BUSINESS MATHEMATICS AND STATISTICS

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Business Mathematics and Statistics

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To my revered father
Late Sri Niranjan Manna

Preface

The University of Calcutta has prescribed a new syllabus for undergraduate courses in Commerce under the framework of Choice-Based Credit System (CBCS) of the UGC from the academic session 2017–2018. The paper entitled *Business Mathematics and Statistics* is included in the third semester (GE 3.3chg) for undergraduate honours and general students of B.Com. The paper consists of two modules, viz. Module I: Business Mathematics, and Module II: Statistics.

So far as examination is concerned, this paper is very scoring, but in the result sheet of the university examination, it is not reflected with regard to the students of commerce. This prompted me to know the reasons behind this scenario and inspired me to write a book in lucid language, showing detailed workings and step-by-step details of solving practical problems, so that the students can gain interest and can score well. The book has been written in such a manner that the students will not find any difficulty in understanding the subject matter and can solve exercise problems without any additional guidance.

The main features of this book are as follow:

- Written strictly as per the syllabus prescribed by the University of Calcutta
- Written in simple and lucid language
- Thematic focus has been introduced at the beginning of each chapter to get glimpse of the chapter
- Pedagogy is arranged sequentially from easy to hard problems
- Exercises contain sufficient number of short answer type, short essay answer type, and multiple choice questions (MCQs)
- Each formula is derived following all the steps
- Good number of worked-out examples are given to comprehend the chapter in depth as well as sufficient number of problems are included in the exercises for the practice of the students
- Theory is explained with the help of illustrations
- Solved C.U. question papers of both honours and general courses for the year 2017 are given at the end of the book

I hope this book will not only cater to the needs of the students of commerce of different universities but will also be useful for the students of other courses such as M.B.A., B.Tech., B.B.A., etc. where Mathematics and Statistics are taught. In its preparation, I consulted many books written by both Indian and foreign authors. I express my gratitude to them. I appreciate the sacrifices and endless support of my mother, wife, daughter and son, during the process of the preparation of the manuscript.

Valuable comments and suggestions for further improvement are earnestly invited from the readers so that these can be taken care of in future editions.

I offer my gratitude to the editorial and production teams at McGraw Hill Education (India) Private Limited, for the best efforts put forth by them in publishing of this book. Specially, I would like to offer my sincere thanks to Mr. Suman Sen, Ms. Laxmi Singh and Ms. Shalini Jha for devoting their valuable time in going through the drafts and providing invaluable suggestions. I would also like to thank all those who are directly or indirectly associated with its publication for their encouragement and support.

Asim Kumar Manna

Syllabus

GE 3.3 Chg
Business Mathematics & Statistics (Honours & General)
Marks 100

Module I

Business Mathematics

Internal Assessment:	**10 marks**
Semester-end Examinations:	**40 marks**
Total	**50 marks**

1. **Permutations and Combinations:** Definition, Factorial Notation, Theorems on Permutation, Permutations with repetitions, Restricted Permutations; Theorems on Combination, Basic identities, Restricted Combinations.

2. **Set Theory:** Definition of set, Presentation of sets, Different types of sets-Null set, Finite and infinite Sets, Universal set, Subset, Power set etc.; Set Operations, Law of algebra of Sets.

3. **Binomial Theorem:** Statement of the theorem for positive integral index, General term, Middle term, Simple properties of binomial coefficients.

4. **Logarithm:** Definition, Base and Index of Logarithm, General properties of Logarithm, Common Problems.

5. **Compound Interest and Annuities:** Simple AP and GP Series, Different types of interest rates, Net present value, Types of annuities, Continuous compounding, Valuation of simple loans and debentures, Problems relating to Sinking Funds.

Module II

Statistics

Internal Assessment:	**10 marks**
Semester-end Examinations:	**40 marks**
Total	**50 marks**

6. **Correlation and Association:** Bivariate data, Scatter diagram, Pearson's correlation coefficient, Spearman's rank correlation, Measures of association of attributes.

7. **Regression Analysis:** Least squares method, Simple regression lines, properties of regression, Identification of regression lines.

8. **Index Numbers:** Meaning and types of index numbers, Problems of constructing index numbers, Construction of price and quantity indices, Test of adequacy, errors in index numbers, Chain base index numbers; Base shifting, Splicing, Deflating, Consumer price index and its uses.

9. **Time Series Analysis:** Causes of variation in time series data, Components of time series, additive and multiplicative models, Determination of trend by semi-average, moving average and least squares (of linear, quadratic and exponential trend) methods; Computation of seasonal Indices by simple average, ratio-to-moving average, ratio-to-trend and link relative methods; Simple forecasting through time series data.

10. **Probability Theory:** Meaning of probability; Different definitions of probability; Conditional probability; Compound probability; Independent events, Simple problems.

Question Pattern

Business Mathematics and Statistics
Paper Code: GE 3.3 Chg
Year 2 : Semester III

Paper Code	Subject/ Paper	Mode of Questions	Marks in each Question	No. of Questions to be Answered	No. of Questions to be Set
GE 3.3 Chg	**Module I: Business Mathematics**	MCQ	2 4	10 5	10 5
	Module II: Statistics	MCQ	2 4	10 5	10 5

One-fourth (1/4th) of marks assigned to any question be deducted for wrong answer to that question.

Allocation of Marks and Lecture (Unit-Wise)

Paper Code	Paper/ Subject	Unit	Marks	Lecture
GE 3.3 Chg	**Module I : Business Mathematics**	1. Permutations and Combinations	8	8
		2. Set Theory	8	8
		3. Binomial Theorem	8	8
		4. Logarithm	8	8
		5. Compound Interest and Annuities	8	8
	Module II : Statistics	1. Correlation and Association	8	8
		2. Regression Analysis	8	8
		3. Index Numbers	8	8
		4. Time Series Analysis	8	8
		5. Probability Theory	8	8

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Chapter 5: Compound Interest and Annuities 5.1–5.75

Chapter 6: Correlation and Association 6.1–6.66

Chapter 7: Regression Analysis 7.1–7.41

CHAPTER

1

Permutations and Combinations

SYLLABUS

Definition, Factorial Notation, Theorems on Permutation, Permutations with Repetitions, Restricted Permutations, Theorem on Combination, Basic Identities, Restricted Combinations

THEMATIC FOCUS

1.1 Introduction
1.2 Meaning of Permutation and Combination
1.3 Differences between Permutation and Combination
1.4 Identification of Permutation and Combination
1.5 Factorials
1.6 Fundamental Principles of Counting
1.7 Different Rules on Permutation
- 1.7.1 Permutations of Different Things
- 1.7.2 Permutation of Things Not All Different
- 1.7.3 Repeated Permutations
- 1.7.4 Circular Permutations
- 1.7.5 Restricted Circular Permutations
- 1.7.6 Restricted Permutations

1.8 Different Rules on Combination
- 1.8.1 Combinations of Different Things
- 1.8.2 Restricted Combinations
- 1.8.3 Total number of Combinations of Different Things
- 1.8.4 Total number of Combinations of Things Not All Different
- 1.8.5 Division of Groups

1.1 INTRODUCTION

Counting the number of points in a sample space is becoming necessary for finding solutions to many statistical experiments. But the process or method of counting the numbers is *tedious*. By using some tools we can make the counting task easier. Permutations and Combinations are such counting tools. They are mostly used in probability, especially in determining the number of favourable outcomes and the number of total outcomes in a given situation.

The concepts of permutation and combination are quite related to each other and due to this students often tend to make a mistake while attempting questions on these topics. So, before taking decision, students must understand the concept of permutation and combination.

1.2 MEANING OF PERMUTATION AND COMBINATION

Let us consider a situation where we need to find out the total number of possible arrangements made out of three objects P, Q and R.

