How many ounces of a 9% alcohol solution must be added to 8 oz of a 3% alcohol solution to obtain a 5% alcohol solution?

11) Write the slope-intercept form of the line containing (7, 2) with slope $\frac{1}{3}$.

12) Solve by graphing.

$$2x + y = -1$$
$$y = 3x - 6$$

Multiply and simplify.

13) $-4p^2(3p^2 - 7p - 1)$

14) $(2k + 5)(2k - 5)$

15) $(t + 8)^2$

16) Divide $\frac{6c^3 + 7c^2 - 38c + 24}{3c - 4}$.

Factor completely.

17) $9m^2 - 121$

18) $z^2 - 14z + 48$

Solve.

19) $a^2 + 6a + 9 = 0$

20) $2(x^2 - 4) = -(7x + 4)$

21) Subtract $\frac{1}{r^2 - 25} - \frac{r + 3}{2r + 10}$.

22) Multiply and simplify $\frac{w^2 - 3w - 54}{w^3 - 8w^2} \cdot \frac{w}{w + 6}$.

23) Solve $\left|\frac{1}{4}q - 7\right| - 8 = -5$.

24) Solve. Graph the solution set and write the answer in interval notation.

$$|9v + 4| > 14$$

25) Graph the compound inequality

$$3x + 4y > 16 \text{ or } y < \frac{1}{5}x + 1.$$

CHAPTER 10

Radicals and Rational Exponents

Algebra at Work: Forensics

Forensic scientists use mathematics in many ways to help them analyze evidence and solve crimes. To help him reconstruct an accident scene, Keith can use this formula containing a radical to estimate the minimum speed of a vehicle when the accident occurred:

$$S = \sqrt{30fd}$$

where f = the drag factor, based on the type of road surface

d = the length of the skid, in feet

S = the speed of the vehicle, in miles per hour

Keith is investigating an accident in a residential neighborhood where the speed limit is 25 mph. The car involved in the accident left skid marks 60 ft long. Tests showed that the drag factor of the asphalt road was 0.80. Was the driver speeding at the time of the accident?

Substitute the values into the equation and evaluate it to determine the minimum speed of the vehicle at the time of the accident:

$$S = \sqrt{30fd}$$
$$S = \sqrt{30(0.80)(60)}$$
$$S = \sqrt{1440} \approx 38 \text{ mph}$$

The driver was going at least 38 mph when the accident occurred. This is well over the speed limit of 25 mph.

We will learn how to simplify radicals in this chapter as well as how to work with equations like the one given here.

Section 10.1 Finding Roots

Objectives

1. Find Square Roots and Principal Square Roots
2. Approximate the Square Root of a Whole Number
3. Find Higher Roots
4. Evaluate $\sqrt[n]{a^n}$

Recall that exponential notation represents repeated multiplication. For example,

$$3^2 \text{ means } 3 \cdot 3, \text{ so } 3^2 = 9.$$
$$2^4 \text{ means } 2 \cdot 2 \cdot 2 \cdot 2, \text{ so } 2^4 = 16.$$

In this chapter, we will study the opposite, or inverse, procedure, finding **roots** of numbers.

1. Find Square Roots and Principal Square Roots

Example 1

Find all square roots of 25.

Solution

To find a *square* root of 25, ask yourself, "What number do I *square* to get 25?" Or, "What number *multiplied by itself* equals 25?" One number is 5 since $5^2 = 25$. Another number is -5 since $(-5)^2 = 25$. So, 5 and -5 are square roots of 25.

You Try 1

Find all square roots of 64.

The $\sqrt{}$ symbol represents the *positive* square root, or the **principal square root,** of a nonnegative number. For example, $\sqrt{25} = 5$.

BE CAREFUL

Notice that $\sqrt{25} = 5$, but $\sqrt{25} \neq -5$. The $\sqrt{}$ symbol represents *only* the principal square root (positive square root).

To find the **negative square root** of a nonnegative number, we must put a $-$ in front of the $\sqrt{}$. For example,

$$-\sqrt{25} = -5$$

Next we will define some terms associated with the $\sqrt{}$ symbol.

The symbol $\sqrt{}$ is the **square root symbol** or the **radical sign**. The number under the radical sign is the **radicand**. The entire expression, $\sqrt{25}$, is called a **radical**.

$$\text{Radical sign} \rightarrow \underbrace{\sqrt{25}}_{\substack{\uparrow \\ \text{Radical}}} \leftarrow \text{Radicand}$$

Example 2

Find each square root, if possible.

a) $\sqrt{100}$ b) $-\sqrt{16}$ c) $\sqrt{\dfrac{4}{25}}$ d) $-\sqrt{\dfrac{81}{49}}$ e) $\sqrt{-9}$

Solution

a) $\sqrt{100} = 10$ since $(10)^2 = 100$.

b) $-\sqrt{16}$ means $-1 \cdot \sqrt{16}$. Therefore, $-\sqrt{16} = -1 \cdot \sqrt{16} = -1 \cdot 4 = -4$.

c) $\sqrt{\dfrac{4}{25}} = \dfrac{2}{5}$ since $\left(\dfrac{2}{5}\right)^2 = \dfrac{4}{25}$.

d) $-\sqrt{\dfrac{81}{49}}$ means $-1 \cdot \sqrt{\dfrac{81}{49}}$. So, $-\sqrt{\dfrac{81}{49}} = -1 \cdot \sqrt{\dfrac{81}{49}} = -1 \cdot \left(\dfrac{9}{7}\right) = -\dfrac{9}{7}$.

e) To find $\sqrt{-9}$, ask yourself, "What number do I *square* to get -9?" *There is no such real number* since $3^2 = 9$ and $(-3)^2 = 9$. Therefore, $\sqrt{-9}$ is not a real number.

You Try 2

Find each square root.

a) $\sqrt{9}$ b) $-\sqrt{144}$ c) $\sqrt{\dfrac{25}{36}}$ d) $-\sqrt{\dfrac{1}{64}}$ e) $\sqrt{-49}$

In Example 2, we found the **principal square roots** of 100 and $\frac{4}{25}$ and the **negative square roots** of 16 and $\frac{81}{49}$.

Let's review what we know about radicands and add a third fact.

Property Radicands and Square Roots

1) If the radicand is a *perfect square*, the square root is a *rational* number.

Example: $\sqrt{16} = 4$ 16 is a perfect square. $\sqrt{\dfrac{100}{49}} = \dfrac{10}{7}$ $\dfrac{100}{49}$ is a perfect square.

2) If the radicand is a *negative number*, the square root is *not* a real number.

Example: $\sqrt{-25}$ is *not* a real number.

3) If the radicand is *positive and not a perfect square*, then the square root is an *irrational* number.

Example: $\sqrt{13}$ is irrational. 13 is not a perfect square.

The square root of such a number is a real number that is a nonrepeating, nonterminating decimal.

It is important to be able to approximate such square roots because sometimes it is necessary to estimate their places on a number line or on a Cartesian coordinate system when graphing. For the purposes of graphing, approximating a radical to the nearest tenth is sufficient. A calculator with a $\sqrt{}$ key will give a better approximation of the radical.

2. Approximate the Square Root of a Whole Number

Example 3

Approximate $\sqrt{13}$ to the nearest tenth and plot it on a number line.

Solution

What is the largest perfect square that is *less than* 13? **9**

What is the smallest perfect square that is *greater than* 13? **16**

Since 13 is between 9 and 16 $(9 < 13 < 16)$, it is true that $\sqrt{13}$ is between $\sqrt{9}$ and $\sqrt{16}$.

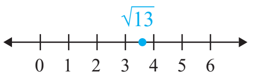
$$\sqrt{9} < \sqrt{13} < \sqrt{16}$$
$$\sqrt{9} = 3$$
$$\sqrt{13} = ?$$
$$\sqrt{16} = 4$$

$\sqrt{13}$ must be between 3 and 4. Numerically, 13 is closer to 16 than it is to 9. So, $\sqrt{13}$ will be closer to $\sqrt{16}$ than to $\sqrt{9}$. Check to see if 3.6 is a good approximation of $\sqrt{13}$. ($\approx$ means approximately equal to.)

$$\text{If } \sqrt{13} \approx 3.6, \text{ then } (3.6)^2 \approx 13.$$
$$(3.6)^2 = (3.6) \cdot (3.6) = 12.96$$

Is 3.7 a better approximation of $\sqrt{13}$? If $\sqrt{13} \approx 3.7$, then $(3.7)^2 \approx 13$.
$$(3.7)^2 = (3.7) \cdot (3.7) = 13.69$$

3.6 is a better approximation of $\sqrt{13}$.



$$\sqrt{13} \approx 3.6$$

A calculator evaluates $\sqrt{13}$ as 3.6055513. Remember that this is only an approximation. ■

You Try 3

Approximate $\sqrt{29}$ to the nearest tenth and plot it on a number line.

3. Find Higher Roots

We saw in Example 2a) that $\sqrt{100} = 10$ since $(10)^2 = 100$. We can also find higher roots of numbers like $\sqrt[3]{a}$ (read as "the cube root of a"), $\sqrt[4]{a}$ (read as "the fourth root of a"), $\sqrt[5]{a}$, etc. We will look at a few roots of numbers before learning important rules.

Example 4

Find each root.

a) $\sqrt[3]{125}$ b) $\sqrt[5]{32}$

Solution

a) To find $\sqrt[3]{125}$ (the cube root of 125), ask yourself, "What number do I *cube* to get 125?" That number is 5.

$$\mathbf{\sqrt[3]{125} = 5} \text{ since } 5^3 = 125$$

Finding the cube root of a number is the *opposite*, or *inverse* procedure, of cubing a number.

b) To find $\sqrt[5]{32}$ (the fifth root of 32), ask yourself, "What number do I raise to the *fifth power* to get 32?" That number is 2.

$$\mathbf{\sqrt[5]{32} = 2} \text{ since } 2^5 = 32$$

Finding the fifth root of a number and raising a number to the fifth power are *opposite*, or *inverse*, procedures. ■

You Try 4

Find each root.

a) $\sqrt[3]{27}$ b) $\sqrt[3]{8}$

The symbol $\sqrt[n]{a}$ is read as "the *n*th root of *a*." If $\sqrt[n]{a} = b$, then $b^n = a$.

$$\text{Index} \rightarrow \underbrace{\sqrt[n]{a}}_{\text{Radical}} \leftarrow \text{Radicand}$$

We call n the **index** of the radical.

Note

When finding square roots, we do not write $\sqrt[2]{a}$. The square root of a is written as $\sqrt{a}$, and the index is understood to be 2.

We know that a positive number, say 36, has a principal square root ($\sqrt{36}$, or 6) and a negative square root ($-\sqrt{36}$, or -6). This is true for all even roots of positive numbers: square roots, fourth roots, sixth roots, etc. For example, 81 has a principal fourth root ($\sqrt[4]{81}$, or 3) and a negative fourth root ($-\sqrt[4]{81}$, or -3).

Definition

***n*th Root**

For any *positive* number a and any *even* index n,
the **principal *n*th root** of a is $\sqrt[n]{a}$.
the **negative *n*th root** of a is $-\sqrt[n]{a}$.

$\sqrt[\text{even}]{\text{positive}} =$ principal (positive) root
$-\sqrt[\text{even}]{\text{positive}} =$ negative root

For any *negative* number a and any *even* index n,
there is **no** real *n*th root of a.

$\sqrt[\text{even}]{\text{negative}} =$ no real root

For any number a and any *odd* index n,
there is **one** real *n*th root of a, $\sqrt[n]{a}$.

$\sqrt[\text{odd}]{\text{any number}} =$ exactly one root

BE CAREFUL

The definition says that $\sqrt[4]{81}$ *cannot* be -3 because $\sqrt[4]{81}$ is *defined* as the principal fourth root of 81, which must be positive.

Section 1.2 contains a table with the powers of numbers that students are expected to know. Knowing these powers is necessary for finding roots, so the student can refer to p. 18 to review this list of powers.

Example 5

Find each root, if possible.

a) $\sqrt[4]{16}$ b) $-\sqrt[4]{16}$ c) $\sqrt[4]{-16}$ d) $\sqrt[3]{64}$ e) $\sqrt[3]{-64}$

Solution

a) To find $\sqrt[4]{16}$, ask yourself, "What *positive* number do I raise to the *fourth power* to get 16?" Since $2^4 = 16$ and 2 is positive, $\sqrt[4]{16} = 2$.

b) In part a) we found that $\sqrt[4]{16} = 2$, so $-\sqrt[4]{16} = -(\sqrt[4]{16}) = -2$.

c) To find $\sqrt[4]{-16}$, ask yourself, "What number do I raise to the *fourth power* to get -16?" There is no such real number since $2^4 = 16$ and $(-2)^4 = 16$. Therefore, $\sqrt[4]{-16}$ has *no real root*. (Recall from the definition that $\sqrt[\text{even}]{\text{negative}}$ has no real root.)

d) To find $\sqrt[3]{64}$, ask yourself, "What number do I *cube* to get 64?" Since $4^3 = 64$ and since $\sqrt[\text{odd}]{\text{any number}}$ gives exactly one root, $\sqrt[3]{64} = 4$.

e) To find $\sqrt[3]{-64}$, ask yourself, "What number do I *cube* to get -64?" Since $(-4)^3 = -64$ and since $\sqrt[\text{odd}]{\text{any number}}$ gives exactly one root, $\sqrt[3]{-64} = -4$. ■

You Try 5

Find each root, if possible.

a) $\sqrt[6]{-64}$ b) $\sqrt[3]{-125}$ c) $-\sqrt[4]{81}$ d) $\sqrt[3]{1}$ e) $\sqrt[4]{81}$

4. Evaluate $\sqrt[n]{a^n}$

Earlier we said that the $\sqrt{}$ symbol represents only the *positive* square root of a number. For example, $\sqrt{9} = 3$. It is also true that $\sqrt{(-3)^2} = \sqrt{9} = 3$.

If a variable is in the radicand and we do not know whether the variable represents a positive number, then we must use the absolute value symbol to evaluate the radical. Then we know that the result will be a positive number. For example, $\sqrt{a^2} = |a|$.

What if the index is greater than 2? Let's look at how to find the following roots:

$$\sqrt[4]{(-2)^4} = \sqrt[4]{16} = 2 \qquad \sqrt[3]{(-4)^3} = \sqrt[3]{-64} = -4$$

When the index on the radical is any positive, even integer and we do not know whether the variable in the radicand represents a positive number, we must use the absolute value symbol to write the root. However, when the index is a positive, odd integer, we do not need to use the absolute value symbol.

Evaluating $\sqrt[n]{a^n}$

If n is a positive, *even* integer, then $\sqrt[n]{a^n} = |a|$.

If n is a positive, *odd* integer, then $\sqrt[n]{a^n} = a$.

Example 6

Simplify.

a) $\sqrt{(-7)^2}$ b) $\sqrt{k^2}$ c) $\sqrt[3]{(-5)^3}$ d) $\sqrt[7]{n^7}$
e) $\sqrt[4]{(y-9)^4}$ f) $\sqrt[5]{(8p+1)^5}$

Solution

a) $\sqrt{(-7)^2} = |-7| = 7$ When the index is even, use the absolute value symbol to be certain that the result is not negative.

b) $\sqrt{k^2} = |k|$ When the index is even, use the absolute value symbol to be certain that the result is not negative.

c) $\sqrt[3]{(-5)^3} = -5$ The index is odd, so the absolute value symbol is not necessary.

d) $\sqrt[7]{n^7} = n$ The index is odd, so the absolute value symbol is not necessary.

e) $\sqrt[4]{(y-9)^4} = |y-9|$ Even index: use the absolute value symbol to be certain that the result is not negative.

f) $\sqrt[5]{(8p+1)^5} = 8p+1$ Odd index: the absolute value symbol is not necessary. ■

You Try 6

Simplify.

a) $\sqrt{(-12)^2}$ b) $\sqrt{w^2}$ c) $\sqrt[3]{(-3)^3}$ d) $\sqrt[5]{r^5}$
e) $\sqrt[6]{(t+4)^6}$ f) $\sqrt[7]{(4h-3)^7}$

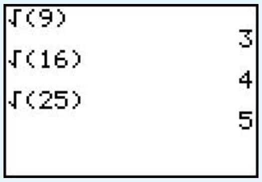
Using Technology

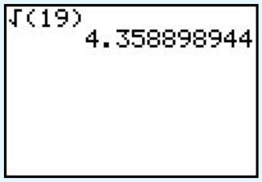
We can evaluate square roots, cube roots, and even higher roots using a graphing calculator. A radical sometimes evaluates to an integer and sometimes must be approximated using a decimal.

To evaluate a square root:

For example, to evaluate $\sqrt{9}$ press $\boxed{2^{nd}}$ $\boxed{x^2}$, enter the radicand $\boxed{9}$, and then press $\boxed{)}$ $\boxed{\text{ENTER}}$. The result is 3, as shown on the screen on the left below. When the radicand is a perfect square such as 9, 16, or 25, then the square root evaluates to a whole number. For example, $\sqrt{16}$ evaluates to 4 and $\sqrt{25}$ evaluates to 5 as shown.

If the radicand of a square root is not a perfect square, then the result is a decimal approximation. For example, to evaluate $\sqrt{19}$ press $\boxed{2^{nd}}$ $\boxed{x^2}$, enter the radicand $\boxed{1}$ $\boxed{9}$, and then press $\boxed{)}$ $\boxed{\text{ENTER}}$. The result is approximately 4.3589, rounded to four decimal places.





To evaluate a cube root:

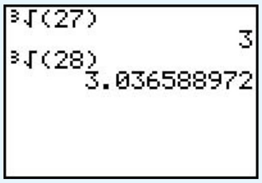
For example, to evaluate $\sqrt[3]{27}$ press $\boxed{\text{MATH}}$ $\boxed{4}$, enter the radicand $\boxed{2}$ $\boxed{7}$, and then press $\boxed{)}$ $\boxed{\text{ENTER}}$. The result is 3 as shown.

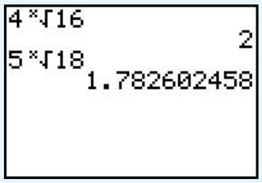
If the radicand is a perfect cube such as 27, then the cube root evaluates to an integer. Since 28 is not a perfect cube, the cube root evaluates to approximately 3.0366.

To evaluate radicals with an index greater than 3:

For example, to evaluate $\sqrt[4]{16}$ enter the index $\boxed{4}$, press $\boxed{\text{MATH}}$ $\boxed{5}$, enter the radicand $\boxed{1}$ $\boxed{6}$, and press $\boxed{\text{ENTER}}$. The result is 2.

Since the fifth root of 18 evaluates to a decimal, the result is an approximation of 1.7826 rounded to four decimal places.



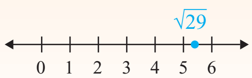


Evaluate each root using a graphing calculator. If necessary, approximate to the nearest tenth.

1) $\sqrt{25}$ 2) $\sqrt[3]{216}$ 3) $\sqrt{29}$ 4) $\sqrt{324}$ 5) $\sqrt[5]{1024}$ 6) $\sqrt[3]{343}$

Answers to You Try Exercises

1) 8 and -8 2) a) 3 b) -12 c) $\frac{5}{6}$ d) $-\frac{1}{8}$ e) not a real number

3) $\sqrt{29} \approx 5.4$ 

4) a) 3 b) 2 5) a) no real root b) -5

c) -3 d) 1 e) 3 6) a) 12 b) $|w|$ c) -3 d) r e) $|t+4|$ f) $4h-3$

Answers to Technology Exercises

1) 5 2) 6 3) 5.4 4) 18 5) 4 6) 7

10.1 Exercises

Objective 1: Find Square Roots and Principal Square Roots

Decide whether each statement is true or false. If it is false, explain why.

1) $\sqrt{121} = 11$ and -11

2) $\sqrt{81} = 9$

3) The square root of a negative number is a negative number.

4) The square root of a nonnegative number is always positive.

Find all square roots of each number.

5) 49

6) 144

7) 1

8) 400

9) 900

10) 2500

11) $\dfrac{4}{9}$

12) $\dfrac{36}{25}$

13) $\dfrac{1}{81}$

14) $\dfrac{1}{16}$

15) 0.25

16) 0.01

Find each square root, if possible.

17) $\sqrt{49}$

18) $\sqrt{144}$

19) $\sqrt{1}$

20) $\sqrt{81}$

21) $\sqrt{169}$

22) $\sqrt{36}$

23) $\sqrt{-4}$

24) $\sqrt{-100}$

25) $\sqrt{\dfrac{81}{25}}$

26) $\sqrt{\dfrac{121}{4}}$

VIDEO 27) $-\sqrt{36}$

28) $-\sqrt{64}$

29) $-\sqrt{\dfrac{1}{121}}$

30) $-\sqrt{\dfrac{1}{100}}$

31) $\sqrt{0.04}$

32) $-\sqrt{0.36}$

Objective 2: Approximate the Square Root of a Whole Number

Approximate each square root to the nearest tenth and plot it on a number line.

VIDEO 33) $\sqrt{11}$

34) $\sqrt{2}$

35) $\sqrt{46}$

36) $\sqrt{22}$

37) $\sqrt{17}$

38) $\sqrt{69}$

39) $\sqrt{5}$

40) $\sqrt{35}$

41) $\sqrt{61}$

42) $\sqrt{8}$

Objective 3: Find Higher Roots

Decide whether each statement is true or false. If it is false, explain why.

43) The cube root of a negative number is a negative number.

44) The even root of a negative number is a negative number.

45) The odd root of a negative number is not a real number.

46) Every nonnegative real number has two real, even roots.

47) Explain how to find $\sqrt[3]{64}$.

48) Explain how to find $\sqrt[4]{16}$.

49) Does $\sqrt[4]{-81} = -3$? Why or why not?

50) Does $\sqrt[3]{-8} = -2$? Why or why not?

Find each root, if possible.

51) $\sqrt[3]{8}$

52) $\sqrt[3]{1}$

53) $\sqrt[3]{125}$

54) $\sqrt[3]{27}$

55) $\sqrt[3]{-1}$

56) $\sqrt[3]{-8}$

57) $\sqrt[4]{81}$

58) $\sqrt[4]{16}$

59) $\sqrt[4]{-1}$

60) $\sqrt[4]{-81}$

61) $-\sqrt[4]{16}$

62) $-\sqrt[4]{1}$

VIDEO 63) $\sqrt[5]{-32}$

64) $-\sqrt[6]{64}$

65) $-\sqrt[3]{-27}$

66) $-\sqrt[3]{-1000}$

67) $\sqrt[6]{-64}$

68) $\sqrt[4]{-16}$

69) $\sqrt[3]{\dfrac{8}{125}}$

70) $\sqrt[4]{\dfrac{81}{16}}$

VIDEO 71) $\sqrt{60 - 11}$

72) $\sqrt{100 + 21}$

73) $\sqrt[3]{100 + 25}$

74) $\sqrt[3]{9 - 36}$

75) $\sqrt{1 - 9}$

76) $\sqrt{25 - 36}$

77) $\sqrt{5^2 + 12^2}$

78) $\sqrt{3^2 + 4^2}$

Objective 4: Evaluate $\sqrt[n]{a^n}$

79) If n is a positive, even integer and we are not certain that $a \geq 0$, then we must use the absolute value symbol to evaluate $\sqrt[n]{a^n}$. That is, $\sqrt[n]{a^n} = |a|$. Why must we use the absolute value symbol?

80) If n is a positive, odd integer, then $\sqrt[n]{a^n} = a$ for any value of a. Why don't we need to use the absolute value symbol?

Simplify.

81) $\sqrt{8^2}$
82) $\sqrt{5^2}$
83) $\sqrt{(-6)^2}$
84) $\sqrt{(-11)^2}$
85) $\sqrt{y^2}$
86) $\sqrt{d^2}$
87) $\sqrt{m^2}$
88) $\sqrt{t^2}$
89) $\sqrt[3]{5^3}$
90) $\sqrt[3]{1^3}$
91) $\sqrt[3]{(-4)^3}$
92) $\sqrt[3]{(-3)^3}$
93) $\sqrt[3]{z^3}$
94) $\sqrt[5]{t^5}$
95) $\sqrt[4]{h^4}$
96) $\sqrt[5]{v^5}$
97) $\sqrt[9]{p^9}$
98) $\sqrt[6]{m^6}$
99) $\sqrt{(x+7)^2}$
100) $\sqrt{(a-9)^2}$
101) $\sqrt[3]{(2t-1)^3}$
102) $\sqrt[5]{(6r+7)^5}$
103) $\sqrt[4]{(3n+2)^4}$
104) $\sqrt[3]{(x-6)^3}$
105) $\sqrt[7]{(d-8)^7}$
106) $\sqrt[6]{(4y+3)^6}$

Section 10.2 Rational Exponents

Objectives

1. Evaluate Expressions of the Form $a^{1/n}$
2. Evaluate Expressions of the Form $a^{m/n}$
3. Evaluate Expressions of the Form $a^{-m/n}$
4. Combine the Rules of Exponents
5. Convert a Radical Expression to Exponential Form and Simplify

1. Evaluate Expressions of the Form $a^{1/n}$

In this section, we will explain the relationship between radicals and rational exponents (fractional exponents). Sometimes, converting between these two forms makes it easier to simplify expressions.

Definition

If n is a positive integer greater than 1 and $\sqrt[n]{a}$ is a real number, then

$$a^{1/n} = \sqrt[n]{a}$$

(The denominator of the fractional exponent is the index of the radical.)

Example 1

Write in radical form and evaluate.

a) $8^{1/3}$ b) $49^{1/2}$ c) $81^{1/4}$

d) $-64^{1/6}$ e) $(-16)^{1/4}$ f) $(-125)^{1/3}$

Solution

a) The denominator of the fractional exponent is the index of the radical. Therefore, $8^{1/3} = \sqrt[3]{8} = 2$.

b) The denominator in the exponent of $49^{1/2}$ is 2, so the index on the radical is 2, meaning *square* root.

$$49^{1/2} = \sqrt{49} = 7$$

c) $81^{1/4} = \sqrt[4]{81} = 3$

d) $-64^{1/6} = -(64^{1/6}) = -\sqrt[6]{64} = -2$

e) $(-16)^{1/4} = \sqrt[4]{-16}$, which is not a real number. Remember, the even root of a negative number is not a real number.

f) $(-125)^{1/3} = \sqrt[3]{-125} = -5$ The odd root of a negative number is a negative number.

You Try 1

Write in radical form and evaluate.

a) $16^{1/4}$ b) $121^{1/2}$ c) $(-121)^{1/2}$

d) $(-27)^{1/3}$ e) $-81^{1/4}$

2. Evaluate Expressions of the Form $a^{m/n}$

We can add another relationship between rational exponents and radicals.

Definition

If m and n are positive integers and $\dfrac{m}{n}$ is in lowest terms, then

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$$

if $a^{1/n}$ is a real number.

The denominator of the fractional exponent is the index of the radical, and the numerator is the power to which we raise the radical expression. We can also think of $a^{m/n}$ this way: $a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$.

Example 2

Write in radical form and evaluate.

a) $25^{3/2}$ b) $-64^{2/3}$ c) $(-81)^{3/2}$ d) $-81^{3/2}$ e) $(-1000)^{2/3}$

Solution

a) The denominator of the fractional exponent is the index of the radical, and the numerator is the power to which we raise the radical expression.

$$\begin{aligned} 25^{3/2} &= (25^{1/2})^3 && \text{Use the definition to rewrite the exponent.} \\ &= (\sqrt{25})^3 && \text{Rewrite as a radical.} \\ &= 5^3 && \sqrt{25} = 5 \\ &= 125 \end{aligned}$$

b) To evaluate $-64^{2/3}$, *first* evaluate $64^{2/3}$, *then* take the negative of that result.

$$\begin{aligned} -64^{2/3} = -(64^{2/3}) &= -(64^{1/3})^2 && \text{Use the definition to rewrite the exponent.} \\ &= -(\sqrt[3]{64})^2 && \text{Rewrite as a radical.} \\ &= -(4)^2 && \sqrt[3]{64} = 4 \\ &= -16 \end{aligned}$$

c) $$\begin{aligned} (-81)^{3/2} &= [(-81)^{1/2}]^3 \\ &= (\sqrt{-81})^3 && \text{The even root of a negative number is not a real number.} \end{aligned}$$

d) $-81^{3/2} = -(81^{1/2})^3 = -(\sqrt{81})^3 = -(9)^3 = -729$

e) $(-1000)^{2/3} = [(-1000)^{1/3}]^2 = (\sqrt[3]{-1000})^2 = (-10)^2 = 100$ ■

You Try 2

Write in radical form and evaluate.

a) $32^{2/5}$ b) $-100^{3/2}$ c) $(-100)^{3/2}$ d) $(-1)^{4/5}$ e) $-1^{5/3}$

BE CAREFUL

In Example 2, notice how the parentheses affect how we evaluate an expression. The base of the expression $(-81)^{3/2}$ is -81, while the base of $-81^{3/2}$ is 81.

3. Evaluate Expressions of the Form $a^{-m/n}$

Recall that if n is any integer and $a \neq 0$, then $a^{-n} = \left(\frac{1}{a}\right)^n = \frac{1}{a^n}$.

That is, to rewrite the expression with a *positive* exponent, take the reciprocal of the base. For example,

$$2^{-4} = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

We can extend this idea to rational exponents.

Definition

If $a^{m/n}$ is a nonzero real number, then

$$a^{-m/n} = \left(\frac{1}{a}\right)^{m/n} = \frac{1}{a^{m/n}}$$

(To rewrite the expression with a *positive* exponent, take the reciprocal of the base.)

Example 3

Rewrite with a positive exponent and evaluate.

a) $36^{-1/2}$ b) $32^{-2/5}$ c) $\left(\frac{125}{64}\right)^{-2/3}$

Solution

a) To write $36^{-1/2}$ with a positive exponent, take the reciprocal of the base.

$$36^{-1/2} = \left(\frac{1}{36}\right)^{1/2}$$ The reciprocal of 36 is $\frac{1}{36}$.

$$= \sqrt{\frac{1}{36}}$$ The denominator of the fractional exponent is the index of the radical.

$$= \frac{1}{6}$$

b) $$32^{-2/5} = \left(\frac{1}{32}\right)^{2/5}$$ The reciprocal of 32 is $\frac{1}{32}$.

$$= \left(\sqrt[5]{\frac{1}{32}}\right)^2$$ The denominator of the fractional exponent is the index of the radical.

$$= \left(\frac{1}{2}\right)^2$$ $\sqrt[5]{\frac{1}{32}} = \frac{1}{2}$

$$= \frac{1}{4}$$

c) $$\left(\frac{125}{64}\right)^{-2/3} = \left(\frac{64}{125}\right)^{2/3}$$ The reciprocal of $\frac{125}{64}$ is $\frac{64}{125}$.

$$= \left(\sqrt[3]{\frac{64}{125}}\right)^2$$ The denominator of the fractional exponent is the index of the radical.

$$= \left(\frac{4}{5}\right)^2$$ $\sqrt[3]{\frac{64}{125}} = \frac{4}{5}$

$$= \frac{16}{25}$$

BE CAREFUL

The negative exponent does not make the expression negative!

You Try 3

Rewrite with a positive exponent and evaluate.

a) $144^{-1/2}$ b) $16^{-3/4}$ c) $\left(\frac{8}{27}\right)^{-2/3}$

4. Combine the Rules of Exponents

We can combine the rules presented in this section with the rules of exponents we learned in Chapter 2 to simplify expressions containing numbers or variables.

Example 4

Simplify completely. The answer should contain only positive exponents.

a) $(6^{1/5})^2$ b) $25^{3/4} \cdot 25^{-1/4}$ c) $\dfrac{8^{2/9}}{8^{11/9}}$

Solution

a) $(6^{1/5})^2 = 6^{2/5}$ Multiply exponents.

b) $25^{3/4} \cdot 25^{-1/4} = 25^{\frac{3}{4}+\left(-\frac{1}{4}\right)}$ Add exponents.

$= 25^{2/4} = 25^{1/2} = 5$

c) $\dfrac{8^{2/9}}{8^{11/9}} = 8^{\frac{2}{9}-\frac{11}{9}}$ Subtract exponents.

$= 8^{-9/9}$ Subtract $\frac{2}{9} - \frac{11}{9}$.

$= 8^{-1}$ Reduce $-\frac{9}{9}$.

$= \left(\frac{1}{8}\right)^1 = \frac{1}{8}$

You Try 4

Simplify completely. The answer should contain only positive exponents.

a) $49^{3/8} \cdot 49^{1/8}$ b) $(16^{1/12})^3$ c) $\dfrac{7^{2/5}}{7^{4/5}}$

Example 5

Simplify completely. Assume the variables represent positive real numbers. The answer should contain only positive exponents.

a) $r^{1/8} \cdot r^{3/8}$ b) $\left(\dfrac{x^{2/3}}{y^{1/4}}\right)^6$ c) $\dfrac{n^{-5/6} \cdot n^{1/3}}{n^{-1/6}}$ d) $\left(\dfrac{a^{-7}b^{1/2}}{a^5b^{1/3}}\right)^{-3/4}$

Solution

a) $r^{1/8} \cdot r^{3/8} = r^{\frac{1}{8}+\frac{3}{8}}$ Add exponents.

$= r^{4/8} = r^{1/2}$

b) $\left(\dfrac{x^{2/3}}{y^{1/4}}\right)^6 = \dfrac{x^{\frac{2}{3}\cdot 6}}{y^{\frac{1}{4}\cdot 6}}$ Multiply exponents.

$= \dfrac{x^4}{y^{3/2}}$ Multiply and reduce.

c) $\dfrac{n^{-5/6} \cdot n^{1/3}}{n^{-1/6}} = \dfrac{n^{-\frac{5}{6}+\frac{1}{3}}}{n^{-1/6}} = \dfrac{n^{-\frac{5}{6}+\frac{2}{6}}}{n^{-1/6}} = \dfrac{n^{-3/6}}{n^{-1/6}}$ Add exponents.

$= n^{-\frac{3}{6}-\left(-\frac{1}{6}\right)} = n^{-2/6} = n^{-1/3} = \dfrac{1}{n^{1/3}}$ Subtract exponents.

d) $\left(\dfrac{a^{-7}b^{1/2}}{a^5b^{1/3}}\right)^{-3/4} = \left(\dfrac{a^5b^{1/3}}{a^{-7}b^{1/2}}\right)^{3/4}$ Eliminate the negative from the outermost exponent by taking the reciprocal of the base.

Simplify the expression inside the parentheses by subtracting the exponents.

$$= (a^{5-(-7)}b^{1/3-1/2})^{3/4} = (a^{5+7}b^{2/6-3/6})^{3/4} = (a^{12}b^{-1/6})^{3/4}$$

Apply the power rule, and simplify.

$$= (a^{12})^{3/4}(b^{-1/6})^{3/4} = a^9b^{-1/8} = \frac{a^9}{b^{1/8}}$$

You Try 5

Simplify completely. Assume the variables represent positive real numbers. The answer should contain only positive exponents.

a) $(a^3b^{1/5})^{10}$ b) $\dfrac{t^{3/10}}{t^{7/10}}$ c) $\dfrac{s^{3/4}}{s^{1/2} \cdot s^{-5/4}}$ d) $\left(\dfrac{x^4y^{3/8}}{x^9y^{1/4}}\right)^{-2/5}$

5. Convert a Radical Expression to Exponential Form and Simplify

Some radicals can be simplified by first putting them into rational exponent form and then converting them back to radicals.

Example 6

Rewrite each radical in exponential form, then simplify. Write the answer in simplest (or radical) form. Assume the variable represents a nonnegative real number.

a) $\sqrt[8]{9^4}$ b) $\sqrt[6]{s^4}$

Solution

a) Since the index of the radical is the denominator of the exponent and the power is the numerator, we can write

$\sqrt[8]{9^4} = 9^{4/8}$ Write with a rational exponent.

$= 9^{1/2} = 3$

b) $\sqrt[6]{s^4} = s^{4/6}$ Write with a rational exponent.

$= s^{2/3} = \sqrt[3]{s^2}$

The expression $\sqrt[6]{s^4}$ is not in simplest form because the 4 and the 6 contain a common factor of 2, but $\sqrt[3]{s^2}$ *is* in simplest form because 2 and 3 do not have any common factors besides 1.

You Try 6

Rewrite each radical in exponential form, then simplify. Write the answer in simplest (or radical) form. Assume the variable represents a nonnegative real number.

a) $\sqrt[6]{125^2}$ b) $\sqrt[10]{p^4}$

In Section 10.1, we said that if a is negative and n is a positive, even number, then $\sqrt[n]{a^n} = |a|$. For example, if we are *not* told that k is positive, then $\sqrt{k^2} = |k|$. However, if we assume that k is positive, then $\sqrt{k^2} = k$. **In the rest of this chapter, we will assume that all variables represent positive, real numbers unless otherwise stated.** When we make this assumption, we do not need to use absolute values when simplifying even roots. And if we consider this together with the relationship between radicals and rational exponents, we have another way to explain why $\sqrt[n]{a^n} = a$.

Example 7

Simplify.

a) $\sqrt[3]{5^3}$ b) $(\sqrt[4]{9})^4$ c) $\sqrt{k^2}$

Solution

a) $\sqrt[3]{5^3} = (5^3)^{1/3} = 5^{3 \cdot \frac{1}{3}} = 5^1 = 5$

b) $(\sqrt[4]{9})^4 = (9^{1/4})^4 = 9^{\frac{1}{4} \cdot 4} = 9^1 = 9$

c) $\sqrt{k^2} = (k^2)^{1/2} = k^{2 \cdot \frac{1}{2}} = k^1 = k$

You Try 7

Simplify.

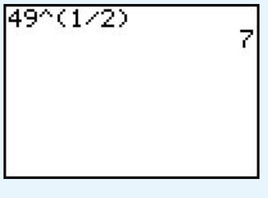
a) $(\sqrt{10})^2$ b) $\sqrt[3]{7^3}$ c) $\sqrt[4]{t^4}$

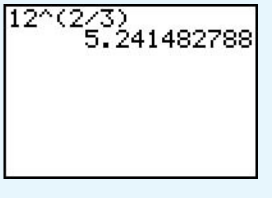
Using Technology

We can evaluate square roots, cube roots, and even higher roots by first rewriting the radical in exponential form and then using a graphing calculator.

For example, to evaluate $\sqrt{49}$, first rewrite the radical as $49^{1/2}$, then enter [4] [9], press [^] [(], enter [1] [÷] [2], and press [)] [ENTER]. The result is 7, as shown on the screen on the left below.

To approximate $\sqrt[3]{12^2}$ rounded to the nearest tenth, first rewrite the radical as $12^{2/3}$, then enter [1] [2], press [^] [(], enter [2] [÷] [3], and press [)], [ENTER]. The result is 5.241482788 as shown on the screen on the right below. The result rounded to the nearest tenth is then 5.2.





To evaluate radicals with an index greater than 3, follow the same procedure explained above. Evaluate by rewriting in exponential form if necessary and then using a graphing calculator. If necessary, approximate to the nearest tenth.

1) $16^{1/2}$ 2) $\sqrt[3]{512}$ 3) $\sqrt{37}$ 4) $361^{1/2}$ 5) $4096^{2/3}$ 6) $2401^{1/4}$

Answers to You Try Exercises

1) a) 2 b) 11 c) not a real number d) -3 e) -3 2) a) 4 b) -1000 c) not a real number d) 1 e) -1 3) a) $\frac{1}{12}$ b) $\frac{1}{8}$ c) $\frac{9}{4}$ 4) a) 7 b) 2 c) $\frac{1}{7^{2/5}}$ 5) a) $a^{30}b^2$ b) $\frac{1}{t^{2/5}}$ c) $s^{3/2}$ d) $\frac{x^2}{y^{1/20}}$ 6) a) 5 b) $\sqrt[5]{p^2}$ 7) a) 10 b) 7 c) t

Answers to Technology Exercises

1) 4 2) 8 3) 6.1 4) 19 5) 256 6) 7

10.2 Exercises

Objective 1: Evaluate Expressions of the Form $a^{1/n}$

1) Explain how to write $25^{1/2}$ in radical form.

2) Explain how to write $1^{1/3}$ in radical form.

Write in radical form and evaluate.

3) $9^{1/2}$
4) $64^{1/2}$
5) $1000^{1/3}$
6) $27^{1/3}$
7) $32^{1/5}$
8) $81^{1/4}$
9) $-125^{1/3}$
10) $-64^{1/6}$
11) $\left(\frac{4}{121}\right)^{1/2}$
12) $\left(\frac{4}{9}\right)^{1/2}$
13) $\left(\frac{125}{64}\right)^{1/3}$
14) $\left(\frac{16}{81}\right)^{1/4}$
15) $-\left(\frac{36}{169}\right)^{1/2}$
16) $-\left(\frac{1000}{27}\right)^{1/3}$
17) $(-81)^{1/4}$
18) $(-169)^{1/2}$
19) $(-1)^{1/7}$
20) $(-8)^{1/3}$

Objective 2: Evaluate Expressions of the Form $a^{m/n}$

21) Explain how to write $16^{3/4}$ in radical form.

22) Explain how to write $100^{3/2}$ in radical form.

Write in radical form and evaluate.

23) $8^{4/3}$
24) $81^{3/4}$
25) $64^{5/6}$
26) $32^{3/5}$
27) $(-125)^{2/3}$
28) $(-1000)^{2/3}$
29) $-36^{3/2}$
30) $-27^{4/3}$
31) $(-81)^{3/4}$
32) $(-25)^{3/2}$
33) $\left(\frac{16}{81}\right)^{3/4}$
34) $-16^{5/4}$
35) $-\left(\frac{1000}{27}\right)^{2/3}$
36) $-\left(\frac{8}{27}\right)^{4/3}$

Objective 3: Evaluate Expressions of the Form $a^{-m/n}$

Decide whether each statement is true or false. Explain your answer.

37) $81^{-1/2} = -9$
38) $\left(\frac{1}{100}\right)^{-3/2} = \left(\frac{1}{100}\right)^{2/3}$

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

39) $64^{-1/2} = \left(\frac{\quad}{\quad}\right)^{1/2}$ The reciprocal of 64 is ___.

$= \sqrt{\frac{1}{64}}$ ______________

$= \underline{\quad}$ Simplify.

40) $\left(\frac{1}{1000}\right)^{-1/3} = (\underline{\quad})^{1/3}$ The reciprocal of $\frac{1}{1000}$ is ___.

$= \sqrt[3]{1000}$ ______________

$= \underline{\quad}$ Simplify.

Rewrite with a positive exponent and evaluate.

41) $49^{-1/2}$
42) $100^{-1/2}$
43) $1000^{-1/3}$
44) $27^{-1/3}$
45) $\left(\frac{1}{81}\right)^{-1/4}$
46) $\left(\frac{1}{32}\right)^{-1/5}$

47) $-\left(\dfrac{1}{64}\right)^{-1/3}$ 48) $-\left(\dfrac{1}{125}\right)^{-1/3}$

VIDEO 49) $64^{-5/6}$ 50) $81^{-3/4}$

51) $125^{-2/3}$ 52) $64^{-2/3}$

53) $\left(\dfrac{25}{4}\right)^{-3/2}$ 54) $\left(\dfrac{9}{100}\right)^{-3/2}$

55) $\left(\dfrac{64}{125}\right)^{-2/3}$ 56) $\left(\dfrac{81}{16}\right)^{-3/4}$

Objective 4: Combine the Rules of Exponents

Simplify completely. The answer should contain only positive exponents, where appropriate.

57) $2^{2/3} \cdot 2^{7/3}$ 58) $5^{3/4} \cdot 5^{5/4}$

59) $(9^{1/4})^2$ 60) $(7^{2/3})^3$

61) $8^{7/5} \cdot 8^{-3/5}$ 62) $6^{-4/3} \cdot 6^{5/3}$

63) $\dfrac{2^{23/4}}{2^{3/4}}$ 64) $\dfrac{5^{3/2}}{5^{9/2}}$

65) $\dfrac{4^{2/5}}{4^{6/5} \cdot 4^{3/5}}$ 66) $\dfrac{6^{-1}}{6^{1/2} \cdot 6^{-5/2}}$

Simplify completely. The answer should contain only positive exponents.

67) $z^{1/6} \cdot z^{5/6}$ 68) $h^{1/6} \cdot h^{-3/4}$

69) $(-9v^{5/8})(8v^{3/4})$ 70) $(-3x^{-1/3})(8x^{4/9})$

71) $\dfrac{a^{5/9}}{a^{4/9}}$ 72) $\dfrac{x^{1/6}}{x^{5/6}}$

73) $\dfrac{20c^{-2/3}}{72c^{5/6}}$ 74) $\dfrac{48w^{3/10}}{10w^{2/5}}$

75) $(x^{-2/9})^3$ 76) $(n^{-2/7})^3$

77) $(z^{1/5})^{2/3}$ 78) $(r^{4/3})^{5/2}$

VIDEO 79) $(81u^{8/3}v^4)^{3/4}$ 80) $(64x^6y^{12/5})^{5/6}$

81) $(32r^{1/3}s^{4/9})^{3/5}$ 82) $(125a^9b^{1/4})^{2/3}$

83) $\left(\dfrac{f^{6/7}}{27g^{-5/3}}\right)^{1/3}$ 84) $\left(\dfrac{16c^{-8}}{b^{-11/3}}\right)^{3/4}$

85) $\left(\dfrac{x^{-5/3}}{w^{3/2}}\right)^{-6}$ 86) $\left(\dfrac{t^{-3/2}}{u^{1/4}}\right)^{-4}$

87) $\dfrac{y^{1/2} \cdot y^{-1/3}}{y^{5/6}}$ 88) $\dfrac{t^5}{t^{1/2} \cdot t^{3/4}}$

89) $\left(\dfrac{a^4b^3}{32a^{-2}b^4}\right)^{2/5}$ 90) $\left(\dfrac{16c^{-8}d^3}{c^4d^5}\right)^{3/2}$

91) $\left(\dfrac{r^{4/5}t^{-2}}{r^{2/3}t^5}\right)^{-3/2}$ 92) $\left(\dfrac{x^{10}y^{1/6}}{x^{-8}y^{2/3}}\right)^{-2/3}$

93) $\left(\dfrac{h^{-2}k^{5/2}}{h^{-8}k^{5/6}}\right)^{-5/6}$ 94) $\left(\dfrac{c^{1/8}d^{-4}}{c^{3/4}d}\right)^{-8/5}$

95) $p^{1/2}(p^{2/3} + p^{1/2})$ 96) $w^{4/3}(w^{1/2} - w^3)$

Objective 5: Convert a Radical Expression to Exponential Form and Simplify

Rewrite each radical in exponential form, then simplify. Write the answer in simplest (or radical) form.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

97) $\sqrt[12]{25^6} =$ ____ Write with a rational exponent.
$=$ ____ Reduce the exponent.
$= 5$ ______

98) $\sqrt[10]{c^4} =$ ___ Write with a rational exponent.
$= c^{2/5}$ ______________
$=$ ___ Write in radical form.

VIDEO 99) $\sqrt[6]{49^3}$ 100) $\sqrt[9]{8^3}$

101) $\sqrt[4]{81^2}$ 102) $\sqrt{3^2}$

103) $(\sqrt{5})^2$ 104) $(\sqrt[3]{10})^3$

105) $(\sqrt[3]{12})^3$ 106) $(\sqrt[4]{15})^4$

107) $\sqrt[3]{x^{12}}$ 108) $\sqrt[4]{t^8}$

109) $\sqrt[6]{k^2}$ 110) $\sqrt[9]{w^6}$

111) $\sqrt[4]{z^2}$ 112) $\sqrt[8]{m^4}$

113) $\sqrt{d^4}$ 114) $\sqrt{s^6}$

The windchill temperature, WC, measures how cold it feels outside (for temperatures under 50 degrees Fahrenheit) when the velocity of the wind, V, is considered along with the air temperature, T. The stronger the wind at a given air temperature, the colder it feels.

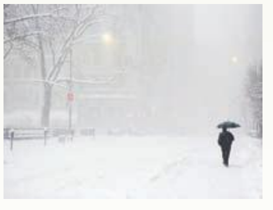
The formula for calculating windchill is

$$WC = 35.74 + 0.6215T - 35.75V^{4/25} + 0.4275TV^{4/25}$$

where WC and T are in degrees Fahrenheit and V is in miles per hour.
(http://www.nws.noaa.gov/om/windchill/windchillglossary.shtml)

Use this information for Exercises 115 and 116, and round all answers to the nearest degree.

115) Determine the windchill when the air temperature is 20 degrees and the wind is blowing at the given speed.
a) 5 mph
b) 15 mph



116) Determine the windchill when the air temperature is 10 degrees and the wind is blowing at the given speed.
a) 12 mph b) 20 mph

Section 10.3 Simplifying Expressions Containing Square Roots

Objectives

1. Multiply Square Roots
2. Simplify the Square Root of a Whole Number
3. Use the Quotient Rule for Square Roots
4. Simplify Square Root Expressions Containing Variables with Even Exponents
5. Simplify Square Root Expressions Containing Variables with Odd Exponents
6. Simplify More Square Root Expressions Containing Variables

In this section, we will introduce rules for finding the product and quotient of square roots as well as for simplifying expressions containing square roots.

1. Multiply Square Roots

Let's begin with the product $\sqrt{4} \cdot \sqrt{9}$. $\sqrt{4} \cdot \sqrt{9} = 2 \cdot 3 = 6$. Also notice that $\sqrt{4} \cdot \sqrt{9} = \sqrt{4 \cdot 9} = \sqrt{36} = 6$.

We obtain the same result. This leads us to the product rule for multiplying expressions containing square roots.

Definition Product Rule for Square Roots

Let a and b be nonnegative real numbers. Then,

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$$

In other words, the product of two square roots equals the square root of the product.

Example 1

Multiply. a) $\sqrt{5} \cdot \sqrt{2}$ b) $\sqrt{3} \cdot \sqrt{x}$

Solution

a) $\sqrt{5} \cdot \sqrt{2} = \sqrt{5 \cdot 2} = \sqrt{10}$ b) $\sqrt{3} \cdot \sqrt{x} = \sqrt{3 \cdot x} = \sqrt{3x}$ ■

BE CAREFUL

We can multiply radicals this way *only if* the indices are the same. We will see later how to multiply radicals with different indices such as $\sqrt{5} \cdot \sqrt[3]{t}$.

You Try 1

Multiply.

a) $\sqrt{6} \cdot \sqrt{5}$ b) $\sqrt{10} \cdot \sqrt{r}$

2. Simplify the Square Root of a Whole Number

Knowing how to simplify radicals is very important in the study of algebra. We begin by discussing how to simplify expressions containing square roots.

How do we know when a square root is simplified?

Property When Is a Square Root Simplified?

An expression containing a square root is simplified when all of the following conditions are met:

1) The radicand does not contain any factors (other than 1) that are perfect squares.

2) The radicand does not contain any fractions.

3) There are no radicals in the denominator of a fraction.

Note: Condition 1) implies that the radical cannot contain variables with exponents greater than or equal to 2, the index of the square root.

We will discuss higher roots in Section 10.4.

To simplify expressions containing square roots, we reverse the process of multiplying. That is, we use the product rule that says $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ where a or b is a perfect square.

Example 2

Simplify completely.

a) $\sqrt{18}$ b) $\sqrt{500}$ c) $\sqrt{21}$ d) $\sqrt{48}$

Solution

a) The radical $\sqrt{18}$ is not in simplest form since 18 contains a factor (other than 1) that is a perfect square. Think of two numbers that multiply to 18 so that at least one of the numbers is a perfect square: $18 = 9 \cdot 2$.

(While it is true that $18 = 6 \cdot 3$, neither 6 nor 3 is a perfect square.)
Rewrite $\sqrt{18}$:

$$\begin{aligned} \sqrt{18} &= \sqrt{9 \cdot 2} && \text{9 is a perfect square.} \\ &= \sqrt{9} \cdot \sqrt{2} && \text{Product rule} \\ &= 3\sqrt{2} && \sqrt{9} = 3 \end{aligned}$$

$3\sqrt{2}$ is completely simplified because 2 does not have any factors that are perfect squares.

b) Does 500 have a factor that is a perfect square? Yes! $500 = 100 \cdot 5$. To simplify $\sqrt{500}$, rewrite it as

$$\begin{aligned} \sqrt{500} &= \sqrt{100 \cdot 5} && \text{100 is a perfect square.} \\ &= \sqrt{100} \cdot \sqrt{5} && \text{Product rule} \\ &= 10\sqrt{5} && \sqrt{100} = 10 \end{aligned}$$

$10\sqrt{5}$ is completely simplified because 5 does not have any factors that are perfect squares.

c) $21 = 3 \cdot 7$ Neither 3 nor 7 is a perfect square.
$21 = 1 \cdot 21$ Although 1 is a perfect square, it will not help us simplify $\sqrt{21}$.

$\sqrt{21}$ is in simplest form.

d) There are different ways to simplify $\sqrt{48}$. We will look at two of them.

i) Two numbers that multiply to 48 are 16 and 3 with 16 being a perfect square. We can write

$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}$$

ii) We can also think of 48 as $4 \cdot 12$ since 4 is a perfect square. We can write

$$\sqrt{48} = \sqrt{4 \cdot 12} = \sqrt{4} \cdot \sqrt{12} = 2\sqrt{12}$$

Therefore, $\sqrt{48} = 2\sqrt{12}$. Is $\sqrt{12}$ in simplest form? *No, because $12 = 4 \cdot 3$ and 4 is a perfect square*. We must continue to simplify.

$$\begin{aligned} \sqrt{48} &= 2\sqrt{12} \\ &= 2\sqrt{4 \cdot 3} = 2\sqrt{4} \cdot \sqrt{3} = 2 \cdot 2 \cdot \sqrt{3} = 4\sqrt{3} \end{aligned}$$

$4\sqrt{3}$ is completely simplified because 3 does not have any factors that are perfect squares.

Example 2(d) shows that using either $\sqrt{48} = \sqrt{16 \cdot 3}$ or $\sqrt{48} = \sqrt{4 \cdot 12}$ leads us to the same result. Furthermore, this example illustrates that a radical is not always *completely* simplified after just one iteration of the simplification process. It is necessary to always examine the radical to determine whether or not it can be simplified more. ■

Note

After simplifying a radical, look at the result and ask yourself, "*Is the radical in simplest form?*" If it is not, simplify again. Asking yourself this question will help you to be sure that the radical *is* completely simplified.

You Try 2

Simplify completely.

a) $\sqrt{28}$ b) $\sqrt{75}$ c) $\sqrt{72}$

3. Use the Quotient Rule for Square Roots

Let's simplify $\frac{\sqrt{36}}{\sqrt{9}}$. We can say $\frac{\sqrt{36}}{\sqrt{9}} = \frac{6}{3} = 2$. It is also true that $\frac{\sqrt{36}}{\sqrt{9}} = \sqrt{\frac{36}{9}} = \sqrt{4} = 2$.

This leads us to the quotient rule for dividing expressions containing square roots.

Definition Quotient Rule for Square Roots

Let a and b be nonnegative real numbers such that $b \neq 0$. Then,

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

The square root of a quotient equals the quotient of the square roots.

Example 3

Simplify completely.

a) $\sqrt{\frac{9}{49}}$ b) $\sqrt{\frac{200}{2}}$ c) $\frac{\sqrt{72}}{\sqrt{6}}$ d) $\sqrt{\frac{5}{81}}$

Solution

a) Since 9 and 49 are each perfect squares, find the square root of each separately.

$$\sqrt{\frac{9}{49}} = \frac{\sqrt{9}}{\sqrt{49}} \quad \text{Quotient rule}$$
$$= \frac{3}{7} \quad \sqrt{9} = 3 \text{ and } \sqrt{49} = 7$$

b) Neither 200 nor 2 is a perfect square, but if we simplify $\frac{200}{2}$ we get 100, which *is* a perfect square.

$$\sqrt{\frac{200}{2}} = \sqrt{100} \quad \text{Simplify } \frac{200}{2}.$$
$$= 10$$

c) We can simplify $\frac{\sqrt{72}}{\sqrt{6}}$ using two different methods.

i) Begin by applying the quotient rule to obtain a fraction under *one* radical and simplify the fraction.

$$\frac{\sqrt{72}}{\sqrt{6}} = \sqrt{\frac{72}{6}} \quad \text{Quotient rule}$$
$$= \sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$$

ii) We can apply the product rule to rewrite $\sqrt{72}$, then simplify the fraction.

$$\frac{\sqrt{72}}{\sqrt{6}} = \frac{\sqrt{6}\cdot\sqrt{12}}{\sqrt{6}} \qquad \text{Product rule}$$
$$= \frac{\overset{1}{\cancel{\sqrt{6}}}\cdot\sqrt{12}}{\underset{1}{\cancel{\sqrt{6}}}} \qquad \text{Divide out the common factor.}$$
$$= \sqrt{12} = \sqrt{4\cdot 3} = \sqrt{4}\cdot\sqrt{3} = 2\sqrt{3}$$

Either method will produce the same result.

d) The fraction $\frac{5}{81}$ does not reduce, but 81 *is* a perfect square. Begin by applying the quotient rule.

$$\sqrt{\frac{5}{81}} = \frac{\sqrt{5}}{\sqrt{81}} \qquad \text{Quotient rule}$$
$$= \frac{\sqrt{5}}{9} \qquad \sqrt{81} = 9$$

You Try 3

Simplify completely.

a) $\sqrt{\frac{100}{169}}$ b) $\sqrt{\frac{27}{3}}$ c) $\frac{\sqrt{250}}{\sqrt{5}}$ d) $\sqrt{\frac{11}{36}}$

4. Simplify Square Root Expressions Containing Variables with Even Exponents

Recall that a square root is not simplified if it contains any factors that are perfect squares. This means that a square root *containing variables is simplified if the power on each variable is less than 2*. For example, $\sqrt{r^6}$ is not in simplified form. If r represents a nonnegative real number, then we can use rational exponents to simplify $\sqrt{r^6}$.

$$\sqrt{r^6} = (r^6)^{1/2} = r^{6\cdot\frac{1}{2}} = r^{6/2} = r^3$$

Multiplying $6\cdot\frac{1}{2}$ is the same as dividing 6 by 2. We can generalize this result with the following statement.

Property $\sqrt{a^m}$

If a is a nonnegative real number and m is an integer, then

$$\sqrt{a^m} = a^{m/2}$$

We can combine this property with the product and quotient rules to simplify radical expressions.

Example 4

Simplify completely.

a) $\sqrt{z^2}$ b) $\sqrt{49t^2}$ c) $\sqrt{18b^{14}}$ d) $\sqrt{\frac{32}{n^{20}}}$

Solution

a) $\sqrt{z^2} = z^{2/2} = z^1 = z$

b) $\sqrt{49t^2} = \sqrt{49}\cdot\sqrt{t^2} = 7\cdot t^{2/2} = 7t$

c) $\sqrt{18b^{14}} = \sqrt{18} \cdot \sqrt{b^{14}}$ Product rule

$= \sqrt{9} \cdot \sqrt{2} \cdot b^{14/2}$ 9 is a perfect square.

$= 3\sqrt{2} \cdot b^7$ Simplify.

$= 3b^7\sqrt{2}$ Rewrite using the commutative property.

d) We begin by using the quotient rule.

$$\sqrt{\frac{32}{n^{20}}} = \frac{\sqrt{32}}{\sqrt{n^{20}}} = \frac{\sqrt{16} \cdot \sqrt{2}}{n^{20/2}} = \frac{4\sqrt{2}}{n^{10}}$$

You Try 4

Simplify completely.

a) $\sqrt{y^{10}}$ b) $\sqrt{144p^{16}}$ c) $\sqrt{\frac{45}{w^4}}$

5. Simplify Square Root Expressions Containing Variables with Odd Exponents

How do we simplify an expression containing a square root if the power under the square root is odd? We can use the product rule for radicals and fractional exponents to help us understand how to simplify such expressions.

Example 5

Simplify completely.

a) $\sqrt{x^7}$ b) $\sqrt{c^{11}}$

Solution

a) To simplify $\sqrt{x^7}$, write x^7 as the product of two factors so that the exponent of one of the factors is the *largest* number less than 7 that is divisible by 2 (the index of the radical).

$\sqrt{x^7} = \sqrt{x^6 \cdot x^1}$ 6 is the largest number less than 7 that is divisible by 2.

$= \sqrt{x^6} \cdot \sqrt{x}$ Product rule

$= x^{6/2} \cdot \sqrt{x}$ Use a fractional exponent to simplify.

$= x^3\sqrt{x}$ $6 \div 2 = 3$

b) To simplify $\sqrt{c^{11}}$, write c^{11} as the product of two factors so that the exponent of one of the factors is the *largest* number less than 11 that is divisible by 2 (the index of the radical).

$\sqrt{c^{11}} = \sqrt{c^{10} \cdot c^1}$ 10 is the largest number less than 11 that is divisible by 2.

$= \sqrt{c^{10}} \cdot \sqrt{c}$ Product rule

$= c^{10/2} \cdot \sqrt{c}$ Use a fractional exponent to simplify.

$= c^5\sqrt{c}$ $10 \div 2 = 5$

You Try 5

Simplify completely.

a) $\sqrt{m^5}$ b) $\sqrt{z^{19}}$

We used the product rule to simplify each radical in Example 5. During the simplification, however, we always divided an exponent by 2. This idea of division gives us another way to simplify radical expressions. Once again let's look at the radicals and their simplified forms in Example 5 to see how we can simplify radical expressions using division.

$$\sqrt{x^7} = x^3\sqrt{x^1} = x^3\sqrt{x} \qquad\qquad \sqrt{c^{11}} = c^5\sqrt{c^1} = c^5\sqrt{c}$$

$$\begin{array}{lr l} & 3 & \rightarrow \text{Quotient} \\ \text{Index of radical} \rightarrow & 2\overline{)\ 7} & \\ & -6 & \\ & \overline{\ \ 1} & \rightarrow \text{Remainder} \end{array} \qquad\qquad \begin{array}{lr l} & 5 & \rightarrow \text{Quotient} \\ \text{Index of radical} \rightarrow & 2\overline{)\ 11} & \\ & -10 & \\ & \overline{\ \ 1} & \rightarrow \text{Remainder} \end{array}$$

Note

To simplify a radical expression containing variables:

1) Divide the original exponent in the radicand by the index of the radical.
2) The exponent on the variable *outside* of the radical will be the *quotient* of the division problem.
3) The exponent on the variable *inside* of the radical will be the *remainder* of the division problem.

Example 6

Simplify completely.

a) $\sqrt{t^9}$ b) $\sqrt{16b^5}$ c) $\sqrt{45y^{21}}$

Solution

a) To simplify $\sqrt{t^9}$, divide:

$$\begin{array}{r l} 4 & \rightarrow \text{Quotient} \\ 2\overline{)\ 9} & \\ -8 & \\ \overline{\ \ 1} & \rightarrow \text{Remainder} \end{array}$$

$$\sqrt{t^9} = t^4\sqrt{t^1} = t^4\sqrt{t}$$

b)
$$\begin{aligned} \sqrt{16b^5} &= \sqrt{16} \cdot \sqrt{b^5} && \text{Product rule} \\ &= 4 \cdot b^2\sqrt{b^1} && 5 \div 2 \text{ gives a quotient of 2 and a remainder of 1.} \\ &= 4b^2\sqrt{b} \end{aligned}$$

c)
$$\begin{aligned} \sqrt{45y^{21}} &= \sqrt{45} \cdot \sqrt{y^{21}} && \text{Product rule} \\ &= \underbrace{\sqrt{9} \cdot \sqrt{5}}_{\text{Product rule}} \cdot \underbrace{y^{10}\sqrt{y^1}}_{21 \div 2 \text{ gives a quotient of 10 and a remainder of 1.}} \\ &= 3\sqrt{5} \cdot y^{10}\sqrt{y} && \sqrt{9} = 3 \\ &= 3y^{10} \cdot \sqrt{5} \cdot \sqrt{y} && \text{Use the commutative property to rewrite the expression.} \\ &= 3y^{10}\sqrt{5y} && \text{Use the product rule to write the expression with one radical.} \end{aligned}$$

You Try 6

Simplify completely.

a) $\sqrt{m^{13}}$ b) $\sqrt{100v^7}$ c) $\sqrt{32a^3}$

If a radical contains more than one variable, apply the product or quotient rule.

Example 7

Simplify completely.

a) $\sqrt{8a^{15}b^3}$ b) $\sqrt{\dfrac{5r^{27}}{s^8}}$

Solution

a)
$$\sqrt{8a^{15}b^3} = \sqrt{8} \cdot \sqrt{a^{15}} \cdot \sqrt{b^3}$$
$$= \sqrt{4} \cdot \sqrt{2} \cdot a^7\sqrt{a^1} \cdot b^1\sqrt{b^1}$$

Product rule 15 ÷ 2 gives a quotient of 7 and a remainder of 1. 3 ÷ 2 gives a quotient of 1 and a remainder of 1.

$$= 2\sqrt{2} \cdot a^7\sqrt{a} \cdot b\sqrt{b}$$ $\sqrt{4} = 2$

$$= 2a^7b \cdot \sqrt{2} \cdot \sqrt{a} \cdot \sqrt{b}$$ Use the commutative property to rewrite the expression.

$$= 2a^7b\sqrt{2ab}$$ Use the product rule to write the expression with one radical.

b)
$$\sqrt{\frac{5r^{27}}{s^8}} = \frac{\sqrt{5r^{27}}}{\sqrt{s^8}}$$ Quotient rule

$$= \frac{\sqrt{5} \cdot \sqrt{r^{27}}}{s^4}$$ → Product rule → 8 ÷ 2 = 4

$$= \frac{\sqrt{5} \cdot r^{13}\sqrt{r^1}}{s^4}$$ 27 ÷ 2 gives a quotient of 13 and a remainder of 1.

$$= \frac{r^{13} \cdot \sqrt{5} \cdot \sqrt{r}}{s^4}$$ Use the commutative property to rewrite the expression.

$$= \frac{r^{13}\sqrt{5r}}{s^4}$$ Use the product rule to write the expression with one radical. ■

You Try 7

Simplify completely.

a) $\sqrt{c^5d^{12}}$ b) $\sqrt{27x^{10}y^9}$ c) $\sqrt{\dfrac{40u^{13}}{v^{20}}}$

6. Simplify More Square Root Expressions Containing Variables

Next we will look at some examples of multiplying and dividing radical expressions that contain variables. Remember to always look at the result and ask yourself, "*Is the radical in simplest form?*" If it is not, simplify completely.

Example 8

Perform the indicated operation and simplify completely.

a) $\sqrt{6t} \cdot \sqrt{3t}$ b) $\sqrt{2a^3b} \cdot \sqrt{8a^2b^5}$ c) $\dfrac{\sqrt{20x^5}}{\sqrt{5x}}$

Solution

a) $$\begin{aligned} \sqrt{6t} \cdot \sqrt{3t} &= \sqrt{6t \cdot 3t} && \text{Product rule} \\ &= \sqrt{18t^2} \\ &= \sqrt{18} \cdot \sqrt{t^2} && \text{Product rule} \\ &= \sqrt{9 \cdot 2} \cdot t = \sqrt{9} \cdot \sqrt{2} \cdot t = 3\sqrt{2} \cdot t = 3t\sqrt{2} \end{aligned}$$

b) $\sqrt{2a^3b} \cdot \sqrt{8a^2b^5}$

There are two good methods for multiplying these radicals.

i) Multiply the radicands to obtain one radical.

$$\begin{aligned} \sqrt{2a^3b} \cdot \sqrt{8a^2b^5} &= \sqrt{2a^3b \cdot 8a^2b^5} && \text{Product rule} \\ &= \sqrt{16a^5b^6} && \text{Multiply.} \end{aligned}$$

Is the radical in simplest form? *No.*

$$\begin{aligned} &= \sqrt{16} \cdot \sqrt{a^5} \cdot \sqrt{b^6} && \text{Product rule} \\ &= 4 \cdot a^2\sqrt{a} \cdot b^3 && \text{Evaluate.} \\ &= 4a^2b^3\sqrt{a} && \text{Commutative property} \end{aligned}$$

ii) Simplify each radical, then multiply.

$$\begin{aligned} \sqrt{2a^3b} &= \sqrt{2} \cdot \sqrt{a^3} \cdot \sqrt{b} \\ &= \sqrt{2} \cdot a\sqrt{a} \cdot \sqrt{b} \\ &= a\sqrt{2ab} \end{aligned} \qquad \begin{aligned} \sqrt{8a^2b^5} &= \sqrt{8} \cdot \sqrt{a^2} \cdot \sqrt{b^5} \\ &= 2\sqrt{2} \cdot a \cdot b^2\sqrt{b} \\ &= 2ab^2\sqrt{2b} \end{aligned}$$

Then, $$\begin{aligned} \sqrt{2a^3b} \cdot \sqrt{8a^2b^5} &= a\sqrt{2ab} \cdot 2ab^2\sqrt{2b} \\ &= a \cdot 2ab^2 \cdot \sqrt{2ab} \cdot \sqrt{2b} && \text{Commutative property} \\ &= 2a^2b^2\sqrt{4ab^2} && \text{Multiply.} \\ &= 2a^2b^2 \cdot 2 \cdot b \cdot \sqrt{a} && \sqrt{4ab^2} = 2b\sqrt{a} \\ &= 4a^2b^3\sqrt{a} && \text{Multiply.} \end{aligned}$$

Both methods give the same result.

c) We can use the quotient rule first or simplify first.

i) $$\begin{aligned} \frac{\sqrt{20x^5}}{\sqrt{5x}} &= \sqrt{\frac{20x^5}{5x}} && \text{Use the quotient rule first.} \\ &= \sqrt{4x^4} = \sqrt{4} \cdot \sqrt{x^4} = 2x^2 \end{aligned}$$

ii) $$\begin{aligned} \frac{\sqrt{20x^5}}{\sqrt{5x}} &= \frac{\sqrt{20} \cdot \sqrt{x^5}}{\sqrt{5x}} && \text{Simplify first by using the product rule.} \\ &= \frac{\sqrt{4} \cdot \sqrt{5} \cdot x^2\sqrt{x}}{\sqrt{5x}} && \text{Product rule; simplify } \sqrt{x^5}. \\ &= \frac{2\sqrt{5} \cdot x^2\sqrt{x}}{\sqrt{5x}} && \sqrt{4} = 2 \\ &= \frac{2x^2\sqrt{5x}}{\sqrt{5x}} && \text{Product rule} \\ &= 2x^2 && \text{Divide out the common factor.} \end{aligned}$$

Both methods give the same result. In this case, the second method was longer. Sometimes, however, this method *can* be more efficient. ■

You Try 8

Perform the indicated operation and simplify completely.

a) $\sqrt{2n^3} \cdot \sqrt{6n}$ b) $\sqrt{15cd^5} \cdot \sqrt{3c^2d}$ c) $\dfrac{\sqrt{128k^9}}{\sqrt{2k}}$

Answers to You Try Exercises

1) a) $\sqrt{30}$ b) $\sqrt{10r}$ 2) a) $2\sqrt{7}$ b) $5\sqrt{3}$ c) $6\sqrt{2}$ 3) a) $\frac{10}{13}$ b) 3 c) $5\sqrt{2}$ d) $\frac{\sqrt{11}}{6}$

4) a) y^5 b) $12p^8$ c) $\frac{3\sqrt{5}}{w^2}$ 5) a) $m^2\sqrt{m}$ b) $z^9\sqrt{z}$ 6) a) $m^6\sqrt{m}$ b) $10v^3\sqrt{v}$ c) $4a\sqrt{2a}$

7) a) $c^2d^6\sqrt{c}$ b) $3x^5y^4\sqrt{3y}$ c) $\frac{2u^6\sqrt{10u}}{v^{10}}$ 8) a) $2n^2\sqrt{3}$ b) $3cd^3\sqrt{5c}$ c) $8k^4$

10.3 Exercises

Unless otherwise stated, assume all variables represent positive real numbers.

Objective 1: Multiply Square Roots

Multiply and simplify.

1) $\sqrt{3}\cdot\sqrt{7}$ 2) $\sqrt{11}\cdot\sqrt{5}$

3) $\sqrt{10}\cdot\sqrt{3}$ 4) $\sqrt{7}\cdot\sqrt{2}$

5) $\sqrt{6}\cdot\sqrt{y}$ 6) $\sqrt{5}\cdot\sqrt{p}$

Objective 2: Simplify the Square Root of a Whole Number

Label each statement as true or false. Give a reason for your answer.

7) $\sqrt{20}$ is in simplest form.

8) $\sqrt{35}$ is in simplest form.

9) $\sqrt{42}$ is in simplest form.

10) $\sqrt{63}$ is in simplest form.

Simplify completely.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

11) $\sqrt{60} = \sqrt{4\cdot 15}$ _____
$= $ ________ Product rule
$= $ _____ Simplify.

12) $\sqrt{200} = $ _____ Factor.
$= \sqrt{100}\cdot\sqrt{2}$ _________
$= $ _____ Simplify.

Simplify completely. If the radical is already simplified, then say so.

VIDEO 13) $\sqrt{20}$ 14) $\sqrt{12}$

15) $\sqrt{54}$ 16) $\sqrt{63}$

17) $\sqrt{33}$ 18) $\sqrt{15}$

19) $\sqrt{80}$ 20) $\sqrt{108}$

21) $\sqrt{98}$ 22) $\sqrt{96}$

23) $\sqrt{38}$ 24) $\sqrt{46}$

25) $\sqrt{400}$ 26) $\sqrt{900}$

27) $\sqrt{750}$ 28) $\sqrt{420}$

Objective 3: Use the Quotient Rule for Square Roots

Simplify completely.

29) $\sqrt{\frac{144}{25}}$ 30) $\sqrt{\frac{16}{81}}$

31) $\frac{\sqrt{4}}{\sqrt{49}}$ 32) $\frac{\sqrt{64}}{\sqrt{121}}$

33) $\frac{\sqrt{54}}{\sqrt{6}}$ 34) $\frac{\sqrt{48}}{\sqrt{3}}$

VIDEO 35) $\sqrt{\frac{60}{5}}$ 36) $\sqrt{\frac{40}{5}}$

37) $\frac{\sqrt{120}}{\sqrt{6}}$ 38) $\frac{\sqrt{54}}{\sqrt{3}}$

39) $\frac{\sqrt{30}}{\sqrt{2}}$ 40) $\frac{\sqrt{35}}{\sqrt{5}}$

41) $\sqrt{\frac{6}{49}}$ 42) $\sqrt{\frac{2}{81}}$

43) $\sqrt{\frac{45}{16}}$ 44) $\sqrt{\frac{60}{49}}$

Objective 4: Simplify Square Root Expressions Containing Variables with Even Exponents

Simplify completely.

45) $\sqrt{x^8}$ 46) $\sqrt{q^6}$

47) $\sqrt{w^{14}}$ 48) $\sqrt{t^{16}}$

49) $\sqrt{100c^2}$ 50) $\sqrt{9z^8}$

51) $\sqrt{64k^6m^{10}}$

52) $\sqrt{25p^{20}q^{14}}$

53) $\sqrt{28r^4}$

54) $\sqrt{27z^{12}}$

55) $\sqrt{300q^{22}t^{16}}$

56) $\sqrt{50n^4y^4}$

57) $\sqrt{\dfrac{81}{c^6}}$

58) $\sqrt{\dfrac{h^2}{169}}$

59) $\dfrac{\sqrt{40}}{\sqrt{t^8}}$

60) $\dfrac{\sqrt{18}}{\sqrt{m^{30}}}$

61) $\sqrt{\dfrac{75x^2}{y^{12}}}$

62) $\sqrt{\dfrac{44}{w^2z^{18}}}$

Objective 5: Simplify Square Root Expressions Containing Variables with Odd Exponents

Simplify completely.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

63) $\sqrt{w^9} = \sqrt{w^8 \cdot w^1}$ _____

$= $ __________ Product rule

$= w^4\sqrt{w}$ _____

64) $\sqrt{z^{19}} = \sqrt{z^{18} \cdot z^1}$ _____

$= \sqrt{z^{18}} \cdot \sqrt{z^1}$ __________

$= $ _____ Simplify.

Simplify completely.

65) $\sqrt{a^5}$

66) $\sqrt{c^7}$

67) $\sqrt{g^{13}}$

68) $\sqrt{k^{15}}$

69) $\sqrt{b^{25}}$

70) $\sqrt{h^{31}}$

71) $\sqrt{72x^3}$

72) $\sqrt{100a^5}$

73) $\sqrt{13q^7}$

74) $\sqrt{20c^9}$

75) $\sqrt{75t^{11}}$

76) $\sqrt{45p^{17}}$

77) $\sqrt{c^8d^2}$

78) $\sqrt{r^4s^{12}}$

79) $\sqrt{a^4b^3}$

80) $\sqrt{x^2y^9}$

81) $\sqrt{u^5v^7}$

82) $\sqrt{f^3g^9}$

83) $\sqrt{36m^9n^4}$

84) $\sqrt{4t^6u^5}$

85) $\sqrt{44x^{12}y^5}$

86) $\sqrt{63c^7d^4}$

87) $\sqrt{32t^5u^7}$

88) $\sqrt{125k^3l^9}$

89) $\sqrt{\dfrac{a^7}{81b^6}}$

90) $\sqrt{\dfrac{x^5}{49y^6}}$

91) $\sqrt{\dfrac{3r^9}{s^2}}$

92) $\sqrt{\dfrac{17h^{11}}{k^8}}$

Objective 6: Simplify More Square Root Expressions Containing Variables

Perform the indicated operation and simplify.

93) $\sqrt{5} \cdot \sqrt{10}$

94) $\sqrt{8} \cdot \sqrt{6}$

95) $\sqrt{21} \cdot \sqrt{3}$

96) $\sqrt{2} \cdot \sqrt{14}$

97) $\sqrt{w} \cdot \sqrt{w^5}$

98) $\sqrt{d^3} \cdot \sqrt{d^{11}}$

99) $\sqrt{n^3} \cdot \sqrt{n^4}$

100) $\sqrt{a^{10}} \cdot \sqrt{a^3}$

101) $\sqrt{2k} \cdot \sqrt{8k^5}$

102) $\sqrt{5z^9} \cdot \sqrt{5z^3}$

103) $\sqrt{6x^4y^3} \cdot \sqrt{2x^5y^2}$

104) $\sqrt{5a^6b^5} \cdot \sqrt{10ab^4}$

105) $\sqrt{8c^9d^2} \cdot \sqrt{5cd^7}$

106) $\sqrt{6t^3u^3} \cdot \sqrt{3t^7u^4}$

107) $\dfrac{\sqrt{18k^{11}}}{\sqrt{2k^3}}$

108) $\dfrac{\sqrt{48m^{15}}}{\sqrt{3m^9}}$

109) $\dfrac{\sqrt{120h^8}}{\sqrt{3h^2}}$

110) $\dfrac{\sqrt{72c^{10}}}{\sqrt{6c^2}}$

111) $\dfrac{\sqrt{50a^{16}b^9}}{\sqrt{5a^7b^4}}$

112) $\dfrac{\sqrt{21y^8z^{18}}}{\sqrt{3yz^{13}}}$

113) The velocity v of a moving object can be determined from its mass m and its kinetic energy KE using the formula $v = \sqrt{\dfrac{2KE}{m}}$, where the velocity is in meters/second, the mass is in kilograms, and the KE is measured in joules. A 600-kg roller coaster car is moving along a track and has kinetic energy of 120,000 joules. What is the velocity of the car?

114) The length of a side s of an equilateral triangle is a function of its area A and can be described by $s(A) = \sqrt{\dfrac{4\sqrt{3}A}{3}}$. If an equilateral triangle has an area of $6\sqrt{3}$ cm^2, how long is each side of the triangle?

Section 10.4 Simplifying Expressions Containing Higher Roots

Objectives

1. Multiply Higher Roots
2. Simplify Higher Roots of Integers
3. Use the Quotient Rule for Higher Roots
4. Simplify Radicals Containing Variables
5. Multiply and Divide Radicals with Different Indices

In Section 10.1, we first discussed finding higher roots like $\sqrt[4]{16} = 2$ and $\sqrt[3]{-27} = -3$. In this section, we will extend what we learned about multiplying, dividing, and simplifying *square* roots to doing the same with higher roots.

1. Multiply Higher Roots

Definition Product Rule for Higher Roots

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$$

This rule enables us to multiply and simplify radicals with any index in a way that is similar to multiplying and simplifying square roots.

Example 1

Multiply.

a) $\sqrt[3]{2} \cdot \sqrt[3]{7}$ b) $\sqrt[4]{t} \cdot \sqrt[4]{10}$

Solution

a) $\sqrt[3]{2} \cdot \sqrt[3]{7} = \sqrt[3]{2 \cdot 7} = \sqrt[3]{14}$ b) $\sqrt[4]{t} \cdot \sqrt[4]{10} = \sqrt[4]{t \cdot 10} = \sqrt[4]{10t}$ ■

You Try 1

Multiply.

a) $\sqrt[4]{6} \cdot \sqrt[4]{5}$ b) $\sqrt[5]{8} \cdot \sqrt[5]{k^2}$

BE CAREFUL

Remember that we can apply the product rule in this way *only* if the indices of the radicals are the same. Later in this section, we will discuss how to multiply radicals with different indices.

2. Simplify Higher Roots of Integers

In Section 10.3, we said that a simplified *square root* cannot contain any *perfect squares*. Next we list the conditions that determine when a radical with *any* index is in simplest form.

Property When Is a Radical Simplified?

Let P be an expression and let n be an integer greater than 1. Then $\sqrt[n]{P}$ is completely simplified when all of the following conditions are met:

1) The radicand does not contain any factors (other than 1) that are perfect nth powers.
2) The exponents in the radicand and the index of the radical do not have any common factors (other than 1).
3) The radicand does not contain any fractions.
4) There are no radicals in the denominator of a fraction.

Note

Condition 1) implies that the radical cannot contain variables with exponents greater than or equal to n, the index of the radical.

To simplify radicals with any index, use the product rule $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$, where a or b is an nth power.

Remember, to be certain that a radical is simplified completely, always look at the radical carefully and ask yourself, "*Is the radical in simplest form?*"

Example 2

Simplify completely.

a) $\sqrt[3]{250}$ b) $\sqrt[4]{48}$

Solution

a) We will look at two methods for simplifying $\sqrt[3]{250}$.

i) Since we must simplify the *cube* root of 250, think of two numbers that multiply to 250 so that at least one of the numbers is a *perfect cube:* $250 = 125 \cdot 2$

$$\begin{aligned} \sqrt[3]{250} &= \sqrt[3]{125 \cdot 2} && \text{125 is a perfect cube.} \\ &= \sqrt[3]{125} \cdot \sqrt[3]{2} && \text{Product rule} \\ &= 5\sqrt[3]{2} && \sqrt[3]{125} = 5 \end{aligned}$$

Is $5\sqrt[3]{2}$ in simplest form? Yes, because 2 does not have any factors that are perfect cubes.

ii) Use a factor tree to find the prime factorization of 250: $250 = 2 \cdot 5^3$

$$\begin{aligned} \sqrt[3]{250} &= \sqrt[3]{2 \cdot 5^3} && 2 \cdot 5^3 \text{ is the prime factorization of 250.} \\ &= \sqrt[3]{2} \cdot \sqrt[3]{5^3} && \text{Product rule} \\ &= \sqrt[3]{2} \cdot 5 && \sqrt[3]{5^3} = 5 \\ &= 5\sqrt[3]{2} && \text{Commutative property} \end{aligned}$$

We obtain the same result using either method.

b) We will use two methods for simplifying $\sqrt[4]{48}$.

i) Since we must simplify the *fourth* root of 48, think of two numbers that multiply to 48 so that at least one of the numbers is a *perfect fourth power:* $48 = 16 \cdot 3$

$$\begin{aligned} \sqrt[4]{48} &= \sqrt[4]{16 \cdot 3} && \text{16 is a perfect fourth power.} \\ &= \sqrt[4]{16} \cdot \sqrt[4]{3} && \text{Product rule} \\ &= 2\sqrt[4]{3} && \sqrt[4]{16} = 2 \end{aligned}$$

Is $2\sqrt[4]{3}$ in simplest form? Yes, because 3 does not have any factors that are perfect fourth powers.

ii) Use a factor tree to find the prime factorization of 48: $48 = 2^4 \cdot 3$

$$\begin{aligned} \sqrt[4]{48} &= \sqrt[4]{2^4 \cdot 3} && 2^4 \cdot 3 \text{ is the prime factorization of 48.} \\ &= \sqrt[4]{2^4} \cdot \sqrt[4]{3} && \text{Product rule} \\ &= 2\sqrt[4]{3} && \sqrt[4]{2^4} = 2 \end{aligned}$$

Once again, both methods give us the same result. ■

You Try 2

Simplify completely.

a) $\sqrt[3]{40}$ b) $\sqrt[5]{63}$

3. Use the Quotient Rule for Higher Roots

Definition Quotient Rule for Higher Roots

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and n is a natural number, then

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

We apply the quotient rule when working with nth roots the same way we apply it when working with square roots.

Example 3

Simplify completely.

a) $\sqrt[3]{-\frac{81}{3}}$ b) $\frac{\sqrt[3]{96}}{\sqrt[3]{2}}$

Solution

a) We can think of $-\frac{81}{3}$ as $\frac{-81}{3}$ or $\frac{81}{-3}$. Let's think of it as $\frac{-81}{3}$.

Neither -81 nor 3 is a perfect cube, but if we simplify $\frac{-81}{3}$ we get -27, which *is* a perfect cube.

$$\sqrt[3]{-\frac{81}{3}} = \sqrt[3]{-27} = -3$$

b) Let's begin by applying the quotient rule to obtain a fraction under *one* radical, then simplify the fraction.

$$\begin{aligned} \frac{\sqrt[3]{96}}{\sqrt[3]{2}} &= \sqrt[3]{\frac{96}{2}} && \text{Quotient rule} \\ &= \sqrt[3]{48} && \text{Simplify } \frac{96}{2}. \\ &= \sqrt[3]{8 \cdot 6} && \text{8 is a perfect cube.} \\ &= \sqrt[3]{8} \cdot \sqrt[3]{6} && \text{Product rule} \\ &= 2\sqrt[3]{6} && \sqrt[3]{8} = 2 \end{aligned}$$

Is $2\sqrt[3]{6}$ in simplest form? Yes, because 6 does not have any factors that are perfect cubes.

You Try 3

Simplify completely.

a) $\sqrt[4]{\frac{1}{81}}$ b) $\frac{\sqrt[3]{162}}{\sqrt[3]{3}}$

4. Simplify Radicals Containing Variables

In Section 10.2, we discussed the relationship between radical notation and fractional exponents. Recall that

Property $\sqrt[n]{a^m}$

If a is a nonnegative number and m and n are integers such that $n > 1$, then

$$\sqrt[n]{a^m} = a^{m/n}.$$

That is, the index of the radical becomes the denominator of the fractional exponent, and the power in the radicand becomes the numerator of the fractional exponent.

This is the principle we use to simplify radicals with indices greater than 2.

Example 4

Simplify completely.

a) $\sqrt[3]{y^{15}}$ b) $\sqrt[4]{16t^{24}u^8}$ c) $\sqrt[5]{\dfrac{c^{10}}{d^{30}}}$

Solution

a) $\sqrt[3]{y^{15}} = y^{15/3} = y^5$

b) $\sqrt[4]{16t^{24}u^8} = \sqrt[4]{16} \cdot \sqrt[4]{t^{24}} \cdot \sqrt[4]{u^8}$ Product rule

$= 2 \cdot t^{24/4} \cdot u^{8/4}$ Write with rational exponents.

$= 2t^6u^2$ Simplify exponents.

c) $\sqrt[5]{\dfrac{c^{10}}{d^{30}}} = \dfrac{\sqrt[5]{c^{10}}}{\sqrt[5]{d^{30}}} = \dfrac{c^{10/5}}{d^{30/5}} = \dfrac{c^2}{d^6}$ Quotient rule; use rational exponents. ■

You Try 4

Simplify completely.

a) $\sqrt[3]{a^3b^{21}}$ b) $\sqrt[4]{\dfrac{m^{12}}{16n^{20}}}$

To simplify a radical expression if the power in the radicand does not divide evenly by the index, we use the same methods we used in Section 10.3 for simplifying similar expressions with square roots. We can use the product rule or we can use the idea of quotient and remainder in a division problem.

Example 5

Simplify $\sqrt[4]{x^{23}}$ completely in two ways: i) use the product rule and ii) divide the exponent by the index and use the quotient and remainder.

Solution

i) Using the product rule:
To simplify $\sqrt[4]{x^{23}}$, write x^{23} as the product of two factors so that the exponent of one of the factors is the *largest* number less than 23 that is divisible by 4 (the index).

$\sqrt[4]{x^{23}} = \sqrt[4]{x^{20} \cdot x^3}$ 20 is the largest number less than 23 that is divisible by 4.

$= \sqrt[4]{x^{20}} \cdot \sqrt[4]{x^3}$ Product rule

$= x^{20/4} \cdot \sqrt[4]{x^3}$ Use a fractional exponent to simplify.

$= x^5\sqrt[4]{x^3}$ $20 \div 4 = 5$

ii) Using the quotient and remainder:

To simplify $\sqrt[4]{x^{23}}$, divide

$$\begin{array}{r} 5 \leftarrow \text{Quotient} \\ 4\overline{)\ 23} \\ -20 \\ \hline 3 \leftarrow \text{Remainder} \end{array}$$

Recall from our work with square roots in Section 10.3 that

i) the exponent on the variable *outside* of the radical will be the *quotient* of the division problem,

and

ii) the exponent on the variable *inside* of the radical will be the *remainder* of the division problem.

$$\sqrt[4]{x^{23}} = x^5\sqrt[4]{x^3}$$

Is $x^5\sqrt[4]{x^3}$ in simplest form? Yes, because the exponent inside of the radical is less than the index, and they contain no common factors other than 1. ■

You Try 5

Simplify $\sqrt[5]{r^{32}}$ completely using both methods shown in Example 5.

We can apply the product and quotient rules together with the methods above to simplify certain radical expressions.

Example 6

Completely simplify $\sqrt[3]{56a^{16}b^8}$.

Solution

$$\sqrt[3]{56a^{16}b^8} = \sqrt[3]{56} \cdot \sqrt[3]{a^{16}} \cdot \sqrt[3]{b^8} \quad \text{Product rule}$$

$$= \sqrt[3]{8} \cdot \sqrt[3]{7} \cdot a^5\sqrt[3]{a^1} \cdot b^2\sqrt[3]{b^2}$$

Product rule

16 ÷ 3 gives a quotient of 5 and a remainder of 1.

8 ÷ 3 gives a quotient of 2 and a remainder of 2.

$$= 2\sqrt[3]{7} \cdot a^5\sqrt[3]{a} \cdot b^2\sqrt[3]{b^2} \quad \text{Simplify } \sqrt[3]{8}.$$

$$= 2a^5b^2 \cdot \sqrt[3]{7} \cdot \sqrt[3]{a} \cdot \sqrt[3]{b^2} \quad \text{Use the commutative property to rewrite the expression.}$$

$$= 2a^5b^2\sqrt[3]{7ab^2} \quad \text{Product rule}$$ ■

You Try 6

Simplify completely.

a) $\sqrt[4]{48x^{15}y^{22}}$ b) $\sqrt[3]{\dfrac{r^{19}}{27s^{12}}}$

5. Multiply and Divide Radicals with Different Indices

The product and quotient rules for radicals apply only when the radicals have the *same* indices. To multiply or divide radicals with *different* indices, we first change the radical expressions to rational exponent form.

Example 7

Multiply the expressions, and write the answer in simplest radical form.

$$\sqrt[3]{x^2} \cdot \sqrt{x}$$

Solution

The indices of $\sqrt[3]{x^2}$ and $\sqrt{x}$ are different, so we *cannot* use the product rule right now. Rewrite each radical as a fractional exponent, use the product rule for *exponents*, then convert the answer back to radical form.

$$\begin{aligned} \sqrt[3]{x^2} \cdot \sqrt{x} &= x^{2/3} \cdot x^{1/2} && \text{Change radicals to fractional exponents.} \\ &= x^{4/6} \cdot x^{3/6} && \text{Get a common denominator to add exponents.} \\ &= x^{\frac{4}{6}+\frac{3}{6}} = x^{7/6} && \text{Add exponents.} \\ &= \sqrt[6]{x^7} = x\sqrt[6]{x} && \text{Rewrite in radical form, and simplify.} \end{aligned}$$

You Try 7

Perform the indicated operation, and write the answer in simplest radical form.

a) $\sqrt[4]{y} \cdot \sqrt[6]{y}$ b) $\dfrac{\sqrt[3]{c^2}}{\sqrt{c}}$

Answers to You Try Exercises

1) a) $\sqrt[4]{30}$ b) $\sqrt[5]{8k^2}$ 2) a) $2\sqrt[3]{5}$ b) simplified 3) a) $\frac{1}{3}$ b) $3\sqrt[3]{2}$ 4) a) ab^7 b) $\frac{m^3}{2n^5}$

5) $r^6\sqrt[5]{r^2}$ 6) a) $2x^3y^5\sqrt[4]{3x^3y^2}$ b) $\frac{r^6\sqrt[3]{r}}{3s^4}$ 7) a) $\sqrt[12]{y^5}$ b) $\sqrt[6]{c}$

10.4 Exercises

Mixed Exercises: Objectives 1–3

1) In your own words, explain the product rule for radicals.

2) In your own words, explain the quotient rule for radicals.

3) How do you know that a radical expression containing a cube root is completely simplified?

4) How do you know that a radical expression containing a fourth root is completely simplified?

Assume all variables represent positive real numbers.

Objective 1: Multiply Higher Roots

Multiply.

5) $\sqrt[3]{5} \cdot \sqrt[3]{4}$
6) $\sqrt[5]{6} \cdot \sqrt[5]{2}$
7) $\sqrt[5]{9} \cdot \sqrt[5]{m^2}$
8) $\sqrt[4]{11} \cdot \sqrt[4]{h^3}$
9) $\sqrt[3]{a^2} \cdot \sqrt[3]{b}$
10) $\sqrt[5]{t^2} \cdot \sqrt[5]{u^4}$

Objectives 2 and 3

Simplify completely.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

11) $\sqrt[3]{56} = \sqrt[3]{8 \cdot 7}$ ______
______ Product rule
______ Simplify.

12) $\sqrt[4]{80} = \sqrt[4]{16 \cdot 5}$ ______
$= \sqrt[4]{16} \cdot \sqrt[4]{5}$ ______
______ Simplify.

Simplify completely.

VIDEO 13) $\sqrt[3]{24}$
14) $\sqrt[3]{48}$
15) $\sqrt[4]{64}$
16) $\sqrt[4]{32}$

17) $\sqrt[3]{54}$
18) $\sqrt[3]{88}$
19) $\sqrt[3]{2000}$
20) $\sqrt[3]{108}$
21) $\sqrt[5]{64}$
22) $\sqrt[4]{162}$
23) $\sqrt[4]{\dfrac{1}{16}}$
24) $\sqrt[3]{\dfrac{1}{125}}$
25) $\sqrt[3]{-\dfrac{54}{2}}$
26) $\sqrt[4]{\dfrac{48}{3}}$
27) $\dfrac{\sqrt[3]{48}}{\sqrt[3]{2}}$
28) $\dfrac{\sqrt[3]{500}}{\sqrt[3]{2}}$
29) $\dfrac{\sqrt[4]{240}}{\sqrt[4]{3}}$
30) $\dfrac{\sqrt[3]{8000}}{\sqrt[3]{4}}$

Objective 4: Simplify Radicals Containing Variables

Simplify completely.

31) $\sqrt[3]{d^6}$
32) $\sqrt[3]{g^9}$
33) $\sqrt[4]{n^{20}}$
34) $\sqrt[4]{t^{36}}$
35) $\sqrt[5]{x^5y^{15}}$
36) $\sqrt[6]{a^{12}b^6}$
37) $\sqrt[3]{w^{14}}$
38) $\sqrt[3]{b^{19}}$
39) $\sqrt[4]{y^9}$
40) $\sqrt[4]{m^7}$
41) $\sqrt[3]{d^5}$
42) $\sqrt[5]{c^{29}}$
43) $\sqrt[3]{u^{10}v^{15}}$
44) $\sqrt[3]{x^9y^{16}}$
45) $\sqrt[3]{b^{16}c^5}$
46) $\sqrt[4]{r^{15}s^9}$
47) $\sqrt[4]{m^3n^{18}}$
48) $\sqrt[3]{a^{11}b}$
49) $\sqrt[3]{24x^{10}y^{12}}$
50) $\sqrt[3]{54y^{10}z^{24}}$
51) $\sqrt[3]{250w^4x^{16}}$
52) $\sqrt[3]{72t^{17}u^7}$
53) $\sqrt[4]{\dfrac{m^8}{81}}$
54) $\sqrt[4]{\dfrac{16}{x^{12}}}$
55) $\sqrt[5]{\dfrac{32a^{23}}{b^{15}}}$
56) $\sqrt[3]{\dfrac{h^{17}}{125k^{21}}}$
57) $\sqrt[4]{\dfrac{t^9}{81s^{24}}}$
58) $\sqrt[5]{\dfrac{32c^9}{d^{20}}}$
59) $\sqrt[3]{\dfrac{u^{28}}{v^3}}$
60) $\sqrt[4]{\dfrac{m^{13}}{n^8}}$

Perform the indicated operation and simplify.

61) $\sqrt[3]{6}\cdot\sqrt[3]{4}$
62) $\sqrt[3]{4}\cdot\sqrt[3]{10}$
63) $\sqrt[3]{9}\cdot\sqrt[3]{12}$
64) $\sqrt[3]{9}\cdot\sqrt[3]{6}$
65) $\sqrt[3]{20}\cdot\sqrt[3]{4}$
66) $\sqrt[3]{28}\cdot\sqrt[3]{2}$
67) $\sqrt[3]{m^4}\cdot\sqrt[3]{m^5}$
68) $\sqrt[3]{t^5}\cdot\sqrt[3]{t}$
69) $\sqrt[4]{k^7}\cdot\sqrt[4]{k^9}$
70) $\sqrt[4]{a^9}\cdot\sqrt[4]{a^{11}}$
71) $\sqrt[3]{r^7}\cdot\sqrt[3]{r^4}$
72) $\sqrt[3]{y^2}\cdot\sqrt[3]{y^{17}}$
73) $\sqrt[5]{p^{14}}\cdot\sqrt[5]{p^9}$
74) $\sqrt[5]{c^{17}}\cdot\sqrt[5]{c^9}$
75) $\sqrt[3]{9z^{11}}\cdot\sqrt[3]{3z^8}$
76) $\sqrt[3]{2h^4}\cdot\sqrt[3]{4h^{16}}$
77) $\sqrt[3]{\dfrac{h^{14}}{h^2}}$
78) $\sqrt[3]{\dfrac{a^{20}}{a^{14}}}$
79) $\sqrt[3]{\dfrac{c^{11}}{c^4}}$
80) $\sqrt[3]{\dfrac{z^{16}}{z^5}}$
81) $\sqrt[4]{\dfrac{162d^{21}}{2d^2}}$
82) $\sqrt[4]{\dfrac{48t^{11}}{3t^6}}$

Objective 5: Multiply and Divide Radicals with Different Indices

The following radical expressions do not have the same indices. Perform the indicated operation, and write the answer in simplest radical form.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

83) $\sqrt{a}\cdot\sqrt[4]{a^3} = a^{1/2}\cdot a^{3/4}$ ________________
$= a^{2/4}\cdot a^{3/4}$ ________________
$=$ ______ Add exponents.
$= \sqrt[4]{a^5}$ ________________
$=$ ____ Simplify.

84) $\sqrt[5]{r^4}\cdot\sqrt[3]{r^2} =$ ______ Change radicals to fractional exponents.
$=$ ________ Rewrite exponents with a common denominator.
$= r^{22/15}$ ________________
$=$ ____ Rewrite in radical form.
$=$ ____ Simplify.

85) $\sqrt{p}\cdot\sqrt[3]{p}$
86) $\sqrt[3]{y^2}\cdot\sqrt[4]{y}$
87) $\sqrt[4]{n^3}\cdot\sqrt{n}$
88) $\sqrt[5]{k^4}\cdot\sqrt{k}$
89) $\sqrt[5]{c^3}\cdot\sqrt[3]{c^2}$
90) $\sqrt[3]{a^2}\cdot\sqrt[5]{a^2}$
91) $\dfrac{\sqrt{w}}{\sqrt[4]{w}}$
92) $\dfrac{\sqrt[4]{m^3}}{\sqrt{m}}$
93) $\dfrac{\sqrt[5]{t^4}}{\sqrt[3]{t^2}}$
94) $\dfrac{\sqrt[4]{h^3}}{\sqrt[3]{h^2}}$

95) A block of candle wax in the shape of a cube has a volume of 64 in^3. The length of a side of the block, s, is given by $s = \sqrt[3]{V}$, where V is the volume of the block of candle wax. How long is each side of the block?

96) The radius $r(V)$ of a sphere is a function of its volume V and can be described by the function $r(V) = \sqrt[3]{\dfrac{3V}{4\pi}}$. If a spherical water tank has a volume of $\dfrac{256\pi}{3}$ ft^3, what is the radius of the tank?

Section 10.5 Adding, Subtracting, and Multiplying Radicals

Objectives

1. Add and Subtract Radical Expressions
2. Simplify Before Adding and Subtracting
3. Multiply a Binomial Containing Radical Expressions by a Monomial
4. Multiply Radical Expressions Using FOIL
5. Square a Binomial Containing Radical Expressions
6. Multiply Radical Expressions of the Form $(a + b)(a - b)$

Just as we can add and subtract like terms such as $4x + 6x = 10x$, we can add and subtract *like radicals* such as $4\sqrt{3} + 6\sqrt{3}$.

Definition

Like radicals have the same index and the same radicand.

Some examples of like radicals are

$$4\sqrt{3} \text{ and } 6\sqrt{3}, \qquad -\sqrt[3]{5} \text{ and } 8\sqrt[3]{5}, \qquad \sqrt{x} \text{ and } 7\sqrt{x}, \qquad 2\sqrt[3]{a^2b} \text{ and } \sqrt[3]{a^2b}$$

In this section, assume that all variables represent nonnegative real numbers.

1. Add and Subtract Radical Expressions

Property Adding and Subtracting Radicals

In order to add or subtract radicals, they must be *like* radicals.

We add and subtract like radicals in the same way we add and subtract like terms—add or subtract the "coefficients" of the radicals and multiply that result by the radical. We are using the distributive property when we combine like terms in this way.

Example 1

Perform the operations and simplify.

a) $4x + 6x$ b) $4\sqrt{3} + 6\sqrt{3}$ c) $\sqrt[4]{5} - 9\sqrt[4]{5}$ d) $3\sqrt{2} + 4\sqrt{3}$

Solution

a) First notice that $4x$ and $6x$ are like terms. Therefore, they can be added.

$$\begin{aligned} 4x + 6x &= (4 + 6)x && \text{Distributive property} \\ &= 10x && \text{Simplify.} \end{aligned}$$

Or, we can say that by just adding the coefficients, $4x + 6x = 10x$.

b) Before attempting to add $4\sqrt{3}$ and $6\sqrt{3}$, we must be certain that they are like radicals. Since they *are* like, they can be added.

$$\begin{aligned} 4\sqrt{3} + 6\sqrt{3} &= (4 + 6)\sqrt{3} && \text{Distributive property} \\ &= 10\sqrt{3} && \text{Simplify.} \end{aligned}$$

Or, we can say that by just adding the coefficients of $\sqrt{3}$, we get $4\sqrt{3} + 6\sqrt{3} = 10\sqrt{3}$.

c) $\sqrt[4]{5} - 9\sqrt[4]{5} = 1\sqrt[4]{5} - 9\sqrt[4]{5} = (1 - 9)\sqrt[4]{5} = -8\sqrt[4]{5}$

d) The radicands in $3\sqrt{2} + 4\sqrt{3}$ are different, so these expressions cannot be combined.

You Try 1

Add.

a) $9c + 7c$ b) $9\sqrt{10} + 7\sqrt{10}$ c) $2\sqrt[3]{4} - 8\sqrt[3]{4}$
d) $5\sqrt{6} - 2\sqrt{3}$

Example 2

Perform the operations and simplify. $6\sqrt{x} + 11\sqrt[3]{x} + 2\sqrt{x} - 6\sqrt[3]{x}$

Solution

Begin by noticing that there are *two* different types of radicals: $\sqrt{x}$ and $\sqrt[3]{x}$. Write the like radicals together.

$$\begin{aligned} 6\sqrt{x} + 11\sqrt[3]{x} + 2\sqrt{x} - 6\sqrt[3]{x} &= 6\sqrt{x} + 2\sqrt{x} + 11\sqrt[3]{x} - 6\sqrt[3]{x} && \text{Commutative property} \\ &= (6+2)\sqrt{x} + (11-6)\sqrt[3]{x} && \text{Distributive property} \\ &= 8\sqrt{x} + 5\sqrt[3]{x} \end{aligned}$$

Is $8\sqrt{x} + 5\sqrt[3]{x}$ in simplest form? *Yes*. The radicals are not like (they have different indices) so they cannot be combined further. Also, each radical, $\sqrt{x}$ and $\sqrt[3]{x}$, is in simplest form.

You Try 2

Perform the operations and simplify. $8\sqrt[3]{2n} - 3\sqrt{2n} + 5\sqrt{2n} + 5\sqrt[3]{2n}$

2. Simplify Before Adding and Subtracting

Sometimes it looks like two radicals cannot be added or subtracted. But if the radicals can be *simplified* and they turn out to be *like* radicals, then we can add or subtract them.

Procedure Adding and Subtracting Radicals

1) Write each radical expression in simplest form.
2) Combine like radicals.

Example 3

Perform the operations and simplify.

a) $8\sqrt{2} + 3\sqrt{50} - \sqrt{45}$

b) $-7\sqrt[3]{40} + \sqrt[3]{5}$

c) $10\sqrt{8t} - 9\sqrt{2t}$

d) $\sqrt[3]{xy^6} + \sqrt[3]{x^7}$

Solution

a) The radicals $8\sqrt{2}$, $3\sqrt{50}$, and $\sqrt{45}$ are not like. The first radical is in simplest form, but $3\sqrt{50}$ and $\sqrt{45}$ should be simplified to determine whether any of the radicals can be combined.

$$\begin{aligned} 8\sqrt{2} + 3\sqrt{50} - \sqrt{45} &= 8\sqrt{2} + 3\sqrt{25 \cdot 2} - \sqrt{9 \cdot 5} && \text{Factor.} \\ &= 8\sqrt{2} + 3\sqrt{25} \cdot \sqrt{2} - \sqrt{9} \cdot \sqrt{5} && \text{Product rule} \\ &= 8\sqrt{2} + 3 \cdot 5 \cdot \sqrt{2} - 3\sqrt{5} && \text{Simplify radicals.} \\ &= 8\sqrt{2} + 15\sqrt{2} - 3\sqrt{5} && \text{Multiply.} \\ &= 23\sqrt{2} - 3\sqrt{5} && \text{Add like radicals.} \end{aligned}$$

b)

$$\begin{aligned} -7\sqrt[3]{40} + \sqrt[3]{5} &= -7\sqrt[3]{8 \cdot 5} + \sqrt[3]{5} && \text{8 is a perfect cube.} \\ &= -7\sqrt[3]{8} \cdot \sqrt[3]{5} + \sqrt[3]{5} && \text{Product rule} \\ &= -7 \cdot 2 \cdot \sqrt[3]{5} + \sqrt[3]{5} && \sqrt[3]{8} = 2 \\ &= -14\sqrt[3]{5} + \sqrt[3]{5} && \text{Multiply.} \\ &= -13\sqrt[3]{5} && \text{Add like radicals.} \end{aligned}$$

c) In the difference $10\sqrt{8t} - 9\sqrt{2t}$ the radical $\sqrt{2t}$ is simplified, but $\sqrt{8t}$ is not. We must simplify $\sqrt{8t}$:

$$\sqrt{8t} = \sqrt{8} \cdot \sqrt{t} = \sqrt{4} \cdot \sqrt{2} \cdot \sqrt{t} = 2\sqrt{2} \cdot \sqrt{t} = 2\sqrt{2t}$$

Substitute $2\sqrt{2t}$ for $\sqrt{8t}$ in the original expression.

$$\begin{aligned} 10\sqrt{8t} - 9\sqrt{2t} &= 10(2\sqrt{2t}) - 9\sqrt{2t} && \text{Substitute } 2\sqrt{2t} \text{ for } \sqrt{8t}. \\ &= 20\sqrt{2t} - 9\sqrt{2t} && \text{Multiply.} \\ &= 11\sqrt{2t} && \text{Subtract.} \end{aligned}$$

d) Each radical in the expression $\sqrt[3]{xy^6} + \sqrt[3]{x^7}$ must be simplified.

$$\sqrt[3]{xy^6} = \sqrt[3]{x} \cdot \sqrt[3]{y^6} = \sqrt[3]{x} \cdot y^2 = y^2\sqrt[3]{x} \qquad \Big| \qquad \sqrt[3]{x^7} = x^2\sqrt[3]{x^1}$$

$7 \div 3$ gives a quotient of 2 and a remainder of 1.

$$\begin{aligned} \sqrt[3]{xy^6} + \sqrt[3]{x^7} &= y^2\sqrt[3]{x} + x^2\sqrt[3]{x} && \text{Substitute the simplified radicals in the original expression.} \\ &= (y^2 + x^2)\sqrt[3]{x} && \text{Factor out } \sqrt[3]{x} \text{ from each term.} \end{aligned}$$

In this problem we cannot *add* $y^2\sqrt[3]{x} + x^2\sqrt[3]{x}$ like we added radicals in previous examples, but we *can* factor out $\sqrt[3]{x}$.

$(y^2 + x^2)\sqrt[3]{x}$ is the completely simplified form of the sum. ■

You Try 3

Perform the operations and simplify.

a) $7\sqrt{3} - \sqrt{12}$ b) $2\sqrt{63} - 11\sqrt{28} + 2\sqrt{21}$ c) $\sqrt[3]{54} + 5\sqrt[3]{16}$

d) $2\sqrt{6k} + 4\sqrt{54k}$ e) $\sqrt[4]{mn^{11}} + \sqrt[4]{81mn^3}$

In the rest of this section, we will learn how to simplify expressions that combine multiplication, addition, and subtraction of radicals.

3. Multiply a Binomial Containing Radical Expressions by a Monomial

Example 4

Multiply and simplify.

a) $4(\sqrt{5} - \sqrt{20})$ b) $\sqrt{2}(\sqrt{10} + \sqrt{15})$ c) $\sqrt{x}(\sqrt{x} + \sqrt{32y})$

Solution

a) Since $\sqrt{20}$ can be simplified, we will do that first.

$$\sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$$

Substitute $2\sqrt{5}$ for $\sqrt{20}$ in the original expression.

$$\begin{aligned} 4(\sqrt{5} - \sqrt{20}) &= 4(\sqrt{5} - 2\sqrt{5}) && \text{Substitute } 2\sqrt{5} \text{ for } \sqrt{20}. \\ &= 4(-\sqrt{5}) && \text{Subtract.} \\ &= -4\sqrt{5} && \text{Multiply.} \end{aligned}$$

b) Neither $\sqrt{10}$ nor $\sqrt{15}$ can be simplified. Begin by applying the distributive property.

$$\begin{aligned} \sqrt{2}(\sqrt{10} + \sqrt{15}) &= \sqrt{2} \cdot \sqrt{10} + \sqrt{2} \cdot \sqrt{15} && \text{Distribute.} \\ &= \sqrt{20} + \sqrt{30} && \text{Product rule} \end{aligned}$$

Is $\sqrt{20} + \sqrt{30}$ in simplest form? *No.* $\sqrt{20}$ can be simplified.

$$\sqrt{20} + \sqrt{30} = \sqrt{4 \cdot 5} + \sqrt{30} = \sqrt{4} \cdot \sqrt{5} + \sqrt{30} = 2\sqrt{5} + \sqrt{30}$$

c) In the expression $\sqrt{x}\,(\sqrt{x} + \sqrt{32y})$, $\sqrt{32y}$ can be simplified. Let's do that first.

$$\sqrt{32y} = \sqrt{32} \cdot \sqrt{y} = \sqrt{16 \cdot 2} \cdot \sqrt{y} = \sqrt{16} \cdot \sqrt{2} \cdot \sqrt{y} = 4\sqrt{2y}$$

Substitute $4\sqrt{2y}$ for $\sqrt{32y}$ in the original expression.

$$\begin{aligned} \sqrt{x}(\sqrt{x} + \sqrt{32y}) &= \sqrt{x}(\sqrt{x} + 4\sqrt{2y}) && \text{Substitute } 4\sqrt{2y} \text{ for } \sqrt{32y}. \\ &= \sqrt{x} \cdot \sqrt{x} + \sqrt{x} \cdot 4\sqrt{2y} && \text{Distribute.} \\ &= x + 4\sqrt{2xy} && \text{Multiply.} \end{aligned}$$

You Try 4

Multiply and simplify.

a) $6(\sqrt{75} + 2\sqrt{3})$ b) $\sqrt{3}(\sqrt{3} + \sqrt{21})$ c) $\sqrt{c}(\sqrt{c^3} - \sqrt{100d})$

4. Multiply Radical Expressions Using FOIL

In Chapter 6, we first multiplied binomials using **FOIL** (**F**irst **O**uter **I**nner **L**ast).

$$\begin{aligned} (2x + 3)(x + 4) &= 2x \cdot x + 2x \cdot 4 + 3 \cdot x + 3 \cdot 4 \\ &= \underset{\text{F}}{2x^2} + \underset{\text{O}}{8x} + \underset{\text{I}}{3x} + \underset{\text{L}}{12} \\ &= 2x^2 + 11x + 12 \end{aligned}$$

We can multiply binomials containing radicals the same way.

Example 5

Multiply and simplify.

a) $(2 + \sqrt{5})(4 + \sqrt{5})$ b) $(2\sqrt{3} + \sqrt{2})(\sqrt{3} - 5\sqrt{2})$

c) $(\sqrt{r} + \sqrt{3s})(\sqrt{r} + 8\sqrt{3s})$

Solution

a) Since we must multiply two binomials, we will use FOIL.

$$\begin{aligned} (2 + \sqrt{5})(4 + \sqrt{5}) &= 2 \cdot 4 + 2 \cdot \sqrt{5} + 4 \cdot \sqrt{5} + \sqrt{5} \cdot \sqrt{5} \\ &= \underset{\text{F}}{8} + \underset{\text{O}}{2\sqrt{5}} + \underset{\text{I}}{4\sqrt{5}} + \underset{\text{L}}{5} && \text{Multiply.} \\ &= 13 + 6\sqrt{5} && \text{Combine like terms.} \end{aligned}$$

b) $(2\sqrt{3} + \sqrt{2})(\sqrt{3} - 5\sqrt{2})$

$$\begin{aligned} &= \underset{\text{F}}{2\sqrt{3} \cdot \sqrt{3}} + \underset{\text{O}}{2\sqrt{3} \cdot (-5\sqrt{2})} + \underset{\text{I}}{\sqrt{2} \cdot \sqrt{3}} + \underset{\text{L}}{\sqrt{2} \cdot (-5\sqrt{2})} \\ &= 2 \cdot 3 + (-10\sqrt{6}) + \sqrt{6} + (-5 \cdot 2) && \text{Multiply.} \\ &= 6 - 10\sqrt{6} + \sqrt{6} - 10 && \text{Multiply.} \\ &= -4 - 9\sqrt{6} && \text{Combine like terms.} \end{aligned}$$

c) $(\sqrt{r} + \sqrt{3s})(\sqrt{r} + 8\sqrt{3s})$

$$\begin{aligned} &= \underset{\text{F}}{\sqrt{r} \cdot \sqrt{r}} + \underset{\text{O}}{\sqrt{r} \cdot 8\sqrt{3s}} + \underset{\text{I}}{\sqrt{3s} \cdot \sqrt{r}} + \underset{\text{L}}{\sqrt{3s} \cdot 8\sqrt{3s}} \\ &= r + 8\sqrt{3rs} + \sqrt{3rs} + 8 \cdot 3s && \text{Multiply.} \\ &= r + 8\sqrt{3rs} + \sqrt{3rs} + 24s && \text{Multiply.} \\ &= r + 9\sqrt{3rs} + 24s && \text{Combine like terms.} \end{aligned}$$

You Try 5

Multiply and simplify.

a) $(6 - \sqrt{7})(5 + \sqrt{7})$ b) $(\sqrt{2} + 4\sqrt{5})(3\sqrt{2} + \sqrt{5})$

c) $(\sqrt{6p} - \sqrt{2q})(\sqrt{6p} - 3\sqrt{2q})$

5. Square a Binomial Containing Radical Expressions

Recall, again, from Chapter 6, that we can use FOIL to square a binomial or we can use these special formulas:

$$(a + b)^2 = a^2 + 2ab + b^2 \qquad (a - b)^2 = a^2 - 2ab + b^2$$

For example,

$$\begin{aligned}(k + 7)^2 &= (k)^2 + 2(k)(7) + (7)^2 \\ &= k^2 + 14k + 49\end{aligned} \quad \text{and} \quad \begin{aligned}(2p - 5)^2 &= (2p)^2 - 2(2p)(5) + (5)^2 \\ &= 4p^2 - 20p + 25\end{aligned}$$

To square a binomial containing radicals, we can either use FOIL or we can use the formulas above. Understanding how to use the special formulas to square a binomial will make it easier to solve radical equations in Section 10.7.

Example 6

Multiply and simplify.

a) $(\sqrt{10} + 3)^2$ b) $(2\sqrt{x} - 6)^2$

Solution

a) Use $(a + b)^2 = a^2 + 2ab + b^2$.

$$\begin{aligned}(\sqrt{10} + 3)^2 &= (\sqrt{10})^2 + 2(\sqrt{10})(3) + (3)^2 && \text{Substitute } \sqrt{10} \text{ for } a \text{ and 3 for } b. \\ &= 10 + 6\sqrt{10} + 9 && \text{Multiply.} \\ &= 19 + 6\sqrt{10} && \text{Combine like terms.}\end{aligned}$$

b) Use $(a - b)^2 = a^2 - 2ab + b^2$.

$$\begin{aligned}(2\sqrt{x} - 6)^2 &= (2\sqrt{x})^2 - 2(2\sqrt{x})(6) + (6)^2 && \text{Substitute } 2\sqrt{x} \text{ for } a \text{ and 6 for } b. \\ &= (4 \cdot x) - (4\sqrt{x})(6) + 36 && \text{Multiply.} \\ &= 4x - 24\sqrt{x} + 36 && \text{Multiply.}\end{aligned}$$

You Try 6

Multiply and simplify.

a) $(\sqrt{6} + 5)^2$ b) $(3\sqrt{2} - 4)^2$ c) $(\sqrt{w} + \sqrt{11})^2$

6. Multiply Radical Expressions of the Form $(a + b)(a - b)$

We will review one last rule from Chapter 6 on multiplying binomials. We will use this in Section 10.7 when we divide radicals.

$$(a + b)(a - b) = a^2 - b^2$$

For example, $(t + 8)(t - 8) = (t)^2 - (8)^2 = t^2 - 64$.

The same rule applies when we multiply binomials containing radicals.

Example 7

Multiply and simplify. $(2\sqrt{x} + \sqrt{y})(2\sqrt{x} - \sqrt{y})$

Solution

Use $(a + b)(a - b) = a^2 - b^2$.

$$(2\sqrt{x} + \sqrt{y})(2\sqrt{x} - \sqrt{y}) = (2\sqrt{x})^2 - (\sqrt{y})^2$$ Substitute $2\sqrt{x}$ for a and $\sqrt{y}$ for b.

$$= 4(x) - y$$ Square each term.

$$= 4x - y$$ Simplify.

Note

When we multiply expressions of the form $(a + b)(a - b)$ containing square roots, the radicals are eliminated. *This will always be true.*

You Try 7

Multiply and simplify.

a) $(4 + \sqrt{10})(4 - \sqrt{10})$ b) $(\sqrt{5h} + \sqrt{k})(\sqrt{5h} - \sqrt{k})$

Answers to You Try Exercises

1) a) $16c$ b) $16\sqrt{10}$ c) $-6\sqrt[3]{4}$ d) $5\sqrt{6} - 2\sqrt{3}$ 2) $13\sqrt[3]{2n} + 2\sqrt{2n}$ 3) a) $5\sqrt{3}$ b) $-16\sqrt{7} + 2\sqrt{21}$ c) $13\sqrt[3]{2}$ d) $14\sqrt{6k}$ e) $(n^2 + 3)\sqrt[4]{mn^3}$ 4) a) $42\sqrt{3}$ b) $3 + 3\sqrt{7}$ c) $c^2 - 10\sqrt{cd}$ 5) a) $23 + \sqrt{7}$ b) $26 + 13\sqrt{10}$ c) $6p - 8\sqrt{3pq} + 6q$ 6) a) $31 + 10\sqrt{6}$ b) $34 - 24\sqrt{2}$ c) $w + 2\sqrt{11w} + 11$ 7) a) 6 b) $5h - k$

10.5 Exercises

Assume all variables represent nonnegative real numbers.

Objective 1: Add and Subtract Radical Expressions

1) How do you know whether two radicals are *like* radicals?

2) Are $5\sqrt{3}$ and $7\sqrt[3]{3}$ like radicals? Why or why not?

Perform the operations and simplify.

3) $5\sqrt{2} + 9\sqrt{2}$

4) $11\sqrt{7} + 7\sqrt{7}$

5) $7\sqrt[3]{4} + 8\sqrt[3]{4}$

6) $10\sqrt[3]{5} - 2\sqrt[3]{5}$

7) $6 - \sqrt{13} + 5 - 2\sqrt{13}$

8) $-8 + 3\sqrt{6} - 4\sqrt{6} + 9$

9) $15\sqrt[3]{z^2} - 20\sqrt[3]{z^2}$

10) $7\sqrt[3]{p} - 4\sqrt[3]{p}$

11) $2\sqrt[5]{n^2} + 9\sqrt[5]{n^2} - 11\sqrt[3]{n^2} + \sqrt[5]{n^2}$

12) $5\sqrt[4]{s} - 3\sqrt[3]{s} + 2\sqrt[3]{s} + 4\sqrt[4]{s}$

13) $\sqrt{5c} - 8\sqrt{6c} + \sqrt{5c} + 6\sqrt{6c}$

14) $10\sqrt{2m} + 6\sqrt{3m} - \sqrt{2m} + 8\sqrt{3m}$

Objective 2: Simplify Before Adding and Subtracting

15) What are the steps for adding or subtracting radicals?

16) Is $6\sqrt{2} + \sqrt{10}$ in simplified form? Explain.

Perform the operations and simplify.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

17) $\sqrt{48} + \sqrt{3}$
$= \sqrt{16 \cdot 3} + \sqrt{3}$ ________
$=$ ________ Product rule
$= 4\sqrt{3} + \sqrt{3}$ ________
$=$ ____ Add like radicals.

18) $\sqrt{44} - 8\sqrt{11}$
$= \sqrt{4 \cdot 11} - 8\sqrt{11}$ ________
$= \sqrt{4} \cdot \sqrt{11} - 8\sqrt{11}$ ________
$=$ ________ Simplify
$=$ ____ Subtract like radicals.

19) $6\sqrt{3} - \sqrt{12}$
20) $\sqrt{45} + 4\sqrt{5}$
21) $\sqrt{32} - 3\sqrt{8}$
22) $3\sqrt{24} + \sqrt{96}$
23) $\sqrt{12} + \sqrt{75} - \sqrt{3}$
24) $\sqrt{96} + \sqrt{24} - 5\sqrt{54}$
25) $8\sqrt[3]{9} + \sqrt[3]{72}$
26) $5\sqrt[3]{88} + 2\sqrt[3]{11}$
27) $\sqrt[3]{6} - \sqrt[3]{48}$
28) $11\sqrt[3]{16} + 7\sqrt[3]{2}$
29) $6q\sqrt{q} + 7\sqrt{q^3}$
30) $11\sqrt{m^3} + 8m\sqrt{m}$
31) $4d^2\sqrt{d} - 24\sqrt{d^5}$
32) $16k^4\sqrt{k} - 13\sqrt{k^9}$
33) $9t^3\sqrt[3]{t} - 5\sqrt[3]{t^{10}}$
34) $8r^4\sqrt[3]{r} - 16\sqrt[3]{r^{13}}$
35) $5a\sqrt[4]{a^7} + \sqrt[4]{a^{11}}$
36) $-3\sqrt[4]{c^{11}} + 6c^2\sqrt[4]{c^3}$
37) $2\sqrt{8p} - 6\sqrt{2p}$
38) $4\sqrt{63t} + 6\sqrt{7t}$
39) $7\sqrt[3]{81a^5} + 4a\sqrt[3]{3a^2}$
40) $3\sqrt[3]{40x} - 12\sqrt[3]{5x}$
41) $\sqrt{xy^3} + 3y\sqrt{xy}$
42) $5a\sqrt{ab} + 2\sqrt{a^3b}$
43) $6c^2\sqrt{8d^3} - 9d\sqrt{2c^4d}$
44) $11v\sqrt{5u^3} - 2u\sqrt{45uv^2}$
45) $18a^5\sqrt[3]{7a^2b} + 2a^3\sqrt[3]{7a^8b}$
46) $8p^2q\sqrt[3]{11pq^2} + 3p^2\sqrt[3]{88pq^5}$
47) $15cd\sqrt[4]{9cd} - \sqrt[4]{9c^5d^5}$
48) $7yz^2\sqrt[4]{11y^4z} + 3z\sqrt[4]{11y^8z^5}$
49) $\sqrt[3]{a^9b} - \sqrt[3]{b^7}$
50) $\sqrt[3]{c^8} + \sqrt[3]{c^2d^3}$

Objective 3: Multiply a Binomial Containing Radical Expressions by a Monomial

Multiply and simplify.

51) $3(x + 5)$
52) $8(k + 3)$
53) $7(\sqrt{6} + 2)$
54) $5(4 - \sqrt{7})$
55) $\sqrt{10}(\sqrt{3} - 1)$
56) $\sqrt{2}(9 + \sqrt{11})$
57) $-6(\sqrt{32} + \sqrt{2})$
58) $10(\sqrt{12} - \sqrt{3})$
59) $4(\sqrt{45} - \sqrt{20})$
60) $-3(\sqrt{18} + \sqrt{50})$
61) $\sqrt{5}(\sqrt{24} - \sqrt{54})$
62) $\sqrt{2}(\sqrt{20} + \sqrt{45})$
63) $\sqrt[4]{3}(5 - \sqrt[4]{27})$
64) $\sqrt[3]{4}(2\sqrt[3]{5} + 7\sqrt[3]{4})$
65) $\sqrt{t}(\sqrt{t} - \sqrt{81u})$
66) $\sqrt{s}(\sqrt{12r} + \sqrt{7s})$
67) $\sqrt{ab}(\sqrt{5a} + \sqrt{27b})$
68) $\sqrt{2xy}(\sqrt{2y} - y\sqrt{x})$
69) $\sqrt[3]{c^2}(\sqrt[3]{c^2} + \sqrt[3]{125cd})$
70) $\sqrt[5]{mn^3}(\sqrt[5]{2m^2n} - n\sqrt[5]{mn^2})$

Mixed Exercises: Objectives 4–6

71) How are the problems *Multiply* $(x + 8)(x + 3)$ and *Multiply* $(3 + \sqrt{2})(1 + \sqrt{2})$ similar? What method can be used to multiply each of them?

72) How are the problems *Multiply* $(y - 5)^2$ and *Multiply* $(\sqrt{7} - 2)^2$ similar? What method can be used to multiply each of them?

73) What formula can be used to multiply $(5 + \sqrt{6})(5 - \sqrt{6})$?

74) What happens to the radical terms whenever we multiply $(a + b)(a - b)$ where the binomials contain square roots?

Objective 4: Multiply Radical Expressions Using FOIL

Multiply and simplify.

75) $(p + 7)(p + 6)$
76) $(z - 8)(z + 2)$

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

77) $(6 + \sqrt{7})(2 + \sqrt{7})$

= ________________________ Use FOIL.

$= 12 + 6\sqrt{7} + 2\sqrt{7} + 7$ ______

= ________ Combine like terms.

78) $(3 + \sqrt{5})(1 + \sqrt{5})$

$= 3 \cdot 1 + 3\sqrt{5} + 1\sqrt{5} + \sqrt{5} \cdot \sqrt{5}$ ______

= ______________ Multiply.

= ________ Combine like terms.

79) $(\sqrt{2} + 8)(\sqrt{2} - 3)$
80) $(\sqrt{6} - 7)(\sqrt{6} + 2)$
81) $(\sqrt{5} - 4\sqrt{3})(2\sqrt{5} - \sqrt{3})$
82) $(5\sqrt{2} - \sqrt{3})(2\sqrt{3} - \sqrt{2})$
83) $(5 + 2\sqrt{3})(\sqrt{7} + \sqrt{2})$
84) $(\sqrt{5} + 4)(\sqrt{3} - 6\sqrt{2})$
85) $(\sqrt[3]{25} - 3)(\sqrt[3]{5} - \sqrt[3]{6})$
86) $(\sqrt[4]{8} - \sqrt[4]{3})(\sqrt[4]{6} + \sqrt[4]{2})$
87) $(\sqrt{6p} - 2\sqrt{q})(8\sqrt{q} + 5\sqrt{6p})$
88) $(4\sqrt{3r} + \sqrt{s})(3\sqrt{s} - 2\sqrt{3r})$

Objective 5: Square a Binomial Containing Radical Expressions

Multiply and simplify.

89) $(\sqrt{3} + 1)^2$
90) $(2 + \sqrt{5})^2$
91) $(\sqrt{11} - \sqrt{5})^2$
92) $(\sqrt{3} + \sqrt{13})^2$
93) $(\sqrt{h} + \sqrt{7})^2$
94) $(\sqrt{m} + \sqrt{3})^2$
95) $(\sqrt{x} - \sqrt{y})^2$
96) $(\sqrt{b} - \sqrt{a})^2$

Objective 6: Multiply Radical Expressions of the Form $(a + b)(a - b)$

Multiply and simplify.

97) $(c + 9)(c - 9)$
98) $(g - 7)(g + 7)$
99) $(6 - \sqrt{5})(6 + \sqrt{5})$
100) $(4 - \sqrt{7})(4 + \sqrt{7})$

VIDEO 101) $(4\sqrt{3} + \sqrt{2})(4\sqrt{3} - \sqrt{2})$

102) $(2\sqrt{2} - 2\sqrt{7})(2\sqrt{2} + 2\sqrt{7})$

103) $(\sqrt[3]{2} - 3)(\sqrt[3]{2} + 3)$

104) $(1 + \sqrt[3]{6})(1 - \sqrt[3]{6})$

105) $(\sqrt{c} + \sqrt{d})(\sqrt{c} - \sqrt{d})$

106) $(\sqrt{2y} + \sqrt{z})(\sqrt{2y} - \sqrt{z})$

107) $(8\sqrt{f} - \sqrt{g})(8\sqrt{f} + \sqrt{g})$

108) $(\sqrt{a} + 3\sqrt{4b})(\sqrt{a} - 3\sqrt{4b})$

Extension

Multiply and simplify.

109) $(1 + 2\sqrt[3]{5})(1 - 2\sqrt[3]{5} + 4\sqrt[3]{25})$

110) $(3 + \sqrt[3]{2})(9 - 3\sqrt[3]{2} + \sqrt[3]{4})$

Let $f(x) = x^2$. Find each function value.

111) $f(\sqrt{7} + 2)$

112) $f(5 - \sqrt{6})$

113) $f(1 - 2\sqrt{3})$

114) $f(3\sqrt{2} + 4)$

Section 10.6 Dividing Radicals

Objectives

1. Rationalize a Denominator: One Square Root
2. Rationalize a Denominator: One Higher Root
3. Rationalize a Denominator Containing Two Terms
4. Rationalize a Numerator
5. Divide Out Common Factors from the Numerator and Denominator

It is generally agreed that a radical expression is *not* in simplest form if its denominator contains a radical. For example, $\dfrac{1}{\sqrt{3}}$ is not simplified, but an equivalent form, $\dfrac{\sqrt{3}}{3}$, is simplified.

Later we will show that $\dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$. The process of eliminating radicals from the denominator of an expression is called **rationalizing the denominator**. We will look at two types of rationalizing problems:

1) Rationalizing a denominator containing one term.
2) Rationalizing a denominator containing two terms.

To rationalize a denominator, we will use the fact that multiplying the numerator and denominator of a fraction by the same quantity results in an equivalent fraction:

$$\frac{2}{3} \cdot \frac{4}{4} = \frac{8}{12} \qquad \frac{2}{3} \text{ and } \frac{8}{12} \text{ are equivalent because } \frac{4}{4} = 1.$$

We use the same idea to rationalize the denominator of a radical expression.

1. Rationalize a Denominator: One Square Root

The goal of rationalizing is to eliminate the radical from the denominator. With regard to square roots, recall that $\sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a$ for $a \geq 0$. For example,

$$\sqrt{2} \cdot \sqrt{2} = \sqrt{2^2} = 2, \ \sqrt{19} \cdot \sqrt{19} = \sqrt{(19)^2} = 19, \ \sqrt{t} \cdot \sqrt{t} = \sqrt{t^2} = t \ (t \geq 0)$$

We will use this property to rationalize the denominators of the following expressions.

Example 1

Rationalize the denominator of each expression.

a) $\dfrac{1}{\sqrt{3}}$ b) $\dfrac{36}{\sqrt{18}}$ c) $\dfrac{5\sqrt{3}}{\sqrt{2}}$

Solution

a) To eliminate the square root from the denominator of $\frac{1}{\sqrt{3}}$, ask yourself, "By what do I multiply $\sqrt{3}$ to get a *perfect square* under the square root?" The answer is $\sqrt{3}$ since $\sqrt{3} \cdot \sqrt{3} = \sqrt{3^2} = \sqrt{9} = 3$. Multiply by $\sqrt{3}$ in the numerator *and* denominator. (We are actually multiplying by 1.)

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{\sqrt{3^2}} = \frac{\sqrt{3}}{\sqrt{9}} = \frac{\sqrt{3}}{3}$$

Rationalize the denominator.

BE CAREFUL

$\frac{\sqrt{3}}{3}$ is in simplest form. We cannot reduce terms inside and outside of the radical.

$$\frac{\sqrt{3}}{3} = \frac{\sqrt{\cancel{3}^{1}}}{\cancel{3}_{1}} = \sqrt{1} = 1 \qquad \textbf{Incorrect!}$$

b) To rationalize the denominator of $\frac{36}{\sqrt{18}}$, begin by simplifying $\sqrt{18}$.

$$\frac{36}{\sqrt{18}} = \frac{36}{3\sqrt{2}} = \frac{12}{\sqrt{2}} = \frac{12}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{12\sqrt{2}}{2} = 6\sqrt{2}$$

Simplify $\sqrt{18}$. Simplify fraction. Rationalize the denominator.

c) To rationalize the denominator of $\frac{5\sqrt{3}}{\sqrt{2}}$, multiply the numerator and denominator by $\sqrt{2}$.

$$\frac{5\sqrt{3}}{\sqrt{2}} = \frac{5\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{6}}{2}$$

You Try 1

Rationalize the denominator of each expression.

a) $\frac{1}{\sqrt{7}}$ b) $\frac{15}{\sqrt{27}}$ c) $\frac{9\sqrt{6}}{\sqrt{5}}$

Sometimes we will apply the quotient or product rule before rationalizing.

Example 2

Simplify completely.

a) $\sqrt{\frac{3}{24}}$ b) $\sqrt{\frac{5}{14}} \cdot \sqrt{\frac{7}{3}}$

Solution

a) Begin by simplifying the fraction $\frac{3}{24}$ under the radical.

$$\sqrt{\frac{3}{24}} = \sqrt{\frac{1}{8}} \qquad \text{Simplify.}$$

$$= \frac{\sqrt{1}}{\sqrt{8}} = \frac{1}{\sqrt{4}\cdot\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2\cdot 2} = \frac{\sqrt{2}}{4}$$

b) Begin by using the product rule to multiply the radicands.

$$\sqrt{\frac{5}{14}}\cdot\sqrt{\frac{7}{3}} = \sqrt{\frac{5}{14}\cdot\frac{7}{3}} \qquad \text{Product rule}$$

Multiply the fractions under the radical.

$$= \sqrt{\frac{5}{\underset{2}{\cancel{14}}}\cdot\frac{\overset{1}{\cancel{7}}}{3}} = \sqrt{\frac{5}{6}} \qquad \text{Multiply.}$$

$$= \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{5}}{\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{30}}{6}$$

You Try 2

Simplify completely.

a) $\sqrt{\frac{10}{35}}$ b) $\sqrt{\frac{21}{10}}\cdot\sqrt{\frac{2}{7}}$

We work with radical expressions containing variables the same way. **In the rest of this section, we will assume that all variables represent positive real numbers.**

Example 3

Simplify completely.

a) $\frac{2}{\sqrt{x}}$ b) $\sqrt{\frac{12m^3}{7n}}$ c) $\sqrt{\frac{6cd^2}{cd^3}}$

Solution

a) Ask yourself, "By what do I multiply $\sqrt{x}$ to get a *perfect square* under the square root?" The perfect square we want to get is $\sqrt{x^2}$.

$$\sqrt{x}\cdot\sqrt{?} = \sqrt{x^2} = x$$
$$\sqrt{x}\cdot\sqrt{x} = \sqrt{x^2} = x$$

$$\frac{2}{\sqrt{x}} = \frac{2}{\sqrt{x}}\cdot\underbrace{\frac{\sqrt{x}}{\sqrt{x}}}_{\text{Rationalize the denominator.}} = \frac{2\sqrt{x}}{\sqrt{x^2}} = \frac{2\sqrt{x}}{x}$$

b) Before rationalizing, apply the quotient rule and simplify the numerator.

$$\sqrt{\frac{12m^3}{7n}} = \frac{\sqrt{12m^3}}{\sqrt{7n}} = \frac{2m\sqrt{3m}}{\sqrt{7n}}$$

Rationalize the denominator. "By what do I multiply $\sqrt{7n}$ to get a *perfect square* under the square root?" The perfect square we want to get is $\sqrt{7^2n^2}$ or $\sqrt{49n^2}$.

$$\sqrt{7n} \cdot \sqrt{?} = \sqrt{7^2n^2} = 7n$$
$$\sqrt{7n} \cdot \sqrt{7n} = \sqrt{7^2n^2} = 7n$$

$$\sqrt{\frac{12m^3}{7n}} = \frac{2m\sqrt{3m}}{\sqrt{7n}}$$
$$= \frac{2m\sqrt{3m}}{\sqrt{7n}} \cdot \frac{\sqrt{7n}}{\sqrt{7n}} = \frac{2m\sqrt{21mn}}{7n}$$

↑ Rationalize the denominator.

c) $\sqrt{\dfrac{6cd^2}{cd^3}} = \sqrt{\dfrac{6}{d}}$ Simplify the radicand using the quotient rule for exponents.

$$= \frac{\sqrt{6}}{\sqrt{d}} = \frac{\sqrt{6}}{\sqrt{d}} \cdot \frac{\sqrt{d}}{\sqrt{d}} = \frac{\sqrt{6d}}{d}$$

■

You Try 3

Simplify completely.

a) $\dfrac{5}{\sqrt{p}}$ b) $\sqrt{\dfrac{18k^5}{10m}}$ c) $\sqrt{\dfrac{20r^3s}{s^2}}$

2. Rationalize a Denominator: One Higher Root

Many students assume that to rationalize denominators we simply multiply the numerator and denominator of the expression by the denominator as in $\dfrac{4}{\sqrt{3}} = \dfrac{4}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{4\sqrt{3}}{3}$.

We will see, however, why this reasoning is incorrect.

To rationalize an expression like $\dfrac{4}{\sqrt{3}}$ we asked ourselves, "By what do I multiply $\sqrt{3}$ to get a *perfect square* under the *square root*?"

To rationalize an expression like $\dfrac{5}{\sqrt[3]{2}}$ we must ask ourselves, "By what do I multiply $\sqrt[3]{2}$ to get a *perfect cube* under the *cube root*?" The perfect cube we want is 2^3 (since we began with 2) so that $\sqrt[3]{2} \cdot \sqrt[3]{2^2} = \sqrt[3]{2^3} = 2$.

We will practice some fill-in-the-blank problems to eliminate radicals before we move on to rationalizing.

Example 4

Fill in the blank.

a) $\sqrt[3]{5} \cdot \sqrt[3]{?} = \sqrt[3]{5^3} = 5$ b) $\sqrt[3]{3} \cdot \sqrt[3]{?} = \sqrt[3]{3^3} = 3$

c) $\sqrt[3]{x^2} \cdot \sqrt[3]{?} = \sqrt[3]{x^3} = x$ d) $\sqrt[5]{8} \cdot \sqrt[5]{?} = \sqrt[5]{2^5} = 2$

e) $\sqrt[4]{27} \cdot \sqrt[4]{?} = \sqrt[4]{3^4} = 3$

Solution

a) Ask yourself, "By what do I multiply $\sqrt[3]{5}$ to get $\sqrt[3]{5^3}$?" The answer is $\sqrt[3]{5^2}$.

$$\sqrt[3]{5} \cdot \sqrt[3]{?} = \sqrt[3]{5^3} = 5$$
$$\sqrt[3]{5} \cdot \sqrt[3]{5^2} = \sqrt[3]{5^3} = 5$$

b) "By what do I multiply $\sqrt[3]{3}$ to get $\sqrt[3]{3^3}$?" $\sqrt[3]{3^2}$

$$\sqrt[3]{3} \cdot \sqrt[3]{?} = \sqrt[3]{3^3} = 3$$
$$\sqrt[3]{3} \cdot \sqrt[3]{3^2} = \sqrt[3]{3^3} = 3$$

c) "By what do I multiply $\sqrt[3]{x^2}$ to get $\sqrt[3]{x^3}$?" $\sqrt[3]{x}$

$$\sqrt[3]{x^2} \cdot \sqrt[3]{?} = \sqrt[3]{x^3} = x$$
$$\sqrt[3]{x^2} \cdot \sqrt[3]{x} = \sqrt[3]{x^3} = x$$

d) In this example, $\sqrt[5]{8} \cdot \sqrt[5]{?} = \sqrt[5]{2^5} = 2$, why are we trying to obtain $\sqrt[5]{2^5}$ instead of $\sqrt[5]{8^5}$? Because in the first radical, $\sqrt[5]{8}$, 8 *is a power of* 2. Before attempting to fill in the blank, rewrite 8 as 2^3.

$$\sqrt[5]{8} \cdot \sqrt[5]{?} = \sqrt[5]{2^5} = 2$$
$$\sqrt[5]{2^3} \cdot \sqrt[5]{?} = \sqrt[5]{2^5} = 2$$
$$\sqrt[5]{2^3} \cdot \sqrt[5]{2^2} = \sqrt[5]{2^5} = 2$$

e)
$$\sqrt[4]{27} \cdot \sqrt[4]{?} = \sqrt[4]{3^4} = 3$$
$$\sqrt[4]{3^3} \cdot \sqrt[4]{?} = \sqrt[4]{3^4} = 3 \qquad \text{Since 27 is a power of 3, rewrite } \sqrt[4]{27} \text{ as } \sqrt[4]{3^3}.$$
$$\sqrt[4]{3^3} \cdot \sqrt[4]{3} = \sqrt[4]{3^4} = 3$$

You Try 4

Fill in the blank.

a) $\sqrt[3]{2} \cdot \sqrt[3]{?} = \sqrt[3]{2^3} = 2$ b) $\sqrt[5]{t^2} \cdot \sqrt[5]{?} = \sqrt[5]{t^5} = t$ c) $\sqrt[4]{125} \cdot \sqrt[4]{?} = \sqrt[4]{5^4} = 5$

We will use the technique presented in Example 4 to rationalize denominators with indices higher than 2.

Example 5

Rationalize the denominator.

a) $\dfrac{7}{\sqrt[3]{3}}$ b) $\sqrt[5]{\dfrac{3}{4}}$ c) $\dfrac{7}{\sqrt[4]{n}}$

Solution

a) *First* identify what we want the denominator to be *after* multiplying. **We want to obtain $\sqrt[3]{3^3}$ since $\sqrt[3]{3^3} = 3$.**

$$\frac{7}{\sqrt[3]{3}} \cdot \frac{\quad}{\quad} = \frac{\quad}{\sqrt[3]{3^3}} \qquad \leftarrow \text{This is what we want to get.}$$

↑
What is needed here?

Ask yourself, "By what do I multiply $\sqrt[3]{3}$ to get $\sqrt[3]{3^3}$?" $\sqrt[3]{3^2}$

$$\frac{7}{\sqrt[3]{3}} \cdot \frac{\sqrt[3]{3^2}}{\sqrt[3]{3^2}} = \frac{7\sqrt[3]{3^2}}{\sqrt[3]{3^3}} \qquad \text{Multiply.}$$
$$= \frac{7\sqrt[3]{9}}{3} \qquad \text{Simplify.}$$

b) Use the quotient rule for radicals to rewrite $\sqrt[5]{\frac{3}{4}}$ as $\frac{\sqrt[5]{3}}{\sqrt[5]{4}}$. Then, write 4 as 2^2 to get

$$\frac{\sqrt[5]{3}}{\sqrt[5]{4}} = \frac{\sqrt[5]{3}}{\sqrt[5]{2^2}}$$

What denominator do we want to get *after* multiplying? **We want to obtain $\sqrt[5]{2^5}$ since $\sqrt[5]{2^5} = 2$.**

$$\frac{\sqrt[5]{3}}{\sqrt[5]{2^2}} \cdot \frac{\quad}{\underline{\qquad}} = \frac{\quad}{\sqrt[5]{2^5}} \qquad \leftarrow \text{This is what we want to get.}$$

$\uparrow$
What is needed here?

"By what do I multiply $\sqrt[5]{2^2}$ to get $\sqrt[5]{2^5}$?" $\sqrt[5]{2^3}$

$$\frac{\sqrt[5]{3}}{\sqrt[5]{2^2}} \cdot \frac{\sqrt[5]{2^3}}{\sqrt[5]{2^3}} = \frac{\sqrt[5]{3} \cdot \sqrt[5]{2^3}}{\sqrt[5]{2^5}} \qquad \text{Multiply.}$$

$$= \frac{\sqrt[5]{3} \cdot \sqrt[5]{8}}{2} = \frac{\sqrt[5]{24}}{2} \qquad \text{Multiply.}$$

c) We must rationalize the denominator of $\frac{7}{\sqrt[4]{n}}$. What denominator do we want to get *after* multiplying? **We want to obtain $\sqrt[4]{n^4}$ since $\sqrt[4]{n^4} = n$.**

$$\frac{7}{\sqrt[4]{n}} \cdot \frac{\quad}{\underline{\qquad}} = \frac{\quad}{\sqrt[4]{n^4}} \qquad \leftarrow \text{This is what we want to get.}$$

$\uparrow$
What is needed here?

Ask yourself, "By what do I multiply $\sqrt[4]{n}$ to get $\sqrt[4]{n^4}$?" $\sqrt[4]{n^3}$

$$\frac{7}{\sqrt[4]{n}} \cdot \frac{\sqrt[4]{n^3}}{\sqrt[4]{n^3}} = \frac{7\sqrt[4]{n^3}}{\sqrt[4]{n^4}} \qquad \text{Multiply.}$$

$$= \frac{7\sqrt[4]{n^3}}{n} \qquad \text{Simplify.}$$

You Try 5

Rationalize the denominator.

a) $\frac{4}{\sqrt[3]{7}}$ b) $\sqrt[4]{\frac{2}{27}}$ c) $\sqrt[5]{\frac{8}{w^3}}$

3. Rationalize a Denominator Containing Two Terms

To rationalize the denominator of an expression like $\frac{1}{5 + \sqrt{3}}$, we multiply the numerator and the denominator of the expression by the *conjugate* of $5 + \sqrt{3}$.

Definition

The **conjugate** of a binomial is the binomial obtained by changing the sign between the two terms.

Expression	Conjugate
$\sqrt{7} - 2\sqrt{5}$	$\sqrt{7} + 2\sqrt{5}$
$\sqrt{a} + \sqrt{b}$	$\sqrt{a} - \sqrt{b}$

In Section 10.5, we applied the formula $(a + b)(a - b) = a^2 - b^2$ to multiply binomials containing square roots. Recall that the terms containing the square roots were eliminated.

Example 6

Multiply $8 - \sqrt{6}$ by its conjugate.

Solution

The conjugate of $8 - \sqrt{6}$ is $8 + \sqrt{6}$. We will first multiply using FOIL to show *why* the radical drops out, then we will multiply using the formula

$$(a + b)(a - b) = a^2 - b^2$$

i) Use FOIL to multiply.

$$\begin{aligned}(8 - \sqrt{6})(8 + \sqrt{6}) &= \underset{F}{8 \cdot 8} + \underset{O}{8 \cdot \sqrt{6}} - \underset{I}{8 \cdot \sqrt{6}} - \underset{L}{\sqrt{6} \cdot \sqrt{6}} \\ &= 64 - 6 \\ &= 58\end{aligned}$$

ii) Use $(a + b)(a - b) = a^2 - b^2$.

$$\begin{aligned}(8 - \sqrt{6})(8 + \sqrt{6}) &= (8)^2 - (\sqrt{6})^2 \\ &= 64 - 6 = 58\end{aligned}$$ Substitute 8 for a and $\sqrt{6}$ for b.

Each method gives the same result.

You Try 6

Multiply $2 + \sqrt{11}$ by its conjugate.

Procedure Rationalizing a Denominator Containing Two Terms

If the denominator of an expression contains two terms, including one or two square roots, then to rationalize the denominator we multiply the numerator and denominator of the expression by the conjugate of the denominator.

Example 7

Rationalize the denominator and simplify completely.

a) $\dfrac{3}{5+\sqrt{3}}$ b) $\dfrac{\sqrt{a}+b}{\sqrt{b}-a}$

Solution

a) The denominator of $\dfrac{3}{5+\sqrt{3}}$ has two terms, so we multiply the numerator and denominator by $5-\sqrt{3}$, the conjugate of the denominator.

$$\frac{3}{5+\sqrt{3}} \cdot \frac{5-\sqrt{3}}{5-\sqrt{3}} \qquad \text{Multiply by the conjugate.}$$
$$= \frac{3(5-\sqrt{3})}{(5)^2-(\sqrt{3})^2} \qquad (a+b)(a-b)=a^2-b^2$$
$$= \frac{15-3\sqrt{3}}{25-3} \qquad \text{Simplify.}$$
$$= \frac{15-3\sqrt{3}}{22} \qquad \text{Subtract.}$$

b) The conjugate of the denominator is $\sqrt{b}+a$.

$$\frac{\sqrt{a}+b}{\sqrt{b}-a} \cdot \frac{\sqrt{b}+a}{\sqrt{b}+a} \qquad \text{Multiply by the conjugate.}$$

In the numerator we must multiply $(\sqrt{a}+b)(\sqrt{b}+a)$. We will use FOIL.

$$\frac{\sqrt{a}+b}{\sqrt{b}-a} \cdot \frac{\sqrt{b}+a}{\sqrt{b}+a} = \frac{\sqrt{ab}+a\sqrt{a}+b\sqrt{b}+ab}{(\sqrt{b})^2-(a)^2} \qquad \begin{array}{l}\leftarrow \text{Use FOIL in the numerator.} \\ \leftarrow (a+b)(a-b)=a^2-b^2\end{array}$$
$$= \frac{\sqrt{ab}+a\sqrt{a}+b\sqrt{b}+ab}{b-a^2} \qquad \text{Square the terms.}$$

You Try 7

Rationalize the denominator and simplify completely.

a) $\dfrac{1}{\sqrt{7}-2}$ b) $\dfrac{c+\sqrt{d}}{c-\sqrt{d}}$

4. Rationalize a Numerator

In higher-level math courses, sometimes it is necessary to rationalize the numerator of a radical expression so that the numerator does not contain a radical.

Example 8

Rationalize the numerator and simplify completely.

a) $\dfrac{\sqrt{7}}{\sqrt{2}}$ b) $\dfrac{8-\sqrt{5}}{3}$

Solution

a) Rationalizing the numerator of $\dfrac{\sqrt{7}}{\sqrt{2}}$ means eliminating the square root from the numerator. Multiply the numerator and denominator by $\sqrt{7}$.

$$\frac{\sqrt{7}}{\sqrt{2}} = \frac{\sqrt{7}}{\sqrt{2}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{7}{\sqrt{14}}$$

b) To rationalize the numerator, we must multiply the numerator and denominator by $8 + \sqrt{5}$, the conjugate of the numerator.

$$\frac{8 - \sqrt{5}}{3} \cdot \frac{8 + \sqrt{5}}{8 + \sqrt{5}} \qquad \text{Multiply by the conjugate.}$$

$$= \frac{8^2 - (\sqrt{5})^2}{3(8 + \sqrt{5})} \qquad (a + b)(a - b) = a^2 - b^2$$

$$= \frac{64 - 5}{24 + 3\sqrt{5}} = \frac{59}{24 + 3\sqrt{5}} \qquad \text{Multiply.}$$

You Try 8

Rationalize the numerator and simplify completely.

a) $\dfrac{\sqrt{3}}{\sqrt{5}}$ b) $\dfrac{6 + \sqrt{7}}{4}$

5. Divide Out Common Factors from the Numerator and Denominator

Sometimes it is necessary to simplify a radical expression by dividing out common factors from the numerator and denominator. This is a skill we will need in Chapter 11 to solve quadratic equations, so we will look at an example here.

Example 9

Simplify completely: $\dfrac{4\sqrt{5} + 12}{4}$.

Solution

It is tempting to do one of the following:

$$\frac{\cancel{4}\sqrt{5} + 12}{\cancel{4}} = \sqrt{5} + 12 \qquad \textbf{Incorrect!}$$

or

$$\frac{4\sqrt{5} + \overset{3}{\cancel{12}}}{\cancel{4}} = 4\sqrt{5} + 3 \qquad \textbf{Incorrect!}$$

Each is incorrect because $4\sqrt{5}$ is a *term* in a sum and 12 is a *term* in a sum.

The correct way to simplify $\dfrac{4\sqrt{5}+12}{4}$ is to begin by factoring out a 4 in the numerator and *then* divide the numerator and denominator by any common factors.

$$\frac{4\sqrt{5}+12}{4} = \frac{4(\sqrt{5}+3)}{4} \quad \text{Factor out 4 from the numerator.}$$

$$= \frac{\overset{1}{\cancel{4}}(\sqrt{5}+3)}{\underset{1}{\cancel{4}}} \quad \text{Divide by 4.}$$

$$= \sqrt{5}+3 \quad \text{Simplify.}$$

We can divide the numerator and denominator by 4 in $\dfrac{4(\sqrt{5}+3)}{4}$ because the 4 in the numerator is part of a *product*, not a sum or difference.

You Try 9

Simplify completely.

a) $\dfrac{5\sqrt{7}-40}{5}$ b) $\dfrac{20+6\sqrt{2}}{4}$

Answers to You Try Exercises

1) a) $\dfrac{\sqrt{7}}{7}$ b) $\dfrac{5\sqrt{3}}{3}$ c) $\dfrac{9\sqrt{30}}{5}$ 2) a) $\dfrac{\sqrt{14}}{7}$ b) $\dfrac{\sqrt{15}}{5}$ 3) a) $\dfrac{5\sqrt{p}}{p}$ b) $\dfrac{3k^2\sqrt{5km}}{5m}$ c) $\dfrac{2r\sqrt{5rs}}{s}$ 4) a) 2^2 or 4 b) t^3 c) 5 5) a) $\dfrac{4\sqrt[3]{49}}{7}$ b) $\dfrac{\sqrt[4]{6}}{3}$ c) $\dfrac{\sqrt[5]{8w^2}}{w}$ 6) -7 7) a) $\dfrac{\sqrt{7}+2}{3}$ b) $\dfrac{c^2+2c\sqrt{d}+d}{c^2-d}$ 8) a) $\dfrac{3}{\sqrt{15}}$ b) $\dfrac{29}{24-4\sqrt{7}}$ 9) a) $\sqrt{7}-8$ b) $\dfrac{10+3\sqrt{2}}{2}$

10.6 Exercises

Assume all variables represent positive real numbers.

Objective 1: Rationalize a Denominator: One Square Root

1) What does it mean to rationalize the denominator of a radical expression?

2) In your own words, explain how to rationalize the denominator of an expression containing one term in the denominator.

Rationalize the denominator of each expression.

3) $\dfrac{1}{\sqrt{5}}$

4) $\dfrac{1}{\sqrt{6}}$

5) $\dfrac{9}{\sqrt{6}}$

6) $\dfrac{25}{\sqrt{10}}$

VIDEO 7) $-\dfrac{20}{\sqrt{8}}$

8) $-\dfrac{18}{\sqrt{45}}$

9) $\dfrac{\sqrt{3}}{\sqrt{28}}$

10) $\dfrac{\sqrt{8}}{\sqrt{27}}$

11) $\sqrt{\dfrac{20}{60}}$

12) $\sqrt{\dfrac{12}{80}}$

13) $\dfrac{\sqrt{56}}{\sqrt{48}}$

14) $\dfrac{\sqrt{66}}{\sqrt{12}}$

Multiply and simplify.

VIDEO 15) $\sqrt{\dfrac{10}{7}} \cdot \sqrt{\dfrac{7}{3}}$

16) $\sqrt{\dfrac{11}{5}} \cdot \sqrt{\dfrac{5}{2}}$

17) $\sqrt{\dfrac{6}{5}} \cdot \sqrt{\dfrac{1}{8}}$

18) $\sqrt{\dfrac{11}{10}} \cdot \sqrt{\dfrac{8}{11}}$

Simplify completely.

19) $\dfrac{8}{\sqrt{y}}$

20) $\dfrac{4}{\sqrt{w}}$

21) $\dfrac{\sqrt{5}}{\sqrt{t}}$

22) $\dfrac{\sqrt{2}}{\sqrt{m}}$

23) $\sqrt{\dfrac{64v^7}{5w}}$

24) $\sqrt{\dfrac{81c^5}{2d}}$

25) $\sqrt{\dfrac{a^3b^3}{3ab^4}}$

26) $\sqrt{\dfrac{m^2n^5}{7m^3n}}$

27) $-\dfrac{\sqrt{75}}{\sqrt{b^3}}$

28) $-\dfrac{\sqrt{24}}{\sqrt{v^3}}$

29) $\dfrac{\sqrt{13}}{\sqrt{j^5}}$

30) $\dfrac{\sqrt{22}}{\sqrt{w^7}}$

Objective 2: Rationalize a Denominator: One Higher Root

Fill in the blank.

31) $\sqrt[3]{2} \cdot \sqrt[3]{?} = \sqrt[3]{2^3} = 2$

32) $\sqrt[3]{5} \cdot \sqrt[3]{?} = \sqrt[3]{5^3} = 5$

33) $\sqrt[3]{9} \cdot \sqrt[3]{?} = \sqrt[3]{3^3} = 3$

34) $\sqrt[3]{4} \cdot \sqrt[3]{?} = \sqrt[3]{2^3} = 2$

35) $\sqrt[3]{c} \cdot \sqrt[3]{?} = \sqrt[3]{c^3} = c$

36) $\sqrt[3]{p} \cdot \sqrt[3]{?} = \sqrt[3]{p^3} = p$

37) $\sqrt[5]{4} \cdot \sqrt[5]{?} = \sqrt[5]{2^5} = 2$

38) $\sqrt[5]{16} \cdot \sqrt[5]{?} = \sqrt[5]{2^5} = 2$

39) $\sqrt[4]{m^3} \cdot \sqrt[4]{?} = \sqrt[4]{m^4} = m$

40) $\sqrt[4]{k} \cdot \sqrt[4]{?} = \sqrt[4]{k^4} = k$

Rationalize the denominator of each expression.

41) $\dfrac{4}{\sqrt[3]{3}}$

42) $\dfrac{26}{\sqrt[3]{5}}$

43) $\dfrac{12}{\sqrt[3]{2}}$

44) $\dfrac{21}{\sqrt[3]{3}}$

45) $\dfrac{9}{\sqrt[3]{25}}$

46) $\dfrac{6}{\sqrt[3]{4}}$

47) $\sqrt[4]{\dfrac{5}{9}}$

48) $\sqrt[4]{\dfrac{2}{25}}$

49) $\sqrt[5]{\dfrac{3}{8}}$

50) $\sqrt[5]{\dfrac{7}{4}}$

51) $\dfrac{10}{\sqrt[3]{z}}$

52) $\dfrac{6}{\sqrt[3]{u}}$

53) $\sqrt[3]{\dfrac{3}{n^2}}$

54) $\sqrt[3]{\dfrac{5}{x^2}}$

55) $\dfrac{\sqrt[3]{7}}{\sqrt[3]{2k^2}}$

56) $\dfrac{\sqrt[3]{2}}{\sqrt[3]{25t}}$

57) $\dfrac{9}{\sqrt[5]{a^3}}$

58) $\dfrac{8}{\sqrt[5]{h^2}}$

59) $\sqrt[4]{\dfrac{5}{2m}}$

60) $\sqrt[4]{\dfrac{2}{3t^2}}$

Objective 3: Rationalize a Denominator Containing Two Terms

61) How do you find the conjugate of an expression with two radical terms?

62) When you multiply a binomial containing a square root by its conjugate, what happens to the radical?

Find the conjugate of each expression. Then multiply the expression by its conjugate.

63) $(5 + \sqrt{2})$

64) $(\sqrt{5} - 4)$

65) $(\sqrt{2} + \sqrt{6})$

66) $(\sqrt{3} - \sqrt{10})$

67) $(\sqrt{t} - 8)$

68) $(\sqrt{p} + 5)$

Rationalize the denominator and simplify completely.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

69) $\dfrac{6}{4 - \sqrt{5}} = \dfrac{6}{4 - \sqrt{5}} \cdot \dfrac{4 + \sqrt{5}}{4 + \sqrt{5}}$ ____________

$= \dfrac{6(4 + \sqrt{5})}{(4)^2 - (\sqrt{5})^2}$ ____________

$=$ ________ Multiply terms in numerator, square terms in denominator.

$=$ ________ Simplify.

70) $\dfrac{\sqrt{6}}{\sqrt{7} + \sqrt{2}} = \dfrac{\sqrt{6}}{\sqrt{7} + \sqrt{2}} \cdot \dfrac{\sqrt{7} - \sqrt{2}}{\sqrt{7} - \sqrt{2}}$ ________

$= \dfrac{\sqrt{6}(\sqrt{7} - \sqrt{2})}{(\sqrt{7})^2 - (\sqrt{2})^2}$ ____________

$=$ ________ Multiply terms in numerator, square terms in denominator.

$=$ ________ Simplify.

71) $\dfrac{3}{2 + \sqrt{3}}$

72) $\dfrac{8}{6 - \sqrt{5}}$

73) $\dfrac{10}{9 - \sqrt{2}}$

74) $\dfrac{5}{4 + \sqrt{6}}$

75) $\dfrac{\sqrt{8}}{\sqrt{3} + \sqrt{2}}$

76) $\dfrac{\sqrt{32}}{\sqrt{5} - \sqrt{7}}$

77) $\dfrac{\sqrt{3} - \sqrt{5}}{\sqrt{10} - \sqrt{3}}$

78) $\dfrac{\sqrt{3} + \sqrt{6}}{\sqrt{2} + \sqrt{5}}$

79) $\dfrac{\sqrt{m}}{\sqrt{m} + \sqrt{n}}$

80) $\dfrac{\sqrt{u}}{\sqrt{u} - \sqrt{v}}$

81) $\dfrac{b - 25}{\sqrt{b} - 5}$

82) $\dfrac{d - 9}{\sqrt{d} + 3}$

83) $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

84) $\dfrac{\sqrt{f}-\sqrt{g}}{\sqrt{f}+\sqrt{g}}$

Objective 4: Rationalize a Numerator

Rationalize the numerator of each expression and simplify.

85) $\dfrac{\sqrt{5}}{3}$

86) $\dfrac{\sqrt{2}}{9}$

87) $\dfrac{\sqrt{x}}{\sqrt{7}}$

88) $\dfrac{\sqrt{8a}}{\sqrt{b}}$

89) $\dfrac{2+\sqrt{3}}{6}$

90) $\dfrac{1+\sqrt{7}}{3}$

91) $\dfrac{\sqrt{x}-2}{x-4}$

92) $\dfrac{3-\sqrt{n}}{n-9}$

93) $\dfrac{4-\sqrt{c+11}}{c-5}$

94) $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

95) Does rationalizing the denominator of an expression change the value of the original expression? Explain your answer.

96) Does rationalizing the numerator of an expression change the value of the original expression? Explain your answer.

Objective 5: Divide Out Common Factors from the Numerator and Denominator

Simplify completely.

97) $\dfrac{5+10\sqrt{3}}{5}$

98) $\dfrac{18-6\sqrt{7}}{6}$

99) $\dfrac{30-18\sqrt{5}}{4}$

100) $\dfrac{36+20\sqrt{2}}{12}$

VIDEO 101) $\dfrac{\sqrt{45}+6}{9}$

102) $\dfrac{\sqrt{48}+28}{4}$

103) $\dfrac{-10-\sqrt{50}}{5}$

104) $\dfrac{-35+\sqrt{200}}{15}$

105) The function $r(A)=\sqrt{\dfrac{A}{\pi}}$ describes the radius of a circle, $r(A)$, in terms of its area, A.

a) If the area of a circle is measured in square inches, find $r(8\pi)$ and explain what it means in the context of the problem.

b) If the area of a circle is measured in square inches, find $r(7)$ and rationalize the denominator. Explain the meaning of $r(7)$ in the context of the problem.

c) Obtain an equivalent form of the function by rationalizing the denominator.

106) The function $r(V)=\sqrt[3]{\dfrac{3V}{4\pi}}$ describes the radius of a sphere, $r(V)$, in terms of its volume, V.

a) If the volume of a sphere is measured in cubic centimeters, find $r(36\pi)$ and explain what it means in the context of the problem.

b) If the volume of a sphere is measured in cubic centimeters, find $r(11)$ and rationalize the denominator. Explain the meaning of $r(11)$ in the context of the problem.

c) Obtain an equivalent form of the function by rationalizing the denominator.

Putting It All Together

Objective

1. Review the Concepts Presented in Sections 10.1–10.6

1. Review the Concepts Presented in Sections 10.1–10.6

In Section 10.1, we learned how to find roots of numbers.

Example 1

Find each root.

a) $\sqrt{64}$ b) $-\sqrt{64}$ c) $\sqrt{-64}$ d) $\sqrt[3]{-64}$

Solution

a) $\sqrt{64}=8$ since $8^2=64$. Remember that the square root symbol, $\sqrt{}$, represents the principal square root (or positive square root) of a number.

b) $-\sqrt{64}$ means $-1\cdot\sqrt{64}$. Therefore, $-\sqrt{64}=-1\cdot\sqrt{64}=-1\cdot 8=-8$.

c) Recall that the *even* root of a negative number is not a real number. Therefore, $\sqrt{-64}$ is not a real number.

d) The *odd* root of a negative number is a negative number. So, $\sqrt[3]{-64}=-4$ since $(-4)^3=-64$.

In Section 10.2, we learned about the relationship between rational exponents and radicals. Recall that if m and n are positive integers and $\frac{m}{n}$ is in lowest terms, then $a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$ provided that $a^{1/n}$ is a real number. For the rest of this section, we will assume that all variables represent positive real numbers.

Example 2

Simplify completely. The answer should contain only positive exponents.

a) $(32)^{4/5}$ b) $\left(\dfrac{a^7 b^{9/8}}{25a^9 b^{3/4}}\right)^{-3/2}$

Solution

a) The denominator of the fractional exponent is the index of the radical, and the numerator is the power to which we raise the radical expression.

$$\begin{aligned} 32^{4/5} &= (\sqrt[5]{32})^4 && \text{Write in radical form.} \\ &= (2)^4 && \sqrt[5]{32} = 2 \\ &= 16 \end{aligned}$$

b) $\left(\dfrac{a^7 b^{9/8}}{25a^9 b^{3/4}}\right)^{-3/2} = \left(\dfrac{25a^9 b^{3/4}}{a^7 b^{9/8}}\right)^{3/2}$ Eliminate the negative from the outermost exponent by taking the reciprocal of the base.

Simplify the expression inside the parentheses by subtracting the exponents.

$$= (25a^{9-7}b^{3/4-9/8})^{3/2} = (25a^2 b^{6/8-9/8})^{3/2} = (25a^2 b^{-3/8})^{3/2}$$

Apply the power rule, and simplify.

$$= (25)^{3/2}(a^2)^{3/2}(b^{-3/8})^{3/2} = (\sqrt{25})^3 a^3 b^{-9/16} = 5^3 a^3 b^{-9/16} = \frac{125a^3}{b^{9/16}}$$

In Sections 10.3–10.6, we learned how to simplify, multiply, divide, add, and subtract radicals. Let's look at these operations together so that we will learn to recognize the techniques needed to perform these operations.

Example 3

Perform the operations and simplify.

a) $\sqrt{3} + 10\sqrt{6} - 4\sqrt{3}$ b) $\sqrt{3}(10\sqrt{6} - 4\sqrt{3})$

Solution

a) This is the *sum and difference* of radicals. Remember that we can add and subtract only radicals that are like radicals.

$$\begin{aligned} \sqrt{3} + 10\sqrt{6} - 4\sqrt{3} &= \sqrt{3} - 4\sqrt{3} + 10\sqrt{6} && \text{Write like radicals together.} \\ &= -3\sqrt{3} + 10\sqrt{6} && \text{Subtract.} \end{aligned}$$

b) This is the *product* of radical expressions. We must multiply the binomial $10\sqrt{6} - 4\sqrt{3}$ by $\sqrt{3}$ using the distributive property.

$$\begin{aligned} \sqrt{3}(10\sqrt{6} - 4\sqrt{3}) &= \sqrt{3} \cdot 10\sqrt{6} - \sqrt{3} \cdot 4\sqrt{3} && \text{Distribute.} \\ &= 10\sqrt{18} - 4 \cdot 3 && \text{Product rule; } \sqrt{3} \cdot \sqrt{3} = 3. \\ &= 10\sqrt{18} - 12 && \text{Multiply.} \end{aligned}$$

Ask yourself, "Is $10\sqrt{18} - 12$ in simplest form?" *No.* $\sqrt{18}$ can be simplified.

$$
\begin{aligned}
&= 10\sqrt{9 \cdot 2} - 12 && \text{9 is a perfect square.}\\
&= 10\sqrt{9} \cdot \sqrt{2} - 12 && \text{Product rule}\\
&= 10 \cdot 3\sqrt{2} - 12 && \sqrt{9} = 3\\
&= 30\sqrt{2} - 12 && \text{Multiply.}
\end{aligned}
$$

The expression is now in simplest form.

Next we will look at multiplication problems involving binomials. Remember that the rules we used to multiply binomials like $(x + 4)(x - 9)$ are the same rules we use to multiply binomials containing radicals.

Example 4

Multiply and simplify.

a) $(8 + \sqrt{2})(9 - \sqrt{11})$ b) $(\sqrt{n} + \sqrt{7})(\sqrt{n} - \sqrt{7})$

c) $(2\sqrt{5} - 3)^2$

Solution

a) Since we must multiply two binomials, we will use FOIL.

$$
\begin{aligned}
(8 + \sqrt{2})(9 - \sqrt{11}) &= \overset{F}{8 \cdot 9} - \overset{O}{8 \cdot \sqrt{11}} + \overset{I}{9 \cdot \sqrt{2}} - \overset{L}{\sqrt{2} \cdot \sqrt{11}} && \text{Use FOIL.}\\
&= 72 - 8\sqrt{11} + 9\sqrt{2} - \sqrt{22} && \text{Multiply.}
\end{aligned}
$$

All radicals are simplified and none of them are like radicals, so this expression is in simplest form.

b) We can multiply $(\sqrt{n} + \sqrt{7})(\sqrt{n} - \sqrt{7})$ using FOIL or, if we notice that this product is in the form $(a + b)(a - b)$, we can apply the rule $(a + b)(a - b) = a^2 - b^2$. Either method will give us the correct answer. We will use the second method.

$$
\begin{aligned}
(a + b)(a - b) &= a^2 - b^2\\
(\sqrt{n} + \sqrt{7})(\sqrt{n} - \sqrt{7}) &= (\sqrt{n})^2 - (\sqrt{7})^2 = n - 7 && \text{Substitute } \sqrt{n} \text{ for } a \text{ and } \sqrt{7} \text{ for } b.
\end{aligned}
$$

c) Once again, either we can use FOIL to expand $(2\sqrt{5} - 3)^2$ or we can use the special formula we learned for squaring a binomial.

We will use $(a - b)^2 = a^2 - 2ab + b^2$.

$$
\begin{aligned}
(2\sqrt{5} - 3)^2 &= (2\sqrt{5})^2 - 2(2\sqrt{5})(3) + (3^2) && \text{Substitute } 2\sqrt{5} \text{ for } a \text{ and 3 for } b.\\
&= (4 \cdot 5) - 4\sqrt{5}(3) + 9 && \text{Multiply.}\\
&= 20 - 12\sqrt{5} + 9 && \text{Multiply.}\\
&= 29 - 12\sqrt{5} && \text{Combine like terms.}
\end{aligned}
$$

Remember that an expression is not considered to be in simplest form if it contains a radical in its denominator. To rationalize the denominator of a radical expression, we must keep in mind the index on the radical and the number of terms in the denominator.

Example 5

Rationalize the denominator of each expression.

a) $\dfrac{10}{\sqrt{2x}}$ b) $\dfrac{10}{\sqrt[3]{2x}}$ c) $\dfrac{\sqrt{10}}{\sqrt{2} - 1}$

Solution

a) First, notice that the denominator of $\dfrac{10}{\sqrt{2x}}$ contains only one term and it is a *square* root. Ask yourself, "By what do I multiply $\sqrt{2x}$ to get a perfect *square* under the radical?" The answer is $\sqrt{2x}$ since $\sqrt{2x} \cdot \sqrt{2x} = \sqrt{4x^2} = 2x$. Multiply the numerator and denominator by $\sqrt{2x}$, and simplify.

$$\frac{10}{\sqrt{2x}} = \frac{10}{\sqrt{2x}} \cdot \frac{\sqrt{2x}}{\sqrt{2x}} \qquad \text{Rationalize the denominator.}$$
$$= \frac{10\sqrt{2x}}{\sqrt{4x^2}} = \frac{10\sqrt{2x}}{2x} = \frac{5\sqrt{2x}}{x}$$

b) The denominator of $\dfrac{10}{\sqrt[3]{2x}}$ contains only one term, but it is a *cube* root. Ask yourself, "By what do I multiply $\sqrt[3]{2x}$ to get a radicand that is a perfect *cube*?" The answer is $\sqrt[3]{4x^2}$ since $\sqrt[3]{2x} \cdot \sqrt[3]{4x^2} = \sqrt[3]{8x^3} = 2x$. Multiply the numerator and denominator by $\sqrt[3]{4x^2}$, and simplify.

$$\frac{10}{\sqrt[3]{2x}} = \frac{10}{\sqrt[3]{2x}} \cdot \frac{\sqrt[3]{4x^2}}{\sqrt[3]{4x^2}} \qquad \text{Rationalize the denominator.}$$
$$= \frac{10\sqrt[3]{4x^2}}{\sqrt[3]{8x^3}} = \frac{10\sqrt[3]{4x^2}}{2x} = \frac{5\sqrt[3]{4x^2}}{x}$$

c) The denominator of $\dfrac{\sqrt{10}}{\sqrt{2} - 1}$ contains two terms, so how do we rationalize the denominator of this expression? We multiply the numerator and denominator by the *conjugate* of the denominator.

$$\frac{\sqrt{10}}{\sqrt{2} - 1} = \frac{\sqrt{10}}{\sqrt{2} - 1} \cdot \frac{\sqrt{2} + 1}{\sqrt{2} + 1} \qquad \text{Multiply by the conjugate.}$$
$$= \frac{\sqrt{10}(\sqrt{2} + 1)}{(\sqrt{2})^2 - (1)^2} \qquad \text{Multiply. } (a + b)(a - b) = a^2 - b^2$$
$$= \frac{\sqrt{20} + \sqrt{10}}{1} \qquad \text{Distribute. Simplify.}$$
$$= 2\sqrt{5} + \sqrt{10} \qquad \sqrt{20} = 2\sqrt{5}\text{; simplify.}$$

You Try 1

a) Perform the operations and simplify.

i) $(\sqrt{w}+8)^2$ ii) $(3-\sqrt{5a})(4+\sqrt{5a})$ iii) $\sqrt{2}(9\sqrt{10}-\sqrt{2})$

iv) $\sqrt{2}+9\sqrt{10}-5\sqrt{2}$ v) $(2\sqrt{3}+y)(2\sqrt{3}-y)$

b) Find each root.

i) $-\sqrt{\dfrac{121}{16}}$ ii) $\sqrt[3]{-1000}$ iii) $\sqrt{0.09}$ iv) $\sqrt{-49}$

c) Simplify completely. The answer should contain only positive exponents.

i) $(-64)^{2/3}$ ii) $\left(\dfrac{81x^3y^{1/2}}{x^{-5}y^6}\right)^{-3/4}$

d) Rationalize the denominator of each expression.

i) $\dfrac{24}{\sqrt[3]{9h}}$ ii) $\dfrac{7+\sqrt{6}}{4+\sqrt{6}}$ iii) $\dfrac{56}{\sqrt{7}}$

Answers to You Try Exercises

1) a) i) $w+16\sqrt{w}+64$ ii) $12-\sqrt{5a}-5a$ iii) $18\sqrt{5}-2$ iv) $-4\sqrt{2}+9\sqrt{10}$ v) $12-y^2$

b) i) $-\dfrac{11}{4}$ ii) -10 iii) 0.3 iv) not a real number

c) i) 16 ii) $\dfrac{y^{33/8}}{27x^6}$ d) i) $\dfrac{8\sqrt[3]{3h^2}}{h}$ ii) $\dfrac{22-3\sqrt{6}}{10}$ iii) $8\sqrt{7}$

Putting It All Together Summary Exercises

Objective 1: Review the Concepts Presented in Sections 10.1–10.6

Assume all variables represent positive real numbers.

Find each root, if possible.

1) $\sqrt[4]{81}$
2) $\sqrt[3]{-1000}$
3) $-\sqrt[6]{64}$
4) $\sqrt{121}$
5) $\sqrt{-169}$
6) $\sqrt{\dfrac{144}{49}}$

Simplify completely. The answer should contain only positive exponents.

7) $(144)^{1/2}$
8) $(-32)^{4/5}$
9) $-1000^{2/3}$
10) $\left(-\dfrac{16}{81}\right)^{3/4}$
11) $125^{-1/3}$
12) $\left(\dfrac{100}{9}\right)^{-3/2}$
13) $k^{-3/5}\cdot k^{3/10}$
14) $(t^{3/8})^{16}$
15) $\left(\dfrac{27a^{-8}}{b^9}\right)^{2/3}$
16) $\left(\dfrac{18x^{-9}y^{4/3}}{2x^3y}\right)^{-5/2}$

Simplify completely.

17) $\sqrt{24}$
18) $\sqrt[4]{32}$
19) $\sqrt[3]{72}$
20) $\sqrt[3]{\dfrac{500}{2}}$
21) $\sqrt[4]{243}$
22) $\sqrt{45t^{11}}$
23) $\sqrt[3]{96m^7n^{15}}$
24) $\sqrt[5]{\dfrac{64x^{19}}{y^{20}}}$

Perform the operations and simplify.

25) $\sqrt[3]{12}\cdot\sqrt[3]{2}$
26) $\sqrt[4]{\dfrac{96k^{11}}{2k^3}}$
VIDEO 27) $(6+\sqrt{7})(2+\sqrt{7})$
28) $4c^2\sqrt[3]{108c}-15\sqrt[3]{32c^7}$

29) $\dfrac{18}{\sqrt{6}}$

30) $\dfrac{5}{\sqrt{3}-\sqrt{2}}$

35) $(2\sqrt{3}+10)^2$

36) $(\sqrt{2}+3)(\sqrt{2}-3)$

VIDEO 31) $3\sqrt{75m^3n}+m\sqrt{12mn}$

32) $\sqrt{6p^7q^3}\cdot\sqrt{15pq^2}$

37) $\dfrac{\sqrt{2}}{4+\sqrt{10}}$

38) $\sqrt[3]{r^2}\cdot\sqrt{r}$

33) $\dfrac{\sqrt{60t^8u^3}}{\sqrt{5t^2u}}$

34) $\dfrac{9}{\sqrt[3]{2}}$

39) $\sqrt[3]{\dfrac{b^2}{9c}}$

40) $\dfrac{\sqrt[4]{32}}{\sqrt[4]{w^{11}}}$

Section 10.7 Solving Radical Equations

Objectives

1. **Understand the Steps for Solving a Radical Equation**
2. **Solve an Equation Containing One Square Root**
3. **Solve an Equation Containing Two Square Roots**
4. **Solve an Equation Containing a Cube Root**

In this section, we will learn how to solve *radical equations.*

An equation containing a variable in the radicand is a **radical equation.** Some examples of radical equations are

$$\sqrt{p}=7 \qquad \sqrt[3]{n}=2 \qquad \sqrt{2x+1}+1=x \qquad \sqrt{5w+6}-\sqrt{4w+1}=1$$

1. Understand the Steps for Solving a Radical Equation

Let's review what happens when we square a square root expression: If $x \ge 0$, then $(\sqrt{x})^2 = x$. That is, to eliminate the radical from $\sqrt{x}$, we *square* the expression. Therefore, to solve equations like those above containing *square roots*, we *square* both sides of the equation to obtain new equations. The solutions of the new equations contain all of the solutions of the original equation and may also contain *extraneous solutions.*

An **extraneous solution** is a value that satisfies one of the new equations but does not satisfy the original equation. Extraneous solutions occur frequently when solving radical equations, so we *must* check all possible solutions in the original equation and discard any that are extraneous.

Procedure Solving Radical Equations Containing Square Roots

Step 1: Get a radical on a side by itself.

Step 2: Square both sides of the equation to eliminate a radical.

Step 3: Combine like terms on each side of the equation.

Step 4: If the equation still contains a radical, repeat steps 1–3.

Step 5: Solve the equation.

Step 6: Check the proposed solutions *in the original equation* and discard extraneous solutions.

2. Solve an Equation Containing One Square Root

Example 1

Solve.

a) $\sqrt{c-2}=3$ b) $\sqrt{t+5}+6=0$

Solution

a) ***Step 1:*** The radical *is* on a side by itself: $\sqrt{c-2}=3$

Step 2: *Square* both sides to eliminate the *square root.*

$$(\sqrt{c-2})^2 = 3^2 \quad \text{Square both sides.}$$
$$c-2=9$$

Steps 3 and 4 do not apply because there are no like terms to combine and no radicals remain.

Step 5: Solve the equation.

$$c = 11 \quad \text{Add 2 to each side.}$$

Step 6: Check $c = 11$ in the *original* equation.

$$\begin{aligned} \sqrt{c - 2} &= 3 \\ \sqrt{11 - 2} &\stackrel{?}{=} 3 \\ \sqrt{9} &= 3 \quad \checkmark \end{aligned}$$

The solution set is $\{11\}$.

b) The first step is to get the radical on a side by itself.

$$\begin{aligned} \sqrt{t + 5} + 6 &= 0 & & \\ \sqrt{t + 5} &= -6 & & \text{Subtract 6 from each side.} \\ (\sqrt{t + 5})^2 &= (-6)^2 & & \text{Square both sides to eliminate the radical.} \\ t + 5 &= 36 & & \text{The square root has been eliminated.} \\ t &= 31 & & \text{Solve the equation.} \end{aligned}$$

Check $t = 31$ in the *original* equation.

$$\begin{aligned} \sqrt{t + 5} + 6 &= 0 \\ \sqrt{31 + 5} + 6 &\stackrel{?}{=} 0 \\ 6 + 6 &\stackrel{?}{=} 0 \quad \text{FALSE} \end{aligned}$$

Because $t = 31$ is an extraneous solution, the equation has no real solution. The solution set is $\varnothing$.

You Try 1

Solve.

a) $\sqrt{a + 4} = 7$ b) $\sqrt{m - 7} + 12 = 9$

Sometimes, we have to square a binomial in order to solve a radical equation. Don't forget that when we square a binomial, we can use either FOIL or one of the following formulas: $(a + b)^2 = a^2 + 2ab + b^2$ or $(a - b)^2 = a^2 - 2ab + b^2$.

Example 2

Solve $\sqrt{2x + 1} + 1 = x$.

Solution

Start by getting the radical on a side by itself.

$$\begin{aligned} \sqrt{2x + 1} &= x - 1 & & \text{Subtract 1 from each side.} \\ (\sqrt{2x + 1})^2 &= (x - 1)^2 & & \text{Square both sides to eliminate the radical.} \\ 2x + 1 &= x^2 - 2x + 1 & & \text{Simplify; square the binomial.} \\ 0 &= x^2 - 4x & & \text{Subtract } 2x\text{; subtract 1.} \\ 0 &= x(x - 4) & & \text{Factor.} \end{aligned}$$

$$\begin{aligned} x = 0 \quad &\text{or} \quad x - 4 = 0 & & \text{Set each factor equal to zero.} \\ x = 0 \quad &\text{or} \quad x = 4 & & \text{Solve.} \end{aligned}$$

Check $x = 0$ and $x = 4$ in the *original* equation.

$x = 0$:
$$\begin{aligned} \sqrt{2x + 1} + 1 &= x \\ \sqrt{2(0) + 1} + 1 &\stackrel{?}{=} 0 \\ \sqrt{1} + 1 &\stackrel{?}{=} 0 \\ 2 &\stackrel{?}{=} 0 \quad \text{FALSE} \end{aligned}$$

$x = 4$:
$$\begin{aligned} \sqrt{2x + 1} + 1 &= x \\ \sqrt{2(4) + 1} + 1 &\stackrel{?}{=} 4 \\ \sqrt{9} + 1 &\stackrel{?}{=} 4 \\ 3 + 1 &\stackrel{?}{=} 4 \quad \text{TRUE} \end{aligned}$$

$x = 4$ *is* a solution but $x = 0$ is *not* because $x = 0$ does not satisfy the original equation. The solution set is $\{4\}$.

You Try 2

Solve

a) $\sqrt{3p+10} - 4 = p$ b) $\sqrt{4h-3} - h = -2$

3. Solve an Equation Containing Two Square Roots

Next, we will take our first look at solving an equation containing two square roots.

Example 3

Solve $\sqrt{2a+4} - 3\sqrt{a-5} = 0$.

Solution

Begin by getting a radical on a side by itself.

$$\begin{aligned} \sqrt{2a+4} &= 3\sqrt{a-5} && \text{Add } 3\sqrt{a-5} \text{ to each side.} \\ (\sqrt{2a+4})^2 &= (3\sqrt{a-5})^2 && \text{Square both sides to eliminate the radicals.} \\ 2a+4 &= 9(a-5) && 3^2 = 9 \\ 2a+4 &= 9a-45 && \text{Distribute.} \\ -7a &= -49 && \\ a &= 7 && \text{Solve.} \end{aligned}$$

Check $a = 7$ in the original equation.

$$\begin{aligned} \sqrt{2a+4} - 3\sqrt{a-5} &= 0 \\ \sqrt{2(7)+4} - 3\sqrt{7-5} &\stackrel{?}{=} 0 \\ \sqrt{14+4} - 3\sqrt{2} &\stackrel{?}{=} 0 \\ \sqrt{18} - 3\sqrt{2} &\stackrel{?}{=} 0 \\ 3\sqrt{2} - 3\sqrt{2} &\stackrel{?}{=} 0 \quad \checkmark \end{aligned}$$

The solution set is $\{7\}$.

You Try 3

Solve $4\sqrt{r-3} - \sqrt{6r+2} = 0$.

Recall from Section 10.5 that we can square binomials containing radical expressions just like we squared $(x-1)^2$ in Example 2: we can use FOIL or the formulas

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \text{or} \qquad (a-b)^2 = a^2 - 2ab + b^2$$

Example 4

Square and simplify $(3 - \sqrt{m+2})^2$.

Solution

Use the formula $(a-b)^2 = a^2 - 2ab + b^2$.

$$\begin{aligned} (3 - \sqrt{m+2})^2 &= (3)^2 - 2(3)(\sqrt{m+2}) + (\sqrt{m+2})^2 && \text{Substitute 3 for } a \text{ and } \sqrt{m+2} \text{ for } b. \\ &= 9 - 6\sqrt{m+2} + (m+2) && \\ &= m + 11 - 6\sqrt{m+2} && \text{Combine like terms.} \end{aligned}$$

You Try 4

Square and simplify each expression.

a) $(\sqrt{z} - 4)^2$ b) $(5 + \sqrt{3d-1})^2$

To solve the next two equations, we will have to square both sides of the equation twice to eliminate the radicals. Be very careful when you are squaring binomials that contain a radical.

Example 5

Solve each equation.

a) $\sqrt{x+5} + \sqrt{x} = 5$ b) $\sqrt{5w+6} - \sqrt{4w+1} = 1$

Solution

a) This equation contains two radicals *and* a constant. Get one of the radicals on a side by itself, then square both sides.

$$\sqrt{x+5} = 5 - \sqrt{x} \qquad \text{Subtract } \sqrt{x} \text{ from each side.}$$
$$(\sqrt{x+5})^2 = (5 - \sqrt{x})^2 \qquad \text{Square both sides.}$$
$$x + 5 = (5)^2 - 2(5)(\sqrt{x}) + (\sqrt{x})^2 \qquad \text{Use the formula } (a-b)^2 = a^2 - 2ab + b^2.$$
$$x + 5 = 25 - 10\sqrt{x} + x \qquad \text{Simplify.}$$

The equation still contains a radical. Therefore, repeat steps 1–3. Begin by getting the radical on a side by itself.

$$5 = 25 - 10\sqrt{x} \qquad \text{Subtract } x \text{ from each side.}$$
$$-20 = -10\sqrt{x} \qquad \text{Subtract 25 from each side.}$$
$$2 = \sqrt{x} \qquad \text{Divide by } -10$$
$$2^2 = (\sqrt{x})^2 \qquad \text{Square both sides.}$$
$$4 = x \qquad \text{Solve.}$$

The check is left to the student. The solution set is $\{4\}$.

b) ***Step 1:*** Get a radical on a side by itself.

$$\sqrt{5w+6} - \sqrt{4w+1} = 1$$
$$\sqrt{5w+6} = 1 + \sqrt{4w+1} \qquad \text{Add } \sqrt{4w+1} \text{ to each side.}$$

Step 2: Square both sides of the equation to eliminate a radical.

$$(\sqrt{5w+6})^2 = (1 + \sqrt{4w+1})^2 \qquad \text{Square both sides.}$$
$$5w + 6 = (1)^2 + 2(1)(\sqrt{4w+1}) + (\sqrt{4w+1})^2 \qquad \text{Use the formula } (a+b)^2 = a^2 + 2ab + b^2.$$
$$5w + 6 = 1 + 2\sqrt{4w+1} + 4w + 1$$

Step 3: Combine like terms on the right side.

$$5w + 6 = 4w + 2 + 2\sqrt{4w+1} \qquad \text{Combine like terms.}$$

Step 4: The equation still contains a radical, so repeat steps 1–3.

Step 1: Get the radical on a side by itself.

$$5w + 6 = 4w + 2 + 2\sqrt{4w+1}$$
$$w + 4 = 2\sqrt{4w+1} \qquad \text{Subtract } 4w \text{ and subtract 2.}$$

We do not need to eliminate the 2 from in front of the radical before squaring both sides. The radical must not be a part of a *sum* or *difference* when we square.

Step 2: Square both sides of the equation to eliminate the radical.

$$(w+4)^2 = (2\sqrt{4w+1})^2 \qquad \text{Square both sides.}$$
$$w^2 + 8w + 16 = 4(4w+1) \qquad \text{Square the binomial; } 2^2 = 4.$$

Steps 3 and 4 no longer apply.

Step 5: Solve the equation.

$$
\begin{aligned}
w^2 + 8w + 16 &= 16w + 4 && \text{Distribute.}\\
w^2 - 8w + 12 &= 0 && \text{Subtract } 16w \text{ and subtract } 4.\\
(w-2)(w-6) &= 0 && \text{Factor.}
\end{aligned}
$$

$$
\begin{aligned}
w - 2 = 0 \quad &\text{or} \quad w - 6 = 0 && \text{Set each factor equal to zero.}\\
w = 2 \quad &\text{or} \quad w = 6 && \text{Solve.}
\end{aligned}
$$

Step 6: The check is left to the student. Verify that $w = 2$ and $w = 6$ each satisfy the original equation. The solution set is $\{2, 6\}$.

You Try 5

Solve each equation.

a) $\sqrt{2y+1} - \sqrt{y} = 1$ b) $\sqrt{3t+4} + \sqrt{t+2} = 2$

BE CAREFUL

Watch out for two common mistakes that students make when solving an equation like the one in Example 5b.

1) Do not square both sides before getting a radical on a side by itself.

This is incorrect: $(\sqrt{5w+6} - \sqrt{4w+1})^2 = 1^2$

$5w + 6 - (4w + 1) = 1$

2) The *second* time we perform step 2, watch out for this common error.

This is incorrect: $(w+4)^2 = (2\sqrt{4w+1})^2$

$w^2 + 16 = 2(4w + 1)$

On the left we must multiply using FOIL or the formula $(a+b)^2 = a^2 + 2ab + b^2$ and on the right we must remember to square the 2.

4. Solve an Equation Containing a Cube Root

We can solve many equations containing cube roots the same way we solve equations containing square roots, except to eliminate a *cube root*, we *cube* both sides of the equation.

Example 6

Solve $\sqrt[3]{7a+1} - 2\sqrt[3]{a-1} = 0$.

Solution

Begin by getting a radical on a side by itself.

$$
\begin{aligned}
\sqrt[3]{7a+1} &= 2\sqrt[3]{a-1} && \text{Add } 2\sqrt[3]{a-1} \text{ to each side.}\\
(\sqrt[3]{7a+1})^3 &= (2\sqrt[3]{a-1})^3 && \text{Cube both sides to eliminate the radicals.}\\
7a + 1 &= 8(a-1) && \text{Simplify; } 2^3 = 8.\\
7a + 1 &= 8a - 8 && \text{Distribute.}\\
9 &= a && \text{Subtract } 7a\text{; add } 8.
\end{aligned}
$$

Check $a = 9$ in the original equation.

$$
\begin{aligned}
\sqrt[3]{7a+1} - 2\sqrt[3]{a-1} &= 0\\
\sqrt[3]{7(9)+1} - 2\sqrt[3]{9-1} &\stackrel{?}{=} 0\\
\sqrt[3]{64} - 2\sqrt[3]{8} &\stackrel{?}{=} 0\\
4 - 2(2) &\stackrel{?}{=} 0\\
4 - 4 &= 0 \quad \checkmark
\end{aligned}
$$

The solution set is $\{9\}$.

You Try 6

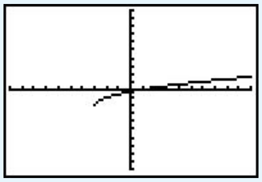
Solve $3\sqrt[3]{r-4} - \sqrt[3]{5r+2} = 0$.

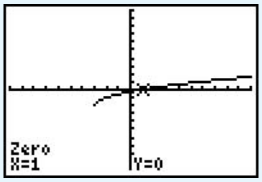
Using Technology

We can use a graphing calculator to solve a radical equation in one variable. First subtract every term on the right side of the equation from both sides of the equation and enter the result in Y_1. Graph the equation in Y_1. The zeros or x-intercepts of the graph are the solutions to the equation.

We will solve $\sqrt{x+3} = 2$ using a graphing calculator.

1) Enter $\sqrt{x+3} - 2$ in Y_1.
2) Press $\boxed{\text{ZOOM}}$ $\boxed{6}$ to graph the function in Y_1 as shown at the left below.
3) Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ 2:zero, move the cursor to the left of the zero and press $\boxed{\text{ENTER}}$, move the cursor to the right of the zero and press $\boxed{\text{ENTER}}$, and move the cursor close to the zero and press $\boxed{\text{ENTER}}$ to display the zero. The solution to the equation is $x = 1$, as shown at the right below.





Solve each equation using a graphing calculator.

1) $\sqrt{x-2} = 1$
2) $\sqrt{3x-2} = 5$
3) $\sqrt{3x-2} = \sqrt{x} + 2$
4) $\sqrt{4x-5} = \sqrt{x+4}$
5) $\sqrt{2x-7} = \sqrt{x} - 1$
6) $\sqrt{\sqrt{x}-1} = 1$

Answers to You Try Exercises

1) a) $\{45\}$ b) $\varnothing$ 2) a) $\{-3, -2\}$ b) $\{7\}$ 3) $\{5\}$ 4) a) $z - 8\sqrt{z} + 16$ b) $3d + 24 + 10\sqrt{3d-1}$ 5) a) $\{0, 4\}$ b) $\{-1\}$ 6) $\{5\}$

Answers to Technology Exercises

1) $\{3\}$ 2) $\{9\}$ 3) $\{9\}$ 4) $\{3\}$ 5) $\{4\}$ 6 $\{4\}$

10.7 Exercises

Objective 1: Understand the Steps for Solving a Radical Equation

1) Why is it necessary to check the proposed solutions to a radical equation in the original equation?