Situation	Arrangements in case of Permutation	No. of Arrangements	Arrangements in case of Combination	No. of Arrangements
(i) one at a time	P, Q, R	3	P, Q, R	3
(ii) two at a time	PQ, QP, QR, RQ, PR, RP	6	PQ, QR, RP	3
(iii) three at a time	PQR, RQP, QPR, RPQ, PRQ, QRP	6	PQR	1

In the above arrangements, in case of permutation, order in which the objects are arranged is considered, i.e. arrangement of PQ and QP of two objects P and Q are different because in PQ, P is at the first place and Q is at the second place from left whereas in QP, Q is at the first place and P is at the second place. Similarly, in case of arrangement of three objects P, Q and R taken all at a time PQR, RQP, QPR, RPQ, PRQ, QRP are considered different arrangements, because in each arrangement order in which the objects are arranged is considered. But in case of combination the order in which the objects are arranged is not considered, so then PQR, RQP, QPR, RPQ, PRQ and QRP are not different but the same. Similarly, PQ and QP are not different but the same. On the basis of the above clarification we can draw the definitions of permutation and combination as follows:

Permutation: A permutation is an arrangement of all or part of a set of objects with regard to the order of arrangement.

Combination: A combination is a selection of all or part of a set of object without regard to the order in which they are selected.

1.3 DIFFERENCES BETWEEN PERMUTATION AND COMBINATION

Permutation	Combination
1. It means arrangement 2. Order of things is taken into consideration. 3. $^nP_r = \dfrac{n!}{(n-r)!}$ 4. $^nP_n = n!$ 5. $^nP_o = 1$	1. It means selection 2. Order of things is not taken into consideration. 3. $^nC_r = \dfrac{n!}{r!(n-r)!}$ 4. $^nC_n = 1$ 5. $^nC_o = 1$

1.4 IDENTIFICATION OF PERMUTATION AND COMBINATION

To understand a question, whether it is related to permutation or combination, students should follow the categories of problems as given below in the chart which help them easily identify the problems on permutations and combinations.

Apply permutation formulae in case of problems	Apply combination formulae in case of problems
(*a*) based on arrangements (*b*) based on arrangement of letters of a word (*c*) based on digits (*d*) based on seated in a row (*e*) based on standing in a line (*f*) based on rank of a word	(*a*) based on selections or choice (*b*) based on groups, committee or team (*c*) based on geometry

If the problem does not fall in the categories mentioned above, then students need to find out whether order is important or not? If order is important, the problem will be related to permutation otherwise it is of combination.

1.5 FACTORIALS

If '*n*' is a natural number then product of all natural numbers upto and including '*n*' is called factorial *n* and is denoted by $n!$ or $\underline{|n}$.

Thus, $n! = n(n-1)(n-2)$ 3.2.1

Note that $n! = n(n-1)! = n(n-1)(n-2)! = n(n-1)(n-2)(n-3)!$, etc.

For example, $6! = 6\times5\times4\times3\times2\times1$

$= 6\times5\times4!$ [As $4! = 4\times3\times2\times1$]

$= 6\times5\times4\times3!$ [As $3! = 3\times2\times1$]

Prove that: $0! = 1$

Solution: We know that $n! = n(n-1)!$

or $1! = 1(1-1)!$ [Putting $n = 1$] or $1 = 1.0!$ or $0! = 1$

1.6 FUNDAMENTAL PRINCIPLES OF COUNTING

1.6.1 Multiplication Principle

If an operation can be performed in '*m*' different ways and following which a second operation performed in '*n*' different ways, then the two operations in succession can be performed in $m \times n$ ways. This can be extended to any finite number of operations.

ILLUSTRATION 1

A company offers 6 different posts in Group D category and 4 in Group C category. Find the possible number of ways with the applicant if he wants to select one post in Group D category and one in Group C category.

Solution: The company has 6 posts in Group D category. Out of which the applicant can select one post in 6 different ways. In Group C category there are 4 different posts out of which the applicant can select one in 4 different ways.

Hence, the required number of ways = 6 × 4 = 24 .

1.6.2 Addition Principle

If an operation can be performed in '*m*' different ways and another operation, which is independent of the first operation, can be performed in '*n*' different ways. Then either of the two operations can be performed in $(m + n)$ ways. This can be extended to any finite number of independent operations.

ILLUSTRATION 2

A company offers 6 different posts in Group D category and 4 in Group C category. Find the possible number of ways an applicant can select exactly one post, either in Group D or in Group C.

Solution: The company has 6 different posts in Group D category out of which the applicant can select one post in 6 ways. In Group C category the company has 4 different posts out of which the applicant can select one in 4 ways.

Hence, the required number of ways = 6 + 4 = 10.

1.7 DIFFERENT RULES ON PERMUTATION

1.7.1 Permutations of Different Things

Rule-1: The number of permutations of 'n' different things taken 'r' at a time is $^nP_r = \frac{n!}{(n-r)!}$ where $r \le n$.

Proof: Say we have 'n' different things $x_1, x_2, x_3, ..., x_n$.

Clearly the first place can be filled up in 'n' ways.

Number of things left after filling-up the first place = $n - 1$.

So the second-place can be filled up in $(n - 1)$ ways. Now number of things left after filling up the first and second places = $n - 2$.

Now the third place can be filled up in $(n - 2)$ ways.

Thus the number of ways of filling-up first place = n

Number of ways of filling-up second-place = $n - 1$

Number of ways of filling-up third-place = $n - 2$

Number of ways of filling-up r-th place = $n - (r - 1) = n - r + 1$

By multiplication-rule of counting, total number of ways of filling up first, second, ……... rth place together—

$$n(n-1)(n-2)...(n-r+1)$$

Hence,

$$\begin{aligned}
^nP_r &= n(n-1)(n-2)...(n-r+1) \\
&= \frac{[n(n-1)(n-2)...(n-r+1)]\,[(n-r).(n-r-1).\ ...3.2.1]}{(n-r).(n-r-1)...3.2.1} \\
&= \frac{n(n-1)(n-2)...(n-r+1)(n-r)(n-r-1)...3.2.1}{(n-r)(n-r-1)...3.2.1} \\
&= \frac{n!}{(n-r)!}
\end{aligned}$$

ILLUSTRATION 3

How many different signals can be made by 5 flags from 8 flags of different colours?

Solution: Required number of ways taking 5 flags out of 8 flags

$$= {}^8P_5 = \frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{8\times7\times6\times5\times4\times3!}{3!}$$

$$= 6,720$$

Rule-2: The number of permutations of n distinct things taking all at a time is ${}^nP_n = n!$

Proof: We know that ${}^nP_r = \dfrac{n!}{(n-r)!}$

Putting $r = n$, we have

$${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! \quad [\text{As } 0! = 1]$$

ILLUSTRATION 4

How many words can be made by using the letters of the word 'SINGLE' taken all at a time?

Solution: There are 6 different letters of the word 'SINGLE'.

Required number of permutations taking all the letters at a time

$$= {}^6P_6 = 6!$$
$$= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

1.7.2 Permutation of Things Not All Different

Rule-3: The number of permutations of n things taken all at a time, in which 'p' of them are of one type, 'q' of them are of second type, 'r' of them are of third type, and rest are all different is $\dfrac{n!}{p!q!r!}$

Proof: Let the n things be represented by n letters and suppose p of them are each equal to a, q of them each equal to b, r of them each equal to c and the rest are all different.

Let x be the required number of permutations.

In each of these x permutations, if the p number of letters 'a' are replaced by p new letters which are all different from one another and different from the rest, then without altering the position of any other letter, they will produce $p!$ permutations. Thus the total number of permutations would become $x \times p!$

Again, if the q number of letters 'b' are replaced by q new letters which are all different from one another and different from the rest, then the total number of permutations would become $x \times p! \times q!$

Again, if the r number of letters 'c' are replaced by r new letters different from one another and different from the rest, then the total number of permutations would become $x \times p! \times q! \times r!$

But the number of permutations of n different things taking all at a time is $n!$

Therefore, $x \times p! \times q! \times r! = n!$

$$\text{or } x = \frac{n!}{p!q!r!}$$

The above principle is also applicable when more than three letters are repeated.

ILLUSTRATION 5

Find the number of arrangements that can be made out of the letters of the word 'ASSASSINATION'. [***C.A. Entrance, May 1975***]

Solution: The word 'ASSASSINATION' has 13 letters. But in these 13 letters, '*A*' occurs three times, '*S*' occurs four times, '*I*' occurs two times and '*N*' occurs two times and the rest are all different.

Hence, the required number of arrangements are

$$\frac{13!}{3!4!2!2!} = \frac{13\times12\times11\times10\times9\times8\times7\times6\times5\times4!}{3\times2\times1\times4!\times2\times1\times2\times1}$$

$$= 1{,}08{,}10{,}800$$

1.7.3 Repeated Permutations

Rule-4: The number of permutations of *n*-things, taken '*r*' at a time when each thing can be repeated *r*-times is n^r.

Proof: The number of ways of filling up first place = n

Since repetition is allowed, so

The number of ways of filling up second place = n

The number of ways of filling up third place = n

The number of ways of filling up r-th place = n

Hence, total number of ways in which first, second, third, … rth places can be filled up

$$= n \times n \times n \times \ldots \, r \text{ times}$$

$$= n^r$$

ILLUSTRATION 6

In how many ways can three prizes, one in recitation, one in music and one in drawing, be given to 5 students, when each student is eligible for all the prizes?

Solution: There are three prizes, one in recitation, one in music and one in drawing and each prize can be given to any one of the 5 students.

Now the prize for the recitation can be won by any one of the 5 students, so it can be given in 5 ways.

Again the prize for the music also can be given in 5 ways, as the student who has won the prize for recitation may also win this prize. Hence, the first two prizes may be given in 5 × 5 ways.

Similarly, the prize for drawing can be given in 5 ways.

Hence, all the three prizes can be given in $5 \times 5 \times 5 = 5^3 = 125$ ways.

1.7.4 Circular Permutations

Rule-5: In circular arrangements, there is no concept of starting point. There are two cases of circular permutations:

(*a*) If clockwise and anti-clockwise orders are different, then total number of circular permutations of *n* things is $(n - 1)!$

(*b*) If clockwise and anti-clockwise orders are taken as the same, i.e. not different then total number of circular permutations of *n* things is $\frac{(n-1)!}{2}$.

NOTE

When the number of circular permutations of *n* things depend on the relative position of a thing other than these *n* things is $n!$

1.7.5 Restricted Circular Permutations

When there is a restriction in a circular permutation then first of all we shall consider the restricted part of the operation and then consider the remaining part treating it similar to a linear permutation.

Rule-6: The number of circular-permutations of '*n*' different things taken '*r*' at a time –

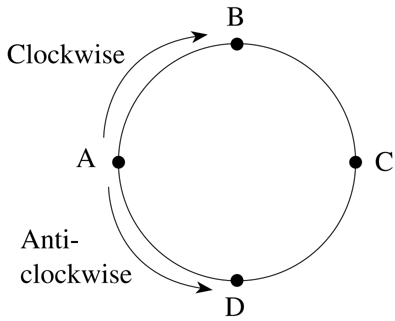
(*a*) If clockwise and anti-clockwise orders are taken as different, then total number of circular permutations $= {}^nP_r / r$

(*b*) If clockwise and anti-clockwise orders are taken as same, i.e. not different, then total number of circular permutation $= {}^nP_r / 2r$

ILLUSTRATION 7

In how many ways can 4 persons be seated at a circular table?

Solution: If 4 persons A, B, C, D are seated at circular table, then the two arrangements ABCD (in clockwise direction) and ADCB (the same order but in anti-clockwise direction) are different.



Hence, the number of arrangements in which 4 different persons can sit in a circular table $= (4-1)! = 3! = 3 \times 2 \times 1 = 6$

ILLUSTRATION 8

In how many ways can 8 flowers of different colours be strung together on a garland?

Solution: The number of circular permutations of 8 flowers of different colours $= (8-1)! = 7!$

But a garland is symmetrical from the right and from the left. So in this case, there is no difference between the clockwise and anti-clockwise arrangements.

Therefore, number of arrangements of 8 flowers of different colours in the garland

$$= \frac{1}{2} \times 7!$$
$$= \frac{1}{2} \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 2{,}520$$

ILLUSTRATION 9

How many necklaces of 6 beads each can be made from 10 beads of different colours?

Solution: In this case there is no difference between the clockwise and anti-clockwise arrangements.

Hence, total number of circular permutations

$$= {}^{10}P_6 / 2 \times 6 = \frac{10!}{4! \times 12} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4! \times 12}$$
$$= 12{,}600$$

ILLUSTRATION 10

In how many ways can 6 persons sit at a round table if the seat arrangements are made with respect to the table.

Solution: As the seat arrangements of 6 persons are made with respect to the table, we can arrange the 6 persons among themselves.

Therefore, the required number of circular permutations = 6! = 6 × 5 × 4 × 3 × 2 × 1 = 720.

1.7.6 Restricted Permutations

Rule-7: The number of permutations of n different things taken r at a time in which 't' particular things never occur is ${}^{n-t}P_r$

As 't' things will never occur, we can keep aside 't' particular things and fill up the 'r' places with the remaining '$n - t$' things.

So, the number of arrangement = ${}^{n-t}P_r$

Rule-8: The number of permutations of 'n' different things taken 'r' at a time in which 't' particular things always occur is ${}^{r}P_t \times {}^{n-t}P_{r-t}$.

Rule-9: The numbers of permutations of 'n' different things taken 'r' at a time in which 't' particular things occupy stated places in ${}^{n-t}P_{r-t}$.

As 't' particular things occupy stated places, no arrangement is required. Then we have only '$n - t$' things to be arranged in '$r - t$' places.

So, the number of arrangements = ${}^{n-t}P_{r-t}$

ILLUSTRATION 11

In how many of the arrangements of 15 different things taken 6 at a time, will 4 specified things (*i*) never occur, (*ii*) always occur?

Solution:

(*i*) Keeping aside the 4 specified things which will never occur, we are now left with 11 different things from which 6 things are to be selected.

So, the number of arrangements

$$= {}^{15-4}P_6 = {}^{11}P_6 = \frac{11!}{(11-6)!} = \frac{11!}{5!}$$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5!} = 3,32,640$$

(*ii*) Keeping aside 4 specified things which will always occur, we have to select the remaining 2 things out of the remaining 11 things.

The number of such arrangements $= {}^{15-4}P_{6-4} = {}^{11}P_2$.

Now, in each of these arrangements there are 2 number of things and they occupy only 2 places. The remaining 4 places are to be filled in by the 4 specified things.

Now, 4 specified things which will always occur, can be placed in one of these 6 places in $^6P_4$ ways.

Therefore, the total number of arrangements

$$= {}^{11}P_2 \times {}^6P_4 = \frac{11!}{9!} \times \frac{6!}{2!} = \frac{11\times 10\times 9!}{9!} \times \frac{6\times 5\times 4\times 3\times 2!}{2!}$$

$$= 11\times 10\times 6\times 5\times 4\times 3 = 39{,}600$$

COMBINATIONS

1.8 DIFFERENT RULES ON COMBINATION

1.8.1 Combinations of Different Things

Rule-1: The number of combination of 'n' different things taken 'r' at a time is $^nC_r = \dfrac{n!}{r!(n-r)!}$ (where $r \le n$)

Proof: Each combination contains 'r' different things, which can be arranged among themselves in $r!$ ways. We denote the number of combinations of 'n' different things taken 'r' at a time by ^nC_r.

For one combination of 'r' different things, number of permutation = $r!$

So for ^nC_r combination number of permutation = $r!\,{}^nC_r$

Therefore, total number of permutations = $r!\,{}^nC_r$ ...(i)

We know that the number of permutation of 'n' different things taken 'r' at a time = ^nP_r ...(ii)

From (i) and (ii) we can write

$$^nP_r = r!\,{}^nC_r$$

$$\text{or } \frac{n!}{(n-r)!} = r!\,{}^nC_r \text{ or } {}^nC_r = \frac{n!}{r!(n-r)!}$$

ILLUSTRATION 12

In how many ways can 5 questions be selected from 8 questions?

Solution: The number of different selections is equal to the number of ways in which 5 questions can be selected from 8 questions.

Therefore, the required number of ways

$$= {}^8C_5 = \frac{8!}{5!(8-5)!} = \frac{8!}{5!3!} = \frac{8\times 7\times 6\times 5!}{5!\times 3\times 2\times 1} = 56$$

Rule-2: The number of combinations of 'n' different things taken 'r' at a time is equal to the number of combinations of 'n' different things taken $(n - r)$ at a time.