2) How do you know, without actually solving and checking the solution, that $\sqrt{y} = -3$ has no solution?

Objective 2: Solve an Equation Containing One Square Root

Solve.

3) $\sqrt{q} = 7$

4) $\sqrt{z} = 10$

5) $\sqrt{w} - \dfrac{2}{3} = 0$

6) $\sqrt{r} - \dfrac{3}{5} = 0$

7) $\sqrt{a} + 5 = 3$

8) $\sqrt{k} + 8 = 2$

9) $\sqrt{b - 11} - 3 = 0$

10) $\sqrt{d + 3} - 5 = 0$

11) $\sqrt{4g - 1} + 7 = 1$

12) $\sqrt{3v + 4} + 10 = 6$

13) $\sqrt{3f + 2} + 9 = 11$

14) $\sqrt{5u - 4} + 12 = 17$

15) $m = \sqrt{m^2 - 3m + 6}$

16) $b = \sqrt{b^2 + 4b - 24}$

17) $\sqrt{9r^2 - 2r + 10} = 3r$

18) $\sqrt{4p^2 - 3p + 6} = 2p$

Square each binomial and simplify.

19) $(n + 5)^2$

20) $(z - 3)^2$

21) $(c - 6)^2$

22) $(2k + 1)^2$

Solve.

23) $p + 6 = \sqrt{12 + p}$

24) $c - 7 = \sqrt{2c + 1}$

25) $6 + \sqrt{c^2 + 3c - 9} = c$

26) $-4 + \sqrt{z^2 + 5z - 8} = z$

27) $w - \sqrt{10w + 6} = -3$

28) $3 - \sqrt{8t + 9} = -t$

29) $3v = 8 + \sqrt{3v + 4}$

30) $4k = 3 + \sqrt{10k + 5}$

31) $m + 4 = 5\sqrt{m}$

32) $b + 5 = 6\sqrt{b}$

33) $y + 2\sqrt{6 - y} = 3$

34) $r - 3\sqrt{r + 2} = 2$

35) $\sqrt{r^2 - 8r - 19} = r - 9$

36) $\sqrt{x^2 + x + 4} = x + 8$

Objective 3: Solve an Equation Containing Two Square Roots

37) $5\sqrt{1 - 5h} = 4\sqrt{1 - 8h}$

38) $3\sqrt{6a - 2} - 4\sqrt{3a + 3} = 0$

39) $3\sqrt{3x + 6} - 2\sqrt{9x - 9} = 0$

40) $5\sqrt{q + 11} = 2\sqrt{8q + 25}$

41) $\sqrt{m} = 3\sqrt{7}$

42) $4\sqrt{3} = \sqrt{p}$

43) $\sqrt{2w - 1} + 2\sqrt{w + 4} = 0$

44) $2\sqrt{3t + 4} + \sqrt{t - 6} = 0$

Square each expression and simplify.

45) $(\sqrt{x} + 5)^2$

46) $(\sqrt{y} - 8)^2$

47) $(9 - \sqrt{a + 4})^2$

48) $(4 + \sqrt{p + 5})^2$

49) $(2\sqrt{3n - 1} + 7)^2$

50) $(5 - 3\sqrt{2k - 3})^2$

Solve.

51) $\sqrt{2y - 1} = 2 + \sqrt{y - 4}$

52) $\sqrt{3n + 4} = \sqrt{2n + 1} + 1$

53) $1 + \sqrt{3s - 2} = \sqrt{2s + 5}$

54) $\sqrt{4p + 12} - 1 = \sqrt{6p - 11}$

55) $\sqrt{5a + 19} - \sqrt{a + 12} = 1$

56) $\sqrt{2u + 3} - \sqrt{5u + 1} = -1$

57) $\sqrt{3k + 1} - \sqrt{k - 1} = 2$

58) $\sqrt{4z - 3} - \sqrt{5z + 1} = -1$

59) $\sqrt{3x + 4} - 5 = \sqrt{3x - 11}$

60) $\sqrt{4c - 7} = \sqrt{4c + 1} - 4$

61) $\sqrt{3v + 3} - \sqrt{v - 2} = 3$

62) $\sqrt{2y + 1} - \sqrt{y} = 1$

Objective 4: Solve an Equation Containing a Cube Root

63) How do you eliminate the radical from an equation like $\sqrt[3]{x} = 2$?

64) Give a reason why $\sqrt[3]{h} = -3$ has no extraneous solutions.

Solve.

65) $\sqrt[3]{y} = 5$

66) $\sqrt[3]{c} = 3$

67) $\sqrt[3]{m} = -4$

68) $\sqrt[3]{t} = -2$

69) $\sqrt[3]{2x - 5} + 3 = 1$

70) $\sqrt[3]{4a + 1} + 7 = 4$

71) $\sqrt[3]{6j - 2} = \sqrt[3]{j - 7}$

72) $\sqrt[3]{w + 3} = \sqrt[3]{2w - 11}$

73) $\sqrt[3]{3y - 1} - \sqrt[3]{2y - 3} = 0$

74) $\sqrt[3]{2 - 2b} + \sqrt[3]{b - 5} = 0$

75) $\sqrt[3]{2n^2} = \sqrt[3]{7n + 4}$

76) $\sqrt[3]{4c^2 - 5c + 11} = \sqrt[3]{c^2 + 9}$

Extension

Solve.

77) $p^{1/2} = 6$

78) $\dfrac{2}{3} = t^{1/2}$

79) $7 = (2z - 3)^{1/2}$

80) $(3k + 1)^{1/2} = 4$

81) $(y + 4)^{1/3} = 3$

82) $-5 = (a - 2)^{1/3}$

83) $\sqrt[4]{n + 7} = 2$

84) $\sqrt[4]{x - 3} = -1$

85) $\sqrt{13 + \sqrt{r}} = \sqrt{r + 7}$

86) $\sqrt{m - 1} = \sqrt{m - \sqrt{m - 4}}$

87) $\sqrt{y + \sqrt{y + 5}} = \sqrt{y + 2}$

88) $\sqrt{2d - \sqrt{d + 6}} = \sqrt{d + 6}$

Mixed Exercises: Objectives 2 and 4

Solve for the indicated variable.

89) $v = \sqrt{\dfrac{2E}{m}}$ for E

90) $V = \sqrt{\dfrac{300VP}{m}}$ for P

91) $c = \sqrt{a^2 + b^2}$ for b^2

92) $r = \sqrt{\dfrac{A}{\pi}}$ for A

93) $T = \sqrt[4]{\dfrac{E}{\sigma}}$ for σ

94) $r = \sqrt[3]{\dfrac{3V}{4\pi}}$ for V

95) The speed of sound is proportional to the square root of the air temperature in still air. The speed of sound is given by the formula.

$$V_S = 20\sqrt{T + 273}$$

where V_S is the speed of sound in meters/second and T is the temperature of the air in °Celsius.

a) What is the speed of sound when the temperature is -17°C (about 1°F)?

b) What is the speed of sound when the temperature is 16°C (about 61°F)?

c) What happens to the speed of sound as the temperature increases?

d) Solve the equation for T.

96) If the area of a square is A and each side has length l, then the length of a side is given by

$$l = \sqrt{A}$$

A square rug has an area of 25 ft^2.

a) Find the dimensions of the rug.

b) Solve the equation for A.

97) Let V represent the volume of a cylinder, h represent its height, and r represent its radius. V, h, and r are related according to the formula

$$r = \sqrt{\frac{V}{\pi h}}$$

a) A cylindrical soup can has a volume of 28π in^3. It is 7 in. high. What is the radius of the can?

b) Solve the equation for V.

98) For shallow water waves, the wave velocity c, in ft/sec, is given by

$$c = \sqrt{gH}$$

where g is the acceleration due to gravity (32 ft/sec^2) and H is the depth of the water (in feet).

a) Find the velocity of a wave in 8 ft of water.

b) Solve the equation for H.

99) Refer to the formula given in Exercise 98.

The catastrophic Indian Ocean tsunami that hit Banda Aceh, Sumatra, Indonesia, on December 26, 2004 was caused by an earthquake whose epicenter was off the coast of northern Sumatra. The tsunami originated in about 14,400 feet of water.

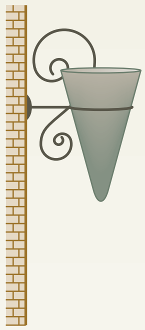
a) Find the velocity of the wave near the epicenter, in miles per hour. Round the answer to the nearest unit. (Hint: 1 mile = 5280 ft.)

b) Banda Aceh, the area hardest hit by the tsunami, was about 60 miles from the tsunami's origin. Approximately how many minutes after the earthquake occurred did the tsunami hit Banda Aceh?

(S. Reynolds et al., *Exploring Geology,* Second Edition, McGraw-Hill, 2008)

100) The radius r of a cone with height h and volume V is given by $r = \sqrt{\frac{3V}{\pi h}}$.

A hanging glass vase in the shape of a cone is 8 inches tall, and the radius of the top of the cone is 2 inches. How much water will the vase hold? Give an exact answer and an approximation to the nearest tenth.



Use the following information for Exercises 101 and 102.

The distance a person can see to the horizon is approximated by the function $D(h) = 1.2\sqrt{h}$, where D is the number of miles a person can see to the horizon from a height of h feet.

101) Sig is the captain of an Alaskan crab fishing boat and can see 4.8 miles to the horizon when he is sailing his ship. Find his height above the sea.

102) Cooper is standing on the deck of a boat and can see 3.6 miles to the horizon. What is his height above the water?

Use the following information for Exercises 103 and 104.

When the air temperature is 0°F, the windchill temperature, W, in degrees Fahrenheit is a function of the velocity of the wind, V, in miles per hour and is given by the formula

$$W(V) = 35.74 - 35.75V^{4/25}$$

103) Calculate the wind speed when the windchill temperature is -10°F. Round to the nearest whole number.

104) Find V so that $W(V) = -20$. Round to the nearest whole number. Explain your result in the context of the problem.

(http://www.nws.noaa.gov/om/windchill/windchillglossary.shtml)

Section 10.8 Complex Numbers

Objectives

1. Find the Square Root of a Negative Number
2. Multiply and Divide Square Roots Containing Negative Numbers
3. Add and Subtract Complex Numbers
4. Multiply Complex Numbers
5. Multiply a Complex Number by Its Conjugate
6. Divide Complex Numbers
7. Simplify Powers of i

1. Find the Square Root of a Negative Number

We have seen throughout this chapter that the square root of a negative number does not exist in the real number system because there is no real number that, when squared, will result in a negative number. For example, $\sqrt{-4}$ is not a real number because there is no real number whose square is -4.

The square roots of negative numbers do exist, however, under another system of numbers called *complex numbers*. Before we define a complex number, we must define the number i. The number i is called an *imaginary number.*

Definition

The **imaginary number i** is defined as $i = \sqrt{-1}$.

Therefore, squaring both sides gives us $i^2 = -1$.

Note

$i = \sqrt{-1}$ and $i^2 = -1$ are two *very* important facts to remember. We will be using them often!

Definition

A **complex number** is a number of the form $a + bi$, where a and b are real numbers; a is called the **real part** and b is called the **imaginary part**.

The following table lists some examples of complex numbers and their real and imaginary parts.

Complex Number	Real Part	Imaginary Part
$5 + 2i$	5	2
$\frac{1}{3} - 7i$	$\frac{1}{3}$	-7
$8i$	0	8
4	4	0

Note

The complex number $8i$ can be written in the form $a + bi$ as $0 + 8i$. Likewise, besides being a real number, 4 is a complex number since it can be written as $4 + 0i$.

Since all real numbers, a, can be written in the form $a + 0i$, all real numbers are also complex numbers.

Property Real Numbers and Complex Numbers

The set of real numbers is a subset of the set of complex numbers.

Since we defined i as $i = \sqrt{-1}$, we can now evaluate square roots of negative numbers.

Example 1

Simplify.

a) $\sqrt{-9}$ b) $\sqrt{-7}$ c) $\sqrt{-12}$

Solution

a) $\sqrt{-9} = \sqrt{-1 \cdot 9} = \sqrt{-1} \cdot \sqrt{9} = i \cdot 3 = 3i$

b) $\sqrt{-7} = \sqrt{-1 \cdot 7} = \sqrt{-1} \cdot \sqrt{7} = i\sqrt{7}$

c) $\sqrt{-12} = \sqrt{-1 \cdot 12} = \sqrt{-1} \cdot \sqrt{12} = i\sqrt{4}\sqrt{3} = i \cdot 2\sqrt{3} = 2i\sqrt{3}$

Note

In Example 1b), we wrote $i\sqrt{7}$ instead of $\sqrt{7}i$, and in Example 1c) we wrote $2i\sqrt{3}$ instead of $2\sqrt{3}i$. We do this to be clear that the i is not under the radical. It is good practice to write the i *before* the radical.

You Try 1

Simplify.

a) $\sqrt{-36}$ b) $\sqrt{-13}$ c) $\sqrt{-20}$

2. Multiply and Divide Square Roots Containing Negative Numbers

When multiplying or dividing radicals with negative radicands, write each radical in terms of i first. Remember, also, that since $i = \sqrt{-1}$, it follows that $i^2 = -1$. We must keep this in mind when simplifying expressions.

Note

Whenever an i^2 appears in an expression, replace it with -1.

Example 2

Multiply and simplify. $\sqrt{-8} \cdot \sqrt{-2}$

Solution

$$\begin{aligned} \sqrt{-8} \cdot \sqrt{-2} &= i\sqrt{8} \cdot i\sqrt{2} && \text{Write each radical in terms of } i \textit{ before} \text{ multiplying.} \\ &= i^2\sqrt{16} && \text{Multiply.} \\ &= (-1)(4) && \text{Replace } i^2 \text{ with } -1. \\ &= -4 \end{aligned}$$

You Try 2

Perform the operation and simplify.

a) $\sqrt{-6} \cdot \sqrt{-3}$ b) $\dfrac{\sqrt{-72}}{\sqrt{-2}}$

3. Add and Subtract Complex Numbers

Just as we can add, subtract, multiply, and divide real numbers, we can perform all of these operations with complex numbers.

Procedure Adding and Subtracting Complex Numbers

1) To add complex numbers, add the real parts and add the imaginary parts.
2) To subtract complex numbers, apply the distributive property and combine the real parts and combine the imaginary parts.

Example 3

Add or subtract.

a) $(8 + 3i) + (4 + 2i)$ b) $(7 + i) - (3 - 4i)$

Solution

a) $(8 + 3i) + (4 + 2i) = (8 + 4) + (3 + 2)i$ Add real parts; add imaginary parts.
$= 12 + 5i$

b) $(7 + i) - (3 - 4i) = 7 + i - 3 + 4i$ Distributive property
$= (7 - 3) + (1 + 4)i$ Add real parts; add imaginary parts.
$= 4 + 5i$

You Try 3

Add or subtract.

a) $(-10 + 6i) + (1 + 8i)$ b) $(2 - 5i) - (-1 + 6i)$

4. Multiply Complex Numbers

We multiply complex numbers just like we would multiply polynomials. There may be an additional step, however. Remember to replace i^2 with -1.

Example 4

Multiply and simplify.

a) $5(-2 + 3i)$ b) $(8 + 3i)(-1 + 4i)$ c) $(6 + 2i)(6 - 2i)$

Solution

a) $5(-2 + 3i) = -10 + 15i$ Distributive property

b) Look carefully at $(8 + 3i)(-1 + 4i)$. Each complex number has two terms, similar to, say, $(x + 3)(x + 4)$. How can we multiply these two complex numbers? We can use FOIL.

$$\begin{aligned}(8 + 3i)(-1 + 4i) &= \overset{F}{(8)(-1)} + \overset{O}{(8)(4i)} + \overset{I}{(3i)(-1)} + \overset{L}{(3i)(4i)} \\ &= -8 + 32i - 3i + 12i^2 \\ &= -8 + 29i + 12(-1) && \text{Replace } i^2 \text{ with } -1. \\ &= -8 + 29i - 12 \\ &= -20 + 29i && \text{Combine like terms.}\end{aligned}$$

c) Use FOIL to find the product $(6 + 2i)(6 - 2i)$.

$$\begin{aligned}(6 + 2i)(6 - 2i) &= \overset{F}{(6)(6)} + \overset{O}{(6)(-2i)} + \overset{I}{(2i)(6)} + \overset{L}{(2i)(-2i)} \\ &= 36 - 12i + 12i - 4i^2 \\ &= 36 - 4(-1) \qquad \text{Replace } i^2 \text{ with } -1. \\ &= 36 + 4 \\ &= 40\end{aligned}$$

You Try 4

Multiply and simplify.

a) $-3(6 - 7i)$ b) $(5 - i)(4 + 8i)$ c) $(-2 - 6i)(-2 + 6i)$

5. Multiply a Complex Number by Its Conjugate

In Section 10.6, we learned about conjugates of radical expressions. For example, the conjugate of $3 + \sqrt{5}$ is $3 - \sqrt{5}$.

The complex numbers in Example 4c, $6 + 2i$ and $6 - 2i$, are **complex conjugates**.

Definition

The **conjugate** of $a + bi$ is $a - bi$.

We found that $(6 + 2i)(6 - 2i) = 40$, which is a real number. The product of a complex number and its conjugate is *always* a real number, as illustrated next.

$$\begin{aligned}(a + bi)(a - bi) &= \overset{F}{(a)(a)} + \overset{O}{(a)(-bi)} + \overset{I}{(bi)(a)} + \overset{L}{(bi)(-bi)} \\ &= a^2 - abi + abi - b^2i^2 \\ &= a^2 - b^2(-1) \qquad \text{Replace } i^2 \text{ with } -1. \\ &= a^2 + b^2\end{aligned}$$

We can summarize these facts about complex numbers and their conjugates as follows:

Summary Complex Conjugates

1) The conjugate of $a + bi$ is $a - bi$.
2) The product of $a + bi$ and $a - bi$ is a real number.
3) We can find the product $(a + bi)(a - bi)$ by using FOIL or by using $(a + bi)(a - bi) = a^2 + b^2$.

Example 5

Multiply $-3 + 4i$ by its conjugate using the formula $(a + bi)(a - bi) = a^2 + b^2$.

Solution

The conjugate of $-3 + 4i$ is $-3 - 4i$.

$$\begin{aligned}(-3 + 4i)(-3 - 4i) &= (-3)^2 + (4)^2 \qquad a = -3, b = 4 \\ &= 9 + 16 \\ &= 25\end{aligned}$$

You Try 5

Multiply $2 - 9i$ by its conjugate using the formula $(a + bi)(a - bi) = a^2 + b^2$.

6. Divide Complex Numbers

To rationalize the denominator of a radical expression like $\frac{2}{3 + \sqrt{5}}$, we multiply the numerator and denominator by $3 - \sqrt{5}$, the conjugate of the denominator. We divide complex numbers in the same way.

Procedure Dividing Complex Numbers

To divide complex numbers, multiply the numerator and denominator by the *conjugate of the denominator*. Write the quotient in the form $a + bi$.

Example 6

Divide. Write the quotient in the form $a + bi$.

a) $\frac{3}{4 - 5i}$ b) $\frac{6 - 2i}{-7 + i}$

Solution

a)

$$\frac{3}{4 - 5i} = \frac{3}{(4 - 5i)} \cdot \frac{(4 + 5i)}{(4 + 5i)}$$ Multiply the numerator and denominator by the conjugate of the denominator.

$$= \frac{12 + 15i}{4^2 + 5^2}$$ Multiply numerators.
$(a + bi)(a - bi) = a^2 + b^2$

$$= \frac{12 + 15i}{16 + 25}$$

$$= \frac{12 + 15i}{41}$$

$$= \frac{12}{41} + \frac{15}{41}i$$ Write the quotient in the form $a + bi$.

Recall that we can find the product $(4 - 5i)(4 + 5i)$ using FOIL *or* by using the formula $(a + bi)(a - bi) = a^2 + b^2$.

b)

$$\frac{6 - 2i}{-7 + i} = \frac{(6 - 2i)}{(-7 + i)} \cdot \frac{(-7 - i)}{(-7 - i)}$$ Multiply the numerator and denominator by the conjugate of the denominator.

$$= \frac{-42 - 6i + 14i + 2i^2}{(-7)^2 + (1)^2}$$ Multiply using FOIL.
$(a + bi)(a - bi) = a^2 + b^2$

$$= \frac{-42 + 8i - 2}{49 + 1} = \frac{-44 + 8i}{50} = -\frac{44}{50} + \frac{8}{50}i = -\frac{22}{25} + \frac{4}{25}i$$

You Try 6

Divide. Write the result in the form $a + bi$.

a) $\frac{6}{-2 + i}$ b) $\frac{5 + 3i}{-6 - 4i}$

7. Simplify Powers of i

All powers of i larger than i^1 (or just i) can be simplified. We use the fact that $i^2 = -1$ to simplify powers of i.

Let's write i through i^4 in their simplest forms.

i is in simplest form.

$$i^2 = -1$$
$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$
$$i^4 = (i^2)^2 = (-1)^2 = 1$$

Let's continue by simplifying i^5 and i^6.

$$\begin{aligned} i^5 &= i^4 \cdot i \\ &= (i^2)^2 \cdot i \\ &= (-1)^2 \cdot i \\ &= 1i \\ &= i \end{aligned}$$

$$\begin{aligned} i^6 &= (i^2)^3 \\ &= (-1)^3 \\ &= -1 \end{aligned}$$

The pattern repeats so that all powers of i can be simplified to i, -1, $-i$, or 1.

Example 7

Simplify each power of i.

a) i^8 b) i^{14} c) i^{11} d) i^{37}

Solution

a) Use the power rule for exponents to simplify i^8. Since the exponent is even, we can rewrite it in terms of i^2.

$$\begin{aligned} i^8 &= (i^2)^4 && \text{Power rule} \\ &= (-1)^4 && i^2 = -1 \\ &= 1 && \text{Simplify.} \end{aligned}$$

b) As in Example 1a), the exponent is even. Rewrite i^{14} in terms of i^2.

$$\begin{aligned} i^{14} &= (i^2)^7 && \text{Power rule} \\ &= (-1)^7 && i^2 = -1 \\ &= -1 && \text{Simplify.} \end{aligned}$$

c) The exponent of i^{11} is odd, so first use the product rule to write i^{11} as a product of i and i^{11-1} or i^{10}.

$$\begin{aligned} i^{11} &= i^{10} \cdot i && \text{Product rule} \\ &= (i^2)^5 \cdot i && \text{10 is even; write } i^{10} \text{ in terms of } i^2. \\ &= (-1)^5 \cdot i && i^2 = -1 \\ &= -1 \cdot i && \text{Simplify.} \\ &= -i && \text{Multiply} \end{aligned}$$

d) The exponent of i^{37} is odd. Use the product rule to write i^{37} as a product of i and i^{37-1} or i^{36}.

$$\begin{aligned} i^{37} &= i^{36} \cdot i && \text{Product rule} \\ &= (i^2)^{18} \cdot i && \text{36 is even; write } i^{36} \text{ in terms of } i^2. \\ &= (-1)^{18} \cdot i && i^2 = -1 \\ &= 1 \cdot i && \text{Simplify.} \\ &= i && \text{Multiply.} \end{aligned}$$

You Try 7

Simplify each power of i.

a) i^{18} b) i^{32} c) i^7 d) i^{25}

Using Technology

We can use a graphing calculator to perform operations on complex numbers or to evaluate square roots of negative numbers.

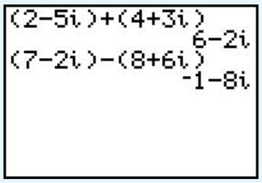
If the calculator is in the default REAL mode, the result is an error message "ERR: NONREAL ANS," which indicates that $\sqrt{-4}$ is a complex number rather than a real number. Before evaluating $\sqrt{-4}$ on the home screen of your calculator, check the mode by pressing MODE and, looking at row 7, change the mode to complex numbers by selecting $a + bi$, as shown at the left below.

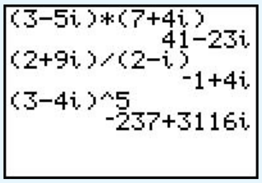
Now evaluating $\sqrt{-4}$ on the home screen results in the correct answer $2i$, as shown on the right below.

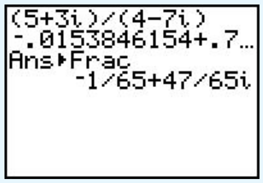
Operations can be performed on complex numbers with the calculator in either REAL or $a + bi$ mode. Simply use the arithmetic operators on the right column on your calculator. To enter the imaginary number i, press 2nd . . To add $2 - 5i$ and $4 + 3i$, enter $(2 - 5i) + (4 + 3i)$ on the home screen and press ENTER as shown on the left screen below. To subtract $8 + 6i$ from $7 - 2i$, enter $(7 - 2i) - (8 + 6i)$ on the home screen and press ENTER as shown.

To multiply $3 - 5i$ and $7 + 4i$, enter $(3 - 5i)/(7 + 4i)$ on the home screen and press ENTER as shown on the middle screen below. To divide $2 + 9i$ by $2 - i$, enter $(2 + 9i)/(2 - i)$ on the home screen and press ENTER as shown.

To raise $3 - 4i$ to the fifth power, enter $(3 - 4i)$^ 5 on the home screen and press ENTER as shown. Consider the quotient $(5 + 3i)/(4 - 7i)$. The exact answer is $-\frac{1}{65} + \frac{47}{65}i$. The calculator automatically displays the decimal result. Press MATH 1 ENTER to convert the decimal result to the exact fractional result, as shown on the right screen below.







Perform the indicated operation using a graphing calculator.

1) Simplify $\sqrt{-36}$.　2) $(3 + 7i) + (5 - 8i)$　3) $(10 - 3i) - (4 + 8i)$

4) $(3 + 2i)(6 - 3i)$　5) $(4 + 3i) \div (1 - i)$　6) $(5 - 3i)^3$

Answers to You Try Exercises

1) a) $6i$ b) $i\sqrt{13}$ c) $2i\sqrt{5}$　2) a) $-3\sqrt{2}$ b) 6　3) a) $-9 + 14i$ b) $3 - 11i$

4) a) $-18 + 21i$ b) $28 + 36i$ c) 40　5) 85　6) a) $-\frac{12}{5} - \frac{6}{5}i$ b) $-\frac{21}{26} + \frac{1}{26}i$

7) a) -1 b) 1 c) $-i$ d) i

Answers to Technology Exercises

1) $6i$　2) $8 - i$　3) $6 - 11i$　4) $24 + 3i$　5) $\frac{1}{2} + \frac{7}{2}i$　6) $-10 - 198i$

10.8 Exercises

Objective 1: Find the Square Root of a Negative Number

Determine whether each statement is true or false.

1) Every complex number is a real number.
2) Every real number is a complex number.
3) Since $i = \sqrt{-1}$, it follows that $i^2 = -1$.
4) In the complex number $-6 + 5i$, -6 is the real part and $5i$ is the imaginary part.

Simplify.

5) $\sqrt{-81}$
6) $\sqrt{-16}$
7) $\sqrt{-25}$
8) $\sqrt{-169}$
9) $\sqrt{-6}$
10) $\sqrt{-30}$
11) $\sqrt{-27}$
12) $\sqrt{-75}$
13) $\sqrt{-60}$
14) $\sqrt{-28}$

Objective 2: Multiply and Divide Square Roots Containing Negative Numbers

Find the error in each of the following exercises, then find the correct answer.

15) $$\begin{aligned}\sqrt{-5}\cdot\sqrt{-10} &= \sqrt{-5\cdot(-10)}\\ &= \sqrt{50}\\ &= \sqrt{25}\cdot\sqrt{2}\\ &= 5\sqrt{2}\end{aligned}$$

16) $$\begin{aligned}(\sqrt{-7})^2 &= \sqrt{(-7)^2}\\ &= \sqrt{49}\\ &= 7\end{aligned}$$

Perform the indicated operation and simplify.

17) $\sqrt{-1}\cdot\sqrt{-5}$
18) $\sqrt{-5}\cdot\sqrt{-15}$
19) $\sqrt{-12}\cdot\sqrt{-3}$
20) $\sqrt{-20}\cdot\sqrt{-5}$
21) $\dfrac{\sqrt{-60}}{\sqrt{-15}}$
22) $\dfrac{\sqrt{-2}}{\sqrt{-128}}$
23) $(\sqrt{-13})^2$
24) $(\sqrt{-1})^2$

Mixed Exercises: Objectives 3–6

25) Explain how to add complex numbers.
26) How is multiplying $(1 + 3i)(2 - 7i)$ similar to multiplying $(x + 3)(2x - 7)$?
27) When i^2 appears in an expression, it should be replaced with what?
28) Explain how to divide complex numbers.

Objective 3: Add and Subtract Complex Numbers

Perform the indicated operations.

29) $(-4 + 9i) + (7 + 2i)$
30) $(6 + i) + (8 - 5i)$
31) $(13 - 8i) - (9 + i)$
32) $(-12 + 3i) - (-7 - 6i)$
33) $\left(-\dfrac{3}{4} - \dfrac{1}{6}i\right) - \left(-\dfrac{1}{2} + \dfrac{2}{3}i\right)$
34) $\left(\dfrac{1}{2} + \dfrac{7}{9}i\right) - \left(\dfrac{7}{8} - \dfrac{1}{6}i\right)$
35) $16i - (3 + 10i) + (3 + i)$
36) $(-6 - 5i) + (2 + 6i) - (-4 + i)$

Objective 4: Multiply Complex Numbers

Multiply and simplify.

37) $3(8 - 5i)$
38) $-6(8 - i)$
39) $\dfrac{2}{3}(-9 + 2i)$
40) $\dfrac{1}{2}(18 + 7i)$
41) $6i(5 + 6i)$
42) $-4i(6 + 11i)$
43) $(2 + 5i)(1 + 6i)$
44) $(2 + i)(10 + 5i)$
45) $(-1 + 3i)(4 - 6i)$
46) $(-4 - 9i)(3 - i)$
47) $(5 - 3i)(9 - 3i)$
48) $(3 - 4i)(6 + 7i)$
49) $\left(\dfrac{3}{4} + \dfrac{3}{4}i\right)\left(\dfrac{2}{5} + \dfrac{1}{5}i\right)$
50) $\left(\dfrac{1}{3} - \dfrac{4}{3}i\right)\left(\dfrac{3}{4} + \dfrac{2}{3}i\right)$

Objective 5: Multiply a Complex Number by Its Conjugate

Identify the conjugate of each complex number, then multiply the number and its conjugate.

51) $11 + 4i$
52) $-1 - 2i$
53) $-3 - 7i$
54) $4 + 9i$
55) $-6 + 4i$
56) $6 - 5i$
57) How are conjugates of complex numbers like conjugates of expressions containing real numbers and radicals?
58) Can the product of two complex numbers be a real number? Explain your answer.

Objective 6: Divide Complex Numbers

Divide. Write the result in the form $a + bi$.

59) $\dfrac{4}{2 - 3i}$
60) $\dfrac{-10}{8 - 9i}$
61) $\dfrac{8i}{4 + i}$
62) $\dfrac{i}{6 - 5i}$
63) $\dfrac{2i}{-3 + 7i}$
64) $\dfrac{9i}{-4 + 10i}$

VIDEO 65) $\dfrac{3 - 8i}{-6 + 7i}$

66) $\dfrac{-5 + 2i}{4 - i}$

67) $\dfrac{2 + 3i}{5 - 6i}$

68) $\dfrac{1 + 6i}{5 + 2i}$

69) $\dfrac{9}{i}$

70) $\dfrac{16 + 3i}{-i}$

Objective 7: Simplify Powers of *i*

Simplify each power of i.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

71) $i^{24} =$ ____ Rewrite i^{24} in terms of i^2 using the power rule.
$= (-1)^{12}$ ______
$=$ _ Simplify.

72) $i^{31} = i^{30} \cdot i$ _________
$= (i^2)^{15} \cdot i$ ________________________
$=$ _______ $i^2 = -1$
$=$ ____ Simplify.
$=$ _ Multiply.

73) i^{24}

74) i^{16}

75) i^{28}

76) i^{30}

77) i^{9}

78) i^{19}

79) i^{35}

80) i^{29}

81) i^{23}

82) i^{40}

83) i^{42}

84) i^{33}

85) $(2i)^5$

86) $(2i)^6$

87) $(-i)^{14}$

88) $(-i)^{15}$

Expand.

89) $(-2 + 5i)^3$

90) $(3 - 4i)^3$

Simplify each expression. Write the result in the form $a + bi$.

91) $1 + \sqrt{-8}$

92) $-7 - \sqrt{-48}$

93) $8 - \sqrt{-45}$

94) $3 + \sqrt{-20}$

95) $\dfrac{-12 + \sqrt{-32}}{4}$

96) $\dfrac{21 - \sqrt{-18}}{3}$

Used in the field of electronics, the **impedance**, Z, is the total opposition to the current flow of an alternating current within an electronic component, circuit, or system. It is expressed as a complex number $\boldsymbol{Z = R + Xj}$, where the i used to represent an imaginary number in most areas of mathematics is replaced by j in electronics. $\boldsymbol{R}$ represents the resistance of a substance, and $\boldsymbol{X}$ represents the **reactance**.

The **total impedance** of components connected in series is the *sum* of the individual impedances of each component.

Each exercise contains the impedance of individual circuits. Find the total impedance of a system formed by connecting the circuits in series by finding the sum of the individual impedances.

97) $Z_1 = 3 + 2j$
$Z_2 = 7 + 4j$

98) $Z_1 = 5 + 3j$
$Z_2 = 9 + 6j$

99) $Z_1 = 5 - 2j$
$Z_2 = 11 + 6j$

100) $Z_1 = 4 - 1.5j$
$Z_2 = 3 + 0.5j$

Chapter 10: Summary

Definition/Procedure	Example
10.1 Finding Roots	
If the radicand is a perfect square, then the square root is a *rational* number. **(p. 567)**	$\sqrt{49} = 7$ since $7^2 = 49$.
If the radicand is a negative number, then the square root is *not* a real number. **(p. 567)**	$\sqrt{-36}$ is not a real number.
If the radicand is positive and not a perfect square, then the square root is an *irrational* number. **(p. 567)**	$\sqrt{7}$ is irrational because 7 is not a perfect square.
$\sqrt[n]{a}$ is read as "the nth root of a." If $\sqrt[n]{a} = b$, then $b^n = a$. We call n the **index** of the radical. **(p. 569)**	$\sqrt[5]{32} = 2$ since $2^5 = 32$.
For any *positive* number a and any *even* index n, the **principal *n*th root** of a is $\sqrt[n]{a}$, and the **negative *n*th root** of a is $-\sqrt[n]{a}$. **(p. 569)**	$\sqrt[4]{16} = 2$ $-\sqrt[4]{16} = -2$
The *odd root* of a negative number is a negative number. **(p. 569)**	$\sqrt[3]{-125} = -5$ since $(-5)^3 = 125$.
The *even root* of a negative number is not a real number. **(p. 569)**	$\sqrt[4]{-16}$ is not a real number.
If n is a positive, *even* integer, then $\sqrt[n]{a^n} = \lvert a \rvert$. **(p. 570)**	$\sqrt[4]{(-2)^4} = \lvert -2 \rvert = 2$
If n is a positive, *odd* integer, then $\sqrt[n]{a^n} = a$. **(p. 570)**	$\sqrt[3]{5^3} = 5$
10.2 Rational Exponents	
If n is a positive integer greater than 1 and $\sqrt[n]{a}$ is a real number, then $a^{1/n} = \sqrt[n]{a}$. **(p. 573)**	$8^{1/3} = \sqrt[3]{8} = 2$
If m and n are positive integers and $\frac{m}{n}$ is in lowest terms, then $a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$ if $a^{1/n}$ is a real number. **(p. 574)**	$16^{3/4} = (\sqrt[4]{16})^3 = 2^3 = 8$
If $a^{m/n}$ is a nonzero real number, then $a^{-m/n} = \left(\frac{1}{a}\right)^{m/n} = \frac{1}{a^{m/n}}$. **(p. 575)**	$25^{-3/2} = \left(\frac{1}{25}\right)^{3/2} = \left(\sqrt{\frac{1}{25}}\right)^3 = \left(\frac{1}{5}\right)^3 = \frac{1}{125}$ **BE CAREFUL** The negative exponent does not make the expression negative.
10.3 Simplifying Expressions Containing Square Roots	
Product Rule for Square Roots Let a and b be nonnegative real numbers. Then $\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$. **(p. 581)**	$\sqrt{5} \cdot \sqrt{7} = \sqrt{5 \cdot 7} = \sqrt{35}$
An expression containing a square root is simplified when all of the following conditions are met: 1) The radicand does not contain any factors (other than 1) that are perfect squares. 2) The radicand does not contain any fractions. 3) There are no radicals in the denominator of a fraction.	

Definition/Procedure	Example
To *simplify square roots,* rewrite using the product rule as $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$, where a or b is a perfect square. After simplifying a radical, look at the result and ask yourself, "*Is the radical in simplest form?*" If it is not, simplify again. **(p. 581)**	Simplify $\sqrt{24}$. $\sqrt{24} = \sqrt{4 \cdot 6}$ 4 is a perfect square. $= \sqrt{4} \cdot \sqrt{6}$ Product rule $= 2\sqrt{6}$ $\sqrt{4} = 2$
Quotient Rule for Square Roots Let a and b be nonnegative real numbers such that $b \neq 0$. Then $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$. **(p. 583)**	$\sqrt{\dfrac{72}{25}} = \dfrac{\sqrt{72}}{\sqrt{25}}$ Quotient rule $= \dfrac{\sqrt{36} \cdot \sqrt{2}}{5}$ Product rule; $\sqrt{25} = 5$ $= \dfrac{6\sqrt{2}}{5}$ $\sqrt{36} = 6$
If a is a nonnegative real number and m is an integer, then $\sqrt{a^m} = a^{m/2}$. **(p. 584)**	$\sqrt{k^{18}} = k^{18/2} = k^9$ (provided k represents a nonnegative real number)
Two Approaches to Simplifying Radical Expressions Containing Variables Let a represent a nonnegative real number. To simplify $\sqrt{a^n}$, where *n is odd and positive,* **i) Method 1:** Write a^n as the product of two factors so that the exponent of one of the factors is the *largest* number less than n that is divisible by 2 (the index of the radical). **(p. 585)**	i) Simplify $\sqrt{x^9}$. $\sqrt{x^9} = \sqrt{x^8 \cdot x^1}$ 8 is the largest number less than 9 that is divisible by 2. $= \sqrt{x^8} \cdot \sqrt{x}$ Product rule $= x^{8/2}\sqrt{x}$ $= x^4\sqrt{x}$ $8 \div 2 = 4$
ii) Method 2: 1) Divide the exponent in the radicand by the index of the radical. 2) The exponent on the variable *outside* of the radical will be the *quotient* of the division problem. 3) The exponent on the variable *inside* of the radical will be the *remainder* of the division problem. **(p. 586)**	ii) Simplify $\sqrt{p^{15}}$. $\sqrt{p^{15}} = p^7\sqrt{p^1}$ $15 \div 2$ gives a quotient of 7 and a remainder of 1. $= p^7\sqrt{p}$

10.4 Simplifying Expressions Containing Higher Roots

Product Rule for Higher Roots If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$. **(p. 591)**	$\sqrt[3]{3} \cdot \sqrt[3]{5} = \sqrt[3]{15}$
Let P be an expression and let n be a positive integer greater than 1. Then $\sqrt[n]{P}$ *is completely simplified when all of the following conditions are met:* 1) The radicand does not contain any factors (other than 1) that are perfect nth powers. 2) The exponents in the radicand and the index of the radical do not have any common factors (other than 1). 3) The radicand does not contain any fractions. 4) There are no radicals in the denominator of a fraction. To *simplify radicals with any index,* reverse the process of multiplying radicals, where a or b is an nth power. **(p. 591)** $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$	Simplify $\sqrt[3]{40}$. **Method 1:** Think of two numbers that multiply to 40 so that one of them is a *perfect cube.* $40 = 8 \cdot 5$ 8 is a perfect cube. Then, $\sqrt[3]{40} = \sqrt[3]{8 \cdot 5}$ $= \sqrt[3]{8} \cdot \sqrt[3]{5}$ Product rule $= 2\sqrt[3]{5}$ $\sqrt[3]{8} = 2$

Definition/Procedure	Example
	Method 2: Begin by using a factor tree to find the prime factorization of 40. $40 = 2^3 \cdot 5$ $\sqrt[3]{40} = \sqrt[3]{2^3 \cdot 5}$ $= \sqrt[3]{2^3} \cdot \sqrt[3]{5}$ Product rule $= 2\sqrt[3]{5}$ $\sqrt[3]{2^3} = 2$
Quotient Rule for Higher Roots If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and n is a natural number, then $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$. **(p. 593)**	$\sqrt[4]{\frac{32}{81}} = \frac{\sqrt[4]{32}}{\sqrt[4]{81}} = \frac{\sqrt[4]{16} \cdot \sqrt[4]{2}}{3} = \frac{2\sqrt[4]{2}}{3}$
Simplifying Higher Roots with Variables in the Radicand If a is a nonnegative number and m and n are integers such that $n > 1$, then $\sqrt[n]{a^m} = a^{m/n}$. **(p. 594)**	Simplify $\sqrt[4]{a^{12}}$. $\sqrt[4]{a^{12}} = a^{12/4} = a^3$
If the exponent does not divide evenly by the index, we can use two methods for simplifying the radical expression. If a is a nonnegative number and m and n are integers such that $n > 1$, then **i) Method 1:** Use the product rule. To simplify $\sqrt[n]{a^m}$, write a^m as the product of two factors so that the exponent of one of the factors is the *largest* number less than m that is divisible by n (the index). **ii) Method 2:** Use the quotient and remainder (presented in Section 10.3). **(p. 594)**	i) Simplify $\sqrt[5]{c^{17}}$. $\sqrt[5]{c^{17}} = \sqrt[5]{c^{15} \cdot c^2}$ 15 is the largest number less than 17 that is divisible by 5. $= \sqrt[5]{c^{15}} \cdot \sqrt[5]{c^2}$ Product rule $= c^{15/5} \cdot \sqrt[5]{c^2}$ $= c^3\sqrt[5]{c^2}$ $15 \div 5 = 3$ ii) Simplify $\sqrt[4]{m^{11}}$. $\sqrt[4]{m^{11}} = m^2\sqrt[4]{m^3}$ $11 \div 4$ gives a quotient of 2 and a remainder of 3.

10.5 Adding, Subtracting, and Multiplying Radicals

Like radicals have the same index and the same radicand. In order to add or subtract radicals, they must be like radicals. **Steps for Adding and Subtracting Radicals** 1) Write each radical expression in simplest form. 2) Combine like radicals. **(p. 599)**	Perform the operations and simplify. a) $5\sqrt{2} + 9\sqrt{7} - 3\sqrt{2} + 4\sqrt{7}$ $= 2\sqrt{2} + 13\sqrt{7}$ b) $\sqrt{72} + \sqrt{18} - \sqrt{45}$ $= \sqrt{36} \cdot \sqrt{2} + \sqrt{9} \cdot \sqrt{2} - \sqrt{9} \cdot \sqrt{5}$ $= 6\sqrt{2} + 3\sqrt{2} - 3\sqrt{5}$ $= 9\sqrt{2} - 3\sqrt{5}$
Combining Multiplication, Addition, and Subtraction of Radicals Multiply expressions containing radicals using the same techniques that are used for multiplying polynomials. **(p. 600)**	Multiply and simplify. a) $\sqrt{m}(\sqrt{2m} + \sqrt{n})$ $= \sqrt{m} \cdot \sqrt{2m} + \sqrt{m} \cdot \sqrt{n}$ $= \sqrt{2m^2} + \sqrt{mn}$ $= m\sqrt{2} + \sqrt{mn}$ b) $(\sqrt{k} + \sqrt{6})(\sqrt{k} - \sqrt{2})$ Since we are multiplying two binomials, multiply using FOIL. $(\sqrt{k} + \sqrt{6})(\sqrt{k} - \sqrt{2})$ $= \underbrace{\sqrt{k} \cdot \sqrt{k}}_{F} - \underbrace{\sqrt{2} \cdot \sqrt{k}}_{O} + \underbrace{\sqrt{6} \cdot \sqrt{k}}_{I} - \underbrace{\sqrt{6} \cdot \sqrt{2}}_{L}$ $= k^2 - \sqrt{2k} + \sqrt{6k} - \sqrt{12}$ Product rule $= k^2 - \sqrt{2k} + \sqrt{6k} - 2\sqrt{3}$ $\sqrt{12} = 2\sqrt{3}$

Definition/Procedure	Example
Squaring a Radical Expression with Two Terms To square a binomial, we can either use FOIL or one of the special formulas from Chapter 6: $(a+b)^2 = a^2 + 2ab + b^2$ $(a-b)^2 = a^2 - 2ab + b^2$ **(p. 602)**	$(\sqrt{7}+5)^2 = (\sqrt{7})^2 + 2(\sqrt{7})(5) + (5)^2$ $= 7 + 10\sqrt{7} + 25$ $= 32 + 10\sqrt{7}$
Multiply $(a+b)(a-b)$ To multiply binomials of the form $(a+b)(a-b)$, use the formula $(a+b)(a-b) = a^2 - b^2$. **(p. 602)**	$(3+\sqrt{10})(3-\sqrt{10}) = (3)^2 - (\sqrt{10})^2$ $= 9 - 10$ $= -1$

10.6 Dividing Radicals

The process of eliminating radicals from the denominator of an expression is called **rationalizing the denominator**. First, we give examples of rationalizing denominators containing one term. **(p. 605)**	Rationalize the denominator of each expression. a) $\dfrac{9}{\sqrt{2}} = \dfrac{9}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{9\sqrt{2}}{2}$ b) $\dfrac{5}{\sqrt[3]{2}} = \dfrac{5}{\sqrt[3]{2}} \cdot \dfrac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}} = \dfrac{5\sqrt[3]{2^2}}{\sqrt[3]{2^3}} = \dfrac{5\sqrt[3]{4}}{2}$
The *conjugate* of an expression of the form $a+b$ is $a-b$. **(p. 611)**	$\sqrt{11} - 4$ conjugate: $\sqrt{11} + 4$ $-8 + \sqrt{5}$ conjugate: $-8 - \sqrt{5}$
Rationalizing a Denominator with Two Terms If the denominator of an expression contains two terms, including one or two square roots, then to rationalize the denominator we multiply the numerator and denominator of the expression by the conjugate of the denominator. **(p. 611)**	Rationalize the denominator of $\dfrac{4}{\sqrt{2}-3}$. $\dfrac{4}{\sqrt{2}-3} = \dfrac{4}{\sqrt{2}-3} \cdot \dfrac{\sqrt{2}+3}{\sqrt{2}+3}$ Multiply by the conjugate of the denominator. $= \dfrac{4(\sqrt{2}+3)}{(\sqrt{2})^2 - (3)^2}$ $(a+b)(a-b) = a^2 - b^2$ $= \dfrac{4(\sqrt{2}+3)}{2-9}$ Square the terms. $= \dfrac{4(\sqrt{2}+3)}{-7} = -\dfrac{4\sqrt{2}+12}{7}$

10.7 Solving Radical Equations

Solving Radical Equations ***Step 1:*** Get a radical on a side by itself. ***Step 2:*** Square both sides of the equation to eliminate a radical. ***Step 3:*** Combine like terms on each side of the equation. ***Step 4:*** If the equation still contains a radical, repeat steps 1–3. ***Step 5:*** Solve the equation. ***Step 6:*** Check the proposed solutions *in the original equation* and discard extraneous solutions. **(p. 621)**	Solve $t = 2 + \sqrt{2t-1}$. $t - 2 = \sqrt{2t-1}$ Get the radical by itself. $(t-2)^2 = (\sqrt{2t-1})^2$ Square both sides. $t^2 - 4t + 4 = 2t - 1$ $t^2 - 6t + 5 = 0$ Get all terms on the same side. $(t-5)(t-1) = 0$ Factor. $t - 5 = 0$ or $t - 1 = 0$ $t = 5$ or $t = 1$ Check $t = 5$ and $t = 1$ in the *original* equation. $t = 5$: $t = 2 + \sqrt{2t-1}$; $5 \overset{?}{=} 2 + \sqrt{2(5)-1}$; $5 \overset{?}{=} 2 + \sqrt{9}$; $5 = 2 + 3$; True $t = 1$: $t = 2 + \sqrt{2t-1}$; $1 \overset{?}{=} 2 + \sqrt{2(1)-1}$; $1 \overset{?}{=} 2 + 1$; $1 = 3$; False $t = 5$ *is* a solution, but $t = 1$ is *not* because $t = 1$ does not satisfy the original equation. The solution set is $\{5\}$.

Definition/Procedure	Example

10.8 Complex Numbers

Definition/Procedure	Example
Definition of *i*: $$i = \sqrt{-1}$$ Therefore, $$i^2 = -1$$ A **complex number** is a number of the form $a + bi$, where a and b are real numbers. a is called the **real part** and b is called the **imaginary part**. The set of real numbers is a subset of the set of complex numbers. **(p. 629)**	*Examples of complex numbers:* $-2 + 7i$ 5 (since it can be written $5 + 0i$) $4i$ (since it can be written $0 + 4i$)
Simplifying Complex Numbers Use the product rule and $i = \sqrt{-1}$. **(p. 629)**	Simplify $\sqrt{-25}$. $$\begin{aligned} \sqrt{-25} &= \sqrt{-1} \cdot \sqrt{25} \\ &= i \cdot 5 \\ &= 5i \end{aligned}$$
When multiplying or dividing radicals with negative radicands, write each radical in terms of i first. **(p. 630)**	Multiply $\sqrt{-12} \cdot \sqrt{-3}$. $$\begin{aligned} \sqrt{-12} \cdot \sqrt{-3} &= i\sqrt{12} \cdot i\sqrt{3} = i^2\sqrt{36} \\ &= -1 \cdot 6 = -6 \end{aligned}$$
Adding and Subtracting Complex Numbers To add and subtract complex numbers, combine the real parts and combine the imaginary parts. **(p. 631)**	Subtract $(10 + 7i) - (-2 + 4i)$. $$\begin{aligned} (10 + 7i) - (-2 + 4i) &= 10 + 7i + 2 - 4i \\ &= 12 + 3i \end{aligned}$$
Multiply complex numbers like we multiply polynomials. Remember to replace i^2 with -1. **(p. 631)**	Multiply and simplify. a) $4(9 + 5i) = 36 + 20i$ b) $$\begin{aligned} (-3 + i)(2 - 7i) &= \underset{\text{F}}{-6} + \underset{\text{O}}{21i} + \underset{\text{I}}{2i} - \underset{\text{L}}{7i^2} \\ &= -6 + 23i - 7(-1) \\ &= -6 + 23i + 7 \\ &= 1 + 23i \end{aligned}$$
Complex Conjugates 1) The conjugate of $a + bi$ is $a - bi$. 2) The product of $a + bi$ and $a - bi$ is a real number. 3) Find the product $(a + bi)(a - bi)$ using FOIL or recall that $(a + bi)(a - bi) = a^2 + b^2$. **(p. 632)**	Multiply $-5 - 3i$ by its conjugate. The conjugate of $-5 - 3i$ is $-5 + 3i$. Use $(a + bi)(a - bi) = a^2 + b^2$. $$\begin{aligned} (-5 - 3i)(-5 + 3i) &= (-5)^2 + (3)^2 \\ &= 25 + 9 \\ &= 34 \end{aligned}$$
Dividing Complex Numbers To divide complex numbers, multiply the numerator and denominator by the *conjugate of the denominator*. Write the result in the form $a + bi$. **(p. 633)**	Divide $\dfrac{6i}{2 + 5i}$. Write the result in the form $a + bi$. $$\begin{aligned} \frac{6i}{2 + 5i} &= \frac{6i}{2 + 5i} \cdot \frac{(2 - 5i)}{(2 - 5i)} \\ &= \frac{12i - 30i^2}{2^2 + 5^2} \\ &= \frac{12i - 30(-1)}{29} \\ &= \frac{30}{29} + \frac{12}{29}i \end{aligned}$$
Simplify Powers of *i* We can simplify powers of i using $i^2 = -1$. **(p. 633)**	Simplify i^{14}. $$\begin{aligned} i^{14} &= (i^2)^7 && \text{Power rule} \\ &= (-1)^7 && i^2 = -1 \\ &= -1 && \text{Simplify.} \end{aligned}$$

Chapter 10: Review Exercises

(10.1) Find each root, if possible.

1) $\sqrt{25}$
2) $\sqrt{-16}$
3) $-\sqrt{81}$
4) $\sqrt{\dfrac{169}{4}}$
5) $\sqrt[3]{64}$
6) $\sqrt[5]{32}$
7) $\sqrt[3]{-1}$
8) $-\sqrt[4]{81}$
9) $\sqrt[6]{-64}$
10) $\sqrt{9-16}$

Simplify. Use absolute values when necessary.

11) $\sqrt{(-13)^2}$
12) $\sqrt[5]{(-8)^5}$
13) $\sqrt{p^2}$
14) $\sqrt[6]{c^6}$
15) $\sqrt[3]{h^3}$
16) $\sqrt[4]{(y+7)^4}$

Approximate each square root to the nearest tenth and plot it on a number line.

17) $\sqrt{34}$
18) $\sqrt{52}$

(10.2)

19) Explain how to write $8^{2/3}$ in radical form.

20) Explain how to eliminate the negative from the exponent in an expression like $9^{-1/2}$.

Evaluate.

21) $36^{1/2}$
22) $32^{1/5}$
23) $\left(\dfrac{27}{125}\right)^{1/3}$
24) $-16^{1/4}$
25) $32^{3/5}$
26) $\left(\dfrac{64}{27}\right)^{2/3}$
27) $81^{-1/2}$
28) $\left(\dfrac{1}{27}\right)^{-1/3}$
29) $81^{-3/4}$
30) $1000^{-2/3}$
31) $\left(\dfrac{27}{1000}\right)^{-2/3}$
32) $\left(\dfrac{25}{16}\right)^{-3/2}$

From this point forward, assume that all variables represent positive real numbers.
Simplify completely. The answer should contain only positive exponents.

33) $3^{6/7}\cdot 3^{8/7}$
34) $(169^4)^{1/8}$
35) $(8^{1/5})^{10}$
36) $\dfrac{8^2}{8^{11/3}}$
37) $\dfrac{7^2}{7^{5/3}\cdot 7^{1/3}}$
38) $(2k^{-5/6})(3k^{1/2})$
39) $(64a^4b^{12})^{5/6}$
40) $\left(\dfrac{t^4u^3}{7t^7u^5}\right)^{-2}$
41) $\left(\dfrac{81c^{-5}d^9}{16c^{-1}d^2}\right)^{-1/4}$

Rewrite each radical in exponential form, then simplify. Write the answer in simplest (or radical) form.

42) $\sqrt[4]{36^2}$
43) $\sqrt[12]{27^4}$
44) $(\sqrt{17})^2$
45) $\sqrt[3]{7^3}$
46) $\sqrt[5]{t^{20}}$
47) $\sqrt[4]{k^{28}}$
48) $\sqrt{x^{10}}$
49) $\sqrt{w^6}$

(10.3) Simplify completely.

50) $\sqrt{28}$
51) $\sqrt{1000}$
52) $\dfrac{\sqrt{63}}{\sqrt{7}}$
53) $\sqrt{\dfrac{18}{49}}$
54) $\dfrac{\sqrt{48}}{\sqrt{121}}$
55) $\sqrt{k^{12}}$
56) $\sqrt{\dfrac{40}{m^4}}$
57) $\sqrt{x^9}$
58) $\sqrt{y^5}$
59) $\sqrt{45t^2}$
60) $\sqrt{80n^{21}}$
61) $\sqrt{72x^7y^{13}}$
62) $\sqrt{\dfrac{m^{11}}{36n^2}}$

Perform the indicated operation and simplify.

63) $\sqrt{5}\cdot\sqrt{3}$
64) $\sqrt{6}\cdot\sqrt{15}$
65) $\sqrt{2}\cdot\sqrt{12}$
66) $\sqrt{b^7}\cdot\sqrt{b^3}$
67) $\sqrt{11x^5}\cdot\sqrt{11x^8}$
68) $\sqrt{5a^2b}\cdot\sqrt{15a^6b^4}$
69) $\dfrac{\sqrt{200k^{21}}}{\sqrt{2k^5}}$
70) $\dfrac{\sqrt{63c^{17}}}{\sqrt{7c^9}}$

(10.4) Simplify completely.

71) $\sqrt[3]{16}$
72) $\sqrt[3]{250}$
73) $\sqrt[4]{48}$
74) $\sqrt[3]{\dfrac{81}{3}}$
75) $\sqrt[4]{z^{24}}$
76) $\sqrt[5]{p^{40}}$
77) $\sqrt[3]{a^{20}}$
78) $\sqrt[5]{x^{14}y^7}$
79) $\sqrt[3]{16z^{15}}$
80) $\sqrt[3]{80m^{17}n^{10}}$
81) $\sqrt[4]{\dfrac{h^{12}}{81}}$
82) $\sqrt[5]{\dfrac{c^{22}}{32d^{10}}}$

Perform the indicated operation and simplify.

83) $\sqrt[3]{3}\cdot\sqrt[3]{7}$
84) $\sqrt[3]{25}\cdot\sqrt[3]{10}$
85) $\sqrt[4]{4t^7}\cdot\sqrt[4]{8t^{10}}$
86) $\sqrt[5]{\dfrac{x^{21}}{x^{16}}}$
87) $\sqrt[3]{n}\cdot\sqrt{n}$
88) $\dfrac{\sqrt[4]{a^3}}{\sqrt[3]{a}}$

(10.5) Perform the operations and simplify.

89) $8\sqrt{5} + 3\sqrt{5}$

90) $\sqrt{125} + \sqrt{80}$

91) $\sqrt{80} - \sqrt{48} + \sqrt{20}$

92) $9\sqrt[3]{72} - 8\sqrt[3]{9}$

93) $3p\sqrt{p} - 7\sqrt{p^3}$

94) $9n\sqrt{n} - 4\sqrt{n^3}$

95) $10d^2\sqrt{8d} - 32d\sqrt{2d^3}$

96) $\sqrt{6}(\sqrt{7} - \sqrt{6})$

97) $3\sqrt{k}(\sqrt{20k} + \sqrt{2})$

98) $(5 - \sqrt{3})(2 + \sqrt{3})$

99) $(\sqrt{2r} + 5\sqrt{s})(3\sqrt{s} + 4\sqrt{2r})$

100) $(2\sqrt{5} - 4)^2$

101) $(1 + \sqrt{y + 1})^2$

102) $(\sqrt{6} - \sqrt{5})(\sqrt{6} + \sqrt{5})$

(10.6) Rationalize the denominator of each expression.

103) $\dfrac{14}{\sqrt{3}}$

104) $\dfrac{20}{\sqrt{6}}$

105) $\dfrac{\sqrt{18k}}{\sqrt{n}}$

106) $\dfrac{\sqrt{45}}{\sqrt{m^5}}$

107) $\dfrac{7}{\sqrt[3]{2}}$

108) $-\dfrac{15}{\sqrt[3]{9}}$

109) $\dfrac{\sqrt[3]{x^2}}{\sqrt[3]{y}}$

110) $\sqrt[4]{\dfrac{3}{4k^2}}$

111) $\dfrac{2}{3 + \sqrt{3}}$

112) $\dfrac{z - 4}{\sqrt{z} + 2}$

Simplify completely.

113) $\dfrac{8 - 24\sqrt{2}}{8}$

114) $\dfrac{-\sqrt{48} - 6}{10}$

(10.7) Solve.

115) $\sqrt{x + 8} = 3$

116) $10 - \sqrt{3r - 5} = 2$

117) $\sqrt{3j + 4} = -\sqrt{4j - 1}$

118) $\sqrt[3]{6d - 14} = -2$

119) $a = \sqrt{a + 8} - 6$

120) $1 + \sqrt{6m + 7} = 2m$

121) $\sqrt{4a + 1} - \sqrt{a - 2} = 3$

122) $\sqrt{6x + 9} - \sqrt{2x + 1} = 4$

123) Solve for V: $r = \sqrt{\dfrac{3V}{\pi h}}$

124) The velocity of a wave in shallow water is given by $c = \sqrt{gH}$, where g is the acceleration due to gravity (32 ft/sec^2) and H is the depth of the water (in feet). Find the velocity of a wave in 10 ft of water.

(10.8) Simplify.

125) $\sqrt{-49}$

126) $\sqrt{-8}$

127) $\sqrt{-2} \cdot \sqrt{-8}$

128) $\sqrt{-6} \cdot \sqrt{-3}$

Perform the indicated operations.

129) $(2 + i) + (10 - 4i)$

130) $(4 + 3i) - (11 - 4i)$

131) $\left(\dfrac{4}{5} - \dfrac{1}{3}i\right) - \left(\dfrac{1}{2} + i\right)$

132) $\left(-\dfrac{3}{8} - 2i\right) + \left(\dfrac{5}{8} + \dfrac{3}{2}i\right) - \left(\dfrac{1}{4} - \dfrac{1}{2}i\right)$

Multiply and simplify.

133) $5(-6 + 7i)$

134) $-8i(4 + 3i)$

135) $3i(-7 + 12i)$

136) $(3 - 4i)(2 + i)$

137) $(4 - 6i)(3 - 6i)$

138) $\left(\dfrac{1}{5} - \dfrac{2}{3}i\right)\left(\dfrac{3}{2} - \dfrac{2}{3}i\right)$

Identify the conjugate of each complex number, then multiply the number and its conjugate.

139) $2 - 7i$

140) $-2 + 3i$

Divide. Write the quotient in the form *a* + *bi*.

141) $\dfrac{6}{2 + 5i}$

142) $\dfrac{-12}{4 - 3i}$

143) $\dfrac{8}{i}$

144) $\dfrac{4i}{1 - 3i}$

145) $\dfrac{9 - 4i}{6 - i}$

146) $\dfrac{5 - i}{-2 + 6i}$

Simplify.

147) i^{10}

148) i^{51}

149) i^{33}

150) i^{24}

Chapter 10: Test

Find each root, if possible.

1) $\sqrt{144}$

2) $\sqrt[3]{-27}$

3) $\sqrt{-16}$

Simplify. Use absolute values when necessary.

4) $\sqrt[4]{w^4}$

5) $\sqrt[5]{(-19)^5}$

Evaluate.

6) $16^{1/4}$

7) $27^{4/3}$

8) $(49)^{-1/2}$

9) $\left(\frac{8}{125}\right)^{-2/3}$

From this point forward, assume that all variables represent positive real numbers.
Simplify completely. The answer should contain only positive exponents.

10) $m^{3/8} \cdot m^{1/4}$

11) $\frac{35a^{1/6}}{14a^{5/6}}$

12) $(2x^{3/10}y^{-2/5})^{-5}$

Simplify completely.

13) $\sqrt{75}$

14) $\sqrt[3]{48}$

15) $\sqrt{\frac{24}{2}}$

Simplify completely.

16) $\sqrt{y^6}$

17) $\sqrt[4]{p^{24}}$

18) $\sqrt{t^9}$

19) $\sqrt{63m^5n^8}$

20) $\sqrt[3]{c^{23}}$

21) $\sqrt[3]{\frac{a^{14}b^7}{27}}$

Perform the operations and simplify.

22) $\sqrt{3} \cdot \sqrt{12}$

23) $\sqrt[3]{z^4} \cdot \sqrt[3]{z^6}$

24) $\frac{\sqrt{120w^{15}}}{\sqrt{2w^4}}$

25) $9\sqrt{7} - 3\sqrt{7}$

26) $\sqrt{12} - \sqrt{108} + \sqrt{18}$

27) $2h^3\sqrt[4]{h} - 16\sqrt[4]{h^{13}}$

Multiply and simplify.

28) $\sqrt{6}(\sqrt{2} - 5)$

29) $(3 - 2\sqrt{5})(\sqrt{2} + 1)$

30) $(\sqrt{7} + \sqrt{3})(\sqrt{7} - \sqrt{3})$

31) $(\sqrt{2p + 1} + 2)^2$

32) $2\sqrt{t}(\sqrt{t} - \sqrt{3u})$

Rationalize the denominator of each expression.

33) $\frac{2}{\sqrt{5}}$

34) $\frac{8}{\sqrt{7} + 3}$

35) $\frac{\sqrt{6}}{\sqrt{a}}$

36) $\frac{5}{\sqrt[3]{9}}$

37) Simplify completely. $\frac{2 - \sqrt{48}}{2}$

Solve.

38) $\sqrt{5h + 4} = 3$

39) $z = \sqrt{1 - 4z} - 5$

40) $\sqrt[3]{n - 5} - \sqrt[3]{2n - 18} = 0$

41) $\sqrt{3k + 1} - \sqrt{2k - 1} = 1$

42) In the formula $r = \sqrt{\frac{V}{\pi h}}$, V represents the volume of a cylinder, h represents the height of the cylinder, and r represents the radius.

a) A cylindrical container has a volume of 72π in^3. It is 8 in. high. What is the radius of the container?

b) Solve the formula for V.

Simplify.

43) $\sqrt{-64}$

44) $\sqrt{-45}$

45) i^{19}

Perform the indicated operation and simplify. Write the answer in the form *a* + *bi*.

46) $(-10 + 3i) - (6 + i)$

47) $(2 - 7i)(-1 + 3i)$

48) $\frac{8 + i}{2 - 3i}$

Cumulative Review: Chapters 1–10

1) Combine like terms.

$$4x - 3y + 9 - \frac{2}{3}x + y - 1$$

2) Write in scientific notation.

8,723,000

3) Solve $3(2c - 1) + 7 = 9c + 5(c + 2)$.

4) Graph $3x + 2y = 12$.

5) Write the equation of the line containing the points (5, 3) and (1, −2). Write the equation in slope-intercept form.

6) Solve by substitution.

$$\begin{aligned} 2x + 7y &= -12 \\ x - 4y &= -6 \end{aligned}$$

7) Multiply.

$$(5p^2 - 2)(3p^2 - 4p - 1)$$

8) Divide.

$$\frac{8n^3 - 1}{2n - 1}$$

Factor completely.

9) $4w^2 + 5w - 6$

10) $8 - 18t^2$

11) Solve $6y^2 - 4 = 5y$.

12) Solve $3(k^2 + 20) - 4k = 2k^2 + 11k + 6$.

13) *Write an equation and solve.* The width of a rectangle is 5 in. less than its length. The area is 84 in^2. Find the dimensions of the rectangle.

Perform the operations and simplify.

14) $\dfrac{5a^2 + 3}{a^2 + 4a} - \dfrac{3a - 2}{a + 4}$

15) $\dfrac{10m^2}{9n} \cdot \dfrac{6n^2}{35m^5}$

16) Solve $\dfrac{3}{r^2 + 8r + 15} - \dfrac{4}{r + 3} = 1$.

17) Solve $|6g + 1| \geq 11$. Write the answer in interval notation.

18) Solve using Gaussian elimination.

$$\begin{aligned} x + 3y + z &= 3 \\ 2x - y - 5z &= -1 \\ -x + 2y + 3z &= 0 \end{aligned}$$

19) Simplify. Assume all variables represent nonnegative real numbers.

a) $\sqrt{500}$ b) $\sqrt[3]{56}$

c) $\sqrt{p^{10}q^7}$ d) $\sqrt[4]{32a^{15}}$

20) Evaluate.

a) $81^{1/2}$ b) $8^{4/3}$

c) $(27)^{-1/3}$

21) Multiply and simplify $2\sqrt{3}(5 - \sqrt{3})$.

22) Rationalize the denominator. Assume the variables represent positive real numbers.

a) $\sqrt{\dfrac{20}{50}}$ b) $\dfrac{6}{\sqrt[3]{2}}$

c) $\dfrac{x}{\sqrt[3]{y^2}}$ d) $\dfrac{\sqrt{a} - 2}{1 - \sqrt{a}}$

23) Solve.

a) $\sqrt{2b - 1} + 7 = 6$

b) $\sqrt{3z + 10} = 2 - \sqrt{z + 4}$

24) Simplify.

a) $\sqrt{-49}$ b) $\sqrt{-56}$

c) i^8

25) Perform the indicated operation and simplify. Write the answer in the form $a + bi$.

a) $(-3 + 4i) + (5 + 3i)$

b) $(3 + 6i)(-2 + 7i)$

c) $\dfrac{2 - i}{-4 + 3i}$

CHAPTER 11

Quadratic Equations

Algebra at Work: Ophthalmology

We have already seen two applications of mathematics to ophthalmology, and here we have a third. An ophthalmologist can use a quadratic equation to convert between a prescription for glasses and a prescription for contact lenses.

After having reexamined her patient for contact lens use, Sarah can use the following quadratic equation to double-check the prescription for the contact lenses based on the prescription her patient currently has for her glasses.

$$D_c = s(D_g)^2 + D_g$$

where D_g = power of the glasses, in diopters

s = distance of the glasses to the eye, in meters

D_c = power of the contact lenses, in diopters

For example, if the power of a patient's eyeglasses is $+9.00$ diopters and the glasses rest 1 cm or 0.01 m from the eye, the power the patient would need in her contact lenses would be

$$D_c = 0.01(9)^2 + 9$$
$$D_c = 0.01(81) + 9$$
$$D_c = 0.81 + 9$$
$$D_c = 9.81 \text{ diopters}$$

An eyeglass power of $+9.00$ diopters would convert to a contact lens power of $+9.81$ diopters.

In this chapter, we will learn different ways to solve quadratic equations.

Section 11.1 Review of Solving Equations by Factoring

Objective

1. Review How to Solve a Quadratic Equation by Factoring

We defined a quadratic equation in Chapter 7. Let's restate the definition:

Definition

A **quadratic equation** can be written in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$.

In Section 7.5, we learned how to solve quadratic equations by factoring. We will not be able to solve all quadratic equations by factoring, however. Therefore, we need to learn other methods. In this chapter, we will discuss the following four methods for solving quadratic equations.

Four Methods for Solving Quadratic Equations

1) Factoring
2) Square root property
3) Completing the square
4) Quadratic formula

1. Review How to Solve a Quadratic Equation by Factoring

We begin by reviewing how to solve an equation by factoring.

Procedure Solving a Quadratic Equation by Factoring

1) Write the equation in the form $ax^2 + bx + c = 0$ so that all terms are on one side of the equal sign and zero is on the other side.
2) Factor the expression.
3) Set each factor equal to zero, and solve for the variable.
4) Check the answer(s).

Example 1

Solve by factoring.

a) $8t^2 + 3 = -14t$ b) $(a - 9)(a + 7) = -15$ c) $3x^3 + 10x^2 = 8x$

Solution

a) Begin by writing $8t^2 + 3 = -14t$ in standard form.

$$8t^2 + 14t + 3 = 0 \quad \text{Standard form}$$
$$(4t + 1)(2t + 3) = 0 \quad \text{Factor.}$$
$$4t + 1 = 0 \quad \text{or} \quad 2t + 3 = 0 \quad \text{Set each factor equal to zero.}$$
$$4t = -1 \qquad\qquad 2t = -3$$
$$t = -\frac{1}{4} \quad \text{or} \quad t = -\frac{3}{2} \quad \text{Solve.}$$

The check is left to the student. The solution set is $\left\{-\frac{3}{2}, -\frac{1}{4}\right\}$.

b) You may want to solve $(a - 9)(a + 7) = -15$ like this:

$$(a - 9)(a + 7) = -15$$
$$\swarrow \qquad \searrow$$
$$a - 9 = -15 \quad \text{or} \quad a + 7 = -15$$
$$a = -6 \quad \text{or} \quad a = -22$$

This is incorrect!

One side of the equation must equal *zero* and the other side must be factored to be able to apply the zero product rule and set each factor equal to zero.

Begin by multiplying the binomials using FOIL.

$$\begin{aligned} (a - 9)(a + 7) &= -15 \\ a^2 - 2a - 63 &= -15 && \text{Multiply using FOIL.} \\ a^2 - 2a - 48 &= 0 && \text{Write in standard form.} \\ (a + 6)(a - 8) &= 0 && \text{Factor.} \end{aligned}$$
$$\swarrow \qquad \searrow$$
$$\begin{aligned} a + 6 = 0 \quad &\text{or} \quad a - 8 = 0 && \text{Set each factor equal to zero.} \\ a = -6 \quad &\text{or} \quad a = 8 && \text{Solve.} \end{aligned}$$

The check is left to the student. The solution set is $\{-6, 8\}$.

c) Although this is a cubic equation and not quadratic, we *can* solve it by factoring.

$$\begin{aligned} 3x^3 + 10x^2 &= 8x \\ 3x^3 + 10x^2 - 8x &= 0 && \text{Get zero on one side of the equal sign.} \\ x(3x^2 + 10x - 8) &= 0 && \text{Factor out } x. \\ x(3x - 2)(x + 4) &= 0 && \text{Factor } 3x^2 + 10x - 8. \end{aligned}$$
$$\swarrow \qquad \downarrow \qquad \searrow$$
$$\begin{aligned} x = 0 \quad \text{or} \quad 3x - 2 &= 0 \quad \text{or} \quad x + 4 = 0 && \text{Set each factor equal to zero.} \\ 3x &= 2 \\ x = 0 \quad \text{or} \quad x &= \frac{2}{3} \quad \text{or} \quad x = -4 && \text{Solve.} \end{aligned}$$

The check is left to the student. The solution set is $\left\{-4, 0, \frac{2}{3}\right\}$.

You Try 1

Solve by factoring.

a) $c^2 - 12 = c$

b) $p(7p + 18) + 8 = 0$

c) $(k - 7)(k - 5) = -1$

d) $2x^3 + 30x = 16x^2$

Answers to You Try Exercises

1) a) $\{-3, 4\}$ b) $\left\{-2, -\frac{4}{7}\right\}$ c) $\{6\}$ d) $\{0, 3, 5\}$

11.1 Exercises

Objective 1: Review How to Solve a Quadratic Equation by Factoring

Solve each equation.

1) $(t + 7)(t - 6) = 0$
2) $3z(2z - 9) = 0$
3) $u^2 + 15u + 44 = 0$
4) $n^2 + 10n - 24 = 0$
5) $x^2 = x + 56$
6) $c^2 + 3c = 54$
7) $1 - 100w^2 = 0$
8) $9j^2 = 49$
9) $5m^2 + 8 = 22m$
10) $19a + 20 = -3a^2$
11) $23d = -10 - 6d^2$
12) $8h^2 + 12 = 35h$
13) $2r = 7r^2$
14) $5n^2 = -6n$

Identify each equation as linear or quadratic.

15) $9m^2 - 2m + 1 = 0$
16) $17 = 3z - z^2$
17) $13 - 4x = 19$
18) $10 - 2(3d + 1) = 5d + 19$
19) $y(2y - 5) = 3y + 1$
20) $3(4y - 3) = y(y + 1)$
21) $-4(b + 7) + 5b = 2b + 9$
22) $6 + 2k(k - 1) = 5k$

In this section, there is a mix of linear and quadratic equations as well as equations of higher degree. Solve each equation.

23) $13c = 2c^2 + 6$
24) $12x - 1 = 2x + 9$
25) $2p(p + 4) = p^2 + 5p + 10$
26) $z^2 - 20 = 22 - z$
27) $5(3n - 2) - 11n = 2n - 1$
28) $5a^2 = 45a$
29) $3t^3 + 5t = -8t^2$
30) $6(2k - 3) + 10 = 3(2k - 5)$
31) $2(r + 5) = 10 - 4r^2$
32) $3d - 4 = d(d + 8)$
33) $9y - 6(y + 1) = 12 - 5y$
34) $3m(2m + 5) - 8 = 2m(3m + 5) + 2$
35) $\frac{1}{16}w^2 + \frac{1}{8}w = \frac{1}{2}$
36) $6h = 4h^3 + 5h^2$
37) $12n + 3 = -12n^2$
38) $u = u^2$
39) $3b^2 - b - 7 = 4b(2b + 3) - 1$
40) $\frac{1}{2}q^2 + \frac{3}{4} = \frac{5}{4}q$
41) $t^3 + 7t^2 - 4t - 28 = 0$
42) $5m^3 + 2m^2 - 5m - 2 = 0$

Write an equation and solve.

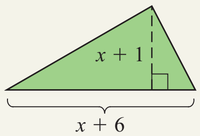
43) The length of a rectangle is 5 in. more than its width. Find the dimensions of the rectangle if its area is 14 in^2.

44) The width of a rectangle is 3 cm shorter than its length. If the area is 70 cm^2, what are the dimensions of the rectangle?

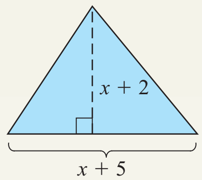
45) The length of a rectangle is 1 cm less than twice its width. The area is 45 cm^2. What are the dimensions of the rectangle?

46) A rectangle has an area of 32 in^2. Its length is 4 in. less than three times its width. Find the length and width.

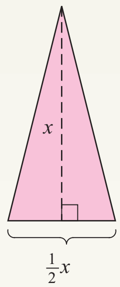
Find the base and height of each triangle.

47)


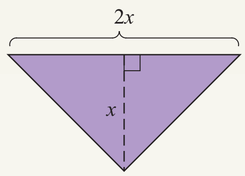
Area = 18 in^2

48)


Area = 27 in^2

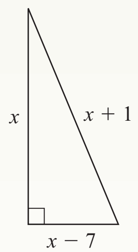
49)


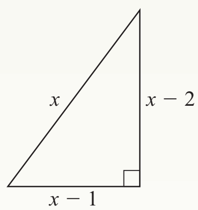
Area = 36 cm^2

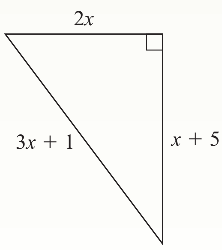
50)


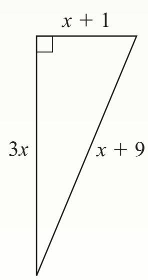
Area = 16 cm^2

Find the lengths of the sides of the following right triangles. (Hint: Use the Pythagorean theorem.)

51)


52)


53)


54)


Section 11.2 The Square Root Property and Completing the Square

Objectives

1. Solve an Equation of the Form $x^2 = k$
2. Solve an Equation of the Form $(ax + b)^2 = k$
3. Use the Distance Formula
4. Complete the Square for an Expression of the Form $x^2 + bx$
5. Solve a Quadratic Equation by Completing the Square

The next method we will discuss for solving quadratic equations is the square root property.

1. Solve an Equation of the Form $x^2 = k$

Definition The Square Root Property

Let k be a constant. If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

(The solution is often written as $x = \pm\sqrt{k}$, read as "x equals plus or minus the square root of k.")

Note

We can use the square root property to solve an equation containing a squared quantity and a constant.

Example 1

Solve using the square root property.

a) $x^2 = 9$ b) $t^2 - 20 = 0$ c) $2a^2 + 21 = 3$

Solution

a)

$$x^2 = 9$$

$$\begin{array}{lcll} x = \sqrt{9} & \text{or} & x = -\sqrt{9} & \text{Square root property} \\ x = 3 & \text{or} & x = -3 & \end{array}$$

The solution set is $\{-3, 3\}$. The check is left to the student.

An equivalent way to solve $x^2 = 9$ is to write it as

$$\begin{aligned} x^2 &= 9 \\ x &= \pm\sqrt{9} && \text{Square root property} \\ x &= \pm 3 \end{aligned}$$

The solution set is $\{-3, 3\}$. We will use this approach when solving using the square root property.

b) To solve $t^2 - 20 = 0$, begin by getting t^2 on a side by itself.

$$\begin{aligned} t^2 - 20 &= 0 \\ t^2 &= 20 && \text{Add 20 to each side.} \\ t &= \pm\sqrt{20} && \text{Square root property} \\ t &= \pm\sqrt{4} \cdot \sqrt{5} && \text{Product rule for radicals} \\ t &= \pm 2\sqrt{5} && \sqrt{4} = 2 \end{aligned}$$

Check:

$t = 2\sqrt{5}$:

$$\begin{aligned} t^2 - 20 &= 0 \\ (2\sqrt{5})^2 - 20 &\stackrel{?}{=} 0 \\ (4 \cdot 5) - 20 &\stackrel{?}{=} 0 \\ 20 - 20 &= 0 \quad \checkmark \end{aligned}$$

$t = -2\sqrt{5}$:

$$\begin{aligned} t^2 - 20 &= 0 \\ (-2\sqrt{5})^2 - 20 &\stackrel{?}{=} 0 \\ (4 \cdot 5) - 20 &\stackrel{?}{=} 0 \\ 20 - 20 &= 0 \quad \checkmark \end{aligned}$$

The solution set is $\{-2\sqrt{5}, 2\sqrt{5}\}$.

c)

$$\begin{aligned} 2a^2 + 21 &= 3 \\ 2a^2 &= -18 && \text{Subtract 21.} \\ a^2 &= -9 && \text{Divide by 2.} \\ a &= \pm\sqrt{-9} && \text{Square root property} \\ a &= \pm 3i \end{aligned}$$

Check:

$$a = 3i: \quad \begin{aligned} 2a^2 + 21 &= 3 \\ 2(3i)^2 + 21 &\stackrel{?}{=} 3 \\ 2(9i^2) + 21 &\stackrel{?}{=} 3 \\ 2(9)(-1) + 21 &\stackrel{?}{=} 3 \\ -18 + 21 &= 3 \quad ✓ \end{aligned}$$

$$a = -3i: \quad \begin{aligned} 2a^2 + 21 &= 3 \\ 2(-3i)^2 + 21 &\stackrel{?}{=} 3 \\ 2(9i^2) + 21 &\stackrel{?}{=} 3 \\ 2(9)(-1) + 21 &\stackrel{?}{=} 3 \\ -18 + 21 &= 3 \quad ✓ \end{aligned}$$

The solution set is $\{-3i, 3i\}$.

You Try 1

Solve using the square root property.

a) $p^2 = 100$ b) $w^2 - 32 = 0$ c) $3m^2 + 19 = 7$

Can we solve $(w - 4)^2 = 25$ using the square root property? Yes. The equation has a *squared quantity* and a *constant*.

2. Solve an Equation of the Form $(ax + b)^2 = k$

Example 2

Solve $x^2 = 25$ and $(w - 4)^2 = 25$ using the square root property.

Solution

While the equation $(w - 4)^2 = 25$ has a *binomial* that is being squared, the two equations are actually in the same form.

$$\underbrace{x^2}_{x \text{ squared}} = \underbrace{25}_{\text{constant}} \qquad \underbrace{(w-4)^2}_{(w-4) \text{ squared}} = \underbrace{25}_{\text{constant}}$$

Solve $x^2 = 25$:

$$\begin{aligned} x^2 &= 25 \\ x &= \pm\sqrt{25} && \text{Square root property} \\ x &= \pm 5 \end{aligned}$$

The solution set is $\{-5, 5\}$.
We solve $(w - 4)^2 = 25$ in the same way with some additional steps.

$$\begin{aligned} (w - 4)^2 &= 25 \\ w - 4 &= \pm\sqrt{25} && \text{Square root property} \\ w - 4 &= \pm 5 \end{aligned}$$

This means $w - 4 = 5$ or $w - 4 = -5$. Solve both equations.

$$\begin{aligned} w - 4 &= 5 \quad \text{or} \quad w - 4 = -5 \\ w &= 9 \quad \text{or} \quad w = -1 && \text{Add 4 to each side.} \end{aligned}$$

Check:

$$w = 9: \quad \begin{aligned} (w - 4)^2 &= 25 \\ (9 - 4)^2 &\stackrel{?}{=} 25 \\ 5^2 &= 25 \quad ✓ \end{aligned}$$

$$w = -1: \quad \begin{aligned} (w - 4)^2 &= 25 \\ (-1 - 4)^2 &\stackrel{?}{=} 25 \\ (-5)^2 &= 25 \quad ✓ \end{aligned}$$

The solution set is $\{-1, 9\}$.

You Try 2

Solve $(c + 6)^2 = 81$ using the square root property.

Example 3

Solve.

a) $(3t + 4)^2 = 9$ b) $(2m - 5)^2 = 12$ c) $(z + 8)^2 + 11 = 7$
d) $(6k - 5)^2 + 20 = 0$

Solution

a) $(3t + 4)^2 = 9$

$$\begin{aligned} 3t + 4 &= \pm\sqrt{9} && \text{Square root property} \\ 3t + 4 &= \pm 3 \end{aligned}$$

This means $3t + 4 = 3$ or $3t + 4 = -3$. Solve both equations.

$$\begin{aligned} 3t + 4 &= 3 \quad \text{or} \quad & 3t + 4 &= -3 \\ 3t &= -1 & 3t &= -7 && \text{Subtract 4 from each side.} \\ t &= -\frac{1}{3} \quad \text{or} & t &= -\frac{7}{3} && \text{Divide by 3.} \end{aligned}$$

The solution set is $\left\{-\frac{7}{3}, -\frac{1}{3}\right\}$.

b) $(2m - 5)^2 = 12$

$$\begin{aligned} 2m - 5 &= \pm\sqrt{12} && \text{Square root property} \\ 2m - 5 &= \pm 2\sqrt{3} && \text{Simplify } \sqrt{12}. \\ 2m &= 5 \pm 2\sqrt{3} && \text{Add 5 to each side.} \\ m &= \frac{5 \pm 2\sqrt{3}}{2} && \text{Divide by 2.} \end{aligned}$$

One solution is $\frac{5 + 2\sqrt{3}}{2}$, and the other is $\frac{5 - 2\sqrt{3}}{2}$.

The solution set, $\left\{\frac{5 - 2\sqrt{3}}{2}, \frac{5 + 2\sqrt{3}}{2}\right\}$, can also be written as $\left\{\frac{5 \pm 2\sqrt{3}}{2}\right\}$.

c) $(z + 8)^2 + 11 = 7$

$$\begin{aligned} (z + 8)^2 &= -4 && \text{Subtract 11 from each side.} \\ z + 8 &= \pm\sqrt{-4} && \text{Square root property} \\ z + 8 &= \pm 2i \\ z &= -8 \pm 2i && \text{Subtract 8 from each side.} \end{aligned}$$

The check is left to the student. The solution set is $\{-8 - 2i, -8 + 2i\}$.

d) $(6k - 5)^2 + 20 = 0$

$$\begin{aligned} (6k - 5)^2 &= -20 && \text{Subtract 20 from each side.} \\ 6k - 5 &= \pm\sqrt{-20} && \text{Square root property} \\ 6k - 5 &= \pm 2i\sqrt{5} && \text{Simplify } \sqrt{-20}. \\ 6k &= 5 \pm 2i\sqrt{5} && \text{Add 5 to each side.} \\ k &= \frac{5 \pm 2i\sqrt{5}}{6} && \text{Divide by 6.} \end{aligned}$$

The check is left to the student. The solution set is $\left\{\frac{5 - 2i\sqrt{5}}{6}, \frac{5 + 2i\sqrt{5}}{6}\right\}$. ■

You Try 3

Solve.

a) $(7q + 1)^2 = 36$ b) $(5a - 3)^2 = 24$
c) $(c - 7)^2 + 100 = 0$ d) $(2y + 3)^2 - 5 = -23$

Did you notice in Examples 1c), 3c), and 3d) that a complex number *and* its conjugate were the solutions to the equations? This will always be true provided that the variables in the equation have real-number coefficients.

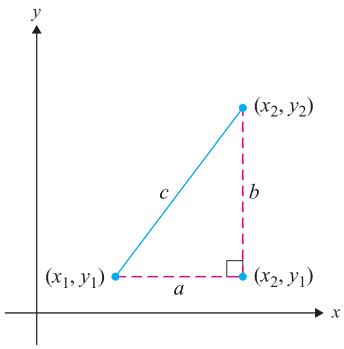
Note

If $a + bi$ is a solution of a quadratic equation having only real coefficients, then $a - bi$ is also a solution.

3. Use the Distance Formula

In mathematics, we sometimes need to find the distance between two points in a plane. The **distance formula** enables us to do that. We can use the Pythagorean theorem and the square root property to develop the distance formula.

Suppose we want to find the distance between any two points with coordinates (x_1, y_1) and (x_2, y_2) as pictured here. [We also include the point (x_2, y_1) in our drawing so that we get a right triangle.]



The lengths of the legs are a and b. The length of the hypotenuse is c. Our goal is to find the *distance* between (x_1, y_1) and (x_2, y_2), *which is the same as* finding the length of c.

How long is side a? $|x_2 - x_1|$

How long is side b? $|y_2 - y_1|$

The Pythagorean theorem states that $a^2 + b^2 = c^2$. Substitute $|x_2 - x_1|$ for a and $|y_2 - y_1|$ for b, then solve for c.

$$a^2 + b^2 = c^2 \qquad \text{Pythagorean theorem}$$
$$|x_2 - x_1|^2 + |y_2 - y_1|^2 = c^2 \qquad \text{Substitute values.}$$
$$\pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = c \qquad \text{Solve for } c \text{ using the square root property.}$$

The distance between the points (x_1, y_1) and (x_2, y_2) is $c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. We want only the positive square root since c is a length.

Since this formula represents the *distance* between two points, we usually use the letter d instead of c.

Definition The Distance Formula

The distance, d, between two points with coordinates (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 4

Find the distance between the points $(-4, 1)$ and $(2, 5)$.

Solution

Begin by labeling the points: $(\overset{x_1,\, y_1}{-4,\ 1}), (\overset{x_2,\, y_2}{2,\ 5})$.

Substitute the values into the distance formula.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{[2 - (-4)]^2 + (5 - 1)^2} \qquad \text{Substitute values.} \\ &= \sqrt{(2 + 4)^2 + (4)^2} \\ &= \sqrt{(6)^2 + (4)^2} = \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13} \end{aligned}$$

You Try 4

Find the distance between the points $(1, 2)$ and $(7, -3)$.

The next method we will learn for solving a quadratic equation is *completing the square*. We need to review an idea first presented in Section 7.4.

A **perfect square trinomial** is a trinomial whose factored form is the square of a binomial. Some examples of perfect square trinomials are

Perfect Square Trinomials	Factored Form
$x^2 + 10x + 25$	$(x + 5)^2$
$d^2 - 8d + 16$	$(d - 4)^2$

In the trinomial $x^2 + 10x + 25$, x^2 is called the *quadratic term*, $10x$ is called the *linear term*, and 25 is called the *constant*.

4. Complete the Square for an Expression of the Form $x^2 + bx$

In a perfect square trinomial where the coefficient of the quadratic term is 1, the constant term is related to the coefficient of the linear term in the following way: *if you find half of the linear coefficient and square the result, you will get the constant term.*

$x^2 + 10x + 25$: The constant, 25, is obtained by

1) finding half of the coefficient of x; then
$$\frac{1}{2}(10) = 5$$

2) squaring the result.
$$5^2 = 25 \text{ (the constant)}$$

$d^2 - 8d + 16$: The constant, 16, is obtained by

1) finding half of the coefficient of d; then
$$\frac{1}{2}(-8) = -4$$

2) squaring the result.
$$(-4)^2 = 16 \text{ (the constant)}$$

We can generalize this procedure so that we can find the constant needed to obtain the perfect square trinomial for any quadratic expression of the form $x^2 + bx$. Finding this perfect square trinomial is called **completing the square** because the trinomial will factor to the square of a binomial.

Procedure Completing the Square for $x^2 + bx$

To find the constant needed to complete the square for $x^2 + bx$:

Step 1: Find half of the coefficient of x: $\frac{1}{2}b$.

Step 2: Square the result: $\left(\frac{1}{2}b\right)^2$

Step 3: Then add it to $x^2 + bx$ to get $x^2 + bx + \left(\frac{1}{2}b\right)^2$. The factored form is $\left(x + \frac{1}{2}b\right)^2$.

BE CAREFUL The coefficient of the squared term *must* be 1 before you complete the square!

Example 5

Complete the square for each expression to obtain a perfect square trinomial. Then, factor.

a) $y^2 + 6y$ b) $t^2 - 14t$

Solution

a) Find the constant needed to complete the square for $y^2 + 6y$.

Step 1: Find half of the coefficient of y:

$$\frac{1}{2}(6) = 3$$

Step 2: Square the result:

$$3^2 = 9$$

Step 3: Add 9 to $y^2 + 6y$:

$$y^2 + 6y + 9$$

The perfect square trinomial is $y^2 + 6y + 9$. The factored form is $(y + 3)^2$.

b) Find the constant needed to complete the square for $t^2 - 14t$.

Step 1: Find half of the coefficient of t:

$$\frac{1}{2}(-14) = -7$$

Step 2: Square the result:

$$(-7)^2 = 49$$

Step 3: Add 49 to $t^2 - 14t$:

$$t^2 - 14t + 49$$

The perfect square trinomial is $t^2 - 14t + 49$. The factored form is $(t - 7)^2$.

You Try 5

Complete the square for each expression to obtain a perfect square trinomial. Then, factor.

a) $w^2 + 2w$ b) $z^2 - 16z$

We've seen the following perfect square trinomials and their factored forms. We will look at the relationship between the constant in the factored form and the coefficient of the linear term.

Perfect Square Trinomial		Factored Form
$x^2 + 10x + 25$	5 is $\frac{1}{2}(10)$.	$(x + 5)^2$
$d^2 - 8d + 16$	-4 is $\frac{1}{2}(-8)$.	$(d - 4)^2$
$y^2 + 6y + 9$	3 is $\frac{1}{2}(6)$.	$(d + 3)^2$
$t^2 - 14t + 49$	-7 is $\frac{1}{2}(-14)$.	$(t - 7)^2$

This pattern will always hold true and can be helpful in factoring some perfect square trinomials.

Example 6

Complete the square for $n^2 + 5n$ to obtain a perfect square trinomial. Then, factor.

Solution

Find the constant needed to complete the square for $n^2 + 5n$.

Step 1: Find half of the coefficient of n: $\frac{1}{2}(5) = \frac{5}{2}$

Step 2: Square the result: $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$

Step 3: Add $\frac{25}{4}$ to $n^2 + 5n$. The perfect square trinomial is $n^2 + 5n + \frac{25}{4}$.

The factored form is $\left(n + \frac{5}{2}\right)^2$

$\frac{5}{2}$ is $\frac{1}{2}(5)$, the coefficient of n.

Check: $\left(n + \frac{5}{2}\right)^2 = n^2 + 2n\left(\frac{5}{2}\right) + \left(\frac{5}{2}\right)^2 = n^2 + 5n + \frac{25}{4}$ ■

You Try 6

Complete the square for $p^2 - 3p$ to obtain a perfect square trinomial. Then, factor.

5. Solve a Quadratic Equation by Completing the Square

Any quadratic equation of the form $ax^2 + bx + c = 0$ $(a \neq 0)$ can be written in the form $(x - h)^2 = k$ by completing the square. Once an equation is in this form, we can use the square root property to solve for the variable.

Procedure Solve a Quadratic Equation ($ax^2 + bx + c = 0$) by Completing the Square

Step 1: **The coefficient of the squared term must be 1.** If it is not 1, divide both sides of the equation by a to obtain a leading coefficient of 1.

Step 2: **Get the variables on one side of the equal sign and the constant on the other side.**

Step 3: **Complete the square.** Find half of the linear coefficient, then square the result. Add that quantity to *both* sides of the equation.

Step 4: **Factor.**

Step 5: **Solve using the square root property.**

Example 7

Solve by completing the square.

a) $x^2 + 6x + 8 = 0$ b) $12h + 4h^2 = -24$

Solution

a) $x^2 + 6x + 8 = 0$

Step 1: The coefficient of x^2 is already 1.

Step 2: Get the variables on one side of the equal sign and the constant on the other side: $x^2 + 6x = -8$

Step 3: Complete the square: $\frac{1}{2}(6) = 3$

$$3^2 = 9$$

Add 9 to both sides of the equation:

$$\begin{aligned} x^2 + 6x + 9 &= -8 + 9 \\ x^2 + 6x + 9 &= 1 \end{aligned}$$

Step 4: Factor: $(x + 3)^2 = 1$

Step 5: Solve using the square root property.

$$\begin{aligned} (x + 3)^2 &= 1 \\ x + 3 &= \pm\sqrt{1} \\ x + 3 &= \pm 1 \end{aligned}$$

$$\begin{aligned} x + 3 &= 1 \quad \text{or} \quad x + 3 = -1 \\ x &= -2 \quad \text{or} \quad x = -4 \end{aligned}$$

The check is left to the student. The solution set is $\{-4, -2\}$.

Note

We would have obtained the same result if we had solved the equation by factoring.

$$\begin{aligned} x^2 + 6x + 8 &= 0 \\ (x + 4)(x + 2) &= 0 \end{aligned}$$

$$\begin{aligned} x + 4 &= 0 \quad \text{or} \quad x + 2 = 0 \\ x &= -4 \quad \text{or} \quad x = -2 \end{aligned}$$

b) $12h + 4h^2 = -24$

Step 1: Since the coefficient of h^2 is *not* 1, divide the whole equation by 4.

$$\begin{aligned} \frac{12h}{4} + \frac{4h^2}{4} &= \frac{-24}{4} \\ 3h + h^2 &= -6 \end{aligned}$$

Step 2: The constant is on a side by itself. Rewrite the left side of the equation.

$$h^2 + 3h = -6$$

Step 3: Complete the square: $\frac{1}{2}(3) = \frac{3}{2}$

$$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

Add $\frac{9}{4}$ to both sides of the equation.

$$h^2 + 3h + \frac{9}{4} = -6 + \frac{9}{4}$$

$$h^2 + 3h + \frac{9}{4} = -\frac{24}{4} + \frac{9}{4} \qquad \text{Get a common denominator.}$$

$$h^2 + 3h + \frac{9}{4} = -\frac{15}{4}$$

Step 4: Factor.

$$\left(h + \frac{3}{2}\right)^2 = -\frac{15}{4}$$

$\uparrow$

$\frac{3}{2}$ is $\frac{1}{2}(3)$, the coefficient of h.

Step 5: Solve using the square root property.

$$\left(h + \frac{3}{2}\right)^2 = -\frac{15}{4}$$

$$h + \frac{3}{2} = \pm\sqrt{-\frac{15}{4}}$$

$$h + \frac{3}{2} = \pm\frac{\sqrt{15}}{2}i \qquad \text{Simplify the radical.}$$

$$h = -\frac{3}{2} \pm \frac{\sqrt{15}}{2}i \qquad \text{Subtract } \frac{3}{2}.$$

The check is left to the student. The solution set is $\left\{-\frac{3}{2} - \frac{\sqrt{15}}{2}i, -\frac{3}{2} + \frac{\sqrt{15}}{2}i\right\}$.

You Try 7

Solve by completing the square.

a) $q^2 + 10q - 24 = 0$ b) $2m^2 + 16 = 10m$

Answers to You Try Exercises

1) a) $\{-10, 10\}$ b) $\{-4\sqrt{2}, 4\sqrt{2}\}$ c) $\{-2i, 2i\}$ 2) $\{-15, 3\}$

3) a) $\left\{-1, \frac{5}{7}\right\}$ b) $\left\{\frac{3 - 2\sqrt{6}}{5}, \frac{3 + 2\sqrt{6}}{5}\right\}$ c) $\{7 - 10i, 7 + 10i\}$

d) $\left\{-\frac{3}{2} - \frac{3\sqrt{2}}{2}i, -\frac{3}{2} + \frac{3\sqrt{2}}{2}i\right\}$ 4) $\sqrt{61}$

5) a) $w^2 + 2w + 1; (w + 1)^2$ b) $z^2 - 16z + 64; (z - 8)^2$

6) $p^2 - 3p + \frac{9}{4}; \left(p - \frac{3}{2}\right)^2$ 7) a) $\{-12, 2\}$ b) $\left\{\frac{5}{2} - \frac{\sqrt{7}}{2}i, \frac{5}{2} + \frac{\sqrt{7}}{2}i\right\}$

11.2 Exercises

Objective 1: Solve an Equation of the Form $x^2 = k$

1) Choose two methods to solve $y^2 - 16 = 0$. Solve the equation using both methods.

2) If k is a negative number and $x^2 = k$, what can you conclude about the solution to the equation?

Solve using the square root property.

3) $b^2 = 36$
4) $h^2 = 64$
5) $r^2 - 27 = 0$
6) $a^2 - 30 = 0$
7) $n^2 = \frac{4}{9}$
8) $v^2 = \frac{121}{16}$
9) $q^2 = -4$
10) $w^2 = -121$
11) $z^2 + 3 = 0$
12) $h^2 + 14 = -23$
13) $z^2 + 5 = 19$
14) $q^2 - 3 = 15$
15) $2d^2 + 5 = 55$
16) $4m^2 + 1 = 37$
17) $5f^2 + 39 = -21$
18) $2y^2 + 56 = 0$

Objective 2: Solve an Equation of the Form $(ax + b)^2 = k$

Solve using the square root property.

19) $(r + 10)^2 = 4$
20) $(x - 5)^2 = 81$
21) $(q - 7)^2 = 1$
22) $(c + 12)^2 = 25$
23) $(p + 4)^2 - 18 = 0$
24) $(d + 2)^2 - 7 = 13$
25) $(c + 3)^2 - 4 = -29$
26) $(u - 15)^2 - 4 = -8$
27) $1 = 15 + (k - 2)^2$
28) $2 = 14 + (g + 4)^2$
29) $20 = (2w + 1)^2$
30) $(5b - 6)^2 = 11$
31) $8 = (3q - 10)^2 - 6$
32) $22 = (6x + 11)^2 + 4$
33) $36 + (4p - 5)^2 = 6$
34) $(3k - 1)^2 + 20 = 4$
35) $(6g + 11)^2 + 50 = 1$
36) $9 = 38 + (9s - 4)^2$
37) $\left(\frac{3}{4}n - 8\right)^2 = 4$
38) $\left(\frac{2}{3}j + 10\right)^2 = 16$
39) $(5y - 2)^2 + 6 = 22$
40) $-6 = 3 - (2q - 9)^2$

Objective 3: Use the Distance Formula

Find the distance between the given points.

41) $(7, -1)$ and $(3, 2)$
42) $(3, 10)$ and $(12, 6)$
43) $(-5, -6)$ and $(-2, -8)$
44) $(5, -2)$ and $(-3, 4)$
45) $(0, 13)$ and $(0, 7)$
46) $(-8, 3)$ and $(2, 1)$
47) $(-4, 11)$ and $(2, 6)$
48) $(0, 3)$ and $(3, -1)$
49) $(3, -3)$ and $(5, -7)$
50) $(-5, -6)$ and $(-1, 2)$

Objective 4: Complete the Square for an Expression of the Form $x^2 + bx$

51) What is a perfect square trinomial? Give an example.

52) Can you complete the square on $3y^2 + 15y$ as it is given? Why or why not?

Complete the square for each expression to obtain a perfect square trinomial. Then, factor.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

53) $w^2 + 8w$

________ Find half of the coefficient of w.

________ Square the result.

________ Add the constant to the expression.

The perfect square trinomial is ________

The factored form of the trinomial is ________

54) $n^2 - n$

$\frac{1}{2}(-1) = -\frac{1}{2}$ ________

$\left(-\frac{1}{2}\right)^2 = \frac{1}{4}$ ________

$n^2 - n + \frac{1}{4}$ ________

The perfect square trinomial is ________

The factored form of the trinomial is ________

55) $a^2 + 12a$
56) $g^2 + 4g$
57) $c^2 - 18c$
58) $k^2 - 16k$

59) $r^2 + 3r$

60) $z^2 - 7z$

61) $b^2 - 9b$

62) $t^2 + 5t$

63) $x^2 + \frac{1}{3}x$

64) $y^2 - \frac{3}{5}y$

Objective 5: Solve a Quadratic Equation by Completing the Square

65) What is the first thing you should do if you want to solve $2p^2 - 7p = 8$ by completing the square?

66) Can $x^3 + 10x - 3 = 0$ be solved by completing the square? Give a reason for your answer.

Solve by completing the square.

VIDEO 67) $x^2 + 6x + 8 = 0$

68) $t^2 + 12t - 13 = 0$

69) $k^2 - 8k + 15 = 0$

70) $v^2 - 6v - 27 = 0$

71) $s^2 + 10 = -10s$

72) $u^2 - 9 = 2u$

73) $t^2 = 2t - 9$

74) $p^2 = -10p - 26$

75) $v^2 + 4v + 8 = 0$

76) $a^2 + 19 = 8a$

VIDEO 77) $m^2 + 3m - 40 = 0$

78) $p^2 + 5p + 4 = 0$

79) $x^2 - 7x + 12 = 0$

80) $d^2 + d - 72 = 0$

81) $r^2 - r = 3$

82) $y^2 - 3y = 7$

83) $c^2 + 5c + 7 = 0$

84) $b^2 + 14 = 7b$

VIDEO 85) $3k^2 - 6k + 12 = 0$

86) $4f^2 + 16f + 48 = 0$

87) $4r^2 + 24r = 8$

88) $3h^2 + 6h = 15$

89) $10d = 2d^2 + 12$

90) $54x - 6x^2 = 48$

91) $2n^2 + 8 = 5n$

92) $2t^2 + 3t + 4 = 0$

93) $4a^2 - 7a + 3 = 0$

94) $n + 2 = 3n^2$

95) $(y + 5)(y - 3) = 5$

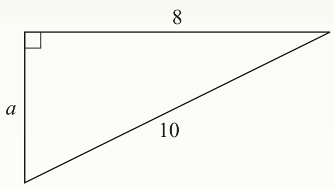
96) $(b - 4)(b + 10) = -17$

97) $(2m + 1)(m - 3) = -7$

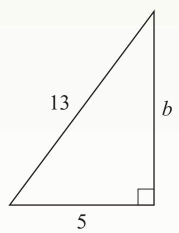
98) $(3c + 4)(c + 2) = 3$

Use the Pythagorean theorem and the square root property to find the length of the missing side.

99)



100)



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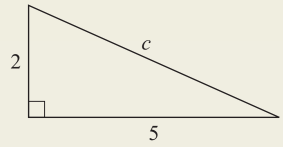

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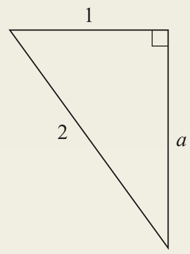


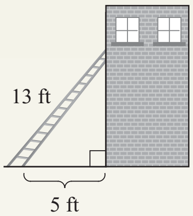
Write an equation and solve. (Hint: Draw a picture.)

103) The width of a rectangle is 4 inches, and its diagonal is $2\sqrt{13}$ inches long. What is the length of the rectangle?

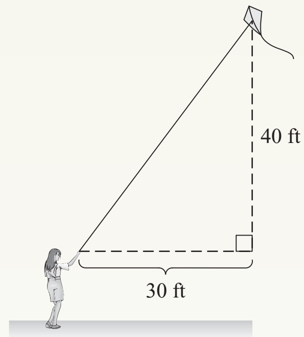
104) Find the length of the diagonal of a rectangle if it has a width of 5 cm and a length of $4\sqrt{2}$ cm.

Write an equation and solve.

VIDEO 105) A 13-foot ladder is leaning against a wall so that the base of the ladder is 5 feet away from the wall. How high on the wall does the ladder reach?



106) Salma is flying a kite. It is 30 feet from her horizontally, and it is 40 feet above her hand. How long is the kite string?



107) Let $f(x) = (x + 3)^2$. Find x so that $f(x) = 49$.

108) Let $g(t) = (t - 5)^2$. Find t so that $g(t) = 12$.

Solve each problem by writing an equation and solving it by completing the square.

109) The length of a rectangular garden is 8 ft. more than its width. Find the dimensions of the garden if it has an area of 153 ft^2.

110) The rectangular screen on a laptop has an area of 375 cm^2. Its width is 10 cm less than its length. What are the dimensions of the screen?

Section 11.3 The Quadratic Formula

Objectives

1. Derive the Quadratic Formula
2. Solve a Quadratic Equation Using the Quadratic Formula
3. Determine the Number and Type of Solutions to a Quadratic Equation Using the Discriminant
4. Solve an Applied Problem Using the Quadratic Formula

1. Derive the Quadratic Formula

In Section 11.2, we saw that any quadratic equation of the form $ax^2 + bx + c = 0$ $(a \neq 0)$ can be solved by completing the square. Therefore, we can solve equations like $x^2 - 8x + 5 = 0$ and $2x^2 + 3x - 1 = 0$ using this method.

We can develop another method for solving quadratic equations by completing the square on the general quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$. This is how we will derive the *quadratic formula*.

The steps we use to complete the square on $ax^2 + bx + c = 0$ are *exactly* the same steps we use to solve an equation like $2x^2 + 3x - 1 = 0$. We will do these steps side by side so that you can more easily understand how we are solving $ax^2 + bx + c = 0$ for x by completing the square.

Solve by Completing the Square

$$2x^2 + 3x - 1 = 0 \qquad\qquad ax^2 + bx + c = 0$$

Step 1: **The coefficient of the squared term must be 1.**

$2x^2 + 3x - 1 = 0$	$ax^2 + bx + c = 0$
$\dfrac{2x^2}{2} + \dfrac{3x}{2} - \dfrac{1}{2} = \dfrac{0}{2}$ Divide by 2.	$\dfrac{ax^2}{a} + \dfrac{bx}{a} + \dfrac{c}{a} = \dfrac{0}{a}$ Divide by a.
$x^2 + \dfrac{3}{2}x - \dfrac{1}{2} = 0$ Simplify.	$x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$ Simplify.

Step 2: **Get the constant on the other side of the equal sign.**

$x^2 + \dfrac{3}{2}x = \dfrac{1}{2}$ Add $\dfrac{1}{2}$.	$x^2 + \dfrac{b}{a}x = -\dfrac{c}{a}$ Subtract $\dfrac{c}{a}$.

Step 3: **Complete the square.**

$\dfrac{1}{2}\left(\dfrac{3}{2}\right) = \dfrac{3}{4}$ $\dfrac{1}{2}$ of x-coefficient	$\dfrac{1}{2}\left(\dfrac{b}{a}\right) = \dfrac{b}{2a}$ $\dfrac{1}{2}$ of x-coefficient
$\left(\dfrac{3}{4}\right)^2 = \dfrac{9}{16}$ Square the result.	$\left(\dfrac{b}{2a}\right)^2 = \dfrac{b^2}{4a^2}$ Square the result.
Add $\dfrac{9}{16}$ to both sides of the equation.	Add $\dfrac{b^2}{4a^2}$ to both sides of the equation.
$x^2 + \dfrac{3}{2}x + \dfrac{9}{16} = \dfrac{1}{2} + \dfrac{9}{16}$	$x^2 + \dfrac{b}{a}x + \dfrac{b^2}{4a^2} = -\dfrac{c}{a} + \dfrac{b^2}{4a^2}$
$x^2 + \dfrac{3}{2}x + \dfrac{9}{16} = \dfrac{8}{16} + \dfrac{9}{16}$ Get a common denominator.	$x^2 + \dfrac{b}{a}x + \dfrac{b^2}{4a^2} = -\dfrac{4ac}{4a^2} + \dfrac{b^2}{4a^2}$ Get a common denominator.
$x^2 + \dfrac{3}{2}x + \dfrac{9}{16} = \dfrac{17}{16}$ Add.	$x^2 + \dfrac{b}{a}x + \dfrac{b^2}{4a^2} = \dfrac{b^2 - 4ac}{4a^2}$ Add.

Step 4: **Factor.**

$$\left(x + \frac{3}{4}\right)^2 = \frac{17}{16}$$

$\frac{3}{4}$ is $\frac{1}{2}\left(\frac{3}{2}\right)$, the coefficient of x.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$\frac{b}{2a}$ is $\frac{1}{2}\left(\frac{b}{a}\right)$, the coefficient of x.

Step 5: **Solve using the square root property.**

$$\left(x + \frac{3}{4}\right)^2 = \frac{17}{16}$$

$$x + \frac{3}{4} = \pm\sqrt{\frac{17}{16}}$$

$$x + \frac{3}{4} = \frac{\pm\sqrt{17}}{4} \qquad \sqrt{16} = 4$$

$$x = -\frac{3}{4} \pm \frac{\sqrt{17}}{4} \qquad \text{Subtract } \frac{3}{4}.$$

$$x = \frac{-3 \pm \sqrt{17}}{4} \qquad \text{Same denominators, add numerators.}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \qquad \sqrt{4a^2} = 2a$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{Subtract } \frac{b}{2a}.$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Same denominators, add numerators.}$$

The result on the right is called the *quadratic formula*.

Definition The Quadratic Formula

The solutions of any quadratic equation of the form $ax^2 + bx + c = 0$ $(a \neq 0)$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note

1) Write the equation to be solved in the form $ax^2 + bx + c = 0$ so that a, b, and c can be identified correctly.

2) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ represents the two solutions $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

3) Notice that the fraction bar runs under $-b$ *and* under the radical.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Correct

$$\xcancel{x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}}$$

Incorrect

4) When deriving the quadratic formula, using the $\pm$ allows us to say that $\sqrt{4a^2} = 2a$.

5) The quadratic formula is a *very* important result and one that we will use often. *It should be memorized!*

2. Solve a Quadratic Equation Using the Quadratic Formula

Example 1

Solve using the quadratic formula.

a) $2x^2 + 3x - 1 = 0$ b) $k^2 = 10k - 29$

Solution

a) Is $2x^2 + 3x - 1 = 0$ in the form $ax^2 + bx + c = 0$? Yes. Identify the values of a, b, and c, and substitute them into the quadratic formula.

$$a = 2 \qquad b = 3 \qquad c = -1$$

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Quadratic formula} \\
&= \frac{-(3) \pm \sqrt{(3)^2 - 4(2)(-1)}}{2(2)} && \text{Substitute } a = 2,\ b = 3, \text{ and } c = -1. \\
&= \frac{-3 \pm \sqrt{9 - (-8)}}{4} && \text{Perform the operations.} \\
&= \frac{-3 \pm \sqrt{17}}{4}
\end{aligned}$$

The solution set is $\left\{\dfrac{-3 - \sqrt{17}}{4}, \dfrac{-3 + \sqrt{17}}{4}\right\}$. This is the same result we obtained when we solved this equation by completing the square at the beginning of the section.

b) Is $k^2 = 10k - 29$ in the form $ax^2 + bx + c = 0$? *No*. Begin by writing the equation in the correct form.

$$\begin{aligned}
&k^2 - 10k + 29 = 0 && \text{Subtract } 10k \text{ and add 29 to both sides.} \\
&a = 1 \qquad b = -10 \qquad c = 29 && \text{Identify } a, b, \text{ and } c.
\end{aligned}$$

$$\begin{aligned}
k &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Quadratic formula} \\
&= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(29)}}{2(1)} && \text{Substitute } a = 1,\ b = -10, \text{ and } c = 29. \\
&= \frac{10 \pm \sqrt{100 - 116}}{2} && \text{Perform the operations.} \\
&= \frac{10 \pm \sqrt{-16}}{2} && 100 - 116 = -16 \\
&= \frac{10 \pm 4i}{2} && \sqrt{-16} = 4i \\
&= \frac{10}{2} \pm \frac{4}{2}i = 5 \pm 2i
\end{aligned}$$

The solution set is $\{5 - 2i, 5 + 2i\}$. ■

You Try 1

Solve using the quadratic formula.

a) $n^2 + 9n + 18 = 0$ b) $5t^2 + t - 2 = 0$

Equations in various forms may be solved using the quadratic formula.

Example 2

Solve using the quadratic formula.

a) $(3p - 1)(3p + 4) = 3p - 5$ b) $\frac{1}{2}w^2 + \frac{2}{3}w - \frac{1}{3} = 0$

Solution

a) Is $(3p - 1)(3p + 4) = 3p - 5$ in the form $ax^2 + bx + c = 0$? *No*. Before we can apply the quadratic formula, we must write it in that form.

$$\begin{aligned} (3p - 1)(3p + 4) &= 3p - 5 \\ 9p^2 + 9p - 4 &= 3p - 5 && \text{Multiply using FOIL.} \\ 9p^2 + 6p + 1 &= 0 && \text{Subtract } 3p \text{ and add 5 to both sides.} \end{aligned}$$

The equation is in the correct form. Identify a, b, and c.

$$a = 9 \qquad b = 6 \qquad c = 1$$

$$\begin{aligned} p &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Quadratic formula} \\ &= \frac{-(6) \pm \sqrt{(6)^2 - 4(9)(1)}}{2(9)} && \text{Substitute } a = 9, b = 6, \text{ and } c = 1. \\ &= \frac{-6 \pm \sqrt{36 - 36}}{18} && \text{Perform the operations.} \\ &= \frac{-6 \pm \sqrt{0}}{18} \\ &= \frac{-6 \pm 0}{18} = \frac{-6}{18} = -\frac{1}{3} \end{aligned}$$

The solution set is $\left\{-\frac{1}{3}\right\}$.

b) Is $\frac{1}{2}w^2 + \frac{2}{3}w - \frac{1}{3} = 0$ in the form $ax^2 + bx + c = 0$? Yes. However, working with fractions in the quadratic formula would be difficult. *Eliminate the fractions by multiplying the equation by 6, the least common denominator of the fractions.*

$$\begin{aligned} 6\left(\frac{1}{2}w^2 + \frac{2}{3}w - \frac{1}{3}\right) &= 6 \cdot 0 && \text{Multiply by 6 to eliminate the fractions.} \\ 3w^2 + 4w - 2 &= 0 \end{aligned}$$

Identify a, b, and c from this form of the equation: $a = 3$, $b = 4$, $c = -2$.

$$\begin{aligned} w &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Quadratic formula} \\ &= \frac{-(4) \pm \sqrt{(4)^2 - 4(3)(-2)}}{2(3)} && \text{Substitute } a = 3, b = 4, \text{ and } c = -2. \\ &= \frac{-4 \pm \sqrt{16 - (-24)}}{6} && \text{Perform the operations.} \\ &= \frac{-4 \pm \sqrt{40}}{6} && 16 - (-24) = 16 + 24 = 40 \\ &= \frac{-4 \pm 2\sqrt{10}}{6} && \sqrt{40} = 2\sqrt{10} \\ &= \frac{2(-2 \pm \sqrt{10})}{6} && \text{Factor out 2 in the numerator.} \\ &= \frac{-2 \pm \sqrt{10}}{3} && \text{Divide numerator and denominator by 2 to simplify.} \end{aligned}$$

The solution set is $\left\{\frac{-2 - \sqrt{10}}{3}, \frac{-2 + \sqrt{10}}{3}\right\}$. ■

You Try 2

Solve using the quadratic formula.

a) $3 - 2z = -2z^2$ b) $(d + 6)(d - 2) = -10$ c) $\frac{5}{4}r^2 + \frac{1}{5} = r$

3. Determine the Number and Type of Solutions to a Quadratic Equation Using the Discriminant

We can find the solutions of any quadratic equation of the form $ax^2 + bx + c = 0$ $(a \neq 0)$ using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The radicand in the quadratic formula determines the type of solution a quadratic equation has.

Property The Discriminant and Solutions

The expression under the radical, $b^2 - 4ac$, is called the **discriminant**. The discriminant tells us what kind of solution a quadratic equation has. If a, b, and c are integers, then

1) If $b^2 - 4ac$ is *positive and the square of an integer*, the equation has *two rational solutions*.
2) If $b^2 - 4ac$ is *positive but not a perfect square*, the equation has *two irrational solutions*.
3) If $b^2 - 4ac$ is *negative*, the equation has *two nonreal, complex solutions of the form* $a + bi$ and $a - bi$.
4) If $b^2 - 4ac = 0$, the equation has *one rational solution*.

Example 3

Find the value of the discriminant. Then, determine the number and type of solutions of each equation.

a) $z^2 + 6z - 4 = 0$ b) $5h^2 = 6h - 2$

Solution

a) Is $z^2 + 6z - 4 = 0$ in the form $ax^2 + bx + c = 0$? *Yes*. Identify a, b, and c.

$$a = 1 \qquad b = 6 \qquad c = -4$$

$$\text{Discriminant} = b^2 - 4ac = (6)^2 - 4(1)(-4) = 36 + 16 = 52$$

Since 52 is positive but *not* a perfect square, the equation will have *two irrational solutions*. ($\sqrt{52}$, or $2\sqrt{13}$, will appear in the solution, and $2\sqrt{13}$ is irrational.)

b) Is $5h^2 = 6h - 2$ in the form $ax^2 + bx + c = 0$? *No*. Rewrite the equation in that form, and identify a, b, and c.

$$5h^2 - 6h + 2 = 0$$

$$a = 5 \qquad b = -6 \qquad c = 2$$

$$\text{Discriminant} = b^2 - 4ac = (-6)^2 - 4(5)(2) = 36 - 40 = -4$$

Since the discriminant is -4, the equation will have *two nonreal, complex solutions of the form* $a + bi$ *and* $a - bi$, where $b \neq 0$.

BE CAREFUL

The discriminant is $b^2 - 4ac$ *not* $\sqrt{b^2 - 4ac}$.

You Try 3

Find the value of the discriminant. Then, determine the number and type of solutions of each equation.

a) $2x^2 + x + 5 = 0$ b) $m^2 + 5m = 24$ c) $-3v^2 = 4v - 1$

d) $4r(2r - 3) = -1 - 6r - r^2$

4. Solve an Applied Problem Using the Quadratic Formula

Example 4

A ball is thrown upward from a height of 20 ft. The height h of the ball (in feet) t sec after the ball is released is given by

$$h = -16t^2 + 16t + 20$$

a) How long does it take the ball to reach a height of 8 ft?

b) How long does it take the ball to hit the ground?

Solution

a) Find the *time* it takes for the ball to reach a height of 8 ft.

Find t when $h = 8$.

$$\begin{aligned} h &= -16t^2 + 16t + 20 && \\ 8 &= -16t^2 + 16t + 20 && \text{Substitute 8 for } h. \\ 0 &= -16t^2 + 16t + 12 && \text{Write in standard form.} \\ 0 &= 4t^2 - 4t - 3 && \text{Divide by } -4. \end{aligned}$$

$$\begin{aligned} t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Quadratic formula} \\ &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(-3)}}{2(4)} && \text{Substitute } a = 4,\ b = -4, \text{ and } c = -3. \\ &= \frac{4 \pm \sqrt{16 + 48}}{8} && \text{Perform the operations.} \\ &= \frac{4 \pm \sqrt{64}}{8} = \frac{4 \pm 8}{8} && \end{aligned}$$

$$t = \frac{4 + 8}{8} \quad \text{or} \quad t = \frac{4 - 8}{8} \qquad \text{The equation has two rational solutions.}$$

$$t = \frac{12}{8} = \frac{3}{2} \quad \text{or} \quad t = \frac{-4}{8} = -\frac{1}{2}$$

Since t represents time, t cannot equal $-\frac{1}{2}$. We reject that as a solution.

Therefore, $t = \frac{3}{2}$ sec or 1.5 sec. The ball will be 8 ft above the ground after 1.5 sec.

b) When the ball hits the ground, it is 0 ft above the ground.

Find t when $h = 0$.

$$\begin{aligned} h &= -16t^2 + 16t + 20 \\ 0 &= -16t^2 + 16t + 20 && \text{Substitute 0 for } h. \\ 0 &= 4t^2 - 4t - 5 && \text{Divide by } -4. \end{aligned}$$

$$\begin{aligned} t &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(-5)}}{2(4)} && \text{Substitute } a = 4,\ b = -4, \text{ and } c = -5. \\ &= \frac{4 \pm \sqrt{16 + 80}}{8} && \text{Perform the operations.} \\ &= \frac{4 \pm \sqrt{96}}{8} \\ &= \frac{4 \pm 4\sqrt{6}}{8} && \sqrt{96} = \sqrt{16} \cdot \sqrt{6} = 4\sqrt{6} \\ &= \frac{4(1 \pm \sqrt{6})}{8} && \text{Factor out 4 in the numerator.} \end{aligned}$$

$$t = \frac{1 \pm \sqrt{6}}{2} \qquad \text{Divide numerator and denominator by 4 to simplify.}$$

$$t = \frac{1 + \sqrt{6}}{2} \quad \text{or} \quad t = \frac{1 - \sqrt{6}}{2} \qquad \text{The equation has two irrational solutions.}$$

$$t \approx \frac{1 + 2.4}{2} \quad \text{or} \quad t \approx \frac{1 - 2.4}{2} \qquad \sqrt{6} \approx 2.4$$

$$t \approx \frac{3.4}{2} = 1.7 \quad \text{or} \quad t \approx -0.7$$

Since t represents time, t cannot equal $\frac{1 - \sqrt{6}}{2}$. We reject this as a solution.

Therefore, $t = \frac{1 + \sqrt{6}}{2}$ sec or $t \approx 1.7$ sec. The ball will hit the ground after about 1.7 sec. ■

You Try 4

An object is thrown upward from a height of 12 ft. The height h of the object (in feet) t sec after the object is thrown is given by

$$h = -16t^2 + 56t + 12$$

a) How long does it take the object to reach a height of 36 ft?

b) How long does it take the object to hit the ground?

Answers to You Try Exercises

1) a) $\{-6, -3\}$ b) $\left\{\frac{-1 - \sqrt{41}}{10}, \frac{-1 + \sqrt{41}}{10}\right\}$ 2) a) $\left\{\frac{1}{2} - \frac{\sqrt{5}}{2}i, \frac{1}{2} + \frac{\sqrt{5}}{2}i\right\}$ b) $\{-2 - \sqrt{6}, -2 + \sqrt{6}\}$ c) $\left\{\frac{2}{5}\right\}$ 3) a) -39; two nonreal, complex solutions b) 121; two rational solutions c) 28; two irrational solutions d) 0; one rational solution 4) a) It takes $\frac{1}{2}$ sec to reach 36 ft on its way up and 3 sec to reach 36 ft on its way down. b) $\frac{7 + \sqrt{61}}{4}$ sec or approximately 3.7 sec.

11.3 Exercises

Mixed Exercises: Objectives 2 and 3

Find the error in each, and correct the mistake.

1) The solution to $ax^2 + bx + c = 0$ $(a \neq 0)$ can be found using the quadratic formula.

$$x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

2) In order to solve $5n^2 - 3n = 1$ using the quadratic formula, a student substitutes a, b, and c into the formula in this way: $a = 5$, $b = -3$, $c = 1$.

$$n = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(5)(1)}}{2(5)}$$

3) $\dfrac{-2 \pm 6\sqrt{11}}{2} = -1 \pm 6\sqrt{11}$

4) The discriminant of $3z^2 - 4z + 1 = 0$ is

$$\begin{aligned}\sqrt{b^2 - 4ac} &= \sqrt{(-4)^2 - 4(3)(1)}\\ &= \sqrt{16 - 12}\\ &= \sqrt{4}\\ &= 2.\end{aligned}$$

Objective 2: Solve a Quadratic Equation Using the Quadratic Formula

Solve using the quadratic formula.

5) $x^2 + 4x + 3 = 0$
6) $v^2 - 8v + 7 = 0$
7) $3t^2 + t - 10 = 0$
8) $6q^2 + 11q + 3 = 0$
VIDEO 9) $k^2 + 2 = 5k$
10) $n^2 = 5 - 3n$
11) $y^2 = 8y - 25$
12) $-4x + 5 = -x^2$
13) $3 - 2w = -5w^2$
14) $2d^2 = -4 - 5d$
15) $r^2 + 7r = 0$
16) $p^2 - 10p = 0$
17) $3v(v + 3) = 7v + 4$
18) $2k(k - 3) = -3$
19) $(2c - 5)(c - 5) = -3$
20) $-11 = (3z - 1)(z - 5)$
21) $\frac{1}{6}u^2 + \frac{4}{3}u = \frac{5}{2}$
22) $\frac{5}{2}r^2 + 3r + 2 = 0$
VIDEO 23) $m^2 + \frac{4}{3}m + \frac{5}{9} = 0$
24) $\frac{1}{6}h + \frac{1}{2} = \frac{3}{4}h^2$
25) $2(p + 10) = (p + 10)(p - 2)$
26) $(t - 8)(t - 3) = 3(3 - t)$
27) $4g^2 + 9 = 0$
28) $25q^2 - 1 = 0$
29) $x(x + 6) = -34$
30) $c(c - 4) = -22$
VIDEO 31) $(2s + 3)(s - 1) = s^2 - s + 6$
32) $(3m + 1)(m - 2) = (2m - 3)(m + 2)$
33) $3(3 - 4y) = -4y^2$
34) $5a(5a + 2) = -1$
35) $-\frac{1}{6} = \frac{2}{3}p^2 + \frac{1}{2}p$
36) $\frac{1}{2}n = \frac{3}{4}n^2 + 2$
37) $4q^2 + 6 = 20q$
38) $4w^2 = 6w + 16$

Objective 3: Determine the Number and Type of Solutions to a Quadratic Equation Using the Discriminant

39) If the discriminant of a quadratic equation is zero, what do you know about the solutions of the equation?

40) If the discriminant of a quadratic equation is negative, what do you know about the solutions of the equation?

Find the value of the discriminant. Then determine the number and type of solutions of each equation. *Do not solve.*

41) $10d^2 - 9d + 3 = 0$
42) $3j^2 + 8j + 2 = 0$
43) $4y^2 + 49 = -28y$
44) $3q = 1 + 5q^2$
45) $-5 = u(u + 6)$
46) $g^2 + 4 = 4g$
VIDEO 47) $2w^2 - 4w - 5 = 0$
48) $3 + 2p^2 - 7p = 0$

Find the value of a, b, or c so that each equation has only one rational solution.

49) $z^2 + bz + 16 = 0$
50) $k^2 + bk + 49 = 0$
51) $4y^2 - 12y + c = 0$
52) $25t^2 - 20t + c = 0$
53) $ap^2 + 12p + 9 = 0$
54) $ax^2 - 6x + 1 = 0$

Objective 4: Solve an Applied Problem Using the Quadratic Formula

Write an equation and solve.

55) One leg of a right triangle is 1 in. more than twice the other leg. The hypotenuse is $\sqrt{29}$ in. long. Find the lengths of the legs.

56) The hypotenuse of a right triangle is $\sqrt{34}$ in. long. The length of one leg is 1 in. less than twice the other leg. Find the lengths of the legs.

Solve.

VIDEO 57) An object is thrown upward from a height of 24 ft. The height h of the object (in feet) t sec after the object is released is given by $h = -16t^2 + 24t + 24$.

a) How long does it take the object to reach a height of 8 ft?

b) How long does it take the object to hit the ground?

58) A ball is thrown upward from a height of 6 ft. The height h of the ball (in feet) t sec after the ball is released is given by $h = -16t^2 + 44t + 6$.

a) How long does it take the ball to reach a height of 16 ft?

b) How long does it take the object to hit the ground?

Putting It All Together

Objective

1. **Decide Which Method to Use to Solve a Quadratic Equation**

We have learned four methods for solving quadratic equations.

Methods for Solving Quadratic Equations

1) Factoring
2) Square root property
3) Completing the square
4) Quadratic formula

While it is true that the quadratic formula can be used to solve *every* quadratic equation of the form $ax^2 + bx + c = 0$ $(a \neq 0)$, it is not always the most *efficient* method. In this section we will discuss how to decide which method to use to solve a quadratic equation.

1. Decide Which Method to Use to Solve a Quadratic Equation

Example 1

Solve.

a) $p^2 - 6p = 16$ b) $m^2 - 8m + 13 = 0$
c) $3t^2 + 8t + 7 = 0$ d) $(2z - 7)^2 - 6 = 0$

Solution

a) Write $p^2 - 6p = 16$ in standard form: $p^2 - 6p - 16 = 0$

Does $p^2 - 6p - 16$ factor? Yes. *Solve by factoring.*

$$(p - 8)(p + 2) = 0$$

$$p - 8 = 0 \quad \text{or} \quad p + 2 = 0 \qquad \text{Set each factor equal to 0.}$$
$$p = 8 \quad \text{or} \quad p = -2 \qquad \text{Solve.}$$

The solution set is $\{-2, 8\}$.

b) To solve $m^2 - 8m + 13 = 0$ ask yourself, "Can I factor $m^2 - 8m + 13$?" No, it does not factor. We could solve this using the quadratic formula, but *completing the square* is also a good method for solving this equation. Why?

Completing the square is a desirable method for solving a quadratic equation when the coefficient of the squared term is 1 *or* -1 *and when the coefficient of the linear term is even.*

We will solve $m^2 - 8m + 13 = 0$ *by completing the square.*

Step 1: The coefficient of m^2 is 1.

Step 2: Get the variables on one side of the equal sign and the constant on the other side.

$$m^2 - 8m = -13$$

Step 3: Complete the square: $\frac{1}{2}(-8) = -4$

$$(-4)^2 = 16$$

Add 16 to both sides of the equation.

$$m^2 - 8m + 16 = -13 + 16$$
$$m^2 - 8m + 16 = 3$$

Step 4: Factor: $(m - 4)^2 = 3$

Step 5: Solve using the square root property:

$$\begin{aligned}(m - 4)^2 &= 3\\ m - 4 &= \pm\sqrt{3}\\ m &= 4 \pm \sqrt{3}\end{aligned}$$

The solution set is $\{4 - \sqrt{3}, 4 + \sqrt{3}\}$.

Note

Completing the square works well when the coefficient of the squared term is 1 or -1 and when the coefficient of the linear term is *even* because when we complete the square in step 3, we will not obtain a fraction. (Half of an even number is an integer.)

c) Ask yourself, "Can I solve $3t^2 + 8t + 7 = 0$ by factoring?" No, $3t^2 + 8t + 7$ does not factor. Completing the square would not be a very efficient way to solve the equation because the coefficient of t^2 is 3, and dividing the equation by 3 would give us $t^2 + \frac{8}{3}t + \frac{7}{3} = 0$.

We will solve $3t^2 + 8t + 7 = 0$ *using the quadratic formula.*

Identify a, b, and c: $a = 3$ $b = 8$ $c = 7$

$$\begin{aligned}
t &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Quadratic formula}\\
&= \frac{-(8) \pm \sqrt{(8)^2 - 4(3)(7)}}{2(3)} && \text{Substitute } a = 3,\ b = 8, \text{ and } c = 7.\\
&= \frac{-8 \pm \sqrt{64 - 84}}{6} && \text{Perform the operations.}\\
&= \frac{-8 \pm \sqrt{-20}}{6}\\
&= \frac{-8 \pm 2i\sqrt{5}}{6} && \sqrt{-20} = i\sqrt{4}\sqrt{5} = 2i\sqrt{5}\\
&= \frac{2(-4 \pm i\sqrt{5})}{6} && \text{Factor out 2 in the numerator.}\\
&= \frac{-4 \pm i\sqrt{5}}{3} && \text{Divide numerator and denominator by 2 to simplify.}\\
&= -\frac{4}{3} \pm \frac{\sqrt{5}}{3}i && \text{Write in the form } a + bi.
\end{aligned}$$

The solution set is $\left\{-\frac{4}{3} - \frac{\sqrt{5}}{3}i, -\frac{4}{3} + \frac{\sqrt{5}}{3}i\right\}$.

d) Which method should we use to solve $(2z - 7)^2 - 6 = 0$?

We *could* square the binomial, combine like terms, then solve, possibly, by factoring or using the quadratic formula. However, this would be very inefficient. The equation contains a squared quantity and a constant.

We will solve $(2z - 7)^2 - 6 = 0$ *using the square root property.*

$$
\begin{aligned}
(2z - 7)^2 - 6 &= 0 \\
(2z - 7)^2 &= 6 && \text{Add 6 to each side.} \\
2z - 7 &= \pm\sqrt{6} && \text{Square root property} \\
2z &= 7 \pm \sqrt{6} && \text{Add 7 to each side.} \\
z &= \frac{7 \pm \sqrt{6}}{2} && \text{Divide by 2.}
\end{aligned}
$$

The solution set is $\left\{\frac{7 - \sqrt{6}}{2}, \frac{7 + \sqrt{6}}{2}\right\}$.

You Try 1

Solve.

a) $2k^2 + 3 = 9k$ b) $2r^2 + 3r - 2 = 0$

c) $(n - 8)^2 + 9 = 0$ d) $y^2 + 4y = -10$

Answers to You Try Exercises

1) a) $\left\{\frac{9 - \sqrt{57}}{4}, \frac{9 + \sqrt{57}}{4}\right\}$ b) $\left\{-2, \frac{1}{2}\right\}$ c) $\{8 - 3i, 8 + 3i\}$ d) $\{-2 - i\sqrt{6}, -2 + i\sqrt{6}\}$

Putting It All Together Summary Exercises

Objective 1: Decide Which Method to Use to Solve a Quadratic Equation

Keep in mind the four methods we have learned for solving quadratic equations: *factoring, the square root property, completing the square, and the quadratic formula*. Solve the equations using one of these methods.

1) $z^2 - 50 = 0$

2) $j^2 - 6j = 8$

VIDEO 3) $a(a + 1) = 20$

4) $2x^2 + 6 = 3x$

5) $u^2 + 7u + 9 = 0$

6) $3p^2 - p - 4 = 0$

7) $2k(2k + 7) = 3(k + 1)$

8) $2 = (w + 3)^2 + 8$

VIDEO 9) $m^2 + 14m + 60 = 0$

10) $\frac{1}{2}y^2 = \frac{3}{4} - \frac{1}{2}y$

VIDEO 11) $10 + (3b - 1)^2 = 4$

12) $c^2 + 8c + 25 = 0$

13) $1 = \frac{x^2}{12} - \frac{x}{3}$

14) $100 = 4d^2$

15) $r^2 - 4r = 3$

16) $2t^3 + 108t = -30t^2$

17) $p(p + 8) = 3(p^2 + 2) + p$

18) $h^2 = h$

19) $\frac{10}{z} = 1 + \frac{21}{z^2}$

20) $2s(2s + 3) = 4s + 5$

VIDEO 21) $(3v + 4)(v - 2) = -9$

22) $34 = 6y - y^2$

23) $(c - 5)^2 + 16 = 0$

24) $(2b + 1)(b + 5) = -7$

25) $3g = g^2$

26) $5z^2 + 15z + 30 = 0$

VIDEO 27) $4m^3 = 9m$

28) $\frac{9}{2a^2} = \frac{1}{6} + \frac{1}{a}$

29) $\frac{1}{3}q^2 + \frac{5}{6}q + \frac{4}{3} = 0$

30) $-3 = (12d + 5)^2 + 6$

Section 11.4 Equations in Quadratic Form

Objectives

1. **Solve Quadratic Equations Resulting from Equations Containing Fractions or Radicals**
2. **Solve an Equation in Quadratic Form by Factoring**
3. **Solve an Equation in Quadratic Form Using Substitution**
4. **Use Substitution for a Binomial to Solve a Quadratic Equation**

In Chapters 8 and 10, we solved some equations that were *not* quadratic but could be rewritten in the form of a quadratic equation, $ax^2 + bx + c = 0$. Two such examples are:

$$\frac{10}{x} - \frac{7}{x+1} = \frac{2}{3} \qquad \text{and} \qquad r + \sqrt{r} = 12$$

Rational equation (Ch. 8) Radical equation (Ch. 10)

We will review how to solve each type of equation.

1. Solve Quadratic Equations Resulting from Equations Containing Fractions or Radicals

Example 1

Solve $\frac{10}{x} - \frac{7}{x+1} = \frac{2}{3}$.

Solution

To solve an equation containing rational expressions, *multiply the equation by the LCD of all of the fractions to eliminate the denominators,* then solve.

$$\text{LCD} = 3x(x+1)$$

$$3x(x+1)\left(\frac{10}{x} - \frac{7}{x+1}\right) = 3x(x+1)\left(\frac{2}{3}\right)$$

Multiply both sides of the equation by the LCD of the fractions.

$$3\cancel{x}(x+1)\cdot\frac{10}{\cancel{x}} - 3x\cancel{(x+1)}\cdot\frac{7}{\cancel{x+1}} = \cancel{3}x(x+1)\cdot\left(\frac{2}{\cancel{3}}\right)$$

Distribute and divide out common factors.

$$\begin{aligned} 30(x+1) - 3x(7) &= 2x(x+1) \\ 30x + 30 - 21x &= 2x^2 + 2x && \text{Distribute.} \\ 9x + 30 &= 2x^2 + 2x && \text{Combine like terms.} \\ 0 &= 2x^2 - 7x - 30 && \text{Write in the form } ax^2 + bx + c = 0. \\ 0 &= (2x+5)(x-6) && \text{Factor.} \end{aligned}$$

$$2x + 5 = 0 \quad \text{or} \quad x - 6 = 0 \qquad \text{Set each factor equal to zero.}$$

$$2x = -5$$

$$x = -\frac{5}{2} \quad \text{or} \quad x = 6 \qquad \text{Solve.}$$

Recall that you *must* check the proposed solutions in the original equation to be certain they do not make a denominator equal zero. The solution set is $\left\{-\frac{5}{2}, 6\right\}$. ■

You Try 1

Solve $\frac{1}{m} = \frac{1}{2} + \frac{m}{m+4}$.

Example 2

Solve $r + \sqrt{r} = 12$.

Solution

The first step in solving a radical equation is getting a radical on a side by itself.

$$\begin{aligned}
r + \sqrt{r} &= 12 \\
\sqrt{r} &= 12 - r && \text{Subtract } r \text{ from each side.} \\
(\sqrt{r})^2 &= (12 - r)^2 && \text{Square both sides.} \\
r &= 144 - 24r + r^2 \\
0 &= r^2 - 25r + 144 && \text{Write in the form } ax^2 + bx + c = 0. \\
0 &= (r - 16)(r - 9) && \text{Factor.}
\end{aligned}$$

$$\begin{aligned}
r - 16 = 0 \quad &\text{or} \quad r - 9 = 0 && \text{Set each factor equal to zero.} \\
r = 16 \quad &\text{or} \quad r = 9 && \text{Solve.}
\end{aligned}$$

Recall that you *must* check the proposed solutions in the original equation.

Check $r = 16$:

$$\begin{aligned}
r + \sqrt{r} &= 12 \\
16 + \sqrt{16} &\stackrel{?}{=} 12 \\
16 + 4 &= 12 && \text{False}
\end{aligned}$$

Check $r = 9$:

$$\begin{aligned}
r + \sqrt{r} &= 12 \\
9 + \sqrt{9} &\stackrel{?}{=} 12 \\
9 + 3 &= 12 && \text{True}
\end{aligned}$$

16 is an extraneous solution. The solution set is $\{9\}$.

You Try 2

Solve $y + 3\sqrt{y} = 10$.

2. Solve an Equation in Quadratic Form by Factoring

Some equations that are not quadratic can be solved using the same methods that can be used to solve quadratic equations. These are called **equations in quadratic form**. Some examples of equations in quadratic form are:

$$x^4 - 10x^2 + 9 = 0, \qquad t^{2/3} + t^{1/3} - 6 = 0, \qquad 2n^4 - 5n^2 = -1$$

Let's compare the equations above to *quadratic equations* to understand why they are said to be in quadratic form.

Note

COMPARE

An Equation in Quadratic Form	**to**	**A Quadratic Equation**
This exponent is *twice* this exponent. $x^4 - 10x^2 + 9 = 0$		This exponent is *twice* this exponent. $x^2 - 10x^1 + 9 = 0$
This exponent is *twice* this exponent. $t^{2/3} + t^{1/3} - 6 = 0$		This exponent is *twice* this exponent. $t^2 + t^1 - 6 = 0$
This exponent is *twice* this exponent. $2n^4 - 5n^2 = -1$		This exponent is *twice* this exponent. $2n^2 - 5n^1 = -1$

This pattern enables us to work with equations in quadratic form like we can work with quadratic equations.

Example 3

Solve.

a) $x^4 - 10x^2 + 9 = 0$ b) $t^{2/3} + t^{1/3} - 6 = 0$

Solution

a) Let's compare $x^4 - 10x^2 + 9 = 0$ to $x^2 - 10x + 9 = 0$.

We can factor $x^2 - 10x + 9$:

$$(x - 9)(x - 1)$$

Confirm by multiplying using FOIL:

$$\begin{aligned}(x - 9)(x - 1) &= x^2 - x - 9x + 9\\ &= x^2 - 10x + 9\end{aligned}$$

Factor $x^4 - 10x^2 + 9$ in a similar way since the exponent, 4, of the first term is twice the exponent, 2, of the second term:

$$x^4 - 10x^2 + 9 = (x^2 - 9)(x^2 - 1)$$

Confirm by multiplying using FOIL:

$$\begin{aligned}(x^2 - 9)(x^2 - 1) &= x^4 - x^2 - 9x^2 + 9\\ &= x^4 - 10x^2 + 9\end{aligned}$$

We can solve $x^4 - 10x^2 + 9 = 0$ by factoring.

$$\begin{aligned}x^4 - 10x^2 + 9 &= 0\\ (x^2 - 9)(x^2 - 1) &= 0 && \text{Factor.}\end{aligned}$$

$$\begin{aligned}x^2 - 9 &= 0 \quad \text{or} \quad x^2 - 1 = 0 && \text{Set each factor equal to 0.}\\ x^2 &= 9 \qquad\qquad x^2 = 1 && \text{Square root property}\\ x &= \pm 3 \qquad\qquad x = \pm 1\end{aligned}$$

The check is left to the student. The solution set is $\{-3, -1, 1, 3\}$.

b) Compare $t^{2/3} + t^{1/3} - 6 = 0$ to $t^2 + t - 6 = 0$.

We can factor $t^2 + t - 6$:

$$(t + 3)(t - 2)$$

Confirm by multiplying using FOIL:

$$\begin{aligned}(t + 3)(t - 2) &= t^2 - 2t + 3t - 6\\ &= t^2 + t - 6\end{aligned}$$

Factor $t^{2/3} + t^{1/3} - 6$ in a similar way since the exponent, $\frac{2}{3}$, of the first term is twice the exponent, $\frac{1}{3}$, of the second term:

$$t^{2/3} + t^{1/3} - 6 = (t^{1/3} + 3)(t^{1/3} - 2)$$

Confirm by multiplying using FOIL:

$$\begin{aligned}(t^{1/3} + 3)(t^{1/3} - 2) &= t^{2/3} - 2t^{1/3} + 3t^{1/3} - 6\\ &= t^{2/3} + t^{1/3} - 6\end{aligned}$$

We can solve $t^{2/3} + t^{1/3} - 6 = 0$ by factoring.

$$\begin{aligned}t^{2/3} + t^{1/3} - 6 &= 0\\ (t^{1/3} + 3)(t^{1/3} - 2) &= 0 && \text{Factor.}\end{aligned}$$

$$\begin{aligned}t^{1/3} + 3 &= 0 \quad \text{or} \quad t^{1/3} - 2 = 0 && \text{Set each factor equal to 0.}\\ t^{1/3} &= -3 \qquad\qquad t^{1/3} = 2 && \text{Isolate the constant.}\\ \sqrt[3]{t} &= -3 \qquad\qquad \sqrt[3]{t} = 2 && t^{1/3} = \sqrt[3]{t}\\ (\sqrt[3]{t})^3 &= (-3)^3 \qquad (\sqrt[3]{t})^3 = 2^3 && \text{Cube both sides.}\\ t &= -27 \quad \text{or} \qquad t = 8 && \text{Solve.}\end{aligned}$$

The check is left to the student. The solution set is $\{-27, 8\}$. ■

You Try 3

Solve.

a) $r^4 - 13r^2 + 36 = 0$ b) $c^{2/3} + 4c^{1/3} - 5 = 0$

3. Solve an Equation in Quadratic Form Using Substitution

The equations in Example 3 can also be solved using a method called **substitution**. We will illustrate the method in Example 4.

Example 4

Solve $x^4 - 10x^2 + 9 = 0$ using substitution.

Solution

$$x^4 - 10x^2 + 9 = 0$$
$$\downarrow$$
$$x^4 = (x^2)^2$$

To rewrite $x^4 - 10x^2 + 9 = 0$ in quadratic form, let $u = x^2$.

$$\text{If } u = x^2, \text{ then}$$
$$u^2 = x^4.$$

$$\begin{aligned} x^4 - 10x^2 + 9 &= 0 \\ u^2 - 10u + 9 &= 0 && \text{Substitute } u^2 \text{ for } x^4 \text{ and } u \text{ for } x^2. \\ (u - 9)(u - 1) &= 0 && \text{Solve by factoring.} \end{aligned}$$

$$\swarrow \qquad \searrow$$

$$\begin{aligned} u - 9 = 0 \quad \text{or} \quad u - 1 &= 0 && \text{Set each factor equal to 0.} \\ u = 9 \quad \text{or} \quad u &= 1 && \text{Solve for } u. \end{aligned}$$

Be careful: $u = 9$ and $u = 1$ are *not* the solutions to $x^4 - 10x^2 + 9 = 0$. We still need to solve for x. Above we let $u = x^2$. *To solve for x, substitute* 9 *for u and solve for x and then substitute* 1 *for* u *and solve for x.*

$$\begin{aligned} & & u &= x^2 & u &= x^2 & & \\ &\text{Substitute 9 for } u. & 9 &= x^2 & 1 &= x^2 & &\text{Substitute 1 for } u. \\ &\text{Square root property} & \pm 3 &= x & \pm 1 &= x & &\text{Square root property} \end{aligned}$$

The solution set is $\{-3, -1, 1, 3\}$. This is the same as the result we obtained in Example 3a).

You Try 4

Solve by substitution.

a) $r^4 - 13r^2 + 36 = 0$ b) $c^{2/3} + 4c^{1/3} - 5 = 0$

If, after substitution, an equation cannot be solved by factoring, we can use the quadratic formula.

Example 5

Solve $2n^4 - 5n^2 = -1$.

Solution

Write the equation in standard form: $2n^4 - 5n^2 + 1 = 0$.

Can we solve the equation by factoring? *No*.

We will solve $2n^4 - 5n^2 + 1 = 0$ *using the quadratic formula.* Begin with substitution.

$$\text{If } u = n^2, \text{ then } u^2 = n^4.$$

$$2n^4 - 5n^2 + 1 = 0$$
$$2u^2 - 5u + 1 = 0 \qquad \text{Substitute } u^2 \text{ for } n^4 \text{ and } u \text{ for } n^2.$$

$$u = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(1)}}{2(2)} \qquad a = 2,\ b = -5,\ c = 1$$

$$u = \frac{5 \pm \sqrt{25 - 8}}{4} = \frac{5 \pm \sqrt{17}}{4}$$

Note that $u = \dfrac{5 \pm \sqrt{17}}{4}$ does not solve the *original* equation. We must solve for x using the fact that $u = x^2$. Since $u = \dfrac{5 \pm \sqrt{17}}{4}$ means $u = \dfrac{5 + \sqrt{17}}{4}$ or $u = \dfrac{5 - \sqrt{17}}{4}$, we get

$$u = x^2 \qquad\qquad u = x^2$$

$$\frac{5 + \sqrt{17}}{4} = x^2 \qquad\qquad \frac{5 - \sqrt{17}}{4} = x^2$$

$$\pm\sqrt{\frac{5 + \sqrt{17}}{4}} = x \qquad\qquad \pm\sqrt{\frac{5 - \sqrt{17}}{4}} = x \qquad \text{Square root property}$$

$$\frac{\pm\sqrt{5 + \sqrt{17}}}{2} = x \qquad\qquad \frac{\pm\sqrt{5 - \sqrt{17}}}{2} = x \qquad \sqrt{4} = 2$$

The solution set is $\left\{\dfrac{\sqrt{5 + \sqrt{17}}}{2}, -\dfrac{\sqrt{5 + \sqrt{17}}}{2}, \dfrac{\sqrt{5 - \sqrt{17}}}{2}, -\dfrac{\sqrt{5 - \sqrt{17}}}{2}\right\}$.

You Try 5

Solve $2k^4 + 3 = 9k^2$.

4. Use Substitution for a Binomial to Solve a Quadratic Equation

We can use substitution to solve an equation like the one in Example 6.

Example 6

Solve $2(3a + 1)^2 - 7(3a + 1) - 4 = 0$.

Solution

The binomial $3a + 1$ appears as a *squared quantity* and as a *linear quantity*. Begin by using substitution.

Let $u = 3a + 1$. Then, $u^2 = (3a + 1)^2$.

Substitute: $2(3a + 1)^2 - 7(3a + 1) - 4 = 0$

$$2u^2 - 7u - 4 = 0$$

Does $2u^2 - 7u - 4 = 0$ factor? *Yes*. Solve by factoring.

$$(2u + 1)(u - 4) = 0 \qquad \text{Factor } 2u^2 - 7u - 4 = 0.$$

$$2u + 1 = 0 \quad \text{or} \quad u - 4 = 0 \qquad \text{Set each factor equal to 0.}$$

$$u = -\frac{1}{2} \quad \text{or} \quad u = 4 \qquad \text{Solve for } u.$$

Solve for a using $u = 3a + 1$.

When $u = -\frac{1}{2}$:

$$u = 3a + 1$$
$$-\frac{1}{2} = 3a + 1$$
$$-\frac{3}{2} = 3a \qquad \text{Subtract 1.}$$
$$-\frac{1}{2} = a \qquad \text{Multiply by } \frac{1}{3}.$$

When $u = 4$:

$$u = 3a + 1$$
$$4 = 3a + 1$$
$$3 = 3a \qquad \text{Subtract 1.}$$
$$1 = a \qquad \text{Divide by 3.}$$

The solution set is $\left\{-\frac{1}{2}, 1\right\}$.

You Try 6

Solve $3(2p - 1)^2 - 11(2p - 1) + 10 = 0$.

BE CAREFUL

Don't forget to solve for the variable in the *original* equation.

Answers to You Try Exercises

1) $\left\{-2, \frac{4}{3}\right\}$ 2) $\{4\}$ 3) a) $\{-3, -2, 2, 3\}$ b) $\{-125, 1\}$ 4) a) $\{-3, -2, 2, 3\}$ b) $\{-125, 1\}$ 5) $\left\{\frac{\sqrt{9 + \sqrt{57}}}{2}, -\frac{\sqrt{9 + \sqrt{57}}}{2}, \frac{\sqrt{9 - \sqrt{57}}}{2}, -\frac{\sqrt{9 - \sqrt{57}}}{2}\right\}$ 6) $\left\{\frac{4}{3}, \frac{3}{2}\right\}$

11.4 Exercises

Objective 1: Solve Quadratic Equations Resulting from Equations Containing Fractions or Radicals.

Solve.

1) $t - \frac{48}{t} = 8$

2) $z + 11 = -\frac{24}{z}$

3) $\frac{2}{x} + \frac{6}{x-2} = -\frac{5}{2}$

4) $\frac{3}{y} - \frac{6}{y-1} = \frac{1}{2}$

VIDEO 5) $1 = \frac{2}{c} + \frac{1}{c-5}$

6) $\frac{2}{g} = 1 + \frac{g}{g+5}$

7) $\frac{3}{2v+2} + \frac{1}{v} = \frac{3}{2}$

8) $\frac{1}{b+3} + \frac{1}{b} = \frac{1}{3}$

9) $\frac{9}{n^2} = 5 + \frac{4}{n}$

10) $3 - \frac{16}{a^2} = \frac{8}{a}$

11) $\frac{5}{6r} = 1 - \frac{r}{6r-6}$

12) $\frac{7}{4} - \frac{x}{4x+4} = \frac{1}{x}$

13) $g = \sqrt{g+20}$

14) $c = \sqrt{7c-6}$

VIDEO 15) $a = \sqrt{\frac{14a-8}{5}}$

16) $k = \sqrt{\frac{6-11k}{2}}$

17) $p - \sqrt{p} = 6$

18) $v + \sqrt{v} = 2$

19) $x = 5\sqrt{x} - 4$

20) $10 = m - 3\sqrt{m}$

21) $2 + \sqrt{2y-1} = y$

22) $1 - \sqrt{5t+1} = -t$

23) $2 = \sqrt{6k+4} - k$

24) $\sqrt{10-3q} - 6 = q$

Mixed Exercises: Objectives 2–3

Determine whether each is an equation in quadratic form. Do *not* solve.

25) $n^4 - 12n^2 + 32 = 0$

26) $p^6 + 8p^3 - 9 = 0$

27) $2t^6 + 3t^3 - 5 = 0$

28) $a^4 - 4a - 3 = 0$

29) $c^{2/3} - 4c - 6 = 0$

30) $3z^{2/3} + 2z^{1/3} + 1 = 0$

31) $m + 9m^{1/2} = 4$

32) $2x^{1/2} - 5x^{1/4} = 2$

33) $5k^4 + 6k - 7 = 0$

34) $r^{-2} = 10 - 4r^{-1}$

Solve.

VIDEO 35) $x^4 - 10x^2 + 9 = 0$

36) $d^4 - 29d^2 + 100 = 0$

37) $p^4 - 11p^2 + 28 = 0$

38) $k^4 - 9k^2 + 8 = 0$

39) $a^4 + 12a^2 = -35$

40) $c^4 + 9c^2 = -18$

41) $b^{2/3} + 3b^{1/3} + 2 = 0$

42) $z^{2/3} + z^{1/3} - 12 = 0$

43) $t^{2/3} - 6t^{1/3} = 40$

44) $p^{2/3} - p^{1/3} = 6$

VIDEO 45) $2n^{2/3} = 7n^{1/3} + 15$

46) $10k^{1/3} + 8 = -3k^{2/3}$

47) $v - 8v^{1/2} + 12 = 0$

48) $j - 6j^{1/2} + 5 = 0$

49) $4h^{1/2} + 21 = h$

50) $s + 12 = -7s^{1/2}$

51) $2a - 5a^{1/2} - 12 = 0$

52) $2w = 9w^{1/2} + 18$

53) $9n^4 = -15n^2 - 4$

54) $4h^4 + 19h^2 + 12 = 0$

55) $z^4 - 2z^2 = 15$

56) $a^4 + 2a^2 = 24$

57) $w^4 - 6w^2 + 2 = 0$

58) $p^4 - 8p^2 + 3 = 0$

59) $2m^4 + 1 = 7m^2$

60) $8x^4 + 2 = 9x^2$

61) $t^{-2} - 4t^{-1} - 12 = 0$

62) $d^{-2} + d^{-1} - 6 = 0$

63) $4 = 13y^{-1} - 3y^{-2}$

64) $14h^{-1} + 3 = 5h^{-2}$

Objective 4: Use Substitution for a Binomial to Solve a Quadratic Equation

Solve.

65) $(x-2)^2 + 11(x-2) + 24 = 0$

66) $(r+1)^2 - 3(r+1) - 10 = 0$

VIDEO 67) $2(3q+4)^2 - 13(3q+4) + 20 = 0$

68) $4(2b-3)^2 - 9(2b-3) - 9 = 0$

69) $(5a-3)^2 + 6(5a-3) = -5$

70) $(3z-2)^2 - 8(3z-2) = 20$

71) $3(k+8)^2 + 5(k+8) = 12$

72) $5(t+9)^2 + 37(t+9) + 14 = 0$

73) $1 - \frac{8}{2w+1} = -\frac{16}{(2w+1)^2}$

74) $1 - \frac{8}{4p+3} = -\frac{12}{(4p+3)^2}$

75) $1 + \frac{2}{h-3} = \frac{1}{(h-3)^2}$

76) $\frac{2}{(c+6)^2} + \frac{2}{(c+6)} = 1$

Write an equation and solve.

77) It takes Kevin 3 hr longer than Walter to build a tree house. Together they can do the job in 2 hr. How long would it take each man to build the tree house on his own?

78) It takes one pipe 4 hours more to empty a pool than it takes another pipe to fill a pool. If both pipes are accidentally left open, it takes 24 hr to fill the pool. How long does it take the single pipe to fill the pool?

79) A boat can travel 9 mi downstream and then 6 mi back upstream in 1 hr. If the speed of the current is 3 mph, what is the speed of the boat in still water?

80) A plane can travel 800 mi with the wind and then 650 mi back against the wind in 5 hr. If the wind blows at 30 mph, what is the speed of the plane?

81) A large fish tank at an aquarium needs to be emptied so that it can be cleaned. When its large and small drains are opened together, the tank can be emptied in 2 hours. By itself, it takes the small drain 3 hours longer to empty the tank than it takes the large drain to empty the tank on its own. How much time would it take for each drain to empty the pool on its own?

82) Working together, a professor and her teaching assistant can grade a set of exams in 1.2 hours. On her own, the professor can grade the tests 1 hour faster than the teaching assistant can grade them on her own. How long would it take for each person to grade the test by herself?

83) Miguel took his son to college in Boulder, Colorado, 600 miles from their hometown. On his way home, he was slowed by a snowstorm so that his speed was 10 mph less than when he was driving to Boulder. His total driving time was 22 hours. How fast did Miguel drive on each leg of the trip?

84) Nariko was training for a race and went out for a run. Her speed was 2 mph faster during the first 6 miles than it was for the last 3 miles. If her total running time was $1\frac{3}{4}$ hours, what was her speed on each part of the run?

Section 11.5 Formulas and Applications

Objectives

1. Solve a Formula for a Variable
2. Solve an Applied Problem Involving Volume
3. Solve an Applied Problem Involving Area
4. Solve an Applied Problem Using a Quadratic Equation

Sometimes, solving a formula for a variable involves using one of the techniques we've learned for solving a quadratic equation or for solving an equation containing a radical.

1. Solve a Formula for a Variable

Example 1

Solve $v = \sqrt{\dfrac{300VP}{m}}$ for m.

Solution

Put a box around the m. The goal is to get m on a side by itself.

$$v = \sqrt{\frac{300VP}{\boxed{m}}}$$

$$v^2 = \frac{300VP}{\boxed{m}} \qquad \text{Square both sides.}$$

Since we are solving for m and it is in the denominator, multiply both sides by m to eliminate the denominator.

$$\boxed{m}\, v^2 = 300VP \qquad \text{Multiply both sides by } m.$$

$$m = \frac{300VP}{v^2} \qquad \text{Divide both sides by } v^2.$$

You Try 1

Solve $v = \sqrt{\dfrac{2E}{m}}$ for m.

We may need to use the quadratic formula to solve a formula for a variable. Compare the following equations. Each equation is *quadratic in x* because each is written in the form $ax^2 + bx + c = 0$.

$$8x^2 + 3x - 2 = 0 \qquad \text{and} \qquad 8x^2 + tx - z = 0$$

$$a = 8 \quad b = 3 \quad c = -2 \qquad\qquad a = 8 \quad b = t \quad c = -z$$

To solve the equations for x, we can use the quadratic formula.

Example 2

Solve for x.

a) $8x^2 + 3x - 2 = 0$ b) $8x^2 + tx - z = 0$

Solution

a) $8x^2 + 3x - 2$ does not factor, so we will solve using the quadratic formula.

$$8x^2 + 3x - 2 = 0$$

$$a = 8 \quad b = 3 \quad c = -2$$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(8)(-2)}}{2(8)} = \frac{-3 \pm \sqrt{9 + 64}}{16} = \frac{-3 \pm \sqrt{73}}{16}$$

The solution set is $\left\{\dfrac{-3 - \sqrt{73}}{16}, \dfrac{-3 + \sqrt{73}}{16}\right\}$.

b) Solve $8x^2 + tx - z = 0$ for x using the quadratic formula.

$$a = 8 \qquad b = t \qquad c = -z$$

$$x = \frac{-t \pm \sqrt{t^2 - 4(8)(-z)}}{2(8)} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-t \pm \sqrt{t^2 + 32z}}{16} \qquad \text{Perform the operations.}$$

The solution set is $\left\{\dfrac{-t - \sqrt{t^2 + 32z}}{16}, \dfrac{-t + \sqrt{t^2 + 32z}}{16}\right\}$.

You Try 2

Solve for n.

a) $3n^2 + 5n - 1 = 0$ b) $3n^2 + pn - r = 0$

2. Solve an Applied Problem Involving Volume

Example 3

A rectangular piece of cardboard is 5 in. longer than it is wide. A square piece that measures 2 in. on each side is cut from each corner, then the sides are turned up to make an uncovered box with volume 252 in^3. Find the length and width of the original piece of cardboard.

Solution

Step 1: **Read** the problem carefully. Draw a picture.

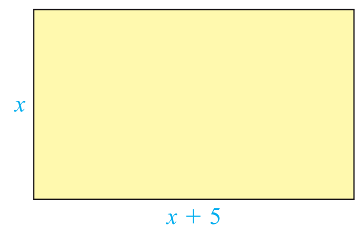
Step 2: **Choose a variable** to represent the unknown, and define the other unknown in terms of this variable.

$$\begin{aligned} \text{Let} \qquad x &= \text{the width of the cardboard} \\ x + 5 &= \text{the length of the cardboard} \end{aligned}$$

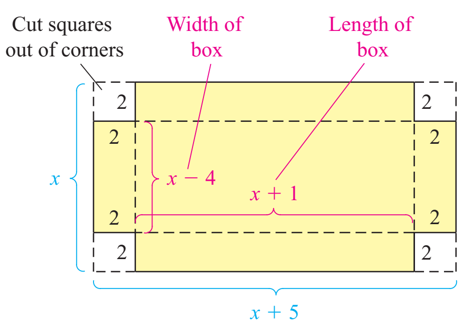
Step 3: **Translate** the information that appears in English into an algebraic equation.

The volume of a box is (length)(width)(height). We will use the formula (length)(width)(height) = 252.

Original Cardboard



Box



The figure on the left shows the original piece of cardboard with the sides labeled. The figure on the right illustrates how to label the box when the squares are cut out of the corners. When the sides are folded along the dotted lines, we must label the length, width, and height of the box.

$$\begin{aligned} \text{Length of box} &= \text{Length of original cardboard} - \text{Length of side cut out on the left} - \text{Length of side cut out on the right} \\ &= x + 5 - 2 - 2 \\ &= x + 1 \\ \text{Width of box} &= \text{Width of original cardboard} - \text{Length of side cut out on top} - \text{Length of side cut out on bottom} \\ &= x - 2 - 2 \\ &= x - 4 \\ \text{Height of box} &= \text{Length of side cut out} \\ &= 2 \end{aligned}$$

Statement: Volume of box = (length)(width)(height)

Equation: $252 = (x + 1)(x - 4)(2)$

Step 4: **Solve** the equation.

$$252 = (x + 1)(x - 4)(2)$$
$$126 = (x + 1)(x - 4) \qquad \text{Divide both sides by 2.}$$
$$126 = x^2 - 3x - 4 \qquad \text{Multiply.}$$
$$0 = x^2 - 3x - 130 \qquad \text{Write in standard form.}$$
$$0 = (x + 10)(x - 13) \qquad \text{Factor.}$$
$$x + 10 = 0 \quad \text{or} \quad x - 13 = 0 \qquad \text{Set each factor equal to zero.}$$
$$x = -10 \quad \text{or} \quad x = 13 \qquad \text{Solve.}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

Because x represents the width, it cannot be negative. Therefore, the width of the original piece of cardboard is 13 in.

The length of the cardboard is $x + 5$, so $13 + 5 = 18$ in.

Width of cardboard $= 13$ in. Length of cardboard $= 18$ in.

Check:

Width of box $= 13 - 4 = 9$ in.; Length of box $= 13 + 1 = 14$ in.;
Height of box $= 2$ in.
Volume of box $= 9(14)(2) = 252 \text{ in}^3$.

You Try 3

The width of a rectangular piece of cardboard is 2 in. less than its length. A square piece that measures 3 in. on each side is cut from each corner, then the sides are turned up to make a box with volume 504 in^3. Find the length and width of the original piece of cardboard.

3. Solve an Applied Problem Involving Area

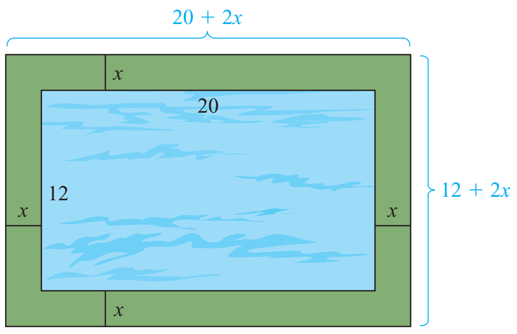
Example 4

A rectangular pond is 20 ft long and 12 ft wide. The pond is bordered by a strip of grass of uniform (the same) width. The area of the grass is 320 ft^2. How wide is the border of grass around the pond?

Solution

Step 1: **Read** the problem carefully. Draw a picture.

Step 2: **Choose a variable** to represent the unknown, and define the other unknowns in terms of this variable.



x = width of the strip of grass
$20 + 2x$ = length of pond plus two strips of grass
$12 + 2x$ = width of pond plus two strips of grass

Step 3: **Translate** from English into an algebraic equation.

We know that the area of the grass border is 320 ft^2. We can calculate the area of the pond since we know its length and width. The pond plus grass border forms a large rectangle of length $20 + 2x$ and width $12 + 2x$. The equation will come from the following relationship:

Statement: Area of pond plus grass − Area of pond = Area of grass border

Equation: $(20 + 2x)(12 + 2x) - 20(12) = 320$

Step 4: **Solve** the equation.

$$
\begin{aligned}
(20 + 2x)(12 + 2x) - 20(12) &= 320 \\
240 + 64x + 4x^2 - 240 &= 320 && \text{Multiply.} \\
4x^2 + 64x &= 320 && \text{Combine like terms.} \\
x^2 + 16x &= 80 && \text{Divide by 4.} \\
x^2 + 16x - 80 &= 0 && \text{Write in standard form.} \\
(x + 20)(x - 4) &= 0 && \text{Factor.} \\
x + 20 = 0 \quad \text{or} \quad x - 4 &= 0 && \text{Set each factor equal to 0.} \\
x = -20 \quad \text{or} \quad x &= 4 && \text{Solve.}
\end{aligned}
$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

x represents the width of the strip of grass, so x cannot equal -20.

The width of the strip of grass is 4 ft.

Check: Substitute $x = 4$ into the equation written in step 3.

$$
\begin{aligned}
[20 + 2(4)][12 + 2(4)] - 20(12) &\stackrel{?}{=} 320 \\
(28)(20) - 240 &\stackrel{?}{=} 320 \\
560 - 240 &= 320 \checkmark
\end{aligned}
$$

■

You Try 4

A rectangular pond is 6 ft wide and 10 ft long and is surrounded by a concrete border of uniform width. The area of the border is 80 ft^2. Find the width of the border.

4. Solve an Applied Problem Using a Quadratic Equation

Example 5

The total tourism-related output in the United States from 2000 to 2004 can be modeled by

$$y = 16.4x^2 - 50.6x + 896$$

where x is the number of years since 2000 and y is the total tourism-related output in billions of dollars. (www.bea.gov)

a) According to the model, how much money was generated in 2002 due to tourism-related output?

b) In what year was the total tourism-related output about \$955 billion?

Solution

a) Since x is the number of years *after* 2000, the year 2002 corresponds to $x = 2$.

$$y = 16.4x^2 - 50.6x + 896$$
$$y = 16.4(2)^2 - 50.6(2) + 896 \qquad \text{Substitute 2 for } x.$$
$$y = 860.4$$

The total tourism-related output in 2002 was approximately \$860.4 billion.

b) Since y represents the total tourism-related output (in billions), substitute 955 for y and solve for x.

$$y = 16.4x^2 - 50.6x + 896$$
$$955 = 16.4x^2 - 50.6x + 896 \qquad \text{Substitute 955 for } y.$$
$$0 = 16.4x^2 - 50.6x - 59 \qquad \text{Write in standard form.}$$

Use the quadratic formula to solve for x.

$$a = 16.4 \quad b = -50.6 \quad c = -59$$
$$x = \frac{50.6 \pm \sqrt{(-50.6)^2 - 4(16.4)(-59)}}{2(16.4)} \qquad \text{Substitute the values into the quadratic formula.}$$
$$x \approx 3.99 \approx 4 \text{ or } x \approx -0.90$$

The negative value of x does not make sense in the context of the problem. Use $x \approx 4$, which corresponds to the year 2004. The total tourism-related output was about \$955 billion in 2004.

Answers to You Try Exercises

1) $m = \frac{2E}{v^2}$ 2) a) $\left\{\frac{-5 - \sqrt{37}}{6}, \frac{-5 + \sqrt{37}}{6}\right\}$ b) $\left\{\frac{-p - \sqrt{p^2 + 12r}}{6}, \frac{-p + \sqrt{p^2 + 12r}}{6}\right\}$

3) length = 20 in., width = 18 in. 4) 2 ft

11.5 Exercises

Objective 1: Solve a Formula for a Variable

Solve for the indicated variable.

1) $A = \pi r^2$ for r

2) $V = \frac{1}{3}\pi r^2 h$ for r

3) $a = \frac{v^2}{r}$ for v

4) $K = \frac{1}{2}Iw^2$ for w

VIDEO 5) $E = \frac{I}{d^2}$ for d

6) $L = \frac{2U}{I^2}$ for I

7) $F = \frac{kq_1q_2}{r^2}$ for r

8) $E = \frac{kq}{r^2}$ for r

9) $d = \sqrt{\frac{4A}{\pi}}$ for A

10) $d = \sqrt{\frac{12V}{\pi h}}$ for V

11) $T_p = 2\pi\sqrt{\frac{l}{g}}$ for l

12) $V = \sqrt{\frac{3RT}{M}}$ for T

VIDEO 13) $T_p = 2\pi\sqrt{\frac{l}{g}}$ for g

14) $V = \sqrt{\frac{3RT}{M}}$ for M

15) Compare the equations $3x^2 - 5x + 4 = 0$ and $rx^2 + 5x + s = 0$.

a) How are the equations alike?

b) How can both equations be solved for x?

16) What method could be used to solve $2t^2 + 7t + 1 = 0$ and $kt^2 + mt + n = 0$ for t? Why?

Solve for the indicated variable.

17) $rx^2 - 5x + s = 0$ for x

18) $cx^2 + dx - 3 = 0$ for x

VIDEO 19) $pz^2 + rz - q = 0$ for z

20) $hr^2 - kr + j = 0$ for r

21) $da^2 - ha = k$ for a

22) $kt^2 + mt = -n$ for t

23) $s = \frac{1}{2}gt^2 + vt$ for t

24) $s = 2\pi rh + \pi r^2$ for r

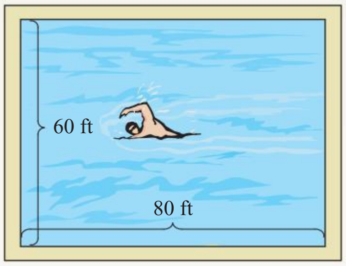
Mixed Exercises: Objectives 2 and 3

Write an equation and solve.

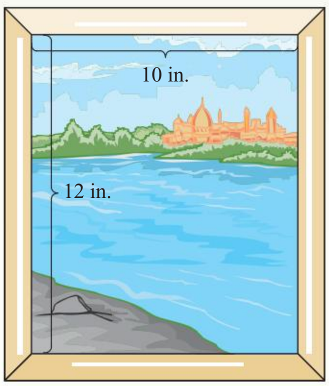
25) The length of a rectangular piece of sheet metal is 3 in. longer than its width. A square piece that measures 1 in. on each side is cut from each corner, then the sides are turned up to make a box with volume 70 in^3. Find the length and width of the original piece of sheet metal.

26) The width of a rectangular piece of cardboard is 8 in. less than its length. A square piece that measures 2 in. on each side is cut from each corner, then the sides are turned up to make a box with volume 480 in^3. Find the length and width of the original piece of cardboard.

27) A rectangular swimming pool is 60 ft wide and 80 ft long. A nonskid surface of uniform width is to be installed around the pool. If there is 576 ft^2 of the nonskid material, how wide can the strip of the nonskid surface be?



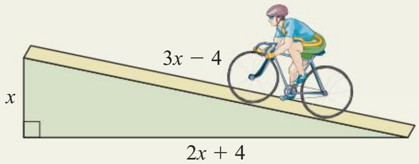
28) A picture measures 10 in. by 12 in. Emilio will get it framed with a border around it so that the total area of the picture plus the frame of uniform width is 168 in^2. How wide is the border?



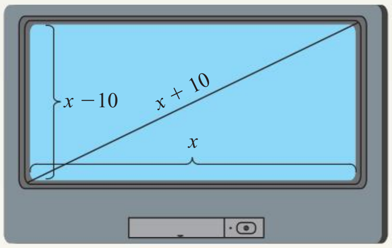
29) The height of a triangular sail is 1 ft less than twice the base of the sail. Find its height and the length of its base if the area of the sail is 60 ft^2.

30) Chandra cuts fabric into isosceles triangles for a quilt. The height of each triangle is 1 in. less than the length of the base. The area of each triangle is 15 in^2. Find the height and base of each triangle.

31) Valerie makes a bike ramp in the shape of a right triangle. The base of the ramp is 4 in. more than twice its height, and the length of the incline is 4 in. less than three times its height. How high is the ramp?



32) The width of a widescreen TV is 10 in. less than its length. The diagonal of the rectangular screen is 10 in. more than the length. Find the length and width of the screen.



Objective 4: Solve an Applied Problem Using a Quadratic Equation

Solve.

VIDEO 33) An object is propelled upward from a height of 4 ft. The height h of the object (in feet) t sec after the object is released is given by

$$h = -16t^2 + 60t + 4$$

a) How long does it take the object to reach a height of 40 ft?

b) How long does it take the object to hit the ground?

34) An object is launched from the ground. The height h of the object (in feet) t sec after the object is released is given by

$$h = -16t^2 + 64t$$

When will the object be 48 ft in the air?

35) Attendance at Broadway plays from 1996 to 2000 can be modeled by

$$y = -0.25x^2 + 1.5x + 9.5$$

where x represents the number of years after 1996 and y represents the number of people who attended a Broadway play (in millions).
(*Statistical Abstracts of the United States*)

a) Approximately how many people saw a Broadway play in 1996?

b) In what year did approximately 11.75 million people see a Broadway play?

36) The illuminance E (measure of the light emitted, in lux) of a light source is given by

$$E = \frac{I}{d^2}$$

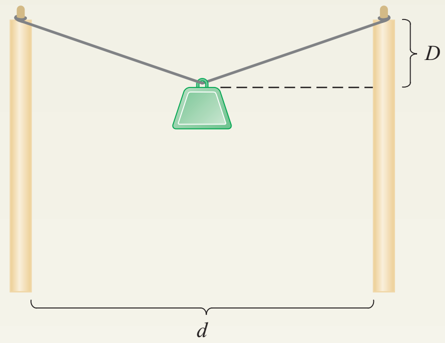
where I is the luminous intensity (measured in candela) and d is the distance, in meters, from the light source. The luminous intensity of a lamp is 2700 candela at a distance of 3 m from the lamp. Find the illuminance, E, in lux.

37) A sandwich shop has determined that the demand for its turkey sandwich is $\frac{65}{P}$ per day, where P is the price of the sandwich in dollars. The daily supply is given by $10P + 3$. Find the price at which the demand for the sandwich equals the supply.

38) A hardware store determined that the demand for shovels one winter was $\frac{2800}{P}$, where P is the price of the shovel in dollars. The supply was given by $12P + 32$. Find the price at which demand for the shovels equals the supply.

Use the following formula for Exercises 39 and 40.

A wire is stretched between two poles separated by a distance d, and a weight is in the center of the wire of length L so that the wire is pulled taut as pictured here. The vertical distance, D, between the weight on the wire and the top of the poles is given by $D = \frac{\sqrt{L^2 - d^2}}{2}$



39) A 12.5-ft clothesline is attached to the top of two poles that are 12 ft apart. A shirt is hanging in the middle of the clothesline. Find the distance, D, that the shirt is hanging down.

40) An 11-ft wire is attached to a ceiling in a loft apartment by hooks that are 10 ft apart. A light fixture is hanging in the middle of the wire. Find the distance, D, between the ceiling and the top of the light fixture. Round the answer to the nearest tenths place.

Chapter 11: Summary

Definition/Procedure	Example

11.1 Review of Solving Equations by Factoring

Steps for Solving a Quadratic Equation by Factoring 1) Write the equation in the form $ax^2 + bx + c = 0$. 2) Factor the expression. 3) Set each factor equal to zero, and solve for the variable. 4) Check the answer(s). **(p. 648)**	Solve $5z^2 - 7z = 6$. $5z^2 - 7z - 6 = 0$ $(5z + 3)(z - 2) = 0$ $5z + 3 = 0$ or $z - 2 = 0$ $5z = -3$ $z = -\frac{3}{5}$ or $z = 2$ The check is left to the student. The solution set is $\left\{-\frac{3}{5}, 2\right\}$.

11.2 The Square Root Property and Completing the Square

The Square Root Property Let k be a constant. If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$. **(p. 651)**	Solve $6p^2 = 54$. $p^2 = 9$ Divide by 6. $p = \pm\sqrt{9}$ Square root property $p = \pm 3$ $\sqrt{9} = 3$ The solution set is $\{-3, 3\}$.
The Distance Formula The **distance,** d, between two points with coordinates (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. **(p. 654)**	Find the distance between the points $(6, -2)$ and $(0, 2)$. Label the points: $(\overset{x_1}{6}, \overset{y_1}{-2})$ $(\overset{x_2}{0}, \overset{y_2}{2})$ Substitute the values into the distance formula. $d = \sqrt{(0 - 6)^2 + [2 - (-2)]^2}$ $= \sqrt{(-6)^2 + (4)^2}$ $= \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}$
A **perfect square trinomial** is a trinomial whose factored form is the square of a binomial. **(p. 655)**	**Perfect Square Trinomial** — **Factored Form** $y^2 + 8y + 16$ — $(y + 4)^2$ $9t^2 - 30t + 25$ — $(3t - 5)^2$
Complete the Square for $x^2 + bx$ To find the constant needed to complete the square for $x^2 + bx$, ***Step 1:*** Find half of the coefficient of x: $\frac{1}{2}b$ ***Step 2:*** Square the result: $\left(\frac{1}{2}b\right)^2$ ***Step 3:*** Add it to $x^2 + bx$: $x^2 + bx + \left(\frac{1}{2}b\right)^2$. **(p. 655)**	Complete the square for $x^2 + 12x$ to obtain a perfect square trinomial. Then, factor. ***Step 1:*** Find half of the coefficient of x: $\frac{1}{2}(12) = 6$ ***Step 2:*** Square the result: $6^2 = 36$ ***Step 3:*** Add 36 to $x^2 + 12x$: $x^2 + 12x + 36$ The perfect square trinomial is $x^2 + 12x + 36$. The factored form is $(x + 6)^2$.
Solve a Quadratic Equation ($ax^2 + bx + c = 0$) by Completing the Square ***Step 1:*** **The coefficient of the squared term must be 1.** If it is not 1, divide both sides of the equation by a to obtain a leading coefficient of 1. ***Step 2:*** **Get the variables on one side of the equal sign and the constant on the other side.**	Solve $x^2 + 6x + 7 = 0$ by completing the square. $x^2 + 6x + 7 = 0$ The coefficient of x^2 is 1. $x^2 + 6x = -7$ Get the constant on the other side of the equal sign. Complete the square: $\frac{1}{2}(6) = 3$ $(3)^2 = 9$

Definition/Procedure	Example
Step 3: **Complete the square.** Find half of the linear coefficient, then square the result. Add that quantity to both sides of the equation. ***Step 4:*** **Factor.** ***Step 5:*** **Solve using the square root property. (p. 657)**	Add 9 to both sides of the equation. $x^2 + 6x + 9 = -7 + 9$ $(x + 3)^2 = 2$ Factor. $x + 3 = \pm\sqrt{2}$ Square root property $x = -3 \pm \sqrt{2}$ The solution set is $\{-3 - \sqrt{2}, -3 + \sqrt{2}\}$.

11.3 The Quadratic Formula

Definition/Procedure	Example
The Quadratic Formula The solutions of any quadratic equation of the form $ax^2 + bx + c = 0$ $(a \neq 0)$ are $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ **(p. 663)**	Solve $2x^2 - 5x - 2 = 0$ using the quadratic formula. $a = 2 \quad b = -5 \quad c = -2$ Substitute the values into the quadratic formula and simplify. $$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-2)}}{2(2)}$$ $$x = \frac{5 \pm \sqrt{25 + 16}}{4} = \frac{5 \pm \sqrt{41}}{4}$$ The solution set is $\left\{\frac{5 - \sqrt{41}}{4}, \frac{5 + \sqrt{41}}{4}\right\}$.
The expression under the radical, $b^2 - 4ac$, is called the **discriminant.** 1) If $b^2 - 4ac$ is **positive and the square of an integer,** the equation has **two rational solutions.** 2) If $b^2 - 4ac$ is **positive but not a perfect square,** the equation has **two irrational solutions.** 3) If $b^2 - 4ac$ is **negative,** the equation has **two nonreal, complex solutions of the form $a + bi$ and $a - bi$.** 4) If $b^2 - 4ac = 0$, the equation has **one rational solution. (p. 666)**	Find the value of the discriminant for $3m^2 + 4m + 5 = 0$, and determine the number and type of solutions of the equation. $a = 3 \quad b = 4 \quad c = 5$ $b^2 - 4ac = (4)^2 - 4(3)(5) = 16 - 60 = -44$ Discriminant $= -44$. The equation has two nonreal, complex solutions of the form $a + bi$ and $a - bi$.

11.4 Equations in Quadratic Form

Definition/Procedure	Example
Some equations that are not quadratic can be solved using the same methods that can be used to solve quadratic equations. These are called **equations in quadratic form. (p. 674)**	Solve $r^4 + 2r^2 - 24 = 0.$ $(r^2 - 4)(r^2 + 6) = 0$ Factor. $r^2 - 4 = 0$ or $r^2 + 6 = 0$ $r^2 = 4$ $\quad$ $r^2 = -6$ $r = \pm\sqrt{4}$ $\quad$ $r = \pm\sqrt{-6}$ $r = \pm 2$ $\quad$ $r = \pm i\sqrt{6}$ The solution set is $\{-i\sqrt{6}, i\sqrt{6}, -2, 2\}$.

Definition/Procedure	Example

11.5 Formulas and Applications

Definition/Procedure	Example
Solve a Formula for a Variable (p. 680)	Solve for s: $g = \frac{10}{s^2}$ $s^2 g = 10$ Multiply both sides by s^2. $s^2 = \frac{10}{g}$ Divide both sides by g. $s = \pm\sqrt{\frac{10}{g}}$ Square root property $s = \frac{\pm\sqrt{10}}{\sqrt{g}} \cdot \frac{\sqrt{g}}{\sqrt{g}}$ Rationalize the denominator. $s = \frac{\pm\sqrt{10g}}{g}$
Solving Application Problems Using a Quadratic Equation (p. 684)	A woman dives off of a cliff 49 m above the ocean. Her height, h, in meters, above the water is given by $h = -9.8t^2 + 49$ where t is the time, in seconds, after she leaves the cliff. When will she hit the water? Let $h = 0$ and solve for t. $h = -9.8t^2 + 49$ $0 = -9.8t^2 + 49$ Substitute 0 for h. $9.8t^2 = 49$ Add $9.8t^2$ to each side. $t^2 = 5$ Divide by 9.8. $t = \pm\sqrt{5}$ Square root property Since t represents time, we discard $-\sqrt{5}$. She will hit the water in $\sqrt{5}$, or about 2.2, sec.

Chapter 11: Review Exercises

(11.1) Solve by factoring.

1) $a^2 - 3a - 54 = 0$

2) $2t^2 + 9t + 10 = 0$

3) $\frac{2}{3}c^2 = \frac{2}{3}c + \frac{1}{2}$

4) $4k = 12k^2$

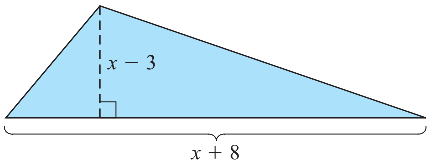
5) $x^3 + 3x^2 - 16x - 48 = 0$

6) $3p - 16 = p(p - 7)$

Write an equation and solve.

7) A rectangle has an area of 96 cm^2. Its width is 4 cm less than its length. Find the length and width.

8) Find the base and height of the triangle if its area is 30 in^2.



(11.2) Solve using the square root property.

9) $d^2 = 144$

10) $m^2 = 75$

11) $v^2 + 4 = 0$

12) $2c^2 - 11 = 25$

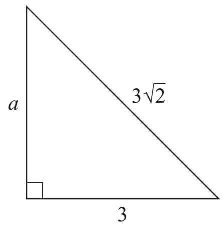
13) $(b - 3)^2 = 49$

14) $(6y + 7)^2 - 15 = 0$

15) $27k^2 - 30 = 0$

16) $(j - 14)^2 + 5 = 0$

17) Find the length of the missing side.



18) A rectangle has a length of $5\sqrt{2}$ in. and a width of 4 in. How long is its diagonal?

Find the distance between the given points.

19) (2, 3) and (7, 5)

20) (−2, 5) and (−1, −3)

21) (3, −1) and (0, 3)

22) (−5, 8) and (2, 3)

Complete the square for each expression to obtain a perfect square trinomial. Then, factor.

23) $r^2 + 10r$

24) $z^2 - 12z$

25) $c^2 - 5c$

26) $x^2 + x$

27) $a^2 + \frac{2}{3}a$

28) $d^2 - \frac{5}{2}d$

Solve by completing the square.

29) $p^2 - 6p - 16 = 0$

30) $w^2 - 2w - 35 = 0$

31) $n^2 + 10n = 6$

32) $t^2 + 9 = -4t$

33) $f^2 + 3f + 1 = 0$

34) $j^2 - 7j = 4$

35) $-3q^2 + 7q = 12$

36) $6v^2 - 15v + 3 = 0$

(11.3) Solve using the quadratic formula.

37) $m^2 + 4m - 12 = 0$

38) $3y^2 = 10y - 8$

39) $10g - 5 = 2g^2$

40) $20 = 4x - 5x^2$

41) $\frac{1}{6}t^2 - \frac{1}{3}t + \frac{2}{3} = 0$

42) $(s - 3)(s - 5) = 9$

43) $(6r + 1)(r - 4) = -2(12r + 1)$

44) $z^2 - \frac{3}{2}z + \frac{13}{16} = 0$

45) If the discriminant of a quadratic equation is positive but not a perfect square, what do you know about the solutions of the equation?

46) If the discriminant of a quadratic equation is negative, what do you know about the solutions of the equation?

Find the value of the discriminant. Then, determine the number and type of solutions of each equation. Do not solve.

47) $3n^2 - 2n - 5 = 0$

48) $5x^2 + 5x + \frac{5}{4} = 0$

49) $t^2 = -3(t + 2)$

50) $3 - 7y = -y^2$

51) Find the value of b so that $4k^2 + bk + 9 = 0$ has only one rational solution.

52) A ball is thrown upward from a height of 4 ft. The height h of the ball (in feet) t sec after the ball is released is given by $h = -16t^2 + 52t + 4$.

a) How long does it take the ball to reach a height of 16 ft?

b) How long does it take the ball to hit the ground?

(11.4) Solve.

53) $z + 2 = \frac{15}{z}$

54) $\frac{5k}{k + 1} = 3k - 4$

55) $\frac{10}{m} = 3 + \frac{8}{m^2}$

56) $f = \sqrt{7f - 12}$

57) $x - 4\sqrt{x} = 5$

58) $n^4 - 17n^2 + 16 = 0$

59) $b^4 + 5b^2 - 14 = 0$

60) $q^{2/3} + 2q^{1/3} - 3 = 0$

61) $y + 2 = 3y^{1/2}$

62) $2r^4 = 7r^2 - 2$

63) $2(v + 2)^2 + (v + 2) - 3 = 0$

64) $(2k - 5)^2 - 5(2k - 5) - 6 = 0$

(11.5) Solve for the indicated variable.

65) $F = \frac{mv^2}{r}$ for v

66) $U = \frac{1}{2}kx^2$ for x

67) $r = \sqrt{\frac{A}{\pi}}$ for A

68) $r = \sqrt{\frac{V}{\pi l}}$ for V

69) $kn^2 - ln - m = 0$ for n

70) $2p^2 + t = rp$ for p

Write an equation and solve.

71) Ayesha is making a pillow sham by sewing a border onto an old pillow case. The rectangular pillow case measures 18 in. by 27 in. When she sews a border of uniform width around the pillowcase, the total area of the surface of the pillow sham will be 792 in^2. How wide is the border?

72) The width of a rectangular piece of cardboard is 4 in. less than its length. A square piece that measures 2 in. on each side is cut from each corner, then the sides are turned up to make a box with volume 280 in^3. Find the length and width of the original piece of cardboard.

73) A flower shop determined that the demand for its tulip bouquet is $\frac{240}{P}$ per week, where P is the price of the bouquet in dollars. The weekly supply is given by $4P - 2$. Find the price at which demand for the tulips equals the supply.

74) U.S. sales of a certain brand of wine can be modeled by

$$y = -0.20x^2 + 4.0x + 8.4$$

for the years 1995–2010. x is the number of years after 1995 and y is the number of bottles sold, in millions.

a) How many bottles were sold in 1995?

b) How many bottles were sold in 2008?

c) In what year did sales reach 28.4 million bottles?

(11.1–11.4) Mixed Exercises

Solve using one of the methods of this chapter.

75) $3k^2 + 4 = 7k$

76) $n^2 - 6n + 11 = 0$

77) $3 = 15 + (y + 8)^2$

78) $(2a + 1)(a + 2) = 14$

79) $c^4 - 26c^2 + 25 = 0$

80) $6 = 2m - 3m^2$

81) $\frac{8}{n - 4} + \frac{n}{n - 3} = -\frac{12}{n^2 - 7n + 12}$

82) $z - 4\sqrt{z} = 12$

83) $\frac{1}{3}w^2 + w = -\frac{5}{6}$

84) $4t^2 + 5 = 7$

85) $6 + p(p - 10) = 2(4p - 15)$

86) $y^{2/3} + 6y^{1/3} = 40$

87) $x^3 = x$

88) $\frac{1}{12}b^2 - \frac{9}{2} = \frac{1}{4}b$

Write an equation and solve.

89) The hypotenuse of a right triangle is 15 cm. One leg is 3 cm shorter than the other leg. Find the lengths of the legs of the triangle.

90) The length of a rectangle is 14 in., and its diagonal is $2\sqrt{65}$ in. long. What is the width of the rectangle?

91) Latrice can organize the stockroom 25 min faster than Erica. If they work together, it takes them 30 min. How long would it take each person to organize the stockroom by herself?

92) A boat can travel 20 mi downstream and then 12 mi back upstream in 1 hr 36 min. If the speed of the current is 5 mph, what is the speed of the boat in still water?

Chapter 11: Test

Solve by factoring.

1) $k^2 - 8k = 48$

2) $16 - 9w^2 = 0$

3) Solve $t^2 + 7 = 25$ using the square root property.

4) If k is a negative number and $x^2 = k$, what can you conclude about the solution set of the equation?

5) Find the distance between the points $(-6, 2)$ and $(4, 3)$.

Solve by completing the square.

6) $b^2 + 4b - 7 = 0$

7) $2x^2 - 6x + 14 = 0$

8) Solve $x^2 - 8x + 17 = 0$ using the quadratic formula.

Solve using any method.

9) $(c + 5)^2 + 8 = 2$

10) $3q^2 + 2q = 8$

11) $y^2 - \dfrac{4}{25} = 0$

12) $(4n + 1)^2 + 9(4n + 1) + 18 = 0$

13) $p^4 + p^2 - 72 = 0$

14) $45a = 54a^2$

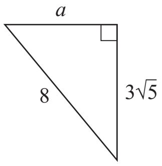
15) $(2t - 3)(t - 2) = 2$

16) $\dfrac{3}{10x} = \dfrac{x}{x - 1} - \dfrac{4}{5}$

17) Find the value of the discriminant. Then, determine the number and type of solutions of the equation. *Do not solve.*

$$5z^2 - 6z - 1 = 0$$

18) Find the length of the missing side.



19) A ball is projected upward from the top of a 200 ft tall building. The height h of the ball (in feet) t sec after the ball is released is given by $h = -16t^2 + 24t + 200$.

a) When will the ball be 40 ft above the ground?

b) When will the ball hit the ground?

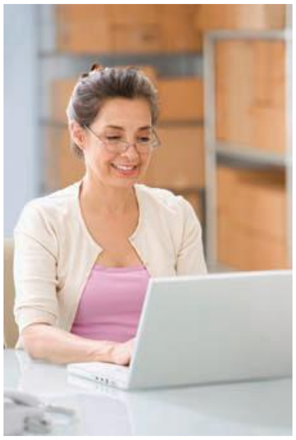
20) Solve $r = \sqrt{\dfrac{3V}{\pi h}}$ for V.

21) Solve $rt^2 - st = 6$ for t.

Write an equation and solve.

22) The length of a rectangular garden is 2 ft more than the width. The diagonal of the garden is 10 ft. Find the length and width of the garden.

23) It takes Kimora 20 min more to type a report than it takes Justine. It would take them 24 min to type the report together. How long would it take each woman to type the report alone?



24) A rectangular piece of sheet metal is 6 in. longer than it is wide. A square piece that measures 3 in. on each side is cut from each corner, then the sides are turned up to make a box with volume 273 in^3. Find the length and width of the original piece of sheet metal.

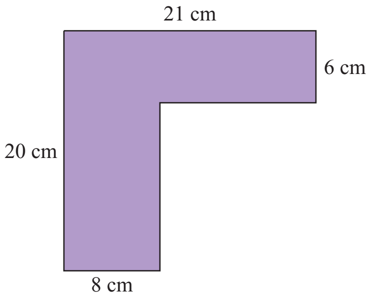
Cumulative Review: Chapters 1–11

Perform the operations and simplify.

1) $\dfrac{4}{15} + \dfrac{1}{6} - \dfrac{3}{5}$

2) $24 - 8 \div 2 - |3 - 10|$

3) Find the area and perimeter of this figure.



Simplify. The final answer should contain only positive exponents.

4) $(-2d^5)^3$

5) $(5x^4y^{-10})(3xy^3)^2$

6) $\left(\dfrac{40a^{-3}b}{8a^{-8}b^4}\right)^{-3}$

7) *Write an equation and solve.*
In December 2010, an electronics store sold 108 digital cameras. This is a 20% increase over their sales in December 2009. How many digital cameras did they sell in December 2009?

8) Solve $y = mx + b$ for m.

9) Find the x- and y-intercepts of $2x - 5y = 8$ and graph.

10) Identify the slope, y-intercept, and graph $y = -\dfrac{3}{4}x + 1$.

11) *Write a system of two equations in two variables and solve.*
Two bags of chips and three cans of soda cost \$3.85 while one bag of chips and two cans of soda cost \$2.30. Find the cost of a bag of chips and a can of soda.

12) Subtract
$(4x^2y^2 - 11x^2y + xy + 2) - (x^2y^2 - 6x^2y + 3xy^2 + 10xy - 6)$.

13) Multiply and simplify $3(r - 5)^2$.

Factor completely.

14) $4p^3 + 14p^2 - 8p$

15) $a^3 + 125$

16) Add $\dfrac{z-8}{z+4} + \dfrac{3}{z}$.

17) Simplify $\dfrac{2 + \dfrac{6}{c}}{\dfrac{2}{c^2} - \dfrac{8}{c}}$.

18) Solve $|4k - 3| = 9$.

19) Solve this system:
$$\begin{aligned} 4x - 2y + z &= -7 \\ -3x + y - 2z &= 5 \\ 2x + 3y + 5z &= 4 \end{aligned}$$

Simplify. Assume all variables represent nonnegative real numbers.

20) $\sqrt{75}$

21) $\sqrt[3]{40}$

22) $\sqrt{63x^7y^4}$

23) Simplify $64^{2/3}$.

24) Rationalize the denominator: $\dfrac{5}{2 + \sqrt{3}}$.

25) Multiply and simplify $(10 + 3i)(1 - 8i)$.

Solve.

26) $3k^2 - 4 = 20$

27) $\dfrac{3}{5}x^2 + \dfrac{1}{5} = \dfrac{1}{5}x$

28) $1 - \dfrac{1}{3h - 2} = \dfrac{20}{(3h - 2)^2}$

29) $p^2 + 6p = 27$

30) Solve $r = \sqrt{\dfrac{V}{\pi h}}$ for V.

CHAPTER 12

Functions and Their Graphs

Algebra at Work: Forensics

When forensic scientists are called to a crime scene where a skeleton has been found, they look at many different features to work with police to piece together a profile of the victim. The hips can reveal whether the person was male or female. Certain facial features indicate the person's ethnicity. And the length of certain bones, considered together with sex and ethnicity, enable forensics experts to estimate a person's height.

The femur, or thigh bone, is the largest bone in the human body and is one that is commonly used to estimate a person's height.

Raul arrives at a crime scene to help identify skeletal remains. After taking measurements on the skull and hips, he determines that the victim was a white female. To estimate her height, Raul can use the linear function

$$H(f) = 2.47f + 54.10$$

where $f =$ the length of the femur, in centimeters

$H(f) =$ the height of the victim, in centimeters

The length of the femur is 44 cm. Substituting 44 for f in the function above, Raul estimates the height of the victim:

$$H(44) = 2.47(44) + 54.10$$
$$H(44) = 162.78 \text{ cm}$$

The measurements that Raul took on the skeletal remains indicate that the victim was a white female and that she was about 162.78 cm or 64 in. tall.

Section 12.1 Relations and Functions

Objectives

1. Identify Relation, Function, Domain, and Range
2. Graph a Linear Function
3. Define a Polynomial Function and a Quadratic Function
4. Evaluate Linear and Quadratic Functions for a Given Variable or Expression
5. Define and Determine the Domain of a Rational Function
6. Define and Determine the Domain of a Square Root Function
7. Summarize Strategies for Determining the Domain of a Given Function

1. Identify Relation, Function, Domain, and Range

We first studied functions in Chapter 4. Let's review some of the concepts first presented in Section 4.6, and we will extend what we have learned to include new functions. Recall the following definitions.

Definition

A **relation** is any set of ordered pairs. The **domain** of a relation is the set of all values of the independent variable (the first coordinates in the set of ordered pairs). The **range** of a relation is the set of all values of the dependent variable (the second coordinates in the set of ordered pairs).

Definition

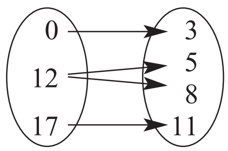
A **function** is a special type of relation. If each element of the domain corresponds to *exactly one* element of the range, then the relation is a function.

Example 1

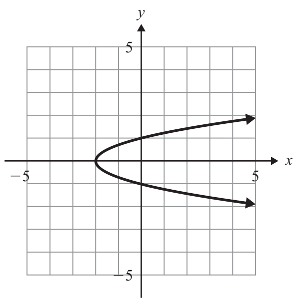
Identify the domain and range of each relation, and determine whether each relation is a function.

a) $\{(-2, -7), (0, -1), (1, 2), (5, 14)\}$

b)



c)



Solution

a) The *domain* is the set of first coordinates, $\{-2, 0, 1, 5\}$.
The *range* is the set of second coordinates, $\{-7, -1, 2, 14\}$.

Ask yourself, "Does every first coordinate correspond to *exactly one* second coordinate?" *Yes*. This relation *is* a function.

b) The *domain* is $\{0, 12, 17\}$.
The *range* is $\{3, 5, 8, 11\}$.

One of the elements in the domain, 12, corresponds to *two* elements in the range, 5 and 8. Therefore, this relation is *not* a function.

c) The domain is $[-2, \infty)$. The range is $(-\infty, \infty)$.

To determine whether this graph represents a function, recall that we can use the *vertical line test.*

The **vertical line test** says that if there is no vertical line that can be drawn through a graph so that it intersects the graph more than once, then the graph represents a function.

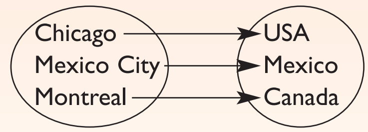
This graph fails the vertical line test because we can draw a vertical line through the graph that intersects it more than once. *This graph does* not *represent a function.*

You Try 1

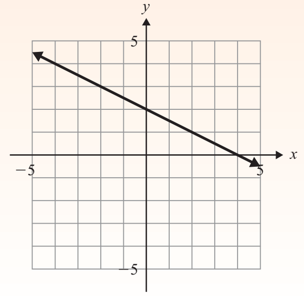
Identify the domain and range of each relation, and determine whether each relation is a function.

a) $\{(-8, 1), (-5, 2), (-5, -2), (7, -4)\}$

b)



c)



Next, we will look at relations and functions written as equations.

If a relation is written as an equation so that y is in terms of x, then the *domain* is the set of all real numbers that can be substituted for the independent variable, x. The resulting set of real numbers that are obtained for y, the dependent variable, is the *range*. To determine the domain, sometimes it is helpful to ask yourself, *"Is there any number that cannot be substituted for x?"*

Example 2

Determine whether each relation describes y as a function of x, and determine the domain of the relation.

a) $y = -3x + 4$ b) $y^2 = x$

Solution

a) *Every* value substituted for x will have *exactly one* corresponding value of y. For example, if we substitute 5 for x, the only value of y is -11. Therefore, $y = -3x + 4$ *is a function*.

To determine the domain, ask yourself, *"Is there any number that cannot be substituted for x in $y = -3x + 4$?" No*. Any real number can be substituted for x, and $y = -3x + 4$ will be defined.

The domain consists of all real numbers. This can be written as $(-\infty, \infty)$.

b) If we substitute a number such as 4 for x and solve for y, we get

$$y^2 = 4$$
$$y = \pm\sqrt{4}$$
$$y = \pm 2$$

The ordered pairs (4, 2) and (4, −2) satisfy $y^2 = x$. Since $x = 4$ corresponds to two *different* y-values, $y^2 = x$ is *not* a function.

To determine the domain of this relation, ask yourself, *"Is there any number that cannot be substituted for x in* $y^2 = x$*?"* In this case, let's first look at y. Since y is squared, any real number substituted for y will produce a number that is greater than or equal to zero. Therefore, in the equation, x will equal a number that is greater than or equal to zero.

The domain is $[0, \infty)$.

You Try 2

Determine whether each relation describes y as a function of x, and determine the domain of the relation.

a) $y = x^2 + 5$ b) $y = x^3$

Recall that if a function describes the relationship between x and y so that x is the independent variable and y is the dependent variable, then *y is a function of x*. That is, *the value of y depends on the value of x.*

Recall also that we can use *function notation* to represent this relationship.

Definition

$y = f(x)$ is called **function notation**, and it is read as, "y equals f of x." $y = f(x)$ means that y is a function of x (that is, y depends on x).

If a relation is a function, then $f(x)$ can be used in place of y. **$f(x)$ is the same as y.**

In Example 2, we concluded that $y = -3x + 4$ is a function. Using function notation, we can write $y = -3x + 4$ as $f(x) = -3x + 4$. *They mean the same thing.*

Types of Functions

2. Graph a Linear Function

In addition to being a function, $y = -3x + 4$ is a linear equation in two variables. Because it is also a function, we call it a *linear function*. We first studied linear functions in Chapter 4, and we restate its definition here.

Definition

A **linear function** has the form $f(x) = mx + b$, where m and b are real numbers, m is the *slope* of the line, and $(0, b)$ is the *y-intercept*.

Let's look at some functions and learn how to determine their domains.

Example 3

$f(x) = \frac{1}{2}x - 3$

a) What is the domain of f? b) Graph the function.

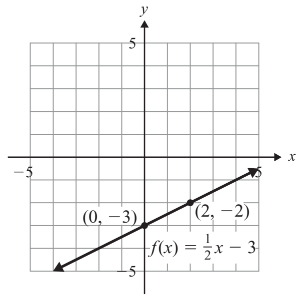
Solution

a) The domain is the set of all real numbers that can be substituted for x. Ask yourself, "*Is there any number that cannot be substituted for x in* $f(x) = \frac{1}{2}x - 3$?"

No. Any real number can be substituted for x, and $f(x) = \frac{1}{2}x - 3$ will be defined.

The domain consists of all real numbers. This can be written as $(-\infty, \infty)$.

b) The y-intercept is $(0, -3)$, and the slope of the line is $\frac{1}{2}$. Use this information to graph the line.



You Try 3

$g(x) = -x + 2$

a) What is the domain of g? b) Graph the function.

3. Define a Polynomial Function and a Quadratic Function

Another function often used in mathematics is a **polynomial function**.

The expression $x^3 + 2x^2 - 9x + 5$ is a polynomial. For each real number that is substituted for x, there will be *only one value* of the expression. For example, if we substitute 2 for x, the *only value* of the expression is 3. (Try this yourself to verify the result!) Since each value substituted for the variable produces *only one value* of the expression, we can use function notation to represent a polynomial like $x^3 + 2x^2 - 9x + 5$.

$f(x) = x^3 + 2x^2 - 9x + 5$ is a **polynomial function** since $x^3 + 2x^2 - 9x + 5$ is a polynomial. The domain of a polynomial function is all real numbers, $(-\infty, \infty)$.

In Chapter 4, we learned how to find a function value. For example, if $f(x) = x^3 + 2x^2 - 9x + 5$, then to find $f(-3)$ means to substitute -3 for x and evaluate.

$$\begin{aligned} f(x) &= x^3 + 2x^2 - 9x + 5 \\ f(-3) &= (-3)^3 + 2(-3)^2 - 9(-3) + 5 && \text{Substitute } -3 \text{ for } x. \\ &= -27 + 18 + 27 + 5 && \text{Evaluate.} \\ &= 23 \end{aligned}$$

We can also say that the ordered pair $(-3, 23)$ satisfies $f(x) = x^3 + 2x^2 - 9x + 5$.

A quadratic function is a special type of polynomial function.

Definition

A polynomial function of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$, is a **quadratic function**.

An example of a *quadratic function* is $f(x) = 3x^2 - x - 8$. (Notice that this is similar to a quadratic equation, an equation of the form $ax^2 + bx + c = 0$.)

4. Evaluate Linear and Quadratic Functions for a Given Variable or Expression

We just found that $f(-3) = 23$ for $f(x) = x^3 + 2x^2 - 9x + 5$. We can also evaluate functions for variables and expressions.

Example 4

Let $g(x) = 4x - 10$ and $h(x) = x^2 + 2x - 11$. Find each of the following and simplify.

a) $g(k)$ b) $g(n + 3)$ c) $h(p)$ d) $h(w - 4)$

Solution

a) Finding $g(k)$ (read as *g of k*) means to substitute k for x in the function g, and simplify the expression as much as possible.

$$\begin{aligned} g(x) &= 4x - 10 \\ g(k) &= 4k - 10 && \text{Substitute } k \text{ for } x. \end{aligned}$$

b) Finding $g(n + 3)$ (read as *g of n plus* 3) means to substitute $n + 3$ for x in the function g, and simplify the expression as much as possible. *Since $n + 3$ contains two terms, we must put it in parentheses.*

$$\begin{aligned} g(x) &= 4x - 10 \\ g(n + 3) &= 4(n + 3) - 10 && \text{Substitute } n + 3 \text{ for } x. \\ g(n + 3) &= 4n + 12 - 10 && \text{Distribute.} \\ g(n + 3) &= 4n + 2 && \text{Combine like terms.} \end{aligned}$$

c) To find $h(p)$, substitute p for x in function h.

$$\begin{aligned} h(x) &= x^2 + 2x - 11 \\ h(p) &= p^2 + 2p - 11 && \text{Substitute } p \text{ for } x. \end{aligned}$$

d) To find $h(w - 4)$, substitute $w - 4$ for x in function h, and simplify the expression as much as possible. *When we substitute, we must put $w - 4$ in parentheses since $w - 4$ consists of more than one term.*

$$\begin{aligned} h(x) &= x^2 + 2x - 11 \\ h(w - 4) &= (w - 4)^2 + 2(w - 4) - 11 && \text{Substitute } w - 4 \text{ for } x. \\ h(w - 4) &= w^2 - 8w + 16 + 2w - 8 - 11 && \text{Multiply.} \\ h(w - 4) &= w^2 - 6w - 3 && \text{Combine like terms.} \end{aligned}$$

You Try 4

Let $f(x) = -9x + 2$ and $k(x) = x^2 + 5x + 8$. Find each of the following and simplify.

a) $f(c)$ b) $f(r - 5)$ c) $k(z)$ d) $k(m + 2)$

5. Define and Determine the Domain of a Rational Function

Some functions contain rational expressions. Like a polynomial, each real number that can be substituted for the variable in a rational expression will produce *only one* value for the expression. Therefore, we can use function notation to represent a rational expression.

$$f(x) = \frac{4x - 3}{x + 5} \text{ is an example of a } \textbf{rational function}.$$

Since a fraction is undefined when its denominator equals zero, it follows that a rational expression is undefined when its denominator equals zero. For example, in the function $f(x) = \dfrac{4x - 3}{x + 5}$, x *cannot* equal -5 because then we would get zero in the denominator. This is important to keep in mind when we are trying to find the domain of a rational function.

Definition

The **domain of a rational function** consists of all real numbers *except* the value(s) of the variable that make the denominator equal zero.

Note

To determine the domain of a rational function we set the denominator equal to zero and solve for the variable. Any values that make the denominator equal to zero are *not* in the domain of the function.

Remember, to determine the domain of a rational function, sometimes it is helpful to ask yourself, *"Is there any number that cannot be substituted for the variable?"*

Example 5

Determine the domain of each rational function.

a) $f(x) = \dfrac{5x}{x - 7}$ b) $g(t) = \dfrac{t + 4}{2t + 1}$ c) $r(x) = \dfrac{2x + 9}{6}$

Solution

a) To determine the domain of $f(x) = \dfrac{5x}{x - 7}$ ask yourself, *"Is there any number that cannot be substituted for x?"* Yes, $f(x)$ is *undefined* when the denominator equals zero. Set the denominator equal to zero, and solve for x.

$$\begin{aligned} x - 7 &= 0 && \text{Set the denominator } = 0. \\ x &= 7 && \text{Solve.} \end{aligned}$$

When $x = 7$, the denominator of $f(x) = \dfrac{5x}{x - 7}$ equals zero. The domain contains all real numbers *except* 7. Write the domain in interval notation as $(-\infty, 7) \cup (7, \infty)$.

b) Ask yourself, "*Is there any number that cannot be substituted for t in* $g(t) = \dfrac{t + 4}{2t + 1}$?" Look at the denominator. When will it equal 0? Set the denominator equal to 0 and solve for t.

$$\begin{aligned} 2t + 1 &= 0 && \text{Set the denominator } = 0. \\ 2t &= -1 \\ t &= -\frac{1}{2} && \text{Solve.} \end{aligned}$$

When $t = -\frac{1}{2}$, the denominator of $g(t) = \frac{t+4}{2t+1}$ equals zero. The domain contains all real numbers *except* $-\frac{1}{2}$. Write the domain in interval notation as $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$.

c) *Is there any number that cannot be substituted for x in* $r(x) = \frac{2x+9}{6}$? The denominator is a constant, 6, and it can never equal zero. Therefore, any real number can be substituted for x and the function will be defined.

The domain consists of all real numbers, which can be written as $(-\infty, \infty)$. ■

You Try 5

Determine the domain of each rational function.

a) $f(x) = \frac{9}{x}$ b) $k(c) = \frac{c}{3c+4}$ c) $g(n) = \frac{n-2}{n^2-9}$

6. Define and Determine the Domain of a Square Root Function

When real numbers are substituted for the variable in radical expressions like $\sqrt{x}$ and $\sqrt{4r+1}$, each value that is substituted will produce *only one* value for the expression. Function notation can be used to represent radical expressions, too.

$f(x) = \sqrt{x}$ and $g(r) = \sqrt{4r+1}$ are examples of **square root functions**.

When we determine the domain of a square root function, we are determining all *real* numbers that may be substituted for the variable so that the range contains only *real* numbers. (Complex numbers are *not* included.) All values substituted for the independent variable that produce complex numbers as function values are *not* in the domain of the function. This means that there may be *many* values that are excluded from the domain of a square root function if it is to be real valued.

Example 6

Determine the domain of each square root function.

a) $f(x) = \sqrt{x}$ b) $g(r) = \sqrt{4r+1}$

Solution

a) Ask yourself, *"Is there any number that cannot be substituted for x in* $f(x) = \sqrt{x}$?*"* Yes. There are many values that cannot be substituted for x. Since we are considering only real numbers in the domain and range, x cannot be a negative number because then $f(x)$ would be imaginary. For example, if $x = -4$, then $f(-4) = \sqrt{-4} = 2i$. Therefore, x must be greater than or equal to 0 in order to produce a real-number value for $f(x)$. The domain consists of $x \geq 0$ or $[0, \infty)$.

b) In part a) we saw that in order for $f(x)$ to be a real number, the *quantity under the radical* (*radicand*) must be 0 or positive. In $g(r) = \sqrt{4r + 1}$, the *radicand* is $4r + 1$. In order for $g(r)$ to be defined, $4r + 1$ must be 0 or positive. Mathematically, we write this as $4r + 1 \geq 0$. To determine the domain of $g(r) = \sqrt{4r + 1}$, solve the inequality $4r + 1 \geq 0$.

$$4r + 1 \geq 0 \qquad \text{The value of the radicand must be } \geq 0.$$
$$4r \geq -1$$
$$r \geq -\frac{1}{4} \qquad \text{Solve.}$$

Any value of r that satisfies $r \geq -\frac{1}{4}$ will make the radicand $4r + 1$ greater than or equal to zero. Write the domain as $\left[-\frac{1}{4}, \infty\right)$. ■

Note

To determine the domain of a square root function, set up an inequality so that the radicand ≥ 0. Solve for the variable. These are the real numbers in the domain of the function.

You Try 6

Determine the domain of each square root function.

a) $h(x) = \sqrt{x - 9}$ b) $k(t) = \sqrt{7t + 2}$

7. Summarize Strategies for Determining the Domain of a Given Function

Let's summarize what we have learned about determining the domain of a function.

Summary Determining the Domain of a Function

The **domain of a function in *x*** is the set of all real numbers that can be substituted for the independent variable, *x*. When determining the domain of a function, it can be helpful to keep these tips in mind.

1) Ask yourself, "Is there any number that *cannot* be substituted for *x*?"
2) The domain of a **linear function** is all real numbers, $(-\infty, \infty)$.
3) The domain of a **polynomial function** is all real numbers, $(-\infty, \infty)$.
4) To find the domain of a **rational function**, determine what value(s) of *x* will make the denominator equal 0 by setting the expression in the denominator equal to zero. Solve for *x*. These *x*-value(s) are *not* in the domain.
5) To determine the domain of a **square root function**, set up an inequality so that the radicand ≥ 0. Solve for *x*. These are the values of *x* in the domain.

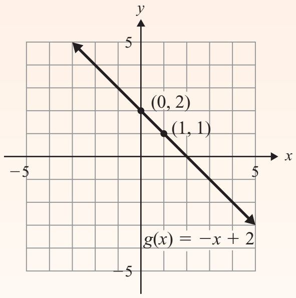
Another function we encounter often in algebra is an absolute value function like $f(x) = |x|$. We will study absolute value functions in Section 12.2.

Answers to You Try Exercises

1) a) domain: $\{-8, -5, 7\}$; range: $\{-4, -2, 1, 2\}$; no b) domain: {Chicago, Mexico City, Montreal}; range: {USA, Mexico, Canada}; yes c) domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$; yes

2) a) is a function; domain: $(-\infty, \infty)$ b) is a function; domain: $(-\infty, \infty)$

3) a) $(-\infty, \infty)$ b)



4) a) $f(c) = -9c + 2$ b) $f(r - 5) = -9r + 47$ c) $k(z) = z^2 + 5z + 8$
d) $k(m + 2) = m^2 + 9m + 22$

5) a) $(-\infty, 0) \cup (0, \infty)$ b) $\left(-\infty, -\frac{4}{3}\right) \cup \left(-\frac{4}{3}, \infty\right)$ c) $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

6) a) $[9, \infty)$ b) $\left[-\frac{2}{7}, \infty\right)$

12.1 Exercises

Objective 1: Identify Relation, Function, Domain, and Range

1) What is a function?

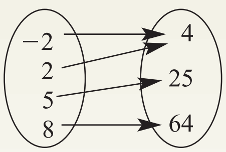
2) What is the domain of a relation?

Identify the domain and range of each relation, and determine whether each relation is a function.

3) $\{(5, 0), (6, 1), (14, 3), (14, -3)\}$

4) $\{(-5, 7), (-4, 5), (0, -3), (0.5, -4), (3, -9)\}$

5)



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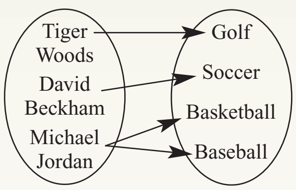

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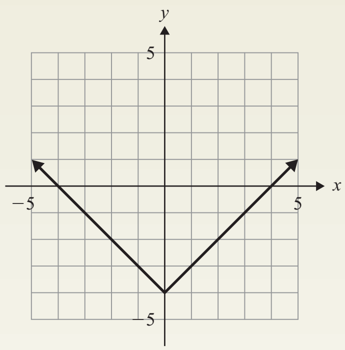

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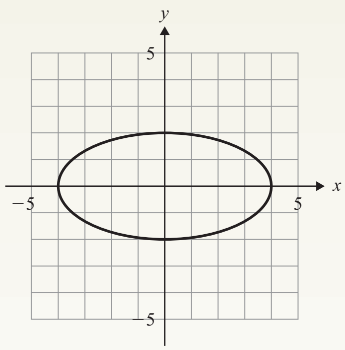


Determine whether each relation describes y as a function of x.

9) $y = 5x + 17$

10) $y = 4x^2 - 10x + 3$

11) $y = \dfrac{x}{x + 6}$

12) $y^2 = x + 2$

VIDEO 13) $y^2 = x - 8$

14) $y = \sqrt{x + 3}$

15) $x = |y|$

16) $y = |x|$

Answer true or false. If the answer is false, explain why.

17) $f(x)$ is read as "f times x."

18) $f(x) = -4x + 1$ is an example of a linear function.

Objective 2: Graph a Linear Function

Graph each function.

19) $f(x) = x - 5$

20) $g(x) = -x + 3$

21) $h(a) = -2a + 1$

22) $r(t) = 3t - 2$

VIDEO 23) $g(x) = -\frac{3}{2}x - 1$

24) $f(x) = \frac{1}{4}x + 2$

25) $k(c) = c$

26) $h(x) = -3$

Mixed Exercises: Objectives 3 and 4

Let $f(x) = 3x - 7$ and $g(x) = x^2 - 4x - 9$. Find each of the following and simplify.

27) $f(6)$

28) $f(0)$

29) $g(3)$

30) $g(-2)$

31) $f(a)$

32) $f(z)$

33) $g(d)$

34) $g(r)$

35) $f(c + 4)$

36) $f(w + 9)$

VIDEO 37) $g(t + 2)$

38) $g(a + 3)$

39) $g(h - 1)$

40) $g(p - 5)$

Let $f(x) = -5x + 2$ and $g(x) = x^2 + 7x + 2$. Find each of the following and simplify.

41) $f(4)$

42) $f(7)$

43) $g(-6)$

44) $g(3)$

45) $f(-3k)$

46) $f(9a)$

47) $g(5t)$

48) $g(-8n)$

49) $f(b + 1)$

50) $f(t - 6)$

51) $g(r + 4)$

52) $g(a - 9)$

53) $f(x) = 4x + 3$. Find x so that $f(x) = 23$.

54) $g(x) = -9x + 1$. Find x so that $g(x) = -17$.

55) $h(x) = -2x - 5$. Find x so that $h(x) = 0$.

56) $k(x) = 8x - 6$. Find x so that $k(x) = 0$.

57) $p(x) = x^2 - 6x - 16$. Find x so that $p(x) = 0$.

58) $g(x) = 2x^2 - 5x - 9$. Find x so that $g(x) = -6$.

Mixed Exercises: Objectives 2–7

59) What is the domain of a linear function?

60) What is the domain of a polynomial function?

61) How do you find the domain of a rational function?

62) How do you find the domain of a square root function?

Determine the domain of each function.

63) $f(x) = x + 10$

64) $h(x) = -8x - 2$

65) $p(a) = 8a^2 + 4a - 9$

66) $r(t) = t^3 - 7t^2 + t + 4$

VIDEO 67) $f(x) = \frac{6}{x + 8}$

68) $k(x) = \frac{2x}{x - 9}$

69) $h(x) = \frac{10}{x}$

70) $Q(r) = \frac{7}{2r}$

71) $g(c) = \frac{3c}{2c - 1}$

72) $f(x) = \frac{4x + 3}{5x + 2}$

73) $R(t) = -\frac{t - 4}{7t + 3}$

74) $k(n) = \frac{8}{1 - 3n}$

75) $h(x) = \frac{9x + 2}{4}$

76) $p(c) = \frac{c - 2}{7}$

77) $k(x) = \frac{1}{x^2 + 11x + 24}$

78) $f(t) = \frac{5}{t^2 - 7t + 6}$

79) $r(c) = \frac{c + 3}{c^2 - 5c - 36}$

80) $g(a) = \frac{4}{2a^2 + 3a}$

81) $f(x) = \sqrt{x}$

82) $r(t) = -\sqrt{t}$

83) $h(n) = \sqrt{n + 2}$

84) $g(c) = \sqrt{c + 10}$

85) $p(a) = \sqrt{a - 8}$

86) $f(a) = \sqrt{a - 1}$

VIDEO 87) $k(x) = \sqrt{2x - 5}$

88) $r(k) = \sqrt{3k + 7}$

89) $g(t) = \sqrt{-t}$

90) $h(x) = \sqrt{3 - x}$

91) $r(a) = \sqrt{9 - a}$

92) $g(c) = \sqrt{8 - 5c}$

93) $f(x) = |x|$

94) $k(t) = |-t|$

95) If a certain carpet costs \$22 per square yard, then the cost C, in dollars, of y yards of carpet is given by the function $C(y) = 22y$.

a) Find the cost of 20 yd^2 of carpet.

b) Find the cost of 56 yd^2 of carpet.

c) If a customer spent \$770 on carpet, how many square yards of carpet did he buy?

d) Graph the function.

96) A freight train travels at a constant speed of 32 mph. The distance D, in miles, that the train travels after t hr is given by the function $D(t) = 32t$.

a) How far will the train travel after 3 hr?

b) How far will the train travel after 8 hr?

c) How long does it take for the train to travel 208 mi?

d) Graph the function.

97) For labor only, the Arctic Air-Conditioning Company charges \$40 to come to the customer's home plus \$50 per hour. These labor charges can be described by the function $L(h) = 50h + 40$, where h is the time, in hours, and L is the cost of labor, in dollars.

a) Find $L(1)$ and explain what this means in the context of the problem.

b) Find $L(1.5)$ and explain what this means in the context of the problem.

c) Find h so that $L(h) = 165$, and explain what this means in the context of the problem.

98) For labor only, a plumber charges \$30 for a repair visit plus \$60 per hour. These labor charges can be described by the function $L(h) = 60h + 30$, where h is the time, in hours, and L is the cost of labor, in dollars.

a) Find $L(2)$ and explain what this means in the context of the problem.

b) Find $L(1)$ and explain what this means in the context of the problem.

c) Find h so that $L(h) = 210$, and explain what this means in the context of the problem.

99) The area, A, of a circle is a function of its radius, r.

a) Write an equation using function notation to describe this relationship between A and r.

b) If the radius is given in centimeters, find $A(3)$ and explain what this means in the context of the problem.

c) If the radius is given in inches, find $A(5)$ and explain what this means in the context of the problem.

d) What is the radius of a circle with an area of 64π in^2?

100) The perimeter, P, of a square is a function of the length of its side, s.

a) Write an equation using function notation to describe this relationship between P and s.

b) If the length of a side is given in feet, find $P(2)$ and explain what this means in the context of the problem.

c) If the length of a side is given in centimeters, find $P(11)$ and explain what this means in the context of the problem.

d) What is the length of each side of a square that has a perimeter of 18 inches?

Section 12.2 Graphs of Functions and Transformations

Objectives

1. Illustrate Vertical Shifts with Absolute Value Functions
2. Illustrate Horizontal Shifts with Quadratic Functions
3. Illustrate Reflecting a Graph About the x-Axis with Square Root Functions
4. Graph a Function Using a Combination of the Transformations
5. Graph a Piecewise Function
6. Define the Greatest Integer Function, $f(x) = [\![x]\!]$
7. Represent an Applied Problem with the Graph of a Greatest Integer Function

Some functions and their graphs appear often when studying algebra. We will look at the basic graphs of

1. the absolute value function, $f(x) = |x|$.
2. the quadratic function, $f(x) = x^2$.
3. the square root function, $f(x) = \sqrt{x}$.

It is possible to obtain the graph of any function by plotting points. But we will also see how we can graph other, similar functions by transforming the graphs of the functions above.

First, we will graph two absolute value functions. We will begin by plotting points so that we can observe the pattern that develops.

1. Illustrate Vertical Shifts with Absolute Value Functions

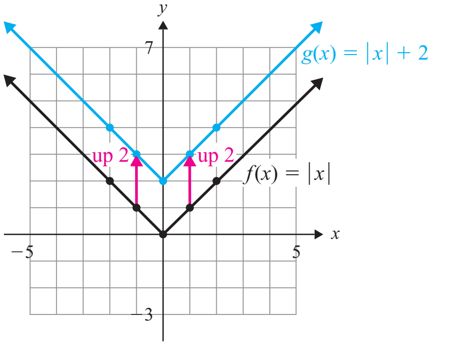
Example 1

Graph $f(x) = |x|$ and $g(x) = |x| + 2$ on the same axes. Identify the domain and range.

Solution

$f(x) = \|x\|$	
x	$f(x)$
0	0
1	1
2	2
−1	1
−2	2

$g(x) = \|x\| + 2$	
x	$g(x)$
0	2
1	3
2	4
−1	3
−2	4



The domain of $f(x) = |x|$ is $(-\infty, \infty)$. The range is $[0, \infty)$.
The domain of $g(x) = |x| + 2$ is $(-\infty, \infty)$. The range is $[2, \infty)$.

Absolute value functions like these have V-shaped graphs. We can see from the tables of values that although the x-values are the same in each table, the corresponding y-values in the table for $g(x)$ are 2 *more than* the y-values in the first table.

$$f(x) = |x| \qquad g(x) = |x| + 2$$
$$g(x) = f(x) + 2 \quad \text{Substitute } f(x) \text{ for } |x|.$$

The y-coordinates of the ordered pairs of $g(x)$ will be 2 *more than* the y-coordinates of the ordered pairs of $f(x)$ when the ordered pairs of f and g have the same x-coordinates. This means that *the graph of g will be the same shape as the graph of f but g will be shifted up* 2 *units*.

Definition

Given the graph of any function $f(x)$, if $g(x) = f(x) + k$, where k is a constant, the graph of $g(x)$ will be the same shape as the graph of $f(x)$, but g will be **shifted vertically** k units.

In Example 1, $k = 2$. $\quad f(x) = |x|$ and $g(x) = |x| + 2$
or $g(x) = f(x) + 2$

The graph of g is the same shape as the graph of f, but the graph of g is shifted *up* 2 units. We say that we can graph $g(x) = |x| + 2$ by *transforming* the graph of $f(x) = |x|$. This vertical shifting works not only for absolute value functions but for any function.

You Try 1

Graph $g(x) = |x| - 1$.

2. Illustrate Horizontal Shifts with Quadratic Functions

In the previous section, we said that a quadratic function can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. Here we begin our discussion of graphing quadratic functions.

The graph of a quadratic function is called a **parabola**. Let's look at the simplest form of a quadratic function, $f(x) = x^2$, and a variation of it. In Section 12.3, we will discuss graphing quadratic functions in much greater detail.

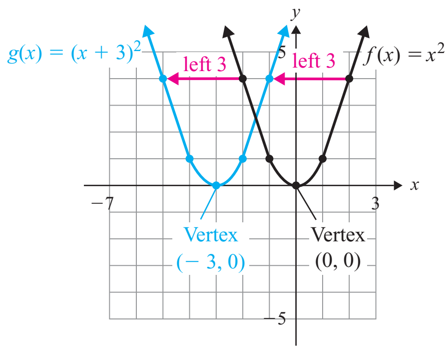
Example 2

Graph $f(x) = x^2$ and $g(x) = (x + 3)^2$ on the same axes. Identify the domain and range.

Solution

$f(x) = x^2$	
x	$f(x)$
0	0
1	1
2	4
-1	1
-2	4

$g(x) = (x + 3)^2$	
x	$g(x)$
-3	0
-2	1
-1	4
-4	1
-5	4



Notice that the graphs of $f(x)$ and $g(x)$ open upward. The lowest point on a parabola that opens upward or the highest point on a parabola that opens downward is called the **vertex**. The vertex of the graph of $f(x)$ is $(0, 0)$, and the vertex of the graph of $g(x)$ is $(-3, 0)$. When graphing a quadratic function by plotting points, it is important to locate the vertex. *The x-coordinate of the vertex is the value of x that makes the expression that is being squared equal to zero.*

The domain of $f(x) = x^2$ is $(-\infty, \infty)$. The range is $[0, \infty)$.

The domain of $g(x) = (x + 3)^2$ is $(-\infty, \infty)$. The range is $[0, \infty)$.

We can see from the tables of values that although the y-values are the same in each table, the corresponding x-values in the table for $g(x)$ are 3 *less than* the x-values in the first table.

The x-coordinates of the ordered pairs of $g(x)$ will be 3 *less than* the x-coordinates of the ordered pairs of $f(x)$ when the ordered pairs of f and g have the same y-coordinates. This means that *the graph of g will be the same shape as the graph of f but the graph of g will be shifted left* 3 *units*.

Definition

Given the graph of any function $f(x)$, if $\mathbf{g(x) = f(x - h)}$, where h is a constant, then the graph of $g(x)$ will be the same shape as the graph of $f(x)$ but the graph of g will be **shifted horizontally** h units.

In Example 2, $h = -3$. $\qquad f(x) = x^2 \quad$ and $\quad g(x) = (x + 3)^2$

or $\quad g(x) = f(x - (-3))$

The graph of g is the same shape as the graph of f but the graph of g is shifted -3 units horizontally or 3 units to the *left*. This horizontal shifting works for any function, not just quadratic functions.

You Try 2

Graph $g(x) = (x + 4)^2$.

BE CAREFUL

It is important to distinguish between the graph of an absolute value function and the graph of a quadratic function. The absolute value functions we will study have V-shaped graphs. The graph of a quadratic function is *not* shaped like a V. It is a parabola.

The next type of transformation we will discuss is reflecting the graph of a function about the x-axis.

3. Illustrate Reflecting a Graph About the x-Axis with Square Root Functions

Example 3

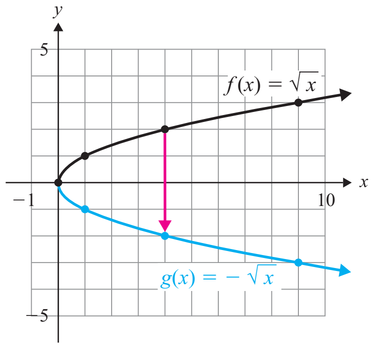
Graph $f(x) = \sqrt{x}$ and $g(x) = -\sqrt{x}$ on the same axes. Identify the domain and range.

Solution

The domain of each function is $[0, \infty)$. From the graphs, we can see that the range of $f(x) = \sqrt{x}$ is $[0, \infty)$, while the range of $g(x) = -\sqrt{x}$ is $(-\infty, 0]$.

$f(x) = \sqrt{x}$	
x	$f(x)$
0	0
1	1
4	2
9	3

$g(x) = -\sqrt{x}$	
x	$g(x)$
0	0
1	-1
4	-2
9	-3



The tables of values show us that although the x-values are the same in each table, the corresponding y-values in the table for $g(x)$ are the *negatives* of the y-values in the first table.

We say that *the graph of g is the reflection of the graph of f about the x-axis*. (g is the mirror image of f with respect to the x-axis.)

Definition

Reflection about the x-axis: Given the graph of any function $f(x)$, if $g(x) = -f(x)$, then the graph of $g(x)$ will be the **reflection of the graph of f about the x-axis**. That is, obtain the graph of g by keeping the x-coordinate of each point on the graph of f the same but take the negative of the y-coordinate.

In Example 3,

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = -\sqrt{x}$$
$$\text{or} \quad g(x) = -f(x)$$

The graph of g is the mirror image of the graph of f with respect to the x-axis. This is true for any function where $g(x) = -f(x)$.

You Try 3

Graph $g(x) = -x^2$.

We can combine the techniques used in the transformation of the graphs of functions to help us graph more complicated functions.

4. Graph a Function Using a Combination of the Transformations

Example 4

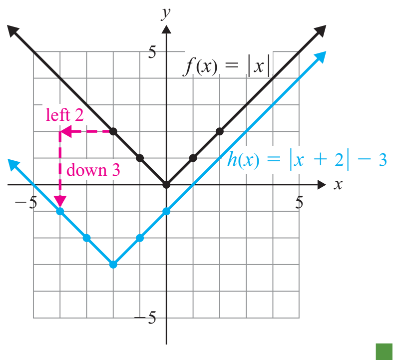
Graph $h(x) = |x + 2| - 3$.

Solution

The graph of $h(x)$ will be the same shape as the graph of $f(x) = |x|$. So, let's see what the constants in $h(x)$ tell us about transforming the graph of $f(x) = |x|$.

$$h(x) = |x \underset{\substack{\text{Shift } f(x) \\ \text{left 2.}}}{\underset{\uparrow}{+ 2}}| \underset{\substack{\text{Shift } f(x) \\ \text{down 3.}}}{\underset{\uparrow}{- 3}}$$

Sketch the graph of $f(x) = |x|$, including some key points, then *move every point on the graph of f left* 2 *and down* 3 *to obtain the graph of h.*



You Try 4

Graph $g(x) = -x^2 + 5$.

Graphs of Other Functions

5. Graph a Piecewise Function

Definition

A **piecewise function** is a single function defined by two or more different rules.

Example 5

Graph the piecewise function

$$f(x) = \begin{cases} 2x - 4, & x \geq 3 \\ -x + 2, & x < 3 \end{cases}$$

Solution

This is a piecewise function because $f(x)$ is defined by two different rules. *The rule we use to find $f(x)$ depends on what value is substituted for x.*

Graph $f(x)$ by making two separate tables of values, one for each rule.

When $x \geq 3$, use the rule

$$f(x) = 2x - 4.$$

The first x-value we will put in the table of values is 3 because it is the smallest number (lower bound) of the domain of $f(x) = 2x - 4$. *The other values we choose for x must be greater than 3 because this is when we use the rule* $f(x) = 2x - 4$. **This part of the graph will not extend to the left of (3, 2).**

$f(x) = 2x - 4$
$(x \geq 3)$

x	f(x)
3	2
4	4
5	6
6	8

When $x < 3$, use the rule

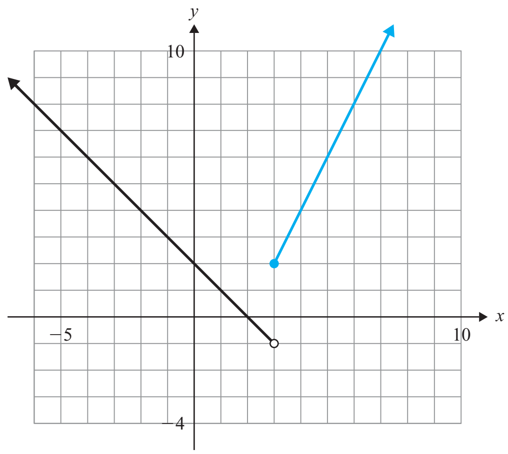
$$f(x) = -x + 2.$$

The first x-value we will put in the table of values is 3 because it is the upper bound of the domain. *Notice that 3 is not included in the domain (the inequality is $<$,* ***not*** *$\leq$) so that the point $(3, f(3))$ will be represented as an open circle on the graph.* The other values we choose for x must be less than 3 because this is when we use the rule $f(x) = -x + 2$. **This part of the graph will not extend to the right of $(3, -1)$.**

$f(x) = -x + 2$
$(x < 3)$

x	f(x)
3	-1
2	0
1	1
0	2

$(3, -1)$ will be an open circle.



You Try 5

Graph.

$$f(x) = \begin{cases} -2x + 3, & x \leq -2 \\ \frac{3}{2}x - 1, & x > -2 \end{cases}$$

6. Define the Greatest Integer Function, $f(x) = [\![x]\!]$

Another function that has many practical applications is the *greatest integer function*. Before we look at a graph and an application, we need to understand what the greatest integer function means.

Definition

The **greatest integer function, $f(x) = [\![x]\!]$,** represents the largest integer less than or equal to x.

Example 6

Let $f(x) = \llbracket x \rrbracket$. Find the following function values.

a) $f\left(9\frac{1}{2}\right)$ b) $f(6)$ c) $f(-2.3)$

Solution

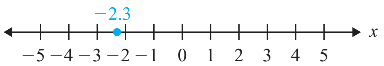
a) $f\left(9\frac{1}{2}\right) = \left\llbracket 9\frac{1}{2} \right\rrbracket$, which means to find the largest integer that is *less than or equal to* $9\frac{1}{2}$. That number is 9.

$$f\left(9\frac{1}{2}\right) = \left\llbracket 9\frac{1}{2} \right\rrbracket = 9$$

b) $f(x) = \llbracket 6 \rrbracket = 6$ since the largest integer *less than or equal to* 6 is 6.

c) $f(-2.3) = \llbracket -2.3 \rrbracket$

To help us understand how to find this function value, we will locate -2.3 on a number line.



The largest integer *less than or equal to* -2.3 is -3. $f(-2.3) = [-2.3] = -3$.

You Try 6

Let $f(x) = \llbracket x \rrbracket$. Find the following function values.

a) $f(5.1)$ b) $f(0)$ c) $f\left(-5\frac{1}{4}\right)$

Let's see what the graph of $f(x) = \llbracket x \rrbracket$ looks like.

Example 7

Graph $f(x) = \llbracket x \rrbracket$.

Solution

To understand what produces the pattern in the graph of this function, we begin by *closely* examining what occurs between $x = 0$ and $x = 1$ (when $0 \le x \le 1$).

$f(x) = \llbracket x \rrbracket$

x	f(x)
0	0
$\frac{1}{4}$	0
$\frac{1}{2}$	0
$\frac{3}{4}$	0
$\vdots$	0
1	1

For all values of x *greater than or equal to 0* and *less than 1*, the function value, $f(x)$, equals zero.

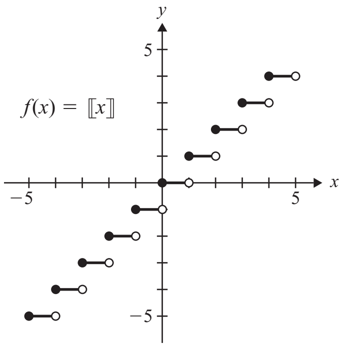
→ When $x = 1$, the function value changes to 1.

The graph will have an open circle at (1, 0) because for $x < 1, f(x) = 0$. That means that x can get *very close to* 1 and the function value will be zero, but $f(1) \neq 0$.

This pattern will continue so that for the x-values in the interval [1, 2), the function values are 1. The graph will have an open circle at (2, 1).

For the x-values in the interval [2, 3), the function values are 2. The graph will have an open circle at (3, 2).

Continuing in this way, we get the graph below.



The domain of the function is $(-\infty, \infty)$.

The range is the set of all integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Because of the appearance of the graph, $f(x) = [\![x]\!]$ is also called a **step function**.

You Try 7

Graph $f(x) = [\![x]\!] - 3$.

7. Represent an Applied Problem with the Graph of a Greatest Integer Function

Example 8

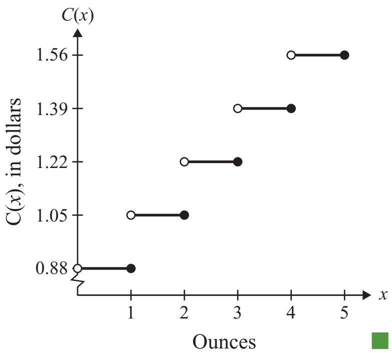
To mail a large envelope within the United States in 2010, the U.S. Postal Service charged \$0.88 for the first ounce and \$0.17 for each additional ounce or fraction of an ounce. Let $C(x)$ represent the cost of mailing a large envelope within the United States, and let x represent the weight of the envelope, in ounces. Graph $C(x)$ for any large envelope weighing up to (and including) 5 ounces. (www.usps.com)

Solution

If a large envelope weighs between 0 and 1 ounce $(0 < x \leq 1)$ the cost, $C(x)$, is \$0.88.

If a large envelope weighs more than 1 oz but less than or equal to 2 oz $(1 < x \leq 2)$, the cost, $C(x)$, is \$0.88 + \$0.17 = \$1.05.

The pattern will continue, and we get the graph at the right.



You Try 8

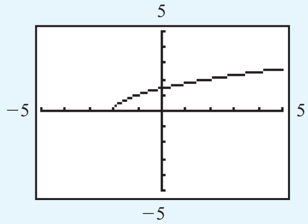
To mail a package within the United States at *book rate* in 2005, the U.S. Postal Service charged \$1.59 for the first pound and \$0.48 for each additional pound or fraction of a pound. Let $C(x)$ represent the cost of mailing a package at book rate and let x represent the weight of the package, in pounds. Graph $C(x)$ for any package weighing up to (and including) 5 pounds. (www.usps.com)

Using Technology

The graphing calculator screens show the graphs of quadratic functions. Match each equation to its graph. Then identify the domain and range of each function.

1) $f(x) = (x - 2)^2$
2) $f(x) = |x| - 2$
3) $f(x) = -x^2$
4) $f(x) = \sqrt{x + 2}$
5) $f(x) = -|x| + 2$
6) $f(x) = (x + 2)^2 - 3$

a)



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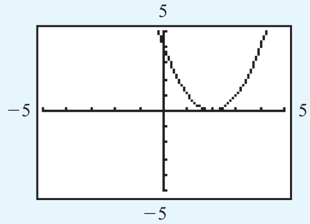
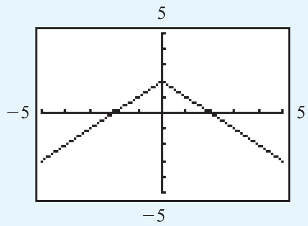
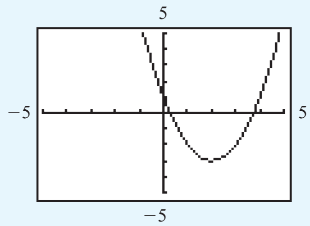


c)



d)



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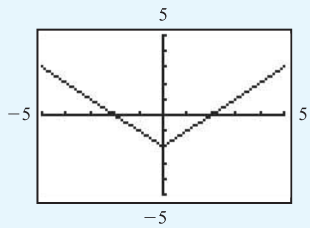

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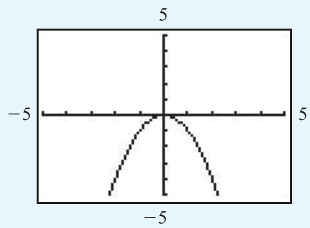
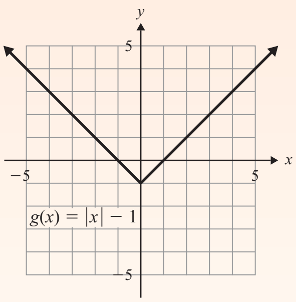


Answers to You Try Exercises

1)



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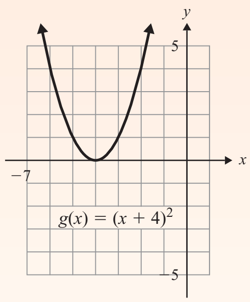

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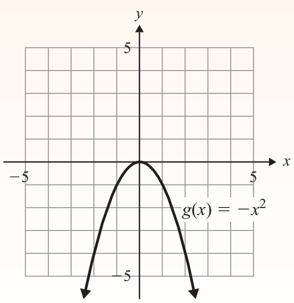
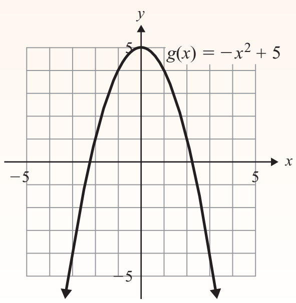
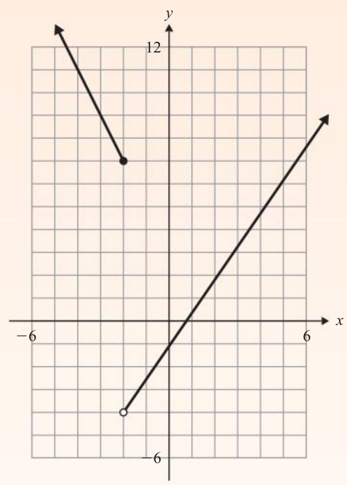


4)

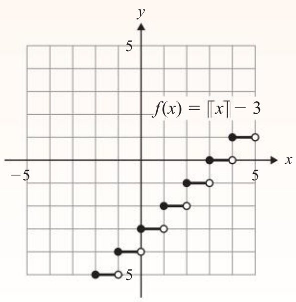


5)

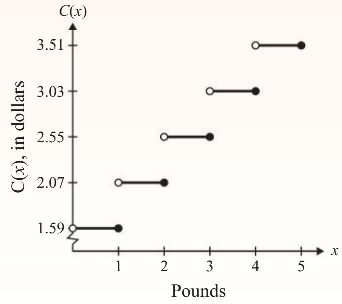


6) a) 5 b) 0 c) −6

7)



8)



Answers to Technology Exercises

1) b); domain $(-\infty, \infty)$; range $[0, \infty)$

2) e); domain $(-\infty, \infty)$; range $[-2, \infty)$

3) f); domain $(-\infty, \infty)$; range $(-\infty, 0]$

4) a); domain $[-2, \infty)$; range $[0, \infty)$

5) c); domain $(-\infty, \infty)$; range $(-\infty, 2]$

6) d); domain $(-\infty, \infty)$; range $[-3, \infty)$

12.2 Exercises

Mixed Exercises: Objectives 1–4

Graph each function by plotting points, and identify the domain and range.

1) $f(x) = |x| + 3$

2) $g(x) = |x - 2|$

VIDEO 3) $k(x) = \frac{1}{2}|x|$

4) $g(x) = 2|x|$

5) $g(x) = x^2 - 4$

6) $h(x) = (x - 2)^2$

7) $f(x) = -x^2 - 1$

8) $f(x) = (x - 2)^2 - 5$

9) $f(x) = \sqrt{x + 3}$

10) $g(x) = \sqrt{x} + 2$

11) $f(x) = 2\sqrt{x}$

12) $h(x) = -\frac{1}{2}\sqrt{x}$

Given the following pairs of functions, explain how the graph of $g(x)$ can be obtained from the graph of $f(x)$ using the transformation techniques discussed in this section.

13) $f(x) = |x|, g(x) = |x| - 2$

14) $f(x) = |x|, g(x) = |x| + 1$

15) $f(x) = x^2, g(x) = (x + 2)^2$

16) $f(x) = x^2, g(x) = (x - 3)^2$

17) $f(x) = x^2, g(x) = -x^2$

18) $f(x) = \sqrt{x}, g(x) = -\sqrt{x}$

Sketch the graph of $f(x)$. Then, graph $g(x)$ on the same axes using the transformation techniques discussed in this section.

19) $f(x) = |x|$
$g(x) = |x| - 2$

20) $f(x) = |x|$
$g(x) = |x| + 1$

21) $f(x) = |x|$
$g(x) = |x| + 3$

22) $f(x) = |x|$
$g(x) = |x| - 4$

VIDEO 23) $f(x) = x^2$
$g(x) = (x + 2)^2$

24) $f(x) = x^2$
$g(x) = (x - 3)^2$

25) $f(x) = x^2$
$g(x) = (x - 4)^2$

26) $f(x) = x^2$
$g(x) = (x + 1)^2$

27) $f(x) = x^2$
$g(x) = -x^2$

28) $f(x) = \sqrt{x}$
$g(x) = -\sqrt{x}$

29) $f(x) = \sqrt{x + 1}$
$g(x) = -\sqrt{x + 1}$

30) $f(x) = \sqrt{x - 2}$
$g(x) = -\sqrt{x - 2}$

31) $f(x) = |x - 3|$
$g(x) = -|x - 3|$

32) $f(x) = |x + 4|$
$g(x) = -|x + 4|$

Use the transformation techniques discussed in this section to graph each of the following functions.

33) $f(x) = |x| - 5$

34) $f(x) = \sqrt{x} + 3$

35) $y = \sqrt{x - 4}$

36) $y = (x - 2)^2$

VIDEO 37) $g(x) = |x + 2| + 3$

38) $h(x) = |x + 1| - 5$

39) $y = (x - 3)^2 + 1$

40) $f(x) = (x + 2)^2 - 3$

41) $f(x) = \sqrt{x + 4} - 2$

42) $y = \sqrt{x - 3} + 2$

43) $h(x) = -x^2 + 6$

44) $y = -(x - 1)^2$

45) $g(x) = -|x - 1| + 3$

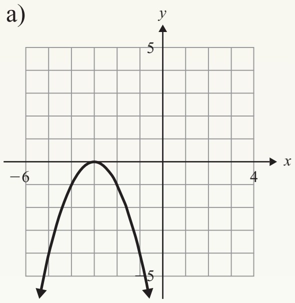
46) $h(x) = -|x + 3| - 2$

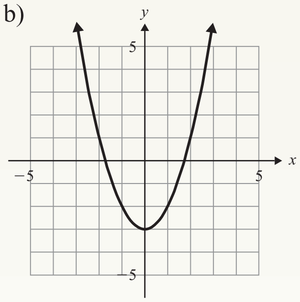
47) $f(x) = -\sqrt{x + 5}$

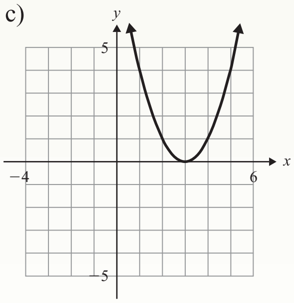
48) $y = -\sqrt{x + 2}$

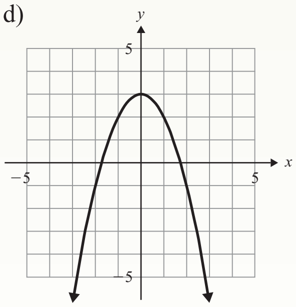
Match each function to its graph.

49) $f(x) = x^2 - 3, g(x) = (x - 3)^2,$
$h(x) = -(x + 3)^2, k(x) = -x^2 + 3$

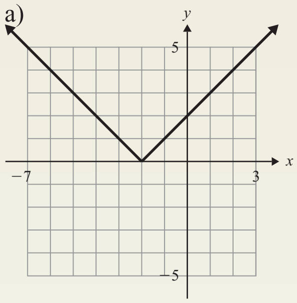
a)


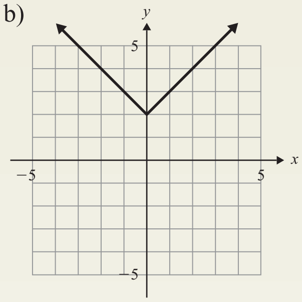
b)


c)


d)


50) $f(x) = -|x - 2|, g(x) = |x + 2|,$
$h(x) = -|x| - 2, k(x) = |x| + 2$

a)


b)


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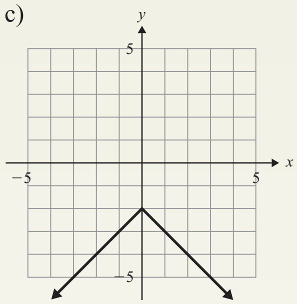
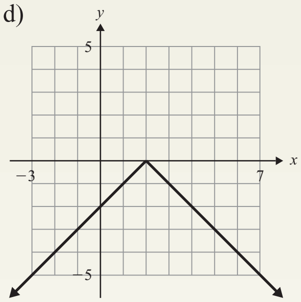


d)


If the following transformations are performed on the graph of $f(x)$ to obtain the graph of $g(x)$, write the equation of $g(x)$.

51) $f(x) = \sqrt{x}$ is shifted 5 units to the left.

52) $f(x) = \sqrt{x}$ is shifted down 6 units.

53) $f(x) = |x|$ is shifted left 2 units and down 1 unit.

54) $f(x) = |x|$ is shifted right 1 unit and up 4 units.

55) $f(x) = x^2$ is shifted left 3 units and up $\frac{1}{2}$ unit.

56) $f(x) = x^2$ is shifted right 5 units and down 1.5 units.

57) $f(x) = x^2$ is reflected about the x-axis.

58) $f(x) = |x|$ is reflected about the x-axis.

59) Graph $f(x) = x^3$ by plotting points. (Hint: Make a table of values and choose 0, positive, and negative numbers for x.) Then, use the transformation techniques discussed in this section to graph each of the following functions.

a) $g(x) = (x + 2)^3$
b) $h(x) = x^3 - 3$
c) $k(x) = -x^3$
d) $r(x) = (x - 1)^3 - 2$

60) Graph $f(x) = \sqrt[3]{x}$ by plotting points. (Hint: Make a table of values and choose 0, positive, and negative numbers for x.) Then, use the transformation techniques discussed in this section to graph each of the following functions.

a) $g(x) = \sqrt[3]{x} + 4$
b) $h(x) = -\sqrt[3]{x}$
c) $k(x) = \sqrt[3]{x - 2}$
d) $r(x) = -\sqrt[3]{x} - 3$

Objective 5: Graph a Piecewise Function

Graph the following piecewise functions.

61) $f(x) = \begin{cases} -x - 3, & x \le -1 \\ 2x + 2, & x > -1 \end{cases}$

62) $g(x) = \begin{cases} x - 1, & x \ge 2 \\ -3x + 3, & x < 2 \end{cases}$

63) $h(x) = \begin{cases} -x + 5, & x \ge 3 \\ \frac{1}{2}x + 1, & x < 3 \end{cases}$

64) $f(x) = \begin{cases} 2x + 13, & x \le -4 \\ -\frac{1}{2}x + 1, & x > -4 \end{cases}$

65) $g(x) = \begin{cases} -\frac{3}{2}x - 3, & x < 0 \\ 1, & x \ge 0 \end{cases}$

66) $h(x) = \begin{cases} -\frac{2}{3}x - \frac{7}{3}, & x \ge -1 \\ 2, & x < -1 \end{cases}$

67) $k(x) = \begin{cases} x + 1, & x \ge -2 \\ 2x + 8, & x < -2 \end{cases}$

68) $g(x) = \begin{cases} x, & x \le 0 \\ 2x + 3, & x > 0 \end{cases}$

69) $f(x) = \begin{cases} 2x - 4, & x > 1 \\ -\frac{1}{3}x - \frac{5}{3}, & x \le 1 \end{cases}$

70) $k(x) = \begin{cases} \frac{1}{2}x + \frac{5}{2}, & x < 3 \\ -x + 7, & x \ge 3 \end{cases}$

Objective 6: Define the Greatest Integer Function, $f(x) = \llbracket x \rrbracket$

Let $f(x) = \llbracket x \rrbracket$. Find the following function values.

71) $f\left(3\frac{1}{4}\right)$

72) $f\left(10\frac{3}{8}\right)$

73) $f(9.2)$

74) $f(7.8)$

75) $f(8)$

76) $f\left(\frac{4}{5}\right)$

77) $f\left(-6\frac{2}{5}\right)$

78) $f\left(-1\frac{3}{4}\right)$

79) $f(-8.1)$

80) $f(-3.6)$

Graph the following greatest integer functions.

81) $f(x) = \llbracket x \rrbracket + 1$

82) $g(x) = \llbracket x \rrbracket - 2$

83) $h(x) = \llbracket x \rrbracket - 4$

84) $k(x) = \llbracket x \rrbracket + 3$

85) $g(x) = \llbracket x + 2 \rrbracket$

86) $h(x) = \llbracket x - 1 \rrbracket$

87) $k(x) = \left\llbracket \frac{1}{2}x \right\rrbracket$

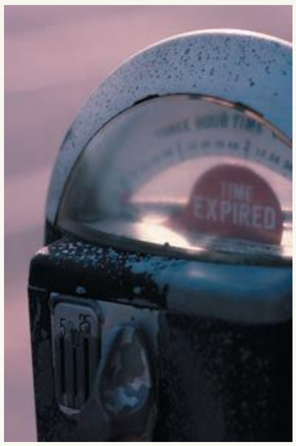
88) $f(x) = \llbracket 2x \rrbracket$

Objective 7: Represent an Applied Problem with the Graph of a Greatest Integer Function

89) To ship small packages within the United States, a shipping company charges \$3.75 for the first pound and \$1.10 for each additional pound or fraction of a pound. Let $C(x)$ represent the cost of shipping a package, and let x represent the weight of the package. Graph $C(x)$ for any package weighing up to (and including) 6 lb.

90) To deliver small packages overnight, an express delivery service charges \$15.40 for the first pound and \$4.50 for each additional pound or fraction of a pound. Let $C(x)$ represent the cost of shipping a package overnight, and let x represent the weight of the package. Graph $C(x)$ for any package weighing up to (and including) 6 lb.

91) Visitors to downtown Hinsdale must pay the parking meters to park their cars. The cost of parking is 5¢ for the first 12 min and 5¢ for each additional 12 min or fraction of this time. Let $P(t)$ represent the cost of parking, and let t represent the number of minutes the car is parked at the meter. Graph $P(t)$ for parking a car for up to (and including) 1 hr.



92) To consult with an attorney costs \$35 for every 10 min or fraction of this time. Let $C(t)$ represent the cost of meeting an attorney, and let t represent the length of the meeting, in minutes. Graph $C(t)$ for meeting with the attorney for up to (and including) 1 hr.

Section 12.3 Quadratic Functions and Their Graphs

Objectives

1. Graph a Quadratic Function by Shifting the Graph of $f(x) = x^2$
2. Graph $f(x) = a(x - h)^2 + k$ Using the Vertex, Axis of Symmetry, and Other Characteristics
3. Graph $f(x) = ax^2 + bx + c$ by Completing the Square
4. Graph $f(x) = ax^2 + bx + c$ Using $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

In this section, we will study quadratic functions in more detail and see how the rules we learned in Section 12.2 apply specifically to these functions. We restate the definition of a quadratic function here.

Definition

A **quadratic function** is a function that can be written in the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are real numbers and $a \neq 0$. An example is $f(x) = x^2 + 6x + 10$. The graph of a quadratic function is called a **parabola**. The lowest point on a parabola that opens upward or the highest point on a parabola that opens downward is called the **vertex**.

Quadratic functions can be written in other forms as well. One common form is $f(x) = a(x - h)^2 + k$. An example is $f(x) = 2(x - 3)^2 + 1$.

We will study the form $f(x) = a(x - h)^2 + k$ first since graphing parabolas from this form comes directly from the transformation techniques we learned earlier.

1. Graph a Quadratic Function by Shifting the Graph of $f(x) = x^2$

Example 1

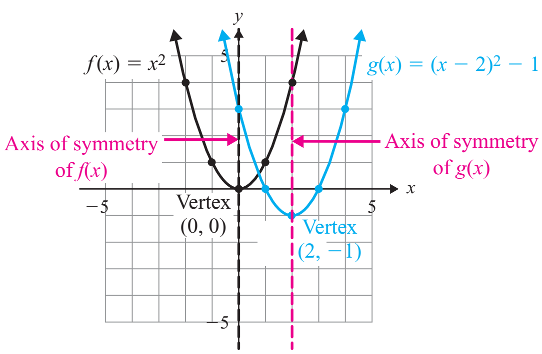
Graph $g(x) = (x - 2)^2 - 1$.

Solution

If we compare $g(x)$ to $f(x) = x^2$, what do the constants that have been added to $g(x)$ tell us about transforming the graph of $f(x)$?

$$g(x) = (\underbrace{x - 2}_{\text{Shift } f(x) \text{ right 2.}})^2 \underbrace{- 1}_{\text{Shift } f(x) \text{ down 1.}}$$

Sketch the graph of $f(x) = x^2$, then move every point on the graph of f right 2 and down 1 to obtain the graph of $g(x)$. This moves the vertex from (0, 0) to (2, −1).



Every parabola has symmetry. If we were to fold the paper along the y-axis, one half of the graph of $f(x) = x^2$ would fall exactly on the other half. The y-axis (the line $x = 0$) is the **axis of symmetry** of $f(x) = x^2$.

Look at the graph of $g(x) = (x - 2)^2 - 1$ in Example 1. This parabola is symmetric with respect to the vertical line $x = 2$ through its vertex (2, −1). If we were to fold the paper along the line $x = 2$, half of the graph of $g(x)$ would fall exactly on the other half. The line $x = 2$ is the *axis of symmetry* of $g(x) = (x - 2)^2 - 1$.

2. Graph $f(x) = a(x - h)^2 + k$ Using the Vertex, Axis of Symmetry, and Other Characteristics

When a quadratic function is in the form $f(x) = a(x - h)^2 + k$, we can read the vertex directly from the equation. Furthermore, the value of a tells us whether the parabola opens upward or downward and whether the graph is narrower, wider, or the same width as $y = x^2$.

Procedure Graphing a Quadratic Function of the Form $f(x) = a(x - h)^2 + k$

1) The vertex of the parabola is (h, k).
2) The axis of symmetry is the vertical line with equation $x = h$.
3) If a is positive, the parabola opens upward.
   If a is negative, the parabola opens downward.
4) If $|a| < 1$, then the graph of $f(x) = a(x - h)^2 + k$ is *wider* than the graph of $y = x^2$.
   If $|a| > 1$, then the graph of $f(x) = a(x - h)^2 + k$ is *narrower* than the graph of $y = x^2$.
   If $a = 1$ or $a = -1$, the graph is the *same* width as $y = x^2$.

Example 2

Graph $y = \frac{1}{2}(x + 3)^2 - 2$. Also find the x- and y-intercepts.

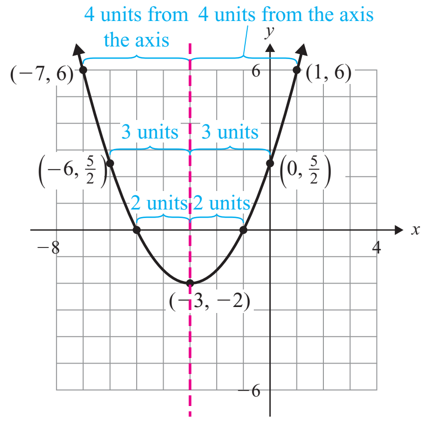
Solution

Here is the information we can get from the equation.

1) $h = -3$ and $k = -2$. The vertex is $(-3, -2)$.
2) The axis of symmetry is $x = -3$.
3) $a = +\frac{1}{2}$. Since a is positive, the parabola opens upward.
4) Since $a = \frac{1}{2}$ and $\frac{1}{2} < 1$, the graph of $y = \frac{1}{2}(x + 3)^2 - 2$ is *wider* than the graph of $y = x^2$.

To graph the function, start by putting the vertex on the axes. Then, choose a couple of values of x to the left or right of the vertex to plot more points. Use the axis of symmetry to find more points on the graph of $y = \frac{1}{2}(x + 3)^2 - 2$.

x	y
−1	0
1	6



We can read the x-intercepts from the graph: $(-5, 0)$ and $(-1, 0)$. To find the y-intercept, let $x = 0$ and solve for y.

$$y = \frac{1}{2}(x + 3)^2 - 2$$
$$y = \frac{1}{2}(0 + 3)^2 - 2$$
$$y = \frac{1}{2}(9) - 2 = \frac{9}{2} - 2 = \frac{9}{2} - \frac{4}{2} = \frac{5}{2}$$

The y-intercept is $\left(0, \frac{5}{2}\right)$.

You Try 1

Graph $y = 2(x - 1)^2 - 2$. Also find the x- and y-intercepts.

Example 3

Graph $f(x) = -(x - 1)^2 + 5$. Find the x- and y-intercepts.

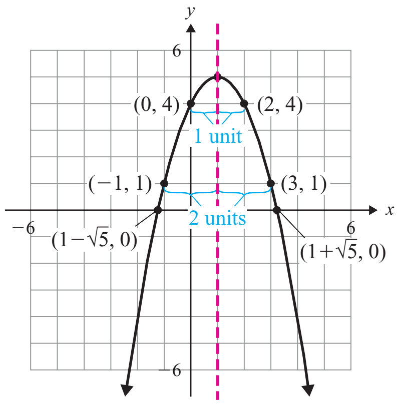
Solution

Here is the information we can get from the equation.

1) $h = 1$ and $k = 5$. The vertex is $(1, 5)$.
2) The axis of symmetry is $x = 1$.
3) $a = -1$. Since a is negative, the parabola opens downward.
4) Since $a = -1$, the graph of $f(x) = -(x - 1)^2 + 5$ is the *same* width as $y = x^2$.

Put the vertex on the axes. Choose a couple of x-values to the left or right of the vertex to plot more points. Use the axis of symmetry to find more points on the graph of $f(x) = -(x - 1)^2 + 5$.

x	y
2	4
3	1



We can read the y-intercept from the graph: $(0, 4)$.

To find the x-intercepts, let $f(x) = 0$ and solve for x.

$$\begin{aligned} f(x) &= -(x - 1)^2 + 5 & \\ 0 &= -(x - 1)^2 + 5 & \text{Substitute 0 for } f(x). \\ -5 &= -(x - 1)^2 & \text{Subtract 5.} \\ 5 &= (x - 1)^2 & \text{Divide by } -1. \\ \pm\sqrt{5} &= x - 1 & \text{Square root property} \\ 1 \pm \sqrt{5} &= x & \text{Add 1.} \end{aligned}$$

The x-intercepts are $(1 + \sqrt{5}, 0)$ and $(1 - \sqrt{5}, 0)$.

You Try 2

Graph $f(x) = -(x + 3)^2 + 2$. Find the x- and y-intercepts.

Procedure Graphing Parabolas from the Form $f(x) = ax^2 + bx + c$

When a quadratic function is written in the form $f(x) = ax^2 + bx + c$, there are two methods we can use to graph the function.

Method 1: Rewrite $f(x) = ax^2 + bx + c$ in the form $f(x) = a(x - h)^2 + k$ by *completing the square.*

Method 2: Use the formula $x = -\frac{b}{2a}$ to find the x-coordinate of the vertex. Then, the vertex has coordinates $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

We will begin with Method 1. We will modify the steps we used in Section 11.2 to solve quadratic equations by completing the square.

3. Graph $f(x) = ax^2 + bx + c$ by Completing the Square

Procedure Rewriting $f(x) = ax^2 + bx + c$ in the Form $f(x) = a(x - h)^2 + k$ by Completing the Square

Step 1: **The coefficient of the square term must be 1.** If it is not 1, multiply or divide both sides of the equation (*including* $f(x)$) by the appropriate value to obtain a leading coefficient of 1.

Step 2: **Separate the constant from the terms containing the variables by grouping the variable terms with parentheses.**

Step 3: **Complete the square for the quantity in the parentheses.** Find half of the linear coefficient, then square the result. *Add* that quantity inside the parentheses and *subtract* the quantity from the constant. (Adding and subtracting the same number on the same side of an equation is like adding 0 to the equation.)

Step 4: **Factor the expression inside the parentheses.**

Step 5: **Solve for $f(x)$.**

Example 4

Graph each function. Begin by completing the square to rewrite each function in the form $f(x) = a(x - h)^2 + k$. Include the intercepts.

a) $f(x) = x^2 + 6x + 10$ b) $g(x) = -\frac{1}{2}x^2 + 4x - 6$

Solution

a) ***Step 1:*** The coefficient of x^2 is 1.

Step 2: Separate the constant from the variable terms using parentheses.

$$f(x) = (x^2 + 6x) + 10$$

Step 3: Complete the square for the quantity in the parentheses.

$$\frac{1}{2}(6) = 3$$
$$3^2 = 9$$

Add 9 inside the parentheses and subtract 9 from the 10. This is like adding 0 to the equation.

$$f(x) = (x^2 + 6x + 9) + 10 - 9$$
$$f(x) = (x^2 + 6x + 9) + 1$$

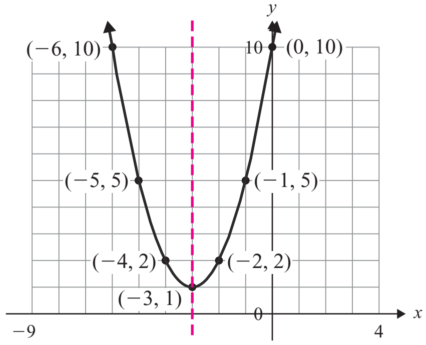
Step 4: Factor the expression inside the parentheses.

$$f(x) = (x + 3)^2 + 1$$

Step 5: The equation *is* solved for $f(x)$.

From the equation $f(x) = (x + 3)^2 + 1$ we can see that

i) The vertex is $(-3, 1)$.
ii) The axis of symmetry is $x = -3$.
iii) $a = 1$ so the parabola opens upward.
iv) Since $a = 1$, the graph is the same width as $y = x^2$.



Find some other points on the parabola. Use the axis of symmetry.

x	$f(x)$
-2	2
-1	5

To find the x-intercepts, let $f(x) = 0$ and solve for x. Use *either* form of the equation. We will use $f(x) = (x + 3)^2 + 1$.

$$0 = (x + 3)^2 + 1 \quad \text{Let } f(x) = 0.$$
$$-1 = (x + 3)^2 \quad \text{Subtract 1.}$$
$$\pm\sqrt{-1} = x + 3 \quad \text{Square root property}$$
$$-3 \pm i = x \quad \sqrt{-1} = i\text{; subtract 3.}$$

Since the solutions to $f(x) = 0$ are *not* real numbers, *there are no x-intercepts*.

To find the y-intercept, let $x = 0$ and solve for $f(0)$.

$$f(x) = (x + 3)^2 + 1$$
$$f(0) = (0 + 3)^2 + 1$$
$$f(0) = 9 + 1 = 10$$

The y-intercept is $(0, 10)$.

b) ***Step 1:*** The coefficient of x^2 is $-\frac{1}{2}$. Multiply both sides of the equation [including the $g(x)$] by -2 so that the coefficient of x^2 will be 1.

$$g(x) = -\frac{1}{2}x^2 + 4x - 6$$
$$-2g(x) = -2\left(-\frac{1}{2}x^2 + 4x - 6\right) \quad \text{Multiply by } -2.$$
$$-2g(x) = x^2 - 8x + 12 \quad \text{Distribute.}$$

Step 2: Separate the constant from the variable terms using parentheses.

$$-2g(x) = (x^2 - 8x) + 12$$

Step 3: Complete the square for the quantity in parentheses.

$$\frac{1}{2}(-8) = -4$$
$$(-4)^2 = 16$$

Add 16 inside the parentheses and subtract 16 from the 12.

$$-2g(x) = (x^2 - 8x + 16) + 12 - 16$$
$$-2g(x) = (x^2 - 8x + 16) - 4$$

Step 4: Factor the expression inside the parentheses.

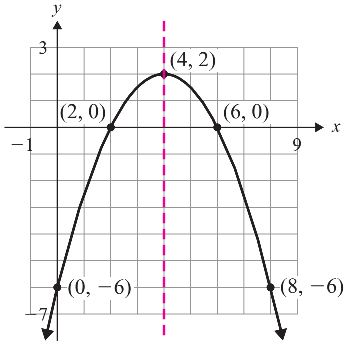
$$-2g(x) = (x - 4)^2 - 4$$

Step 5: Solve the equation for $g(x)$ by dividing by -2.

$$\frac{-2g(x)}{-2} = \frac{(x-4)^2}{-2} - \frac{4}{-2}$$
$$g(x) = -\frac{1}{2}(x-4)^2 + 2$$

From $g(x) = -\frac{1}{2}(x-4)^2 + 2$ we can see that

i) The vertex is (4, 2).
ii) The axis of symmetry is $x = 4$.
iii) $a = -\frac{1}{2}$ [the same as in the form $g(x) = -\frac{1}{2}x^2 + 4x - 6$] so the parabola opens downward.
iv) Since $a = -\frac{1}{2}$, the graph of $g(x)$ will be *wider* than $y = x^2$.



x	$g(x)$
6	0
8	−6

Find some other points on the parabola.
Use the axis of symmetry.

Using the axis of symmetry, we can see that the x-intercepts are (6, 0) and (2, 0) and that the y-intercept is $(0, -6)$.

You Try 3

Graph each function. Begin by completing the square to rewrite each function in the form $f(x) = a(x - h)^2 + k$. Include the intercepts.

a) $f(x) = x^2 + 4x + 3$ b) $g(x) = -2x^2 + 12x - 8$

4. Graph $f(x) = ax^2 + bx + c$ Using $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

We can also graph quadratic functions of the form $f(x) = ax^2 + bx + c$ by using $h = -\frac{b}{2a}$ to find the x-coordinate of the vertex. The formula comes from completing the square on $f(x) = ax^2 + bx + c$.

Although there is a formula for k, it is only necessary to remember the formula for h. The y-coordinate of the vertex, then, is $k = f\left(-\frac{b}{2a}\right)$. The axis of symmetry is $x = h$.

Example 5

Graph $f(x) = x^2 - 6x + 3$ using the vertex formula. Include the intercepts.

Solution

$a = 1, b = -6, c = 3$. Since $a = +1$, the graph opens upward. The x-coordinate, h, of the vertex is

$$h = -\frac{b}{2a} = -\frac{(-6)}{2(1)} = \frac{6}{2} = 3$$

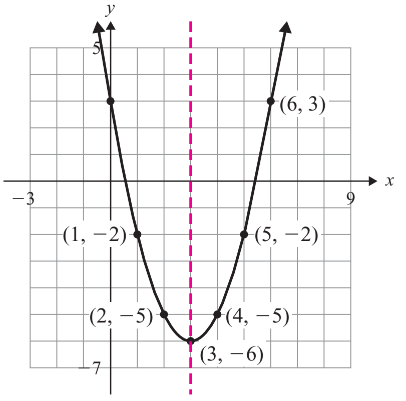
$h = 3$. Then the y-coordinate, k, of the vertex is $k = f(3)$.

$$\begin{aligned} f(x) &= x^2 - 6x + 3 \\ f(3) &= 3^2 - 6(3) + 3 \\ &= 9 - 18 + 3 = -6 \end{aligned}$$

The vertex is $(3, -6)$. The axis of symmetry is $x = 3$.

Find more points on the graph of $f(x) = x^2 - 6x + 3$, then use the axis of symmetry to find other points on the parabola.

x	$f(x)$
4	-5
5	-2
6	3



To find the x-intercepts, let $f(x) = 0$ and solve for x.

$$0 = x^2 - 6x + 3$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(3)}}{2(1)} \quad \text{Solve using the quadratic formula.}$$

$$x = \frac{6 \pm \sqrt{24}}{2} = \frac{6 \pm 2\sqrt{6}}{2} \quad \text{Simplify.}$$

$$x = 3 \pm \sqrt{6}$$

The x-intercepts are $(3 + \sqrt{6}, 0)$ and $(3 - \sqrt{6}, 0)$. We can see from the graph that the y-intercept is $(0, 3)$.

You Try 4

Graph $f(x) = -x^2 - 8x - 13$ using the vertex formula. Include the intercepts.

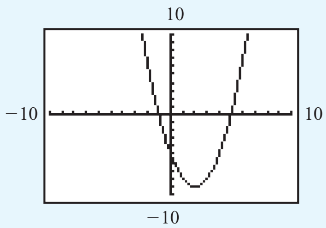
Using Technology

In Section 7.5, we said that the solutions of the equation $x^2 - x - 6 = 0$ are the x-intercepts of the graph of $y = x^2 - x - 6$. The x-intercepts are also called the *zeros* of the function since they are the values of x that make $y = 0$.

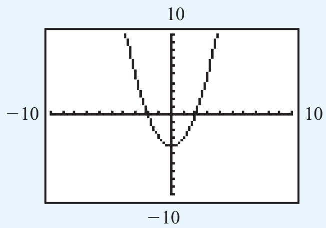
Use the zeros of each function and its transformation from $y = x^2$ to match each equation with its graph:

1) $f(x) = x^2 - 2x + 2$
2) $f(x) = x^2 - 4x - 5$
3) $f(x) = -(x + 1)^2 + 4$
4) $f(x) = x^2 - 4$
5) $f(x) = x^2 - 6x + 9$
6) $f(x) = -x^2 - 8x - 19$

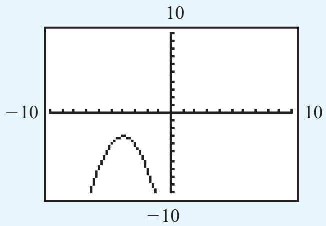
a)



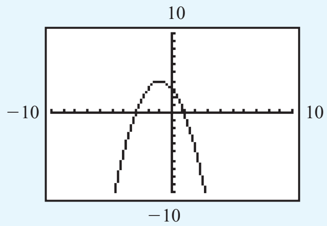
b)



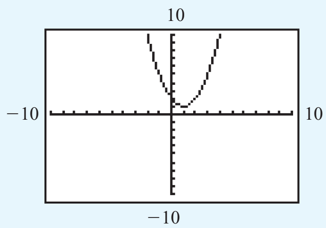
c)



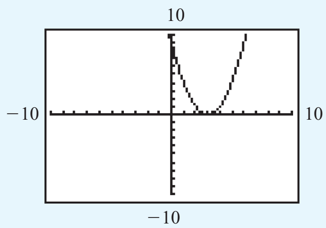
d)



e)

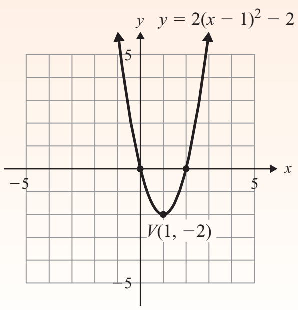


f)

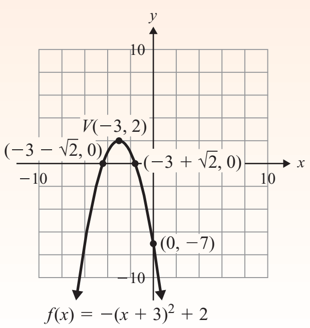


Answers to You Try Exercises

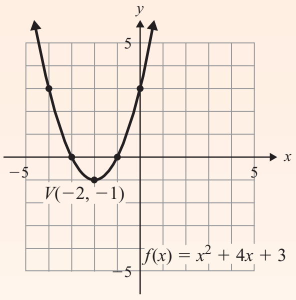
1) x-ints: (0, 0), (2, 0); y-int: (0, 0)



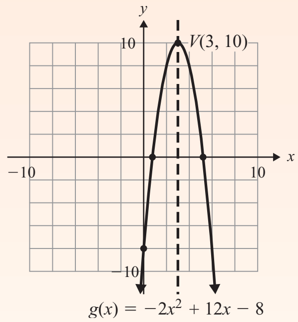
2) x-ints: $(-3 + \sqrt{2}, 0)$, $(-3 - \sqrt{2}, 0)$; y-int: $(0, -7)$



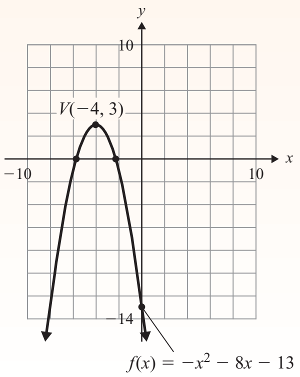
3) a) x-ints: $(-3, 0)$, $(-1, 0)$; y-int: $(0, 3)$



b) x-ints: $(3 + \sqrt{5}, 0)$, $(3 - \sqrt{5}, 0)$; y-int: $(0, -8)$



4) x-ints: $(-4 + \sqrt{3}, 0)$, $(-4 - \sqrt{3}, 0)$; y-int: $(0, -13)$



Answers to Technology Exercises

1) e) 2) a) 3) d) 4) b) 5) f) 6) c)

12.3 Exercises

Mixed Exercises: Objectives 1 and 2

Given a quadratic function of the form $f(x) = a(x - h)^2 + k$, answer the following.

1) What is the vertex?

2) What is the equation of the axis of symmetry?

3) How do you know whether the parabola opens upward?

4) How do you know whether the parabola opens downward?

5) How do you know whether the parabola is narrower than the graph of $y = x^2$?

6) How do you know whether the parabola is wider than the graph of $y = x^2$?

For each quadratic function, identify the vertex, axis of symmetry, and x- and y-intercepts. Then graph the function.

VIDEO 7) $f(x) = (x + 1)^2 - 4$

8) $g(x) = (x - 3)^2 - 1$

9) $g(x) = (x - 2)^2 + 3$

10) $h(x) = (x + 2)^2 + 7$

11) $y = (x - 4)^2 - 2$

12) $y = (x + 1)^2 - 5$

VIDEO 13) $f(x) = -(x + 3)^2 + 6$

14) $g(x) = -(x - 3)^2 + 2$

15) $y = -(x + 1)^2 - 5$

16) $f(x) = -(x - 2)^2 - 4$

17) $f(x) = 2(x - 1)^2 - 8$

18) $y = 2(x + 1)^2 - 2$

19) $h(x) = \frac{1}{2}(x + 4)^2$

20) $g(x) = \frac{1}{4}x^2 - 1$

21) $y = -x^2 + 5$

22) $h(x) = -(x - 3)^2$

23) $f(x) = -\frac{1}{3}(x + 4)^2 + 3$

24) $y = -\frac{1}{2}(x - 4)^2 + 2$

25) $g(x) = 3(x + 2)^2 + 5$

26) $f(x) = 2(x - 3)^2 + 3$

Objective 3: Graph $f(x) = ax^2 + bx + c$ by Completing the Square

Rewrite each function in the form $f(x) = a(x - h)^2 + k$.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

27) $f(x) = x^2 + 8x + 11$
______ Group the variable terms together using parentheses.
______ Find the number that completes the square in the parentheses.
$f(x) = (x^2 + 8x + 16) + 11 - 16$ ______
______ Factor and simplify.

28) $f(x) = x^2 - 4x - 7$ ______
$f(x) = (x^2 - 4x) - 7$ ______
______ Find the number that completes the square in the parentheses.
______ Add and subtract the number above to the same side of the equation.
$f(x) = (x - 2)^2 - 11$ ______

Rewrite each function in the form $f(x) = a(x - h)^2 + k$ by completing the square. Then graph the function. Include the intercepts.

VIDEO 29) $f(x) = x^2 - 2x - 3$

30) $g(x) = x^2 + 6x + 8$

31) $y = x^2 + 6x + 7$

32) $h(x) = x^2 - 4x + 1$

33) $g(x) = x^2 + 4x$

34) $y = x^2 - 8x + 18$

35) $h(x) = -x^2 - 4x + 5$

36) $f(x) = -x^2 - 2x + 3$

VIDEO 37) $y = -x^2 + 6x - 10$

38) $g(x) = -x^2 - 4x - 6$

39) $f(x) = 2x^2 - 8x + 4$

40) $y = 2x^2 - 8x + 2$

41) $g(x) = -\frac{1}{3}x^2 - 2x - 9$

42) $h(x) = -\frac{1}{2}x^2 - 3x - \frac{19}{2}$

43) $y = x^2 - 3x + 2$

44) $f(x) = x^2 + 5x + \frac{21}{4}$

Objective 4: Graph $f(x) = ax^2 + bx + c$ Using $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

Graph each function using the vertex formula. Include the intercepts.

45) $y = x^2 + 2x - 3$

46) $g(x) = x^2 - 6x + 8$

47) $f(x) = -x^2 - 8x - 13$

48) $y = -x^2 + 2x + 2$

VIDEO 49) $g(x) = 2x^2 - 4x + 4$

50) $f(x) = -4x^2 - 8x - 6$

51) $y = -3x^2 + 6x + 1$

52) $h(x) = 2x^2 - 12x + 9$

53) $f(x) = \frac{1}{2}x^2 - 4x + 5$

54) $y = \frac{1}{2}x^2 + 2x - 3$

55) $h(x) = -\frac{1}{3}x^2 - 2x - 5$

56) $g(x) = \frac{1}{5}x^2 - 2x + 8$

Section 12.4 Applications of Quadratic Functions and Graphing Other Parabolas

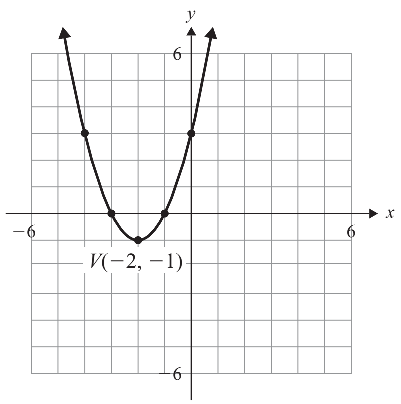
Objectives

1. Find the Maximum or Minimum Value of a Quadratic Function
2. Given a Quadratic Function, Solve an Applied Problem Involving a Maximum or Minimum Value
3. Write a Quadratic Function to Solve an Applied Problem Involving a Maximum or Minimum Value
4. Graph Parabolas of the Form $x = a(y - k)^2 + h$
5. Rewrite $x = ay^2 + by + c$ as $x = a(y - k)^2 + h$ by Completing the Square
6. Find the Vertex of the Graph of $x = ay^2 + by + c$ Using $y = -\frac{b}{2a}$, and Graph the Equation

1. Find the Maximum or Minimum Value of a Quadratic Function

From our work with quadratic functions, we have seen that the vertex is either the lowest point or the highest point on the graph depending on whether the parabola opens upward or downward.

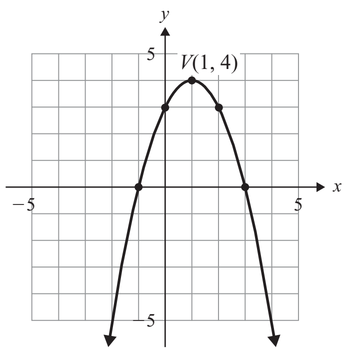
If the parabola opens upward, the vertex is the *lowest* point on the parabola.



$f(x) = x^2 + 4x + 3$

The y-coordinate of the vertex, -1, is the *smallest* y-value the function will have. We say that **-1 is the minimum value of the function**. $f(x)$ has no maximum because the graph continues upward indefinitely—the y-values get larger without bound.

If the parabola opens downward, the vertex is the *highest* point on the parabola.



$g(x) = -x^2 + 2x + 3$

The y-coordinate of the vertex, 4, is the *largest* y-value the function will have. We say that **4 is the maximum value of the function**. $g(x)$ has no minimum because the graph continues downward indefinitely—the y-values get smaller without bound.

Property Maximum and Minimum Values of a Quadratic Function

Let $f(x) = ax^2 + bx + c$.

1) If a is **positive,** the graph of $f(x)$ opens upward, so the vertex is the lowest point on the parabola. The y-coordinate of the vertex is the **minimum** value of the function $f(x)$.
2) If a is **negative,** the graph of $f(x)$ opens downward, so the vertex is the highest point on the parabola. The y-coordinate of the vertex is the **maximum** value of the function $f(x)$.

We can use this information about the vertex to help us solve problems.

Example 1

Let $f(x) = -x^2 + 4x + 2$.

a) Does the function attain a minimum or maximum value at its vertex?

b) Find the vertex of the graph of $f(x)$.

c) What is the minimum or maximum value of the function?

d) Graph the function to verify parts a)–c).

Solution

a) Since $a = -1$, the graph of $f(x)$ will open downward. Therefore, the vertex will be the *highest* point on the parabola. The function will attain its *maximum* value at the vertex.

b) Use $x = -\dfrac{b}{2a}$ to find the x-coordinate of the vertex. For $f(x) = -x^2 + 4x + 2$,

$$x = -\frac{b}{2a} = -\frac{(4)}{2(-1)} = 2$$

The y-coordinate of the vertex is $f(2)$.

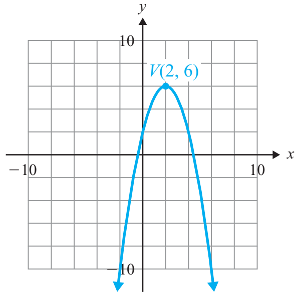
$$\begin{aligned} f(2) &= -(2)^2 + 4(2) + 2 \\ &= -4 + 8 + 2 = 6 \end{aligned}$$

The vertex is (2, 6).

c) $f(x)$ has no minimum value. The *maximum* value of the function is 6, the y-coordinate of the vertex. (The largest y-value of the function is 6.)

We say that the maximum value of the function is 6 and that it occurs at $x = 2$ (the x-coordinate of the vertex).

d) From the graph of $f(x)$, we can see that our conclusions in parts a)–c) make sense.



You Try 1

Let $f(x) = x^2 + 6x + 7$. Repeat parts a)–d) from Example 1.

2. Given a Quadratic Function, Solve an Applied Problem Involving a Maximum or Minimum Value

Example 2

A ball is thrown upward from a height of 24 ft. The height h of the ball (in feet) t sec after the ball is released is given by

$$h(t) = -16t^2 + 16t + 24.$$

a) How long does it take the ball to reach its maximum height?

b) What is the maximum height attained by the ball?

Solution

a) Begin by understanding what the function $h(t)$ tells us: $a = -16$, so the graph of h would open downward. Therefore, the vertex is the highest point on the parabola. The maximum value of the function occurs at the vertex. The ordered pairs that satisfy $h(t)$ are of the form $(t, h(t))$.

To determine how long it takes the ball to reach its maximum height, we must find the t-coordinate of the vertex.

$$t = -\frac{b}{2a} = -\frac{16}{2(-16)} = \frac{1}{2}$$

The ball will reach its maximum height after $\dfrac{1}{2}$ sec.

b) The maximum height the ball reaches is the y-coordinate (or $h(t)$-coordinate) of the vertex. Since the ball attains its maximum height when $t = \frac{1}{2}$, find $h\left(\frac{1}{2}\right)$.

$$\begin{aligned} h\left(\frac{1}{2}\right) &= -16\left(\frac{1}{2}\right)^2 + 16\left(\frac{1}{2}\right) + 24 \\ &= -16\left(\frac{1}{4}\right) + 8 + 24 \\ &= -4 + 32 = 28 \end{aligned}$$

The ball reaches a maximum height of 28 ft.

You Try 2

An object is propelled upward from a height of 10 ft. The height h of the object (in feet) t sec after the ball is released is given by

$$h(t) = -16t^2 + 32t + 10$$

a) How long does it take the object to reach its maximum height?

b) What is the maximum height attained by the object?

3. Write a Quadratic Function to Solve an Applied Problem Involving a Maximum or Minimum Value

Example 3

Ayesha plans to put a fence around her rectangular garden. If she has 32 ft of fencing, what is the maximum area she can enclose?

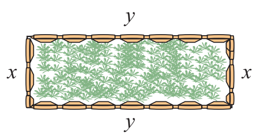
Solution

Begin by drawing a picture.

Let x = the width of the garden

Let y = the length of the garden

Label the picture.



We will write two equations for a problem like this:

1) *The maximize or minimize equation;* this equation describes what we are trying to maximize or minimize.

2) *The constraint equation;* this equation describes the restrictions on the variables or the conditions the variables must meet.

Here is how we will get the equations.

1) We will write a *maximize* equation because we are trying to find the *maximum area* of the garden.

$$\text{Let } A = \text{area of the garden}$$

The area of the rectangle above is xy. Our equation is

$$\text{Maximize: } A = xy$$

2) To write the *constraint* equation, think about the restriction put on the variables. We cannot choose *any* two numbers for x and y. Since Ayesha has 32 ft of fencing, the distance around the garden is 32 ft. This is the perimeter of the rectangular garden. The perimeter of the rectangle drawn above is $2x + 2y$, and it must equal 32 ft.

The constraint equation is

$$\text{Constraint: } 2x + 2y = 32$$

Set up this maximization problem as

$$\begin{aligned}\text{Maximize: } A &= xy\\ \text{Constraint: } 2x + 2y &= 32\end{aligned}$$

Solve the constraint for a variable, and then substitute the expression into the maximize equation.

$$\begin{aligned}2x + 2y &= 32\\ 2y &= 32 - 2x\\ y &= 16 - x \qquad \text{Solve the constraint for } y.\end{aligned}$$

Substitute $y = 16 - x$ into $A = xy$.

$$\begin{aligned}A &= x(16 - x)\\ A &= 16x - x^2 \qquad \text{Distribute.}\\ A &= -x^2 + 16x \qquad \text{Write in descending powers.}\end{aligned}$$

Look carefully at $A = -x^2 + 16x$. This is a quadratic function! Its graph is a parabola that opens downward (since $a = -1$). At the vertex, the function attains its maximum. The ordered pairs that satisfy this function are of the form $(x, A(x))$, where x represents the width and $A(x)$ represents the area of the rectangular garden. *The second coordinate of the vertex is the maximum area we are looking for.*

$$A = -x^2 + 16x$$

Use $x = -\dfrac{b}{2a}$ with $a = -1$ and $b = 16$ to find the x-coordinate of the vertex (the width of the rectangle that produces the maximum area).

$$x = -\frac{16}{2(-1)} = 8$$

Substitute $x = 8$ into $A = -x^2 + 16x$ to find the maximum area.

$$\begin{aligned}A &= -(8)^2 + 16(8)\\ A &= -64 + 128\\ A &= 64\end{aligned}$$

The graph of $A = -x^2 + 16x$ is a parabola that opens downward with vertex (8, 64).

The maximum area of the garden is 64 ft^2, and this will occur when the width of the garden is 8 ft. (The length will be 8 ft as well.) ■

Procedure Steps for Solving a Max/Min Problem Like Example 3

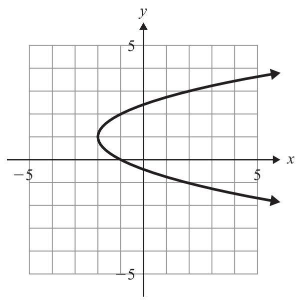
1) Draw a picture, if applicable.
2) Define the unknowns. Label the picture.
3) Write the max/min equation.
4) Write the constraint equation.
5) Solve the constraint for a variable. Substitute the expression into the max/min equation to obtain a quadratic function.
6) Find the vertex of the parabola using the vertex formula, $x = -\dfrac{b}{2a}$.
7) Answer the question being asked.

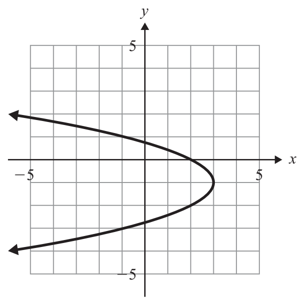
You Try 3

Find the maximum area of a rectangle that has a perimeter of 28 in.

4. Graph Parabolas of the Form $x = a(y - k)^2 + h$

Not all parabolas are functions. Parabolas can open in the x-direction as illustrated below. Clearly, these fail the vertical line test for functions.





Parabolas that open in the y-direction, or vertically, result from the functions

$$y = a(x - h)^2 + k \quad \text{or} \quad y = ax^2 + bx + c.$$

If we interchange the x and y, we obtain the equations

$$x = a(y - k)^2 + h \quad \text{or} \quad x = ay^2 + by + c.$$

The graphs of these equations are parabolas that open in the x-direction, or horizontally.

Procedure Graphing an Equation of the Form $x = a(y - k)^2 + h$

1) The vertex of the parabola is (h, k). (Notice, however, that h and k have changed their positions when compared to a quadratic function.)
2) The axis of symmetry is the horizontal line $y = k$.
3) If a is positive, the graph opens to the right.
   If a is negative, the graph opens to the left.

Example 4

Graph each equation. Find the x- and y-intercepts.

a) $x = (y + 2)^2 - 1$ b) $x = -2(y - 2)^2 + 4$

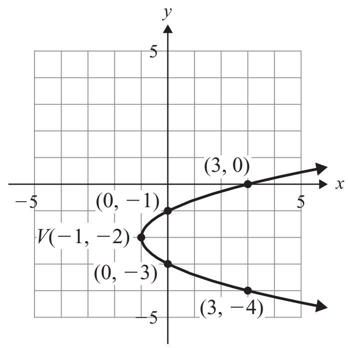
Solution

a) 1) $h = -1$ and $k = -2$. The vertex is $(-1, -2)$.
2) The axis of symmetry is $y = -2$.
3) $a = +1$, so the parabola opens to the right. It is the same width as $y = x^2$.

To find the x-intercept, let $y = 0$ and solve for x.

$$x = (y + 2)^2 - 1$$
$$x = (0 + 2)^2 - 1$$
$$x = 4 - 1 = 3$$

The x-intercept is $(3, 0)$.



Find the y-intercepts by substituting 0 for x and solving for y.

$$\begin{aligned} x &= (y+2)^2 - 1 & \\ 0 &= (y+2)^2 - 1 & \text{Substitute 0 for } x. \\ 1 &= (y+2)^2 & \text{Add 1.} \\ \pm 1 &= y + 2 & \text{Square root property} \end{aligned}$$

$$\begin{aligned} 1 &= y + 2 \quad \text{or} \quad -1 = y + 2 \\ -1 &= y \qquad\qquad\;\; -3 = y \qquad \text{Solve.} \end{aligned}$$

The y-intercepts are $(0, -3)$ and $(0, -1)$.
Use the axis of symmetry to locate the point $(3, -4)$ on the graph.

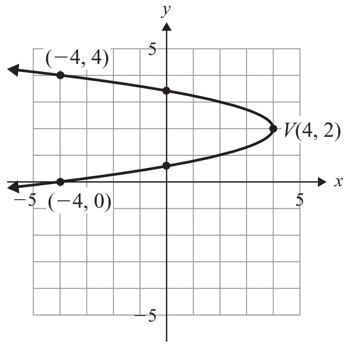
b) $x = -2(y-2)^2 + 4$

1) $h = 4$ and $k = 2$. The vertex is $(4, 2)$.
2) The axis of symmetry is $y = 2$.
3) $a = -2$, so the parabola opens to the left. It is narrower than $y = x^2$.

To find the x-intercept, let $y = 0$ and solve for x.

$$\begin{aligned} x &= -2(y-2)^2 + 4 \\ x &= -2(0-2)^2 + 4 \\ x &= -2(4) + 4 = -4 \end{aligned}$$

The x-intercept is $(-4, 0)$.



Find the y-intercepts by substituting 0 for x and solving for y.

$$\begin{aligned} x &= -2(y-2)^2 + 4 & \\ 0 &= -2(y-2)^2 + 4 & \text{Substitute 0 for } x. \\ -4 &= -2(y-2)^2 & \text{Subtract 4.} \\ 2 &= (y-2)^2 & \text{Divide by } -2. \\ \pm\sqrt{2} &= y - 2 & \text{Square root property} \\ 2 \pm \sqrt{2} &= y & \text{Add 2.} \end{aligned}$$

The y-intercepts are $(0, 2 - \sqrt{2})$ and $(0, 2 + \sqrt{2})$.
Use the axis of symmetry to locate the point $(-4, 4)$ on the graph. ■

You Try 4

Graph $x = -(y+1)^2 - 3$. Find the x- and y-intercepts.

Procedure Graphing Parabolas from the Form $x = ay^2 + by + c$

We can use two methods to graph $x = ay^2 + by + c$.

Method 1: Rewrite $x = ay^2 + by + c$ in the form $x = a(y-k)^2 + h$ by *completing the square*.

Method 2: Use the formula $y = -\dfrac{b}{2a}$ to find the *y-coordinate* of the vertex. Find the x-coordinate by substituting the y-value into the equation $x = ay^2 + by + c$.

5. Rewrite $x = ay^2 + by + c$ as $x = a(y - k)^2 + h$ by Completing the Square

Example 5

Rewrite $x = 2y^2 - 4y + 8$ in the form $x = a(y - k)^2 + h$ by completing the square.

Solution

To complete the square, follow the same procedure used for quadratic functions. (This is outlined on p. 721 in Section 12.3.)

Step 1: Divide the equation by 2 so that the coefficient of y^2 is 1.

$$\frac{x}{2} = y^2 - 2y + 4$$

Step 2: Separate the constant from the variable terms using parentheses.

$$\frac{x}{2} = (y^2 - 2y) + 4$$

Step 3: Complete the square for the quantity in parentheses. Add 1 *inside* the parentheses and *subtract* 1 from the 4.

$$\frac{x}{2} = (y^2 - 2y + 1) + 4 - 1$$
$$\frac{x}{2} = (y^2 - 2y + 1) + 3$$

Step 4: Factor the expression inside the parentheses.

$$\frac{x}{2} = (y - 1)^2 + 3$$

Step 5: Solve the equation for x by multiplying by 2.

$$2\left(\frac{x}{2}\right) = 2[(y - 1)^2 + 3]$$
$$x = 2(y - 1)^2 + 6$$

You Try 5

Rewrite $x = -y^2 - 6y - 1$ in the form $x = a(y - k)^2 + h$ by completing the square.

6. Find the Vertex of the Graph of $x = ay^2 + by + c$ Using $y = -\frac{b}{2a}$, and Graph the Equation

Example 6

Graph $x = y^2 - 2y + 5$. Find the vertex using the vertex formula. Find the x- and y-intercepts.

Solution

Since this equation is solved for x and is quadratic in y, it opens in the x-direction. $a = 1$, so it opens to the right. Use the vertex formula to find the *y-coordinate* of the vertex.

$$y = -\frac{b}{2a}$$
$$y = -\frac{-2}{2(1)} = 1 \qquad a = 1, b = -2$$

Substitute $y = 1$ into $x = y^2 - 2y + 5$ to find the x-coordinate of the vertex.

$$x = (1)^2 - 2(1) + 5$$
$$x = 1 - 2 + 5 = 4$$

The vertex is (4, 1). Since the vertex is (4, 1) and the parabola opens to the right, the graph has *no y-intercepts*.

To find the x-intercept, let $y = 0$ and solve for x.

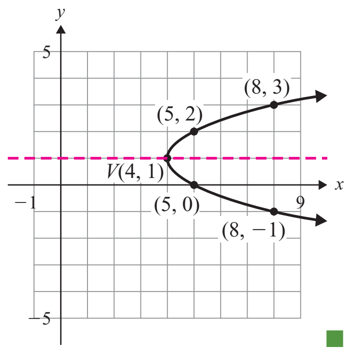
$$x = y^2 - 2y + 5$$
$$x = 0^2 - 2(0) + 5$$
$$x = 5$$

The x-intercept is (5, 0).

Find another point on the parabola by choosing a value for y that is close to the y-coordinate of the vertex. Let $y = -1$. Find x.

$$x = (-1)^2 - 2(-1) + 5$$
$$x = 1 + 2 + 5 = 8$$

Another point on the parabola is $(8, -1)$. Use the axis of symmetry to locate the additional points (5, 2) and (8, 3).



You Try 6

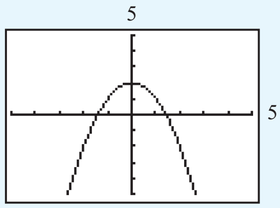
Graph $x = y^2 + 6y + 3$. Find the vertex using the vertex formula. Find the x- and y-intercepts.

Using Technology

To graph a parabola that is a function, just enter the equation and press $\boxed{\text{GRAPH}}$.

Example 1: Graph $f(x) = -x^2 + 2$.

Enter $Y_1 = -x^2 + 2$ to graph the function on a calculator.



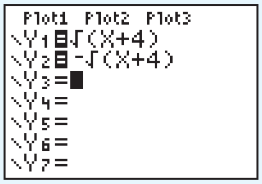
To graph an equation on a calculator, it must be entered so that y is a function of x. Since a parabola that opens horizontally is not a function, we must solve for y in terms of x so that the equation is represented by two different functions.

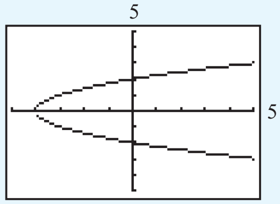
Example 2: Graph $x = y^2 - 4$ on a calculator.

Solve for y.

$$x = y^2 - 4$$
$$x + 4 = y^2$$
$$\pm\sqrt{x + 4} = y$$

Now the equation $x = y^2 - 4$ is rewritten so that y is in terms of x. In the graphing calculator, enter $y = \sqrt{x + 4}$ as Y_1. This represents the top half of the parabola since the y-values are positive above the x-axis. Enter $y = -\sqrt{x + 4}$ as Y_2. This represents the bottom half of the parabola since the y-values are negative below the x-axis. Set an appropriate window and press $\boxed{\text{GRAPH}}$.



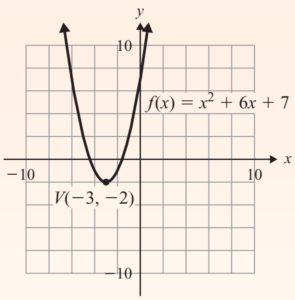


Graph each parabola on a graphing calculator. Where appropriate, rewrite the equation for y in terms of x. These problems come from the homework exercises so that the graphs can be found in the Answers to Exercises appendix.

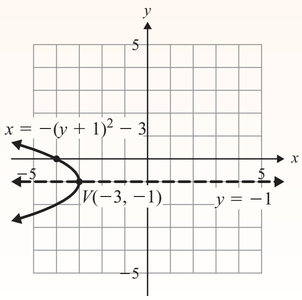
1) $f(x) = x^2 + 6x + 9$; Exercise 9
2) $x = y^2 + 2$; Exercise 33
3) $x = \frac{1}{4}(y + 2)^2$; Exercise 39
4) $f(x) = -\frac{1}{2}x^2 + 4x - 6$; Exercise 11
5) $x = -(y - 4)^2 + 5$; Exercise 35
6) $x = y^2 - 4y + 5$; Exercise 41

Answers to You Try Exercises

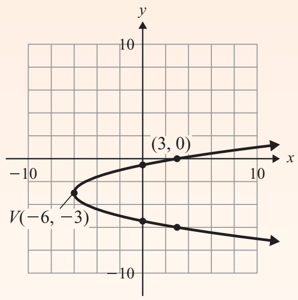
1) a) minimum value b) vertex $(-3, -2)$
c) The minimum value of the function is -2.
d)



2) a) 1 sec b) 26 ft 3) 49 in^2
4) $V(-3, -1)$; x-int: $(-4, 0)$; y-int: none



5) $x = -(y + 3)^2 + 8$
6) $V(-6, -3)$; x-int: $(3, 0)$;
y-ints: $(0, -3 - \sqrt{6})$, $(0, -3 + \sqrt{6})$



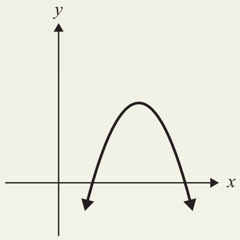
Answers to Technology Exercises

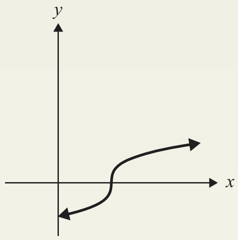
1) The equation can be entered as it is.
2) $Y_1 = \sqrt{x - 2}$; $Y_2 = -\sqrt{x - 2}$
3) $Y_1 = -2 + \sqrt{4x}$; $Y_2 = -2 - \sqrt{4x}$
4) The equation can be entered as it is.
5) $Y_1 = 4 + \sqrt{5 - x}$; $Y_2 = 4 - \sqrt{5 - x}$
6) $Y_1 = 2 + \sqrt{x - 1}$; $Y_2 = 2 - \sqrt{x - 1}$

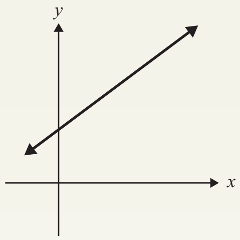
12.4 Exercises

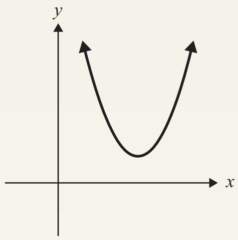
Objective 1: Find the Maximum or Minimum Value of a Quadratic Function

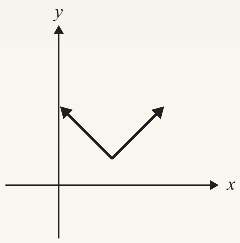
For Exercises 1–6, determine whether the function has a maximum value, minimum value, or neither.

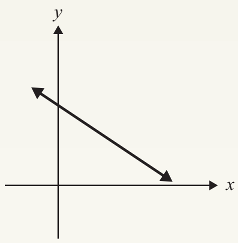
1) 

2) 

3) 

4) 

5) 

6) 

7) Let $f(x) = ax^2 + bx + c$. How do you know whether the function has a maximum or minimum value at the vertex?

8) Is there a maximum value of the function $y = 2x^2 + 12x + 11$? Explain your answer.

For Problems 9–12, answer parts a)–d) for each function, $f(x)$.

a) Does the function attain a minimum or maximum value at its vertex?

b) Find the vertex of the graph of $f(x)$.

c) What is the minimum or maximum value of the function?

d) Graph the function to verify parts a)–c).

9) $f(x) = x^2 + 6x + 9$

10) $f(x) = -x^2 + 2x + 4$

VIDEO 11) $f(x) = -\dfrac{1}{2}x^2 + 4x - 6$

12) $f(x) = 2x^2 + 4x$

Objective 2: Given a Quadratic Function, Solve an Applied Problem Involving a Maximum or Minimum Value

Solve.

13) An object is fired upward from the ground so that its height h (in feet) t sec after being fired is given by

$$h(t) = -16t^2 + 320t$$

a) How long does it take the object to reach its maximum height?

b) What is the maximum height attained by the object?

c) How long does it take the object to hit the ground?

14) An object is thrown upward from a height of 64 ft so that its height h (in feet) t sec after being thrown is given by

$$h(t) = -16t^2 + 48t + 64$$

a) How long does it take the object to reach its maximum height?

b) What is the maximum height attained by the object?

c) How long does it take the object to hit the ground?

15) The number of guests staying at the Cozy Inn from January to December 2010 can be approximated by

$$N(x) = -10x^2 + 120x + 120$$

where x represents the number of months after January 2010 ($x = 0$ represents January, $x = 1$ represents February, etc.), and $N(x)$ represents the number of guests who stayed at the inn. During which month did the inn have the greatest number of guests? How many people stayed at the inn during that month?

16) The average number of traffic tickets issued in a city on any given day Sunday–Saturday can be approximated by

$$T(x) = -7x^2 + 70x + 43$$

where x represents the number of days after Sunday ($x = 0$ represents Sunday, $x = 1$ represents Monday, etc.), and $T(x)$ represents the number of traffic tickets issued. On which day are the most tickets written? How many tickets are issued on that day?

17) The number of babies born to teenage mothers from 1989 to 2002 can be approximated by

$$N(t) = -0.721t^2 + 2.75t + 528$$

where t represents the number of years after 1989 and $N(t)$ represents the number of babies born (in thousands). According to this model, in what year was the number of babies born to teen mothers the greatest? How many babies were born that year? (U.S. Census Bureau)

18) The number of violent crimes in the United States from 1985 to 1999 can be modeled by

$$C(x) = -49.2x^2 + 636x + 12{,}468$$

where x represents the number of years after 1985 and $C(x)$ represents the number of violent crimes (in thousands). During what year did the greatest number of violent crimes occur, and how many were there? (U.S. Census Bureau)

Objective 3: Write a Quadratic Function to Solve an Applied Problem Involving a Maximum or Minimum Value

Solve.

19) Every winter Rich makes a rectangular ice rink in his backyard. He has 100 ft of material to use as the border. What is the maximum area of the ice rink?

20) Find the dimensions of the rectangular garden of greatest area that can be enclosed with 40 ft of fencing.

VIDEO 21) The Soo family wants to fence in a rectangular area to hold their dogs. One side of the pen will be their barn. Find the dimensions of the pen of greatest area that can be enclosed with 48 ft of fencing.

22) A farmer wants to enclose a rectangular area with 120 ft of fencing. One side is a river and will not require a fence. What is the maximum area that can be enclosed?

23) Find two integers whose sum is 18 and whose product is a maximum.

24) Find two integers whose sum is 26 and whose product is a maximum.

25) Find two integers whose difference is 12 and whose product is a minimum.

26) Find two integers whose difference is 30 and whose product is a minimum.

Objective 4: Graph Parabolas of the Form $x = a(y - k)^2 + h$

Given a quadratic equation of the form $x = a(y - k)^2 + h$, answer the following.

27) What is the vertex?

28) What is the equation of the axis of symmetry?

29) If a is negative, which way does the parabola open?

30) If a is positive, which way does the parabola open?

For each equation, identify the vertex, axis of symmetry, and x- and y-intercepts. Then, graph the equation.

VIDEO 31) $x = (y - 1)^2 - 4$

32) $x = (y + 3)^2 - 1$

33) $x = y^2 + 2$

34) $x = (y - 4)^2$

35) $x = -(y - 4)^2 + 5$

36) $x = -(y + 1)^2 - 7$

37) $x = -2(y - 2)^2 - 9$

38) $x = -\frac{1}{2}(y - 4)^2 + 7$

39) $x = \frac{1}{4}(y + 2)^2$

40) $x = 2y^2 + 3$

Objective 5: Rewrite $x = ay^2 + by + c$ as $x = a(y - k)^2 + h$ by Completing the Square

Rewrite each equation in the form $x = a(y - k)^2 + h$ by completing the square and graph it.

41) $x = y^2 - 4y + 5$

42) $x = y^2 + 4y - 6$

43) $x = -y^2 + 6y + 6$

44) $x = -y^2 - 2y - 5$

VIDEO 45) $x = \frac{1}{3}y^2 + \frac{8}{3}y - \frac{5}{3}$

46) $x = 2y^2 - 4y + 5$

47) $x = -4y^2 - 8y - 10$

48) $x = \frac{1}{2}y^2 + 4y - 1$

Objective 6: Find the Vertex of $x = ay^2 + by + c$ Using $y = -\frac{b}{2a}$, and Graph the Equation

Graph each equation using the vertex formula. Find the x- and y-intercepts.

49) $x = y^2 - 4y + 3$

50) $x = -y^2 + 4y$

51) $x = -y^2 + 2y + 2$

52) $x = y^2 + 6y - 4$

VIDEO 53) $x = -2y^2 + 4y - 6$

54) $x = 3y^2 + 6y - 1$

55) $x = 4y^2 - 16y + 13$

56) $x = 2y^2 + 4y + 8$

57) $x = \frac{1}{4}y^2 - \frac{1}{2}y + \frac{25}{4}$

58) $x = -\frac{3}{4}y^2 + \frac{3}{2}y - \frac{11}{4}$

Mixed Exercises

Exercises 59–68 contain parabolas that open either horizontally or vertically. Graph each equation.

59) $h(x) = -x^2 + 6$

60) $y = x^2 - 6x - 1$

61) $x = y^2$

62) $f(x) = -3x^2 + 12x - 8$

63) $x = -\frac{1}{2}y^2 - 4y - 5$

64) $x = (y - 4)^2 + 3$

65) $y = x^2 + 2x - 3$

66) $x = -3(y + 2)^2 + 11$

67) $f(x) = -2(x - 4)^2 + 3$

68) $g(x) = \frac{3}{2}x^2 - 12x + 20$

Section 12.5 The Algebra of Functions

Objectives

1. Add, Subtract, Multiply, and Divide Functions
2. Solve an Applied Problem Using Operations on Functions
3. Find the Composition of Functions
4. Solve an Applied Problem Using the Composition of Functions

1. Add, Subtract, Multiply, and Divide Functions

We have learned that we can add, subtract, multiply, and divide polynomials. These same operations can be performed with functions.

Properties Operations Involving Functions

Given the functions $f(x)$ and $g(x)$, the sum, difference, product, and quotient of f and g are defined by

1) $(f + g)(x) = f(x) + g(x)$
2) $(f - g)(x) = f(x) - g(x)$
3) $(fg)(x) = f(x) \cdot g(x)$
4) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$, where $g(x) \neq 0$

The domain of $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, and $\left(\dfrac{f}{g}\right)(x)$ is the *intersection* of the domains of $f(x)$ and $g(x)$.

Example 1

Let $f(x) = x^2 - 2x + 7$ and $g(x) = 4x - 3$. Find each of the following.

a) $(f + g)(x)$ b) $(f - g)(x)$ and $(f - g)(-1)$ c) $(fg)(x)$ d) $\left(\dfrac{f}{g}\right)(x)$

Solution

a) $(f + g)(x) = f(x) + g(x)$

$= (x^2 - 2x + 7) + (4x - 3)$ Substitute the functions.

$= x^2 + 2x + 4$ Combine like terms.

b) $(f - g)(x) = f(x) - g(x)$

$= (x^2 - 2x + 7) - (4x - 3)$ Substitute the functions.

$= x^2 - 2x + 7 - 4x + 3$ Distribute.

$= x^2 - 6x + 10$ Combine like terms.

Use the result above to find $(f - g)(-1)$.

$(f - g)(x) = x^2 - 6x + 10$

$(f - g)(-1) = (-1)^2 - 6(-1) + 10$ Substitute -1 for x.

$= 1 + 6 + 10$

$= 17$

We can also find $(f - g)(-1)$ using the rule this way:

$(f - g)(-1) = f(-1) - g(-1)$

$= [(-1)^2 - 2(-1) + 7] - [4(-1) - 3]$ Substitute -1 for x in $f(x)$ and $g(x)$.

$= (1 + 2 + 7) - (-4 - 3)$

$= 10 - (-7)$

$= 17$

c) $(fg)(x) = f(x) \cdot g(x)$

$= (x^2 - 2x + 7)(4x - 3)$ Substitute the functions.

$= 4x^3 - 3x^2 - 8x^2 + 6x + 28x - 21$ Multiply.

$= 4x^3 - 11x^2 + 34x - 21$ Combine like terms.

d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$, where $g(x) \neq 0$

$= \dfrac{x^2 - 2x + 7}{4x - 3}$, where $x \neq \dfrac{3}{4}$ Substitute the functions.

You Try 1

Let $f(x) = 3x^2 - 8$ and $g(x) = 2x + 1$. Find

a) $(f + g)(x)$ and $(f + g)(-2)$ b) $(f - g)(x)$ c) $(fg)(x)$ d) $\left(\frac{f}{g}\right)(3)$

2. Solve an Applied Problem Using Operations on Functions

Example 2

A publisher sells paperback romance novels to a large bookstore chain for $4.00 per book. Therefore, the publisher's revenue, in dollars, is defined by the function

$$R(x) = 4x$$

where x is the number of books sold to the retailer. The publisher's cost, in dollars, to produce x books is

$$C(x) = 2.5x + 1200$$

In business, profit is defined as revenue − cost. In terms of functions this is written as $P(x) = R(x) - C(x)$, where $P(x)$ is the profit function.

a) Find the profit function, $P(x)$, that describes the publisher's profit from the sale of x books.

b) If the publisher sells 10,000 books to this chain of bookstores, what is the publisher's profit?

Solution

a)
$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 4x - (2.5x + 1200) \quad \text{Substitute the functions.} \\ &= 1.5x - 1200 \\ P(x) &= 1.5x - 1200 \end{aligned}$$

b) Find $P(10{,}000)$.

$$\begin{aligned} P(10{,}000) &= 1.5(10{,}000) - 1200 \\ &= 15{,}000 - 1200 \\ &= 13{,}800 \end{aligned}$$

The publisher's profit is $13,800. ■

You Try 2

A candy company sells its Valentine's Day candy to a grocery store retailer for $6.00 per box. The candy company's revenue, in dollars, is defined by $R(x) = 6x$, where x is the number of boxes sold to the retailer. The company's cost, in dollars, to produce x boxes of candy is $C(x) = 4x + 900$.

a) Find the profit function, $P(x)$, that defines the company's profit from the sale of x boxes of candy.

b) Find the candy company's profit from the sale of 2000 boxes of candy.

3. Find the Composition of Functions

Another operation that can be performed with functions is called the **composition of functions**. We can use composite functions when we are given certain two-step processes and want to combine them into a single step.

For example, if you work x hours per week earning $8 per hour, your earnings before taxes and other deductions can be described by the function $g(x) = 8x$. Your take-home pay is different, however, because of taxes and other deductions. So if your take-home pay is 75% of your earnings before deductions, then $f(x) = 0.75x$ can be used to compute your take-home pay when x is your earnings before deductions.

We can describe what is happening with two tables of values.

$g(x) = 8x$

Hours Worked x	**Earnings Before Deductions** $g(x)$
6	48
10	80
20	160
40	320

$$x \longrightarrow g(x)$$

$f(x) = 0.75x$

Earnings Before Deductions x	**Take-Home Pay** $f(x)$
48	36
80	60
160	120
320	240

$$x \longrightarrow f(x)$$

We have one function, $g(x)$, that describes total earnings before deductions in terms of the number of hours worked. We have another function, $f(x)$, that describes take-home pay in terms of total earnings before deductions. It would be convenient to have a function that would allow us to compute, directly, the take-home pay in terms of the number of hours worked.

$g(x) = 8x$ $\qquad$ $f(x) = 0.75x$

Hours Worked	**Earnings Before Deductions**	**Take-Home Pay**
6	48	36
10	80	60
20	160	120
40	320	240

$$x \xrightarrow{g} g(x) \xrightarrow{f} h(x) = f(g(x))$$

$$h(x) = (f \circ g)(x) = f(g(x))$$

If we substitute the function $g(x)$ into the function $f(x)$, we will get a new function, $h(x)$, where $h(x) = f(g(x))$. The take-home pay in terms of the number of hours worked, $h(x)$, is given by the composition function $f(g(x))$, read as "f of g of x" and given by

$$\begin{aligned} h(x) = f(g(x)) &= f((8x)) \\ &= 0.75(8x) \\ &= 6x \end{aligned}$$

Therefore, $h(x) = 6x$ allows us to directly compute the take-home pay from the number of hours worked. To find out your take-home pay when you work 20 hours in a week, find $h(20)$.

$$\begin{aligned} h(x) &= 6x \\ h(20) &= 6(20) = 120 \end{aligned}$$

Working 20 hours will result in take-home pay of \$120. Notice that this is the same as the take-home pay computed in the tables.

Another way to write $f(g(x))$ is $(f \circ g)(x)$, and both can be read as "f of g of x," or "f composed with g," or "the composition of f and g." Likewise, $g(f(x)) = (g \circ f)(x)$, and these can be read as "g of f of x," or "g composed with f," or "the composition of g and f."

Definition

Given the functions $f(x)$ and $g(x)$, the **composition function** $f \circ g$ (read "f of g") is defined as

$$(f \circ g)(x) = f(g(x))$$

where $g(x)$ is in the domain of f.

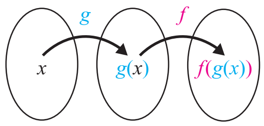
Example 3

Let $f(x) = 3x + 5$ and $g(x) = x - 2$. Find $(f \circ g)(x)$.

Solution

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f(x - 2) && \text{Substitute } x - 2 \text{ for } g(x). \\
&= 3(x - 2) + 5 && \text{Substitute } x - 2 \text{ for } x \text{ in } f(x). \\
&= 3x - 6 + 5 && \text{Distribute.} \\
&= 3x - 1
\end{aligned}$$

The composition of functions can also be explained this way. Finding $(f \circ g)(x) = f(g(x))$ in Example 3 meant that the function $g(x)$ was substituted into the function $f(x)$. $g(x)$ was the innermost function. Therefore, *first* the function g performs an operation on x. The result is $g(x)$. Then the function f performs an operation on $g(x)$. The result is $f(g(x))$.



You Try 3

Let $f(x) = -6x + 2$ and $g(x) = 5x + 1$. Find $(f \circ g)(x)$.

Example 4

Let $f(x) = x + 8$ and $g(x) = 2x - 5$. Find

a) $g(3)$ b) $(f \circ g)(3)$ c) $(f \circ g)(x)$

Solution

a)
$$\begin{aligned}
g(x) &= 2x - 5 \\
g(3) &= 2(3) - 5 \\
&= 1
\end{aligned}$$

b)
$$\begin{aligned}
(f \circ g)(3) &= f(g(3)) \\
&= f(1) && g(3) = 1 \text{ from a)} \\
&= 1 + 8 && \text{Substitute 1 for } x \text{ in } f(x) = x + 8. \\
&= 9
\end{aligned}$$

c)
$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f(2x - 5) && \text{Substitute } 2x - 5 \text{ for } g(x). \\
&= (2x - 5) + 8 && \text{Substitute } 2x - 5 \text{ for } x \text{ in } f(x). \\
&= 2x + 3
\end{aligned}$$

We can also find $(f \circ g)(3)$ by substituting 3 into the expression for $(f \circ g)(x)$.

$$\begin{aligned}
(f \circ g)(x) &= 2x + 3 \\
(f \circ g)(3) &= 2(3) + 3 = 9
\end{aligned}$$

Notice that this is the same as the result we obtained in b).

You Try 4

Let $f(x) = x - 10$ and $g(x) = 3x + 4$. Find

a) $g(-1)$ b) $(f \circ g)(-1)$ c) $(f \circ g)(x)$

Example 5

Let $f(x) = 4x - 1$, $g(x) = x^2$, and $h(x) = x^2 + 5x - 2$. Find

a) $(f \circ g)(x)$ b) $(g \circ f)(x)$ c) $(h \circ f)(x)$

Solution

a)
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) & \\ &= f(x^2) & \text{Substitute } x^2 \text{ for } g(x). \\ &= 4(x^2) - 1 & \text{Substitute } x^2 \text{ for } x \text{ in } f(x). \\ &= 4x^2 - 1 &\end{aligned}$$

b)
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) & \\ &= g(4x - 1) & \text{Substitute } 4x - 1 \text{ for } f(x). \\ &= (4x - 1)^2 & \text{Substitute } 4x - 1 \text{ for } x \text{ in } g(x). \\ &= 16x^2 - 8x + 1 & \text{Expand the binomial.}\end{aligned}$$

Note

In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

c)
$$\begin{aligned}(h \circ f)(x) &= h(f(x)) & \\ &= h(4x - 1) & \text{Substitute } 4x - 1 \text{ for } f(x). \\ &= (4x - 1)^2 + 5(4x - 1) - 2 & \text{Substitute } 4x - 1 \text{ for } x \text{ in } h(x). \\ &= 16x^2 - 8x + 1 + 20x - 5 - 2 & \text{Distribute.} \\ &= 16x^2 + 12x - 6 & \text{Combine like terms.}\end{aligned}$$

You Try 5

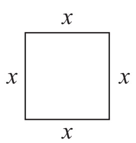
Let $f(x) = x^2 + 6$, $g(x) = 2x - 3$, and $h(x) = x^2 - 4x + 9$. Find

a) $(g \circ f)(x)$ b) $(f \circ g)(x)$ c) $(h \circ g)(x)$

4. Solve an Applied Problem Using the Composition of Functions

Example 6

The area, A, of a square expressed in terms of its perimeter, P, is defined by the function $A(P) = \frac{1}{16}P^2$. The perimeter of a square that has a side of length x is defined by the function $P(x) = 4x$.



a) Find $(A \circ P)(x)$ and explain what it represents.

b) Find $(A \circ P)(3)$ and explain what it represents.

Solution

a)
$$\begin{aligned}(A \circ P)(x) &= A(P(x)) \\ &= A(4x) && \text{Substitute } 4x \text{ for } P(x). \\ &= \frac{1}{16}(4x)^2 && \text{Substitute } 4x \text{ for } P \text{ in } A(P) = \frac{1}{16}P^2. \\ &= \frac{1}{16}(16x^2) \\ &= x^2\end{aligned}$$

$(A \circ P)(x) = x^2$. This is the formula for the area of a square in terms of the length of a side, x.

b) To find $(A \circ P)(3)$, use the result obtained in a).

$$\begin{aligned}(A \circ P)(x) &= x^2 \\ (A \circ P)(3) &= 3^2 = 9\end{aligned}$$

A square that has a side of length 3 units has an area of 9 square units. ■

You Try 6

Let $f(x) = 100x$ represent the number of centimeters in x meters. Let $g(y) = 1000y$ represent the number of meters in y kilometers.

a) Find $(f \circ g)(y)$ and explain what it represents.

b) Find $(f \circ g)(4)$ and explain what it represents.

Answers to You Try Exercises

1) a) $3x^2 + 2x - 7$; 1 b) $3x^2 - 2x - 9$ c) $6x^3 + 3x^2 - 16x - 8$ d) $\frac{19}{7}$
2) a) $P(x) = 2x - 900$ b) \$3100 3) $(f \circ g)(x) = -30x - 4$ 4) a) 1 b) -9 c) $3x - 6$
5) a) $2x^2 + 9$ b) $4x^2 - 12x + 15$ c) $4x^2 - 20x + 30$ 6) a) $(f \circ g)(y) = 100{,}000y$; this tells us the number of centimeters in y km. b) $(f \circ g)(4) = 400{,}000$; there are 400,000 cm in 4 km.

12.5 Exercises

Objective 1: Add, Subtract, Multiply, and Divide Functions

For each pair of functions, find a) $(f + g)(x)$, b) $(f + g)(5)$, c) $(f - g)(x)$, and d) $(f - g)(2)$.

1) $f(x) = -3x + 1,\ g(x) = 2x - 11$

2) $f(x) = 5x - 9,\ g(x) = x + 4$

3) $f(x) = 4x^2 - 7x - 1,\ g(x) = x^2 + 3x - 6$

4) $f(x) = -2x^2 + x + 8,\ g(x) = 3x^2 - 4x - 6$

For each pair of functions, find a) $(fg)(x)$ and b) $(fg)(-3)$.

5) $f(x) = x,\ g(x) = -x + 5$

6) $f(x) = -2x,\ g(x) = 3x + 1$

7) $f(x) = 2x + 3,\ g(x) = 3x + 1$

8) $f(x) = 4x + 7,\ g(x) = x - 5$

For each pair of functions, find a) $\left(\frac{f}{g}\right)(x)$ and b) $\left(\frac{f}{g}\right)(-2)$.

Identify any values that are not in the domain of $\left(\frac{f}{g}\right)(x)$.

9) $f(x) = 6x + 9,\ g(x) = x + 4$

10) $f(x) = 3x - 8,\ g(x) = x - 1$

11) $f(x) = x^2 - 5x - 24,\ g(x) = x - 8$

12) $f(x) = x^2 - 15x + 54,\ g(x) = x - 9$

13) $f(x) = 3x^2 + 14x + 8,\ g(x) = 3x + 2$

14) $f(x) = 2x^2 + x - 15,\ g(x) = 2x - 5$

15) Find two polynomial functions $f(x)$ and $g(x)$ so that $(f + g)(x) = 5x^2 + 8x - 2$.

16) Let $f(x) = 6x^3 - 9x^2 - 4x + 10$. Find $g(x)$ so that $(f - g)(x) = x^3 + 3x^2 + 8$.

17) Let $f(x) = 4x - 5$. Find $g(x)$ so that $(fg)(x) = 8x^2 - 22x + 15$.

18) Let $f(x) = 12x^3 - 18x^2 + 2x$. Find $g(x)$ so that $\left(\dfrac{f}{g}\right)(x) = 6x^2 - 9x + 1$.

Objective 2: Solve an Applied Problem Using Operations on Functions

VIDEO 19) A manufacturer's revenue, $R(x)$ in dollars, from the sale of x calculators is given by $R(x) = 12x$. The company's cost, $C(x)$ in dollars, to produce x calculators is $C(x) = 8x + 2000$.

a) Find the profit function, $P(x)$, that defines the manufacturer's profit from the sale of x calculators.

b) What is the profit from the sale of 1500 calculators?

20) $R(x) = 80x$ is the revenue function for the sale of x bicycles, in dollars. The cost to manufacture x bikes, in dollars, is $C(x) = 60x + 7000$.

a) Find the profit function, $P(x)$, that describes the manufacturer's profit from the sale of x bicycles.

b) What is the profit from the sale of 500 bicycles?

21) $R(x) = 18x$ is the revenue function for the sale of x toasters, in dollars. The cost to manufacture x toasters, in dollars, is $C(x) = 15x + 2400$.

a) Find the profit function, $P(x)$, that describes the profit from the sale of x toasters.

b) What is the profit from the sale of 800 toasters?

22) A company's revenue, $R(x)$ in dollars, from the sale of x doghouses is given by $R(x) = 60x$. The company's cost, $C(x)$ in dollars, to produce x doghouses is $C(x) = 45x + 6000$.

a) Find the profit function, $P(x)$, that describes the company's profit from the sale of x doghouses.

b) What is the profit from the sale of 300 doghouses?

For Exercises 23 and 24, let x be the number of items sold (in hundreds), and let $R(x)$ and $C(x)$ be in thousands of dollars.

23) A manufacturer's revenue, $R(x)$, from the sale of flat-screen TVs is given by $R(x) = -0.2x^2 + 23x$, while the cost, $C(x)$, is given by $C(x) = 4x + 9$.

a) Find the profit function, $P(x)$, that describes the company's profit from the sale of x hundred flat-screen TVs.

b) What is the profit from the sale of 2000 TVs?

24) A manufacturer's revenue, $R(x)$, from the sale of laptop computers is given by $R(x) = -0.4x^2 + 30x$, while the cost, $C(x)$, is given by $C(x) = 3x + 11$.

a) Find the profit function, $P(x)$, that describes the company's profit from the sale of x hundred laptop computers.

b) What is the profit from the sale of 1500 laptop computers?

Objective 3: Find the Composition of Functions

25) Given two functions $f(x)$ and $g(x)$, explain how to find $(f \circ g)(x)$.

26) Given two functions $f(x)$ and $g(x)$, explain the difference between $(f \circ g)(x)$ and $(f \cdot g)(x)$.

For Exercises 27–28, find

a) $g(4)$ b) $(f \circ g)(4)$ using the result in part a)

c) $(f \circ g)(x)$ d) $(f \circ g)(4)$ using the result in part c)

27) $f(x) = 3x + 1, g(x) = 2x - 9$

28) $f(x) = x^2 - 5, g(x) = x + 3$

VIDEO 29) Let $f(x) = 5x - 4$ and $g(x) = x + 7$. Find

a) $(f \circ g)(x)$ b) $(g \circ f)(x)$

c) $(f \circ g)(3)$

30) Let $r(x) = 6x + 2$ and $v(x) = -7x - 5$. Find

a) $(v \circ r)(x)$ b) $(r \circ v)(x)$

c) $(r \circ v)(2)$

VIDEO 31) Let $g(x) = x^2 - 6x + 11$ and $h(x) = x - 4$. Find

a) $(h \circ g)(x)$ b) $(g \circ h)(x)$

c) $(g \circ h)(4)$

32) Let $f(x) = x^2 + 7x - 9$ and $g(x) = x + 2$. Find

a) $(g \circ f)(x)$ b) $(f \circ g)(x)$

c) $(g \circ f)(3)$

33) Let $f(x) = 2x^2 + 3x - 10$ and $g(x) = 3x - 5$. Find

a) $(f \circ g)(x)$ b) $(g \circ f)(x)$

c) $(f \circ g)(1)$

34) Let $h(x) = 3x^2 - 8x + 2$ and $k(x) = 2x - 3$. Find

a) $(h \circ k)(x)$ b) $(k \circ h)(x)$

c) $(k \circ h)(0)$

35) Let $m(x) = x + 8$ and $n(x) = -x^2 + 3x - 8$. Find

a) $(n \circ m)(x)$ b) $(m \circ n)(x)$

c) $(m \circ n)(0)$

36) Let $f(x) = -x^2 + 10x + 4$ and $g(x) = x + 1$. Find

a) $(g \circ f)(x)$ b) $(f \circ g)(x)$

c) $(f \circ g)(-2)$

37) Let $f(x) = \sqrt{x + 10}$, $g(x) = x^2 - 6$. Find

a) $(f \circ g)(x)$ b) $(g \circ f)(x)$

c) $(f \circ g)(-3)$

38) Let $h(x) = x^2 + 7$, $k(x) = \sqrt{x - 1}$. Find

a) $(h \circ k)(x)$ b) $(k \circ h)(x)$

c) $(k \circ h)(0)$

39) Let $P(t) = \dfrac{1}{t + 8}$, $Q(t) = t^2$. Find

a) $(P \circ Q)(t)$ b) $(Q \circ P)(t)$

c) $(Q \circ P)(-5)$

40) Let $F(a) = \dfrac{1}{5a}$, $G(a) = a^2$. Find

a) $(G \circ F)(a)$ b) $(F \circ G)(a)$

c) $(G \circ F)(-2)$

Objective 4: Solve an Applied Problem Using the Composition of Functions

VIDEO 41) Oil spilled from a ship off the coast of Alaska with the oil spreading out in a circle across the surface of the water. The radius of the oil spill is given by $r(t) = 4t$, where t is the number of minutes after the leak began and $r(t)$ is in feet. The area of the spill is given by $A(r) = \pi r^2$ where r represents the radius of the oil slick. Find each of the following and explain their meanings.

a) $r(5)$ b) $A(20)$ c) $A(r(t))$ d) $A(r(5))$

42) The radius of a circle is half its diameter. We can express this with the function $r(d) = \dfrac{1}{2}d$, where d is the diameter of a circle and r is the radius. The area of a circle in terms of its radius is $A(r) = \pi r^2$. Find each of the following and explain their meanings.

a) $r(6)$ b) $A(3)$ c) $A(r(d))$ d) $A(r(6))$

43) The sales tax on goods in a major metropolitan area is 7% so that the final cost of an item, $f(x)$, is given by $f(x) = 1.07x$, where x is the cost of the item. A women's clothing store is having a sale so that all of its merchandise is 20% off. If the regular price of an item is x dollars, then the sale price, $s(x)$, is given by $s(x) = 0.80x$. Find each of the following and explain their meanings.

a) $s(40)$ b) $f(32)$ c) $(f \circ s)(x)$ d) $(f \circ s)(40)$

44) The function $C(F) = \dfrac{5}{9}(F - 32)$ can be used to convert a temperature from degrees Fahrenheit, F, to degrees Celsius, C. The relationship between the Celsius scale, C, and the Kelvin scale, K, is given by $K(C) = C + 273$. Find each of the following and explain their meanings.

a) $C(59)$ b) $K(15)$ c) $K(C(F))$ d) $K(C(59))$

Extension

For Exercises 45–50, find $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x)$.

45) $h(x) = \sqrt{x^2 + 13}$

46) $h(x) = \sqrt{2x^2 + 7}$

47) $h(x) = (8x - 3)^2$

48) $h(x) = (4x + 9)^2$

49) $h(x) = \dfrac{1}{6x + 5}$

50) $h(x) = \dfrac{2}{x - 10}$

Section 12.6 Variation

Objectives

1. Solve Direct Variation Problems
2. Solve Inverse Variation Problems
3. Solve Joint Variation Problems
4. Solve Combined Variation Problems

1. Solve Direct Variation Problems

In Section 4.6, we discussed the following situation:

If you are driving on a highway at a constant speed of 60 mph, the distance you travel depends on the amount of time spent driving.

Let y = the distance traveled, in miles, and let x = the number of hours spent driving. An equation relating x and y is $y = 60x$.

A table of values relating x and y is to the right.

As the value of x increases, the value of y also increases. (The more hours you drive, the farther you will go.) Likewise, as the value of x decreases, the value of y also decreases. We can say that the distance traveled, y, is *directly proportional to* the time spent traveling, x. Or *y varies directly as x.*

x	y
1	60
1.5	90
2	120
3	180

Definition

Direct Variation: For $k > 0$, **y varies directly as x** (or **y is directly proportional to x**) means $y = kx$.

k is called the constant of variation.

If two quantities vary directly, then as one quantity increases, the other increases as well. And, as one quantity decreases, the other decreases.

In our example of driving distance, $y = 60x$, 60 *is the constant of variation*.

Given information about how variables are related, we can write an equation and solve a variation problem.

Example 1

Suppose y varies directly as x. If $y = 18$ when $x = 3$,

a) find the constant of variation, k.

b) write a variation equation relating x and y using the value of k found in a).

c) find y when $x = 11$.

Solution

a) To find the constant of variation, write a *general* variation equation relating x and y. *y varies directly as x means $y = kx$.*

We are told that $y = 18$ when $x = 3$. Substitute these values into the equation and solve for k.

$$\begin{aligned} y &= kx \\ 18 &= k(3) && \text{Substitute 3 for } x \text{ and 18 for } y. \\ 6 &= k && \text{Divide by 3.} \end{aligned}$$

b) The *specific* variation equation is the equation obtained when we substitute 6 for k in $y = kx$: $y = 6x$.

c) To find y when $x = 11$, substitute 11 for x in $y = 6x$ and evaluate.

$$\begin{aligned} y &= 6x \\ &= 6(11) && \text{Substitute 11 for } x. \\ &= 66 && \text{Multiply.} \end{aligned}$$

Procedure Steps for Solving a Variation Problem

Step 1: Write the *general* variation equation.

Step 2: Find k by substituting the known values into the equation and solving for k.

Step 3: Write the *specific* variation equation by substituting the value of k into the *general* variation equation.

Step 4: Use the specific variation equation to solve the problem.

You Try 1

Suppose y varies directly as x. If $y = 40$ when $x = 5$,

a) find the constant of variation.

b) write the specific variation equation relating x and y.

c) find y when $x = 3$.

Example 2

Suppose p varies directly as the square of z. If $p = 12$ when $z = 2$, find p when $z = 5$.

Solution

Step 1: Write the *general* variation equation.

p varies directly as the *square* of z means $p = kz^2$.

Step 2: Find k using the known values: $p = 12$ when $z = 2$.

$$\begin{aligned} p &= kz^2 \\ 12 &= k(2)^2 && \text{Substitute 2 for } z \text{ and 12 for } p. \\ 12 &= k(4) \\ 3 &= k \end{aligned}$$

Step 3: Substitute $k = 3$ into $p = kz^2$ to get the *specific* variation equation: $p = 3z^2$.

Step 4: We are asked to find p when $z = 5$. Substitute $z = 5$ into $p = 3z^2$ to get p.

$$\begin{aligned} p &= 3z^2 \\ &= 3(5)^2 && \text{Substitute 5 for } z. \\ &= 3(25) \\ &= 75 \end{aligned}$$

You Try 2

Suppose w varies directly as the cube of n. If $w = 135$ when $n = 3$, find w when $n = 2$.

Example 3

A theater's nightly revenue varies directly as the number of tickets sold. If the revenue from the sale of 80 tickets is \$3360, find the revenue from the sale of 95 tickets.

Solution

Let $n =$ the number of tickets sold
$R =$ revenue

We will follow the four steps for solving a variation problem.

Step 1: Write the *general* variation equation, $R = kn$.

Step 2: Find k using the known values: $R = 3360$ when $n = 80$.

$$\begin{aligned} R &= kn && \text{General variation equation} \\ 3360 &= k(80) && \text{Substitute 80 for } n \text{ and 3360 for } R. \\ 42 &= k && \text{Divide by 80.} \end{aligned}$$

Step 3: Substitute $k = 42$ into $R = kn$ to get the *specific* variation equation, $R = 42n$.

Step 4: We must find the revenue from the sale of 95 tickets. Substitute $n = 95$ into $R = 42n$ to find R.

$$R = 42n \qquad \text{Specific variation equation}$$
$$R = 42(95)$$
$$R = 3990$$

The revenue is \$3990.

You Try 3

The cost to carpet a room varies directly as the area of the room. If it costs \$525.00 to carpet a room of area 210 ft^2, how much would it cost to carpet a room of area 288 ft^2?

2. Solve Inverse Variation Problems

If two quantities vary *inversely* (are *inversely* proportional), then as one value increases, the other decreases. Likewise, as one value decreases, the other increases.

Definition

Inverse Variation: For $k > 0$, **y varies inversely as x** (or y is **inversely proportional to x**) means $y = \dfrac{k}{x}$.

k is the constant of variation.

A good example of inverse variation is the relationship between the time, t, it takes to travel a given distance, d, as a function of the rate (or speed), r. We can define this relationship as $t = \dfrac{d}{r}$. As the rate, r, increases, the time, t, that it takes to travel d mi decreases. Likewise, as r decreases, the time, t, that it takes to travel d mi increases. Therefore, t varies *inversely* as r.

Example 4

Suppose q varies inversely as h. If $q = 4$ when $h = 15$, find q when $h = 10$.

Solution

Step 1: Write the *general* variation equation, $q = \dfrac{k}{h}$.

Step 2: Find k using the known values: $q = 4$ when $h = 15$.

$$q = \frac{k}{h}$$
$$4 = \frac{k}{15} \qquad \text{Substitute 15 for } h \text{ and 4 for } q.$$
$$60 = k \qquad \text{Multiply by 15.}$$

Step 3: Substitute $k = 60$ into $q = \dfrac{k}{h}$ to get the *specific* variation equation, $q = \dfrac{60}{h}$.

Step 4: Substitute 10 for h in $q = \dfrac{60}{h}$ to find q.

$$q = \frac{60}{10}$$
$$q = 6$$

You Try 4

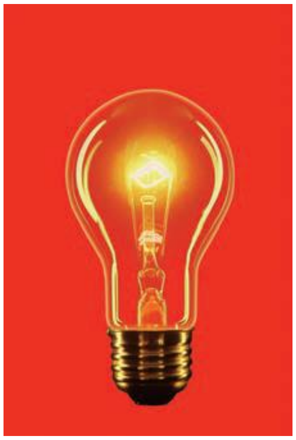
Suppose m varies inversely as the square of v. If $m = 1.5$ when $v = 4$, find m when $v = 2$.

Example 5

The intensity of light (in lumens) varies inversely as the square of the distance from the source. If the intensity of the light is 40 lumens 5 ft from the source, what is the intensity of the light 4 ft from the source?

Solution

Let d = distance from the source (in feet)
I = intensity of the light (in lumens)



Step 1: Write the *general* variation equation, $I = \dfrac{k}{d^2}$.

Step 2: Find k using the known values: $I = 40$ when $d = 5$.

$$
\begin{aligned}
I &= \frac{k}{d^2} && \text{General variation equation} \\
40 &= \frac{k}{(5)^2} && \text{Substitute 5 for } d \text{ and 40 for } I. \\
40 &= \frac{k}{25} \\
1000 &= k && \text{Multiply by 25.}
\end{aligned}
$$

Step 3: Substitute $k = 1000$ into $I = \dfrac{k}{d^2}$ to get the *specific* variation equation, $I = \dfrac{1000}{d^2}$.

Step 4: Find the intensity, I, of the light 4 ft from the source. Substitute $d = 4$ into $I = \dfrac{1000}{d^2}$ to find I.

$$
\begin{aligned}
I &= \frac{1000}{(4)^2} \\
&= \frac{1000}{16} = 62.5
\end{aligned}
$$

The intensity of the light is 62.5 lumens. ■

You Try 5

If the voltage in an electrical circuit is held constant (stays the same), then the current in the circuit varies inversely as the resistance. If the current is 40 amps when the resistance is 3 ohms, find the current when the resistance is 8 ohms.

3. Solve Joint Variation Problems

If a variable varies directly as the *product* of two or more other variables, the first variable *varies jointly* as the other variables.

Definition

Joint Variation: For $k > 0$, **y varies jointly as x and z** means $y = kxz$.

Example 6

For a given amount invested in a bank account (called the principal), the interest earned varies jointly as the interest rate (expressed as a decimal) and the time the principal is in the account. If Graham earns \$80 in interest when he invests his money for 1 yr at 4%, how much interest would the same principal earn if he invested it at 5% for 2 yr?

Solution

Let r = interest rate (as a decimal)
t = the number of years the principal is invested
I = interest earned

Step 1: Write the *general* variation equation, $I = krt$.

Step 2: Find k using the known values: $I = 80$ when $t = 1$ and $r = 0.04$.

$$
\begin{aligned}
I &= krt && \text{General variation equation} \\
80 &= k(0.04)(1) && \text{Substitute the values into } I = krt. \\
80 &= 0.04k \\
2000 &= k && \text{Divide by 0.04.}
\end{aligned}
$$

(The amount he invested, the principal, is \$2000.)

Step 3: Substitute $k = 2000$ into $I = krt$ to get the *specific* variation equation, $I = 2000rt$.

Step 4: Find the interest Graham would earn if he invested \$2000 at 5% interest for 2 yr. Let $r = 0.05$ and $t = 2$. Solve for I.

$$
\begin{aligned}
I &= 2000(0.05)(2) && \text{Substitute 0.05 for } r \text{ and 2 for } t. \\
&= 200 && \text{Multiply.}
\end{aligned}
$$

Graham would earn \$200.

You Try 6

The volume of a box of constant height varies jointly as its length and width. A box with a volume of 9 ft^3 has a length of 3 ft and a width of 2 ft. Find the volume of a box with the same height if its length is 4 ft and its width is 3 ft.

4. Solve Combined Variation Problems

A combined variation problem involves both direct and inverse variation.

Example 7

Suppose y varies directly as the square root of x and inversely as z. If $y = 12$ when $x = 36$ and $z = 5$, find y when $x = 81$ and $z = 15$.

Solution

Step 1: Write the *general* variation equation.

$$
y = \frac{k\sqrt{x}}{z} \qquad \begin{aligned} &\leftarrow y \text{ varies directly as the square root of } x. \\ &\leftarrow y \text{ varies inversely as } z. \end{aligned}
$$

Step 2: Find k using the known values: $y = 12$ when $x = 36$ and $z = 5$.

$$
\begin{aligned}
12 &= \frac{k\sqrt{36}}{5} && \text{Substitute the values.} \\
60 &= 6k && \text{Multiply by 5; } \sqrt{36} = 6. \\
10 &= k && \text{Divide by 6.}
\end{aligned}
$$

Step 3: Substitute $k = 10$ into $y = \dfrac{k\sqrt{x}}{z}$ to get the specific variation equation.

$$y = \frac{10\sqrt{x}}{z}$$

Step 4: Find y when $x = 81$ and $z = 15$.

$$y = \frac{10\sqrt{81}}{15} \qquad \text{Substitute 81 for } x \text{ and 15 for } z.$$

$$y = \frac{10 \cdot 9}{15} = \frac{90}{15} = 6$$

You Try 7

Suppose a varies directly as b and inversely as the square of c. If $a = 28$ when $b = 12$ and $c = 3$, find a when $b = 36$ and $c = 4$.

Answers to You Try Exercises

1) a) 8 b) $y = 8x$ c) 24 2) 40 3) \$720.00 4) 6 5) 15 amps
6) 18 ft^3 7) 47.25

12.6 Exercises

Mixed Exercises: Objectives 1–4

1) If z varies directly as y, then as y increases, the value of z ________.

2) If a varies inversely as b, then as b increases, the value of a ________.

Decide whether each equation represents direct, inverse, joint, or combined variation.

3) $y = 6x$

4) $c = 4ab$

5) $f = \dfrac{15}{t}$

6) $z = 3\sqrt{x}$

7) $p = \dfrac{8q^2}{r}$

8) $w = \dfrac{11}{v^2}$

Write a general variation equation using k as the constant of variation.

9) M varies directly as n.

10) q varies directly as r.

11) h varies inversely as j.

12) R varies inversely as B.

13) T varies inversely as the square of c.

14) b varies directly as the cube of w.

15) s varies jointly as r and t.

16) C varies jointly as A and D.

17) Q varies directly as the square root of z and inversely as m.

18) r varies directly as d and inversely as the square of L.

VIDEO 19) Suppose z varies directly as x. If $z = 63$ when $x = 7$,

a) find the constant of variation.

b) write the specific variation equation relating z and x.

c) find z when $x = 6$.

20) Suppose A varies directly as D. If $A = 12$ when $D = 3$,

a) find the constant of variation.

b) write the specific variation equation relating A and D.

c) find A when $D = 11$.

21) Suppose N varies inversely as y. If $N = 4$ when $y = 12$,

a) find the constant of variation.

b) write the specific variation equation relating N and y.

c) find N when $y = 3$.

22) Suppose j varies inversely as m. If $j = 7$ when $m = 9$,

a) find the constant of variation.

b) write the specific variation equation relating j and m.

c) find j when $m = 21$.

23) Suppose Q varies directly as the square of r and inversely as w. If $Q = 25$ when $r = 10$ and $w = 20$,

a) find the constant of variation.

b) write the specific variation equation relating Q, r, and w.

c) find Q when $r = 6$ and $w = 4$.

24) Suppose y varies jointly as a and the square root of b. If $y = 42$ when $a = 3$ and $b = 49$,

a) find the constant of variation.

b) write the specific variation equation relating y, a, and b.

c) find y when $a = 4$ and $b = 9$.

Solve.

25) If B varies directly as R, and $B = 35$ when $R = 5$, find B when $R = 8$.

26) If q varies directly as p, and $q = 10$ when $p = 4$, find q when $p = 10$.

27) If L varies inversely as the square of h, and $L = 8$ when $h = 3$, find L when $h = 2$.

28) If w varies inversely as d, and $w = 3$ when $d = 10$, find w when $d = 5$.

29) If y varies jointly as x and z, and $y = 60$ when $x = 4$ and $z = 3$, find y when $x = 7$ and $z = 2$.

30) If R varies directly as P and inversely as the square of Q, and $R = 5$ when $P = 10$ and $Q = 4$, find R when $P = 18$ and $Q = 3$.

Solve each problem by writing a variation equation.

31) Kosta is paid hourly at his job. His weekly earnings vary directly as the number of hours worked. If Kosta earned \$437.50 when he worked 35 hr, how much would he earn if he worked 40 hr?

32) The cost of manufacturing a certain brand of spiral notebook is inversely proportional to the number produced. When 16,000 notebooks are produced, the cost per notebook is \$0.60. What is the cost of each notebook when 12,000 are produced?

33) If distance is held constant, the time it takes to travel that distance is inversely proportional to the speed at which one travels. If it takes 14 hr to travel the given distance at 60 mph, how long would it take to travel the same distance at 70 mph?

34) The surface area of a cube varies directly as the square of the length of one of its sides. A cube has a surface area of 54 in^2 when the length of each side is 3 in. What is the surface area of a cube with a side of length 6 in.?

35) The power in an electrical system varies jointly as the current and the square of the resistance. If the power is 100 watts when the current is 4 amps and the resistance is 5 ohms, what is the power when the current is 5 amps and the resistance is 6 ohms?

36) The force exerted on an object varies jointly as the mass and acceleration of the object. If a 20-newton force is exerted on an object of mass 10 kg and an acceleration of 2 m/sec^2, how much force is exerted on a 50-kg object with an acceleration of 8 m/sec^2?

37) The kinetic energy of an object varies jointly as its mass and the square of its speed. When a roller coaster car with a mass of 1000 kg is traveling at 15 m/sec, its kinetic energy is 112,500 J (joules). What is the kinetic energy of the same car when it travels at 18 m/sec?

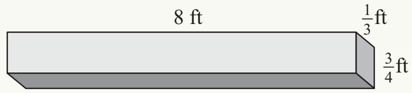
38) The volume of a cylinder varies jointly as its height and the square of its radius. The volume of a cylindrical can is 108π cm^3 when its radius is 3 cm and it is 12 cm high. Find the volume of a cylindrical can with a radius of 4 cm and a height of 3 cm.

39) The frequency of a vibrating string varies inversely as its length. If a 5-ft-long piano string vibrates at 100 cycles/sec, what is the frequency of a piano string that is 2.5 ft long?

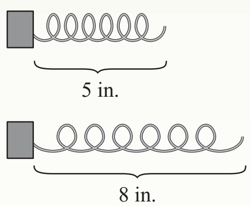
40) The amount of pollution produced varies directly as the population. If a city of 500,000 people produces 800,000 tons of pollutants, how many tons of pollutants would be produced by a city of 1,000,000 people?

41) The resistance of a wire varies directly as its length and inversely as its cross-sectional area. A wire of length 40 cm and cross-sectional area 0.05 cm^2 has a resistance of 2 ohms. Find the resistance of 60 cm of the same type of wire.

42) When a rectangular beam is positioned horizontally, the maximum weight that it can support varies jointly as its width and the square of its thickness and inversely as its length. A beam is $\frac{3}{4}$ ft wide, $\frac{1}{3}$ ft thick, and 8 ft long, and it can support 17.5 tons. How much weight can a similar beam support if it is 1 ft wide, $\frac{1}{2}$ ft thick and 12 ft long?

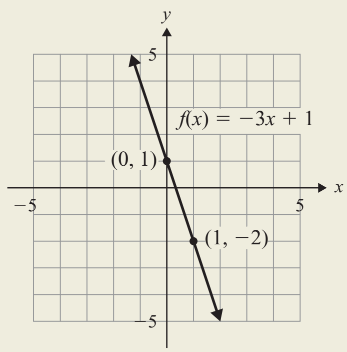


43) Hooke's law states that the force required to stretch a spring is proportional to the distance that the spring is stretched from its original length. A force of 200 lb is required to stretch a spring 5 in. from its natural length. How much force is needed to stretch the spring 8 in. beyond its natural length?



44) The weight of an object on Earth varies inversely as the square of its distance from the center of the earth. If an object weighs 300 lb on the surface of the earth (4000 mi from the center), what is the weight of the object if it is 800 mi above the earth? (Round to the nearest pound.)

Chapter 12: Summary

Definition/Procedure	Example
12.1 Relations and Functions	
A **relation** is any set of ordered pairs. A relation can also be represented as a correspondence or mapping from one set to another or as an equation. **(p. 696)**	a) $\{(-6, 2), (-3, 1), (0, 0), (9, -3)\}$ b) $y^2 = x$
The **domain** of a relation is the set of values of the independent variable (the first coordinates in the set of ordered pairs). The **range** of a relation is the set of all values of the dependent variable (the second coordinates in the set of ordered pairs). If a relation is written as an equation so that y is in terms of x, then the **domain** is the set of all real numbers that can be substituted for the independent variable, x. The resulting set of real numbers that are obtained for y, the dependent variable, is the **range**. **(p. 696)**	In a) above, the domain is $\{-6, -3, 0, 9\}$, and the range is $\{-3, 0, 1, 2\}$. In b) above, the domain is $[0, \infty)$, and the range is $(-\infty, \infty)$.
A **function** is a relation in which each element of the domain corresponds to *exactly one* element of the range. **(p. 696)**	The relation in a) *is* a function. The relation in b) *is not* a function since there are elements of the domain that correspond to more than one element in the range. For example, if $x = 4$, then $y = 2$ or $y = -2$.
$y = f(x)$ is called **function notation**, and it is read as, "y equals f of x." **(p. 698)**	Let $f(x) = 7x + 2$. Find $f(-2)$. $f(-2) = 7(-2) + 2$ Substitute -2 for x. $= -14 + 2$ Multiply. $= -12$ Add.
A **linear function** has the form $f(x) = mx + b$, where m and b are real numbers, m is the *slope* of the line, and $(0, b)$ is the *y-intercept*. **(p. 698)**	Given $f(x) = -3x + 1$, determine the domain of f and graph the function. To determine the domain, ask yourself, *"Is there any number that cannot be substituted for x in f(x) = −3x + 1?"* No. Any number can be substituted for x and the function will be defined. Therefore, the domain consists of all real numbers. This is written as $(-\infty, \infty)$. This is a linear function, so its graph is a line. 
Polynomial Functions The **domain of a polynomial function** is all real numbers, $(-\infty, \infty)$, since any real number can be substituted for the variable. **(p. 699)**	$f(x) = 2x^3 - 9x^2 + 8x + 4$ is an example of a **polynomial function** since $2x^3 - 9x^2 + 8x + 4$ is a polynomial.
Rational Functions The **domain of a rational function** consists of all real numbers *except* the value(s) of the variable that make the denominator equal zero. **(p. 701)**	$g(x) = \dfrac{7x}{x-2}$ is an example of a **rational function** since $\dfrac{7x}{x-2}$ is a rational expression. To determine the domain, ask yourself, *"Is there any number that cannot be substituted for x in* $g(x) = \dfrac{7x}{x-2}$*?"* Yes. We cannot substitute 2 for x because the denominator would equal zero so the function would be undefined. The domain of $g(x)$ is $(-\infty, 2) \cup (2, \infty)$.

Definition/Procedure	Example

Square Root Functions

When we determine the **domain of a square root function**, we are determining all *real* numbers that may be substituted for the variable so that the range contains only *real* numbers. Therefore, a square root function is defined only when its radicand is nonnegative. This means that there may be *many* values that are excluded from the domain of a square root function. **(p. 702)**

$h(x) = \sqrt{x + 8}$ is an example of a **square root function**.

Determine the domain of $h(x) = \sqrt{x + 8}$.

Solve the inequality $x + 8 \geq 0$.

$x + 8 \geq 0$ The value of the radicand must be ≥ 0.

$x \geq -8$ Solve.

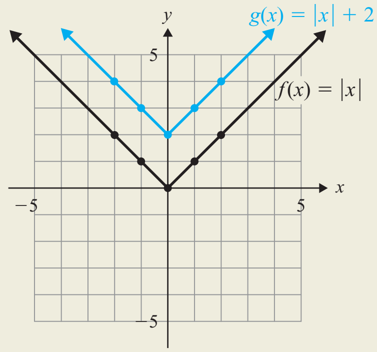
The domain is $[-8, \infty)$.

12.2 Graphs of Functions and Transformations

Vertical Shifts

Given the graph of $f(x)$, if $g(x) = f(x) + k$, where k is a constant, then the graph of $g(x)$ will be the same shape as the graph of $f(x)$, but the graph of g will be shifted *vertically* k units. **(p. 707)**

The graph of $g(x) = |x| + 2$ will be the same shape as the graph of $f(x) = |x|$, but $g(x)$ is shifted *up* 2 units.

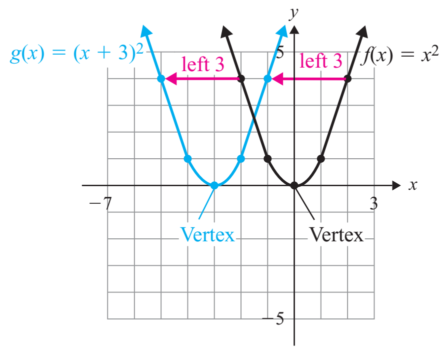


Functions f and g are **absolute value functions**.

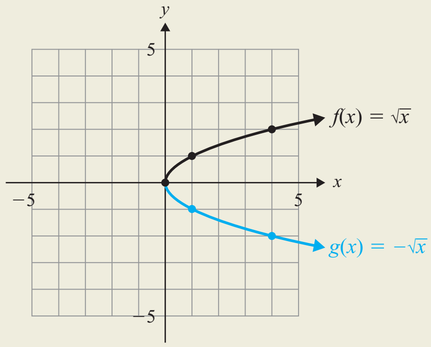
Horizontal Shifts

Given the graph of $f(x)$, if $g(x) = f(x - h)$, where h is a constant, the graph of $g(x)$ will be the same shape as the graph of $f(x)$, but the graph of g will be shifted *horizontally* h units. **(p. 708)**

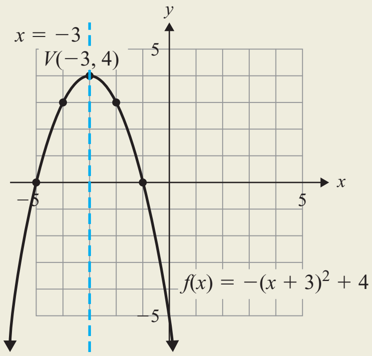
The graph of $g(x) = (x + 3)^2$ will be the same shape as the graph of $f(x) = x^2$, but g is shifted *left* 3 units.

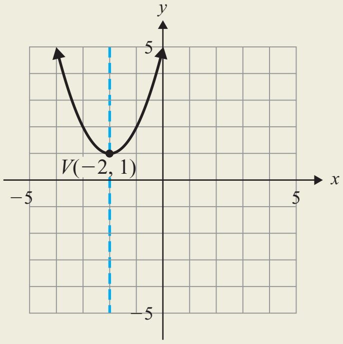


Functions f and g are **quadratic functions**, and their graphs are called **parabolas**.

Definition/Procedure	Example
Reflection About the *x*-Axis Given the graph of $f(x)$, if $g(x) = -f(x)$, then the graph of $g(x)$ will be the *reflection of the graph of f about the x-axis*. That is, obtain the graph of g by keeping the x-coordinate of each point on the graph of f the same but take the negative of the y-coordinate. **(p. 709)**	Let $f(x) = \sqrt{x}$ and $g(x) = -\sqrt{x}$. The graph of $g(x)$ will be the reflection of $f(x)$ about the x-axis. 
A **piecewise function** is a single function defined by two or more different rules. **(p. 710)**	$f(x) = \begin{cases} x + 3, & x > -2 \\ -\frac{1}{2}x + 2, & x \le -2 \end{cases}$
The **greatest integer function**, $f(x) = [\![x]\!]$, represents the largest integer less than or equal to x. **(p. 711)**	$[\![8.3]\!] = 8$ $\left[\!\!\left[-4\frac{3}{8}\right]\!\!\right] = -5$

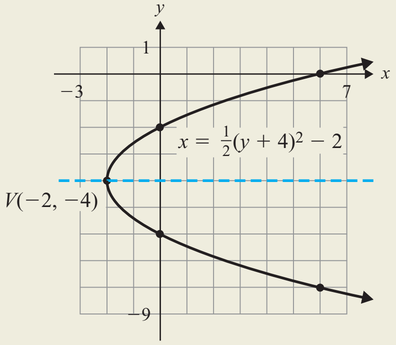
12.3 Quadratic Functions and Their Graphs

A **quadratic function** is a function that can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \ne 0$. The graph of a quadratic function is called a **parabola**. **(p. 718)**	$f(x) = 5x^2 + 7x - 9$
A quadratic function can also be written in the form $f(x) = a(x - h)^2 + k$. From this form, we can learn a great deal of information. 1) The vertex of the parabola is (h, k). 2) The axis of symmetry is the vertical line with equation $x = h$. 3) If a is positive, the parabola opens upward. If a is negative, the parabola opens downward. 4) If $\lvert a \rvert < 1$, then the graph of $f(x) = a(x - h)^2 + k$ is *wider* than the graph of $y = x^2$. If $\lvert a \rvert > 1$, then the graph of $f(x) = a(x - h)^2 + k$ is *narrower* than the graph of $y = x^2$. **(p. 719)**	Graph $f(x) = -(x + 3)^2 + 4$.  Vertex: $(-3, 4)$ Axis of symmetry: $x = -3$ $a = -1$, so the graph opens downward.

Definition/Procedure	Example
When a quadratic function is written in the form $f(x) = ax^2 + bx + c$, there are two methods we can use to graph the function. **Method 1:** Rewrite $f(x) = ax^2 + bx + c$ in the form $f(x) = a(x - h)^2 + k$ by *completing the square.* **Method 2:** Use the formula $h = -\frac{b}{2a}$ to find the x-coordinate of the vertex. Then, the vertex has coordinates $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. **(p. 721)**	Graph $f(x) = x^2 + 4x + 5$. **Method 1:** Complete the square. $f(x) = x^2 + 4x + 5$ $f(x) = (x^2 + 4x + 2^2) + 5 - 2^2$ $f(x) = (x^2 + 4x + 4) + 5 - 4$ $f(x) = (x + 2)^2 + 1$ The vertex of the parabola is $(-2, 1)$. **Method 2:** Use the formula $h = -\frac{b}{2a}$. $h = -\frac{4}{2(1)} = -2$. Then, $f(-2) = 1$. The vertex of the parabola is $(-2, 1)$. 

12.4 Applications of Quadratic Functions and Graphing Other Parabolas

Maximum and Minimum Values of a Quadratic Function Let $f(x) = ax^2 + bx + c$. 1) If a is **positive,** the y-coordinate of the vertex is the **minimum** value of the function $f(x)$. 2) If a is **negative,** the y-coordinate of the vertex is the **maximum** value of the function $f(x)$. **(p. 728)**	Find the minimum value of the function $f(x) = 2x^2 + 12x + 7$. $a = 2$. Since a is positive, the function attains its *minimum* value at the vertex. x-coordinate of vertex: $h = -\frac{b}{2a} = -\frac{12}{2(2)} = -3$ y-coordinate of vertex: $f(-3) = 2(-3)^2 + 12(-3) + 7$ $= 18 - 36 + 7 = -11$ The minimum value of the function is -11.

Definition/Procedure	Example
The graph of the quadratic equation $x = ay^2 + by + c$ is a parabola that opens in the x-direction, or horizontally. The quadratic equation $x = ay^2 + by + c$ can also be written in the form $x = a(y - k)^2 + h$. When it is written in this form we can find the following. 1) The vertex of the parabola is (h, k). 2) The axis of symmetry is the horizontal line $y = k$. 3) If a is positive, the graph opens to the right. If a is negative, the graph opens to the left. **(p. 732)**	Graph $x = \frac{1}{2}(y + 4)^2 - 2$.  Vertex: $(-2, -4)$ Axis of symmetry: $y = -4$ $a = \frac{1}{2}$, so the graph opens to the right.

12.5 The Algebra of Functions

Operations Involving Functions Given the functions $f(x)$ and $g(x)$, we can find their sum, difference, product, and quotient. **(p. 739)**	Let $f(x) = 5x - 1$ and $g(x) = x + 4$. $$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ &= (5x - 1) + (x + 4)\\ &= 6x + 3\end{aligned}$$
Composition of Functions Given the functions $f(x)$ and $g(x)$, the **composition function** $f \circ g$ (read "f of g") is defined as $$(f \circ g)(x) = f(g(x))$$ where $g(x)$ is in the domain of f. **(p. 740)**	$f(x) = 4x - 10$ and $g(x) = -3x + 2$. $$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f(-3x + 2)\\ &= 4(-3x + 2) - 10 && \text{Substitute } -3x + 2 \text{ for } x \text{ in } f(x).\\ &= -12x + 8 - 10\\ &= -12x - 2\end{aligned}$$

12.6 Variation

Direct Variation For $k > 0$, **y varies directly as x** (or *y is directly proportional to x*) means $y = kx$. k is called the **constant of variation. (p. 747)**	The circumference, C, of a circle is given by $C = 2\pi r$. C varies directly as r where $k = 2\pi$.
Inverse Variation For $k > 0$, **y varies inversely as x** (or *y is inversely proportional to x*) means $y = \frac{k}{x}$. **(p. 749)**	The time, t (in hours), it takes to drive 600 mi is inversely proportional to the rate, r, at which you drive. $$t = \frac{600}{r}$$ The constant of variation, k, equals 600.

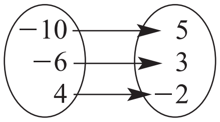
Definition/Procedure	Example
Joint Variation For $k > 0$, **y varies jointly as x and z** means $y = kxz$. **(p. 750)**	For a given amount, called the principal, deposited in a bank account, the interest earned, I, varies jointly as the interest rate, r, and the time, t, the principal is in the account. $I = 1000rt$ The constant of variation, k, equals 1000. This is the amount of money deposited in the account, the principal.
Steps for Solving a Variation Problem **Step 1:** Write the **general** variation equation. **Step 2:** Find k by substituting the known values into the equation and solving for k. **Step 3:** Write the **specific** variation equation by substituting the value of k into the *general* variation equation. **Step 4:** Substitute the given values into the *specific* equation to find the required value. **(p. 747)**	The cost of manufacturing a certain soccer ball is inversely proportional to the number produced. When 15,000 are made, the cost per ball is \$4.00. What is the cost to manufacture each soccer ball when 25,000 are produced? Let n = number of soccer balls produced C = cost of producing each ball **Step 1:** *General* variation equation: $C = \frac{k}{n}$ **Step 2:** Find k using $C = 4$ when $n = 15{,}000$. $4 = \frac{k}{15{,}000}$ $60{,}000 = k$ **Step 3:** *Specific* variation equation: $C = \frac{60{,}000}{n}$ **Step 4:** Find the cost, C, per ball when $n = 25{,}000$. $C = \frac{60{,}000}{25{,}000}$ Substitute 25,000 for n. $= 2.4$ The cost per ball is \$2.40.

Chapter 12: Review Exercises

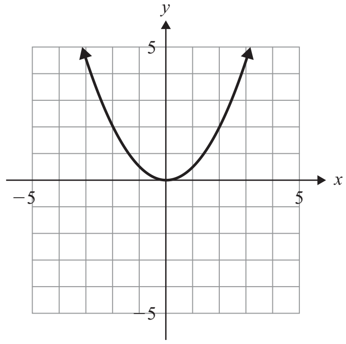
(12.1) Identify the domain and range of each relation, and determine whether each relation is a function.

1) $\{(-7, -4), (-5, -1), (2, 3), (2, 5), (4, 9)\}$

2)



3)



4) $y = 6x - 4$

Determine whether each relation describes y as a function of *x*.

5) $y = -\dfrac{5}{2}x + 3$

6) $y = \dfrac{x + 4}{2x - 3}$

7) $x = |y|$

8) $y = |x|$

9) $y = -\sqrt{3x - 8}$

10) $y^2 = x - 1$

11) $y = x^2 - 10x + 7$

12) $y = \sqrt{-x}$

Graph each function.

13) $f(x) = 2x - 5$

14) $h(x) = -\dfrac{1}{3}x + 4$

15) $g(t) = t$

16) $r(a) = 2$

17) Let $f(x) = -8x + 3$ and $g(x) = x^2 + 7x - 12$. Find each of the following and simplify.

a) $f(5)$
b) $f(-4)$
c) $g(-2)$
d) $g(3)$
e) $f(c)$
f) $g(r)$
g) $f(p - 3)$
h) $g(t + 4)$

18) $h(x) = 3x + 10$. Find x so that $h(x) = -8$.

19) $k(x) = -\dfrac{2}{3}x + 8$. Find x so that $k(x) = 0$.

20) $g(x) = x^2 + 3x - 28$. Find x so that $g(x) = 0$.

21) $p(x) = x^2 - 8x + 15$. Find x so that $p(x) = 3$.

22) How do you find the domain of a

a) square root function?

b) rational function?

Determine the domain of each function.

23) $f(x) = \dfrac{9x}{x - 5}$

24) $P(a) = 3a^4 - 7a^3 + 12a^2 + a - 8$

25) $C(n) = -4n - 9$

26) $g(x) = \sqrt{x}$

27) $h(t) = \sqrt{5t - 7}$

28) $R(c) = \dfrac{c + 8}{2c + 3}$

29) $g(x) = \dfrac{6}{x}$

30) $f(p) = \sqrt{3 - 4p}$

31) $k(c) = c^2 + 11c + 2$

32) $Q(x) = -\dfrac{4x + 1}{10}$

33) $h(a) = \dfrac{a + 6}{a^2 - 7a - 8}$

34) $s(r) = \sqrt{r + 12}$

35) To rent a car for one day, a company charges its customers a flat fee of \$26 plus \$0.20 per mile. This can be described by the function $C(m) = 0.20m + 26$, where m is the number of miles driven and C is the cost of renting a car, in dollars.

a) What is the cost of renting a car that is driven 30 mi?

b) What is the cost of renting a car that is driven 100 mi?

c) If a customer paid \$56 to rent a car, how many miles did she drive?

d) If a customer paid \$42 to rent a car, how many miles did he drive?

36) The area, A, of a square is a function of the length of its side, s.

a) Write an equation using function notation to describe this relationship between A and s.

b) If the length of a side is given in inches, find $A(4)$ and explain what this means in the context of the problem.

c) If the length of a side is given in feet, find $A(1)$ and explain what this means in the context of the problem.

d) What is the length of each side of a square that has an area of 49 cm^2?

(12.2) Graph each function, and identify the domain and range.

37) $f(x) = \sqrt{x}$

38) $g(x) = |x|$

39) $h(x) = (x + 4)^2$

40) $f(x) = \sqrt{x} - 3$

41) $k(x) = -|x| + 5$

42) $h(x) = |x - 1| + 2$

43) $g(x) = \sqrt{x - 2} - 1$

44) $f(x) = \dfrac{1}{2}\sqrt{x}$

Graph each piecewise function.

45) $f(x) = \begin{cases} -\frac{1}{2}x - 2, & x \le 2 \\ x - 3, & x > 2 \end{cases}$

46) $g(x) = \begin{cases} 1, & x < -3 \\ x + 4, & x \ge -3 \end{cases}$

Let $f(x) = [\![x]\!]$. Find the following function values.

47) $f\left(7\frac{2}{3}\right)$

48) $f(2.1)$

49) $f\left(-8\frac{1}{2}\right)$

50) $f(-5.8)$

51) $f\left(\frac{3}{8}\right)$

Graph each greatest integer function.

52) $f(x) = [\![x]\!]$

53) $g(x) = \left[\!\left[\frac{1}{2}x\right]\!\right]$

54) To mail a letter from the United States to Mexico in 2005 cost \$0.60 for the first ounce, \$0.85 over one ounce but less than or equal to 2 oz, then \$0.40 for each additional ounce or fraction of an ounce. Let $C(x)$ represent the cost of mailing a letter from the U.S. to Mexico, and let x represent the weight of the letter, in ounces. Graph $C(x)$ for any letter weighing up to (and including) 5 oz. (www.usps.com)

If the following transformations are performed on the graph of $f(x)$ to obtain the graph of $g(x)$, write the equation of $g(x)$.

55) $f(x) = |x|$ is shifted right 5 units.

56) $f(x) = \sqrt{x}$ is shifted left 2 units and up 1 unit.

(12.3 and 12.4)

57) Given a quadratic function in the form $f(x) = a(x - h)^2 + k$, answer the following.

a) What is the vertex?

b) What is the equation of the axis of symmetry?

c) What does the sign of a tell us about the graph of f?

58) What are two ways to find the vertex of the graph of $f(x) = ax^2 + bx + c$?

59) Given a quadratic equation of the form $x = a(y - k)^2 + h$, answer the following.

a) What is the vertex?

b) What is the equation of the axis of symmetry?

c) What does the sign of a tell us about the graph of the equation?

60) What are two ways to find the vertex of the graph of $x = ay^2 + by + c$?

For each quadratic equation, identify the vertex, axis of symmetry, and x- and y-intercepts. Then, graph the equation.

61) $f(x) = (x + 2)^2 - 1$

62) $g(x) = -\frac{1}{2}(x - 3)^2 - 2$

63) $x = -y^2 - 1$

64) $y = 2x^2$

65) $x = -(y - 3)^2 + 11$

66) $x = (y + 1)^2 - 5$

Rewrite each equation in the form $f(x) = a(x - h)^2 + k$ or $x = a(y - k)^2 + h$ by completing the square. Then graph the function. Include the intercepts.

67) $x = y^2 + 8y + 7$

68) $f(x) = -2x^2 - 8x + 2$

69) $y = \frac{1}{2}x^2 - 4x + 9$

70) $x = -y^2 + 4y - 4$

Graph each equation using the vertex formula. Include the intercepts.

71) $f(x) = x^2 - 2x - 4$

72) $x = 3y^2 - 12y$

73) $x = -\frac{1}{2}y^2 - 3y - \frac{5}{2}$

74) $y = -x^2 - 6x - 10$

Solve.

75) An object is thrown upward from a height of 240 ft so that its height h (in feet) t sec after being thrown is given by

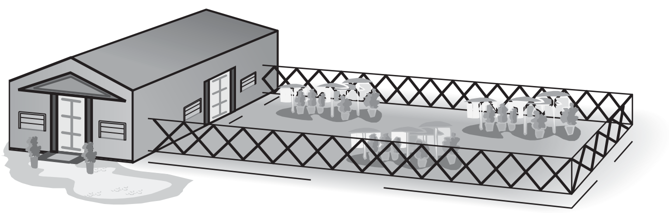
$$h(t) = -16t^2 + 32t + 240$$

a) How long does it take the object to reach its maximum height?

b) What is the maximum height attained by the object?

c) How long does it take the object to hit the ground?

76) A restaurant wants to add outdoor seating to its inside service. It has 56 ft of fencing to enclose a rectangular, outdoor café. Find the dimensions of the outdoor café of maximum area if the building will serve as one side of the café.



(12.5) Let $f(x) = 5x + 2$, $g(x) = -x + 4$, $h(x) = 3x^2 - 7$, and $k(x) = x^2 - 7x - 8$. Find each of the following.

77) $(f + g)(x)$

78) $(h - k)(x)$

79) $(g - h)(2)$

80) $(f + k)(-3)$

81) $(fg)(x)$

82) $(gk)(1)$

For each pair of functions, find a) $\left(\dfrac{f}{g}\right)(x)$ and b) $\left(\dfrac{f}{g}\right)(3)$. Identify any values that are not in the domain of $\left(\dfrac{f}{g}\right)(x)$.

83) $f(x) = 6x - 5, g(x) = x + 4$

84) $f(x) = 3x^2 - 5x + 2, g(x) = 3x - 2$

85) $R(x) = 20x$ is the revenue function for the sale of x children's soccer uniforms, in dollars. The cost to produce x soccer uniforms, in dollars, is

$$C(x) = 14x + 400$$

a) Find the profit function, $P(x)$, that describes the profit from the sale of x uniforms.

b) What is the profit from the sale of 200 uniforms?

86) Let $f(x) = x + 6$ and $g(x) = 2x - 9$. Find

a) $(g \circ f)(x)$

b) $(f \circ g)(x)$

c) $(f \circ g)(5)$

87) Let $h(x) = 2x - 1$ and $k(x) = x^2 + 5x - 4$. Find

a) $(k \circ h)(x)$

b) $(h \circ k)(x)$

c) $(h \circ k)(-3)$

88) Let $g(x) = x^2 + 10$ and $h(x) = \sqrt{x - 7}$. Find

a) $(g \circ h)(x)$

b) $(h \circ g)(x)$

c) $(h \circ g)(6)$

89) Antoine's gross weekly pay, G, in terms of the number of hours, h, he worked is given by $G(h) = 12h$. His net weekly pay, N, in terms of his gross pay is given by $N(G) = 0.8G$.

a) Find $(N \circ G)(h)$ and explain what it represents.

b) Find $(N \circ G)(30)$ and explain what it represents.

c) What is his net pay if he works 40 hr in 1 week?

(12.6)

90) Suppose c varies directly as m. If $c = 56$ when $m = 8$, find c when $m = 3$.

91) Suppose A varies jointly as t and r. If $A = 15$ when $t = \dfrac{1}{2}$ and $r = 5$, find A when $t = 3$ and $r = 4$.

92) Suppose p varies directly as n and inversely as the square of d. If $p = 42$ when $n = 7$ and $d = 2$, find p when $n = 12$ and $d = 3$.

Solve each problem by writing a variation equation.

93) The weight of a ball varies directly as the cube of its radius. If a ball with a radius of 2 in. weighs 0.96 lb, how much would a ball made out of the same material weigh if it had a radius of 3 in.?

94) If the temperature remains the same, the volume of a gas is inversely proportional to the pressure. If the volume of a gas is 10 L (liters) at a pressure of 1.25 atm (atmospheres), what is the volume of the gas at 2 atm?

Chapter 12: Test

1) What is a function?

2) Given the relation $\{(-8, -1), (2, 3), (5, 3), (7, 10)\}$,

a) determine the domain.

b) determine the range.

c) is this a function?

3) For $y = \sqrt{3x + 7}$,

a) determine the domain.

b) is y a function of x?

Determine the domain of each function.

4) $f(x) = \frac{4}{9}x - 2$

5) $g(t) = \frac{t + 10}{7t - 8}$

Let $f(x) = 4x + 3$ and $g(x) = x^2 - 6x + 10$. Find each of the following and simplify.

6) $g(4)$

7) $f(c)$

8) $f(n - 7)$

9) $g(k + 5)$

10) Let $h(x) = -2x + 6$. Find x so that $h(x) = 9$.

11) A garden supply store charges \$50 per cubic yard plus a \$60 delivery fee to deliver cedar mulch. This can be described by the function $C(m) = 50m + 60$, where m is the amount of cedar mulch delivered, in cubic yards, and C is the cost, in dollars.

a) Find $C(3)$ and explain what it means in the context of the problem.

b) If a customer paid \$360 to have cedar mulch delivered to his home, how much did he order?

Graph each function and identify the domain and range.

12) $f(x) = |x| - 4$

13) $g(x) = \sqrt{x + 3}$

14) $h(x) = -\frac{1}{3}x - 1$

15) Graph $f(x) = \begin{cases} x + 3, & x > -1 \\ -2x - 5, & x \le -1 \end{cases}$

Graph each equation. Identify the vertex, axis of symmetry, and intercepts.

16) $f(x) = -(x + 2)^2 + 4$

17) $x = y^2 - 3$

18) $x = 3y^2 - 6y + 5$

19) $g(x) = x^2 - 6x + 8$

20) A rock is thrown upward from a cliff so that it falls into the ocean below. The height h (in feet) of the rock t sec after being thrown is given by

$$h(t) = -16t^2 + 64t + 80$$

a) What is the maximum height attained by the rock?

b) How long does it take the rock to hit the water?

Let $f(x) = 2x + 7$ and $g(x) = x^2 + 5x - 3$. Find each of the following.

21) $(g - f)(x)$

22) $(f + g)(-1)$

23) $(f \circ g)(x)$

24) $(g \circ f)(x)$

25) Suppose n varies jointly as r and the square of s. If $n = 72$ when $r = 2$ and $s = 3$, find n when $r = 3$ and $s = 5$.

26) The loudness of a sound is inversely proportional to the square of the distance between the source of the sound and the listener. If the sound level measures 112.5 decibels (dB) 4 ft from a speaker, how loud is the sound 10 ft from the speaker?

Cumulative Review: Chapters 1–12

What property is illustrated by each statement? Choose from commutative, associative, distributive, inverse, and identity.

1) $4 \cdot \frac{1}{4} = 1$

2) $16 + 5 = 5 + 16$

Evaluate.

3) $\left(\frac{1}{2}\right)^5$

4) 5^{-3}

5) 10^0

6) Solve $6(2y + 1) - 4y = 5(y + 2)$.

7) Solve the compound inequality

$$x + 8 \le 6 \text{ or } 1 - 2x \le -5$$

Graph the solution set and write the answer in interval notation.

8) Write the equation of the line parallel to $4x + 3y = 15$ containing the point $(-5, 6)$. Express it in standard form.

9) Solve this system using any method.

$$x - \frac{1}{4}y = \frac{5}{2}$$
$$\frac{1}{2}x + \frac{1}{3}y = \frac{13}{6}$$

10) Multiply and simplify $(p - 8)(p + 7)$.

11) Divide $\dfrac{12r - 40r^2 + 6r^3 + 4}{4r^2}$.

Factor completely.

12) $k^2 - 15k + 54$

13) $100 - 9m^2$

14) Solve $(a + 4)(a + 1) = 18$.

15) Divide $\dfrac{c - 8}{2c^2 - 5c - 12} \div \dfrac{3c - 24}{c^2 - 16}$.

16) Solve $\dfrac{4x + 2}{x + 5} = 10$.

17) Solve $|7y + 6| \le -8$.

18) Graph the compound inequality

$$y \le -\frac{1}{2}x + 4 \text{ and } 2x - y \le 2$$

Simplify. Assume all variables represent nonnegative real numbers.

19) $\sqrt{60}$

20) $\sqrt[4]{16}$

21) $\sqrt{18c^6d^{11}}$

22) $(100)^{-3/2}$

23) Add $\sqrt{12} + \sqrt{3} + \sqrt{48}$.

24) Solve $x = -6 + \sqrt{x + 8}$.

25) Divide $\dfrac{4 - 2i}{2 + 3i}$.

Solve.

26) $(3n - 4)^2 + 9 = 0$

27) $4(y^2 + 2y) = 5$

28) $q - 8q^{1/2} + 7 = 0$

29) Let $g(x) = x + 1$ and $h(x) = x^2 + 4x + 3$.

a) Find $g(7)$.

b) Find $\left(\frac{h}{g}\right)(x)$. Identify any values that are not in the domain of $\left(\frac{h}{g}\right)(x)$.

c) Find x so that $g(x) = 5$.

d) Find $(g \circ h)(x)$.

30) Graph $f(x) = -x^2 + 4$ and identify the domain and range.

31) Graph $x = -y^2 - 2y - 3$.

32) Suppose y varies inversely as the square of x. If $y = 12$ when $x = 2$, find y when $x = 4$.

CHAPTER 13

Exponential and Logarithmic Functions

Algebra at Work: Finances

Financial planners use mathematical formulas involving exponential functions every day to help clients invest their money. Interest on investments can be paid in various ways—monthly, weekly, continuously, etc.

Isabella has a client who wants to invest \$5000 for 4 yr. He would like to know whether it is better to invest it at 4.8% compounded continuously or at 5% compounded monthly.

To determine how much the investment will be worth after 4 yr if it is put in the account with interest compounded continuously, Isabella uses the formula $A = Pe^{rt}$ ($e \approx 2.71828$). To determine how much money her client will have after 4 yr if he puts his \$5000 into the account earning 5% interest compounded monthly, Isabella uses the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$.

Compounded Continuously

$$A = Pe^{rt}$$
$$A = 5000e^{(0.048)(4)}$$
$$A = \$6058.35$$

Compounded Monthly

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$A = (5000)\left(1 + \frac{0.05}{12}\right)^{(12)(4)}$$
$$A = \$6104.48$$

If her client invests \$5000 in the account paying 4.8% interest compounded continuously, his investment will grow to \$6058.35 after 4 yr. If he puts the money in the account paying 5% compounded monthly, he will have \$6104.48. Isabella advises him to invest his money in the 5% account.

We will work with these and other exponential functions in this chapter.

Section 13.1 Inverse Functions

Objectives

1. Decide Whether a Function Is One-to-One
2. Use the Horizontal Line Test to Determine Whether a Function Is One-to-One
3. Find the Inverse of a One-to-One Function
4. Given the Graph of $f(x)$, Graph $f^{-1}(x)$
5. Show That $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$

In this chapter, we will study inverse functions and two very useful types of functions in mathematics: exponential and logarithmic functions. But first, we must learn about one-to-one and inverse functions. This is because exponential and logarithmic functions are related in a special way: they are *inverses* of one another.

One-to-One Functions

1. Decide Whether a Function Is One-to-One

Recall from Sections 4.6 and 12.1 that a relation is a *function* if each x-value corresponds to exactly one y-value. Let's look at two functions, f and g.

$$f = \{(1, -3), (2, -1), (4, 3), (7, 9)\} \qquad g = \{(0, 3), (1, 4), (-1, 4), (2, 7)\}$$

In functions f and g, each x-value corresponds to exactly one y-value. That is why they are functions. In function f, each *y-value also corresponds to exactly one x-value*. Therefore, f is a *one-to-one function*. In function g, however, each y-value does *not* correspond to exactly one x-value. (The y-value of 4 corresponds to $x = 1$ and $x = -1$.) Therefore, g is *not* a one-to-one function.

Definition

In order for a function to be a **one-to-one function**, each x-value corresponds to exactly one y-value, and each y-value corresponds to exactly one x-value.

Alternatively, we can say that a function is one-to-one if each value in its domain corresponds to exactly one value in its range *and* if each value in its range corresponds to exactly one value in its domain.

Example 1

Determine whether each function is one-to-one.