Proof: We know, ${}^nC_r = \frac{n!}{r!(n-r)!}$

and $${}^nC_{n-r} = \frac{n!}{(n-r)!\{n-(n-r)\}!} = \frac{n!}{(n-r)!(n-n+r)!}$$

$$= \frac{n!}{(n-r)!r!}$$

Therefore, ${}^nC_r = {}^nC_{n-r}$

Rule-3: If ${}^nC_p = {}^nC_q$ then either $p = q$ or $p + q = n$

Proof: It is clear that when $p = q$ then ${}^nC_p = {}^nC_q$.

Again, we know ${}^nC_p = {}^nC_{n-p}$

or ${}^nC_q = {}^nC_{n-p}$ [As ${}^nC_p = {}^nC_q$]

or $q = n - p$

or $p + q = n$

ILLUSTRATION 13

If ${}^{20}C_r = {}^{20}C_{r+4}$, find r.

Solution: ${}^{20}C_r = {}^{20}C_{r+4}$

or $r + r + 4 = 20$

or $2r + 4 = 20$

or $2r = 20 - 4 = 16$

or $r = 8$

Rule-4: ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ [***C.U. B. Com. (H) 2017***]

Proof: L.H.S $= {}^nC_r + {}^nC_{r-1}$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= n!\left[\frac{1}{r!(n-r)!} + \frac{1}{(r-1)!(n-r+1)!}\right]$$

$$= n!\left[\frac{n-r+1}{r!(n-r+1)(n-r)!} + \frac{r}{r(r-1)!(n-r+1)!}\right]$$

$$= n!\left[\frac{n-r+1}{r!(n-r+1)!} + \frac{r}{r!(n-r+1)!}\right]$$

$$= n!\left[\frac{n-r+1+r}{r!(n-r+1)!}\right]$$

$$= n!\left[\frac{n+1}{r!(n-r+1)!}\right] = \frac{(n+1)n!}{r!(n-r+1)!}$$

$$= \frac{(n+1)!}{r!(n-r+1)!} = {}^{n+1}C_r$$

Rule-5: ${}^nC_r \,/\, {}^nC_{r-1} = \dfrac{n-r+1}{r}$

Proof: L.H.S $= {}^nC_r \,/\, {}^nC_{r-1} = \dfrac{\dfrac{n!}{r!(n-r)!}}{\dfrac{n!}{(r-1)!(n-r+1)!}}$

$$= \frac{n!}{r!(n-r)!} \times \frac{(r-1)!(n-r+1)!}{n!}$$

$$= \frac{(r-1)!(n-r+1)!}{r!(n-r)!} = \frac{(r-1)!(n-r+1)(n-r)!}{r(r-1)!(n-r)!}$$

$$= \frac{n-r+1}{r}$$

1.8.2 Restricted Combinations

Rule-6: The number of combinations of '*n*' different things taken '*r*' at a time, when '*t*' particular things always occur is ${}^{n-t}C_{r-t}$

Rule-7: The number of combinations of '*n*' different things taken '*r*' at a time, when '*t*' particular things never occur is ${}^{n-t}C_r$

ILLUSTRATION 14

In how many ways can a cricket-eleven be chosen out of 15 players? If

(*i*) A particular player is always chosen

(*ii*) A particular player is never chosen

Solution: (*i*) A particular player is always chosen, it means that remaining 10 players are to be selected out of the remaining 14 players.

Therefore, required number of ways $= {}^{14}C_{10} = \dfrac{14!}{10!4!}$

$$= \frac{14 \times 13 \times 12 \times 11 \times 10!}{10!4 \times 3 \times 2 \times 1} = 1001$$

(*ii*) A particular player is never chosen, it means that 11 players are to be selected out of 14 players.

Therefore, required number of ways $= {}^{14}C_{11} = \dfrac{14!}{11!3!}$

$$= \frac{14 \times 13 \times 12 \times 11!}{11!\ 3 \times 2 \times 1} = 364$$

1.8.3 Total Number of Combinations of Different Things

Rule-8: The total number of combinations of 'n' different things taken some or all of them at a time is $2^n - 1$.

Proof: Number of ways of selecting 1 thing out of 'n' things = ${}^nC_1$

Number of ways of selecting 2 things out of 'n' things = ${}^nC_2$

Number of ways of selecting 3 things out of 'n' things = ${}^nC_3$

……………………………………………………………………………...

……………………………………………………………………………...

Number of ways of selecting 'n' things out of 'n' things = ${}^nC_n$

Therefore, total number of ways of selecting one or more things out of n different things

$$= {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n$$

$$= ({}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n) - {}^nC_0$$

$$= 2^n - 1 \quad [\text{As } {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n \text{ and } {}^nC_0 = 1]$$

ILLUSTRATION 15

Kabir has 8 friends. In how many ways can he invites one or more of them to dinner?

Solution: Kabir can invite one or more than one of his 8 friends at a time to dinner.

Therefore, the required number of ways

$$= {}^8C_1 + {}^8C_2 + {}^8C_3 + \ldots + {}^8C_8 = 2^8 - 1 = 255$$

1.8.4 Total Number of Combinations of Things Not All Different

Rule-9: The total number of combinations of $(p + q + r + \ldots)$ things of which 'p' things are of one kind, 'q' of the second kind and 'r' of the third kind and so on, taken any number of things at a time

$$= [(p+1)(q+1)(r+1).........] - 1$$

ILLUSTRATION 16

A box contains 12 black balls, 7 red balls and 6 blue balls. In how many ways can one or more balls be selected?

Solution: The number of ways in which one or more objects can be selected out of 'p' alike subjects of one kind, 'q' alike subjects of second kind and 'r' alike subjects of third kind

$$= (p+1)(q+1)(r+1) - 1 = (12+1)(7+1)(6+1) - 1$$
$$= 13 \times 8 \times 7 - 1 = 728 - 1 = 727$$

Rule-10: The number of ways of selecting one or more things from 'p' identical things of one type, 'q' identical thing of another type, 'r' identical things of the third type and 'n' different things is $(p+1)(q+1)(r+1)2^n - 1$.

ILLUSTRATION 17

Find the number of different choices that can be made from 5 apples, 4 bananas and 3 mangoes, if at least one fruit is to be chosen.

Solution: Number of ways of selecting apples = (5 + 1) = 6

Number of ways of selecting bananas = (4 + 1) = 5

Number of ways of selecting mangoes = (3 + 1) = 4

Therefore, total number of ways of selecting fruits = 6 × 5 × 4

But this includes, when no fruits, i.e. zero fruits is selected.

Therefore, number of ways of selecting at least one fruit = (6 × 5 × 4) – 1 = 119

[As there was no fruit of a different type, hence, $n = 0$, so $2^n = 2^0 = 1$]

1.8.5 Division of Groups

Rule-11: The number of ways in which different things can be divided into two groups which contain 'm' and 'n' things respectively is

$$^{m+n}C_m \cdot {}^nC_n = \frac{(m+n)!}{m!n!}, \text{where } m \neq n$$

NOTE

If $m = n$, then the groups are equal size. Division of these groups can be given by two types.

Type-1: If order of group is not important: The number of ways in which '$2n$' different things can be divided equally into two groups is $\frac{(2n)!}{2!(n!)^2}$.

Type-2: If order of group is important: The number of ways in which '$2n$' different things can be divided equally into two distinct groups is

$$\frac{(2n)!}{2!(n!)^2}\times 2! = \frac{(2n)!}{(n!)^2} = \frac{2n\cdot(2n-1)\cdot(2n-2)\cdot(2n-3)\cdot(2n-4)\cdot......4.3.2.1}{(n!)^2}$$

$$= \frac{2n\cdot(2n-1)\cdot 2(n-1)\cdot(2n-3)\cdot 2(n-2)\cdot......4.3.2.1}{(n!)^2}$$

$$= \frac{2^n[n(n-1)(n-2).....3.2.1][(2n-1)\cdot(2n-3)\cdot(2n-5)\cdot.....5.3.1]}{(n!)^2}$$

$$= \frac{2^n\cdot n![(2n-1)\cdot(2n-3)\cdot(2n-5)\cdot......5.3.1]}{(n!)^2}$$

$$= \frac{2^n[(2n-1)(2n-3)(2n-5)\cdot......5.3.1]}{n!}$$

Rule-12: The number of ways in which $(m + n + p)$ different things can be divided into three groups which contain m, n and p things respectively is

$$^{m+n+p}C_m \cdot {}^{n-p}C_n \cdot {}^{p}C_p = \frac{(m+n+p)!}{m!n!p!}, \text{where } m \neq n \neq p.$$

NOTE

If $m = n = p$, then the groups are equal size. Division of these groups can be given by two types.