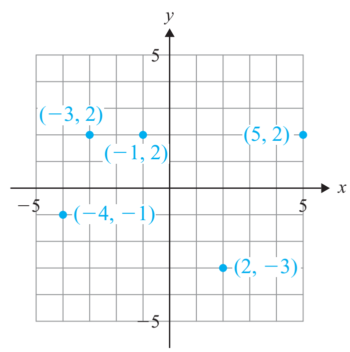
a) $f = \{-1, 9), (1, -3), (2, -6), (4, -6)\}$

b) $g = \{(-3, 13), (-1, 5), (5, -19), (8, -31)\}$

c)

State	Number of Representatives in U.S. House of Representatives (2010)
Alaska	1
California	53
Connecticut	5
Delaware	1
Ohio	18

d)



Solution

a) f is *not* a one-to-one function since the y-value -6 corresponds to two different x-values: $(2, -6)$ and $(4, -6)$.

b) g *is* a one-to-one function since each y-value corresponds to exactly one x-value.

c) The information in the table does *not* represent a one-to-one function since the value 1 in the range corresponds to two different values in the domain, Alaska and Delaware.

d) The graph does *not* represent a one-to-one function since three points have the same y-value: $(-3, 2)$, $(-1, 2)$, and $(5, 2)$.

You Try 1

Determine whether each function is one-to-one.

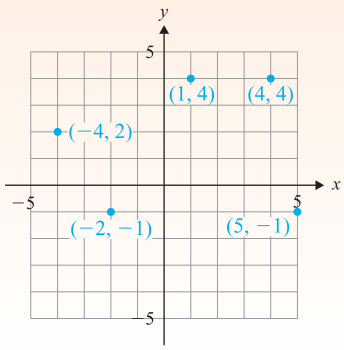
a) $f = \{(-2, -13), (0, -7), (4, 5), (5, 8)\}$

b) $g = \{(-4, 2), (-1, 1), (0, 2), (3, 5)\}$

c)

Element	Atomic Mass (in amu)
Hydrogen	1.00794
Lithium	6.941
Sulfur	32.066
Lead	207.2

d)

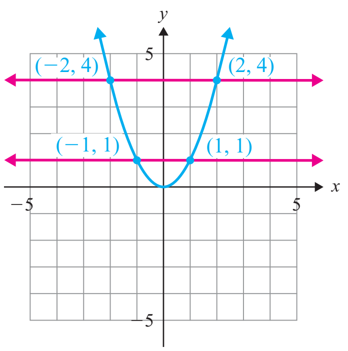


2. Use the Horizontal Line Test to Determine Whether a Function Is One-to-One

Just as we can use the vertical line test to determine whether a graph represents a function, we can use the *horizontal line test* to determine whether a function is one-to-one.

Definition

Horizontal Line Test: If every horizontal line that could be drawn through a function would intersect the graph at most once, then the function is one-to-one.

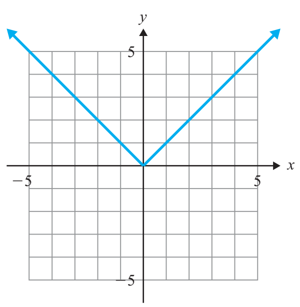


Look at the graph of the function on the left. We can see that if a horizontal line intersects the graph more than once, then one y-value corresponds to more than one x-value. This means that the function is not one-to-one. For example, the y-value of 1 corresponds to $x = 1$ and $x = -1$.

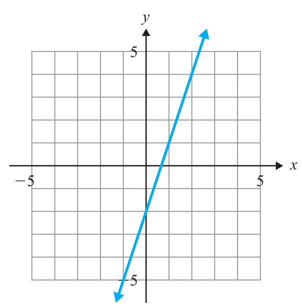
Example 2

Determine whether each graph represents a one-to-one function.

a)



b)



Solution

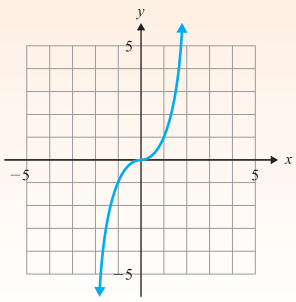
a) *Not* one-to-one. It is possible to draw a horizontal line through the graph so that it intersects the graph more than once.

b) *Is* one-to-one. Every horizontal line that could be drawn through the graph would intersect the graph at most once. ■

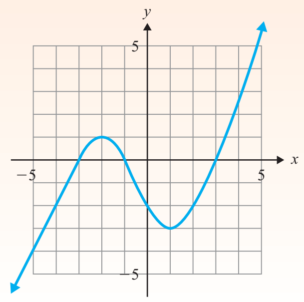
You Try 2

Determine whether each graph represents a one-to-one function.

a)



b)



Inverse Functions

3. Find the Inverse of a One-to-One Function

One-to-one functions lead to other special functions—inverse functions. *A one-to-one function has an inverse function.*

To find the inverse of a one-to-one function, we interchange the coordinates of the ordered pairs.

Example 3

Find the inverse function of $f = \{(4, 2), (9, 3), (36, 6)\}$.

Solution

To find the inverse of f, switch the x- and y-coordinates of each ordered pair. The inverse of f is $\{(2, 4), (3, 9), (6, 36)\}$. ■

You Try 3

Find the inverse function of $f = \{(-5, -1), (-3, 2), (0, 7), (4, 13)\}$.

We use special notation to represent the inverse of a function. If f is a one-to-one function, then f^{-1} (read "f inverse") represents the inverse of f. For Example 3, we can write the inverse as $f^{-1} = \{(2, 4), (3, 9), (6, 36)\}$.

Definition

Inverse Function: Let f be a one-to-one function. The **inverse** of f, denoted by f^{-1}, is a one-to-one function that contains the set of all ordered pairs (y, x), where (x, y) belongs to f.

BE CAREFUL

1) f^{-1} is read "*f* inverse" *not* "*f* to the negative one."

2) f^{-1} does *not* mean $\frac{1}{f}$.

3) If a function is not one-to-one, it does not have an inverse.

We said that if (x, y) belongs to the one-to-one function $f(x)$, then (y, x) belongs to its inverse, $f^{-1}(x)$ (read as *f inverse of x*). We use this idea to find the equation for the inverse of $f(x)$.

Procedure How to Find an Equation of the Inverse of $y = f(x)$

Step 1: Replace $f(x)$ with y.

Step 2: Interchange x and y.

Step 3: Solve for y.

Step 4: Replace y with the inverse notation, $f^{-1}(x)$.

Example 4

Find an equation of the inverse of $f(x) = 3x + 4$.

Solution

$$\begin{aligned} f(x) &= 3x + 4 \\ y &= 3x + 4 && \text{Replace } f(x) \text{ with } y. \\ x &= 3y + 4 && \text{Interchange } x \text{ and } y. \end{aligned}$$

Solve for y.

$$\begin{aligned} x - 4 &= 3y && \text{Subtract 4.} \\ \frac{x-4}{3} &= y && \text{Divide by 3.} \\ \frac{1}{3}x - \frac{4}{3} &= y && \text{Simplify.} \\ f^{-1}(x) &= \frac{1}{3}x - \frac{4}{3} && \text{Replace } y \text{ with } f^{-1}(x). \end{aligned}$$

You Try 4

Find an equation of the inverse of $f(x) = -5x + 10$.

In Example 5, we will look more closely at the relationship between a function and its inverse.

Example 5

Find the equation of the inverse of $f(x) = 2x - 4$. Then, graph $f(x)$ and $f^{-1}(x)$ on the same axes.

Solution

$$\begin{aligned} f(x) &= 2x - 4 \\ y &= 2x - 4 && \text{Replace } f(x) \text{ with } y. \\ x &= 2y - 4 && \text{Interchange } x \text{ and } y. \end{aligned}$$

Solve for y.

$$x + 4 = 2y \qquad \text{Add 4.}$$

$$\frac{x+4}{2} = y \qquad \text{Divide by 2.}$$

$$\frac{1}{2}x + 2 = y \qquad \text{Simplify.}$$

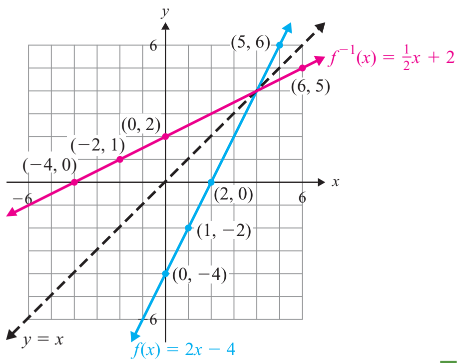
$$f^{-1}(x) = \frac{1}{2}x + 2 \qquad \text{Replace } y \text{ with } f^{-1}(x).$$

We will graph $f(x)$ and $f^{-1}(x)$ by making a table of values for each. Then we can see another relationship between the two functions.

$f(x) = 2x - 4$	
x	$y = f(x)$
0	−4
1	−2
2	0
5	6

$f^{-1}(x) = \frac{1}{2}x + 2$	
x	$y = f^{-1}(x)$
−4	0
−2	1
0	2
6	5

Notice that the x- and y-coordinates have switched when we compare the tables of values. Graph $f(x)$ and $f^{-1}(x)$.



You Try 5

Find the equation of the inverse of $f(x) = -3x + 1$. Then graph $f(x)$ and $f^{-1}(x)$ on the same axes.

4. Given the Graph of $f(x)$, Graph $f^{-1}(x)$

Look again at the tables in Example 5. The x-values for $f(x)$ become the y-values of $f^{-1}(x)$, and the y-values of $f(x)$ become the x-values of $f^{-1}(x)$. This is true not only for the values in the tables but for *all* values of x and y. That is, for all ordered pairs (x, y) that belong to $f(x)$, (y, x) belongs to $f^{-1}(x)$. Another way to say this is *the domain of f becomes the range of f^{-1}, and the range of f becomes the domain of f^{-1}.*

Let's turn our attention to the graph in Example 5. The graphs of $f(x)$ and $f^{-1}(x)$ are mirror images of one another with respect to the line $y = x$. We say that *the graphs of $f(x)$ and $f^{-1}(x)$ are symmetric with respect to the line $y = x$.* This is true for every function $f(x)$ and its inverse, $f^{-1}(x)$.

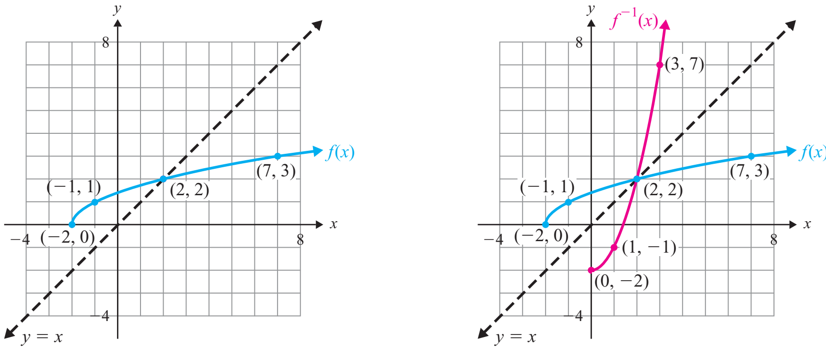
Note

The graphs of $f(x)$ and $f^{-1}(x)$ are symmetric with respect to the line $y = x$.

Example 6

Given the graph of $f(x)$, graph $f^{-1}(x)$.

Solution

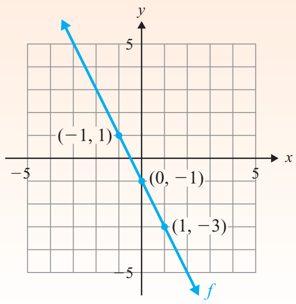


Some points on the graph of $f(x)$ are $(-2, 0)$, $(-1, 1)$, $(2, 2)$, and $(7, 3)$. We can obtain points on the graph of $f^{-1}(x)$ by interchanging the x- and y-values.

Some points on the graph of $f^{-1}(x)$ are $(0, -2)$, $(1, -1)$, $(2, 2)$, and $(3, 7)$. Plot these points to get the graph of $f^{-1}(x)$. Notice that the graphs are symmetric with respect to the line $y = x$.

You Try 6

Given the graph of $f(x)$, graph $f^{-1}(x)$.



5. Show That $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$

Going back to the tables in Example 5, we see from the first table that $f(0) = -4$ and from the second table that $f^{-1}(-4) = 0$. The first table also shows that $f(1) = -2$ while the second table shows that $f^{-1}(-2) = 1$. That is, putting x into the function f produces $f(x)$. And putting $f(x)$ into $f^{-1}(x)$ produces x.

$$\begin{array}{cccc} f(0) = -4 & \text{and} & f^{-1}(-4) = 0 \\ f(\underset{\substack{\uparrow \\ x}}{1}) = \underset{\substack{\uparrow \\ f(x)}}{-2} & \text{and} & f^{-1}(\underset{\uparrow}{-2}) = \underset{\uparrow}{1} \\ & & f^{-1}(f(x)) = x \end{array}$$

This leads us to another fact about functions and their inverses.

Note

Let f be a one-to-one function. Then f^{-1} is the inverse of f such that $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$.

Example 7

If $f(x) = 4x + 3$, show that $f^{-1}(x) = \frac{1}{4}x - \frac{3}{4}$.

Solution

Show that $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$.

$$\begin{aligned}
(f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\
&= f^{-1}(4x + 3) && \text{Substitute } 4x + 3 \text{ for } f(x). \\
&= \frac{1}{4}(4x + 3) - \frac{3}{4} && \text{Evaluate.} \\
&= x + \frac{3}{4} - \frac{3}{4} && \text{Distribute.} \\
&= x
\end{aligned}$$

$$\begin{aligned}
(f \circ f^{-1})(x) &= f(f^{-1}(x)) \\
&= f\left(\frac{1}{4}x - \frac{3}{4}\right) && \text{Substitute } \frac{1}{4}x - \frac{3}{4} \text{ for } f^{-1}(x). \\
&= 4\left(\frac{1}{4}x - \frac{3}{4}\right) + 3 && \text{Evaluate.} \\
&= x - 3 + 3 && \text{Distribute.} \\
&= x
\end{aligned}$$

You Try 7

If $f(x) = -6x + 2$, show that $f^{-1}(x) = -\frac{1}{6}x + \frac{1}{3}$.

Using Technology

A graphing calculator can list tables of values on one screen for more than one equation. The graphing calculator screen shown here is the table of values generated when the equation of one line is entered as Y_1 and the equation of another line is entered as Y_2.

X	Y1	Y2
0	4	-2
2	8	-1
4	12	0
6	16	1
8	20	2
10	24	3
12	28	4

X=0

We read the table as follows:

- The points (0, 4), (2, 8), (4, 12), (6, 16), (8, 20), (10, 24), and (12, 28) are points on the line entered as Y_1.
- The points (0, −2), (2, −1) (4, 0), (6, 1), (8, 2), (10, 3), and (12, 4) are points on the line entered as Y_2.

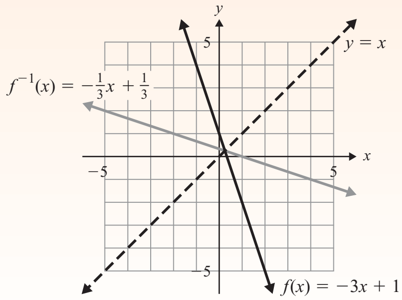
Equations Y_1 and Y_2 are linear functions, and they are inverses.

1) Looking at the table of values, what evidence is there that the functions Y_1 and Y_2 are inverses of each other?
2) Find the equations of the lines Y_1 and Y_2.
3) Graph Y_1 and Y_2. Is there evidence from their graphs that they are inverses?
4) Using the methods of this chapter, show that Y_1 and Y_2 are inverses.

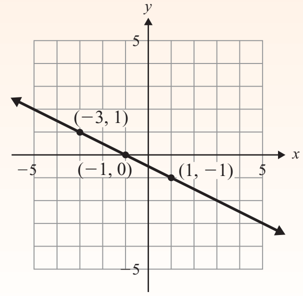
Answers to You Try Exercises

1) a) yes b) no c) yes d) no 2) a) yes b) no 3) $\{(-1, -5), (2, -3), (7, 0), (13, 4)\}$

4) $f^{-1}(x) = -\frac{1}{5}x + 2$ 5) $f^{-1}(x) = -\frac{1}{3}x + \frac{1}{3}$



6)



7) Show that $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$.

Answers to Technology Exercises

1) If Y_1 and Y_2 are inverses, then if (x, y) is a point on Y_1, (y, x) is a point on Y_2. We see this is true with $(0, 4)$ on Y_1 and $(4, 0)$ on Y_2, with $(2, 8)$ on Y_1 and $(8, 2)$ on Y_2, and with $(4, 12)$ on Y_1 and $(12, 4)$ on Y_2.
2) $Y_1 = 2x + 4$, $Y_2 = 0.5x - 2$
3) Yes. They appear to be symmetric with respect to the line $y = x$.
4) Let $f(x) = Y_1$ and $f^{-1}(x) = Y_2$. We can show that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

13.1 Exercises

Objective 1: Decide Whether a Function is One-to-One

Determine whether each function is one-to-one. If it is one-to-one, find its inverse.

1) $f = \{(-4, 3), (-2, -3), (2, -3), (6, 13)\}$
2) $g = \{(0, -7), (1, -6), (4, -5), (25, -2)\}$
3) $h = \{(-5, -16), (-1, -4), (3, 8)\}$
4) $f = \{(-6, 3), (-1, 8), (4, 3)\}$
5) $g = \{(2, 1), (5, 2), (7, 14), (10, 19)\}$
6) $h = \{(-1, 4), (0, -2), (5, 1), (9, 4)\}$

Determine whether each function is one-to-one.

7) The table shows the average temperature during selected months in Tulsa, Oklahoma. The function matches each month with the average temperature, in °F. Is it one-to-one? (www.noaa.gov)

Month	Average Temp. (°F)
Jan.	36.4
Apr.	60.8
July	83.5
Oct.	62.6

8) The table shows some NCAA conferences and the number of schools in the conference as of August 2010. The function matches each conference with the number of schools it contains. Is it one-to-one?

Conference	Number of Member Schools
ACC	12
Big 10	11
Big 12	12
MVC	10
Pac10	10

Mixed Exercises: Objectives 1, 2, and 4

9) Do all functions have inverses? Explain your answer.

10) What test can be used to determine whether the graph of a function has an inverse?

Determine whether each statement is true or false. If it is false, rewrite the statement so that it is true.

11) $f^{-1}(x)$ is read as "f to the negative one of x."

12) If f^{-1} is the inverse of f, then $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$.

13) The domain of f is the range of f^{-1}.

14) If f is one-to-one and (5, 9) is on the graph of f, then $(-5, -9)$ is on the graph of f^{-1}.

15) The graphs of $f(x)$ and $f^{-1}(x)$ are symmetric with respect to the x-axis.

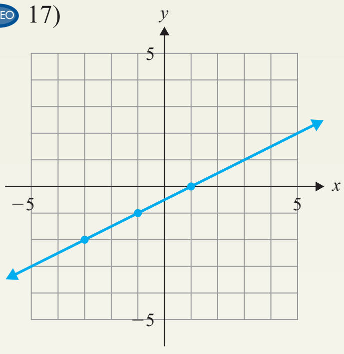
16) Let $f(x)$ be one-to-one. If $f(7) = 2$, then $f^{-1}(2) = 7$.

For each function graphed here, answer the following.

a) Determine whether it is one-to-one.

b) If it is one-to-one, graph its inverse.

17)



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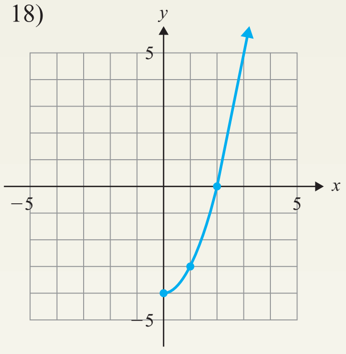

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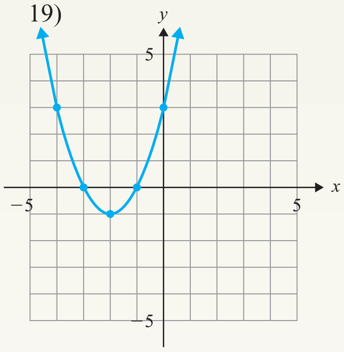

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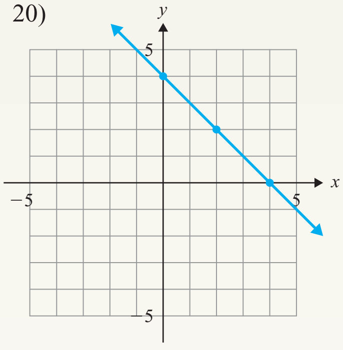

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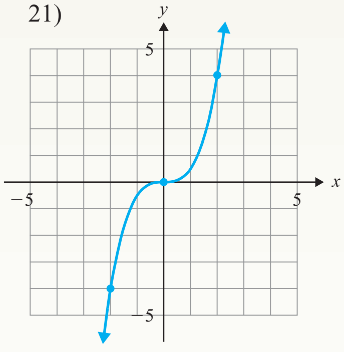

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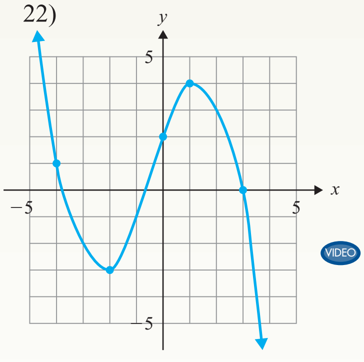


Objective 3: Find the Inverse of a One-to-One Function

Find the inverse of each one-to-one function.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

23) $f(x) = 2x - 10$

$y = 2x - 10$ ________

________ Interchange x and y.

Solve for y.

$x + 10 = 2y$ ________

________ Divide by 2 and simplify.

$f^{-1}(x) = \frac{1}{2}x + 5$ ________

24) $g(x) = \frac{1}{3}x + 4$

________ Replace $g(x)$ with y.

$x = \frac{1}{3}y + 4$ ________

Solve for y.

________ Subtract 4.

$3x - 12 = y$ ________

________ Replace y with $g^{-1}(x)$.

Find the inverse of each one-to-one function. Then graph the function and its inverse on the same axes.

25) $g(x) = x - 6$

26) $h(x) = x + 3$

27) $f(x) = -2x + 5$

28) $g(x) = 4x - 9$

29) $g(x) = \frac{1}{2}x$

30) $h(x) = -\frac{1}{3}x$

31) $f(x) = x^3$

32) $g(x) = \sqrt[3]{x} + 4$

Find the inverse of each one-to-one function.

33) $f(x) = 2x - 6$

34) $g(x) = -4x + 8$

35) $h(x) = -\frac{3}{2}x + 4$

36) $f(x) = \frac{2}{5}x + 1$

37) $g(x) = \sqrt[3]{x + 2}$

38) $h(x) = \sqrt[3]{x - 7}$

39) $f(x) = \sqrt{x}, x \geq 0$

40) $g(x) = \sqrt{x + 3}, x \geq -3$

Objective 5: Show That $(f^{-1} \circ f)(x) = x$ and $(f \circ f^{-1})(x) = x$

Given the one-to-one function $f(x)$, find the function values *without* finding the equation of $f^{-1}(x)$. Find the value in a) before b).

41) $f(x) = 5x - 2$
 a) $f(1)$ b) $f^{-1}(3)$

42) $f(x) = 3x + 7$
 a) $f(-4)$ b) $f^{-1}(-5)$

43) $f(x) = -\frac{1}{3}x + 5$
 a) $f(9)$ b) $f^{-1}(2)$

44) $f(x) = \frac{1}{2}x - 1$
 a) $f(6)$ b) $f^{-1}(2)$

45) $f(x) = -x + 3$
 a) $f(-7)$ b) $f^{-1}(10)$

46) $f(x) = -\frac{5}{4}x + 2$
 a) $f(8)$ b) $f^{-1}(-8)$

47) $f(x) = 2^x$
 a) $f(3)$ b) $f^{-1}(8)$

48) $f(x) = 3^x$
 a) $f(-2)$ b) $f^{-1}\left(\frac{1}{9}\right)$

49) If $f(x) = x + 9$, show that $f^{-1}(x) = x - 9$.

50) If $f(x) = x - 12$, show that $f^{-1}(x) = x + 12$.

51) If $f(x) = -6x + 4$, show that $f^{-1}(x) = -\frac{1}{6}x + \frac{2}{3}$.

52) If $f(x) = -\frac{1}{7}x + \frac{2}{7}$, show that $f^{-1}(x) = -7x + 2$.

53) If $f(x) = \frac{3}{2}x - 9$, show that $f^{-1}(x) = \frac{2}{3}x + 6$.

54) If $f(x) = -\frac{5}{8}x + 10$, show that $f^{-1}(x) = -\frac{8}{5}x + 16$.

55) If $f(x) = \sqrt[3]{x - 10}$, show that $f^{-1}(x) = x^3 + 10$.

56) If $f(x) = x^3 - 1$, show that $f^{-1}(x) = \sqrt[3]{x + 1}$.

Section 13.2 Exponential Functions

Objectives

1. Define an Exponential Function
2. Graph $f(x) = a^x$
3. Graph $f(x) = a^{x+c}$
4. Define the Number e and Graph $f(x) = e^x$
5. Solve an Exponential Equation
6. Solve an Applied Problem Using a Given Exponential Function

In Chapter 12, we studied the following types of functions:

Linear functions like $f(x) = 2x + 5$
Quadratic functions like $g(x) = x^2 - 6x + 8$
Absolute value functions like $h(x) = |x|$
Square root functions like $k(x) = \sqrt{x - 3}$

1. Define an Exponential Function

In this section, we will learn about *exponential functions*.

Definition

An **exponential function** is a function of the form

$$f(x) = a^x$$

where $a > 0$, $a \neq 1$, and x is a real number.

Note

1) We stipulate that a is a positive number ($a > 0$) because if a were a negative number, some expressions would not be real numbers.

 Example: If $a = -2$ and $x = \frac{1}{2}$, we get $f(x) = (-2)^{1/2} = \sqrt{-2}$ (not real).

 Therefore, a *must* be a positive number.

2) We add the condition that $a \neq 1$ because if $a = 1$, the function would be linear, not exponential.

 Example: If $a = 1$, then $f(x) = 1^x$. This is equivalent to $f(x) = 1$, which is a linear function.

2. Graph $f(x) = a^x$

We can graph exponential functions by plotting points. *It is important to choose many values for the variable so that we obtain positive numbers, negative numbers, and zero in the exponent.*

Example 1

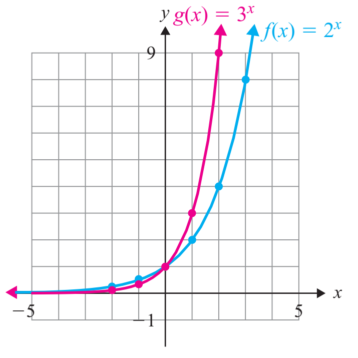
Graph $f(x) = 2^x$ and $g(x) = 3^x$ on the same axes. Determine the domain and range.

Solution

Make a table of values for each function. Be sure to choose values for x that will give us *positive numbers, negative numbers, and zero* in the exponent.

$f(x) = 2^x$	
x	$f(x)$
0	1
1	2
2	4
3	8
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$

$g(x) = 3^x$	
x	$g(x)$
0	1
1	3
2	9
3	27
-1	$\frac{1}{3}$
-2	$\frac{1}{9}$



Plot each set of points and connect them with a smooth curve. Note that the larger the value of a, the more rapidly the y-values increase. Additionally, as x increases, the value of y also increases. Here are some other interesting facts to note about the graphs of these functions.

1) Each graph passes the vertical line test so the graphs *do* represent functions.
2) Each graph passes the horizontal line test, so the functions are one-to-one.
3) The y-intercept of each function is $(0, 1)$.
4) The domain of each function is $(-\infty, \infty)$, and the range is $(0, \infty)$.

You Try 1

Graph $f(x) = 4^x$. Determine the domain and range.

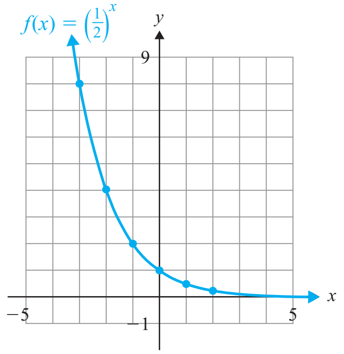
Example 2

Graph $f(x) = \left(\frac{1}{2}\right)^x$. Determine the domain and range.

Solution

Make a table of values and be sure to choose values for x that will give us *positive numbers, negative numbers, and zero* in the exponent.

$f(x) = \left(\frac{1}{2}\right)^x$	
x	$f(x)$
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
-1	2
-2	4
-3	8



Like the graphs of $f(x) = 2^x$ and $g(x) = 3^x$ in Example 1, the graph of $f(x) = \left(\frac{1}{2}\right)^x$ passes both the vertical and horizontal line tests, making it a one-to-one function. The y-intercept is (0, 1). The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

In the case of $f(x) = \left(\frac{1}{2}\right)^x$, however, as the value of x increases, the value of y *decreases*. This is because $0 < a < 1$.

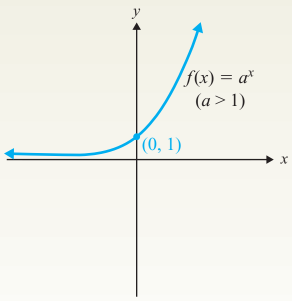
You Try 2

Graph $g(x) = \left(\frac{1}{3}\right)^x$. Determine the domain and range.

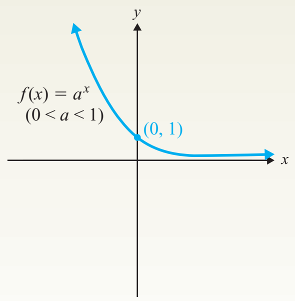
We can summarize what we have learned so far about exponential functions:

Summary Characteristics of $f(x) = a^x$, where $a > 0$ and $a \neq 1$

1) If $f(x) = a^x$ where $a > 1$, the value of y increases as the value of x increases.



2) If $f(x) = a^x$, where $0 < a < 1$, the value of y decreases as the value of x increases.



3) The function is one-to-one.
4) The y-intercept is (0, 1).
5) The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

3. Graph $f(x) = a^{x+c}$

Next we will graph an exponential function with an expression other than x as its exponent.

Example 3

Graph $f(x) = 3^{x-2}$. Determine the domain and range.

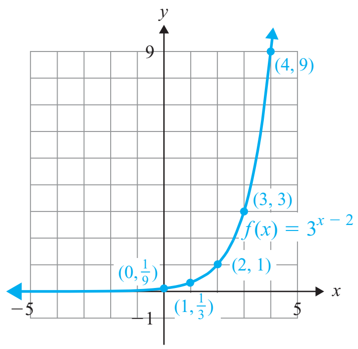
Solution

Remember, for the table of values we want to choose values of x that will give us positive numbers, negative numbers, and zero *in the exponent*. First we will determine what value of x will make the exponent equal zero.

$$\begin{aligned} x - 2 &= 0 \\ x &= 2 \end{aligned}$$

If $x = 2$, the exponent equals zero. Choose a couple of numbers *greater than* 2 and a couple that are *less than* 2 to get positive and negative numbers in the exponent.

	x	$x - 2$	$f(x) = 3^{x-2}$	Plot
	2	0	$3^0 = 1$	$(2, 1)$
Values greater than 2	3	1	$3^1 = 3$	$(3, 3)$
	4	2	$3^2 = 9$	$(4, 9)$
Values less than 2	1	-1	$3^{-1} = \frac{1}{3}$	$\left(1, \frac{1}{3}\right)$
	0	-2	$3^{-2} = \frac{1}{9}$	$\left(0, \frac{1}{9}\right)$



Note that the y-intercept is not $(0, 1)$ because the exponent is $x - 2$, *not* x, as in $f(x) = a^x$. The graph of $f(x) = 3^{x-2}$ is the same shape as the graph of $g(x) = 3^x$ except that the graph of f is shifted 2 units to the right. This is because $f(x) = g(x - 2)$. The domain of f is $(-\infty, \infty)$, and the range is $(0, \infty)$. ■

You Try 3

Graph $f(x) = 2^{x+4}$. Determine the domain and range.

4. Define the Number e and Graph $f(x) = e^x$

Next we will introduce a special exponential function, one with a base of e.

Like the number π, e is an irrational number that has many uses in mathematics. In the 1700s, the work of Swiss mathematician Leonhard Euler led him to the approximation of e.

Definition

Approximation of e

$$e \approx 2.718281828459045235$$

One of the questions Euler set out to answer was, what happens to the value of $\left(1 + \frac{1}{n}\right)^n$ as n gets larger and larger? He found that as n gets larger, $\left(1 + \frac{1}{n}\right)^n$ gets closer to a fixed number. This number is e. Euler approximated e to the 18 decimal places in the definition, and the letter e was chosen to represent this number in his honor. It should be noted that there are other ways to generate e. Finding the value that $\left(1 + \frac{1}{n}\right)^n$ approaches as n gets larger and larger is just one way. Also, since e is irrational, it is a nonterminating, nonrepeating decimal.

Example 4

Graph $f(x) = e^x$. Determine the domain and range.

Solution

A calculator is needed to generate a table of values. We will use either the $\boxed{e^x}$ key or the two keys $\boxed{\text{INV}}$ (or $\boxed{\text{2ND}}$) and $\boxed{\ln x}$ to find powers of e. (Calculators will approximate powers of e to a few decimal places.)

For example, if a calculator has an $\boxed{e^x}$ key, find e^2 by pressing the following keys:

$$\boxed{2}\ \boxed{e^x} \qquad \text{or} \qquad \boxed{e^x}\ \boxed{2}\ \boxed{\text{ENTER}}$$

To four decimal places, $e^2 \approx 7.3891$.

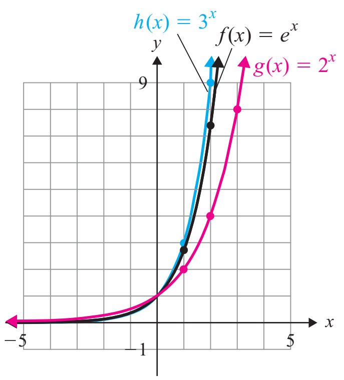
If a calculator has an $\boxed{\ln x}$ key with e^x written above it, find e^2 by pressing the following keys:

$$\boxed{2}\ \boxed{\text{INV}}\ \boxed{\ln x} \qquad \text{or} \qquad \boxed{\text{INV}}\ \boxed{\ln x}\ \boxed{2}\ \boxed{\text{ENTER}}$$

The same approximation for e^2 is obtained.

Remember to choose positive numbers, negative numbers, and zero for x when making the table of values. We will approximate the values of e^x to four decimal places.

$f(x) = e^x$	
x	$f(x)$
0	1
1	2.7183
2	7.3891
3	20.0855
−1	0.3679
−2	0.1353



Notice that the graph of $f(x) = e^x$ is between the graphs of $g(x) = 2^x$ and $h(x) = 3^x$. This is because $2 < e < 3$, so e^x grows more quickly than 2^x, but e^x grows more slowly than 3^x. The domain of $f(x) = e^x$ is $(-\infty, \infty)$, and the range is $(0, \infty)$.

We will study e^x and its special properties in more detail later in the chapter.

5. Solve an Exponential Equation

An **exponential equation** is an equation that has a variable in the exponent. Some examples of exponential equations are

$$2^x = 8, \qquad 3^{a-5} = \frac{1}{9}, \qquad e^t = 14, \qquad 5^{2y-1} = 6^{y+4}$$

In this section, we will learn how to solve exponential equations like the first two examples. We can solve those equations by getting the same base.

We know that the exponential function $f(x) = a^x$ $(a > 0, a \neq 1)$ is one-to-one. This leads to the following property that enables us to solve many exponential equations.

$$\text{If } a^x = a^y, \text{ then } x = y. \ (a > 0, a \neq 1)$$

This property says that if two sides of an equation have the same base, set the exponents equal and solve for the unknown variable.

Procedure Solving an Exponential Equation

Step 1: **If possible, express each side of the equation with the same base.** If it is *not* possible to get the same base, a different method must be used. (This is presented in Section 13.6.)

Step 2: **Use the rules of exponents to simplify the exponents.**

Step 3: **Set the exponents equal and solve for the variable.**

Example 5

Solve each equation.

a) $2^x = 8$ b) $49^{c+3} = 7^{3c}$ c) $9^{6n} = 27^{n-4}$ d) $3^{a-5} = \frac{1}{9}$

Solution

a) ***Step 1:*** Express each side of the equation with the same base.

$$\begin{aligned} 2^x &= 8 \\ 2^x &= 2^3 && \text{Rewrite 8 with a base of 2: } 8 = 2^3. \end{aligned}$$

Step 2: The exponents are simplified.

Step 3: Since the bases are the same, set the exponents equal and solve.

$$x = 3$$

The solution set is $\{3\}$.

b) ***Step 1:*** Express each side of the equation with the same base.

$$\begin{aligned} 49^{c+3} &= 7^{3c} \\ (7^2)^{c+3} &= 7^{3c} && \text{Both sides are powers of 7; } 49 = 7^2. \end{aligned}$$

Step 2: Use the rules of exponents to simplify the exponents.

$$\begin{aligned} 7^{2(c+3)} &= 7^{3c} && \text{Power rule for exponents} \\ 7^{2c+6} &= 7^{3c} && \text{Distribute.} \end{aligned}$$

Step 3: Since the bases are the same, set the exponents equal and solve.

$$\begin{aligned} 2c + 6 &= 3c && \text{Set the exponents equal.} \\ 6 &= c && \text{Subtract } 2c. \end{aligned}$$

The solution set is $\{6\}$.

c) ***Step 1:*** Express each side of the equation with the same base. 9 *and* 27 *are each powers of* 3.

$$9^{6n} = 27^{n-4}$$
$$(3^2)^{6n} = (3^3)^{n-4} \qquad 9 = 3^2;\ 27 = 3^3$$

Step 2: Use the rules of exponents to simplify the exponents.

$$3^{2(6n)} = 3^{3(n-4)} \qquad \text{Power rule for exponents}$$
$$3^{12n} = 3^{3n-12} \qquad \text{Multiply.}$$

Step 3: Since the bases are the same, set the exponents equal and solve.

$$12n = 3n - 12 \qquad \text{Set the exponents equal.}$$
$$9n = -12 \qquad \text{Subtract } 3n.$$
$$n = -\frac{12}{9} = -\frac{4}{3} \qquad \text{Divide by 9; simplify.}$$

The solution set is $\left\{-\frac{4}{3}\right\}$.

d) ***Step 1:*** Express each side of the equation $3^{a-5} = \frac{1}{9}$ with the same base. $\frac{1}{9}$ *can be expressed with a base of* 3: $\frac{1}{9} = \left(\frac{1}{3}\right)^2 = 3^{-2}$.

$$3^{a-5} = \frac{1}{9}$$
$$3^{a-5} = 3^{-2} \qquad \text{Rewrite } \frac{1}{9} \text{ with a base of 3.}$$

Step 2: The exponents are simplified.

Step 3: Set the exponents equal and solve.

$$a - 5 = -2 \qquad \text{Set the exponents equal.}$$
$$a = 3 \qquad \text{Add 5.}$$

The solution set is $\{3\}$.

You Try 4

Solve each equation.

a) $(12)^x = 144$ b) $6^{t-5} = 36^{t+4}$ c) $32^{2w} = 8^{4w-1}$ d) $8^k = \frac{1}{64}$

6. Solve an Applied Problem Using a Given Exponential Function

Example 6

The value of a car depreciates (decreases) over time. The value, $V(t)$, in dollars, of a sedan t yr after it is purchased is given by

$$V(t) = 18{,}200(0.794)^t$$

a) What was the purchase price of the car?

b) What will the car be worth 5 yr after purchase?

Solution

a) To find the purchase price of the car, let $t = 0$.
Evaluate $V(0)$ given that $V(t) = 18{,}200(0.794)^t$.

$$\begin{aligned} V(0) &= 18{,}200(0.794)^0 \\ &= 18{,}200(1) \\ &= 18{,}200 \end{aligned}$$

The purchase price of the car was \$18,200.

b) To find the value of the car after 5 yr, let $t = 5$. Use a calculator to find $V(5)$.

$$\begin{aligned} V(5) &= 18{,}200(0.794)^5 \\ &= 5743.46 \end{aligned}$$

The car will be worth about \$5743.46. ■

You Try 5

The value, $V(t)$, in dollars, of a pickup truck t yr after it is purchased is given by

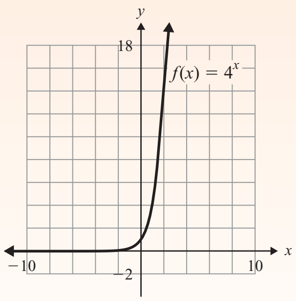
$$V(t) = 23{,}500(0.785)^t$$

a) What was the purchase price of the pickup?

b) What will the pickup truck be worth 4 yr after purchase?

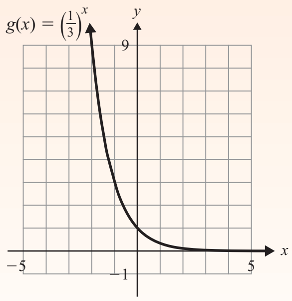
Answers to You Try Exercises

1)



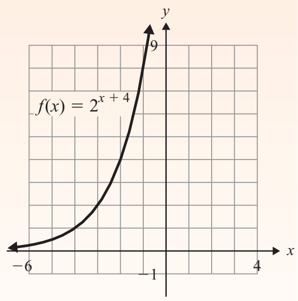
domain: $(-\infty, \infty)$; range: $(0, \infty)$

2)



domain: $(-\infty, \infty)$; range: $(0, \infty)$

3)



domain: $(-\infty, \infty)$; range: $(0, \infty)$

4) a) $\{2\}$ b) $\{-13\}$ c) $\left\{\frac{3}{2}\right\}$ d) $\{-2\}$ 5) a) \$23,500 b) \$8923.73

13.2 Exercises

Mixed Exercises: Objectives 1 and 2

1) When making a table of values to graph an exponential function, what kind of values should be chosen for the variable?

2) What is the y-intercept of the graph of $f(x) = a^x$ where $a > 0$ and $a \neq 1$?

Graph each exponential function. Determine the domain and range.

3) $f(x) = 5^x$

4) $g(x) = 4^x$

VIDEO 5) $y = 2^x$

6) $f(x) = 3^x$

7) $h(x) = \left(\dfrac{1}{3}\right)^x$

8) $y = \left(\dfrac{1}{4}\right)^x$

For an exponential function of the form $f(x) = a^x$ ($a > 0, a \neq 1$), answer the following.

9) What is the domain?

10) What is the range?

Objective 3: Graph $f(x) = a^{x+c}$

Graph each exponential function. State the domain and range.

11) $g(x) = 2^{x+1}$

12) $y = 3^{x+2}$

13) $f(x) = 3^{x-4}$

14) $h(x) = 2^{x-3}$

15) $y = 4^{x+3}$

16) $g(x) = 4^{x-1}$

17) $f(x) = 2^{2x}$

18) $h(x) = 3^{\frac{1}{2}x}$

19) $y = 2^x + 1$

20) $f(x) = 2^x - 3$

21) $g(x) = 3^x - 2$

22) $h(x) = 3^x + 1$

23) $y = -2^x$

24) $f(x) = -\left(\frac{1}{3}\right)^x$

25) As the value of x gets larger, would you expect $f(x) = 2x$ or $g(x) = 2^x$ to grow faster? Why?

26) Let $f(x) = \left(\frac{1}{5}\right)^x$. The graph of $f(x)$ gets very close to the line $y = 0$ (the x-axis) as the value of x gets larger. Why?

27) If you are given the graph of $f(x) = a^x$, where $a > 0$ and $a \neq 1$, how would you obtain the graph of $g(x) = a^x - 2$?

28) If you are given the graph of $f(x) = a^x$, where $a > 0$ and $a \neq 1$, how would you obtain the graph of $g(x) = a^{x-3}$?

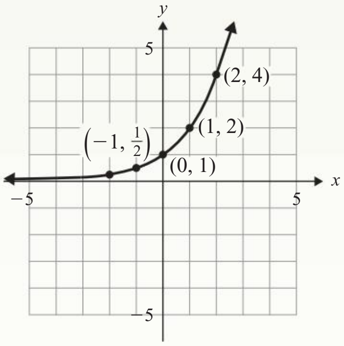
Objective 4: Define the Number e and Graph $f(x) = e^x$

29) What is the approximate value of e to four decimal places?

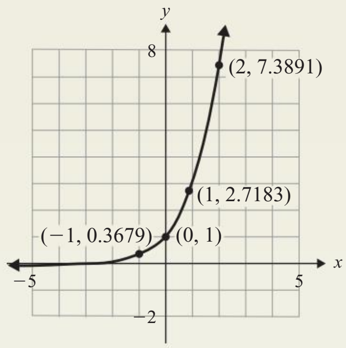
30) Is e a rational or an irrational number? Explain your answer.

For Exercises 31–34, match each exponential function with its graph.

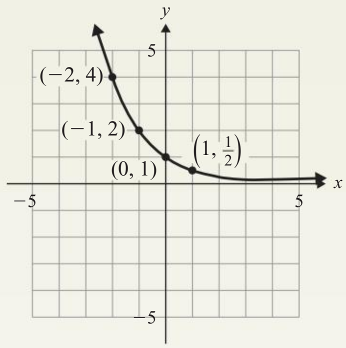
A)



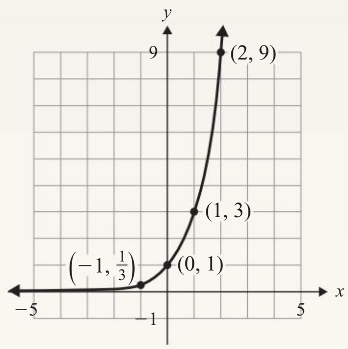
B)



C)



D)



31) $f(x) = e^x$

32) $g(x) = 2^x$

33) $h(x) = 3^x$

34) $k(x) = \left(\frac{1}{2}\right)^x$

Graph each function. State the domain and range.

35) $f(x) = e^x - 2$

36) $g(x) = e^x + 1$

37) $y = e^{x+1}$

38) $h(x) = e^{x-3}$

39) $g(x) = \frac{1}{2}e^x$

40) $y = 2e^x$

41) $h(x) = -e^x$

42) $f(x) = e^{-x}$

43) Graph $y = e^x$, and compare it with the graph of $h(x) = -e^x$ in Exercise 41. What can you say about these graphs?

44) Graph $y = e^x$, and compare it with the graph of $f(x) = e^{-x}$ in Exercise 42. What can you say about these graphs?

Objective 5: Solve an Exponential Equation

Solve each exponential equation.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

45) $6^{3n} = 36^{n-4}$

________	Express each side with the same base.
$6^{3n} = 6^{2(n-4)}$	________________
$6^{3n} = 6^{2n-8}$	________
________	Set the exponents equal.
$n = -8$	Solve for n.

The solution set is ____.

46) $125^{2w} = 5^{w+2}$

$(5^3)^{2w} = 5^{w+2}$	____________________
________	Power rule for exponents
$5^{6w} = 5^{w+2}$	______
$6w = w + 2$	________________
	Solve for w.
________	

The solution set is ____.

Solve each exponential equation.

47) $9^x = 81$

48) $4^y = 16$

49) $5^{4d} = 125$

50) $4^{3a} = 64$

51) $16^{m-2} = 2^{3m}$

52) $3^{5t} = 9^{t+4}$

53) $7^{2k-6} = 49^{3k+1}$

54) $(1000)^{2p-3} = 10^{4p+1}$

55) $32^{3c} = 8^{c+4}$

56) $(125)^{2x-9} = 25^{x-3}$

57) $100^{5z-1} = (1000)^{2z+7}$

58) $32^{y+1} = 64^{y+2}$

59) $81^{3n+9} = 27^{2n+6}$

60) $27^{5v} = 9^{v+4}$

61) $6^x = \dfrac{1}{36}$

62) $11^t = \dfrac{1}{121}$

63) $2^a = \dfrac{1}{8}$

64) $3^z = \dfrac{1}{81}$

65) $9^r = \dfrac{1}{27}$

66) $16^c = \dfrac{1}{8}$

67) $\left(\dfrac{3}{4}\right)^{5k} = \left(\dfrac{27}{64}\right)^{k+1}$

68) $\left(\dfrac{3}{2}\right)^{y+4} = \left(\dfrac{81}{16}\right)^{y-2}$

69) $\left(\dfrac{5}{6}\right)^{3x+7} = \left(\dfrac{36}{25}\right)^{2x}$

70) $\left(\dfrac{7}{2}\right)^{5w} = \left(\dfrac{4}{49}\right)^{4w+3}$

Objective 6: Solve an Applied Problem Using a Given Exponential Function

Solve each application.

71) The value of a car depreciates (decreases) over time. The value, $V(t)$, in dollars, of an SUV t yr after it is purchased is given by

$$V(t) = 32{,}700(0.812)^t$$

a) What was the purchase price of the SUV?

b) What will the SUV be worth 3 yr after purchase?

72) The value, $V(t)$, in dollars, of a compact car t yr after it is purchased is given by

$$V(t) = 10{,}150(0.784)^t$$

a) What was the purchase price of the car?

b) What will the car be worth 5 yr after purchase?

73) The value, $V(t)$, in dollars, of a minivan t yr after it is purchased is given by

$$V(t) = 16{,}800(0.803)^t$$

a) What was the purchase price of the minivan?

b) What will the minivan be worth 6 yr after purchase?

74) The value, $V(t)$, in dollars, of a sports car t yr after it is purchased is given by

$$V(t) = 48{,}600(0.820)^t$$

a) What was the purchase price of the sports car?

b) What will the sports car be worth 4 yr after purchase?

75) From 1995 to 2005, the value of homes in a suburb increased by 3% per year. The value, $V(t)$, in dollars, of a particular house t yr after 1995 is given by

$$V(t) = 185{,}200(1.03)^t$$

a) How much was the house worth in 1995?

b) How much was the house worth in 2002?

76) From 2000 to 2010, the value of condominiums in a big city high-rise building increased by 2% per year. The value, $V(t)$, in dollars, of a particular condo t yr after 2000 is given by

$$V(t) = 420{,}000(1.02)^t$$

a) How much was the condominium worth in 2000?

b) How much was the condominium worth in 2010?

An *annuity* is an account into which money is deposited every year. The amount of money, A in dollars, in the account after t yr of depositing c dollars at the beginning of every year earning an interest rate r (as a decimal) is

$$A = c\left[\frac{(1+r)^t - 1}{r}\right](1+r)$$

Use the formula for Exercises 77–80.

77) After Fernando's daughter is born, he decides to begin saving for her college education. He will deposit \$2000 every year in an annuity for 18 yr at a rate of 9%. How much will be in the account after 18 yr?

78) To save for retirement, Susan plans to deposit \$6000 per year in an annuity for 30 yr at a rate of 8.5%. How much will be in the account after 30 yr?

79) Patrice will deposit \$4000 every year in an annuity for 10 yr at a rate of 7%. How much will be in the account after 10 yr?

80) Haeshin will deposit \$3000 every year in an annuity for 15 yr at a rate of 8%. How much will be in the account after 15 yr?

81) After taking a certain antibiotic, the amount of amoxicillin $A(t)$, in milligrams, remaining in the patient's system t hr after taking 1000 mg of amoxicillin is

$$A(t) = 1000e^{-0.5332t}$$

How much amoxicillin is in the patient's system 6 hr after taking the medication?

82) Some cockroaches can reproduce according to the formula

$$y = 2(1.65)^t$$

where y is the number of cockroaches resulting from the mating of two cockroaches and their offspring t months after the first two cockroaches mate.

If Morris finds two cockroaches in his kitchen (assuming one is male and one is female) how large can the cockroach population become after 12 months?

Section 13.3 Logarithmic Functions

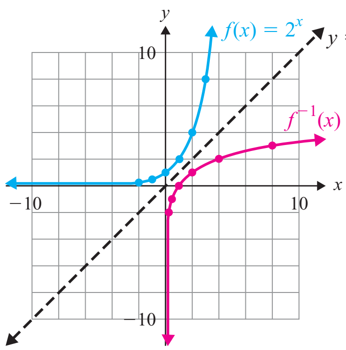
Objectives
1. Define a Logarithm
2. Convert from Logarithmic Form to Exponential Form
3. Convert from Exponential Form to Logarithmic Form
4. Solve an Equation of the Form $\log_a b = c$
5. Evaluate a Logarithm
6. Evaluate Common Logarithms, and Solve Equations of the Form $\log b = c$
7. Use the Properties $\log_a a = 1$ and $\log_a 1 = 0$
8. Define and Graph a Logarithmic Function
9. Solve an Applied Problem Using a Logarithmic Equation

1. Define a Logarithm

In Section 13.2, we graphed $f(x) = 2^x$ by making a table of values and plotting the points. The graph passes the horizontal line test, making the function one-to-one. Recall that if (x, y) is on the graph of a function, then (y, x) is on the graph of its inverse. We can graph the inverse of $f(x) = 2^x$, $f^{-1}(x)$, by switching the x- and y-coordinates in the table of values and plotting the points.

$f(x) = 2^x$	
x	$y = f(x)$
0	1
1	2
2	4
3	8
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$

$f^{-1}(x)$	
x	$y = f^{-1}(x)$
1	0
2	1
4	2
8	3
$\frac{1}{2}$	-1
$\frac{1}{4}$	-2



Above is the graph of $f(x) = 2^x$ and its inverse. Notice that, like the graphs of all functions and their inverses, they are symmetric with respect to the line $y = x$.

What is the equation of $f^{-1}(x)$ if $f(x) = 2^x$? We will use the procedure outlined in Section 13.1 to find the equation of $f^{-1}(x)$.

If $f(x) = 2^x$, then find the equation of $f^{-1}(x)$ as follows.

Step 1: Replace $f(x)$ with y.

$$y = 2^x$$

Step 2: Interchange x and y.

$$x = 2^y$$

Step 3: Solve for y.

How do we solve $x = 2^y$ for y? To answer this question, we must introduce another concept called *logarithms*.

Definition

Definition of Logarithm: If $a > 0$, $a \neq 1$, and $x > 0$, then for every real number y,

$$y = \log_a x \text{ means } x = a^y$$

The word **log** is an abbreviation for **logarithm**. We read $\log_a x$ as "log of x to the base a" or "log to the base a of x." *This definition of a logarithm should be memorized!*

Note

It is very important to note that the base of the logarithm must be positive and not equal to 1, and that x must be positive as well.

The relationship between the logarithmic form of an equation ($y = \log_a x$) and the exponential form of an equation ($x = a^y$) is one that has many uses. Notice the relationship between the two forms.

Logarithmic Form	Exponential Form
Value of the logarithm $\downarrow$ $y = \log_a x$ $\uparrow$ Base	Exponent $\downarrow$ $x = a^y$ $\uparrow$ Base

From the above, you can see that *a logarithm is an exponent*. $\log_a x$ is the power to which we raise a to get x.

2. Convert from Logarithmic Form to Exponential Form

Much of our work with logarithms involves converting between logarithmic and exponential notation. After working with logs and exponential form, we will come back to the question of how to solve $x = 2^y$ for y.

Example 1

Write in exponential form.

a) $\log_6 36 = 2$ b) $\log_4 \dfrac{1}{64} = -3$ c) $\log_7 1 = 0$

Solution

a) $\log_6 36 = 2$ means that 2 is the power to which we raise 6 to get 36. The exponential form is $6^2 = 36$.

$$\log_6 36 = 2 \text{ means } 6^2 = 36.$$

b) $\log_4 \frac{1}{64} = -3$ means $4^{-3} = \frac{1}{64}$.

c) $\log_7 1 = 0$ means $7^0 = 1$.

You Try 1

Write in exponential form.

a) $\log_3 81 = 4$ b) $\log_5 \frac{1}{25} = -2$ c) $\log_{64} 8 = \frac{1}{2}$ d) $\log_{13} 13 = 1$

3. Convert from Exponential Form to Logarithmic Form

Example 2

Write in logarithmic form.

a) $10^4 = 10{,}000$ b) $9^{-2} = \frac{1}{81}$ c) $8^1 = 8$

d) $\sqrt{25} = 5$

Solution

a) $10^4 = 10{,}000$ means $\log_{10} 10{,}000 = 4$.

b) $9^{-2} = \frac{1}{81}$ means $\log_9 \frac{1}{81} = -2$.

c) $8^1 = 8$ means $\log_8 8 = 1$.

d) To write $\sqrt{25} = 5$ in logarithmic form, rewrite $\sqrt{25}$ as $25^{1/2}$.

$$\sqrt{25} = 5 \text{ is the same as } 25^{1/2} = 5$$

$25^{1/2} = 5$ means $\log_{25} 5 = \frac{1}{2}$.

Note

When working with logarithms, we will often change radical notation to the equivalent fractional exponent. This is because a logarithm *is* an exponent.

You Try 2

Write in logarithmic form.

a) $7^2 = 49$ b) $5^{-4} = \frac{1}{625}$ c) $19^0 = 1$ d) $\sqrt{144} = 12$

Solving Logarithmic Equations

4. Solve an Equation of the Form $\log_a b = c$

A **logarithmic equation** is an equation in which at least one term contains a logarithm. In this section, we will learn how to solve a logarithmic equation of the form $\log_a b = c$. We will learn how to solve other types of logarithmic equations in Sections 13.5 and 13.6.

Procedure Solve an Equation of the Form $\log_a b = c$

To solve a logarithmic equation of the form $\log_a b = c$, write the equation in exponential form ($a^c = b$) and solve for the variable.

Example 3

Solve each logarithmic equation.

a) $\log_{10} r = 3$ b) $\log_3(7a + 18) = 4$ c) $\log_w 25 = 2$

d) $\log_2 16 = c$ e) $\log_{36} \sqrt[4]{6} = x$

Solution

a) Write the equation in exponential form and solve for r.

$$\log_{10} r = 3 \quad \text{means} \quad \begin{aligned} 10^3 &= r \\ 1000 &= r \end{aligned}$$

The solution set is $\{1000\}$.

b) Write $\log_3(7a + 18) = 4$ in exponential form and solve for a.

$$\log_3(7a + 18) = 4 \quad \text{means} \quad \begin{aligned} 3^4 &= 7a + 18 \\ 81 &= 7a + 18 \\ 63 &= 7a && \text{Subtract 18.} \\ 9 &= a && \text{Divide by 7.} \end{aligned}$$

The solution set is $\{9\}$.

c) Write $\log_w 25 = 2$ in exponential form and solve for w.

$$\log_w 25 = 2 \quad \text{means} \quad \begin{aligned} w^2 &= 25 \\ w &= \pm 5 && \text{Square root property} \end{aligned}$$

Although we get $w = 5$ or $w = -5$ when we solve $w^2 = 25$, recall that the base of a logarithm must be a positive number. Therefore, $w = -5$ is *not* a solution.

The solution set is $\{5\}$.

d) Write $\log_2 16 = c$ in exponential form and solve for c.

$$\log_2 16 = c \quad \text{means} \quad \begin{aligned} 2^c &= 16 \\ c &= 4 \end{aligned}$$

The solution set is $\{4\}$.

e) $\log_{36}\sqrt[4]{6} = x$ means $36^x = \sqrt[4]{6}$

$$\begin{aligned} (6^2)^x &= 6^{1/4} && \text{Express each side with the same base; rewrite the radical as a fractional exponent.} \\ 6^{2x} &= 6^{1/4} && \text{Power rule for exponents} \\ 2x &= \frac{1}{4} && \text{Set the exponents equal.} \\ x &= \frac{1}{8} && \text{Divide by 2.} \end{aligned}$$

The solution set is $\left\{\frac{1}{8}\right\}$.

You Try 3

Solve each logarithmic equation.

a) $\log_2 y = 5$ b) $\log_5(3p + 11) = 3$ c) $\log_x 169 = 2$

d) $\log_6 36 = n$ e) $\log_{64}\sqrt[5]{8} = k$

5. Evaluate a Logarithm

Often when working with logarithms, we are asked to *evaluate* them or to find the value of a log.

Example 4

Evaluate.

a) $\log_3 9$ b) $\log_2 8$ c) $\log_{10}\frac{1}{10}$ d) $\log_{25} 5$

Solution

a) To evaluate (or find the value of) $\log_3 9$ means to find the power to which we raise 3 to get 9. That power is **2**.

$$\log_3 9 = 2 \quad \text{since} \quad 3^2 = 9$$

b) To evaluate $\log_2 8$ means to find the power to which we raise 2 to get 8. That power is **3**.

$$\log_2 8 = 3 \quad \text{since} \quad 2^3 = 8$$

c) To evaluate $\log_{10}\frac{1}{10}$ means to find the power to which we raise 10 to get $\frac{1}{10}$. That power is $\mathbf{-1}$.

If you don't see that this is the answer, set the expression $\log_{10}\frac{1}{10}$ equal to x, write the equation in exponential form, and solve for x as in Example 3.

$\log_{10}\frac{1}{10} = x$ means $10^x = \frac{1}{10}$

$$\begin{aligned} 10^x &= 10^{-1} && \frac{1}{10} = 10^{-1} \\ x &= -1 \end{aligned}$$

Then, $\log_{10}\frac{1}{10} = -1$.

d) To evaluate $\log_{25} 5$ means to find the power to which we raise 25 to get 5. That power is $\mathbf{\frac{1}{2}}$.

Once again, we can also find the value of $\log_{25} 5$ by setting it equal to x, writing the equation in exponential form, and solving for x.

$$\begin{aligned} \log_{25} 5 = x \quad \text{means} \quad 25^x &= 5 \\ (5^2)^x &= 5 && \text{Express each side with the same base.} \\ 5^{2x} &= 5^1 && \text{Power rule; } 5 = 5^1 \\ 2x &= 1 && \text{Set the exponents equal.} \\ x &= \frac{1}{2} && \text{Divide by 2.} \end{aligned}$$

Therefore, $\log_{25} 5 = \frac{1}{2}$. ■

You Try 4

Evaluate.

a) $\log_{10} 100$ b) $\log_3 81$ c) $\log_8 \frac{1}{8}$ d) $\log_{144} 12$

6. Evaluate Common Logarithms, and Solve Equations of the Form $\log b = c$

Logarithms have many applications not only in mathematics but also in other areas such as chemistry, biology, engineering, and economics.

Since our number system is a base 10 system, logarithms to the base 10 are very widely used and are called **common logarithms** or **common logs**. A base 10 log has a special notation—$\log_{10} x$ is written as $\log x$. *When a log is written in this way, the base is assumed to be 10.*

$$\log x \text{ means } \log_{10} x$$

We must keep this in mind when evaluating logarithms and when solving logarithmic equations.

Example 5

Evaluate log 100.

Solution

log 100 is equivalent to $\log_{10} 100$. To evaluate log 100 means to find the power to which we raise 10 to get 100. That power is **2**.

$$\log 100 = 2$$

■

You Try 5

Evaluate log 1000.

Example 6

Solve $\log(3x - 8) = 1$.

Solution

$\log(3x - 8) = 1$ is equivalent to $\log_{10}(3x - 8) = 1$. Write the equation in exponential form and solve for x.

$$\begin{aligned} \log(3x - 8) = 1 \text{ means } 10^1 &= 3x - 8 \\ 10 &= 3x - 8 \\ 18 &= 3x && \text{Add 8.} \\ 6 &= x && \text{Divide by 3.} \end{aligned}$$

The solution set is $\{6\}$.

You Try 6

Solve $\log(12q + 16) = 2$.

We will study common logs in more depth in Section 13.5.

7. Use the Properties $\log_a a = 1$ and $\log_a 1 = 0$

There are a couple of properties of logarithms that can simplify our work.

If a is any real number, then $a^1 = a$. Furthermore, if $a \neq 0$, then $a^0 = 1$. Write $a^1 = a$ and $a^0 = 1$ in logarithmic form to obtain these two properties of logarithms:

Properties of Logarithms

If $a > 0$ and $a \neq 1$,

1) $\log_a a = 1$
2) $\log_a 1 = 0$

Example 7

Use the properties of logarithms to evaluate each.

a) $\log_{12} 1$ b) $\log_3 3$ c) $\log 10$ d) $\log_{\sqrt{5}} 1$

Solution

a) By property 2, $\log_{12} 1 = 0$.
b) By property 1, $\log_3 3 = 1$.
c) The base of $\log 10$ is 10. Therefore, $\log 10 = \log_{10} 10$. By property 1, $\log 10 = 1$.
d) By property 2, $\log_{\sqrt{5}} 1 = 0$.

You Try 7

Use the properties of logarithms to evaluate each.

a) $\log_{16} 16$ b) $\log_{1/3} 1$ c) $\log_{\sqrt{11}} \sqrt{11}$

8. Define and Graph a Logarithmic Function

Next we define a logarithmic function.

Definition

For $a > 0$, $a \neq 1$, and $x > 0$, $f(x) = \log_a x$ is the **logarithmic function with base a.**

Note

$f(x) = \log_a x$ can also be written as $y = \log_a x$. Changing $y = \log_a x$ to exponential form, we get $a^y = x$. Remembering that *a is a positive number not equal to* 1, it follows that

1) any real number may be substituted for y. Therefore, **the range of $y = \log_a x$ is $(-\infty, \infty)$.**
2) x must be a positive number. So, **the domain of $y = \log_a x$ is $(0, \infty)$.**

Let's return to the problem of finding the equation of the inverse of $f(x) = 2^x$ that was first introduced on p. 785.

Example 8

Find the equation of the inverse of $f(x) = 2^x$.

Solution

Step 1: Replace $f(x)$ with y: $\quad y = 2^x$

Step 2: Interchange x and y: $\quad x = 2^y$

Step 3: Solve for y.

To solve $x = 2^y$ for y, write the equation in logarithmic form.

$$x = 2^y \quad \text{means} \quad y = \log_2 x$$

Step 4: Replace y with $f^{-1}(x)$.

$$f^{-1}(x) = \log_2 x$$

The inverse of the exponential function $f(x) = 2^x$ is $f^{-1}(x) = \log_2 x$. ■

You Try 8

Find the equation of the inverse of $f(x) = 6^x$.

Note

The inverse of the exponential function $f(x) = a^x$ (where $a > 0$, $a \neq 1$, and x is any real number) is $f^{-1}(x) = \log_a x$. Furthermore,

1) the domain of $f(x)$ is the range of $f^{-1}(x)$.
2) the range of $f(x)$ is the domain of $f^{-1}(x)$.

Their graphs are symmetric with respect to $y = x$.

To graph a logarithmic function, write it in exponential form first. Then make a table of values, plot the points, and draw the curve through the points.

Example 9

Graph $f(x) = \log_2 x$.

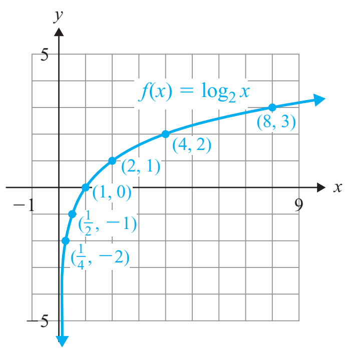
Solution

Substitute y for $f(x)$ and write the equation in exponential form.

$$y = \log_2 x \quad \text{means} \quad 2^y = x$$

To make a table of values, it will be easier to *choose values for* y and compute the corresponding values of x. Remember to choose values of y that will give positive numbers, negative numbers, and zero in the exponent.

$2^y = x$	
x	**y**
1	0
2	1
4	2
8	3
$\frac{1}{2}$	-1
$\frac{1}{4}$	-2



From the graph, we can see that the domain of f is $(0, \infty)$, and the range of f is $(-\infty, \infty)$. ■

You Try 9

Graph $f(x) = \log_4 x$.

Example 10

Graph $f(x) = \log_{1/3} x$.

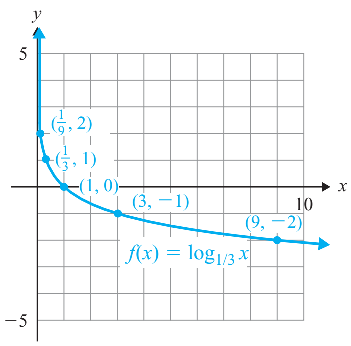
Solution

Substitute y for $f(x)$ and write the equation in exponential form.

$$y = \log_{1/3} x \quad \text{means} \quad \left(\frac{1}{3}\right)^y = x$$

For the table of values, *choose values for* y and compute the corresponding values of x.

$\left(\frac{1}{3}\right)^y = x$	
x	**y**
1	0
$\frac{1}{3}$	1
$\frac{1}{9}$	2
3	-1
9	-2



The domain of f is $(0, \infty)$, and the range is $(-\infty, \infty)$. ■

You Try 10

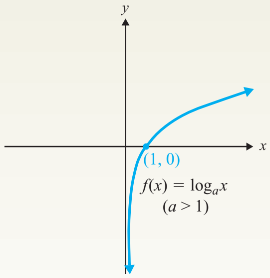
Graph $f(x) = \log_{1/4} x$.

The graphs in Examples 9 and 10 are typical of the graphs of logarithmic functions—Example 9 for functions where $a > 1$ and Example 10 for functions where $0 < a < 1$.

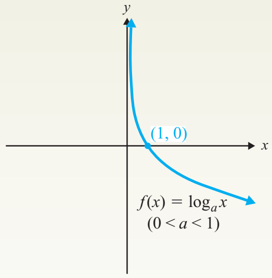
Next is a summary of some characteristics of logarithmic functions.

Summary Characteristics of a Logarithmic Function $f(x) = \log_a x$, where $a > 0$ and $a \neq 1$

1) If $f(x) = \log_a x$ where $a > 1$, the value of y increases as the value of x increases.



2) If $f(x) = \log_a x$ where $0 < a < 1$, the value of y decreases as the value of x increases.



3) The function is one-to-one.
4) The x-intercept is $(1, 0)$.
5) The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$.
6) The inverse of $f(x) = \log_a x$ is $f^{-1}(x) = a^x$.

Compare these characteristics of logarithmic functions to the characteristics of exponential functions on p. 777 in Section 13.2. The domain and range of logarithmic and exponential functions are interchanged since they are inverse functions.

9. Solve an Applied Problem Using a Logarithmic Equation

Example 11

A hospital has found that the function $A(t) = 50 + 8\log_2(t + 2)$ approximates the number of people treated each year since 1995 for severe allergic reactions to peanuts. If $t = 0$ represents the year 1995, answer the following.

a) How many people were treated in 1995?
b) How many people were treated in 2001?
c) In what year were approximately 82 people treated for allergic reactions to peanuts?

Solution

a) The year 1995 corresponds to $t = 0$. Let $t = 0$, and find $A(0)$.

$$\begin{aligned} A(0) &= 50 + 8\log_2(0 + 2) && \text{Substitute 0 for } t. \\ &= 50 + 8\log_2 2 \\ &= 50 + 8(1) && \log_2 2 = 1 \\ &= 58 \end{aligned}$$

In 1995, 58 people were treated for peanut allergies.

b) The year 2001 corresponds to $t = 6$. Let $t = 6$, and find $A(6)$.

$$\begin{aligned} A(6) &= 50 + 8\log_2(6 + 2) && \text{Substitute 6 for } t. \\ &= 50 + 8\log_2 8 \\ &= 50 + 8(3) && \log_2 8 = 3 \\ &= 50 + 24 \\ &= 74 \end{aligned}$$

In 2001, 74 people were treated for peanut allergies.

c) To determine in what year 82 people were treated, let $A(t) = 82$ and solve for t.

$$82 = 50 + 8\log_2(t + 2) \qquad \text{Substitute 82 for } A(t).$$

To solve for t, we first need to get the term containing the logarithm on a side by itself. Subtract 50 from each side.

$$\begin{aligned} 32 &= 8\log_2(t + 2) && \text{Subtract 50.} \\ 4 &= \log_2(t + 2) && \text{Divide by 8.} \\ 2^4 &= t + 2 && \text{Write in exponential form.} \\ 16 &= t + 2 \\ 14 &= t \end{aligned}$$

$t = 14$ corresponds to the year 2009. (Add 14 to the year 1995.)

82 people were treated for peanut allergies in 2009.

You Try 11

The amount of garbage (in millions of pounds) collected in a certain town each year since 1990 can be approximated by $G(t) = 6 + \log_2(t + 1)$, where $t = 0$ represents the year 1990.

a) How much garbage was collected in 1990?

b) How much garbage was collected in 1997?

c) In what year would it be expected that 11,000,000 pounds of garbage will be collected? [Hint: Let $G(t) = 11$.]

Answers to You Try Exercises

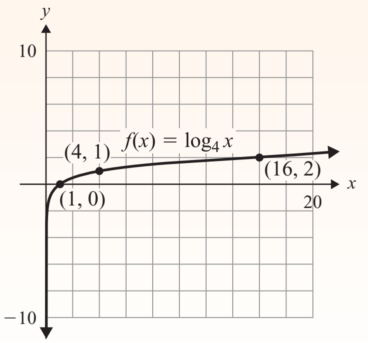
1) a) $3^4 = 81$ b) $5^{-2} = \frac{1}{25}$ c) $64^{1/2} = 8$ d) $13^1 = 13$

2) a) $\log_7 49 = 2$ b) $\log_5 \frac{1}{625} = -4$ c) $\log_{19} 1 = 0$ d) $\log_{144} 12 = \frac{1}{2}$

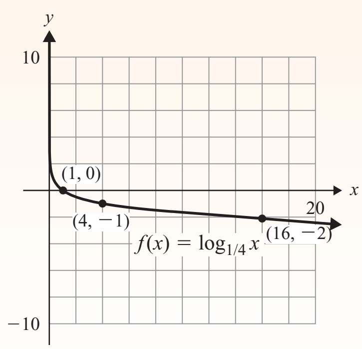
3) a) $\{32\}$ b) $\{38\}$ c) $\{13\}$ d) $\{2\}$ e) $\left\{\frac{1}{10}\right\}$

4) a) 2 b) 4 c) -1 d) $\frac{1}{2}$ 5) 3 6) $\{7\}$ 7) a) 1 b) 0 c) 1 8) $f^{-1}(x) = \log_6 x$

9)



10)



11) a) 6,000,000 lb b) 9,000,000 lb c) 2021

13.3 Exercises

Mixed Exercises: Objectives 1 and 2

1) In the equation $y = \log_a x$, a must be what kind of number?

2) In the equation $y = \log_a x$, x must be what kind of number?

3) What is the base of $y = \log x$?

4) A base 10 logarithm is called a ________________ logarithm.

Write in exponential form.

5) $\log_7 49 = 2$

6) $\log_{11} 121 = 2$

7) $\log_2 8 = 3$

8) $\log_2 32 = 5$

9) $\log_9 \frac{1}{81} = -2$

10) $\log_8 \frac{1}{64} = -2$

11) $\log 1{,}000{,}000 = 6$

12) $\log 10{,}000 = 4$

13) $\log_{25} 5 = \frac{1}{2}$

14) $\log_{64} 4 = \frac{1}{3}$

15) $\log_{13} 13 = 1$

16) $\log_9 1 = 0$

Objective 3: Convert from Exponential Form to Logarithmic Form

Write in logarithmic form.

17) $9^2 = 81$

18) $12^2 = 144$

19) $10^2 = 100$

20) $10^3 = 1000$

21) $3^{-4} = \frac{1}{81}$

22) $2^{-5} = \frac{1}{32}$

23) $10^0 = 1$

24) $10^1 = 10$

25) $169^{1/2} = 13$

26) $27^{1/3} = 3$

27) $\sqrt{9} = 3$

28) $\sqrt{64} = 8$

29) $\sqrt[3]{64} = 4$

30) $\sqrt[4]{81} = 3$

Mixed Exercises: Objectives 4 and 6

31) Explain how to solve a logarithmic equation of the form $\log_a b = c$.

32) A student solves $\log_x 9 = 2$ and gets the solution set $\{-3, 3\}$. Is this correct? Why or why not?

Solve each logarithmic equation.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

33) $\log_2 x = 6$
$2^6 = x$ ________________
_____ Solve for x.
The solution set is _____.

34) $\log_5 t = -3$
_____ Rewrite in exponential form.
Solve for t.
_____
The solution set is _____.

Solve each logarithmic equation.

35) $\log_{11} x = 2$

36) $\log_5 k = 3$

37) $\log_4 r = 3$

38) $\log_2 y = 4$

39) $\log p = 5$

40) $\log w = 2$

41) $\log_m 49 = 2$

42) $\log_x 4 = 2$

43) $\log_6 h = -2$

44) $\log_4 b = -3$

45) $\log_2(a + 2) = 4$

46) $\log_6(5y + 1) = 2$

47) $\log_3(4t - 3) = 3$

48) $\log_2(3n + 7) = 5$

49) $\log_{81} \sqrt[4]{9} = x$

50) $\log_{49} \sqrt[3]{7} = d$

51) $\log_{125} \sqrt{5} = c$

52) $\log_{16} \sqrt[5]{4} = k$

53) $\log_{144} w = \frac{1}{2}$

54) $\log_{64} p = \frac{1}{3}$

55) $\log_8 x = \frac{2}{3}$

56) $\log_{16} t = \frac{3}{4}$

57) $\log_{(3m-1)} 25 = 2$

58) $\log_{(y-1)} 4 = 2$

Mixed Exercises: Objectives 5–7

Evaluate each logarithm.

59) $\log_5 25$

60) $\log_9 81$

61) $\log_2 32$

62) $\log_4 64$

63) $\log 100$

64) $\log 1000$

65) $\log_{49} 7$

66) $\log_{36} 6$

67) $\log_8 \frac{1}{8}$

68) $\log_3 \frac{1}{3}$

69) $\log_5 5$

70) $\log_2 1$

VIDEO 71) $\log_{1/4} 16$

72) $\log_{1/3} 27$

Objective 8: Define and Graph a Logarithmic Function

73) Explain how to graph a logarithmic function of the form $f(x) = \log_a x$.

74) What are the domain and range of $f(x) = \log_a x$?

Graph each logarithmic function.

75) $f(x) = \log_3 x$

76) $f(x) = \log_4 x$

77) $f(x) = \log_2 x$

78) $f(x) = \log_5 x$

VIDEO 79) $f(x) = \log_{1/2} x$

80) $f(x) = \log_{1/3} x$

81) $f(x) = \log_{1/4} x$

82) $f(x) = \log_{1/5} x$

Find the inverse of each function.

83) $f(x) = 3^x$

84) $f(x) = 4^x$

85) $f(x) = \log_2 x$

86) $f(x) = \log_5 x$

Objective 9: Solve an Applied Problem Using a Logarithmic Equation

Solve each problem.

VIDEO 87) The function $L(t) = 1800 + 68 \log_3(t + 3)$ approximates the number of dog licenses issued by a city each year since 1980. If $t = 0$ represents the year 1980, answer the following.

a) How many dog licenses were issued in 1980?

b) How many were issued in 2004?

c) In what year would it be expected that 2072 dog licenses will be issued?

88) Until the 1990s, Rock Glen was a rural community outside of a large city. In 1994, subdivisions of homes began to be built. The number of houses in Rock Glen t years after 1994 can be approximated by

$$H(t) = 142 + 58 \log_2(t + 1)$$

where $t = 0$ represents 1994.

a) Determine the number of homes in Rock Glen in 1994.

b) Determine the number of homes in Rock Glen in 1997.

c) In what year were there approximately 374 homes?

89) A company plans to introduce a new type of cookie to the market. The company predicts that its sales over the next 24 months can be approximated by

$$S(t) = 14 \log_3(2t + 1)$$

where t is the number of months after the product is introduced, and $S(t)$ is in thousands of boxes of cookies.

a) How many boxes of cookies were sold after they were on the market for 1 month?

b) How many boxes were sold after they were on the market for 4 months?

c) After 13 months, sales were approximately 43,000. Does this number fall short of, meet, or exceed the number of sales predicted by the formula?

90) Based on previous data, city planners have calculated that the number of tourists (in millions) to their city each year can be approximated by

$$N(t) = 10 + 1.2 \log_2(t + 2)$$

where t is the number of years after 1995.

a) How many tourists visited the city in 1995?

b) How many tourists visited the city in 2001?

c) In 2009, actual data put the number of tourists at 14,720,000. How does this number compare to the number predicted by the formula?

Section 13.4 Properties of Logarithms

Objectives

1. Use the Product Rule for Logarithms
2. Use the Quotient Rule for Logarithms
3. Use the Power Rule for Logarithms
4. Use the Properties $\log_a a^x = x$ and $a^{\log_a x} = x$
5. Combine the Properties of Logarithms

Logarithms have properties that are very useful in applications and in higher mathematics.

In this section, we will learn more properties of logarithms, and we will practice using them because they can make some very difficult mathematical calculations much easier. *The properties of logarithms come from the properties of exponents.*

1. Use the Product Rule for Logarithms

The product rule for logarithms can be derived from the product rule for exponents.

Property The Product Rule for Logarithms

Let x, y, and a be positive real numbers where $a \neq 1$. Then,

$$\log_a xy = \log_a x + \log_a y$$

The logarithm of a product, xy, is the same as the sum of the logarithms of each factor, x and y.

BE CAREFUL

$\log_a xy \neq (\log_a x)(\log_a y)$

Example 1

Rewrite as the sum of logarithms and simplify, if possible. Assume the variables represent positive real numbers.

a) $\log_6(4 \cdot 7)$ b) $\log_4 16t$ c) $\log_8 y^3$ d) $\log 10pq$

Solution

a) The logarithm of a product equals the *sum* of the logs of the factors. Therefore,

$$\log_6(4 \cdot 7) = \log_6 4 + \log_6 7 \qquad \text{Product rule}$$

b) $$\begin{aligned} \log_4 16t &= \log_4 16 + \log_4 t && \text{Product rule} \\ &= 2 + \log_4 t && \log_4 16 = 2 \end{aligned}$$

Evaluate logarithms, like $\log_4 16$, when possible.

c) $$\begin{aligned} \log_8 y^3 &= \log_8(y \cdot y \cdot y) && \text{Write } y^3 \text{ as } y \cdot y \cdot y. \\ &= \log_8 y + \log_8 y + \log_8 y && \text{Product rule} \\ &= 3\log_8 y \end{aligned}$$

d) Recall that if no base is written, then it is assumed to be 10.

$$\begin{aligned} \log 10pq &= \log 10 + \log p + \log q && \text{Product rule} \\ &= 1 + \log p + \log q && \log 10 = 1 \end{aligned}$$

You Try 1

Rewrite as the sum of logarithms and simplify, if possible. Assume the variables represent positive real numbers.

a) $\log_9(2 \cdot 5)$ b) $\log_2 32k$ c) $\log_6 c^4$ d) $\log 100yz$

We can use the product rule for exponents in the "opposite" direction, too. That is, given the sum of logarithms we can write a single logarithm.

Example 2

Write as a single logarithm. Assume the variables represent positive real numbers.

a) $\log_8 5 + \log_8 3$ b) $\log 7 + \log r$ c) $\log_3 x + \log_3(x + 4)$

Solution

a) $\log_8 5 + \log_8 3 = \log_8(5 \cdot 3)$ Product rule
$= \log_8 15$ $5 \cdot 3 = 15$

b) $\log 7 + \log r = \log 7r$ Product rule

c) $\log_3 x + \log_3(x + 4) = \log_3 x(x + 4)$ Product rule
$= \log_3(x^2 + 4x)$ Distribute.

BE CAREFUL

$\log_a(x + y) \neq \log_a x + \log_a y$. Therefore, $\log_3(x^2 + 4x)$ does *not* equal $\log_3 x^2 + \log_3 4x$.

You Try 2

Write as a single logarithm. Assume the variables represent positive real numbers.

a) $\log_5 9 + \log_5 4$ b) $\log_6 13 + \log_6 c$ c) $\log y + \log(y - 6)$

2. Use the Quotient Rule for Logarithms

The quotient rule for logarithms can be derived from the quotient rule for exponents.

Property The Quotient Rule for Logarithms

Let x, y, and a be positive real numbers where $a \neq 1$. Then,

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

The logarithm of a quotient, $\frac{x}{y}$, is the same as the logarithm of the numerator *minus* the logarithm of the denominator.

BE CAREFUL

$\log_a \frac{x}{y} \neq \frac{\log_a x}{\log_a y}$.

Example 3

Write as the difference of logarithms and simplify, if possible. Assume $w > 0$.

a) $\log_7 \frac{3}{10}$ b) $\log_3 \frac{81}{w}$

Solution

a) $\log_7 \frac{3}{10} = \log_7 3 - \log_7 10$ Quotient rule

b) $\log_3 \frac{81}{w} = \log_3 81 - \log_3 w$ Quotient rule

$= 4 - \log_3 w$ $\log_3 81 = 4$

You Try 3

Write as the difference of logarithms and simplify, if possible. Assume $n > 0$.

a) $\log_6 \frac{2}{9}$ b) $\log_5 \frac{n}{25}$

Example 4

Write as a single logarithm. Assume the variable is defined so that the expressions are positive.

a) $\log_2 18 - \log_2 6$ b) $\log_4(z - 5) - \log_4(z^2 + 9)$

Solution

a) $\log_2 18 - \log_2 6 = \log_2 \frac{18}{6}$ Quotient rule

$= \log_2 3$ $\frac{18}{6} = 3$

b) $\log_4(z - 5) - \log_4(z^2 + 9) = \log_4 \frac{z - 5}{z^2 + 9}$ Quotient rule

BE CAREFUL

$\log_a(x - y) \neq \log_a x - \log_a y$

You Try 4

Write as a single logarithm. Assume the variable is defined so that the expressions are positive.

a) $\log_4 36 - \log_4 3$ b) $\log_5(c^2 - 2) - \log_5(c + 1)$

3. Use the Power Rule for Logarithms

In Example 1c), we saw that $\log_8 y^3 = 3 \log_8 y$ since

$$\begin{aligned}\log_8 y^3 &= \log_8(y \cdot y \cdot y)\\ &= \log_8 y + \log_8 y + \log_8 y\\ &= 3 \log_8 y\end{aligned}$$

This result can be generalized as the next property and comes from the power rule for exponents.

Property The Power Rule for Logarithms

Let x and a be positive real numbers, where $a \neq 1$, and let r be any real number. Then,

$$\log_a x^r = r \log_a x$$

BE CAREFUL

The rule applies to $\log_a x^r$ *not* $(\log_a x)^r$. Be sure you can distinguish between the two expressions.

Example 5

Rewrite each expression using the power rule and simplify, if possible. Assume the variables represent positive real numbers and that the variable bases are positive real numbers not equal to 1.

a) $\log_9 y^4$ b) $\log_2 8^5$ c) $\log_a \sqrt{3}$ d) $\log_w \frac{1}{w}$

Solution

a) $\log_9 y^4 = 4 \log_9 y$ Power rule

b) $\log_2 8^5 = 5 \log_2 8$ Power rule
$= 5(3)$ $\log_2 8 = 3$
$= 15$ Multiply.

c) *It is common practice to rewrite radicals as fractional exponents when applying the properties of logarithms*. This will be our first step.

$$\log_a \sqrt{3} = \log_a 3^{1/2} \quad \text{Rewrite as a fractional exponent.}$$
$$= \frac{1}{2} \log_a 3 \quad \text{Power rule}$$

d) Rewrite $\frac{1}{w}$ as w^{-1}: $\log_w \frac{1}{w} = \log_w w^{-1}$ $\frac{1}{w} = w^{-1}$
$= -1 \log_w w$ Power rule
$= -1(1)$ $\log_w w = 1$
$= -1$ Multiply.

You Try 5

Rewrite each expression using the power rule and simplify, if possible. Assume the variables represent positive real numbers and that the variable bases are positive real numbers not equal to 1.

a) $\log_8 t^9$ b) $\log_3 9^7$ c) $\log_a \sqrt[3]{5}$ d) $\log_m \frac{1}{m^8}$

The next properties we will look at can be derived from the power rule and from the fact that $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions.

4. Use the Properties $\log_a a^x = x$ and $a^{\log_a x} = x$

Other Properties of Logarithms

Let a be a positive real number such that $a \neq 1$. Then,

1) $\log_a a^x = x$ for any real number x.

2) $a^{\log_a x} = x$ for $x > 0$.

Example 6

Evaluate each expression.

a) $\log_6 6^7$ b) $\log 10^8$ c) $5^{\log_5 3}$

Solution

a) $\log_6 6^7 = 7$ $\log_a a^x = x$

b) $\log 10^8 = 8$ The base of the log is 10.

c) $5^{\log_5 3} = 3$ $a^{\log_a x} = x$

You Try 6

Evaluate each expression.

a) $\log_3 3^{10}$ b) $\log 10^{-6}$ c) $7^{\log_7 9}$

Next is a summary of the properties of logarithms. The properties presented in Section 13.3 are included as well.

Summary Properties of Logarithms

Let x, y, and a be positive real numbers where $a \neq 1$, and let r be any real number. Then,

1) $\log_a a = 1$
2) $\log_a 1 = 0$
3) $\log_a xy = \log_a x + \log_a y$ Product rule
4) $\log_a \dfrac{x}{y} = \log_a x - \log_a y$ Quotient rule
5) $\log_a x^r = r \log_a x$ Power rule
6) $\log_a a^x = x$ for any real number x
7) $a^{\log_a x} = x$

Many students make the same mistakes when working with logarithms. Keep in mind the following to avoid these common errors.

1) $\log_a xy \neq (\log_a x)(\log_a y)$
2) $\log_a(x + y) \neq \log_a x + \log_a y$
3) $\log_a \dfrac{x}{y} \neq \dfrac{\log_a x}{\log_a y}$
4) $\log_a(x - y) \neq \log_a x - \log_a y$
5) $(\log_a x)^r \neq r \log_a x$

5. Combine the Properties of Logarithms

Not only can the properties of logarithms simplify some very complicated computations, they are also needed for solving some types of logarithmic equations. The properties of logarithms are also used in calculus and many areas of science.

Next, we will see how to use different properties of logarithms together to rewrite logarithmic expressions.

Example 7

Write each expression as the sum or difference of logarithms in simplest form. Assume all variables represent positive real numbers and that the variable bases are positive real numbers not equal to 1.

a) $\log_8 r^5 t$ b) $\log_3 \dfrac{27}{ab^2}$ c) $\log_7 \sqrt{7p}$ d) $\log_a(4a + 5)$

Solution

a)
$$\begin{aligned} \log_8 r^5 t &= \log_8 r^5 + \log_8 t && \text{Product rule} \\ &= 5 \log_8 r + \log_8 t && \text{Power rule} \end{aligned}$$

b)
$$\begin{aligned} \log_3 \frac{27}{ab^2} &= \log_3 27 - \log_3 ab^2 && \text{Quotient rule} \\ &= 3 - (\log_3 a + \log_3 b^2) && \log_3 27 = 3;\ \text{product rule} \\ &= 3 - (\log_3 a + 2 \log_3 b) && \text{Power rule} \\ &= 3 - \log_3 a - 2 \log_3 b && \text{Distribute.} \end{aligned}$$

c)
$$\begin{aligned} \log_7 \sqrt{7p} &= \log_7 (7p)^{1/2} && \text{Rewrite radical as fractional exponent.} \\ &= \frac{1}{2} \log_7 (7p) && \text{Power rule} \\ &= \frac{1}{2}(\log_7 7 + \log_7 p) && \text{Product rule} \\ &= \frac{1}{2}(1 + \log_7 p) && \log_7 7 = 1 \\ &= \frac{1}{2} + \frac{1}{2} \log_7 p && \text{Distribute.} \end{aligned}$$

d) $\log_a(4a + 5)$ is in simplest form and cannot be rewritten using any properties of logarithms. [Recall that $\log_a(x + y) \neq \log_a x + \log_a y$.] ■

You Try 7

Write each expression as the sum or difference of logarithms in simplest form. Assume all variables represent positive real numbers and that the variable bases are positive real numbers not equal to 1.

a) $\log_2 8s^2t^5$ b) $\log_a \dfrac{4c^2}{b^3}$ c) $\log_5 \sqrt[3]{\dfrac{25}{n}}$ d) $\dfrac{\log_4 k}{\log_4 m}$

Example 8

Write each as a single logarithm in simplest form. Assume the variable represents a positive real number.

a) $2 \log_7 5 + 3 \log_7 2$ b) $\dfrac{1}{2} \log_6 s - 3 \log_6(s^2 + 1)$

Solution

a)
$$\begin{aligned} 2 \log_7 5 + 3 \log_7 2 &= \log_7 5^2 + \log_7 2^3 && \text{Power rule} \\ &= \log_7 25 + \log_7 8 && 5^2 = 25;\ 2^3 = 8 \\ &= \log_7 (25 \cdot 8) && \text{Product rule} \\ &= \log_7 200 && \text{Multiply.} \end{aligned}$$

b) $\frac{1}{2}\log_6 s - 3\log_6(s^2+1) = \log_6 s^{1/2} - \log_6(s^2+1)^3$ Power rule

$= \log_6\sqrt{s} - \log_6(s^2+1)^3$ Write in radical form.

$= \log_6 \frac{\sqrt{s}}{(s^2+1)^3}$ Quotient rule ■

You Try 8

Write each as a single logarithm in simplest form. Assume the variables are defined so that the expressions are positive.

a) $2\log 4 + \log 5$ b) $\frac{2}{3}\log_5 c + \frac{1}{3}\log_5 d - 2\log_5(c-6)$

Given the values of logarithms, we can compute the values of other logarithms using the properties we have learned in this section.

Example 9

Given that $\log 6 \approx 0.7782$ and $\log 4 \approx 0.6021$, use the properties of logarithms to approximate the following.

a) $\log 24$ b) $\log\sqrt{6}$

Solution

a) To find the value of log 24, we must determine how to write 24 in terms of 6 or 4 or some combination of the two. Since $24 = 6 \cdot 4$, we can write

$$\begin{aligned}\log 24 &= \log(6\cdot 4) && 24 = 6\cdot 4\\ &= \log 6 + \log 4 && \text{Product rule}\\ &\approx 0.7782 + 0.6021 && \text{Substitute.}\\ &= 1.3803 && \text{Add.}\end{aligned}$$

b) We can write $\sqrt{6}$ as $6^{1/2}$.

$$\begin{aligned}\log\sqrt{6} &= \log 6^{1/2} && \sqrt{6} = 6^{1/2}\\ &= \frac{1}{2}\log 6 && \text{Power rule}\\ &\approx \frac{1}{2}(0.7782) && \log 6 \approx 0.7782\\ &= 0.3891 && \text{Multiply.}\end{aligned}$$ ■

You Try 9

Using the values given in Example 9, approximate each of the following.

a) $\log 16$ b) $\log\frac{6}{4}$ c) $\log\sqrt[3]{4}$ d) $\log\frac{1}{6}$

Answers to You Try Exercises

1) a) $\log_9 2 + \log_9 5$ b) $5 + \log_2 k$ c) $4 \log_6 c$ d) $2 + \log y + \log z$ 2) a) $\log_5 36$ b) $\log_6 13c$ c) $\log(y^2 - 6y)$ 3) a) $\log_6 2 - \log_6 9$ b) $\log_5 n - 2$ 4) a) $\log_4 12$ b) $\log_5 \dfrac{c^2 - 2}{c + 1}$ 5) a) $9 \log_8 t$ b) 14 c) $\dfrac{1}{3}\log_a 5$ d) -8 6) a) 10 b) -6 c) 9

7) a) $3 + 2\log_2 s + 5\log_2 t$ b) $\log_a 4 + 2\log_a c - 3\log_a b$ c) $\dfrac{2}{3} - \dfrac{1}{3}\log_5 n$

d) cannot be simplified 8) a) $\log 80$ b) $\log_5 \dfrac{\sqrt[3]{c^2 d}}{(c - 6)^2}$ 9) a) 1.2042 b) 0.1761 c) 0.2007 d) -0.7782

13.4 Exercises

Mixed Exercises: Objectives 1–5

Decide whether each statement is true or false.

1) $\log_6 8c = \log_6 8 + \log_6 c$

2) $\log_5 \dfrac{m}{3} = \log_5 m - \log_5 3$

3) $\log_9 \dfrac{7}{2} = \dfrac{\log_9 7}{\log_9 2}$

4) $\log 1000 = 3$

5) $(\log_4 k)^2 = 2 \log_4 k$

6) $\log_2(x^2 + 8) = \log_2 x^2 + \log_2 8$

7) $5^{\log_5 4} = 4$

8) $\log_3 4^5 = 5 \log_3 4$

Write as the sum or difference of logarithms and simplify, if possible. Assume all variables represent positive real numbers.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

9) $\log_5 25y$

$\log_5 25y = \log_5 25 + \log_5 y$ ________

$=$ ________ Evaluate $\log_5 25$.

10) $\log_3 \dfrac{81}{n^2}$

$\log_3 \dfrac{81}{n^2} = \log_3 81 - \log_3 n^2$ ________

$=$ ________ Evaluate $\log_3 81$; use power rule.

11) $\log_8(3 \cdot 10)$

12) $\log_2(6 \cdot 5)$

13) $\log_7 5d$

14) $\log_4 6w$

15) $\log_9 \dfrac{4}{7}$

16) $\log_5 \dfrac{20}{17}$

17) $\log_5 2^3$

18) $\log_8 10^4$

19) $\log p^8$

20) $\log_3 z^5$

VIDEO 21) $\log_3 \sqrt{7}$

22) $\log_7 \sqrt[3]{4}$

23) $\log_5 25t$

24) $\log_2 16p$

25) $\log_2 \dfrac{8}{k}$

26) $\log_3 \dfrac{x}{9}$

27) $\log_7 49^3$

28) $\log_8 64^{12}$

29) $\log 1000b$

30) $\log_3 27m$

31) $\log_2 32^7$

32) $\log_2 2^9$

33) $\log_5 \sqrt{5}$

34) $\log \sqrt[3]{10}$

35) $\log \sqrt[3]{100}$

36) $\log_2 \sqrt{8}$

37) $\log_6 w^4 z^3$

38) $\log_5 x^2 y$

39) $\log_7 \dfrac{a^2}{b^5}$

40) $\log_4 \dfrac{s^4}{t^6}$

41) $\log \dfrac{\sqrt[5]{11}}{y^2}$

42) $\log_3 \dfrac{\sqrt{x}}{y^4}$

43) $\log_2 \dfrac{4\sqrt{n}}{m^3}$

44) $\log_9 \dfrac{gf^2}{h^3}$

VIDEO 45) $\log_4 \dfrac{x^3}{yz^2}$

46) $\log \dfrac{3}{ab^2}$

47) $\log_5 \sqrt{5c}$

48) $\log_8 \sqrt[3]{\dfrac{z}{8}}$

49) $\log k(k - 6)$

50) $\log_2 \dfrac{m^5}{m^2 + 3}$

Write as a single logarithm. Assume the variables are defined so that the variable expressions are positive and so that the bases are positive real numbers not equal to 1.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

51) $2 \log_6 x + \log_6 y$

$2 \log_6 x + \log_6 y = \log_6 x^2 + \log_6 y$ ________

$=$ ________ Product rule

52) $5 \log 2 + \log c - 3 \log d$

$5 \log 2 + \log c - 3 \log d$

$=$ ________ Power rule

$= \log 32 + \log c - \log d^3$ ________

$=$ ________ Product rule

$= \log \dfrac{32c}{d^3}$ ________

53) $\log_a m + \log_a n$

54) $\log_4 7 + \log_4 x$

55) $\log_7 d - \log_7 3$

56) $\log_p r - \log_p s$

57) $4 \log_3 f + \log_3 g$

58) $5 \log_y m + 2 \log_y n$

59) $\log_8 t + 2 \log_8 u - 3 \log_8 v$

60) $3 \log a + 4 \log c - 6 \log b$

61) $\log(r^2 + 3) - 2 \log(r^2 - 3)$

62) $2 \log_2 t - 3 \log_2(5t + 1)$

63) $3 \log_n 2 + \dfrac{1}{2}\log_n k$

64) $2 \log_z 9 + \dfrac{1}{3}\log_z w$

65) $\dfrac{1}{3}\log_d 5 - 2 \log_d z$

66) $\dfrac{1}{2}\log_5 a - 4 \log_5 b$

VIDEO 67) $\log_6 y - \log_6 3 - 3 \log_6 z$

68) $\log_7 8 - 4 \log_7 x - \log_7 y$

69) $4 \log_3 t - 2 \log_3 6 - 2 \log_3 u$

70) $2 \log_9 m - 4 \log_9 2 - 4 \log_9 n$

71) $\dfrac{1}{2} \log_b (c + 4) - 2 \log_b (c + 3)$

72) $\dfrac{1}{2} \log_a r + \dfrac{1}{2} \log_a (r - 2) - \log_a (r + 2)$

73) $\log(a^2 + b^2) - \log(a^4 - b^4)$

74) $\log_n (x^3 - y^3) - \log_n (x - y)$

Given that $\log 5 \approx 0.6990$ and $\log 9 \approx 0.9542$, use the properties of logarithms to approximate the following. **Do not use a calculator.**

VIDEO 75) $\log 45$

76) $\log 25$

77) $\log 81$

78) $\log \dfrac{9}{5}$

79) $\log \dfrac{5}{9}$

80) $\log \sqrt{5}$

81) $\log 3$

82) $\log \dfrac{1}{9}$

83) $\log \dfrac{1}{5}$

84) $\log 5^8$

85) $\log \dfrac{1}{81}$

86) $\log 90$

87) $\log 50$

88) $\log \dfrac{25}{9}$

89) Since $8 = (-4)(-2)$, can we use the properties of logarithms in the following way? Explain.

$$\begin{aligned} \log_2 8 &= \log_2(-4)(-2) \\ &= \log_2(-4) + \log_2 (-2) \end{aligned}$$

90) **Derive the product rule for logarithms from the product rule for exponents.** Assume a, x, and y are positive real numbers with $a \neq 1$. Let $a^m = x$ so that $\log_a x = m$, and let $a^n = y$ so that $\log_a y = n$. Since $a^m \cdot a^n = xy$, show that $\log_a xy = \log_a x + \log_a y$.

Section 13.5 Common and Natural Logarithms and Change of Base

Objectives

1. Evaluate Common Logarithms Without a Calculator
2. Evaluate Common Logarithms Using a Calculator
3. Solve an Equation Containing a Common Logarithm
4. Solve an Applied Problem Given an Equation Containing a Common Logarithm
5. Define and Evaluate a Natural Logarithm
6. Graph a Natural Logarithm Function
7. Solve an Equation Containing a Natural Logarithm
8. Solve Applied Problems Using Exponential Functions
9. Use the Change-of-Base Formula

In this section, we will focus our attention on two widely used logarithmic bases—base 10 and base *e*.

Common Logarithms

1. Evaluate Common Logarithms Without a Calculator

In Section 13.3, we said that a base 10 logarithm is called a **common logarithm**. It is often written as $\log x$.

$$\log x \text{ means } \log_{10} x$$

We can evaluate many logarithms without the use of a calculator because we can write them in terms of a base of 10.

Example 1

Evaluate.

a) $\log 1000$ b) $\log \dfrac{1}{100}$

Solution

a) $\log 1000 = 3$ since $10^3 = 1000$

b) $\log \dfrac{1}{100} = \log 10^{-2}$ $\quad \dfrac{1}{100} = 10^{-2}$

$= \log_{10} 10^{-2}$

$= -2$ $\quad \log_a a^x = x$

You Try 1

Evaluate.

a) $\log 100{,}000$ b) $\log \dfrac{1}{10}$

2. Evaluate Common Logarithms Using a Calculator

Common logarithms are used throughout mathematics and other fields to make calculations easier to solve in applications. Often, however, we need a calculator to evaluate the logarithms. Next we will learn how to use a calculator to find the value of a base 10 logarithm. *We will approximate the value to four decimal places.*

Example 2

Find log 12.

Solution

Enter $\boxed{12}\ \boxed{\text{LOG}}$ or $\boxed{\text{LOG}}\ \boxed{12}\ \boxed{\text{ENTER}}$ into your calculator.

$$\log 12 \approx 1.0792$$

(Note that $10^{1.0792} \approx 12$. Press $\boxed{10}\ \boxed{y^x}\ \boxed{1.0792}\ \boxed{=}$ to evaluate $10^{1.0792}$.)

You Try 2

Find log 3.

We can solve logarithmic equations with or without the use of a calculator.

3. Solve an Equation Containing a Common Logarithm

Example 3

Solve $\log x = -3$.

Solution

Change to exponential form, and solve for x.

$$
\begin{aligned}
\log x = -3 \quad \text{means} \quad \log_{10} x &= -3 \\
10^{-3} &= x \qquad \text{Exponential form} \\
\frac{1}{1000} &= x
\end{aligned}
$$

The solution set is $\left\{\frac{1}{1000}\right\}$. This is the *exact* solution.

You Try 3

Solve $\log x = 2$.

For the equation in Example 4, we will give an exact solution *and* a solution that is approximated to four decimal places. This will give us an idea of the size of the exact solution.

Example 4

Solve $\log x = 2.4$. Give an exact solution and a solution that is approximated to four decimal places.

Solution

Change to exponential form, and solve for x.

$$
\begin{aligned}
\log x = 2.4 \quad \text{means} \quad \log_{10} x &= 2.4 \\
10^{2.4} &= x \qquad \text{Exponential form} \\
251.1886 &\approx x \qquad \text{Approximation}
\end{aligned}
$$

The exact solution is $\{10^{2.4}\}$. This is approximately $\{251.1886\}$.

You Try 4

Solve $\log x = 0.7$. Give an exact solution and a solution that is approximated to four decimal places.

4. Solve an Applied Problem Given an Equation Containing a Common Logarithm

Example 5

The loudness of sound, $L(I)$ in decibels (dB), is given by

$$L(I) = 10 \log \frac{I}{10^{-12}}$$

where I is the intensity of sound in watts per square meter (W/m^2). Fifty meters from the stage at a concert, the intensity of sound is 0.01 W/m^2. Find the loudness of the music at the concert 50 m from the stage.

Solution

Substitute 0.01 for I and find $L(0.01)$.

$$\begin{aligned} L(0.01) &= 10 \log \frac{0.01}{10^{-12}} & \\ &= 10 \log \frac{10^{-2}}{10^{-12}} & 0.01 = 10^{-2} \\ &= 10 \log 10^{10} & \text{Quotient rule for exponents} \\ &= 10(10) & \log 10^{10} = 10 \\ &= 100 & \end{aligned}$$

The sound level of the music 50 m from the stage is 100 dB. (To put this in perspective, a normal conversation has a loudness of about 50 dB.) ■

You Try 5

The intensity of sound from a thunderstorm is about 0.001 W/m^2. Find the loudness of the storm, in decibels.

Natural Logarithms

5. Define and Evaluate a Natural Logarithm

Another base that is often used for logarithms is the number e. In Section 13.2, we said that e, like π, is an irrational number. To four decimal places, $e \approx 2.7183$.

A base e logarithm is called a **natural logarithm** or **natural log.** The notation used for a base e logarithm is $\ln x$ (read as "*the natural log of x*" or "*ln of x*"). Since it is a base e logarithm, it is important to remember that

$$\ln x \text{ means } \log_e x$$

Using the properties $\log_a a^x = x$ and $\log_a a = 1$, we can find the value of some natural logarithms without using a calculator.

Example 6

Evaluate.

a) $\ln e$ b) $\ln e^2$

Solution

a) To evaluate $\ln e$, remember that $\ln e = \log_e e = 1$ since $\log_a a = 1$.

$$\ln e = 1$$

This is a value you should remember. We will use this in Section 13.6 to solve exponential equations with base e.

b) $\ln e^2 = \log_e e^2 = 2$ $\quad \log_a a^x = x$

You Try 6

Evaluate.

a) $5 \ln e$ b) $\ln e^8$

We can use a calculator to *approximate natural logarithms to four decimal places* if the properties do not give us an exact value.

Example 7

Find ln 5.

Solution

Enter $\boxed{5}$ $\boxed{\text{LN}}$ or $\boxed{\text{LN}}$ $\boxed{5}$ $\boxed{\text{ENTER}}$ into your calculator.

$$\ln 5 \approx 1.6094$$

You Try 7

Find ln 9.

6. Graph a Natural Logarithm Function

We can graph $y = \ln x$ by substituting values for x and using a calculator to approximate the values of y.

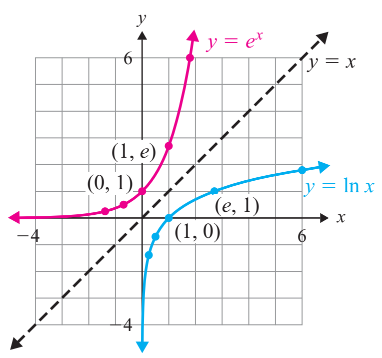
Example 8

Graph $y = \ln x$. Determine the domain and range.

Solution

Choose values for x, and use a calculator to approximate the corresponding values of y. Remember that $\ln e = 1$, so e is a good choice for x.

x	y
1	0
$e \approx 2.72$	1
6	1.79
0.5	−0.69
0.25	−1.39



The domain of $y = \ln x$ is $(0, \infty)$, and the range is $(-\infty, \infty)$.

The graph of the inverse of $y = \ln x$ is also shown. We can obtain the graph of the inverse of $y = \ln x$ by reflecting the graph about the line $y = x$. The inverse of $y = \ln x$ is $y = e^x$.

Notice that the domain of $y = e^x$ is $(-\infty, \infty)$, while the range is $(0, \infty)$, the opposite of the domain and range of $y = \ln x$. This is a direct result of the relationship between a function and its inverse. ■

You Try 8

Graph $y = \ln(x + 4)$. Determine the domain and range.

It is important to remember that $y = \ln x$ means $y = \log_e x$. Understanding this relationship allows us to make the following connections:

$$y = \ln x \text{ is equivalent to } y = \log_e x$$

and

$$y = \log_e x \text{ can be written in exponential form as } e^y = x.$$

Therefore, $y = \ln x$ can be written in exponential form as $e^y = x$.

We can use this relationship to show that the inverse of $y = \ln x$ is $y = e^x$ (this is Exercise 97) and also to verify the result in Example 7 when we found that $\ln 5 \approx 1.6094$. We can express $\ln 5 \approx 1.6094$ in exponential form as $e^{1.6094} \approx 5$. Evaluate $e^{1.6094}$ on a calculator to confirm that $e^{1.6094} \approx 5$.

7. Solve an Equation Containing a Natural Logarithm

Note

To solve an equation containing a natural logarithm, like $\ln x = 4$, we change to exponential form and solve for the variable. We can give an exact solution and a solution that is approximated to four decimal places.

Example 9

Solve each equation. Give an exact solution and a solution that is approximated to four decimal places.

a) $\ln x = 4$ b) $\ln(2x + 5) = 3.8$

Solution

a) $\ln x = 4$ means $\log_e x = 4$

$$\begin{aligned} e^4 &= x && \text{Exponential form} \\ 54.5982 &\approx x && \text{Approximation} \end{aligned}$$

The exact solution is $\{e^4\}$. This is approximately $\{54.5982\}$.

b) $\ln(2x + 5) = 3.8$ means $\log_e(2x + 5) = 3.8$

$$\begin{aligned} e^{3.8} &= 2x + 5 && \text{Exponential form} \\ e^{3.8} - 5 &= 2x && \text{Subtract 5.} \\ \frac{e^{3.8} - 5}{2} &= x && \text{Divide by 2.} \\ 19.8506 &\approx x && \text{Approximation} \end{aligned}$$

The exact solution is $\left\{\dfrac{e^{3.8} - 5}{2}\right\}$. This is approximately $\{19.8506\}$. ■

You Try 9

Solve each equation. Give an exact solution and a solution that is approximated to four decimal places.

a) $\ln y = 2.7$ b) $\ln(3a - 1) = 0.5$

8. Solve Applied Problems Using Exponential Functions

One of the most practical applications of exponential functions is for compound interest.

Definition

Compound Interest: The amount of money, A, in dollars, in an account after t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where P (the principal) is the amount of money (in dollars) deposited in the account, r is the annual interest rate, and n is the number of times the interest is compounded (paid) per year.

Note

We can also think of this formula in terms of the amount of money owed, A, after t yr when P is the amount of money loaned.

Example 10

If \$2000 is deposited in an account paying 4% per year, find the total amount in the account after 5 yr if the interest is compounded

a) quarterly. b) monthly.

(We assume no withdrawals or additional deposits are made.)

Solution

a) If interest compounds quarterly, then interest is paid four times per year. Use

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

with $P = 2000$, $r = 0.04$, $t = 5$, $n = 4$.

$$\begin{aligned} A &= 2000\left(1 + \frac{0.04}{4}\right)^{4(5)} \\ &= 2000(1.01)^{20} \\ &\approx 2440.3801 \end{aligned}$$

Since A is an amount of money, round to the nearest cent. The account will contain \$2440.38 after 5 yr.

b) If interest is compounded monthly, then interest is paid 12 times per year. Use

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

with $P = 2000$, $r = 0.04$, $t = 5$, $n = 12$.

$$\begin{aligned} A &= 2000\left(1 + \frac{0.04}{12}\right)^{12(5)} \\ &\approx 2441.9932 \end{aligned}$$

Round A to the nearest cent. The account will contain \$2441.99 after 5 yr. ■

You Try 10

If \$1500 is deposited in an account paying 5% per year, find the total amount in the account after 8 yr if the interest is compounded

a) monthly. b) weekly.

In Example 10 we saw that the account contained more money after 5 yr when the interest compounded monthly (12 times per year) versus quarterly (four times per year). This will always be true. The more often interest is compounded each year, the more money that accumulates in the account.

If interest *compounds continuously*, we obtain the formula for *continuous compounding,* $A = Pe^{rt}$.

Definition

Continuous Compounding: If P dollars is deposited in an account earning interest rate r compounded continuously, then the amount of money, A (in dollars), in the account after t years is given by

$$A = Pe^{rt}$$

Example 11

Determine the amount of money in an account after 5 yr if \$2000 was initially invested at 4% compounded continuously.

Solution

Use $A = Pe^{rt}$ with $P = 2000$, $r = 0.04$, and $t = 5$.

$$\begin{aligned} A &= 2000e^{0.04(5)} && \text{Substitute values.} \\ &= 2000e^{0.20} && \text{Multiply (0.04)(5).} \\ &\approx 2442.8055 && \text{Evaluate using a calculator.} \end{aligned}$$

Round A to the nearest cent.

The account will contain \$2442.81 after 5 yr. Note that, as expected, this is more than the amounts obtained in Example 10 when the same amount was deposited for 5 yr at 4% but the interest was compounded quarterly and monthly. ■

You Try 11

Determine the amount of money in an account after 8 yr if \$1500 was initially invested at 5% compounded continuously.

9. Use the Change-of-Base Formula

Sometimes, we need to find the value of a logarithm with a base other than 10 or e—like $\log_3 7$. Some calculators, however, do not calculate logarithms other than common logs (base 10) and natural logs (base e). In such cases, we can use the change-of-base formula to evaluate logarithms with bases other than 10 or e.

Definition

Change-of-Base Formula: If a, b, and x are positive real numbers and $a \neq 1$ and $b \neq 1$, then

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Note

We can choose any positive real number not equal to 1 for b, but it is most convenient to choose 10 or e since these will give us common logarithms and natural logarithms, respectively.

Example 12

Find the value of $\log_3 7$ to four decimal places using

a) common logarithms. b) natural logarithms.

Solution

a) The base we will use to evaluate $\log_3 7$ is 10—this is the base of a common logarithm. Then

$$\log_3 7 = \frac{\log_{10} 7}{\log_{10} 3} \quad \text{Change-of-base formula}$$
$$\approx 1.7712 \quad \text{Use a calculator.}$$

b) The base of a natural logarithm is e. Then

$$\log_3 7 = \frac{\log_e 7}{\log_e 3}$$
$$= \frac{\ln 7}{\ln 3}$$
$$\approx 1.7712 \quad \text{Use a calculator.}$$

Using either base 10 or base e gives us the same result. ■

You Try 12

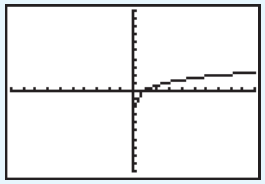
Find the value of $\log_5 38$ to four decimal places using

a) common logarithms. b) natural logarithms.

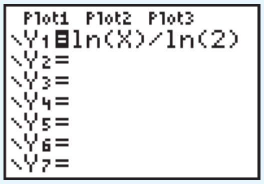
Using Technology

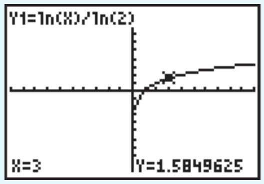
Graphing calculators will graph common logarithmic functions and natural logarithmic functions directly using the **log** or **LN** keys.

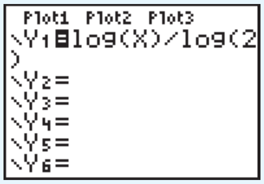
For example, let's graph $f(x) = \ln x$.

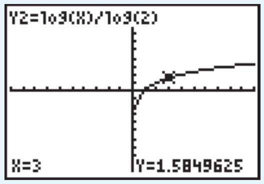


To graph a logarithmic function with a base other than 10 or e, it is necessary to use the change-of-base formula. For example, to graph the function $f(x) = \log_2 x$, first rewrite the function as a quotient of natural logarithms or common logarithms: $f(x) = \log_2 x = \dfrac{\ln x}{\ln 2}$ or $\dfrac{\log x}{\log 2}$. Enter one of these quotients in Y_1 and press $\boxed{\text{GRAPH}}$ to graph as shown below. To illustrate that the same graph results in either case, trace to the point where $x = 3$.









Graph the following functions using a graphing calculator.

1) $f(x) = \log_3 x$ 2) $f(x) = \log_5 x$ 3) $f(x) = 4 \log_2 x + 1$

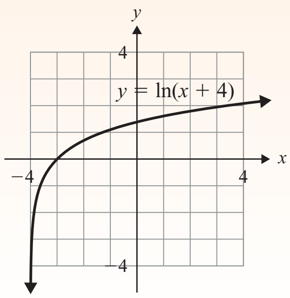
4) $f(x) = \log_2 (x - 3)$ 5) $f(x) = 2 - \log_4 x$ 6) $f(x) = 3 - \log_2 (x + 1)$

Answers to You Try Exercises

1) a) 5 b) -1 2) 0.4771 3) {100} 4) $\{10^{0.7}\}$; {5.0119} 5) 90 dB

6) a) 5 b) 8 7) 2.1972

8) domain: $(-4, \infty)$; range: $(-\infty, \infty)$

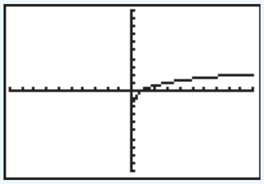


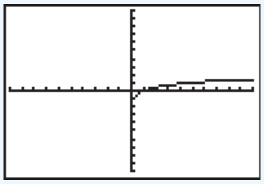
9) a) $\{e^{2.7}\}$; {14.8797} b) $\left\{\dfrac{e^{0.5}+1}{3}\right\}$; {0.8829}

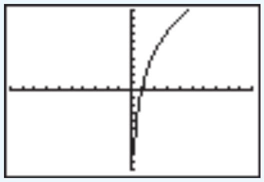
10) a) $2235.88 b) $2237.31 11) $2237.74

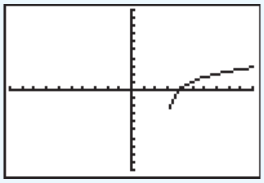
12) a) 2.2602 b) 2.2602

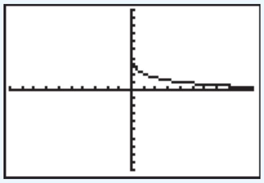
Answers to Technology Exercises

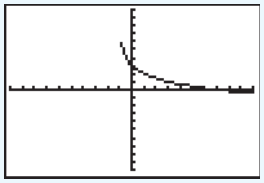
1) 

2) 

3) 

4) 

5) 

6) 

13.5 Exercises

Mixed Exercises: Objectives 1 and 5

1) What is the base of $\ln x$?

2) What is the base of $\log x$?

Evaluate each logarithm. Do *not* use a calculator.

3) $\log 100$

4) $\log 10{,}000$

5) $\log \frac{1}{1000}$

6) $\log \frac{1}{100{,}000}$

7) $\log 0.1$

8) $\log 0.01$

9) $\log 10^9$

10) $\log 10^7$

11) $\log \sqrt[4]{10}$

12) $\log \sqrt[5]{10}$

13) $\ln e^6$

14) $\ln e^{10}$

15) $\ln \sqrt{e}$

16) $\ln \sqrt[3]{e}$

17) $\ln \frac{1}{e^5}$

18) $\ln \frac{1}{e^2}$

19) $\ln 1$

20) $\log 1$

Mixed Exercises: Objectives 2 and 5

Use a calculator to find the approximate value of each logarithm to four decimal places.

21) $\log 16$

22) $\log 23$

23) $\log 0.5$

24) $\log 627$

25) $\ln 3$

26) $\ln 6$

27) $\ln 1.31$

28) $\ln 0.218$

Objective 3: Solve an Equation Containing a Common Logarithm

Solve each equation. Do *not* use a calculator.

29) $\log x = 3$

30) $\log z = 5$

31) $\log k = -1$

32) $\log c = -2$

33) $\log(4a) = 2$

34) $\log(5w) = 1$

35) $\log(3t + 4) = 1$

36) $\log(2p + 12) = 2$

Mixed Exercises: Objectives 3 and 7

Solve each equation. Give an exact solution and a solution that is approximated to four decimal places.

37) $\log a = 1.5$

38) $\log y = 1.8$

39) $\log r = 0.8$

40) $\log c = 0.3$

41) $\ln x = 1.6$

42) $\ln p = 1.1$

43) $\ln t = -2$

44) $\ln z = 0.25$

45) $\ln(3q) = 2.1$

46) $\ln\left(\frac{1}{4}m\right) = 3$

47) $\log\left(\frac{1}{2}c\right) = 0.47$

48) $\log(6k) = -1$

49) $\log(5y - 3) = 3.8$

50) $\log(8x + 15) = 2.7$

51) $\ln(10w + 19) = 1.85$

52) $\ln(7a - 4) = 0.6$

53) $\ln(2d - 5) = 0$

54) $\log(3t + 14) = 2.4$

Objective 6: Graph a Natural Logarithm Function

Graph each function. State the domain and range.

55) $y = \ln x + 2$

56) $f(x) = \ln x - 3$

57) $h(x) = \ln x - 1$

58) $g(x) = \ln x + 1$

59) $f(x) = \ln(x - 2)$

60) $y = \ln(x - 1)$

61) $g(x) = \ln(x + 3)$

62) $h(x) = \ln(x + 2)$

63) $y = -\ln x$

64) $f(x) = \ln(-x)$

65) $h(x) = \log x$

66) $k(x) = \log(x + 4)$

67) If you are given the graph of $f(x) = \ln x$, how could you obtain the graph of $g(x) = \ln(x + 5)$ without making a table of values and plotting points?

68) If you are given the graph of $f(x) = \ln x$, how could you obtain the graph of $h(x) = \ln x + 4$ without making a table of values and plotting points?

Objective 9: Use the Change-of-Base Formula

Use the change-of-base formula with either base 10 or base e to approximate each logarithm to four decimal places.

69) $\log_2 13$
70) $\log_6 25$
71) $\log_9 70$
72) $\log_3 52$
73) $\log_{1/3} 16$
74) $\log_{1/2} 23$
75) $\log_5 3$
76) $\log_7 4$

Mixed Exercises: Objectives 4 and 8

For Exercises 77–80, use the formula

$$L(I) = 10 \log \frac{I}{10^{-12}}$$

where I is the intensity of sound, in watts per square meter, and $L(I)$ is the loudness of sound in decibels. Do *not* use a calculator.

77) The intensity of sound from fireworks is about 0.1 W/m^2. Find the loudness of the fireworks, in decibels.

78) The intensity of sound from a dishwasher is about 0.000001 W/m^2. Find the loudness of the dishwasher, in decibels.

79) The intensity of sound from a refrigerator is about 0.00000001 W/m^2. Find the loudness of the refrigerator, in decibels.

80) The intensity of sound from the takeoff of a space shuttle is 1,000,000 W/m^2. Find the loudness of the sound made by the space shuttle at takeoff, in decibels.

Use the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ to solve each problem. See Example 10.

81) Isabel deposits \$3000 in an account earning 5% per year compounded monthly. How much will be in the account after 3 yr?

82) How much money will Pavel have in his account after 8 yr if he initially deposited \$6000 at 4% interest compounded quarterly?

83) Find the amount Christopher owes at the end of 5 yr if he borrows \$4000 at a rate of 6.5% compounded quarterly.

84) How much will Anna owe at the end of 4 yr if she borrows \$5000 at a rate of 7.2% compounded weekly?

Use the formula $A = Pe^{rt}$ to solve each problem. See Example 11.

85) If \$3000 is deposited in an account earning 5% compounded continuously, how much will be in the account after 3 yr?

86) If \$6000 is deposited in an account earning 4% compounded continuously, how much will be in the account after 8 yr?

87) How much will Cyrus owe at the end of 6 yr if he borrows \$10,000 at a rate of 7.5% compounded continuously?

88) Find the amount Nadia owes at the end of 5 yr if she borrows \$4500 at a rate of 6.8% compounded continuously.

89) The number of bacteria, $N(t)$, in a culture t hr after the bacteria are placed in a dish is given by

$$N(t) = 5000e^{0.0617t}$$

a) How many bacteria were originally in the culture?

b) How many bacteria are present after 8 hr?

90) The number of bacteria, $N(t)$, in a culture t hr after the bacteria are placed in a dish is given by

$$N(t) = 8000e^{0.0342t}$$

a) How many bacteria were originally in the culture?

b) How many bacteria are present after 10 hr?

91) The function $N(t) = 10{,}000e^{0.0492t}$ describes the number of bacteria in a culture t hr after 10,000 bacteria were placed in the culture. How many bacteria are in the culture after 1 day?

92) How many bacteria are present 2 days after 6000 bacteria are placed in a culture if the number of bacteria in the culture is

$$N(t) = 6000e^{0.0285t}$$

t hr after the bacteria are placed in a dish?

In chemistry, the pH of a solution is given by

$$\text{pH} = -\log[H^+]$$

where $[H^+]$ is the molar concentration of the hydronium ion. A neutral solution has pH = 7. *Acidic solutions* have pH < 7, and *basic solutions* have pH > 7.

For Exercises 93–96, the hydronium ion concentrations, $[H^+]$, are given for some common substances. Find the pH of each substance (to the tenths place), and determine whether each substance is acidic or basic.

93) Cola: $[H^+] = 2 \times 10^{-3}$

94) Tomatoes: $[H^+] = 1 \times 10^{-4}$

95) Ammonia: $[H^+] = 6 \times 10^{-12}$

96) Egg white: $[H^+] = 2 \times 10^{-8}$

Extension

97) Show that the inverse of $y = \ln x$ is $y = e^x$.

Section 13.6 Solving Exponential and Logarithmic Equations

Objectives

1. Solve an Exponential Equation
2. Solve Logarithmic Equations Using the Properties of Logarithms
3. Solve Applied Problems Involving Exponential Functions Using a Calculator
4. Solve an Applied Problem Involving Exponential Growth or Decay

In this section, we will learn another property of logarithms that will allow us to solve additional types of exponential and logarithmic equations.

Properties for Solving Exponential and Logarithmic Equations

Let a, x, and y be positive, real numbers, where $a \neq 1$.

1) If $x = y$, then $\log_a x = \log_a y$.

2) If $\log_a x = \log_a y$, then $x = y$.

For example, 1) tells us that if $x = 3$, then $\log_a x = \log_a 3$. Likewise, 2) tells us that if $\log_a 5 = \log_a y$, then $5 = y$. We can use the properties above to solve exponential and logarithmic equations that we could not solve previously.

1. Solve an Exponential Equation

We will look at two types of exponential equations—equations where both sides *can* be expressed with the same base and equations where both sides *cannot* be expressed with the same base. If the two sides of an exponential equation *cannot* be expressed with the same base, we will use logarithms to solve the equation.

Example 1

Solve.

a) $2^x = 8$ b) $2^x = 12$

Solution

a) Since 8 is a power of 2, we can solve $2^x = 8$ by expressing each side of the equation with the same base and setting the exponents equal to each other.

$$\begin{aligned} 2^x &= 8 \\ 2^x &= 2^3 && 8 = 2^3 \\ x &= 3 && \text{Set the exponents equal.} \end{aligned}$$

The solution set is $\{3\}$.

b) Can we express both sides of $2^x = 12$ with the same base? *No. We will use property* 1) *to solve* $2^x = 12$ *by taking the logarithm of each side.*

We can use a logarithm of *any* base. It is most convenient to use base 10 (common logarithm) or base e (natural logarithm) because this is what we can find most easily on our calculators. *We will take the natural log of both sides*.

$$\begin{aligned} 2^x &= 12 \\ \ln 2^x &= \ln 12 && \text{Take the natural log of each side.} \\ x \ln 2 &= \ln 12 && \log_a x^r = r \log_a x \\ x &= \frac{\ln 12}{\ln 2} && \text{Divide by } \ln 2. \end{aligned}$$

The exact solution is $\left\{\frac{\ln 12}{\ln 2}\right\}$. Use a calculator to get an approximation to four decimal places: $x \approx 3.5850$.

The approximation is $\{3.5850\}$. We can verify the solution by substituting it for x in $2^x = 12$: $2^{3.5850} \approx 12$.

BE CAREFUL

$\frac{\ln 12}{\ln 2} \neq \ln 6$

Procedure Solving an Exponential Equation

Begin by asking yourself, "*Can I express each side with the same base?*"

1) If the answer is **yes,** then write each side of the equation with the same base, set the exponents equal, and solve for the variable.

2) If the answer is **no,** then take the natural logarithm of each side, use the properties of logarithms, and solve for the variable.

You Try 1

Solve.

a) $3^{a-5} = 9$ b) $3^t = 24$

Example 2

Solve $5^{x-2} = 16$.

Solution

Ask yourself, *"Can I express each side with the same base?"* **No**. Therefore, take the natural log of each side.

$$5^{x-2} = 16$$
$$\ln 5^{x-2} = \ln 16 \qquad \text{Take the natural log of each side.}$$
$$(x-2)\ln 5 = \ln 16 \qquad \log_a x^r = r\log_a x$$

$(x - 2)$ *must* be in parentheses since it contains two terms.

$$x\ln 5 - 2\ln 5 = \ln 16 \qquad \text{Distribute.}$$
$$x\ln 5 = \ln 16 + 2\ln 5 \qquad \text{Add } 2\ln 5 \text{ to get the } x\text{-term by itself.}$$
$$x = \frac{\ln 16 + 2\ln 5}{\ln 5} \qquad \text{Divide by } \ln 5.$$

The exact solution is $\left\{\frac{\ln 16 + 2\ln 5}{\ln 5}\right\}$. This is approximately $\{3.7227\}$.

You Try 2

Solve $9^{k+4} = 2$.

Recall that $\ln e = 1$. This property is the reason it is convenient to take the *natural logarithm* of both sides of an equation when a base is e.

Example 3

Solve $e^{5n} = 4$.

Solution

Begin by taking the natural log of each side.

$$\begin{aligned} e^{5n} &= 4 \\ \ln e^{5n} &= \ln 4 && \text{Take the natural log of each side.} \\ 5n \ln e &= \ln 4 && \log_a x^r = r \log_a x \\ 5n(1) &= \ln 4 && \ln e = 1 \\ 5n &= \ln 4 \\ n &= \frac{\ln 4}{5} && \text{Divide by 5.} \end{aligned}$$

The exact solution is $\left\{\frac{\ln 4}{5}\right\}$. The approximation is $\{0.2773\}$.

You Try 3

Solve $e^{6c} = 2$.

2. Solve Logarithmic Equations Using the Properties of Logarithms

We learned earlier that to solve a logarithmic equation like $\log_2(t + 5) = 4$, we write the equation in exponential form and solve for the variable.

$$\begin{aligned} \log_2(t + 5) &= 4 \\ 2^4 &= t + 5 && \text{Write in exponential form.} \\ 16 &= t + 5 && 2^4 = 16 \\ 11 &= t && \text{Subtract 5.} \end{aligned}$$

In this section, we will learn how to solve other types of logarithmic equations as well. We will look at equations where

1) each term in the equation contains a logarithm.
2) one term in the equation does *not* contain a logarithm.

Procedure How to Solve an Equation Where Each Term Contains a Logarithm

1) Use the properties of logarithms to write the equation in the form $\log_a x = \log_a y$.
2) Set $x = y$ and solve for the variable.
3) Check the proposed solution(s) in the original equation to be sure the values satisfy the equation.

Example 4

Solve.

a) $\log_5(m - 4) = \log_5 9$ b) $\log x + \log(x + 6) = \log 16$

Solution

a) To solve $\log_5(m - 4) = \log_5 9$, use the property that states if $\log_a x = \log_a y$, then $x = y$.

$$\begin{aligned} \log_5(m - 4) &= \log_5 9 \\ m - 4 &= 9 \\ m &= 13 && \text{Add 4.} \end{aligned}$$

Check to be sure that $m = 13$ satisfies the original equation.

$$\begin{aligned} \log_5(13 - 4) &\stackrel{?}{=} \log_5 9 \\ \log_5 9 &= \log_5 9 \quad \checkmark \end{aligned}$$

The solution set is $\{13\}$.

b) To solve $\log x + \log(x + 6) = \log 16$, we must begin by using the product rule for logarithms to obtain one logarithm on the left side.

$$\begin{aligned} \log x + \log(x + 6) &= \log 16 & & \\ \log x(x + 6) &= \log 16 & & \text{Product rule} \\ x(x + 6) &= 16 & & \text{If } \log_a x = \log_a y \text{, then } x = y. \\ x^2 + 6x &= 16 & & \text{Distribute.} \\ x^2 + 6x - 16 &= 0 & & \text{Subtract 16.} \\ (x + 8)(x - 2) &= 0 & & \text{Factor.} \\ x + 8 = 0 \quad \text{or} \quad x - 2 &= 0 & & \text{Set each factor equal to 0.} \\ x = -8 \quad \text{or} \quad x &= 2 & & \text{Solve.} \end{aligned}$$

Check to be sure that $x = -8$ and $x = 2$ satisfy the original equation.

Check $x = -8$:

$$\begin{aligned} \log x + \log(x + 6) &= \log 16 \\ \log(-8) + \log(-8 + 6) &\stackrel{?}{=} \log 16 \\ &\text{FALSE} \end{aligned}$$

We reject $x = -8$ as a solution because it leads to $\log(-8)$, which is undefined.

Check $x = 2$:

$$\begin{aligned} \log x + \log(x + 6) &= \log 16 \\ \log 2 + \log(2 + 6) &\stackrel{?}{=} \log 16 \\ \log 2 + \log 8 &\stackrel{?}{=} \log 16 \\ \log(2 \cdot 8) &\stackrel{?}{=} \log 16 \\ \log 16 &= \log 16 \quad \checkmark \end{aligned}$$

$x = 2$ satisfies the original equation.

The solution set is $\{2\}$.

BE CAREFUL

Just because a proposed solution is a negative number does *not* mean it should be rejected. You *must* check it in the original equation; it may satisfy the equation.

You Try 4

Solve.

a) $\log_8(z + 3) = \log_8 5$ b) $\log_3 c + \log_3(c - 1) = \log_3 12$

Procedure How to Solve an Equation Where One Term Does *Not* Contain a Logarithm

1) Use the properties of logarithms to get one logarithm on one side of the equation and a constant on the other side. That is, write the equation in the form $\log_a x = y$.
2) Write $\log_a x = y$ in exponential form, $a^y = x$, and solve for the variable.
3) Check the proposed solution(s) in the original equation to be sure the values satisfy the equation.

Example 5

Solve $\log_2 3w - \log_2(w - 5) = 3$.

Solution

Notice that one term in the equation $\log_2 3w - \log_2(w - 5) = 3$ does *not* contain a logarithm. Therefore, we want to use the properties of logarithms to get *one* logarithm on the left. Then, write the equation in exponential form and solve.

$$\begin{aligned}
\log_2 3w - \log_2(w - 5) &= 3 \\
\log_2 \frac{3w}{w-5} &= 3 && \text{Quotient rule} \\
2^3 &= \frac{3w}{w-5} && \text{Write in exponential form.} \\
8 &= \frac{3w}{w-5} && 2^3 = 8 \\
8(w - 5) &= 3w && \text{Multiply by } w - 5. \\
8w - 40 &= 3w && \text{Distribute.} \\
-40 &= -5w && \text{Subtract } 8w. \\
8 &= w && \text{Divide by } -5.
\end{aligned}$$

Verify that $w = 8$ satisfies the original equation. The solution set is $\{8\}$.

You Try 5

Solve.

a) $\log_4(7p + 1) = 3$ b) $\log_3 2x - \log_3(x - 14) = 2$

Let's look at the two types of equations we have discussed side by side. Notice the difference between them.

Solve each equation

1) $\log_3 x + \log_3(2x + 5) = \log_3 12$

Use the properties of logarithms to get one log on the left.

$$\log_3 x(2x + 5) = \log_3 12$$

Since both terms contain logarithms, use the property that states if $\log_a x = \log_a y$, then $x = y$.

$$\begin{aligned}
x(2x + 5) &= 12 \\
2x^2 + 5x &= 12 \\
2x^2 + 5x - 12 &= 0 \\
(2x - 3)(x + 4) &= 0
\end{aligned}$$

$$2x - 3 = 0 \quad \text{or} \quad x + 4 = 0$$

$$x = \frac{3}{2} \quad \text{or} \quad x = -4$$

Reject -4 as a solution. The solution set is $\left\{\frac{3}{2}\right\}$.

2) $\log_3 x + \log_3(2x + 5) = 1$

Use the properties of logarithms to get one log on the left.

$$\log_3 x(2x + 5) = 1$$

The term on the right does *not* contain a logarithm. Write the equation in exponential form and solve.

$$\begin{aligned}
3^1 &= x(2x + 5) \\
3 &= 2x^2 + 5x \\
0 &= 2x^2 + 5x - 3 \\
0 &= (2x - 1)(x + 3)
\end{aligned}$$

$$2x - 1 = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = \frac{1}{2} \quad \text{or} \quad x = -3$$

Reject $x = -3$ as a solution. The solution set is $\left\{\frac{1}{2}\right\}$.

3. Solve Applied Problems Involving Exponential Functions Using a Calculator

Recall that $A = Pe^{rt}$ is the formula for continuous compound interest where P (the principal) is the amount invested, r is the interest rate, and A is the amount (in dollars) in the account after t yr. Here we will look at how we can use the formula to solve a different problem from the type we solved in Section 13.5.

Example 6

If \$3000 is invested at 5% interest compounded continuously, how long would it take for the investment to grow to \$4000?

Solution

In this problem, we are asked to find t, the amount of *time* it will take for \$3000 to grow to \$4000 when invested at 5% compounded continuously.

Use $A = Pe^{rt}$ with $P = 3000$, $A = 4000$, and $r = 0.05$.

$$\begin{aligned}
A &= Pe^{rt} \\
4000 &= 3000e^{0.05t} && \text{Substitute the values.} \\
\frac{4}{3} &= e^{0.05t} && \text{Divide by 3000.} \\
\ln\frac{4}{3} &= \ln e^{0.05t} && \text{Take the natural log of both sides.} \\
\ln\frac{4}{3} &= 0.05t \ln e && \log_a x^r = r\log_a x \\
\ln\frac{4}{3} &= 0.05t(1) && \ln e = 1 \\
\ln\frac{4}{3} &= 0.05t \\
\frac{\ln\frac{4}{3}}{0.05} &= t && \text{Divide by 0.05.} \\
5.75 &\approx t && \text{Use a calculator to get the approximation.}
\end{aligned}$$

It would take about 5.75 yr for \$3000 to grow to \$4000. ■

You Try 6

If \$4500 is invested at 6% interest compounded continuously, how long would it take for the investment to grow to \$5000?

The amount of time it takes for a quantity to double in size are called the *doubling time*. We can use this in many types of applications.

Example 7

The number of bacteria, $N(t)$, in a culture t hr after the bacteria are placed in a dish is given by

$$N(t) = 5000e^{0.0462t}$$

where 5000 bacteria are initially present. How long will it take for the number of bacteria to double?

Solution

If there are 5000 bacteria present initially, there will be $2(5000) = 10{,}000$ bacteria when the number doubles.

Find t when $N(t) = 10{,}000$.

$$\begin{aligned}
N(t) &= 5000e^{0.0462t} \\
10{,}000 &= 5000e^{0.0462t} && \text{Substitute 10,000 for } N(t). \\
2 &= e^{0.0462t} && \text{Divide by 5000.} \\
\ln 2 &= \ln e^{0.0462t} && \text{Take the natural log of both sides.} \\
\ln 2 &= 0.0462t \ln e && \log_a x^r = r \log_a x \\
\ln 2 &= 0.0462t(1) && \ln e = 1 \\
\ln 2 &= 0.0462t \\
\frac{\ln 2}{0.0462} &= t && \text{Divide by 0.0462.} \\
15 &\approx t
\end{aligned}$$

It will take about 15 hr for the number of bacteria to double. ■

You Try 7

The number of bacteria, $N(t)$, in a culture t hr after the bacteria are placed in a dish is given by

$$N(t) = 12{,}000e^{0.0385t}$$

where 12,000 bacteria are initially present. How long will it take for the number of bacteria to double?

4. Solve an Applied Problem Involving Exponential Growth or Decay

We can generalize the formulas used in Examples 6 and 7 with a formula widely used to model situations that grow or decay exponentially. That formula is

$$y = y_0e^{kt}$$

where y_0 is the initial amount or quantity at time $t = 0$, y is the amount present after time t, and k is a constant. If k is positive, it is called a *growth constant* because the quantity will *increase* over time. If k is negative, it is called a *decay constant* because the quantity will *decrease* over time.

Example 8

In April 1986, an accident at the Chernobyl nuclear power plant released many radioactive substances into the environment. One such substance was cesium-137. Cesium-137 decays according to the equation

$$y = y_0e^{-0.0230t}$$

where y_0 is the initial amount present at time $t = 0$ and y is the amount present after t yr. If a sample of soil contains 10 g of cesium-137 immediately after the accident,

a) how many grams will remain after 15 yr?

b) how long would it take for the initial amount of cesium-137 to decay to 2 g?

c) the **half-life** of a substance is the amount of time it takes for a substance to decay to half its original amount. What is the half-life of cesium-137?

Solution

a) The initial amount of cesium-137 is 10 g, so $y_0 = 10$. We must find y when $y_0 = 10$ and $t = 15$.

$$\begin{aligned} y &= y_0 e^{-0.0230t} & & \\ &= 10e^{-0.0230(15)} & & \text{Substitute the values.} \\ &\approx 7.08 & & \text{Use a calculator to get the approximation.} \end{aligned}$$

There will be about 7.08 g of cesium-137 remaining after 15 yr.

b) The initial amount of cesium-137 is $y_0 = 10$. To determine how long it will take to decay to 2 g, let $y = 2$ and solve for t.

$$\begin{aligned} y &= y_0 e^{-0.0230t} & & \\ 2 &= 10e^{-0.0230t} & & \text{Substitute 2 for } y \text{ and 10 for } y. \\ 0.2 &= e^{-0.0230t} & & \text{Divide by 10.} \\ \ln 0.2 &= \ln e^{-0.0230t} & & \text{Take the natural log of both sides.} \\ \ln 0.2 &= -0.0230t \ln e & & \log_a x^r = r \log_a x \\ \ln 0.2 &= -0.0230t & & \ln e = 1 \\ \frac{\ln 0.2}{-0.0230} &= t & & \text{Divide by } -0.0230. \\ 69.98 &\approx t & & \text{Use a calculator to get the approximation.} \end{aligned}$$

It will take about 69.98 yr for 10 g of cesium-137 to decay to 2 g.

c) Since there are 10 g of cesium-137 in the original sample, to determine the half-life we will determine how long it will take for the 10 g to decay to 5 g because $\frac{1}{2}(10) = 5$.

Let $y_0 = 10$, $y = 5$, and solve for t.

$$\begin{aligned} y &= y_0 e^{-0.0230t} & & \\ 5 &= 10e^{-0.0230t} & & \text{Substitute the values.} \\ 0.5 &= e^{-0.0230t} & & \text{Divide by 10.} \\ \ln 0.5 &= \ln e^{-0.0230t} & & \text{Take the natural log of both sides.} \\ \ln 0.5 &= -0.0230t \ln e & & \log_a x^r = r \log_a x \\ \ln 0.5 &= -0.0230t & & \ln e = 1 \\ \frac{\ln 0.5}{-0.0230} &= t & & \text{Divide by } -0.0230. \\ 30.14 &\approx t & & \text{Use a calculator to get the approximation.} \end{aligned}$$

The half-life of cesium-137 is about 30.14 yr. This means that it would take about 30.14 yr for any quantity of cesium-137 to decay to half of its original amount.

You Try 8

Radioactive strontium-90 decays according to the equation

$$y = y_0 e^{-0.0244t}$$

where t is in years. If a sample contains 40 g of strontium-90,

a) how many grams will remain after 8 yr?

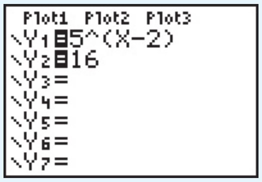
b) how long would it take for the initial amount of strontium-90 to decay to 30 g?

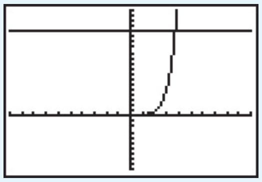
c) what is the half-life of strontium-90?

Using Technology

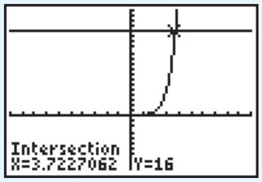
We can solve exponential and logarithmic equations in the same way that we solved other equations—by graphing both sides of the equation and finding where the graphs intersect.

In Example 2 of this section, we learned how to solve $5^{x-2} = 16$. Because the right side of the equation is 16, the graph will have to go at least as high as 16. So set the Y_{max} to be 20, enter the left side of the equation as Y_1 and the right side as Y_2, and press $\boxed{\text{GRAPH}}$:





Recall that the x-coordinate of the point of intersection is the solution to the equation. To find the point of intersection, press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ and then highlight 5:intersect and press $\boxed{\text{ENTER}}$. Press $\boxed{\text{ENTER}}$ three more times to see that the x-coordinate of the point of intersection is approximately 3.723.



Remember, while the calculator can sometimes save you time, it will often give an approximate answer and not an exact solution.

Use a graphing calculator to solve each equation. Round your answer to the nearest thousandth.

1) $7^x = 49$
2) $6^{2b+1} = 13$
3) $5^{4a+7} = 8^{2a}$
4) $\ln x = 1.2$
5) $\log(k + 9) = \log 11$
6) $\ln(x + 3) = \ln(x - 2)$

Answers to You Try Exercises

1) a) {7} b) $\left\{\dfrac{\ln 24}{\ln 3}\right\}$; {2.8928}
2) $\left\{\dfrac{\ln 2 - 4\ln 9}{\ln 9}\right\}$; {−3.6845}
3) $\left\{\dfrac{\ln 2}{6}\right\}$; {0.1155}
4) a) {2} b) {4}
5) a) {9} b) {18}
6) 1.76 yr
7) 18 hr
8) a) 32.91 g b) 11.79 yr c) 28.41 yr

Answers to Technology Exercises

1) {2} 2) {0.216} 3) {−4.944} 4) {3.320} 5) {2} 6) $\varnothing$

13.6 Exercises

Objective 1: Solve an Exponential Equation

Solve each equation. Give the exact solution. If the answer contains a logarithm, approximate the solution to four decimal places.

1) $7^x = 49$
2) $5^c = 125$
3) $7^n = 15$
4) $5^a = 38$
5) $8^z = 3$
6) $4^y = 9$
7) $6^{5p} = 36$
8) $2^{3t} = 32$
9) $4^{6k} = 2.7$
10) $3^{2x} = 7.8$
11) $2^{4n+1} = 5$
12) $6^{2b+1} = 13$
13) $5^{3a-2} = 8$
14) $3^{2x-3} = 14$
15) $4^{2c+7} = 64^{3c-1}$
16) $27^{5m-2} = 3^{m+6}$
17) $9^{5d-2} = 4^{3d}$
18) $5^{4a+7} = 8^{2a}$

Solve each equation. Give the exact solution and the approximation to four decimal places.

19) $e^y = 12.5$
20) $e^t = 0.36$
21) $e^{-4x} = 9$
22) $e^{3p} = 4$

VIDEO 23) $e^{0.01r} = 2$

24) $e^{-0.08k} = 10$

25) $e^{0.006t} = 3$

26) $e^{0.04a} = 12$

27) $e^{-0.4y} = 5$

28) $e^{-0.005c} = 16$

Objective 2: Solve Logarithmic Equations Using the Properties of Logarithms

Solve each equation.

29) $\log_6(k + 9) = \log_6 11$

30) $\log_5(d - 4) = \log_5 2$

31) $\log_7(3p - 1) = \log_7 9$

32) $\log_4(5y + 2) = \log_4 10$

VIDEO 33) $\log x + \log(x - 2) = \log 15$

34) $\log_9 r + \log_9(r + 7) = \log_9 18$

35) $\log_3 n + \log_3(12 - n) = \log_3 20$

36) $\log m + \log(11 - m) = \log 24$

37) $\log_2(-z) + \log_2(z - 8) = \log_2 15$

38) $\log_5 8y - \log_5(3y - 4) = \log_5 2$

39) $\log_6(5b - 4) = 2$

40) $\log_3(4c + 5) = 3$

41) $\log(3p + 4) = 1$

42) $\log(7n - 11) = 1$

43) $\log_3 y + \log_3(y - 8) = 2$

44) $\log_4 k + \log_4(k - 6) = 2$

VIDEO 45) $\log_2 r + \log_2(r + 2) = 3$

46) $\log_9(z + 8) + \log_9 z = 1$

47) $\log_4 20c - \log_4(c + 1) = 2$

48) $\log_6 40x - \log_6(1 + x) = 2$

49) $\log_2 8d - \log_2(2d - 1) = 4$

50) $\log_6(13 - x) + \log_6 x = 2$

Mixed Exercises: Objectives 3 and 4

Use the formula $A = Pe^{rt}$ to solve Exercises 51–58.

51) If \$2000 is invested at 6% interest compounded continuously, how long would it take

a) for the investment to grow to \$2500?

b) for the initial investment to double?

52) If \$5000 is invested at 7% interest compounded continuously, how long would it take

a) for the investment to grow to \$6000?

b) for the initial investment to double?

53) How long would it take for an investment of \$7000 to earn \$800 in interest if it is invested at 7.5% compounded continuously?

54) How long would it take for an investment of \$4000 to earn \$600 in interest if it is invested at 6.8% compounded continuously?

55) Cynthia wants to invest some money now so that she will have \$5000 in the account in 10 yr. How much should she invest in an account earning 8% compounded continuously?

56) How much should Leroy invest now at 7.2% compounded continuously so that the account contains \$8000 in 12 yr?

57) Raj wants to invest \$3000 now so that it grows to \$4000 in 4 yr. What interest rate should he look for? (Round to the nearest tenth of a percent.)

58) Marisol wants to invest \$12,000 now so that it grows to \$20,000 in 7 yr. What interest rate should she look for? (Round to the nearest tenth of a percent.)

59) The number of bacteria, $N(t)$, in a culture t hr after the bacteria are placed in a dish is given by

$$N(t) = 4000e^{0.0374t}$$

where 4000 bacteria are initially present.

a) After how many hours will there be 5000 bacteria in the culture?

b) How long will it take for the number of bacteria to double?

60) The number of bacteria, $N(t)$, in a culture t hr after the bacteria are placed in a dish is given by

$$N(t) = 10{,}000e^{0.0418t}$$

where 10,000 bacteria are initially present.

a) After how many hours will there be 15,000 bacteria in the culture?

b) How long will it take for the number of bacteria to double?

61) VIDEO The population of an Atlanta suburb is growing at a rate of 3.6% per year. If 21,000 people lived in the suburb in 2004, determine how many people will live in the town in 2012. Use $y = y_0e^{0.036t}$.

62) The population of a Seattle suburb is growing at a rate of 3.2% per year. If 30,000 people lived in the suburb in 2008, determine how many people will live in the town in 2015. Use $y = y_0e^{0.032t}$.

63) A rural town in South Dakota is losing residents at a rate of 1.3% per year. The population of the town was 2470 in 1990. Use $y = y_0e^{-0.013t}$ to answer the following questions.

a) What was the population of the town in 2005?

b) In what year would it be expected that the population of the town is 1600?

64) In 1995, the population of a rural town in Kansas was 1682. The population is decreasing at a rate of 0.8% per year. Use $y = y_0e^{-0.008t}$ to answer the following questions.

a) What was the population of the town in 2000?

b) In what year would it be expected that the population of the town is 1000?

65) Radioactive carbon-14 is a substance found in all living organisms. After the organism dies, the carbon-14 decays according to the equation

$$y = y_0 e^{-0.000121t}$$

where t is in years, y_0 is the initial amount present at time $t = 0$, and y is the amount present after t yr.

a) If a sample initially contains 15 g of carbon-14, how many grams will be present after 2000 yr?

b) How long would it take for the initial amount to decay to 10 g?

c) What is the half-life of carbon-14?

66) Plutonium-239 decays according to the equation

$$y = y_0 e^{-0.0000287t}$$

where t is in years, y_0 is the initial amount present at time $t = 0$, and y is the amount present after t yr.

a) If a sample initially contains 8 g of plutonium-239, how many grams will be present after 5000 yr?

b) How long would it take for the initial amount to decay to 5 g?

c) What is the half-life of plutonium-239?

67) Radioactive iodine-131 is used in the diagnosis and treatment of some thyroid-related illnesses. The concentration of the iodine in a patient's system is given by

$$y = 0.4e^{-0.086t}$$

where t is in days and y is in the appropriate units.

a) How much iodine-131 is given to the patient?

b) How much iodine-131 remains in the patient's system 7 days after treatment?

68) The amount of cobalt-60 in a sample is given by

$$y = 30e^{-0.131t}$$

where t is in years and y is in grams.

a) How much cobalt-60 is originally in the sample?

b) How long would it take for the initial amount to decay to 10 g?

Extension

Solve. Where appropriate, give the exact solution and the approximation to four decimal places.

69) $\log_2(\log_2 x) = 2$

70) $\log_3(\log y) = 1$

71) $\log_3 \sqrt{n^2 + 5} = 1$

72) $\log(p - 7)^2 = 4$

73) $e^{|t|} = 13$

74) $e^{r^2 - 25} = 1$

75) $e^{2y} + 3e^y - 4 = 0$

76) $e^{2x} - 9e^x + 8 = 0$

77) $5^{2c} - 4 \cdot 5^c - 21 = 0$

78) $9^{2a} + 5 \cdot 9^a - 24 = 0$

79) $(\log x)^2 = \log x^3$

80) $\log 6^y = y^2$

Chapter 13: Summary

Definition/Procedure	Example

13.1 Inverse Functions

One-to-One Function

In order for a function to be a **one-to-one function**, each x-value corresponds to exactly one y-value, and each y-value corresponds to exactly one x-value.

The **horizontal line test** tells us how we can determine whether a graph represents a one-to-one function:

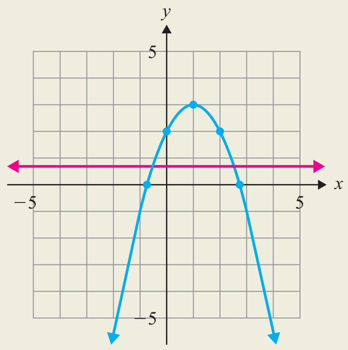
If every horizontal line that could be drawn through a function would intersect the graph at most once, then the function is one-to-one. **(p. 766)**

Determine whether each function is one-to-one.

a) $f = \{(-2, 9), (1, 3), (3, -1), (7, -9)\}$ *is* one-to-one.

b) $g = \{(0, 9), (2, 1), (4, 1), (5, 4)\}$ is *not* one-to-one since the y-value 1 corresponds to two different x-values.

c)



No. It fails the horizontal line test.

Inverse Function

Let f be a one-to-one function. The **inverse** of f, denoted by f^{-1}, is a one-to-one function that contains the set of all ordered pairs (y, x) where (x, y) belongs to f.

How to Find an Equation of the Inverse of $y = f(x)$

Step 1: Replace $f(x)$ with y.

Step 2: Interchange x and y.

Step 3: Solve for y.

Step 4: Replace y with the inverse notation, $f^{-1}(x)$.

The graphs of $f(x)$ and $f^{-1}(x)$ are symmetric with respect to the line $y = x$. **(p. 769)**

Find an equation of the inverse of $f(x) = 2x - 4$.

Step 1: $y = 2x - 4$ — Replace $f(x)$ with y.

Step 2: $x = 2y - 4$ — Interchange x and y.

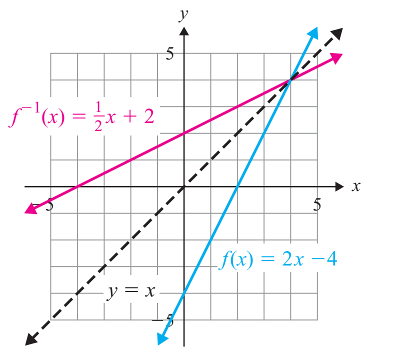
Step 3: Solve for y.

$x + 4 = 2y$ — Add 4.

$\dfrac{x + 4}{2} = y$ — Divide by 2.

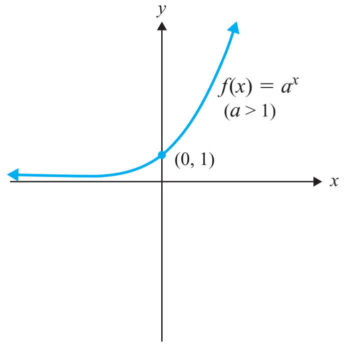
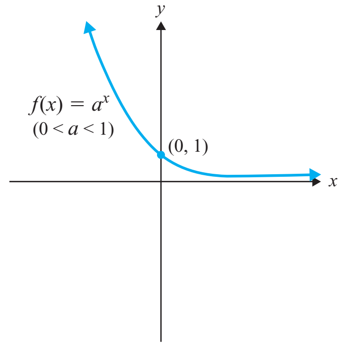
$\dfrac{1}{2}x + 2 = y$ — Simplify.

Step 4: $f^{-1}(x) = \dfrac{1}{2}x + 2$ — Replace y with $f^{-1}(x)$.



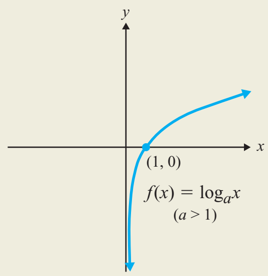
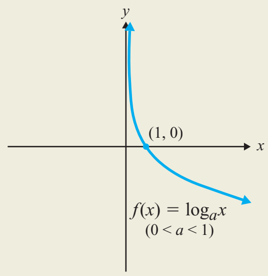
Definition/Procedure	Example

13.2 Exponential Functions

An **exponential function** is a function of the form $f(x) = a^x$ where $a > 0$, $a \neq 1$, and x is a real number. **(p. 775)**	$f(x) = 3^x$
Characteristics of an Exponential Function $f(x) = a^x$ 1) If $f(x) = a^x$, where $a > 1$, the value of y increases as the value of x increases. 2) If $f(x) = a^x$ where $0 < a < 1$, the value of y decreases as the value of x increases. 3) The function is one-to-one. 4) The y-intercept is $(0, 1)$. 5) The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$. **(p. 777)** $f(x) = e^x$ is a special exponential function that has many uses in mathematics. Like the number π, e is an irrational number. **(p. 778)** $e \approx 2.7183$	1)  2) 
Solving an Exponential Equation ***Step 1:*** **If possible, express each side of the equation with the same base.** If it is not possible to get the same base, a method in Section 13.6 can be used. ***Step 2:*** **Use the rules of exponents to simplify the exponents.** ***Step 3:*** **Set the exponents equal and solve for the variable. (p. 780)**	Solve $5^{4x-1} = 25^{3x+4}$. ***Step 1:*** $5^{4x-1} = (5^2)^{(3x+4)}$ — Both sides are powers of 5. ***Step 2:*** $5^{4x-1} = 5^{2(3x+4)}$ — Power rule for exponents $5^{4x-1} = 5^{6x+8}$ — Distribute. ***Step 3:*** $4x - 1 = 6x + 8$ — The bases are the same. Set the exponents equal. $-2x = 9$ — Subtract $6x$; add 1. $x = -\frac{9}{2}$ — Divide by -2. The solution set is $\left\{-\frac{9}{2}\right\}$.

Definition/Procedure	Example

13.3 Logarithmic Functions

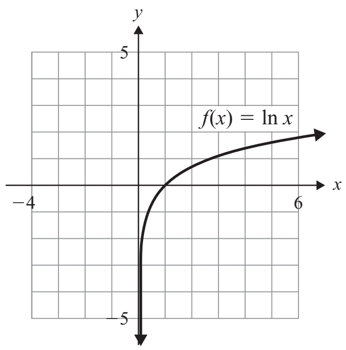
Definition/Procedure	Example
Definition of Logarithm If $a > 0$, $a \neq 1$, and $x > 0$, then for every real number y, $y = \log_a x$ means $x = a^y$. **(p. 786)**	Write $\log_5 125 = 3$ in exponential form. $\log_5 125 = 3$ means $5^3 = 125$
A **logarithmic equation** is an equation in which at least one term contains a logarithm. To solve a logarithmic equation of the form $\log_a b = c$ write the equation in exponential form ($a^c = b$) and solve for the variable. **(p. 788)**	Solve $\log_2 k = 3$. Write the equation in exponential form and solve for k. $\log_2 k = 3$ means $2^3 = k$. $8 = k$ The solution set is $\{8\}$.
To evaluate $\log_a b$ means *to find the power to which we raise a to get b.* **(p. 789)**	Evaluate $\log_7 49$. $\log_7 49 = 2$ since $7^2 = 49$
A base 10 logarithm is called a **common logarithm.** A base 10 logarithm is often written without the base. **(p. 790)**	$\log x$ means $\log_{10} x$.
Characteristics of a Logarithmic Function $f(x) = \log_a x$, **where** $a > 0$ **and** $a \neq 1$ 1) If $f(x) = \log_a x$, where $a > 1$, the value of y increases as the value of x increases. 2) If $f(x) = \log_a x$, where $0 < a < 1$, the value of y decreases as the value of x increases. 3) The function is one-to-one. 4) The x-intercept is $(1, 0)$. 5) The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$. 6) The inverse of $f(x) = \log_a x$ is $f^{-1}(x) = a^x$. **(p. 794)**	1)  2) 

13.4 Properties of Logarithms

Definition/Procedure	Example
Let x, y, and a be positive real numbers where $a \neq 1$ and let r be any real number. Then, 1) $\log_a a = 1$ 2) $\log_a 1 = 0$ 3) $\log_a xy = \log_a x + \log_a y$ — Product rule 4) $\log_a \frac{x}{y} = \log_a x - \log_a y$ — Quotient rule 5) $\log_a x^r = r \log_a x$ — Power rule 6) $\log_a a^x = x$ for any real number x 7) $a^{\log_a x} = x$ **(p. 802)**	Write $\log_4 \frac{c^5}{d^2}$ as the sum or difference of logarithms in simplest form. Assume c and d represent positive real numbers. $\log_4 \frac{c^5}{d^2} = \log_4 c^5 - \log_4 d^2$ — Quotient rule $= 5 \log_4 c - 2 \log_4 d$ — Power rule

Definition/Procedure	Example

13.5 Common and Natural Logarithms and Change of Base

Definition/Procedure	Example
We can evaluate common logarithms with or without a calculator. **(p. 807)**	Find the value of each. a) log 100 b) log 53 a) $\log 100 = \log_{10} 100 = \log_{10} 10^2 = 2$ b) Using a calculator, we get $\log 53 \approx 1.7243$.
The number e is approximately equal to 2.7183. A base e logarithm is called a **natural logarithm**. The notation used for a natural logarithm is $\ln x$. The domain of $f(x) = \ln x$ is $(0, \infty)$, and the range is $(-\infty, \infty)$. **(p. 809)**	$f(x) = \ln x$ means $f(x) = \log_e x$ The graph of $f(x) = \ln x$ looks like this: 
ln e = 1 since $\ln e = 1$ means $\log_e e = 1$. We can find the values of some natural logarithms using the properties of logarithms. We can approximate the values of other natural logarithms using a calculator. **(p. 810)**	Find the value of each. a) $\ln e^{12}$ b) $\ln 18$ a) $\ln e^{12} = 12 \ln e$ Power rule $= 12(1)$ $\ln e = 1$ $= 12$ b) Using a calculator, we get $\ln 18 \approx 2.8904$.
To solve an equation such as $\ln x = 1.6$, change to exponential form and solve for the variable. **(p. 811)**	Solve $\ln x = 1.6$. $\ln x = 1.6$ means $\log_e x = 1.6$. $\log_e x = 1.6$ $e^{1.6} = x$ Exponential form $4.9530 \approx x$ Approximation The exact solution is $\{e^{1.6}\}$. The approximation is $\{4.9530\}$.
Applications of Exponential Functions **Continuous Compounding** If P dollars is deposited in an account earning interest rate r compounded continuously, then the amount of money, A (in dollars), in the account after t years is given by $A = Pe^{rt}$. **(p. 813)**	Determine the amount of money in an account after 6 yr if \$3000 was initially invested at 5% compounded continuously. $A = Pe^{rt}$ $= 3000e^{0.05(6)}$ Substitute values. $= 3000e^{0.30}$ Multiply (0.05)(6). ≈ 4049.5764 Evaluate using a calculator. $\approx \$4049.58$ Round to the nearest cent.
Change-of-Base Formula If a, b, and x are positive real numbers and $a \neq 1$ and $b \neq 1$, then $\log_a x = \dfrac{\log_b x}{\log_b a}$ **(p. 814)**	Find $\log_2 75$ to four decimal places. $\log_2 75 = \dfrac{\log_{10} 75}{\log_{10} 2} \approx 6.2288$

Definition/Procedure	Example
13.6 Solving Exponential and Logarithmic Equations	
Let a, x, and y be positive real numbers, where $a \neq 1$. 1) If $x = y$, then $\log_a x = \log_a y$. 2) If $\log_a x = \log_a y$, then $x = y$. **Solving an Exponential Equation** Begin by asking yourself, "*Can I express each side with the same base?*" 1) If the answer is **yes,** then write each side of the equation with the same base, set the exponents equal, and solve for the variable. 2) If the answer is **no,** then take the natural logarithm of each side, use the properties of logarithms, and solve for the variable. **(p. 819)**	Solve each equation. a) $4^x = 64$ Ask yourself, "*Can I express both sides with the same base?*" **Yes.** $4^x = 64$ $4^x = 4^3$ $x = 3$ Set the exponents equal. The solution set is $\{3\}$. b) $4^x = 9$ Ask yourself, "*Can I express both sides with the same base?*" **No.** Take the natural logarithm of each side. $4^x = 9$ $\ln 4^x = \ln 9$ Take the natural log of each side. $x \ln 4 = \ln 9$ $\log_a x^r = r \log_a x$ $x = \dfrac{\ln 9}{\ln 4}$ Divide by ln 4. $x \approx 1.5850$ Use a calculator to get the approximation. The exact solution is $\left\{\dfrac{\ln 9}{\ln 4}\right\}$. The approximation is $\{1.5850\}$.
Solve an exponential equation with base e by taking the natural logarithm of each side. **(p. 819)**	Solve $e^y = 35.8$. $\ln e^y = \ln 35.8$ Take the natural log of each side. $y \ln e = \ln 35.8$ $\log_a x^r = r \log_a x$ $y(1) = \ln 35.8$ $\ln e = 1$ $y = \ln 35.8$ $y \approx 3.5779$ Approximation The exact solution is $\{\ln 35.8\}$. The approximation is $\{3.5779\}$.
Solving Logarithmic Equations Sometimes we must use the properties of logarithms to solve logarithmic equations. **(p. 820)**	Solve $\log x + \log(x - 3) = \log 28$. $\log x + \log(x - 3) = \log 28$ $\log x(x - 3) = \log 28$ Product rule $x(x - 3) = 28$ If $\log_a x = \log_a y$, then $x = y$. $x^2 - 3x = 28$ Distribute. $x^2 - 3x - 28 = 0$ Subtract 28. $(x - 7)(x + 4) = 0$ Factor. $x - 7 = 0$ or $x + 4 = 0$ Set each factor equal to 0. $x = 7$ or $x = -4$ Solve. Verify that only 7 satisfies the original equation. The solution set is $\{7\}$.

Chapter 13: Review Exercises

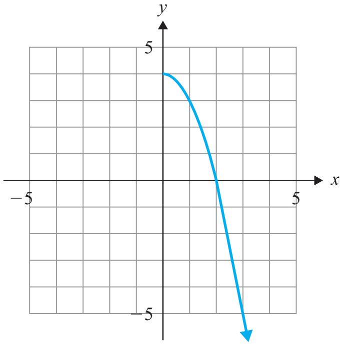
(13.1) Determine whether each function is one-to-one. If it is one-to-one, find its inverse.

1) $f = \{(-7, -4), (-2, 1), (1, 5), (6, 11)\}$

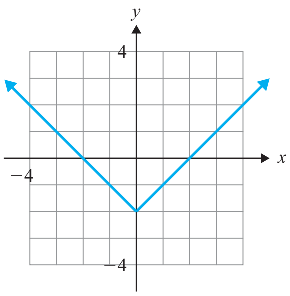
2) $g = \{(1, 4), (3, 7), (6, 4), (10, 9)\}$

Determine whether each function is one-to-one. If it is one-to-one, graph its inverse.

3)



4)



Find the inverse of each one-to-one function. Graph each function and its inverse on the same axes.

5) $f(x) = x + 4$

6) $g(x) = 2x - 10$

7) $h(x) = \frac{1}{3}x - 1$

8) $f(x) = \sqrt[3]{x} + 2$

Given each one-to-one function $f(x)$, find the following function values *without* finding an equation of $f^{-1}(x)$. Find the value in a) before b).

9) $f(x) = 6x - 1$
   a) $f(2)$
   b) $f^{-1}(11)$

10) $f(x) = \sqrt[3]{x + 5}$
   a) $f(-13)$
   b) $f^{-1}(-2)$

(13.2) Graph each exponential function. State the domain and range.

11) $f(x) = 2^x$

12) $h(x) = \left(\frac{1}{3}\right)^x$

13) $y = 2^x - 4$

14) $f(x) = 3^{x-2}$

15) $f(x) = e^x$

16) $g(x) = e^x + 2$

Solve each exponential equation.

17) $2^c = 64$

18) $7^{m+5} = 49$

19) $16^{3z} = 32^{2z-1}$

20) $9^y = \frac{1}{81}$

21) $\left(\frac{3}{2}\right)^{x+4} = \left(\frac{4}{9}\right)^{x-3}$

22) The value, $V(t)$, in dollars, of a luxury car t yr after it is purchased is given by $V(t) = 38{,}200(0.816)^t$.
   a) What was the purchase price of the car?
   b) What will the car be worth 4 yr after purchase?

(13.3)

23) What is the domain of $y = \log_a x$?

24) In the equation $y = \log_a x$, a must be what kind of number?

Write in exponential form.

25) $\log_5 125 = 3$

26) $\log_{16} \frac{1}{4} = -\frac{1}{2}$

27) $\log 100 = 2$

28) $\log 1 = 0$

Write in logarithmic form.

29) $3^4 = 81$

30) $\left(\frac{2}{3}\right)^{-2} = \frac{9}{4}$

31) $10^3 = 1000$

32) $\sqrt{121} = 11$

Solve.

33) $\log_2 x = 3$

34) $\log_9(4x + 1) = 2$

35) $\log_{32} 16 = x$

36) $\log(2x + 5) = 1$

Evaluate.

37) $\log_8 64$

38) $\log_3 27$

39) $\log 1000$

40) $\log 1$

41) $\log_{1/2} 16$

42) $\log_{1/5} \frac{1}{25}$

Graph each logarithmic function.

43) $f(x) = \log_2 x$

44) $g(x) = \log_3 x$

45) $h(x) = \log_{1/3} x$

46) $f(x) = \log_{1/4} x$

Find the inverse of each function.

47) $f(x) = 5^x$

48) $g(x) = 3^x$

49) $h(x) = \log_6 x$

Solve.

50) A company plans to test market its new dog food in a large metropolitan area before taking it nationwide. The company predicts that its sales over the next 12 months can be approximated by

$$S(t) = 10 \log_3(2t + 1)$$

where t is the number of months after the dog food is introduced, and $S(t)$ is in thousands of bags of dog food.

a) How many bags of dog food were sold after 1 month on the market?

b) How many bags of dog food were sold after 4 months on the market?

(13.4) Decide whether each statement is true or false.

51) $\log_5(x + 4) = \log_5 x + \log_5 4$

52) $\log_2 \frac{k}{6} = \log_2 k - \log_2 6$

Write as the sum or difference of logarithms and simplify, if possible. Assume all variables represent positive real numbers.

53) $\log_8 3z$

54) $\log_7 \frac{49}{t}$

55) $\log_4 \sqrt{64}$

56) $\log \frac{1}{100}$

57) $\log_5 c^4d^3$

58) $\log_4 m\sqrt{n}$

59) $\log_a \frac{xy}{z^3}$

60) $\log_4 \frac{a^2}{bc^4}$

61) $\log p(p + 8)$

62) $\log_6 \frac{r^3}{r^2 - 5}$

Write as a single logarithm. Assume the variables are defined so that the variable expressions are positive and so that the bases are positive real numbers not equal to 1.

63) $\log c + \log d$

64) $\log_4 n - \log_4 7$

65) $9 \log_2 a + 3 \log_2 b$

66) $\log_5 r - 2 \log_5 t$

67) $\log_3 5 + 4 \log_3 m - 2 \log_3 n$

68) $\frac{1}{2} \log_z a - \log_z b$

69) $3 \log_5 c - \log_5 d - 2 \log_5 f$

70) $2 \log_6 x + \frac{1}{3} \log_6(x - 4)$

Given that log 7 ≈ 0.8451 and log 9 ≈ 0.9542, use the properties of logarithms to approximate the following. Do NOT use a calculator.

71) $\log 49$

72) $\log 63$

73) $\log \frac{7}{9}$

74) $\log \frac{1}{7}$

(13.5)

75) What is the base of $\ln x$?

76) Evaluate $\ln e$.

Evaluate each logarithm. Do not use a calculator.

77) $\log 10$

78) $\log 100$

79) $\log \sqrt{10}$

80) $\log \frac{1}{100}$

81) $\log 0.001$

82) $\ln e^4$

83) $\ln 1$

84) $\ln \sqrt[3]{e}$

Use a calculator to find the approximate value of each logarithm to four decimal places.

85) $\log 8$

86) $\log 0.3$

87) $\ln 1.75$

88) $\ln 0.924$

Solve each equation. Do not use a calculator.

89) $\log p = 2$

90) $\log(5n) = 3$

91) $\log\left(\frac{1}{2}c\right) = -1$

92) $\log(6z - 5) = 1$

Solve each equation. Give an exact solution and a solution that is approximated to four decimal places.

93) $\log x = 2.1$

94) $\log k = -1.4$

95) $\ln y = 2$

96) $\ln c = -0.5$

97) $\log(4t) = 1.75$

98) $\ln(2a - 3) = 1$

Graph each function. State the domain and range.

99) $f(x) = \ln (x - 3)$

100) $g(x) = \ln x - 2$

Use the change-of-base formula with either base 10 or base e to approximate each logarithm to four decimal places.

101) $\log_4 19$

102) $\log_9 42$

103) $\log_{1/2} 38$

104) $\log_6 0.82$

For Exercises 105 and 106, use the formula $L(I) = 10 \log \frac{I}{10^{-12}}$, where I is the intensity of sound, in watts per square meter, and $L(I)$ is the loudness of sound in decibels. Do *not* use a calculator.

105) The intensity of sound from the crowd at a college basketball game reached 0.1 W/m^2. Find the loudness of the crowd, in decibels.

106) Find the intensity of the sound of a jet taking off if the noise level can reach 140 dB 25 m from the jet.

Use the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ and a calculator to solve Exercises 107 and 108.

107) Pedro deposits \$2500 in an account earning 6% interest compounded quarterly. How much will be in the account after 5 yr?

108) Find the amount Chelsea owes at the end of 6 yr if she borrows \$18,000 at a rate of 7% compounded monthly.

Use the formula $A = Pe^{rt}$ and a calculator to solve Exercises 109 and 110.

109) Find the amount Liang will owe at the end of 4 yr if he borrows \$9000 at a rate of 6.2% compounded continuously.

110) If \$4000 is deposited in an account earning 5.8% compounded continuously, how much will be in the account after 7 yr?

111) The number of bacteria, $N(t)$, in a culture t hr after the bacteria are placed in a dish is given by

$$N(t) = 6000e^{0.0514t}$$

a) How many bacteria were originally in the culture?

b) How many bacteria are present after 12 hr?

112) The pH of a solution is given by $\text{pH} = -\log[H^+]$, where $[H^+]$ is the molar concentration of the hydronium ion. Find the ideal pH of blood if $[H^+] = 3.98 \times 10^{-8}$.

(13.6) Solve each equation. Give the exact solution. If the answer contains a logarithm, approximate the solution to four decimal places. ***Some of these exercises require the use of a calculator to obtain a decimal approximation.***

113) $2^y = 16$

114) $3^n = 7$

115) $9^{4k} = 2$

116) $125^{m-4} = 25^{1-m}$

117) $6^{2c} = 8^{c-5}$

118) $e^z = 22$

119) $e^{5p} = 8$

120) $e^{0.03t} = 19$

Solve each logarithmic equation.

121) $\log_3(5w + 3) = 2$

122) $\log(3n - 5) = 3$

123) $\log_2 x + \log_2(x + 2) = \log_2 24$

124) $\log_7 10p - \log_7(p - 8) = \log_7 6$

125) $\log_4 k + \log_4(k - 12) = 3$

126) $\log_3 12m - \log_3(1 + m) = 2$

Use the formula $A = Pe^{rt}$ to solve Exercises 127 and 128.

127) Jamar wants to invest some money now so that he will have \$10,000 in the account in 6 yr. How much should he invest in an account earning 6.5% compounded continuously?

128) Samira wants to invest \$6000 now so that it grows to \$9000 in 5 yr. What interest rate (compounded continuously) should she look for? (Round to the nearest tenth of a percent.)

129) The population of a suburb is growing at a rate of 1.6% per year. The population of the suburb was 16,410 in 1990. Use $y = y_0e^{0.016t}$ to answer the following questions.

a) What was the population of the town in 1995?

b) In what year would it be expected that the population of the town is 23,000?

130) Radium-226 decays according to the equation

$$y = y_0e^{-0.000436t}$$

where t is in years, y_0 is the initial amount present at time $t = 0$, and y is the amount present after t yr.

a) If a sample initially contains 80 g of radium-226, how many grams will be present after 500 yr?

b) How long would it take for the initial amount to decay to 20 g?

c) What is the half-life of radium-226?

Chapter 13: Test

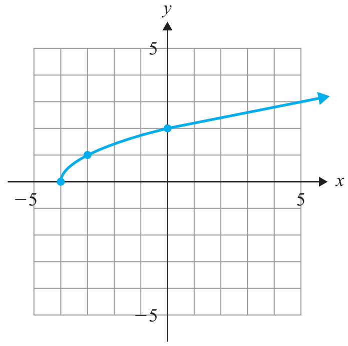
Use a calculator only where indicated.

Determine whether each function is one-to-one. If it is one-to-one, find its inverse.

1) $f = \{(-4, 5), (-2, 7), (0, 3), (6, 5)\}$

2) $g = \left\{(2, 4), (6, 6), \left(9, \frac{15}{2}\right), (14, 10)\right\}$

3) Is this function one-to-one? If it is one-to-one, graph its inverse.



4) Find an equation of the inverse of $f(x) = -3x + 12$.

Use $f(x) = 2^x$ and $g(x) = \log_2 x$ for Exercises 5–8.

5) Graph $f(x)$.

6) Graph $g(x)$.

7) a) What is the domain of $g(x)$?

b) What is the range of $g(x)$?

8) How are the functions $f(x)$ and $g(x)$ related?

9) Write $3^{-2} = \frac{1}{9}$ in logarithmic form.

Solve each equation.

10) $9^{4x} = 81$

11) $125^{2c} = 25^{c-4}$

12) $\log_5 y = 3$

13) $\log(3r + 13) = 2$

14) $\log_6(2m) + \log_6(2m - 3) = \log_6 40$

15) Evaluate.

a) $\log_2 16$

b) $\log_7 \sqrt{7}$

16) Find $\ln e$.

Write as the sum or difference of logarithms and simplify, if possible. Assume all variables represent positive real numbers.

17) $\log_8 5n$

18) $\log_3 \frac{9a^4}{b^5 c}$

19) Write as a single logarithm.

$2 \log x - 3 \log(x + 1)$

Use a calculator for the rest of the problems.

Solve each equation. Give an exact solution and a solution that is approximated to four decimal places.

20) $\log w = 0.8$

21) $e^{0.3t} = 5$

22) $\ln x = -0.25$

23) $4^{4a+3} = 9$

Graph each function. State the domain and range.

24) $y = e^x - 4$

25) $f(x) = \ln(x + 1)$

26) Approximate $\log_5 17$ to four decimal places.

27) If \$6000 is deposited in an account earning 7.4% interest compounded continuously, how much will be in the account after 5 yr? Use $A = Pe^{rt}$.

28) Polonium-210 decays according to the equation

$$y = y_0 e^{-0.00495t}$$

where t is in days, y_0 is the initial amount present at time $t = 0$, and y is the amount present after t days.

a) If a sample initially contains 100 g of polonium-210, how many grams will be present after 30 days?

b) How long would it take for the initial amount to decay to 20 g?

c) What is the half-life of polonium-210?

Cumulative Review: Chapters 1–13

1) Evaluate $40 + 8 \div 2 - 3^2$.

2) Evaluate $\frac{5}{6} - \frac{14}{15} \cdot \frac{10}{7}$.

Simplify. The answer should not contain any negative exponents.

3) $(-5a^2)(3a^4)$

4) $\frac{40z^3}{10z^{-5}}$

5) $\left(\frac{2c^{10}}{d^3}\right)^{-3}$

6) Write 0.00009231 in scientific notation.

7) *Write an equation and solve.*
A watch is on sale for \$38.40. This is 20% off of the regular price. What was the regular price of the watch?

8) Solve $-4x + 7 < 13$. Graph the solution set and write the answer in interval notation.

9) Solve using the elimination method.

$$\begin{aligned} x + 4y &= -2 \\ -2x + 3y &= 15 \end{aligned}$$

10) Solve using the substitution method.

$$\begin{aligned} 6x + 5y &= -8 \\ 3x - y &= 3 \end{aligned}$$

11) Write the equation of a line containing the points $(-2, 5)$ and $(2, -1)$. Write it in slope-intercept form.

12) Divide $(6c^3 - 7c^2 - 22c + 5) \div (2c - 5)$.

For Exercises 13–15, factor completely.

13) $4w^2 + w - 18$

14) $3p^3 + 2p^2 - 3p - 2$

15) $y^2 - 6y + 9$

16) Solve $x^2 + 14x = -48$.

17) Subtract $\frac{r}{r^2 - 49} - \frac{3}{r^2 - 2r - 63}$.

18) Solve $\frac{9}{y + 6} + \frac{4}{y - 6} = \frac{-4}{y^2 - 36}$.

19) Graph the compound inequality $x + 2y \geq 6$ and $y - x \leq -2$.

Simplify. Assume all variables represent positive real numbers.

20) $\sqrt{120}$

21) $\sqrt{45t^9}$

22) $\sqrt{\frac{36a^5}{a^3}}$

23) $(27)^{2/3}$

24) Solve $\sqrt{h^2 + 2h - 7} = h - 3$.

25) Multiply and simplify $(2 - 7i)(3 + i)$.

26) Solve by completing the square $k^2 - 8k + 4 = 0$.

Solve.

27) $r^2 + 5r = -2$

28) $t^2 = 10t - 41$

29) $4m^4 + 4 = 17m^2$

30) Find the domain of $f(x) = \frac{4}{3x - 2}$.

31) Graph $f(x) = |x| - 4$ and identify the domain and range.

32) Graph $g(x) = 2x^2 + 4x + 4$.

33) Let $f(x) = x^2 - 6x + 2$ and $g(x) = x - 3$.

a) Find $f(-1)$.

b) Find $(f \circ g)(x)$.

c) Find x so that $g(x) = -7$.

34) Graph $f(x) = 2^x - 3$. State the domain and range.

35) Solve $16^y = \frac{1}{64}$.

36) Solve $\log_4(5x + 1) = 2$.

37) Write as a single logarithm.
$\log a + 2 \log b - 5 \log c$

38) Solve $\log 5r - \log(r + 6) = \log 2$.

39) Solve $e^{-0.04t} = 6$. Give an exact solution and an approximation to four decimal places.

40) Graph $f(x) = -\ln x$. State the domain and range.

CHAPTER 14

Conic Sections, Nonlinear Inequalities, and Nonlinear Systems

Algebra at Work: Forensics

We will look at one more application of mathematics to forensic science. Conic sections are used to help solve crimes.

Vanessa is a forensics expert and is called to the scene of a shooting. Blood is spattered everywhere. She uses algebra and trigonometry to help her analyze the blood stain patterns to determine how far the shooter was standing from the victim and the angle at which the victim was shot. The location and angle help police determine the height of the gun when the shots were fired.

Forensics experts perform blood stain pattern analysis. Pictures are taken of the crime scene, and scientists measure the distance between each of the blood stains and the pool of blood. The individual blood stains are roughly elliptical in shape with tails at the ends. On the pictures, scientists draw a best-fit ellipse into each blood stain, measure the long axis of the ellipse (the major axis) and the short axis of the ellipse (the minor axis) and do calculations that reveal where the shooter was standing at the time of the shooting.

In this chapter, we will learn about the ellipse and other conic sections.

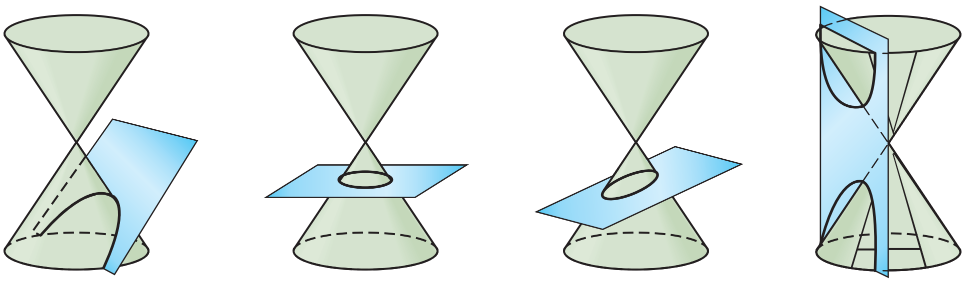
Section 14.1 The Circle

Objectives

1. Define a Conic Section
2. Use the Midpoint Formula
3. Graph a Circle Given in the Form $(x - h)^2 + (y - k)^2 = r^2$
4. Graph a Circle of the Form $Ax^2 + Ay^2 + Cx + Dy + E = 0$

1. Define a Conic Section

In this chapter, we will study the *conic sections*. When a right circular cone is intersected by a plane, the result is a **conic section**. The conic sections are parabolas, circles, ellipses, and hyperbolas. The following figures show how each conic section is obtained from the intersection of a cone and a plane.



Parabola   Circle   Ellipse   Hyperbola

In Chapter 12, we learned how to graph parabolas. The graph of a quadratic function, $f(x) = ax^2 + bx + c$, is a *parabola* that opens vertically. Another form this function may take is $f(x) = a(x - h)^2 + k$. The graph of a quadratic equation of the form $x = ay^2 + by + c$, or $x = a(y - k)^2 + h$, is a *parabola* that opens horizontally. The next conic section we will discuss is the circle.

We will use the distance formula, presented in Section 11.2, to derive the equation of a circle. But first, let's learn the **midpoint formula.**

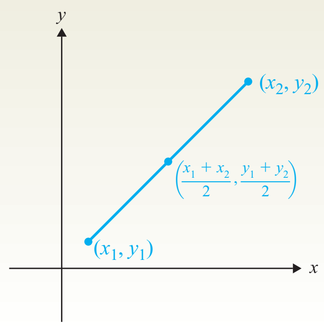
2. Use the Midpoint Formula

The **midpoint** of a line segment is the point that is exactly halfway between the endpoints of a line segment. We use the *midpoint formula* to find the midpoint.

Definition The Midpoint Formula

If (x_1, y_1) and (x_2, y_2) are the endpoints of a line segment, then the **midpoint** of the segment has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



Note

The x-coordinate of the midpoint is the *average* of the x-coordinates of the endpoints. The y-coordinate of the midpoint is the *average* of the y-coordinates of the endpoints.

Example 1

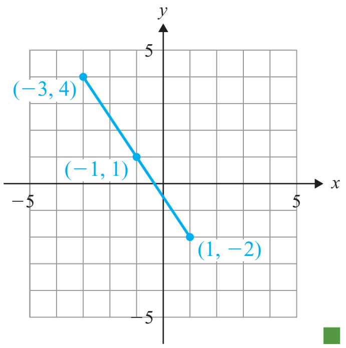
Find the midpoint of the line segment with endpoints $(-3, 4)$ and $(1, -2)$.

Solution

Begin by labeling the points: $\overset{x_1,\, y_1}{(-3, 4)}$, $\overset{x_2,\, y_2}{(1, -2)}$.

Substitute the values into the midpoint formula.

$$
\begin{aligned}
\text{Midpoint} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\
&= \left(\frac{-3 + 1}{2}, \frac{4 + (-2)}{2}\right) && \text{Substitute values.} \\
&= \left(\frac{-2}{2}, \frac{2}{2}\right) \\
&= (-1, 1) && \text{Simplify.}
\end{aligned}
$$



You Try 1

Find the midpoint of the line segment with endpoints $(5, 2)$ and $(1, -3)$.

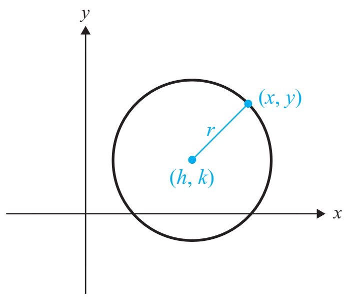
The midpoint of a diameter of a circle is the **center** of the circle.

3. Graph a Circle Given in the Form $(x - h)^2 + (y - k)^2 = r^2$

A **circle** is defined as the set of all points in a plane equidistant (the same distance) from a fixed point. The fixed point is the **center** of the circle. The distance from the center to a point on the circle is the **radius** of the circle.

Let the center of a circle have coordinates (h, k) and let (x, y) represent any point on the circle. Let r represent the distance between these two points. r is the radius of the circle.

We will use the distance formula to find the distance between the center, (h, k), and the point (x, y) on the circle.



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \text{Distance formula}$$

Substitute (x, y) for (x_2, y_2), (h, k) for (x_1, y_1), and r for d.

$$
\begin{aligned}
r &= \sqrt{(x - h)^2 + (y - k)^2} \\
r^2 &= (x - h)^2 + (y - k)^2 && \text{Square both sides.}
\end{aligned}
$$

This is the **standard form** for the equation of a circle.

Definition

Standard Form for the Equation of a Circle: The standard form for the equation of a circle with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

Example 2

Graph $(x - 2)^2 + (y + 1)^2 = 9$.

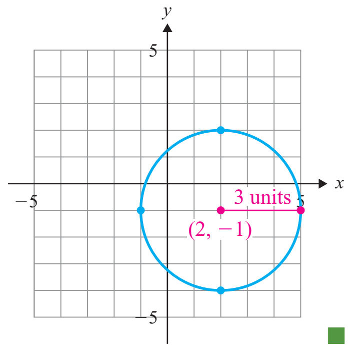
Solution

Standard form is $(x - h)^2 + (y - k)^2 = r^2$.
Our equation is $(x - 2)^2 + (y + 1)^2 = 9$.

$$h = 2 \qquad k = -1 \qquad r = \sqrt{9} = 3$$

The center is $(2, -1)$. The radius is 3.

To graph the circle, first plot the center $(2, -1)$. Use the radius to locate four points on the circle. From the center, move 3 units up, down, left, and right. Draw a circle through the four points.



You Try 2

Graph $(x + 3)^2 + (y - 1)^2 = 16$.

Example 3

Graph $x^2 + y^2 = 1$.

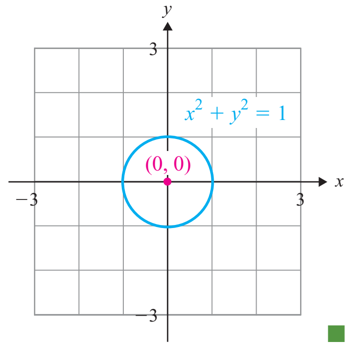
Solution

Standard form is $(x - h)^2 + (y - k)^2 = r^2$.
Our equation is $x^2 + y^2 = 1$.

$$h = 0 \qquad k = 0 \qquad r = \sqrt{1} = 1$$

The center is $(0, 0)$. The radius is 1. Plot $(0, 0)$, then use the radius to locate four points on the circle. From the center, move 1 unit up, down, left, and right. Draw a circle through the four points.

The circle $x^2 + y^2 = 1$ is used often in other areas of mathematics such as trigonometry. $x^2 + y^2 = 1$ is called the **unit circle**.



You Try 3

Graph $x^2 + y^2 = 25$.

If we are told the center and radius of a circle, we can write its equation.

Example 4

Find an equation of the circle with center $(-5, 0)$ and radius $\sqrt{7}$.

Solution

The x-coordinate of the center is h. $\quad h = -5$
The y-coordinate of the center is k. $\quad k = 0$

$$r = \sqrt{7}$$

Substitute these values into $(x - h)^2 + (y - k)^2 = r^2$.

$$[x - (-5)]^2 + (y - 0)^2 = (\sqrt{7})^2 \qquad \text{Substitute } -5 \text{ for } x, 0 \text{ for } k, \text{ and } \sqrt{7} \text{ for } r.$$
$$(x + 5)^2 + y^2 = 7$$

You Try 4

Find an equation of the circle with center (4, 7) and radius 5.

4. Graph a Circle of the Form $Ax^2 + Ay^2 + Cx + Dy + E = 0$

The equation of a circle can take another form—general form.

Definition

General Form for the Equation of a Circle: An equation of the form $Ax^2 + Ay^2 + Cx + Dy + E = 0$, where A, C, D, and E are real numbers, is the **general form** for the equation of a circle.

The coefficients of x^2 and y^2 must be the same in order for this to be the equation of a circle.

To graph a circle given in this form, we complete the square on x and on y to put it into standard form.

After we learn *all* of the conic sections, it is very important that we understand how to identify each one. To do this, we will usually look at the coefficients of the square terms.

Example 5

Graph $x^2 + y^2 + 6x + 2y + 6 = 0$.

Solution

The coefficients of x^2 and y^2 are each 1. Therefore, this is the equation of a circle.

Our goal is to write the given equation in standard form, $(x - h)^2 + (y - k)^2 = r^2$, so that we can identify its center and radius. To do this, we will group x^2 and $6x$ together, group y^2 and $2y$ together, then complete the square on each group of terms.

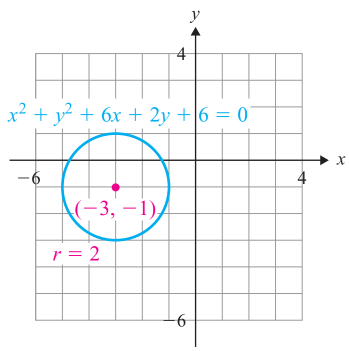
$$x^2 + y^2 + 6x + 2y + 6 = 0 \qquad \text{Group } x^2 \text{ and } 6x \text{ together.}$$
$$(x^2 + 6x) + (y^2 + 2y) = -6 \qquad \text{Group } y^2 \text{ and } 2y \text{ together. Move the constant to the other side.}$$

Complete the square for each group of terms.

$$(x^2 + 6x + 9) + (y^2 + 2y + 1) = -6 + 9 + 1$$
$$(x + 3)^2 + (y + 1)^2 = 4$$

Since 9 and 1 are added on the left, they must also be added on the right. Factor; add.

The center of the circle is $(-3, -1)$.
The radius is 2.



You Try 5

Graph $x^2 + y^2 + 10x - 4y + 20 = 0$.

Note

If we rewrite $Ax^2 + Ay^2 + Cx + Dy + E = 0$ in standard form and get $(x - h)^2 + (y - k)^2 = 0$, then the graph is just the point (h, k). If the constant on the right side of the standard form equation is a negative number, then the equation has no graph.

Using Technology

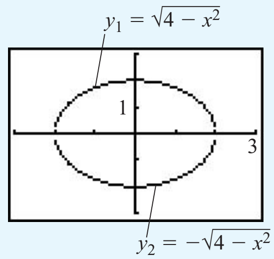
Recall that the equation of a circle is not a function. However, if we want to graph an equation on a graphing calculator, it must be entered as a function or a pair of functions. Therefore, to graph a circle we must solve the equation for y in terms of x.

Let's discuss how to graph $x^2 + y^2 = 4$ on a graphing calculator.

We must solve the equation for y.

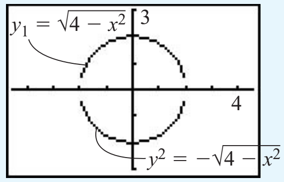
$$x^2 + y^2 = 4$$
$$y^2 = 4 - x^2$$
$$y = \pm\sqrt{4 - x^2}$$

Now the equation of the circle $x^2 + y^2 = 4$ is rewritten so that y is in terms of x. In the graphing calculator, enter $y = \sqrt{4 - x^2}$ as Y_1. This represents the top half of the circle since the y-values are positive above the x-axis. Enter $y = -\sqrt{4 - x^2}$ as Y_2. This represents the bottom half of the circle since the y-values are negative below the x-axis. Here we have the window set from -3 to 3 in both the x- and y-directions. Press $\boxed{\text{GRAPH}}$.



The graph is distorted and does not actually look like a circle! This is because the screen is rectangular, and the graph is longer in the x-direction. We can "fix" this by squaring the window.

To square the window and get a better representation of the graph of $x^2 + y^2 = 4$, press $\boxed{\text{ZOOM}}$ and choose 5:ZSquare. The graph reappears on a "squared" window and now looks like a circle.



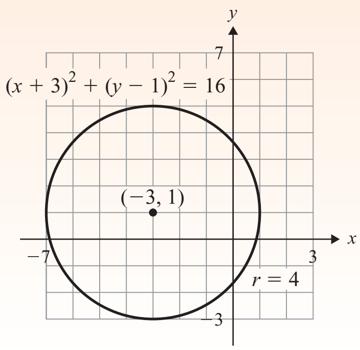
Identify the center and radius of each circle. Then rewrite each equation for y in terms of x, and graph each circle on a graphing calculator. These problems come from the homework exercises so that the graphs can be found in the Answers to Exercises appendix at the back of the book.

1) $x^2 + y^2 = 36$; Exercise 23
2) $x^2 + y^2 = 9$; Exercise 25
3) $(x + 3)^2 + y^2 = 4$; Exercise 19
4) $x^2 + (y - 1)^2 = 25$; Exercise 27

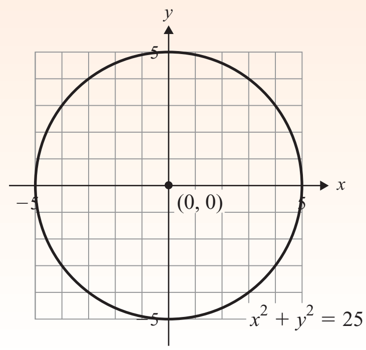
Answers to You Try Exercises

1) $\left(3, -\frac{1}{2}\right)$

2)

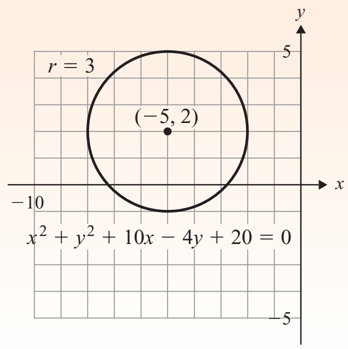


3)



4) $(x - 4)^2 + (y - 7)^2 = 25$

5)



Answers to Technology Exercises

1) Center (0, 0); radius = 6; $Y_1 = \sqrt{36 - x^2}$, $Y_2 = -\sqrt{36 - x^2}$
2) Center (0, 0); radius = 3; $Y_1 = \sqrt{9 - x^2}$, $Y_2 = -\sqrt{9 - x^2}$
3) Center (−3, 0); radius = 2; $Y_1 = \sqrt{4 - (x + 3)^2}$, $Y_2 = -\sqrt{4 - (x + 3)^2}$
4) Center (0, 1); radius = 5; $Y_1 = 1 + \sqrt{25 - x^2}$, $Y_2 = 1 - \sqrt{25 - x^2}$

14.1 Exercises

Objective 2: Use the Midpoint Formula

1) (1, 3) and (7, 9)
2) (2, 10) and (8, 4)
3) (−5, 2) and (−1, −8)
4) (6, −3) and (0, 5)
5) (−3, −7) and (1, −2)
6) (−1, 3) and (2, −9)
7) (4, 0) and (−3, −5)
8) (−2, 4) and (9, 3)
9) $\left(\frac{3}{2}, -1\right)$ and $\left(\frac{5}{2}, \frac{7}{2}\right)$
10) $\left(\frac{9}{2}, \frac{3}{2}\right)$ and $\left(-\frac{7}{2}, -5\right)$
11) (−6.2, 1.5) and (4.8, 5.7)
12) (−3.7, −1.8) and (3.7, −3.6)

Objective 3: Graph a Circle Given in the Form $(x - h)^2 + (y - k)^2 = r^2$

13) Is the equation of a circle a function? Explain your answer.

14) The standard form for the equation of a circle is
$$(x - h)^2 + (y - k)^2 = r^2$$
Identify the center and the radius.

Identify the center and radius of each circle and graph.

VIDEO 15) $(x + 2)^2 + (y - 4)^2 = 9$
16) $(x + 1)^2 + (y + 3)^2 = 25$
17) $(x - 5)^2 + (y - 3)^2 = 1$
18) $x^2 + (y - 5)^2 = 9$
19) $(x + 3)^2 + y^2 = 4$
20) $(x - 2)^2 + (y - 2)^2 = 36$
21) $(x - 6)^2 + (y + 3)^2 = 16$
22) $(x + 8)^2 + (y - 4)^2 = 4$
23) $x^2 + y^2 = 36$
24) $x^2 + y^2 = 16$
25) $x^2 + y^2 = 9$
26) $x^2 + y^2 = 25$
27) $x^2 + (y - 1)^2 = 25$
28) $(x + 3)^2 + y^2 = 1$

Find an equation of the circle with the given center and radius.

VIDEO 29) Center (4, 1); radius = 5
30) Center (3, 5); radius = 2
31) Center (−3, 2); radius = 1
32) Center (4, −6); radius = 3
33) Center (−1, −5); radius = $\sqrt{3}$
34) Center (−2, −1); radius = $\sqrt{5}$
35) Center (0, 0); radius = $\sqrt{10}$
36) Center (0, 0); radius = $\sqrt{6}$
37) Center (6, 0); radius = 4
38) Center (0, −3); radius = 5
39) Center (0, −4); radius = $2\sqrt{2}$
40) Center (1, 0); radius = $3\sqrt{2}$

Objective 4: Graph a Circle of the Form $Ax^2 + By^2 + Cx + Dy + E = 0$

Write the equation of the circle in standard form.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

41) $x^2 + y^2 - 8x + 2y + 8 = 0$
$x^2 - 8x + y^2 + 2y = -8$ ______________
______________ Complete the square.
______________ Factor.

42) $x^2 + y^2 + 2x + 10y + 10 = 0$
$x^2 + 2x + y^2 + 10y = -10$ ______________
______________ Complete the square.
______________ Factor.

Put the equation of each circle in the form $(x - h)^2 + (y - k)^2 = r^2$, identify the center and the radius, and graph.

VIDEO 43) $x^2 + y^2 + 2x + 10y + 17 = 0$

44) $x^2 + y^2 - 4x - 6y + 9 = 0$

45) $x^2 + y^2 + 8x - 2y - 8 = 0$

46) $x^2 + y^2 - 6x + 8y + 24 = 0$

47) $x^2 + y^2 - 10x - 14y + 73 = 0$

48) $x^2 + y^2 + 12x + 12y + 63 = 0$

49) $x^2 + y^2 + 6y + 5 = 0$

50) $x^2 + y^2 + 2x - 24 = 0$

51) $x^2 + y^2 - 4x - 1 = 0$

52) $x^2 + y^2 - 10y + 22 = 0$

53) $x^2 + y^2 - 8x + 8y - 4 = 0$

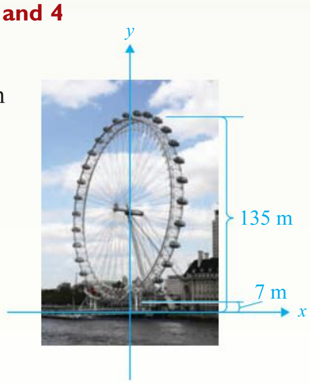
54) $x^2 + y^2 - 6x + 2y - 6 = 0$

VIDEO 55) $4x^2 + 4y^2 - 12x - 4y - 6 = 0$
(Hint: Begin by dividing the equation by 4.)

56) $16x^2 + 16y^2 + 16x - 24y - 3 = 0$
(Hint: Begin by dividing the equation by 16.)

Mixed Exercises: Objectives 3 and 4

57) The London Eye is a Ferris wheel that opened in London in March 2000. It is 135 m high, and the bottom of the wheel is approximately 7 m off the ground.



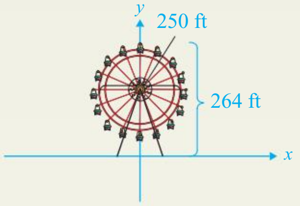
a) What is the diameter of the wheel?

b) What is the radius of the wheel?

c) Using the axes in the illustration, what are the coordinates of the center of the wheel?

d) Write the equation of the wheel.
(www.londoneye.com)

58) The first Ferris wheel was designed and built by George W. Ferris in 1893 for the Chicago World's Fair. It was 264 ft tall, and the wheel had a diameter of 250 ft.

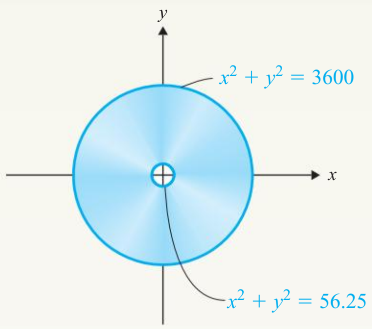


a) What is the radius of the wheel?

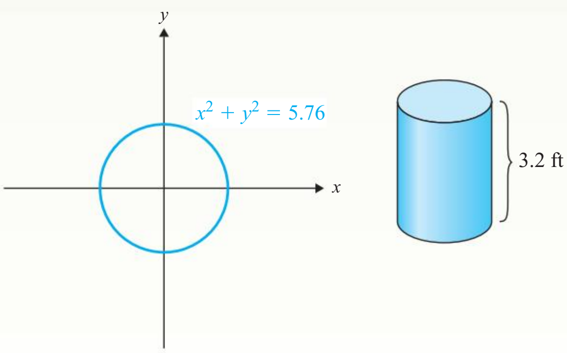
b) Using the axes in the illustration, what are the coordinates of the center of the wheel?

c) Write the equation of the wheel.

59) A CD is placed on axes as shown in the figure, where the units of measurement for x and y are millimeters. Using $\pi \approx 3.14$, what is the surface area of a CD (to the nearest square millimeter)?



60) A storage container is in the shape of a right circular cylinder. The top of the container may be described by the equation $x^2 + y^2 = 5.76$, as shown in the figure (x and y are in feet). If the container is 3.2 ft tall, what is the storage capacity of the container (to the nearest ft^3)?



Section 14.2 The Ellipse and the Hyperbola

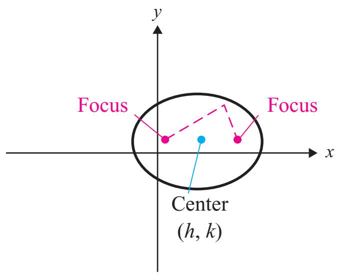
Objectives

1. Graph an Ellipse
2. Graph a Hyperbola
3. Graph Other Square Root Functions

The Ellipse

1. Graph an Ellipse

The next conic section we will study is the *ellipse*. An **ellipse** is the set of all points in a plane such that the *sum* of the distances from a point on the ellipse to two fixed points is constant. Each fixed point is called a **focus** (plural: **foci**). The point halfway between the foci is the **center** of the ellipse.



The orbits of planets around the sun as well as satellites around the earth are elliptical. Statuary Hall in the U.S. Capitol building is an ellipse. If a person stands at one focus of this ellipse and whispers, a person standing across the room on the other focus can clearly hear what was said. Properties of the ellipse are used in medicine as well. One procedure for treating kidney stones involves immersing the patient in an elliptical tub of water. The kidney stone is at one focus, while at the other focus, high energy shock waves are produced, which destroy the kidney stone.

Definition

Standard Form for the Equation of an Ellipse: The standard form for the equation of an ellipse is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The center of the ellipse is (h, k).

It is important to remember that the terms on the left are *both* positive quantities.

Example 1

Graph $\dfrac{(x-3)^2}{16} + \dfrac{(y-1)^2}{4} = 1$.

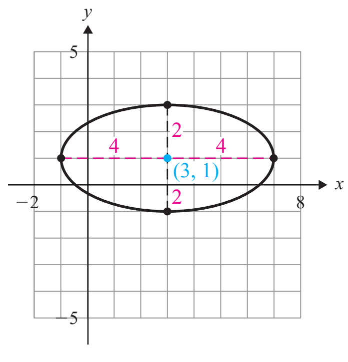
Solution

Standard form is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$.

Our equation is $\dfrac{(x-3)^2}{16} + \dfrac{(y-1)^2}{4} = 1$.

$h = 3 \qquad k = 1$

$a = \sqrt{16} = 4 \qquad b = \sqrt{4} = 2$



The center is (3, 1).

To graph the ellipse, first plot the center (3, 1). Since $a = 4$ and a^2 is under the squared quantity containing the $\boldsymbol{x}$, move 4 units each way in the x-direction from the center. These are two points on the ellipse.

Since $b = 2$ and b^2 is under the squared quantity containing the y, move 2 units each way in the y-direction from the center. These are two more points on the ellipse. Sketch the ellipse through the four points.

You Try 1

Graph $\dfrac{(x+2)^2}{25} + \dfrac{(y-3)^2}{16} = 1$.

Example 2

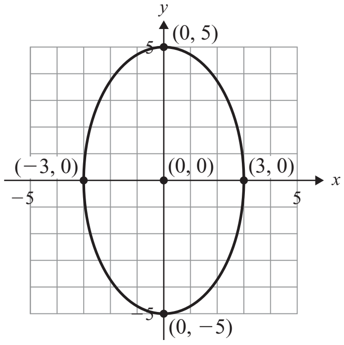
Graph $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$.

Solution

Standard form is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$.

Our equation is $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$.

$$h = 0 \qquad k = 0$$
$$a = \sqrt{9} = 3 \qquad b = \sqrt{25} = 5$$



The center is (0, 0).

Plot the center (0, 0). Since $a = 3$ and a^2 is under the x^2, move 3 units each way in the x-direction from the center. These are two points on the ellipse.

Since $b = 5$ and b^2 is under the y^2, move 5 units each way in the y-direction from the center. These are two more points on the ellipse. Sketch the ellipse through the four points.

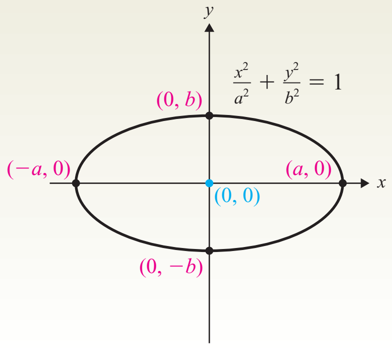
You Try 2

Graph $\dfrac{x^2}{36} + \dfrac{y^2}{9} = 1$.

In Example 2, note that the *origin*, (0, 0), is the center of the ellipse. Notice also that $a = 3$ and the x-intercepts are (3, 0) and (−3, 0); $b = 5$ and the y-intercepts are (0, 5) and (0, −5). We can generalize these relationships as follows.

Definition

Equation of an Ellipse with Center at Origin: The graph of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is an ellipse with center at the origin, x-intercepts $(a, 0)$ and $(-a, 0)$, and y-intercepts $(0, b)$ and $(0, -b)$.



Looking at Examples 1 and 2, we can make another interesting observation.

Example 1

$$\frac{(x-3)^2}{16} + \frac{(y-1)^2}{4} = 1$$
$$a^2 = 16 \quad b^2 = 4$$
$$a^2 > b^2$$

The number under $(x-3)^2$ is greater than the number under $(y-1)^2$. *The ellipse is longer in the x-direction.*

Example 2

$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$
$$a^2 = 9 \quad b^2 = 25$$
$$b^2 > a^2$$

The number under y^2 is greater than the number under x^2. *The ellipse is longer in the y-direction.*

This relationship between a^2 and b^2 will always produce the same result.

The equation of an ellipse can take other forms.

Example 3

Graph $4x^2 + 25y^2 = 100$.

Solution

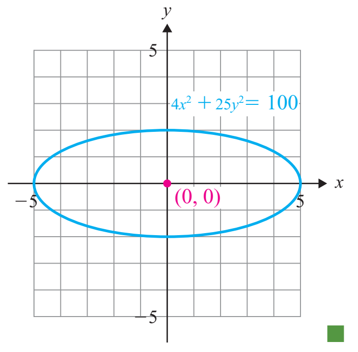
How can we tell whether this is a circle or an ellipse? We look at the coefficients of x^2 and y^2. Both of the coefficients are positive, *and* they are different. *This is an ellipse.* (If this were a circle, the coefficients would be the same.)

Since the standard form for the equation of an ellipse has a 1 on one side of the = sign, divide both sides of $4x^2 + 25y^2 = 100$ by 100 to obtain a 1 on the right.

$$4x^2 + 25y^2 = 100$$
$$\frac{4x^2}{100} + \frac{25y^2}{100} = \frac{100}{100} \qquad \text{Divide both sides by 100.}$$
$$\frac{x^2}{25} + \frac{y^2}{4} = 1 \qquad \text{Simplify.}$$

The center is (0, 0). $a = \sqrt{25} = 5$ and $b = \sqrt{4} = 2$. Plot (0, 0). Move 5 units each way from the center in the x-direction. Move 2 units each way from the center in the y-direction.

Notice that the x-intercepts are (5, 0) and $(-5, 0)$. The y-intercepts are (2, 0) and $(-2, 0)$.



You Try 3

Graph $x^2 + 4y^2 = 4$.

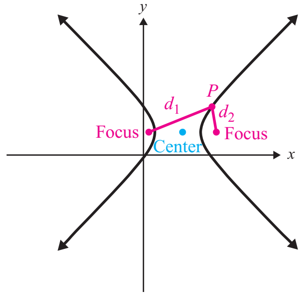
Note

You may have noticed that if $a^2 = b^2$, then the ellipse is a circle.

The Hyperbola

2. Graph a Hyperbola

The last of the conic sections is the *hyperbola*. A **hyperbola** is the set of all points in a plane such that the absolute value of the *difference* of the distances from two fixed points is constant. Each fixed point is called a **focus**. The point halfway between the foci is the **center** of the hyperbola.



Some navigation systems used by ships are based on the properties of hyperbolas. A lamp casts a hyperbolic shadow on a wall, and many telescopes use hyperbolic lenses.

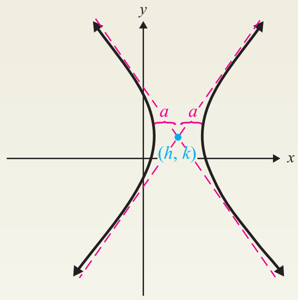
A hyperbola is a graph consisting of two branches. The hyperbolas we will consider will have branches that open either in the x-direction or in the y-direction.

Definition Standard Form for the Equation of a Hyperbola

1) A hyperbola with center (h, k) and branches that open in the *x-direction* has equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

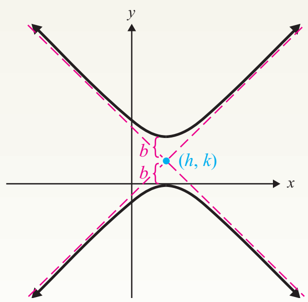
Its graph is to the right.



2) A hyperbola with center (h, k) and branches that open in the *y-direction* has equation

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1.$$

Its graph is to the right.



Notice in 1) that $\dfrac{(x-h)^2}{a^2}$ is the positive quantity, and the branches open in the x-direction.

In 2), the positive quantity is $\dfrac{(y-k)^2}{b^2}$, and the branches open in the y-direction.

In 1) and 2) of the definition notice how the branches of the hyperbola get closer to the dotted lines as the branches continue indefinitely. These dotted lines are called **asymptotes**. They are not an actual part of the graph of the hyperbola, but we can use them to help us obtain the hyperbola.

Example 4

Graph $\dfrac{(x+2)^2}{9} - \dfrac{(y-1)^2}{4} = 1$.

Solution

How do we know that this is a hyperbola and not an ellipse? *It is a hyperbola because there is a subtraction sign between the two quantities on the left*. If it were addition, it would be an ellipse.

Standard form is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$.

Our equation is $\dfrac{(x+2)^2}{9} - \dfrac{(y-1)^2}{4} = 1$.

$h = -2 \qquad k = 1$

$a = \sqrt{9} = 3 \qquad b = \sqrt{4} = 2$

The center is $(-2, 1)$. *Since the quantity* $\dfrac{(x-h)^2}{a^2}$ *is the* positive *quantity, the branches of the hyperbola will open in the x-direction.*

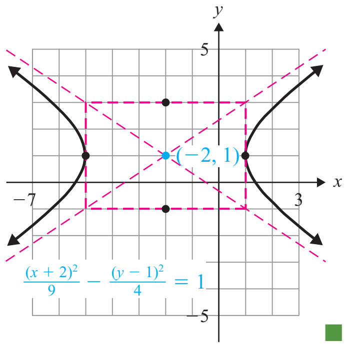
We will use the center, $(-2, 1)$, $a = 3$, and $b = 2$ to draw a *reference rectangle*. The diagonals of this rectangle are the asymptotes of the hyperbola.

First, plot the center $(-2, 1)$. Since $a = 3$ and a^2 is under the squared quantity containing the x, move 3 units each way in the x-direction from the center. These are two points on the rectangle.

Since $b = 2$ and b^2 is under the squared quantity containing the y, move 2 units each way in the y-direction from the center. These are two more points on the rectangle.

Draw the rectangle containing these four points, then draw the diagonals of the rectangle as dotted lines. These are the asymptotes of the hyperbola.

Sketch the branches of the hyperbola opening in the x-direction with the branches approaching the asymptotes.



You Try 4

Graph $\dfrac{(x+1)^2}{9} - \dfrac{(y+1)^2}{16} = 1$.

Example 5

Graph $\dfrac{y^2}{4} - \dfrac{x^2}{25} = 1$.

Solution

Standard form is $\dfrac{(y-k)^2}{b^2} - \dfrac{(x-h)^2}{a^2} = 1$. Our equation is $\dfrac{y^2}{4} - \dfrac{x^2}{25} = 1$.

$$k = 0 \qquad h = 0$$
$$b = \sqrt{4} = 2 \qquad a = \sqrt{25} = 5$$

The center is (0, 0). *Since the quantity* $\dfrac{y^2}{4}$ *is the* positive *quantity, the branches of the hyperbola will open in the y-direction.*

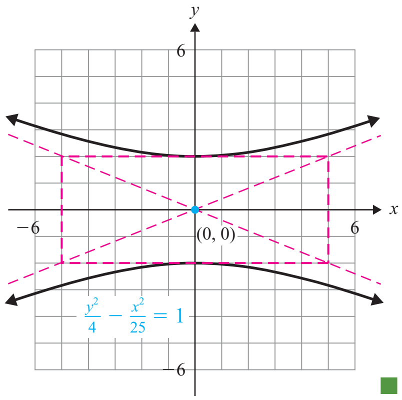
Use the center, (0, 0), $a = 5$, and $b = 2$ to draw the reference rectangle and its diagonals.

Plot the center (0, 0). Since $a = 5$ and a^2 is under the x^2, move 5 units each way in the x-direction from the center to get two points on the rectangle.

Since $b = 2$ and b^2 is under the y^2, move 2 units each way in the y-direction from the center to get two more points on the rectangle.

Draw the rectangle and its diagonals as dotted lines. These are the asymptotes of the hyperbola.

Sketch the branches of the hyperbola opening in the y-direction approaching the asymptotes.

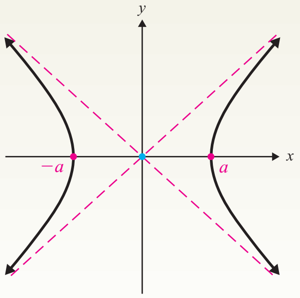


You Try 5

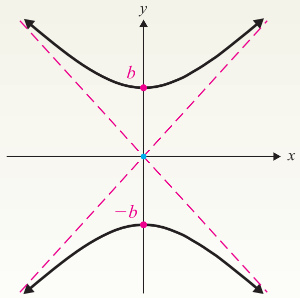
Graph $\dfrac{y^2}{16} - \dfrac{x^2}{16} = 1$.

Definition Equation of a Hyperbola with Center at the Origin

1) The graph of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is a hyperbola with center (0, 0) and x-intercepts $(a, 0)$ and $(-a, 0)$, as shown below.



2) The graph $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$ is a hyperbola with center (0, 0) and y-intercepts $(0, b)$ and $(0, -b)$, as shown below.



The equations of the asymptotes are $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$.

Let's look at another example.

Example 6

Graph $y^2 - 9x^2 = 9$.

Solution

This is a hyperbola since there is a subtraction sign between the two terms.

Since the standard form for the equation of the hyperbola has a 1 on one side of the equal sign, divide both sides of $y^2 - 9x^2 = 9$ by 9 to obtain a 1 on the right.

$$y^2 - 9x^2 = 9$$

$$\frac{y^2}{9} - \frac{9x^2}{9} = \frac{9}{9} \quad \text{Divide both sides by 9.}$$

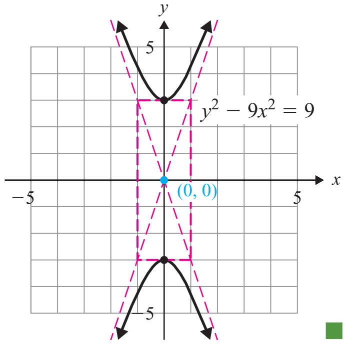
$$\frac{y^2}{9} - x^2 = 1 \quad \text{Simplify.}$$

The center is (0, 0). *The branches of the hyperbola will open in the y-direction since* $\frac{y^2}{9}$ *is a positive quantity.*

x^2 is the same as $\frac{x^2}{1}$, so $a = \sqrt{1} = 1$ and $b = \sqrt{9} = 3$.

Plot the center at the origin. Move 1 unit each way in the x-direction from the center and 3 units each way in the y-direction. Draw the rectangle and the asymptotes.

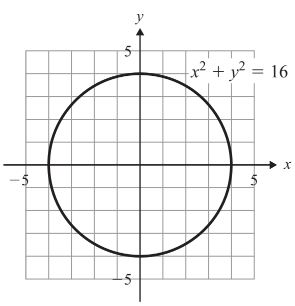
Sketch the branches of the hyperbola opening in the y-direction approaching the asymptotes. The y-intercepts are (0, 3) and (0, −3). There are no x-intercepts.



You Try 6

Graph $4x^2 - 9y^2 = 36$.

3. Graph Other Square Root Functions

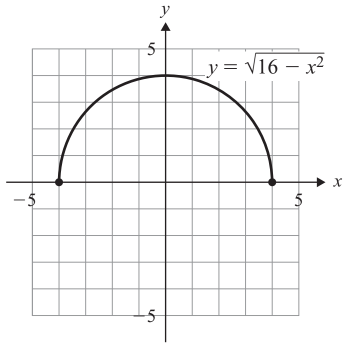


We have already learned how to graph square root functions like $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x - 3}$. Next, we will learn how to graph other square root functions by relating them to the graphs of conic sections.

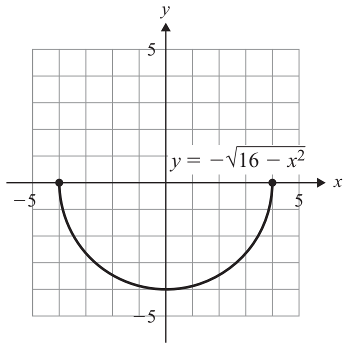
The vertical line test shows that horizontal parabolas, circles, ellipses, and some hyperbolas are not the graphs of functions. What happens, however, if we look at a portion of the graph of a conic section? Let's start with a circle.

The graph of $x^2 + y^2 = 16$, at the left, is a circle with center (0, 0) and radius 4. If we solve this equation for y, we get $y = \pm\sqrt{16 - x^2}$. This represents two equations, $y = \sqrt{16 - x^2}$ and $y = -\sqrt{16 - x^2}$.

The graph of $y = \sqrt{16 - x^2}$ is the **top half of the circle** since the y-coordinates of all points on the graph will be nonnegative. The domain is $[-4, 4]$, and the range is $[0, 4]$.



Because of the negative sign in front of the radical, the graph of $y = -\sqrt{16 - x^2}$ is the **bottom half of the circle** since the y-coordinates of all points on the graph will be nonpositive. The domain is $[-4, 4]$, and the range is $[-4, 0]$.



Therefore, to graph $y = \sqrt{16 - x^2}$, it is helpful if we recognize that it is the top half of the circle with equation $x^2 + y^2 = 16$. Likewise, if we are asked to graph $y = -\sqrt{16 - x^2}$, we should recognize that it is the bottom half of the graph of $x^2 + y^2 = 16$.

Let's graph another square root function by first relating it to a conic section.

Example 7

Graph $f(x) = -5\sqrt{1 - \frac{x^2}{9}}$. Identify the domain and range.

Solution

The graph of this function is half of the graph of a conic section. First, notice that $f(x)$ is always a nonpositive quantity. Since all nonpositive values of y are on or below the x-axis, the graph of this function will be only on or below the x-axis.

Replace $f(x)$ with y, and rearrange the equation into a form we recognize as a conic section.

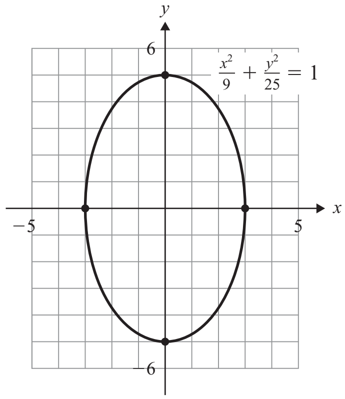
$$y = -5\sqrt{1 - \frac{x^2}{9}} \qquad \text{Replace } f(x) \text{ with } y.$$

$$-\frac{y}{5} = \sqrt{1 - \frac{x^2}{9}} \qquad \text{Divide by } -5.$$

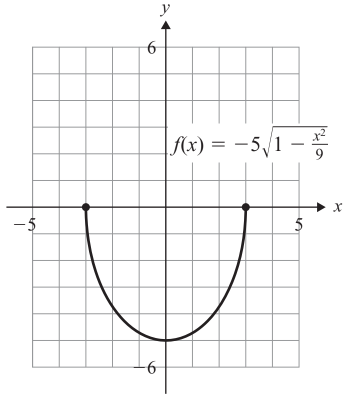
$$\frac{y^2}{25} = 1 - \frac{x^2}{9} \qquad \text{Square both sides.}$$

$$\frac{x^2}{9} + \frac{y^2}{25} = 1 \qquad \text{Add } \frac{x^2}{9}.$$

The equation $\frac{x^2}{9} + \frac{y^2}{25} = 1$ represents an ellipse centered at the origin. Its domain is $[-3, 3]$, and its range is $[-5, 5]$. It is not a function.



The graph of $f(x) = -5\sqrt{1 - \frac{x^2}{9}}$ is the bottom half of the ellipse, and it is a function. Its domain is $[-3, 3]$, and its range is $[-5, 0]$.



You Try 7

Graph $f(x) = \sqrt{\frac{x^2}{4} - 1}$. Identify the domain and range.

More about the conic sections and their characteristics are studied in later mathematics courses.

Using Technology

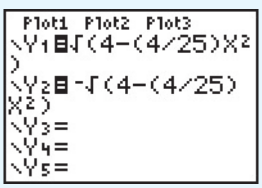
We graph ellipses and hyperbolas on a graphing calculator in the same way that we graphed circles: solve the equation for y in terms of x, enter both values of y, and graph both equations.

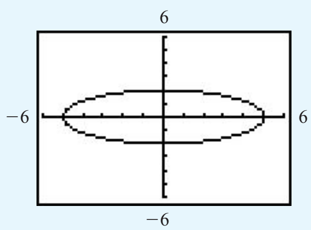
Let's graph the ellipse given in Example 3: $4x^2 + 25y^2 = 100$.

Solve for y.

$$25y^2 = 100 - 4x^2$$
$$y^2 = \frac{100 - 4x^2}{25}$$
$$y^2 = 4 - \frac{4x^2}{25}$$
$$y = \pm\sqrt{4 - \frac{4x^2}{25}}$$

Enter $y = \sqrt{4 - \frac{4x^2}{25}}$ as Y_1. This represents the top half of the ellipse since the y-values are positive above the x-axis. Enter $y = -\sqrt{4 - \frac{4x^2}{25}}$ as Y_2. This represents the bottom half of the ellipse since the y-values are negative below the x-axis. Set an appropriate window, and press GRAPH.



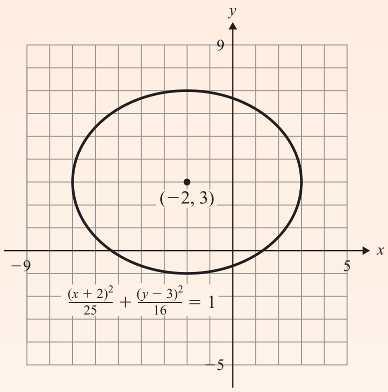


Use the same technique to graph a hyperbola on a graphing calculator.

Identify each conic section as either an ellipse or a hyperbola. Identify the center of each, rewrite each equation for y in terms of x, and graph each equation on a graphing calculator. These problems come from the homework exercises so that the graphs can be found in the Answers to Exercises appendix at the back of the book.

1) $4x^2 + 9y^2 = 36$; Exercise 21
2) $x^2 + \dfrac{y^2}{4} = 1$; Exercise 15
3) $\dfrac{x^2}{25} + (y + 4)^2 = 1$; Exercise 17
4) $9x^2 - y^2 = 36$; Exercise 37
5) $\dfrac{y^2}{16} - \dfrac{x^2}{4} = 1$; Exercise 27
6) $y^2 - \dfrac{(x - 1)^2}{9} = 1$; Exercise 33

Answers to You Try Exercises

1)


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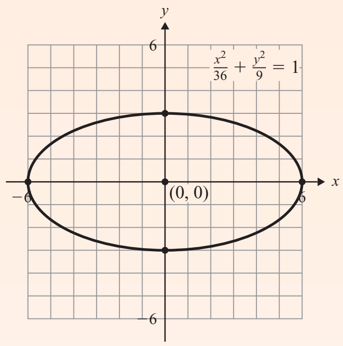

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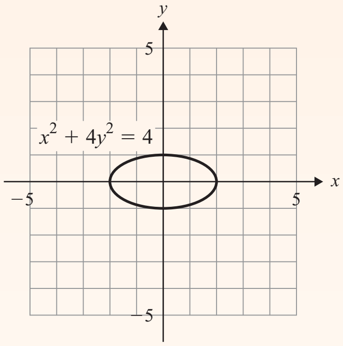

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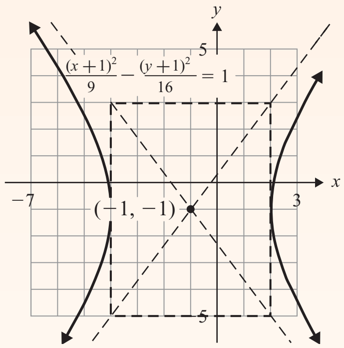

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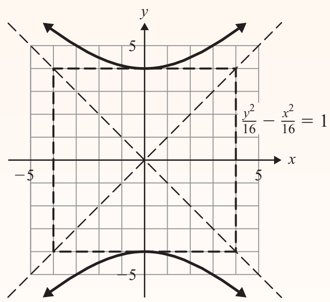
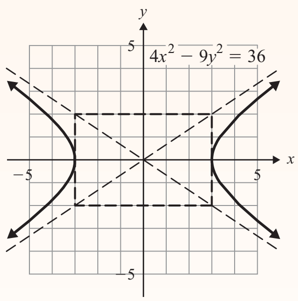
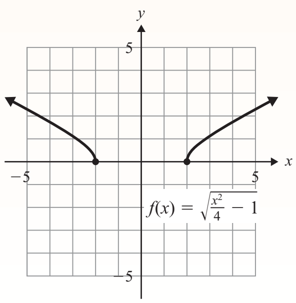


6)


7) domain: $(-\infty, -2] \cup [2, \infty)$; range: $[0, \infty)$



Answers to Technology Exercises

1) ellipse with center $(0, 0)$; $Y_1 = \sqrt{4 - \dfrac{4x^2}{9}}$, $Y_2 = -\sqrt{4 - \dfrac{4x^2}{9}}$

2) ellipse with center $(0, 0)$; $Y_1 = \sqrt{4 - 4x^2}$, $Y_2 = -\sqrt{4 - 4x^2}$

3) ellipse with center $(0, -4)$; $Y_1 = -4 + \sqrt{1 - \dfrac{x^2}{25}}$, $Y_2 = -4 - \sqrt{1 - \dfrac{x^2}{25}}$

4) hyperbola with center $(0, 0)$; $Y_1 = \sqrt{9x^2 - 36}$, $Y_2 = -\sqrt{9x^2 - 36}$

5) hyperbola with center $(0, 0)$; $Y_1 = \sqrt{16 + 4x^2}$, $Y_2 = -\sqrt{16 + 4x^2}$

6) hyperbola with center $(1, 0)$; $Y_1 = \sqrt{1 + \dfrac{(x-1)^2}{9}}$, $Y_2 = -\sqrt{1 + \dfrac{(x-1)^2}{9}}$

14.2 Exercises

Mixed Exercises: Objectives 1 and 2

Identify each equation as an ellipse or a hyperbola.

1) $\dfrac{x^2}{36} + \dfrac{y^2}{4} = 1$

2) $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$

3) $\dfrac{(y-3)^2}{4} - \dfrac{(x+5)^2}{9} = 1$

4) $\dfrac{(x-4)^2}{16} + \dfrac{(y-1)^2}{9} = 1$

5) $16x^2 - y^2 = 16$

6) $4x^2 + 25y^2 = 100$

7) $\dfrac{x^2}{25} + y^2 = 1$

8) $-\dfrac{(x+6)^2}{4} + \dfrac{y^2}{10} = 1$

Objective 1: Graph an Ellipse

Identify the center of each ellipse and graph the equation.

9) $\dfrac{(x+2)^2}{9} + \dfrac{(y-1)^2}{4} = 1$

10) $\dfrac{(x-4)^2}{4} + \dfrac{(y-3)^2}{16} = 1$

11) $\dfrac{(x-3)^2}{9} + \dfrac{(y+2)^2}{16} = 1$

12) $\dfrac{(x+4)^2}{25} + \dfrac{(y-5)^2}{16} = 1$

13) $\dfrac{x^2}{36} + \dfrac{y^2}{16} = 1$

14) $\dfrac{x^2}{36} + \dfrac{y^2}{4} = 1$

15) $x^2 + \dfrac{y^2}{4} = 1$

16) $\dfrac{x^2}{9} + y^2 = 1$

17) $\dfrac{x^2}{25} + (y+4)^2 = 1$

18) $(x+3)^2 + \dfrac{(y+4)^2}{9} = 1$

19) $\dfrac{(x+1)^2}{4} + \dfrac{(y+3)^2}{9} = 1$

20) $\dfrac{(x-2)^2}{16} + \dfrac{y^2}{25} = 1$

21) $4x^2 + 9y^2 = 36$

22) $x^2 + 4y^2 = 16$

23) $25x^2 + y^2 = 25$

24) $9x^2 + y^2 = 36$

Objective 2: Graph a Hyperbola

Identify the center of each hyperbola and graph the equation.

25) $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$

26) $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$

27) $\dfrac{y^2}{16} - \dfrac{x^2}{4} = 1$

28) $\dfrac{y^2}{4} - \dfrac{x^2}{4} = 1$

29) $\dfrac{(x-2)^2}{9} - \dfrac{(y+3)^2}{16} = 1$

30) $\dfrac{(x+3)^2}{4} - \dfrac{(y+1)^2}{16} = 1$

31) $\dfrac{(y+1)^2}{25} - \dfrac{(x+4)^2}{4} = 1$

32) $\dfrac{(y-1)^2}{36} - \dfrac{(x+1)^2}{9} = 1$

33) $y^2 - \dfrac{(x-1)^2}{9} = 1$

34) $\dfrac{(y+4)^2}{4} - x^2 = 1$

35) $\dfrac{(x-1)^2}{25} - \dfrac{(y-2)^2}{25} = 1$

36) $\dfrac{(x-2)^2}{16} - \dfrac{(y-3)^2}{9} = 1$

37) $9x^2 - y^2 = 36$

38) $4y^2 - x^2 = 16$

39) $y^2 - x^2 = 1$

40) $x^2 - y^2 = 25$

Objective 3: Graph Other Square Root Functions

Graph each square root function. Identify the domain and range.

41) $f(x) = \sqrt{9 - x^2}$

42) $g(x) = \sqrt{25 - x^2}$

43) $h(x) = -\sqrt{1 - x^2}$

44) $k(x) = -\sqrt{9 - x^2}$

45) $g(x) = -2\sqrt{1 - \dfrac{x^2}{9}}$

46) $f(x) = 3\sqrt{1 - \dfrac{x^2}{16}}$

47) $h(x) = -3\sqrt{\dfrac{x^2}{4} - 1}$

48) $k(x) = 2\sqrt{\dfrac{x^2}{16} - 1}$

Sketch the graph of each equation.

49) $x = \sqrt{16 - y^2}$

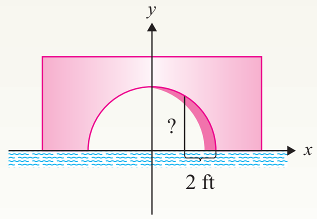
50) $x = -\sqrt{4 - y^2}$

51) $x = -3\sqrt{1 - \dfrac{y^2}{4}}$

52) $x = \sqrt{1 - \dfrac{y^2}{9}}$

53) The Oval Office in the White House is an ellipse about 36 ft long and 29 ft wide. If the center of the room is at the origin of a Cartesian coordinate system and the length of the room is along the x-axis, write an equation of the elliptical room. (www.whitehousehistory.org)

54) The arch of a bridge over a canal in Amsterdam is half of an ellipse. At water level, the arch is 14 ft wide, and it is 6 ft tall at its highest point.



a) Write an equation of the arch.

b) What is the height of the arch (to the nearest foot) 2 ft from the bottom edge?

Putting It All Together

Objective

1. Identify and Graph Different Types of Conic Sections

1. Identify and Graph Different Types of Conic Sections

Sometimes the most difficult part of graphing a conic section is identifying which type of graph will result from the given equation. In this section, we will discuss how to look at an equation and decide what type of conic section it represents.

Example 1

Graph $x^2 + y^2 + 4x - 6y + 9 = 0$.

Solution

First, notice that this equation has two squared terms. Therefore, its graph cannot be a parabola since the equation of a parabola contains only one squared term. Next, observe that the coefficients of x^2 and y^2 are each 1. Since the coefficients are the same, *this is the equation of a circle.*

Write the equation in the form $(x - h)^2 + (y - k)^2 = r^2$ by completing the square on the x-terms and on the y-terms.

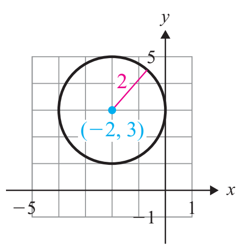
$$x^2 + y^2 + 4x - 6y + 9 = 0$$

$$(x^2 + 4x) + (y^2 - 6y) = -9 \quad \text{Group the } x\text{-terms together and group the } y\text{-terms together. Move the constant to the other side.}$$

$$(x^2 + 4x + 4) + (y^2 - 6y + 9) = -9 + 4 + 9 \quad \text{Complete the square for each group of terms.}$$

$$(x + 2)^2 + (y - 3)^2 = 4 \quad \text{Factor; add.}$$

The center of the circle is $(-2, 3)$. The radius is 2.



Example 2

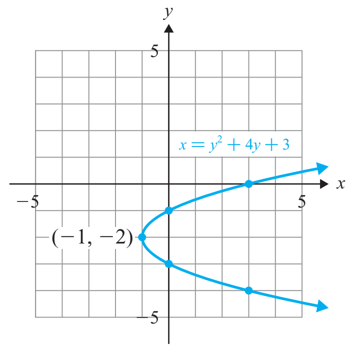
Graph $x = y^2 + 4y + 3$.

Solution

This equation contains only one squared term. Therefore, *this is the equation of a parabola.* Since the squared term is y^2 and $a = 1$, the parabola will open to the right.

Use the formula $y = -\dfrac{b}{2a}$ to find the y-coordinate of the vertex.

$$a = 1 \quad b = 4 \quad c = 3$$
$$y = -\frac{4}{2(1)} = -2$$
$$x = (-2)^2 + 4(-2) + 3 = -1$$



The vertex is $(-1, -2)$. Make a table of values to find other points on the parabola, and use the axis of symmetry to find more points.

x	y
0	−1
3	0

Plot the points in the table. Locate the points $(0, -3)$ and $(3, -4)$ using the axis of symmetry, $y = -2$.

Example 3

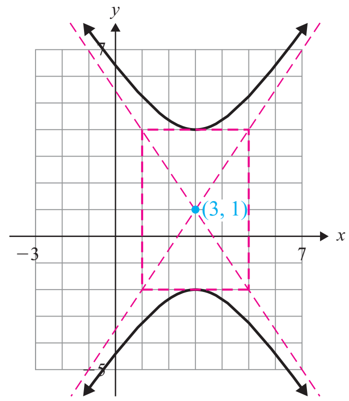
Graph $\dfrac{(y-1)^2}{9} - \dfrac{(x-3)^2}{4} = 1$.

Solution

In this equation, we see the *difference* of two squares. *The graph of this equation is a hyperbola. The branches of the hyperbola will open in the y-direction since the quantity containing the variable y,* $\dfrac{(y-1)^2}{9}$, *is the positive, squared quantity.*

The center is $(3, 1)$; $a = \sqrt{4} = 2$ and $b = \sqrt{9} = 3$. Draw the reference rectangle and its diagonals, the asymptotes of the graph.

The branches of the hyperbola approach the asymptotes.



Example 4

Graph $x^2 + 9y^2 = 36$.

Solution

This equation contains the *sum* of two squares with *different* coefficients. *This is the equation of an ellipse.* (If the coefficients were the same, the graph would be a circle.)

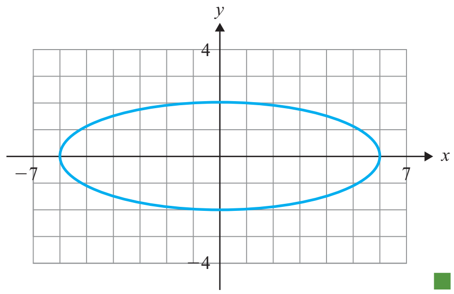
Divide both sides of the equation by 36 to get 1 on the right side of the = sign.

$$x^2 + 9y^2 = 36$$

$$\frac{x^2}{36} + \frac{9y^2}{36} = \frac{36}{36} \quad \text{Divide both sides by 36.}$$

$$\frac{x^2}{36} + \frac{y^2}{4} = 1 \quad \text{Simplify.}$$

The center is (0, 0), $a = 6$, and $b = 2$.



You Try 1

Determine whether the graph of each equation is a parabola, circle, ellipse, or hyperbola. Then, graph each equation.

a) $4x^2 - 25y^2 = 100$

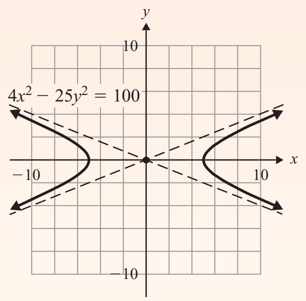
b) $x^2 + y^2 - 6x - 12y + 9 = 0$

c) $y = -x^2 - 2x + 4$

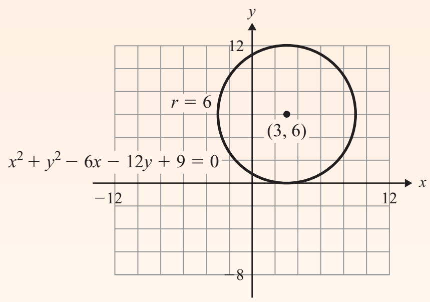
d) $x^2 + \frac{(y + 4)^2}{9} = 1$

Answers to You Try Exercises

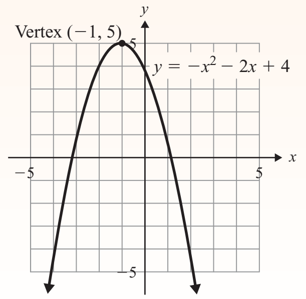
1) a) hyperbola



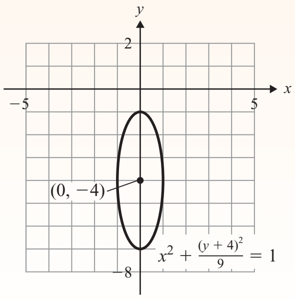
b) circle



c) parabola



d) ellipse



Putting It All Together Summary Exercises

Objective 1: Identify and Graph Different Types of Conic Sections

Determine whether the graph of each equation is a parabola, circle, ellipse, or hyperbola. Then graph each equation.

1) $y = x^2 + 4x + 8$

2) $(x + 5)^2 + (y - 3)^2 = 25$

VIDEO 3) $\dfrac{(y + 4)^2}{9} - \dfrac{(x + 1)^2}{4} = 1$

4) $x = (y - 1)^2 + 8$

VIDEO 5) $16x^2 + 9y^2 = 144$

6) $x^2 - 4y^2 = 36$

VIDEO 7) $x^2 + y^2 + 8x - 6y - 11 = 0$

8) $\dfrac{(x - 2)^2}{25} + \dfrac{(y - 2)^2}{36} = 1$

9) $(x - 1)^2 + \dfrac{y^2}{16} = 1$

10) $x^2 + y^2 + 8y + 7 = 0$

VIDEO 11) $x = -(y + 4)^2 - 3$

12) $\dfrac{(y + 4)^2}{4} - (x + 2)^2 = 1$

13) $25x^2 - 4y^2 = 100$

14) $4x^2 + y^2 = 16$

15) $(x - 3)^2 + y^2 = 16$

16) $y = -x^2 + 6x - 7$

17) $x = \dfrac{1}{2}y^2 + 2y + 3$

18) $\dfrac{(x - 3)^2}{16} + y^2 = 1$

VIDEO 19) $(x - 2)^2 - (y + 1)^2 = 9$

20) $x^2 + y^2 + 6x - 8y + 9 = 0$

Section 14.3 Nonlinear Systems of Equations

Objectives

1. Define a Nonlinear System of Equations
2. Solve a Nonlinear System by Substitution
3. Solve a Nonlinear System Using the Elimination Method

1. Define a Nonlinear System of Equations

In Chapter 5, we learned to solve systems of linear equations by graphing, substitution, and the elimination method. We can use these same techniques for solving a *nonlinear* system of equations in two variables. A **nonlinear system of equations** is a system in which at least one of the equations is not linear.

Solving a nonlinear system by graphing is not practical since it would be very difficult (if not impossible) to accurately read the points of intersection. Therefore, we will solve the systems using substitution and the elimination method. We will graph the equations, however, so that we can visualize the solution(s) as the point(s) of intersection of the graphs.

We are interested only in real-number solutions. If a system has imaginary solutions, then the graphs of the equations do not intersect in the real-number plane.

2. Solve a Nonlinear System by Substitution

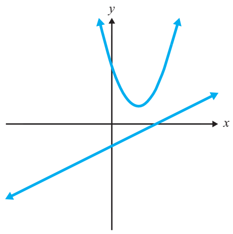
When one of the equations in a system is linear, it is often best to use the substitution method to solve the system.

Example 1

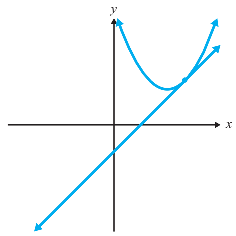
Solve the system
$$\begin{aligned} x^2 - 2y &= 2 \quad (1) \\ -x + y &= 3 \quad (2) \end{aligned}$$

Solution

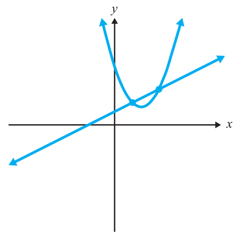
The graph of equation (1) is a parabola, and the graph of equation (2) is a line. Let's begin by thinking about the number of possible points of intersection the graphs can have.



No points of intersection
The system has no solution.



One point of intersection
The system has one solution.



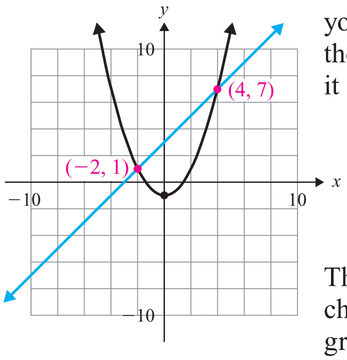
Two points of intersection
The system has two solutions.

Solve the linear equation for one of the variables.

$$\begin{aligned} -x + y &= 3 \\ y &= x + 3 \quad (3) \qquad \text{Solve for } y. \end{aligned}$$

Substitute $x + 3$ for y in equation (1).

$$\begin{aligned} x^2 - 2y &= 2 && \text{Equation (1).} \\ x^2 - 2(x + 3) &= 2 && \text{Substitute.} \\ x^2 - 2x - 6 &= 2 && \text{Distribute.} \\ x^2 - 2x - 8 &= 0 && \text{Subtract 2.} \\ (x - 4)(x + 2) &= 0 && \text{Factor.} \\ x = 4 \text{ or } x &= -2 && \text{Solve.} \end{aligned}$$



To find the corresponding value of y for each value of x, we can substitute $x = 4$ and then $x = -2$ into *either* equation (1), (2), or (3). No matter which equation you choose, you should always check the solutions in *both* of the original equations. We will substitute the values into equation (3) because this is just an alternative form of equation (2), and it is already solved for y.

Substitute each value into equation (3) to find y.

$x = 4$: $y = x + 3$ $y = 4 + 3$ $y = 7$	$x = -2$: $y = x + 3$ $y = -2 + 3$ $y = 1$

The proposed solutions are (4, 7) and (−2, 1). Verify that they solve the system by checking them in equation (1). The solution set is $\{(4, 7), (-2, 1)\}$. We can see on the graph to the left that these are the points of intersection of the graphs. ■

You Try 1

Solve the system $x^2 + 3y = 6$
$x + y = 2$

Example 2

Solve the system $x^2 + y^2 = 1$ (1)
$x + 2y = -1$ (2)

Solution

The graph of equation (1) is a circle, and the graph of equation (2) is a line. These graphs can intersect at zero, one, or two points. Therefore, this system will have zero, one, or two solutions.

We will not solve equation (1) for a variable because doing so would give us a radical in the expression. It will be easiest to solve equation (2) for x because its coefficient is 1.

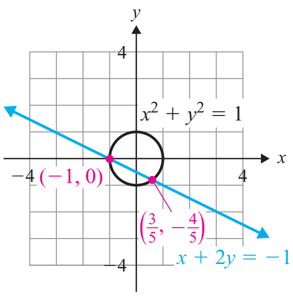
$$\begin{aligned} x + 2y &= -1 \quad (2) \\ x &= -2y - 1 \quad (3) \qquad \text{Solve for } x. \end{aligned}$$

Substitute $-2y - 1$ for x in equation (1).

$$
\begin{aligned}
x^2 + y^2 &= 1 && (1)\\
(-2y - 1)^2 + y^2 &= 1 && \text{Substitute.}\\
4y^2 + 4y + 1 + y^2 &= 1 && \text{Expand } (-2y-1)^2.\\
5y^2 + 4y &= 0 && \text{Combine like terms; subtract 1.}\\
y(5y + 4) &= 0 && \text{Factor.}\\
y = 0 \quad \text{or} \quad 5y + 4 &= 0 && \text{Set each factor equal to zero.}\\
y &= -\frac{4}{5} && \text{Solve for } y.
\end{aligned}
$$

Substitute $y = 0$ and then $y = -\frac{4}{5}$ into equation (3) to find their corresponding values of x.

$$
\begin{aligned}
y = 0:\ x &= -2y - 1\\
x &= -2(0) - 1\\
x &= -1
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
y = -\frac{4}{5}:\ x &= -2y - 1\\
x &= -2\left(-\frac{4}{5}\right) - 1\\
x &= \frac{8}{5} - 1 = \frac{3}{5}
\end{aligned}
$$



The proposed solutions are $(-1, 0)$ and $\left(\frac{3}{5}, -\frac{4}{5}\right)$. Check them in equations (1) and (2).

The solution set is $\left\{(-1, 0), \left(\frac{3}{5}, -\frac{4}{5}\right)\right\}$. The graph at left shows that these are the points of intersection of the graphs. ■

You Try 2

Solve the system $x^2 + y^2 = 25$
$x - y = 7$

Note

We must always check the proposed solutions in *each* equation in the system.

3. Solve a Nonlinear System Using the Elimination Method

The elimination method can be used to solve a system when both equations are second-degree equations.

Example 3

Solve the system
$$
\begin{aligned}
5x^2 + 3y^2 &= 21 \quad (1)\\
4x^2 - y^2 &= 10 \quad (2)
\end{aligned}
$$

Solution

Each equation is a second-degree equation. The first is an ellipse and the second is a hyperbola. They can have zero, one, two, three, or four points of intersection. Multiply equation (2) by 3. Then adding the two equations will eliminate the y^2-terms.

Original System		**Rewrite the System**
$5x^2 + 3y^2 = 21$ $4x^2 - y^2 = 10$	$\rightarrow$	$5x^2 + 3y^2 = 21$ $12x^2 - 3y^2 = 30$

$$\begin{aligned} 5x^2 + 3y^2 &= 21 \\ + \underline{12x^2 - 3y^2} &\underline{= 30} \\ 17x^2 &= 51 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \end{aligned}$$

Add the equations to eliminate y^2.

Find the corresponding values of y for $x = \sqrt{3}$ and $x = -\sqrt{3}$.

$x = \sqrt{3}$:

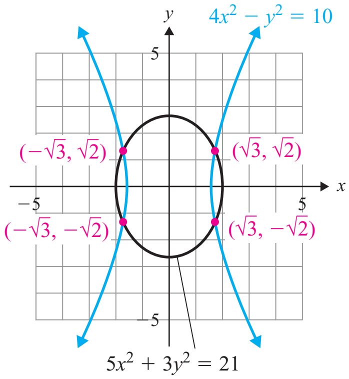
$$\begin{aligned} 4x^2 - y^2 &= 10 \quad (2) \\ 4(\sqrt{3})^2 - y^2 &= 10 \\ 12 - y^2 &= 10 \\ -y^2 &= -2 \\ y^2 &= 2 \\ y &= \pm\sqrt{2} \end{aligned}$$

This gives us $(\sqrt{3}, \sqrt{2})$ and $(\sqrt{3}, -\sqrt{2})$.

$x = -\sqrt{3}$:

$$\begin{aligned} 4x^2 - y^2 &= 10 \quad (2) \\ 4(-\sqrt{3})^2 - y^2 &= 10 \\ 12 - y^2 &= 10 \\ -y^2 &= -2 \\ y^2 &= 2 \\ y &= \pm\sqrt{2} \end{aligned}$$

This gives us $(-\sqrt{3}, \sqrt{2})$ and $(-\sqrt{3}, -\sqrt{2})$.



Check the proposed solutions in equation (1) to verify that they satisfy that equation as well.

The solution set is $\{(\sqrt{3}, \sqrt{2}), (\sqrt{3}, -\sqrt{2}), (-\sqrt{3}, \sqrt{2}), (-\sqrt{3}, -\sqrt{2})\}$. ■

You Try 3

Solve the system $2x^2 - 13y^2 = 20$
$-x^2 + 10y^2 = 4$

For solving some systems, using *either* substitution or the elimination method works well. Look carefully at each system to decide which method to use.

We will see in Example 4 that not all systems have solutions.

Example 4

Solve the system
$$\begin{aligned} y &= \sqrt{x} \quad (1) \\ y^2 - 4x^2 &= 4 \quad (2) \end{aligned}$$

Solution

The graph of the square root function $y = \sqrt{x}$ is half of a parabola. The graph of equation (2) is a hyperbola. Solve this system by substitution. Replace y in equation (2) with $\sqrt{x}$ from equation (1).

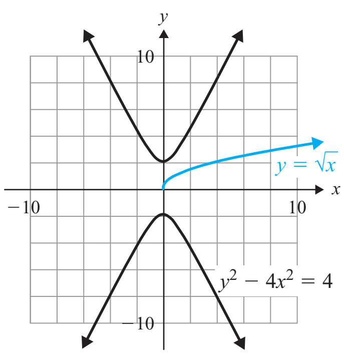
$$\begin{aligned} y^2 - 4x^2 &= 4 \quad (2) \\ (\sqrt{x})^2 - 4x^2 &= 4 \\ x - 4x^2 &= 4 \\ 0 &= 4x^2 - x + 4 \end{aligned}$$

Substitute $y = \sqrt{x}$ into equation (2).

Since the right-hand side does not factor, solve it using the quadratic formula.

$4x^2 - x + 4 = 0 \qquad a = 4 \qquad b = -1 \qquad c = 4$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(4)}}{2(4)} = \frac{1 \pm \sqrt{1 - 64}}{8} = \frac{1 \pm \sqrt{-63}}{8}$$



Since $\sqrt{-63}$ is not a real number, there are no real-number values for x. The system has no solution, so the solution set is $\varnothing$. The graph is shown in the margin. ■

You Try 4

Solve the system $\begin{aligned} 4x^2 + y^2 &= 4 \\ x - y &= 3 \end{aligned}$

Using Technology

We can solve systems of nonlinear equations on the graphing calculator just like we solved systems of linear equations in Chapter 5—graph the equations and find their points of intersection.

Let's look at Example 3:

$$\begin{aligned} 5x^2 + 3y^2 &= 21 \\ 4x^2 - y^2 &= 10 \end{aligned}$$

Solve each equation for y and enter them into the calculator.

Solve $5x^2 + 3y^2 = 21$ for y:

$$y = \pm\sqrt{7 - \frac{5}{3}x^2}$$

Enter $\sqrt{7 - \frac{5}{3}x^2}$ as Y_1.

Enter $-\sqrt{7 - \frac{5}{3}x^2}$ as Y_2.

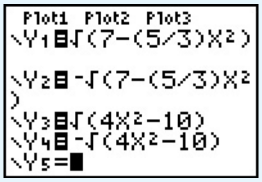
Solve $4x^2 - y^2 = 10$ for y:

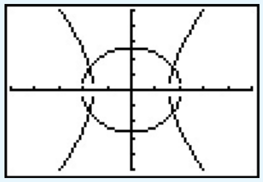
$$y = \pm\sqrt{4x^2 - 10}$$

Enter $\sqrt{4x^2 - 10}$ as Y_3.

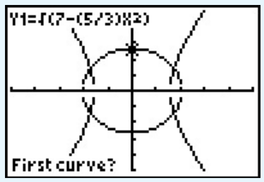
Enter $-\sqrt{4x^2 - 10}$ as Y_4.

After entering the equations, press $\boxed{\text{GRAPH}}$.

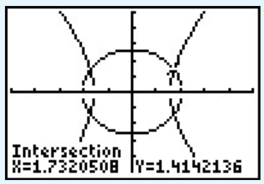




The system has four real solutions since the graphs have four points of intersection. We can use the INTERSECT option to find the solutions. Since we graphed four functions, we must tell the calculator which point of intersection we want to find. Note that the point where the graphs intersect in the first quadrant comes from the intersection of equations Y_1 and Y_3. Press $\boxed{\text{2nd}}$ $\boxed{\text{TRACE}}$ and choose 5:intersect and you will see the screen to the right.



Notice that the top left of the screen to the right displays the function Y_1. Since we want to find the intersection of Y_1 and Y_3, press $\boxed{\text{ENTER}}$ when Y_1 is displayed. Now Y_2 appears at the top left, but we do not need this function. Press the down arrow to see the equation for Y_3 and be sure that the cursor is close to the intersection point in quadrant I. Press $\boxed{\text{ENTER}}$ twice. You will see the approximate solution (1.732, 1.414), as shown to the right.



In Example 3 we found the exact solutions algebraically. The calculator solution, (1.732, 1.414), is an approximation of the exact solution $(\sqrt{3}, \sqrt{2})$.

The other solutions of the system can be found in the same way.

Use the graphing calculator to find all real-number solutions of each system. These are taken from the examples in the section and from the Chapter Summary.

1) $\begin{aligned} x^2 - 2y &= 2 \\ -x + y &= 3 \end{aligned}$

2) $\begin{aligned} x^2 + y^2 &= 1 \\ x + 2y &= -1 \end{aligned}$

3) $\begin{aligned} x - y^2 &= 3 \\ x - 2y &= 6 \end{aligned}$

4) $\begin{aligned} y &= \sqrt{x} \\ y^2 - 4x^2 &= 4 \end{aligned}$

Answers to You Try Exercises

1) $\{(0, 2), (3, -1)\}$ 2) $\{(4, -3), (3, -4)\}$ 3) $\{(6, 2), (6, -2), (-6, 2), (-6, -2)\}$ 4) $\varnothing$

Answers to Technology Exercises

1) $\{(4, 7), (-2, 1)\}$ 2) $\{(-1, 0), (0.6, -0.8)\}$ 3) $\{(12, 3), (4, -1)\}$ 4) $\varnothing$

14.3 Exercises

Objective 1: Define a Nonlinear System of Equations

If a nonlinear system consists of equations with the following graphs,

a) sketch the different ways in which the graphs can intersect.

b) make a sketch in which the graphs do not intersect.

c) how many possible solutions can each system have?

1) circle and line
2) parabola and line
3) parabola and ellipse
4) ellipse and hyperbola
5) parabola and hyperbola
6) circle and ellipse

Mixed Exercises: Objectives 2 and 3

Solve each system using either substitution or the elimination method.

7) $x^2 + 4y = 8$, $x + 2y = -8$

8) $x^2 + y = 1$, $-x + y = -5$

VIDEO 9) $x + 2y = 5$, $x^2 + y^2 = 10$

10) $y = 2$, $x^2 + y^2 = 8$

11) $y = x^2 - 6x + 10$, $y = 2x - 6$

12) $y = x^2 - 10x + 22$, $y = 4x - 27$

13) $x^2 + 2y^2 = 11$, $x^2 - y^2 = 8$

14) $2x^2 - y^2 = 7$, $2y^2 - 3x^2 = 2$

VIDEO 15) $x^2 + y^2 = 6$, $2x^2 + 5y^2 = 18$

16) $5x^2 - y^2 = 16$, $x^2 + y^2 = 14$

17) $3x^2 + 4y = -1$, $x^2 + 3y = -12$

18) $2x^2 + y = 9$, $y = 3x^2 + 4$

19) $y = 6x^2 - 1$, $2x^2 + 5y = -5$

20) $x^2 + 2y = 5$, $-3x^2 + 2y = 5$

21) $x^2 + y^2 = 4$, $-2x^2 + 3y = 6$

22) $x^2 + y^2 = 49$, $x - 2y^2 = 7$

VIDEO 23) $x^2 + y^2 = 3$, $x + y = 4$

24) $y - x = 1$, $4y^2 - 16x^2 = 64$

25) $x = \sqrt{y}$, $x^2 - 9y^2 = 9$

26) $x = \sqrt{y}$, $x^2 - y^2 = 4$

27) $9x^2 + y^2 = 9$, $x^2 + y^2 = 5$

28) $x^2 + y^2 = 6$, $5x^2 + y^2 = 10$

29) $y = -x^2 - 2$, $x^2 + y^2 = 4$

30) $x^2 + y^2 = 1$, $y = x^2 + 1$

Write a system of equations and solve.

31) Find two numbers whose product is 40 and whose sum is 13.

32) Find two numbers whose product is 28 and whose sum is 11.

VIDEO 33) The perimeter of a rectangular computer screen is 38 in. Its area is 88 in^2. Find the dimensions of the screen.

34) The area of a rectangular bulletin board is 180 in^2, and its perimeter is 54 in. Find the dimensions of the bulletin board.

35) A sporting goods company estimates that the cost y, in dollars, to manufacture x thousands of basketballs is given by

$$y = 6x^2 + 33x + 12$$

The revenue y, in dollars, from the sale of x thousands of basketballs is given by

$$y = 15x^2$$

The company breaks even on the sale of basketballs when revenue equals cost. The point, (x, y), at which this occurs is called the *break-even point*. Find the break-even point for the manufacture and sale of the basketballs.

36) A backpack manufacturer estimates that the cost y, in dollars, to make x thousands of backpacks is given by

$$y = 9x^2 + 30x + 18$$

The revenue y, in dollars, from the sale of x thousands of backpacks is given by

$$y = 21x^2$$

Find the break-even point for the manufacture and sale of the backpacks. (See Exercise 35 for explanation.)

Section 14.4 Quadratic and Rational Inequalities

Objectives

1. Solve a Quadratic Inequality by Graphing
2. Solve a Quadratic Inequality Using Test Points
3. Solve Quadratic Inequalities with Special Solutions
4. Solve an Inequality of Higher Degree
5. Solve a Rational Inequality

In Chapter 3, we learned how to solve *linear* inequalities such as $3x - 5 \leq 16$. In this section, we will discuss how to solve *quadratic* and *rational* inequalities.

Definition

A **quadratic inequality** can be written in the form

$$ax^2 + bx + c \leq 0 \quad \text{or} \quad ax^2 + bx + c \geq 0$$

where a, b, and c are real numbers and $a \neq 0$. ($<$ and $>$ may be substituted for $\leq$ and $\geq$.)

1. Solve a Quadratic Inequality by Graphing

To understand how to solve a quadratic inequality, let's look at the graph of a quadratic function.

Example 1

a) Graph $y = x^2 - 2x - 3$.
b) Solve $x^2 - 2x - 3 < 0$.
c) Solve $x^2 - 2x - 3 \geq 0$.

Solution

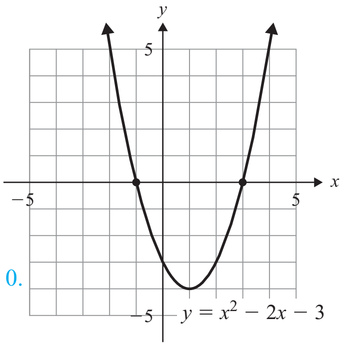
a) The graph of the quadratic function $y = x^2 - 2x - 3$ is a parabola that opens upward. Use the vertex formula to confirm that the vertex is $(1, -4)$.

To find the y-intercept, let $x = 0$ and solve for y.

$$y = 0^2 - 2(0) - 3$$
$$y = -3$$

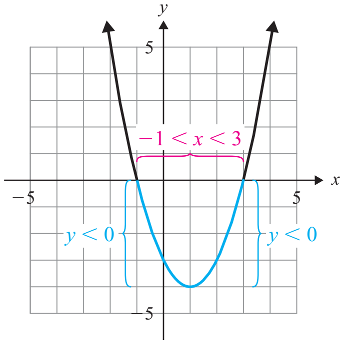
The y-intercept is $(0, -3)$.
To find the x-intercepts, let $y = 0$ and solve for x.

$$0 = x^2 - 2x - 3$$
$$0 = (x - 3)(x + 1) \quad \text{Factor.}$$
$$x - 3 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{Set each factor equal to 0.}$$
$$x = 3 \quad \text{or} \quad x = -1 \quad \text{Solve.}$$



b) We will use the graph of $y = x^2 - 2x - 3$ to solve the inequality $x^2 - 2x - 3 < 0$. That is, to solve $x^2 - 2x - 3 < 0$ we must ask ourselves, "Where are the *y-values* of the function *less than* zero?"

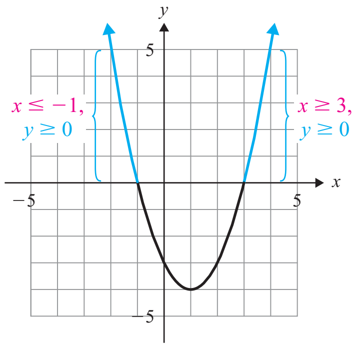
The y-values of the function are less than zero when the x-values are greater than -1 and less than 3, as shown to the right.



The solution set of $x^2 - 2x - 3 < 0$ (in interval notation) is $(-1, 3)$.

c) To solve $x^2 - 2x - 3 \geq 0$ means to find the x-values for which the *y-values* of the function $y = x^2 - 2x - 3$ are *greater than or equal to* zero. (Recall that the x-intercepts are where the function equals zero.)

The y-values of the function are greater than or equal to zero when $x \leq -1$ or $x \geq 3$. The solution set of $x^2 - 2x - 3 \geq 0$ is $(-\infty, -1] \cup [3, \infty)$.



When $x \leq -1$ or $x \geq 3$, the y-values are greater than or equal to 0.

You Try 1

a) Graph $y = x^2 + 6x + 5$. b) Solve $x^2 + 6x + 5 \leq 0$. c) Solve $x^2 + 6x + 5 > 0$.

2. Solve a Quadratic Inequality Using Test Points

Example 1 illustrates how the x-intercepts of $y = x^2 - 2x - 3$ break up the x-axis into the three separate intervals: $x < -1$, $-1 < x < 3$, and $x > 3$. We can use this idea of intervals to solve a quadratic inequality without graphing.

Example 2

Solve $x^2 - 2x - 3 < 0$.

Solution

Begin by solving the equation $x^2 - 2x - 3 = 0$.

$$\begin{aligned} x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 && \text{Factor.} \\ x - 3 = 0 \quad \text{or} \quad x + 1 &= 0 && \text{Set each factor equal to 0.} \\ x = 3 \quad \text{or} \quad x &= -1 && \text{Solve.} \end{aligned}$$

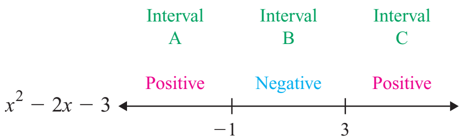
(These are the x-intercepts of $y = x^2 - 2x - 3$.)

Note

The $<$ indicates that we want to find the values of x that will make $x^2 - 2x - 3 < 0$; that is, find the values of x that make $x^2 - 2x - 3$ a *negative* number.

Put $x = 3$ and $x = -1$ on a number line with the smaller number on the left. This breaks up the number line into three intervals: $x < -1$, $-1 < x < 3$, and $x > 3$.

Choose a test number in each interval and substitute it into $x^2 - 2x - 3$ to determine whether that value makes $x^2 - 2x - 3$ positive or negative. (If one number in the interval makes $x^2 - 2x - 3$ positive, then *all* numbers in that interval will make $x^2 - 2x - 3$ positive.) Indicate the result on the number line.



Interval A: $(x < -1)$ As a test number, choose any number less than -1. We will choose -2. Evaluate $x^2 - 2x - 3$ for $x = -2$.

$$\begin{aligned} x^2 - 2x - 3 &= (-2)^2 - 2(-2) - 3 && \text{Substitute } -2 \text{ for } x. \\ &= 4 + 4 - 3 \\ &= 8 - 3 = 5 \end{aligned}$$

When $x = -2$, $x^2 - 2x - 3$ is *positive*. Therefore, $x^2 - 2x - 3$ will be positive for all values of x in this interval. Indicate this on the number line as seen above.

Interval B: $(-1 < x < 3)$ As a test number, choose any number between -1 and 3. We will choose 0. Evaluate $x^2 - 2x - 3$ for $x = 0$.

$$\begin{aligned} x^2 - 2x - 3 &= (0)^2 - 2(0) - 3 && \text{Substitute 0 for } x. \\ &= 0 - 0 - 3 = -3 \end{aligned}$$

When $x = 0$, $x^2 - 2x - 3$ is *negative*. Therefore, $x^2 - 2x - 3$ will be negative for all values of x in this interval. Indicate this on the number line above.

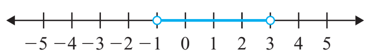
Interval C: $(x > 3)$ As a test number, choose any number greater than 3. We will choose 4. Evaluate $x^2 - 2x - 3$ for $x = 4$.

$$\begin{aligned} x^2 - 2x - 3 &= (4)^2 - 2(4) - 3 && \text{Substitute 4 for } x. \\ &= 16 - 8 - 3 \\ &= 8 - 3 = 5 \end{aligned}$$

When $x = 4$, $x^2 - 2x - 3$ is *positive*. Therefore, $x^2 - 2x - 3$ will be positive for all values of x in this interval. Indicate this on the number line.

Look at the number line. The solution set of $x^2 - 2x - 3 < 0$ consists of the interval(s) where $x^2 - 2x - 3$ is *negative*. This is in interval B, $(-1, 3)$.

The graph of the solution set is



The solution set is $(-1, 3)$. This is the same as the result we obtained in Example 1 by graphing. ■

You Try 2

Solve $x^2 + 5x + 4 \le 0$. Graph the solution set and write the solution in interval notation.

Next we will summarize how to solve a quadratic inequality.

Summary How to Solve a Quadratic Inequality

Step 1: **Write the inequality in the form $ax^2 + bx + c \le 0$ or $ax^2 + bx + c \ge 0$.** ($<$ and $>$ may be substituted for $\le$ and ≥ 0.) If the inequality symbol is $<$ or $\le$, we are looking for a *negative* quantity in the interval on the number line. If the inequality symbol is $>$ or $\ge$, we are looking for a *positive* quantity in the interval.

Step 2: **Solve** the equation $ax^2 + bx + c = 0$.

Step 3: **Put the solutions of $ax^2 + bx + c = 0$ on a number line.** These values break up the number line into intervals.

Step 4: **Choose a test number in each interval** to determine whether $ax^2 + bx + c$ is positive or negative in each interval. Indicate this on the number line.

Step 5: **If the inequality is in the form $ax^2 + bx + c \le 0$ or $ax^2 + bx + c < 0$, then the solution set contains the numbers in the interval where $ax^2 + bx + c$ is *negative*.**
If the inequality is in the form $ax^2 + bx + c \ge 0$ or $ax^2 + bx + c > 0$, then the solution set contains the numbers in the interval where $ax^2 + bx + c$ is *positive*.

Step 6: **If the inequality symbol is $\le$ or $\ge$, then the endpoints of the interval(s) (the numbers found in step 3) are included in the solution set.** Indicate this with brackets in the interval notation.
If the inequality symbol is $<$ or $>$, then the endpoints of the interval(s) are not included in the solution set. Indicate this with parentheses in interval notation.

3. Solve Quadratic Inequalities with Special Solutions

We should look carefully at the inequality before trying to solve it. Sometimes, it is not necessary to go through all of the steps.

Example 3

Solve.

a) $(y + 4)^2 \ge -5$ b) $(t - 8)^2 < -3$

Solution

a) The inequality $(y + 4)^2 \ge -5$ says that a squared quantity, $(y + 4)^2$, is greater than or equal to a *negative* number, -5. *This is always true.* (A squared quantity will *always* be greater than or equal to zero.) Any real number, y, will satisfy the inequality.

The solution set is (∞, ∞).

b) The inequality $(t - 8)^2 < -3$ says that a squared quantity, $(t - 8)^2$, is less than a *negative* number, -3. *There is no real number value for t so that* $(t - 8)^2 < -3$.

The solution set is $\varnothing$.

You Try 3

Solve.

a) $(k + 2)^2 \le -4$ b) $(z - 9)^2 > -1$

4. Solve an Inequality of Higher Degree

Other polynomial inequalities in factored form can be solved in the same way that we solve quadratic inequalities.

Example 4

Solve $(c - 2)(c + 5)(c - 4) < 0$.

Solution

This is the factored form of a third-degree polynomial. Since the inequality is $<$, the solution set will contain the intervals where $(c - 2)(c + 5)(c - 4)$ is *negative*.

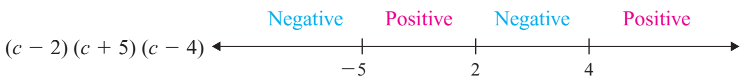
Solve $(c - 2)(c + 5)(c - 4) = 0$.

$$\begin{array}{lllll} c - 2 = 0 & \text{or} & c + 5 = 0 & \text{or} & c - 4 = 0 \\ c = 2 & \text{or} & c = -5 & \text{or} & c = 4 \end{array}$$

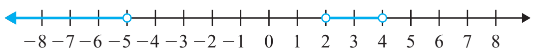
Set each factor equal to 0.
Solve.

Put $c = 2$, $c = -5$, and $c = 4$ on a number line, and test a number in each interval.

Interval	$c < -5$	$-5 < c < 2$	$2 < c < 4$	$c > 4$
Test number	$c = -6$	$c = 0$	$c = 3$	$c = 5$
Evaluate $(c - 2)(c + 5)(c - 4)$	$(-6 - 2)(-6 + 5)(-6 - 4)$ $= (-8)(-1)(-10)$ $= -80$	$(0 - 2)(0 + 5)(0 - 4)$ $= (-2)(5)(-4)$ $= 40$	$(3 - 2)(3 + 5)(3 - 4)$ $= (1)(8)(-1)$ $= -8$	$(5 - 2)(5 + 5)(5 - 4)$ $= (3)(10)(1)$ $= 30$
Sign	Negative	Positive	Negative	Positive



We can see that the intervals where $(c - 2)(c + 5)(c - 4)$ is negative are $(-\infty, -5)$ and $(2, 4)$. The endpoints are not included since the inequality is $<$.

The graph of the solution set is 

The solution set of $(c - 2)(c + 5)(c - 4) < 0$ is $(-\infty, -5) \cup (2, 4)$.

You Try 4

Solve $(y + 3)(y - 1)(y + 1) \geq 0$. Graph the solution set and write the solution in interval notation.

5. Solve a Rational Inequality

An inequality containing a rational expression, $\frac{p}{q}$, where p and q are polynomials, is called a **rational inequality**. The way we solve rational inequalities is very similar to the way we solve quadratic inequalities.

Procedure How to Solve a Rational Inequality

Step 1: **Write the inequality so that there is a 0 on one side and only one rational expression on the other side.** If the inequality symbol is $<$ or $\leq$, we are looking for a *negative* quantity in the interval on the number line. If the inequality symbol is $>$ or $\geq$, we are looking for a *positive* quantity in the interval.

Step 2: **Find the numbers that make the numerator equal 0 and any numbers that make the denominator equal 0.**

Step 3: **Put the numbers found in step 2 on a number line.** These values break up the number line into intervals.

Step 4: **Choose a test number in each interval** to determine whether the rational inequality is positive or negative in each interval. Indicate this on the number line.

Step 5: **If the inequality is in the form $\frac{p}{q} \leq 0$ or $\frac{p}{q} < 0$, then the solution set contains the numbers in the interval where $\frac{p}{q}$ is *negative*.**

If the inequality is in the form $\frac{p}{q} \geq 0$ or $\frac{p}{q} > 0$, then the solution set contains the numbers in the interval where $\frac{p}{q}$ is *positive*.

Step 6: **Determine whether the endpoints of the intervals are included in or excluded from the solution set.** Do not include any values that make the denominator equal 0.

Example 5

Solve $\dfrac{5}{x+3} > 0$.

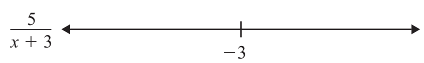
Solution

Step 1: The inequality is in the correct form—zero on one side and only one rational expression on the other side. Since the inequality symbol is > 0, the solution set will contain the interval(s) where $\dfrac{5}{x+3}$ is *positive*.

Step 2: Find the numbers that make the numerator equal 0 and any numbers that make the denominator equal 0.

Numerator: 5	Denominator: $x + 3$
The numerator is a constant, 5, so it cannot equal 0.	Set $x + 3 = 0$ and solve for x. $x + 3 = 0$ $x = -3$

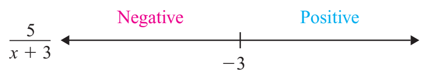
Step 3: Put -3 on a number line to break it up into intervals.



Step 4: Choose a test number in each interval to determine whether $\dfrac{5}{x+3}$ is positive or negative in each interval.

Interval	$x < -3$	$x > -3$
Test number	$x = -4$	$x = 0$
Evaluate $\dfrac{5}{x+3}$	$\dfrac{5}{-4+3} = \dfrac{5}{-1} = -5$	$\dfrac{5}{0+3} = \dfrac{5}{3}$
Sign	Negative	Positive

Step 5: The solution set of $\dfrac{5}{x+3} > 0$ contains the numbers in the interval where $\dfrac{5}{x+3}$ is positive. This interval is $(-3, \infty)$.



Step 6: Since the inequality symbol is $>$, the endpoint of the interval, -3, is not included in the solution set.

The graph of the solution set is 

The solution set is $(-3, \infty)$.

You Try 5

Solve $\dfrac{2}{y-6} < 0$. Graph the solution set and write the solution in interval notation.

Example 6

Solve $\dfrac{7}{a+2} \le 3$.

Solution

Step 1: Get a zero on one side and only one rational expression on the other side.

$$\frac{7}{a+2} \le 3$$

$$\frac{7}{a+2} - 3 \le 0 \qquad \text{Subtract 3.}$$

$$\frac{7}{a+2} - \frac{3(a+2)}{a+2} \le 0 \qquad \text{Get a common denominator.}$$

$$\frac{7}{a+2} - \frac{3a+6}{a+2} \le 0 \qquad \text{Distribute.}$$

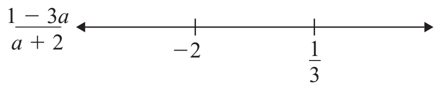
$$\frac{1-3a}{a+2} \le 0 \qquad \text{Combine numerators and combine like terms.}$$

From this point forward, we will work with the inequality $\dfrac{1-3a}{a+2} \le 0$. It is equivalent to the original inequality. Since the inequality symbol is $\le$, the solution set contains the interval(s) where $\dfrac{1-3a}{a+2}$ is *negative*.

Step 2: Find the numbers that make the numerator equal 0 and any numbers that make the denominator equal 0.

Numerator	Denominator
$1 - 3a = 0$ $-3a = -1$ $a = \dfrac{1}{3}$	$a + 2 = 0$ $a = -2$

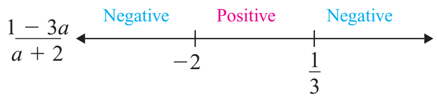
Step 3: Put $\frac{1}{3}$ and -2 on a number line to break it up into intervals.



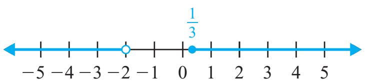
Step 4: Choose a test number in each interval.

Interval	$a < -2$	$-2 < a < \frac{1}{3}$	$a > \frac{1}{3}$
Test number	$a = -3$	$a = 0$	$a = 1$
Evaluate $\frac{1-3a}{a+2}$	$\frac{1-3(-3)}{-3+2} = \frac{10}{-1} = -10$	$\frac{1-3(0)}{0+2} = \frac{1}{2}$	$\frac{1-3(1)}{1+2} = -\frac{2}{3}$
Sign	Negative	Positive	Negative

Step 5: The solution set of $\frac{1-3a}{a+2} \le 0$ $\left(\text{and therefore } \frac{7}{a+2} \le 3\right)$ will contain the numbers in the interval where $\frac{1-3a}{a+2}$ is negative. These are the first and last intervals.



Step 6: Determine whether the endpoints of the intervals, -2 and $\frac{1}{3}$, are included in the solution set. The endpoint $\frac{1}{3}$ is included since it does not make the denominator equal 0. *But* -2 *is not included because it makes the denominator equal* 0.

The graph of the solution set of $\frac{7}{a+2} \le 3$ is 

The solution set is $(-\infty, -2) \cup \left[\frac{1}{3}, \infty\right)$.

BE CAREFUL

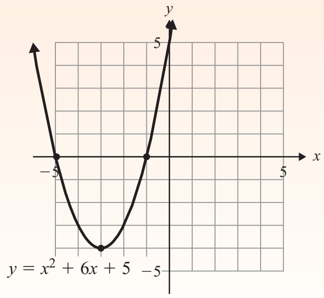
Although an inequality symbol may be $\le$ or $\ge$, an endpoint cannot be included in the solution set if it makes the denominator equal 0.

You Try 6

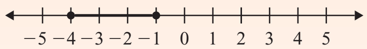
Solve $\frac{3}{z+4} \ge 2$. Graph the solution set and write the solution in interval notation.

Answers to You Try Exercises

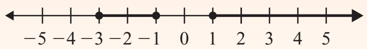
1) a)



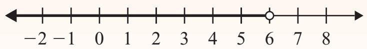
b) $[-5, -1]$ c) $(-\infty, -5) \cup (-1, \infty)$

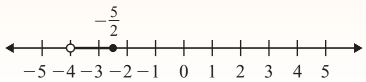
2)  ; $[-4, -1]$

3) a) $\varnothing$ b) $(-\infty, \infty)$

4)  ;

$[-3, -1] \cup [1, \infty)$

5)  ; $(-\infty, 6)$

6)  ; $\left(-4, -\frac{5}{2}\right]$

14.4 Exercises

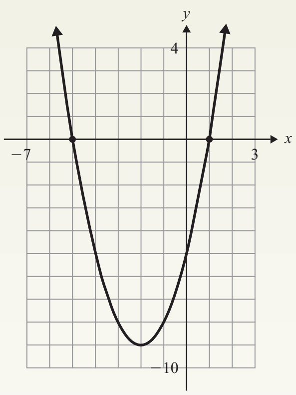
1) When solving a quadratic inequality, how do you know when to include and when to exclude the endpoints in the solution set?

2) If a rational inequality contains a $\leq$ or $\geq$ symbol, will the endpoints of the solution set always be included? Explain your answer.

Objective 1: Solve a Quadratic Inequality by Graphing

For Exercises 3–6, use the graph of the function to solve each inequality.

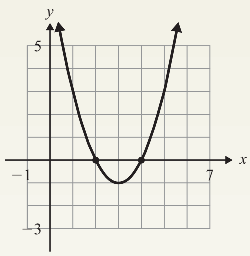
3) $y = x^2 + 4x - 5$



a) $x^2 + 4x - 5 \leq 0$

b) $x^2 + 4x - 5 > 0$

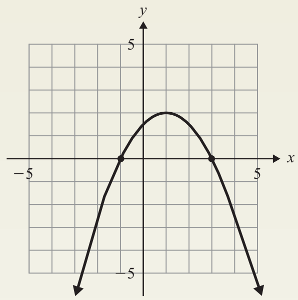
4) $y = x^2 - 6x + 8$



a) $x^2 - 6x + 8 > 0$

b) $x^2 - 6x + 8 \leq 0$

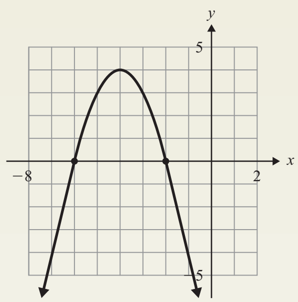
5) $y = -\frac{1}{2}x^2 + x + \frac{3}{2}$



a) $-\frac{1}{2}x^2 + x + \frac{3}{2} \geq 0$

b) $-\frac{1}{2}x^2 + x + \frac{3}{2} < 0$

6) $y = -x^2 - 8x - 12$



a) $-x^2 - 8x - 12 < 0$

b) $-x^2 - 8x - 12 \geq 0$

Objective 2: Solve a Quadratic Inequality Using Test Points

Solve each quadratic inequality. Graph the solution set and write the solution in interval notation.

7) $x^2 + 6x - 7 \geq 0$

8) $m^2 - 2m - 24 > 0$

9) $c^2 + 5c < 36$

10) $t^2 + 36 \leq 15t$

VIDEO 11) $r^2 - 13r > -42$

12) $v^2 + 10v < -16$

13) $3z^2 + 14z - 24 \leq 0$

14) $5k^2 + 36k + 7 \geq 0$

15) $7p^2 - 4 > 12p$

16) $4w^2 - 19w < 30$

17) $b^2 - 9b > 0$

18) $c^2 + 12c \leq 0$

19) $4y^2 \leq -5y$

20) $2a^2 \geq 7a$

21) $m^2 - 64 < 0$

22) $p^2 - 144 > 0$

23) $121 - h^2 \leq 0$

24) $1 - d^2 > 0$

VIDEO 25) $144 \geq 9s^2$

26) $81 \leq 25q^2$

Objective 3: Solve Quadratic Inequalities with Special Solutions

Solve each inequality.

27) $(k + 7)^2 \geq -9$

28) $(h + 5)^2 \geq -2$

29) $(3v - 11)^2 > -20$

30) $(r + 4)^2 < -3$

31) $(2y - 1)^2 < -8$

32) $(4d - 3)^2 > -1$

33) $(n + 3)^2 \leq -10$

34) $(5s - 2)^2 \leq -9$

Objective 4: Solve an Inequality of Higher Degree

Solve each inequality. Graph the solution set and write the solution in interval notation.

VIDEO 35) $(r + 2)(r - 5)(r - 1) \leq 0$

36) $(b + 2)(b - 3)(b - 12) > 0$

37) $(j - 7)(j - 5)(j + 9) \geq 0$

38) $(m + 4)(m - 7)(m + 1) \leq 0$

39) $(6c + 1)(c + 7)(4c - 3) < 0$

40) $(t + 2)(4t - 7)(5t - 1) \geq 0$

Objective 5: Solve a Rational Inequality

Solve each rational inequality. Graph the solution set and write the solution in interval notation.

41) $\dfrac{7}{p + 6} > 0$

42) $\dfrac{3}{v - 2} < 0$

43) $\dfrac{5}{z + 3} \leq 0$

44) $\dfrac{9}{m - 4} \geq 0$

45) $\dfrac{x - 4}{x - 3} > 0$

46) $\dfrac{a - 2}{a + 1} < 0$

VIDEO 47) $\dfrac{h - 9}{3h + 1} \leq 0$

48) $\dfrac{2c + 1}{c + 4} \geq 0$

49) $\dfrac{k}{k + 3} \leq 0$

50) $\dfrac{r}{r - 7} \geq 0$

51) $\dfrac{7}{t + 6} < 3$

52) $\dfrac{3}{x + 7} < -2$

53) $\dfrac{3}{a + 7} \geq 1$

54) $\dfrac{5}{w - 3} \leq 1$

VIDEO 55) $\dfrac{2y}{y - 6} \leq -3$

56) $\dfrac{3z}{z + 4} \geq 2$

57) $\dfrac{3w}{w + 2} > -4$

58) $\dfrac{4h}{h + 3} < 1$

59) $\dfrac{(6d + 1)^2}{d - 2} \leq 0$

60) $\dfrac{(x + 2)^2}{x + 7} \geq 0$

61) $\dfrac{(4t - 3)^2}{t - 5} > 0$

62) $\dfrac{(2y + 3)^2}{y + 3} < 0$

63) $\dfrac{n + 6}{n^2 + 4} < 0$

64) $\dfrac{b - 3}{b^2 + 2} > 0$

65) $\dfrac{m + 1}{m^2 + 3} \geq 0$

66) $\dfrac{w - 7}{w^2 + 8} \leq 0$

67) $\dfrac{s^2 + 2}{s - 4} \leq 0$

68) $\dfrac{z^2 + 10}{z + 6} \leq 0$

Mixed Exercises: Objectives 2 and 5

Write an inequality and solve.

69) Compu Corp. estimates that its total profit function, $P(x)$, for producing x thousand units is given by $P(x) = -2x^2 + 32x - 96$.

a) At what level of production does the company make a profit?

b) At what level of production does the company lose money?

70) A model rocket is launched from the ground with an initial velocity of 128 ft/s. The height $s(t)$, in feet, of the rocket t seconds after liftoff is given by the function $s(t) = -16t^2 + 128t$.

a) When is the rocket more than 192 feet above the ground?

b) When does the rocket hit the ground?

71) A designer purse company has found that the average cost, $\overline{C}(x)$, of producing x purses per month can be described by the function $\overline{C}(x) = \dfrac{10x + 100{,}000}{x}$. How many purses must the company produce each month so that the average cost of producing each purse is no more than \$20?

72) A company that produces clay pigeons for target shooting has determined that the average cost, $\overline{C}(x)$, of producing x cases of clay pigeons per month can be described by the function $\overline{C}(x) = \dfrac{2x + 15{,}000}{x}$. How many cases of clay pigeons must the company produce each month so that the average cost of producing each case is no more than \$3?

Chapter 14: Summary

Definition/Procedure	Example

14.1 The Circle

The Midpoint Formula

If (x_1, y_1) and (x_2, y_2) are the endpoints of a line segment, then the **midpoint** of the segments has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right). \textbf{ (p. 840)}$$

Find the midpoint of the line segment with endpoints $(-2, 5)$ and $(6, 3)$.

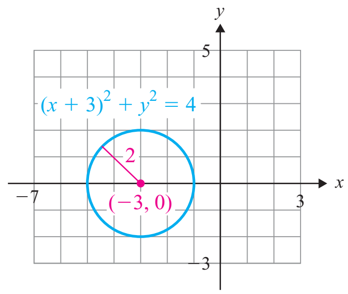
$$\text{Midpoint} = \left(\frac{-2 + 6}{2}, \frac{5 + 3}{2}\right) = \left(\frac{4}{2}, \frac{8}{2}\right) = (2, 4)$$

Parabolas, circles, ellipses, and hyperbolas are called **conic sections.** The **standard form for the equation of a circle** with center (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2 \quad \textbf{(p. 841)}$$

Graph $(x + 3)^2 + y^2 = 4$.

The center is $(-3, 0)$. The radius is $\sqrt{4} = 2$.



The **general form for the equation of a circle** is

$$Ax^2 + Ay^2 + Cx + Dy + E = 0$$

where A, C, D, and E are real numbers.

To rewrite the equation in the form $(x - h)^2 + (y - k)^2 = r^2$, divide the equation by A so that the coefficient of each squared term is 1, then complete the square on x and on y to put it into standard form. **(p. 843)**

Write $x^2 + y^2 - 16x + 4y + 67 = 0$ in the form $(x - h)^2 + (y - k)^2 = r^2$.

Group the x-terms together and group the y-terms together.

$$(x^2 - 16x) + (y^2 + 4y) = -67$$

Complete the square for each group of terms.

$$(x^2 - 16x + 64) + (y^2 + 4y + 4) = -67 + 64 + 4$$

$$(x - 8)^2 + (y + 2)^2 = 1$$

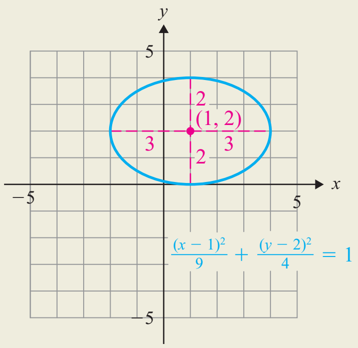
14.2 The Ellipse and the Hyperbola

The **standard form for the equation of an ellipse** is

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

The center of the ellipse is (h, k). **(p. 847)**

Graph $\frac{(x - 1)^2}{9} + \frac{(y - 2)^2}{4} = 1$.



The center is $(1, 2)$.
$a = \sqrt{9} = 3$
$b = \sqrt{4} = 2$

Definition/Procedure	Example

Standard Form for the Equation of a Hyperbola

1) A hyperbola with center (h, k) with branches that open in the *x-direction* has equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

2) A hyperbola with center (h, k) with branches that open in the *y-direction* has equation

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

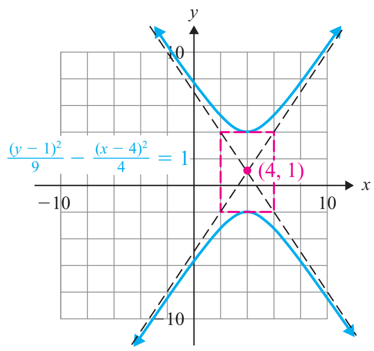
Notice in 1) that $\frac{(x-h)^2}{a^2}$ is the positive quantity, and the branches open in the x-direction.

In 2), the positive quantity is $\frac{(y-k)^2}{b^2}$, and the branches open in the y-direction. **(p. 850)**

Graph $\frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1$.

The center is $(4, 1)$, $a = \sqrt{4} = 2$, and $b = \sqrt{9} = 3$.

Use the center, $a = 2$, and $b = 3$ to draw the reference rectangle. The diagonals of the rectangle are the asymptotes of the hyperbola.



14.3 Nonlinear Systems of Equations

A **nonlinear system of equations** is a system in which at least one of the equations is not linear. We can solve nonlinear systems by substitution or the elimination method. **(p. 861)**

Solve
$$x - y^2 = 3 \quad (1)$$
$$x - 2y = 6 \quad (2)$$

$$x - y^2 = 3 \quad (1) \qquad \text{Solve equation}$$
$$x = y^2 + 3 \quad (3) \qquad \text{(1) for } x.$$

Substitute $x = y^2 + 3$ into equation (2).

$$\begin{aligned}
(y^2 + 3) - 2y &= 6 && \\
y^2 - 2y - 3 &= 0 && \text{Subtract 6.} \\
(y-3)(y+1) &= 0 && \text{Factor.} \\
y - 3 = 0 \quad \text{or} \quad y + 1 &= 0 && \text{Set each factor equal to 0.} \\
y = 3 \quad \text{or} \quad y &= -1 && \text{Solve.}
\end{aligned}$$

Substitute each value into equation (3).

$y = 3$:	$y = -1$:
$x = y^2 + 3$	$x = y^2 + 3$
$x = (3)^2 + 3$	$x = (-1)^2 + 3$
$x = 12$	$x = 4$

The proposed solutions are $(12, 3)$ and $(4, -1)$.
Verify that they also satisfy (2).
The solution set is $\{(12, 3), (4, -1)\}$.

Definition/Procedure	Example

14.4 Quadratic and Rational Inequalities

A **quadratic inequality** can be written in the form

$$ax^2 + bx + c \le 0 \quad \text{or} \quad ax^2 + bx + c \ge 0$$

where a, b, and c are real numbers and $a \ne 0$. ($<$ and $>$ may be substituted for $\le$ and $\ge$.)

An inequality containing a rational expression, like $\dfrac{c-5}{c+1} \le 0$, is called a **rational inequality**. **(p. 867)**

Solve $r^2 - 4r \ge 12$.

Step 1: $r^2 - 4r - 12 \ge 0$ Subtract 12.

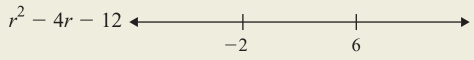
Since the inequality symbol is $\ge$, the solution set will contain the interval(s) where the quantity $r^2 - 4r - 12$ is *positive*.

Step 2: Solve $r^2 - 4r - 12 = 0$.

$$(r-6)(r+2) = 0 \quad \text{Factor.}$$

$$r - 6 = 0 \quad \text{or} \quad r + 2 = 0$$
$$r = 6 \quad \text{or} \quad r = -2$$

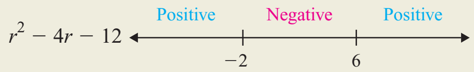
Step 3: Put $r = 6$ and $r = -2$ on a number line.



The solution set will contain the intervals where $r^2 - 4r - 12$ is positive.

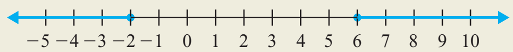
Step 4: Choose a test number in each interval to determine the sign of $r^2 - 4r - 12$.

Step 5: The solution set will contain the numbers in the intervals where $r^2 - 4r - 12$ is positive.



Step 6: The endpoints of the intervals are included since the inequality is $\ge$.

The graph of the solution set is



The solution set of $r^2 - 4r - 12$ is $(-\infty, -2] \cup [6, \infty)$.

Chapter 14: Review Exercises

(14.1) Find the midpoint of the line segment with the given endpoints.

1) $(3, 8)$ and $(5, 2)$

2) $(-6, 1)$ and $(-2, -1)$

3) $(7, -3)$ and $(6, -4)$

4) $\left(\frac{2}{3}, \frac{1}{4}\right)$ and $\left(-\frac{1}{6}, \frac{5}{8}\right)$

Identify the center and radius of each circle and graph.

5) $(x + 3)^2 + (y - 5)^2 = 36$

6) $x^2 + (y + 4)^2 = 9$

7) $x^2 + y^2 - 10x - 4y + 13 = 0$

8) $x^2 + y^2 + 4x + 16y + 52 = 0$

Find an equation of the circle with the given center and radius.

9) Center $(3, 0)$; radius $= 4$

10) Center $(1, -5)$; radius $= \sqrt{7}$

(14.2)

11) How can you distinguish between the equation of an ellipse and the equation of a hyperbola?

12) When is an ellipse also a circle?

Identify the center of the ellipse and graph the equation.

13) $\dfrac{x^2}{25} + \dfrac{y^2}{36} = 1$

14) $\dfrac{(x + 3)^2}{9} + \dfrac{(y - 3)^2}{4} = 1$

15) $(x - 4)^2 + \dfrac{(y - 2)^2}{16} = 1$

16) $25x^2 + 4y^2 = 100$

Identify the center of the hyperbola and graph the equation.

17) $\dfrac{y^2}{9} - \dfrac{x^2}{25} = 1$

18) $\dfrac{(y - 3)^2}{4} - \dfrac{(x + 2)^2}{9} = 1$

19) $\dfrac{(x + 1)^2}{4} - \dfrac{(y + 2)^2}{4} = 1$

20) $16x^2 - y^2 = 16$

(14.1–14.2) Determine whether the graph of each equation is a parabola, circle, ellipse, or hyperbola. Then, graph each equation.

21) $x^2 + 9y^2 = 9$

22) $x^2 + y^2 = 25$

23) $x = -y^2 + 6y - 5$

24) $x^2 - y = 3$

25) $\dfrac{(x - 3)^2}{16} - \dfrac{(y - 4)^2}{25} = 1$

26) $\dfrac{(x + 3)^2}{16} + \dfrac{(y + 1)^2}{25} = 1$

27) $x^2 + y^2 - 2x + 2y - 2 = 0$

28) $4y^2 - 9x^2 = 36$

29) $y = \frac{1}{2}(x + 2)^2 + 1$

30) $x^2 + y^2 - 6x - 8y + 16 = 0$

Graph each function. Identify the domain and range.

31) $h(x) = 2\sqrt{1 - \dfrac{x^2}{9}}$

32) $f(x) = -\sqrt{4 - x^2}$

(14.3)

33) If a nonlinear system of equations consists of an ellipse and a hyperbola, how many possible solutions can the system have?

34) If a nonlinear system of equations consists of a line and a circle, how many possible solutions can the system have?

Solve each system.

35) $-4x^2 + 3y^2 = 3$
$7x^2 - 5y^2 = 7$

36) $y - x^2 = 7$
$3x^2 + 4y = 28$

37) $y = 3 - x^2$
$x - y = -1$

38) $x^2 + y^2 = 9$
$8x + y^2 = 21$

39) $4x^2 + 9y^2 = 36$
$y = \frac{1}{3}x - 5$

40) $4x + 3y = 0$
$4x^2 + 4y^2 = 25$

Write a system of equations and solve.

41) Find two numbers whose product is 36 and whose sum is 13.

42) The perimeter of a rectangular window is 78 in., and its area is 378 in^2. Find the dimensions of the window.

(14.4) Solve each inequality. Graph the solution set and write the solution in interval notation.

43) $a^2 + 2a - 3 < 0$

44) $4m^2 + 8m \geq 21$

45) $6h^2 + 7h > 0$

46) $64v^2 \geq 25$

47) $36 - r^2 > 0$

48) $(5c + 2)(c - 4)(3c - 1) < 0$

49) $(6x - 5)^2 \geq -2$

50) $(p - 6)^2 \leq -5$

51) $\dfrac{t + 7}{2t - 3} > 0$

52) $\dfrac{6}{g - 7} \leq 0$

53) $\dfrac{4w + 3}{5w - 6} \leq 0$

54) $\dfrac{z}{z - 2} \leq 3$

55) $\dfrac{1}{n - 4} > -3$

56) $\dfrac{(3y - 4)^2}{y - 1} > 0$

57) $\dfrac{r^2 + 4}{r - 7} \geq 0$

58) $\dfrac{2k + 1}{k^2 + 5} \leq 0$

Chapter 14: Test

1) Find the midpoint of the line segment with endpoints $(2, 1)$ and $(10, -6)$.

Determine whether the graph of each equation is a parabola, circle, ellipse, or hyperbola. Then, graph each equation.

2) $\dfrac{(x-2)^2}{25} + \dfrac{(y+3)^2}{4} = 1$

3) $y = -2x^2 + 6$

4) $y^2 - 4x^2 = 16$

5) $x^2 + (y-1)^2 = 9$

6) Write $x^2 + y^2 + 2x - 6y - 6 = 0$ in the form $(x-h)^2 + (y-k)^2 = r^2$. Identify the center, radius, and graph the equation.

7) Write an equation of the circle with center $(5, 2)$ and radius $\sqrt{11}$.

8) The Colosseum in Rome is an ellipse measuring 188 m long and 156 m wide. If the Colosseum is represented on a Cartesias coordinate system with the center of the ellipse at the origin and the longer axis along the x-axis, write an equation of this elliptical structure.

(www.romaviva.com/Colosseo/colosseum.htm)

9) Graph $f(x) = -\sqrt{25 - x^2}$. Identify the domain and range.

10) If a nonlinear system consists of the equation of a parabola and a circle,

a) sketch the different ways in which the graphs can intersect.

b) make a sketch in which the graphs do not intersect.

c) how many possible solutions can the system have?

Solve each system.

11) $\begin{aligned} x - 2y^2 &= -1 \\ x + 4y &= -1 \end{aligned}$

12) $\begin{aligned} 2x^2 + 3y^2 &= 21 \\ -x^2 + 12y^2 &= 3 \end{aligned}$

13) $\begin{aligned} x^2 + y^2 &= 7 \\ 3x - 2y^2 &= 0 \end{aligned}$

14) The perimeter of a rectangular picture frame is 44 in. The area is 112 in^2. Find the dimensions of the frame.

Solve each inequality. Graph the solution set and write the solution in interval notation.

15) $y^2 + 4y - 45 \ge 0$

16) $2w^2 + 11w < -12$

17) $49 - 9p^2 \le 0$

18) $\dfrac{m-5}{m+3} \ge 0$

19) $\dfrac{6}{n-2} > 2$

Cumulative Review: Chapters 1–14

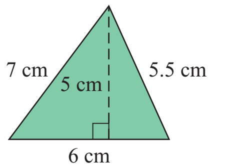
Perform the indicated operations and simplify.

1) $\frac{1}{6} - \frac{11}{12}$

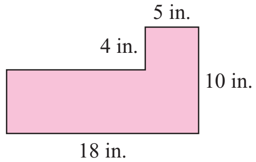
2) $16 + 20 \div 4 - (5 - 2)^2$

Find the area and perimeter of each figure.

3)



4)



Evaluate.

5) $(-1)^5$

6) 2^4

7) Simplify $\left(\frac{2a^8b}{a^2b^{-4}}\right)^{-3}$

8) Solve $\frac{3}{8}k + 11 = -4$.

9) Solve for n. $an + z = c$

10) Solve $8 - 5p \le 28$

11) *Write an equation and solve.*
The sum of three consecutive odd integers is 13 more than twice the largest integer. Find the numbers.

12) Find the slope of the line containing the points $(-6, 4)$ and $(2, -4)$.

13) What is the slope of the line with equation $y = 3$?

14) Graph $y = -2x + 5$.

15) Write the slope-intercept form of the line containing the points $(-4, 7)$ and $(4, 1)$.

16) Solve the system $\begin{aligned} 3x + 4y &= 3 \\ 5x + 6y &= 4 \end{aligned}$

17) *Write a system of equations and solve.*
How many milliliters of an 8% alcohol solution and how many milliliters of a 16% alcohol solution must be mixed to make 20 mL of a 14% alcohol solution?

18) Subtract $5p^2 - 8p + 4$ from $2p^2 - p + 10$.

19) Multiply and simplify. $(4w - 3)(2w^2 + 9w - 5)$

20) Divide. $(x^3 - 7x - 36) \div (x - 4)$

Factor completely.

21) $6c^2 - 14c + 8$

22) $m^3 - 8$

23) Solve $(x + 1)(x + 2) = 2(x + 7) + 5x$.

24) Multiply and simplify. $\frac{a^2 + 3a - 54}{4a + 36} \cdot \frac{10}{36 - a^2}$

25) Simplify $\dfrac{\frac{t^2 - 9}{4}}{\frac{t - 3}{24}}$.

26) Solve $|3n + 11| = 7$.

27) Solve $|5r + 3| > 12$.

Simplify. Assume all variables represent nonnegative real numbers.

28) $\sqrt{75}$

29) $\sqrt[3]{48}$

30) $\sqrt[3]{27a^5b^{13}}$

31) $(16)^{-3/4}$

32) $\frac{18}{\sqrt{12}}$

33) Rationalize the denominator of $\frac{5}{\sqrt{3} + 4}$.

Solve.

34) $(2p - 1)^2 + 16 = 0$

35) $y^2 = -7y - 3$

36) Given the relation $\{(4, 0), (3, 1), (3, -1), (0, 2)\}$,

a) what is the domain?

b) what is the range?

c) is the relation a function?

37) Graph $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x + 3}$ on the same axes.

38) $f(x) = 3x - 7$ and $g(x) = 2x + 5$.

a) Find $g(-4)$.

b) Find $(g \circ f)(x)$.

c) Find x so that $f(x) = 11$.

39) Given the function
$f = \{(-5, 9), (-2, 11), (3, 14), (7, 9)\}$,

a) is f one-to-one?

b) does f have an inverse?

40) Find an equation of the inverse of $f(x) = \frac{1}{3}x + 4$.

Solve.

41) $8^{5t} = 4^{t-3}$

42) $\log_3(4n - 11) = 2$

43) Evaluate log 100.

44) Graph $f(x) = \log_2 x$

45) Solve $e^{3k} = 8$. Give an exact solution and an approximation to four decimal places.

46) Graph $\frac{y^2}{4} - \frac{x^2}{9} = 1$.

47) Graph $x^2 + y^2 - 2x + 6y - 6 = 0$.

48) Solve the system. $\begin{aligned} y - 5x^2 &= 3 \\ x^2 + 2y &= 6 \end{aligned}$

49) Solve $25p^2 \le 144$.

50) Solve $\frac{t - 3}{2t + 5} > 0$.

CHAPTER 15

Sequences and Series

Algebra at Work: Finance

Many people deposit money in an account on a regular basis to save for their children's college educations. The formula used to determine how much money will be in such an account results from a geometric series since it is found from a sum of deposits and interest payments over a period of time.

Mr. and Mrs. Chesbit consult a financial planner, George, for advice on how much money they will save for their daughter's college education if they deposit \$200 per month for 18 yr into an account earning 6% annual interest compounded monthly. Using a formula based on a geometric series, George calculates that they will have saved \$77,470.64. Of this amount, \$43,200 is money they have invested, and the rest is interest.

We will learn more about geometric series in this chapter.

Section 15.1 Sequences and Series

Objectives

1. Define Infinite Sequence, Finite Sequence, and General Term of a Sequence
2. Write the Terms of a Sequence
3. Find the General Term of a Sequence
4. Solve an Applied Problem Using a Sequence
5. Understand the Meaning of Series and Summation Notation
6. Evaluate a Series
7. Write a Series Using Summation Notation
8. Use Summation Notation to Represent the Average of a Group of Numbers

1. Define Infinite Sequence, Finite Sequence, and General Term of a Sequence

Suppose that a math class is conducting an experiment. On the first day, 2 pennies will be placed in a jar. On the second day, 4 pennies will be placed in the jar. On the third day, students will put 6 pennies in the jar, on the fourth day they will put 8 pennies in the jar, and so on. The number of pennies placed in the jar each day forms a **sequence**.

If the students continue to deposit pennies into the jar in this way indefinitely, we obtain the sequence

$$2, 4, 6, 8, 10, 12, 14, 16, \ldots$$

This sequence has an infinite number of terms and is called an *infinite sequence*.

Suppose, however, that the students will stop putting pennies in the jar after 5 days. Then we obtain the sequence

$$2, 4, 6, 8, 10$$

This sequence has a finite number of terms and is called a *finite sequence*.

The number of pennies placed in the jar on a given day is related to how many days into the experiment the class is.

$$\begin{array}{ccccc} 2 & 4 & 6 & 8 & 10 \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \text{Day 1} & \text{Day 2} & \text{Day 3} & \text{Day 4} & \text{Day 5} \end{array}$$

The number of pennies placed in the jar on day n is 2n, where n is a natural number beginning with 1.

Each number in the sequence 2, 4, 6, 8, 10, . . . is called a **term** of the sequence. The terms of a sequence are related to the set of natural numbers, and a sequence is a function.

Definition

An **infinite sequence** is a function whose domain is the set of natural numbers.

A **finite sequence** is a function whose domain is the set of the first n natural numbers.

Previously, we used the notation $f(x)$ to denote a function. When describing sequences, however, we use the notation a_n (read *a sub n*).

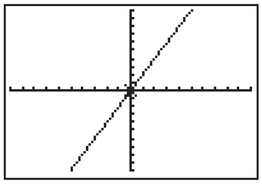
To describe the sequence 2, 4, 6, 8, 10, . . . , we use the notation $a_n = 2n$.
The first term of the sequence is denoted by a_1, the second term is denoted by a_2, the third term by a_3, and so on. Therefore, using the formula $a_n = 2n$

$$\begin{aligned} a_1 &= 2(1) = 2 && \text{Let } n = 1 \text{ to find the first term.} \\ a_2 &= 2(2) = 4 && \text{Let } n = 2 \text{ to find the second term.} \\ a_3 &= 2(3) = 6 && \text{Let } n = 3 \text{ to find the third term.} \\ a_4 &= 2(4) = 8 && \text{Let } n = 4 \text{ to find the fourth term.} \\ &\text{etc.} \end{aligned}$$

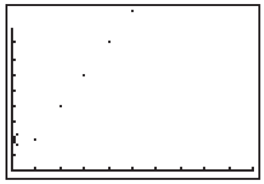
The nth term of the sequence, a_n, is called the **general term** of the sequence. Therefore, the general term of the sequence 2, 4, 6, 8, 10, . . . is $a_n = 2n$.

What is the difference between the two functions $f(x) = 2x$ and $a_n = 2n$? The domain of $f(x) = 2x$ is the set of real numbers, while the domain of $a_n = 2n$ is the set of natural numbers.

We can see how their graphs differ. The graphing calculator boxes illustrate that $f(x) = 2x$ is a line, while $a_n = 2n$ consists of function values found when n is a *natural number* so that the points are not connected.



The graph of $f(x) = 2x$



The graph of the first five terms of $a_n = 2n$

2. Write the Terms of a Sequence

Example 1

Write the first five terms of each sequence with general term a_n.

a) $a_n = 4n - 1$ b) $a_n = 3 \cdot \left(\frac{1}{2}\right)^n$ c) $a_n = (-1)^{n+1} \cdot 2n$

Solution

a) Evaluate $a_n = 4n - 1$ for $n = 1, 2, 3, 4$, and 5.

n	$a_n = 4n - 1$
1	$a_1 = 4(1) - 1 = 3$
2	$a_2 = 4(2) - 1 = 7$
3	$a_3 = 4(3) - 1 = 11$
4	$a_4 = 4(4) - 1 = 15$
5	$a_5 = 4(5) - 1 = 19$

The first five terms of the sequence are 3, 7, 11, 15, 19.

b) Evaluate $a_n = 3 \cdot \left(\frac{1}{2}\right)^n$ for $n = 1, 2, 3, 4$, and 5.

n	$a_n = 3 \cdot \left(\frac{1}{2}\right)^n$
1	$a_1 = 3 \cdot \left(\frac{1}{2}\right)^1 = \frac{3}{2}$
2	$a_2 = 3 \cdot \left(\frac{1}{2}\right)^2 = \frac{3}{4}$
3	$a_3 = 3 \cdot \left(\frac{1}{2}\right)^3 = \frac{3}{8}$
4	$a_4 = 3 \cdot \left(\frac{1}{2}\right)^4 = \frac{3}{16}$
5	$a_5 = 3 \cdot \left(\frac{1}{2}\right)^5 = \frac{3}{32}$

The first five terms of the sequence are $\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}$.

c) Evaluate $a_n = (-1)^{n+1} \cdot 2n$ for $n = 1, 2, 3, 4,$ and 5.

n	$a_n = (-1)^{n+1} \cdot 2n$
1	$a_1 = (-1)^{1+1} \cdot 2(1) = (-1)^2 \cdot 2 = 2$
2	$a_2 = (-1)^{2+1} \cdot 2(2) = (-1)^3 \cdot 4 = -4$
3	$a_3 = (-1)^{3+1} \cdot 2(3) = (-1)^4 \cdot 6 = 6$
4	$a_4 = (-1)^{4+1} \cdot 2(4) = (-1)^5 \cdot 8 = -8$
5	$a_5 = (-1)^{5+1} \cdot 2(5) = (-1)^6 \cdot 10 = 10$

The first five terms of the sequence are $2, -4, 6, -8, 10$. Notice that the terms of this sequence have alternating signs. A sequence in which the signs of the terms alternate is called an **alternating sequence**.

You Try 1

Write the first five terms of each sequence with general term a_n.

a) $a_n = 2n - 5$ b) $a_n = \left(\frac{1}{3}\right)^{n-1}$ c) $a_n = (-1)^n \cdot n$

Example 2

The general term of a sequence is given by $a_n = (-1)^n \cdot (2n + 1)$. Find each of the following.

a) the first term of the sequence

b) a_6

c) the 49th term of the sequence

Solution

a) The first term of the sequence is a_1. To find a_1, let $n = 1$ and evaluate.

$$\begin{aligned} a_n &= (-1)^n \cdot (2n + 1) \\ a_1 &= (-1)^1 \cdot [2(1) + 1] \qquad \text{Substitute 1 for } n. \\ a_1 &= -1 \cdot (3) \\ a_1 &= -3 \end{aligned}$$

b) a_6 is the sixth term of the sequence. Substitute 6 for n and evaluate.

$$\begin{aligned} a_n &= (-1)^n \cdot (2n + 1) \\ a_6 &= (-1)^6 \cdot [2(6) + 1] \qquad \text{Substitute 6 for } n. \\ a_6 &= 1 \cdot (13) \\ a_6 &= 13 \end{aligned}$$

c) The 49th term of the sequence is a_{49}. To find a_{49}, let $n = 49$ and evaluate.

$$\begin{aligned} a_n &= (-1)^n \cdot (2n + 1) \\ a_{49} &= (-1)^{49} \cdot [2(49) + 1] \qquad \text{Substitute 49 for } n. \\ a_{49} &= -1 \cdot (99) \\ a_{49} &= -99 \end{aligned}$$

You Try 2

The general term of a sequence is given by $a_n = \dfrac{(-1)^{n+1}}{5n}$. Find each of the following.

a) the first term of the sequence b) a_{10} c) the 32nd term of the sequence

3. Find the General Term of a Sequence

Next we will look at the opposite procedure. Given the first few terms of a sequence we will find a formula for the general term, a_n. To do this, look for a pattern and try to find a relationship between the term and the term number.

Example 3

Find a formula for the general term, a_n, of each sequence.

a) $1, 2, 3, 4, 5, \ldots$ b) $7, 14, 21, 28, 35, \ldots$

c) $\dfrac{1}{1}, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \dfrac{1}{25}, \ldots$ d) $-3, 9, -27, 81, -243, \ldots$

Solution

a) It is helpful to write each term with its term number below. Ask yourself, *"What is the relationship between the term and the term number?"*

Term:	1	2	3	4	5
Term number:	a_1	a_2	a_3	a_4	a_5

Each term is the same as the subscript on its term number. The nth term can be written as $a_n = n$.

b) Write each term with its term number below. *What is the relationship between the term and the term number?*

Term:	7	14	21	28	35
Term number:	a_1	a_2	a_3	a_4	a_5

Each term is 7 times the subscript on its term number. The nth term may be written as $a_n = 7n$.

c) Write each term with its term number below. *What is the relationship between the term and the term number?*

Term:	$\frac{1}{1}$	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$
Term number:	a_1	a_2	a_3	a_4	a_5

Each numerator is 1. Each denominator is the square of the subscript on its term number. The nth term may be written as $a_n = \dfrac{1}{n^2}$.

d) Write each term with its term number below. *What is the relationship between the term and the term number?*

Term:	-3	9	-27	81	-243
Term number:	a_1	a_2	a_3	a_4	a_5

The terms alternate in sign with the first term being negative. $(-1)^n$ will give us the desired alternating signs.

Disregarding the signs, each term is a power of 3: $3 = 3^1, 9 = 3^2, 27 = 3^3, 81 = 3^4, 243 = 3^5$, with the exponent being the subscript of the term number.

The nth term may be written as $a_n = (-1)^n \cdot 3^n$. ■

BE CAREFUL

There may be more than one general term, a_n, that produces the first few terms of a sequence.

You Try 3

Find a formula for the general term, a_n, of each sequence.

a) $11, 22, 33, 44, 55, \ldots$ b) $5, 6, 7, 8, 9, \ldots$

c) $\frac{1}{6}, \frac{1}{12}, \frac{1}{18}, \frac{1}{24}, \frac{1}{30}, \ldots$ d) $4, -16, 64, -256, 1024, \ldots$

4. Solve an Applied Problem Using a Sequence

Sequences can be used to model many real-life situations.

Example 4

A university anticipates that it will need to increase its tuition 10% per year for the next several years. If tuition is currently \$8000 per year, write a sequence that represents the amount of tuition students will pay each of the next 4 years.

Solution

Next year Tuition: $\$8000 + 0.10(\$8000) = \$8800$
In 2 years Tuition: $\$8800 + 0.10(\$8800) = \$9680$
In 3 years Tuition: $\$9680 + 0.10(\$9680) = \$10{,}648$
In 4 years Tuition: $\$10{,}648 + 0.10(\$10{,}648) = \$11{,}712.80$

The amount of tuition students will pay each of the next 4 years is given by the sequence

$$\$8800, \$9680, \$10{,}648, \$11{,}712.80$$

You Try 4

The number of customers paying their utility bills online has been increasing by 5% per year for the last several years. This year, 1200 customers of a particular utility company pay their bills online. Write a sequence that represents the number of customers paying their bills online each of the next 4 years.

Series

5. Understand the Meaning of Series and Summation Notation

If we add the terms of a sequence, we get a *series*.

Definition

A sum of the terms of a sequence is called a **series**.

Just like a sequence can be finite or infinite, a series can be finite or infinite. In this section, we will discuss only finite series.

Suppose the *sequence*

$$\$10, \$10.05, \$10.10, \$10.15, \$10.20, \$10.25$$

represents the amount of interest earned in an account each month over a 6-month period. Then the *series*

$$\$10 + \$10.05 + \$10.10 + \$10.15 + \$10.20 + \$10.25$$

represents the total amount of interest earned during the 6-month period. The total amount of interest earned is \$60.75.

We use a shorthand notation, called **summation notation**, to denote a series when we know the general term, a_n, of the sequence. The Greek letter Σ (sigma) is used to mean *sum*. Instead of using the letter n, the letters i, j, or k are usually used with Σ.

For example, $\sum_{i=1}^{4}(3i-5)$ means "find the sum of the first four terms of the sequence defined by $a_n = 3n - 5$." We read $\sum_{i=1}^{4}(3i-5)$ as "the sum of $3i - 5$ as i goes from 1 to 4." The variable i is called the **index of summation**. The $i = 1$ under the Σ tells us that 1 is the first value to substitute for i, and the 4 above the Σ tells us that 4 is the last value to substitute for i. To evaluate $\sum_{i=1}^{4}(3i-5)$, find the first four terms of the sequence by first substituting 1 for i, then 2, then 3, and finally let $i = 4$. Then add the terms of the sequence.

$$\begin{aligned}\sum_{i=1}^{4}(3i-5) &= (3(1)-5) + (3(2)-5) + (3(3)-5) + (3(4)-5)\\ &= \underset{i=1}{-2} + \underset{i=2}{1} + \underset{i=3}{4} + \underset{i=4}{7} = 10\end{aligned}$$

BE CAREFUL The i used in summation notation has no relationship with the complex number i.

6. Evaluate a Series

Example 5

Evaluate each series.

a) $\displaystyle\sum_{i=1}^{5}\frac{i+1}{3}$ b) $\displaystyle\sum_{i=3}^{6}(-1)^i \cdot 2^i$

Solution

a) i will start with 1 and end with 5. To evaluate the series, find the terms and then find their sum.

$$\begin{aligned}\sum_{i=1}^{5}\frac{i+1}{3} &= \underset{i=1}{\frac{1+1}{3}} + \underset{i=2}{\frac{2+1}{3}} + \underset{i=3}{\frac{3+1}{3}} + \underset{i=4}{\frac{4+1}{3}} + \underset{i=5}{\frac{5+1}{3}}\\ &= \frac{2}{3} + \frac{3}{3} + \frac{4}{3} + \frac{5}{3} + \frac{6}{3} = \frac{20}{3}\end{aligned}$$

b) i will start with 3 and end with 6. Find the terms, then find their sum.

$$\sum_{i=3}^{6} (-1)^i \cdot 2^i = \underset{i=3}{(-1)^3 \cdot 2^3} + \underset{i=4}{(-1)^4 \cdot 2^4} + \underset{i=5}{(-1)^5 \cdot 2^5} + \underset{i=6}{(-1)^6 \cdot 2^6}$$

$$= -8 + 16 + (-32) + 64 = 40$$

You Try 5

Evaluate each series.

a) $\displaystyle\sum_{i=1}^{6} (-1)^{i+1} \cdot (2i^2)$ b) $\displaystyle\sum_{i=4}^{8} (10 - i)$

7. Write a Series Using Summation Notation

To write a series using summation notation, we need to find a general term, a_n, that will produce the terms in the sum. Remember, try to find a relationship between the term and the term number. There may be more than one way to represent a series in summation notation.

Example 6

Write each series using summation notation.

a) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7}$ b) $9 + 16 + 25 + 36$

Solution

a) Find a general term, a_n, that will produce the terms in the series. Write each term with its term number below.

Term:	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$
Term number:	a_1	a_2	a_3	a_4	a_5	a_6

The numerator of each term is the *same* as the subscript on its term number. The denominator of each term is *one more than* the subscript on its term number.

Therefore, the terms of this series can be produced if $a_n = \frac{n}{n+1}$ for $n = 1$ to 6.

In summation notation, we can write $\displaystyle\sum_{i=1}^{6} \frac{i}{i+1}$.

b) We will use this example to illustrate how two different summation notations can represent the same series.

i) Write each term with its term number below. Note that each term is a perfect square.

Term:	$9 = 3^2$	$16 = 4^2$	$25 = 5^2$	$36 = 6^2$
Term number:	a_1	a_2	a_3	a_4

The base of each exponential expression is *two more than* its term number.

The terms of this series can be produced if $a_n = (n + 2)^2$ for $n = 1$ to 4.

In summation notation, we can write $\displaystyle\sum_{i=1}^{4} (i + 2)^2$.

ii) Write each term in the series as a perfect square:

$$9 + 16 + 25 + 36 = 3^2 + 4^2 + 5^2 + 6^2$$

If we let i begin at 3 and end at 6, we can write $\displaystyle\sum_{i=3}^{6} i^2$.

The series $9 + 16 + 25 + 36$ is a good example of one that can be written in summation notation in more than one way.

You Try 6

Write each series using summation notation.

a) $7 + \frac{7}{2} + \frac{7}{3} + \frac{7}{4} + \frac{7}{5}$ b) $8 + 9 + 10 + 11 + 12 + 13$

8. Use Summation Notation to Represent the Average of a Group of Numbers

Summation notation is widely used in statistics. To find the average of a group of numbers, for example, we find the sum of the numbers and divide by the number of numbers in the group.

In statistics, we use *summation notation* to represent the **average** or **arithmetic mean** of a group of numbers. The arithmetic mean is represented by $\bar{x}$, and is given by the formula

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

where $x_1, x_2, x_3, \ldots, x_n$ are the numbers in the group and n is the number of numbers in the group.

Example 7

The number of calories in 12-oz cans of five different soft drinks is 150, 120, 170, 150, and 140. What is the average number of calories in a 12-oz can of soda?

Solution

Since we must find the average of five numbers, the formula for $\bar{x}$ will be

$$\bar{x} = \frac{\sum_{i=1}^{5} x_i}{5}$$

where $x_1 = 150$, $x_2 = 120$, $x_3 = 170$, $x_4 = 150$, and $x_5 = 140$.

$$\bar{x} = \frac{150 + 120 + 170 + 150 + 140}{5}$$
$$= 146$$

The average number of calories in a 12-oz can of soda is 146.

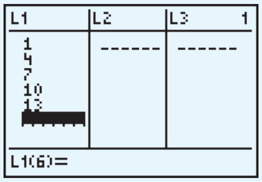
You Try 7

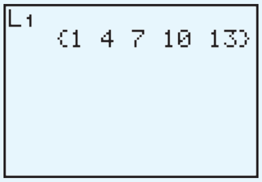
A movie theater complex has eight separate theaters. They had movies starting between 7 P.M. and 8 P.M. on a Friday night. Attendance figures in these theaters were as follows: 138, 58, 79, 178, 170, 68, 115, and 94. Find the average attendance per theater.

Using Technology

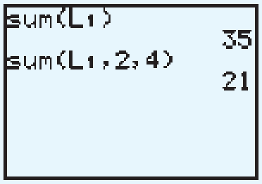
We can use a graphing calculator to display the terms of a sequence or to find the sum of a finite number of terms of a sequence, which is the same as evaluating a series. For example, consider the sequence 1, 4, 7, 10, 13, ... A sequence of numbers is represented as a list on a graphing calculator. The list variables are L1, L2, L3, L4, L5, and L6 and can be accessed by pressing 2nd 1, 2nd 2, 2nd 3, 2nd 4, 2nd 5, and 2nd 6, respectively. Store the first five numbers in the sequence 1, 4, 7, 10, 13, . . . in L1 by pressing STAT ENTER and then entering each number followed by ENTER, as shown at the left below.

The numbers stored in L1 can be displayed on the home screen by pressing 2nd 1 ENTER, as shown on the right screen below.



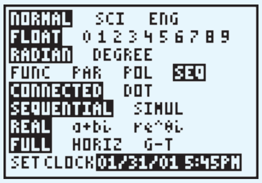


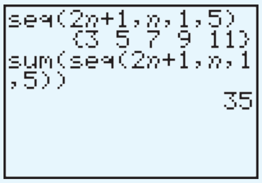
To find the sum of the terms of this sequence stored in L1, press 2nd STAT, then press the right arrow twice to MATH, select 5:sum(then enter 2nd 1) on the home screen as shown. To find the sum of the 2nd, 3rd, and 4th terms of this sequence, enter "sum(L1, 2, 4)" as shown here.



Given the nth term of an arithmetic sequence, a graphing calculator can display a finite number of terms. First change the mode of the calculator to sequence mode by pressing MODE and selecting SEQ in row 4, as shown at the left below.

The syntax for displaying terms of an arithmetic sequence is seq(nth term, n, starting index, ending index). For example, to display the first five terms of the arithmetic sequence with nth term $2n + 1$, enter seq$(2n + 1, n, 1, 5)$ on the home screen. To do this, press 2nd STAT, right arrow to OPS menu, and select 5:seq(2 X, T, θ, n + 1, X, T, θ, n, 1, 5). To find the sum if the first five terms of the arithmetic sequence, press 2nd STAT and right arrow twice to MATH menu and select 5:sum(followed by the sequence definition shown above, as shown at the right below.





Evaluate each sum using a graphing calculator.

1) $\sum_{i=1}^{6} (3i + 8)$

2) $\sum_{i=1}^{9} (2i - 14)$

3) $\sum_{i=1}^{12} (-4i + 9)$

4) $\sum_{i=1}^{215} (i - 10)$

5) $\sum_{i=1}^{100} \left(\frac{1}{2}i + 6\right)$

6) $\sum_{i=1}^{20} (4i - 5)$

Answers to You Try Exercises

1) a) $-3, -1, 1, 3, 5$ b) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$ c) $-1, 2, -3, 4, -5$ 2) a) $a_1 = \frac{1}{5}$ b) $a_{10} = -\frac{1}{50}$ c) $a_{32} = -\frac{1}{160}$ 3) a) $a_n = 11n$ b) $a_n = n + 4$ c) $a_n = \frac{1}{6n}$ d) $a_n = (-1)^{n+1} \cdot 4^n$
4) 1260, 1323, 1389, 1458 5) a) -42 b) 20 6) a) $\sum_{i=1}^{5} \frac{7}{i}$ b) $\sum_{i=8}^{13} i$ or $\sum_{i=1}^{6} (i + 7)$
7) 112.5

Answers to Technology Exercises

1) 111 2) -36 3) -204 4) 21,070 5) 3125 6) 740

15.1 Exercises

Objective 2: Write the Terms of a Sequence

Write out the first five terms of each sequence.

1) $a_n = n + 2$
2) $a_n = n - 4$
3) $a_n = 3n - 4$
4) $a_n = 4n + 1$
5) $a_n = 2n^2 - 1$
6) $a_n = 2n^2 + 3$
7) $a_n = 3^{n-1}$
8) $a_n = 2^n$
9) $a_n = 5 \cdot \left(\frac{1}{2}\right)^n$
10) $a_n = 6 \cdot \left(\frac{1}{3}\right)^{n-1}$
VIDEO 11) $a_n = (-1)^{n+1} \cdot 7n$
12) $a_n = (-1)^n \cdot (n + 1)$
13) $a_n = \frac{n-4}{n+3}$
14) $a_n = \frac{n^2 - 1}{n}$

Given the general term of each sequence, find each of the following.

VIDEO 15) $a_n = 3n + 2$
a) the first term of the sequence
b) a_5
c) the 28th term

16) $a_n = 3n - 11$
a) the first term of the sequence
b) a_7
c) the 32nd term

17) $a_n = \frac{n-4}{n+6}$
a) a_1
b) a_2
c) the 16th term

18) $a_n = \frac{3n-1}{4n+5}$
a) a_1
b) a_{10}
c) the 21st term

19) $a_n = 10 - n^2$
a) the first term of the sequence
b) the 6th term
c) a_{20}

20) $a_n = 4n^2 - 9$
a) the first term of the sequence
b) the fourth term
c) the 13th term

Objective 3: Find the General Term of a Sequence

Find a formula for the general term, a_n, of each sequence.

21) 2, 4, 6, 8, . . .
22) 9, 18, 27, 36, . . .
23) 1, 4, 9, 16, . . .
24) 1, 8, 27, 64, . . .
25) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$
26) $\frac{4}{5}, \frac{4}{25}, \frac{4}{125}, \frac{4}{625}, \ldots$
VIDEO 27) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
28) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
29) $5, -10, 15, -20, \ldots$
30) $-2, 4, -6, 8, \ldots$
31) $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$
32) $\frac{1}{4}, -\frac{1}{16}, \frac{1}{64}, -\frac{1}{256}, \ldots$

Objective 4: Solve an Applied Problem Using a Sequence

33) A television's value decreases by $\frac{1}{3}$ each year. If it was purchased for $2592, write a sequence that represents the value of the TV at the beginning of each of the next 4 years.

34) Due to an increase in costs, Hillcrest Health Club has decided to increase dues by 10% each year for the next 5 years. A membership currently costs $1000 per year. Write a sequence that represents the cost of a membership each of the next 5 years.

35) Carlton wants to improve his bench press. He plans on adding 10 lb to the bar each week. If he can lift 100 lb this week, how much will he lift 6 weeks from now?

36) Currently, Sierra earns $8.80 per hour, and she can get a raise of $0.50 per hour every 6 months. What will be her hourly wage 18 months from now?

Mixed Exercises: Objectives 5 and 6

37) What is the difference between a sequence and a series?

38) Explain what $\sum_{i=1}^{5}(7i + 2)$ means.

Evaluate each series.

39) $\sum_{i=1}^{6}(2i + 1)$

40) $\sum_{i=1}^{5}(4i + 3)$

41) $\sum_{i=1}^{5}(i - 8)$

42) $\sum_{i=1}^{4}(5 - 2i)$

43) $\sum_{i=1}^{4}(4i^2 - 2i)$

44) $\sum_{i=1}^{6}(3i^2 - 4i)$

45) $\sum_{i=1}^{6}\frac{i}{2}$

46) $\sum_{i=1}^{3}\frac{2i}{i + 3}$

47) $\sum_{i=1}^{5}(-1)^{i+1}\cdot(i)$

48) $\sum_{i=1}^{6}(-1)^{i}\cdot(i)$

49) $\sum_{i=5}^{9}(i - 2)$

50) $\sum_{i=6}^{10}(2i - 3)$

51) $\sum_{i=3}^{6}(i^2)$

52) $\sum_{i=2}^{7}(i - 1)^2$

Objective 7: Write a Series Using Summation Notation

Write each series using summation notation.

53) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

54) $11 + \frac{11}{2} + \frac{11}{3} + \frac{11}{4} + \frac{11}{5} + \frac{11}{6}$

55) $3 + 6 + 9 + 12$

56) $4 + 8 + 12 + 16 + 20 + 24 + 28$

57) $5 + 6 + 7 + 8 + 9 + 10$

58) $4 + 5 + 6 + 7$

59) $-1 + 2 - 3 + 4 - 5 + 6 - 7$

60) $2 - 4 + 8 - 16 + 32$

61) $3 - 9 + 27 - 81$

62) $-1 + 4 - 9 + 16 - 25$

Objective 8: Use Summation Notation to Represent the Average of a Group of Numbers

Find the arithmetic mean of each group of numbers.

63) 19, 24, 20, 17, 23, 17

64) 38, 31, 43, 40, 33

65) 8, 7, 11, 9, 12

66) 5, 9, 6, 5, 8, 3, 1, 7

67) Corey's credit card balance each month from January through June of 2011 is given in this table. Find the average monthly credit card balance during this period.

Month	Balance
January	$1431.60
February	$1117.82
March	$985.43
April	$1076.22
May	$900.00
June	$813.47

68) The annual rainfall amounts (in inches) at Lindbergh Field in San Diego from 1999 to 2005 are listed below. Find the average annual rainfall during this period. (Round the answer to the hundredths place.)
(www.sdcwa.org)

Year	Total Rainfall (inches)
1999	6.51
2000	5.77
2001	8.82
2002	3.44
2003	10.24
2004	5.31
2005	22.81

Section 15.2 Arithmetic Sequences and Series

Objectives

1. Define Arithmetic Sequence and Common Difference
2. Find the Common Difference for an Arithmetic Sequence
3. Write the Terms of a Sequence
4. Find the General Term of an Arithmetic Sequence
5. Find a Specified Term of an Arithmetic Sequence
6. Determine the Number of Terms in an Arithmetic Sequence
7. Solve an Applied Problem Involving an Arithmetic Sequence
8. Find the Sum of Terms of an Arithmetic Sequence
9. Solve an Applied Problem Involving an Arithmetic Series

1. Define Arithmetic Sequence and Common Difference

In this section, we will discuss a special type of sequence called an *arithmetic sequence*.

Definition

An **arithmetic sequence** is a sequence in which each term after the first differs from the preceding term by a constant amount, d. d is called the **common difference**.

Note

An arithmetic sequence is also called an **arithmetic progression**.

For example, 3, 7, 11, 15, 19, . . . is an arithmetic sequence. If we subtract each term from the term that follows it, we see that the common difference d, equals 4.

$$\begin{aligned} 7 - 3 &= 4 \\ 11 - 7 &= 4 \\ 15 - 11 &= 4 \\ 19 - 15 &= 4 \\ &\vdots \\ \underset{\substack{\uparrow \\ n\text{th term}}}{a_n} - \underset{\substack{\uparrow \\ \text{Previous term}}}{a_{n-1}} &= 4 \end{aligned}$$

For this sequence $a_n - a_{n-1} = 4$. That is, choose a term in the sequence (a_n) and subtract the term before it (a_{n-1}) to get 4.

Note

For any arithmetic sequence, $d = a_n - a_{n-1}$.

2. Find the Common Difference for an Arithmetic Sequence

Example 1

Find d for each arithmetic sequence.

a) $-4, -1, 2, 5, 8, \ldots$ b) $35, 27, 19, 11, 3, \ldots$

Solution

a) To find d, choose any term and subtract the preceding term: $d = 5 - 2 = 3$.

You can see that choosing a different pair of terms will produce the same result.

$$d = 2 - (-1) = 3$$

b) Choose any term and subtract the preceding term: $d = 27 - 35 = -8$. ■

You Try 1

Find d for each arithmetic sequence.

a) $-2, 3, 8, 13, 18, \ldots$ b) $25, 13, 1, -11, -23, \ldots$

3. Write the Terms of a Sequence

If we are given the first term, a_1, of an arithmetic sequence and the common difference, d, we can write the terms of the sequence.

Example 2

Write the first five terms of the arithmetic sequence with first term 5 and common difference 6.

Solution

Since the first term is 5, $a_1 = 5$. Add 6 to get the second term, a_2. Continue adding 6 to get each term.

$$\begin{aligned} a_1 &= 5 \\ a_2 &= 5 + 6 = 11 \\ a_3 &= 11 + 6 = 17 \\ a_4 &= 17 + 6 = 23 \\ a_5 &= 23 + 6 = 29 \end{aligned}$$

The first five terms of the sequence are 5, 11, 17, 23, 29. ■

You Try 2

Write the first five terms of the arithmetic sequence with first term -3 and common difference 2.

4. Find the General Term of an Arithmetic Sequence

Given the first term, a_1, of an arithmetic sequence and the common difference, d, we can find the general term of the sequence, a_n.

Consider the sequence in Example 2: 5, 11, 17, 23, 29. We will show that each term can be written in terms of a_1 and d, leading us to a formula for a_n.

5	11	17	23	29
$a_1 = 5$	$a_2 = 5 + 6$	$a_3 = 11 + 6$	$a_4 = 17 + 6$	$a_5 = 23 + 6$
	$a_2 = a_1 + d$	$a_3 = a_2 + d$	$a_4 = a_3 + d$	$a_5 = a_4 + d$
		$a_3 = (a_1 + d) + d$	$a_4 = (a_1 + 2d) + d$	$a_5 = (a_1 + 3d) + d$
		$a_3 = a_1 + 2d$	$a_4 = a_1 + 3d$	$a_5 = a_1 + 4d$

For each term after the first, the coefficient of d is one less than the term number.

This pattern applies for any arithmetic sequence. Therefore, the general term, a_n, of an arithmetic sequence is given by $a_n = a_1 + (n - 1)d$.

Definition

General Term of an Arithmetic Sequence: The general term of an arithmetic sequence with first term a_1 and common difference d is given by

$$a_n = a_1 + (n - 1)d$$

Example 3

Given the arithmetic sequence 9, 13, 17, 21, 25, . . . , find

a) a_1 and d. b) a formula for a_n. c) the 31st term of the sequence.

Solution

a) a_1 is the first term of the sequence. $a_1 = 9$. To find d, choose any term and subtract the preceding term.

$$d = 13 - 9 = 4$$

Therefore, $a_1 = 9$ and $d = 4$.

b) To find a_n, begin with the formula and substitute 9 for a_1 and 4 for d.

$$\begin{aligned} a_n &= a_1 + (n-1)d && \text{Formula for } a_n \\ a_n &= 9 + (n-1)(4) && \text{Substitute 9 for } a_1 \text{ and 4 for } d. \\ a_n &= 9 + 4n - 4 && \text{Distribute.} \\ a_n &= 4n + 5 && \text{Combine like terms.} \end{aligned}$$

c) Finding the 31st term of the sequence means finding a_{31}. Let $n = 31$ in the formula for a_n.

$$\begin{aligned} a_n &= 4n + 5 \\ a_{31} &= 4(31) + 5 \\ a_{31} &= 129 \end{aligned}$$

You Try 3

Given the arithmetic sequence 4, 11, 18, 25, 32, . . . , find

a) a_1 and d. b) a formula for a_n. c) the 25th term of the sequence.

Example 4

Find the general term, a_n, and the 54th term of the arithmetic sequence 49, 41, 33, 25, 17, ...

Solution

To find a_n, we must use the formula $a_n = a_1 + (n-1)d$. Therefore, identify a_1 and find d.

$$a_1 = 49 \qquad d = 41 - 49 = -8$$

Substitute 49 for a_1 and -8 for d in the formula for a_n.

$$\begin{aligned} a_n &= a_1 + (n-1)d && \text{Formula for } a_n \\ a_n &= 49 + (n-1)(-8) && \text{Substitute 49 for } a_1 \text{ and } -8 \text{ for } d. \\ a_n &= 49 - 8n + 8 && \text{Distribute.} \\ a_n &= -8n + 57 && \text{Combine like terms.} \end{aligned}$$

The fifty-fourth term of the sequence is a_{54}. Let $n = 54$ in the formula for a_n.

$$\begin{aligned} a_n &= -8n + 57 \\ a_{54} &= -8(54) + 57 \\ a_{54} &= -375 \end{aligned}$$

You Try 4

Find the general term, a_n, and the 41st term of the arithmetic sequence $-1, -6, -11, -16, -21, \ldots$.

5. Find a Specified Term of an Arithmetic Sequence

If we know a_1 and d for an arithmetic sequence, we can find any term without having the formula for a_n.

Example 5

Find the 15th term of the arithmetic sequence with first term -17 and common difference 3.

Solution

The first term is -17, so $a_1 = -17$. The common difference is 3, so $d = 3$. Finding the 15th term means finding a_{15}. Use the formula $a_n = a_1 + (n - 1)d$ and substitute 15 for n, -17 for a_1, and 3 for d.

$$
\begin{aligned}
a_n &= a_1 + (n-1)d && n = 15,\ a_1 = -17,\ d = 3 \\
a_{15} &= -17 + (15 - 1)(3) && \\
a_{15} &= -17 + 14(3) && \text{Subtract.} \\
a_{15} &= -17 + 42 && \text{Multiply.} \\
a_{15} &= 25 && \text{Add.}
\end{aligned}
$$

The 15th term is 25.

You Try 5

Find the 23rd term of the arithmetic sequence with first term 18 and common difference 4.

Sometimes, we can write and solve a system of equations to find the general term, a_n.

Example 6

The fourth term of an arithmetic sequence is -10 and the ninth term is 25. Find

a) the general term, a_n. b) the 16th term.

Solution

a) Since the fourth term is -10, $a_4 = -10$. Since the ninth term is 25, $a_9 = 25$.

Use the formula for a_n along with $a_4 = -10$ and $a_9 = 25$ to obtain two equations containing the two variables a_1 and d.

$a_4 = -10$		$a_9 = 25$	
$a_n = a_1 + (n-1)d$		$a_n = a_1 + (n-1)d$	
$a_4 = a_1 + (4-1)d$	$n = 4$	$a_9 = a_1 + (9-1)d$	$n = 9$
$-10 = a_1 + 3d$	$a_4 = -10$	$25 = a_1 + 8d$	$a_9 = 25$

We obtain the system of equations

$$
\begin{aligned}
-10 &= a_1 + 3d \\
25 &= a_1 + 8d
\end{aligned}
$$

Multiply the first equation by -1 and add the two equations. This will eliminate a_1 and enable us to solve for d.

$$
\begin{aligned}
10 &= -a_1 - 3d && \text{Multiply first equation by } -1. \\
+\ 25 &= a_1 + 8d && \\
\hline
35 &= 5d && \\
7 &= d &&
\end{aligned}
$$

Substitute $d = 7$ into $-10 = a_1 + 3d$ to solve for a_1.

$$\begin{aligned} -10 &= a_1 + 3(7) \\ -10 &= a_1 + 21 \\ -31 &= a_1 \end{aligned}$$

$a_1 = -31$ and $d = 7$. Substitute these values into $a_n = a_1 + (n-1)d$ to find a formula for a_n.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ &= -31 + (n-1)(7) && d = 7 \\ &= -31 + 7n - 7 && \text{Distribute.} \\ &= 7n - 38 && \text{Combine like terms.} \end{aligned}$$

The general term is $a_n = 7n - 38$.

b) The 16th term is a_{16}. Substitute 16 for n in $a_n = 7n - 38$ to find the 16th term.

$$\begin{aligned} a_n &= 7n - 38 \\ a_{16} &= 7(16) - 38 && \text{Let } n = 16. \\ &= 74 \end{aligned}$$

$a_{16} = 74$.

You Try 6

The third term of an arithmetic sequence is 11 and the tenth term is -31. Find

a) the general term, a_n. b) the 22nd term.

6. Determine the Number of Terms in an Arithmetic Sequence

We can determine the number of terms in a sequence using the formula for a_n.

Example 7

Find the number of terms in the arithmetic sequence 18, 14, 10, 6, . . . , -54.

Solution

The first term in the sequence is 18, so $a_1 = 18$.

Let $n =$ the number of terms in the sequence.

Then $a_n = -54$, since -54 is the last (or nth) term.

$d = -4$ since $14 - 18 = -4$.

Substitute 18 for a_1, -4 for d, and -54 for a_n into $a_n = a_1 + (n-1)d$, and solve for n.

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ -54 &= 18 + (n-1)(-4) && \text{Let } a_n = -54,\ a_1 = 18, \text{ and } d = -4. \\ -54 &= 18 - 4n + 4 && \text{Distribute.} \\ -54 &= 22 - 4n && \text{Combine like terms.} \\ -76 &= -4n && \text{Subtract 22.} \\ 19 &= n && \text{Divide by } -4. \end{aligned}$$

This sequence has 19 terms.

You Try 7

Find the number of terms in the arithmetic sequence 3, 8, 13, 18, . . . , 48.

7. Solve an Applied Problem Involving an Arithmetic Sequence

Example 8

Due to a decrease in sales, a company decides to decrease the number of employees at one of its manufacturing plants by 25 per month for the next 6 months. The plant currently has 483 workers.

a) Find the general term of an arithmetic sequence, a_n, that models the number of employees working at the manufacturing plant.

b) How many employees remain after 6 months?

Solution

a) The first term of the arithmetic sequence is the current number of employees. So $a_1 = 483$.

The number of workers will decrease by 25 per month, so $d = -25$. Substitute $a_1 = 483$ and $d = -25$ into $a_n = a_1 + (n - 1)d$.

$$
\begin{aligned}
a_n &= a_1 + (n - 1)d \\
a_n &= 483 + (n - 1)(-25) && a_1 = 483,\ d = -25 \\
a_n &= 483 - 25n + 25 && \text{Distribute.} \\
a_n &= -25n + 508 && \text{Combine like terms.}
\end{aligned}
$$

b) The number of employees remaining after 6 months is the sixth term of the sequence, a_6.

Substitute 6 for n in $a_n = -25n + 508$.

$$
\begin{aligned}
a_n &= -25n + 508 \\
a_6 &= -25(6) + 508 \\
a_6 &= -150 + 508 \\
a_6 &= 358
\end{aligned}
$$

There will be 358 employees. ■

You Try 8

Arianna wants to save money for a car. She makes an initial deposit of \$2000, and then she will deposit \$300 at the beginning of each month.

a) Find the general term for an arithmetic sequence, a_n, that models the amount of money (ignoring interest) that she has saved.

b) How much has she saved (ignoring interest) after 12 months?

Arithmetic Series

8. Find the Sum of Terms of an Arithmetic Sequence

We first defined a series in Section 15.1 as a sum of the terms of a sequence.

Therefore, an **arithmetic series** is a sum of terms of an arithmetic sequence.

It would not be difficult to find the sum of, say, the first 5 terms of an arithmetic sequence. But if we were asked to find the sum of the first 50 terms, using a formula would be more convenient.

Let S_n represent the first n terms of an arithmetic sequence. Then,

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \cdots + [a_1 + (n-1)d]$$

We can write the sum with the terms in reverse order as

$$S_n = a_n + (a_n - d) + (a_n - 2d) + (a_n - 3d) + \cdots + [a_n - (n-1)d]$$

Next, add the two expressions by adding the corresponding terms. We get

$$\begin{aligned} 2S_n &= (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) \\ 2S_n &= n(a_1 + a_n) && \text{There are } n\ (a_1 + a_n)\text{-terms.} \\ S_n &= \frac{n}{2}(a_1 + a_n) && \text{Divide by 2.} \end{aligned}$$

$S_n = \frac{n}{2}(a_1 + a_n)$ is one formula for the sum of the first n terms of an arithmetic sequence.

We can derive another formula for the sum if we substitute $a_n = a_1 + (n-1)d$ for a_n in the formula above.

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) \\ &= \frac{n}{2}[a_1 + (a_1 + (n-1)d)] && a_n = a_1 + (n-1)d \\ &= \frac{n}{2}[2a_1 + (n-1)d] && \text{Combine like terms.} \end{aligned}$$

Another formula for the sum of the first n terms of an arithmetic sequence is

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

Formula Sum of the First n Terms of an Arithmetic Sequence

The sum of the first n terms, S_n, of an arithmetic sequence with first term a_1, nth term a_n, and common difference d is given by

1) $S_n = \frac{n}{2}(a_1 + a_n)$

2) $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

Note

It is convenient to use 1) when the first term, the last term, and the number of terms are known. If the last term is not known, then 2) may be a better choice for finding the sum.

Example 9

Find the sum of the first 17 terms of the arithmetic sequence with first term 41 and last term -23.

Solution

We are given the first term, the last term, and the number of terms. *We will use formula 1)* $S_n = \frac{n}{2}(a_1 + a_n)$ to find S_{17}, the sum of the first 17 terms.

$a_1 = 41, a_{17} = -23, \quad n = 17$

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad \text{Formula 1)}$$

$$S_{17} = \frac{17}{2}(41 + a_{17}) \qquad \text{Let } n = 17 \text{ and } a_1 = 41.$$

$$S_{17} = \frac{17}{2}[41 + (-23)] \qquad a_{17} = -23$$

$$S_{17} = \frac{17}{2}(18) \qquad \text{Add.}$$

$$S_{17} = 153 \qquad \text{Multiply.}$$

The sum of the first 17 terms is 153.

You Try 9

Find the sum of the first 15 terms of the arithmetic sequence with first term 2 and last term 72.

Example 10

Find the sum of the first 12 terms of the arithmetic sequence with first term 7 and common difference 3.

Solution

We are given the first term ($a_1 = 7$), the common difference ($d = 3$), and the number of terms ($n = 12$). The last term is not known. *We will use formula 2)* $S_n = \frac{n}{2}[2a_1 + (n-1)d]$ to find S_{12}, the sum of the first 12 terms.

$a_1 = 7, d = 3, n = 12$

$$S_n = \frac{n}{2}[2a_1 + (n-1)d] \qquad \text{Formula 2)}$$

$$S_{12} = \frac{12}{2}[2(7) + (12-1)(3)] \qquad \text{Let } a_1 = 7, d = 3, \text{ and } n = 12.$$

$$S_{12} = 6[14 + (11)(3)]$$

$$S_{12} = 6(47)$$

$$S_{12} = 282$$

The sum of the first 12 terms is 282.

You Try 10

Find the sum of the first 10 terms of the arithmetic sequence with first term 6 and common difference 4.

The general term of an arithmetic sequence has the form $a_n = bn + c$, where b and c are constants. Therefore, $\sum_{i=1}^{n}(bn + c)$ represents an arithmetic series or the sum of the first n terms of an arithmetic sequence. We can use formula 1) to evaluate $\sum_{i=1}^{n}(bn + c)$.

Example 11

Evaluate $\sum_{i=1}^{16}(3i - 19)$.

Solution

Since i begins at 1 and ends at 16, to evaluate $\sum_{i=1}^{16}(3i - 19)$ means to find the sum of the first 16 terms, S_{16}, of the arithmetic sequence with general term $a_n = 3n - 19$.

Find the first term by substituting 1 for i: $a_1 = 3(1) - 19 = -16$

Find the last (the sixteenth) term by substituting 16 for i: $a_{16} = 3(16) - 19 = 29$.

There are 16 terms, so $n = 16$.

Because we know that $a_1 = -16$, $a_{16} = 29$, and $n = 16$, use formula 1) to evaluate $\sum_{i=1}^{16}(3i - 19)$.

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad \text{Formula 1)}$$
$$S_{16} = \frac{16}{2}(-16 + a_{16}) \qquad \text{Let } n = 16 \text{ and } a_1 = -16.$$
$$S_{16} = 8(-16 + 29) \qquad a_{16} = 29$$
$$S_{16} = 8(13)$$
$$S_{16} = 104$$

Therefore, $\sum_{i=1}^{16}(3i - 19) = 104$.

You Try 11

Evaluate $\sum_{i=1}^{19}(2i + 1)$.

9. Solve an Applied Problem Involving an Arithmetic Series

Example 12

An acrobatic troupe forms a human pyramid with six people in the bottom row, five people in the second row, four people in the third row, and so on. How many people are in the pyramid if the pyramid has six rows?

Solution

The information in the problem suggests the arithmetic sequence 6, 5, 4, . . . , 1, where each term represents the number of people in a particular row of the pyramid.

We are asked to find the *total* number of people in the pyramid, so we are finding S_6, the sum of the six terms of the sequence.

Since there are six rows, $n = 6$.

There are six people in the first row, so $a_1 = 6$.

There is 1 person in the last row, so $a_6 = 1$. Use formula 1).

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_6 = \frac{6}{2}(6 + a_6) \qquad \text{Let } n = 6 \text{ and } a_1 = 6.$$

$$S_6 = 3(6 + 1) \qquad \text{Let } a_6 = 1.$$

$$S_6 = 3(7) = 21$$

There are 21 people in the pyramid. ■

You Try 12

A child builds a tower with blocks so that the bottom row contains nine blocks, the next row contains seven blocks, the next row contains five blocks, and so on. If the tower has five rows, how many blocks are in the tower?

Answers to You Try Exercises

1) a) $d = 5$ b) $d = -12$ 2) $-3, -1, 1, 3, 5$ 3) a) $a_1 = 4, d = 7$ b) $a_n = 7n - 3$ c) 172
4) $a_n = -5n + 4$; $a_{41} = -201$ 5) 106 6) a) $a_n = -6n + 29$ b) -103 7) 10
8) a) $a_n = 300n + 1700$ b) \$5300 9) 555 10) 240 11) 399 12) 25

15.2 Exercises

Mixed Exercises: Objectives 1 and 2

1) What is an arithmetic sequence? Give an example.

2) How do you find the common difference for an arithmetic sequence?

Determine whether each sequence is arithmetic. If it is, find the common difference, d.

3) 3, 11, 19, 27, 35, . . .

4) 4, 7, 10, 13, 16, . . .

5) 10, 6, 2, -2, -6, . . .

6) 27, 20, 13, 6, -1, . . .

7) 4, 8, 16, 32, 64, . . .

8) 1, 3, 6, 10, 15, . . .

9) $-17, -14, -11, -8, -5, \ldots$

10) $-12, -10, -8, -6, -4, \ldots$

Objective 3: Write the Terms of a Sequence

Write the first five terms of each arithmetic sequence with the given first term and common difference.

11) $a_1 = 7, d = 2$

12) $a_1 = 20, d = 4$

13) $a_1 = 15, d = -8$

14) $a_1 = -3, d = -2$

15) $a_1 = -10, d = 3$

16) $a_1 = -19, d = 5$

Write the first five terms of the arithmetic sequence with general term a_n.

17) $a_n = 6n + 7$

18) $a_n = 2n + 7$

19) $a_n = 5 - n$

20) $a_n = 3 - 4n$

Mixed Exercises: Objectives 4 and 5

VIDEO 21) Given the arithmetic sequence
4, 7, 10, 13, 16, . . .

a) Find a_1 and d.

b) Find a formula for the general term of the sequence, a_n.

c) Find the 35th term of the sequence.

22) Given the arithmetic sequence
$-5, -3, -1, 1, 3, \ldots$

a) Find a_1 and d.

b) Find a formula for the general term of the sequence, a_n.

c) Find the 35th term of the sequence.

23) Given the arithmetic sequence
$4, -1, -6, -11, -16, \ldots$

a) Find a_1 and d.

b) Find a formula for the general term of the sequence, a_n.

c) Find a_{19}.

24) Given the arithmetic sequence
$-9, -21, -33, -45, -57, \ldots$

a) Find a_1 and d.

b) Find a formula for the general term of the sequence, a_n.

c) Find a_{15}.

For each arithmetic sequence, find a_n and then use a_n to find the indicated term.

25) $-7, -5, -3, -1, 1, \ldots; a_{25}$

26) $13, 19, 25, 31, 37, \ldots; a_{30}$

27) $1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots; a_{18}$

28) $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, \ldots; a_{21}$

29) $a_1 = 0, d = -5; a_{23}$

30) $a_1 = -5, d = -7; a_{14}$

Find the indicated term for each arithmetic sequence.

31) $a_1 = -5, d = 4; a_{16}$

32) $a_1 = 10, d = 3; a_{29}$

33) $a_1 = -7, d = -5; a_{21}$

34) $a_1 = 27, d = -4; a_{32}$

Two terms of an arithmetic sequence are given in each problem. Find the general term of the sequence, a_n, and find the indicated term.

VIDEO 35) $a_3 = 11, a_7 = 19; a_{11}$

36) $a_5 = 13, a_{11} = 31; a_{16}$

37) $a_2 = 7, a_6 = -13; a_{14}$

38) $a_3 = -9, a_7 = -25; a_{10}$

39) $a_4 = -5, a_{11} = 16; a_{18}$

40) $a_4 = -10, a_9 = 0; a_{17}$

Objective 6: Determine the Number of Terms in an Arithmetic Sequence

Find the number of terms in each arithmetic sequence.

41) $8, 13, 18, 23, \ldots, 63$

42) $8, 11, 14, 17, \ldots, 50$

43) $9, 7, 5, 3, \ldots, -27$

44) $-7, -11, -15, -19, \ldots, -91$

Objective 8: Find the Sum of Terms of an Arithmetic Sequence

45) For a particular sequence, suppose you are asked to find S_{15}. What are you finding?

46) Write down the two formulas for S_n, and explain when to use each formula.

VIDEO 47) Find the sum of the first 10 terms of the arithmetic sequence with first term 14 and last term 68.

48) Find the sum of the first nine terms of the arithmetic sequence with first term 2 and last term 34.

49) Find the sum of the first seven terms of the arithmetic sequence with first term 3 and last term -9.

50) Find the sum of the first 11 terms of the arithmetic sequence with first term -8 and last term -58.

Find S_8 for each arithmetic sequence described below.

51) $a_1 = -1, a_8 = -29$

52) $a_1 = -5, a_8 = 9$

53) $a_1 = 3, d = 5$

54) $a_1 = 2, d = 3$

55) $a_1 = 10, d = -6$

56) $a_1 = -1, d = -3$

57) $a_n = -4n - 1$

58) $a_n = 4n + 1$

59) $a_n = 3n + 4$

60) $a_n = -6n + 5$

61) a) Evaluate $\sum_{i=1}^{10} (2i + 7)$ by writing out each term and finding the sum.

b) Evaluate $\sum_{i=1}^{10} (2i + 7)$ using a formula for S_n.

c) Which method do you prefer and why?

Evaluate each sum using a formula for S_n.

62) $\sum_{i=1}^{6} (3i + 8)$

63) $\sum_{i=1}^{5} (8i - 5)$

64) $\sum_{i=1}^{9} (2i - 14)$

VIDEO 65) $\sum_{i=1}^{7} (-2i + 7)$

66) $\sum_{i=1}^{12} (-4i + 9)$

67) $\sum_{i=1}^{18} (3i - 11)$

68) $\sum_{i=1}^{215} (i - 10)$

69) $\sum_{i=1}^{500} i$

70) $\sum_{i=1}^{100} \left(\frac{1}{2}i + 6\right)$

Mixed Exercises: Objectives 7 and 9

Solve each application.

71) This month, Warren deposited \$1500 into a bank account. He will deposit \$100 into the account at the beginning of each month. Disregarding interest, how much money will Warren have saved after 9 months?

72) When Antoinnette is hired for a job, she signs a contract for a salary of \$34,000 plus a raise of \$1800 each year for the next 4 years. What will be her salary in the last year of her contract?

73) Beginning the first week of June, Noor will begin to deposit money in her bank. She will deposit \$1 the first week, \$2 the second week, \$3 the third week, \$4 the fourth week, and she will continue to deposit money in this way for 24 weeks. How much money will she have saved after 24 weeks?

74) Refer to Exercise 73. If Tracy deposits money weekly but deposits \$1 then \$3 then \$5, etc., how much will Tracy have saved after 24 weeks?

75) A stack of logs has 12 logs in the bottom row (the first row), 11 logs in the second row, 10 logs in the third row, and so on, until the last row contains one log.

a) How many logs are in the eighth row?

b) How many logs are in the stack?

76) A landscaper plans to put a pyramid design in a brick patio so that the bottom row of the pyramid contains 9 bricks and every row above it contains two fewer bricks. How many bricks does she need to make the design?

VIDEO 77) A lecture hall has 14 rows. The first row has 12 seats, and each row after that has 2 more seats than the previous row. How many seats are in the last row? How many seats are in the lecture hall?

78) A theater has 23 rows. The first row contains 10 seats, the next row has 12 seats, the next row has 14 seats, and so on. How many seats are in the last row? How many seats are in the theater?

79) The main floor of a concert hall seats 860 people. The first row contains 24 seats, and the last row contains 62 seats. If each row has 2 more seats than the previous row, how many rows of seats are on the main floor of the concert hall?

80) A child builds a tower with blocks so that the bottom row contains 9 blocks and the top row contains 1 block. If he uses 45 blocks, how many rows are in the tower?

Section 15.3 Geometric Sequences and Series

Objectives

1. Define Geometric Sequence and Common Ratio
2. Find the Common Ratio for a Geometric Sequence
3. Find the Terms of a Geometric Sequence
4. Find the General Term of a Geometric Sequence
5. Find a Specified Term of a Geometric Sequence
6. Solve an Applied Problem Involving a Geometric Sequence
7. Find the Sum of Terms of a Geometric Sequence
8. Find the Sum of Terms of an Infinite Geometric Sequence
9. Solve an Applied Problem Involving an Infinite Series
10. Distinguish Between an Arithmetic and a Geometric Sequence

1. Define Geometric Sequence and Common Ratio

In Section 15.2, we learned that a sequence such as

$$5, 9, 13, 17, 21, \ldots$$

is an *arithmetic sequence* because each term differs from the previous term by a constant amount, called the common difference, d. In this case, $d = 4$.

The terms of the sequence

$$3, 6, 12, 24, 48, \ldots$$

do not differ by a constant amount, but each term after the first is obtained by *multiplying* the preceding term by 2. Such a sequence is called a *geometric sequence*.

Definition

A **geometric sequence** is a sequence in which each term after the first is obtained by multiplying the preceding term by a constant, r. r is called the **common ratio**.

Note

A geometric sequence is also called a **geometric progression**.

In the sequence 3, 6, 12, 24, 48, . . . the common ratio, r, is 2. We can find the value of r by dividing any term after the first by the preceding term. For example,

$$r = \frac{6}{3} = \frac{12}{6} = \frac{24}{12} = \frac{48}{24} = 2$$

2. Find the Common Ratio for a Geometric Sequence

Example 1

Find the common ratio, r, for each geometric sequence.

a) 5, 15, 45, 135, . . .   b) 12, 6, 3, $\frac{3}{2}$, . . .

Solution

a) To find r, choose any term and divide it by the term preceding it: $r = \frac{15}{5} = 3$.

It is important to realize that dividing *any* term by the term immediately before it will give the same result.

b) Choose any term and divide by the term preceding it: $r = \frac{3}{6} = \frac{1}{2}$.

You Try 1

Find the common ratio, r, for each geometric sequence.

a) 2, 12, 72, 432, . . .   b) 15, 5, $\frac{5}{3}$, $\frac{5}{9}$, . . .

3. Find the Terms of a Geometric Sequence

Example 2

Write the first five terms of the geometric sequence with first term 10 and common ratio 2.

Solution

Each term after the first is obtained by multiplying by 2.

$$\begin{aligned} a_1 &= 10 \\ a_2 &= 10(2) = 20 \\ a_3 &= 20(2) = 40 \\ a_4 &= 40(2) = 80 \\ a_5 &= 80(2) = 160 \end{aligned}$$

The first five terms of the sequence are 10, 20, 40, 80, 160. ■

You Try 2

Write the first five terms of the geometric sequence with first term 2 and common ratio 2.

4. Find the General Term of a Geometric Sequence

Example 2 suggests a pattern that enables us to find a formula for the general term, a_n, of a geometric sequence. The common ratio, r, is 2. The first five terms of the sequence are

10	20	40	80	160
$a_1 = 10$	$a_2 = 10 \cdot 2$	$a_3 = 10 \cdot 4$	$a_4 = 10 \cdot 8$	$a_5 = 10 \cdot 16$
	$a_2 = a_1 \cdot r$	$a_3 = a_1 \cdot r^2$	$a_4 = a_1 \cdot r^3$	$a_5 = a_1 \cdot r^4$

The exponent on r is one less than the term number.

This pattern applies for any geometric sequence. Therefore, the general term, a_n, of a geometric sequence is given by $a_n = a_1 r^{n-1}$.

Definition

General Term of a Geometric Sequence: The general term of a geometric sequence with first term a_1 and common ratio r is given by

$$a_n = a_1 r^{n-1}$$

Example 3

Find the general term, a_n, and the eighth term of the geometric sequence 3, 6, 12, 24, 48, ...

Solution

$a_1 = 3$ and $r = \frac{6}{3} = 2$. Substitute these values into $a_n = a_1 r^{n-1}$.

$$a_n = a_1 r^{n-1}$$
$$a_n = 3(2)^{n-1} \qquad \text{Let } a_1 = 3 \text{ and } r = 2.$$

The general term, $a_n = 3(2)^{n-1}$, is in simplest form.

To find the eighth term, a_8, substitute 8 for n and simplify.

$$\begin{aligned} a_n &= 3(2)^{n-1} & \\ a_8 &= 3(2)^{8-1} & \text{Let } n = 8. \\ a_8 &= 3(2)^7 & \text{Subtract.} \\ a_8 &= 3(128) & 2^7 = 128 \\ a_8 &= 384 & \text{Multiply.} \end{aligned}$$

The eighth term is 384. ■

BE CAREFUL

Remember the order of operations when evaluating $3(2)^7$. We evaluate exponents before we do multiplication to find $2^7 = 128$ *before* multiplying by 3.

You Try 3

Find the general term, a_n, and the fifth term of the geometric sequence 2, 6, 18, ...

5. Find a Specified Term of a Geometric Sequence

If we know a_1 and r for a geometric sequence, we can find any term using $a_n = a_1 r^{n-1}$.

Example 4

Find the sixth term of the geometric sequence $12, -4, \frac{4}{3}, \ldots$.

Solution

$a_1 = 12$ and $r = -\frac{4}{12} = -\frac{1}{3}$. Find a_6 using $a_n = a_1 r^{n-1}$.

$$a_n = a_1 r^{n-1}$$

$$a_6 = (12)\left(-\frac{1}{3}\right)^{6-1} \qquad \text{Let } n = 6,\ a_1 = 12, \text{ and } r = -\frac{1}{3}.$$

$$a_6 = 12\left(-\frac{1}{3}\right)^5$$

$$a_6 = 12\left(-\frac{1}{243}\right)$$

$$a_6 = -\frac{4}{81}$$

The sixth term is $-\frac{4}{81}$.

You Try 4

Find the seventh term of the geometric sequence $-50, 25, -\frac{25}{2}, \ldots$.

6. Solve an Applied Problem Involving a Geometric Sequence

Example 5

A pickup truck purchased for \$24,000 (This is its value at the beginning of year 1.) depreciates 25% each year. That is, its value each year is 75% of its value the previous year.

a) Find the general term, a_n, of the geometric sequence that models the value of the truck at the beginning of each year.

b) What is the pickup truck worth at the beginning of the fourth year?

Solution

a) To find the value of the pickup each year, we *multiply* how much it was worth the previous year by 0.75. Therefore, the value of the truck each year can be modeled by a *geometric* sequence.

$a_1 = 24{,}000$ since this is the value at the beginning of year 1. $r = 0.75$ since we will *multiply* the value each year by 0.75 to find the value the next year.

Substitute $a_1 = 24{,}000$ and $r = 0.75$ into $a_n = a_1 r^{n-1}$ to find the general term.

$$a_n = a_1 r^{n-1}$$

$$a_n = 24{,}000(0.75)^{n-1} \qquad \text{Let } a_1 = 24{,}000 \text{ and } r = 0.75.$$

b) To find the value of the pickup truck at the beginning of the fourth year, use a_n from a) and let $n = 4$.

$$\begin{aligned} a_n &= 24{,}000(0.75)^{n-1} \\ a_4 &= 24{,}000(0.75)^{4-1} && \text{Let } n = 4. \\ a_4 &= 24{,}000(0.75)^{3} \\ a_4 &= 10{,}125 \end{aligned}$$

The truck is worth \$10,125 at the beginning of the fourth year. ■

You Try 5

A minivan purchased for \$27,000 depreciates 30% each year. That is, its value each year is 70% of its value the previous year.

a) Find the general term, a_n, of the geometric sequence that models the value of the minivan at the beginning of each year.

b) What is the minivan worth at the beginning of the third year?

Geometric Series

7. Find the Sum of Terms of a Geometric Sequence

A **geometric series** is a sum of terms of a geometric sequence.

Just as we can use a formula to find the sum of the first n terms of an arithmetic sequence, there is a formula to find the sum of the first n terms of a geometric sequence.

Let S_n represent the sum of the first n terms of a geometric sequence. Then,

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}$$

Multiply both sides of the equation by r.

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + \cdots + a_1 r^n$$

Subtract rS_n from S_n.

$$\begin{aligned} S_n - rS_n = {} & (a_1 - a_1 r) + (a_1 r - a_1 r^2) + (a_1 r^2 - a_1 r^3) + (a_1 r^3 - a_1 r^4) + \cdots \\ & + (a_1 r^{n-1} - a_1 r^n) \end{aligned}$$

Regrouping the right-hand side gives us

$$\begin{aligned} S_n - rS_n = {} & a_1 + (a_1 r - a_1 r) + (a_1 r^2 - a_1 r^2) + (a_1 r^3 - a_1 r^3) + \cdots \\ & + (a_1 r^{n-1} - a_1 r^{n-1}) - a_1 r^n \end{aligned}$$

The differences in parentheses equal zero, and we get $S_n - rS_n = a_1 - a_1 r^n$.

Factor out S_n on the left-hand side and a_1 on the right-hand side.

$$\begin{aligned} S_n(1 - r) &= a_1(1 - r^n) \\ S_n &= \frac{a_1(1 - r^n)}{1 - r} && \text{Divide by } (1 - r). \end{aligned}$$

Definition

Sum of the First n Terms of a Geometric Sequence: The sum of the first n terms, S_n, of a geometric sequence with first term a_1 and common ratio r is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

where $r \neq 1$.

Example 6

Find the sum of the first four terms of the geometric sequence with first term 2 and common ratio 5.

Solution

$a_1 = 2$, $r = 5$, and $n = 4$. We are asked to find S_4, the sum of the first four terms of the geometric sequence. Use the formula

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

$$S_4 = \frac{2[1 - (5)^4]}{1 - 5} \qquad \text{Let } n = 4, r = 5, \text{ and } a_1 = 2.$$

$$S_4 = \frac{2(1 - 625)}{-4} \qquad 5^4 = 625$$

$$S_4 = \frac{2(-624)}{-4}$$

$$S_4 = 312$$

We will verify that this result is the same as the result we would obtain by finding the first four terms of the sequence and then finding their sum.

$$a_1 = 2 \qquad a_2 = 2 \cdot 5 = 10 \qquad a_3 = 10 \cdot 5 = 50 \qquad a_4 = 50 \cdot 5 = 250$$

The terms are 2, 10, 50, and 250. Their sum is $2 + 10 + 50 + 250 = 312$. ■

You Try 6

Find the sum of the first five terms of the geometric sequence with first term 3 and common ratio 4.

Using summation notation, $\sum_{i=1}^{n} a \cdot b^i$ (where a and b are constants) represents a geometric series or the sum of the first n terms of a geometric sequence. Furthermore, the first term is found by substituting 1 for i (so that the first term is ab) and the common ratio is b. We can evaluate $\sum_{i=1}^{n} a \cdot b^i$ using the formula for S_n.

Example 7

Evaluate $\sum_{i=1}^{5} 6(2)^i$.

Solution

Use the formula $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$ to find the sum. If we let $i = 1$, we obtain $a_1 = 12$, $r = 2$, and $n = 5$.

Substitute these values into the formula for S_n.

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

$$S_5 = \frac{12(1 - 2^5)}{1 - 2} \qquad \text{Let } n = 5, a_1 = 12, \text{ and } r = 2.$$

$$S_5 = \frac{12(1 - 32)}{-1} \qquad 2^5 = 32$$

$$S_5 = -12(-31)$$

$$S_5 = 372$$

$$\sum_{i=1}^{5} 6 \cdot 2^i = 372$$

You Try 7

Evaluate $\sum_{i=1}^{4} 3(5)^i$.

8. Find the Sum of Terms of an Infinite Geometric Sequence

Until now, we have considered only the sum of the first n terms of a geometric sequence. That is, we have discussed the sum of a *finite* series. Is it possible to find the sum of an *infinite* series?

Consider a geometric series with common ratio $r = \frac{1}{2}$. What happens to the value of $\left(\frac{1}{2}\right)^n$, or r^n, as n gets larger?

We will make a table of values containing n and $\left(\frac{1}{2}\right)^n$.

n	$r^n = \left(\frac{1}{2}\right)^n$
1	$\frac{1}{2} = 0.5$
2	$\frac{1}{4} = 0.25$
3	$\frac{1}{8} = 0.125$
4	$\frac{1}{16} = 0.0625$
5	$\frac{1}{32} = 0.03125$
6	$\frac{1}{64} = 0.015625$
$\vdots$	
15	$\frac{1}{32{,}768} \approx 0.0000305$

At the left you can see that as the value of n gets larger, the value of r^n gets smaller. In fact, the value of r^n gets closer and closer to zero.

We say that as n approaches infinity, r^n approaches zero.

How does this affect the formula for the sum of the first n terms of a geometric sequence?

The formula is $S_n = \frac{a_1(1 - r^n)}{1 - r}$.

If n approaches infinity and r^n approaches zero, we get $S = \frac{a_1(1 - 0)}{1 - r} = \frac{a_1}{1 - r}$.

This formula will hold for $|r| < 1$.

Definition

Sum of the Terms of an Infinite Geometric Sequence: The sum of the terms, S, of an infinite geometric sequence with first term a_1 and common ratio r, where $|r| < 1$, is given by

$$S = \frac{a_1}{1 - r}.$$

If $|r| \geq 1$, then the sum does not exist.

Example 8

Find the sum of the terms of the infinite geometric sequence $6, 4, \frac{8}{3}, \frac{16}{9}, \ldots$.

Solution

We will use the formula $S = \frac{a_1}{1 - r}$, so we must identify a_1 and find r. $a_1 = 6$.

$$r = \frac{4}{6} = \frac{2}{3}$$

Since $|r| = \left|\frac{2}{3}\right| < 1$, the sum exists. Substitute $a_1 = 6$ and $r = \frac{2}{3}$ into the formula.

$$S = \frac{a_1}{1 - r} = \frac{6}{1 - \frac{2}{3}} = \frac{6}{\frac{1}{3}} = 6 \cdot 3 = 18$$

The sum is 18.

You Try 8

Find the sum of the terms of the infinite geometric sequence $1, \frac{3}{5}, \frac{9}{25}, \frac{27}{125}, \ldots$.

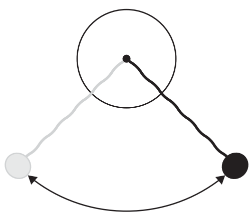
BE CAREFUL

Remember, if $|r| \geq 1$ then the sum does not exist!

9. Solve an Applied Problem Involving an Infinite Series

Example 9

Each time a certain pendulum swings, it travels 80% of the distance it traveled on the previous swing. If it travels 2 ft on its first swing, find the total distance the pendulum travels before coming to rest.



Solution

The geometric series that models this problem is

$$2 + 1.6 + 1.28 + \cdots$$

where

$$\begin{aligned} 2 &= \text{number of feet traveled on the first swing} \\ 2(0.80) = 1.6 &= \text{number of feet traveled on the second swing} \\ 1.6(0.80) = 1.28 &= \text{number of feet traveled on the third swing} \\ &\ \text{etc.} \end{aligned}$$

We can use the formula $S = \frac{a_1}{1 - r}$ with $a_1 = 2$ and $r = 0.80$ to find the total distance the pendulum travels before coming to rest.

$$S = \frac{a_1}{1 - r} = \frac{2}{1 - 0.80} = \frac{2}{0.20} = 10$$

The pendulum travels 10 ft before coming to rest.

You Try 9

Each time a certain pendulum swings, it travels 90% of the distance it traveled on the previous swing. If it travels 20 in. on its first swing, find the total distance the pendulum travels before coming to rest.

10. Distinguish Between an Arithmetic and a Geometric Sequence

Example 10

Determine whether each sequence is arithmetic or geometric. Then find the sum of the first six terms of each sequence.

a) $-8, -4, -2, -1, \ldots$ b) $-9, -4, 1, 6, \ldots$

Solution

a) If every term of the sequence is obtained by *adding* the same constant to the previous term, then the sequence is arithmetic. (The sequence has a common difference, d.)

If every term is obtained by *multiplying* the previous term by the same constant, then the sequence is geometric. (The sequence has a common ratio, r.)

By inspection we can see that the terms of the sequence

$$-8, -4, -2, -1, \ldots$$

are *not* obtained by adding the same amount to each term. For example, $-4 = -8 + 4$, but $-2 = -4 + 2$.

Is there a common ratio?

$$\frac{-4}{-8} = \frac{1}{2}, \quad \frac{-2}{-4} = \frac{1}{2}, \quad \frac{-1}{-2} = \frac{1}{2}$$

Yes, $r = \frac{1}{2}$. *The sequence is geometric.*

Use $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$ with $a_1 = -8$, $r = \frac{1}{2}$, and $n = 6$ to find S_6, the sum of the first six terms of this geometric sequence.

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

$$S_6 = \frac{-8\left[1 - \left(\frac{1}{2}\right)^6\right]}{1 - \left(\frac{1}{2}\right)} \qquad \text{Let } a_1 = -8,\ r = \tfrac{1}{2}, \text{ and } n = 6.$$

$$S_6 = \frac{-8\left(1 - \frac{1}{64}\right)}{\frac{1}{2}} \qquad \left(\tfrac{1}{2}\right)^6 = \tfrac{1}{64}$$

$$S_6 = \frac{-8\left(\frac{63}{64}\right)}{\frac{1}{2}} \qquad \text{Subtract.}$$

$$S_6 = \frac{-\frac{63}{8}}{\frac{1}{2}} = -\frac{63}{8} \cdot 2 = -\frac{63}{4}$$

The sum of the first six terms of the geometric sequence $-8, -4, -2, -1, \ldots$ is $-\frac{63}{4}$.

b) Each term in the sequence $-9, -4, 1, 6,$ is obtained by adding 5 to the previous term. *This is an arithmetic sequence with* $a_1 = -9$ *and common difference* $d = 5$.
Since we know a_1, d, and n ($n = 6$), we will use the formula

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

to find S_6, the sum of the first six terms of this arithmetic sequence.

$$S_6 = \frac{6}{2}[2(-9) + (6-1)5] \qquad \text{Let } a_1 = -9,\ d = 5, \text{ and } n = 6.$$
$$S_6 = 3[-18 + (5)(5)]$$
$$S_6 = 3[-18 + 25] = 3(7) = 21$$

The sum of the first six terms of the arithmetic sequence $-9, -4, 1, 6, \ldots$ is 21.

You Try 10

Determine whether each sequence is arithmetic or geometric. Then, find the sum of the first seven terms of each sequence.

a) $25, 22, 19, 16, \ldots$ b) $\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \ldots$

Answers to You Try Exercises

1) a) 6 b) $\frac{1}{3}$ 2) 2, 4, 8, 16, 32 3) $a_n = 2(3)^{n-1}$; $a_5 = 162$ 4) $a_7 = -\frac{25}{32}$

5) a) $a_n = 27{,}000(0.70)^{n-1}$ b) \$13,230 6) $S_5 = 1023$ 7) 2340 8) $\frac{5}{2}$

9) 200 in. 10) a) arithmetic; $S_7 = 112$ b) geometric; $S_7 = \frac{127}{6}$

15.3 Exercises

Mixed Exercises: Objectives 1 and 2

1) What is the difference between an arithmetic and a geometric series?

2) Give an example of a geometric sequence.

Find the common ratio, r, for each geometric sequence.

3) $1, 2, 4, 8, \ldots$

4) $3, 12, 48, 192, \ldots$

5) $9, 3, 1, \frac{1}{3}, \ldots$

6) $8, 4, 2, 1, \ldots$

7) $-2, \frac{1}{2}, -\frac{1}{8}, \frac{1}{32}, \ldots$

8) $2, -6, 18, -54, \ldots$

Objective 3: Find the Terms of a Geometric Sequence

Write the first five terms of the geometric sequence with the given first term and common ratio.

9) $a_1 = 2, r = 5$

10) $a_1 = 3, r = 2$

11) $a_1 = \frac{1}{4}, r = -2$

12) $a_1 = 250, r = \frac{1}{5}$

13) $a_1 = 72, r = \frac{2}{3}$

14) $a_1 = -20, r = -\frac{3}{2}$

Mixed Exercises: Objectives 4 and 5

Find the general term, a_n, for each geometric sequence. Then, find the indicated term.

15) $a_1 = 4, r = 7; a_3$

16) $a_1 = 3, r = 8; a_3$

17) $a_1 = -1, r = 3; a_5$

18) $a_1 = -5, r = -\frac{1}{3}; a_4$

19) $a_1 = 2, r = \frac{1}{5}; a_4$

20) $a_1 = 7, r = 3; a_5$

21) $a_1 = -\frac{1}{2}, r = -\frac{3}{2}; a_4$

22) $a_1 = \frac{3}{5}, r = 2; a_6$

Find the general term of each geometric sequence.

23) $5, 10, 20, 40, \ldots$

24) $4, 12, 36, 108, \ldots$

25) $-3, -\frac{3}{5}, -\frac{3}{25}, -\frac{3}{125}, \ldots$

26) $-1, 4, -16, 64, \ldots$

27) $3, -6, 12, -24, \ldots$

28) $2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots$

29) $\frac{1}{3}, \frac{1}{12}, \frac{1}{48}, \frac{1}{192}, \ldots$

30) $-\frac{1}{5}, -\frac{3}{10}, -\frac{9}{20}, -\frac{27}{40}, \ldots$

Find the indicated term of each geometric sequence.

31) $1, 2, 4, 8, \ldots; a_{12}$

32) $1, 3, 9, 27, \ldots; a_{10}$

33) $27, -9, 3, -1, \ldots; a_8$

34) $-\frac{1}{125}, -\frac{1}{25}, -\frac{1}{5}, -1, \ldots; a_7$

35) $-\frac{1}{64}, -\frac{1}{32}, -\frac{1}{16}, -\frac{1}{8}, \ldots; a_{12}$

36) $-5, 10, -20, 40, \ldots; a_8$

Objective 10: Distinguish Between an Arithmetic and a Geometric Sequence

Determine whether each sequence is arithmetic or geometric. Then, find the general term, a_n, of the sequence.

37) $15, 24, 33, 42, 51, \ldots$

38) $-1, -3, -9, -27, -81, \ldots$

39) $-2, 6, -18, 54, -162, \ldots$

40) $8, 3, -2, -7, -12, \ldots$

41) $\frac{1}{9}, \frac{1}{18}, \frac{1}{36}, \frac{1}{72}, \frac{1}{144}, \ldots$

42) $11, 22, 44, 88, 176, \ldots$

43) $-31, -24, -17, -10, -3, \ldots$

44) $\frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$

Objective 6: Solve an Applied Problem Involving a Geometric Sequence

Solve each application.

45) A sports car purchased for \$40,000 depreciates 20% each year.

a) Find the general term, a_n, of the geometric sequence that models the value of the sports car at the beginning of each year.

b) How much is the sports car worth at the beginning of the fifth year?

46) A luxury car purchased for \$64,000 depreciates 15% each year.

a) Find the general term, a_n, of the geometric sequence that models the value of the car at the beginning of each year.

b) How much is the luxury car worth at the beginning of the fourth year?

47) A company's advertising budget is currently \$500,000 per year. For the next several years, the company will cut the budget by 10% per year.

a) Find the general term, a_n, of the geometric sequence that models the company's advertising budget for each of the next several years.

b) What is the advertising budget 3 years from now?

48) In January 2011, approximately 1000 customers at a grocery store used the self-checkout lane. The owners predict that number will increase by 20% per month for the next year.

a) Find the general term, a_n, of the geometric sequence that models the number of customers expected to use the self-checkout lane each month for the next year.

b) Predict how many people will use the self-checkout lane in September 2011. Round to the nearest whole number.

49) A home purchased for \$160,000 increases in value by 4% per year.

a) Find the general term of the geometric sequence that models the future value of the house.

b) How much is the home worth 5 years after it is purchased? (Hint: Think carefully about what number to substitute for n.) Round the answer to the nearest dollar.

50) A home purchased for \$140,000 increases in value by 5% per year.

a) Find the general term of the geometric sequence that models the future value of the house.

b) How much is the home worth 8 years after it is purchased? (Hint: Think carefully about what number to substitute for n.) Round the answer to the nearest dollar.

Objective 7: Find the Sum of Terms of a Geometric Sequence

51) Find the sum of the first six terms of the geometric sequence with $a_1 = 9$ and $r = 2$.

52) Find the sum of the first four terms of the geometric sequence with $a_1 = 6$ and $r = 3$.

Use the formula for S_n to find the sum of the terms of each geometric sequence.

53) 7, 28, 112, 448, 1792, 7168, 28672

54) $-5, -30, -180, -1080, -6480$

55) $-\frac{1}{4}, -\frac{1}{2}, -1, -2, -4, -8$

56) $\frac{3}{8}, \frac{3}{2}, 6, 24, 96, 384$

57) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$

58) $-3, \frac{6}{5}, -\frac{12}{25}, \frac{24}{125}, -\frac{48}{625}$

59) $\sum_{i=1}^{7} 9(2)^i$

60) $\sum_{i=1}^{8} 5(2)^i$

61) $\sum_{i=1}^{5} (-4)(3^i)$

62) $\sum_{i=1}^{6} (-7)(-2)^i$

63) $\sum_{i=1}^{6} 3\left(-\frac{1}{2}\right)^i$

64) $\sum_{i=1}^{5} 2\left(\frac{1}{3}\right)^i$

65) $\sum_{i=1}^{4} (-18)\left(-\frac{2}{3}\right)^i$

66) $\sum_{i=1}^{4} 10\left(-\frac{2}{5}\right)^i$

67) Gemma decides to save some pennies so that she'll put 1¢ in her bank on the first day, 2¢ on the second day, 4¢ on the third day, 8¢ on the fourth day, and so on. If she continued in this way,

a) how many pennies would she have to put in her bank on the tenth day to continue the pattern?

b) how much money will she have saved after 10 days?

68) The number of bacteria in a culture doubles every day. If a culture begins with 1000 bacteria, how many bacteria are present after 7 days?

Objective 8: Find the Sum of Terms of an Infinite Geometric Sequence

Find the sum of the terms of the infinite geometric sequence, if possible.

69) $a_1 = 8, r = \frac{1}{4}$

70) $a_1 = 18, r = \frac{1}{3}$

71) $a_1 = 5, r = -\frac{4}{5}$

72) $a_1 = 20, r = -\frac{3}{4}$

73) $a_1 = 9, r = \frac{5}{3}$

74) $a_1 = 3, r = \frac{3}{2}$

75) $8, \frac{16}{3}, \frac{32}{9}, \frac{64}{27}, \dots$

76) $\frac{7}{2}, \frac{7}{4}, \frac{7}{8}, \frac{7}{16}, \dots$

77) $-\frac{15}{2}, \frac{15}{4}, -\frac{15}{8}, \frac{15}{16}, \dots$

78) $-12, 8, -\frac{16}{3}, \frac{32}{9}, \dots$

79) $\frac{1}{25}, \frac{1}{5}, 1, 5, \dots$

80) $36, 6, 1, \frac{1}{6}, \dots$

81) $-40, -30, -\frac{45}{2}, -\frac{135}{8}, \dots$

82) $4, -12, 36, -108, \dots$

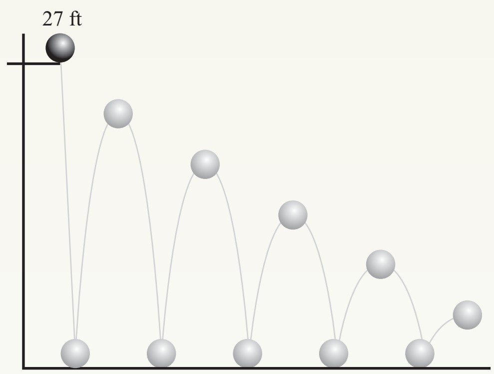
Objective 9: Solve an Applied Problem Involving an Infinite Series

Solve each application.

83) Each time a certain pendulum swings, it travels 75% of the distance it traveled on the previous swing. If it travels 3 ft on its first swing, find the total distance the pendulum travels before coming to rest.

84) Each time a certain pendulum swings, it travels 70% of the distance it traveled on the previous swing. If it travels 42 in. on its first swing, find the total distance the pendulum travels before coming to rest.

85) A ball is dropped from a height of 27 ft. Each time the ball bounces, it rebounds to $\frac{2}{3}$ of its previous height.



a) Find the height the ball reaches after the fifth bounce.

b) Find the total vertical distance the ball has traveled when it comes to rest.