Type-1: If order of group is not important: The number of ways in which '$3p$' different things can be divided equally into three groups is

$$\frac{(3p)!}{3!(p!)^3}$$

Type-2: If order of group is important: The number of ways in which '$3p$' different things can be divided equally into three distinct groups is

$$\frac{(3p)!}{3!(p!)^3}\times 3! = \frac{(3p)!}{(p!)^3}$$

ILLUSTRATIVE EXAMPLES OF PERMUTATIONS

A. SHORT TYPE

EXAMPLE 1

Find the value of ${}^{12}P_4$

Solution: ${}^{12}P_4 = \dfrac{12!}{(12-4)!} = \dfrac{12!}{8!} = \dfrac{12 \times 11 \times 10 \times 9 \times 8!}{8!} = 12 \times 11 \times 10 \times 9 = 11{,}880.$

EXAMPLE 2

Show that ${}^{2n}P_n = 2^n[1.3.5.7\ldots\ldots\ldots.., (2n-1)]$ [***C.U. B.Com. 2011***]

Solution: ${}^{2n}P_n = \dfrac{(2n)!}{(2n-n)!} = \dfrac{(2n)!}{n!}$

$$= \frac{2n(2n-1)(2n-2)(2n-3)\ldots\ldots 7.6.5.4.3.2.1}{n!}$$

$$= \frac{[2n(2n-2)(2n-4)\ldots\ldots\ldots..6.4.2], [(2n-1)(2n-3)\ldots.7.5.3.1]}{n!}$$

$$= \frac{[2n.2(n-1)\cdot 2(n-2)\cdot\ldots\ldots..6.4.2][1.3.5.7\ldots\ldots\ldots(2n-1)]}{n!}$$

$$= \frac{2^n[n(n-1)(n-2)\ldots\ldots\ldots\ldots3.2.1][1.3.5.7\ldots\ldots\ldots(2n-1)]}{n!}$$

$$= \frac{2^n.n![1.3.5.7\ldots\ldots..(2n-1)]}{n!}$$

$= 2^n[1.3.5.7\ldots\ldots\ldots(2n-1)]$ [Proved].

EXAMPLE 3

If ${}^{2n-1}P_n : {}^{2n+1}P_{n-1} = 5:3$, find n. [***C.U. B.Com. (H) 2011, 2014***]

Solution: ${}^{2n-1}P_n : {}^{2n+1}P_{n-1} = 5:3$

or $\dfrac{(2n-1)!}{(2n-1-n)!} : \dfrac{(2n+1)!}{(2n+1-n+1)!} = 5:3$

or $\dfrac{(2n-1)!}{(n-1)!} : \dfrac{(2n+1)!}{(n+2)!} = 5:3$

or $\dfrac{(2n-1)!}{(n-1)!} \times \dfrac{(n+2)(n+1)\cdot n\cdot(n-1)!}{(2n+1).2n.(2n-1)!} = \dfrac{5}{3}$

or $\dfrac{n(n+1)(n+2)}{2n(2n+1)} = \dfrac{5}{3}$

or $\dfrac{n^2+3n+2}{4n+2} = \dfrac{5}{3}$

or $3n^2+9n+6 = 20n+10$

or $3n^2+9n-20n+6-10 = 0$

or $3n^2-11n-4 = 0$

or $3n^2-12n+n-4 = 0$

or $3n(n-4)+1(n-4) = 0$

or $(n-4)(3n+1) = 0$

either $n - 4 = 0$ or $n = 4$

or $3n + 1 = 0$ or $n = -\dfrac{1}{3}$

Value of 'n' cannot be negative

Therefore, the required value of $n = 4$.

EXAMPLE 4

Show that ${}^{n}P_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$

Solution: We have, ${}^{n-1}P_r + r.{}^{n-1}P_{r-1}$

$$= \frac{(n-1)!}{(n-1-r)!} + r.\frac{(n-1)!}{(n-1-r+1)!}$$

$$= (n-1)!\left[\frac{1}{(n-r-1)!} + r.\frac{1}{(n-r)!}\right]$$

$$= (n-1)!\left[\frac{(n-r)}{(n-r)\cdot(n-r-1)!} + r.\frac{1}{(n-r)!}\right]$$

$$= (n-1)!\left[\frac{(n-r)}{(n-r)!} + \frac{r}{(n-r)!}\right]$$

$$= (n-1)!\left[\frac{n-r+r}{(n-r)!}\right]$$

$$= (n-1)!\,\frac{n}{(n-r)!} = \frac{n.(n-1)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!} = {}^{n}P_r \quad \text{[Proved]}$$

EXAMPLE 5

Find the value of 'n', if ${}^{n+1}P_3 = 2.\,{}^{n}P_3$ [***C.U. B.Com. (H) 2013***]

Solution: ${}^{n+1}P_3 = 2.\,{}^{n}P_3$

or $\dfrac{(n+1)!}{(n+1-3)!} = 2.\dfrac{n!}{(n-3)!}$

or $\dfrac{(n+1).n!}{(n-2)!} = 2.\dfrac{n!}{(n-3)!}$

or $\dfrac{(n+1)}{(n-2).(n-3)!} = \dfrac{2}{(n-3)!}$

or $\dfrac{n+1}{n-2} = 2$

or $2n-4 = n+1$

or $2n-n = 1+4$

or $n = 5.$

EXAMPLE 6

For what value of n, ${}^{n}P_4 = 12 \times {}^{n}P_2$? [***C.U. B.Com. (H) 2016***]

Solution: ${}^{n}P_4 = 12 \times {}^{n}P_2$

or $\dfrac{n!}{(n-4)!} = 12.\dfrac{n!}{(n-2)!}$

or $\dfrac{n!}{(n-4)!} = 12.\dfrac{n!}{(n-2).(n-3).(n-4)!}$

or $1 = \dfrac{12}{(n-2)(n-3)}$ or $n^2 - 5n + 6 = 12$

or $n^2 - 5n - 6 = 0$ or $n^2 - 6n + n - 6 = 0$

or $n(n-6) + 1(n-6) = 0$ or $(n-6)(n+1) = 0$

either, $n - 6 = 0$ or $n = 6$
or $n + 1 = 0$ or $n = -1$ [It is impossible]
Therefore, The required value of $n = 6$.

EXAMPLE 7

Aiswariya has 6 tops, 7 skirts and 5 caps from which to choose an outfit. In how many ways can she select one top, one skirt and one cap?

Solution: Aiswariya can select one top from 6 tops in 6 ways

one skirt from 7 skirts in 7 ways, and
one cap from 5 caps in 5 ways.
Therefore, required number of ways = 6 × 7 × 5 = 210.

EXAMPLE 8

A person wants to go from station K to station M via station L. There are 6 routes from K to L and 7 routes from L to M. In how many ways can he travel from K to M?

Solution: He can go from K to L in 6 ways and L to M in 7 ways.
Therefore, number of ways of travel from K to M is 6 × 7 = 42.

EXAMPLE 9

A college offers 7 courses in the evening and 5 in the morning. Find the number of ways a student can select exactly one course, either in the evening or in the morning.

Solution: The college has 7 courses in the evening out of which the student can select one course in 7 ways.

In the morning the college has 5 courses out of which the student can select one in 5 ways.

Therefore, the required number of ways = 7 + 5 = 12.

EXAMPLE 10

How many words can be made by using the letters of the word 'FORMULA' taken all at a time?

Solution: There are '7' different letters of the word 'FORMULA'.

Number of permutations taking all the letters at a time = ${}^7P_7 = 7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040$.

EXAMPLE 11

How many numbers of 4 digits can be formed with digits 1, 2, 3, 4 and 5?

Solution: Here total number of digits (n) = 5
Number of 4 digits to be formed, i.e. $r = 4$
Therefore, number of 4 digits numbers =

$${}^5P_4 = \frac{5!}{(5-4)!} = \frac{5!}{1!} = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

EXAMPLE 12

At a dog show, 3 awards are given: best in show, first runner-up, and second runner-up. A group of 12 dogs are competing in the competition. In how many different ways can the prizes be awarded?

Solution: The same three dogs could get best in show, first runner-up, and second runner-up and each arrangement is different. So, the required number of ways = ${}^{12}P_3 = \frac{12!}{(12-3)!} = \frac{12!}{9!} = \frac{12 \times 11 \times 10 \times 9!}{9!} = 1,320.$

EXAMPLE 13

How many 3-letter words, with or without meaning, can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed?

Solution: In the word 'LOGARITHMS', there are 10 different letters.

3-letter words to be formed by using these letters.

Therefore, number of ways = ${}^{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \times 9 \times 8 \times 7!}{7!} = 720.$

EXAMPLE 14

In how many different ways can the letters of the word 'RUMOUR' be arranged?

Solution: The word 'RUMOUR' has 6 letters.

In these 6 letters, 'R' occurs 2 times, 'U' occurs 2 times and rest of the letters are different.

Therefore, number of ways = $\frac{6!}{2!2!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!\,2 \times 1} = 180.$

EXAMPLE 15

A coin is tossed 3 times. Find out the number of possible outcomes.

Solution: When a coin is tossed once, there are two possible outcomes: Head (H) and Tail (T)

Therefore, when a coin is tossed 3 times, the number of possible outcomes $= 2 \times 2 \times 2 = 8$

EXAMPLE 16

Find out the number of ways in which 6 rings of different types can be worn in 3 fingers?

Solution: The first ring can be worn in any of the 3 fingers in 3 ways.

Similarly, each of the remaining 5 rings can also be worn in 3 ways.

Therefore, total number of ways = $3^6 = 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 729$.

EXAMPLE 17

How many words can be formed with the letters of the word 'MONDAY' such that the words will start with 'M' but do not end with 'Y'.

[***C.U. B.Com. (H) 2013***]

Solution: The word 'MONDAY' has six different letters. These letters can arrange among themselves in 6! ways = 6 × 5 × 4 × 3 × 2 × 1 = 720 ways.

Therefore, the total number of words = 720.

To find the number of words which start with M, we keep M fixed at the beginning and arrange the remaining 5 letters among themselves, which can be done in 5! ways = 120 ways.

Therefore, the number of words starting with M = 120.

Again, to find the number of words starting with M and ending with Y, we keep M in the first and Y in the last places and arrange the remaining four letters, among themselves, which can be done in 4! ways = 24 ways.

Therefore, the number of words starting with M and ending with Y = 24.

Now, the number of words starting with M and not ending with Y = Number of words starting with M – number of words starting with M and ending with Y = 120 – 24 = 96.

EXAMPLE 18

In how many ways can 7 boys be seated in a circular order?

Solution: If 7 boys seated are in a circular order, then the arrangements in clockwise and anti-clockwise directions are different.

Therefore, the required number of arrangements = (7 – 1)! = 6! = 6 × 5 × 4 × 3 × 2 × 1 = 720.

EXAMPLE 19

In how many ways can 7 beads can be arranged to form a necklace?

Solution: In this case there is no difference between clockwise and anticlockwise arrangements.

Therefore, number of arrangements

$$= \frac{1}{2}\times(7-1)! = \frac{1}{2}\times 6! = \frac{1}{2}\times 6\times 5\times 4\times 3\times 2\times 1 = 360.$$

B. SHORT ESSAY TYPE

EXAMPLE 20

In how many ways can 5 men draw water from 5 taps if no tap can be used more than once?

Solution: First man can draw water from any of the 5 taps in 5 different ways.

Second man can draw water from any of the remaining 4 taps in 4 different ways.

Third man can draw water from any of the remaining 3 taps in 3 different ways.

Fourth man can draw water from any of the remaining 2 taps in 2 different ways.
Fifth man can draw water from remaining 1 tap in one way.
Therefore, total number of ways = $5 \times 4 \times 3 \times 2 \times 1 = 120$.

EXAMPLE 21

How many (*i*) odd number and (*ii*) even numbers of five digits can be formed with the digits 0, 1, 2, 3, 4? [***C.U. B.Com. 2013(G), 2014(G)***]

Solution:

(*i*) In the given digits, odd numbers are 1 and 3. For odd numbers of five digits, the unit's place must be filled up by the above odd numbers, i.e. 1 or 3 and it can be done in ${}^2P_1 = 2$ ways. The remaining four places can be filled up from the remaining four digits in ${}^4P_4 = 4!$ ways. Again the digit 0 cannot be placed in the beginning, as it is not counted in this place.

Now placing 0 at the beginning, and odd digit at the unit's place, the remaining three places can be filled up in 3! ways.

So, the number of odd numbers placing any one odd number at the unit place = $4! - 3! = 4 \times 3 \times 2 \times 1 - 3 \times 2 \times 1 = 24 - 6 = 18$.

Therefore, the required number of odd numbers of five digits = $18 \times 2 = 36$ [As there are two odd digit numbers].

(*ii*) For getting even numbers the units place can be filled up by the digits 0, 2 or 4.

Keeping 0 at the unit place, the remaining four places can be filled up by the remaining four digits in 4! ways = $4 \times 3 \times 2 \times 1 = 24$ ways.

Again, keeping 2 at unit place, remaining four places can be filled up by the remaining four digits in 4! ways. But the digit 0 cannot be placed in the beginning, in which case the number will be of four digits.

Now placing 0 at the beginning, and 2 at the unit place, the remaining three places can be filled up in 3! ways.

So, the required number of ways = $4! - 3! = 4.3.2.1 - 3.2.1 = 24 - 6 = 18$.

Similarly, keeping 4 at the units place, required no. of ways = 18

Therefore, the required number of even numbers of five digits = $24 + 18 + 18 = 60$

EXAMPLE 22

How many numbers lying between 5000 and 6000 can be formed with digits 3, 4, 5, 6, 7, 8, each digit occurring once in each number?

Solution: Numbers lying between 5000 and 6000 represent 4-digit numbers starting with 5.

Placing 5 at the beginning, the remaining 3 places can be filled up by the remaining 5 digits in ${}^5P_3$ ways.

Therefore, the required number of numbers = $^5P_3$

$$= \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\times4\times3\times2!}{2!} = 60.$$

EXAMPLE 23

How many numbers greater than 500 can be formed with the digits 1, 2, 3, 4 and 5, no repetitions being allowed?

Solution: Numbers greater than 500 formed with the digits 1, 2, 3, 4 and 5 can be either (*i*) 3-digit numbers having 5 in hundred's place.

or, (*ii*) 4-digit numbers

or, (*iii*) 5-digit numbers

(*i*) **No. of 3-digit numbers having 5 in the hundred's place**

Placing 5 in the hundred's place, remaining two places can be filled up by the remaining four digits in $^4P_2$ ways = 12 ways.

(*ii*) **No. of 4-digit numbers**

Four places can be filled up by the five digits in $^5P_4$ ways = 120 ways.

(*iii*) **No. of 5-digit numbers**

Similarly, 5 places can be filled up by the 5 digits in $^5P_5$ = 120 ways.

Therefore, the required no. of numbers greater than 500 which can be formed with the given digits = 12 + 120 + 120 = 252.

EXAMPLE 24

How many numbers not exceeding 10,000 can be made using the digits 2, 4, 5, 6, 8 if repetitions of digits is allowed?

Solution: Given that the numbers should not exceed 10,000. Hence, the numbers can be

either (*i*) 1-digit numbers

or (*ii*) 2-digit numbers

or (*iii*) 3-digit numbers

or (*iv*) 4-digit numbers

(*i*) **No. of 1-digit numbers**

The unit place can be filled up by any of the given 5 digits = $^5P_1$ = 5 ways.

(*ii*) **No. of 2-digit numbers**

Since repetition is allowed, any of the given 5 digits, can be placed in units place and tens place in 5^2 = 25 ways.

(*iii*) **No. of 3-digit numbers**

Similarly, any of the given 5 digits can be placed in unit place, tens place and hundreds place in 5^3 = 125 ways.

(*iv*) **No. of 4-digit numbers**

Similarly, any of the given 5 digits can be placed in unit place, tens place, hundreds place and thousands place in 5^4 = 625 ways.

Therefore, total no. of numbers not exceeding 10,000 that can be made using the digits 2, 4, 5, 6, 8 with repetition of digits = 5 + 25 + 125 + 625 = 780.