86) A ball is dropped from a height of 16 ft. Each time the ball bounces, it rebounds to $\frac{3}{4}$ of its previous height.

a) Find the height the ball reaches after the fourth bounce.

b) Find the total vertical distance the ball has traveled when it comes to rest.

Section 15.4 The Binomial Theorem

Objectives

1. Expand $(a + b)^n$ Using Pascal's Triangle
2. Evaluate Factorials
3. Evaluate $\binom{n}{r}$
4. Expand $(a + b)^n$ Using the Binomial Theorem
5. Find a Specified Term in the Expansion of $(a + b)^n$

In this section, we will learn how to expand a binomial, $(a + b)^n$, where n is a nonnegative integer. We first encountered expansion of binomials in Chapter 6 when we learned to expand binomials such as $(a + b)^2$ and $(a + b)^3$. To expand $(a + b)^2$ means to find the product $(a + b)(a + b)$. Multiplying the binomials using FOIL gives us

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b) \\ &= a^2 + ab + ab + b^2 && \text{Multiply using FOIL.} \\ &= a^2 + 2ab + b^2\end{aligned}$$

To expand $(a + b)^3$ means to find the product $(a + b)(a + b)(a + b)$ or $(a + b)(a + b)^2$. Expanding $(a + b)^3$ gives us

$$\begin{aligned}(a + b)^3 &= (a + b)(a + b)(a + b) \\ &= (a + b)(a + b)^2 \\ &= (a + b)(a^2 + 2ab + b^2) \\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 && \text{Distribute.} \\ &= a^3 + 3a^2b + 3ab^2 + b^3 && \text{Combine like terms.}\end{aligned}$$

Expanding, for example, $(a + b)^5$ in this way would be an extremely long process. There are other ways to expand binomials, and the first one we will discuss is *Pascal's triangle*.

1. Expand $(a + b)^n$ Using Pascal's Triangle

Here are the expansions of $(a + b)^n$ for several values of n:

$$\begin{aligned}(a + b)^0 &= 1 \\ (a + b)^1 &= a + b \\ (a + b)^2 &= a^2 + 2ab + b^2 \\ (a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ (a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ (a + b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

Notice the following patterns in the expansion of $(a + b)^n$:

1) There are $n + 1$ terms in the expansion of $(a + b)^n$. For example, in the expansion of $(a + b)^4$, $n = 4$ and the expansion contains $4 + 1 = 5$ terms.
2) The first term is a^n and the last term is b^n.
3) Reading the expansion from left to right, the exponents on a **decrease by 1** from one term to the next, while the exponents on b **increase by 1** from one term to the next.
4) In each term in the expansion, the sum of the exponents of the variables is n.

The coefficients of the terms in the expansion follow a pattern too. If we write the coefficients in triangular form, we obtain **Pascal's triangle**, named after seventeenth-century French mathematician Blaise Pascal. *The numbers in the nth row of the triangle tell us the coefficients of the terms in the expansion of* $(a + b)^n$.

Coefficients of the Terms in the Expansion of:	Pascal's Triangle
$(a + b)^0$:	1
$(a + b)^1$:	1 1
$(a + b)^2$:	1 2 1
$(a + b)^3$:	1 3 3 1
$(a + b)^4$:	1 4 6 4 1
$(a + b)^5$:	1 5 10 10 5 1
	etc.

Notice that the first and last numbers of each row in the triangle are 1. The other numbers in the triangle are obtained by adding the two numbers above it. For example, here is how to obtain the sixth row from the fifth:

$$\begin{array}{lcccccccccccc}
\text{Fifth row } (n = 4): & & 1 & & 4 & & 6 & & 4 & & 1 & \\
& & & + & & + & & + & & + & & \\
\text{Sixth row } (n = 5): & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Example 1

Expand $(a + b)^6$.

Solution

The coefficients of the terms in $(a + b)^5$ are given by the last row of the triangle above. We must find the next row of the triangle to find the coefficients of the terms in the expansion of $(a + b)^6$.

$$\begin{array}{lccccccccccccc}
(a + b)^5: & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
& & & + & & + & & + & & + & & + & & \\
(a + b)^6: & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Recall that the first term is a^n and the last term is b^n. Since $n = 6$, the first term will be a^6, and the exponent of a will decrease by 1 for each term. The variable b will appear in the second term and increase by 1 for each term until the last term, b^6.

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

You Try 1

Expand $(a + b)^7$.

Although Pascal's triangle is a better way to expand $(a + b)^n$ than doing repeated polynomial multiplication, it can be tedious for large values of n. A more practical way to expand a binomial is by using the **binomial theorem**. Before learning this method, we need to learn about **factorials** and **binomial coefficients**.

2. Evaluate Factorials

The notation $n!$ is read as "n factorial."

Definition

$n! = n(n - 1)(n - 2)(n - 3) \cdots (1)$, where n is a positive integer.

Note

By definition, $0! = 1$.

Example 2

Evaluate.

a) $4!$ b) $7!$

Solution

a) $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ b) $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

You Try 2

Evaluate.

a) $3!$ b) $6!$

3. Evaluate $\binom{n}{r}$

Factorials are used to evaluate binomial coefficients. A **binomial coefficient** has the form $\binom{n}{r}$, read as "the number of combinations of n items taken r at a time" or as "n choose r." $\binom{n}{r}$ is used extensively in many areas of mathematics including probability. Another notation for $\binom{n}{r}$ is nCr.

Definition

Binomial Coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where n and r are positive integers and $r \le n$.

Example 3

Evaluate.

a) $\binom{5}{3}$ b) $\binom{9}{2}$ c) $\binom{3}{3}$ d) $\binom{4}{0}$

Solution

a) To evaluate $\binom{5}{3}$, substitute 5 for n and 3 for r.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} \qquad \text{Let } n = 5 \text{ and } r = 3.$$

$$= \frac{5!}{3!2!} \qquad \text{Subtract.}$$

$$= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(2 \cdot 1)} \qquad \text{Rewrite each factorial as a product.}$$

At this point, do *not* find the products in the numerator and denominator. Instead, divide out common factors in the numerator and denominator.

$$= \frac{5 \cdot 4 \cdot \cancel{3 \cdot 2 \cdot 1}}{\cancel{(3 \cdot 2 \cdot 1)}(2 \cdot 1)} \quad \text{Divide out common factors.}$$

$$= \frac{20}{2} \quad \text{Multiply.}$$

$$= 10 \quad \text{Simplify.}$$

$$\binom{5}{3} = 10$$

b) To evaluate $\binom{9}{2}$, substitute 9 for n and 2 for r.

$$\binom{9}{2} = \frac{9!}{2!(9-2)!} \quad \text{Let } n = 9 \text{ and } r = 2.$$

$$= \frac{9!}{2!\,7!} \quad \text{Subtract.}$$

$$= \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)} \quad \text{Rewrite each factorial as a product.}$$

$$= \frac{9 \cdot 8 \cdot \cancel{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}}{(2 \cdot 1)\cancel{(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)}} \quad \text{Divide out common factors.}$$

$$= \frac{72}{2}$$

$$= 36$$

$$\binom{9}{2} = 36$$

c) To evaluate $\binom{3}{3}$, substitute 3 for n and for r.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\binom{3}{3} = \frac{3!}{3!(3-3)!} \quad \text{Let } n = 3 \text{ and } r = 3.$$

$$= \frac{3!}{3!\,0!} \quad \text{Subtract.}$$

$$= \frac{\cancel{3!}}{\cancel{3!}(1)} \quad \text{Divide out common factors; } 0! = 1.$$

$$= \frac{1}{1} = 1 \quad \text{Simplify.}$$

$$\binom{3}{3} = 1$$

d) To evaluate $\binom{4}{0}$, substitute 4 for n and 0 for r.

$$\binom{4}{0} = \frac{4!}{0!(4-0)!} \quad \text{Let } n = 4 \text{ and } r = 0.$$

$$= \frac{4!}{0!\,4!} \quad \text{Subtract.}$$

$$= \frac{\cancel{4!}}{(1)\cancel{(4!)}} \quad \text{Divide out common factors; } 0! = 1.$$

$$= \frac{1}{1} = 1 \quad \text{Simplify.}$$

$$\binom{4}{0} = 1$$

Note

We can extend the results of c) and d) and say that for any natural number n,

$$\binom{n}{n} = 1 \text{ and } \binom{n}{0} = 1$$

You Try 3

Evaluate.

a) $\binom{4}{1}$ b) $\binom{8}{5}$ c) $\binom{3}{0}$ d) $\binom{6}{6}$

4. Expand $(a + b)^n$ Using the Binomial Theorem

Now that we can evaluate a binomial coefficient, we state the binomial theorem for expanding $(a + b)^n$.

Definition

Binomial Theorem: For any positive integer n,

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

The same patterns that emerged in the expansion of $(a + b)^n$ using Pascal's triangle appear when using the binomial theorem. Keep in mind that

1) there are $n + 1$ terms in the expansion.
2) the first term in the expansion is a^n and the last term is b^n.
3) after a^n, the exponents on *a decrease by 1* from one term to the next, while b is introduced in the second term and then the exponents on *b increase by 1* from one term to the next.
4) in each term in the expansion, the sum of the exponents of the variables is n.

Example 4

Use the binomial theorem to expand $(a + b)^4$.

Solution

Let $n = 4$ in the binomial theorem.

$$(a + b)^4 = a^4 + \binom{4}{1}a^{4-1}b + \binom{4}{2}a^{4-2}b^2 + \binom{4}{3}a^{4-3}b^3 + b^4$$
$$= a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + b^4$$

Notice that the exponents of a decrease by 1 while the exponents of b increase by 1.

$$= a^4 + \frac{4!}{1!3!}a^3b + \frac{4!}{2!2!}a^2b^2 + \frac{4!}{3!1!}ab^3 + b^4$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

This is the same result as the expansion on p. 918.

You Try 4

Use the binomial theorem to expand $(a+b)^3$.

Example 5

Use the binomial theorem to expand $(x+6)^3$.

Solution

Substitute x for a, 6 for b, and 3 for n in the binomial theorem to expand $(x+6)^3$.

$$\begin{aligned}(x+6)^3 &= (x)^3 + \binom{3}{1}(x)^{3-1}(6) + \binom{3}{2}(x)^{3-2}(6)^2 + (6)^3 \\ &= x^3 + 3(x^2)(6) + (3)x(36) + 216 \\ &= x^3 + 18x^2 + 108x + 216\end{aligned}$$

You Try 5

Use the binomial theorem to expand $(y+5)^4$.

BE CAREFUL

When expanding a binomial containing the *difference* of two terms, rewrite the expression in terms of addition.

Example 6

Use the binomial theorem to expand $(2x-3y)^5$.

Solution

Since the binomial theorem applies to the expansion of $(a+b)^n$, rewrite $(2x-3y)^5$ as $[2x+(-3y)]^5$.

Substitute $2x$ for a, $-3y$ for b, and 5 for n in the binomial theorem. *Be sure to put $2x$ and $-3y$ in parentheses to find the expansion correctly.*

$$\begin{aligned}[2x+(-3y)]^5 &= (2x)^5 + \binom{5}{1}(2x)^{5-1}(-3y) + \binom{5}{2}(2x)^{5-2}(-3y)^2 \\ &\quad + \binom{5}{3}(2x)^{5-3}(-3y)^3 + \binom{5}{4}(2x)^{5-4}(-3y)^4 + (-3y)^5 \\ &= 32x^5 + (5)(2x)^4(-3y) + (10)(2x)^3(9y^2) + (10)(2x)^2(-27y^3) \\ &\quad + (5)(2x)^1(81y^4) + (-243y^5) \\ &= 32x^5 + (5)(16x^4)(-3y) + (10)(8x^3)(9y^2) \\ &\quad + (10)(4x^2)(-27y^3) + (5)(2x)(81y^4) + (-243y^5) \\ &= 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5\end{aligned}$$

You Try 6

Use the binomial theorem to expand $(3x-4y)^4$.

5. Find a Specified Term in the Expansion of $(a + b)^n$

If we want to find a specific term of a binomial expansion without writing out the entire expansion, we can use the following formula.

Definition

The *k*th Term of a Binomial Expansion: The kth term of the expansion of $(a + b)^n$ is given by

$$\frac{n!}{(n - k + 1)!(k - 1)!} a^{n-k+1} b^{k-1}$$

where $k \le n + 1$.

Example 7

Find the fifth term in the expansion of $(c^2 + 2d)^8$.

Solution

Since we want to find the fifth term, $k = 5$, use the formula above with $a = c^2$, $b = 2d$, $n = 8$, and $k = 5$. The fifth term is

$$\begin{aligned} \frac{8!}{(8 - 5 + 1)!(5 - 1)!}(c^2)^{8-5+1}(2d)^{5-1} &= \frac{8!}{4!4!}(c^2)^4(2d)^4 \\ &= 70c^8(16d^4) \\ &= 1120c^8d^4 \end{aligned}$$

You Try 7

Find the sixth term in the expansion of $(2m + n^2)^9$.

Using Technology

We will discuss how to compute factorials and the binomial coefficient on a graphing calculator. Sometimes it is quicker to calculate them by hand, and sometimes a calculator will make our work easier.

Evaluating 3! can be done very easily by multiplying: $3! = 3 \times 2 \times 1 = 6$. To find 10! by hand we would multiply: $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3{,}628{,}800$. On a graphing calculator, we could find 10! either by performing this multiplication or we can use a special function.

Graphing calculators have a factorial key built in. It is found using the MATH key. When you press MATH, move the arrow over to the PRB column, and you will see this menu:

```
MATH NUM CPX PRB
1:rand
2:nPr
3:nCr
4:!
5:randInt(
6:randNorm(
7:randBin(
```

Notice that choice 4 is the factorial symbol.

To compute 10!, enter 10 and then press MATH. Highlight PRB so that you see the screen at above right. Choose 4: ! and press ENTER. The screen displays 10!. Press ENTER to see that $10! = 3{,}628{,}800$.

```
10!
          3628800
```

Because the binomial coefficient, $\binom{n}{r}$, is used so often in mathematical applications, most graphing calculators have a built-in key that performs the calculations for you. It is located on the same menu as the factorial. Refer to the first calculator screen to see that ${}_nC_r$ is choice number 3.

To find the value of $\binom{9}{4}$, press $\boxed{9}$ $\boxed{\text{MATH}}$, and then highlight PRB. Choose 3: nCr and press $\boxed{\text{ENTER}}$. Now enter 4 and press $\boxed{\text{ENTER}}$. The screen will look like the next screen here. The value of $\binom{9}{4}$ is 126.

```
9 nCr 4
                126
```

Although the calculator has functions to evaluate factorials and binomial coefficients, sometimes it is actually quicker to evaluate them by hand. Think about this as you evaluate the following problems.

Evaluate each of the following using the methods discussed in this section. Verify the result using a graphing calculator. Think about which method you prefer for each problem.

1) 4!	2) 6!	3) 9!	4) $\binom{5}{2}$
5) $\binom{7}{4}$	6) $\binom{15}{8}$	7) $\binom{18}{14}$	8) $\binom{25}{24}$

Answers to You Try Exercises

1) $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

2) a) 6 b) 720 3) a) 4 b) 56 c) 1 d) 1 4) $a^3 + 3a^2b + 3ab^2 + b^3$

5) $y^4 + 20y^3 + 150y^2 + 500y + 625$ 6) $81x^4 - 432x^3y + 864x^2y^2 - 768xy^3 + 256y^4$

7) $2016m^4n^{10}$

Answers to Technology Exercises

1) 24 2) 720 3) 362,880 4) 10 5) 35 6) 6435 7) 3060 8) 25

15.4 Exercises

Objective 1: Expand $(a + b)^n$ Using Pascal's Triangle

1) In your own words, explain how to construct Pascal's triangle.

2) What are the first and last terms in the expansion of $(a + b)^n$?

Use Pascal's triangle to expand each binomial.

3) $(r + s)^3$

4) $(m + n)^4$

5) $(y + z)^5$

6) $(c + d)^6$

7) $(x + 5)^4$

8) $(k + 2)^5$

Objective 2: Evaluate Factorials

9) In your own words, explain how to evaluate $n!$ for any positive integer.

10) Evaluate 0!.

Evaluate.

11) 2!

12) 3!

VIDEO 13) 5!

14) 6!

Objective 3: Evaluate $\binom{n}{r}$

Evaluate each binomial coefficient.

15) $\binom{5}{2}$

16) $\binom{4}{2}$

17) $\binom{7}{3}$

18) $\binom{8}{5}$

19) $\binom{10}{4}$

20) $\binom{9}{3}$

21) $\binom{9}{7}$

22) $\binom{11}{8}$

23) $\binom{4}{4}$

24) $\binom{5}{5}$

25) $\binom{6}{1}$

26) $\binom{3}{1}$

27) $\binom{5}{0}$

28) $\binom{7}{0}$

Objective 4: Expand $(a + b)^n$ Using the Binomial Theorem

29) How many terms are in the expansion of $(a + b)^9$?

30) Before expanding $(t - 4)^6$ using the binomial theorem, how should the binomial be rewritten?

Use the binomial theorem to expand each expression.

31) $(f + g)^3$

32) $(c + d)^5$

33) $(w + 2)^4$

34) $(h + 4)^4$

35) $(b + 3)^5$

36) $(t + 9)^3$

37) $(a - 3)^4$

38) $(p - 2)^3$

39) $(u - v)^3$

40) $(p - q)^5$

41) $(3m + 2)^4$

42) $(2k + 1)^4$

43) $(3a - 2b)^5$

44) $(4c - 3d)^4$

45) $(x^2 + 1)^3$

46) $(w^3 + 2)^3$

47) $\left(\frac{1}{2}m - 3n\right)^4$

48) $\left(\frac{1}{3}a + 2b\right)^5$

49) $\left(\frac{1}{3}y + 2z^2\right)^3$

50) $\left(t^2 - \frac{1}{2}u\right)^4$

Objective 5: Find a Specified Term in the Expansion of $(a + b)^n$

Find the indicated term of each binomial expansion.

51) $(k + 5)^8$; third term

52) $(y + 4)^7$; fifth term

53) $(w + 1)^{15}$; tenth term

54) $(z + 3)^9$; seventh term

55) $(q - 3)^9$; second term

56) $(u - 2)^7$; fourth term

57) $(3x - 2)^6$; fifth term

58) $(2w - 1)^9$; seventh term

59) $(2y^2 + z)^{10}$; eighth term

60) $(p + 3q^2)^8$; fifth term

61) $(c^3 - 3d^2)^7$; third term

62) $(2r^3 - s^4)^6$; sixth term

63) $(5u + v^3)^{11}$; last term

64) $(4h - k^4)^{12}$; last term

65) Show that $\binom{n}{n} = 1$ for any positive integer n.

66) Show that $\binom{n}{1} = n$ for any positive integer n.

Chapter 15: Summary

Definition/Procedure	Example
15.1 Sequences and Series	
Finite Sequence A **finite sequence** is a function whose domain is the set of the first n natural numbers. **(p. 884)**	$a_n = \{1, 2, 3, \ldots, n\}$ for $n \geq 0$
Infinite Sequence An **infinite sequence** is a function whose domain is the set of natural numbers. **(p. 884)**	$a = \{1, 2, 3, \ldots\}$
General Term The nth term of the sequence, a_n, is the **general term** of the sequence. **(p. 884)**	Write the first five terms of the sequence with general term $a_n = 5n - 9$. Evaluate $a_n = 5n - 9$ for $n = 1, 2, 3, 4,$ and 5. The first five terms are $-4, 1, 6, 11, 16$.
Given some terms of a sequence, we can find a formula for a_n. **(p. 887)**	Find the general term, a_n, for the sequence $3, 9, 27, 81, 243, \ldots$ Write each term as a power of 3. Term: $3^1, \quad 3^2, \quad 3^3, \quad 3^4, \quad 3^5$ Term number: $a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$ The nth term may be written as $a_n = 3^n$.
Series A sum of the terms of a sequence is called a **series.** A series can be finite or infinite. **(p. 888)**	$S_n = a_1 + a_2 + \cdots + a_n$
Summation Notation $\sum$ (sigma) is shorthand notation for a series. The letters i, j, and k are often used as variables for the **index of summation. (p. 889)**	Evaluate $\displaystyle\sum_{i=1}^{3} (-1)^i \cdot 5^i$. $\displaystyle\sum_{i=1}^{3} (-1)^i \cdot 5^i = (-1)^1 \cdot 5^1 + (-1)^2 \cdot 5^2 + (-1)^3 \cdot 5^3$ $= -5 + 25 + (-125)$ $= -105$
Arithmetic Mean The **average** of a group of numbers is represented by $\bar{x}$, and is given by the formula $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$, where $x_1, x_2, x_3, \ldots, x_n$ are the numbers in the group and n is the number of numbers in the group. **(p. 891)**	Jin's grades on his five sociology tests were 86, 91, 83, 78, and 88. What is his test average? $\bar{x} = \dfrac{\sum_{i=1}^{5} x_i}{5}$, where $x_1 = 86, x_2 = 91, x_3 = 83, x_4 = 78$, and $x_5 = 88$. $\bar{x} = \dfrac{86 + 91 + 83 + 78 + 88}{5} = 85.2$

Definition/Procedure	Example
15.2 Arithmetic Sequences and Series	
Arithmetic Sequence An **arithmetic sequence** is a sequence in which each term after the first differs from the preceding term by a constant amount d. d is called the **common difference. (p. 895)**	10, 13, 16, 19, 22, ... is an arithmetic sequence since each term is 3 more than the previous term. The common difference, d, is 3.
Sum of the First *n* Terms of an Arithmetic Sequence An **arithmetic series** is a sum of terms of an arithmetic sequence. The sum of the first n terms, S_n, is given by $S_n = \frac{n}{2}(a_1 + a_n)$ or $S_n = \frac{n}{2}[2a_1 + (n-1)d]$ **(p. 901)**	Find the sum of the first 20 terms of the arithmetic sequence with first term -9 and last term 29. Since we are given $n = 20$, $a_1 = -9$, and $a_{20} = 29$, find S_{20}. $S_n = \frac{n}{2}(a_1 + a_n)$ $S_{20} = \frac{20}{2}(-9 + a_{20})$ Let $n = 20$ and $a_1 = -9$. $S_{20} = 10(-9 + 29)$ Let $a_{20} = 29$. $S_{20} = 10(20) = 200$
15.3 Geometric Sequences and Series	
A **geometric sequence** is a sequence in which each term after the first is obtained by multiplying the preceding term by a constant, r. r is called the **common ratio. (p. 906)**	$-2, -6, -18, -54, -162, \ldots$ is a geometric sequence since each term after the first is obtained by multiplying the previous term by 3. The common ratio, r, is 3. We can find r by dividing any term by the preceding term: $r = \frac{-6}{-2} = \frac{-18}{-6} = \frac{-54}{-18} = \frac{-162}{-54} = 3$
The **general term of a geometric sequence** with first term a_1 and common ratio r is given by $a_n = a_1 r^{n-1}$. **(p. 908)**	Find a_n for the geometric sequence $-2, -6, -18, -54, -162, \ldots$ $a_1 = -2$ $r = 3$ $a_n = a_1 r^{n-1}$ $a_n = (-2)(3)^{n-1}$ Let $a_1 = -2$ and $r = 3$.
Sum of the First *n* Terms of a Geometric Sequence A **geometric series** is a sum of terms of a geometric sequence. The sum of the first n terms, S_n, of a geometric sequence with first term a_1 and common ratio r is given by $S_n = \frac{a_1(1 - r^n)}{1 - r}$ where $r \neq 1$ **(p. 910)**	Find the sum of the first six terms of the geometric sequence with first term 7 and common ratio 2. $n = 6$, $a_1 = 7$, $r = 2$. Find S_6. $S_n = \frac{a_1(1 - r^n)}{1 - r}$ $S_6 = \frac{7(1 - 2^6)}{1 - 2}$ Let $n = 6$, $a_1 = 7$, $r = 2$. $S_6 = \frac{7(-63)}{-1} = 441$
Sum of the Terms of an Infinite Geometric Sequence The sum of the terms, S, of an infinite geometric sequence with first term a_1 and common ratio r, where $\lvert r \rvert < 1$, is given by $S = \frac{a_1}{1 - r}$. If $\lvert r \rvert \geq 1$, then the sum does not exist. **(p. 912)**	Find the sum of the terms of the infinite geometric sequence $75, 15, 3, \frac{3}{5}, \ldots$ $a_1 = 75$, $r = \frac{15}{75} = \frac{1}{5}$ Since $\lvert r \rvert = \left\lvert \frac{1}{5} \right\rvert < 1$, the sum exists. $S = \frac{a_1}{1 - r} = \frac{75}{1 - \frac{1}{5}} = \frac{75}{\frac{4}{5}} = 75 \cdot \frac{5}{4} = \frac{375}{4}$

Definition/Procedure	Example

15.4 The Binomial Theorem

Pascal's Triangle

The numbers in the rows of Pascal's triangle tell us the coefficients of the terms in the expansion of $(a + b)^n$. **(p. 918)**

Expand $(a + b)^4$.

The coefficients of the terms in the expansion are in the fifth row of Pascal's triangle.

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Binomial Coefficient

$\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$, where n and r are positive integers and $r \le n$.

$$\binom{n}{n} = 1 \quad \text{and} \quad \binom{n}{0} = 1 \text{ **(p. 920)**}$$

Evaluate $\binom{6}{2}$.

$$\binom{6}{2} = \frac{6!}{2!(6-2)!} = \frac{6!}{2!\,4!} = \frac{6 \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{(2 \cdot 1)\cancel{(4 \cdot 3 \cdot 2 \cdot 1)}} \quad \text{Divide out common factors.}$$

$$= \frac{30}{2} = 15$$

Binomial Theorem

For any positive integer n,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n \text{ **(p. 922)**}$$

Use the binomial theorem to expand $(2c + 5)^4$.

Let $a = 2c$, $b = 5$, and $n = 4$.

$$(2c+5)^4 = (2c)^4 + \binom{4}{1}(2c)^3(5) + \binom{4}{2}(2c)^2(5)^2 + \binom{4}{3}(2c)(5)^3 + (5)^4$$

$$= 16c^4 + (4)(8c^3)(5) + 6(4c^2)(25) + (4)(2c)(125) + 625$$

$$= 16c^4 + 160c^3 + 600c^2 + 1000c + 625$$

The *k*th Term of a Binomial Expansion

The kth term of the expansion of $(a + b)^n$ is given by

$\dfrac{n!}{(n-k+1)!(k-1)!}a^{n-k+1}b^{k-1}$, where $k \le n + 1$. **(p. 924)**

Find the sixth term in the expansion of $(x - 2y)^9$.

$$n = 9,\ k = 6,\ a = x,\ b = -2y$$

The sixth term is

$$\frac{9!}{(9-6+1)!(6-1)!}(x)^{9-6+1}(-2y)^{6-1}$$

$$= \frac{9!}{4!5!}(x)^4(-2y)^5$$

$$= 126x^4(-32y^5)$$

$$= -4032x^4y^5$$

Chapter 15: Review Exercises

(15.1) Write out the first five terms of each sequence.

1) $a_n = 7n + 1$

2) $a_n = n^2 - 7$

3) $a_n = (-1)^{n+1} \cdot \left(\frac{1}{n^2}\right)$

4) $a_n = \frac{n+1}{n^2}$

Find a formula for the general term, a_n, of each sequence.

5) $5, 10, 15, 20, \ldots$

6) $1, \frac{1}{8}, \frac{1}{27}, \frac{1}{64}, \ldots$

7) $-2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, \ldots$

8) $-1, 2, -3, 4, \ldots$

9) Currently, Maura earns \$8.25 per hour, and she can get a raise of \$0.25 per hour every 4 months. Write a sequence that represents her current wage as well as her hourly wage 4, 8, and 12 months from now.

10) Dorian wants to increase the number of students at his martial arts school by 20% every 6 months. If he currently has 40 students, how many does he hope to have 18 months from now? Round to the nearest whole number.

11) What is the difference between a sequence and a series?

12) Write in summation notation. *Find the sum of the first eight terms in the sequence given by $a_n = 6n - 1$.* Do **not** evaluate.

Write out the terms in each series and evaluate.

13) $\sum_{i=1}^{5} (2i^2 + 1)$

14) $\sum_{i=1}^{6} (-1)^i(2i)$

Write each series using summation notation.

15) $13 + \frac{13}{2} + \frac{13}{3} + \frac{13}{4}$

16) $2 - 4 + 8 - 16 + 32 - 64$

17) Find the arithmetic mean of this group of numbers: 18, 25, 26, 20, 22

18) The Shiu family's heating bills from December 2010 to March 2011 are given in this table. Find the average amount they paid per month to heat their home during this time period.

Month	Heating Bill
December	\$143.88
January	\$210.15
February	\$227.90
March	\$178.11

(15.2) Write the first five terms of each arithmetic sequence with the given first term and common difference.

19) $a_1 = 11, d = 7$

20) $a_1 = 0, d = -4$

21) $a_1 = -58, d = 8$

22) $a_1 = -1, d = \frac{1}{2}$

For each arithmetic sequence, find a) a_1 and d, b) a formula for the general term of the sequence, a_n, and c) a_{20}.

23) $6, 10, 14, 18, 22, \ldots$

24) $-13, -6, 1, 8, 15, \ldots$

25) $-8, -13, -18, -23, -28, \ldots$

26) $14, 17, 20, 23, 26, \ldots$

Find the indicated term for each arithmetic sequence.

27) $-15, -9, -3, 3, 9, \ldots; a_{15}$

28) $5, 2, -1, -4, -7, \ldots; a_{22}$

29) $a_1 = -4, d = -\frac{3}{2}; a_{21}$

30) $a_1 = -6, d = 7; a_{23}$

Two terms of an arithmetic sequence are given in each problem. Find the general term of the sequence, a_n, and find the indicated term.

31) $a_6 = 24, a_9 = 36; a_{12}$

32) $a_4 = 1, a_{10} = 13; a_{15}$

33) $a_5 = -5, a_{10} = -15; a_{17}$

34) $a_3 = -13, a_8 = -28; a_{20}$

Find the number of terms in each arithmetic sequence.

35) $-4, -2, 0, 2, \ldots, 34$

36) $2, -3, -8, -13, \ldots, -158$

37) $-8, -7, -6, -5, \ldots, 43$

38) $7, 13, 19, 25, \ldots, 109$

39) Find the sum of the first eight terms of the arithmetic sequence with first term 5 and last term -27.

40) Find the sum of the first 15 terms of the arithmetic sequence with the first term -9 and last term 19.

Find S_{10} for each arithmetic sequence described below.

41) $a_1 = -6, d = 7$

42) $a_1 = 8, a_{10} = 35$

43) $a_1 = 13, a_{10} = -59$

44) $a_1 = -11, d = -5$

45) $a_n = 2n - 5$

46) $a_n = n - 7$

47) $a_n = -7n + 16$

48) $a_n = 9n - 2$

Evaluate each sum using a formula for S_n.

49) $\sum_{i=1}^{4} (-11i - 2)$

50) $\sum_{i=1}^{8} (2i - 7)$

51) $\sum_{i=1}^{13} (3i + 4)$

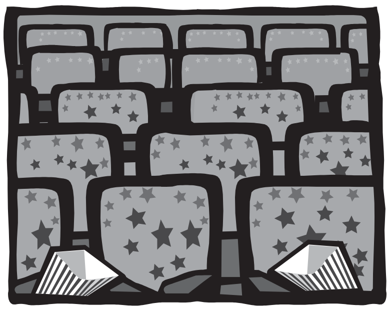
52) $\sum_{i=1}^{15} (-5i + 1)$

53) $\sum_{i=1}^{11} (-4i + 2)$

54) $\sum_{i=1}^{19} (-6i - 2)$

Solve each application.

55) A theater has 20 rows. The first row contains 15 seats, the next row has 17 seats, the next row has 19 seats, and so on. How many seats are in the last row? How many seats are in the theater?



56) A stack of logs has 16 logs in the bottom row, 15 logs in the second row, 14 logs in the third row, and so on until the last row contains one log.

a) How many logs are in the 12th row?

b) How many logs are in the stack?

57) On March 1, Darshan deposits \$2 into his bank. The next week he deposits \$4, the week after that he saves \$6, the next week he deposits \$8, and so on. If he saves his money in this way for a total of 30 weeks, how much money will he have saved?

58) Laura's company currently employs 14 people. She plans to expand her business, so starting next month she will hire three new workers per month for the next 6 months. How many employees will she have after that hiring period?

(15.3) Find the common ratio, r, for each geometric sequence.

59) 4, 20, 100, 500, . . .

60) $8, 2, \frac{1}{2}, \frac{1}{8}, \ldots$

Write the first five terms of the geometric sequence with the given first term and common ratio.

61) $a_1 = 7, r = 2$

62) $a_1 = 5, r = -3$

63) $a_1 = 48, r = \frac{1}{4}$

64) $a_1 = -16, r = \frac{3}{2}$

Find the general term, a_n, for each geometric sequence. Then, find the indicated term.

65) $a_1 = 3, r = 2; a_6$

66) $a_1 = 4, r = 3; a_5$

67) $a_1 = 8, r = \frac{1}{3}; a_4$

68) $a_1 = -\frac{1}{2}, r = -3; a_4$

Find the general term of each geometric sequence.

69) 7, 42, 252, 1512, . . .

70) $-4, -\frac{4}{5}, -\frac{4}{25}, -\frac{4}{125}, \ldots$

71) $-15, 45, -135, 405, \ldots$

72) $\frac{1}{9}, \frac{4}{9}, \frac{16}{9}, \frac{64}{9}, \ldots$

Find the indicated term of each geometric sequence.

73) $1, 3, 9, 27, \ldots; a_8$

74) $4, 2, 1, \frac{1}{2}, \ldots; a_9$

75) Find the sum of the first five terms of the geometric sequence with $a_1 = 8$ and $r = 3$.

76) Find the sum of the first four terms of the geometric sequence with $a_1 = -2$ and $r = 5$.

Use the formula for S_n to find the sum of each geometric sequence.

77) 8, 40, 200, 1000, 5000

78) $-\frac{1}{3}, 1, -3, 9, -27$

79) $8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}$

80) 7, 14, 28, 56, 112, 224

81) $\sum_{i=1}^{5} 7\left(\frac{1}{2}\right)^i$

82) $\sum_{i=1}^{4} 2(3)^i$

Solve each application.

83) An SUV purchased for \$28,000 depreciates 20% each year.

a) Find the general term, a_n, of the geometric sequence that models the value of the SUV at the beginning of each year.

b) How much is the SUV worth at the beginning of the third year?

84) A home purchased for \$180,000 increases in value by 4% per year.

a) Find the general term of the geometric sequence that models the future value of the house.

b) How much is the home worth 6 years after it is purchased? Round the answer to the nearest dollar.

For each sequence,

a) determine whether it is arithmetic or geometric.

b) find a_n.

c) find S_8.

85) $7, 9, 11, 13, \dots$

86) $33, 28, 23, 18, \dots$

87) $9, \frac{9}{2}, \frac{9}{4}, \frac{9}{8}, \dots$

88) $-\frac{9}{2}, -4, -\frac{7}{2}, -3, \dots$

89) $-1, -3, -9, -27, \dots$

90) $5, -10, 20, -40, \dots$

Solve each application.

91) The number of bacteria in a culture increases by 50% each day. If a culture begins with 2000 bacteria, how many bacteria are present after 5 days?

92) Trent will put 2¢ in his bank on one day, 4¢ on the second day, 8¢ on the third day, 16¢ on the fourth day, and so on. If he continues in this way,

a) how much money will Trent have to put in his bank on the 11th day to continue the pattern?

b) how much money will he have saved after 11 days?

93) When does the sum of an infinite geometric sequence exist?

Find the sum of the terms of each infinite geometric sequence, if possible.

94) $a_1 = 24, r = \frac{3}{7}$

95) $a_1 = -3, r = \frac{1}{8}$

96) $a_1 = 9, r = \frac{4}{3}$

97) $-15, 10, -\frac{20}{3}, \frac{40}{9}, \dots$

98) $20, -5, \frac{5}{4}, -\frac{5}{16}, \dots$

99) $-4, 12, -36, 108, \dots$

Solve.

100) Each time a certain pendulum swings, it travels 80% of the distance it traveled on the previous swing. If it travels 4 ft on its first swing, find the total distance the pendulum travels before coming to rest.

(15.4) Use Pascal's triangle to expand each binomial.

101) $(y + z)^4$

102) $(c + d)^3$

Evaluate.

103) $6!$

104) $8!$

105) $\binom{5}{3}$

106) $\binom{7}{2}$

107) $\binom{9}{1}$

108) $\binom{6}{6}$

Use the binomial theorem to expand each expression.

109) $(m + n)^4$

110) $(k + 2)^6$

111) $(h - 9)^3$

112) $(2w - 5)^3$

113) $(2p^2 - 3r)^5$

114) $\left(\frac{1}{3}b + c\right)^4$

Find the indicated term of each binomial expansion.

115) $(z + 4)^8$; fifth term

116) $(y - 6)^8$; sixth term

117) $(2k - 1)^{13}$; 11th term

118) $(3p + q)^9$; seventh term

Chapter 15: Test

Write out the first five terms of each sequence.

1) $a_n = 2n - 3$

2) $a_n = (-1)^{n+1}\left(\frac{n}{n+2}\right)$

3) What is the difference between an arithmetic and a geometric sequence?

4) Write the first five terms of the geometric sequence with first term 32 and common ratio $-\frac{1}{2}$.

5) Find the common difference for the arithmetic sequence $-17, -11, -5, 1, 7, \ldots$

Determine whether each sequence is arithmetic or geometric, and find a_n.

6) $4, 12, 36, 108, 324, \ldots$

7) $5, 2, -1, -4, -7, \ldots$

8) Find the forty-first term of the arithmetic sequence with $a_6 = -3$ and $a_{11} = 7$.

9) Write out each term of $\sum_{i=1}^{4} (5i^2 + 6)$ and find the sum.

10) Use the formula for S_n to find the sum of the first six terms of the geometric sequence with $a_1 = 9$ and $r = 2$.

11) Use a formula for S_n to find the sum of the first 11 terms of the arithmetic sequence with $a_1 = 5$ and $d = 3$.

12) Evaluate $\sum_{i=1}^{100} (-4i + 3)$.

13) Find the sum of the terms of the infinite geometric sequence with $a_1 = 7$ and $r = \frac{3}{10}$.

Solve each application.

14) At a construction site, pipes are stacked so that there are 14 in the bottom row, 13 in the next row, 12 in the next row, and so on until the top row contains one pipe. How many pipes are in the stack?

15) In 2007, a bank found that 11,000 of its customers used online banking. That number continued to increase by 10% per year. How many customers used online banking in 2010? Round the answer to the nearest whole number.

16) Each time a certain pendulum swings, it travels 75% of the distance it traveled on the previous swing. If it travels 40 in. on its first swing, find the total distance the pendulum travels before coming to rest.

Evaluate.

17) $5!$

18) $\binom{6}{2}$

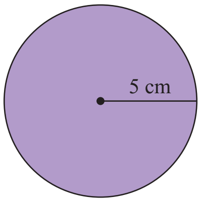
19) Expand using the binomial theorem: $(3x + 1)^4$.

20) Find the fourth term in the expansion of $(k - 2)^9$.

Cumulative Review: Chapters 1–15

1) Add $\frac{5}{8} + \frac{1}{6} + \frac{3}{4}$.

2) Find the a) area and b) circumference of the circle. Give the exact answer for each in terms of π, and give an approximation using 3.14 for π. Include the correct units.



3) Simplify completely. The answer should contain only positive exponents.

a) $(-9k^4)^2$ b) $(-7z^3)(8z^{-9})$

c) $\left(\frac{40a^{-7}b^3}{8ab^{-2}}\right)^{-3}$

4) Write 0.00008215 in scientific notation.

5) Solve each equation.

a) $\frac{4}{3}y - 7 = 13$

b) $3(n - 4) + 11 = 5n - 2(3n + 8)$

6) *Write an equation and solve.*
The width of a rectangle is 5 in. less than its length. If the perimeter of the rectangle is 26 in., find its length and width.

7) *Write an equation and solve.*
Kathleen invests a total of \$9000. She invests some of it at 4% simple interest and the rest at 6% simple interest. If she earns \$480 in interest after 1 year, how much did she invest at each rate?

8) Solve $5 - 8x \geq 9$. Write the answer in interval notation.

9) Solve the compound inequality $2c + 9 < 3$ or $c - 7 > -2$. Write the answer in interval notation.

10) Graph each line.

a) $3x + y = 4$ b) $y = \frac{1}{2}x - 3$

c) $x = 1$

11) Write the equation of the line (in slope-intercept form) containing the points $(-3, 5)$ and $(6, 2)$.

12) Write an equation of the line perpendicular to $2x + 3y = 15$ containing the point $(-4, 2)$. Write the equation in standard form.

13) Solve using the elimination method.

$$\begin{aligned} 3x + 5y &= 12 \\ 2x - 3y &= 8 \end{aligned}$$

14) *Write a system of two equations and solve.*
How many milliliters of a 7% acid solution and how many milliliters of a 15% acid solution must be mixed to make 60 mL of a 9% acid solution?

15) Perform the operations and simplify.

a) $(9m^3 - 7m^2 + 3m + 2) - (4m^3 + 11m^2 - 7m - 1)$

b) $5(2p + 3) + \frac{2}{3}(4p - 9)$

16) Multiply and simplify.

a) $(2d - 7)(3d + 4)$

b) $(h - 6)(3h^2 - 7h + 2)$

17) Divide.

a) $\frac{12x^3 + 7x^2 - 37x + 18}{3x - 2}$

b) $\frac{20a^3b^3 - 45a^2b + 10ab + 60}{10ab}$

18) Factor completely.

a) $b^2 + 4b - 12$ b) $20xy + 5x + 4y + 1$

c) $5c^2 + 27cd - 18d^2$ d) $4r^2 - 36$

e) $27a^3 + 125b^3$ f) $w^2 - 16w + 64$

19) Solve each equation.

a) $m^2 - 15m + 54 = 0$ b) $x^3 = 3x^2 + 28x$

20) Divide $\frac{3z^2 + 22z - 16}{z + 8} \div \frac{8 - 12z}{7z + 14}$.

21) Add $\frac{k}{k^2 - 11k + 18} + \frac{k + 2}{2k^2 - 17k - 9}$.

22) Simplify each complex fraction.

a) $\dfrac{\frac{8v^2w}{3}}{\frac{16vw^2}{21}}$ b) $\dfrac{\frac{9}{c} + 4c}{5 + \frac{6}{c}}$

23) Solve $\frac{1}{a + 1} - \frac{a}{6} = \frac{a - 4}{a + 1}$.

24) Solve $|8y + 3| = 19$.

25) Solve $\left|\frac{1}{4}t - 5\right| \leq 2$. Write the answer in interval notation.

26) Graph $x + 3y < 6$.

27) Solve the system.
$$\begin{aligned} -x + 3y + 3z &= 5 \\ 3x - 2y - z &= -9 \\ -3x + y + 3z &= 5 \end{aligned}$$

28) Evaluate.

a) $\sqrt{25}$ b) $\sqrt[3]{27}$

c) $32^{3/5}$ d) $125^{-2/3}$

29) Simplify. Assume all variables represent positive real numbers.

a) $\sqrt{63}$ b) $\sqrt[4]{48}$

c) $\sqrt{20x^2y^9}$ d) $\sqrt[3]{250c^{17}d^{12}}$

30) Perform the operations and simplify.
$\sqrt{18} + \sqrt{2} - \sqrt{72}$

31) Rationalize each denominator.

a) $\dfrac{5}{\sqrt{t}}$ b) $\dfrac{4}{\sqrt{6} - 2}$

c) $\dfrac{n}{\sqrt[3]{4}}$

32) Solve $\sqrt{3x - 2} + \sqrt{x + 2} = 4$.

33) Evaluate $\sqrt{-16}$.

34) Solve by completing the square.
$r^2 - 8r + 11 = 0$

35) Solve each equation.

a) $9h^2 + 2h + 1 = 0$

b) $(w + 11)^2 + 4 = 0$

c) $(b + 4)^2 - (b + 4) = 12$

d) $k^4 + 15 = 8k^2$

36) Let $f(x) = 3x + 1$ and $g(x) = x^2 - 6x + 2$.

a) Find $f(-4)$. b) Find $g(5)$.

c) Find $(g \circ f)(x)$. d) Find $(f \circ g)(x)$.

e) Find the domain of $f(x)$. f) Graph $f(x)$.

37) Determine the domain of each function.

a) $f(x) = \dfrac{6}{5x - 10}$

b) $h(x) = \sqrt{2x + 3}$

38) Graph $f(x) = |x + 2|$.

39) Sketch the graph of each equation. Identify the vertex.

a) $x = (y - 1)^2 - 3$

b) $f(x) = -\dfrac{1}{2}x^2 + 2x + 1$

40) Write an equation of the inverse of $f(x) = -6x + 8$.

41) Evaluate.

a) $\log_2 16$ b) $\log 100$

c) $\ln e$ d) $\log_3 \sqrt{3}$

42) Write as a single logarithm: $\log_a 5 + 2 \log_a r - 3 \log_a s$

43) Solve.

a) $5^{2y} = 125^{y+4}$

b) $4^{x-3} = 3^{2x}$

c) $e^{-6t} = 8$

44) Solve.

a) $\log_3(2x + 1) = 2$

b) $\log_6 9m - \log_6(2m + 5) = \log_6 2$

45) Graph $(x - 2)^2 + (y + 1)^2 = 9$.

46) Solve the system.
$$\begin{aligned} x^2 + y^2 &= 20 \\ x + y &= -2 \end{aligned}$$

47) Solve each inequality. Write the answer in interval notation.

a) $w^2 + 5w > -6$

b) $\dfrac{c + 1}{c + 7} \le 0$

48) Determine whether each sequence is arithmetic or geometric, and find a_n.

a) 7, 11, 15, 19, 23, . . .

b) 2, −6, 18, −54, 162, . . .

49) Use a formula for S_n to find S_6 for each sequence.

a) the geometric sequence with $a_1 = 48$ and $r = \dfrac{1}{2}$

b) the arithmetic sequence with $a_1 = -10$ and $a_6 = -25$

50) Expand $(2t + 5)^3$ using the binomial theorem.

Appendix: Beginning Algebra Review

Section A1 The Real Number System and Geometry

Objectives

1. Multiply, Divide, Add, and Subtract Fractions
2. The Order of Operations
3. Review Concepts from Geometry
4. Define and Identify Sets of Numbers
5. Define Absolute Value and Perform Operations on Real Numbers
6. Learn the Vocabulary for Algebraic Expressions
7. Learn the Properties of Real Numbers
8. Combine Like Terms

1. Multiply, Divide, Add, and Subtract Fractions

To **multiply** fractions, we can divide out the common factors, then multiply numerators and denominators. To **divide** fractions, multiply the first fraction by the reciprocal of the second fraction. To **add** or **subtract** fractions, they must have a common denominator.

Example 1

Perform the operations and simplify. Write the answer in lowest terms.

a) $\frac{5}{9} \div \frac{4}{7}$ b) $\frac{3}{8} + \frac{1}{6}$

Solution

a) $\frac{5}{9} \div \frac{4}{7} = \frac{5}{9} \cdot \frac{7}{4}$ Multiply $\frac{5}{9}$ by the reciprocal of $\frac{4}{7}$.

$= \frac{35}{36}$ Multiply.

b) $\frac{3}{8} + \frac{1}{6}$ Identify the least common denominator (LCD): LCD = 24.

$$\frac{3}{8} \cdot \frac{3}{3} = \frac{9}{24} \qquad \frac{1}{6} \cdot \frac{4}{4} = \frac{4}{24} \qquad \frac{3}{8} + \frac{1}{6} = \frac{9}{24} + \frac{4}{24} = \frac{13}{24}$$

2. The Order of Operations

We evaluate expressions using the following rules called the **order of operations.**

Summary The Order of Operations—simplify expressions in the following order:

1) If parentheses or other grouping symbols appear in an expression, simplify what is in these grouping symbols first.
2) Simplify expressions with exponents.
3) Perform multiplication and division from left to right.
4) Perform addition and subtraction from left to right.

Remember that multiplication and division are at the same "level" in the process of performing operations, and that addition and subtraction are at the same "level."

Example 2

Evaluate $15 \div 3 + (5 - 2)^3$.

Solution

$$
\begin{aligned}
15 \div 3 + (5-2)^3 &= 15 \div 3 + 3^3 && \text{Perform operations in parentheses.}\\
&= 15 \div 3 + 27 && \text{Evaluate } 3^3.\\
&= 5 + 27 && \text{Perform the division.}\\
&= 32 && \text{Add.}
\end{aligned}
$$

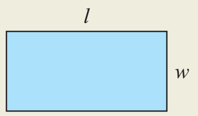
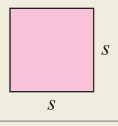
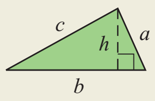
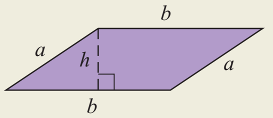
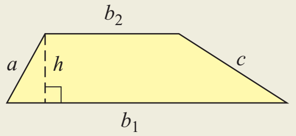
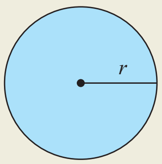
A good way to remember the order of operations is to remember the sentence, "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally." (**P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition, **S**ubtraction)

3. Review Concepts from Geometry

Recall that two angles are **complementary** if the sum of their measures is 90°. Two angles are **supplementary** if their measures add up to 180°.

Perimeter and Area

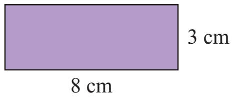
Students should be familiar with the following area and perimeter formulas.

Figure		Perimeter	Area
Rectangle:		$P = 2l + 2w$	$A = lw$
Square:		$P = 4s$	$A = s^2$
Triangle: h = height		$P = a + b + c$	$A = \frac{1}{2}bh$
Parallelogram: h = height		$P = 2a + 2b$	$A = bh$
Trapezoid: h = height		$P = a + c + b_1 + b_2$	$A = \frac{1}{2}h(b_1 + b_2)$
Circle:		Circumference $C = 2\pi r$	$A = \pi r^2$

The **radius, *r*,** is the distance from the center of the circle to a point on the circle. A line segment that passes through the center of the circle and has its endpoints on the circle is called a **diameter.**

"π" is the ratio of the circumference of any circle to its diameter. $\pi \approx 3.14159265\ldots$, but we will use 3.14 as an approximation for π.

Example 3

For the rectangle  find,

a) the perimeter. b) the area.

Solution

a) $P = 2l + 2w$
$= 2(8\text{ cm}) + 2(3\text{ cm})$
$= 16\text{ cm} + 6\text{ cm}$
$= 22\text{ cm}$

b) $A = lw$
$= (8\text{ cm})(3\text{ cm})$
$= 24\text{ cm}^2$

4. Define and Identify Sets of Numbers

We can define the following sets of numbers:

Natural numbers: $\{1, 2, 3, \ldots\}$ Whole numbers: $\{0, 1, 2, 3, \ldots\}$

Integers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

A **rational number** is any number of the form $\dfrac{p}{q}$, where p and q are integers and $q \neq 0$. The set of rational numbers includes terminating decimals and repeating decimals. The set of numbers that *cannot* be written as the quotient of two integers is the set of **irrational numbers.** Written in decimal form, an irrational number is a nonrepeating, nonterminating decimal. The set of **real numbers** consists of the rational and irrational numbers.

Example 4

Given this set of real numbers $\left\{-8, 0, 3, \sqrt{19}, 1.4, \dfrac{5}{9}, 0.\overline{26}, 7.68412\ldots\right\}$, list the

a) natural numbers b) integers c) rational numbers

d) irrational numbers

Solution

a) 3 b) $-8, 0, 3$ c) $-8, 0, 3, 1.4, \dfrac{5}{9}, 0.\overline{26}$

d) $\sqrt{19}, 7.68412\ldots$

5. Define Absolute Value and Perform Operations on Real Numbers

Recall that on a number line, positive numbers are to the right of zero, and negative numbers are to the left of zero. The **absolute value** of a number is the distance between that number and 0 on the number line. The absolute value of a number is always positive or zero.

If a is any real number, then the *absolute value* of a is denoted by $|a|$. For example, $|2| = 2$ since 2 is two units from 0. It is also true that $|-2| = 2$, since -2 is two units from 0.

Example 5

Perform the operations.

a) $-6 + (-4)$ b) $5 - 14$ c) $4 \cdot (-3)$
d) $-42 \div (-6)$

Solution

a) To add two numbers with the same sign, find the absolute value of each number and add them. The sum will have the same sign as the numbers being added. $-6 + (-4) = -(|-6| + |-4|) = -(6 + 4) = -10$

b) To subtract two numbers, $a - b$, change subtraction to addition and add the additive inverse of b. $5 - 14 = 5 + (-14) = -9$

c) The product or quotient of two real numbers with *different signs* is *negative.* $4 \cdot (-3) = -12$

d) The product or quotient of two real numbers with the *same sign* is positive. $-42 \div (-6) = 7$

6. Learn the Vocabulary for Algebraic Expressions

An **algebraic expression** is a collection of numbers, variables, and grouping symbols connected by operations symbols such as $+$, $-$, $\times$, and $\div$.

Example 6

List the terms and coefficients of the expression $t^3 - t^2 - 4t + 7$. What is the constant term?

Solution

Term	Coefficient
t^3	1
$-t^2$	-1
$-4t$	-4
7	7

The constant term is 7.

We can **evaluate** an algebraic expression by substituting value(s) for the variable(s) and simplifying.

7. Learn the Properties of Real Numbers

Next, we will summarize some properties of real numbers.

Properties of Real Numbers

Assume a, b, and c are real numbers. Then the following properties are true for a, b, and c:

1) Commutative properties: $a + b = b + a$; $ab = ba$
2) Associative properties: $(a + b) + c = a + (b + c)$; $(ab)c = a(bc)$
3) Identity properties: $a + 0 = a$; $a \cdot 1 = a$
4) Inverse properties: $a + (-a) = 0$; $b \cdot \frac{1}{b} = 1$ $(b \neq 0)$
5) Distributive properties: $a(b + c) = ab + ac$
   $a(b - c) = ab - ac$

Example 7

Rewrite each expression using the indicated property.

a) $5 + (8 + 1)$; associative b) $3(n + 7)$; distributive

Solution

a) $5 + (8 + 1) = (5 + 8) + 1$

b) $$\begin{aligned} 3(n + 7) &= 3n + 3(7) \\ &= 3n + 21 \end{aligned}$$

8. Combine Like Terms

Like terms contain the same variables with the same exponents. We can add and subtract only those terms that are like terms. To simplify an expression containing parentheses, we use the distributive property to clear the parentheses, and then combine like terms.

Example 8

Simplify by combining like terms: $5(4x + 3y) - 3(x - 2y)$

Solution

$$\begin{aligned} 5(4x + 3y) - 3(x - 2y) &= 20x + 15y - 3x + 6y && \text{Distributive property} \\ &= 17x + 21y && \text{Combine like terms.} \end{aligned}$$

A1 Exercises

Evaluate.

1) $-|6|$

2) $-|-1.4|$

3) 2^5

4) -7^2

Decide whether each statement is true or false.

5) $\frac{2}{3} < \frac{8}{9}$

6) $0.06 > 0.6$

Graph the numbers on a number line. Label each.

7) $3, -4, \frac{2}{3}, 1.5, -2\frac{1}{4}$

8) $5, \frac{3}{4}, -2, -3.2, 2\frac{1}{5}$

Perform the indicated operations and simplify.

9) $\frac{5}{8} \cdot \frac{6}{7}$

10) $8 - (-9)$

11) $\frac{2}{7} \div \frac{4}{21}$

12) $-14.6 - (-21.4)$

13) $\dfrac{9 \cdot 2 - 7 \cdot 4}{\sqrt{81} + (7 - 4)^4}$

14) $\dfrac{-48}{-3}$

15) $\frac{5}{9} \cdot \frac{9}{5}$

16) $-5\frac{1}{3} \div 1\frac{3}{7}$

17) $2\frac{1}{2} + \frac{1}{6}$

18) $\dfrac{20 - 2^3 + 9}{2 + 5 \cdot 4 - 10}$

19) $16 - 30$

20) $-7(4.3)$

21) $8 \cdot 7 - (1 + 3)^3 + 6 \cdot 2$

22) $\frac{2}{5} + \frac{3}{4} - \frac{3}{20}$

23) $5 - 16 + 2 - 24 - (-11)$

24) $\sqrt{16} + |8 - 11| - 45 \div 5$

25) $-\frac{2}{3} \cdot \frac{4}{5}$

26) $-\frac{3}{4} + \frac{2}{9}$

For Exercises 27–30, write a mathematical expression and simplify.

27) 10 less than 16

28) The quotient of -24 and 3

29) Twice the sum of -19 and 4

30) 9 less than the product of 7 and 6

For Exercises 31–34, use this set to list the indicated numbers.

$$A = \left\{\frac{8}{11}, -14, 3.7, 5, \sqrt{7}, 0, -1\frac{1}{2}, 6.\overline{2}, 2.8193\ldots\right\}$$

31) The integers in set A

32) The whole numbers in set A

33) The rational numbers in set A

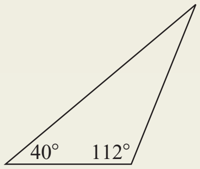
34) The irrational numbers in set A

35) The supplement of 41° is ________.

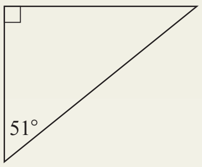
36) The complement of 32° is ________.

For Exercises 37 and 38, find the measure of the missing angle, and classify each triangle as acute, obtuse, or right.

37)

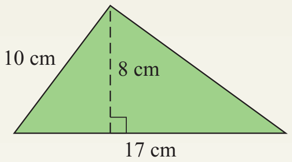


38)

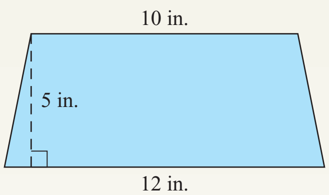


For Exercises 39 and 40, find the area of each figure. Include the correct units.

39)

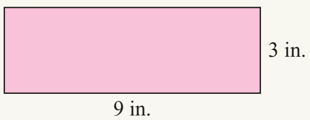


40)

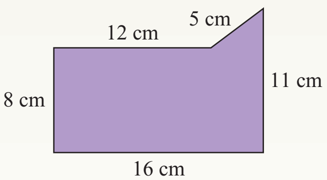


For Exercises 41 and 42, find the area and perimeter of each figure. Include the correct units.

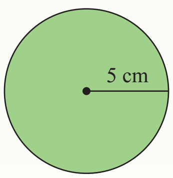
41)



42)



43) Find the a) area and b) circumference of the circle. Give an exact answer for each, and give an approximation using 3.14 for π. Include the correct units.



List the terms and their coefficients. Also, identify the constant.

44) $7z^3 - 3z^2 + z + 2.8$

45) Evaluate $5p^2 - 4p + 2$ when

a) $p = 3$

b) $p = -2$

46) Evaluate $\dfrac{a - b^2}{2a + 3b}$ when $a = 6$ and $b = -3$.

47) What is the multiplicative inverse of -8?

48) What is the additive inverse of 7?

Which property of real numbers is illustrated by each example? Choose from the commutative, associative, identity, inverse, or distributive properties.

49) $4(3 - 5) = 4 \cdot 3 - 4 \cdot 5$

50) $9 + 2 = 2 + 9$

51) $-18 + (6 + 1) = (-18 + 6) + 1$

52) $\left(\dfrac{6}{5}\right)\left(\dfrac{5}{6}\right) = 1$

53) Is $x - 4$ equivalent to $4 - x$? Why or why not?

54) Is $9 + 2w$ equivalent to $2w + 9$? Why or why not?

Rewrite each expression using the indicated property.

55) $8 + 3$; commutative

56) $(2 + 7) + 4$; associative

57) $-5p$; identity

58) $6t + 1$; commutative

Rewrite each expression using the distributive property.

59) $-7(2w + 1)$

60) $-2(5m + 6)$

61) $6(5 - 7r)$

62) $5(1 - 9h)$

63) $-8(3a - 4b - c)$

64) $-(t - 8)$

Determine whether the following groups of terms are like terms.

65) $-6a^2, 5a^3, a$

66) $3p^2, -\dfrac{1}{4}p^2, -8p^2$

Simplify by combining like terms.

67) $5z^2 - 7z + 10 - 2z^2 - z + 3$

68) $6(2k + 1) + 5(k - 3)$

69) $10 - 4(3n - 2) + 7(2n - 1)$

70) $t^2 + t + 3 - (7t + 2) - 6t^2$

Section A2 The Rules of Exponents

Objectives

1. Use the Rules of Exponents
2. Use Scientific Notation

1. Use the Rules of Exponents

Exponential notation is used as a shorthand way to represent repeated multiplication.

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 \qquad x \cdot x \cdot x = x^3$$

Next, we will review some rules for working with expressions containing exponents.

Summary Rules of Exponents

In the rules below, a and b are any real numbers, and m and n are positive integers.

Rule	Example
Product Rule: $a^m \cdot a^n = a^{m+n}$	$p^4 \cdot p^{11} = p^{4+11} = p^{15}$
Basic Power Rule: $(a^m)^n = a^{mn}$	$(c^8)^3 = c^{8 \cdot 3} = c^{24}$
Power Rule for a Product: $(ab)^n = a^n b^n$	$(3z)^4 = 3^4 \cdot z^4 = 81z^4$
Power Rule for a Quotient: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, where $b \neq 0$	$\left(\frac{w}{2}\right)^4 = \frac{w^4}{2^4} = \frac{w^4}{16}$
Zero Exponent: If $a \neq 0$, then $a^0 = 1$.	$(-3)^0 = 1$
Negative Exponent: For $a \neq 0$, $a^{-n} = \left(\frac{1}{a}\right)^n = \frac{1}{a^n}$.	$5^{-3} = \left(\frac{1}{5}\right)^3 = \frac{1^3}{5^3} = \frac{1}{125}$
If $a \neq 0$ and $b \neq 0$, then $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$.	$\frac{r^{-6}}{s^{-3}} = \frac{s^3}{r^6}$
Quotient Rule: If $a \neq 0$, then $\frac{a^m}{a^n} = a^{m-n}$.	$\frac{m^7}{m^5} = m^{7-5} = m^2$

Sometimes, it is necessary to combine the rules of exponents to simplify a product or a quotient. We must also remember to use the order of operations.

Example 1

Simplify.

a) $(2t^4)^3(5t)^2$ b) $\dfrac{(4n^6)^3}{(10m^3)^2}$ c) $\left(\dfrac{7}{4}\right)^{-2}$

Solution

a) $(2t^4)^3(5t)^2 = (2t^4)^3 \cdot (5t)^2$

$= (2^3)(t^4)^3 \cdot (5)^2(t)^2$ Power rule for products

$= 8t^{12} \cdot 25t^2$ Basic power rule

$= 200t^{14}$ Multiply coefficients and use the product rule.

b) $\dfrac{(4n^6)^3}{(10m^3)^2} = \dfrac{64n^{18}}{100m^6}$ Power rule for products

$= \dfrac{\overset{16}{\cancel{64}}n^{18}}{\underset{25}{\cancel{100}}\,m^6}$ Divide out the common factor of 4.

$= \dfrac{16n^{18}}{25m^6}$

BE CAREFUL

$(ab)^n = a^n b^n$ is different from $(a + b)^n$. $(a + b)^n \neq a^n + b^n$. See Section A6 or Chapter 6.

c) The reciprocal of $\frac{7}{4}$ is $\frac{4}{7}$, so $\left(\frac{7}{4}\right)^{-2} = \left(\frac{4}{7}\right)^2 = \frac{4^2}{7^2} = \frac{16}{49}$.

Next we will simplify more expressions containing negative exponents.

Example 2

Rewrite each expression with positive exponents. Assume the variables do not equal zero.

a) $5k^{-3}$ b) $\dfrac{7a^4b^{-1}}{c^{-5}d^{-2}}$

Solution

a) The base in $5k^{-3}$ is k. The 5 is not part of the base since it is not in parentheses.

$$5k^{-3} = 5 \cdot \left(\frac{1}{k}\right)^3 = 5 \cdot \frac{1}{k^3} = \frac{5}{k^3}$$

b) $\dfrac{7a^4b^{-1}}{c^{-5}d^{-2}} = \dfrac{7a^4c^5d^2}{b}$ The 7 and the a^4 keep their positions within the expression. b^{-1}, c^{-5}, and d^{-2} switch their positions, and the exponents become positive.

Example 3

Simplify $\left(\dfrac{c^4d^{-2}}{10c^9d^{-8}}\right)^{-3}$. Assume that the variables represent nonzero real numbers. The final answer should not contain negative exponents.

Solution

To begin, eliminate the negative exponent *outside* of the parentheses by taking the reciprocal of the base. Notice that we have *not* eliminated the negatives on the exponents *inside* the parentheses.

$$\begin{aligned}
\left(\frac{c^4d^{-2}}{10c^9d^{-8}}\right)^{-3} &= \left(\frac{10c^9d^{-8}}{c^4d^{-2}}\right)^3 \\
&= (10c^5d^{-6})^3 && \text{Subtract the exponents.} \\
&= 1000c^{15}d^{-18} && \text{Power rule} \\
&= \frac{1000c^{15}}{d^{18}} && \text{Write the answer using positive exponents.}
\end{aligned}$$

2. Use Scientific Notation

Scientific notation is a shorthand method for writing very large and very small numbers.

Definition

A number is in **scientific notation** if it is written in the form $a \times 10^n$, where $1 \leq |a| < 10$ and n is an integer.

First, we will convert a number from scientific notation to a number without exponents.

Example 4

Write without exponents. 7.382×10^4

Solution

Since $10^4 = 10{,}000$, multiplying 7.382 by 10,000 will give us a result that is *larger* than 7.382. Move the decimal point in 7.382 four places to the right.

$$7.382 \times 10^4 = 7.\underbrace{3820}_{\text{Four places to the right}} = 73{,}820$$

To write a number in scientific notation, $a \times 10^n$, remember that a must have one number to the left of the decimal point.

Example 5

Write in scientific notation. 0.00000974

Solution

$$0.00000974$$

The decimal point will be here.

$$0.\underbrace{000009}74 = 9.74 \times 10^{-6}$$ Move the decimal point six places to the right.

A2 Exercises

Write in exponential form.

1) $y \cdot y \cdot y \cdot y$ 2) $6 \cdot k \cdot k \cdot k$

Simplify using the rules of exponents. Assume the variables do not equal zero.

3) $2^2 \cdot 2^4$ 4) $\left(\frac{1}{4}\right) \cdot \left(\frac{1}{4}\right)^2$

5) $4m^9 \cdot 3m$ 6) $-6w^8 \cdot 9w^5 \cdot w^2$

7) $(-7k^5)^2$ 8) $\left(\frac{1}{5}z^9\right)^3$

9) $\left(\frac{x}{y}\right)^9$ 10) $\left(\frac{12}{n}\right)^2$

11) $\frac{w^{10}}{w^7}$ 12) $\frac{m^9}{m^4}$

13) $\frac{2p^4q^7}{p^3q^2}$ 14) $\frac{8a^6b^4}{3ab^2}$

Evaluate.

15) 6^0 16) $(-3)^0$

17) $\left(\frac{1}{5}\right)^{-3}$ 18) $\left(\frac{3}{2}\right)^{-4}$

Rewrite each expression with only positive exponents. Assume the variables do not equal zero.

19) x^{-8} 20) h^{-5}

21) $\left(\frac{t}{4}\right)^{-3}$ 22) $\left(\frac{10}{k}\right)^{-2}$

23) $\frac{a^{-4}}{7b^{-2}}$ 24) $\frac{20x^{-1}}{y^{-6}}$

25) $r^{-3}s$ 26) $\frac{8c^{-5}}{24d^2}$

Simplify. Assume the variables do not equal zero. The final answer should not contain negative exponents.

27) $\left(-\frac{8a^5b}{4c^3}\right)^3$ 28) $\left(\frac{m^3n^8}{m^7n^2}\right)^2$

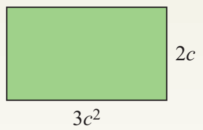
29) $(3c^5d^2)^3(-c^4d)^4$ 30) $2(3a^6b^2)^3$

31) $\left(\frac{r^{-3}s}{r^{-2}s^6}\right)^3$ 32) $(-3k^5)^2\left(\frac{2}{7}k^{-3}\right)^2\left(\frac{1}{6}k\right)$

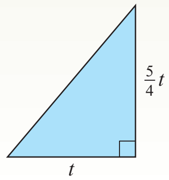
33) $\left(\frac{v^{-3}}{v}\right)^{-5}$ 34) $(-c^{-8}d^2)^3(2c^2d^{-1})^4$

35) $\left(\frac{14a^{-3}b}{7a^5b^{-6}}\right)^{-4}$ 36) $\left(\frac{4x^5y^{-1}}{12x^2y^2}\right)^{-2}$

37) Find an algebraic expression for the area and perimeter of this rectangle.



38) Find an algebraic expression for the area of this triangle.



Write each number without an exponent.

39) 5.07×10^{-4}

40) 2.8×10^{3}

Write each number in scientific notation.

41) 94,000

42) 0.00000295

43) Divide. Write the final answer without an exponent.
$$\frac{3.6 \times 10^{11}}{9 \times 10^{6}}$$

44) Suppose 1,000,000 atoms were lined up end-to-end. If the diameter of an atom is about 1×10^{-8} cm, find the length of the chain of atoms. Write the final answer without an exponent.

Section A3 Linear Equations and Inequalities

Objectives

1. Solve a Linear Equation
2. Solve Linear Equations with No Solution or an Infinite Number of Solutions
3. Solve Applied Problems
4. Solve an Equation for a Specific Variable
5. Solve a Linear Inequality in One Variable
6. Solve Compound Inequalities Containing *And* or *Or*

1. Solve a Linear Equation

Definition

An **equation** is a mathematical statement that two expressions are equal.

An *equation* contains an equal (=) sign, and an *expression* does not. We can **solve** equations, and we can **simplify** expressions.

There are many different types of equations. In this section, we will discuss how to solve a linear equation in one variable.

Definition

A **linear equation in one variable** is an equation that can be written in the form

$$ax + b = 0$$

where a and b are real numbers and $a \neq 0$.

Examples of linear equations in one variable include $7t + 6 = 27$ and $4(m + 1) - 9 = 3m + 10$. Notice that the exponent on each variable is 1. For this reason, these equations are also known as first-degree equations.

To **solve an equation** means to find the value or values of the variable that make the equation true. We use the following properties of equality to help us solve equations.

Properties of Equality

Let a, b, and c be expressions representing real numbers. Then the following properties hold.

1) *Addition Property of Equality* — If $a = b$, then $a + c = b + c$.
2) *Subtraction Property of Equality* — If $a = b$, then $a - c = b - c$.
3) *Multiplication Property of Equality* — If $a = b$, then $ac = bc$.
4) *Division Property of Equality* — If $a = b$, then $\frac{a}{c} = \frac{b}{c}$ $(c \neq 0)$.

Example 1

Solve $7n - 9 = 5$.

Solution

Use the properties of equality to solve for the variable.

$$
\begin{aligned}
7n - 9 &= 5 \\
7n - 9 + 9 &= 5 + 9 && \text{Add 9 to each side.} \\
7n &= 14 \\
\frac{7n}{7} &= \frac{14}{7} && \text{Divide each side by 7.} \\
n &= 2
\end{aligned}
$$

The solution set is $\{2\}$. The check is left to the student.

When a linear equation contains variables on both sides of the equal sign, we need to get the variables on one side of the equal sign and get the constants on the other side. With this in mind, we summarize the steps used to solve a linear equation in one variable.

Procedure How to Solve a Linear Equation

Step 1: **Clear parentheses** and **combine like terms** on each side of the equation.

Step 2: **Get the variable on one side of the equal sign and the constant on the other side of the equal sign** (isolate the variable) using the addition or subtraction property of equality.

Step 3: **Solve for the variable** using the multiplication or division property of equality.

Step 4: **Check the solution** in the original equation.

Example 2

Solve $3a + 10 - 2(4a + 7) = 6(a - 1) - 5a$.

Solution

Follow the steps listed above.

$$
\begin{aligned}
3a + 10 - 2(4a + 7) &= 6(a - 1) - 5a \\
3a + 10 - 8a - 14 &= 6a - 6 - 5a && \text{Distribute.} \\
-5a - 4 &= a - 6 && \text{Combine like terms.} \\
-5a - a - 4 &= a - a - 6 && \text{Subtract } a \text{ from each side to get the } a\text{-terms on the same side.} \\
-6a - 4 &= -6 && \text{Combine like terms.} \\
-6a - 4 + 4 &= -6 + 4 && \text{Add 4 to each side.} \\
-6a &= -2 \\
\frac{-6a}{-6} &= \frac{-2}{-6} && \text{Divide by } -6. \\
a &= \frac{1}{3} && \text{Simplify.}
\end{aligned}
$$

The solution set is $\left\{\frac{1}{3}\right\}$. The check is left to the student.

Some equations contain several fractions or decimals. Before applying the steps for solving a linear equation, we can eliminate the fractions and decimals from the equation. To eliminate the fractions, determine the least common denominator (LCD) for all of the fractions in the equation. Then multiply both sides of the equation by the LCD. To eliminate decimals, multiply both sides of the equation by the appropriate power of 10.

2. Solve Linear Equations with No Solution or an Infinite Number of Solutions

Some equations have no solution while others have an infinite number of solutions.

Example 3

Solve each equation.

a) $3(2k - 1) = 6k + 5$ b) $z + 8 - 5z = -4(z - 2)$

Solution

a)
$$
\begin{aligned}
3(2k - 1) &= 6k + 5 \\
6k - 3 &= 6k + 5 && \text{Distribute.} \\
6k - 6k - 3 &= 6k - 6k + 5 && \text{Subtract } 6k \text{ from each side.} \\
-3 &= 5 && \text{False}
\end{aligned}
$$

There is no solution to this equation. An equation that has no solution is called a **contradiction**. We say that the solution set is the **empty set,** denoted by $\varnothing$.

b)
$$
\begin{aligned}
z + 8 - 5z &= -4(z - 2) \\
-4z + 8 &= -4z + 8 && \text{Distribute.} \\
-4z + 4z + 8 &= -4z + 4z + 8 && \text{Add } 4z \text{ to each side.} \\
8 &= 8 && \text{True}
\end{aligned}
$$

The variable was eliminated, and we are left with a true statement. This means that any real number we substitute for z will make the original equation true. An equation that has all real numbers in its solution set is called an **identity.** This equation has an **infinite number of solutions,** and the solution set is {all real numbers}.

3. Solve Applied Problems

Next we will discuss how to translate information presented in English into an algebraic equation. The following approach is suggested to help in the problem-solving process.

Procedure Steps for Solving Applied Problems

Step 1: **Read** the problem carefully, more than once if necessary, until you understand it. Draw a picture, if applicable. Identify what you are being asked to find.

Step 2: **Choose a variable** to represent an unknown quantity. If there are any other unknowns, define them in terms of the variable.

Step 3: **Translate** the problem from English into an equation using the chosen variable. Some suggestions for doing so are:
- Restate the problem in your own words.
- Read and think of the problem in "small parts."
- Make a chart to separate these "small parts" of the problem to help you translate into mathematical terms.
- Write an equation in English, then translate it into an algebraic equation.

Step 4: **Solve** the equation.

Step 5: **Check** the answer in the original problem, and **interpret** the solution as it relates to the problem. Be sure your answer makes sense in the context of the problem.

Example 4

Write an equation and solve.

Roy advises 23 more students than Tobi. Together, they advise a total of 459 students. How many students are assigned to each advisor?

Solution

We will follow the given steps to solve this problem.

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

Find the number of students assigned to each advisor.

Step 2: **Choose** a variable to represent an unknown, and define the other unknown in terms of this variable.

$$x = \text{the number of Tobi's advisees}$$
$$x + 23 = \text{the number of Roy's advisees}$$

Step 3: **Translate** the information from English into an algebraic equation.

Statement:	Number of Tobi's advisees	+	Number of Roy's advisees	=	Total number of advisees
Equation:	x	$+$	$(x + 23)$	$=$	459

The equation is $x + (x + 23) = 459$.

Step 4: **Solve** the equation.

$$x + (x + 23) = 459$$
$$2x + 23 = 459 \quad \text{Combine like terms.}$$
$$2x = 436 \quad \text{Subtract 23 from each side.}$$
$$x = 218 \quad \text{Divide by 2.}$$

Step 5: **Check** the answer and **interpret** the solution. Find the other unknown. Since $x = 218$, $x + 23 = 241$.

Tobi advises 218 students, and Roy advises 241 students. The answer makes sense since $218 + 241 = 459$, the total number of students both Tobi and Roy advise.

Investment Problems

There are different ways to calculate the amount of interest earned from an investment. Here we will discuss **simple interest.**

The initial amount of money deposited in an account is called the *principal.* The formula used to calculate simple interest, I, is $I = PRT$, where

$$I = \text{interest earned (simple)}$$
$$P = \text{principal (initial amount invested)}$$
$$R = \text{annual interest rate (expressed as a decimal)}$$
$$T = \text{amount of time the money is invested (in years)}$$

We can use this to solve an application.

Example 5

Write an equation and solve.

When Francisco received an \$8000 bonus at work, he decided to invest some of it in an account paying 4% simple interest, and he invested the rest of it in a certificate of deposit that paid 7% simple interest. He earned a total of \$410 in interest after 1 year. How much did Francisco invest in each account?

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

Find the amount Francisco invested in the 4% account and the amount he invested in the certificate of deposit that paid 7% simple interest.

Step 2: **Choose** a variable to represent an unknown, and define the other unknown in terms of this variable.

$$\begin{aligned} x &= \text{amount invested in the 4\% account} \\ 8000 - x &= \text{amount invested in the CD paying 7\% simple interest} \end{aligned}$$

Step 3: **Translate** from English into an algebraic equation.

$$\begin{array}{ccccc} \text{Total interest earned} & = & \text{Interest from 4\% account} & + & \text{Interest from 7\% account} \\ & & \overset{P\ \ R\ \ T}{} & & \overset{P\ \ \ \ R\ \ \ T}{} \\ 410 & = & x(0.04)(1) & + & (8000 - x)(0.07)(1) \end{array}$$

The equation can be rewritten as

$$410 = 0.04x + 0.07(8000 - x)$$

Step 4: **Solve** the equation.

Begin by multiplying both sides of the equation by 100 to eliminate the decimals.

$$\begin{aligned} 410 &= 0.04x + 0.07(8000 - x) & \\ 100(410) &= 100[0.04x + 0.07(8000 - x)] & &\text{Multiply by 100.} \\ 41{,}000 &= 4x + 7(8000 - x) & \\ 41{,}000 &= 4x + 56{,}000 - 7x & &\text{Distribute.} \\ 41{,}000 &= -3x + 56{,}000 & &\text{Combine like terms.} \\ -15{,}000 &= -3x & \\ 5{,}000 &= x & \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem. $x = 5000$ and $8000 - x = 3000$

Francisco invested \$5000 in the account earning 4% interest and \$3000 in the certificate of deposit earning 7% interest. The check is left to the student. ■

4. Solve an Equation for a Specific Variable

Formulas are widely used not only in mathematics but also in disciplines such as business, economics, and the sciences. Often, it is necessary to solve a formula for a particular variable.

Example 6

Solve $A = P + PRT$ for T.

Solution

We will put a box around the T to remind us that this is the variable for which we are solving. The goal is to get the T on a side by itself.

$$A = P + PR\boxed{T}$$

$$A - P = P - P + PR\boxed{T} \qquad \text{Subtract } P \text{ from each side.}$$

$$A - P = PR\boxed{T} \qquad \text{Simplify.}$$

$$\frac{A - P}{PR} = \frac{PR\boxed{T}}{PR} \qquad \text{Since } T \text{ is being multiplied by } PR \text{, divide both sides by } PR.$$

$$\frac{A - P}{PR} = T \qquad \text{Simplify.}$$

5. Solve a Linear Inequality in One Variable

While an equation states that two expressions are equal, an **inequality** states that two expressions are not necessarily equal.

> **Definition**
>
> **A linear inequality in one variable** can be written in the form $ax + b < c$, $ax + b \leq c$, $ax + b > c$, or $ax + b \geq c$, where a, b, and c are real numbers and $a \neq 0$.

We solve linear inequalities in very much the same way we solve linear equations *except that, when we multiply or divide by a negative number, we must reverse the direction of the inequality symbol.* To represent the solution to an inequality we can:

1) Graph the solution set.
2) Write the answer in set notation.
3) Write the answer in interval notation.

Example 7

Solve each inequality. Graph the solution set and write the answer in both set and interval notations.

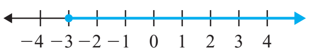
a) $2 - y \leq 5$ b) $-2 \leq \frac{3}{4}c - 5 < 1$

Solution

a) $2 - y \leq 5$

$-y \leq 3$ Subtract 2 from each side.

$y \geq -3$ Divide by -1 and reverse the inquality symbol.

The graph of the solution set is 

In set notation, we represent the solution as $\{y \mid y \geq -3\}$. In interval notation, we represent the solution as $[-3, \infty)$.

b) $-2 \leq \frac{3}{4}c - 5 < 1$

This is a *compound inequality.* A **compound inequality** contains more than one inequality symbol. To solve this type of inequality, we must remember that *whatever operation we perform on one part of the inequality, we must perform on all parts of the inequality.*

$$-2 \le \frac{3}{4}c - 5 < 1$$

$$-2 + 5 \le \frac{3}{4}c - 5 + 5 < 1 + 5 \qquad \text{Add 5 to each part of the inequality.}$$

$$3 \le \frac{3}{4}c < 6 \qquad \text{Combine like terms.}$$

$$\frac{4}{3} \cdot 3 \le \frac{4}{3} \cdot \frac{3}{4}c < \frac{4}{3} \cdot 6 \qquad \text{Multiply each part by } \frac{4}{3}.$$

$$4 \le c < 8 \qquad \text{Simplify.}$$

The graph of the solution set is 

In set notation, we write the solution as $\{c \mid 4 \le c < 8\}$. In interval notation, we write the solution as $[4, 8)$.

In Example 7b), we said that a compound inequality contains more than one inequality symbol. Next we will look at compound inequalities containing the words *and* or *or.*

6. Solve Compound Inequalities Containing *And* or *Or*

The solution set of a compound inequality joined by *and* is the *intersection* of the solution sets of the individual inequalities. The solution set of a compound inequality joined by *or* is the *union* of the solution sets of the individual inequalities.

Example 8

Solve each compound inequality, and write the answer in interval notation.

a) $3x < 12$ and $x + 6 > 4$ b) $\frac{3}{2}w - 1 \ge 2$ or $2w - 5 \le -5$

Solution

a) ***Step 1:*** These two inequalities are connected by "**and.**" The solution set will consist of the values of x that make *both* inequalities true. The solution set will be the **intersection** of the solution sets of $3x < 12$ and $x + 6 > 4$.

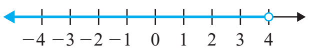
Step 2: Solve each inequality separately.

$$3x < 12 \quad \text{and} \quad x + 6 > 4$$

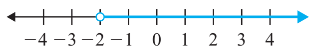
$$\text{Divide by 3.} \quad x < 4 \quad \text{and} \quad x > -2 \quad \text{Subtract 6.}$$

Step 3: Graph the solution set to each inequality on its own number line even if the problem does not require you to graph the solution set. This will help you visualize the solution set of the compound inequality.

$x < 4$:

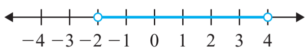


$x > -2$:



Step 4: The *intersection* of the two solution sets is the region where their graphs intersect. Ask yourself, *"If I were to put the number lines on top of each other, where would they intersect?"* They would intersect between -2 and 4.

The graph of $x < 4$ and $x > -2$ is



Step 5: In interval notation, we can represent the shaded region above as $(-2, 4)$. Every real number in this interval will satisfy each inequality.

b) ***Step 1:*** The solution to the compound inequality $\frac{3}{2}w - 1 \geq 2$ or $2w - 5 \leq -5$ is the **union** of the solution sets of the individual inequalities.

Step 2: Solve each inequality separately.

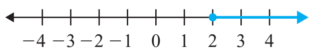
$$\frac{3}{2}w - 1 \geq 2 \quad \text{or} \quad 2w - 5 \leq -5$$

Add 1. $\quad \frac{3}{2}w \geq 3 \quad$ or $\quad 2w \leq 0 \quad$ Add 5.

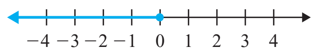
Multiply by $\frac{2}{3}$. $\quad w \geq 2 \quad$ or $\quad w \leq 0 \quad$ Divide by 2.

Step 3: Graph the solution set to each inequality on its own number line.

$w \geq 2$:



$w \leq 0$:



Step 4: The *union* of the two solution sets consists of the *total* of what would be shaded if the number lines above were placed on top of each other.

The graph of $w \geq 2$ or $w \leq 0$ is 

Step 5: In interval notation, we can represent the shaded region above as $(-\infty, 0] \cup [2, \infty)$. (Use the union symbol, $\cup$, for **or.**)

A3 Exercises

Determine whether the given value is a solution to the equation.

1) $8b + 5 = 1; b = -\frac{1}{2}$

2) $3 - 2(t + 1) = 1 + t; t = 3$

Solve and check each equation.

3) $y + 6 = 10$

4) $n - 7 = -2$

5) $-\frac{3}{4}x = 3$

6) $-16 = -\frac{4}{9}c$

7) $7b + 2 = 23$

8) $3v - 8 = 16$

9) $4 - \frac{1}{3}h = 6$

10) $0.2q + 7 = 8$

11) $3y + 4 - 9y + 7 = 14$

12) $11 + 2m + 6m + 3 = 12$

13) $9n - 4 = 3n + 14$

14) $10b + 9 = 2b + 25$

15) $2(4z - 3) - 5z = z - 6$

16) $8 - 3d + 9d - 1 = 4(d - 2) + 3$

17) $4(k + 1) + 3k = 7k + 9$

18) $x - 3(2x - 7) = 5(4 - x) + 1$

19) $\frac{2}{3}r - 1 = \frac{2}{5}r + \frac{3}{5}$

20) $\frac{1}{6}t + \frac{3}{8} = \frac{1}{8}t + \frac{1}{12}$

21) $0.4(p + 5) + 0.2(p - 3) = 0.8$

22) $0.12(v - 2) - 0.03(2v - 5) = 0.09$

Solve using the five steps for solving applied problems.

23) Eleven less than a number is 15. Find the number.

24) An electrician must cut a 60-ft cable into two pieces so that one piece is twice as long as the other. Find the length of each piece.

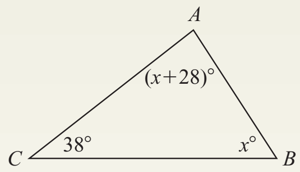
25) In the 2006 FIFA World Cup, Sweden received two fewer yellow cards than Mexico. The players on the two teams received a total of 20 yellow cards. How many yellow cards did each team receive?

26) The sum of three consecutive even integers is 102. Find the numbers.

27) Gonzalo inherited \$8000 and invested some of it in an account earning 6% simple interest and the rest of it in an account earning 7% simple interest. After 1 year, he earned \$500 in interest. How much did Gonzalo deposit in each account?

28) A pair of running shoes is on sale for \$60.80. This is 20% off of the original price. Find the original price of the shoes.

29) Find the measures of angles A and B.



30) How many milliliters of a 12% acid solution and how many milliliters of a 4% acid solution must be mixed to make 100 mL of a 10% acid solution?

31) A collection of coins contains twice as many quarters as dimes. The coins are worth \$18.60. How many quarters are in the collection?

32) Nick and Jessica leave home going in opposite directions. Nick is walking at 3 mph, and Jessica is jogging at 6 mph. After how long will they be 3 mi apart?

Solve each formula for the indicated variable.

33) $V = \frac{AH}{3}$ for H

34) $A = P + PRT$ for R

35) $A = \frac{1}{2}h(b_1 + b_2)$ for b_2

36) $A = \pi(R^2 - r^2)$ for R^2

Solve each inequality. Graph the solution set and write the answer in interval notation.

37) $r - 9 \geq -3$

38) $4t < 12$

39) $-5n + 7 \geq 22$

40) $8 - a \leq 11$

41) $3(2z - 1) - z > 2z - 15$

42) $\frac{2}{3}(k - 2) \geq \frac{5}{4}(2k + 1) - \frac{5}{3}$

43) $-1 \leq 4 - 3p < 10$

44) $-3 \leq \frac{1 - w}{2} \leq 1$

Solve each compound inequality. Graph the solution set, and write the answer in interval notation.

45) $6z + 4 > 1$ and $z - 3 < -1$

46) $5 - r \leq 3$ and $2r + 5 \geq 1$

47) $2c - 5 > -3$ or $-4c \geq 12$

48) $t + 7 \leq 7$ or $3t + 4 \geq 10$

Section A4 Linear Equations in Two Variables

Objectives

1. Define a Linear Equation in Two Variables
2. Graph a Line by Plotting Points and Finding Intercepts
3. Find the Slope of a Line
4. Graph a Line Given in Slope-Intercept Form
5. Write an Equation of a Line
6. Determine Whether a Relation Is a Function, and Find the Domain and Range
7. Given an Equation, Determine Whether y Is a Function of x and Find the Domain
8. Use Function Notation and Find Function Values

1. Define a Linear Equation in Two Variables

Definition

A **linear equation in two variables** can be written in the form $Ax + By = C$ where A, B, and C are real numbers and where both A and B do not equal zero.

Two examples of such equations are $5x + 4y = 8$ and $y = -2x + 1$. A **solution** to a linear equation in two variables is an ordered pair, (x, y), that satisfies the equation.

2. Graph a Line by Plotting Points and Finding Intercepts

Property Graph of a Linear Equation

The graph of a linear equation in two variables, $Ax + By = C$, is a straight line. Each point on the line is a solution to the equation, and every linear equation has an infinite number of solutions.

We can graph a line by making a table of values, finding the intercepts, or using the slope and y-intercept of the line. We will look at a couple of examples.

Example 1

Graph $y = \frac{2}{3}x - 2$ by finding the intercepts and one other point.

Solution

We will begin by finding the intercepts.

x-intercept: Let $y = 0$, and solve for x.

$$y = \frac{2}{3}x - 2$$
$$0 = \frac{2}{3}x - 2$$
$$2 = \frac{2}{3}x$$
$$3 = x$$

The x-intercept is $(3, 0)$.

y-intercept: Let $x = 0$, and solve for y.

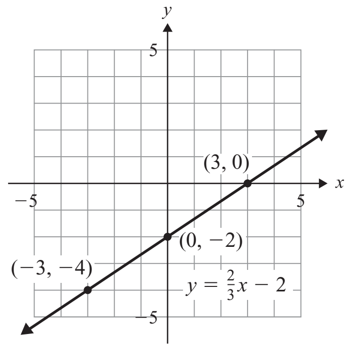
$$y = \frac{2}{3}x - 2$$
$$y = \frac{2}{3}(0) - 2$$
$$y = 0 - 2$$
$$y = -2$$

The y-intercept is $(0, -2)$.

We must find another point. If we choose a value for x that is a multiple of 3 (the denominator of the fraction), then $\frac{2}{3}x$ will not be a fraction.

Let $x = -3$.
$$\begin{aligned} y &= \frac{2}{3}x - 2 \\ &= \frac{2}{3}(-3) - 2 \\ &= -2 - 2 \\ &= -4 \end{aligned}$$

The third point is $(-3, -4)$.
Plot the points, and draw the line through them.



To graph a vertical or horizontal line, follow these rules:

If c is a constant, then the graph of $x = c$ is a *vertical line* going through the point $(c, 0)$.

If d is a constant, then the graph of $y = d$ is a *horizontal line* going through the point $(0, d)$.

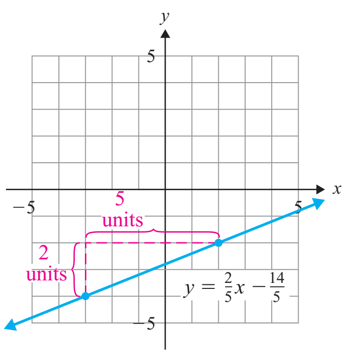
3. Find the Slope of a Line

The slope of a line measures its steepness. It is the ratio of the vertical change (the change in y) to the horizontal change (the change in x). Slope is denoted by m.

For example, if a line has a slope of $\frac{2}{5}$, then the rate of change between two points on the line is a vertical change (change in y) of 2 units for every horizontal change (change in x) of 5 units.

If a line has a *positive slope*, then it slopes *upward* from left to right. This means that as the value of x increases, the value of y increases as well. If a line has a *negative slope*, then it slopes *downward* from left to right. This means that as the value of x increases, the value of y decreases.

If we know two points on a line, we can find its slope.



Formula Slope of a Line

The slope, m, of a line containing the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{y_2 - y_1}{x_2 - x_1}$$

We can also think of slope as $\frac{\text{rise}}{\text{run}}$ or $\frac{\text{change in } y}{\text{change in } x}$.

Example 2

Find the slope of the line containing $(-2, 3)$ and $(1, 9)$.

Solution

The slope formula is $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Let $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (1, 9)$. Then $m = \frac{9 - 3}{1 - (-2)} = \frac{6}{3} = 2$.

We have graphed lines by finding the intercepts and plotting another point. We can also use the slope to help us graph a line.

4. Graph a Line Given in Slope-Intercept Form

If we require that A, B, and C are integers and that A is positive, then $Ax + By = C$ is called the **standard form** of the equation of a line. Lines can take other forms, too.

Definition

The **slope-intercept form of a line** is $\mathbf{y = mx + b}$, where m is the slope and $(0, b)$ is the y-intercept.

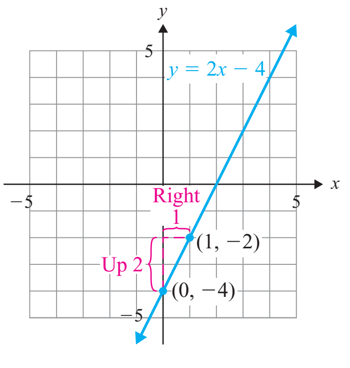
Example 3

Graph $y = 2x - 4$.

Solution

Identify the slope and y-intercept. The slope is $m = 2$, and the y-intercept is $(0, -4)$.

Graph the line by first plotting the y-intercept and then by using the slope to locate another point on the line. Think of the slope as $m = \frac{2}{1} = \frac{\text{change in } y}{\text{change in } x}$. To get from the point $(0, -4)$ to another point on the line, move up 2 units and right 1 unit.



5. Write an Equation of a Line

In addition to knowing how to graph a line, it is important that we know how to write the equation of a line given certain information. Here are some guidelines to follow when we want to write the equation of a line:

Procedure Writing Equations of Lines

If you are given

1) **the slope and y-intercept of the line,** use $y = mx + b$ and substitute those values into the equation.
2) **the slope of the line and a point on the line,** use the point-slope formula:

$$y - y_1 = m(x - x_1)$$

Substitute the slope for m and the point you are given for (x_1, y_1). Write your answer in slope-intercept or standard form.

3) **two points on the line,** find the slope of the line and then use the slope and *either* one of the points in the point-slope formula. Write your answer in slope-intercept or standard form.

The equation of a *horizontal line* containing the point (c, d) is $y = d$.

The equation of a *vertical line* containing the point (c, d) is $x = c$.

Example 4

Find an equation of the line containing the point (5, 1) with slope $= -3$. Express the answer in slope-intercept form.

Solution

First, ask yourself, "*What kind of information am I given?*" Since the problem tells us the slope of the line and a point on the line, we will use the point-slope formula.

Use $y - y_1 = m(x - x_1)$.

Substitute -3 for m. Substitute (5, 1) for (x_1, y_1).

$$y - 1 = -3(x - 5) \qquad \text{Substitute } -3 \text{ for } m, 5 \text{ for } x_1, \text{ and } 1 \text{ for } y_1.$$
$$y - 1 = -3x + 15 \qquad \text{Distribute.}$$
$$y = -3x + 16 \qquad \text{Solve for } y.$$

The equation is $y = -3x + 16$.

Often we need to write the equations of parallel or perpendicular lines.

Note

Parallel lines have the same slopes and different y-intercepts.

Two lines are *perpendicular* if their slopes are negative reciprocals of each other.

6. Determine Whether a Relation Is a Function, and Find the Domain and Range

A set of ordered pairs like $\{(1, 40), (2.5, 100), (3, 120)\}$ is a *relation.*

Definition

A **relation** is any set of ordered pairs.

The **domain** of a relation is the set of all values of the independent variable (the first coordinates in the set of ordered pairs). The **range** of a relation is the set of all values of the dependent variable (the second coordinates in the set of ordered pairs). The relation $\{(1, 40), (2.5, 100), (3, 120)\}$ is also a *function*.

Definition

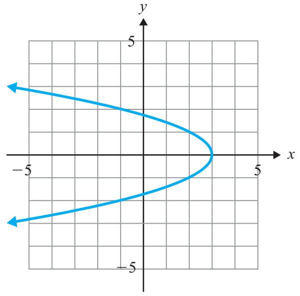
A **function** is a special type of relation. If each element of the domain corresponds to *exactly one* element of the range, then the relation is a function.

Example 5

Identify the domain and range of each relation, and determine whether each relation is a function.

a) $\{(-5, 4), (-2, 1), (0, -2), (6, -7)\}$

b)



Solution

a) The *domain* is the set of first coordinates, $\{-5, -2, 0, 6\}$.

The *range* is the set of second coordinates, $\{4, 1, -2, -7\}$.

To determine whether $\{(-5, 4), (-2, 1), (0, -2), (6, -7)\}$ is a function, ask yourself, "Does every first coordinate correspond to *exactly one* second coordinate?" *Yes.* So, this relation is a function.

b) The domain is $(-\infty, 3]$. The range is $(-\infty, \infty)$.

To determine whether this graph represents a function, recall that we can use the *vertical line test*. The **vertical line test** says that if there is no vertical line that can be drawn through a graph so that it intersects the graph more than once, then the graph represents a function.

This graph fails the vertical line test because we can draw a vertical line through the graph that intersects it more than once. *This graph does* not *represent a function.* ■

7. Given an Equation, Determine Whether *y* Is a Function of *x* and Find the Domain

In an equation like $y = 5x$, we say that x is the independent variable and y is the dependent variable. That is, the value of y depends on the value of x.

If a relation is written as an equation so that y is in terms of x, then the **domain** is the set of all real numbers that can be substituted for the independent variable, x. The resulting set of real numbers obtained for y, the dependent variable, is the **range.**

Procedure Finding the Domain of a Relation

The domain of a relation that is written as an equation, where y is in terms of x, is the set of all real numbers that can be substituted for the independent variable, x. When determining the domain of a relation, it can be helpful to keep these tips in mind.

1) Ask yourself, "Is there any number that *cannot* be substituted for x?"

2) If x is in the denominator of a fraction, determine what value of x will make the denominator equal 0 by setting the denominator equal to zero. Solve for x. This x-value is *not* in the domain.

Example 6

Determine whether $y = \dfrac{8}{x + 5}$ describes y as a function of x, and determine its domain.

Solution

Ask yourself, "Is there any number that *cannot* be substituted for x in $y = \dfrac{8}{x + 5}$?" Look at the denominator. When will it equal 0? Set the denominator equal to 0 and solve for x to determine what value of x will make the denominator equal 0.

$$x + 5 = 0 \qquad \text{Set the denominator} = 0.$$
$$x = -5 \qquad \text{Solve.}$$

When $x = -5$, the denominator of $y = \dfrac{8}{x + 5}$ equals zero. The domain contains all real numbers *except* -5. Write the domain in interval notation as $(-\infty, -5) \cup (-5, \infty)$.

$y = \dfrac{8}{x + 5}$ *is a function.* For every value that can be substituted for x, there is only one corresponding value of y.

8. Use Function Notation and Find Function Values

If y is a function of x, then we can use *function notation* to represent this relationship.

Definition

$y = f(x)$ is called **function notation,** and it is read as, "y equals f of x." $y = f(x)$ means that y is a function of x (that is, y depends on x).

If such a relation is a function, then $f(x)$ can be used in place of y. *$f(x)$ is the same as y.* One special type of function is a *linear function.*

Definition

A **linear function** has the form $f(x) = mx + b$, where m and b are real numbers, m is the *slope* of the line, and $(0, b)$ is the *y-intercept.*

We graph linear functions just like we graph lines in the form $y = mx + b$, and we can evaluate a linear function for values of the variable. We call this **finding function values.** (We can find function values for any type of function, not just linear functions.)

Example 7

Let $f(x) = -2x + 5$. Find $f(3)$.

Solution

To find $f(3)$ (read as "f of 3") means to find the value of the function when $x = 3$.

$$f(x) = -2x + 5$$
$$f(3) = -2(3) + 5 \qquad \text{Substitute 3 for } x.$$
$$f(3) = -1$$

We can also say that the ordered pair $(3, -1)$ satisfies $f(x) = -2x + 5$, where the ordered pair represents $(x, f(x))$.

A4 Exercises

Determine whether each ordered pair is a solution of the equation $3x - 5y = 7$.

1) $(-1, -2)$

2) $(2, 3)$

Make a table of values, and graph each equation.

3) $y = x - 4$

4) $2x - 3y = 9$

5) $x = -3$

6) $y = 2$

Graph each equation by finding the intercepts and at least one other point.

7) $y = -\frac{3}{2}x + 3$

8) $y = x - 4$

9) $3x - y = 2$

10) $3x + 4y = 4$

Use the slope formula to find the slope of the line containing each pair of points.

11) $(5, 2)$ and $(-2, -3)$

12) $(-6, 2)$ and $(4, -3)$

Graph the line containing the given point and with the given slope.

13) $(-2, -5)$; $m = \frac{1}{4}$

14) $(0, 1)$; $m = -\frac{2}{5}$

15) $(2, 3)$; slope is undefined.

16) $(-3, -2)$; $m = 0$

Each of the following equations is in slope-intercept form. Identify the slope and the y-intercept, then graph each line using this information.

17) $y = -4x + 1$

18) $y = -x + 3$

19) $y = -x$

20) $y = \frac{2}{3}x$

Graph each line using any method.

21) $x - y = 5$

22) $3x + 5y = 10$

23) $7x + 2y = 6$

24) $y - 2x = 3$

25) $y = \frac{5}{3}x - 2$

26) $y = x$

Write the slope-intercept form of the equation of the line, if possible, given the following information.

27) $m = \frac{3}{8}$ and y-intercept $(0, 4)$

28) $m = -2$ and contains $(-3, 5)$

29) contains $(-1, 6)$ and $(7, 2)$

30) contains $(-8, -1)$ and $(-2, -11)$

31) $m = 0$ and contains $(4, 7)$

32) slope undefined and contains $(-3, 2)$

Write the standard form of the equation of the line given the following information.

33) $m = -6$ and contains $(0, 2)$

34) $m = 1$ and contains $(3, 8)$

35) contains $(-4, -1)$ and $(-1, 8)$

36) contains $(0, -3)$ and $(2, 7)$

Write the slope-intercept form, if possible, of the equation of the line meeting the given conditions.

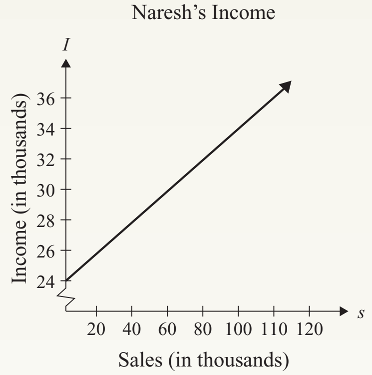
37) perpendicular to $y = \frac{3}{2}x - 6$ and containing $(9, -1)$

38) parallel to $y = \frac{1}{4}x - 5$ and containing $(8, 2)$

39) parallel to $x = -4$ and containing $(-1, 4)$

40) perpendicular to $y = 3$ and containing $(5, -2)$

41) Naresh works in sales, and his income is a combination of salary and commission. He earns \$24,000 per year plus 10% of his total sales. The equation $I = 0.10s + 24,000$ represents his total income I, in dollars, when his sales total s dollars.



a) What is the I-intercept? What does it mean in the context of the problem?

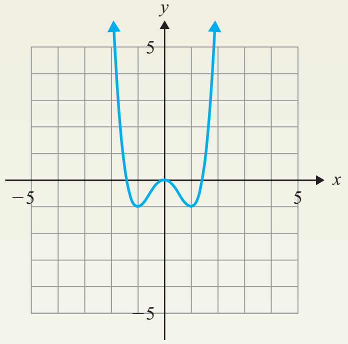
b) What is the slope? What does it mean in the context of the problem?

c) Use the graph to find Naresh's income if his total sales are \$100,000. Confirm your answer using the equation.

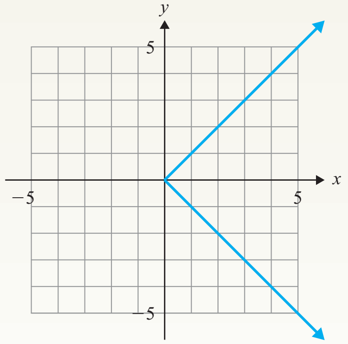
Identify the domain and range of each relation, and determine whether each relation is a function.

42) $\{(-4, 16), (-1, 7), (1, 1), (3, -5)\}$

43)



44)



Determine whether each relation describes y as a function of x, and determine the domain of the relation.

45) $y = x + 8$

46) $y = \frac{7}{x}$

47) $y^2 = x$

48) $y = -\frac{2}{5x + 6}$

Let $f(x) = 2x - 9$ and $g(x) = x^2 - 6x - 4$. Find the following function values.

49) $f(3)$

50) $g(-3)$

51) $f(z)$

52) $g(t)$

53) $f(c + 3)$

54) $f(m - 4)$

55) $f(x) = -3x + 5$. Find x so that $f(x) = -7$.

56) $k(x) = 4x - 3$. Find x so that $k(x) = 9$.

Graph each function by making a table of values and plotting points.

57) $f(x) = x - 3$

58) $g(x) = 3x - 2$

Graph each function by finding the x- and y-intercepts and one other point.

59) $f(x) = -x + 3$

60) $g(x) = -2x + 3$

Graph each function using the slope and y-intercept.

61) $h(x) = -2x - 1$

62) $g(x) = 4x - 5$

63) A plane travels at a constant speed of 420 mph. The distance D (in miles) that the plane travels after t hours can be defined by the function

$$D(t) = 420t$$

a) How far will the plane travel after 2 hr?

b) How long does it take the plane to travel 1890 mi?

c) Graph the function.

Section A5 Solving Systems of Linear Equations

Objectives

1. Solve a System of Equations by Graphing
2. Solve a System of Equations by Substitution
3. Solve a System of Equations Using the Elimination Method
4. Solve an Applied Problem Using a System of Two Equations in Two Variables
5. Solve a System of Linear Equations in Three Variables

A **system of linear equations** consists of two or more linear equations with the same variables. We begin by learning how to solve systems of two equations in two variables. Later in this section, we will discuss how to solve a system of three equations in three variables.

A **solution of a system** of two equations in two variables is an ordered pair that is a solution of each equation in the system.

We will review three methods for solving a system of linear equations:

1. Graphing
2. Substitution
3. Elimination

1. Solve a System of Equations by Graphing

When solving a system of linear equations by graphing, *the point of intersection of the two lines is the solution of the system.* A system that has one solution is called a **consistent** system.

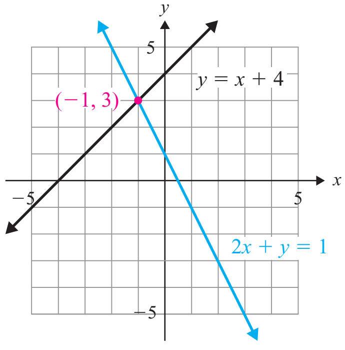
Example 1

Solve the system by graphing.

$$\begin{aligned} 2x + y &= 1 \\ y &= x + 4 \end{aligned}$$

Solution

Graph each line on the same axes. The lines intersect at the point $(-1, 3)$. Therefore, the solution of the system is $(-1, 3)$. You can verify by substituting the ordered pair into each equation to see that it satisfies each equation.



The lines in Example 1 intersect at one point, so the system has one solution. Some systems, however, have no solution while others have an infinite number of solutions.

Note

When solving a system of equations by graphing, if the lines are parallel, then the system has **no solution.** We write this as $\varnothing$. Furthermore, a system that has no solution is an **inconsistent system,** and the equations are **independent.**

When solving a system of equations by graphing, if the graph of each equation is the same line, then the system has an **infinite number of solutions.** The system is **consistent,** and the equations are **dependent.**

2. Solve a System of Equations by Substitution

Another method we can use to solve a system of equations is *substitution*. This method is especially good when one of the variables has a coefficient of 1 or -1. Here are the steps we can follow to solve a system by substitution:

Procedure Solving a System of Equations by Substitution

Step 1: Solve one of the equations for one of the variables. If possible, solve for a variable that has a coefficient of 1 or -1.

Step 2: Substitute the expression found in *Step 1* into the *other* equation. The equation you obtain should contain only one variable.

Step 3: Solve the equation you obtained in *Step 2*.

Step 4: Substitute the value found found in *Step 3* into either of the equations to obtain the value of the other variable.

Step 5: Check the values in each of the original equations, and write the solution as an ordered pair.

Example 2

Solve the system by substitution.

$$\begin{aligned} x + 3y &= -5 \qquad (1) \\ 3x + 2y &= 6 \qquad (2) \end{aligned}$$

Solution

We number the equations to make the process easier to follow. Let's follow the steps.

Step 1: For which variable should we solve? The x in the first equation is the only variable with a coefficient of 1 or -1. Therefore, we will solve the first equation for x.

$$\begin{aligned} x + 3y &= -5 && \text{First equation (1)} \\ x &= -5 - 3y && \text{Subtract } 3y. \end{aligned}$$

Step 2: Substitute $-5 - 3y$ for the x in equation (2).

$$\begin{aligned} 3x + 2y &= 6 && \text{(2)} \\ 3(-5 - 3y) + 2y &= 6 && \text{Substitute.} \end{aligned}$$

Step 3: Solve the equation above for y.

$$\begin{aligned} 3(-5 - 3y) + 2y &= 6 \\ -15 - 9y + 2y &= 6 && \text{Distribute.} \\ -7y - 15 &= 6 \\ -7y &= 21 && \text{Add 15.} \\ y &= -3 && \text{Divide by } -7. \end{aligned}$$

Step 4: To determine the value of x, we can substitute -3 for y in either equation. We will substitute it in equation (1).

$$\begin{aligned} x + 3y &= -5 && \text{(1)} \\ x + 3(-3) &= -5 && \text{Substitute.} \\ x - 9 &= -5 \\ x &= 4 \end{aligned}$$

Step 5: The check is left to the reader. The solution of the system is $(4, -3)$.

When we were solving systems by graphing, we learned that some systems are inconsistent (have no solution) and some systems have equations that are dependent (have an infinite number of solutions).

If we are trying to solve a system by substitution and it is either inconsistent or its equations are dependent, then somewhere in the process of solving the system, the variables are eliminated.

When we are solving a system of equations and the variables are eliminated:

1) if we get a **false statement,** like $3 = 5$, then the system has **no solution.**
2) if we get a **true statement,** like $-4 = -4$, then the system has an **infinite number of solutions.**

3. Solve a System of Equations Using the Elimination Method

The **elimination method** (or **addition method**) is another technique we can use to solve a system of equations. It is based on the addition property of equality, which says that we can add the same quantity to each side of an equation and preserve the equality.

Addition Property of Equality: If $a = b$, then $a + c = b + c$.

We can extend this idea by saying that we can add *equal* quantities to each side of an equation and still preserve the equality.

If $a = b$ and $c = d$, then $a + c = b + d$.

The object of the elimination method is to add the equations (or multiples of one or both of the equations) so that one variable is eliminated. Then we can solve for the remaining variable.

Procedure Solving a System of Two Linear Equations by the Elimination Method

Step 1: Write each equation in the form $Ax + By = C$.

Step 2: Determine which variable to eliminate. If necessary, multiply one or both of the equations by a number so that the coefficients of the variable to be eliminated are negatives of one another.

Step 3: Add the equations, and solve for the remaining variable.

Step 4: Substitute the value found in *Step 3* into either of the original equations to find the value of the other variable.

Step 5: Check the solution in each of the original equations.

Example 3

Solve the system using the elimination method.

$$\begin{aligned} -3x + 4y &= -8 \quad (1) \\ 2x + 5y &= 13 \quad (2) \end{aligned}$$

Solution

Step 1: **Write each equation in the form $Ax + By = C$.**

Each equation is written in the form $Ax + By = C$.

Step 2: **Determine which variable to eliminate from equations (1) and (2).** Often, it is easier to eliminate the variable with the smaller coefficients. Therefore, *we will eliminate x.*

In equation (1), the coefficient of x is -3, and in equation (2) the coefficient of x is 2. *Multiply equation* (1) *by* 2 *and equation* (2) *by* 3. The x-coefficient in equation (1) will be -6, and the x-coefficient in equation (2) will be 6. When we add the equations, the x will be eliminated.

$$\begin{aligned} 2(-3x + 4y) &= 2(-8) \quad \text{2 times equation (1)} \longrightarrow & -6x + 8y &= -16 \\ 3(2x + 5y) &= 3(13) \quad \text{3 times equation (2)} & 6x + 15y &= 39 \end{aligned}$$

Step 3: **Add the resulting equations to eliminate x. Solve for y.**

$$\begin{aligned} -6x + 8y &= -16 \\ + \quad 6x + 15y &= 39 \\ \hline 23y &= 23 \\ y &= 1 \end{aligned}$$

Step 4: **Substitute $y = 1$ into equation (1) and solve for x.**

$$\begin{aligned} -3x + 4y &= -8 && (1) \\ -3x + 4(1) &= -8 && \text{Substitute 1 for } y. \\ -3x + 4 &= -8 \\ -3x &= -12 && \text{Subtract 4.} \\ x &= 4 && \text{Divide by } -3. \end{aligned}$$

Step 5: **Check** to verify that (4, 1) satisfies each of the original equations. The solution is (4, 1).

When solving a system by graphing and by substitution, we learned that some systems have no solution and some have an infinite number of solutions. The same can be true when solving using the elimination method.

4. Solve an Applied Problem Using a System of Two Equations in Two Variables

In Section A3 (and in Chapter 3) we introduced the five steps for solving applied problems. We defined unknown quantities in terms of *one* variable to write a linear equation to solve a problem.

Sometimes, it is easier to use *two* variables and a system of *two* equations to solve an applied problem. Here are some steps you can follow to solve an applied problem using a system of **two equations in two variables.** These steps can be applied to solving applied problems involving a system of three equations as well.

Procedure Solving an Applied Problem Using a System of Equations

Step 1: **Read** the problem carefully, more than once if necessary. Draw a picture, if applicable. Identify what you are being asked to find.

Step 2: **Choose variables** to represent the unknown quantities. Label any pictures with the variables.

Step 3: **Write a system of equations using two variables.** It may be helpful to begin by writing the equations in words.

Step 4: **Solve** the system.

Step 5: **Check** the answer in the original problem, and **interpret** the solution as it relates to the problem. Be sure your answer makes sense in the context of the problem.

Example 4

Write a system of equations and solve.

On her iPod, Shelby has 27 more hip-hop songs than reggae songs. If she has a total of 85 hip-hop and reggae tunes on her iPod, how many of each type does she have?

Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

We must find the number of hip-hop and reggae songs on Shelby's iPod.

Step 2: **Choose variables** to represent the unknown quantities.

x = number of hip-hop songs
y = number of reggae songs

Step 3: **Write a system of equations using two variables.**

Let's think of the equations in English first. Then, we can translate them into algebraic equations.

To get one equation, use the information that Shelby has 27 more hip-hop songs than reggae songs.

English:	Number of hip-hop songs	is	27 more than	Number of reggae songs
	↓	↓	↓	↓
Equation:	x	$=$	$27 +$	y

One equation is $x = 27 + y$.

To get the second equation, use the information that Shelby has a total of 85 hip-hop and reggae songs on her iPod. Write an equation in words, then translate it into an algebraic equation.

English:	Number of hip-hop songs	+	Number of reggae songs	=	Number of hip-hop *and* reggae songs
Equation:	x	+	y	=	85

The second equation is $x + y = 85$.

The system of equations is
$$x = 27 + y$$
$$x + y = 85$$

Step 4: **Solve** the system.

$$\begin{aligned} x + y &= 85 && \text{Second equation} \\ (27 + y) + y &= 85 && \text{Substitute } 27 + y \text{ for } x. \\ 27 + 2y &= 85 && \text{Combine like terms.} \\ 2y &= 58 && \text{Subtract 27 from each side.} \\ y &= 29 \end{aligned}$$

Find x by substituting $y = 29$ into $x = 27 + y$.
$$x = 27 + 29$$
$$x = 56$$

The solution to the system is (56, 29).

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

Shelby has 56 hip-hop songs and 29 reggae songs on her iPod.

Does the answer make sense? Yes. The total number of hip-hop and reggae songs is 85 and $56 + 29 = 85$, and the number of hip-hop songs is 27 more than the number of reggae songs: $56 = 27 + 29$.

5. Solve a System of Linear Equations in Three Variables

We will extend our study of solving a system of two equations in two variables to solving a system of three equations in three variables.

Definition

A **linear equation in three variables** is an equation of the form $Ax + By + Cz = D$, where A, B, and C are not all zero and where A, B, C, and D are real numbers. Solutions to this type of an equation are **ordered triples** of the form (x, y, z).

An example of a linear equation in three variables is $x + 2y - 3z = -8$. One solution to the equation is (2, 1, 4), since $2 + 2(1) - 3(4) = -8$.

Like systems of linear equations in two variables, systems of linear equations in *three* variables can have *one solution* (the system is consistent), *no solution* (the system is inconsistent), or *infinitely many solutions* (the equations are dependent).

Example 5

Solve.
$$\begin{aligned} (1)\quad -x + 2y + z &= 3 \\ (2)\quad 3x + y - 4z &= 13 \\ (3)\quad 2x - 2y + 3z &= -5 \end{aligned}$$

Solution

Step 1: **Label** the equations (1), (2), and (3).

Step 2: **Choose a variable to eliminate.**

We will eliminate y from *two* sets of *two* equations.

a) *Equations* ① *and* ③. Add the equations to eliminate y. Label the resulting equation $\boxed{A}$.

$$\begin{array}{rrrrrrl} ① & & -x & + 2y & + z & = & 3 \\ ③ & + & 2x & - 2y & + 3z & = & -5 \\ \hline \boxed{A} & & x & + & 4z & = & -2 \end{array}$$

b) *Equations* ② *and* ③. To eliminate y, multiply equation ② by 2 and add it to equation ③. Label the resulting equation $\boxed{B}$.

$$\begin{array}{rrrrrl} 2 \times ② & 6x & + 2y & - 8z & = & 26 \\ ③ & + 2x & - 2y & + 3z & = & -5 \\ \hline \boxed{B} & 8x & & - 5z & = & 21 \end{array}$$

Note

Equations $\boxed{A}$ and $\boxed{B}$ contain only *two* variables, and they are the *same* variables, x and z.

Step 3: **Use the elimination method to eliminate a variable from equations $\boxed{A}$ and $\boxed{B}$.**

We will eliminate x from equations $\boxed{A}$ and $\boxed{B}$. Multiply equation $\boxed{A}$ by -8 and add it to equation $\boxed{B}$.

$$\begin{array}{rrrrl} -8 \times \boxed{A} & & -8x - 32z & = & 16 \\ \boxed{B} & + & 8x - 5z & = & 21 \\ \hline & & -37z & = & 37 \\ & & & & \boxed{z = -1} \quad \text{Divide by } -37. \end{array}$$

Step 4: **Find the value of another variable** by substituting $z = -1$ into either equation $\boxed{A}$ or $\boxed{B}$.

We will substitute $z = -1$ into equation $\boxed{A}$.

$$\begin{array}{lrll} \boxed{A} & x + 4z & = -2 & \\ & x + 4(-1) & = -2 & \text{Substitute } -1 \text{ for } z. \\ & x - 4 & = -2 & \text{Multiply.} \\ & & \boxed{x = 2} & \text{Add 4.} \end{array}$$

Step 5: **Find the value of the third variable** by substituting $x = 2$ and $z = -1$ into either equation ①, ②, or ③.

We will substitute $x = 2$ and $z = -1$ into equation ① to solve for y.

$$\begin{array}{lrll} ① & -x + 2y + z & = 3 & \\ & -(2) + 2y + (-1) & = 3 & \text{Substitute 2 for } x \text{ and } -1 \text{ for } z. \\ & -2 + 2y - 1 & = 3 & \\ & 2y - 3 & = 3 & \text{Combine like terms.} \\ & 2y & = 6 & \text{Add 3.} \\ & & \boxed{y = 3} & \text{Divide by 2.} \end{array}$$

Step 6: **Check** the solution, $(2, 3, -1)$, in each of the original equations. This will be left to the student.

The solution is $(2, 3, -1)$.

When one or more of the equations in the system contain only two variables, we can modify the steps we have just used to solve the system.

To solve applications involving a system of three linear equations in three variables, we use the same process that is outlined for systems of two equations on p. A-29 of this section.

A5 Exercises

Determine whether the ordered pair is a solution of the system of equations.

1) $\begin{aligned} 2x - 5y &= 4 \\ x + y &= 9 \end{aligned}$
(7, 2)

2) $\begin{aligned} -x + 4y &= 7 \\ 2x - y &= -5 \end{aligned}$
(−3, 1)

Solve each system of equations by graphing. If the system is inconsistent or if the equations are dependent, so indicate.

3) $\begin{aligned} y &= \frac{2}{3}x - 1 \\ x + y &= 4 \end{aligned}$

4) $\begin{aligned} x + 2y &= -2 \\ y &= -2x + 2 \end{aligned}$

5) $\begin{aligned} 2y - 2x &= -3 \\ x &= y - 2 \end{aligned}$

6) $\begin{aligned} 2x - 6y &= 5 \\ 18y &= 6x - 15 \end{aligned}$

Solve each system by substitution.

7) $\begin{aligned} y &= 2x - 15 \\ -x + 3y &= -5 \end{aligned}$

8) $\begin{aligned} 4x + 5y &= -13 \\ x &= 3y + 1 \end{aligned}$

9) $\begin{aligned} x &= 2 - 5y \\ -4x - 20y &= -8 \end{aligned}$

10) $\begin{aligned} 4x - y &= 7 \\ -8x + 2y &= 9 \end{aligned}$

Solve each system using the elimination method.

11) $\begin{aligned} 12x + 5y &= 2 \\ -2x + 3y &= 15 \end{aligned}$

12) $\begin{aligned} 7x - 2y &= -8 \\ 5x + 4y &= -22 \end{aligned}$

13) $\begin{aligned} -8x + 12y &= 3 \\ 6x - 9y &= 1 \end{aligned}$

14) $\begin{aligned} \frac{1}{4}x + \frac{3}{2}y &= -\frac{1}{2} \\ \frac{1}{6}x + \frac{2}{3}y &= \frac{1}{3} \end{aligned}$

Solve each system using any method.

15) $\begin{aligned} \frac{1}{2}x &= -\frac{1}{3}y - \frac{4}{3} \\ \frac{3}{4}x + \frac{1}{2}y &= \frac{1}{2} \end{aligned}$

16) $\begin{aligned} 3x + 2y &= 6 \\ 3x - 8y &= 1 \end{aligned}$

17) $\begin{aligned} 7(x + 1) + 5y &= 6(x + 2) - 4 \\ 2x + 3(3y + 2) &= 2(y + 1) \end{aligned}$

18) $\begin{aligned} y + 4 &= 2(2y - x) + 1 \\ 7x - 2(2y + 5) &= 2(x - 4) + 1 \end{aligned}$

19) $\begin{aligned} y &= \frac{2}{5}x \\ x - 5y &= 15 \end{aligned}$

20) $\begin{aligned} 4y - 3(x - 1) &= 2(y + 2x) \\ 6(x + y) + 4x &= x + 2y - 6 \end{aligned}$

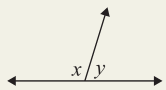
21) $\begin{aligned} 0.04x - 0.07y &= 0.22 \\ 0.6x + 0.5y &= 0.2 \end{aligned}$

22) $\begin{aligned} 6x - 3(2y + 4) &= 3(y + 2) \\ 6y &= 4(x - 3) \end{aligned}$

Write a system of equations and solve.

23) The tray table on the back of a seat on an airplane is 7 in. longer than it is wide. Find the length and width of the tray table if its perimeter is 54 in.

24) Find the measures of angles x and y if the measure of angle y is 34° less than the measure of angle x and if the angles are related according to the figure below.



25) How many ounces of an 18% alcohol solution and how many ounces of an 8% solution should be mixed to obtain 60 oz of a 12% alcohol solution?

26) For a round-trip flight between Chicago and Denver, two economy-class tickets and one business-class ticket cost \$1050 while one economy ticket and two business tickets cost \$1500. Find the cost of an economy-class ticket and the cost of a business-class ticket.

27) A passenger train and a freight train leave cities that are 300 mi apart and travel toward each other. The passenger train is traveling 15 mph faster than the freight train. Find the speed of each train if they pass each other after 4 hr.

28) Lydia inherited \$12,000 and invested some of it in an account earning 5% simple interest and put the rest of it in an account earning 8% simple interest. If she earned a total of \$690 in interest after 1 year, how much did she invest in each account?

Solve each system.

29) $\begin{aligned} x + 4y + 2z &= 12 \\ 3x - y + z &= -2 \\ -2x + 3y - z &= 2 \end{aligned}$

30) $\begin{aligned} 2x + 6y - z &= -2 \\ -x + 9y + 2z &= -1 \\ 2x - 3y + z &= 12 \end{aligned}$

31) $\begin{aligned} a - 4b + 2c &= 1 \\ -2a + 3b - c &= -8 \\ -a + 2b + c &= -7 \end{aligned}$

32) $\begin{aligned} -3a + b + 2c &= 1 \\ 4a - b - 6c &= 5 \\ 6a - 2b - 4c &= 3 \end{aligned}$

33) $\begin{aligned} -3a + 2b + c &= 5 \\ 5a - b - 2c &= -6 \\ 4a - c &= 1 \end{aligned}$

34) $\begin{aligned} 2x + 9y &= -7 \\ 3y - 4z &= 11 \\ 2x - z &= -14 \end{aligned}$

Write a system of three equations and solve.

35) During their senior year of high school, Ryan, Seth, and Summer applied to colleges. Summer applied to three more schools than Seth while Seth applied to five fewer schools than Ryan. All together, the friends sent out 11 applications. To how many schools did each one apply?

36) The measure of the smallest angle of a triangle is half the measure of the largest angle. The third angle measures 10° less than the largest angle. Find the measures of each angle of the triangle. (Hint: Recall that the sum of the measures of the angles of a triangle is 180°.)

Section A6 Polynomials

Objectives

1. Learn the Vocabulary Associated with Polynomials
2. Add and Subtract Polynomials
3. Define a Polynomial Function and Find Function Values
4. Multiply Polynomials
5. Divide Polynomials

1. Learn the Vocabulary Associated with Polynomials

Definition

A **polynomial in *x*** is the sum of a finite number of terms of the form ax^n, where n is a whole number and a is a real number. (The exponents must be whole numbers.)

An example of a polynomial is $7x^3 + x^2 - 8x + \frac{2}{3}$. Let's look closely at some characteristics of this polynomial.

1) The polynomial is written in **descending powers of *x*,** since the powers of x decrease from left to right. Generally, we write polynomials in descending powers of the variable.
2) The **degree of a term** equals the exponent on its variable. (If a term has more than one variable, the degree equals the *sum* of the exponents on the variables.)

Term	Degree
$7x^3$	3
x^2	2
$-8x$	1
$\frac{2}{3}$	0

$\frac{2}{3} = \frac{2}{3}x^0$ since $x^0 = 1$.

3) The **degree of the polynomial** equals the highest degree of any nonzero term. The degree of $7x^3 + x^2 - 8x + \frac{2}{3}$ is 3. Or, we say that this is a *third-degree polynomial.*

A **monomial** is a polynomial that consists of one term. A **binomial** is a polynomial that consists of exactly two terms. A polynomial that consists of exactly three terms is called a **trinomial.**

2. Add and Subtract Polynomials

To add polynomials, add like terms. Polynomials can be added horizontally or vertically.

Example 1

Add the polynomials $(3a^2b^2 + 8a^2b - 10ab - 9) + (2a^2b^2 + 3ab - 1)$.

Solution

We will add these polynomials vertically. Line up like terms in columns and add.

$$\begin{array}{rrrrr} & 3a^2b^2 & + 8a^2b & - 10ab & - 9 \\ + & 2a^2b^2 & & + 3ab & - 1 \\ \hline & 5a^2b^2 & + 8a^2b & - 7ab & - 10 \end{array}$$

To subtract two polynomials, change the sign of each term in the second polynomial. Then add the polynomials.

3. Define a Polynomial Function and Find Function Values

We can use function notation to represent a polynomial like $x^2 - 12x - 7$ since each value substituted for the variable produces *only one value* of the expression.

$f(x) = x^2 - 12x - 7$ is a **polynomial function** since $x^2 - 12x - 7$ is a polynomial.

Finding $f(-3)$ when $f(x) = x^2 - 12x - 7$ is the same as evaluating $x^2 - 12x - 7$ when $x = -3$.

Example 2

If $f(x) = x^2 - 12x - 7$, find $f(-3)$.

Solution

Substitute -3 for x.

$$\begin{aligned} f(x) &= x^2 - 12x - 7 \\ f(-3) &= (-3)^2 - 12(-3) - 7 = 9 + 36 - 7 = 38 \end{aligned}$$

4. Multiply Polynomials

When we multiply polynomials, we use the distributive property. The *way* in which we use it, however, may vary depending on what types of polynomials are in the product.

Multiplying a Monomial and a Polynomial

To find the product of a monomial and a larger polynomial, multiply each term of the polynomial by the monomial.

Example 3

Multiply $7z^3(z^2 - 5z + 3)$.

Solution

$$\begin{aligned} 7z^3(z^2 - 5z + 3) &= (7z^3)(z^2) + (7z^3)(-5z) + (7z^3)(3) && \text{Distribute.} \\ &= 7z^5 - 35z^4 + 21z^3 && \text{Multiply.} \end{aligned}$$

Multiplying Two Polynomials

Procedure Multiplying Two Polynomials

To multiply two polynomials, multiply each term in the second polynomial by each term in the first polynomial. Then combine like terms. The answer should be written in descending powers.

Example 4

Multiply $(4c - 5)(3c^2 + 8c - 2)$.

Solution

We will multiply each term in the second polynomial by the $4c$ in the first polynomial. Then, multiply each term in $(3c^2 + 8c - 2)$ by the -5 in $(4c - 5)$. Then, add like terms.

$$\begin{aligned} &(4c - 5)(3c^2 + 8c - 2) \\ &\quad = (4c)(3c^2) + (4c)(8c) + (4c)(-2) + (-5)(3c^2) && \text{Distribute.} \\ &\qquad + (-5)(8c) + (-5)(-2) \\ &\quad = 12c^3 + 32c^2 - 8c - 15c^2 - 40c + 10 && \text{Multiply.} \\ &\quad = 12c^3 + 17c^2 - 48c + 10 && \text{Combine like terms.} \end{aligned}$$

Multiplying Two Binomials

We can use the distributive property as we did in Example 4 to multiply two binomials such as $(x + 6)(x + 2)$.

$$\begin{aligned}(x + 6)(x + 2) &= (x)(x) + (x)(2) + (6)(x) + (6)(2) && \text{Distribute.}\\ &= x^2 + 2x + 6x + 12 && \text{Multiply.}\\ &= x^2 + 8x + 12 && \text{Combine like terms.}\end{aligned}$$

Or, we can apply the distributive property in a different way—we can use FOIL. **FOIL** stands for **F**irst **O**utside **I**nner **L**ast. The letters in the word FOIL tell us how to multiply the terms in the binomials. Then we add like terms.

Example 5

Use FOIL to multiply $(x + 6)(x + 2)$.

Solution

Last

First

$$\begin{aligned}(x + 6)(x + 2) = (x + 6)(x + 2) &= \overset{F}{x(x)} + \overset{O}{x(2)} + \overset{I}{6(x)} + \overset{L}{6(2)} && \text{Use FOIL.}\\ &= x^2 + 2x + 6x + 12 && \text{Multiply.}\\ &= x^2 + 8x + 12 && \text{Combine like terms.}\end{aligned}$$

Inner

Outer

Special Products

Binomial multiplication is very common in algebra, and there are three types of special products we explain here. The first special product we will review is

$$(a + b)(a - b) = a^2 - b^2$$

Example 6

Multiply $(q + 7)(q - 7)$.

Solution

$(q + 7)(q - 7)$ is in the form $(a + b)(a - b)$, where $a = q$ and $b = 7$.

$$\begin{aligned}(q + 7)(q - 7) &= q^2 - 7^2\\ &= q^2 - 49\end{aligned}$$

Two more special products involve squaring a binomial like $(a + b)^2$. We have the following formulas for the square of binomials:

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$

We can think of the formulas in words as:

> *To square a binomial, you square the first term, square the second term, then multiply 2 times the first term times the second term and add.*

Finding the product of a binomial raised to a power is also called *expanding* the binomial.

Example 7

Expand $(k + 8)^2$.

Solution

Using the formula $(a + b)^2 = a^2 + 2ab + b^2$ with $a = k$ and $b = 8$, we get

$$(k+8)^2 = \underbrace{k^2}_{\text{Square the first term.}} + \underbrace{2(k)(8)}_{\text{Two times first term times second term}} + \underbrace{8^2}_{\text{Square the second term.}} = k^2 + 16k + 64$$

5. Divide Polynomials

The last operation we will discuss is division of polynomials. We will break up this topic into two parts, division by a monomial and division by a polynomial containing two or more terms.

Procedure Dividing a Polynomial by a Monomial

To divide a polynomial by a monomial, divide each term in the polynomial by the monomial and simplify.

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \quad (c \neq 0)$$

Example 8

Divide $\dfrac{12x^3 - 8x^2 + 2x}{2x}$.

Solution

Since the polynomial in the numerator is being divided by a *monomial*, we will divide each term in the numerator by $2x$.

$$\begin{aligned}\frac{12x^3 - 8x^2 + 2x}{2x} &= \frac{12x^3}{2x} - \frac{8x^2}{2x} + \frac{2x}{2x} \\ &= 6x^2 - 4x + 1 \qquad \text{Simplify.}\end{aligned}$$

We can check our answer the same way we can check any division problem—we can multiply the quotient by the divisor, and the result should be the dividend.

Check: $2x(6x^2 - 4x + 1) = 12x^3 - 8x^2 + 2x$ ✓

Dividing a Polynomial by a Polynomial

When dividing a polynomial by a polynomial containing two or more terms, we use *long division of polynomials.* When we perform long division, the polynomials must be written so that the exponents are in descending order.

Example 9

Divide $\dfrac{2x^2 + 17x + 30}{x + 6}$.

Solution

First, notice that we are dividing by more than one term. That tells us to use long division of polynomials.

$$
\begin{array}{r}
2x \phantom{{}+30} \\
x + 6 \overline{)\; 2x^2 + 17x + 30} \\
\underline{-(2x^2 + 12x)} \downarrow \\
5x + 30
\end{array}
$$

1) By what do we multiply x to get $2x^2$? $2x$
Line up terms in the quotient according to exponents, so write $2x$ above $17x$.
2) Multiply $2x$ by $(x + 6)$. $2x(x + 6) = 2x^2 + 12x$
3) Subtract: $(2x^2 + 17x) - (2x^2 + 12x) = 5x$.
4) Bring down the $+30$.

Start the process again.

$$
\begin{array}{r}
2x + 5 \\
x + 6 \overline{)\; 2x^2 + 17x + 30} \\
\underline{-(2x^2 + 12x)} \phantom{{}+30} \\
5x + 30 \\
\underline{-(5x + 30)} \\
0
\end{array}
$$

1) By what do we multiply x to get $5x$? 5
Write $+5$ above $+30$.
2) Multiply 5 by $(x + 6)$. $5(x + 6) = 5x + 30$
3) Subtract: $(5x + 30) - (5x + 30) = 0$.

$\dfrac{2x^2 + 17x + 30}{x + 6} = 2x + 5$ There is no remainder.

Check: $(x + 6)(2x + 5) = 2x^2 + 5x + 12x + 30 = 2x^2 + 17x + 30$ ✓

When we are dividing polynomials, we must watch out for missing terms. If a dividend is missing one or more terms, we put them into the dividend with coefficients of zero.

Synthetic Division

When we divide a polynomial by a binomial of the form $x - c$, another method called *synthetic division* can be used. Synthetic division uses only the numerical coefficients of the variables to find the quotient.

Example 10

Use synthetic division to divide $(2x^3 - x^2 - 16x + 8)$ by $(x - 3)$.

Solution

Remember, in order to be able to use synthetic division, the divisor must be in the form $x - c$. $x - 3$ is in the form $x - c$, and $c = 3$.

Write the 3 in the open box, and then write the coefficients of the dividend. Skip a line and draw a horizontal line under the first coefficient. Bring down the 2. Then, multiply the 2 by 3 to get 6. Add the -1 and 6 to get 5.

$$
\begin{array}{r|rrrr}
3 & 2 & -1 & -16 & 8 \\
 & & 6 & & \\
\hline
 & 2 & 5 & &
\end{array}
\qquad 3 \cdot 2 = 6; \quad -1 + 6 = 5
$$

We continue working in this way until we get

$$\begin{array}{r|rrrr} \multicolumn{5}{c}{\overbrace{2x^3 \; -x^2 \; -16x \; + \; 8}^{\text{Dividend}}} \\ 3 & 2 & -1 & -16 & 8 \\ & & 6 & 15 & -3 \\ \hline & 2 & 5 & -1 & 5 \rightarrow \text{Remainder} \\ \multicolumn{5}{c}{\underbrace{2x^2 \; +5x \; -1}_{\text{Quotient}}} \end{array}$$

The numbers in the last row represent the quotient and the remainder. The last number is the remainder. The numbers before it are the coefficients of the quotient. *The degree of the quotient is one less than the degree of the dividend.*

$$(2x^3 - x^2 - 16x + 8) \div (x - 3) = 2x^2 + 5x - 1 + \frac{5}{x - 3}$$ ■

A6 Exercises

Evaluate.

1) $(-2)^3 \cdot (-2)^2$

2) $\dfrac{5^{11}}{5^{14}}$

3) $\left(\dfrac{3}{4}\right)^{-3}$

Simplify. Assume all variables represent nonzero real numbers. The answer should not contain negative exponents.

4) $(w^3)^5$

5) $(9z^8)^2$

6) $\left(-\dfrac{3}{2}xy^7\right)^3$

7) $\dfrac{40t^{11}}{8t}$

8) $\dfrac{9a^4}{18a^{-5}}$

9) $\left(\dfrac{5n^6}{m^2}\right)^{-3}$

10) Identify each term in the polynomial, the coefficient and degree of each term, and the degree of the polynomial.

$$5x^3 - x^2 - 8x + 6$$

11) Evaluate $-c^2 + 7c + 10$ for $c = -2$.

Perform the operations and simplify.

12) $(8y^2 - 6y + 5) + (3y^2 + 2y - 3)$

13) $(7a^2b^2 - a^2b + 9ab + 14)$
$\qquad - (3a^2b^2 + 14a^2b - 5ab^2 - 8ab + 20)$

14) Subtract $\left(\dfrac{4}{15}w^2 + \dfrac{8}{9}w - \dfrac{3}{4}\right)$ from $\left(\dfrac{4}{5}w^2 + \dfrac{1}{6}w + \dfrac{11}{4}\right)$.

15) If $f(x) = x^2 + 8x - 11$, find

a) $f(5)$

b) $f(-3)$

Multiply and simplify.

16) $3x(2x + 1)$

17) $-7q^3(2q^4 - 9q^2 - 6)$

18) $0.4h^2(12h^4 + 8h^2 - 4h - 9)$

19) $(5b + 2)(3b^3 - 2b^2 - b + 9)$

20) $(6n^3 - 9n + 5)(2n^2 - 1)$

21) $(-5d^2 + 12d + 1)(6d^4 - d - 2)$

22) $(t - 11)(t - 2)$

23) $(3r + 7)(2r + 1)$

24) $(5a + 2b)(a - b)$

25) $3d^2(2d - 5)(4d + 1)$

26) $(c + 12)(c - 12)$

27) $(3a + 4)(3a - 4)$

28) $\left(t + \dfrac{1}{4}\right)\left(t - \dfrac{1}{4}\right)$

29) $\left(h^2 - \dfrac{3}{8}\right)\left(h^2 + \dfrac{3}{8}\right)$

Expand.

30) $(x + 9)^2$

31) $(r - 10)^2$

32) $(11j + 2)^2$

33) $\left(\dfrac{3}{4}c - 6\right)^2$

34) $(a - 2)^3$

35) $(v + 5)^3$

Divide.

36) $\dfrac{14n^4 - 28n^3 - 35n^2}{7n^2}$

37) $(-110h^5 + 40h^4 + 10h^3) \div 10h^3$

38) $\dfrac{30t^5 - 15t^4 + 20t^3 - 6t}{6t^4}$

39) $\dfrac{3a^2b^2 + ab^2 - 27ab}{-3a^2b}$

40) $\dfrac{c^2 + 10c + 16}{c + 8}$

41) $\dfrac{2a^2 + a - 15}{2a - 5}$

42) $(35t^3 + 31t^2 + 16t + 6) \div (5t + 3)$

43) $(6x^2 + 11x - 25) \div (2x + 3)$

44) $\dfrac{11w - 19w^2 - 8 + 25w^4}{5w - 2}$

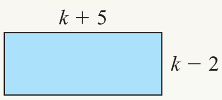
45) $\dfrac{b^2 - 9 + 20b^3}{4b - 3}$

46) $\dfrac{c^3 - 64}{c - 4}$

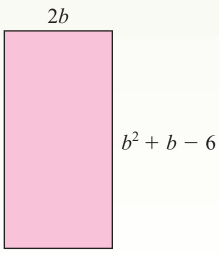
47) $(12h^4 + 8h^3 - 23h^2 + 4h - 6) \div (2h^2 + 3h - 1)$

For each figure, find a polynomial that represents a) its perimeter and b) its area.

48)

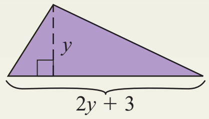


49)

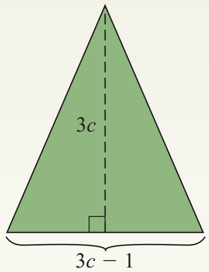


Find a polynomial that represents the area of each triangle.

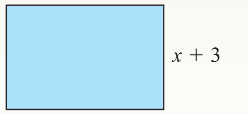
50)



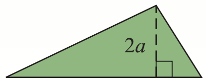
51)



52) Find a polynomial that represents the length of the rectangle if its area is given by $4x^2 + 7x - 15$ and its width is $x + 3$.



53) Find a polynomial that represents the base of the triangle if the area is given by $3a^3 - a^2 - 7a$ and the height is $2a$.



Section A7 Factoring Polynomials

Objectives

1. Factor Out the Greatest Common Factor
2. Factor by Grouping
3. Factor Trinomials of the Form $x^2 + bx + c$
4. Factor Trinomials of the Form $ax^2 + bx + c$ $(a \neq 1)$
5. Factor a Perfect Square Trinomial
6. Factor the Difference of Two Squares
7. Factor the Sum and Difference of Two Cubes
8. Solve a Quadratic Equation by Factoring
9. Use the Pythagorean Theorem to Solve an Applied Problem

In this section, we will review different techniques for factoring polynomials, and then we will discuss how to solve equations by factoring.

Recall that the **greatest common factor (GCF)** of a group of monomials is the *largest* common factor of the terms in the group. For example, the greatest common factor of $24t^5$, $56t^3$, and $40t^8$ is $8t^3$.

1. Factor Out the Greatest Common Factor

To **factor an integer** is to write it as the product of two or more integers. For example, a factorization of 15 is $3 \cdot 5$ since $15 = 3 \cdot 5$.

Likewise, to **factor a polynomial** is to write it as a product of two or more polynomials. It is important to understand that factoring a polynomial is the opposite of multiplying polynomials. Example 1 shows how these procedures are related.

Example 1

Factor out the GCF from $3p^2 + 12p$.

Solution

Use the distributive property to factor out the greatest common factor from $3p^2 + 12p$.

$$\begin{aligned} \text{GCF} &= 3p \\ 3p^2 + 12p &= (3p)(p) + (3p)(4) \\ &= 3p(p + 4) \qquad \text{Distributive property} \end{aligned}$$

We can check our result by multiplying. $3p(p + 4) = 3p^2 + 12p$ ✓

Sometimes we can take out a binomial factor. This leads us to our next method of factoring a polynomial—factoring by grouping.

2. Factor by Grouping

When we are asked to factor a polynomial containing four terms, we often try to **factor by grouping.**

Note

The first step in factoring *any* polynomial is to ask yourself, *"Can I factor out a GCF?"* If you can, then factor it out.

Example 2

Factor completely. $5z^5 - 10z^4 + 15z^3 - 30z^2$

Solution

Notice that this polynomial has four terms. This is a clue for us to try factoring by grouping. *The first step in factoring this polynomial is to factor out* $5z^2$.

$$5z^5 - 10z^4 + 15z^3 - 30z^2 = 5z^2(z^3 - 2z^2 + 3z - 6) \qquad \text{Factor out the GCF, } 5z^2.$$

The polynomial in parentheses has four terms. Try to factor it by grouping.

$$\begin{aligned} &5z^2(\underbrace{z^3 - 2z^2} + \underbrace{3z - 6}) \\ &= 5z^2[z^2(z - 2) + 3(z - 2)] \qquad \text{Take out the common factor in each group.} \\ &= 5z^2(z - 2)(z^2 + 3) \qquad \text{Factor out } (z - 2) \text{ using the distributive property.} \end{aligned}$$

3. Factor Trinomials of the Form $x^2 + bx + c$

One of the factoring problems encountered most often in algebra is the factoring of trinomials. Next we will discuss how to factor a trinomial of the form $x^2 + bx + c$.

Understanding that factoring is the opposite of multiplying will help us understand how to factor this type of trinomial. Consider the following multiplication problem:

$$\begin{aligned} (x + 3)(x + 5) &= x^2 + 5x + 3x + 3 \cdot 5 \\ &= x^2 + (5 + 3)x + 15 \\ &= x^2 + 8x + 15 \end{aligned}$$

So, if we were asked to *factor* $x^2 + 8x + 15$, we need to think of two integers whose *product* is 15 and whose *sum* is 8. Those numbers are 3 and 5. The *factored form* of $x^2 + 8x + 15$ is $(x + 3)(x + 5)$.

Example 3

Factor completely.

a) $n^2 + 9n + 18$ b) $2k^3 - 8k^2 - 24k$

Solution

a) $n^2 + 9n + 18$

Begin by asking yourself, *"Can I factor out a GCF?" No.* We must find the two integers whose *product* is 18 and whose *sum* is 9. Those numbers are 3 and 6.

$$n^2 + 9n + 18 = (n + 3)(n + 6)$$

Check: $(n + 3)(n + 6) = n^2 + 6n + 3n + 18$
$= n^2 + 9n + 18$ ✓

b) $2k^3 - 8k^2 - 24k$

Ask yourself, *"Can I factor out a GCF?" Yes.* The GCF is $2k$.

$$2k^3 - 8k^2 - 24k = 2k(k^2 - 4k - 12)$$

Look at the trinomial and ask yourself, *"Can I factor again?" Yes.* The integers whose product is -12 and whose sum is -4 are -6 and 2. Therefore,

$$\begin{aligned} 2k^3 - 8k^2 - 24k &= 2k(k^2 - 4k - 12) \\ &= 2k(k - 6)(k + 2) \end{aligned}$$

We cannot factor again. The check is left to the student.

The completely factored form of $2k^3 - 8k^2 - 24k$ is $2k(k - 6)(k + 2)$. ■

Note

After performing one factorization, you should always ask yourself, *"Can I factor again?"* If you can, then factor the polynomial again. If not, then you know that the polynomial has been completely factored.

4. Factor Trinomials of the Form $ax^2 + bx + c$ ($a \neq 1$)

We will discuss two methods for factoring a trinomial of the form $ax^2 + bx + c$ when $a \neq 1$ and when we cannot factor out the leading coefficient of a.

Factoring $ax^2 + bx + c$ ($a \neq 1$) by Grouping

Example 4

Factor $4t^2 - 7t - 2$ completely.

Solution

Ask yourself, *"Can I factor out a GCF?" No.* Multiply the 4 and -2 to get -8.

$$\underbrace{4t^2 - 7t - 2}_{\text{Product: } 4(-2) = -8}$$

Find two integers whose *product* is -8 and whose *sum* is -7. The numbers are -8 and 1. Rewrite $-7t$ as $-8t + 1t$.

$$\begin{aligned} 4t^2 - 7t - 2 &= \underbrace{4t^2 - 8t} + \underbrace{1t - 2} \\ &= 4t(t - 2) + 1(t - 2) && \text{Take out the common factor from each group.} \\ &= (t - 2)(4t + 1) && \text{Factor out } t - 2. \\ 4t^2 - 7t - 2 &= (t - 2)(4t + 1) \end{aligned}$$

The check is left to the student.

Factoring $ax^2 + bx + c$ ($a \neq 1$) by Trial and Error

Example 5

Factor $3r^2 - 29r + 18$ completely.

Solution

Can we factor out a GCF? *No.* To get a product of $3r^2$, we will use $3r$ and r.

$$3r^2 - 29r + 18 = \underbrace{(3r \qquad)(r}_{3r^2} \qquad)$$

Since the last term is positive and the middle term is negative, we want pairs of *negative* integers that multiply to 18. The pairs are **−1** and **−18, −2** and **−9,** and **−3** and **−6**.

When we try -2 and -9, we get $3r^2 - 29r + 18 \stackrel{?}{=} (3r - 2)(r - 9)$

$$\begin{array}{r} -2r \\ +\;(-27r) \\ \hline -29r \end{array} \quad \text{Correct!}$$

$$3r^2 - 29r + 18 = (3r - 2)(r - 9)$$

5. Factor a Perfect Square Trinomial

We can use the following formulas to factor perfect square trinomials:

$$a^2 + 2ab + b^2 = (a + b)^2 \qquad a^2 - 2ab + b^2 = (a - b)^2$$

In order for a trinomial to be a perfect square, two of its terms must be perfect squares.

Example 6

Factor $16c^2 - 24c + 9$ completely.

Solution

We cannot take out a GCF. Let's see whether this trinomial fits the pattern of a perfect square trinomial.

$$\begin{array}{ccc} 16c^2 & - 24c + & 9 \\ \downarrow & & \downarrow \\ (4c)^2 & & (3)^2 \end{array}$$

What do you square to get $16c^2$? $4c$

What do you square to get 9? 3

Does the middle term equal $2 \cdot 4c \cdot 3$? Yes. $2 \cdot 4c \cdot 3 = 24c$

$$\begin{aligned} \text{Therefore, } 16c^2 - 24c + 9 &= (4c)^2 - 2(4c)(3) + (3)^2 \\ &= (4c - 3)^2 \end{aligned}$$

6. Factor the Difference of Two Squares

Another common type of factoring problem is a **difference of two squares.** Some examples of these types of binomials are

$$x^2 - 25, \qquad 4t^2 - 49u^2, \qquad 100 - n^2, \text{ and} \qquad z^4 - 1$$

Notice that in each binomial, the terms are being *subtracted*, and each term is a perfect square. To factor the difference of two squares, we use the following formula:

$$a^2 - b^2 = (a + b)(a - b)$$

Example 7

Factor $x^2 - 25$ completely.

Solution

Since each term is a perfect square, we can use the formula $a^2 - b^2 = (a + b)(a - b)$.

Identify a and b.

$$\begin{array}{ccc} & x^2 - 25 & \\ & \downarrow \quad \downarrow & \\ \text{What do you square to get } x^2\text{? } x & (x)^2 \quad (5)^2 & \text{What do you square to get 25? 5} \end{array}$$

Then, $a = x$ and $b = 5$. $x^2 - 25 = (x + 5)(x - 5)$

7. Factor the Sum and Difference of Two Cubes

A binomial that is either the sum of two cubes or the difference of two cubes can be factored using the following formulas:

Formula

Factoring the Sum and Difference of Two Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Notice that each factorization is the product of a binomial and a trinomial.

Example 8

Factor $r^3 + 27$ completely.

Solution

Identify a and b: $a = r$ and $b = 3$. Using $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, we get

$$\begin{aligned} r^3 + 27 &= (r + 3)[r^2 - (r)(3) + 3^2] && \text{Let } a = r \text{ and } b = 3. \\ &= (r + 3)(r^2 - 3r + 9) \end{aligned}$$

In addition to learning the different factoring techniques, it is important to remember these two things:

1. The first thing you should do when factoring is ask yourself, *"Can I factor out a GCF?"*
2. The last thing you should do when factoring is look at the result and ask yourself, *"Can I factor again?"*

8. Solve a Quadratic Equation by Factoring

A **quadratic equation** can be written in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. There are many different ways to solve quadratic equations. Here, we will review how to solve them by factoring.

Solving an equation by factoring is based on the **zero product rule,** which states:

$$\text{If } ab = 0, \text{ then } a = 0 \text{ or } b = 0.$$

Here are the steps to use to solve a quadratic equation by factoring:

Procedure Solving a Quadratic Equation by Factoring

1) Write the equation in the form $ax^2 + bx + c = 0$ so that all terms are on one side of the equal sign and zero is on the other side.
2) Factor the expression.
3) Set each factor equal to zero, and solve for the variable. (Use the zero product rule.)
4) Check the answer(s).

Example 9

Solve by factoring. $3w^2 + 7w = 6$

Solution

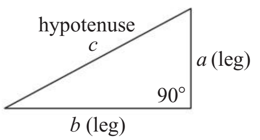
Begin by writing $3w^2 + 7w = 6$ in standard form, $ax^2 + bx + c = 0$.

$$3w^2 + 7w - 6 = 0 \qquad \text{Standard form}$$
$$(3w - 2)(w + 3) = 0 \qquad \text{Factor.}$$
$$3w - 2 = 0 \quad \text{or} \quad w + 3 = 0 \qquad \text{Set each factor equal to zero.}$$
$$3w = 2$$
$$w = \frac{2}{3} \quad \text{or} \quad w = -3 \qquad \text{Solve.}$$

Check the solutions by substituting them back into the original equation. The solution set is $\left\{-3, \frac{2}{3}\right\}$.

9. Use the Pythagorean Theorem to Solve an Applied Problem

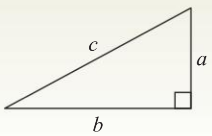
A **right triangle** is a triangle that contains a 90° (*right*) angle. We label a right triangle as follows.



The side opposite the 90° angle is the longest side of the triangle and is called the **hypotenuse.** The other two sides are called the **legs.** The Pythagorean theorem states a relationship between the lengths of the sides of a right triangle. This is a very important relationship in mathematics and one that is used in many different ways.

Definition

Pythagorean Theorem: Given a right triangle with legs of length a and b and hypotenuse of length c,

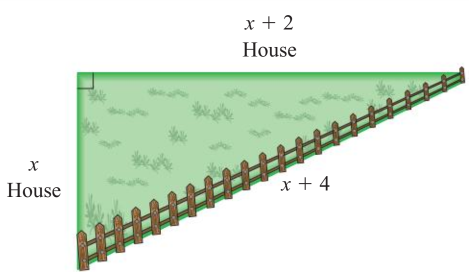


the Pythagorean theorem states that $a^2 + b^2 = c^2$ [or $(\text{leg})^2 + (\text{leg})^2 = (\text{hypotenuse})^2$].

Example 10

Write an equation and solve.

A garden situated between two walls of a house will be in the shape of a right triangle with a fence on the third side. The side with the fence will be 4 ft longer than the shortest side, and the other side will be 2 ft longer than the shortest side. How long is the fence?



Solution

Step 1: **Read** the problem carefully, and identify what we are being asked to find.

Find the length of the fence.

Step 2: **Define** the unknowns.

$$\begin{aligned} x &= \text{length of shortest side} \\ x + 2 &= \text{length of the other side along the house} \\ x + 4 &= \text{length of the fence} \end{aligned}$$

Label the picture.

Step 3: **Translate** from English into math.

The garden is in the shape of a right triangle. The hypotenuse is $x + 4$ since it is the side across from the right angle. The legs are x and $x + 2$. From the Pythagorean theorem, we get

$$\begin{aligned} a^2 + b^2 &= c^2 && \text{Pythagorean theorem} \\ x^2 + (x + 2)^2 &= (x + 4)^2 && \text{Substitute.} \end{aligned}$$

Step 4: **Solve** the equation.

$$\begin{aligned} x^2 + (x + 2)^2 &= (x + 4)^2 \\ x^2 + x^2 + 4x + 4 &= x^2 + 8x + 16 && \text{Expand.} \\ 2x^2 + 4x + 4 &= x^2 + 8x + 16 \\ x^2 - 4x - 12 &= 0 && \text{Write in standard form.} \\ (x - 6)(x + 2) &= 0 && \text{Factor.} \end{aligned}$$

$$\begin{aligned} x - 6 = 0 \quad &\text{or} \quad x + 2 = 0 && \text{Set each factor equal to 0.} \\ x = 6 \quad &\text{or} \quad x = -2 && \text{Solve.} \end{aligned}$$

Step 5: **Check** the answer and **interpret** the solution as it relates to the problem.

Since x represents the length of the shortest side, it cannot equal -2.

Therefore, the length of the shortest side must be 6 ft.

The length of the fence $= x + 4 = 6 + 4 = 10$ ft. The check is left to the student. ■

A7 Exercises

Factor out the greatest common factor.

1) $28k + 8$
2) $12p^3 - 6p$
3) $26y^3z^3 + 8y^3z^2 - 12yz^3$
4) $a(b + 7) + 4(b + 7)$

Factor by grouping.

5) $vw + 3v + 12w + 36$
6) $rs - 6r + 10s - 60$
7) $21ab - 18a - 56b + 48$
8) $18pq - 66p + 15q - 55$

Completely factor the following trinomials.

9) $x^2 + 7x + 10$
10) $z^2 + 4z - 32$
11) $2d^4 - 2d^3 - 24d^2$
12) $x^2 + 10xy + 24y^2$
13) $3n^2 + 10n + 8$
14) $2b^2 + 17b + 21$
15) $8z^2 - 14z - 9$
16) $7m^2 + 4m - 6$
17) $12c^2 - 28c + 16$
18) $h^2 + 4h + 4$
19) $x^2 - 8xy + 16y^2$
20) $100w^2 + 40w + 4$

Completely factor each binomial.

21) $m^2 - 9$
22) $\frac{1}{4} - a^2$
23) $25q^6 - 9q^4$
24) $x^4 - 1$
25) $u^2 + 25$
26) $h^3 - 64$
27) $64s^3 - 27t^3$
28) $5a^4 - 40a$

Factor completely. These exercises consist of all of the different types of polynomials presented in this section.

29) $45m - 54$
30) $a^2 - 10a + 21$
31) $4t^2 - 81$
32) $m^4n + 15m^3n + 36m^2n$

33) $w^2 + 13w - 48$

34) $n^2 + \frac{2}{3}n + \frac{1}{9}$

35) $2k^4 - 10k^3 + 8k^2$

36) $h^2 + 9$

37) $10r^2 - 43r + 12$

38) $9k^2 - 30k + 25$

39) $cd - c + d - 1$

40) $28 - 7a^2$

41) $m^2 + 5m - 50$

42) $r^2 - 7rs - 30s^2$

Solve each equation.

43) $(y + 8)(y - 3) = 0$

44) $(5m - 2)(m - 1) = 0$

45) $r^2 + 7r + 12 = 0$

46) $x^2 + 15x + 54 = 0$

47) $4w^2 = 5w$

48) $10a - 8 = 3a^2$

49) $k^2 - 169 = 0$

50) $18c^2 = 8$

51) $(y + 4)(y + 8) = 5$

52) $a(a - 1) = 56$

53) $6q^2 = -24q$

54) $2(d + 8) = (d + 9)^2 - 5$

55) $2(h - 5)^2 - 11 = 2(15 - h^2)$

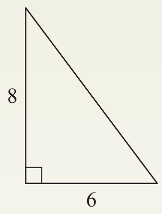
56) $16q^3 = 4q$

57) $z^3 + 7z^2 + 6z = 0$

58) $(7r - 3)(r^2 + 12r + 36) = 0$

Use the Pythagorean theorem to find the length of the missing side.

59)



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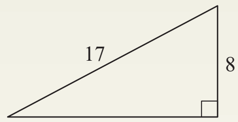
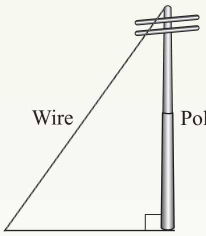
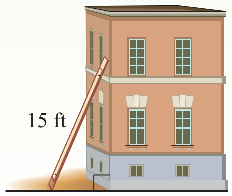


Write an equation and solve.

61) A wire is attached to the top of a pole. The pole is 2 ft shorter than the wire, and the distance from the wire on the ground to the bottom of the pole is 9 ft less than the length of the wire. Find the length of the wire and the height of the pole.



62) A 15-ft board is leaning against a wall. The distance from the top of the board to the bottom of the wall is 3 ft more than the distance from the bottom of the board to the wall. Find the distance from the bottom of the board to the wall.



Section A8 Rational Expressions

Objectives

1. Evaluate a Rational Expression, and Determine Where It Equals Zero and Where It Is Undefined
2. Define a Rational Function and Determine the Domain of a Rational Function
3. Write a Rational Expression in Lowest Terms
4. Multiply and Divide Rational Expressions
5. Add and Subtract Rational Expressions
6. Simplify Complex Fractions
7. Solve Rational Equations
8. Solve an Equation for a Specific Variable

1. Evaluate a Rational Expression, and Determine Where It Equals Zero and Where It Is Undefined

We begin by defining a rational expression.

Definition

A **rational expression** is an expression of the form $\frac{P}{Q}$, where P and Q are polynomials and where $Q \neq 0$.

We can *evaluate* rational expressions for given values of the variable(s), but just as a fraction like $\frac{9}{0}$ is undefined because its denominator equals zero, a rational expression is undefined when its denominator equals zero.

Note

1) A fraction (rational expression) equals zero when its *numerator* equals zero.
2) A fraction (rational expression) is *undefined* when its denominator equals zero.

Example 1

Given the rational expression $\dfrac{3w + 4}{w + 5}$, answer the following.

a) Evaluate the expression for $w = -2$.
b) For what value of the variable does the expression equal zero?
c) For what value of the variable is the expression undefined?

Solution

a) Substitute -2 for w in the expression and simplify.

$$\frac{3(-2) + 4}{-2 + 5} = \frac{-6 + 4}{3} = \frac{-2}{3} = -\frac{2}{3}$$

b) $\dfrac{3w + 4}{w + 5} = 0$ when its *numerator* equals zero. Set the numerator equal to zero, and solve for w.

$$\begin{aligned} 3w + 4 &= 0 \\ 3w &= -4 && \text{Subtract 4 from each side.} \\ w &= -\frac{4}{3} && \text{Divide by 3.} \end{aligned}$$

$\dfrac{3w + 4}{w + 5} = 0$ when $w = -\dfrac{4}{3}$.

c) $\dfrac{3w + 4}{w + 5}$ is *undefined* when its *denominator* equals zero. Set the denominator equal to zero, and solve for w.

$$\begin{aligned} w + 5 &= 0 \\ w &= -5 \end{aligned}$$

$\dfrac{3w + 4}{w + 5}$ is *undefined* when $w = -5$. So, w cannot equal -5 in the expression. ■

2. Define a Rational Function and Determine the Domain of a Rational Function

$f(x) = \dfrac{x + 10}{x - 3}$ is an example of a **rational function** since $\dfrac{x + 10}{x - 3}$ is a rational expression and each value that can be substituted for x will produce *only one* value for the expression.

The domain of a rational function consists of all real numbers except the value(s) of the variable that make the denominator equal zero. Therefore, to determine the domain of a rational function, we set the denominator equal to zero and solve for the variable. The value(s) that make the denominator equal to zero are *not* in the domain of the function. To determine the domain of a rational function, sometimes it is helpful to ask yourself, "Is there any number that *cannot* be substituted for the variable?"

Example 2

Determine the domain of $f(x) = \dfrac{6x - 1}{x^2 + 12x + 32}$.

Solution

To determine the domain of $f(x) = \dfrac{6x - 1}{x^2 + 12x + 32}$, ask yourself, "Is there any number that *cannot* be substituted for x? Yes, $f(x)$ is *undefined* when its *denominator* equals zero. Set the denominator equal to zero and solve for x.

$$
\begin{array}{rcll}
x^2 + 12x + 32 = 0 & & & \text{Set the denominator} = 0. \\
(x + 8)(x + 4) = 0 & & & \text{Factor.} \\
x + 8 = 0 & \text{or} & x + 4 = 0 & \text{Set each factor equal to 0.} \\
x = -8 & \text{or} & x = -4 & \text{Solve.}
\end{array}
$$

When $x = -8$ or $x = -4$, the denominator of $f(x) = \dfrac{6x - 1}{x^2 + 12x + 32}$ equals zero. The domain contains all real numbers *except* -8 and -4. Write the domain in interval notation as $(-\infty, -8) \cup (-8, -4) \cup (-4, \infty)$. ■

3. Write a Rational Expression in Lowest Terms

A rational expression is in lowest terms when its numerator and denominator contain no common factors except 1. We can use the fundamental property of rational expressions to write a rational expression in lowest terms.

Definition

Fundamental Property of Rational Expressions: If P, Q, and C are polynomials such that $Q \neq 0$ and $C \neq 0$, then $\dfrac{PC}{QC} = \dfrac{P}{Q}$.

Procedure Writing a Rational Expression in Lowest Terms

1) Completely *factor* the numerator and denominator.
2) *Divide* the numerator and denominator by the greatest common factor.

Example 3

Write $\dfrac{d^2 + 3d - 18}{5d^2 - 15d}$ in lowest terms.

Solution

$$
\begin{aligned}
\frac{d^2 + 3d - 18}{5d^2 - 15d} &= \frac{(d - 3)(d + 6)}{5d(d - 3)} && \text{Factor.} \\
&= \frac{\cancel{(d - 3)}(d + 6)}{5d\cancel{(d - 3)}} && \text{Divide out the common factor, } d - 3. \\
&= \frac{d + 6}{5d}
\end{aligned}
$$

■

4. Multiply and Divide Rational Expressions

We multiply and divide rational expressions the same way that we multiply and divide rational numbers.

Procedure Multiplying Rational Expressions:

If $\frac{P}{Q}$ and $\frac{R}{T}$ are rational expressions, then $\frac{P}{Q} \cdot \frac{R}{T} = \frac{PR}{QT}$.

To multiply two rational expressions, multiply their numerators, multiply their denominators, and simplify.

Use the following steps to multiply two rational expressions:

1) Factor.
2) Divide out common factors and multiply.

All products must be written in lowest terms.

Example 4

Multiply $\frac{k^2 - 7k + 12}{3k^2 - 8k - 3} \cdot \frac{3k^2 + 3k}{2k^2 - 32}$.

Solution

$$\frac{k^2 - 7k + 12}{3k^2 - 8k - 3} \cdot \frac{3k^2 + 3k}{2k^2 - 32} = \frac{\cancel{(k-3)}\,\cancel{(k-4)}}{(3k+1)\,\cancel{(k-3)}} \cdot \frac{3k(k+1)}{2(k+4)\cancel{(k-4)}} \quad \text{Factor.}$$

$$= \frac{3k(k+1)}{2(3k+1)(k+4)} \quad \text{Divide out common factors and multiply.}$$

To divide rational expressions, we multiply the first rational expression by the reciprocal of the second expression.

5. Add and Subtract Rational Expressions

In order to add or subtract fractions, they must have a common denominator. The same is true for rational expressions.

To find the least common denominator (LCD) of a group of rational expressions, begin by factoring the denominators. The LCD will contain each unique factor the *greatest* number of times it appears in any single factorization. The LCD is the product of these factors.

Once we have identified the least common denominator for a group of rational expressions, we must be able to rewrite each expression with this LCD.

Procedure Writing Rational Expressions as Equivalent Expressions with the Least Common Denominator

Step 1: Identify and write down the LCD.

Step 2: Look at each rational expression (with its denominator in factored form) and compare its denominator with the LCD. Ask yourself, *"What factors are missing?"*

Step 3: Multiply the numerator and denominator by the "missing" factors to obtain an equivalent rational expression with the desired LCD. Multiply the terms in the numerator, but leave the denominator as the product of factors.

Example 5

Identify the least common denominator of $\frac{8}{5t^2 - 10t}$ and $\frac{3t}{t^2 - 6t + 8}$, and rewrite each as an equivalent fraction with the LCD as its denominator.

Solution

Follow the steps.

Step 1: Identify and write down the LCD of $\frac{8}{5t^2 - 10t}$ and $\frac{3t}{t^2 - 6t + 8}$. First, we must factor the denominators.

$$\frac{8}{5t^2 - 10t} = \frac{8}{5t(t - 2)}, \quad \frac{3t}{t^2 - 6t + 8} = \frac{3t}{(t - 2)(t - 4)}$$

We will work with the factored forms of the expressions.

$$\text{LCD} = 5t(t - 2)(t - 4)$$

Step 2: Compare the denominators of $\frac{8}{5t(t - 2)}$ and $\frac{3t}{(t - 2)(t - 4)}$ to the LCD and ask yourself, *"What's missing from each denominator?"*

$\frac{8}{5t(t - 2)}$: $5t(t - 2)$ is "missing" the factor $t - 4$.

$\frac{3t}{(t - 2)(t - 4)}$: $(t - 2)(t - 4)$ is "missing" $5t$.

Step 3: Multiply the numerator and denominator by $t - 4$.

$$\frac{8}{5t(t - 2)} \cdot \frac{t - 4}{t - 4} = \frac{8(t - 4)}{5t(t - 2)(t - 4)} = \frac{8t - 32}{5t(t - 2)(t - 4)}$$

Multiply the numerator and denominator by $5t$.

$$\frac{3t}{(t - 2)(t - 4)} \cdot \frac{5t}{5t} = \frac{15t^2}{5t(t - 2)(t - 4)}$$

Multiply the factors in the numerators.

$$\frac{8}{5t(t - 2)} = \frac{8t - 32}{5t(t - 2)(t - 4)} \quad \text{and} \quad \frac{3t}{(t - 2)(t - 4)} = \frac{15t^2}{5t(t - 2)(t - 4)}$$

Now that we have reviewed how to rewrite rational expressions with a least common denominator, we summarize the steps we can use to add and subtract rational expressions.

Procedure Adding and Subtracting Rational Expressions with Different Denominators

Step 1: Factor the denominators.

Step 2: Write down the LCD.

Step 3: Rewrite each rational expression as an equivalent rational expression with the LCD.

Step 4: Add or subtract the numerators and keep the common denominator in factored form.

Step 5: After combining like terms in the numerator, ask yourself, *"Can I factor it?"* If so, factor.

Step 6: Reduce the rational expression, if possible.

Example 6

Add $\dfrac{8x - 24}{x^2 - 36} + \dfrac{x}{x + 6}$.

Solution

Step 1: Factor the denominator of $\dfrac{8x - 24}{x^2 - 36}$.

$$\frac{8x - 24}{x^2 - 36} = \frac{8x - 24}{(x + 6)(x - 6)}$$

Step 2: Identify the LCD of $\dfrac{8x - 24}{(x + 6)(x - 6)}$ and $\dfrac{x}{x + 6}$: LCD $= (x + 6)(x - 6)$.

Step 3: Rewrite $\dfrac{x}{x + 6}$ with the LCD.

$$\frac{x}{x + 6} \cdot \frac{x - 6}{x - 6} = \frac{x(x - 6)}{(x + 6)(x - 6)}$$

Step 4:

$$\begin{aligned}
\frac{8x - 24}{x^2 - 36} + \frac{x}{x + 6} &= \frac{8x - 24}{(x + 6)(x - 6)} + \frac{x}{x + 6} && \text{Factor the denominator.}\\
&= \frac{8x - 24}{(x + 6)(x - 6)} + \frac{x(x - 6)}{(x + 6)(x - 6)} && \text{Write each expression with the LCD.}\\
&= \frac{8x - 24 + x(x - 6)}{(x + 6)(x - 6)} && \text{Add the expressions.}\\
&= \frac{8x - 24 + x^2 - 6x}{(x + 6)(x - 6)} && \text{Distribute.}\\
&= \frac{x^2 + 2x - 24}{(x + 6)(x - 6)} && \text{Combine like terms.}
\end{aligned}$$

Steps 5 and 6: Ask yourself, *"Can I factor the numerator?"* Yes.

$$\begin{aligned}
\frac{x^2 + 2x - 24}{(x + 6)(x - 6)} &= \frac{\cancel{(x + 6)}(x - 4)}{\cancel{(x + 6)}(x - 6)} && \text{Factor.}\\
&= \frac{x - 4}{x - 6} && \text{Reduce.}
\end{aligned}$$

■

6. Simplify Complex Fractions

A **complex fraction** is a rational expression that contains one or more fractions in its numerator, its denominator, or both. A complex fraction is not considered to be an expression in simplest form. Let's review how to simplify complex fractions.

Example 7

Simplify each complex fraction.

a) $\dfrac{\dfrac{2a + 12}{9}}{\dfrac{a + 6}{5a}}$ b) $\dfrac{1 - \dfrac{3}{4}}{\dfrac{1}{2} + \dfrac{1}{3}}$

Solution

a) We can think of $\dfrac{\frac{2a+12}{9}}{\frac{a+6}{5a}}$ as a division problem.

$$\dfrac{\frac{2a+12}{9}}{\frac{a+6}{5a}} = \frac{2a+12}{9} \div \frac{a+6}{5a}$$ Rewrite the complex fraction as a division problem.

$$= \frac{2a+12}{9} \cdot \frac{5a}{a+6}$$ Change division to multiplication by the reciprocal of $\frac{a+6}{5a}$.

$$= \frac{2\cancel{(a+6)}}{9} \cdot \frac{5a}{\cancel{a+6}}$$ Factor and divide the numerator and denominator by $a+6$ to simplify.

$$= \frac{10a}{9}$$ Multiply.

If a complex fraction contains one term in the numerator and one term in the denominator, it can be simplified by rewriting it as a division problem and then performing the division.

b) We can simplify the complex fraction $\dfrac{1-\frac{3}{4}}{\frac{1}{2}+\frac{1}{3}}$ in two different ways. We can combine the terms in the numerator, combine the terms in the denominator, and then proceed as in part a). Or, we can follow the steps below.

Step 1: Look at all of the fractions in the complex fraction. They are $\frac{3}{4}$, $\frac{1}{2}$, and $\frac{1}{3}$. Write down their LCD: LCD $= 12$.

Step 2: Multiply the numerator and denominator of the complex fraction by the LCD, 12.

$$\dfrac{1-\frac{3}{4}}{\frac{1}{2}+\frac{1}{3}} = \dfrac{12\left(1-\frac{3}{4}\right)}{12\left(\frac{1}{2}+\frac{1}{3}\right)}$$

Step 3: Simplify.

$$\dfrac{12\left(1-\frac{3}{4}\right)}{12\left(\frac{1}{2}+\frac{1}{3}\right)} = \dfrac{12\cdot 1 - 12\cdot\frac{3}{4}}{12\cdot\frac{1}{2} + 12\cdot\frac{1}{3}}$$ Distribute.

$$= \frac{12-9}{6+4}$$ Multiply.

$$= \frac{3}{10}$$ Simplify.

If a complex fraction contains more than one term in the numerator and/or denominator, we can multiply the numerator and denominator by the LCD of all of the fractions in the expression and simplify.

7. Solve Rational Equations

When we add and subtract rational expressions, we must write each of them with the least common denominator. When we solve rational equations, however, we must *multiply* the entire equation by the LCD to eliminate the denominators. Here is a summary of the steps we use to solve rational equations:

Procedure How to Solve a Rational Equation

1) If possible, factor all denominators.
2) Write down the LCD of all of the expressions.
3) Multiply both sides of the equation by the LCD to *eliminate* the denominators.
4) Solve the equation.
5) Check the solution(s) in the original equation. If a proposed solution makes a denominator equal 0, then it is rejected as a solution.

Example 8

Solve $\dfrac{m}{m-2} - \dfrac{1}{4} = \dfrac{5}{m-2}$.

Solution

We do not need to factor the denominators, so identify the LCD of the expressions. Then, multiply both sides of the equation by the LCD to eliminate the denominators.

LCD $= 4(m-2)$

$$4(m-2)\cdot\left(\frac{m}{m-2}-\frac{1}{4}\right) = 4(m-2)\cdot\frac{5}{m-2}$$ Multiply both sides of the equation by the LCD, $4(m-2)$.

$$4\cancel{(m-2)}\cdot\left(\frac{m}{\cancel{m-2}}\right) - \cancel{4}(m-2)\cdot\left(\frac{1}{\cancel{4}}\right) = 4\cancel{(m-2)}\cdot\frac{5}{\cancel{m-2}}$$ Distribute and divide out common factors.

$$4m-(m-2) = 20$$ Multiply.

$$4m-m+2 = 20$$ Distribute.

$$3m+2 = 20$$ Combine like terms.

$$3m = 18$$ Subtract 2 from each side.

$$m = 6$$ Divide by 3.

Check: $\dfrac{6}{6-2} - \dfrac{1}{4} \stackrel{?}{=} \dfrac{5}{6-2}$ Substitute 6 for m in the original equation.

$$\frac{6}{4} - \frac{1}{4} \stackrel{?}{=} \frac{5}{4}$$

$$\frac{5}{4} = \frac{5}{4} \checkmark$$

The solution is $\{6\}$. ■

Always check what *appears* to be the solution or solutions to an equation containing rational expressions. If a value makes a denominator zero, then it *cannot* be a solution to the equation.

Example 9 shows how to solve a special type of rational equation.

Example 9

Solve $\dfrac{15}{c+3} = \dfrac{6}{c}$.

Solution

This equation is a *proportion.* A **proportion** is a statement that two ratios are equal. We can solve this proportion as we have solved the other equations in this section, by multiplying both sides of the equation by the LCD. Or, *we can solve a proportion by setting the cross products equal to each other.*

$$\frac{15}{c+3} = \frac{6}{c}$$

Multiply. Multiply.

$$\begin{aligned} 15c &= 6(c+3) && \text{Set the cross products equal to each other.} \\ 15c &= 6c + 18 && \text{Distribute.} \\ 9c &= 18 && \text{Subtract } 6c. \\ c &= 2 && \text{Divide by 9.} \end{aligned}$$

The check is left to the student. The solution is $\{2\}$.

8. Solve an Equation for a Specific Variable

When an equation contains more than one letter and we are asked to solve for a specific variable, we use the same ideas that we used in the previous examples.

Example 10

Solve $n = \dfrac{3k - t}{r}$ for r.

Solution

Since we are asked to solve for r, put the r in a box to indicate that it is the variable for which we must solve. Multiply both sides of the equation by r to eliminate the denominator.

$$\begin{aligned} n &= \frac{3k-t}{\boxed{r}} && \text{Put } r \text{ in a box.} \\ \boxed{r}(n) &= \boxed{r}\left(\frac{3k-t}{\boxed{r}}\right) && \text{Multiply both sides by } r \text{ to eliminate the denominator.} \\ \boxed{r}n &= 3k - t && \text{Multiply.} \\ r &= \frac{3k-t}{n} && \text{Divide by } n. \end{aligned}$$

A8 Exercises

Evaluate for a) $x = 5$ and b) $x = -2$.

1) $\dfrac{x^2 - 4}{3x - 1}$

Determine the value(s) of the variable for which a) the expression equals zero and b) the expression is undefined.

2) $\dfrac{h - 4}{3h + 10}$

3) $\dfrac{x^2 - 9}{7}$

4) $\dfrac{8}{t - 7}$

Determine the domain of each rational function.

5) $f(x) = \dfrac{x - 1}{x + 9}$

6) $g(c) = \dfrac{10}{2c - 3}$

7) $h(t) = \dfrac{t}{t^2 - 5t - 6}$

8) $k(n) = \dfrac{3n + 7}{n^2 + 1}$

Write each rational expression in lowest terms.

9) $\dfrac{40z^3}{48z}$

10) $\dfrac{56m^{11}}{7m}$

11) $\dfrac{12k + 48}{7k + 28}$

12) $\dfrac{c^2 + 4}{c + 2}$

13) $\dfrac{a^2 + 2a - 8}{a^2 + 11a + 28}$

14) $\dfrac{14 - 18q}{9q^2 - 16q + 7}$

Multiply or divide as indicated.

15) $\dfrac{9t^3 - 3t^2 + t}{t + 2} \div \dfrac{27t^3 + 1}{9t^2 - 1}$

16) $\dfrac{z^2 + 8z + 12}{z^3} \cdot \dfrac{3z^2 + 4z}{z^2 + 12z + 36}$

17) $\dfrac{16 - r^2}{12} \div \dfrac{2r^2 - 9r + 4}{6r - 3}$

18) $\dfrac{2a^2 - 162}{ab - 5a + 9b - 45} \cdot \dfrac{b^3 - 125}{4a - 36}$

Identify the least common denominator of each pair of fractions, and rewrite each as an equivalent fraction with the LCD as its denominator.

19) $\dfrac{3}{8w^3}, \dfrac{1}{6w}$

20) $\dfrac{c}{c - 2}, \dfrac{5}{c + 3}$

Add or subtract as indicated.

21) $\dfrac{5}{4j} + \dfrac{11}{12j}$

22) $\dfrac{3}{8z^4} - \dfrac{1}{12z}$

23) $\dfrac{7}{w} + \dfrac{4}{w - 2}$

24) $\dfrac{6}{x + 5} - \dfrac{1}{x}$

25) $\dfrac{a}{4a - 3} + \dfrac{6}{a + 2}$

26) $\dfrac{5}{c - 6} + \dfrac{2c}{c + 8}$

27) $\dfrac{k^2 + 21}{k^2 - 49} + \dfrac{5}{7 - k}$

28) $\dfrac{n + 4}{3n^2 - 5n - 2} + \dfrac{4}{3n - 1} - \dfrac{7n}{9n^2 - 1}$

Perform the indicated operations and simplify.

29) $\dfrac{3r}{4r^2 - 12r + 9} \div \dfrac{9r + 9}{2r^2 - r - 3}$

30) $\dfrac{5}{d - 6} - \dfrac{3}{d}$

31) $\dfrac{w - 4}{w^2 - 2w - 8} \cdot \dfrac{w^2 - 16}{2w^2 + 5w - 12}$

32) $\dfrac{2a - 3}{a^2 + 4a - 5} - \dfrac{a}{a^2 - 2a + 1}$

33) $\dfrac{x^2 - y^2}{x + y} \cdot \dfrac{2x^2 - 8xy}{x^2 - 5xy + 4y^2}$

34) $\dfrac{2}{5z + 4} + \dfrac{z - 1}{3z - 2}$

35) $\dfrac{9 - m^2}{5m^2 + 15m} \div (m^3 - 27)$

36) $\dfrac{h + 1}{4h^3 + h^2} - \dfrac{2}{12h + 3}$

37) $\dfrac{6x}{x - 8} + \dfrac{1}{8 - x}$

38) $\dfrac{6}{2p - 1} - \dfrac{5}{1 - 4p^2}$

Simplify completely.

39) $\dfrac{\frac{7}{9}}{\frac{5}{6}}$

40) $\dfrac{\frac{3s^4t}{7}}{\frac{9s^2t^4}{28}}$

41) $\dfrac{\frac{r - 9}{r}}{\frac{r - 9}{8}}$

42) $\dfrac{\frac{1}{8} + \frac{2}{5}}{\frac{3}{4} - \frac{3}{10}}$

43) $\dfrac{m + \frac{5}{m}}{2 + \frac{3}{m}}$

44) $\dfrac{\frac{9}{c} + 1}{\frac{7}{c} - c}$

45) $\dfrac{\frac{w^2 - 1}{4}}{\frac{3w + 3}{8}}$

46) $\dfrac{\frac{1}{h + 1} - \frac{h}{h^2 - 1}}{\frac{4}{h^2 - 1} - \frac{1}{h - 1}}$

Solve each equation.

47) $\dfrac{r}{2} + \dfrac{1}{3} = -\dfrac{7}{6}$

48) $\dfrac{2}{5}m - \dfrac{4}{15} = \dfrac{1}{3}$

49) $2 + \dfrac{z}{z - 6} = \dfrac{3z}{4}$

50) $\dfrac{x}{x + 9} + \dfrac{x}{2} = -2$

51) $\dfrac{2}{a} + \dfrac{1}{2} = \dfrac{6}{a}$

52) $\dfrac{8w + 15}{18w - 6} - \dfrac{1}{3w - 1} = \dfrac{2w}{3}$

53) $\dfrac{11}{3z - 15} + \dfrac{z + 3}{z - 5} = \dfrac{z}{3}$

54) $\dfrac{k}{k + 6} + \dfrac{4}{k - 3} = \dfrac{11k}{k^2 + 3k - 18}$

55) $\dfrac{x - 1}{2x^2 - 3x} + \dfrac{3}{2x^2 + 3x} = \dfrac{2x + 1}{4x^2 - 9}$

56) $\dfrac{b}{2b^2 + 7b + 3} - \dfrac{b - 1}{b^2 - 9} = \dfrac{1}{2b^2 - 5b - 3}$

Solve for the indicated variable.

57) $\dfrac{y - b}{x} = m$ for y

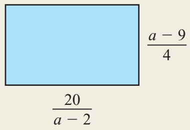
58) $V = \dfrac{nRT}{P}$ for P

59) $C = \dfrac{ab}{d - t}$ for d

60) $\dfrac{1}{R_1} + \dfrac{1}{R_2} = \dfrac{1}{R_3}$ for R_1

For each rectangle, find a rational expression in simplest form to represent its a) area and b) perimeter.

61)



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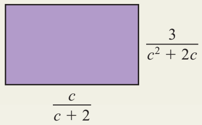


Write an equation for each and solve.

63) The ratio of jazz music to reggae in Consuela's music library is 5 to 3. If there are 18 fewer reggae songs than jazz songs, how many songs of each type are in her music library?

64) A cold water faucet can fill a sink in 4 min, while it takes a hot water faucet 6 min. How long would it take to fill the sink if both faucets are on?

Answers to Exercises

Chapter 1

Section 1.1

1) a) $\frac{2}{5}$ b) $\frac{2}{3}$ c) 1

3) $\frac{1}{2}$ 

5) a) 1, 2, 3, 6, 9, 18 b) 1, 2, 4, 5, 8, 10, 20, 40 c) 1, 23

7) a) composite b) composite c) prime

9) Composite. It is divisible by 2 and has other factors as well.

11) a) $2 \cdot 3 \cdot 3$ b) $2 \cdot 3 \cdot 3 \cdot 3$ c) $2 \cdot 3 \cdot 7$
d) $2 \cdot 3 \cdot 5 \cdot 5$

13) a) $\frac{3}{4}$ b) $\frac{3}{4}$ c) $\frac{12}{5}$ or $2\frac{2}{5}$ d) $\frac{3}{7}$

15) a) $\frac{6}{35}$ b) $\frac{10}{39}$ c) $\frac{7}{15}$ d) $\frac{12}{25}$ e) $\frac{1}{2}$ f) $\frac{7}{4}$ or $1\frac{3}{4}$

17) She multiplied the whole numbers and multiplied the fractions. She should have converted the mixed numbers to improper fractions before multiplying. Correct answer: $\frac{77}{6}$ or $12\frac{5}{6}$.

19) a) $\frac{1}{12}$ b) $\frac{15}{44}$ c) $\frac{4}{7}$ d) 7 e) $\frac{24}{7}$ or $3\frac{3}{7}$ f) $\frac{1}{14}$

21) 30 23) a) 30 b) 24 c) 36

25) a) $\frac{8}{11}$ b) $\frac{3}{5}$ c) $\frac{3}{5}$ d) $\frac{7}{18}$ e) $\frac{29}{30}$ f) $\frac{1}{18}$
g) $\frac{71}{63}$ or $1\frac{8}{63}$ h) $\frac{7}{12}$ i) $\frac{7}{5}$ or $1\frac{2}{5}$ j) $\frac{41}{54}$

27) a) $14\frac{7}{11}$ b) $11\frac{2}{5}$ c) $6\frac{1}{2}$ d) $5\frac{9}{20}$ e) $1\frac{2}{5}$ f) $4\frac{13}{40}$
g) $11\frac{5}{28}$ h) $8\frac{9}{20}$

29) four bears; $\frac{1}{3}$ yd remaining 31) 50

33) $16\frac{1}{2}$ in. by $22\frac{5}{8}$ in. 35) $3\frac{5}{12}$ cups 37) $5\frac{3}{20}$ gal

39) 35 41) $7\frac{23}{24}$ in. 43) 240

Section 1.2

1) a) base: 6; exponent: 4 b) base: 2; exponent: 3
c) base: $\frac{9}{8}$; exponent: 5

3) a) 9^4 b) 2^8 c) $\left(\frac{1}{4}\right)^3$

5) a) 64 b) 121 c) 16 d) 125 e) 81 f) 144 g) 1
h) $\frac{9}{100}$ i) $\frac{1}{64}$ j) 0.09

7) $(0.5)^2 = 0.5 \cdot 0.5 = 0.25$ or $(0.5)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

9) Answers may vary.

11) 19 13) 38 15) 23 17) 17 19) $\frac{13}{20}$ 21) $\frac{19}{18}$ or $1\frac{1}{18}$

23) 4 25) 15 27) 19 29) 37 31) 11 33) $\frac{5}{6}$

35) $\frac{27}{7}$ or $3\frac{6}{7}$

Section 1.3

1) acute 3) straight 5) supplementary; complementary

7) 31° 9) 78° 11) 37° 13) 142°

15) $m\angle A = m\angle C = 149°$, $m\angle B = 31°$ 17) 180

19) 39°; obtuse 21) 39°; right 23) equilateral 25) isosceles

27) true 29) $A = 80\ \text{ft}^2$; $P = 36$ ft

31) $A = 42\ \text{cm}^2$; $P = 29.25$ cm

33) $A = 42.25\ \text{mi}^2$; $P = 26$ mi 35) $A = 162\ \text{in}^2$; $P = 52$ in.

37) a) $A = 25\pi\ \text{in}^2$; $A \approx 78.5\ \text{in}^2$ b) $C = 10\pi$ in; $C \approx 31.4$ in.

39) a) $A = 6.25\pi\ \text{m}^2$; $A \approx 19.625\ \text{m}^2$
b) $C = 5\pi$ m; $C = 15.7$ m

41) $A = \frac{1}{4}\pi\ \text{m}^2$; $C = \pi$ m 43) $A = 49\pi\ \text{ft}^2$; $C = 14\pi$ ft

45) $A = 376\ \text{m}^2$; $P = 86$ m 47) $A = 201.16\ \text{in}^2$; $P = 67.4$ in.

49) 88 in^2 51) 25.75 ft^2 53) 177.5 cm^2 55) 70 m^3

57) $288\pi\ \text{in}^3$ 59) $\frac{500}{3}\pi\ \text{ft}^3$ 61) $136\pi\ \text{cm}^3$

63) a) 58.5 ft^2 b) No, it would cost \$1170 to use this glass.

65) a) 226.08 ft^3 b) 1691 gal 67) a) 62.8 in. b) 314 in^2

69) 6395.4 gal

71) No. This granite countertop would cost \$2970.00.

73) 28.9 in. 75) a) 44 ft^2 b) \$752 77) 140.4 in^2

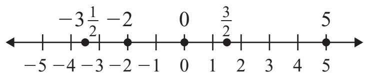
79) 1205.76 ft^3

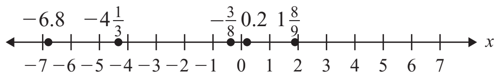
Section 1.4

1) Answers may vary.

3) a) 17 b) 17, 0 c) 17, 0, −25
d) 17, 3.8, $\frac{4}{5}$, 0, −25, $6.\overline{7}$, $-2\frac{1}{8}$
e) $\sqrt{10}$, 9.721983… f) all numbers in the set

5) true 7) false 9) true

11) 

13) 

15) the distance of the number from zero 17) −8

19) 15 21) $\frac{3}{4}$ 23) 10 25) $\frac{9}{4}$ 27) −14 29) 13

31) $-4\frac{1}{7}$ 33) −10, −2, 0, $\frac{9}{10}$, 3.8, 7

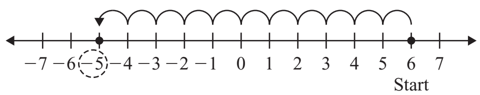
35) −6.51, −6.5, −5, 2, $7\frac{1}{3}$, $7\frac{5}{6}$ 37) true 39) false

41) false 43) false 45) −53 47) 1.4 million

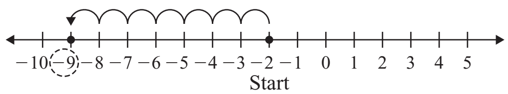
49) −419,000

Section 1.5

1) Answers may vary. 3) Answers may vary.

5) 

6 − 11 = −5

7) 

−2 + (−7) = −9

9) −7 11) −14 13) 23 15) −11 17) −850 19) $\frac{1}{6}$

21) $-\frac{25}{24}$ or $-1\frac{1}{24}$ 23) $-\frac{8}{45}$ 25) 2.7 27) −9.23

29) 15 31) −11 33) −2 35) −19 37) 23

39) $-\frac{19}{18}$ or $-1\frac{1}{18}$ 41) $\frac{5}{24}$ 43) 12 45) 11 47) −7

49) false 51) false 53) true

55) −18 + 6 = −12. His score in the 2005 Masters was −12.

57) 6,110,000 − 5,790,000 = 320,000. The carbon emissions of China were 320,000 thousand metric tons more than those of the United States.

59) 881,566 + 45,407 = 926,973. There were 926,973 flights at O' Hare in 2007.

61) a) −3 b) −134 c) −131 d) −283

63) a) −0.7 b) 0.4 c) 0.1 d) −0.2

65) 5 + 7; 12 67) 10 − 16; −6 69) 9 − (−8); 17

71) −21 + 13; −8 73) −20 + 30; 10 75) 23 − 19; 4

77) (−5 + 11) − 18; −12

Section 1.6

1) negative 3) −56 5) 45 7) 84 9) $-\frac{2}{15}$ 11) 1.4

13) 135 15) −84 17) when k is negative

19) when $k \neq 0$ 21) 36 23) −125 25) 9 27) −49

29) −32 31) positive 33) 10 35) −4 37) −8

39) $\frac{10}{13}$ 41) 0 43) $-\frac{3}{2}$ or $-1\frac{1}{2}$ 45) −33 47) 43

49) 16 51) 16 53) $\frac{1}{4}$ 55) $-12 \cdot 6$; −72

57) (−7)(−5) + 9; 44 59) $\frac{63}{-9} + 7$; 0

61) (−4)(−8) − 19; 13 63) $\frac{-100}{4} - (-7 + 2)$; −20

65) 2[18 + (−31)]; −26 67) $\frac{2}{3}(-27)$; −18

69) $12(-5) + \frac{1}{2}(36)$; −42

Section 1.7

1)

Term	Coeff.
$7p^2$	7
$-6p$	−6
4	4

The constant is 4.

3)

Term	Coeff.
x^2y^2	1
$2xy$	2
$-y$	−1
11	11

The constant is 11.

5)

Term	Coeff.
$-2g^5$	−2
$\frac{g^4}{5}$	$\frac{1}{5}$
$3.8g^2$	3.8
g	1
−1	−1

The constant is −1.

7) a) 11 b) -17 9) -17 11) 0 13) -3 15) $\frac{3}{2}$

17) No. The exponents are different. 19) Yes. Both are a^3b-terms.

21) 1 23) -5 25) distributive 27) identity

29) commutative 31) associative 33) $19 + p$

35) $(8 + 1) + 9$ 37) $3k - 21$ 39) y

41) No. Subtraction is not commutative.

43) $2 \cdot 1 + 2 \cdot 9 = 2 + 18 = 20$

45) $-2 \cdot 5 + (-2) \cdot 7 = -10 + (-14) = -24$

47) $4 \cdot 8 - 4 \cdot (3) = 32 - 12 = 20$

49) $-10 + 4 = -6$

51) $8y + 8 \cdot 3 = 8y + 24$ 53) $-10z + (-10) \cdot 6 = -10z - 60$

55) $-3x - (-3) \cdot (4y) - (-3) \cdot (6) = -3x + 12y + 18$

57) $8c - 9d + 14$ 59) $24p + 7$ 61) $-19y^2 + 30$

63) $\frac{16}{5}r - \frac{2}{9}$ 65) $7w + 10$ 67) 0 69) $-5g + 2$ 71) $-3t$

73) $26x + 37$ 75) $\frac{11}{10}z + \frac{13}{2}$ 77) $\frac{65}{8}t - \frac{19}{16}$

79) $-1.1x - 19.6$ 81) $x + 18$ 83) $x - 6$ 85) $x - 3$

87) $12 + 2x$ 89) $(3 + 2x) - 7; 2x - 4$

91) $(x + 15) - 5; x + 10$

Chapter 1 Review Exercises

1) a) 1, 2, 4, 8, 16 b) 1, 37 3) a) $\frac{2}{5}$ b) $\frac{23}{39}$ 5) $\frac{3}{10}$

7) 40 9) $\frac{3}{14}$ 11) $\frac{11}{12}$ 13) $\frac{7}{10}$ 15) $\frac{19}{56}$ 17) $6\frac{13}{24}$ 19) 81

21) $\frac{27}{64}$ 23) 10 25) $\frac{1}{4}$ 27) 102°

29) $A = 6\frac{9}{16}$ mi^2; $P = 10\frac{3}{4}$ mi 31) $A = 100$ in^2; $P = 40$ in.