EXAMPLE 25

In how many different ways can the letters of the word 'OPTICAL' be arranged so that the vowels always come together?

Solution: The word 'OPTICAL' has 7 letters, which includes 3 vowels (O, I, A) and these vowels should always come together.

Hence, considering these three vowels as a single letter, i.e. PTCL (OIA), we can assume total letters as 5 and all these letters are different.

Now, these 5 letters can be arranged in 5! ways = 5 × 4 × 3 × 2 × 1 = 120 ways.

Again, all the 3 vowels (OIA) are different, they can be arranged among themselves in 3! ways = 3 × 2 × 1 = 6 ways.

Hence, total number of ways = 120 × 6 = 720.

EXAMPLE 26

In how many different ways can the letters of the word 'MATHEMATICS' be arranged such that the vowels must always come together?

Solution: There are 11 letters in the word 'MATHEMATICS', which includes 4 vowels (A, E, A, I) and these 4 vowels must always come together.

Considering these 4 vowels as a single letter, i.e. MTHMTCS (AEAI), we can assume total letters as 8. But in these 8 letters, 'M' occurs 2 times, 'T' occurs 2 times and rest of the letters are different.

Therefore, the number of arrangement $= \frac{8!}{2!2!}$

$$= \frac{8\times7\times6\times5\times4\times3\times2!}{2\times1\times2!} = 10,080.$$

Again, in the vowels (AEAI), 'A' occurs 2 times and rest of the vowels are different.

Therefore, the number of arrangements $= \frac{4!}{2!} = \frac{4\times3\times2!}{2!} = 12$

Therefore, required no. of ways = 10,080 × 12 = 1,20,960.

EXAMPLE 27

How many permutations can be made out of the letters of the word 'SUNDAY'?

(*i*) How many of these will begin with S?

(*ii*) How many of these will end with Y?

(*iii*) How many of these will begin with S and end with Y?

(*iv*) How many of these will begin with S but will not end with Y?

(*v*) How many of these will not begin with S?

Solution: There are 6 letters in the word SUNDAY, which are all different. So, they can be arranged in 6! = 720 ways.

(*i*) Placing S at the beginning, the remaining 5 places can be filled up by 5 letters in 5! = 120 ways.
Therefore, the number of permutations with S in the beginning = 120.

(*ii*) Placing Y in the last place, the remaining 5 letters can be arranged in 5 places in 5! = 120 ways.
Therefore, the number of permutations with Y in the end = 120.

(*iii*) Placing S at the beginning and Y at the end, the remaining 4 places can be filled in by the remaining 4 letters in 4! = 24 ways.
Therefore, the number of permutations with S at the beginning and Y at the end = 24.

(*iv*) The number of permutations which begin with S but do not end with Y = The number of permutations which begin with S – The number of permutations which begin with S and end with Y = 120 – 24 = 96.

(*v*) The number of permutations which will not begin with S = The total number of permutations – The number of permutations which will begin with S = 720 – 120 = 600.

EXAMPLE 28

In how many of the arrangements of 10 different things, taken 6 at a time, will 2 particular things (*i*) always occur, and (*ii*) never occur?

Solution:

(*i*) The number of arrangements of 10 different things, taken 6 at a time is the same as the number of ways in which 6 places can be filled up from 10 different things. Now the 2 particular things, which are always to occur can be arranged in 2 places out of 6 places in ${}^6P_2 = 6 \times 5 = 30$ ways. In each of these ways, the remaining 4 places can be filled up from the remaining 8 different things in ${}^8P_4 = 8 \times 7 \times 6 \times 5 = 1680$ ways.
Therefore, the required number of arrangements = 30 × 1680 = 50,400.

(*ii*) Keeping aside 2 particular things which will never occur, we are left with only 8 different things. The problem is then that of finding the number of arrangements in which 6 places can be filled up with 8 different things and the number is ${}^8P_6$.
Therefore, the required number of arrangements = ${}^8P_6 = 8 \times 7 \times 6 \times 5 \times 4 \times 3 = 20{,}160$.

EXAMPLE 29

In how many ways can 5 boys and 5 girls be seated at a round table so that no two girls may be together?

Solution: Leaving one seat vacant between 2 boys, 5 boys may be seated at a round table in (5 – 1)! i.e. 4! ways. Then on the remaining 5 seats, 5 girls can sit in 5! ways.

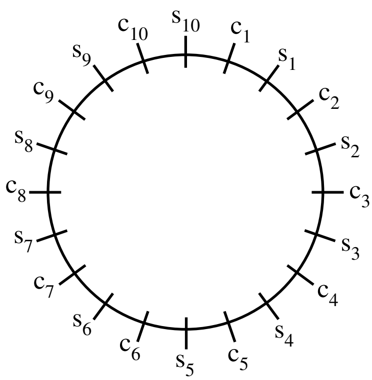
Hence, the required no. of ways = 4!5! = 24 × 120 = 2880.

EXAMPLE 30

A company has 10 software engineers and 6 civil engineers. In how many ways can they be seated around a round table so that no two of the civil engineers will sit together?

Solution: 10 software engineers ($S_1, S_2, S_3, \ldots S_{10}$) can be seated around a round table in (10 – 1)!, i.e. 9! ways.

Now we have to arrange civil engineers such that no two civil engineers can be seated together, i.e. we can arrange 6 civil engineers in any of the 10 vacant seats ($c_1, c_2, c_3, \ldots c_{10}$)



This can be done in $^{10}P_6$ ways.

Therefore, the required number of ways = 9! × $^{10}P_6$

$$= 9! \times \frac{10!}{4!}$$

EXAMPLE 31

A board meeting of a company is organised in a room for 24 persons along the two sides of a table with 12 chairs on each side. 6 persons want to sit on a particular side and 3 persons want to sit on the other side. In how many ways can they be seated?

Solution: 6 persons can be arranged on the 12 chairs on the particular sides in $^{12}P_6$ ways and 3 persons can be arranged on the 12 chairs on the other side in $^{12}P_3$ ways.

Remaining persons = 24 – 6 – 3 = 15 and remaining chairs = 24 – 6 – 3 = 15.

Now, the remaining 15 persons can be arranged on the remaining 15 chairs in $^{15}P_{15}$, i.e. 15! ways.

Therefore, the required number of ways = $^{12}P_6 \times {}^{12}P_3 \times 15!$

ILLUSTRATIVE EXAMPLES OF COMBINATIONS

A. SHORT TYPE

EXAMPLE 1

Find the value of $^{16}C_{13}$

Solution: $$^{16}C_{13} = \frac{16!}{13!\,(16-13)!} = \frac{16!}{13!\,3!} = \frac{16\times15\times14\times13!}{13!\times3\times2\times1} = 560.$$

EXAMPLE 2

How many combinations of 4 letters can be made of the letters of the word 'JAIPUR'?

Solution: 4 letters are to be selected out of 6 different letters.

So, the number of combinations = $$^{6}C_4 = \frac{6!}{4!\,2!}$$
$$= \frac{6\times5\times4!}{4!\times2\times1} = 15.$$

EXAMPLE 3

How many ways are there in selecting 5 members consisting 3 males and 2 females from 6 males and 5 females?

Solution: 3 males to be selected from 6 males and 2 females to be selected from 5 females.

Therefore, required number of ways = $^{6}C_3 \times {}^{5}C_2$
$$= \frac{6!}{3!\,3!}\times\frac{5!}{2!\,3!} = \frac{6\times5\times4\times3!}{3\times2\times1\times3!}\times\frac{5\times4\times3!}{2\times1\times3!}$$
$$= 20\times10 = 200.$$

EXAMPLE 4

A question paper has two parts A and B, each containing 10 questions. If a student needs to choose 8 from Part A and 4 from Part B, in how many ways can he do that?

Solution: Number of ways to choose 8 questions from Part A = $^{10}C_8$

Number of ways to choose 4 questions from Part B = ${}^{10}C_4$

Therefore, the required number of ways = ${}^{10}C_8 \times {}^{10}C_4$

$$= \frac{10!}{8!\,2!} \times \frac{10!}{4!\,6!} = \frac{10 \times 9 \times 8!}{2 \times 1 \times 8!} \times \frac{10 \times 9 \times 8 \times 7 \times 6!}{4 \times 3 \times 2 \times 1 \times 6!}$$

$$= 45 \times 210 = 9450.$$

EXAMPLE 5

Tulu has 10 friends and he wants to invite 6 of them to a party. How many times will 3 particular friends never attend the party?