33) a) $A = 9\pi$ in^2; $A \approx 28.26$ in^2
b) $C = 6\pi$ in.; $C \approx 18.84$ in.

35) 360.66 cm^2 37) 1.3π ft^3 39) $\frac{125}{8}$ in^3 or $15\frac{5}{8}$ in^3

41) a) $\{-16, 0, 4\}$ b) $\left\{\frac{7}{15}, -16, 0, 3.\overline{2}, 8.5, 4\right\}$

c) $\{4\}$ d) $\{0, 4\}$ e) $\{\sqrt{31}, 6.01832\ldots\}$

43) a) 18 b) -7 45) 19 47) $-\frac{5}{24}$ 49) -12 51) 72

53) -12 55) $\frac{15}{4}$ or $3\frac{3}{4}$ 57) -36 59) 64 61) 27

63) -9 65) -31 67) $\frac{-120}{-3}$; 40 69) $(-4) \cdot 7 - 15; -43$

71)

Term	Coeff.
$5z^4$	5
$-8z^3$	-8
$\frac{3}{5}z^2$	$\frac{3}{5}$
$-z$	-1
14	14

73) $\frac{1}{52}$ 75) inverse 77) distributive

79) $7 \cdot 3 - 7 \cdot 9 = 21 - 63 = -42$ 81) $-15 + 3 = -12$

83) $12m - 10$ 85) $17y^2 - 3y - 3$ 87) $\frac{31}{4}n - \frac{9}{2}$

Chapter 1 Test

1) $2 \cdot 3 \cdot 5 \cdot 7$ 2) a) $\frac{5}{8}$ b) $\frac{3}{4}$ 3) $\frac{5}{24}$ 4) $\frac{23}{36}$ 5) $7\frac{5}{12}$

6) $\frac{1}{27}$ 7) $-\frac{1}{4}$ 8) -17 9) 20 10) $-\frac{1}{12}$ 11) 60

12) -3.7 13) -49 14) $\frac{2}{3}$ 15) 15,211 ft

16) a) 125 b) -16 c) 43 d) -37 17) 149°

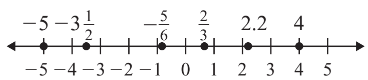
18) 49°; acute

19) a) $A = 9$ mm^2; $P = 14.6$ mm b) $A = 105$ cm^2; $P = 44$ cm
c) $A = 200$ in^2; $P = 68$ in.

20) 9 ft^3 21) a) 81π ft^2 b) 254.34 ft^2

22) a) 22, 0 b) 22 c) $\sqrt{43}$, 8.0934 …

d) 22, -7, 0 e) $3\frac{1}{5}$, 22 -7, 0, 6.2, $1.\overline{5}$

23) 

24) a) $-4 + 27$; 23 b) $17-5(-6)$; 47

25)

Term	Coeff.
$4p^3$	4
$-p^2$	-1
$\frac{1}{3}p$	$\frac{1}{3}$
-10	-10

26) $\frac{1}{3}$ 27) a) commutative b) associative

c) inverse d) distributive

28) a) $(-4) \cdot 2 + (-4) \cdot 7 = -8 + (-28) = -36$
b) $3 \cdot 8m - 3 \cdot 3n + 3 \cdot 11 = 24m - 9n + 33$

29) a) $-6k^2 + 4k - 14$ b) $6c - \frac{49}{6}$ 30) $2x - 9$

Chapter 2

Section 2.1A

1) 9^6 3) $\left(\frac{1}{7}\right)^4$ 5) $(-5)^7$ 7) $(-3y)^8$ 9) base: 6; exponent: 8

11) base: 0.05; exponent: 7 13) base: -8; exponent: 5

15) base: $9x$; exponent: 8 17) base: $-11a$; exponent: 2

19) base: p; exponent: 4 21) base: y; exponent: 2

23) $(3 + 4)^2 = 49$, $3^2 + 4^2 = 25$. They are not equivalent because when evaluating $(3 + 4)^2$, first add $3 + 4$ to get 7, then square the 7.

25) Answers may vary. 27) No. $3t^4 = 3 \cdot t^4$; $(3t)^4 = 3^4 \cdot t^4 = 81t^4$

29) 32 31) 121 33) 16 35) -81 37) -8 39) $\frac{1}{125}$

41) 32 43) 81 45) 200 47) $\frac{1}{64}$ 49) 8^{12} 51) 5^{11}

53) $(-7)^8$ 55) b^6 57) k^6 59) $8y^5$ 61) $54m^{15}$ 63) $-42r^5$

65) $28t^{16}$ 67) $-40x^6$ 69) $8b^{15}$ 71) y^{12} 73) w^{77} 75) 729

77) $(-5)^6$ 79) $\frac{1}{81}$ 81) $\frac{36}{a^2}$ 83) $\frac{m^5}{n^5}$ 85) $10{,}000y^4$

87) $81p^4$ 89) $-64a^3b^3$ 91) $6x^3y^3$ 93) $-9t^4u^4$

95) a) $A = 3w^2$ sq units; $P = 8w$ units
b) $A = 5k^5$ sq units; $P = 10k^3 + 2k^2$ units

97) $\frac{3}{8}x^2$ sq units

Section 2.1B

1) operations 3) k^{24} 5) $200z^{26}$ 7) $-6a^{31}b^7$ 9) 121

11) $-64t^{18}u^{26}$ 13) $288k^{14}t^4$ 15) $\frac{3}{4g^{15}}$ 17) $\frac{49}{4}n^{22}$ 19) $900h^{28}$

21) $-147w^{45}$ 23) $\frac{36x^6}{25y^{10}}$ 25) $\frac{d^{18}}{4c^{30}}$ 27) $\frac{2a^{36}b^{63}}{9c^2}$ 29) $\frac{r^{39}}{242t^5}$

31) $\frac{2}{3}x^{24}y^{14}$ 33) $-\frac{1}{10}c^{29}d^{18}$ 35) $\frac{125x^{15}y^6}{z^{12}}$ 37) $\frac{81t^{16}u^{36}}{16v^{28}}$

39) $\frac{9w^{10}}{x^6y^{12}}$ 41) a) $20l^2$ units b) $25l^4$ sq units

43) a) $\frac{3}{8}x^2$ sq units b) $\frac{11}{4}x$ units

Section 2.2A

1) false 3) true 5) 1 7) -1 9) 0 11) 2 13) $\frac{1}{36}$

15) $\frac{1}{16}$ 17) $\frac{1}{125}$ 19) 64 21) 32 23) $\frac{27}{64}$ 25) $\frac{49}{81}$

27) -64 29) $\frac{64}{9}$ 31) $-\frac{1}{64}$ 33) -1 35) $\frac{1}{16}$

37) $\frac{13}{36}$ 39) $\frac{83}{81}$

Section 2.2B

1) a) w b) n c) $2p$ d) c 3) 1 5) -2 7) 2 9) $\frac{1}{d^3}$

11) $\frac{1}{p}$ 13) $\frac{b^3}{a^{10}}$ 15) $\frac{x^5}{y^8}$ 17) $\frac{t^5u^3}{8}$ 19) $\frac{5m^6}{n^2}$ 21) $2t^{11}u^5$

23) $\frac{8a^6c^{10}}{5bd}$ 25) $2x^7y^6z^4$ 27) $\frac{36}{a^2}$ 29) $\frac{q^5}{32n^5}$ 31) $\frac{c^2d^2}{144b^2}$

33) $-\frac{9}{k^2}$ 35) $\frac{3}{t^3}$ 37) $-\frac{1}{m^9}$ 39) z^{10} 41) j 43) $5n^2$ 45) cd^3

Section 2.3

1) You must subtract the denominator's exponent from the numerator's exponent; a^2.

3) d^5 5) m^4 7) $8t^7$ 9) 36 11) 81 13) $\frac{1}{16}$ 15) $\frac{1}{125}$

17) $10d^2$ 19) $\frac{2}{3}c^5$ 21) $\frac{1}{y^5}$ 23) $\frac{1}{x^9}$ 25) $\frac{1}{t^3}$ 27) $\frac{1}{a^{10}}$ 29) t^3

31) $\frac{15}{w^8}$ 33) $-\frac{6}{k^3}$ 35) a^3b^7 37) $\frac{2k^3}{3l^8}$ 39) $\frac{10}{x^5y^5}$

41) $\frac{w^6}{9v^3}$ 43) $\frac{3}{8}c^4d$ 45) $(x + y)^7$ 47) $(c + d)^6$

Putting It All Together

1) $\frac{16}{81}$ 2) 64 3) 1 4) -125 5) $\frac{9}{100}$ 6) $\frac{49}{9}$ 7) 9

8) -125 9) $\frac{1}{100}$ 10) $\frac{1}{8}$ 11) $\frac{1}{32}$ 12) 81 13) $-\frac{27}{125}$

14) 64 15) $\frac{1}{36}$ 16) $\frac{13}{36}$ 17) $270g^{12}$ 18) $56d^9$ 19) $\frac{33}{s^{11}}$

20) $\frac{1}{c^5}$ 21) $\frac{16}{81}x^{40}y^{24}$ 22) $\frac{a^9b^{15}}{1000}$ 23) $\frac{n^6}{81m^{16}}$ 24) $\frac{r^8s^{24}}{81}$

25) $-b^{15}$ 26) h^{88} 27) $-27m^{15}n^6$ 28) $169a^{12}b^2$

29) $-6z^3$ 30) $-9w^9$ 31) $\frac{t^{18}}{s^{42}}$ 32) $\frac{1}{m^3n^{14}}$ 33) $a^{14}b^3c^7$

34) $\frac{4}{9v^{10}}$ 35) $\frac{27u^{30}}{64v^{21}}$ 36) $\frac{81}{x^6y^8}$ 37) $-27t^6u^{15}$

38) $\frac{1}{144}k^{16}m^2$ 39) $\frac{1}{h^{18}}$ 40) $-\frac{1}{d^{20}}$ 41) $\frac{h^4}{16}$ 42) $\frac{13}{f^2}$

43) $56c^{10}$ 44) $80p^{15}$ 45) $\frac{3}{a^5}$ 46) $\frac{1}{9r^2s^2}$ 47) $\frac{6}{55}r^{10}$

48) $\frac{1}{f^{36}}$ 49) $\frac{a^2b^9}{c}$ 50) $\frac{x^9y^{24}}{z^3}$ 51) $\frac{72n^3}{5m^5}$ 52) $\frac{t^7}{100s^7}$ 53) 1

54) 1 55) $\frac{9}{49d^6}$ 56) $\frac{1}{100x^{10}y^8}$ 57) p^{12c} 58) $25d^{8t}$

59) y^{4m} 60) x^{4c} 61) $\frac{1}{t^{3b}}$ 62) $\frac{1}{a^{7y}}$ 63) $\frac{5}{8c^{7x}}$ 64) $-\frac{3}{8y^{8a}}$

Section 2.4

1) yes 3) no 5) no 7) yes 9) Answers may vary.

11) Answers may vary. 13) 7176.5 15) 0.0406

17) 0.1200006 19) -0.000068 21) $-52{,}600$ 23) 0.000008

25) 602,196.7 27) 3,000,000 29) -0.000744

31) 24,428,000 33) 0.000000000025 meters

35) 2.1105×10^3 37) 9.6×10^{-5} 39) -7×10^6

41) 3.4×10^3 43) 8×10^{-4} 45) -7.6×10^{-2} 47) 6×10^3

49) 3.808×10^8 kg 51) 1×10^{-8} cm 53) 30,000

55) 690,000 57) -1200 59) -0.06 61) -0.0005

63) 160,000 65) 0.0001239 67) 5,256,000,000 particles

69) 17,000 lb/cow 71) 26,400,000 droplets 73) \$6083

75) 1.34784×10^9 m 77) 20 metric tons

Chapter 2 Review Exercises

1) a) 8^6 b) $(-7)^4$ 3) a) 32 b) $\frac{1}{27}$ c) 7^{12} d) k^{30}

5) a) $125y^3$ b) $-14m^{16}$ c) $\frac{a^6}{b^6}$ d) $6x^2y^2$ e) $\frac{25}{3}c^8$

7) a) z^{22} b) $-18c^{10}d^{16}$ c) 125 d) $\frac{25t^6}{2u^{21}}$

9) a) 1 b) -1 c) $\frac{1}{9}$ d) $-\frac{5}{36}$ e) $\frac{125}{64}$

11) a) $\frac{1}{v^9}$ b) $\frac{c^2}{81}$ c) y^8 d) $-\frac{7}{k^9}$ e) $\frac{19a}{z^4}$ f) $\frac{20n^5}{m^6}$ g) $\frac{k^5}{32j^5}$

13) a) 9 b) r^8 c) $\frac{3}{2t^5}$ d) $\frac{3x^7}{5y}$

15) a) $81s^{16}t^{20}$ b) $2a^{16}$ c) $\frac{y^{18}}{z^{24}}$ d) $-36x^{11}y^{11}$ e) $\frac{d^{25}}{c^{35}}$

f) $8m^3n^{12}$ g) $\frac{125t^9}{27k^{18}}$ h) 14

17) a) y^{10k} b) x^{10p} c) z^{7c} d) $\frac{1}{t^{5d}}$ 19) -418.5

21) 0.00067 23) 20,000 25) 5.75×10^{-5} 27) 3.2×10^7

29) 1.78×10^5 31) 9.315×10^{-4} 33) 0.0000004

35) 3.6 37) 7500 39) 30,000 quills

41) 0.00000000000000299 g 43) 25,740,000

Chapter 2 Test

1) $(-3)^3$ 2) x^5 3) 125 4) $\frac{1}{x^7}$ 5) 8^{36} 6) p^5 7) 81

8) 1 9) $\frac{1}{32}$ 10) $\frac{3}{16}$ 11) $-\frac{27}{64}$ 12) $\frac{49}{100}$ 13) $125n^{18}$

14) $-30p^{12}$ 15) m^6 16) $\frac{a^4}{b^6}$ 17) $-\frac{t^{33}}{27u^{27}}$ 18) $\frac{8}{y^9}$

19) 1 20) $2m + n$ 21) $\frac{3a^4c^2}{5b^3d^3}$ 22) y^{18} 23) t^{13k}

24) 728,300 25) 1.65×10^{-4} 26) $-50{,}000$

27) 21,800,000 28) 0.0000000000000000182 g

Cumulative Review for Chapters 1–2

1) $\frac{3}{5}$ 2) $\frac{7}{12}$ 3) $\frac{7}{25}$ 4) 12 5) -28 6) -81 7) -1

8) 42 9) a) $346\frac{2}{3}$ yd b) \$11,520 10) 62 11) $V = \frac{4}{3}\pi r^3$

12) a) $-4, 3$ b) $\sqrt{11}$ c) 3 d) $-4, 3, -2.1\overline{3}, 2\frac{2}{3}$ e) 3

13) 261 14) $\frac{9}{2}m - 15n + \frac{21}{4}$ 15) $-3t^2 + 37t - 25$

16) $\frac{1}{2}x - 13$ 17) 4^{10} 18) $\frac{y^3}{x^3}$ 19) $\frac{1}{4x^5}$ 20) $-\frac{81r^4}{t^{12}}$

21) $-28z^8$ 22) $\frac{1}{n^7}$ 23) $-\frac{32b^5}{a^{30}}$ 24) 7.29×10^{-4}

25) 58,280

Chapter 3

Section 3.1

1) equation 3) expression 5) No, it is an expression.

7) b, d 9) no 11) yes 13) yes 15) $\{17\}$ 17) $\{-6\}$

19) $\{-4\}$ 21) $\left\{-\frac{1}{8}\right\}$ 23) $\{5.9\}$ 25) Answers may vary.

27) $\{10\}$ 29) $\{-7\}$ 31) $\{8\}$ 33) $\{48\}$ 35) $\{-48\}$

37) $\{-15\}$ 39) $\{18\}$ 41) $\left\{\frac{11}{5}\right\}$ 43) $\{12\}$ 45) $\{7\}$

47) $\{0\}$ 49) $\{8\}$ 51) $\{2\}$ 53) $\{0\}$ 55) $\left\{-\frac{2}{5}\right\}$

57) $\{-1\}$ 59) $\{27\}$ 61) $\left\{-\frac{7}{5}\right\}$ 63) $\{10\}$ 65) $\left\{\frac{24}{5}\right\}$

67) $\left\{-\frac{3}{2}\right\}$ 69) $\{6\}$ 71) $\{-3.8\}$

73) Combine like terms; $8x + 11 - 11 = 27 - 11$; Combine like terms; $\frac{8x}{8} = \frac{16}{8}$; $x = 2$; $\{2\}$

75) $\{-3\}$ 77) $\{19\}$ 79) $\left\{-\frac{5}{4}\right\}$ 81) $\{-3\}$ 83) $\{0\}$

85) $\{-1\}$ 87) $\{2\}$ 89) $\left\{\frac{1}{4}\right\}$

Section 3.2

1) Answers may vary. 3) $\{3\}$ 5) $\{-3\}$ 7) $\left\{\frac{7}{3}\right\}$

9) $\{-6\}$ 11) $\{4\}$ 13) $\{0\}$ 15) $\left\{\frac{5}{4}\right\}$

17) Eliminate the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

19) Multiply both sides of the equation by 8.

21) $\{5\}$ 23) $\{3\}$ 25) $\{-8\}$ 27) $\left\{\frac{20}{9}\right\}$ 29) $\{-2\}$

31) $\{3\}$ 33) $\{-20\}$ 35) $\{1.2\}$ 37) $\{300\}$ 39) $\{24\}$

41) The variable is eliminated, and you get a false statement like $5 = 12$.

43) $\varnothing$ 45) {all real numbers} 47) {all real numbers}

49) $\varnothing$ 51) $\{0\}$ 53) $\{6000\}$ 55) {all real numbers}

57) $\left\{\frac{3}{4}\right\}$ 59) $\varnothing$ 61) Answers may vary.

63) $x + 12 = 5; -7$ 65) $x - 9 = 12; 21$ 67) $2x + 5 = 17; 6$

69) $2x + 18 = 8; -5$ 71) $3x - 8 = 40; 16$ 73) $\frac{3}{4}x = 33; 44$

75) $\frac{1}{2}x - 9 = 3; 24$ 77) $x + 6 = 8; 2$ 79) $2x - 3 = x + 8; 11$

81) $\frac{1}{3}x + 10 = x - 2; 18$ 83) $x - 45 = \frac{x}{4}; 60$

85) $x + \frac{2}{3}x = 25; 15$ 87) $x - 2x = 13; -13$

Section 3.3

1) $c + 14$ 3) $c - 37$ 5) $\frac{1}{2}s$ 7) $14 - x$

9) The number of children must be a whole number.

11) It is an even number. 13) 1905: 4.2 inches, 2004: 3.0 inches

15) Lance: 7, Miguel: 5 17) regular: 260 mg, decaf: 20 mg

19) Spanish: 186, French: 124 21) 11 in., 25 in.

23) bracelet: 9.5 in., necklace: 19 in.

25) Derek: 2 ft, Cory: 3 ft, Tamara: 1 ft 27) 41, 42, 43

29) 18, 20 31) $-15, -13, -11$ 33) 107, 108

35) Jimmy: 13, Kelly: 7 37) 5 ft, 11 ft

39) Bonnaroo: 70,000, Lollapalooza: 225, 000 41) 57, 58, 59

43) Helen: 140 lb, Tara: 155 lb, Mike: 207 lb

45) 12 in., 24 in., 36 in.

47) Lil Wayne: 2.88 million, Coldplay: 2.15 million, Taylor Swift: 2.11 million

49) 72, 74, 76

Section 3.4

1) \$42.50 3) \$20.65 5) \$19.60 7) \$140.00 9) \$17.60

11) \$66.80 13) 1800 acres 15) 1015 17) 4400 people

19) \$9 21) \$6955 23) \$315 25) \$9000 at 6%, \$6000 at 7%

27) \$1650 at 6%, \$2100 at 5%

29) \$3000 at 9.5%, \$4500 at 6.5% 31) 3 oz 33) 8.25 mL

35) 16 oz of the 4% acid solution, 8 oz of the 10% acid solution

37) 3 L 39) 2 lb 41) 250 g 43) $1\frac{1}{5}$ gallons 45) \$8.75

47) \$8500 49) CD: \$2000, IRA: \$4000, mutual fund: \$3000

51) \$32.00 53) \$38,600 55) 9%: 3 oz, 17%: 9 oz

57) peanuts: 7 lb; cashews: 3 lb 59) 4%: \$9000, 7%: \$11,000

61) 16 oz of orange juice, 60 oz of the fruit drink 63) 2,100,000

Section 3.5

1) No. The height of a triangle cannot be a negative number.

3) cubic centimeters 5) $\frac{11}{4}$ 7) 3000 9) 2.5 11) 9.2π

13) 4 15) 4 17) 9 19) 4 21) 10 23) 78 ft 25) 8 in.

27) 314 yd^2 29) 67 mph 31) 24 in. 33) 3 ft 35) 2.5%

37) 18 in. $\times$ 28 in. 39) 12 ft $\times$ 19 ft 41) 2 in., 8 in.

43) 1.5 ft, 1.5 ft, 2.5 ft 45) $m\angle A = 35°$, $m\angle C = 62°$

47) $m\angle A = 26°$, $m\angle B = 52°$

49) $m\angle A = 44°$, $m\angle B = m\angle C = 68°$

51) 43°, 43° 53) 172°, 172° 55) 38°, 38° 57) 144°, 36°

59) 120°, 60° 61) 73°, 107° 63) $180 - x$ 65) 63° 67) 24°

69) angle: 20°, comp: 70°, supp: 160° 71) 72° 73) 45°

75) a) $x = 21$ b) $x = y - h$ c) $x = c - r$

77) a) $c = 7$ b) $c = \frac{d}{a}$ c) $c = \frac{v}{m}$

79) a) $a = 44$ b) $a = ry$ c) $a = dw$

81) a) $d = 3$ b) $d = \frac{z + a}{k}$ 83) a) $h = -\frac{2}{3}$ b) $h = \frac{n - v}{q}$

85) $m = \frac{F}{a}$ 87) $c = nv$ 89) $\sigma = \frac{E}{T^4}$ 91) $h = \frac{3V}{\pi r^2}$

93) $E = IR$ 95) $R = \frac{I}{PT}$ 97) $l = \frac{P - 2w}{2}$ or $l = \frac{P}{2} - w$

99) $N = \frac{2.5H}{D^2}$ 101) $b_2 = \frac{2A}{h} - b_1$ or $b_2 = \frac{2A - hb_1}{h}$

103) $h^2 = \frac{S}{\pi} - \frac{c^2}{4}$ or $h^2 = \frac{1}{4}\left(\frac{4S}{\pi} - c^2\right)$

105) a) $w = \frac{P - 2l}{2}$ or $w = \frac{P}{2} - l$ b) 3 cm

107) a) $F = \frac{9}{5}C + 32$ b) 68°F

Section 3.6

1) Answers may vary, but some possible answers are $\frac{6}{8}$, $\frac{9}{12}$, and $\frac{12}{16}$.

3) Yes, a percent can be written as a fraction with a denominator of 100. For example, 25% can be written as $\frac{25}{100}$ or $\frac{1}{4}$.

5) $\frac{4}{3}$ 7) $\frac{2}{25}$ 9) $\frac{1}{4}$ 11) $\frac{2}{3}$ 13) $\frac{3}{8}$

15) package of 8: \$0.786 per battery

17) 48-oz jar: \$0.177 per oz 19) 24-oz box: \$0.262 per oz

21) A ratio is a quotient of two quantities. A proportion is a statement that two ratios are equal.

23) true 25) false 27) true 29) $\{2\}$ 31) $\{40\}$

33) $\{18\}$ 35) $\left\{\frac{8}{3}\right\}$ 37) $\{-2\}$ 39) $\{11\}$ 41) $\{-1\}$

43) $\left\{\frac{5}{2}\right\}$ 45) \$3.54 47) $\frac{1}{2}$ cup 49) 82.5 mg 51) 360

53) 8 lb 55) 35.75 Euros 57) $x = 10$ 59) $x = 13$

61) $x = 63$ 63) a) \$0.70 b) 70¢ 65) a) \$4.22 b) 422¢

67) a) \$2.20 b) 220¢ 69) a) $0.25q$ b) $25q$

71) a) $0.10d$ b) $10d$ 73) a) $0.01p + 0.05n$ b) $p + 5n$

75) 9 dimes, 17 quarters

77) 11 \$5 bills, 18 \$1 bills 79) 19 adults, 38 children

81) Marc Anthony: \$86, Santana: \$66.50 83) miles

85) northbound: 200 mph, southbound: 250 mph 87) 5 hours

89) $1\frac{2}{3}$ hr 91) 36 minutes 93) Nick: 14 mph, Scott: 12 mph

95) passenger train: 50 mph, freight train: 30 mph

97) 7735 yen 99) $\frac{1}{4}$ hour or 15 min

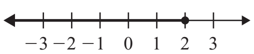
101) 27 dimes, 16 quarters

103) jet: 400 mph, small plane: 200 mph 105) 240

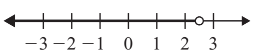
Section 3.7

1) Use brackets when there is a $\le$ or $\ge$ symbol.

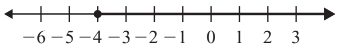
3) $(-\infty, 4)$ 5) $[-3, \infty)$

7) 

a) $\{k \mid k \le 2\}$ b) $(-\infty, 2]$

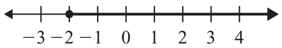
9) 

a) $\left\{c \,\middle|\, c < \frac{5}{2}\right\}$ b) $\left(-\infty, \frac{5}{2}\right)$

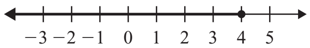
11) 

a) $\{a \mid a \ge -4\}$ b) $[-4, \infty)$

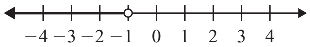
13) when you multiply or divide the inequality by a negative number

15) 

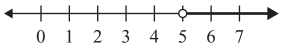
a) $\{k \mid k \ge -2\}$ b) $[-2, \infty)$

17) 

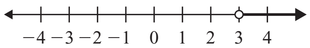
a) $\{c \mid c \le 4\}$ b) $(-\infty, 4]$

19) 

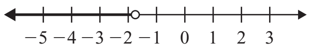
a) $\{d \mid d < -1\}$ b) $(-\infty, -1)$

21) 

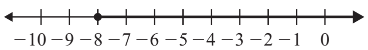
a) $\{z \mid z > 5\}$ b) $(5, \infty)$

23) 

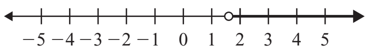
a) $\{m \mid m > 3\}$ b) $(3, \infty)$

25) 

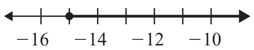
a) $\left\{x \,\middle|\, x < -\frac{7}{4}\right\}$ b) $\left(-\infty, -\frac{7}{4}\right)$

27) 

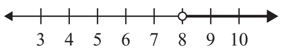
a) $\{b \mid b \ge -8\}$ b) $[-8, \infty)$

29) 

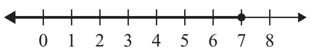
a) $\left\{a \,\middle|\, a > \frac{5}{3}\right\}$ b) $\left(\frac{5}{3}, \infty\right)$

31) 

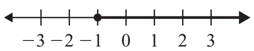
a) $\{k \mid k \ge -15\}$ b) $[-15, \infty)$

33) 

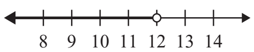
a) $\{c \mid c > 8\}$ b) $(8, \infty)$

35) 

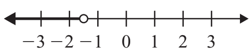
$(-\infty, 7]$

37) 

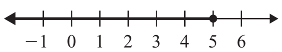
$[-1, \infty)$

39) 

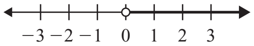
$(-\infty, 12)$

41) 

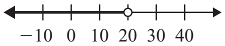
$\left(-\infty, -\frac{3}{2}\right)$

43) 

$(-\infty, 5]$

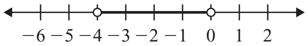
45) 

$(0, \infty)$

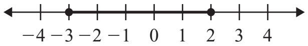
47) 

$(-\infty, 20)$

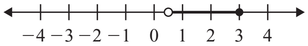
49) $[-5, 3]$ 51) $(-3, 0]$

53) 

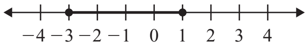
a) $\{y \mid -4 < y < 0\}$ b) $(-4, 0)$

55) 

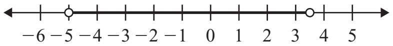
a) $\{k \mid -3 \le k \le 2\}$ b) $[-3, 2]$

57) 

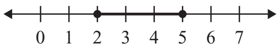
a) $\left\{n \,\middle|\, \frac{1}{2} < n \le 3\right\}$ b) $\left(\frac{1}{2}, 3\right]$

59) 

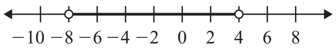
$[-3, 1]$

61) 

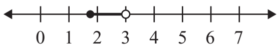
$\left(-5, \frac{7}{2}\right)$

63) 

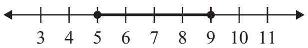
$[2, 5]$

65) 

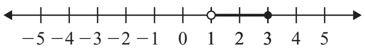
$(-8, 4)$

67) 

$\left[\frac{7}{4}, 3\right)$

69) 

$[5, 9]$

71) 

$(1, 3]$

73) 

$[5, 8)$

75) 17 77) 26 79) at most 6 mi 81) 89 or higher

Section 3.8

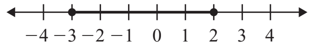
1) $A \cap B$ means "A intersect B." $A \cap B$ is the set of all numbers that are in set A *and* in set B.

3) $\{8\}$ 5) $\{2, 4, 5, 6, 7, 8, 9, 10\}$ 7) $\varnothing$

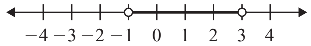
9) $\{1, 2, 3, 4, 5, 6, 8, 10\}$

11) {Liliane Bettencourt, Alice Walton}

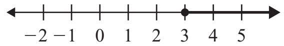
13) {Liliane Bettencourt, J. K. Rowling, Oprah Winfrey}

15) 

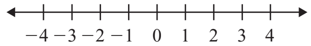
$[-3, 2]$

17) 

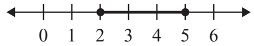
$(-1, 3)$

19) 

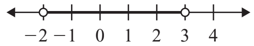
$[3, \infty)$

21) 

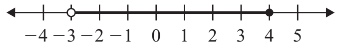
$\varnothing$

23) 

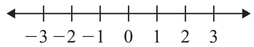
$[2, 5]$

25) 

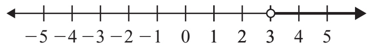
$(-2, 3)$

27) 

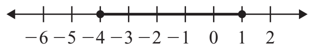
$(-3, 4]$

29) 

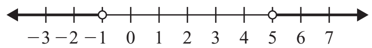
$\varnothing$

31) 

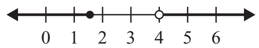
$(3, \infty)$

33) 

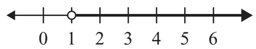
$[-4, 1]$

35) 

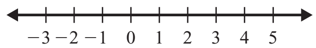
$(-\infty, -1) \cup (5, \infty)$

37) 

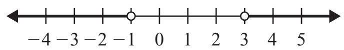
$\left(-\infty, \frac{5}{3}\right] \cup (4, \infty)$

39) 

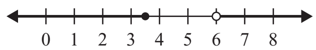
$(1, \infty)$

41) 

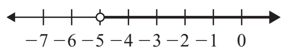
$(-\infty, \infty)$

43) 

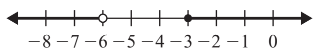
$(-\infty, -1) \cup (3, \infty)$

45) 

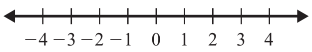
$\left(-\infty, \frac{7}{2}\right] \cup (6, \infty)$

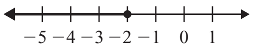
47) 

$(-5, \infty)$

49) 

$(-\infty, -6) \cup [-3, \infty)$

51) 
$(-\infty, \infty)$

53) 
$(-\infty, -2]$

Chapter 3 Review Exercises

1) no

3) The variables are eliminated and you get a false statement like $5 = 13$.

5) $\{-19\}$ 7) $\{-8\}$ 9) $\{36\}$ 11) $\left\{\frac{15}{2}\right\}$ 13) $\{5\}$

15) $\left\{-\frac{2}{3}\right\}$ 17) $\{0\}$ 19) $\{10\}$ 21) {all real numbers}

23) $2x - 9 = 25$; 17 25) Thursday: 75, Friday: 51

27) 14 in., 22 in. 29) 6 lb 31) \$1500 at 2%, \$4500 at 4%

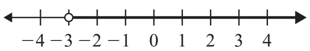
33) 7 35) 7 in. 37) $m\angle A = 55°, m\angle B = 55°, m\angle C = 70°$

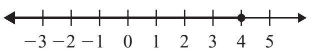
39) 61°, 61° 41) $p = z + n$ 43) $b = \frac{2A}{h}$

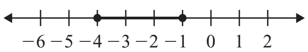
45) Yes. It can be written as $\frac{15}{100}$ or $\frac{3}{20}$.

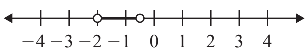
47) $\frac{4}{5}$ 49) $\{12\}$ 51) 1125 53) 12 \$20 bills, 10 \$10 bills

55) Jared: 5 mph, Meg: 6 mph

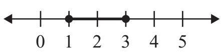
57) 
$(-3, \infty)$

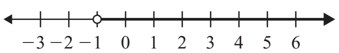
59) 
$(-\infty, 4]$

61) 
$[-4, -1]$

63) 
$\left(-2, -\frac{1}{2}\right)$

65) {Toyota} 67) {10, 20, 25, 30, 35, 40, 50}

69) 
$[1, 3]$

71) 
$(-1, \infty)$

73) $\left\{\frac{5}{2}\right\}$ 75) $\{-42\}$ 77) $\varnothing$ 79) $\{13\}$ 81) $\{-1\}$

83) 120 oz 85) 17, 19 87) 8 cm, 11 cm, 16 cm 89) 360

Chapter 3 Test

1) $\left\{-\frac{7}{9}\right\}$ 2) $\{-9\}$ 3) $\{6\}$ 4) $\{-3\}$ 5) $\left\{\frac{2}{3}\right\}$

6) $\varnothing$ 7) $\{1\}$

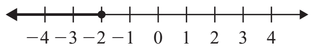
8) A ratio is a quotient of two quantities. A proportion is a statement that two ratios are equal.

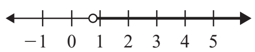
9) 36, 38, 40 10) 250 mL 11) \$31.90 12) 10 in. × 15 in.

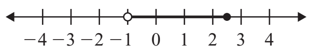
13) eastbound: 66 mph, westbound: 72 mph

14) $a = \frac{4B}{n}$ 15) $h = \frac{S - 2\pi r^2}{2\pi r}$ or $h = \frac{S}{2\pi r} - r$

16) $m\angle A = 26°, m\angle B = 115°, m\angle C = 39°$

17)  $(-\infty, -2]$

18)  $\left(\frac{3}{4}, \infty\right)$

19)  $\left(-1, \frac{5}{2}\right]$

20) at least 77 21) $(-\infty, -8) \cup \left(\frac{7}{3}, \infty\right)$

22) $[0, 3]$ 23) $(-\infty, \infty)$

Cumulative Review for Chapters 1–3

1) $-\frac{11}{24}$ 2) $\frac{15}{2}$ 3) 54 4) -87 5) -47 6) 27 cm^2

7) $\{-5, 0, 9\}$ 8) $\left\{\frac{3}{4}, -5, 2.5, 0, 0.\overline{4}, 9\right\}$

9) $\{0, 9\}$ 10) distributive

11) No. For example, $10 - 3 \neq 3 - 10$.

12) $17y^2 - 18y$ 13) $\frac{5}{4}r^{12}$ 14) $-72\,m^{33}$ 15) $-\frac{9}{2z^6}$

16) $\frac{1}{4c^6d^{10}}$ 17) 8.95×10^{-6} 18) $\left\{-\frac{23}{2}\right\}$

19) $\{4\}$ 20) {all real numbers} 21) $\left\{\frac{11}{8}\right\}$

22) $\{18\}$ 23) car: 60 mph, train: 70 mph

24) $(-\infty, -6]$ 25) $(-\infty, -3] \cup \left[\frac{11}{4}, \infty\right)$

Chapter 4

Section 4.1

1) 16.1 gallons 3) 2004 and 2006; 15.9 gallons

5) Consumption was increasing. 7) New Jersey; 86.3%

9) Florida's graduation rate is about 32.4% less than New Jersey's.

11) Answers may vary. 13) yes

15) yes 17) no 19) yes 21) 5 23) $-\frac{7}{2}$ 25) 5

27)

x	y
0	−4
1	−2
−1	−6
−2	−8

29)

x	y
0	0
$\frac{1}{2}$	2
3	12
−5	−20

31)

x	y
0	−2
$-\frac{8}{5}$	0
1	$-\frac{13}{4}$
$-\frac{12}{5}$	1

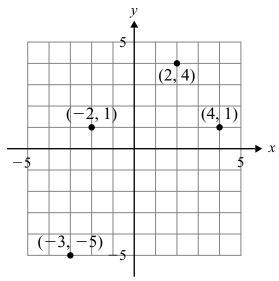
33)

x	y
0	−2
−3	−2
8	−2
17	−2

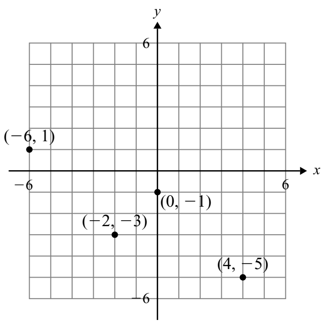
35) Answers may vary.

37) *A*: (−2, 1), quadrant II; *B*: (5, 0), no quadrant; *C:* (−2, −1), quadrant III; *D*: (0, −1), no quadrant; *E*: (2, −2); quadrant IV; *F*: (3, 4); quadrant I

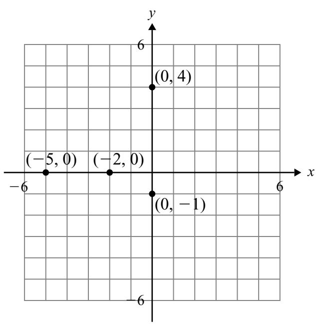
39–42)



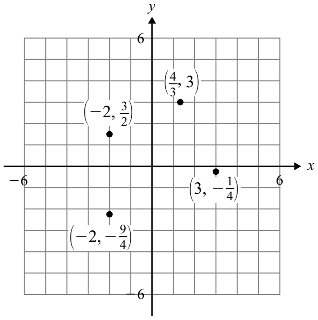
43–46)



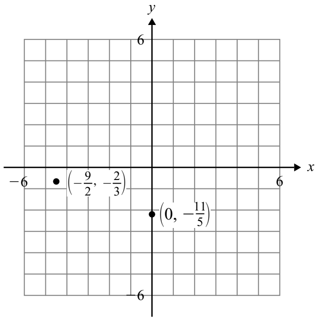
47–50)



51–54)

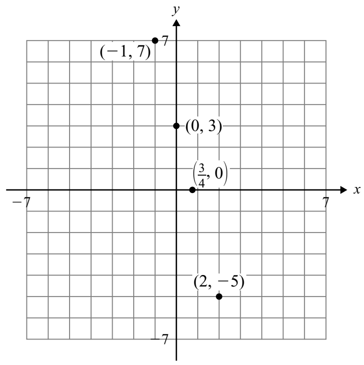


55)



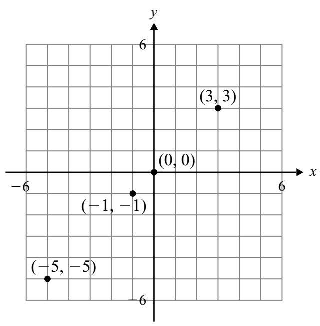
57)

x	y
0	3
$\frac{3}{4}$	0
2	−5
−1	7



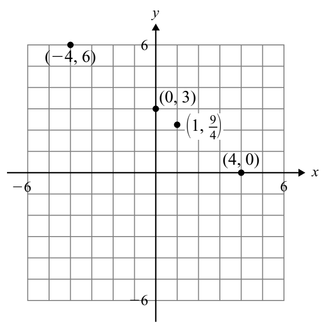
59)

x	y
0	0
−1	−1
3	3
−5	−5



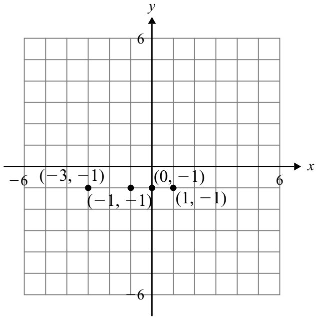
61)

x	y
0	3
4	0
1	$\frac{9}{4}$
−4	6



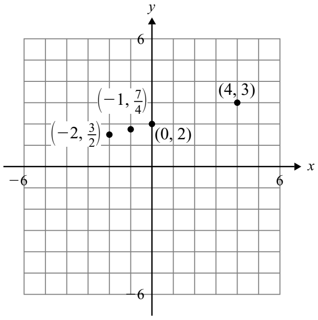
63)

x	y
0	−1
1	−1
−3	−1
−1	−1



65)

x	y
0	2
−2	$\frac{3}{2}$
4	3
−1	$\frac{7}{4}$

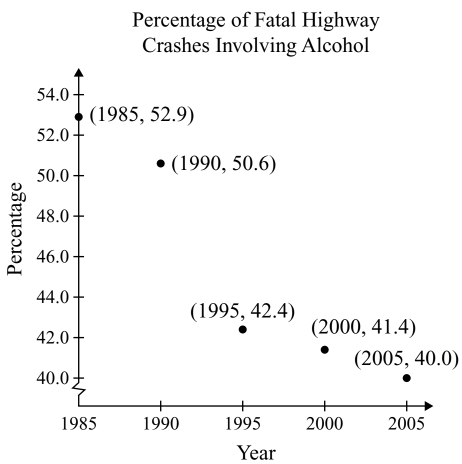


67) a) (3, −5), (6, −3), (−3, −9)
b) $\left(1, -\frac{19}{3}\right), \left(5, -\frac{11}{3}\right), \left(-2, -\frac{25}{3}\right)$
c) The x-values in part a) are multiples of the denominator of $\frac{2}{3}$. When you multiply $\frac{2}{3}$ by a multiple of 3, the fraction is eliminated.

69) negative 71) negative 73) positive 75) zero

77) a) x represents the year; y represents the number of visitors in millions.
b) In 2004, there were 37.4 million visitors to Las Vegas.
c) 38.9 million d) 2005 e) 2 million f) (2007, 39.2)

79) a) (1985, 52.9), (1990, 50.6), (1995, 42.4), (2000, 41.4), (2005, 40.0)
b)

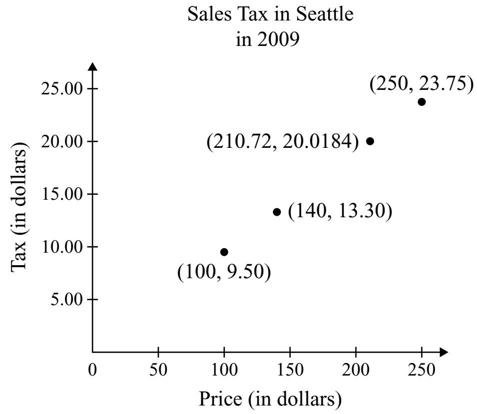


c) In the year 2000, 41.4% of all fatal highway accidents involved alcohol.

81) a)

x	y
100.00	9.50
140.00	13.30
210.72	20.0184
250.00	23.75

b)



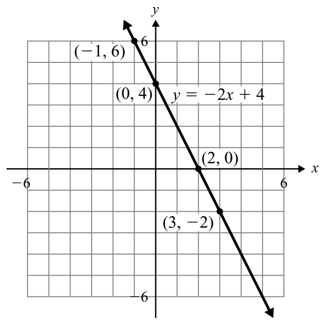
c) If a bill totals \$140.00, the sales tax will be \$13.30.
d) \$20.02 e) They lie on a straight line. f) \$200.00

Section 4.2

1) line

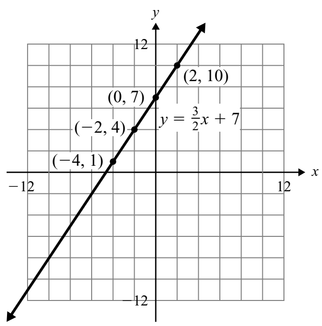
3)

x	y
0	4
−1	6
2	0
3	−2



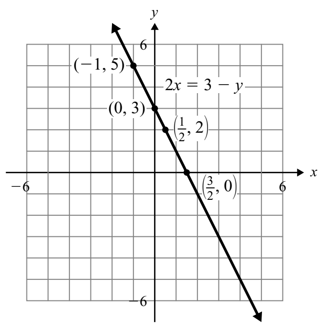
5)

x	y
0	7
2	10
−2	4
−4	1



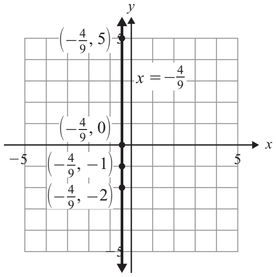
7)

x	y
$\frac{3}{2}$	0
0	3
$\frac{1}{2}$	2
−1	5



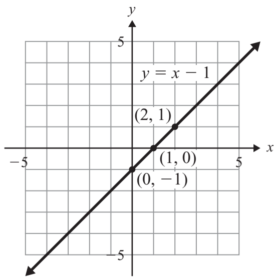
9)

x	y
$-\frac{4}{9}$	5
$-\frac{4}{9}$	0
$-\frac{4}{9}$	-1
$-\frac{4}{9}$	-2

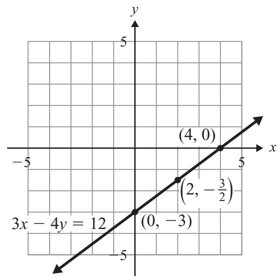


11) It is the point where the graph intersects the y-axis. Let $x = 0$ in the equation and solve for y.

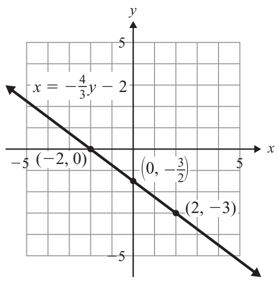
13) $(1, 0), (0, -1), (2, 1)$



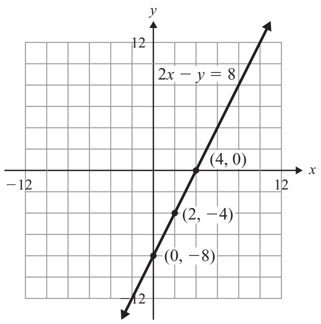
15) $(4, 0), (0, -3), (2, -\frac{3}{2})$



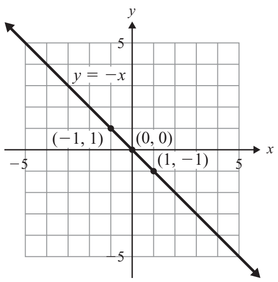
17) $(-2, 0), (0, -\frac{3}{2},) (2, -3)$



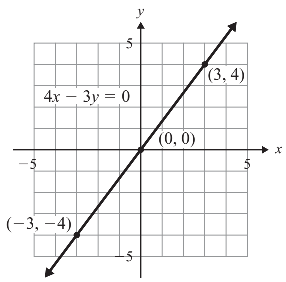
19) $(4, 0), (0, -8), (2, -4)$



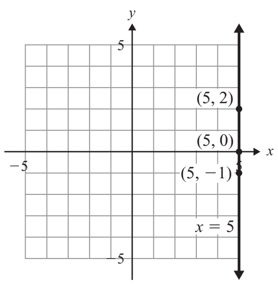
21) $(0, 0), (1, -1), (-1, 1)$



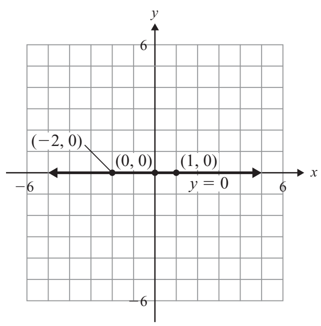
23) $(0, 0), (3, 4), (-3, -4)$



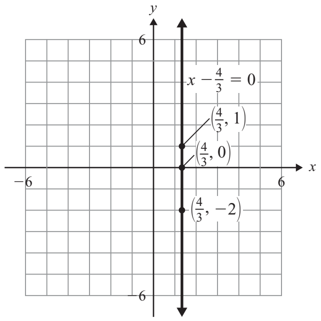
25) $(5, 0), (5, 2), (5, -1)$



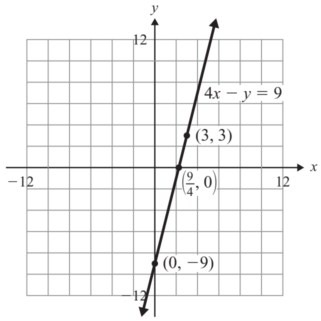
27) $(0, 0), (1, 0), (-2, 0)$



29) $(\frac{4}{3}, 0), (\frac{4}{3}, 1), (\frac{4}{3}, -2)$



31) $(\frac{9}{4}, 0), (0, -9), (3, 3)$



33) $(0, 0)$

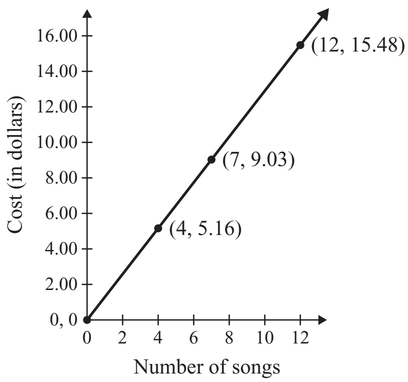
35) a)

x	y
0	0
4	5.16
7	9.03
12	15.48

(0, 0), (4, 5.16), (7, 9.03), (12, 15.48)

b) (0, 0): If no songs are purchased, the cost is \$0. (4, 5.16): The cost of downloading 4 songs is \$5.16. (7, 9.03): The cost of downloading 7 songs is \$9.03. (12, 15.48): The cost of downloading 12 songs is \$15.48.

c) Cost of Downloading Popular Songs from iTunes

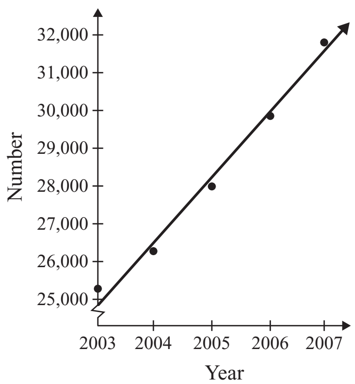


d) 9

37) a) 2004: 26,275; 2007: 31,801

b) 2004: 26,578; 2007: 31,564; yes, they are close.

c) Number of Science and Engineering Doctorates Awarded in the U.S.



d) The y-intercept is 24,916. In 2003, approximately 24,916 science and engineering doctorates were awarded. It looks like it is within about 300 units of the plotted point.
e) 39,874

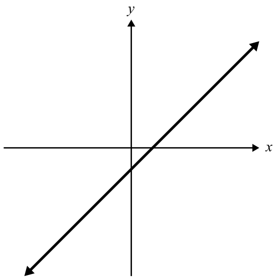
Section 4.3

1) The slope of a line is the ratio of vertical change to horizontal change. It is $\dfrac{\text{Change in } y}{\text{Change in } x}$ or $\dfrac{\text{Rise}}{\text{Run}}$ or $\dfrac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) and (x_2, y_2) are points on the line.

3) It slants upward from left to right.

5) undefined 7) $m = \dfrac{3}{4}$ 9) $m = -\dfrac{2}{3}$ 11) $m = -3$

13) Slope is undefined.

15)


17) 2 19) -4 21) $-\dfrac{11}{5}$ 23) undefined 25) 0 27) $\dfrac{3}{7}$

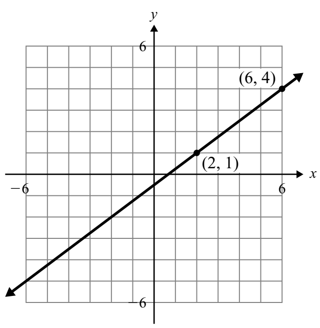
29) 0.75 31) $\dfrac{12}{79}$

33) No. The slope of the slide is $0.\overline{6}$. This is more than the recommended slope.

35) Yes. The slope of the driveway is 0.0375. This is less than the maximum slope allowed.

37) $\dfrac{6}{13}$

39) a) 2.89 mil; 2.70 mil
b) negative
c) The number of injuries is decreasing.
d) $m = -0.1$; the number of injuries is decreasing by about 0.1 million, or 100,000 per year.

41)


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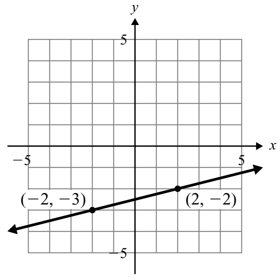

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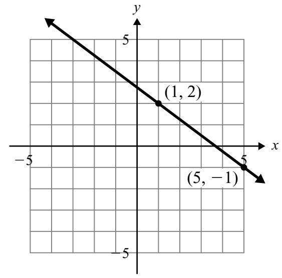

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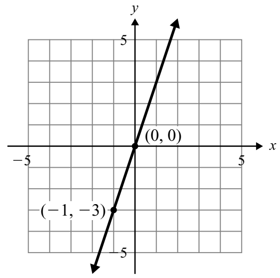

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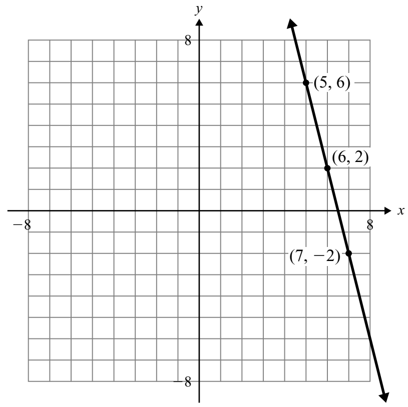

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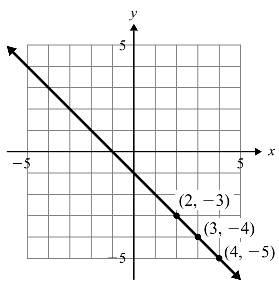

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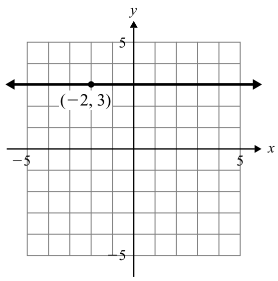

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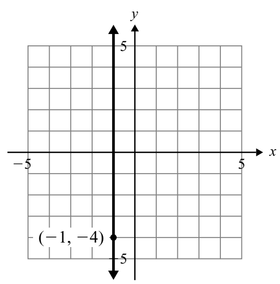

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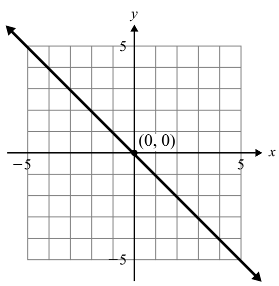
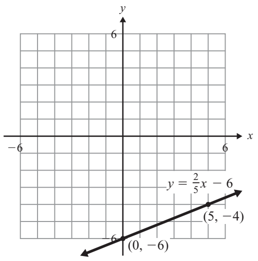


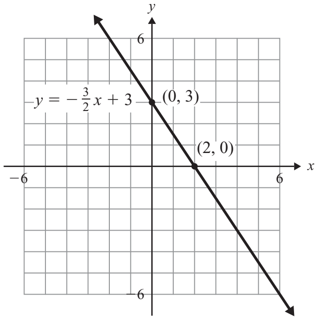
Section 4.4

1) The slope is m, and the y-intercept is $(0, b)$.

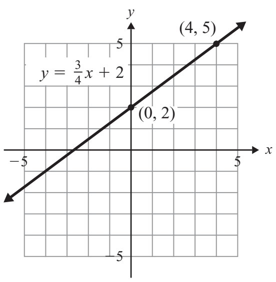
3) $m = \frac{2}{5}$, y-int: $(0, -6)$



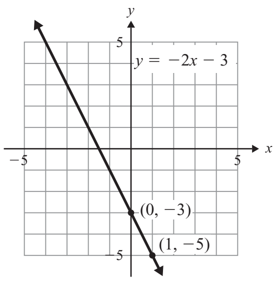
5) $m = -\frac{3}{2}$, y-int: $(0, 3)$



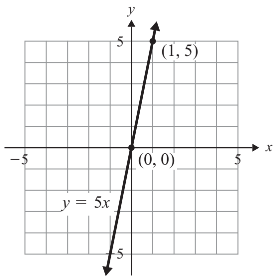
7) $m = \frac{3}{4}$, y-int: $(0, 2)$



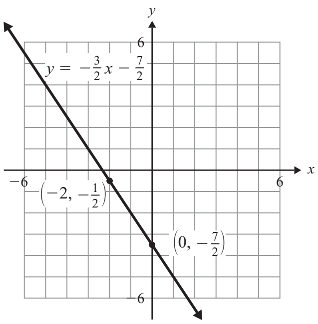
9) $m = -2$, y-int: $(0, -3)$



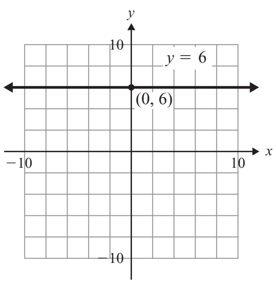
11) $m = 5$, y-int: $(0, 0)$



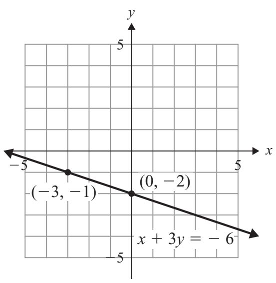
13) $m = -\frac{3}{2}$, y-int: $\left(0, -\frac{7}{2}\right)$



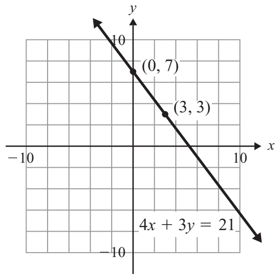
15) $m = 0$, y-int: $(0, 6)$



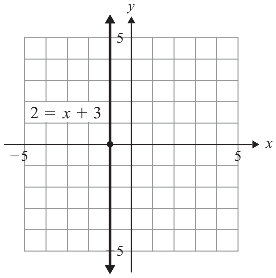
17) $y = -\frac{1}{3}x - 2$



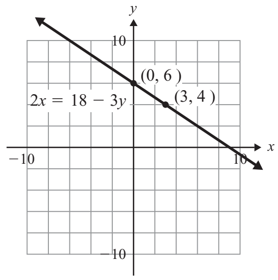
19) $y = -\frac{4}{3}x + 7$



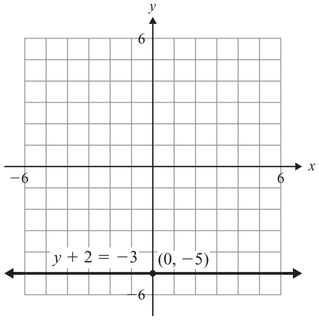
21) This cannot be written in slope-intercept form.



23) $y = -\frac{2}{3}x + 6$



25) $y = -5$



27) a) (0, 0); if Kolya works 0 hr, he earns \$0.
b) $m = 8.50$; Kolya earns \$8.50 per hour.
c) \$102.00

29) a) (0, 18); when the joey comes out of the pouch, it weighs 18 oz.
b) 24 oz
c) A joey gains 2 oz per week after coming out of its mother's pouch.
d) 7 weeks

31) a) (0, 0); \$0 = 0 rupees
b) 48.2; each American dollar is worth 48.2 rupees.
c) 3856 rupees
d) \$50.00

33) $y = -4x + 7$ 35) $y = \frac{9}{5}x - 3$ 37) $y = -\frac{5}{2}x - 1$

39) $y = x + 2$ 41) $y = 0$

43) Their slopes are negative reciprocals, or one line is vertical and one is horizontal.

45) perpendicular 47) parallel 49) neither 51) neither

53) perpendicular 55) parallel 57) perpendicular

59) parallel 61) perpendicular 63) neither 65) parallel

67) parallel 69) perpendicular

Section 4.5

1) $2x + y = -4$ 3) $x - y = 1$ 5) $4x - 5y = -5$

7) $4x + 12y = -15$

9) Substitute the slope and y-intercept into $y = mx + b$.

11) $y = -7x + 2$ 13) $4x + y = 6$

15) $2x - 7y = 21$ 17) $y = -x$

19) a) $y - y_1 = m(x - x_1)$
b) Substitute the slope and point into the point-slope formula.

21) $y = x + 2$ 23) $y = -5x + 19$ 25) $4x - y = -7$

27) $2x - 5y = -50$ 29) $y = -\frac{5}{4}x + \frac{29}{4}$ 31) $5x - 6y = -15$

33) Find the slope and use it and one of the points in the point-slope formula.

35) $y = -3x + 4$ 37) $y = 2x - 3$ 39) $y = -\frac{1}{3}x + \frac{10}{3}$

41) $x + 3y = -2$ 43) $5x - 3y = 18$ 45) $y = -3.0x + 1.4$

47) $y = \frac{3}{4}x - 1$ 49) $y = -3x - 4$ 51) $y = 3$

53) $y = -\frac{4}{3}x + \frac{5}{3}$ 55) $y = x + 2$ 57) $y = 7x + 6$

59) $x = 3$ 61) $y = 3$ 63) $y = -4x - 4$

65) $y = -3x + 20$ 67) $y = \frac{1}{2}x - 2$

69) They have the same slopes and different y-intercepts.

71) $y = 4x + 2$ 73) $4x - y = 0$ 75) $x + 2y = 10$

77) $y = 5x - 2$ 79) $y = \frac{3}{2}x - 4$ 81) $x - 5y = 10$

83) $y = -x - 5$ 85) $3x - y = 10$ 87) $y = -3x - 12$

89) $y = -x + 11$ 91) $y = 4$ 93) $y = 2$ 95) $y = -\frac{2}{7}x + \frac{1}{7}$

97) $y = -\frac{3}{2}$

99) a) $y = 4603.3x + 81{,}150$
b) The average salary of a mathematician is increasing by \$4603.30 per year.
c) \$122,580

101) a) $y = -15{,}000x + 500{,}000$
b) The budget is being cut by \$15,000 per year.
c) \$455,000
d) 2016

103) a) $y = 8x + 100$
b) A kitten gains about 8 g per day.
c) 140 g; 212 g
d) 23 days

105) a) $E = 1.6A + 28.4$
b) 40

Section 4.6

1) a) any set of ordered pairs
b) Answers may vary.
c) Answers may vary.

3) domain: $\{-8, -2, 1, 5\}$
range: $\{-3, 4, 6, 13\}$
function

5) domain: $\{1, 9, 25\}$
range: $\{-3, -1, 1, 5, 7\}$
not a function

7) domain: $\{-1, 2, 5, 8\}$
range: $\{-7, -3, 12, 19\}$
not a function

9) domain: $(-\infty, \infty)$
range: $(-\infty, \infty)$
function

11) domain: $(-\infty, 4]$
range: $(-\infty, \infty)$
not a function

13) domain: $(-\infty, \infty)$
range: $(-\infty, 6]$
function

15) yes 17) yes 19) no 21) no 23) $(-\infty, \infty)$; function

25) $(-\infty, \infty)$; function 27) $[0, \infty)$; not a function

29) $(-\infty, 0) \cup (0, \infty)$; function

31) $(-\infty, -4) \cup (-4, \infty)$; function

33) $(-\infty, 5) \cup (5, \infty)$; function

35) $\left(-\infty, \frac{3}{5}\right) \cup \left(\frac{3}{5}, \infty\right)$; function

37) $\left(-\infty, -\frac{4}{3}\right) \cup \left(-\frac{4}{3}, \infty\right)$; function

39) $(-\infty, 3) \cup (3, \infty)$; function

41) $(-\infty, \infty)$; function

43) y is a function, and y is a function of x.

45) a) $y = 7$ b) $f(3) = 7$ 47) -13 49) 7 51) 50

53) -10 55) $-\frac{25}{4}$ 57) -105 59) $f(-1) = 10, f(4) = -5$

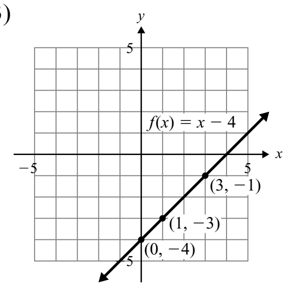
61) $f(-1) = 6, f(4) = 2$ 63) $f(-1) = 7, f(4) = 3$

65) -4 67) 6

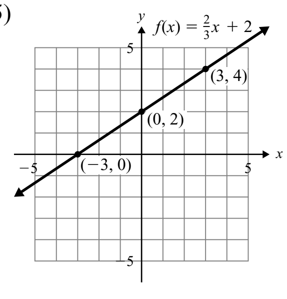
69) Substitute $k + 6$ for x; $= 4k + 24 - 5$; $= 4k + 19$

71) a) $f(c) = -7c + 2$ b) $f(t) = -7t + 2$
c) $f(a + 4) = -7a - 26$ d) $f(z - 9) = -7z + 65$
e) $g(k) = k^2 - 5k + 12$ f) $g(m) = m^2 - 5m + 12$
g) $-7x - 7h + 2$ h) $-7h$

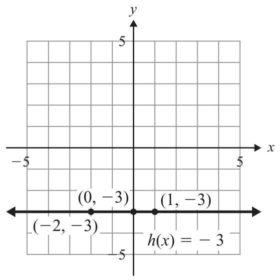
73)



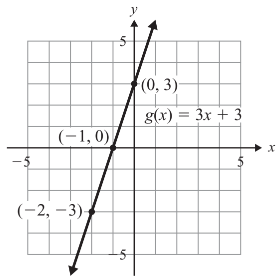
75)



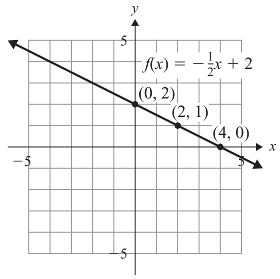
77)



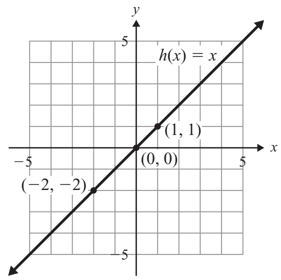
79) x-int: $(-1, 0)$; y-int: $(0, 3)$



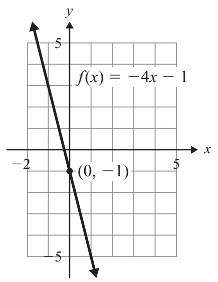
81) x-int: $(4, 0)$; y-int: $(0, 2)$



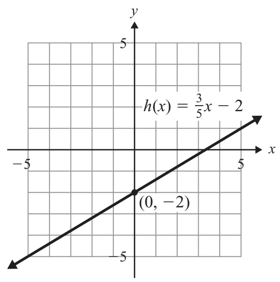
83) intercept: $(0, 0)$



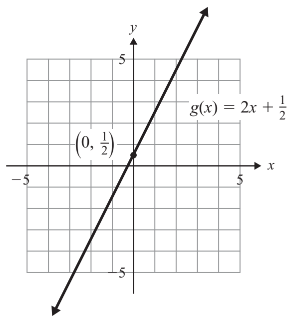
85) $m = -4$; y-int: $(0, -1)$



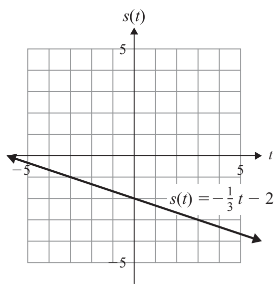
87) $m = \dfrac{3}{5}$; y-int: $(0, -2)$



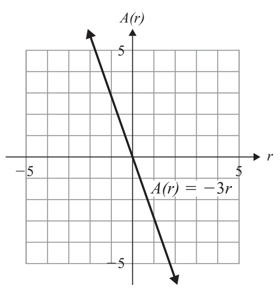
89) $m = 2$; y-int: $\left(0, \dfrac{1}{2}\right)$



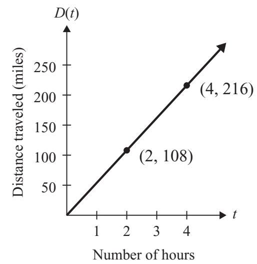
91)



93)

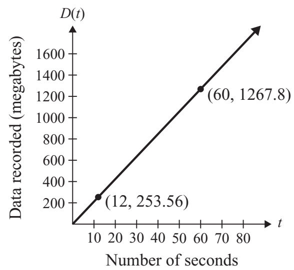


95) a) 108 mi b) 2.5 hr
c)



97) a) $E(10) = 75$; when Jenelle works for 10 hr, she earns \$75.00.
b) $t = 28$; for Jenelle to earn \$210.00, she must work 28 hr.

99) a) 253.56 MB b) 1267.8 MB c) 20 sec
d)



101) a) 60,000 b) 3.5 sec

103) a) $S(50) = 2{,}205{,}000$; in 50 sec, the CD player reads 2,205,000 samples of sound.
b) $t = 60$; the CD player reads 2,646,000 samples of sound in 60 seconds (or 1 min).

105) a) 2 hours; 400 mg
b) after about 30 min and after 6 hours c) 200 mg
d) $A(8) = 50$. After 8 hours, there are 50 mg of ibuprofen in Sasha's bloodstream.

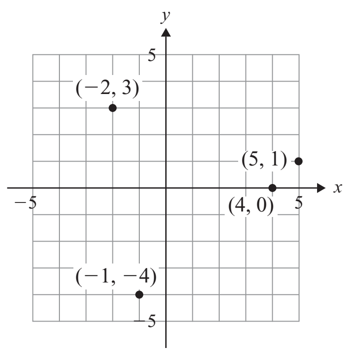
Chapter 4 Review Exercises

1) yes 3) yes 5) 14 7) −9

9)

x	y
0	−14
6	−8
−3	−17
−8	−22

11)

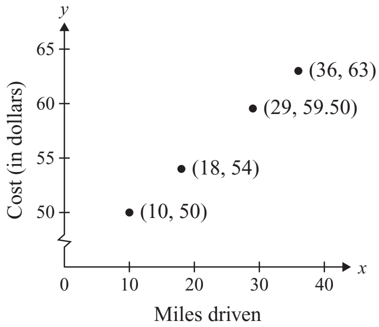


13) a)

x	y
10	50
18	54
29	59.50
36	63

(10, 50), (18, 54), (29, 59.50), (36, 63)

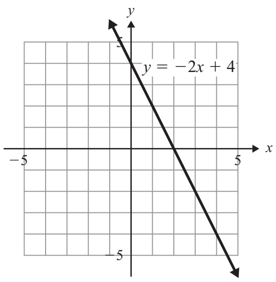
b) Cost of Renting a Pick-Up



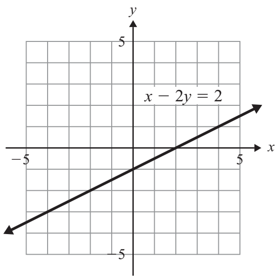
c) The cost of renting the pick-up is $74.00 if it is driven 58 miles.

15)

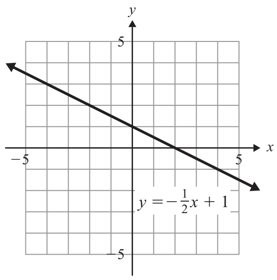
x	y
0	4
1	2
2	0
3	-2



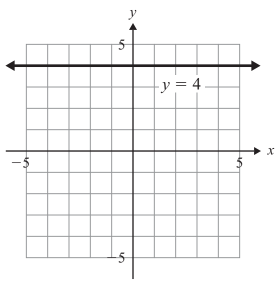
17) (2, 0), (0, -1); (4, 1) may vary.



19) (2, 0), (0, 1); (-2, 2) may vary.



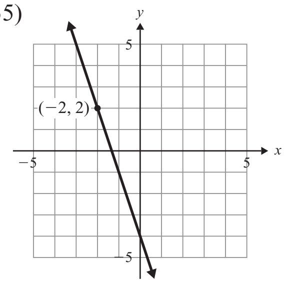
21) (0, 4); (2, 4), (-1, 4) may vary.



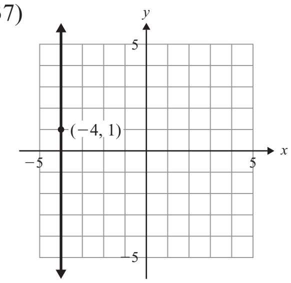
23) $\frac{3}{2}$ 25) 5 27) $\frac{2}{3}$ 29) -7 31) 0

33) a) $4.00
b) The slope is positive, so the value of the album is increasing over time.
c) $m = 1$; the value of the album is increasing by $1.00 per year.

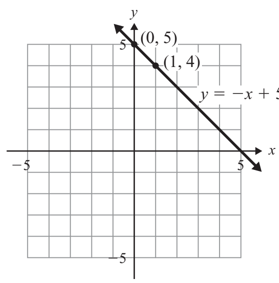
35)



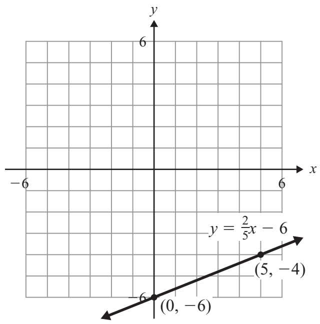
37)



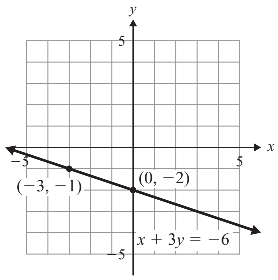
39) $m = -1$, y-int: (0, 5)



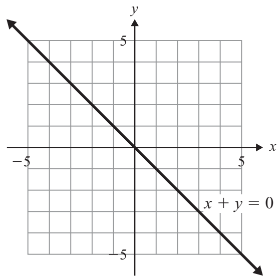
41) $m = \frac{2}{5}$, y-int: (0, -6)



43) $m = -\frac{1}{3}$, y-int: (0, -2)



45) $m = -1$, y-int: (0, 0)



47) a) (0, 197.6); in 2003, the value of the squash crop was about $197.6 million.
b) It has been increasing by $7.9 million per year.
c) $213 million; $213.4 million

49) parallel 51) perpendicular 53) $y = 6x + 10$

55) $y = -\frac{3}{4}x + 7$ 57) $y = -2x + 9$ 59) $y = 7$

61) $3x - y = 7$ 63) $5x - 2y = 8$ 65) $4x + y = 0$

67) $x + y = 7$

69) a) $y = 3500x + 62{,}000$
b) Mr. Romanski's salary is increasing by $3500 per year.
c) $72,500 d) 2014

71) $y = -8x + 6$ 73) $x - 2y = -18$ 75) $y = -\frac{1}{5}x + 10$

77) $y = x - 12$ 79) $y = \frac{3}{2}x + 2$ 81) $y = 8$

83) domain: $\{-3, 5, 12\}$
range: $\{-3, 1, 3, 4\}$
not a function

85) domain: {Beagle, Siamese, Parrot}
range: {Dog, Cat, Bird}
function

87) domain: $[0, 4]$
range: $[0, 2]$
not a function

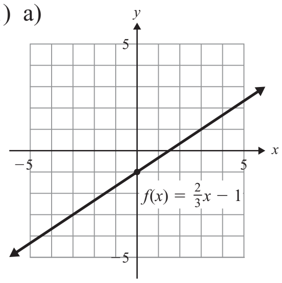
89) $(-\infty, 3) \cup (-3, \infty)$; function 91) $[0, \infty)$; not a function

93) $\left(-\infty, \frac{2}{7}\right) \cup \left(\frac{2}{7}, \infty\right)$; function 95) $f(3) = 27, f(-2) = -8$

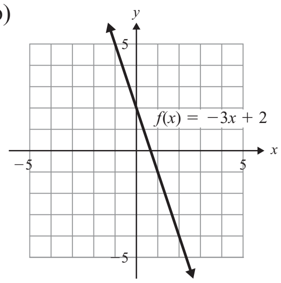
97) a) 8 b) -27 c) 32 d) 5 e) $5a - 12$ f) $t^2 + 6t + 5$
g) $5k + 28$ h) $5c - 22$ i) $5x + 5h - 12$ j) $5h$

99) $\frac{1}{3}$

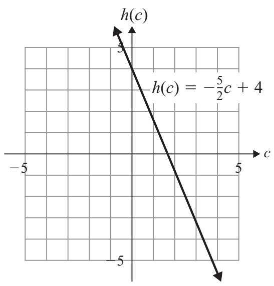
101) a)



b)



103)



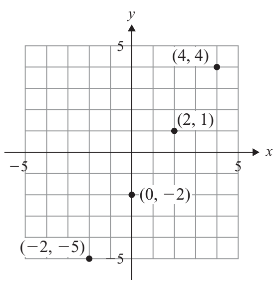
105) a) 960 MB; 2880 MB
b) 2.5 sec

Chapter 4 Test

1) Yes

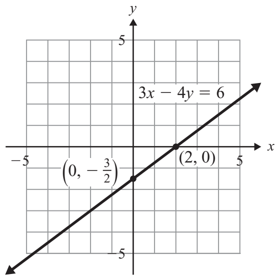
2)

x	y
0	-2
-2	-5
4	4
2	1

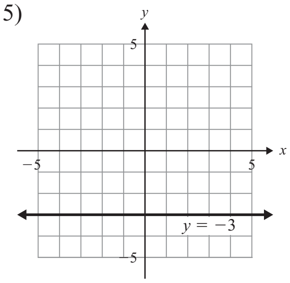


3) positive; negative

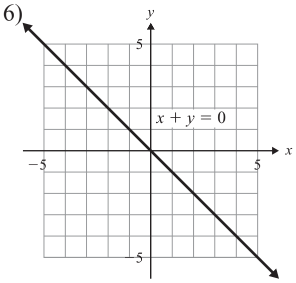
4) a) $(2, 0)$ b) $(0, -\frac{3}{2})$ c) Answers may vary.
d)



5)

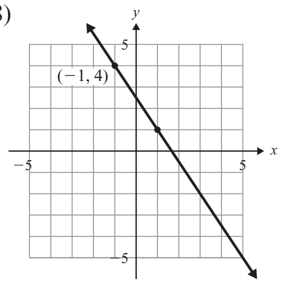


6)

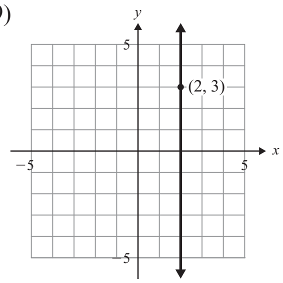


7) a) $-\frac{5}{4}$ b) 0

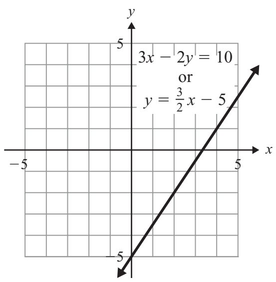
8)



9)



10) $y = \frac{3}{2}x - 5$



11) $y = 7x - 10$ 12) $x + 3y = 12$ 13) perpendicular

14) a) $y = -\frac{1}{2}x + 7$ b) $y = \frac{3}{4}x - \frac{1}{4}$

15) a) 399 b) $y = -9x + 419$
c) According to the equation, 401 students attended the school in 2007. The actual number was 399.
d) The school is losing 9 students per year.
e) $(0, 419)$; in 2005, 419 students attended this school.
f) 347

16) It is a relation in which each element of the domain corresponds to exactly one element of the range.

17) domain: $\{-2, 1, 3, 8\}$
range: $\{-5, -1, 1, 4\}$
function

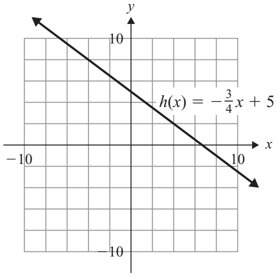
18) domain: $[-3, \infty)$; range: $(-\infty, \infty)$
not a function

19) a) $(-\infty, \infty)$ b) yes 20) a) $\left(-\infty, \frac{5}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$ b) yes

21) -3 22) 5 23) -22 24) 5 25) $t^2 - 3t + 7$

26) $-4h + 30$

27)



28) a) 36 MB b) 11 sec

Cumulative Review for Chapters 1–4

1) $\frac{14}{33}$ 2) 39 in. 3) -81 4) $\frac{14}{25}$ 5) $-\frac{13}{8}$ 6) 12

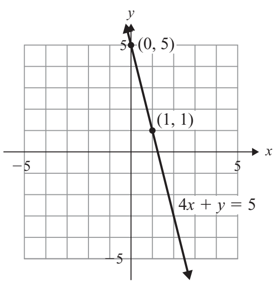
7) $2(17) - 9$; 25 8) $-20k^{15}$ 9) $\frac{1}{16w^{32}}$ 10) $\{-15\}$

11 $\{1\}$ 12) $\varnothing$ 13) $\left[\frac{5}{2}, \infty\right)$ 14) \$340,000

15) $m\angle A = 29°$, $m\angle B = 131°$

16) Lynette's age = 41; daughter's age = 16 17) $m = 1$

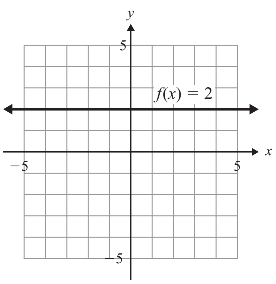
18)



19) $5x + 4y = -36$ 20) $y = -3x$

21) $(-\infty, -7) \cup (-7, \infty)$ 22) -37 23) $8a + 3$ 24) $8t + 19$

25)



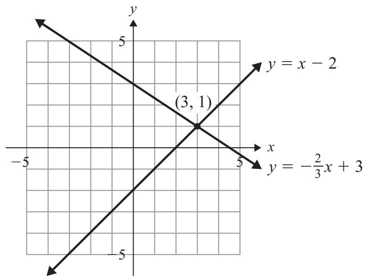
Chapter 5

Section 5.1

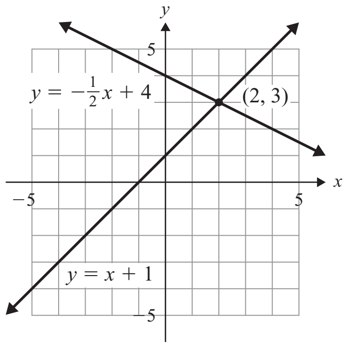
1) yes 3) no 5) yes 7) no

9) The lines are parallel.

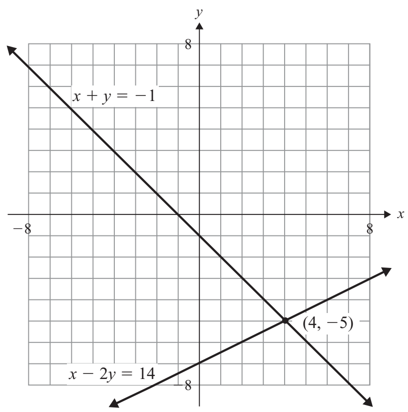
11) (3, 1)



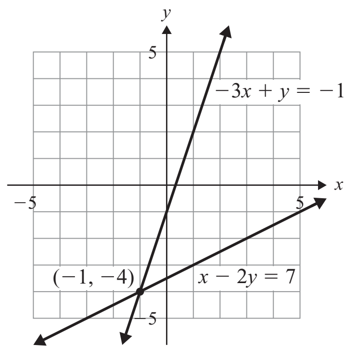
13) (2, 3)



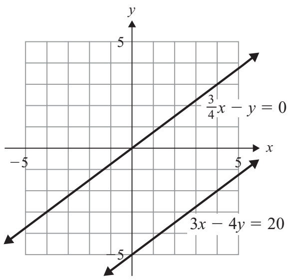
15) (4, -5)



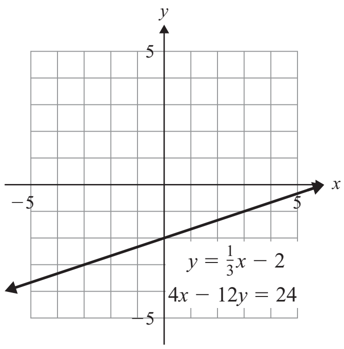
17) (-1, -4)



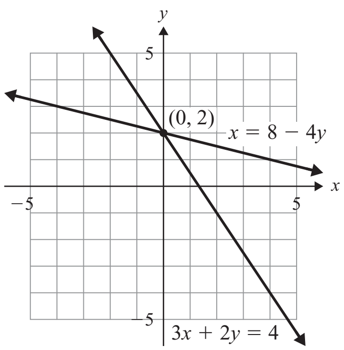
19) $\varnothing$; inconsistent system



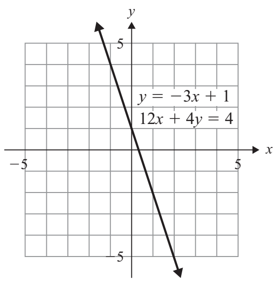
21) infinite number of solutions of the form $\left\{(x, y) \middle| y = \frac{1}{3}x - 2\right\}$; dependent equations



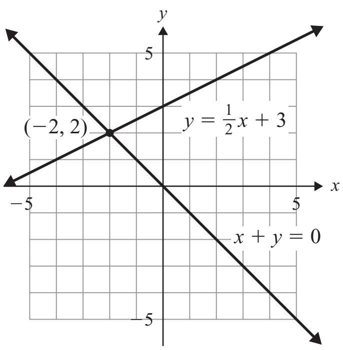
23) (0, 2)



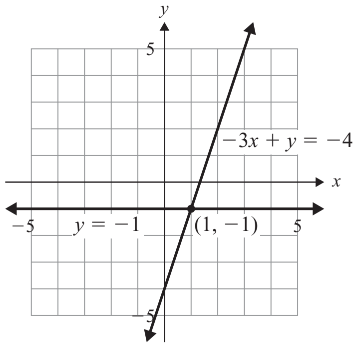
25) infinite number of solutions of the form $\{(x, y)|y = -3x + 1\}$; dependent equations



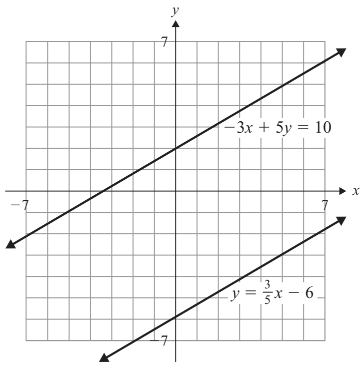
27) (−2, 2)



29) (1, −1)



31) $\varnothing$; inconsistent system



33–38) Answers may vary. 39) C; (−3, 4) is in quadrant II.

41) B; (4.1, 0) is the only point on the positive x-axis.

43) The slopes are different.

45) one solution 47) no solution

49) infinite number of solutions 51) one solution

53) no solution

55) a) 1985–2000
b) (2000, 1.4); in the year 2000, 1.4% of foreign students were from Hong Kong and 1.4% were from Malaysia.
c) 1985–1990 and 2000–2005
d) 1985–1990; this line segment has the most negative slope.

57) (3, −4) 59) (4, 1) 61) (−2.25, −1.6)

Section 5.2

1) It is the only variable with a coefficient of 1.

3) The variables are eliminated, and you get a false statement.

5) (2, 5) 7) (−3, −2) 9) (1, −2)

11) (0, −7) 13) $\varnothing$

15) infinite number of solutions of the form $\{(x, y)|x - 2y = 10\}$

17) $\left(-\frac{4}{5}, 3\right)$ 19) (4, 5)

21) infinite number of solutions of the form $\{(x, y)|-x + 2y = 2\}$

23) (−3, 4) 25) $\left(\frac{5}{3}, 2\right)$ 27) $\varnothing$

29) Multiply the equation by the LCD of the fractions to eliminate the fractions.

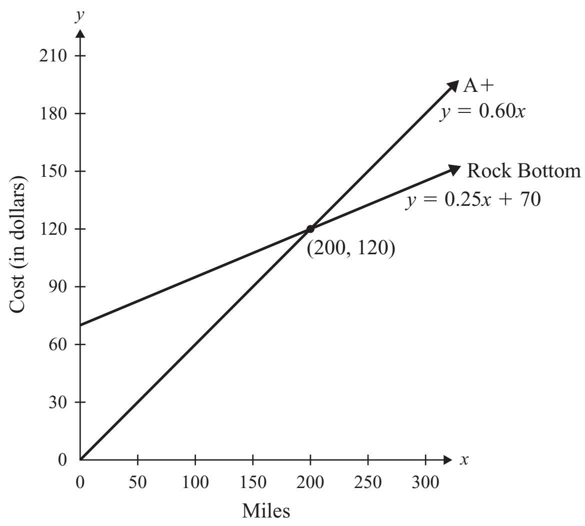
31) (6, 1) 33) (−6, 4) 35) (3, −2)

37) infinite number of solutions of the form $\{(x, y)|y - \frac{5}{2}x = -2\}$

39) (8, 0) 41) (3, 5) 43) (1.5, −1) 45) (−16, −12)

47) ∅ 49) (6, 4) 51) (−4, −9) 53) (0, −8)

55) a) A+: \$96.00; Rock Bottom: \$110.00
b) A+: \$180.00; Rock Bottom: \$145.00
c) (200, 120); if the cargo trailer is driven 200 miles, the cost would be the same from each company: \$120.00.
d) If it is driven less than 200 miles, it is cheaper to rent from A+. If it is driven more than 200 miles, it is cheaper to rent from Rock Bottom Rental. If the trailer is driven exactly 200 miles, the cost is the same from each company.



Section 5.3

1) Add the equations.

3) (5, 8) 5) (−6, −2) 7) (−7, 2) 9) (2, 0) 11) (7, −1)

13) (0, 2) 15) $\left(-\frac{2}{3}, -5\right)$

17) infinite number of solutions of the form $\{(x, y)|9x - y = 2\}$

19) (8, 1) 21) $\left(5, -\frac{3}{2}\right)$ 23) ∅ 25) (9, 5) 27) ∅

29) Eliminate the fractions. Multiply the first equation by 4, and multiply the second equation by 24.

31) $\left(\frac{5}{8}, 4\right)$ 33) $\left(-\frac{9}{2}, -13\right)$ 35) (−6, 1)

37) infinite number of solutions of the form $\left\{(x, y) \,\middle|\, y = \frac{2}{3}x - 7\right\}$

39) (1, 1) 41) (−7, −4) 43) (12, −1) 45) ∅

47) (0.25, 5) 49) (4, 3) 51) $\left(-\frac{3}{2}, 4\right)$ 53) (1, 1)

55) $\left(-\frac{123}{17}, \frac{78}{17}\right)$ 57) $\left(-\frac{203}{10}, \frac{49}{5}\right)$ 59) 3 61) −8

63) (a) 5 b) c can be any real number except 5.

65) a) 3 b) a can be any real number except 3.

67) $\left(\frac{2}{5}, \frac{2}{b}\right)$ 69) $\left(-\frac{1}{4a}, \frac{19}{4b}\right)$

Chapter 5 Putting It All Together

1) Elimination method; none of the coefficients is 1 or −1; (5, 6).

2) Substitution; the first equation is solved for x and does not contain any fractions; (3, 5).

3) Since the coefficient of y in the second equation is 1, you can solve for y and use substitution. Or, multiply the second equation by 5 and use the elimination method. Either method will work well; $(\frac{1}{4}, -3)$.

4) Elimination method; none of the coefficients is 1 or −1; $(0, -\frac{2}{5})$.

5) Substitution; the second equation is solved for x and does not contain any fractions; (1, −7).

6) The second equation is solved for y, but it contains two fractions. Multiply this equation by 4 to eliminate the fractions, then write it in the form $Ax + By = C$. Use the elimination method to solve the system; (6, 4).

7) (6, 0) 8) (−2, 2) 9) $\left(-\frac{2}{3}, \frac{1}{5}\right)$ 10) (1, 4) 11) ∅

12) infinite number of solutions of the form $\{(x, y)|y = -6x + 5\}$

13) $\left(-\frac{1}{2}, 1\right)$ 14) $\left(\frac{5}{6}, -\frac{3}{2}\right)$ 15) (4, 3) 16) $\left(\frac{1}{6}, 6\right)$

17) (9, −7) 18) (10, −9) 19) (0, 4) 20) (−3, −1)

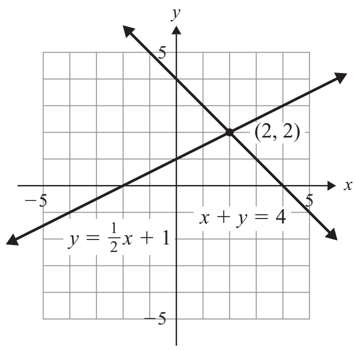
21) infinite number of solutions of the form $\{(x, y)|3x - y = 5\}$

22) ∅ 23) (−9, −14) 24) $\left(2, \frac{4}{3}\right)$ 25) $\left(-\frac{68}{41}, \frac{64}{41}\right)$

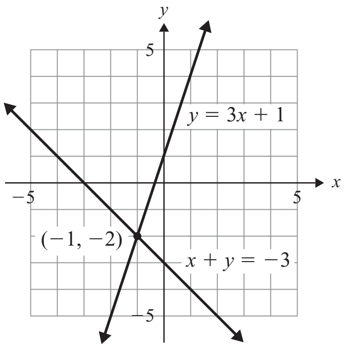
26) $\left(-\frac{53}{34}, \frac{24}{17}\right)$ 27) $\left(\frac{3}{4}, 0\right)$ 28) $\left(5, \frac{3}{4}\right)$ 29) (1, 1)

30) (−2, 4) 31) $\left(-\frac{5}{2}, 10\right)$ 32) (−4, −6)

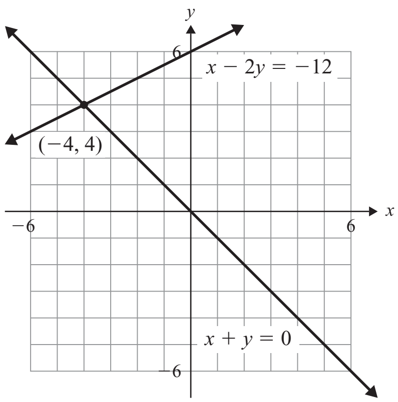
33) (2, 2)



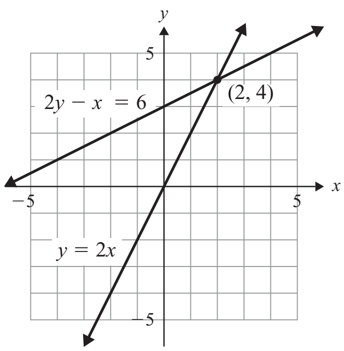
34) $(-1, -2)$



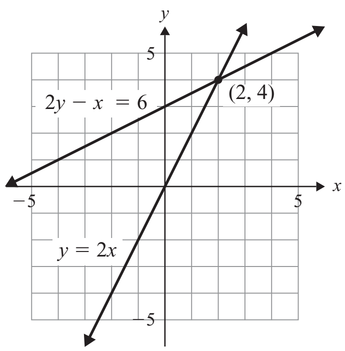
35) $(-4, 4)$



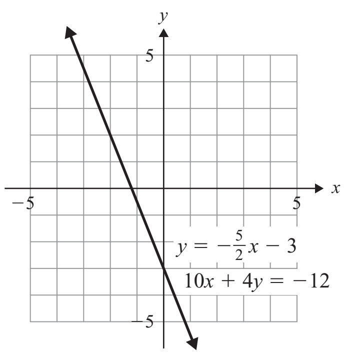
36) $(2, 4)$



37) $\varnothing$



38) infinite number of solutions of the form $\{(x, y) | y = -\frac{5}{2}x - 3\}$



39) $(-1.25, -0.5)$ 40) $(7.5, -13)$

Section 5.4

1) 38 and 49

3) *The Dark Knight:* \$67.2 million; *Transformers:* \$60.6 million

5) Beyonce: 16; T.I.: 11 7) Urdu: 325,000; Polish: 650,000

9) Gagarin: 108 min; Shepard: 15 min

11) width: 30 in.; height: 80 in.

13) length: 110 mm; width: 61.8 mm

15) width: 34 ft; length: 51 ft

17) $m\angle x = 67.5°$; $m\angle y = 112.5°$

19) T-shirt: \$20.00; hockey puck: \$8.00

21) bobblehead: \$19.00; mug: \$12.00

23) hamburger: \$0.61; small fries: \$1.39

25) wrapping paper: \$7.00; gift bags: \$8.00

27) 9%: 3 oz; 17%: 9 oz 29) pure acid: 2 L; 25%: 8 L

31) Asian Treasure: 24 oz; Pearadise: 36 oz

33) taco: 330 mg; chalupa: 650 mg

35) 2%: \$2500; 4%: \$3500 37) \$0.44: 12; \$0.28: 4

39) Michael: 9 mph; Jan: 8 mph

41) small plane: 240 mph; jet: 400 mph

43) Pam: 8 mph; Jim: 10 mph

45) speed of boat in still water: 6 mph; speed of the current: 1 mph

47) speed of boat in still water: 13 mph; speed of the current: 3 mph

49) speed of jet in still air: 450 mph; speed of the wind: 50 mph

Section 5.5

1) yes 3) no 5) Answers may vary. 7) $(-2, 0, 5)$

9) $(1, -1, 4)$ 11) $\left(2, -\frac{1}{2}, \frac{5}{2}\right)$ 13) $\varnothing$; inconsistent

15) $\{(x, y, z) | 5x + y - 3z = -1\}$; dependent

17) $\{(a, b, c) | -a + 4b - 3c = -1\}$; dependent

19) $(2, 5, -5)$ 21) $\left(-4, \frac{3}{5}, 4\right)$ 23) $(0, -7, 6)$

25) $(1, 5, 2)$ 27) $\left(-\frac{1}{4}, -5, 3\right)$ 29) $\varnothing$; inconsistent

31) $\left(4, -\frac{3}{2}, 0\right)$ 33) $(4, 4, 4)$

35) $\{(x, y, z) | -4x + 6y + 3z = 3\}$; dependent 37) $\left(1, -7, \frac{1}{3}\right)$

39) $(0, -1, 2)$ 41) $(-3, -1, 1)$

43) Answers may vary.

45) hot dog: \$2.00, fries: \$1.50, soda: \$2.00

47) Clif Bar: 11g, Balance Bar: 15 g, PowerBar: 24 g

49) Knicks: \$160 million, Lakers: \$149 million, Bulls: \$119 million

51) value: \$22; regular: \$36; prime: \$45 53) 104°, 52°, 24°

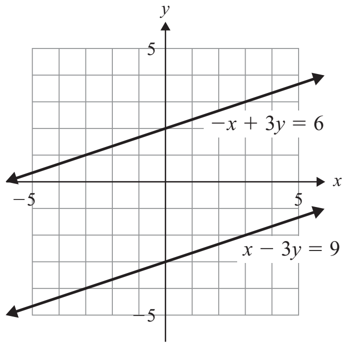
55) 80°, 64°, 36° 57) 12 cm, 10 cm, 7 cm

59) (3, 1, 2, 1) 61) (0, −3, 1, −4)

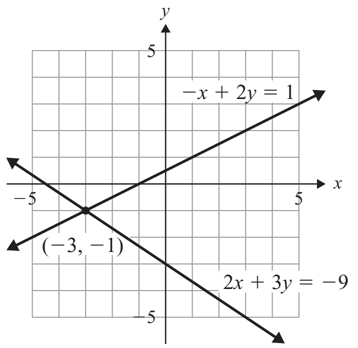
Chapter 5 Review Exercises

1) no 3) The lines are parallel.

5) ∅



7) (−3, −1)



9) infinite number of solutions 11) (2, 5)

13) (5, 3) 15) (−4, −1)

17) infinite number of solutions of the form $\{(x, y)|5x - 2y = 4\}$

19) $\left(\frac{33}{43}, -\frac{36}{43}\right)$

21) when one of the variables has a coefficient of 1 or −1

23) $\left(-\frac{5}{3}, 2\right)$ 25) (0, 3) 27) $\left(\frac{3}{4}, 0\right)$ 29) ∅

31) white: 94; chocolate: 47 33) Edwin: 8 mph; Camille: 6 mph

35) length: 12 cm; width: 7 cm 37) quarters: 35; dimes: 28

39) hand warmers: \$4.50; socks: \$18.50

41) no 43) (3, −1, 4) 45) $\left(-1, 2, \frac{1}{2}\right)$

47) $\left(3, \frac{2}{3}, -\frac{1}{2}\right)$ 49) ∅; inconsistent

51) $\{(a, b, c)|3a - 2b + c = 2\}$; dependent

53) (1, 0, 3) 55) $\left(\frac{3}{4}, -2, 1\right)$

57) Propel: 35 mg; Powerade: 52 mg; Gatorade: 110 mg

59) Blair: 65; Serena: 50; Chuck: 25

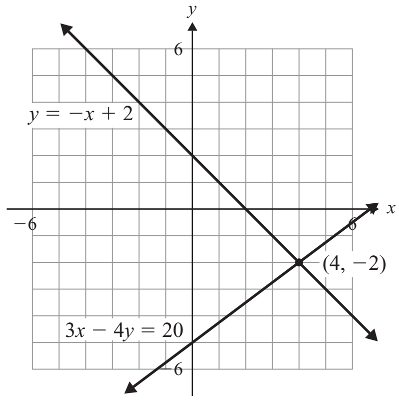
61) ice cream cone: \$1.50; shake: \$2.50; sundae: \$3.00

63) 92°, 66°, 22°

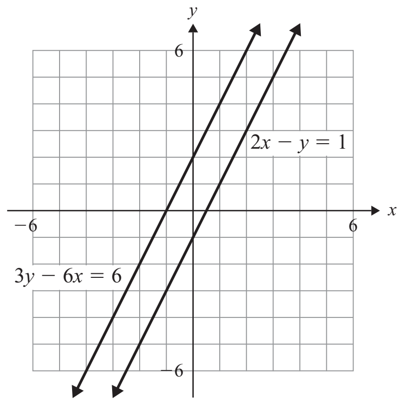
Chapter 5 Test

1) yes

2) (4, −2)



3) ∅



4) a) 2001; approximately 4.6% of the population was unemployed.
b) (2003, 4.3): in 2003, 4.3% of the population of Hawaii and New Hampshire was unemployed.
c) Hawaii from 2003 to 2005; this means that during this time, Hawaii experienced the largest decrease in the unemployment rate during all years represented on the graph whether for Hawaii or for New Hampshire.

5) $\left(-5, -\frac{1}{2}\right)$ 6) infinite number of solutions of the form $\left\{(x, y)\,\middle|\, y = \frac{1}{2}x - 3\right\}$ 7) (3, 1) 8) (0, 6) 9) ∅

10) (−2, −8) 11) (2, −4) 12) $\left(-\frac{4}{3}, 0\right)$ 13) (0, 3, −2)

14) Answers may vary. 15) Yellowstone: 2.2 mil acres; Death Valley: 3.3 mil acres 16) adult: \$45.00; child: \$20.00

17) length: 38 cm; width: 19 cm 18) 12%: 40 mL; 30%: 32 mL

19) Rory: 42 mph; Lorelei: 38 mph 20) 105°, 42°, 33°

Cumulative Review for Chapters 1–5

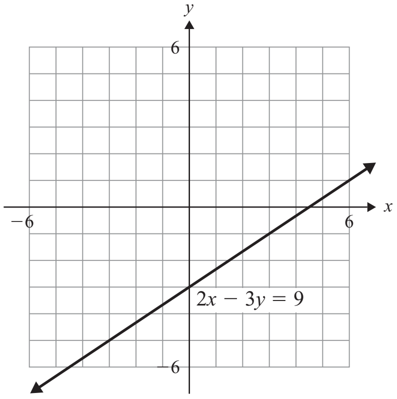
1) $\frac{41}{30}$ 2) $10\frac{2}{3}$ 3) -29 4) 30 in^2 5) $-12x^2 - 15x + 3$

6) $32p^{20}$ 7) $\frac{63}{x^4}$ 8) $\frac{3n^4}{2m^{10}}$ 9) 7.319×10^{-4} 10) $\varnothing$

11) $\left\{\frac{1}{2}\right\}$ 12) (1, 6) 13) 15.8 mpg

14) a) $h = \frac{2A}{b_1 + b_2}$ b) 6 cm

15)



16) x-int: (16, 0); y-int: (0, −2) 17) $y = \frac{1}{4}x + \frac{5}{4}$

18) perpendicular 19) (4, 10) 20) (3, 1) 21) $\varnothing$

22) $\{(x, y) | 3x + 9y = -2\}$ 23) $(-2, -1, 0)$

24) 4-ft boards: 16; 6-ft boards: 32

25) Juanes: 17; Alejandro Sanz: 14; Shakira: 7

Chapter 6

Section 6.1

1) quotient rule; k^6 3) power rule for a product; $16h^4$

5) 64 7) -64 9) $\frac{1}{6}$ 11) 81 13) $\frac{16}{81}$ 15) -31 17) $\frac{1}{64}$

19) t^{13} 21) $-16c^9$ 23) z^{24} 25) $125p^{30}$ 27) $-\frac{8}{27}a^{21}b^3$

29) f^4 31) $7v$ 33) $\frac{d^4}{6}$ 35) $\frac{1}{x^6}$ 37) $\frac{1}{m}$ 39) $\frac{3}{2k^4}$

41) $20m^8n^{14}$ 43) $24y^8$ 45) $\frac{1}{49a^8b^2}$ 47) $\frac{b^5}{a^3}$ 49) $\frac{x^3}{y^{20}}$

51) $\frac{4a^2b^{15}}{3c^{17}}$ 53) $\frac{y^{15}}{x^5}$ 55) $\frac{64c^6}{a^6b^3}$ 57) $\frac{9}{h^8k^8}$

59) $\frac{c^6}{27d^{18}}$ 61) $\frac{u^{13}v^4}{32}$

63) $A = 10x^2$ sq units; $P = 14x$ units

65) $A = \frac{3}{16}p^2$ sq units; $P = 2p$ units

67) k^{6a} 69) g^{8x} 71) x^{3b} 73) $\frac{1}{8r^{18m}}$

Section 6.2

1) Yes; the coefficients are real numbers and the exponents are whole numbers.

3) No; one of the exponents is a negative number.

5) No; two of the exponents are fractions.

7) binomial 9) trinomial 11) monomial

13) It is the same as the degree of the term in the polynomial with the highest degree.

15) Add the exponents on the variables.

17)

Term	Coeff.	Degree
$3y^4$	3	4
$7y^3$	7	3
$-2y$	-2	1
8	8	0

Degree of polynomial is 4.

19)

Term	Coeff.	Degree
$-4x^2y^3$	-4	5
$-x^2y^2$	-1	4
$\frac{2}{3}xy$	$\frac{2}{3}$	2
$5y$	5	1

Degree of polynomial is 5.

21) a) 1 b) 13 23) 37 25) 146 27) -23

29) a) $y = 680$; if he rents the equipment for 5 hours, the cost of building the road will be \$680.00.
b) \$920.00 c) 8 hours

31) $13z$ 33) $-11c^2 + 7c$ 35) $-4.3t^6 + 4.2t^2$

37) $6a^2b^2 - 7ab^2$ 39) $9n - 5$ 41) $-5a^3 + 7a$

43) $12r^2 + 6r + 11$ 45) $4b^2 - 3$

47) $\frac{7}{18}w^4 - \frac{1}{2}w^2 - \frac{3}{8}w - \frac{3}{2}$ 49) $2m^2 + 3m + 19$

51) $10c^4 - \frac{1}{5}c^3 - \frac{1}{4}c + \frac{25}{9}$ 53) $1.2d^3 + 7.7d^2 - 11.3d + 0.6$

55) $12w - 4$ 57) $-y + 2$ 59) $-2b^2 - 10b + 19$

61) $4f^4 - 14f^3 + 6f^2 - 8f + 9$ 63) $5.8r^2 + 6.5r + 11.8$

65) $7j^2 + 9j - 5$ 67) $8s^5 - 4s^4 - 4s^2 + 1$

69) $\frac{1}{16}r^2 + \frac{7}{9}r - \frac{5}{6}$ 71) Answers may vary.

73) No. If the coefficients of the like terms are opposite in sign, their sum will be zero. Example: $(3x^2 + 4x + 5) + (2x^2 - 4x + 1) = 5x^2 + 6$

75) $-7a^4 - 12a^2 + 14$ 77) $7n + 8$ 79) $4w^3 + 7w^2 - w - 2$

81) $\frac{4}{3}y^3 - \frac{7}{4}y^2 + 3y - \frac{1}{14}$ 83) $m^3 - 7m^2 - 4m - 7$

85) $9p^2 + 2p - 8$ 87) $5z^6 + 9z^2 - 4$ 89) $-7p^2 + 9p$

91) $4w + 14z$ 93) $-5ac + 12a + 5c$ 95) $4u^2v^2 + 31uv - 4$

97) $17x^3y^2 - 11x^2y^2 - 20$ 99) $6x + 6$ units

101) $10p^2 - 2p - 6$ units

103) a) 16 b) 4 105) a) 32 b) 8 107) $-\frac{2}{3}$ 109) 30

Section 6.3

1) Answers may vary. 3) $24m^8$ 5) $-32c^6$

7) $10a^2 - 35a$ 9) $-42c^2 - 12c$ 11) $6v^5 - 24v^4 - 12v^3$

13) $-36b^5 + 18b^4 + 54b^3 + 81b^2$

15) $3a^3b^3 + 18a^3b^2 - 39a^2b^2 + 21a^2b$

17) $-9k^6 - 12k^5 + \frac{9}{5}k^4$ 19) $6c^3 + 11c^2 - 45c + 28$

21) $3f^3 - 13f^2 - 14f + 20$

23) $8x^4 - 22x^3 + 17x^2 - 26x - 10$

25) $4y^4 + \frac{7}{3}y^3 + 45y^2 + 28y - 36$

27) $s^4 + 3s^3 - 5s^2 + 11s - 6$

29) $-24h^5 + 26h^4 - 53h^3 + 19h^2 - 18h$

31) $15y^3 - 22y^2 + 17y - 6$ 33) First, Outer, Inner, Last

35) $w^2 + 12w + 35$ 37) $r^2 + 6r - 27$ 39) $y^2 - 8y + 7$

41) $3p^2 + p - 14$ 43) $21n^2 + 19n + 4$

45) $4w^2 - 17w + 15$ 47) $12a^2 + ab - 20b^2$

49) $48x^2 + 74xy + 21y^2$ 51) $v^2 + \frac{13}{12}v + \frac{1}{4}$

53) $\frac{1}{3}a^2 + \frac{17}{6}ab - 5b^2$

55) a) $4y + 4$ units
b) $y^2 + 2y - 15$ sq units

57) a) $2m^2 + 2m + 14$ units
b) $3m^3 - 6m^2 + 21m$ sq units

59) $3n^2 - \frac{5}{2}n$ sq units 61) Both are correct.

63) $8n^2 + 14n - 30$ 65) $-5z^4 + 50z^3 - 80z^2$

67) $c^3 + 6c^2 + 5c - 12$ 69) $3x^3 - 25x^2 + 44x - 12$

71) $2p^5 + 34p^3 + 120p$ 73) $y^2 - 25$ 75) $a^2 - 49$

77) $9 - p^2$ 79) $u^2 - \frac{1}{25}$ 81) $\frac{4}{9} - k^2$ 83) $4r^2 - 49$

85) $k^2 - 64j^2$ 87) $d^2 + 8d + 16$ 89) $n^2 - 26n + 169$

91) $h^2 - 1.2h + 0.36$ 93) $9u^2 + 6u + 1$

95) $4d^2 - 20d + 25$ 97) $9c^2 + 12cd + 4d^2$

99) $\frac{9}{4}k^2 + 24km + 64m^2$

101) $4a^2 + 4ab + b^2 + 12a + 6b + 9$

103) $f^2 - 6fg + 9g^2 - 16$

105) No. The order of operations tells us to perform exponents, $(r + 2)^2$, before multiplying by 3.

107) $7y^2 + 28y + 28$ 109) $4c^3 + 24c^2 + 36c$

111) $r^3 + 15r^2 + 75r + 125$ 113) $g^3 - 12g^2 + 48g - 64$

115) $8a^3 - 12a^2 + 6a - 1$

117) $h^4 + 12h^3 + 54h^2 + 108h + 81$

119) $625t^4 - 1000t^3 + 600t^2 - 160t + 16$

121) No; $(x + 2)^2 = x^2 + 4x + 4$

123) $c^2 - 5c - 84$ 125) $-40a^2 + 68a - 24$

127) $10k^3 - 37k^2 - 38k + 9$ 129) $\frac{1}{36} - h^2$

131) $27c^3 + 27c^2 + 9c + 1$ 133) $\frac{9}{32}p^{11}$

135) $a^4 + 14a^2b^2 + 49b^4$ 137) $-5z^3 + 30z^2 - 45z$

139) $x^2 - 8xy + 16y^2 - 25$

141) $a^3 + 12a^2 + 48a + 64$ cubic units

143) $\pi k^2 + 10\pi k + 25\pi$ sq units 145) $9c^2 - 15c + 4$ sq units

Section 6.4

1) dividend: $6c^3 + 15c^2 - 9c$; divisor: $3c$; quotient: $2c^2 + 5c - 3$ 3) Answers may vary.

5) $7p^4 + 3p^3 + 4p^2$ 7) $3w^2 - 10w - 9$

9) $11z^5 + 7z^4 - 19z^2 + 1$ 11) $h^6 + 6h^4 - 12h$

13) $3r^6 - r^3 + \frac{1}{2r}$ 15) $4d^2 - 6d + 9$

17) $7k^5 + 2k^3 - 11k^2 - 9$ 19) $-5d^3 + 1$

21) $\frac{5}{3}w^3 + 2w - 1 + \frac{1}{3w}$ 23) $6k^3 - 2k - \frac{15}{2} - \frac{3}{2k} + \frac{1}{2k^5}$

25) $8p^3q^2 + 10p^2q - 9p + 3$ 27) $2s^4t^5 - 4s^3t^3 - \frac{1}{7}st^2 + \frac{3}{s}$

29) The answer is incorrect. When you divide $5p$ by $5p$, you get 1. The quotient should be $8p^2 - 2p + 1$.

31) dividend: $12w^3 - 2w^2 - 23w - 7$; divisor: $3w + 1$; quotient: $4w^2 - 2w - 7$

33) 2 35) $158\frac{1}{6}$ 37) $437\frac{4}{9}$ 39) $g + 4$ 41) $a + 6$

43) $k - 6$ 45) $2h^2 + 5h + 1$ 47) $3p^2 + 5p - 1$

49) $7m + 12 + \frac{7}{m - 4}$ 51) $4a^2 - 7a + 2 - \frac{8}{5a - 2}$

53) $n^2 - 3n + 9$ 55) $2r^2 + 4r + 5$

57) $6x^2 - 9x + 5 - \dfrac{11}{2x + 3}$ 59) $k^2 + k + 5$

61) $3t^2 - 8t - 6 + \dfrac{2t - 4}{5t^2 - 1}$

63) No. For example, $\dfrac{12x + 8}{3x} = 4 + \dfrac{8}{3x}$. The quotient is not a polynomial because one term has a variable in the denominator.

65) Use synthetic division when the divisor is in the form $x - c$.

67) $t + 9$ 69) $5n + 6 + \dfrac{2}{n + 3}$ 71) $2y^2 - 3y + 5 - \dfrac{4}{y + 5}$

73) $2c^2 + 2c - 5$ 75) $w^3 - 2w^2 + 3w + 7 + \dfrac{6}{w - 2}$

77) $m^3 + 3m^2 + 9m + 27$ 79) $2x^2 + 8x - 12$

81) $5a^2b^2 + 3a^2b - \dfrac{1}{10} + \dfrac{1}{5ab}$

83) $-3f^3 + 6f^2 - 2f + 9 + \dfrac{23}{5f - 2}$ 85) $8t + 5 + \dfrac{11}{t - 3}$

87) $16p^2 + 12p + 9$ 89) $6x^2 + x - 7$

91) $5h^2 - 3h - 2 + \dfrac{1}{2h^2 - 9}$ 93) $j^2 + 1$

95) $14q^2 - \dfrac{8}{3}q + 3 + \dfrac{2}{q} - \dfrac{3}{q^2}$

97) $3p^2 - 5p + 1 + \dfrac{6p + 20}{7p^2 + 2p - 4}$ 99) $3x + 7$

101) $30n^2 - 36n + 12$ 103) $3x^2 - 7x + 2$ mph

Chapter 6 Review Exercises

1) 32 3) $\dfrac{125}{8}$ 5) p^{28} 7) $5t^6$ 9) $-42c^9$ 11) $\dfrac{1}{k^5}$

13) $-\dfrac{48s^4}{r^3}$ 15) $\dfrac{9y^{28}}{4x^6}$ 17) $\dfrac{1}{m^2n^6}$

19) $A = 12f^2$ sq units; $P = 14f$ units 21) y^{7a} 23) r^{9x}

25)

Term	Coeff.	Degree
$7s^3$	7	3
$-9s^2$	-9	2
s	1	1
6	6	0

Degree of polynomial is 3.

27) 31 29) a) 20 b) -6 31) $-2c^2 + c + 5$

33) $9.8j^3 + 4.3j^2 + 3.4j - 0.5$ 35) $\dfrac{1}{2}k^2 - k + 6$

37) $-3x^2y^2 + 9x^2y - 5xy + 6$ 39) $3m + 4n - 3$

41) $6x^2 - 11x - 28$ 43) $4d^2 + 6d + 6$ units 45) $24r^2 - 39r$

47) $-32w^4 - 24w^3 - 8w^2 - 2w + 3$ 49) $y^2 - 12y + 27$

51) $6n^2 - 29n + 28$ 53) $-a^2 + 3a + 130$

55) $42p^4q^4 + 66p^3q^4 - 6p^2q^3 + 24pq^2$ 57) $4x^2 - 16xy - 9y^2$

59) $10x^6 + 50x^5 - 123x^4 - 15x^3 + 42x^2 + 30x - 72$

61) $8f^4 - 76f^3 + 168f^2$ 63) $z^3 + 8z^2 + 19z + 12$

65) $\dfrac{1}{7}d^2 - \dfrac{11}{14}d - 24$ 67) $c^2 + 8c + 16$

69) $16p^2 - 24p + 9$ 71) $x^3 - 9x^2 + 27x - 27$

73) $m^2 - 6m + 9 + 2mn - 6n + n^2$ 75) $p^2 - 169$

77) $\dfrac{81}{4} - \dfrac{25}{36}x^2$ 79) $9a^2 - \dfrac{1}{4}b^2$ 81) $3u^3 + 24u^2 + 48u$

83) a) $2n^2 + 7n - 22$ sq units
b) $6n + 18$ units

85) $4t^2 - 10t - 5$ 87) $w + 5$ 89) $4r^2 + r - 3$

91) $t + 2 - \dfrac{3}{2t} + \dfrac{10}{7t^2}$ 93) $2v - 1 + \dfrac{6}{4v + 9}$

95) $3v^2 - 7v + 8$ 97) $c^2 + 2c + 4$

99) $6k^2 - 4k + 7 - \dfrac{18}{3k + 2}$ 101) $-\dfrac{5}{3}x^3y^2 - 4xy^2 + 1 - \dfrac{5}{4y^2}$

103) $8a + 2$ 105) $20c^3 - 12c^2 - 11c + 1$ 107) $144 - 49w^2$

109) $-40r^{21}t^{27}$ 111) $13a^3b^3 + 7ab^2 - \dfrac{5}{3}b + \dfrac{1}{3ab^2}$

113) $h^3 - 15h^2 + 75h - 125$ 115) $2c^2 - 8c + 9 + \dfrac{5}{c + 4}$

117) $\dfrac{y^{12}}{125}$ 119) $2p^2 + 3p - 5$

Chapter 6 Test

1) $\dfrac{64}{27}$ 2) 625 3) $-32p^9$ 4) $32t^{15}$ 5) $\dfrac{g^4}{h^{10}}$ 6) $\dfrac{25a^6}{9b^{18}}$

7) a) -1 b) 3 8) 9 9) 3 10) $24h^5 - 12h^4 + 4h^3$

11) $12a^3b^2 - 3a^2b^2 - 3ab + 9$ 12) $9y^2 - 13y - 7$

13) $-11c^3 - 12c^2 - 19c - 14$ 14) $u^2 - 14u + 45$

15) $8g^2 + 10g + 3$ 16) $v^2 - \dfrac{4}{25}$ 17) $6x^2 - 11xy - 7y^2$

18) $-12n^3 - 8n^2 + 63n - 40$ 19) $2y^3 + 24y^2 + 72y$

20) $9m^2 - 24m + 16$ 21) $\dfrac{16}{9}x^2 + \dfrac{8}{3}xy + y^2$

22) $t^3 - 6t^2 + 12t - 8$ 23) $w + 3$

24) $3m^2 - 5m + 1 - \dfrac{3}{4m}$ 25) $6p^2 - p + 5 - \dfrac{15}{3p - 7}$

26) $y^2 + 3y + 9$ 27) $2r^2 + 3r - 4$

28) a) $3d^2 - 14d - 5$ sq units
b) $8d - 8$ units

29) $4n + 3$

Cumulative Review for Chapters 1–6

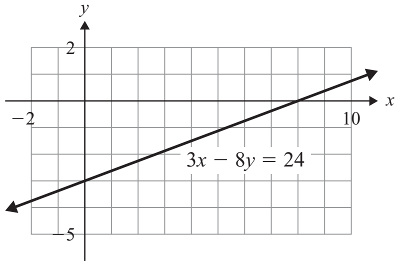
1) a) $\{41, 0\}$
b) $\{-15, 41, 0\}$
c) $\left\{\frac{3}{8}, -15, 2.1, 41, 0.\overline{52}, 0\right\}$

2) -87 3) $\frac{75}{31}$ or $2\frac{13}{31}$ 4) $-32a^{14}$ 5) c^{17} 6) $\frac{64}{p^{21}}$

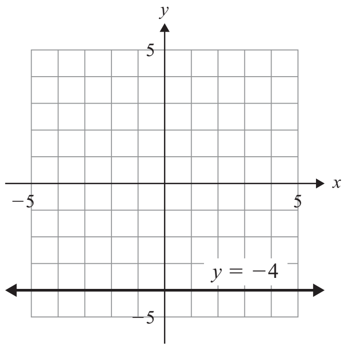
7) $\left\{-\frac{35}{3}\right\}$ 8) $\varnothing$ 9) $(-\infty, -5]$

10) 40 mL of 15% solution; 30 mL of 8% solution

11) x-int: $(8, 0)$; y-int: $(0, -3)$



12) $y = -4$



13) $3x + y = -5$ 14) $y = -\frac{1}{4}x + 3$ 15) $\left(-5, \frac{1}{2}\right)$

16) width: 12 cm; length: 35 cm 17) $-2q^2 \ -45$

18) $n^2 + n - 56$ 19) $9a^2 - 121$

20) $ab^2 - \frac{3}{2b} + \frac{5}{a^2b} + \frac{1}{2a^3b}$

21) $5p^2 + p - 7 - \frac{16}{p - 3}$

22) $12n^4 + 4n^3 - 35n^2 - 9n + 18$

23) $5c^3 - 40c^2 + 80c$

24) $4z^2 - 2z + 1$ 25) -7

Chapter 7

Section 7.1

1) 7 3) $6p^2$ 5) $4n^6$ 7) $5a^2b$ 9) $21r^3s^2$ 11) ab

13) $(k - 9)$ 15) Answers may vary. 17) yes 19) no

21) yes 23) $2(w + 5)$ 25) $9(2z^2 - 1)$ 27) $10m(10m^2 - 3)$

29) $r^2(r^7 + 1)$ 31) $\frac{1}{5}y(y + 4)$ 33) does not factor

35) $5n^3(2n^2 - n + 8)$ 37) $8p^3(5p^3 + 5p^2 - p + 1)$

39) $9a^2b(7ab^2 - 4ab + 1)$ 41) $-6(5n + 7)$

43) $-4w^3(3w^2 + 4)$ 45) $-1(k - 3)$ 47) $(t - 5)(u + 6)$

49) $(6x + 1)(y - z)$ 51) $(q + 12)(p + 1)$

53) $(9k + 8)(5h^2 - 1)$ 55) $(b + 2)(a + 7)$

57) $(3t + 4)(r - 9)$ 59) $(2b + 5c)(4b + c^2)$

61) $(g - 7)(f + 4)$ 63) $(t - 10)(s - 6)$

65) $(5u + 6)(t - 1)$ 67) $(12g^3 + h)(3g - 8h)$

69) Answers may vary.

71) $4(xy + 3x + 5y + 15)$; Group the terms and factor out the GCF from each group; $4(y + 3)(x + 5)$

73) $3(c + 7)(d + 2)$ 75) $2p(p - 4)(q - 5)$

77) $(5s - 6)(2t + 1)$ 79) $(3a^2 - 2b)(a - 7b)$

81) $2uv(4u + 5)(v + 2)$ 83) $(n + 7)(3m + 10)$

85) $8(2b - 3)$ 87) $(d + 6)(c - 4)$ 89) $2a^3(3a - 4)(b + 2)$

91) $(d + 4)(7c + 3)$ 93) $(g - 1)(d + 1)$ 95) $x^3y^2(x + 12y)$

97) $4(m + 3)(n + 2)$ 99) $-2(3p^2 + 10p - 1)$

Section 7.2

1) a) 5, 2 b) $-8, 7$ c) $5, -1$ d) $-9, -4$

3) They are negative. 5) Can I factor out a GCF?

7) Can I factor again? 9) $n + 2$ 11) $c - 10$ 13) $x + 4$

15) $(g + 6)(g + 2)$ 17) $(y + 8)(y + 2)$ 19) $(w - 9)(w - 8)$

21) $(b - 4)(b + 1)$ 23) prime 25) $(c - 9)(c - 4)$

27) $(m + 10)(m - 6)$ 29) $(r - 12)(r + 8)$ 31) prime

33) $(x + 8)(x + 8)$ or $(x + 8)^2$ 35) $(n - 1)(n - 1)$ or $(n - 1)^2$

37) $(d + 12)(d + 2)$ 39) prime 41) $2(k - 3)(k - 8)$

43) $5h(h + 5)(h + 2)$ 45) $r^2(r + 12)(r - 11)$

47) $7q(q^2 - 7q - 6)$ 49) $3z^2(z + 4)(z + 4)$ or $3z^2(z + 4)^2$

51) $xy(y - 9)(y + 7)$ 53) $-(m + 5)(m + 7)$

55) $-(c + 7)(c - 4)$ 57) $-(z - 3)(z - 10)$

59) $-(p - 8)(p + 7)$ 61) $(x + 4y)(x + 3y)$

63) $(c - 8d)(c + d)$ 65) $(u - 5v)(u - 9v)$

67) $(m - 3n)(m + 7n)$ 69) $(a + 12b)(a + 12b)$ or $(a + 12b)^2$

71) No; from $(3x + 6)$ you can factor out a 3. The correct answer is $3(x + 2)(x + 5)$.

73) yes 75) $2(x + 5)(x + 3)$ 77) $(n - 4)(n - 2)$

79) $(m - 4n)(m + 11n)$ 81) prime 83) $4q(q - 3)(q - 4)$

85) $-(k + 9)(k + 9)$ or $-(k + 9)^2$ 87) $4h^3(h + 7)(h + 1)$

89) $(k + 12)(k + 9)$ 91) $pq(p - 7q)(p - 10q)$

93) prime 95) $(x - 12y)(x - y)$ 97) $5v^3(v^2 + 11v - 9)$

99) $6xy^2(x - 9)(x + 1)$ 101) $(z - 9)(z - 4)$

103) $(ab + 6)(ab + 7)$ 105) $(x + y)(z + 10)(z - 3)$

107) $(a - b)(c - 7)(c - 4)$

109) $(p + q)(r + 12)(r + 12)$ or $(p + q)(r + 12)^2$

Section 7.3

1) a) $10, -5$ b) $-27, -1$ c) $6, 2$ d) $-12, 6$

3) $(3c + 8)(c + 4)$ 5) $(6k - 7)(k - 1)$

7) $(2x - 9y)(3x + 4y)$ 9) Can I factor out a GCF?

11) $4k^2 + 17k + 18$ 13) $t + 2$

15) $3a + 2$ 17) $3x - y$ 19) $(2h + 3)(h + 5)$

21) $(7y - 4)(y - 1)$ 23) $(5b - 6)(b + 3)$

25) $(3p + 2)(2p - 1)$ 27) $(2t + 3)(2t + 5)$

29) $(9x - 4y)(x - y)$

31) because 2 can be factored out of $2x - 4$, but 2 cannot be factored out of $2x^2 + 13x - 24$

33) $(2r + 5)(r + 2)$ 35) $(3u - 5)(u - 6)$

37) $(7a - 4)(a + 5)$ 39) $(3y + 10)(2y + 1)$

41) $(9w - 7)(w + 3)$ 43) $(4c - 3)(2c - 9)$

45) $(2k + 11)(2k + 9)$ 47) $(10b + 9)(2b - 5)$

49) $(2r - 3t)(r + 8t)$ 51) $(6a - b)(a - 4b)$

53) $(4z - 3)(z + 2)$; the answer is the same.

55) $(3p + 2)(p - 6)$ 57) $(4k + 3)(k + 3)$

59) $2w(5w + 1)(3w + 7)$ 61) $3(7r - 2)(r - 4)$ 63) prime

65) $(7b + 3)(6b - 1)$ 67) $(7x - 3y)(x - 2y)$

69) $2(d + 5)(d - 4)$ 71) $r^2t^2(6r + 1)(5r + 3)$ 73) $(3k - 7)^2$

75) $(n + 9)(2m - 7)(m + 1)$ 77) $(u + 4)^2(3v + 4)(2v + 5)$

79) $(2a - 1)^4(5b - 6)(3b - 2)$ 81) $-(n + 12)(n - 4)$

83) $-(7a + 3)(a - 1)$ 85) $-(5z - 2)(2z - 3)$

87) $-5m(2m + 9)(2m + 3)$ 89) $-ab(6a + b)(a - 2b)$

Section 7.4

1) a) 49 b) 81 c) 36 d) 100 e) 25 f) 16 g) 121 h) $\frac{1}{9}$ i) $\frac{9}{64}$

3) a) c^2 b) $3r$ c) $9p$ d) $6m^2$ e) $\frac{1}{2}$ f) $\frac{12}{5}$

5) $y^2 + 12y + 36$

7) The middle term does not equal $2(2a)(-3)$. It would have to equal $-12a$ to be a perfect square trinomial.

9) $(h + 5)^2$ 11) $(b - 7)^2$ 13) $(2w + 1)^2$ 15) $(3k - 4)^2$

17) $\left(c + \frac{1}{2}\right)^2$ 19) $\left(k - \frac{7}{5}\right)^2$ 21) $(a + 4b)^2$ 23) $(5m - 3n)^2$

25) $4(f + 3)^2$ 27) $5a^2(a + 3)^2$ 29) $-4(2y + 5)^2$

31) $3h(25h^2 - 2h + 4)$ 33) a) $x^2 - 81$ b) $81 - x^2$

35) $w - 8$ 37) $11 - p$ 39) $8c - 5b$ 41) $(k + 2)(k - 2)$

43) $(c + 5)(c - 5)$ 45) prime 47) $\left(x + \frac{1}{3}\right)\left(x - \frac{1}{3}\right)$

49) $\left(a + \frac{2}{7}\right)\left(a - \frac{2}{7}\right)$ 51) $(12 + v)(12 - v)$

53) $(1 + h)(1 - h)$ 55) $\left(\frac{6}{5} + b\right)\left(\frac{6}{5} - b\right)$

57) $(10m + 7)(10m - 7)$ 59) $(13k + 1)(13k - 1)$

61) prime 63) $\left(\frac{1}{3}t + \frac{5}{2}\right)\left(\frac{1}{3}t - \frac{5}{2}\right)$ 65) $(u^2 + 10)(u^2 - 10)$

67) $(6c + d^2)(6c - d^2)$ 69) $(r^2 + 1)(r + 1)(r - 1)$

71) $(r^2 + 9t^2)(r + 3t)(r - 3t)$ 73) $5(u + 3)(u - 3)$

75) $2(n + 12)(n - 12)$ 77) $3z^2(2z + 5)(2z - 5)$

79) a) 64 b) 1 c) 1000 d) 27 e) 125 f) 8

81) a) m b) $3t$ c) $2b$ d) h^2 83) $y^2 - 2y + 4$

85) $(t + 4)(t^2 - 4t + 16)$ 87) $(z - 1)(z^2 + z + 1)$

89) $(3m - 5)(9m^2 + 15m + 25)$ 91) $(5y - 2)(25y^2 + 10y + 4)$

93) $(10c - d)(100c^2 + 10cd + d^2)$

95) $(2j + 3k)(4j^2 - 6jk + 9k^2)$

97) $(4x + 5y)(16x^2 - 20xy + 25y^2)$

99) $6(c + 2)(c^2 - 2c + 4)$

101) $7(v - 10w)(v^2 + 10vw + 100w^2)$

103) $(h + 2)(h - 2)(h^2 - 2h + 4)(h^2 + 2h + 4)$

105) $7(2d + 1)$ 107) $4(2k + 3)(k - 2)$

109) $(r + 1)(r^2 - 7r + 19)$ 111) $(c - 1)(c^2 + 13c + 61)$

Chapter 7 Putting It All Together

1) $(c + 8)(c + 7)$ 2) $(r + 10)(r - 10)$ 3) $(u + 9)(v + 6)$

4) $(5t + 4)(t - 8)$ 5) $(2p - 7)(p - 3)$ 6) $(h - 11)^2$

7) $9v^3(v^2 + 10v - 6)$ 8) $(m + 10n)(m - 4n)$

9) $4q(3q + 8)(2q - 1)$ 10) $5(k - 2)(k^2 + 2k + 4)$

11) $(g + 5)(g^2 - 5g + 25)$ 12) $(x - 9)(y - 1)$

13) $(12 + w)(12 - w)$ 14) prime 15) $(3r + 2t)^2$

16) $5(8b - 7)$ 17) $7n^2(n - 10)(n + 1)$

18) $(2x - 3)(2x + 5)$ 19) prime 20) $4a(b + 3)(c - 6)$

21) $5(2x - 3)(4x^2 + 6x + 9)$ 22) $(7c + 4)^2$

23) $\left(m + \frac{1}{10}\right)\left(m - \frac{1}{10}\right)$ 24) prime

25) $(5y - 6)(2x + 1)(2x - 1)$ 26) $4ab(5a^2 + 3b)(5a^2 - 3b)$

27) $(p + 15q)(p + 2q)$ 28) $8(k + 2)(k^2 - 2k + 4)$

29) prime 30) $6g^2h(2g^2h^2 + 9g + 5)$ 31) $2(5n - 2)^2$

32) $(8a - 9)(a + 1)$ 33) $3(12r + 7s)(r + s)$

34) $\left(t + \frac{9}{13}\right)\left(t - \frac{9}{13}\right)$ 35) $(9x^2 + y^2)(3x + y)(3x - y)$

36) $(v - 12)(v - 11)$ 37) $2(a - 9)(a + 4)$

38) $(p + 1)(p - 1)(q - 6)$ 39) $\left(h - \frac{1}{5}\right)^2$

40) $(m - 4)(m^2 + 4m + 16)$ 41) $(8u - 5)(2v + 3)$

42) $-9r(3r - 10)(r - 2)$ 43) $(4b + 3)(2b - 5)$

44) $3(2b + 3)^2$ 45) $4y^2z^2(2y^2z - 7yz - 10y + 1)$

46) $(7 + p)(7 - p)$ 47) prime 48) $6hk(h + k)(h + 8k)$

49) $(4u + 5v)^2$ 50) $(b^2 + 4)(b + 2)(b - 2)$

51) $(8k - 3)(3k + 5)$ 52) prime

53) $5(s - 4t)(s^2 + 4st + 16t^2)$ 54) $12w^3(3w^3 - 7w + 1)$

55) $(a - 1)(b - 1)$ 56) $(d + 8)^2$ 57) $7(h + 1)(h - 1)$

58) $(3p - 4q)(3p - 2q)$ 59) $6(m - 5)^2$ 60) prime

61) $(11z + 13)(11z - 13)$

62) $(4a - 5b)(16a^2 + 20ab + 25b^2)$ 63) $-3(4r + 1)(r + 6)$

64) $9(c + 4)(c + 2)$ 65) $(n + 1)(n^2 - n + 1)$ 66) $(4t + 1)^2$

67) $(9u^2 + v^2)(3u + v)(3u - v)$ 68) $2(3u + 7v^2)(2u + v)$

69) $(13h + 2)(h + 1)$ 70) $2g(g - 8)(g + 7)$

71) $t^4(5t^3 - 8)$ 72) $\left(m + \frac{12}{5}\right)\left(m - \frac{12}{5}\right)$

73) $(d - 10)(d + 3)$ 74) $(5k - 6)^2$ 75) prime

76) $2(3w + 2)(9w^2 - 6w + 4)$ 77) $(r + 1)^2$

78) $(b - 12)(b - 7)$ 79) $(7n + 10)(7n - 10)$

80) $9y^2(y + 3)(y - 3)$ 81) $(2z + 1)(y + 11)(y - 5)$

82) $(2k - 7)(h + 5)(h - 9)$ 83) $(t + 1)(t - 4)$

84) $v(v + 2)$ 85) $z(z + 3)$ 86) $(3n + 7)(3n - 10)$

87) $4ab$ 88) $-4y(x + 2y)$ 89) $3(7p - q)(p - q)$

90) $(7s - t)(s + 3t)$ 91) $(r + 5)(r^2 + r + 7)$

92) $(d - 3)(d^2 - 12d + 39)$ 93) $(k - 8)(k^2 - 13k + 43)$

94) $2(w - 1)(4w^2 + 22w + 49)$ 95) $(a + b - 4)(a - b - 4)$

96) $(x + y + 3)(x - y + 3)$ 97) $(s + t + 9)(s - t + 9)$

98) $(m + n - 1)(m - n - 1)$

Section 7.5

1) $ax^2 + bx + c = 0$

3) a) quadratic b) linear c) quadratic d) linear

5) If the product of two quantities equals 0, then one or both of the quantities must be zero.

7) $\{-11, 4\}$ 9) $\left\{\frac{3}{2}, 10\right\}$ 11) $\{0, 12\}$ 13) $\left\{-\frac{5}{3}\right\}$

15) $\left\{-\frac{1}{2}, -\frac{2}{9}\right\}$ 17) $\left\{-\frac{1}{4}, \frac{2}{5}\right\}$ 19) $\{0, 4.6\}$

21) No; the product of the factors must equal zero.

23) $\{-6, -2\}$ 25) $\{-10, 11\}$ 27) $\left\{\frac{4}{3}, 2\right\}$ 29) $\left\{-\frac{3}{4}, \frac{2}{3}\right\}$

31) $\{-12, 5\}$ 33) $\{6, 9\}$ 35) $\{-8, 8\}$ 37) $\left\{-\frac{7}{10}, \frac{7}{10}\right\}$

39) $\left\{-\frac{6}{5}, -1\right\}$ 41) $\{0, 4\}$ 43) $\{-6, 2\}$ 45) $\{7, 12\}$

47) $\{-9, -7\}$ 49) $\{8\}$ 51) $\left\{-3, -\frac{4}{5}\right\}$

53) No. You cannot divide an equation by a variable because you may eliminate a solution and may be dividing by zero.

55) $\left\{-6, 0, \frac{9}{8}\right\}$ 57) $\left\{-\frac{7}{6}, 2, 3\right\}$ 59) $\{0, -7, 7\}$

61) $\{-4, 0, 9\}$ 63) $\left\{0, \frac{5}{2}, 3\right\}$ 65) $\left\{0, -\frac{2}{9}, \frac{2}{9}\right\}$

67) $\left\{\frac{7}{2}, \frac{9}{2}\right\}$ 69) $\left\{\frac{1}{3}\right\}$ 71) $\{-5, 6\}$ 73) $\{0, -4, 4\}$

75) $\left\{-\frac{1}{3}, 9\right\}$ 77) $\{-11, 11\}$ 79) $\left\{\frac{5}{2}, 3\right\}$ 81) $\left\{-\frac{1}{3}, 11\right\}$

83) $\left\{-\frac{1}{13}, 1\right\}$ 85) $\left\{-3, -\frac{3}{2}, 4\right\}$ 87) $\left\{-\frac{1}{5}, \frac{7}{2}\right\}$

89) $\{-8, -1, 1\}$ 91) $-8, -2$ 93) $\frac{5}{2}, 4$ 95) $-5, 5$

97) 0, 1, 6

Section 7.6

1) length = 7 in.; width = 4 in.

3) base = 11 cm; height = 8 cm

5) base = 9 in.; height = 4 in.

7) height = 6 in.; width = 9 in.

9) length = 4 ft; width = 2 ft 11) 2.5 ft by 6 ft

13) length = 6 in.; width = 5 in.

15) height = 7 cm; base = 6 cm 17) 8 and 9, or 1 and 2

19) 6, 8, 10; or $-2, 0, 2$ 21) 7, 9, 11

23) Answers may vary. 25) 9 27) 5 29) 20 31) 3, 4, 5

33) 5, 12, 13 35) 10 cm 37) 5 ft 39) 5 mi

41) a) 144 ft b) after 2 sec c) 3 sec

43) a) 288 ft b) 117 ft c) 324 ft d) 176 ft

45) a) 0 ft b) after 2 sec and after 4 sec c) 144 ft d) after 6 sec

47) a) \$2835 b) \$2880 c) \$18

Chapter 7 Review Exercises

1) 8 3) $5h^3$ 5) $9(7t + 5)$ 7) $2p^4(p^2 - 10p + 1)$

9) $(m + 8)(n - 5)$ 11) $-5r(3r^2 + 8r - 1)$

13) $(a + 9)(b + 2)$ 15) $(x - 7)(4y - 3)$ 17) $(q + 6)(q + 4)$

19) $(z - 12)(z + 6)$ 21) $(m - 3n)(m - 10n)$

23) $4(v - 8)(v + 2)$ 25) $9w^2(w - 1)(w + 2)$

27) $(3r - 2)(r - 7)$ 29) $(2p - 5)(2p + 1)$

31) $2(3c + 2)(2c + 5)$ 33) $(5x - 3y)(2x + 9y)$

35) $(w + 7)(w - 7)$ 37) $(8t + 5u)(8t - 5u)$ 39) prime

41) $4x(4 + x)(4 - x)$ 43) $(r + 6)^2$ 45) $5(2k - 3)^2$

47) $(v - 3)(v^2 + 3v + 9)$ 49) $(5x + 4y)(25x^2 - 20xy + 16y^2)$

51) $(5z + 4)(2z - 3)$ 53) $k^2(3k + 4)(3k - 4)$

55) $(d - 12)(d - 5)$ 57) $(3b + 1)(a + 2)(a - 2)$

59) $6(2p - q)(4p^2 + 2pq + q^2)$ 61) $(x + y - 1)(x - y + 9)$

63) $(5c - 2)^2$ 65) $\left\{-\frac{7}{3}, 0\right\}$ 67) $\left\{2, \frac{9}{2}\right\}$ 69) $\{-9, -8\}$

71) $\left\{-\frac{11}{9}, \frac{11}{9}\right\}$ 73) $\{-4, 10\}$ 75) $\{-6, -5\}$

77) $\left\{-\frac{4}{5}, 1, \frac{4}{3}\right\}$ 79) $\left\{\frac{1}{4}, 4\right\}$ 81) $\left\{0, -\frac{2}{3}, \frac{2}{3}\right\}$

83) base = 9 cm; height = 4 cm 85) base = 6 ft; height = 2 ft

87) 15 89) length = 4 ft; width = 2.5 ft

91) $-1, 0, 1$; or 4, 5, 6 93) 3 miles

Chapter 7 Test

1) See whether you can factor out a GCF.

2) $(h - 8)(h - 6)$

3) $(6 + v)(6 - v)$ 4) $(7p - 8)(p + 2)$

5) $4ab^2(5a^2b^2 + 9ab + 1)$ 6) prime

7) $(4t - 3u)(16t^2 + 12tu + 9u^2)$ 8) $4z(z + 4)(z + 3)$

9) $(6r - 5)^2$ 10) $(n + 7)(n + 2)(n - 2)$

11) $(x + 3y)(x - 6y)$ 12) $(9c^2 + d^2)(3c + d)(3c - d)$

13) $(q - 4)^2(p + 15)(p + 2)$ 14) $2(8w - 5)(2w + 3)$

15) $k^5(k + 1)(k^2 - k + 1)$ 16) $\{-9, 4\}$ 17) $\{0, -12, 12\}$

18) $\left\{-\frac{4}{7}, \frac{4}{7}\right\}$ 19) $\{4, 8\}$ 20) $\left\{-\frac{8}{3}, -\frac{1}{2}\right\}$ 21) $\left\{-\frac{2}{5}, 3\right\}$

22) length = 6 in.; width = 2 in. 23) 9, 11, 13 24) 9 miles

25) length = 42 ft; width = 6 ft

26) a) 184 ft b) 544 ft

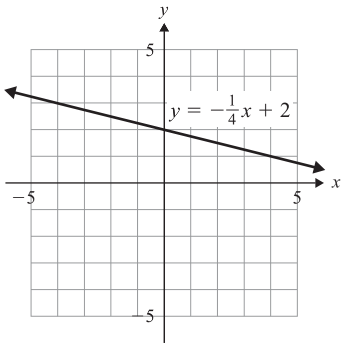
c) when $t = 2\frac{1}{2}$ sec and when $t = 10$ sec d) $12\frac{1}{2}$ sec

Cumulative Review for Chapters 1–7

1) $-\frac{5}{18}$ 2) $\frac{10}{27}$ 3) $18p^8q^2$ 4) $\frac{7a^{11}}{10b^5}$ 5) 8.39×10^{-5}

6) $\{-2\}$ 7) $b_2 = \frac{2A - hb_1}{h}$ 8) $[-12, \infty)$

9)



10) $y = \frac{5}{3}x + \frac{4}{3}$ 11) $\varnothing$ 12) adults: 120; students: 345

13) $8w^2 - 2w - 21$ 14) $9n^2 + 24n + 16$

15) $12z^3 + 32z^2 - 53z + 15$ 16) $13v^2 - 13v - 5$

17) $4x^2 + 7x - 2$ 18) $3m - 1 + \frac{1}{2m}$

19) $3(2c + 9)(c - 2)$ 20) prime

21) $(x + 4)(y + 1)(y - 1)$ 22) $\left(\frac{1}{2} + b\right)\left(\frac{1}{2} - b\right)$

23) $(h + 5)(h^2 - 5h + 25)$ 24) $\left\{\frac{4}{3}, 3\right\}$ 25) $\left\{0, -\frac{3}{2}, \frac{3}{2}\right\}$

Chapter 8

Section 8.1

1) when its denominator equals zero

3) a) $\frac{5}{17}$ b) $\frac{5}{8}$ 5) a) $-\frac{4}{3}$ b) undefined 7) a) $\frac{4}{3}$ b) 0

9) Set the denominator equal to zero and solve for the variable. That value cannot be substituted into the expression because it will make the denominator equal to zero.

11) a) -4 b) 0 13) a) $\frac{7}{2}$ b) $-\frac{1}{4}$ 15) a) 0, 11 b) $\frac{9}{5}$

17) a) never equals 0 b) 0 19) a) 0 b) $-4, -5$

21) a) -20 b) $\frac{3}{2}, -3$ 23) a) $-6, -3$ b) 0

25) a) 0
b) never undefined—any real number may be substituted for y

27) $\frac{7x}{3}$ 29) $\frac{3}{7g^2}$ 31) $\frac{4}{5}$ 33) $-\frac{7}{3}$ 35) $\frac{13}{10}$ 37) $g - 8$

39) $\frac{1}{t+5}$ 41) $\frac{3c+4}{c+2}$ 43) $\frac{q+5}{2q+3}$ 45) $\frac{w+5}{5}$

47) $\frac{9}{c+3}$ 49) $\frac{4(u-5)}{13}$ 51) $\frac{m+n}{m^2+mn+n^2}$ 53) -1

55) -1 57) $-m-11$ 59) $-\frac{6}{x+2}$ 61) $-4(b+2)$

63) $-\frac{y^2+2}{7}$ 65) $\frac{3}{2(c-1)}$ 67) $-\frac{r^2+rt+t^2}{r+t}$

69) $\frac{b-6}{4(b+1)}$ 71) $4h^3-8h^2+1$ 73) $-\frac{1}{2w^2}$

75) $-\frac{5}{(v-2)(v-1)}$

77) possible answers:
$\frac{-u-7}{u-2}, \frac{-(u+7)}{u-2}, \frac{u+7}{2-u}, \frac{u+7}{-(u-2)}, \frac{u+7}{-u+2}$

79) possible answers:
$\frac{-9+5t}{2t-3}, \frac{5t-9}{2t-3}, \frac{-(9-5t)}{2t-3}, \frac{9-5t}{-2t+3}, \frac{9-5t}{3-2t}, \frac{9-5t}{-(2t-3)}$

81) possible answers:
$-\frac{12m}{m^2-3}, \frac{12m}{-(m^2-3)}, \frac{12m}{-m^2+3}, \frac{12m}{3-m^2}$

83) $4y-3$ 85) $4a^2-10a+25$ 87) $3x+2$ 89) $2c+4$

91) $3k+1$ 93) $(-\infty, 7)\cup(7, \infty)$

95) $\left(-\infty, -\frac{2}{5}\right)\cup\left(-\frac{2}{5}, \infty\right)$ 97) $(-\infty, 1)\cup(1, 8)\cup(8, \infty)$

99) $(-\infty, -9)\cup(-9, 9)\cup(9, \infty)$ 101) $(-\infty, \infty)$

103) Answers may vary.

Section 8.2

1) $\frac{35}{54}$ 3) $\frac{5}{21}$ 5) $\frac{16b^4}{27}$ 7) $\frac{t^2}{6s^6}$ 9) $\frac{5}{2}$ 11) $\frac{1}{2t(3t-2)}$

13) $\frac{2u^2}{3(4u-5)^2}$ 15) 6 17) $\frac{3(y-5)}{2y^2}$ 19) $\frac{5}{c}$

21) $-7x(x+11)$ 23) 6 25) $\frac{28}{3}$ 27) $\frac{32m^7}{35}$ 29) $-\frac{20}{3g^3h^2}$

31) $\frac{8}{3k^6(k-2)}$ 33) $8q(p+7)$ 35) $\frac{5}{q(q-7)}$

37) $\frac{z+10}{(2z+1)(z+8)}$ 39) $\frac{3}{4(3a+1)}$ 41) $\frac{7}{2}$ 43) $\frac{c+1}{6c}$

45) Answers may vary. 47) $h^2+3h-10$ 49) $7-2z$

51) $\frac{25}{16}$ 53) $\frac{1}{18}$ 55) $\frac{1}{4}$ 57) $\frac{4}{3r^2}$ 59) $\frac{a-5}{12a^8}$

61) $\frac{2(4x+5)}{3x^4}$ 63) $\frac{(c+6)(c+1)}{9(c+5)}$ 65) $\frac{5x+1}{4x}$

67) $\frac{k(2k+3)}{2}$ 69) $\frac{t^4(3t-1)}{2(t+2)}$ 71) $-\frac{h^4}{3(h+8)}$ 73) $\frac{3x^6}{4y^6}$

75) $\frac{7}{3}$ 77) $-\frac{a}{6}$ 79) $\frac{a^2}{a-4}$ 81) $-\frac{3}{4(x+y)}$ 83) $\frac{m+2}{16}$

85) $\frac{4j-1}{3j+2}$ 87) $\frac{x}{12(x-9)}$ 89) $\frac{12x^5}{y^3}$

Section 8.3

1) 60 3) 120 5) n^{11} 7) $28r^7$ 9) $36z^5$ 11) $110m^4$

13) $24x^3y^2$ 15) $11(z-3)$ 17) $w(2w+1)$

19) Factor the denominators. 21) $10(c-1)$

23) $3p^5(3p-2)$ 25) $(m-7)(m-3)$

27) $(z+3)(z+8)(z-3)$ 29) $(t+6)(t-6)(t+3)$

31) $a-8$ or $8-a$ 33) $x-y$ or $y-x$

35) Answers may vary. 37) $\frac{28}{48}$ 39) $\frac{72}{9z}$ 41) $\frac{21k^3}{56k^4}$

43) $\frac{12t^2u^3}{10t^7u^5}$ 45) $\frac{7r}{r(3r+4)}$ 47) $\frac{4v^6}{16v^5(v-3)}$

49) $\frac{9x^2-45x}{(x+6)(x-5)}$ 51) $\frac{z^2+5z-24}{(2z-5)(z+8)}$ 53) $-\frac{5}{p-3}$

55) $\frac{8c}{7-6c}$ 57) $\frac{8}{15}=\frac{16}{30}; \frac{1}{6}=\frac{5}{30}$ 59) $\frac{4}{u}=\frac{4u^2}{u^3}; \frac{8}{u^3}=\frac{8}{u^3}$

61) $\frac{9}{8n^6}=\frac{27}{24n^6}; \frac{2}{3n^2}=\frac{16n^4}{24n^6}$

63) $\frac{6}{4a^3b^5}=\frac{6a}{4a^4b^5}; \frac{6}{a^4b}=\frac{24b^4}{4a^4b^5}$

65) $\frac{r}{5}=\frac{r^2-4r}{5(r-4)}; \frac{2}{r-4}=\frac{10}{5(r-4)}$

67) $\frac{3}{d}=\frac{3d-27}{d(d-9)}; \frac{7}{d-9}=\frac{7d}{d(d-9)}$

69) $\frac{m}{m+7}=\frac{m^2}{m(m+7)}; \frac{3}{m}=\frac{3m+21}{m(m+7)}$

71) $\frac{a}{30a-15}=\frac{2a}{30(2a-1)}; \frac{1}{12a-6}=\frac{5}{30(2a-1)}$

73) $\frac{8}{k-9}=\frac{8k+24}{(k-9)(k+3)}; \frac{5k}{k+3}=\frac{5k^2-45k}{(k-9)(k+3)}$

75) $\frac{3}{a+2}=\frac{9a+12}{(a+2)(3a+4)}; \frac{2a}{3a+4}=\frac{2a^2+4a}{(a+2)(3a+4)}$

77) $\frac{9y}{y^2-y-42}=\frac{18y^2}{2y(y+6)(y-7)};$
$\frac{3}{2y^2+12y}=\frac{3y-21}{2y(y+6)(y-7)}$

79) $\frac{c}{c^2+9c+18}=\frac{c^2+6c}{(c+6)^2(c+3)};$
$\frac{11}{c^2+12c+36}=\frac{11c+33}{(c+6)^2(c+3)}$

81) $\frac{11}{g-3}=\frac{11g+33}{(g+3)(g-3)}; \frac{4}{9-g^2}=-\frac{4}{(g+3)(g-3)}$

83) $\frac{4}{3x-4}=\frac{12x+16}{(3x+4)(3x-4)}$;
$\frac{7x}{16-9x^2}=-\frac{7x}{(3x+4)(3x-4)}$

85) $\frac{2}{z^2+3z}=\frac{6z+18}{3z(z+3)^2}$; $\frac{6}{3z^2+9z}=\frac{6z+18}{3z(z+3)^2}$;
$\frac{8}{z^2+6z+9}=\frac{24z}{3z(z+3)^2}$

87) $\frac{t}{t^2-13t+30}=\frac{t^2+3t}{(t+3)(t-3)(t-10)}$;
$\frac{6}{t-10}=\frac{6t^2-54}{(t+3)(t-3)(t-10)}$;
$\frac{7}{t^2-9}=\frac{7t-70}{(t+3)(t-3)(t-10)}$

89) $-\frac{9}{h^3+8}=-\frac{45}{5(h+2)(h^2-2h+4)}$;
$\frac{2h}{5h^2-10h+20}=\frac{2h^2+4}{5(h+2)(h^2-2h+4)}$

Section 8.4

1) $\frac{7}{8}$ 3) $\frac{4}{7}$ 5) $-\frac{18}{p}$ 7) $\frac{5}{c}$ 9) $\frac{z+6}{z-1}$ 11) 2

13) $\frac{5}{t}$ 15) $\frac{d+6}{d+5}$

17) a) $18b^4$
b) Multiply the numerator and denominator of $\frac{4}{9b^2}$ by $2b^2$, and multiply the numerator and denominator of $\frac{5}{6b^4}$ by 3.
c) $\frac{4}{9b^2}=\frac{8b^2}{18b^4}$; $\frac{5}{6b^4}=\frac{15}{18b^4}$

19) $\frac{31}{40}$ 21) $\frac{8t+9}{6}$ 23) $\frac{6h^2+50}{15h^3}$ 25) $\frac{3-14f}{2f^2}$

27) $\frac{16y+9}{y(y+3)}$ 29) $\frac{11d+32}{d(d-8)}$ 31) $\frac{3(5c+18)}{(c-4)(c+8)}$

33) $\frac{m^2-16m-10}{(3m+5)(m-10)}$ 35) $\frac{3u+2}{u-1}$

37) $\frac{7g^2-53g+6}{(g+2)(g+8)(g-8)}$ 39) $\frac{3(a+1)}{(a-9)(a-1)}$

41) $\frac{2(x^2-5x+8)}{(x-4)(x+5)(x-3)}$ 43) $\frac{4b^2+28b+3}{3(b-4)(b+3)}$

45) No. If the sum is rewritten as $\frac{9}{x-6}-\frac{4}{x-6}$, then the LCD $=x-6$. If the sum is rewritten as $\frac{-9}{6-x}+\frac{4}{6-x}$, then the LCD is $6-x$.

47) $\frac{6}{q-4}$ or $-\frac{6}{4-q}$ 49) $\frac{26}{f-7}$ or $-\frac{26}{7-f}$

51) $\frac{8-x}{x-4}$ or $\frac{x-8}{4-x}$ 53) 1 55) $\frac{3(1+2u)}{2u-3v}$ or $-\frac{3(1+2u)}{3v-2u}$

57) $-\frac{2(x-1)}{(x+3)(x-3)}$ 59) $\frac{3(3a+4)}{(2a+3)(2a-3)}$

61) $\frac{-10a^2+8a-11}{a(a-2)}$ 63) $\frac{10b^2+2b+15}{b(b+8)(3b+1)}$

65) $\frac{c^2-2c+20}{(c-4)^2(c+3)}$ 67) $\frac{17a+23b}{4(a+b)(a-b)}$

69) $\frac{2(9v+2)}{(6v+1)(3v+2)(v-5)}$ 71) $\frac{7g^2-45g+205}{5g(g-6)(2g-5)}$

73) a) $\frac{2(k-4)}{k+1}$ b) $\frac{k^2-3k+28}{2(k+1)}$

75) a) $\frac{6h}{(h+5)^2(h+4)}$ b) $\frac{2(h^2+4h+6)}{(h+5)(h+4)}$ 77) $\frac{49x+6}{4x^2}$

Chapter 8 Putting It All Together

1) a) 0 b) $\frac{1}{2}$ 2) a) $\frac{3}{5}$ b) undefined

3) a) undefined b) $\frac{7}{45}$ 4) a) 3 b) $\frac{1}{2}$

5) a) $-6, 6$ b) 0 6) a) $-3, -\frac{5}{2}$ b) 4

7) a) $-4, 2$ b) $\frac{3}{5}$ 8) a) $-8, 8$ b) $\frac{8}{5}$

9) a) 0 b) never equals 0

10) a) never undefined—any real number may be substituted for t b) 15

11) $4w^{11}$ 12) $\frac{7}{3n^5}$ 13) $\frac{m+9}{2(m+4)}$ 14) $\frac{1}{j-5}$ 15) $-\frac{3}{n+2}$

16) $-\frac{1}{y+5}$ 17) $\frac{32}{3}$ 18) $\frac{2(2f-11)}{f(f+11)}$

19) $\frac{4j^2+27j+27}{(j+9)(j-9)(j+6)}$ 20) $\frac{5a^2b}{3}$ 21) $\frac{y}{3z^2}$

22) $\frac{8q^2-37q+21}{(q-5)(q+4)(q+7)}$ 23) $\frac{x^2+2x+12}{(2x+1)^2(x-4)}$

24) $-\frac{n-4}{4}$ or $\frac{4-n}{4}$ 25) $-\frac{m+7}{8}$ 26) $\frac{12}{r-7}$

27) $\frac{9x}{(y+8)(9x^2+15x+25)}$ 28) $\frac{6}{d^6}$ 29) $\frac{9d^2+8d+24}{d^2(d+3)}$

30) $\frac{3}{5}$ 31) $\frac{3k^2(3k+1)}{2}$ 32) $\frac{35}{12z}$

33) $-\frac{(w-15)(w-1)}{(w+5)(w-5)(w-7)}$ 34) $\frac{3a^3(a^2+5)}{10}$

35) $\frac{2(7x-1)}{(x-8)(x+3)}$ 36) $\frac{3y^3+16}{12y^4}$ 37) $\frac{h+5}{8(2h+1)}$

38) $(b-3)(b+4)$ 39) $\dfrac{3m+20n}{7m-4n}$ 40) -1

41) $\dfrac{-2p^2-8p+11}{p(p+7)(p-8)}$ 42) $\dfrac{5u^2+37u-19}{u(3u-2)(u+1)}$ 43) $\dfrac{2(t+1)^2}{t}$

44) $\dfrac{1}{5r^2(3r-7)}$ 45) $\dfrac{c^2}{24}$ 46) $\dfrac{9}{2}$ 47) 1

48) $\dfrac{12p^2-92p-15}{(4p+3)(p+2)(p-6)}$ 49) $\dfrac{3m+1}{7(m+4)}$

50) $\dfrac{4(c^2+16)}{3(c-2)}$ 51) $-\dfrac{1}{3k}$ 52) $-\dfrac{11}{5w}$

53) a) $\dfrac{6z}{(z+5)(z+2)}$ b) $\dfrac{2(z^2+8z+30)}{(z+5)(z+2)}$

54) $\dfrac{111n+8}{36n^2}$ 55) $\left(-\infty,-\dfrac{1}{2}\right)\cup\left(-\dfrac{1}{2},\dfrac{1}{2}\right)\cup\left(\dfrac{1}{2},\infty\right)$

56) $(-\infty,-7)\cup(-7,\infty)$ 57) $(-\infty,\infty)$

58) $(-\infty,0)\cup(0,10)\cup(10,\infty)$ 59) $(-\infty,0)\cup(0,\infty)$

60) $(-\infty,-9)\cup(-9,6)\cup(6,\infty)$

Section 8.5

1) Method 1: Rewrite it as a division problem, then simplify.

$\dfrac{2}{9}\div\dfrac{5}{18}=\dfrac{2}{\cancel{9}_1}\cdot\dfrac{\cancel{18}^2}{5}=\dfrac{4}{5}$

Method 2: Multiply the numerator and denominator by 18, the LCD of $\dfrac{2}{9}$ and $\dfrac{5}{18}$. Then simplify.

$$\frac{\cancel{18}^2\left(\dfrac{2}{\cancel{9}}\right)}{\cancel{18}_1\left(\dfrac{5}{\cancel{18}}\right)}=\frac{4}{5}$$

3) $\dfrac{20}{63}$ 5) $\dfrac{u}{v}$ 7) x^2y 9) $\dfrac{2m^4}{15n^2}$ 11) $\dfrac{18}{m}$ 13) $3(g-6)$

15) $\dfrac{5d^2}{2}$ 17) $\dfrac{c-8}{11}$ 19) $\dfrac{1}{28}$ 21) $\dfrac{r(r-4s)}{3r+s}$

23) $\dfrac{8}{r(3t-r^2)}$ 25) $\dfrac{8w+17}{10(w+1)}$ 27) $\dfrac{1}{28}$ 29) $\dfrac{r(r-4s)}{3r+s}$

31) $\dfrac{8}{r(3t-r^2)}$ 33) $\dfrac{8w+17}{10(w+1)}$ 35) Answers may vary.

37) $\dfrac{a}{12}$ 39) $-\dfrac{n+6}{2(3n-5)}$ 41) $\dfrac{6-w^2}{w+6}$ 43) 2 45) $\dfrac{t-5}{t+4}$

47) $\dfrac{9xy}{2(x+y)}$ 49) $\dfrac{3c^2}{5}$ 51) $-\dfrac{38}{21}$ 53) $\dfrac{4}{35}$ 55) $\dfrac{1}{v(3u-2v^2)}$

57) $a+b$ 59) $\dfrac{4(x+2)}{x-6}$ 61) $\dfrac{z}{y^2}$ 63) $\dfrac{11}{m}$

65) $\dfrac{2}{(h-2)(h+2)}$ 67) $\dfrac{2(v-9)(v+2)}{3(v+3)(v+1)}$

Section 8.6

1) Eliminate the denominators. 3) difference; $\dfrac{8r+15}{6}$

5) equation; $\left\{-\dfrac{2}{5}\right\}$ 7) sum; $\dfrac{a^2+3a+33}{a^2(a+11)}$

9) equation; $\{12\}$ 11) 0, 2 13) 0, 3, -3 15) -4, 9

17) $\{-1\}$ 19) $\left\{\dfrac{16}{3}\right\}$ 21) $[-4]$ 23) $\{-11\}$ 25) $\{1\}$

27) $\{-6\}$ 29) $\left\{\dfrac{7}{3}\right\}$ 31) $\varnothing$ 33) $\{-15\}$ 35) $\{2\}$

37) $\{-6,-2\}$ 39) $\{3,5\}$ 41) $\{-8\}$ 43) $\{-10,12\}$

45) $\{3\}$ 47) $\left\{\dfrac{21}{5}\right\}$ 49) $\varnothing$ 51) $\{-1\}$ 53) $\{0,-12\}$

55) $\{8\}$ 57) $\{3,1\}$ 59) $\varnothing$ 61) $\{-5,-3\}$ 63) $\{-3\}$

65) $\left\{-\dfrac{2}{3}\right\}$ 67) $\{-5,4\}$ 69) $\{5\}$ 71) $\{-2,8\}$ 73) $\varnothing$

75) $m=\dfrac{CA}{W}$ 77) $b=\dfrac{rt}{2a}$ 79) $x=\dfrac{t+u}{3B}$

81) $n=\dfrac{dz-t}{d}$ 83) $s=\dfrac{3A-hr}{h}$ 85) $y=\dfrac{kx+raz}{r}$

87) $r=\dfrac{st}{s+t}$ 89) $z=\dfrac{4xy}{x-5y}$

Section 8.7

1) $\{60\}$ 3) $\{12\}$ 5) 12 ft 7) 4 9) 800

11) length: 48 ft; width: 30 ft

13) used tutoring: 9; did not use tutoring: 24

15) a) 6 mph b) 10 mph

17) a) $x+40$ mph b) $x-30$ mph 19) 10 mph

21) 225 mph 23) 4 mph 25) 40 mph 27) 60 mph

29) $\dfrac{1}{3}$ homework/hour 31) $\dfrac{1}{t}$ job/hr 33) $2\dfrac{2}{9}$ hr 35) $2\dfrac{2}{5}$ hr

37) 20 min 39) 7.5 hr 41) 2 hr

Chapter 8 Review Exercises

1) a) 0 b) 0 3) a) 0 b) $-\dfrac{11}{4}$

5) a) never equals 0 b) $-\dfrac{3}{2},\dfrac{3}{2}$

7) a) $-\dfrac{5}{3}, 2$
b) never undefined—any real number may be substituted for m.

9) $11k^6$ 11) $\dfrac{r-8}{4r}$ 13) $\dfrac{1}{2z+1}$ 15) $-\dfrac{1}{x+11}$

17) possible answers: $\dfrac{-4n-1}{5-3n}, \dfrac{-(4n+1)}{5-3n}, \dfrac{4n+1}{3n-5}, \dfrac{4n+1}{-5+3n}, \dfrac{4n+1}{-(5-3n)}$

19) $b+3$ 21) $(-\infty, 2)\cup(2, \infty)$

23) $(-\infty, -8)\cup(-8, 8)\cup(8, \infty)$ 25) $\dfrac{24}{35}$ 27) $\dfrac{t+2}{2(t+6)}$

29) $\dfrac{x-3}{45}$ 31) $\dfrac{1}{2r^2(r-7)}$ 33) $\dfrac{q}{35p^2}$ 35) $\dfrac{1}{3}$ 37) $\dfrac{3}{10}$

39) 30 41) k^5 43) $(4x+9)(x-7)$ 45) $w-5$ or $5-w$

47) $(c+4)(c+5)(c-7)$ 49) $\dfrac{12y^2}{20y^3}$ 51) $\dfrac{6z}{z(2z+5)}$

53) $\dfrac{t^2+t-12}{(3t+1)(t+4)}$

55) $\dfrac{8c}{c^2+5c-24} = \dfrac{8c^2-24c}{(c-3)^2(c+8)}$; $\dfrac{5}{c^2-6c+9} = \dfrac{5c+40}{(c-3)^2(c+8)}$

57) $\dfrac{7}{2q^2-12q} = \dfrac{7q+42}{2q(q+6)(q-6)}$; $\dfrac{3q}{36-q^2} = -\dfrac{6q^2}{2q(q+6)(q-6)}$; $\dfrac{q-5}{2q^2+12q} = \dfrac{q^2-11q+30}{2q(q+6)(q-6)}$

59) $\dfrac{4}{3c}$ 61) $\dfrac{36u-5v}{40u^3v^2}$ 63) $\dfrac{n^2-12n+20}{n(3n-5)}$

65) $\dfrac{4y-37}{(y-3)(y+2)}$ 67) $\dfrac{(k-7)(k+2)}{k(k+7)^2}$ 69) $\dfrac{t+20}{t-18}$

71) $\dfrac{5w^2-51w+8}{(2w-7)(w+8)(w+3)}$ 73) $\dfrac{2b^2+7b+2}{2b(3b+2)(3b-2)}$

75) a) $\dfrac{2}{x(x+2)}$ b) $\dfrac{2x^3+4x+8}{x^2(x+2)}$ 77) $\dfrac{y}{x^2}$ 79) $\dfrac{p^2+4}{p^2+9}$

81) $\dfrac{1}{5}$ 83) $\dfrac{(y+4)(y-9)}{(y-8)(y+6)}$ 85) $r+t$ 87) $\{4\}$

89) $\{-7, 5\}$ 91) $\{-3\}$ 93) $\{-1, 5\}$ 95) $\varnothing$

97) $D = \dfrac{s+T}{R}$ 99) $k = \dfrac{cw-N}{aw}$ 101) $R_1 = \dfrac{R_2R_3}{R_2-R_3}$

103) 2 mph 105) $3\frac{1}{13}$ hr 107) $\dfrac{n^2+6n+3}{(2n-1)(n+2)}$

109) $\dfrac{a+2}{(4a-7)(2a+5)}$ 111) $-\dfrac{cd}{(c+d)(c-d)}$

113) $\{1, 3\}$ 115) $\{-10\}$

Chapter 8 Test

1) $-\dfrac{3}{8}$ 2) a) -10 b) $\dfrac{9}{2}$

3) a) $-4, 9$ b) never equals zero

4) $\dfrac{1}{3t^4u^3}$ 5) $\dfrac{h-8}{9h^2+3h+1}$

6) possible answers: $-\dfrac{m-7}{4m-5}, \dfrac{m-7}{5-4m}, \dfrac{7-m}{4m-5}, \dfrac{m-7}{-4m+5}$

7) $z(z+6)$ 8) $\dfrac{2}{3r}$ 9) $\dfrac{7b}{5a^6}$ 10) $-\dfrac{13h}{36}$

11) $\dfrac{c^2+20c+30}{(3c+5)(c+2)}$ 12) $-\dfrac{k^2+2}{4}$ 13) $\dfrac{2d}{5(d+3)}$

14) $\dfrac{t-14}{t-7}$ or $\dfrac{14-t}{7-t}$ 15) $\dfrac{-2v^2-2v+39}{(2v-3)(v-2)(v+9)}$

16) $\dfrac{m}{m-2}$ 17) $\dfrac{x+y}{4xy}$ 18) $\{-2\}$ 19) $\varnothing$ 20) $\{-7, 3\}$

21) $b = \dfrac{ac}{a-c}$ 22) $\dfrac{2(x-2)}{x+1}$ 23) $\dfrac{2(x^2-x+48)}{5(x+1)}$

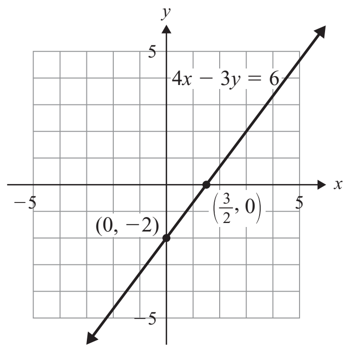
24) Ricardo: 3 hr, Michael: 6 hr 25) 12 mph

Cumulative Review Chapters 1–8

1) 45 cm^2 2) 63 3) $32p^{15}$ 4) $\dfrac{1}{125y^6}$ 5) 5 ft by 9 ft

6) $\left(-\infty, \dfrac{7}{4}\right)$ 7) $[0, 15]$

8) x-int: $\left(\dfrac{3}{2}, 0\right)$; y-int: $(0, -2)$



9) $-\dfrac{4}{3}$ 10) $\left(3, -\dfrac{5}{2}\right)$ 11) $4n^2-12n+9$

12) $64a^2-b^2$ 13) $3h^2-\dfrac{5}{3}h+1-\dfrac{2}{3h^2}$

14) $5k^2-2k-3+\dfrac{4}{k+4}$ 15) $(2d+5)(2d-3)$

16) $3(z^2+4)(z+2)(z-2)$

17) $3m(m-2)(m^2+2m+4)$ 18) $(r+8)(t-1)$

19) $\{-12, -3\}$ 20) a) 0, 6 b) $-\dfrac{2}{7}$ 21) $\dfrac{3(c-2)}{c-6}$

22) $\dfrac{2n}{4-n}$ 23) $\dfrac{3(y-5)}{y(y+5)}$ 24) $\dfrac{r-6}{r-11}$ 25) $\{6\}$

Chapter 9

Section 9.1

1) Answers may vary. 3) $|x| = 9$, may vary 5) $\{-6, 6\}$

7) $\{2, 8\}$ 9) $\left\{-\frac{1}{2}, 3\right\}$ 11) $\left\{-\frac{1}{2}, -\frac{1}{3}\right\}$ 13) $\left\{-1, \frac{5}{4}\right\}$

15) $\{-24, 15\}$ 17) $\{0, 12\}$ 19) $\left\{\frac{28}{15}, \frac{52}{15}\right\}$

21) $|x| = \frac{1}{2}$, may vary 23) $\{-10, 22\}$ 25) $\left\{-\frac{16}{5}, 2\right\}$

27) $\left\{1, \frac{8}{3}\right\}$ 29) $\left\{\frac{1}{4}, \frac{3}{2}\right\}$ 31) $\{-12, 0\}$ 33) $\left\{-\frac{14}{3}, 4\right\}$

35) $\left\{\frac{1}{2}, 4\right\}$ 37) $\left\{\frac{2}{5}, 2\right\}$ 39) $\{-4, -1\}$ 41) $\{10\}$

43) $\{-5, 1.2\}$ 45) $\varnothing$ 47) $\left\{-\frac{1}{5}\right\}$ 49) $\{-14\}$ 51) $\varnothing$

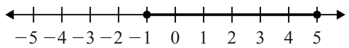
53) $\varnothing$ 55) $\{0, 14\}$ 57) $\left\{-\frac{25}{3}, -\frac{5}{3}\right\}$ 59) $\left\{\frac{3}{10}\right\}$

61) $\{-2, 2.4\}$ 63) $\varnothing$ 65) $\left\{\frac{16}{3}, \frac{20}{3}\right\}$ 67) $\left\{-\frac{39}{8}, \frac{33}{20}\right\}$

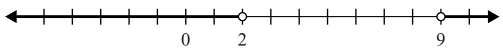
69) $\{-1.25\}$

Section 9.2

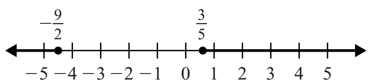
1) $[-1, 5]$



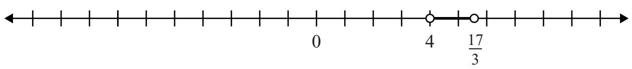
3) $(-\infty, 2) \cup (9, \infty)$



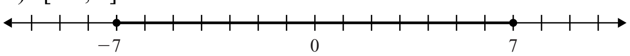
5) $\left(-\infty, -\frac{9}{2}\right] \cup \left[\frac{3}{5}, \infty\right)$



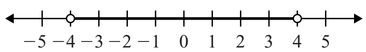
7) $\left(4, \frac{17}{3}\right)$



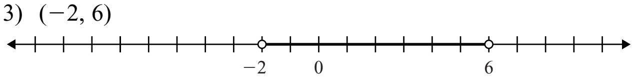
9) $[-7, 7]$



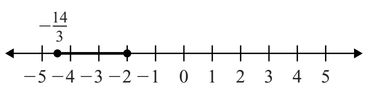
11) $(-4, 4)$



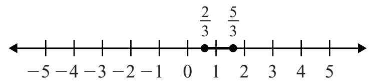
13) $(-2, 6)$



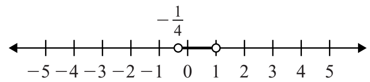
15) $\left[-\frac{14}{3}, -2\right]$



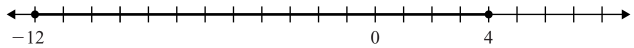
17) $\left[\frac{2}{3}, \frac{5}{3}\right]$



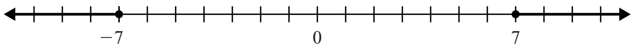
19) $\left(-\frac{1}{4}, 1\right)$



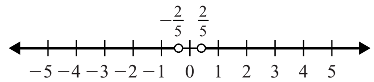
21) $[-12, 4]$



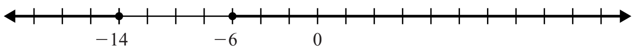
23) $(-\infty, -7] \cup [7, \infty)$



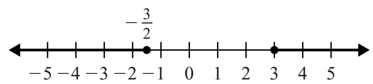
25) $\left(-\infty, -\frac{2}{5}\right) \cup \left(\frac{2}{5}, \infty\right)$



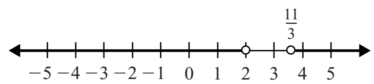
27) $(-\infty, -14] \cup [-6, \infty)$



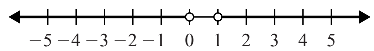
29) $\left(-\infty, -\frac{3}{2}\right] \cup [3, \infty)$



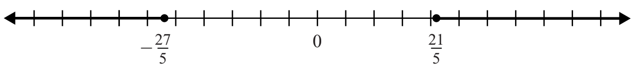
31) $(-\infty, 2) \cup \left(\frac{11}{3}, \infty\right)$



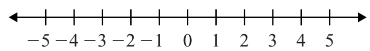
33) $(-\infty, 0) \cup (1, \infty)$



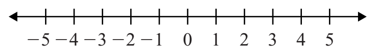
35) $\left(-\infty, -\frac{27}{5}\right] \cup \left[\frac{21}{5}, \infty\right)$



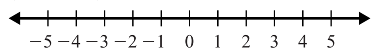
37) $\varnothing$



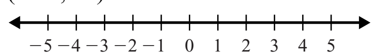
39) $\varnothing$



41) $(-\infty, \infty)$



43) $(-\infty, \infty)$



45) The absolute value of a quantity is always 0 or positive; it cannot be less than 0.

47) The absolute value of a quantity is always 0 or positive, so for any real number, x, the quantity $|2x + 1|$ will be greater than -3.

49) $(-\infty, -6) \cup (-3, \infty)$ 51) $\left\{-2, -\frac{1}{2}\right\}$

53) $\left(-\infty, -\frac{1}{4}\right] \cup [2, \infty)$ 55) $(-3, \infty)$ 57) $[3, 13]$

59) $\varnothing$ 61) $\{-21, -3\}$ 63) $(-\infty, 0) \cup \left(\frac{16}{3}, \infty\right)$

65) $\varnothing$ 67) $\left(-\infty, -\frac{1}{25}\right]$ 69) $(-\infty, \infty)$ 71) $[-15, -1]$

73) $\left(-\infty, \frac{1}{5}\right) \cup (3, \infty)$ 75) $\{-4, 20\}$ 77) $\left[-\frac{11}{5}, 1\right]$

79) $|a - 128| \le 0.75$; $127.25 \le a \le 128.75$; there is between 127.25 oz and 128.75 oz of milk in the container.

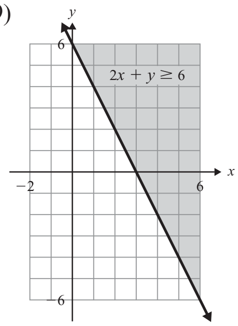
81) $|b - 38| \le 5$; $33 \le b \le 43$; he will spend between \$33 and \$43 on his daughter's gift.

Section 9.3

1) Answers may vary. 3) Answers may vary.

5) Answers may vary. 7) dotted

9)



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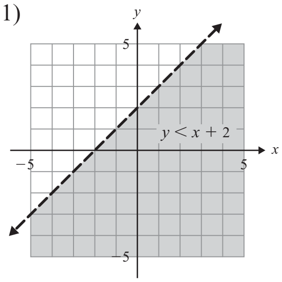

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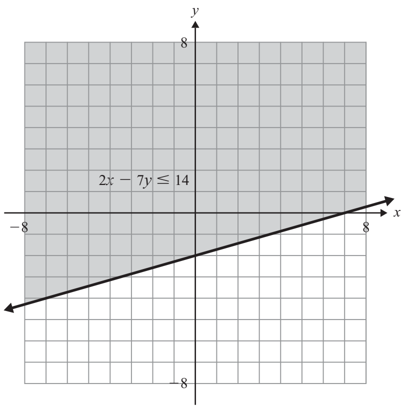

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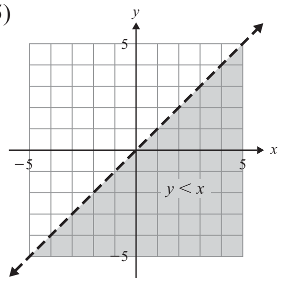

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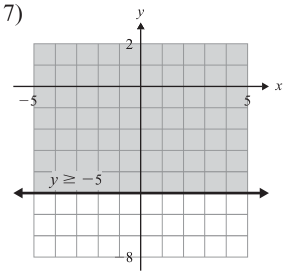
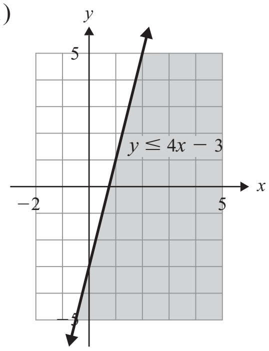


19) below

21)



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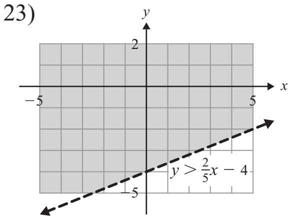

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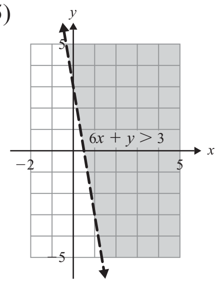

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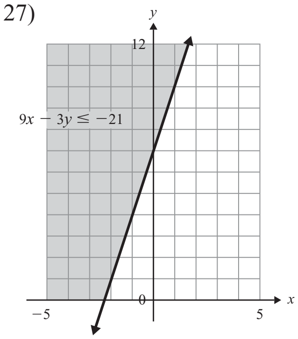

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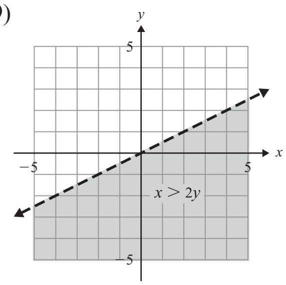
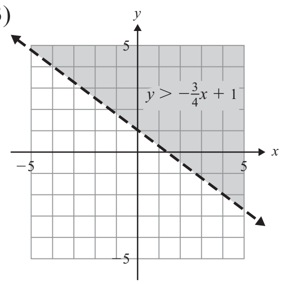


31) Answers may vary.

33)



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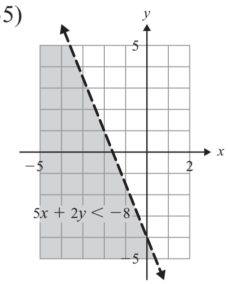
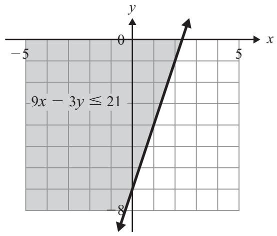
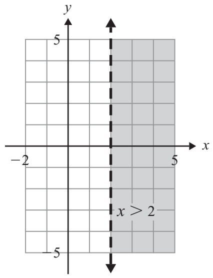
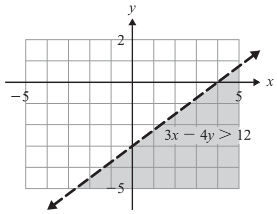
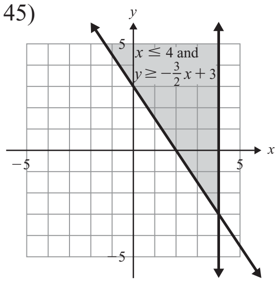


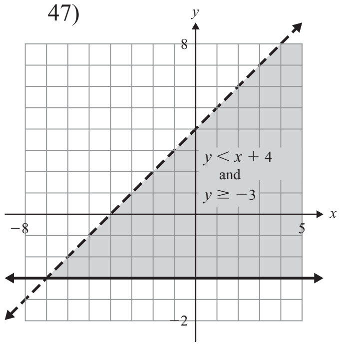
37) 

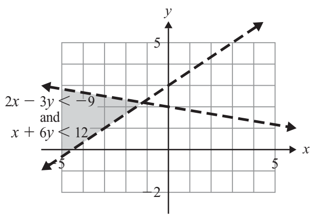
39) 

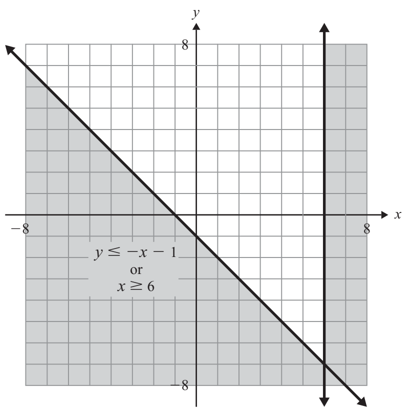
41) 

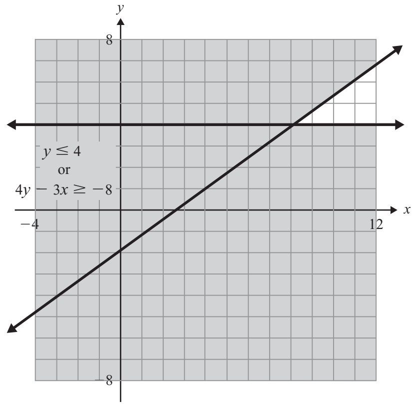
43) No; (3, 5) satisfies $x - y \ge -6$ but not $2x + y < 7$. Since the inequality contains *and*, it must satisfy *both* inequalities.

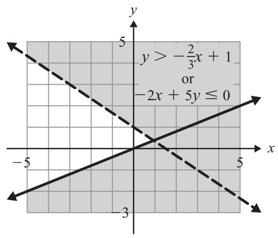
45) 

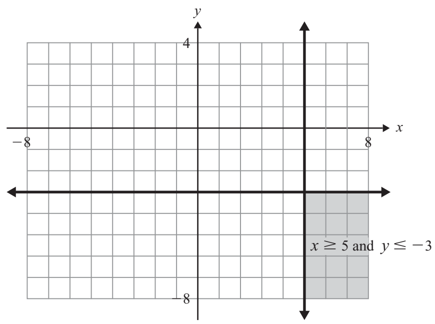
47) 

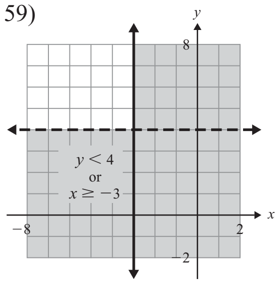
49) 

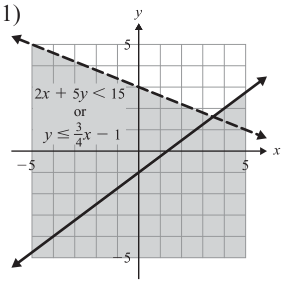
51) 

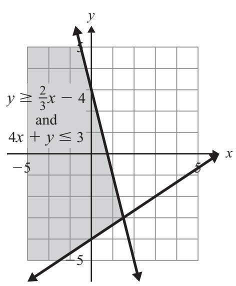
53) 

55) 

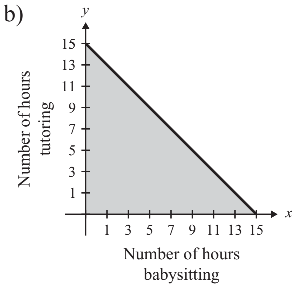
57) 

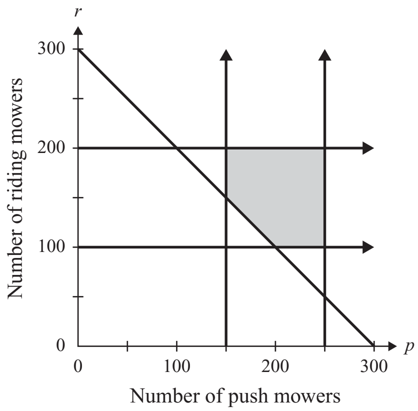
59) 

61) 

63) 

65) a) $x \geq 0$
$y \geq 0$
$x + y \leq 15$
c) Answers may vary.
d) Answers may vary.

b) 

67) a) $150 \leq p \leq 250$ and $100 \leq r \leq 200$ and $p + r \geq 300$
b) 

c) It represents the production of 175 push mowers and 110 riding mowers per day. This does not meet the level of production needed because it is not a total of at least 300 mowers per day and is not in the feasible region.
d) Answers may vary. e) Answers may vary.

Section 9.4

1) $\left[\begin{array}{cc|c} 1 & -7 & 15 \\ 4 & 3 & -1 \end{array}\right]$

3) $\left[\begin{array}{ccc|c} 1 & 6 & -1 & -2 \\ 3 & 1 & 4 & 7 \\ -1 & -2 & 3 & 8 \end{array}\right]$

5) $3x + 10y = -4$
$x - 2y = 5$

7) $x - 6y = 8$
$y = -2$

9) $x - 3y + 2z = 7$
$4x - y + 3z = 0$
$-2x + 2y - 3z = -9$

11) $x + 5y + 2z = 14$
$y - 8z = 2$
$z = -3$

13) $(3, -1)$ 15) $(-6, -5)$ 17) $(0, -2)$ 19) $(-1, 4, 8)$

21) $(10, 1, -4)$ 23) $(0, 1, 8)$ 25) $\varnothing$; inconsistent

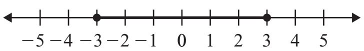
27) $(-5, 2, 1, -1)$ 29) $(3, 0, -2, 1)$

Chapter 9 Review Exercises

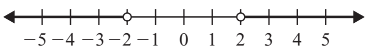
1) $\{-9, 9\}$ 3) $\left\{-1, \frac{1}{7}\right\}$ 5) $\left\{-\frac{15}{8}, -\frac{7}{8}\right\}$ 7) $\left\{\frac{11}{5}, \frac{13}{5}\right\}$

9) $\left\{-8, \frac{4}{15}\right\}$ 11) $\varnothing$ 13) $\left\{-\frac{4}{9}\right\}$ 15) $|a| = 4$, may vary

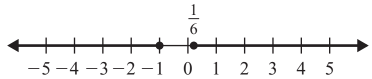
17) $[-3, 3]$



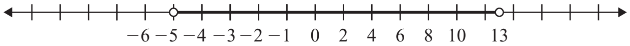
19) $(-\infty, -2) \cup (2, \infty)$



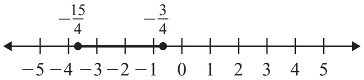
21) $(-\infty, -1] \cup \left[\frac{1}{6}, \infty\right)$



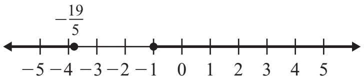
23) $(-5, 13)$



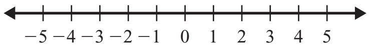
25) $\left[-\frac{15}{4}, -\frac{3}{4}\right]$



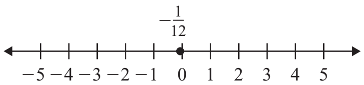
27) $\left(-\infty, -\frac{19}{5}\right] \cup [-1, \infty)$

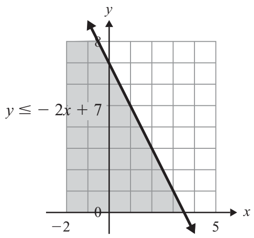


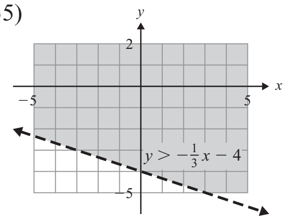
29) $(-\infty, \infty)$

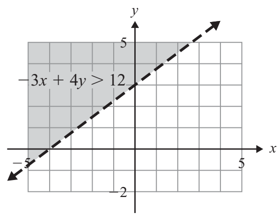


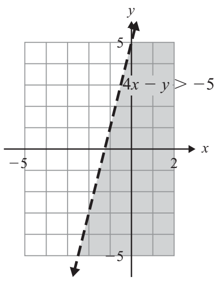
31) $\left\{-\frac{1}{12}\right\}$

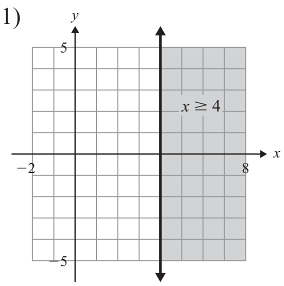


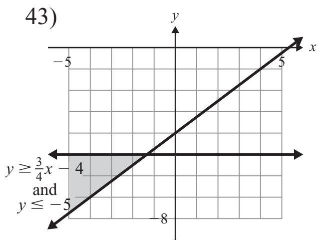
33) 

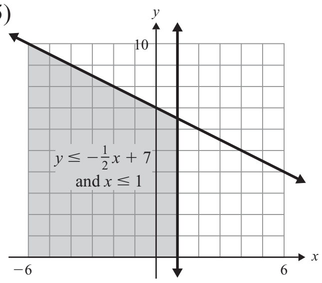
35) 

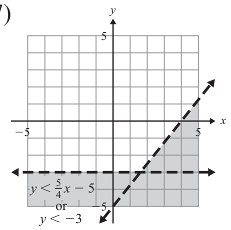
37) 

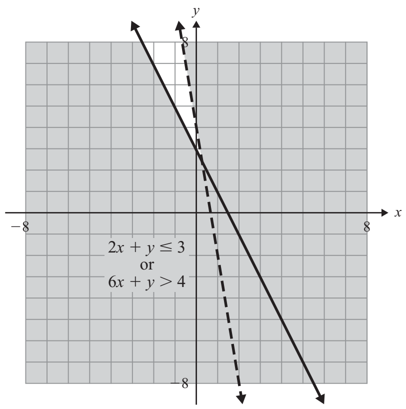
39) 

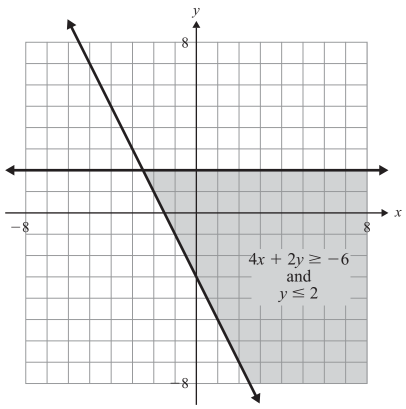
41) 

43) 

45) 

47) 

49) 

51) 

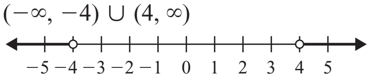
53) $(-9, 2)$ 55) $(1, 0)$ 57) $(5, -2, 6)$

Chapter 9 Test

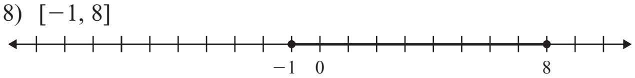
1) $\left\{-\frac{1}{2}, 5\right\}$ 2) $\{-16, 48\}$ 3) $\left\{-8, \frac{3}{2}\right\}$ 4) $\varnothing$

5) $|x| = 8$, may vary 6) The absolute value of a quantity is always greater than or equal to zero, so for any real number a, the quantity $|0.8a + 1.3| \ge 0$.

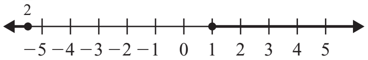
7) $(-\infty, -4) \cup (4, \infty)$

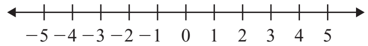


8) $[-1, 8]$

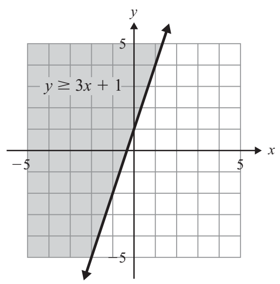


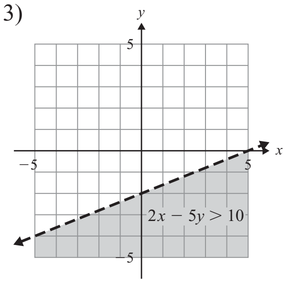
9) $\left(-\infty, -\frac{11}{2}\right] \cup [1, \infty)$

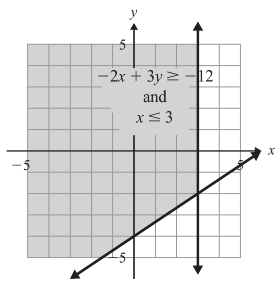


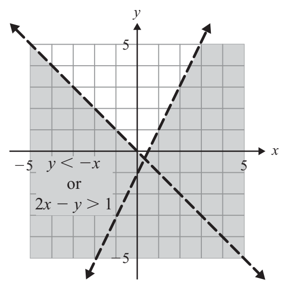
10) $\varnothing$ 

11) $|w - 168| \le 0.75$; $167.25 \le w \le 168.75$; Thanh's weight is between 167.25 lb and 168.75 lb.

12) 

13) 

14) 

15) 

16) $(6, -2)$ 17) $(1, -1, 1)$

Cumulative Review Chapters 1–9

1) 26 2) $-\frac{11}{24}$ 3) 81 4) 32 5) $\frac{1}{64}$ 6) $\frac{1}{64}$

7) 9.14×10^{-6} 8) $\left\{\frac{11}{5}\right\}$ 9) $(-\infty, -21]$ 10) 4 oz

11) $y = \frac{1}{3}x - \frac{1}{3}$ 12) $(1, -3)$ 13) $-12p^4 + 28p^3 + 4p^2$

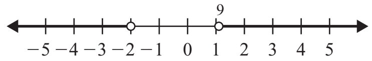
14) $4k^2 - 25$ 15) $t^2 + 16t + 64$ 16) $2c^2 + 5c - 6$

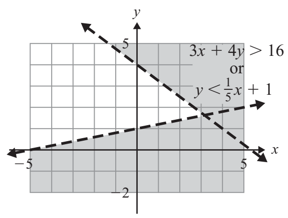
17) $(3m + 11)(3m - 11)$ 18) $(z - 6)(z - 8)$ 19) $\{-3\}$

20) $\left\{-4, \frac{1}{2}\right\}$ 21) $\frac{-r^2 + 2r + 17}{2(r + 5)(r - 5)}$ 22) $\frac{w - 9}{w(w - 8)}$

23) $\{16, 40\}$

24) $(-\infty, -2) \cup \left(\frac{10}{9}, \infty\right)$



25) 

Chapter 10

Section 10.1

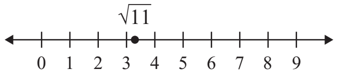
1) False; the $\sqrt{\ }$ symbol means to find only the positive square root of 121. $\sqrt{121} = 11$

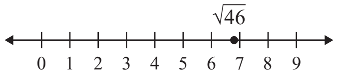
3) False; the square root of a negative number is not a real number.

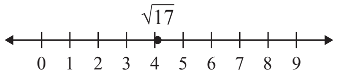
5) 7 and -7 7) 1 and -1 9) 30 and -30 11) $\frac{2}{3}$ and $-\frac{2}{3}$

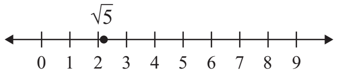
13) $\frac{1}{9}$ and $-\frac{1}{9}$ 15) 0.5 and -0.5 17) 7 19) 1 21) 13

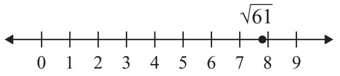
23) not real 25) $\frac{9}{5}$ 27) -6 29) $-\frac{1}{11}$ 31) 0.2

33) 3.3


35) 6.8


37) 4.1


39) 2.2


41) 7.8


43) true

45) False; the odd root of a negative number is a negative number.

47) $\sqrt[3]{64}$ is the number you cube to get 64. $\sqrt[3]{64} = 4$

49) No; the even root of a negative number is not a real number.

51) 2 53) 5 55) -1 57) 3 59) not real 61) -2

63) -2 65) 3 67) not real 69) $\frac{2}{5}$ 71) 7 73) 5

75) not real 77) 13

79) If a is negative and we didn't use the absolute values, the result would be negative. This is incorrect because if a is negative and n is even, then $a^n > 0$ so that $\sqrt[n]{a^n} > 0$. Using absolute values ensures a positive result.

81) 8 83) 6 85) $|y|$ 87) $|m|$ 89) 5 91) -4 93) z

95) $|h|$ 97) p 99) $|x + 7|$ 101) $2t - 1$ 103) $|3n + 2|$

105) $d - 8$

Section 10.2

1) The denominator of 2 becomes the index of the radical. $25^{1/2} = \sqrt{25}$

3) 3 5) 10 7) 2 9) -5 11) $\frac{2}{11}$ 13) $\frac{5}{4}$ 15) $-\frac{6}{13}$

17) not a real number 19) -1

21) The denominator of 4 becomes the index of the radical. The numerator of 3 is the power to which we raise the radical expression. $16^{3/4} = (\sqrt[4]{16})^3$

23) 16 25) 32 27) 25 29) -216 31) not a real number

33) $\frac{8}{27}$ 35) $-\frac{100}{9}$

37) False; the negative exponent does not make the result negative. $81^{-1/2} = \frac{1}{9}$

39) $\frac{1}{64}$; $\frac{1}{64}$; The denominator of the fractional exponent is the index of the radical; $\frac{1}{8}$

41) $\frac{1}{7}$ 43) $\frac{1}{10}$ 45) 3 47) -4 49) $\frac{1}{32}$ 51) $\frac{1}{25}$ 53) $\frac{8}{125}$

55) $\frac{25}{16}$ 57) 8 59) 3 61) $8^{4/5}$ 63) 32 65) $\frac{1}{4^{7/5}}$ 67) z

69) $-72v^{11/8}$ 71) $a^{1/9}$ 73) $\frac{5}{18c^{3/2}}$ 75) $\frac{1}{x^{2/3}}$ 77) $z^{2/15}$

79) $27u^2v^3$ 81) $8r^{1/5}s^{4/15}$ 83) $\frac{f^{2/7}g^{5/9}}{3}$ 85) $x^{10}w^9$

87) $\frac{1}{y^{2/3}}$ 89) $\frac{a^{12/5}}{4b^{2/5}}$ 91) $\frac{t^{21/2}}{r^{1/5}}$ 93) $\frac{1}{h^5k^{25/18}}$ 95) $p^{7/6} + p$

97) $25^{6/12}$; $25^{1/2}$; Evaluate. 99) 7 101) 9 103) 5

105) 12 107) x^4 109) $\sqrt[3]{k}$ 111) $\sqrt{z}$ 113) d^2

115) a) 13 degrees b) 6 degrees

Section 10.3

1) $\sqrt{21}$ 3) $\sqrt{30}$ 5) $\sqrt{6y}$

7) False; 20 contains a factor of 4, which is a perfect square.

9) True; 42 does not have any factors (other than 1) that are perfect squares.

11) Factor; $\sqrt{4} \cdot \sqrt{15}$; $2\sqrt{15}$ 13) $2\sqrt{5}$ 15) $3\sqrt{6}$

17) simplified 19) $4\sqrt{5}$ 21) $7\sqrt{2}$ 23) simplified 25) 20

27) $5\sqrt{30}$ 29) $\frac{12}{5}$ 31) $\frac{2}{7}$ 33) 3 35) $2\sqrt{3}$ 37) $2\sqrt{5}$

39) $\sqrt{15}$ 41) $\frac{\sqrt{6}}{7}$ 43) $\frac{3\sqrt{5}}{4}$ 45) x^4 47) w^7 49) $10c$

51) $8k^3m^5$ 53) $2r^2\sqrt{7}$ 55) $10q^{11}t^8\sqrt{3}$ 57) $\frac{9}{c^3}$ 59) $\frac{2\sqrt{10}}{t^4}$

61) $\frac{5x\sqrt{3}}{y^6}$ 63) Factor; $\sqrt{w^8} \cdot \sqrt{w^1}$; Simplify. 65) $a^2\sqrt{a}$

67) $g^6\sqrt{g}$ 69) $b^{12}\sqrt{b}$ 71) $6x\sqrt{2x}$ 73) $q^3\sqrt{13q}$

75) $5t^5\sqrt{3t}$ 77) c^4d 79) $a^2b\sqrt{b}$ 81) $u^2v^3\sqrt{uv}$

83) $6m^4n^2\sqrt{m}$ 85) $2x^6y^2\sqrt{11y}$ 87) $4t^2u^3\sqrt{2tu}$ 89) $\dfrac{a^3\sqrt{a}}{9b^3}$

91) $\dfrac{r^4\sqrt{3r}}{s}$ 93) $5\sqrt{2}$ 95) $3\sqrt{7}$ 97) w^3 99) $n^3\sqrt{n}$

101) $4k^3$ 103) $2x^4y^2\sqrt{3xy}$ 105) $2c^5d^4\sqrt{10d}$ 107) $3k^4$

109) $2h^3\sqrt{10}$ 111) $a^4b^2\sqrt{10ab}$ 113) 20 m/sec

Section 10.4

1) Answers may vary.

3) i) Its radicand will not contain any factors that are perfect cubes.
ii) The radicand will not contain fractions.
iii) There will be no radical in the denominator of a fraction.

5) $\sqrt[3]{20}$ 7) $\sqrt[5]{9m^2}$ 9) $\sqrt[3]{a^2b}$ 11) Factor; $\sqrt[3]{8}\cdot\sqrt[3]{7}$; $2\sqrt[3]{7}$

13) $2\sqrt[3]{3}$ 15) $2\sqrt[4]{4}$ 17) $3\sqrt[3]{2}$ 19) $10\sqrt[3]{2}$ 21) $2\sqrt[5]{2}$

23) $\dfrac{1}{2}$ 25) -3 27) $2\sqrt[3]{3}$ 29) $2\sqrt[4]{5}$ 31) d^2 33) n^5

35) xy^3 37) $w^4\sqrt[3]{w^2}$ 39) $y^2\sqrt[4]{y}$ 41) $d\sqrt[3]{d^2}$ 43) $u^3v^5\sqrt[3]{u}$

45) $b^5c\sqrt[3]{bc^2}$ 47) $n^4\sqrt[4]{m^3n^2}$ 49) $2x^3y^4\sqrt[3]{3x}$

51) $5wx^5\sqrt[3]{2wx}$ 53) $\dfrac{m^2}{3}$ 55) $\dfrac{2a^4\sqrt[5]{a^3}}{b^3}$ 57) $\dfrac{t^2\sqrt[4]{t}}{3s^6}$

59) $\dfrac{u^9\sqrt[3]{u}}{v}$ 61) $2\sqrt[3]{3}$ 63) $3\sqrt[3]{4}$ 65) $2\sqrt[3]{10}$ 67) m^3

69) k^4 71) $r^3\sqrt[3]{r^2}$ 73) $p^4\sqrt[5]{p^3}$ 75) $3z^6\sqrt[3]{z}$ 77) h^4

79) $c^2\sqrt[3]{c}$ 81) $3d^4\sqrt[4]{d^3}$

83) Change radicals to fractional exponents; Rewrite exponents with a common denominator; $a^{5/4}$; Rewrite in radical form; $a\sqrt[4]{a}$

85) $\sqrt[6]{p^5}$ 87) $n\sqrt[4]{n}$ 89) $c\sqrt[15]{c^4}$ 91) $\sqrt[4]{w}$ 93) $\sqrt[15]{t^2}$

95) 4 in.

Section 10.5

1) They have the same index and the same radicand.

3) $14\sqrt{2}$ 5) $15\sqrt[3]{4}$ 7) $11-3\sqrt{13}$ 9) $-5\sqrt[3]{z^2}$

11) $-9\sqrt[3]{n^2}+10\sqrt[5]{n^2}$ 13) $2\sqrt{5c}-2\sqrt{6c}$

15) i) Write each radical expression in simplest form.
ii) Combine like radicals.

17) Factor; $\sqrt{16}\cdot\sqrt{3}+\sqrt{3}$; Simplify; $5\sqrt{3}$ 19) $4\sqrt{3}$

21) $-2\sqrt{2}$ 23) $6\sqrt{3}$ 25) $10\sqrt[3]{9}$ 27) $-\sqrt[3]{6}$ 29) $13q\sqrt{q}$

31) $-20d^2\sqrt{d}$ 33) $4t^3\sqrt[3]{t}$ 35) $6a^2\sqrt[4]{a^3}$ 37) $-2\sqrt{2p}$

39) $25a\sqrt[3]{3a^2}$ 41) $4y\sqrt{xy}$ 43) $3c^2d\sqrt{2d}$ 45) $20a^5\sqrt[3]{7a^2b}$

47) $14cd\sqrt[4]{9cd}$ 49) $\sqrt[3]{b}(a^3-b^2)$ 51) $3x+15$

53) $7\sqrt{6}+14$ 55) $\sqrt{30}-\sqrt{10}$ 57) $-30\sqrt{2}$ 59) $4\sqrt{5}$

61) $-\sqrt{30}$ 63) $5\sqrt[4]{3}-3$ 65) $t-9\sqrt{tu}$

67) $a\sqrt{5b}+3b\sqrt{3a}$ 69) $c\sqrt[3]{c}+5c\sqrt[3]{d}$

71) Both are examples of multiplication of two binomials. They can be multiplied using FOIL.

73) $(a+b)(a-b)=a^2-b^2$ 75) $p^2+13p+42$

77) $6\cdot 2+6\sqrt{7}+2\sqrt{7}+\sqrt{7}\cdot\sqrt{7}$; Multiply; $19+8\sqrt{7}$

79) $-22+5\sqrt{2}$ 81) $22-9\sqrt{15}$

83) $5\sqrt{7}+5\sqrt{2}+2\sqrt{21}+2\sqrt{6}$

85) $5-\sqrt[3]{150}-3\sqrt[3]{5}+3\sqrt[3]{6}$ 87) $-2\sqrt{6pq}+30p-16q$

89) $4+2\sqrt{3}$ 91) $16-2\sqrt{55}$ 93) $h+2\sqrt{7h}+7$

95) $x-2\sqrt{xy}+y$ 97) c^2-81 99) 31 101) 46

103) $\sqrt[3]{4}-9$ 105) $c-d$ 107) $64f-g$ 109) 41

111) $11+4\sqrt{7}$ 113) $13-4\sqrt{3}$

Section 10.6

1) Eliminate the radical from the denominator.

3) $\dfrac{\sqrt{5}}{5}$ 5) $\dfrac{3\sqrt{6}}{2}$ 7) $-5\sqrt{2}$ 9) $\dfrac{\sqrt{21}}{14}$ 11) $\dfrac{\sqrt{3}}{3}$

13) $\dfrac{\sqrt{42}}{6}$ 15) $\dfrac{\sqrt{30}}{3}$ 17) $\dfrac{\sqrt{15}}{10}$ 19) $\dfrac{8\sqrt{y}}{y}$ 21) $\dfrac{\sqrt{5t}}{t}$

23) $\dfrac{8v^3\sqrt{5vw}}{5w}$ 25) $\dfrac{a\sqrt{3b}}{3b}$ 27) $-\dfrac{5\sqrt{3b}}{b^2}$ 29) $\dfrac{\sqrt{13j}}{j^3}$

31) 2^2 or 4 33) 3 35) c^2 37) 2^3 or 8 39) m

41) $\dfrac{4\sqrt[3]{9}}{3}$ 43) $6\sqrt[3]{4}$ 45) $\dfrac{9\sqrt[3]{5}}{5}$ 47) $\dfrac{\sqrt[4]{45}}{3}$ 49) $\dfrac{\sqrt[5]{12}}{2}$

51) $\dfrac{10\sqrt[3]{z^2}}{z}$ 53) $\dfrac{\sqrt[3]{3n}}{n}$ 55) $\dfrac{\sqrt[3]{28k}}{2k}$ 57) $\dfrac{9\sqrt[5]{a^2}}{a}$

59) $\dfrac{\sqrt[4]{40m^3}}{2m}$ 61) Change the sign between the two terms.

63) $(5-\sqrt{2})$; 23 65) $(\sqrt{2}-\sqrt{6})$; -4

67) $(\sqrt{t}+8)$; $t-64$

69) Multiply by the conjugate; $(a+b)(a-b)=a^2-b^2$; $\dfrac{24+6\sqrt{5}}{16-5}$; $\dfrac{24+6\sqrt{5}}{11}$

71) $6-3\sqrt{3}$ 73) $\dfrac{90+10\sqrt{2}}{79}$ 75) $2\sqrt{6}-4$

77) $\dfrac{\sqrt{30}-5\sqrt{2}+3-\sqrt{15}}{7}$ 79) $\dfrac{m-\sqrt{mn}}{m-n}$

81) $\sqrt{b}+5$ 83) $\dfrac{x+2\sqrt{xy}+y}{x-y}$ 85) $\dfrac{5}{3\sqrt{5}}$ 87) $\dfrac{x}{\sqrt{7x}}$

89) $\dfrac{1}{12-6\sqrt{3}}$ 91) $\dfrac{1}{\sqrt{x}+2}$ 93) $-\dfrac{1}{4+\sqrt{c+11}}$

95) No, because when we multiply the numerator and denominator by the conjugate of the denominator, we are multiplying the original expression by 1.

97) $1+2\sqrt{3}$ 99) $\dfrac{15-9\sqrt{5}}{2}$ 101) $\dfrac{\sqrt{5}+2}{3}$

103) $-2-\sqrt{2}$

105) a) $r(8\pi) = 2\sqrt{2}$; When the area of a circle is 8π in^2, its radius is $2\sqrt{2}$ in.
b) $r(7) = \dfrac{\sqrt{7\pi}}{\pi}$; When the area of a circle is 7 in^2, its radius is $\dfrac{\sqrt{7\pi}}{\pi}$ in. (This is approximately 1.5 in.)
c) $r(A) = \sqrt{\dfrac{A\pi}{\pi}}$

Chapter 10 Putting It All Together

1) 3 2) -10 3) -2 4) 11 5) not a real number 6) $\dfrac{12}{7}$

7) 12 8) 16 9) -100 10) not a real number 11) $\dfrac{1}{5}$

12) $\dfrac{27}{1000}$ 13) $\dfrac{1}{k^{3/10}}$ 14) t^6 15) $\dfrac{9}{a^{16/3}b^6}$ 16) $\dfrac{x^{30}}{243y^{5/6}}$

17) $2\sqrt{6}$ 18) $2\sqrt[4]{2}$ 19) $2\sqrt[3]{9}$ 20) $5\sqrt[3]{2}$ 21) $3\sqrt[4]{3}$

22) $3t^5\sqrt{5t}$ 23) $2m^2n^5\sqrt[3]{12m}$ 24) $\dfrac{2x^3\sqrt[5]{2x^4}}{y^4}$ 25) $2\sqrt[3]{3}$

26) $2k^2\sqrt[4]{3}$ 27) $19 + 8\sqrt{7}$ 28) $-18c^2\sqrt[3]{4c}$ 29) $3\sqrt{6}$

30) $5\sqrt{3} + 5\sqrt{2}$ 31) $17m\sqrt{3mn}$ 32) $3p^4q^2\sqrt{10q}$

33) $2t^3u\sqrt{3}$ 34) $\dfrac{9\sqrt[3]{4}}{2}$ 35) $112 + 40\sqrt{3}$ 36) -7

37) $\dfrac{2\sqrt{2} - \sqrt{5}}{3}$ 38) $r\sqrt[6]{r}$ 39) $\dfrac{\sqrt[3]{3b^2c^2}}{3c}$ 40) $\dfrac{2\sqrt[4]{2w}}{w^3}$

Section 10.7

1) Sometimes there are extraneous solutions.

3) $\{49\}$ 5) $\left\{\dfrac{4}{9}\right\}$ 7) $\varnothing$ 9) $\{20\}$ 11) $\varnothing$ 13) $\left\{\dfrac{2}{3}\right\}$

15) $\{2\}$ 17) $\{5\}$ 19) $n^2 + 10n + 25$ 21) $c^2 - 12c + 36$

23) $\{-3\}$ 25) $\varnothing$ 27) $\{1, 3\}$ 29) $\{4\}$ 31) $\{1, 16\}$

33) $\{-3\}$ 35) $\{10\}$ 37) $\{-3\}$ 39) $\{10\}$ 41) $\{63\}$

43) $\varnothing$ 45) $x + 10\sqrt{x} + 25$ 47) $85 - 18\sqrt{a+4} + a$

49) $12n + 28\sqrt{3n-1} + 45$ 51) $\{5, 13\}$ 53) $\{2\}$

55) $\left\{\dfrac{1}{4}\right\}$ 57) $\{1, 5\}$ 59) $\varnothing$ 61) $\{2, 11\}$

63) Raise both sides of the equation to the third power.

65) $\{125\}$ 67) $\{-64\}$ 69) $\left\{-\dfrac{3}{2}\right\}$ 71) $\{-1\}$ 73) $\{-2\}$

75) $\left\{-\dfrac{1}{2}, 4\right\}$ 77) $\{36\}$ 79) $\{26\}$ 81) $\{23\}$ 83) $\{9\}$

85) $\{9\}$ 87) $\{-1\}$ 89) $E = \dfrac{mv^2}{2}$ 91) $b^2 = c^2 - a^2$

93) $\sigma = \dfrac{E}{T^4}$

95) a) 320 m/sec b) 340 m/sec
c) The speed of sound increases. d) $T = \dfrac{V_s^2}{400} - 273$

97) a) 2 in. b) $V = \pi r^2 h$

99) a) 463 mph b) about 8 minutes

101) 16 ft 103) 5 mph

Section 10.8

1) false 3) true 5) $9i$ 7) $5i$ 9) $i\sqrt{6}$ 11) $3i\sqrt{3}$

13) $2i\sqrt{15}$

15) Write each radical in terms of i *before* multiplying.

$$\begin{aligned}\sqrt{-5} \cdot \sqrt{-10} &= i\sqrt{5} \cdot i\sqrt{10} \\ &= i^2\sqrt{50} \\ &= (-1)\sqrt{25} \cdot \sqrt{2} \\ &= -5\sqrt{2}\end{aligned}$$

17) $-\sqrt{5}$ 19) -6 21) 2 23) -13

25) Add the real parts and add the imaginary parts.

27) -1 29) $3 + 11i$ 31) $4 - 9i$ 33) $-\dfrac{1}{4} - \dfrac{5}{6}i$ 35) $7i$

37) $24 - 15i$ 39) $-6 + \dfrac{4}{3}i$ 41) $-36 + 30i$

43) $-28 + 17i$ 45) $14 + 18i$ 47) $36 - 42i$ 49) $\dfrac{3}{20} + \dfrac{9}{20}i$

51) conjugate: $11 - 4i$; product: 137

53) conjugate: $-3 + 7i$; product: 58

55) conjugate: $-6 - 4i$; product: 52

57) Answers may vary.

59) $\dfrac{8}{13} + \dfrac{12}{13}i$ 61) $\dfrac{8}{17} + \dfrac{32}{17}i$ 63) $\dfrac{7}{29} - \dfrac{3}{29}i$

65) $-\dfrac{74}{85} + \dfrac{27}{85}i$ 67) $-\dfrac{8}{61} + \dfrac{27}{61}i$ 69) $-9i$

71) $(i^2)^{12}$; $i^2 = -1$; 1 73) 1 75) 1 77) i 79) $-i$

81) $-i$ 83) -1 85) $32i$ 87) -1 89) $142 - 65i$

91) $1 + 2i\sqrt{2}$ 93) $8 - 3i\sqrt{5}$ 95) $-3 + i\sqrt{2}$

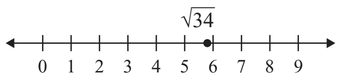
97) $Z = 10 + 6j$ 99) $Z = 16 + 4j$

Chapter 10 Review Exercises

1) 5 3) -9 5) 4 7) -1 9) not real 11) 13

13) $|p|$ 15) h

17) 5.8



19) The denominator of the fractional exponent becomes the index on the radical. The numerator is the power to which we raise the radical expression. $8^{2/3} = (\sqrt[3]{8})^2$

21) 6 23) $\dfrac{3}{5}$ 25) 8 27) $\dfrac{1}{9}$ 29) $\dfrac{1}{27}$ 31) $\dfrac{100}{9}$ 33) 9

35) 64 37) 1 39) $32a^{10/3}b^{10}$ 41) $\dfrac{2c}{3d^{7/4}}$ 43) 3 45) 7

47) k^7 49) w^3 51) $10\sqrt{10}$ 53) $\dfrac{3\sqrt{2}}{7}$ 55) k^6 57) $x^4\sqrt{x}$

59) $3t\sqrt{5}$ 61) $6x^3y^6\sqrt{2xy}$ 63) $\sqrt{15}$ 65) $2\sqrt{6}$

67) $11x^6\sqrt{x}$ 69) $10k^8$ 71) $2\sqrt[3]{2}$ 73) $2\sqrt[4]{3}$ 75) z^6

77) $a^6\sqrt[3]{a^2}$ 79) $2z^5\sqrt[3]{2}$ 81) $\dfrac{h^3}{3}$ 83) $\sqrt[3]{21}$ 85) $2t^4\sqrt[4]{2t}$

87) $\sqrt[6]{n^5}$ 89) $11\sqrt{5}$ 91) $6\sqrt{5} - 4\sqrt{3}$ 93) $-4p\sqrt{p}$

95) $-12d^2\sqrt{2d}$ 97) $6k\sqrt{5} + 3\sqrt{2k}$

99) $23\sqrt{2rs} + 8r + 15s$ 101) $2 + 2\sqrt{y+1} + y$

103) $\dfrac{14\sqrt{3}}{3}$ 105) $\dfrac{3\sqrt{2kn}}{n}$ 107) $\dfrac{7\sqrt[3]{4}}{2}$ 109) $\dfrac{\sqrt[3]{x^2y^2}}{y}$

111) $\dfrac{3 - \sqrt{3}}{3}$ 113) $1 - 3\sqrt{2}$ 115) $\{1\}$ 117) $\varnothing$

119) $\{-4\}$ 121) $\{2, 6\}$ 123) $V = \dfrac{1}{3}\pi r^2 h$ 125) $7i$

127) -4 129) $12 - 3i$ 131) $\dfrac{3}{10} - \dfrac{4}{3}i$ 133) $-30 + 35i$

135) $-36 - 21i$ 137) $-24 - 42i$

139) conjugate: $2 + 7i$;
product: 53

141) $\dfrac{12}{29} - \dfrac{30}{29}i$ 143) $-8i$ 145) $\dfrac{58}{37} - \dfrac{15}{37}i$ 147) -1

149) i

Chapter 10 Test

1) 12 2) -3 3) not real 4) $|w|$ 5) -19 6) 2 7) 81

8) $\dfrac{1}{7}$ 9) $\dfrac{25}{4}$ 10) $m^{5/8}$ 11) $\dfrac{5}{2a^{2/3}}$ 12) $\dfrac{y^2}{32x^{3/2}}$ 13) $5\sqrt{3}$

14) $2\sqrt[3]{6}$ 15) $2\sqrt{3}$ 16) y^3 17) p^6 18) $t^4\sqrt{t}$

19) $3m^2n^4\sqrt{7m}$ 20) $c^7\sqrt[3]{c^2}$ 21) $\dfrac{a^4b^2\sqrt[3]{a^2b}}{3}$ 22) 6

23) $z^3\sqrt[3]{z}$ 24) $2w^5\sqrt{15w}$ 25) $6\sqrt{7}$ 26) $3\sqrt{2} - 4\sqrt{3}$

27) $-14h^3\sqrt[4]{h}$ 28) $2\sqrt{3} - 5\sqrt{6}$

29) $3\sqrt{2} + 3 - 2\sqrt{10} - 2\sqrt{5}$ 30) 4

31) $2p + 5 + 4\sqrt{2p+1}$ 32) $2t - 2\sqrt{3tu}$ 33) $\dfrac{2\sqrt{5}}{5}$

34) $12 - 4\sqrt{7}$ 35) $\dfrac{\sqrt{6a}}{a}$ 36) $\dfrac{5\sqrt[3]{3}}{3}$ 37) $1 - 2\sqrt{3}$

38) $\{1\}$ 39) $\{-2\}$ 40) $\{13\}$ 41) $\{1, 5\}$

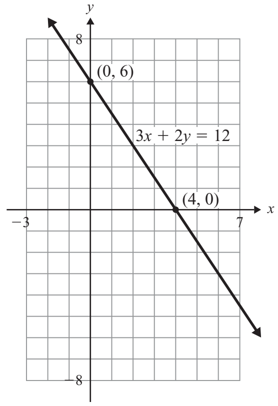
42) a) 3 in. b) $V = \pi r^2 h$ 43) $8i$ 44) $3i\sqrt{5}$ 45) $-i$

46) $-16 + 2i$ 47) $19 + 13i$ 48) $1 + 2i$

Cumulative Review for Chapters 1–10

1) $\dfrac{10}{3}x - 2y + 8$ 2) 8.723×10^6 3) $\left\{-\dfrac{3}{4}\right\}$

4)



5) $y = \dfrac{5}{4}x - \dfrac{13}{4}$ 6) $(-6, 0)$

7) $15p^4 - 20p^3 - 11p^2 + 8p + 2$ 8) $4n^2 + 2n + 1$

9) $(4w - 3)(w + 2)$ 10) $2(2 + 3t)(2 - 3t)$

11) $\left\{-\dfrac{1}{2}, \dfrac{4}{3}\right\}$ 12) $\{6, 9\}$ 13) length = 12 in., width = 7 in.

14) $\dfrac{2a^2 + 2a + 3}{a(a+4)}$ 15) $\dfrac{4n}{21m^3}$ 16) $\{-8, -4\}$

17) $(-\infty, -2] \cup \left[\dfrac{5}{3}, \infty\right)$ 18) $(4, -1, 2)$

19) a) $10\sqrt{5}$ b) $2\sqrt[3]{7}$ c) $p^5q^3\sqrt{q}$ d) $2a^3\sqrt[4]{2a^3}$

20) a) 9 b) 16 c) $\dfrac{1}{3}$ 21) $10\sqrt{3} - 6$

22) a) $\dfrac{\sqrt{10}}{5}$ b) $3\sqrt[3]{4}$ c) $\dfrac{x\sqrt[3]{y}}{y}$ d) $\dfrac{a - 2 - \sqrt{a}}{1 - a}$

23) a) $\varnothing$ b) $\{-3\}$ 24) a) $7i$ b) $2i\sqrt{14}$ c) 1

25) a) $2 + 7i$ b) $-48 + 9i$ c) $-\dfrac{11}{25} - \dfrac{2}{25}i$

Chapter 11

Section 11.1

1) $\{-7, 6\}$ 3) $\{-11, -4\}$ 5) $\{-7, 8\}$ 7) $\left\{-\dfrac{1}{10}, \dfrac{1}{10}\right\}$

9) $\left\{\dfrac{2}{5}, 4\right\}$ 11) $\left\{-\dfrac{10}{3}, -\dfrac{1}{2}\right\}$ 13) $\left\{0, \dfrac{2}{7}\right\}$ 15) quadratic

17) linear 19) quadratic 21) linear 23) $\left\{\dfrac{1}{2}, 6\right\}$

25) $\{-5, 2\}$ 27) $\left\{\dfrac{9}{2}\right\}$ 29) $\left\{-\dfrac{5}{3}, -1, 0\right\}$ 31) $\left\{-\dfrac{1}{2}, 0\right\}$

33) $\left\{\dfrac{9}{4}\right\}$ 35) $\{-4, 2\}$ 37) $\left\{-\dfrac{1}{2}\right\}$ 39) $\left\{-2, -\dfrac{3}{5}\right\}$

41) $\{-7, -2, 2\}$ 43) width = 2 in., length = 7 in.

45) width = 5 cm, length = 9 cm

47) base = 9 in.; height = 4 in.

49) base = 6 cm; height = 12 cm

51) legs = 5, 12; hypotenuse = 13

53) legs = 6, 8; hypotenuse = 10

Section 11.2

1) Methods may vary: $\{-4, 4\}$ 3) $\{-6, 6\}$

5) $\{-3\sqrt{3}, 3\sqrt{3}\}$ 7) $\left\{-\frac{2}{3}, \frac{2}{3}\right\}$ 9) $\{-2i, 2i\}$

11) $\{-i\sqrt{3}, i\sqrt{3}\}$ 13) $\{-\sqrt{14}, \sqrt{14}\}$ 15) $\{-5, 5\}$

17) $\{-2i\sqrt{3}, 2i\sqrt{3}\}$ 19) $\{-12, -8\}$ 21) $\{6, 8\}$

23) $\{-4 - 3\sqrt{2}, -4 + 3\sqrt{2}\}$ 25) $\{-3 - 5i, -3 + 5i\}$

27) $\{2 - i\sqrt{14}, 2 + i\sqrt{14}\}$ 29) $\left\{\frac{-1 - 2\sqrt{5}}{2}, \frac{-1 + 2\sqrt{5}}{2}\right\}$

31) $\left\{\frac{10 - \sqrt{14}}{3}, \frac{10 + \sqrt{14}}{3}\right\}$

33) $\left\{\frac{5}{4} - \frac{\sqrt{30}}{4}i, \frac{5}{4} + \frac{\sqrt{30}}{4}i\right\}$ 35) $\left\{-\frac{11}{6} - \frac{7}{6}i, -\frac{11}{6} + \frac{7}{6}i\right\}$

37) $\left\{8, \frac{40}{3}\right\}$ 39) $\left\{-\frac{2}{5}, \frac{6}{5}\right\}$ 41) 5 43) $\sqrt{13}$ 45) 6

47) $\sqrt{61}$ 49) $2\sqrt{5}$

51) A trinomial whose factored form is the square of a binomial; examples may vary.

53) $\frac{1}{2}(8) = 4$; $4^2 = 16$; $w^2 + 8w + 16$; $w^2 + 8w + 16$; $(w + 4)^2$

55) $a^2 + 12a + 36$; $(a + 6)^2$ 57) $c^2 - 18c + 81$; $(c - 9)^2$

59) $r^2 + 3r + \frac{9}{4}; \left(r + \frac{3}{2}\right)^2$ 61) $b^2 - 9b + \frac{81}{4}; \left(b - \frac{9}{2}\right)^2$

63) $x^2 + \frac{1}{3}x + \frac{1}{36}; \left(x + \frac{1}{6}\right)^2$

65) Divide both sides of the equation by 2. 67) $\{-4, -2\}$

69) $\{3, 5\}$ 71) $\{-5 - \sqrt{15}, -5 + \sqrt{15}\}$

73) $\{1 - 2i\sqrt{2}, 1 + 2i\sqrt{2}\}$ 75) $\{-2 - 2i, -2 + 2i\}$

77) $\{-8, 5\}$ 79) $\{3, 4\}$ 81) $\left\{\frac{1}{2} - \frac{\sqrt{13}}{2}, \frac{1}{2} + \frac{\sqrt{13}}{2}\right\}$

83) $\left\{-\frac{5}{2} - \frac{\sqrt{3}}{2}i, -\frac{5}{2} + \frac{\sqrt{3}}{2}i\right\}$ 85) $\{1 - i\sqrt{3}, 1 + i\sqrt{3}\}$

87) $\{-3 - \sqrt{11}, -3 + \sqrt{11}\}$ 89) $\{2, 3\}$

91) $\left\{\frac{5}{4} - \frac{\sqrt{39}}{4}i, \frac{5}{4} + \frac{\sqrt{39}}{4}i\right\}$ 93) $\left\{\frac{3}{4}, 1\right\}$

95) $\{-1 - \sqrt{21}, -1 + \sqrt{21}\}$ 97) $\left\{\frac{5}{4} - \frac{\sqrt{7}}{4}i, \frac{5}{4} + \frac{\sqrt{7}}{4}i\right\}$

99) 6 101) $\sqrt{29}$ 103) 6 in. 105) 12 ft 107) $-10, 4$

109) width = 9 ft, length = 17 ft

Section 11.3

1) The fraction bar should also be under $-b$:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3) You cannot divide only the -2 by 2.
$$\frac{-2 \pm 6\sqrt{11}}{2} = \frac{2(-1 \pm 3\sqrt{11})}{2} = -1 \pm 3\sqrt{11}$$

5) $\{-3, -1\}$ 7) $\left\{-2, \frac{5}{3}\right\}$ 9) $\left\{\frac{5 - \sqrt{17}}{2}, \frac{5 + \sqrt{17}}{2}\right\}$

11) $\{4 - 3i, 4 + 3i\}$ 13) $\left\{\frac{1}{5} - \frac{\sqrt{14}}{5}i, \frac{1}{5} + \frac{\sqrt{14}}{5}i\right\}$

15) $\{-7, 0\}$ 17) $\left\{\frac{-1 - \sqrt{13}}{3}, \frac{-1 + \sqrt{13}}{3}\right\}$ 19) $\left\{\frac{7}{2}, 4\right\}$

21) $\{-4 - \sqrt{31}, -4 + \sqrt{31}\}$ 23) $\left\{-\frac{2}{3} - \frac{1}{3}i, -\frac{2}{3} + \frac{1}{3}i\right\}$

25) $\{-10, 4\}$ 27) $\left\{-\frac{3}{2}i, \frac{3}{2}i\right\}$

29) $\{-3 - 5i, -3 + 5i\}$ 31) $\{-1 - \sqrt{10}, -1 + \sqrt{10}\}$

33) $\left\{\frac{3}{2}\right\}$ 35) $\left\{-\frac{3}{8} - \frac{\sqrt{7}}{8}i, -\frac{3}{8} + \frac{\sqrt{7}}{8}i\right\}$

37) $\left\{\frac{5 - \sqrt{19}}{2}, \frac{5 + \sqrt{19}}{2}\right\}$ 39) There is one rational solution.

41) -39; two nonreal, complex solutions

43) 0; one rational solution 45) 16; two rational solutions

47) 56; two irrational solutions 49) -8 or 8 51) 9 53) 4

55) 2 in., 5 in. 57) a) 2 sec b) $\frac{3 + \sqrt{33}}{4}$ sec or about 2.2 sec

Chapter 11 Putting It All Together

1) $\{-5\sqrt{2}, 5\sqrt{2}\}$ 2) $\{3 - \sqrt{17}, 3 + \sqrt{17}\}$

3) $\{-5, 4\}$ 4) $\left\{\frac{3}{4} - \frac{\sqrt{39}}{4}i, \frac{3}{4} + \frac{\sqrt{39}}{4}i\right\}$

5) $\left\{\frac{-7 - \sqrt{13}}{2}, \frac{-7 + \sqrt{13}}{2}\right\}$ 6) $\left\{-1, \frac{4}{3}\right\}$ 7) $\left\{-3, \frac{1}{4}\right\}$

8) $\{-3 - i\sqrt{6}, -3 + i\sqrt{6}\}$ 9) $\{-7 - i\sqrt{11}, -7 + i\sqrt{11}\}$

10) $\left\{\frac{-1 - \sqrt{7}}{2}, \frac{-1 + \sqrt{7}}{2}\right\}$ 11) $\left\{\frac{1}{3} - \frac{\sqrt{6}}{3}i, \frac{1}{3} + \frac{\sqrt{6}}{3}i\right\}$

12) $\{-4 - 3i, -4 + 3i\}$ 13) $\{-2, 6\}$ 14) $\{-5, 5\}$

15) $\{2 - \sqrt{7}, 2 + \sqrt{7}\}$ 16) $\{-9, -6, 0\}$ 17) $\left\{\frac{3}{2}, 2\right\}$

18) $\{0, 1\}$ 19) $\{3, 7\}$ 20) $\left\{\frac{-1 - \sqrt{21}}{4}, \frac{-1 + \sqrt{21}}{4}\right\}$

21) $\left\{\frac{1}{3} - \frac{\sqrt{2}}{3}i, \frac{1}{3} + \frac{\sqrt{2}}{3}i\right\}$ 22) $\{3 - 5i, 3 + 5i\}$

23) $\{5 - 4i, 5 + 4i\}$ 24) $\left\{-4, -\frac{3}{2}\right\}$ 25) $\{0, 3\}$

26) $\left\{-\frac{3}{2} - \frac{\sqrt{15}}{2}i, -\frac{3}{2} + \frac{\sqrt{15}}{2}i\right\}$ 27) $\left\{-\frac{3}{2}, 0, \frac{3}{2}\right\}$

28) $\{-9, 3\}$ 29) $\left\{-\frac{5}{4} - \frac{\sqrt{39}}{4}i, -\frac{5}{4} + \frac{\sqrt{39}}{4}i\right\}$

30) $\left\{-\frac{5}{12} - \frac{1}{4}i, -\frac{5}{12} + \frac{1}{4}i\right\}$

Section 11.4

1) $\{-4, 12\}$ 3) $\left\{-2, \frac{4}{5}\right\}$ 5) $\{4 - \sqrt{6}, 4 + \sqrt{6}\}$

7) $\left\{\frac{1 - \sqrt{7}}{3}, \frac{1 + \sqrt{7}}{3}\right\}$ 9) $\left\{-\frac{9}{5}, 1\right\}$

11) $\left\{\frac{11 - \sqrt{21}}{10}, \frac{11 + \sqrt{21}}{10}\right\}$ 13) $\{5\}$ 15) $\left\{\frac{4}{5}, 2\right\}$

17) $\{9\}$ 19) $\{1, 16\}$ 21) $\{5\}$ 23) $\{0, 2\}$ 25) yes

27) yes 29) no 31) yes 33) no 35) $\{-3, -1, 1, 3\}$

37) $\{-\sqrt{7}, -2, 2, \sqrt{7}\}$ 39) $\{-i\sqrt{7}, -i\sqrt{5}, i\sqrt{5}, i\sqrt{7}\}$

41) $\{-8, -1\}$ 43) $\{-64, 1000\}$ 45) $\left\{-\frac{27}{8}, 125\right\}$

47) $\{4, 36\}$ 49) $\{49\}$ 51) $\{16\}$

53) $\left\{-\frac{2\sqrt{3}}{3}i, -\frac{\sqrt{3}}{3}i, \frac{\sqrt{3}}{3}i, \frac{2\sqrt{3}}{3}i\right\}$

55) $\{-\sqrt{5}, \sqrt{5}, -i\sqrt{3}, i\sqrt{3}\}$

57) $\{-\sqrt{3 + \sqrt{7}}, \sqrt{3 + \sqrt{7}}, -\sqrt{3 - \sqrt{7}}, \sqrt{3 - \sqrt{7}}\}$

59) $\left\{-\frac{\sqrt{7 + \sqrt{41}}}{2}, \frac{\sqrt{7 + \sqrt{41}}}{2}, -\frac{\sqrt{7 - \sqrt{41}}}{2}, \frac{\sqrt{7 - \sqrt{41}}}{2}\right\}$

61) $\left\{-\frac{1}{2}, \frac{1}{6}\right\}$ 63) $\left\{\frac{1}{4}, 3\right\}$ 65) $\{-6, -1\}$ 67) $\left\{-\frac{1}{2}, 0\right\}$

69) $\left\{-\frac{2}{5}, \frac{2}{5}\right\}$ 71) $\left\{-11, -\frac{20}{3}\right\}$ 73) $\left\{\frac{3}{2}\right\}$

75) $\{2 - \sqrt{2}, 2 + \sqrt{2}\}$ 77) Walter: 3 hr; Kevin: 6 hr

79) 15 mph 81) large drain: 3 hr; small drain: 6 hr

83) to Boulder: 60 mph; going home: 50 mph

Section 11.5

1) $r = \frac{\pm\sqrt{A\pi}}{\pi}$ 3) $v = \pm\sqrt{ar}$ 5) $d = \frac{\pm\sqrt{IE}}{E}$

7) $r = \frac{\pm\sqrt{kq_1q_2F}}{F}$ 9) $A = \frac{1}{4}\pi d^2$ 11) $l = \frac{gT_p^2}{4\pi^2}$

13) $g = \frac{4\pi^2 l}{T_p^2}$

15) a) Both are written in the standard form for a quadratic equation, $ax^2 + bx + c = 0$.
b) Use the quadratic formula.

17) $x = \frac{5 \pm \sqrt{25 - 4rs}}{2r}$ 19) $z = \frac{-r \pm \sqrt{r^2 + 4pq}}{2p}$

21) $a = \frac{h \pm \sqrt{h^2 + 4dk}}{2d}$ 23) $t = \frac{-v \pm \sqrt{v^2 + 2gs}}{g}$

25) length = 12 in., width = 9 in. 27) 2 ft

29) base = 8 ft, height = 15 ft 31) 10 in.

33) a) 0.75 sec on the way up, 3 sec on the way down
b) $\frac{15 + \sqrt{241}}{8}$ sec or about 3.8 sec

35) a) 9.5 million b) 1999 37) \$2.40 39) 1.75 ft

Chapter 11 Review Exercises

1) $\{-6, 9\}$ 3) $\left\{-\frac{1}{2}, \frac{3}{2}\right\}$ 5) $\{-4, -3, 4\}$

7) width = 8 cm, length = 12 cm 9) $\{-12, 12\}$

11) $\{-2i, 2i\}$ 13) $\{-4, 10\}$ 15) $\left\{-\frac{\sqrt{10}}{3}, \frac{\sqrt{10}}{3}\right\}$ 17) 3

19) $\sqrt{29}$ 21) 5 23) $r^2 + 10r + 25$; $(r + 5)^2$

25) $c^2 - 5c + \frac{25}{4}$; $\left(c - \frac{5}{2}\right)^2$ 27) $a^2 + \frac{2}{3}a + \frac{1}{9}$; $\left(a + \frac{1}{3}\right)^2$

29) $\{-2, 8\}$ 31) $\{-5 - \sqrt{31}, -5 + \sqrt{31}\}$

33) $\left\{-\frac{3}{2} - \frac{\sqrt{5}}{2}, -\frac{3}{2} + \frac{\sqrt{5}}{2}\right\}$ 35) $\left\{\frac{7}{6} - \frac{\sqrt{95}}{6}i, \frac{7}{6} + \frac{\sqrt{95}}{6}i\right\}$

37) $\{-6, 2\}$ 39) $\left\{\frac{5 - \sqrt{15}}{2}, \frac{5 + \sqrt{15}}{2}\right\}$

41) $\{1 - i\sqrt{3}, 1 + i\sqrt{3}\}$ 43) $\left\{-\frac{2}{3}, \frac{1}{2}\right\}$

45) There are two irrational solutions.

47) 64; two rational solutions

49) -15; two nonreal, complex solutions 51) -12 or 12

53) $\{-5, 3\}$ 55) $\left\{\frac{4}{3}, 2\right\}$ 57) $\{25\}$

59) $\{-\sqrt{2}, \sqrt{2}, -i\sqrt{7}, i\sqrt{7}\}$ 61) $\{1, 4\}$ 63) $\left\{-\frac{7}{2}, -1\right\}$

65) $v = \frac{\pm\sqrt{Frm}}{m}$ 67) $A = \pi r^2$ 69) $n = \frac{l \pm \sqrt{l^2 + 4km}}{2k}$

71) 3 in. 73) \$8.00 75) $\left\{1, \frac{4}{3}\right\}$

77) $\{-8 - 2i\sqrt{3}, -8 + 2i\sqrt{3}\}$ 79) $\{-5, -1, 1, 5\}$

81) $\{-6, 2\}$ 83) $\left\{-\frac{3}{2} - \frac{1}{2}i, -\frac{3}{2} + \frac{1}{2}i\right\}$

85) $\{9 - 3\sqrt{5}, 9 + 3\sqrt{5}\}$ 87) $\{-1, 0, 1\}$

89) 9 cm and 12 cm 91) Latrice: 50 min; Erica: 75 min

Chapter 11 Test

1) $\{-4, 12\}$ 2) $\left\{-\frac{4}{3}, \frac{4}{3}\right\}$ 3) $\{-3\sqrt{2}, 3\sqrt{2}\}$

4) The solution set contains two nonreal, complex numbers.

5) $\sqrt{101}$ 6) $\{-2 - \sqrt{11}, -2 + \sqrt{11}\}$

7) $\left\{\frac{3}{2} - \frac{\sqrt{19}}{2}i, \frac{3}{2} + \frac{\sqrt{19}}{2}i\right\}$ 8) $\{4 - i, 4 + i\}$

9) $\{-5 - i\sqrt{6}, -5 + i\sqrt{6}\}$ 10) $\left\{-2, \frac{4}{3}\right\}$ 11) $\left\{-\frac{2}{5}, \frac{2}{5}\right\}$

12) $\left\{-\frac{7}{4}, -1\right\}$ 13) $\{-2\sqrt{2}, 2\sqrt{2}, -3i, 3i\}$ 14) $\left\{0, \frac{5}{6}\right\}$

15) $\left\{\frac{7 - \sqrt{17}}{4}, \frac{7 + \sqrt{17}}{4}\right\}$ 16) $\left\{-\frac{3}{2}, -1\right\}$

17) 56; two irrational solutions 18) $\sqrt{19}$

19) a) after 4 sec b) after $\frac{3 + \sqrt{209}}{4}$ sec or about 4.4 sec

20) $V = \frac{1}{3}\pi r^2 h$ 21) $t = \frac{s \pm \sqrt{s^2 + 24r}}{2r}$

22) width = 6 ft; length = 8 ft

23) Justine: 40 min; Kimora: 60 min

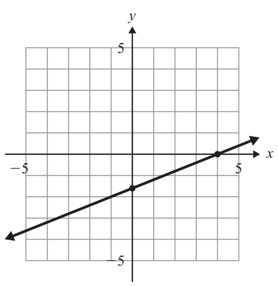
24) width $= 13$ in.; length $= 19$ in.

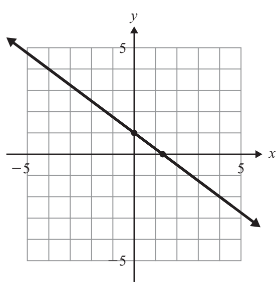
Cumulative Review for Chapters 1–11

1) $-\frac{1}{6}$ 2) 13 3) area: 238 cm^2; perimeter: 82 cm

4) $-8d^{15}$ 5) $\frac{45x^6}{y^4}$ 6) $\frac{b^9}{125a^{15}}$ 7) 90 8) $m = \frac{y - b}{x}$

9) x-int: (4, 0); y-int: $\left(0, -\frac{8}{5}\right)$ 10) $m = -\frac{3}{4}$, y-int: (0, 1)





11) chips: \$0.80; soda: \$0.75

12) $3x^2y^2 - 5x^2y - 3xy^2 - 9xy + 8$ 13) $3r^2 - 30r + 75$

14) $2p(2p - 1)(p + 4)$ 15) $(a + 5)(a^2 - 5a + 25)$

16) $\frac{z^2 - 5z + 12}{z(z + 4)}$ 17) $\frac{c(c + 3)}{1 - 4c}$ 18) $\left\{-\frac{3}{2}, 3\right\}$

19) $(0, 3, -1)$ 20) $5\sqrt{3}$ 21) $2\sqrt[3]{5}$ 22) $3x^3y^2\sqrt{7x}$

23) 16 24) $10 - 5\sqrt{3}$ 25) $34 - 77i$ 26) $\{-2\sqrt{2}, 2\sqrt{2}\}$

27) $\left\{\frac{1}{6} - \frac{\sqrt{11}}{6}i, \frac{1}{6} + \frac{\sqrt{11}}{6}i\right\}$ 28) $\left\{-\frac{2}{3}, \frac{7}{3}\right\}$

29) $\{-9, 3\}$ 30) $V = \pi r^2 h$

Chapter 12

Section 12.1

1) It is a special type of relation in which each element of the domain corresponds to exactly one element in the range.

3) domain: $\{5, 6, 14\}$; range: $\{-3, 0, 1, 3\}$; not a function

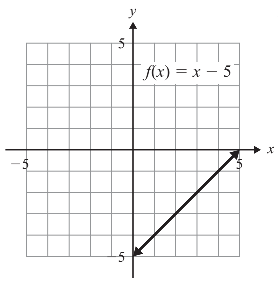
5) domain: $\{-2, 2, 5, 8\}$; range: $\{4, 25, 64\}$; is a function

7) domain: $(-\infty, \infty)$; range: $[-4, \infty)$; is a function

9) yes 11) yes 13) no 15) no

17) False; it is read as "f of x."

19)



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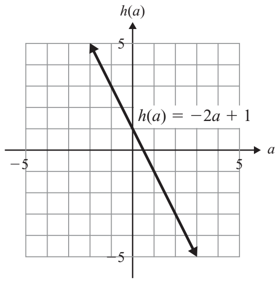

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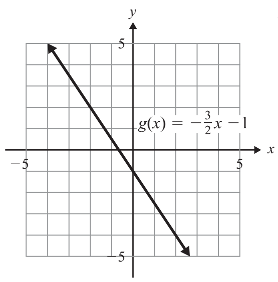
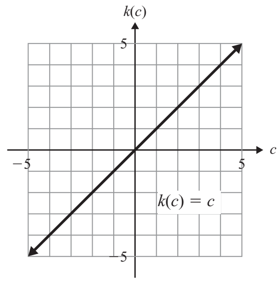


25)



27) 11 29) -12 31) $3a - 7$ 33) $d^2 - 4d - 9$

35) $3c + 5$ 37) $t^2 - 13$ 39) $h^2 - 6h - 4$ 41) -18

43) -4 45) $15k + 2$ 47) $25t^2 + 35t + 2$ 49) $-5b - 3$

51) $r^2 + 15r + 46$ 53) 5 55) $-\frac{5}{2}$

57) -2 or 8 59) $(-\infty, \infty)$

61) Set the denominator equal to 0 and solve for the variable. The domain consists of all real numbers *except* the values that make the denominator equal to 0.

63) $(-\infty, \infty)$ 65) $(-\infty, \infty)$ 67) $(-\infty, -8) \cup (-8, \infty)$

69) $(-\infty, 0) \cup (0, \infty)$ 71) $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

73) $\left(-\infty, -\frac{3}{7}\right) \cup \left(-\frac{3}{7}, \infty\right)$ 75) $(-\infty, \infty)$

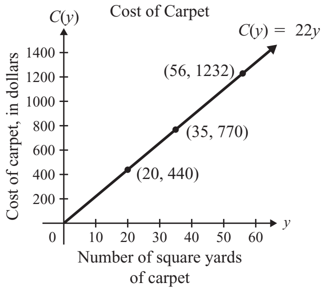
77) $(-\infty, -8) \cup (-8, -3) \cup (-3, \infty)$

79) $(-\infty, -4) \cup (-4, 9) \cup (9, \infty)$

81) $[0, \infty)$ 83) $[-2, \infty)$ 85) $[8, \infty)$ 87) $\left[\dfrac{5}{2}, \infty\right)$

89) $(-\infty, 0]$ 91) $(-\infty, 9]$ 93) $(-\infty, \infty)$

95) a) \$440 b) \$1232 c) 35 yd^2
d)

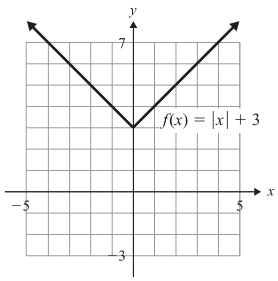


97) a) $L(1) = 90$. The labor charge for a 1-hr repair job is \$90.
b) $L(1.5) = 115$. The labor charge for a 1.5-hr repair job is \$115.
c) $h = 2.5$. If the labor charge is \$165, the repair job took 2.5 hr.

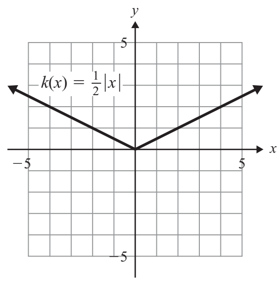
99) a) $A(r) = \pi r^2$
b) $A(3) = 9\pi$. When the radius of the circle is 3 cm, the area of the circle is 9π cm^2.
c) $A(5) = 25\pi$. When the radius of the circle is 5 in., the area of the circle is 25π in^2.
d) $r = 8$ in.

Section 12.2

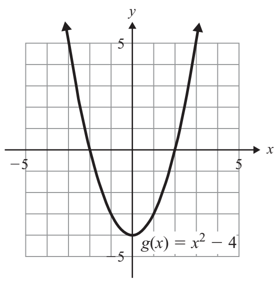
1) domain: $(-\infty, \infty)$; range: $[3, \infty)$



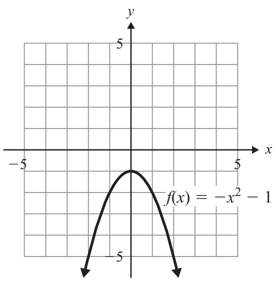
3) domain: $(-\infty, \infty)$; range: $[0, \infty)$



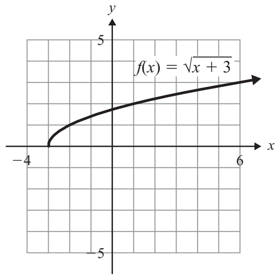
5) domain: $(-\infty, \infty)$; range: $[-4, \infty)$



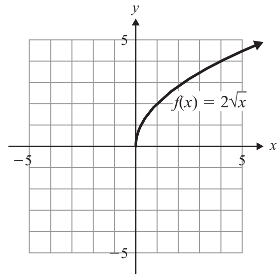
7) domain: $(-\infty, \infty)$; range: $(-\infty, -1]$



9) domain: $[-3, \infty)$; range: $[0, \infty)$



11) domain: $[0, \infty)$; range: $[0, \infty)$

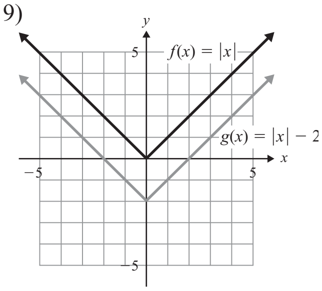


13) The graph of $g(x)$ is the same shape as $f(x)$, but g is shifted down 2 units.

15) The graph of $g(x)$ is the same shape as $f(x)$, but g is shifted left 2 units.

17) The graph of $g(x)$ is the reflection of $f(x)$ about the x-axis.

19)



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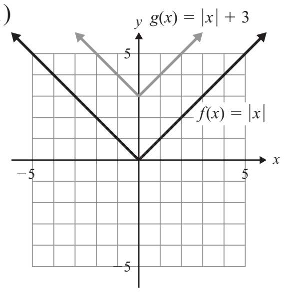

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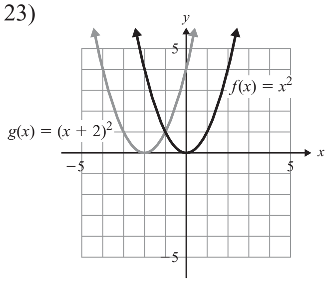

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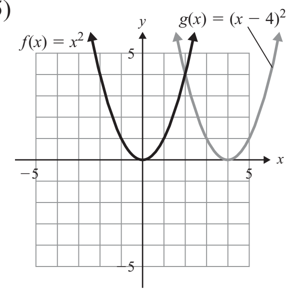

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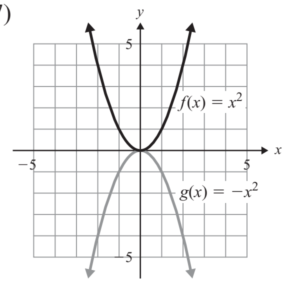

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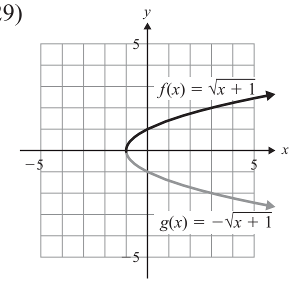
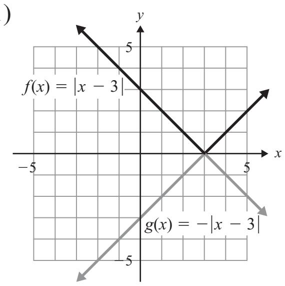
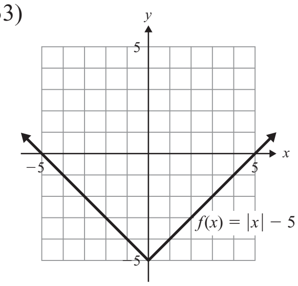


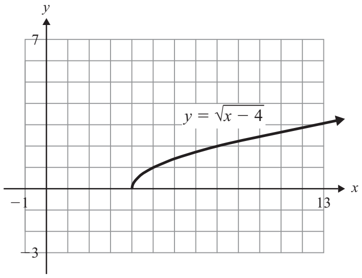
31)



33)



35)



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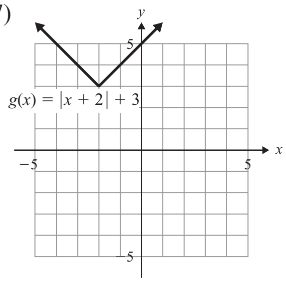

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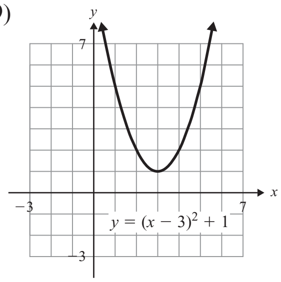
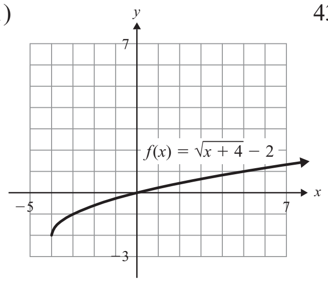


41)



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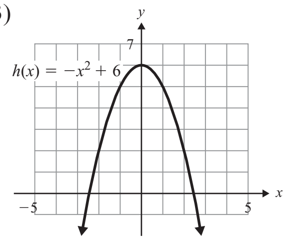
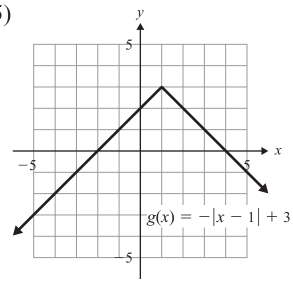
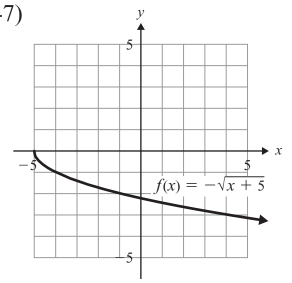


45)



47)

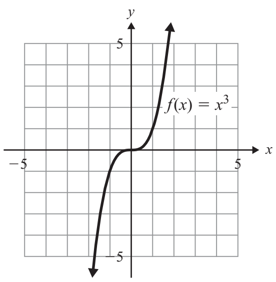


49) a) $h(x)$ b) $f(x)$ c) $g(x)$ d) $k(x)$ 51) $g(x) = \sqrt{x + 5}$

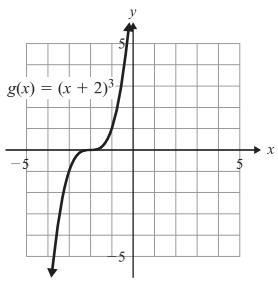
53) $g(x) = |x + 2| - 1$ 55) $g(x) = (x + 3)^2 + \frac{1}{2}$

57) $g(x) = -x^2$

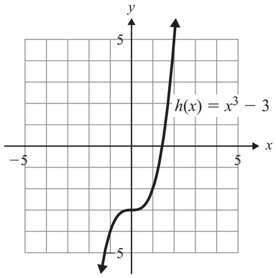
59)



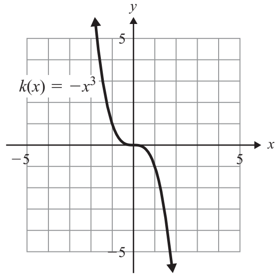
a)



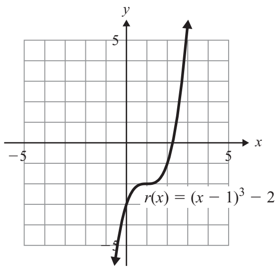
b)



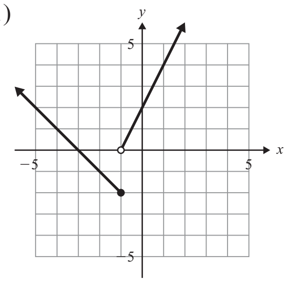
c)



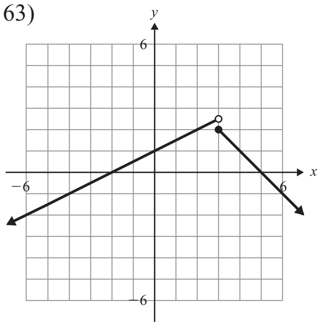
d)



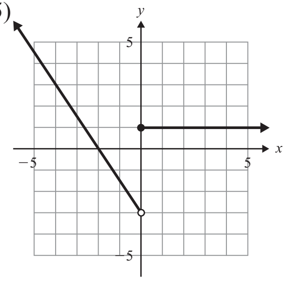
61)



63)



65)



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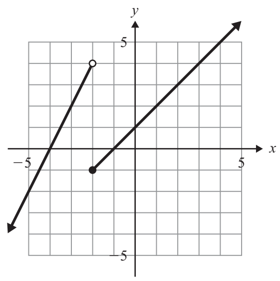
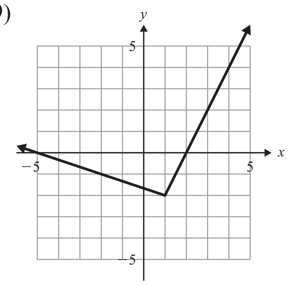
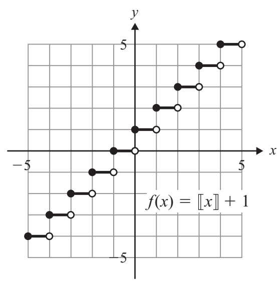


69)

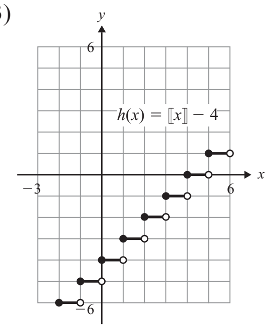


71) 3 73) 9 75) 8 77) -7 79) -9

81)



83)



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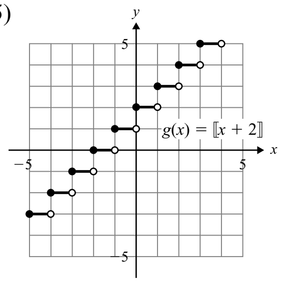
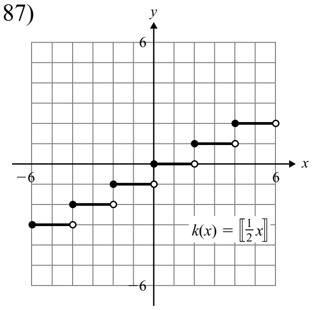


87)



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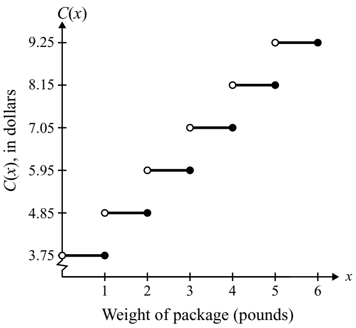

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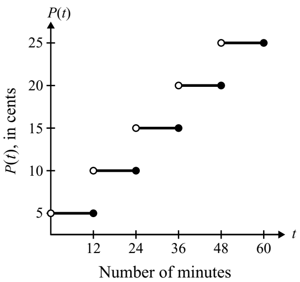
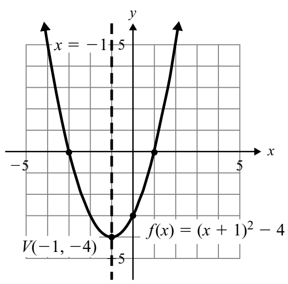


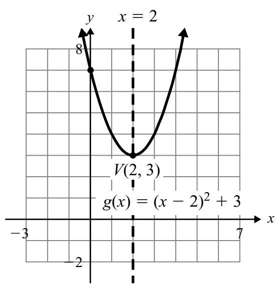
Section 12.3

1) (h, k) 3) a is positive. 5) $|a| > 1$

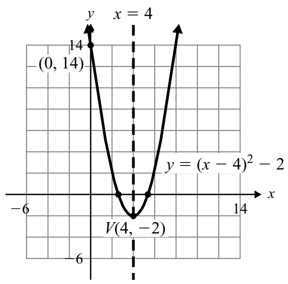
7) $V(-1, -4)$; $x = -1$; x-ints: $(-3, 0)$, $(1, 0)$; y-int: $(0, -3)$



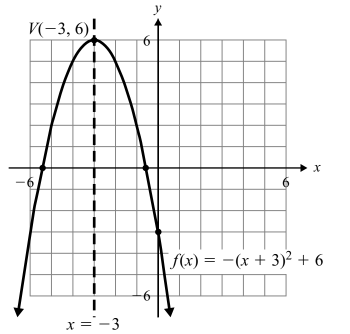
9) $V(2, 3)$; $x = 2$; x-ints: none; y-int: $(0, 7)$



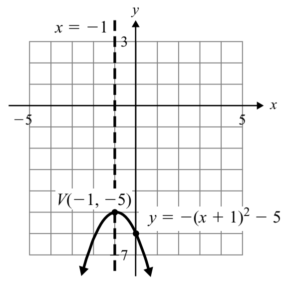
11) $V(4, -2)$; $x = 4$; x-ints: $(4 - \sqrt{2}, 0)$, $(4 + \sqrt{2}, 0)$; y-int: $(0, 14)$



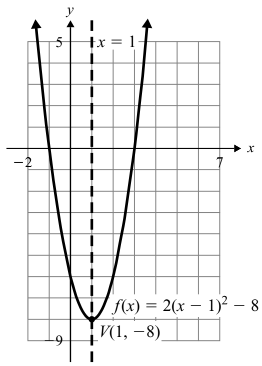
13) $V(-3, 6)$; $x = -3$; x-ints: $(-3 - \sqrt{6}, 0)$, $(-3 + \sqrt{6}, 0)$; y-int: $(0, -3)$



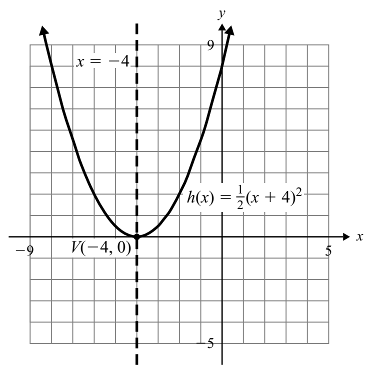
15) $V(-1, -5)$; $x = -1$; x-ints: none; y-int: $(0, -6)$



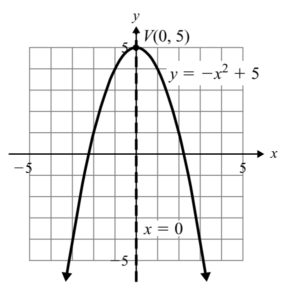
17) $V(1, -8)$; $x = 1$; x-ints: $(-1, 0)$, $(3, 0)$; y-int: $(0, -6)$



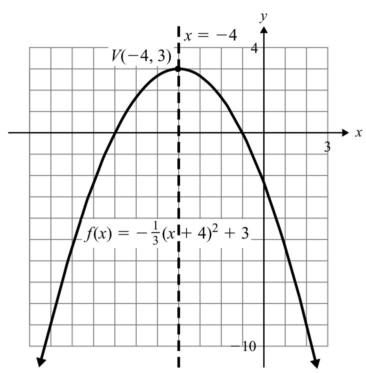
19) $V(-4, 0)$; $x = -4$; x-int: $(-4, 0)$; y-int: $(0, 8)$



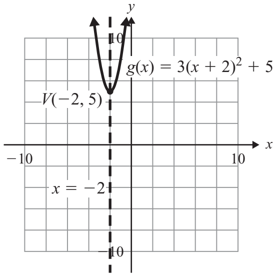
21) $V(0, 5)$; $x = 0$; x-ints: $(-\sqrt{5}, 0)$, $(\sqrt{5}, 0)$; y-int: $(0, 5)$



23) $V(-4, 3)$; $x = -4$; x-ints: $(-7, 0)$, $(-1, 0)$; y-int: $\left(0, -\dfrac{7}{3}\right)$

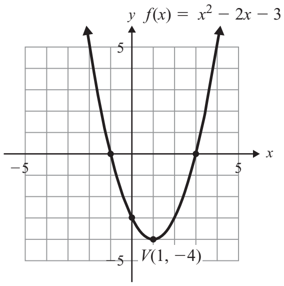


25) $V(-2, 5)$; $x = -2$; x-int: none; y-int: $(0, 17)$

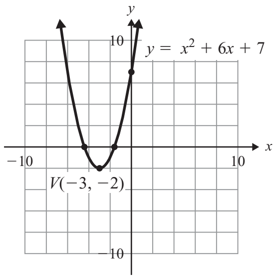


27) $f(x) = (x^2 + 8x) + 11$; $\frac{1}{2}(8) = 4$; $4^2 = 16$; Add and subtract the number above to the same side of the equation; $f(x) = (x + 4)^2 - 5$

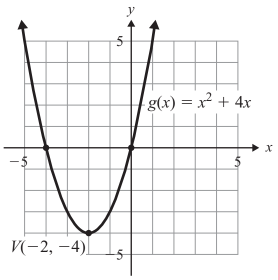
29) $f(x) = (x - 1)^2 - 4$; x-ints: $(-1, 0)$, $(3, 0)$; y-int: $(0, -3)$



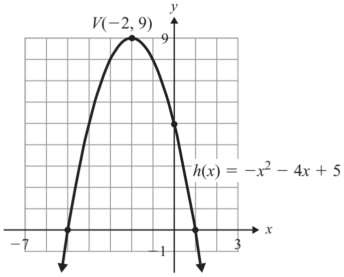
31) $y = (x + 3)^2 - 2$; x-ints: $(-3 - \sqrt{2}, 0)$, $(-3 + \sqrt{2}, 0)$; y-int: $(0, 7)$



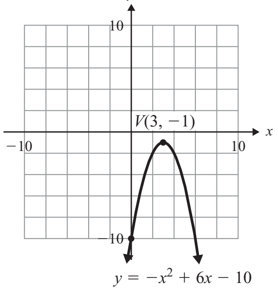
33) $g(x) = (x + 2)^2 - 4$; x-ints: $(-4, 0)$, $(0, 0)$; y-int: $(0, 0)$



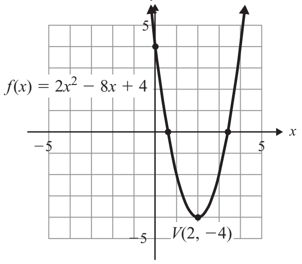
35) $h(x) = -(x + 2)^2 + 9$; x-ints: $(-5, 0)$, $(1, 0)$; y-int: $(0, 5)$



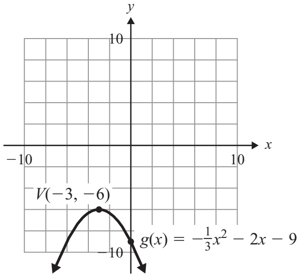
37) $y = -(x - 3)^2 - 1$; x-ints: none; y-int: $(0, -10)$



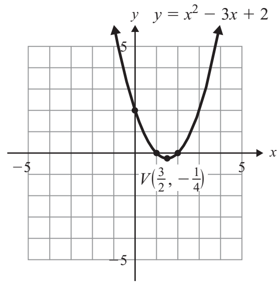
39) $f(x) = 2(x - 2)^2 - 4$; x-ints: $(2 - \sqrt{2}, 0)$, $(2 + \sqrt{2}, 0)$; y-int: $(0, 4)$



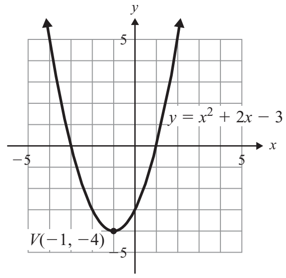
41) $g(x) = -\frac{1}{3}(x + 3)^2 - 6$; x-ints: none; y-int: $(0, -9)$



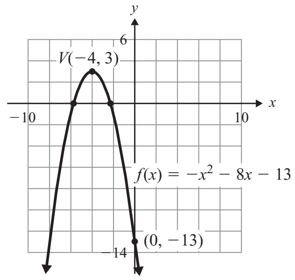
43) $y = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$; x-ints: $(1, 0)$, $(2, 0)$; y-int: $(0, 2)$



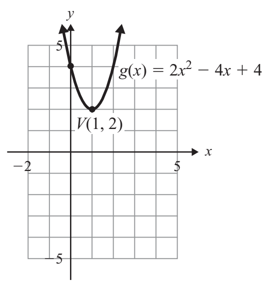
45) $V(-1, -4)$; x-ints: $(-3, 0)$, $(1, 0)$; y-int: $(0, -3)$



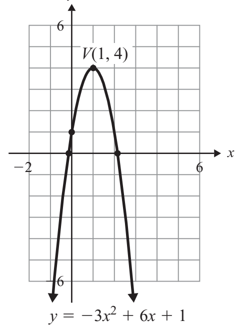
47) $V(-4, 3)$; x-ints: $(-4 - \sqrt{3}, 0)$, $(-4 + \sqrt{3}, 0)$; y-int: $(0, -13)$



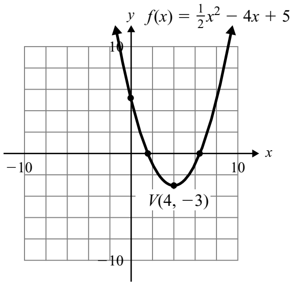
49) $V(1, 2)$; x-int: none; y-int: $(0, 4)$



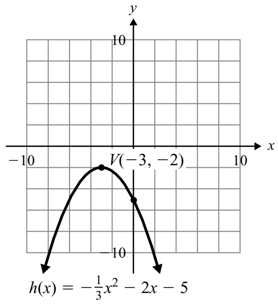
51) $V(1, 4)$; x-ints: $\left(1 + \frac{2\sqrt{3}}{3}, 0\right)$; $\left(1 - \frac{2\sqrt{3}}{3}, 0\right)$, y-int: $(0, 1)$



53) $V(4, -3)$; x-ints: $(4 - \sqrt{6}, 0)$, $(4 + \sqrt{6}, 0)$; y-int: $(0, 5)$



55) $V(-3, -2)$; x-int: none; y-int: $(0, -5)$

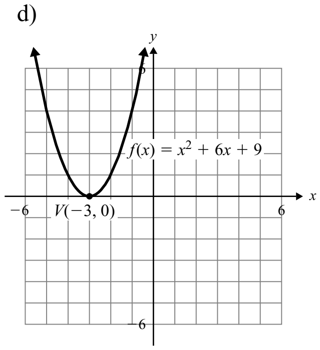


Section 12.4

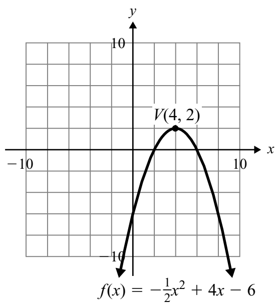
1) maximum 3) neither 5) minimum

7) If a is positive the graph opens upward, so the y-coordinate of the vertex is the minimum value of the function. If a is negative the graph opens downward, so the y-coordinate of the vertex is the maximum value of the function.

9) a) minimum
b) $(-3, 0)$ c) 0
d)



11) a) maximum
b) $(4, 2)$ c) 2
d)

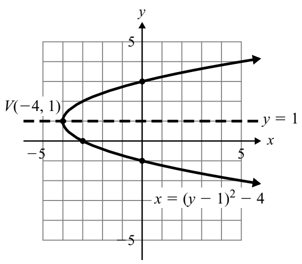


13) a) 10 sec b) 1600 ft c) 20 sec 15) July; 480 people

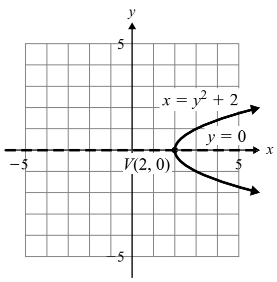
17) 1991; 531,000 19) 625 ft^2 21) 12 ft $\times$ 24 ft

23) 9 and 9 25) 6 and -6 27) (h, k) 29) to the left

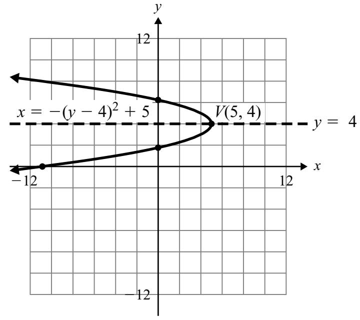
31) $V(-4, 1)$; $y = 1$; x-int: $(-3, 0)$; y-ints: $(0, -1)$, $(0, 3)$



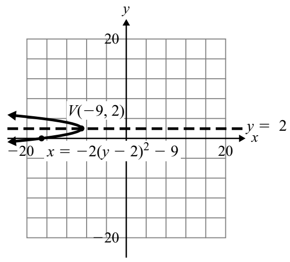
33) $V(2, 0)$; $y = 0$; x-int: $(2, 0)$; y-int: none



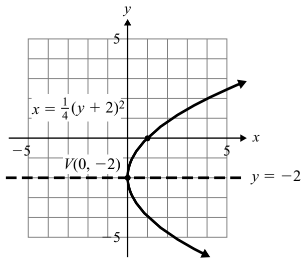
35) $V(5, 4)$; $y = 4$; x-int: $(-11, 0)$; y-ints: $(0, 4 - \sqrt{5})$, $(0, 4 + \sqrt{5})$



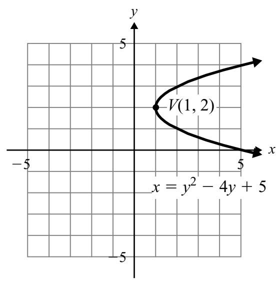
37) $V(-9, 2)$; $y = 2$; x-int: $(-17, 0)$; y-int: none



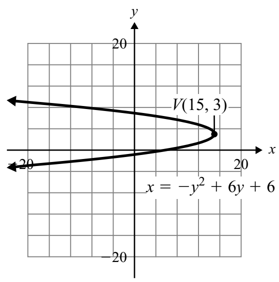
39) $V(0, -2)$; $y = -2$; x-int: $(1, 0)$; y-int: $(0, -2)$



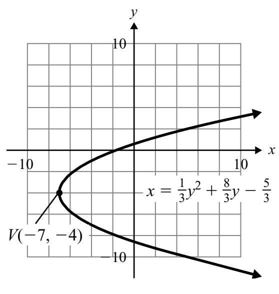
41) $x = (y - 2)^2 + 1$



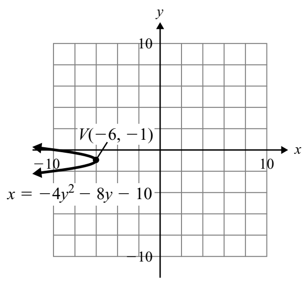
43) $x = -(y - 3)^2 + 15$



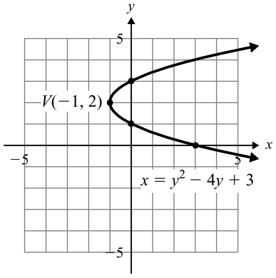
45) $x = \dfrac{1}{3}(y + 4)^2 - 7$



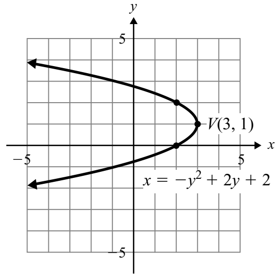
47) $x = -4(y + 1)^2 - 6$



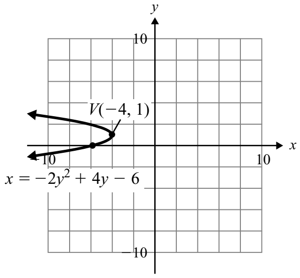
49) $V(-1, 2)$; x-int: $(3, 0)$; y-ints: $(0, 1)$, $(0, 3)$



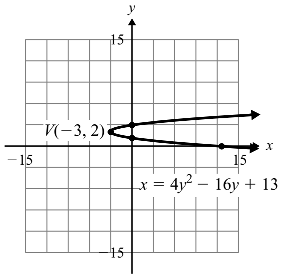
51) $V(3, 1)$; x-int: $(2, 0)$; y-ints: $(0, 1 - \sqrt{3})$, $(0, 1 + \sqrt{3})$



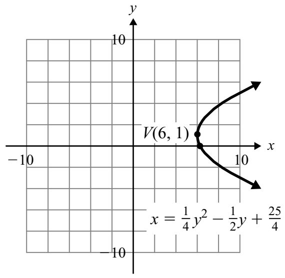
53) $V(-4, 1)$; x-int: $(-6, 0)$; y-int: none



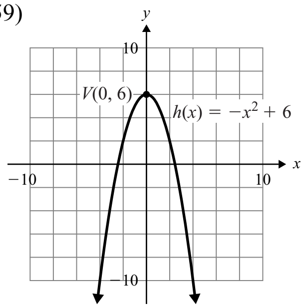
55) $V(-3, 2)$; x-int: $(13, 0)$; y-ints: $\left(0, 2 - \dfrac{\sqrt{3}}{2}\right)$, $\left(0, 2 + \dfrac{\sqrt{3}}{2}\right)$



57) $V(6, 1)$; x-int: $\left(\dfrac{25}{4}, 0\right)$; y-int: none



59)



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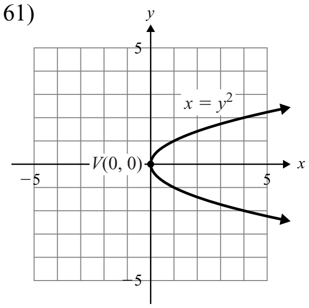

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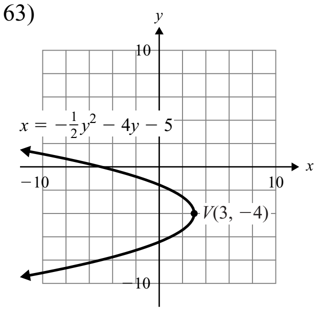

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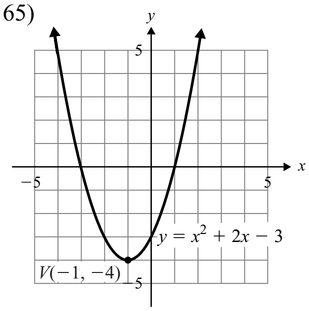
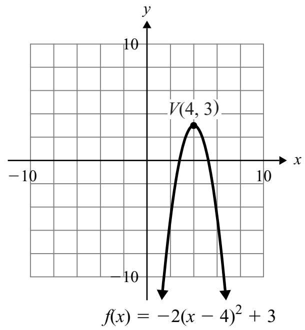


67)



Section 12.5

1) a) $-x - 10$ b) -15 c) $-5x + 12$ d) 2

3) a) $5x^2 - 4x - 7$ b) 98 c) $3x^2 - 10x + 5$ d) -3

5) a) $-x^2 + 5x$ b) -24 7) a) $6x^2 + 11x + 3$ b) 24

9) a) $\dfrac{6x + 9}{x + 4}, x \neq -4$ b) $-\dfrac{3}{2}$ 11) a) $x + 3, x \neq 8$ b) 1

13) a) $x + 4, x \neq -\dfrac{2}{3}$ b) 2 15) Answers may vary.

17) $g(x) = 2x - 3$ 19) a) $P(x) = 4x - 2000$ b) $\$4000$

21) a) $P(x) = 3x - 2400$ b) $\$0$

23) a) $P(x) = -0.2x^2 + 19x - 9$ b) $\$291{,}000$

25) $(f \circ g)(x) = f(g(x))$ so substitute the function $g(x)$ into the function $f(x)$ and simplify.

27) a) -1 b) -2 c) $6x - 26$ d) -2

29) a) $5x + 31$ b) $5x + 3$ c) 46

31) a) $x^2 - 6x + 7$ b) $x^2 - 14x + 51$ c) 11

33) a) $18x^2 - 51x + 25$ b) $6x^2 + 9x - 35$ c) -8

35) a) $-x^2 - 13x - 48$ b) $-x^2 + 3x$ c) 0

37) a) $\sqrt{x^2 + 4}$ b) $x + 4, x \geq 10$ c) $\sqrt{13}$

39) a) $\dfrac{1}{t^2 + 8}$ b) $\dfrac{1}{(t + 8)^2}$ c) $\dfrac{1}{9}$

41) a) $r(5) = 20$. The radius of the spill 5 min after the ship started leaking was 20 ft.
b) $A(20) = 400\pi$. The area of the oil slick is 400π ft^2 when its radius is 20 ft.
c) $A(r(t)) = 16\pi t^2$. This is the area of the oil slick in terms of t, the number of minutes after the leak began.
d) $A(r(5)) = 400\pi$. The area of the oil slick 5 min after the ship began leaking was 400π ft^2.

43) a) $s(40) = 32$. When the regular price of an item is \$40, the sale price is \$32.
b) $f(32) = 34.24$. When the cost of an item is \$32, the final cost after sales tax is \$34.24.
c) $(f \circ s)(x) = 0.856x$. This is the final cost of the item after the discount and sales tax.
d) $(f \circ s)(40) = 34.24$. When the original cost of an item is \$40, the final cost after the discount and sales tax is \$34.24.

45) $f(x) = 1\overline{x}$, $g(x) = x^2 + 13$; answers may vary

47) $f(x) = x^2$, $g(x) = 8x - 3$; answers may vary

49) $f(x) = \dfrac{1}{x}$, $g(x) = 6x + 5$; answers may vary

Section 12.6

1) increases 3) direct 5) inverse 7) combined

9) $M = kn$ 11) $h = \dfrac{k}{j}$ 13) $T = \dfrac{k}{c^2}$ 15) $s = krt$

17) $Q = \dfrac{k\sqrt{z}}{m}$ 19) a) 9 b) $z = 9x$ c) 54

21) a) 48 b) $N = \dfrac{48}{y}$ c) 16 23) a) 5 b) $Q = \dfrac{5r^2}{w}$ c) 45

25) 56 27) 18 29) 70 31) \$500.00 33) 12 hr

35) 180 watts 37) 162,000 J 39) 200 cycles/sec

41) 3 ohms 43) 320 lb

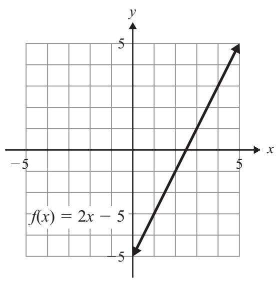
Chapter 12 Review Exercises

1) domain: $\{-7, -5, 2, 4\}$; range: $\{-4, -1, 3, 5, 9\}$; not a function

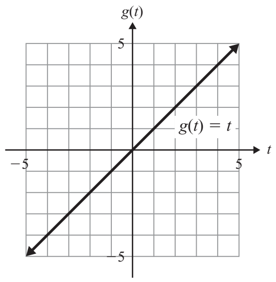
3) domain: $(-\infty, \infty)$; range: $[0, \infty)$; is a function

5) yes 7) no 9) yes 11) yes

13)



15)



17) a) -37 b) 35 c) -22 d) 18 e) $-8c + 3$
f) $r^2 + 7r - 12$ g) $-8p + 27$ h) $t^2 + 15t + 32$

19) 12 21) 2 or 6

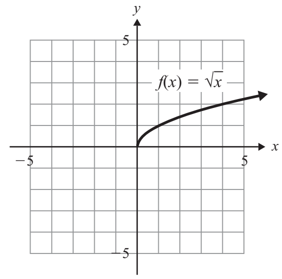
23) $(-\infty, 5) \cup (5, \infty)$ 25) $(-\infty, \infty)$ 27) $\left[\dfrac{7}{5}, \infty\right)$

29) $(-\infty, 0) \cup (0, \infty)$ 31) $(-\infty, \infty)$

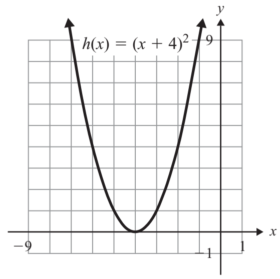
33) $(-\infty, -1) \cup (-1, 8) \cup (8, \infty)$

35) a) \$32 b) \$46 c) 150 mi d) 80 mi

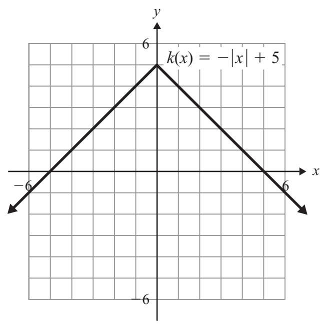
37) domain: $[0, \infty)$; range: $[0, \infty)$



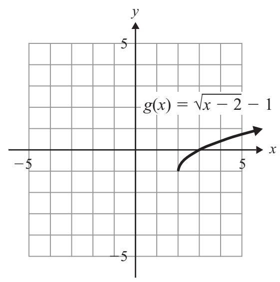
39) domain: $(-\infty, \infty)$; range: $[0, \infty)$



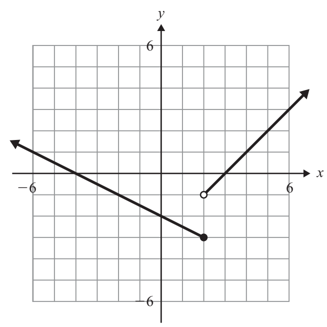
41) domain: $(-\infty, \infty)$; range: $(-\infty, 5]$



43) domain: $[2, \infty)$; range: $[-1, \infty)$

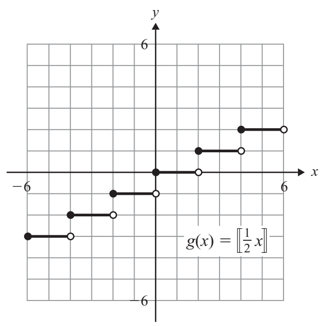


45)



47) 7 49) -9 51) 0

53)

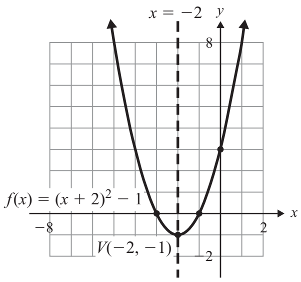


55) $g(x) = |x - 5|$

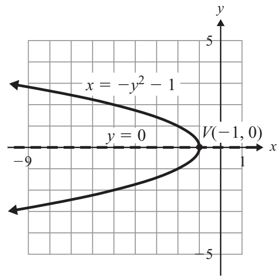
57) a) (h, k) b) $x = h$
c) If a is positive, the parabola opens upward. If a is negative, the parabola opens downward.

59) a) (h, k) b) $y = k$
c) If a is positive, the parabola opens to the right. If a is negative, the parabola opens to the left.

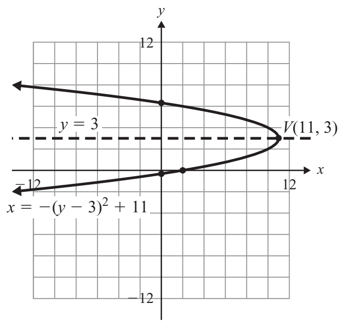
61) $V(-2, -1)$; $x = -2$; x-ints: $(-3, 0)$, $(-1, 0)$; y-int: $(0, 3)$



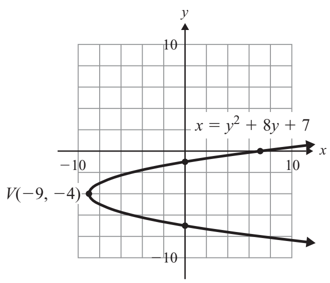
63) $V(-1, 0)$; $y = 0$; x-int: $(-1, 0)$; y-int: none



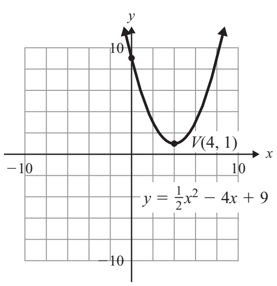
65) $V(11, 3)$; $y = 3$; x-int: $(2, 0)$; y-ints: $(0, 3 - \sqrt{11})$, $(0, 3 + \sqrt{11})$



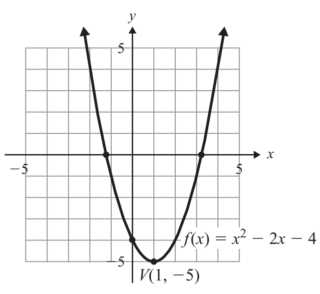
67) $x = (y + 4)^2 - 9$; x-int: $(7, 0)$; y-ints: $(0, -1)$, $(0, -7)$



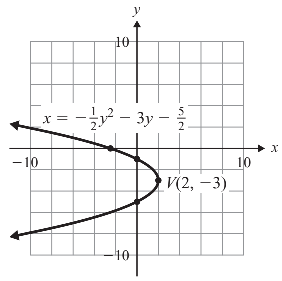
69) $y = \dfrac{1}{2}(x - 4)^2 + 1$; x-int: none; y-int: $(0, 9)$



71) $V(1, -5)$; x-ints: $(1 - \sqrt{5}, 0)$, $(1 + \sqrt{5}, 0)$; y-int: $(0, -4)$



73) $V(2, -3)$; x-int: $\left(-\dfrac{5}{2}, 0\right)$; y-ints: $(0, -5)$, $(0, -1)$



75) a) 1 sec b) 256 ft c) 5 sec 77) $4x + 6$ 79) -3

81) $-5x^2 + 18x + 8$ 83) a) $\dfrac{6x - 5}{x + 4}, x \neq -4$ b) $\dfrac{13}{7}$

85) a) $P(x) = 6x - 400$ b) \$800

87) a) $4x^2 + 6x - 8$ b) $2x^2 + 10x - 9$ c) -21

89) a) $(N \circ G)(h) = 9.6h$. This is Antoine's net pay in terms of how many hours he has worked.
b) $(N \circ G)(30) = 288$. When Antoine works 30 hr, his net pay is \$288. c) \$384

91) 72 93) 3.24 lb

Chapter 12 Test

1) It is a special type of relation in which each element of the domain corresponds to exactly one element of the range.

2) a) $\{-8, 2, 5, 7\}$ b) $\{-1, 3, 10\}$ c) yes

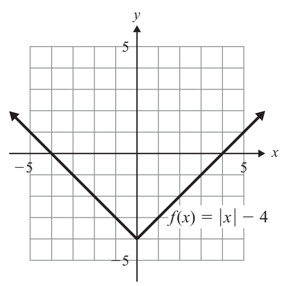
3) a) $\left[-\dfrac{7}{3}, \infty\right)$ b) yes 4) $(-\infty, \infty)$

5) $\left(-\infty, \dfrac{8}{7}\right) \cup \left(\dfrac{8}{7}, \infty\right)$ 6) 2 7) $4c + 3$ 8) $4n - 25$

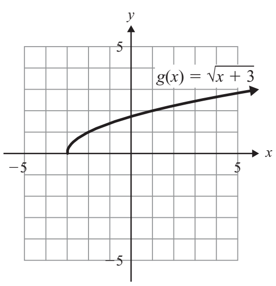
9) $k^2 + 4k + 5$ 10) $-\dfrac{3}{2}$

11) a) $C(3) = 210$. The cost of delivering 3 yd^3 of cedar mulch is \$210.
b) 6 yd^3

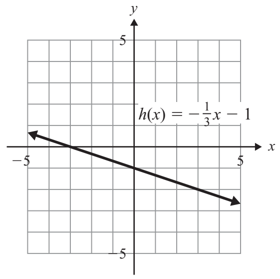
12) domain: $(-\infty, \infty)$; range: $[-4, \infty)$



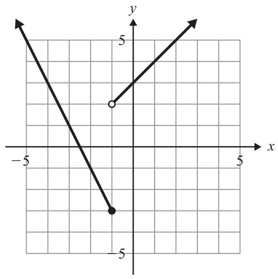
13) domain: $[-3, \infty)$; range: $[0, \infty)$



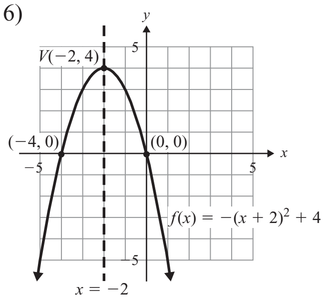
14) domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$



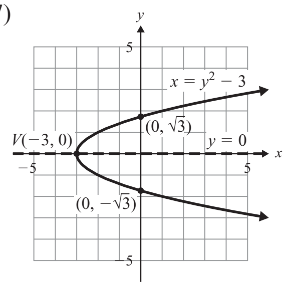
15)



16)



17)



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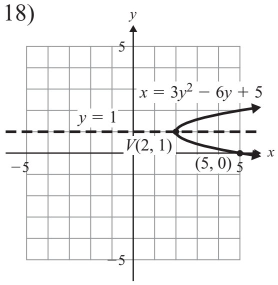

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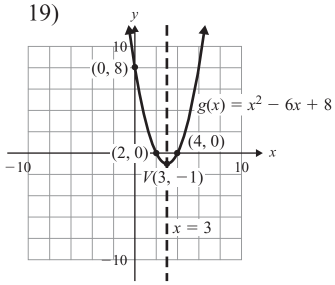


20) a) 144 ft b) 5 sec 21) $x^2 + 3x - 10$ 22) -2

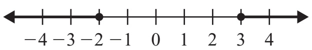
23) $2x^2 + 10x + 1$ 24) $4x^2 + 38x + 81$

25) 300 26) 18 dB

Cumulative Review for Chapters 1–12

1) inverse 2) commutative 3) $\frac{1}{32}$ 4) $\frac{1}{125}$ 5) 1 6) $\left\{\frac{4}{3}\right\}$

7) $(-\infty, -2] \cup [3, \infty)$



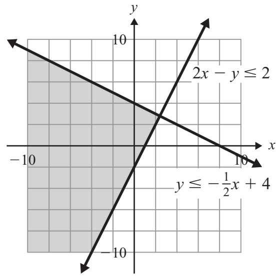
8) $4x + 3y = -2$ 9) $(3, 2)$ 10) $p^2 - p - 56$

11) $\frac{3}{2}r - 10 + \frac{3}{r} + \frac{1}{r^2}$ 12) $(k - 9)(k - 6)$

13) $(10 + 3m)(10 - 3m)$ 14) $\{-7, 2\}$ 15) $\frac{c + 4}{3(2c + 3)}$

16) $\{-8\}$ 17) $\varnothing$

18)



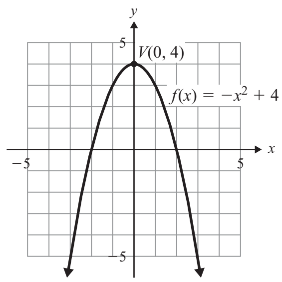
19) $2\sqrt{15}$ 20) 2 21) $3c^3d^5\sqrt{2d}$ 22) $\frac{1}{1000}$ 23) $7\sqrt{3}$

24) $\{-4\}$ 25) $\frac{2}{13} - \frac{16}{13}i$ 26) $\left\{\frac{4}{3} - i, \frac{4}{3} + i\right\}$

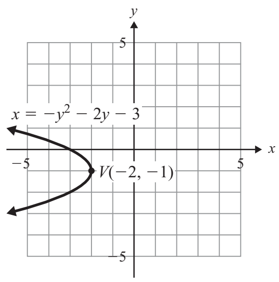
27) $\left\{-\frac{5}{2}, \frac{1}{2}\right\}$ 28) $\{1, 49\}$

29) a) 8 b) $x + 3, x \neq -1$ c) 4 d) $x^2 + 4x + 4$

30) domain: $(-\infty, \infty)$; range: $(-\infty, 4]$



31)



32) 3

Chapter 13

Section 13.1

1) no 3) yes; $h^{-1} = \{(-16, -5), (-4, -1), (8, 3)\}$

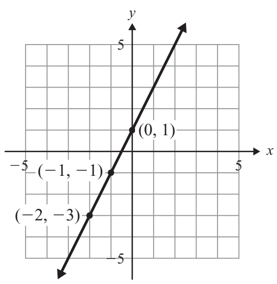
5) yes; $g^{-1} = \{(1, 2), (2, 5), (14, 7), (19, 10)\}$

7) yes 9) No; only one-to-one functions have inverses.

11) False; it is read "f inverse of x." 13) true

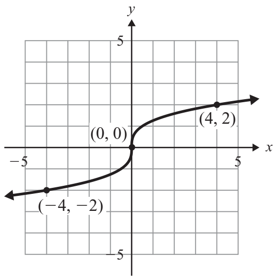
15) False; they are symmetric with respect to $y = x$.

17) a) yes
b)



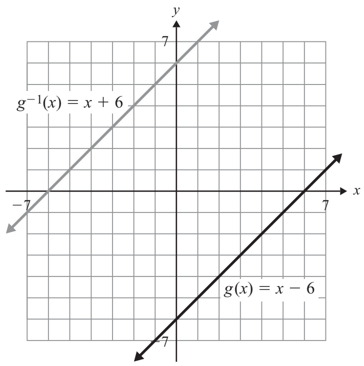
19) no

21) a) yes
b)

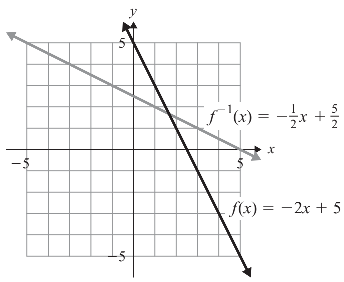


23) Replace $f(x)$ with y; $x = 2y - 10$; Add 10; $\frac{1}{2}x + 5 = y$; Replace y with $f^{-1}(x)$.

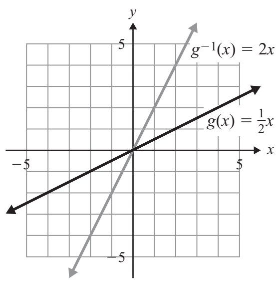
25) $g^{-1}(x) = x + 6$



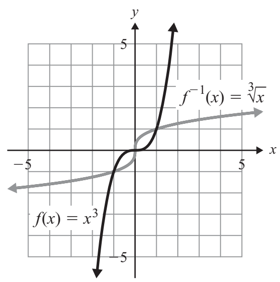
27) $f^{-1}(x) = -\frac{1}{2}x + \frac{5}{2}$



29) $g^{-1}(x) = 2x$



31) $f^{-1}(x) = \sqrt[3]{x}$



33) $f^{-1}(x) = \frac{1}{2}x + 3$ 35) $h^{-1}(x) = -\frac{2}{3}x + \frac{8}{3}$

37) $g^{-1}(x) = x^3 - 2$ 39) $f^{-1}(x) = x^2, x \geq 0$

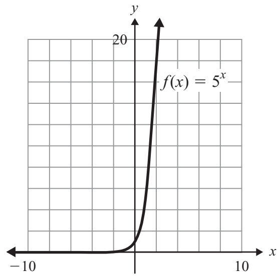
41) a) 3 b) 1 43) a) 2 b) 9 45) a) 10 b) −7

47) a) 8 b) 3 49–56) Answers may vary.

Section 13.2

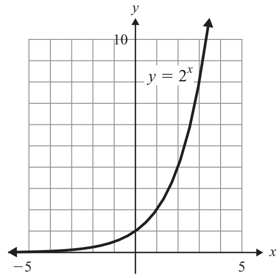
1) Choose values for the variable that will give positive numbers, negative numbers, and zero in the exponent.

3)



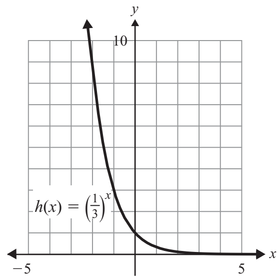
domain: $(-\infty, \infty)$;
range: $(0, \infty)$

5)



domain: $(-\infty, \infty)$;
range: $(0, \infty)$

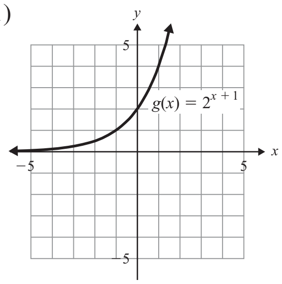
7)



domain: $(-\infty, \infty)$;
range: $(0, \infty)$

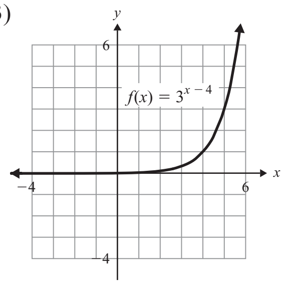
9) $(-\infty, \infty)$

11)



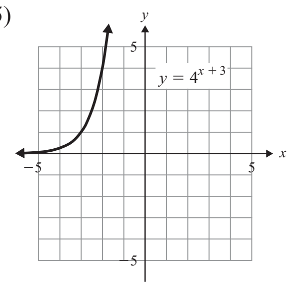
domain: $(-\infty, \infty)$;
range: $(0, \infty)$

13)



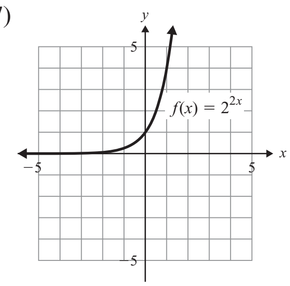
domain: $(-\infty, \infty)$;
range: $(0, \infty)$

15)

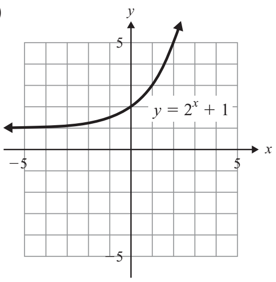


domain: $(-\infty, \infty)$;
range: $(0, \infty)$

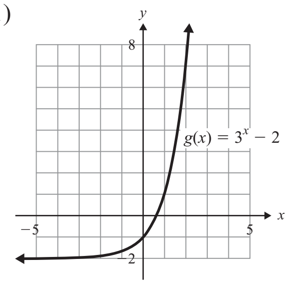
17)



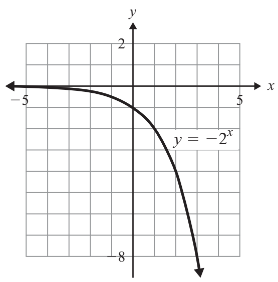
domain: $(-\infty, \infty)$;
range: $(0, \infty)$

19)


domain: $(-\infty, \infty)$;
range: $(1, \infty)$

21)


domain: $(-\infty, \infty)$;
range: $(-2, \infty)$

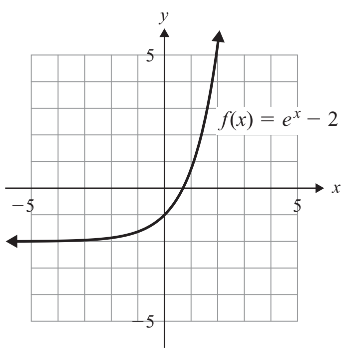
23)


domain: $(-\infty, \infty)$;
range: $(-\infty, 0)$

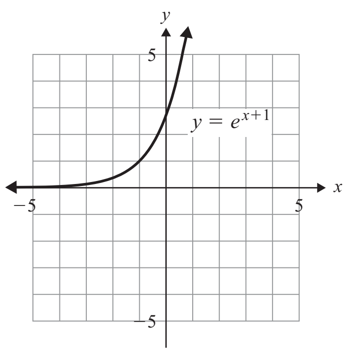
25) $g(x) = 2^x$ would grow faster because for values of $x > 2$, $2^x > 2x$.

27) Shift the graph of $f(x)$ down 2 units. 29) 2.7183

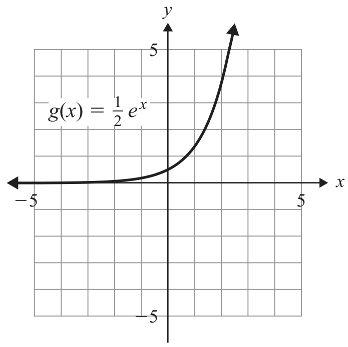
31) B 33) D

35)


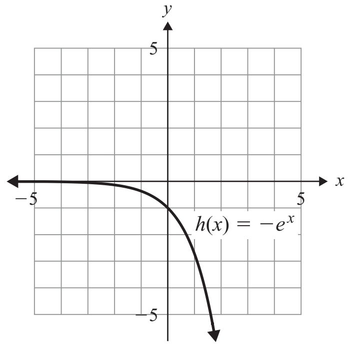
domain: $(-\infty, \infty)$; range: $(-2, \infty)$

37)


domain: $(-\infty, \infty)$; range: $(0, \infty)$

39)


domain: $(-\infty, \infty)$; range: $(0, \infty)$

41)


domain: $(-\infty, \infty)$; range: $(-\infty, 0)$

43) They are symmetric with respect to the x-axis.

45) $6^{3n} = (6^2)^{n-4}$; Power rule for exponents; Distribute; $3n = 2n - 8$; $n = -8$; $\{-8\}$

47) $\{2\}$ 49) $\left\{\frac{3}{4}\right\}$ 51) $\{8\}$ 53) $\{-2\}$ 55) $\{1\}$

57) $\left\{\frac{23}{4}\right\}$ 59) $\{-3\}$ 61) $\{-2\}$ 63) $\{-3\}$ 65) $\left\{-\frac{3}{2}\right\}$

67) $\left\{\frac{3}{2}\right\}$ 69) $\{-1\}$ 71) a) \$32,700 b) \$17,507.17

73) a) \$16,800 b) \$4504.04

75) a) \$185,200 b) \$227,772.64 77) \$90,036.92

79) \$59,134.40 81) 40.8 mg

Section 13.3

1) a must be a positive real number that is not equal to 1.

3) 10 5) $7^2 = 49$ 7) $2^3 = 8$

9) $9^{-2} = \frac{1}{81}$ 11) $10^6 = 1{,}000{,}000$ 13) $25^{1/2} = 5$

15) $13^1 = 13$ 17) $\log_9 81 = 2$ 19) $\log_{10} 100 = 2$

21) $\log_3 \frac{1}{81} = -4$ 23) $\log_{10} 1 = 0$ 25) $\log_{169} 13 = \frac{1}{2}$

27) $\log_9 3 = \frac{1}{2}$ 29) $\log_{64} 4 = \frac{1}{3}$

31) Write the equation in exponential form, then solve for the variable.

33) Rewrite in exponential form; $64 = x$; $\{64\}$

35) $\{121\}$ 37) $\{64\}$ 39) $\{100{,}000\}$ 41) $\{7\}$ 43) $\left\{\frac{1}{36}\right\}$

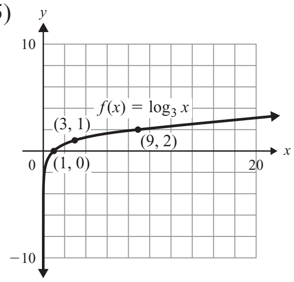
45) $\{14\}$ 47) $\left\{\frac{15}{2}\right\}$ 49) $\left\{\frac{1}{8}\right\}$ 51) $\left\{\frac{1}{6}\right\}$ 53) $\{12\}$

55) $\{4\}$ 57) $\{2\}$ 59) 2 61) 5 63) 2 65) $\frac{1}{2}$ 67) -1

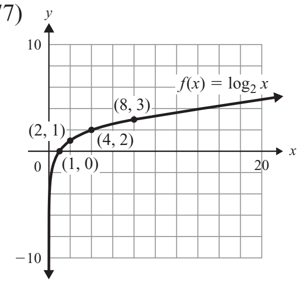
69) 1 71) -2

73) Replace $f(x)$ with y, write $y = \log_a x$ in exponential form, make a table of values, then plot the points and draw the curve.

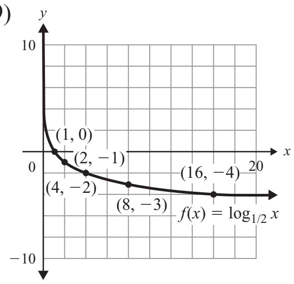
75)



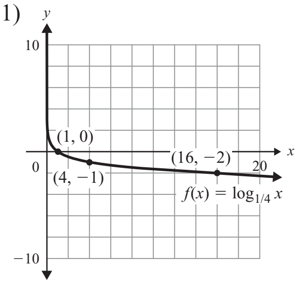
77)



79)



81)



83) $f^{-1}(x) = \log_3 x$ 85) $f^{-1}(x) = 2^x$

87) a) 1868 b) 2004 c) 2058

89) a) 14,000 b) 28,000
c) It is 1000 more than what was predicted by the formula.

Section 13.4

1) true 3) false 5) false 7) true

9) Product rule; $2 + \log_5 y$ 11) $\log_8 3 + \log_8 10$

13) $\log_7 5 + \log_7 d$ 15) $\log_9 4 - \log_9 7$ 17) $3 \log_5 2$

19) $8 \log p$ 21) $\frac{1}{2}\log_3 7$ 23) $2 + \log_5 t$ 25) $3 - \log_2 k$

27) 6 29) $3 + \log b$ 31) 35 33) $\frac{1}{2}$ 35) $\frac{2}{3}$

37) $4 \log_6 w + 3 \log_6 z$ 39) $2 \log_7 a - 5 \log_7 b$

41) $\frac{1}{5} \log 11 - 2 \log y$ 43) $2 + \frac{1}{2} \log_2 n - 3 \log_2 m$

45) $3 \log_4 x - \log_4 y - 2 \log_4 z$ 47) $\frac{1}{2} + \frac{1}{2} \log_5 c$

49) $\log k + \log(k - 6)$ 51) Power rule; $\log_6 x^2 y$

53) $\log_a mn$ 55) $\log_7 \frac{d}{3}$ 57) $\log_3 f^4 g$ 59) $\log_8 \frac{tu^2}{v^3}$

61) $\log \frac{r^2 + 3}{(r^2 - 3)^2}$ 63) $\log_n 8\sqrt{k}$ 65) $\log_d \frac{\sqrt[3]{5}}{z^2}$

67) $\log_6 \frac{y}{3z^3}$ 69) $\log_3 \frac{t^4}{36u^2}$ 71) $\log_b \frac{\sqrt{c + 4}}{(c + 3)^2}$

73) $-\log(a^2 - b^2)$ 75) 1.6532 77) 1.9084

79) -0.2552 81) 0.4771 83) -0.6990 85) -1.9084

87) 1.6990 89) No. $\log_a xy$ is defined only if x and y are positive.

Section 13.5

1) e 3) 2 5) -3 7) -1 9) 9 11) $\frac{1}{4}$ 13) 6

15) $\frac{1}{2}$ 17) -5 19) 0 21) 1.2041 23) -0.3010

25) 1.0986 27) 0.2700 29) $\{1000\}$ 31) $\left\{\frac{1}{10}\right\}$ 33) $\{25\}$

35) $\{2\}$ 37) $\{10^{1.5}\}$; $\{31.6228\}$ 39) $\{10^{0.8}\}$; $\{6.3096\}$

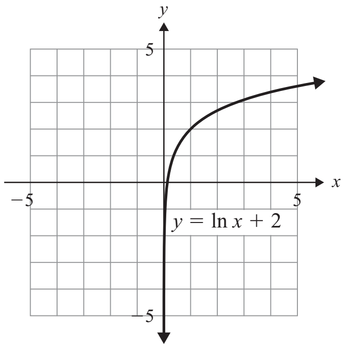
41) $\{e^{1.6}\}$; $\{4.9530\}$ 43) $\left\{\frac{1}{e^2}\right\}$; $\{0.1353\}$

45) $\left\{\frac{e^{2.1}}{3}\right\}$; $\{2.7221\}$ 47) $\{2 \cdot 10^{0.47}\}$; $\{5.9024\}$

49) $\left\{\frac{3 + 10^{3.8}}{5}\right\}$; $\{1262.5147\}$

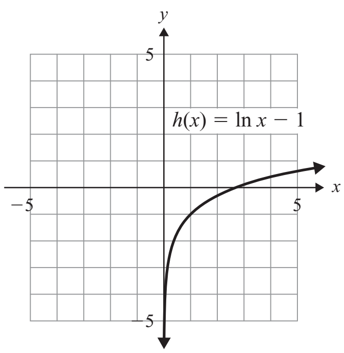
51) $\left\{\frac{e^{1.85} - 19}{10}\right\}$; $\{-1.2640\}$ 53) $\{3\}$

55)



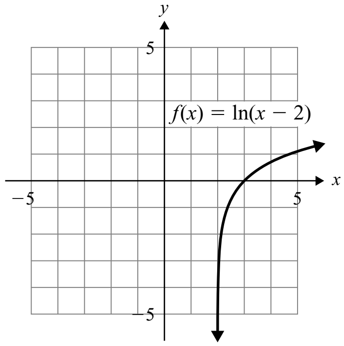
domain: $(0, \infty)$; range: $(-\infty, \infty)$

57)



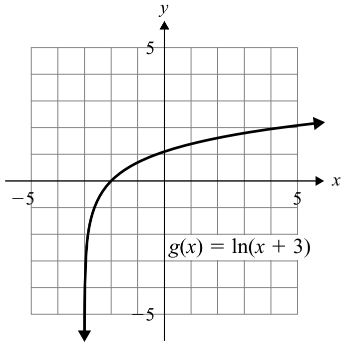
domain: $(0, \infty)$; range: $(-\infty, \infty)$

59)



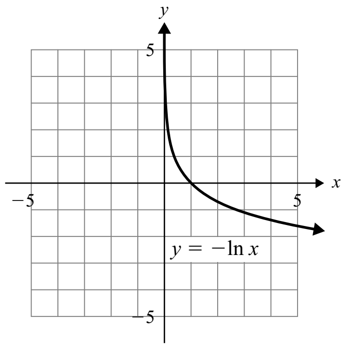
domain: $(2, \infty)$; range: $(-\infty, \infty)$

61)



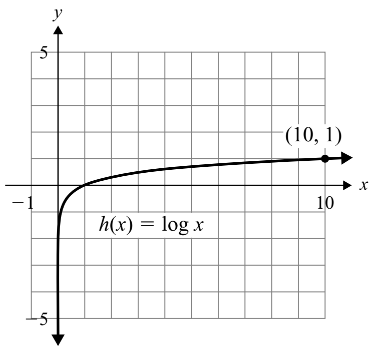
domain: $(-3, \infty)$; range: $(-\infty, \infty)$

63)



domain: $(0, \infty)$; range: $(-\infty, \infty)$

65)



domain: $(0, \infty)$; range: $(-\infty, \infty)$

67) Shift the graph of $f(x)$ left 5 units. 69) 3.7004

71) 1.9336 73) -2.5237 75) 0.6826 77) 110 dB

79) 40 dB 81) \$3484.42 83) \$5521.68 85) \$3485.50

87) \$15,683.12 89) a) 5000 b) 8191 91) 32,570

93) 2.7; acidic 95) 11.2; basic 97) Answers may vary.

Section 13.6

1) $\{2\}$ 3) $\left\{\frac{\ln 15}{\ln 7}\right\}$; $\{1.3917\}$ 5) $\left\{\frac{\ln 3}{\ln 8}\right\}$; $\{0.5283\}$

7) $\left\{\frac{2}{5}\right\}$ 9) $\left\{\frac{\ln 2.7}{6 \ln 4}\right\}$; $\{0.1194\}$

11) $\left\{\frac{\ln 5 - \ln 2}{4 \ln 2}\right\}$; $\{0.3305\}$ 13) $\left\{\frac{\ln 8 + 2 \ln 5}{3 \ln 5}\right\}$; $\{1.0973\}$

15) $\left\{\frac{10}{7}\right\}$ 17) $\left\{\frac{2 \ln 9}{5 \ln 9 - 3 \ln 4}\right\}$; $\{0.6437\}$

19) $\{\ln 12.5\}$; $\{2.5257\}$ 21) $\left\{-\frac{\ln 9}{4}\right\}$; $\{-0.5493\}$

23) $\left\{\frac{\ln 2}{0.01}\right\}$; $\{69.3147\}$ 25) $\left\{\frac{\ln 3}{0.006}\right\}$; $\{183.1021\}$

27) $\left\{-\frac{\ln 5}{0.4}\right\}$; $\{-4.0236\}$ 29) $\{2\}$ 31) $\left\{\frac{10}{3}\right\}$ 33) $\{5\}$

35) $\{2, 10\}$ 37) $\varnothing$ 39) $\{8\}$ 41) $\{2\}$ 43) $\{9\}$ 45) $\{2\}$

47) $\{4\}$ 49) $\left\{\frac{2}{3}\right\}$ 51) a) 3.72 yr b) 11.55 yr

53) 1.44 yr 55) \$2246.64 57) 7.2%

59) a) 6 hr b) 18.5 hr 61) 28,009 63) a) 2032 b) 2023

65) a) 11.78 g b) 3351 yr c) 5728 yr

67) a) 0.4 units b) 0.22 units 69) $\{16\}$ 71) $\{-2, 2\}$

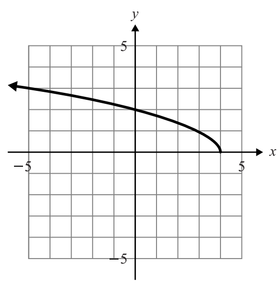
73) $\{-\ln 13, \ln 13\}$; $\{-2.5649, 2.5649\}$ 75) $\{0\}$

77) $\left\{\frac{\ln 7}{\ln 5}\right\}$; $\{1.2091\}$ 79) $\{1, 1000\}$

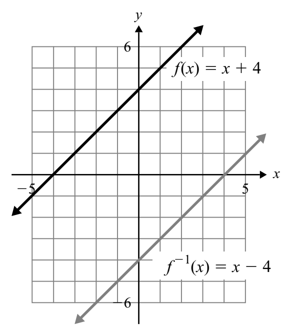
Chapter 13 Review Exercises

1) yes; $\{(-4, -7), (1, -2), (5, 1), (11, 6)\}$

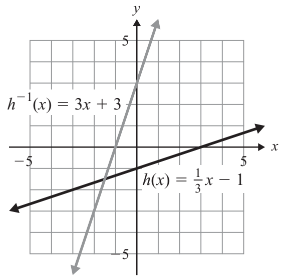
3) yes



5) $f^{-1}(x) = x - 4$

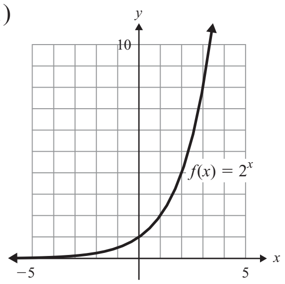


7) $h^{-1}(x) = 3x + 3$



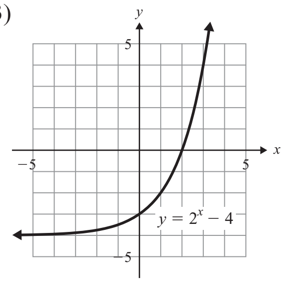
9) a) 11 b) 2

11)



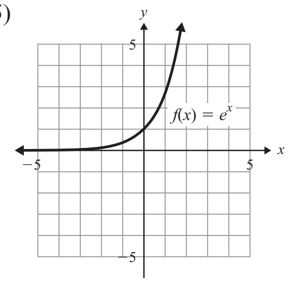
domain: $(-\infty, \infty)$;
range: $(0, \infty)$

13)



domain: $(-\infty, \infty)$;
range: $(-4, \infty)$

15)



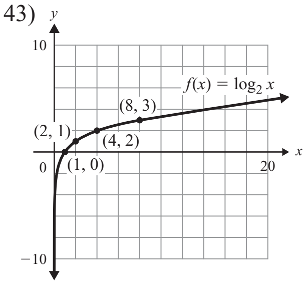
domain: $(-\infty, \infty)$;
range: $(0, \infty)$

17) $\{6\}$ 19) $\left\{-\frac{5}{2}\right\}$ 21) $\left\{\frac{2}{3}\right\}$ 23) $(0, \infty)$ 25) $5^3 = 125$

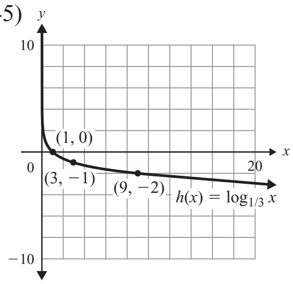
27) $10^2 = 100$ 29) $\log_3 81 = 4$ 31) $\log 1000 = 3$ 33) $\{8\}$

35) $\left\{\frac{4}{5}\right\}$ 37) 2 39) 3 41) -4

43)



45)



47) $f^{-1}(x) = \log_5 x$ 49) $h^{-1}(x) = 6^x$ 51) false

53) $\log_8 3 + \log_8 z$ 55) $\frac{3}{2}$ 57) $4 \log_5 c + 3 \log_5 d$

59) $\log_a x + \log_a y - 3 \log_a z$ 61) $\log p + \log(p + 8)$

63) $\log cd$ 65) $\log_2 a^9 b^3$ 67) $\log_3 \frac{5m^4}{n^2}$ 69) $\log_5 \frac{c^3}{df^2}$

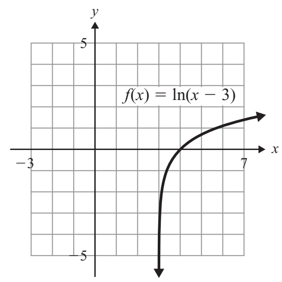
71) 1.6902 73) -0.1091 75) e 77) 1 79) $\frac{1}{2}$ 81) -3

83) 0 85) 0.9031 87) 0.5596 89) $\{100\}$ 91) $\left\{\frac{1}{5}\right\}$

93) $\{10^{2.1}\}$; $\{125.8925\}$ 95) $\{e^2\}$; $\{7.3891\}$

97) $\left\{\frac{10^{1.75}}{4}\right\}$; $\{14.0585\}$

99)



domain: $(3, \infty)$;
range: $(-\infty, \infty)$

101) 2.1240 103) -5.2479 105) 110 dB 107) \$3367.14

109) \$11,533.14 111) a) 6000 b) 11,118 113) $\{4\}$

115) $\left\{\frac{\ln 2}{4 \ln 9}\right\}$; $\{0.0789\}$ 117) $\left\{\frac{5 \ln 8}{\ln 8 - 2 \ln 6}\right\}$; $\{-6.9127\}$

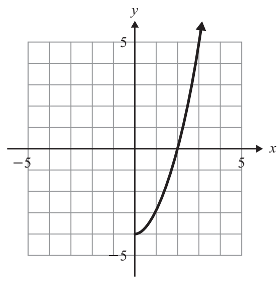
119) $\left\{\frac{\ln 8}{5}\right\}$; $\{0.4159\}$ 121) $\left\{\frac{6}{5}\right\}$ 123) $\{4\}$ 125) $\{16\}$

127) \$6770.57 129) a) 17,777 b) 2011

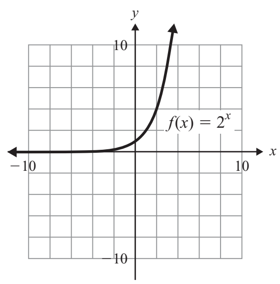
Chapter 13 Test

1) no 2) yes; $g^{-1} = \left\{(4, 2), (6, 6), \left(\frac{15}{2}, 9\right), (10, 14)\right\}$

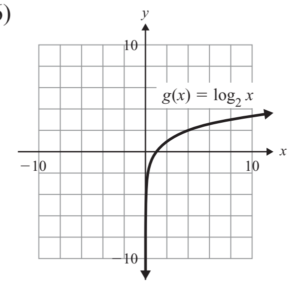
3) yes 4) $f^{-1}(x) = -\frac{1}{3}x + 4$



5)



6)



7) a) $(0, \infty)$ b) $(-\infty, \infty)$ 8) They are inverses.

9) $\log_3 \frac{1}{9} = -2$ 10) $\left\{\frac{1}{2}\right\}$ 11) $\{-2\}$ 12) $\{125\}$

13) $\{29\}$ 14) $\{4\}$ 15) a) 4 b) $\frac{1}{2}$

16) 1 17) $\log_8 5 + \log_8 n$

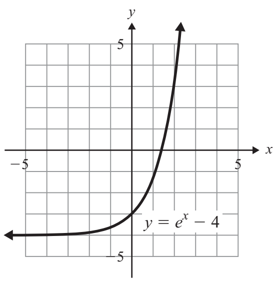
18) $2 + 4\log_3 a - 5\log_3 b - \log_3 c$

19) $\log \frac{x^2}{(x+1)^3}$ 20) $\{10^{0.08}\}$; $\{6.3096\}$

21) $\left\{\frac{\ln 5}{0.3}\right\}$; $\{5.3648\}$ 22) $\{e^{-0.25}\}$; $\{0.7788\}$

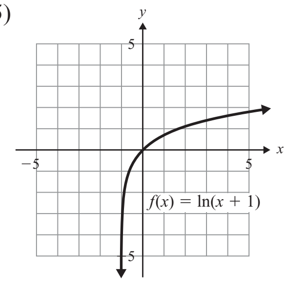
23) $\left\{\frac{\ln 9 - 3\ln 4}{4\ln 4}\right\}$; $\{-0.3538\}$

24)



domain: $(-\infty, \infty)$;
range: $(-4, \infty)$

25)



domain: $(-1, \infty)$;
range: $(-\infty, \infty)$

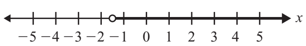
26) 1.7604 27) \$8686.41

28) a) 86.2 g b) 325.1 days c) 140 days

Cumulative Review for Chapters 1–13

1) 35 2) $-\frac{1}{2}$ 3) $-15a^6$ 4) $4z^8$ 5) $\frac{d^9}{8c^{30}}$

6) 9.231×10^{-5} 7) \$48.00

8) 

; $\left(-\frac{3}{2}, \infty\right)$

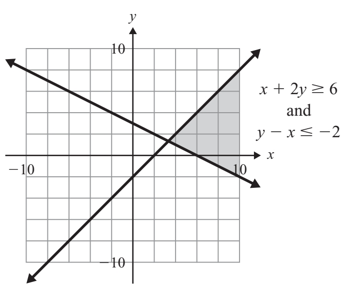
9) $(-6, 1)$ 10) $\left(\frac{1}{3}, -2\right)$ 11) $y = -\frac{3}{2}x + 2$

12) $3c^2 + 4c - 1$ 13) $(4w + 9)(w - 2)$

14) $(p + 1)(p - 1)(3p + 2)$ 15) $(y - 3)^2$ 16) $\{-8, -6\}$

17) $\frac{r^2 - 12r + 21}{(r + 7)(r - 7)(r - 9)}$ 18) $\{2\}$

19)



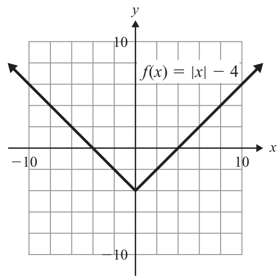
20) $2\sqrt{30}$ 21) $3t^4\sqrt[4]{5t}$ 22) $6a$ 23) 9 24) $\varnothing$

25) $13 - 19i$ 26) $\{4 - 2\sqrt{3}, 4 + 2\sqrt{3}\}$

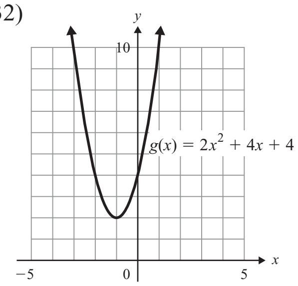
27) $\left\{-\frac{5}{2} - \frac{\sqrt{17}}{2}, -\frac{5}{2} + \frac{\sqrt{17}}{2}\right\}$ 28) $\{5 + 4i, 5 - 4i\}$

29) $\left\{-2, -\frac{1}{2}, \frac{1}{2}, 2\right\}$ 30) $\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \infty\right)$

31) domain: $(-\infty, \infty)$;
range: $[-4, \infty)$

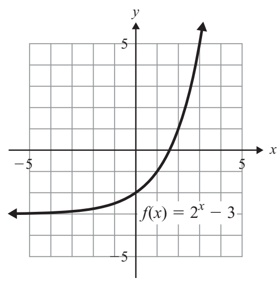


32)



33) a) 9 b) $x^2 - 12x + 29$ c) -4

34)

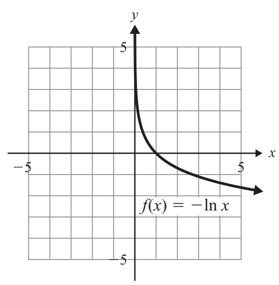


domain: $(-\infty, \infty)$; range: $(-3, \infty)$

35) $\left\{-\frac{3}{2}\right\}$ 36) $\{3\}$ 37) $\log \frac{ab^2}{c^5}$ 38) $\{4\}$

39) $\left\{-\frac{\ln 6}{0.04}\right\}$; $\{-44.7940\}$

40)



domain: $(0, \infty)$; range: $(-\infty, \infty)$

Chapter 14

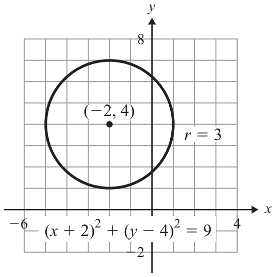
Section 14.1

1) $(4, 6)$ 3) $(-3, -3)$ 5) $\left(-1, -\frac{9}{2}\right)$ 7) $\left(\frac{1}{2}, -\frac{5}{2}\right)$

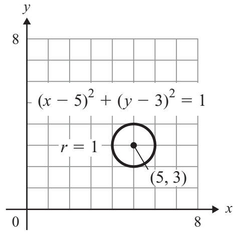
9) $\left(2, \frac{5}{4}\right)$ 11) $(-0.7, 3.6)$

13) No; there are values in the domain that give more than one value in the range. The graph fails the vertical line test.

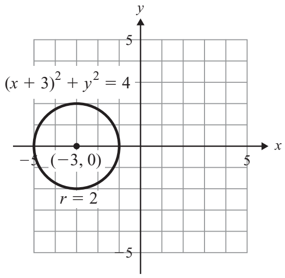
15) center: $(-2, 4)$; $r = 3$



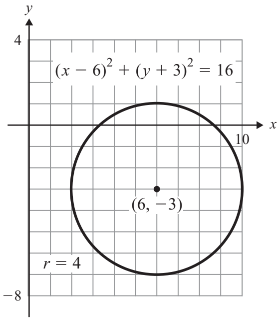
17) center: $(5, 3)$; $r = 1$



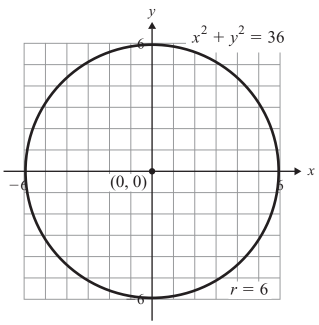
19) center: $(-3, 0)$; $r = 2$



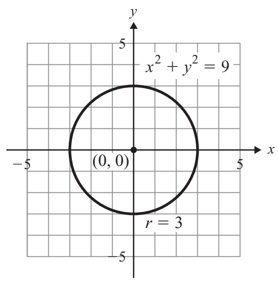
21) center: $(6, -3)$; $r = 4$



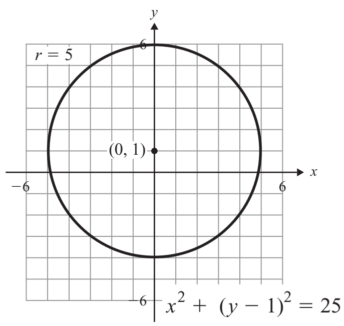
23) center: $(0, 0)$; $r = 6$



25) center: $(0, 0)$; $r = 3$



27) center: $(0, 1)$; $r = 5$



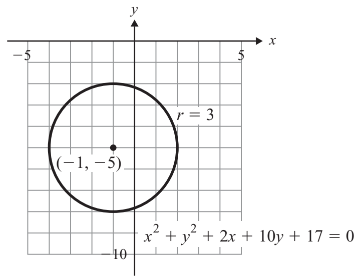
29) $(x - 4)^2 + (y - 1)^2 = 25$ 31) $(x + 3)^2 + (y - 2)^2 = 1$

33) $(x + 1)^2 + (y + 5)^2 = 3$ 35) $x^2 + y^2 = 10$

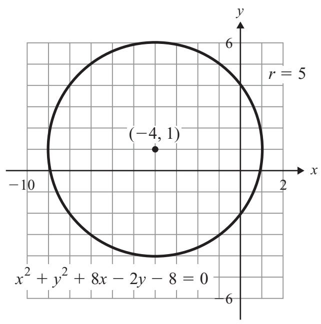
37) $(x - 6)^2 + y^2 = 16$ 39) $x^2 + (y + 4)^2 = 8$

41) Group x and y terms separately;
$(x^2 - 8x + 16) + (y^2 + 2y + 1) = -8 + 16 + 1$;
$(x - 4)^2 + (y + 1)^2 = 9$

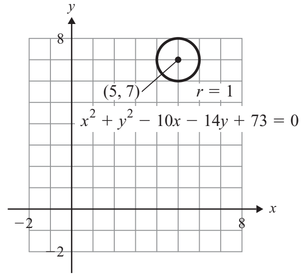
43) $(x + 1)^2 + (y + 5)^2 = 9$;
center: $(-1, -5)$; $r = 3$



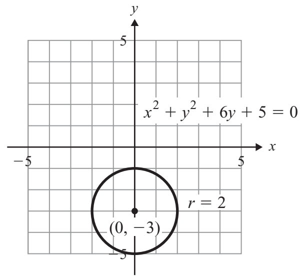
45) $(x + 4)^2 + (y - 1)^2 = 25$;
center: $(-4, 1)$; $r = 5$



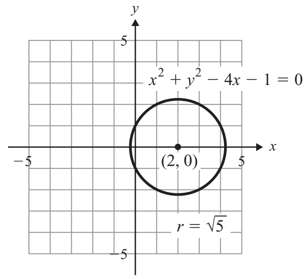
47) $(x - 5)^2 + (y - 7)^2 = 1$;
center: $(5, 7)$; $r = 1$



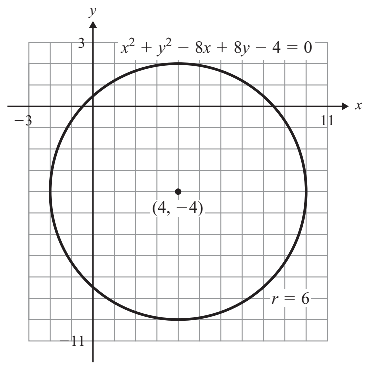
49) $x^2 + (y + 3)^2 = 4$;
center: $(0, -3)$; $r = 2$



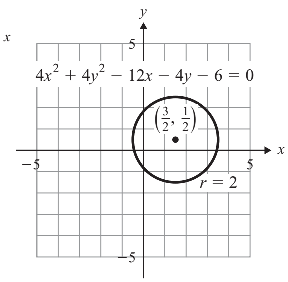
51) $(x - 2)^2 + y^2 = 5$;
center: $(2, 0)$; $r = \sqrt{5}$



53) $(x - 4)^2 + (y + 4)^2 = 36$;
center: $(4, -4)$; $r = 6$



55) $\left(x - \dfrac{3}{2}\right)^2 + \left(y - \dfrac{1}{2}\right)^2 = 4$;
center: $\left(\dfrac{3}{2}, \dfrac{1}{2}\right)$; $r = 2$



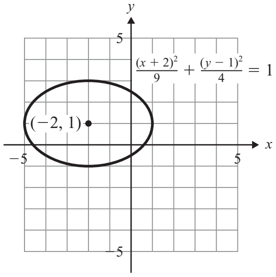
57) a) 128 m b) 64 m c) $(0, 71)$ d) $x^2 + (y - 71)^2 = 4096$

59) 11,127 mm^2

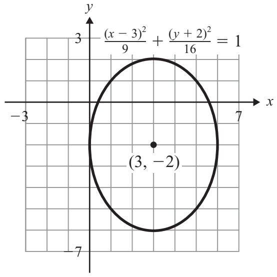
Section 14.2

1) ellipse 3) hyperbola 5) hyperbola 7) ellipse

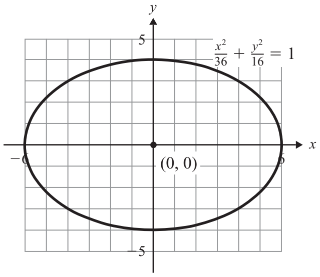
9) center: $(-2, 1)$



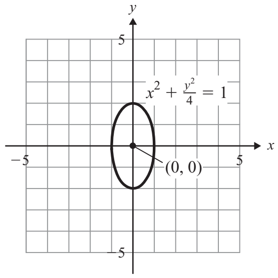
11) center: $(3, -2)$



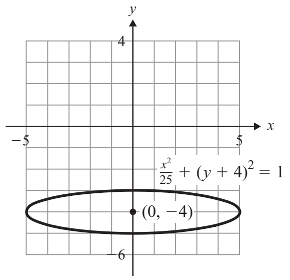
13) center: $(0, 0)$



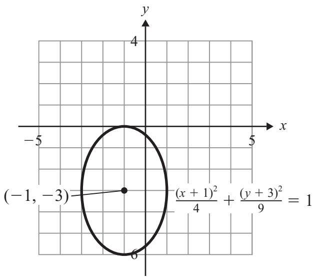
15) center: $(0, 0)$



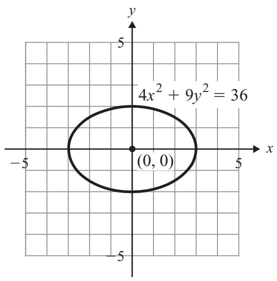
17) center: $(0, -4)$



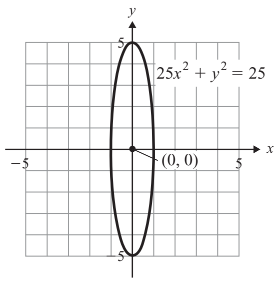
19) center: $(-1, -3)$



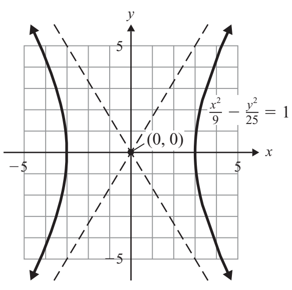
21) center: $(0, 0)$



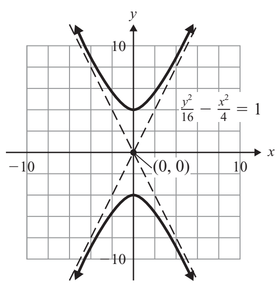
23) center: $(0, 0)$



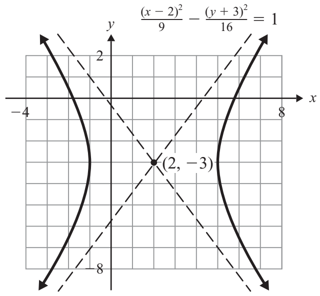
25) center: $(0, 0)$



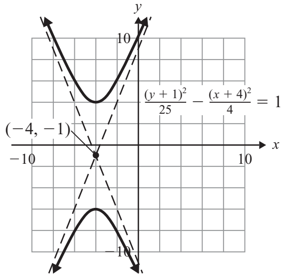
27) center: $(0, 0)$



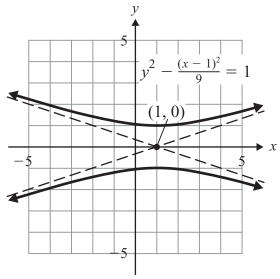
29) center: $(2, -3)$



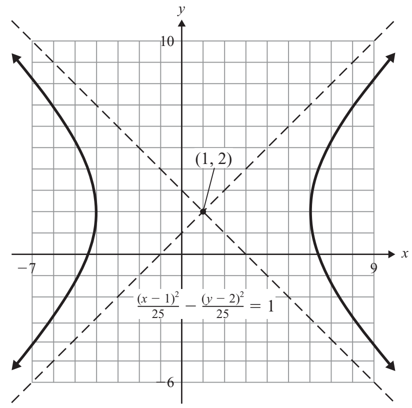
31) center: $(-4, -1)$



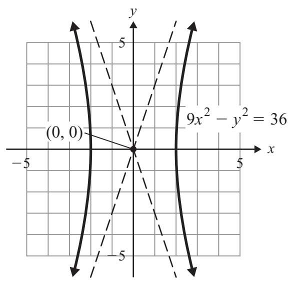
33) center: $(1, 0)$



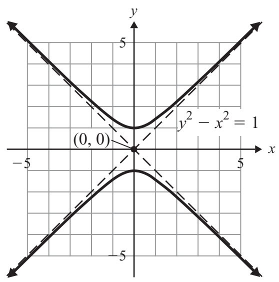
35) center: $(1, 2)$



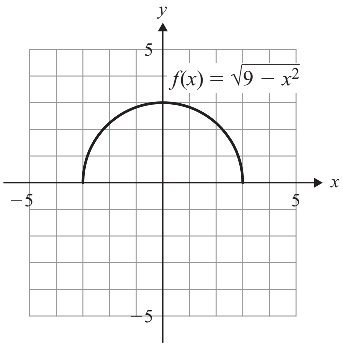
37) center: $(0, 0)$



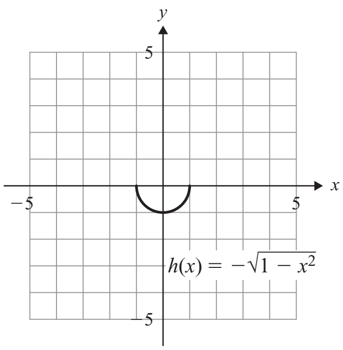
39) center: $(0, 0)$



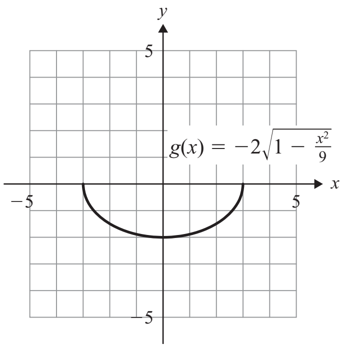
41) domain: $[-3, 3]$; range: $[0, 3]$



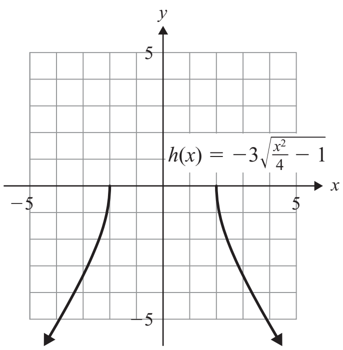
43) domain: $[-1, 1]$; range: $[-1, 0]$



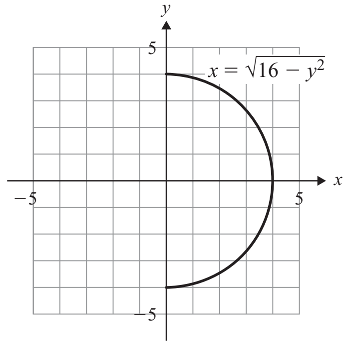
45) domain: $[-3, 3]$; range: $[-2, 0]$



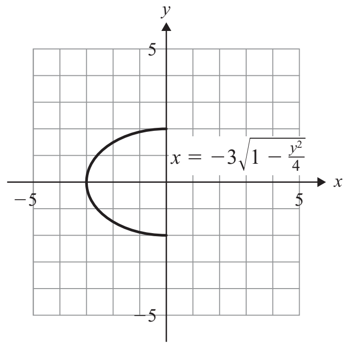
47) domain: $(-\infty, -2] \cup [2, \infty)$; range: $(-\infty, 0]$



49)



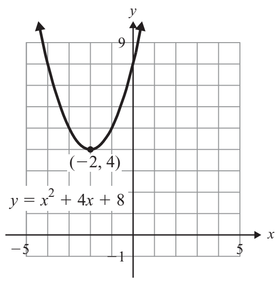
51)



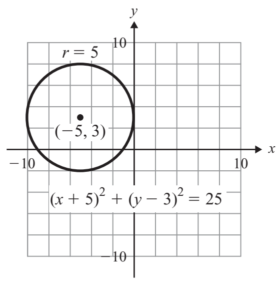
53) $\dfrac{x^2}{324} + \dfrac{y^2}{210.25} = 1$

Chapter 14 Putting It All Together

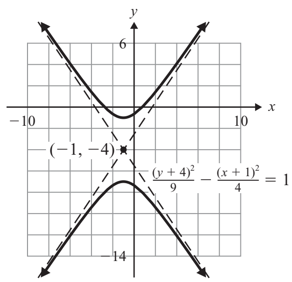
1) parabola



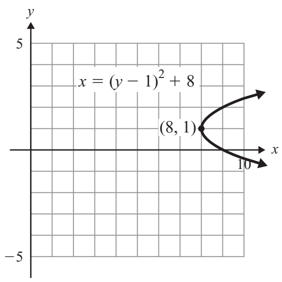
2) circle



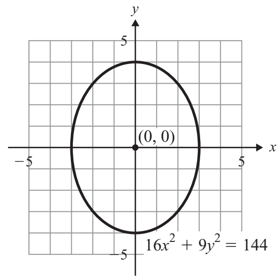
3) hyperbola



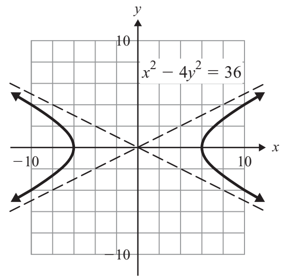
4) parabola



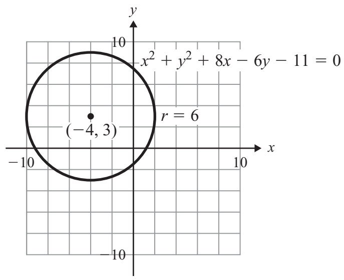
5) ellipse



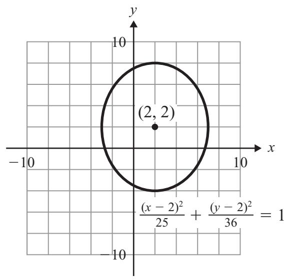
6) hyperbola



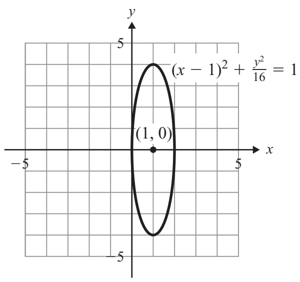
7) circle



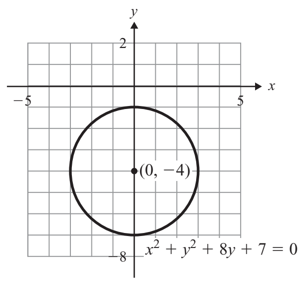
8) ellipse



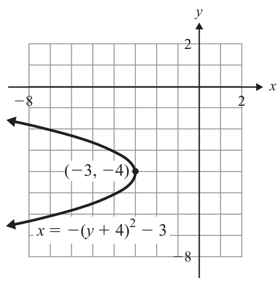
9) ellipse



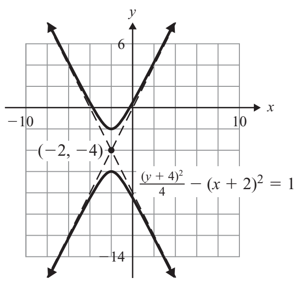
10) circle



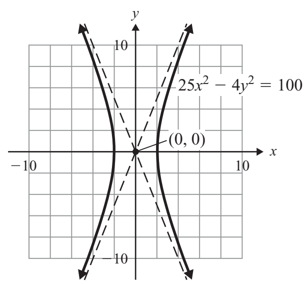
11) parabola



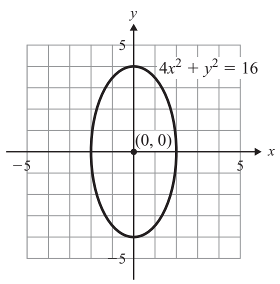
12) hyperbola



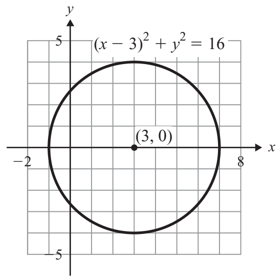
13) hyperbola



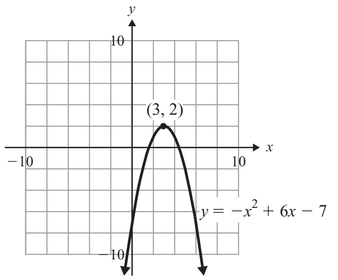
14) ellipse



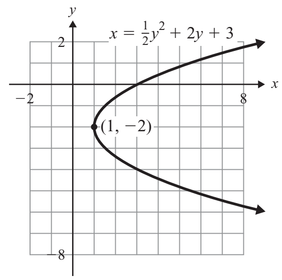
15) circle



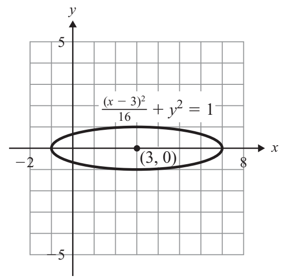
16) parabola



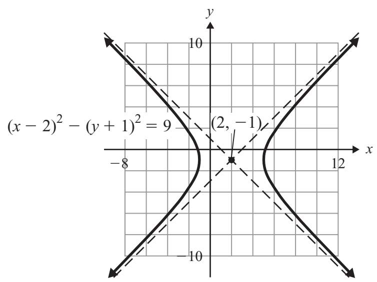
17) parabola



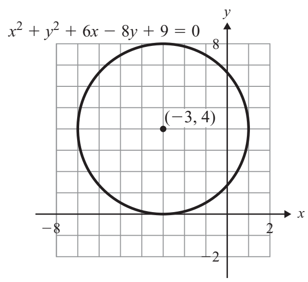
18) ellipse



19) hyperbola

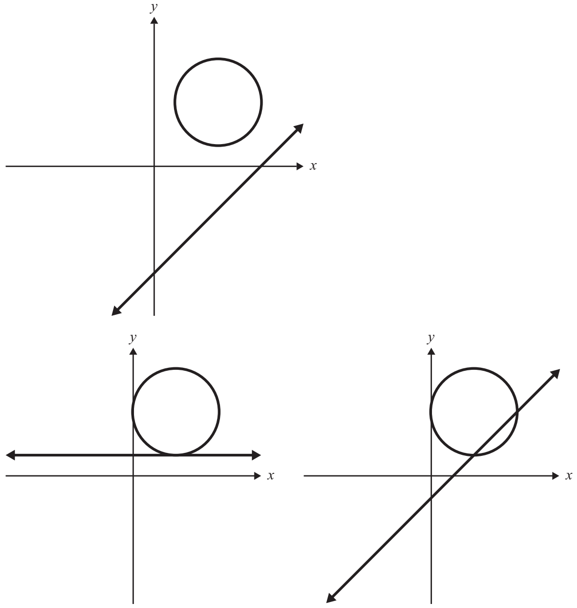


20) circle

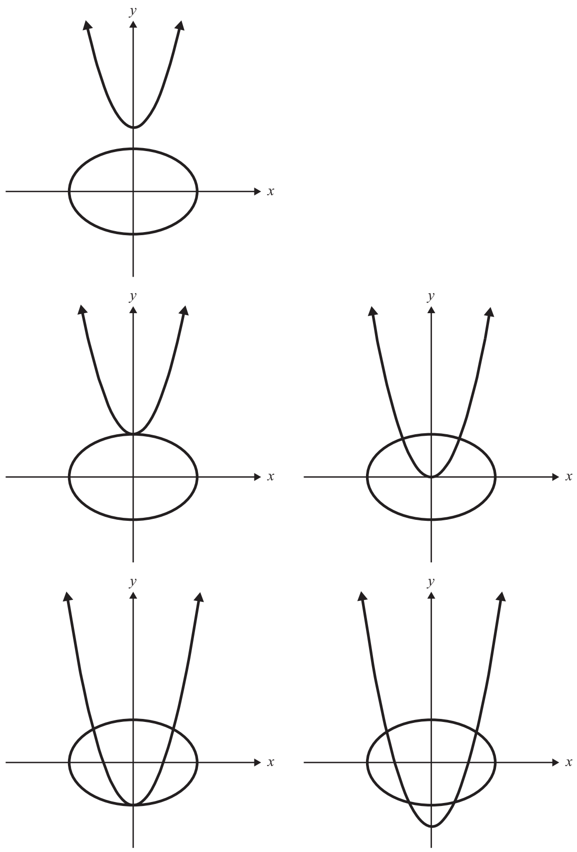


Section 14.3

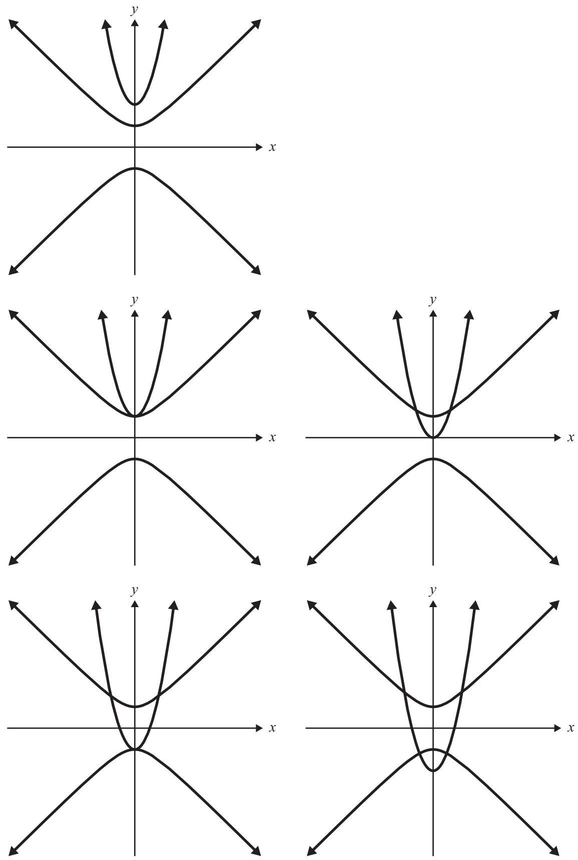
1) c) 0, 1, or 2



3) c) 0, 1, 2, 3, or 4



5) c) 0, 1, 2, 3, or 4



7) $\{(-4, -2), (6, -7)\}$ 9) $\{(-1, 3), (3, 1)\}$ 11) $\{(4, 2)\}$

13) $\{(3, 1), (3, -1), (-3, 1), (-3, -1)\}$

15) $\{(2, \sqrt{2}), (2, -\sqrt{2}), (-2, \sqrt{2}), (-2, -\sqrt{2})\}$

17) $\{(3, -7), (-3, -7)\}$ 19) $\{(0, -1)\}$ 21) $\{(0, 2)\}$

23) $\varnothing$ 25) $\varnothing$

27) $\left\{\left(\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)\right\}$

29) $\{(0, -2)\}$ 31) 8 and 5 33) 8 in. $\times$ 11 in.

35) 4000 basketballs; $240

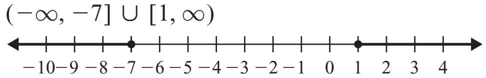
Section 14.4

1) The endpoints are included when the inequality symbol is $\le$ or $\ge$. The endpoints are not included when the symbol is $<$ or $>$.

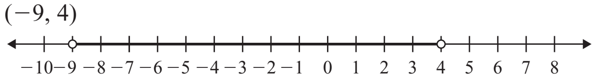
3) a) $[-5, 1]$ b) $(-\infty, -5) \cup (1, \infty)$

5) a) $[-1, 3]$ b) $(-\infty, -1) \cup (3, \infty)$

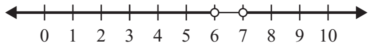
7) $(-\infty, -7] \cup [1, \infty)$



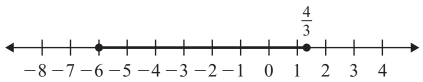
9) $(-9, 4)$



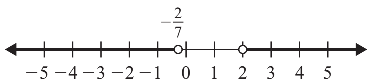
11) $(-\infty, 6) \cup (7, \infty)$



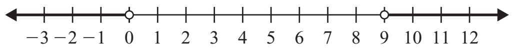
13) $\left[-6, \frac{4}{3}\right]$



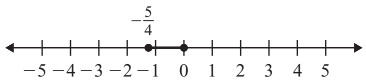
15) $\left(-\infty, -\frac{2}{7}\right) \cup (2, \infty)$



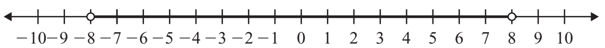
17) $(-\infty, 0) \cup (9, \infty)$



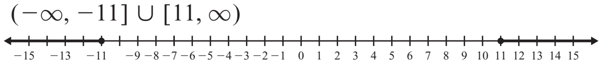
19) $\left[-\frac{5}{4}, 0\right]$



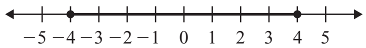
21) $(-8, 8)$



23) $(-\infty, -11] \cup [11, \infty)$

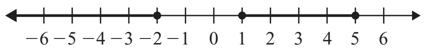


25) $[-4, 4]$

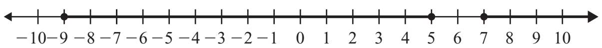


27) $(-\infty, \infty)$ 29) $(-\infty, \infty)$ 31) $\varnothing$ 33) $\varnothing$

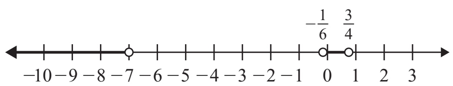
35) $(-\infty, -2] \cup [1, 5]$



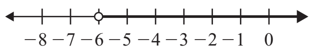
37) $[-9, 5] \cup [7, \infty)$



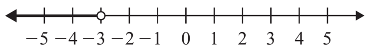
39) $(-\infty, -7) \cup \left(-\frac{1}{6}, \frac{3}{4}\right)$



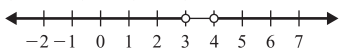
41) $(-6, \infty)$



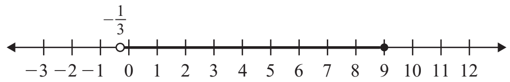
43) $(-\infty, -3)$



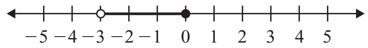
45) $(-\infty, 3) \cup (4, \infty)$



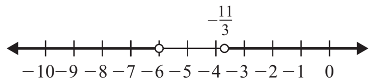
47) $\left(-\frac{1}{3}, 9\right]$



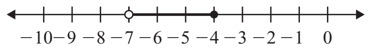
49) $(-3, 0]$



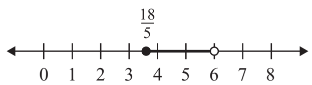
51) $(-\infty, -6) \cup \left(-\frac{11}{3}, \infty\right)$



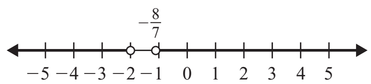
53) $(-7, -4]$



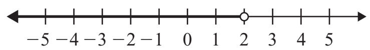
55) $\left[\frac{18}{5}, 6\right)$



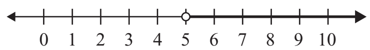
57) $(-\infty, -2) \cup \left(-\frac{8}{7}, \infty\right)$



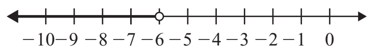
59) $(-\infty, 2)$



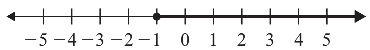
61) $(5, \infty)$



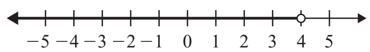
63) $(-\infty, -6)$



65) $[-1, \infty)$



67) $(-\infty, 4)$



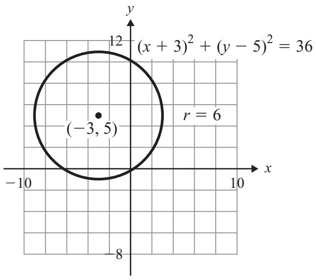
69) a) between 4000 and 12,000 units
b) when it produces less than 4000 units or more than 12,000 units

71) 10,000 or more

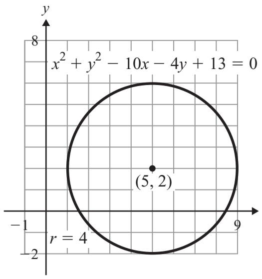
Chapter 14 Review Exercises

1) $(4, 5)$ 3) $\left(\frac{13}{2}, -\frac{7}{2}\right)$

5) center: $(-3, 5)$; $r = 6$



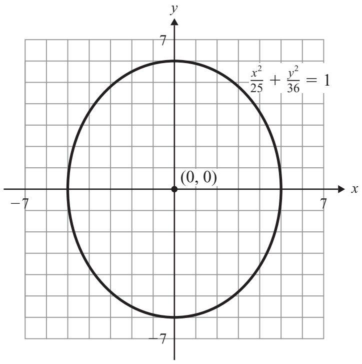
7) center: $(5, 2)$; $r = 4$



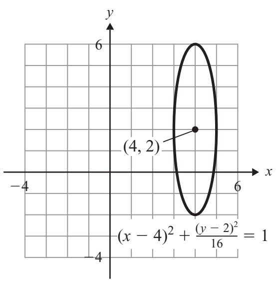
9) $(x - 3)^2 + y^2 = 16$

11) The equation of an ellipse contains the sum of two squares, $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$, but the equation of a hyperbola contains the difference of two squares, $\dfrac{(x - h)^2}{a^2} - \dfrac{(y - k)^2}{b^2} = 1$ or $\dfrac{(y - k)^2}{b^2} - \dfrac{(x - h)^2}{a^2} = 1$.

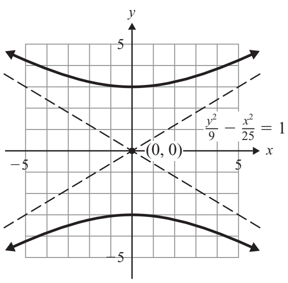
13) center: $(0, 0)$



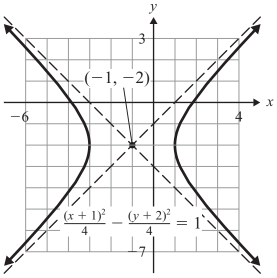
15) center: $(4, 2)$



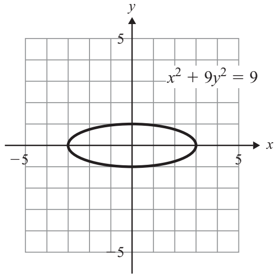
17) center: $(0, 0)$



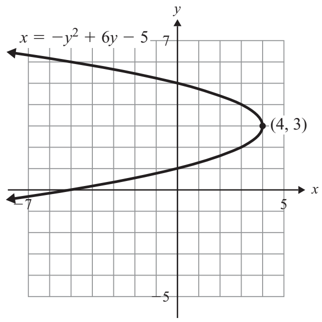
19) center: $(-1, -2)$



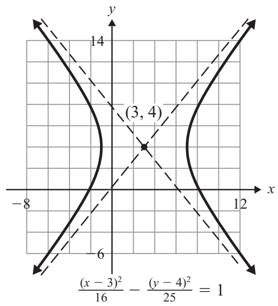
21) ellipse



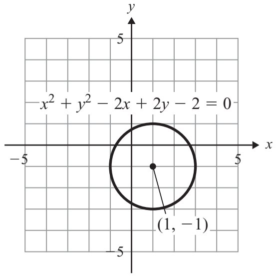
23) parabola



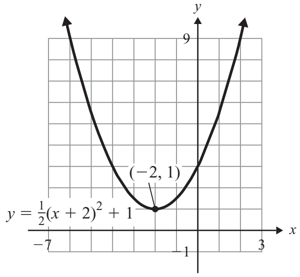
25) hyperbola



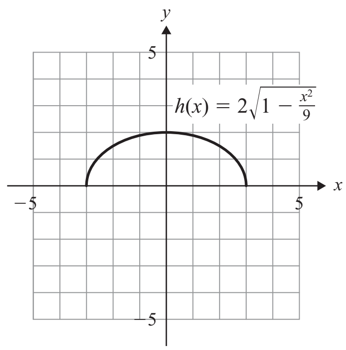
27) circle



29) parabola



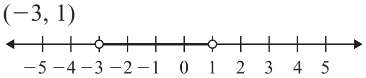
31) domain: $[-3, 3]$; range: $[0, 2]$



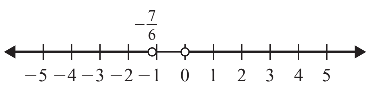
33) 0, 1, 2, 3, or 4 35) $\{(6, 7), (6, -7), (-6, 7), (-6, -7)\}$

37) $\{(1, 2), (-2, -1)\}$ 39) $\varnothing$ 41) 9 and 4

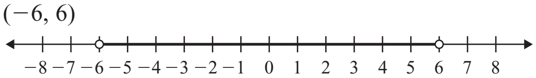
43) $(-3, 1)$



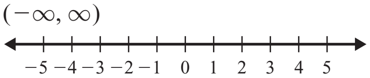
45) $\left(-\infty, -\dfrac{7}{6}\right) \cup (0, \infty)$



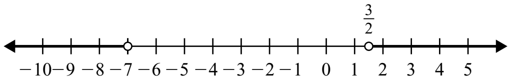
47) $(-6, 6)$



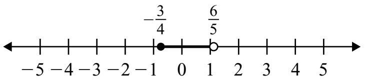
49) $(-\infty, \infty)$



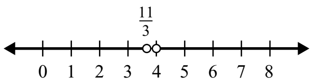
51) $(-\infty, -7) \cup \left(\dfrac{3}{2}, \infty\right)$



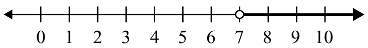
53) $\left[-\dfrac{3}{4}, \dfrac{6}{5}\right)$



55) $\left(-\infty, \dfrac{11}{3}\right) \cup (4, \infty)$



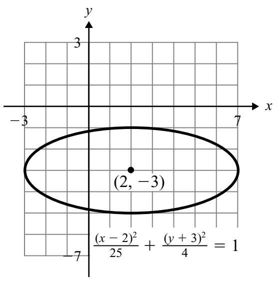
57) $(7, \infty)$



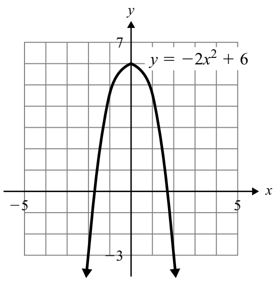
Chapter 14 Test

1) $\left(6, -\dfrac{5}{2}\right)$

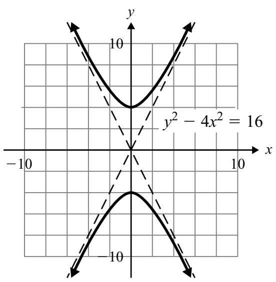
2) ellipse



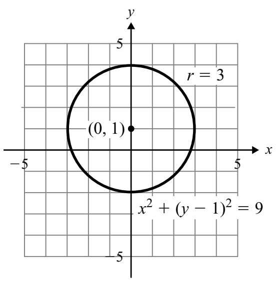
3) parabola



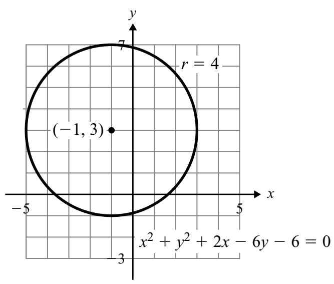
4) hyperbola



5) circle

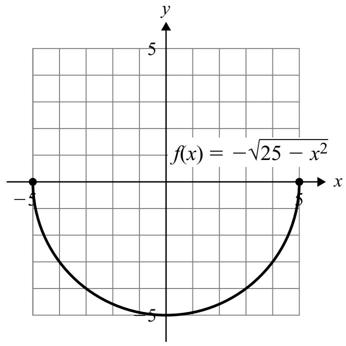


6) $(x + 1)^2 + (y - 3)^2 = 16$; center $(-1, 3)$; $r = 4$

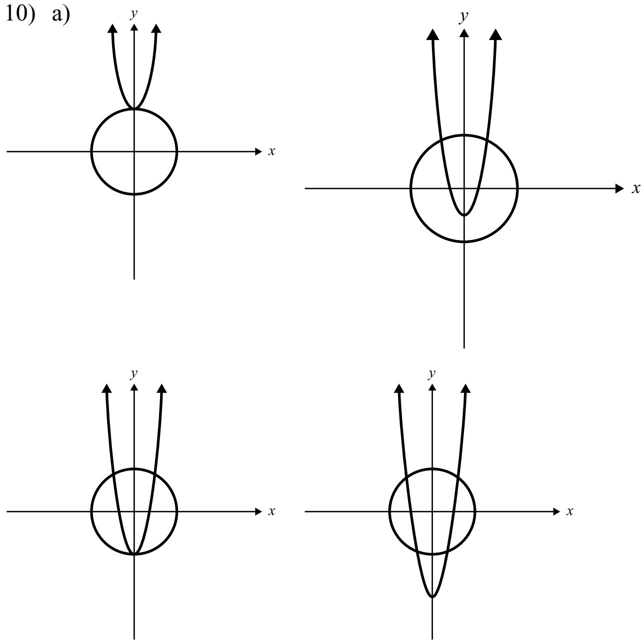


7) $(x - 5)^2 + (y - 2)^2 = 11$ 8) $\dfrac{x^2}{8836} + \dfrac{y^2}{6084} = 1$

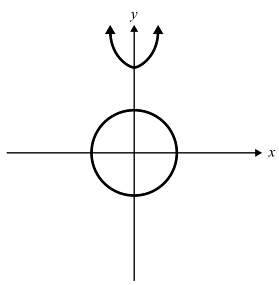
9) domain: $[-5, 5]$; range: $[-5, 0]$



10) a)



b)



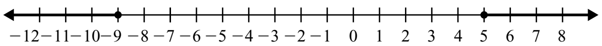
c) 0, 1, 2, 3, or 4

11) $\{(-1, 0), (7, -2)\}$

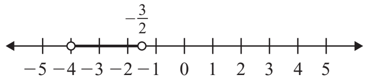
12) $\{(3, 1), (3, -1), (-3, 1), (-3, -1)\}$

13) $\{(2, \sqrt{3}), (2, -\sqrt{3})\}$ 14) 8 in. × 14 in.

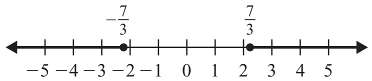
15) $(-\infty, -9] \cup [5, \infty)$



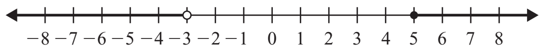
16) $\left(-4, -\frac{3}{2}\right)$



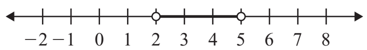
17) $\left(-\infty, -\frac{7}{3}\right] \cup \left[\frac{7}{3}, \infty\right)$



18) $(-\infty, -3) \cup [5, \infty)$



19) $(2, 5)$



Cumulative Review for Chapters 1–14

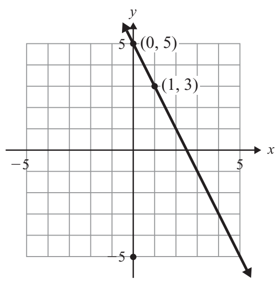
1) $-\frac{3}{4}$ 2) 12 3) $A = 15\text{ cm}^2$; $P = 18.5$ cm

4) $A = 128\text{ in}^2$; $P = 56$ in. 5) -1 6) 16 7) $\frac{1}{8a^{18}b^{15}}$

8) $\{-40\}$ 9) $n = \frac{c - z}{a}$ 10) $[-4, \infty)$ 11) 15, 17, 19

12) -1 13) 0

14)



15) $y = -\frac{3}{4}x + 4$ 16) $\left(-1, \frac{3}{2}\right)$

17) 5 mL of 8%, 15 mL of 16% 18) $-3p^2 + 7p + 6$

19) $8w^3 + 30w^2 - 47w + 15$ 20) $x^2 + 4x + 9$

21) $2(3c - 4)(c - 1)$ 22) $(m - 2)(m^2 + 2m + 4)$

23) $\{-2, 6\}$ 24) $-\frac{5}{2(a + 6)}$ 25) $6(t + 3)$

26) $\left\{-6, -\frac{4}{3}\right\}$ 27) $(-\infty, -3) \cup \left(\frac{9}{5}, \infty\right)$ 28) $5\sqrt{3}$

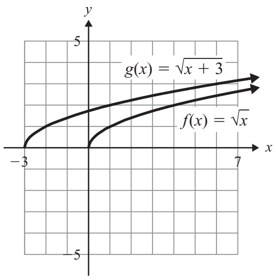
29) $2\sqrt[3]{6}$ 30) $3ab^4\sqrt[3]{a^2b}$ 31) $\frac{1}{8}$ 32) $3\sqrt{3}$

33) $\frac{20 - 5\sqrt{3}}{13}$ 34) $\left\{\frac{1}{2} + 2i, \frac{1}{2} - 2i\right\}$

35) $\left\{-\frac{7}{2} + \frac{\sqrt{37}}{2}, -\frac{7}{2} - \frac{\sqrt{37}}{2}\right\}$

36) a) $\{0, 3, 4\}$ b) $\{-1, 0, 1, 2\}$ c) no

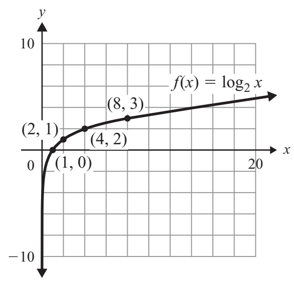
37)



38) a) -3 b) $6x - 9$ c) 6 39) a) no b) no

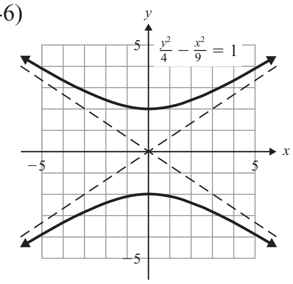
40) $f^{-1}(x) = 3x - 12$ 41) $\left\{-\frac{6}{13}\right\}$ 42) $\{5\}$ 43) 2

44)

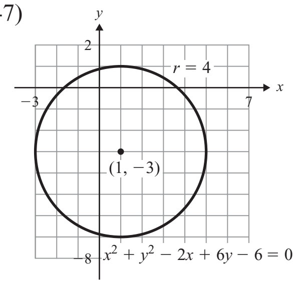


45) $\left\{\frac{\ln 8}{3}\right\}$; $\{0.6931\}$

46)



47)



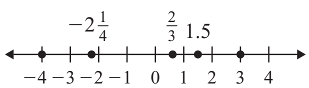
48) $(0, 3)$ 49) $\left[-\frac{12}{5}, \frac{12}{5}\right]$ 50) $\left(-\infty, -\frac{5}{2}\right) \cup (3, \infty)$

Appendix

Section A1

1) -6 3) 32 5) true

7)



9) $\frac{15}{28}$ 11) $\frac{3}{2}$ 13) $-\frac{1}{9}$ 15) 1 17) $2\frac{2}{3}$ 19) -14 21) 4

23) -22 25) $-\frac{8}{15}$ 27) $16 - 10$; 6 29) $2(-19 + 4)$; -30

31) $-14, 0, 5$ 33) $\frac{8}{11}, -14, 3.7, 5, 0, -1\frac{1}{2}, 6.\overline{2}$

35) 139° 37) 28°; obtuse 39) 68 cm^2

41) $A = 27\text{ in}^2$; $P = 24$ in.

43) a) $A = 25\pi\text{ cm}^2$; $A \approx 78.5\text{ cm}^2$
b) $C = 10\pi$ cm; $C \approx 31.4$ cm

45) a) 35 b) 30 47) $-\frac{1}{8}$ 49) distributive 51) associative

53) No; subtraction is not commutative. 55) $3 + 8$

57) $-5p$ 59) $-14w - 7$ 61) $30 - 42r$

63) $-24a + 32b + 8c$ 65) no 67) $3z^2 - 8z + 13$

69) $2n + 11$

Section A2

1) y^4 3) 64 5) $12m^{10}$ 7) $49k^{10}$ 9) $\frac{x^9}{y^9}$ 11) w^3

13) $2pq^5$ 15) 1 17) 125 19) $\frac{1}{x^8}$ 21) $\frac{64}{t^3}$ 23) $\frac{b^2}{7a^4}$

25) $\frac{s}{r^3}$ 27) $-\frac{8a^{15}b^3}{c^9}$ 29) $27c^{31}d^{10}$ 31) $\frac{1}{r^3s^{15}}$ 33) v^{20}

35) $\frac{a^{32}}{16b^{28}}$ 37) $A = 6c^3$ square units; $P = 6c^2 + 4c$ units

39) 0.000507 41) 9.4×10^4 43) 40,000

Section A3

1) yes 3) $\{4\}$ 5) $\{-4\}$ 7) $\{3\}$ 9) $\{-6\}$ 11) $\left\{-\frac{1}{2}\right\}$

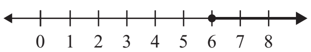
13) $\{3\}$ 15) $\{0\}$ 17) $\varnothing$ 19) $\{6\}$ 21) $\{-1\}$ 23) 26

25) Sweden: 9; Mexico: 11 27) \$6000 at 6% and \$2000 at 7%

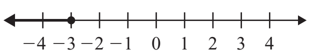
29) $m\angle A = 85°$, $m\angle B = 57°$ 31) 62

33) $H = \frac{3V}{A}$ 35) $b_2 = \frac{2A}{h} - b_1$ or $b_2 = \frac{2A - hb_1}{h}$

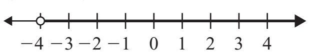
37) $[6, \infty)$



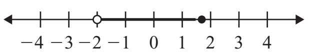
39) $(-\infty, -3]$



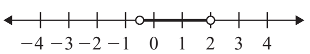
41) $(-4, \infty)$



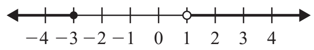
43) $\left(-2, \frac{5}{3}\right]$



45) $\left(-\frac{1}{2}, 2\right)$



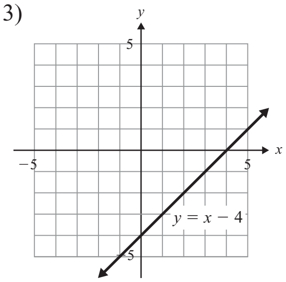
47) $(-\infty, -3] \cup (1, \infty)$



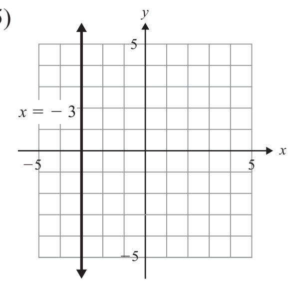
Section A4

1) yes

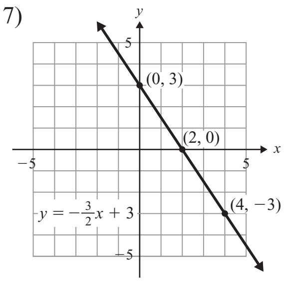
3)



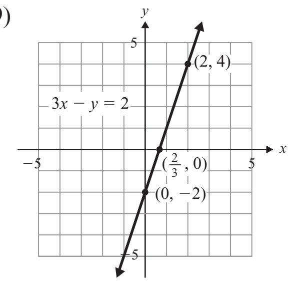
5)



7)

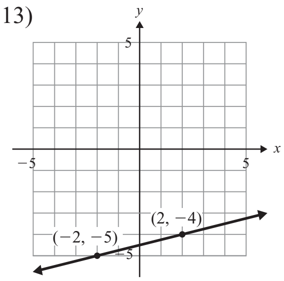


9)

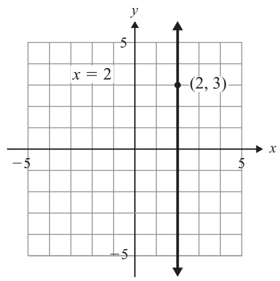


11) $\frac{5}{7}$

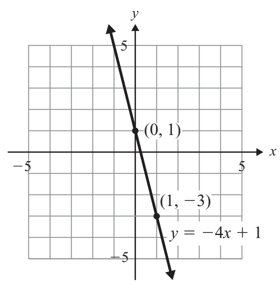
13)



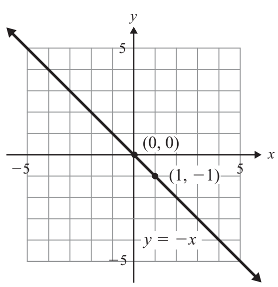
15)



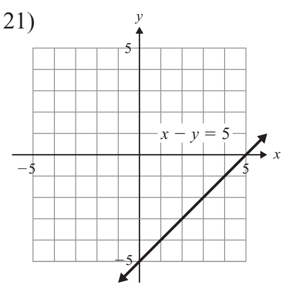
17) $m = -4$, y-int: $(0, 1)$



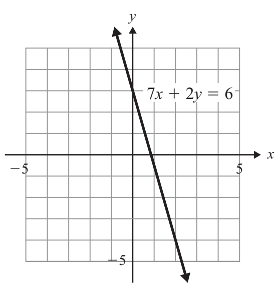
19) $m = -1$, y-int: $(0, 0)$



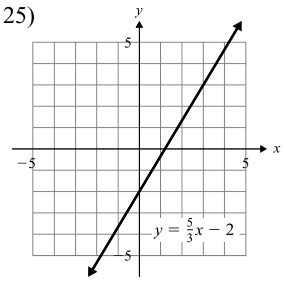
21)



23)



25)



27) $y = \dfrac{3}{8}x + 4$ 29) $y = -\dfrac{1}{2}x + \dfrac{11}{2}$ 31) $y = 7$

33) $6x + y = 2$ 35) $3x - y = -11$ 37) $y = -\dfrac{2}{3}x + 5$

39) $x = -1$

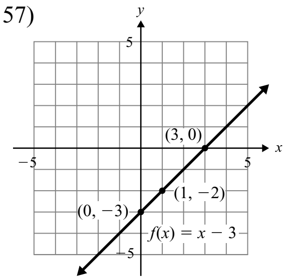
41) a) (0, 24000); If Naresh has \$0 in sales, his income is \$24,000.
b) $m = 0.10$; Naresh earns \$0.10 for \$1.00 in sales
c) \$34,000

43) domain: $(-\infty, \infty)$; range: $[-1, \infty)$; function

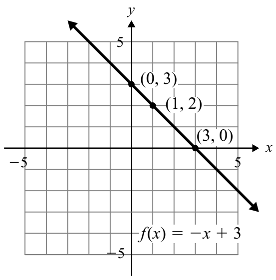
45) function; $(-\infty, \infty)$ 47) not a function; $[0, \infty)$

49) -3 51) $2z - 9$ 53) $2c - 3$ 55) 4 56) 3

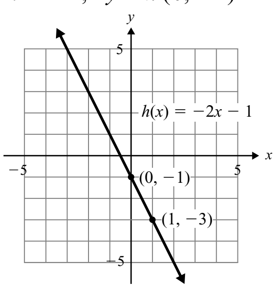
57)



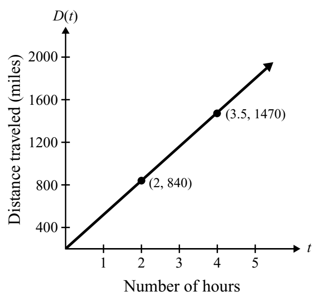
59)



61) $m = -2$, y-int: $(0, -1)$



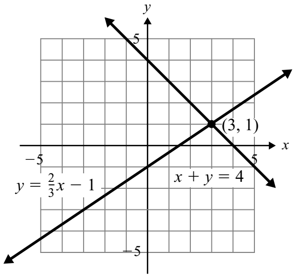
63) a) 840 mi b) 4.5 hours
c)



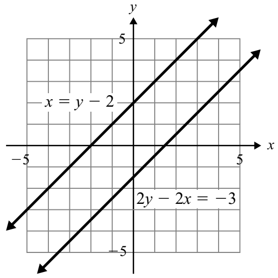
Section A5

1) yes

3) (3, 1)



5) $\varnothing$



7) (8, 1)

9) infinite number of solutions of the form $\{(x, y) | x = 2 - 5y\}$

11) $\left(-\dfrac{3}{2}, 4\right)$ 13) $\varnothing$ 15) $\varnothing$ 17) $(-9, 2)$ 19) $(-15, -6)$

21) $(2, -2)$ 23) length = 17 in., width = 10 in.

25) 24 oz of 18% solution and 36 oz of 8% solution

27) passenger train: 45 mph; freight train: 30 mph

29) $(-2, 1, 5)$ 31) $(5, 0, -2)$ 33) $(2, 2, 7)$

35) Ryan: 6; Seth: 1; Summer: 4

Section A6

1) -32 3) $\dfrac{64}{27}$ 5) $81z^{16}$ 7) $5t^{10}$ 9) $\dfrac{m^6}{125n^{18}}$

11) -8 13) $4a^2b^2 - 15a^2b + 5ab^2 + 17ab - 6$

15) a) 54 b) -26 17) $-14q^7 + 63q^5 + 42q^3$

19) $15b^4 - 4b^3 - 9b^2 + 43b + 18$

21) $-30d^6 + 72d^5 + 6d^4 - 2d^2 - 25d - 2$

23) $6r^2 + 17r + 7$ 25) $24d^4 - 54d^3 - 15d^2$ 27) $9a^2 - 16$

29) $h^4 - \dfrac{9}{64}$ 31) $r^2 - 20r + 100$ 33) $\dfrac{9}{16}c^2 - 9c + 36$

35) $v^3 + 15v^2 + 75v + 125$ 37) $-11h^2 + 4h + 1$

39) $-b - \dfrac{b}{3a} + \dfrac{9}{a}$ 41) $a + 3$ 43) $3x + 1 - \dfrac{28}{2x + 3}$

45) $5b^2 + 4b + 3$ 47) $6h^2 - 5h - 1 + \dfrac{2h - 7}{2h^2 + 3h - 1}$

49) a) $2b^2 + 6b - 12$ units b) $2b^3 + 2b^2 - 12b$ square units

51) $\dfrac{9}{2}c^2 - \dfrac{3}{2}c$ square units 53) $3a^2 - a - 7$ units

Section A7

1) $4(7k + 2)$ 3) $2yz^2(13y^2z + 4y^2 - 6z)$

5) $(v + 12)(w + 3)$ 7) $(3a - 8)(7b - 6)$

9) $(x + 5)(x + 2)$ 11) $2d^2(d - 4)(d + 3)$

13) $(3n + 4)(n + 2)$ 15) $(2z + 1)(4z - 9)$

17) $4(3c - 4)(c - 1)$ 19) $(x - 4y)^2$

21) $(m + 3)(m - 3)$ 23) $q^4(5q + 3)(5q - 3)$

25) prime 27) $(4s - 3t)(16s^2 + 12st + 9t^2)$

29) $9(5m - 6)$ 31) $(2t + 9)(2t - 9)$ 33) $(w + 16)(w - 3)$

35) $2k^2(k - 4)(k - 1)$ 37) $(10r - 3)(r - 4)$

39) $(c + 1)(d - 1)$ 41) $(m + 10)(m - 5)$

43) $\{-8, 3\}$ 45) $\{-4, -3\}$ 47) $\left\{0, \frac{5}{4}\right\}$

49) $\{-13, 13\}$ 51) $\{-9, -3\}$ 53) $\{-4, 0\}$ 55) $\left\{\frac{1}{2}, \frac{9}{2}\right\}$

57) $\{-6, -1, 0\}$ 59) 10

61) length of wire $=$ 17 ft; height of pole $=$ 15 ft

Section A8

1) a) $\frac{3}{2}$ b) 0

3) a) -3 or 3
b) never undefined—any real number may be substituted for x.

5) $(-\infty, -9) \cup (-9, \infty)$

7) $(-\infty, -1) \cup (-1, 6) \cup (6, \infty)$

9) $\frac{5z^2}{6}$ 11) $\frac{12}{7}$ 13) $\frac{a - 2}{a + 7}$ 15) $\frac{t(3t - 1)}{t + 2}$ 17) $-\frac{r + 4}{4}$

19) $\frac{3}{8w^3} = \frac{9}{24w^3}; \frac{1}{6w} = \frac{4w^2}{24w^3}$ 21) $\frac{13}{6j}$ 23) $\frac{11w - 14}{w(w - 2)}$

25) $\frac{a^2 + 26a - 18}{(4a - 3)(a + 2)}$ 27) $\frac{k + 2}{k + 7}$ 29) $\frac{r}{3(2r - 3)}$

31) $\frac{w - 4}{(2w - 3)(w + 2)}$ 33) $2x$ 35) $-\frac{1}{5m(m^2 + 3m + 9)}$

37) $\frac{6x - 1}{x - 8}$ 39) $\frac{14}{15}$ 41) $\frac{8}{r}$ 43) $\frac{m^2 + 5}{2m + 3}$ 45) $\frac{2(w - 1)}{3}$

47) $\{-3\}$ 49) $\{2, 8\}$ 51) $\{8\}$ 53) $\{-2, 10\}$ 55) $\{2\}$

57) $y = mx + b$ 59) $d = \frac{ab + Ct}{C}$

61) a) $\frac{5(a - 9)}{a - 2}$ square units b) $\frac{a^2 - 11a + 98}{2(a - 2)}$ units

63) 45 jazz and 27 reggae

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