Solution: Excluding the 3 particular friends, Tulu can invite 6 friends from the remaining 7 friends. This can be done in ${}^7C_6$ ways = 7 ways.

EXAMPLE 6

How many triangles can be formed by joining the vertices of an octagon?

Solution: 3 points are required to form a triangle and number of vertices of an octagon = 8

Therefore, number of triangles that can be formed by joining the vertices of an octagon = ${}^8C_3 = \frac{8!}{3!5!}$

$$= \frac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 5!} = 56.$$

EXAMPLE 7

How many straight lines can be formed by joining 12 points on a plane out of which no points are collinear?

Solution: There are 12 points in a plane out of which no points are collinear.

2 points are required to form a straight line.

Therefore, number of straight lines that can be formed by joining these 12 points =

$${}^{12}C_2 = \frac{12!}{2!\,10!} = \frac{12 \times 11 \times 10!}{2 \times 1 \times 10!} = 66.$$

EXAMPLE 8

If ${}^{18}C_r = {}^{18}C_{r+2}$; find r. [***C.U. B.Com. 1985, 2012***]

Solution: ${}^{18}C_r = {}^{18}C_{r+2}$

or $r + r + 2 = 18$ [If ${}^nC_p = {}^nC_q$, then $p + q = n$]

or $2r + 2 = 18$

or $2r = 18 - 2 = 16$

or $\quad r=\frac{16}{2}=8.$

EXAMPLE 9

If ${}^{n}C_3=120,$ find n. **[*C.U. B.Com. 2011, 2015(H)*]**

Solution: ${}^{n}C_3=120$

or $\frac{n!}{3!(n-3)!}=120$

or $\frac{n.(n-1).(n-2).(n-3)!}{3\times 2\times 1\times (n-3)!}=120$

or $n.(n-1).(n-2)=120\times 6=720$

or $n(n-1)(n-2)=10\times 9\times 8$

or $n=10.$

EXAMPLE 10

If ${}^{n}P_r=6.{}^{n}C_r$ and ${}^{n}C_r=56$ then find n and r. **[*C.U. B.Com. 2014(G)*]**

Solution: ${}^{n}P_r=6.{}^{n}C_r=6.56$ [As ${}^{n}C_r=56$]

or $\frac{n!}{(n-r)!}=336$ ...(*i*)

Again, ${}^{n}C_r=56$

or $\frac{n!}{r!(n-r)!}=56$ ...(*ii*)

Dividing (*i*) by (*ii*) we get.

$$\frac{n!}{(n-r)!}\Big/\frac{n!}{r!(n-r)!}=\frac{336}{56}$$

or $\frac{n!}{(n-r)!}\times\frac{r!(n-r)!}{n!}=6$

or $r!=6=3\times 2\times 1=3!$

Therefore, $r=3.$

Putting $r=3$ in equation no. (*i*) we get,

$$\frac{n!}{(n-3)!}=336$$

or $\frac{n(n-1).(n-2).(n-3)!}{(n-3)!}=336$

or $n(n-1)(n-2) = 8.7.6$
Therefore, $n = 8$.
Therefore, required value of $n = 8$ and $r = 3$.

EXAMPLE 11

6 persons meet and shake hands. How many handshakes are there in all?
[***C.U. B.Com. 1995***]

Solution: To make a handshake two persons are required.

Hence, for 6 persons number of handshakes = $^6C_2 = \frac{6!}{2!\,4!} = \frac{6\times5\times4!}{2\times1\times4!} = 15.$

EXAMPLE 12

If $^{2n}C_3 : {}^nC_2 = 12 : 1$ then find n. [***C.U. B.Com. 2013(G)***]

Solution: $^{2n}C_3 : {}^nC_2 = 12 : 1$

or $\frac{2n!}{3!(2n-3)!} \Big/ \frac{n!}{2!(n-2)!} = 12:1$

or $\frac{2n(2n-1)(2n-2).(2n-3)!}{3\times2\times1\times(2n-3)!} \times \frac{2\times1\times(n-2)!}{n.(n-1).(n-2)!} = 12$

or $\frac{2n.(2n-1).2(n-1)}{6} \times \frac{2}{n(n-1)} = 12$

or $4(2n-1) = 36$

or $2n - 1 = 9$

or $2n = 10$

Therefore, $n = 5$.

EXAMPLE 13

A box contains 4 red, 3 white and 2 blue balls. Three balls are drawn at random. Find out the number of ways of selecting the balls of different colours?

Solution: 1 red ball can be selected in $^4C_1$ ways
1 white ball can be selected in $^3C_1$ ways
1 blue ball can be selected in $^2C_1$ ways
Therefore, total number of ways = $^4C_1 \times {}^3C_1 \times {}^2C_1$
$= 4 \times 3 \times 2 = 24.$

EXAMPLE 14

There are 8 English books, 6 Hindi books and 5 Bengali books in a library. In how many ways can one or more than one book be selected?

Solution: We know that the total number of combinations of $(p + q + r + \ldots\ldots)$ things of which 'p' things are of one kind, 'q' of the second kind and 'r' of the third kind and so on, taken any number of things at a time.

$= [(p + 1)(q + 1)(r + 1)\ldots\ldots\ldots] - 1$

Here, $p = 8$, $q = 6$, $r = 5$

Hence, required number of ways

$= (8 + 1)(6 + 1)(5 + 1) - 1$

$= 9 \times 7 \times 6 - 1 = 378 - 1 = 377.$

EXAMPLE 15

From 8 boys and 5 girls, how many different selections can be made so as to include at least one boy and one girl?

Solution: One or more boys can be selected from 8 boys in $(2^8 - 1)$ ways; for each such selection, one or more girls can be selected from 5 girls in $(2^5 - 1)$ ways.

Hence, the required number of selections with at least one boy and one girl = $(2^8 - 1) \times (2^5 - 1) = 255 \times 31 = 7905$.

EXAMPLE 16

In an examination a candidate has to secure minimum marks in each of the 5 subjects to pass the examination. In how many cases can a student fail?

Solution: The candidate will fail in the examination, if he fails in any number of 5 subjects, i.e. fails in one or two or three or four or five subjects.

Therefore, the number of cases in which the candidate may fail

$$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$$

$$= 2^5 - 1 = 32 - 1 = 31$$

EXAMPLE 17

In how many ways can 12 different mangoes be divided into three groups containing 3, 4 and 5 mangoes?

Solution: 12 different mangoes can be divided into three groups containing 3, 4 and 5 mangoes in

$$\frac{12!}{3!4!5!} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1 \times 5!} = 27{,}720$$

EXAMPLE 18

In how many way can 6 different things be divided into three equal groups?

Solution: 6 different things can be divided into three equal groups in

$$\frac{6!}{3!(2!)^3} = \frac{6\times5\times4\times3!}{3!\times8} = 15$$

EXAMPLE 19

In how many ways can 9 different apples be distributed equally among 3 boys?

Solution: 9 different apples can be divided among 3 boys in $\frac{9!}{(3!)^3}$

$$= \frac{9\times8\times7\times6\times5\times4\times3!}{6\times6\times3!} = 1680$$

EXAMPLE 20

If ${}^nC_r = x.\, {}^nP_r$, what is the value of x? [***C.U. B.Com. 2016(H)***]

Solution: ${}^nC_r = x.\, {}^nP_r$

or

$$x = \frac{{}^nC_r}{{}^nP_r} = \frac{n!}{r!(n-r)!} \times \frac{(n-r)!}{n!}$$

$$= \frac{1}{r!}$$

B. SHORT ESSAY TYPE

EXAMPLE 21

A man has 20 acquaintances of which 12 are relatives. In how many ways he may invite 13 guests out of them so that 9 would be relatives and one his best friend? [***C.U. B.Com. 1992***]

Solution: Out of 20 acquaintances 12 are relatives.

Therefore, number of non-relative friends = 20 – 12 = 8, which include his best friend.

So, he can select 13 guests of which 9 are relatives, one his best friend and 3 are non-relatives (excluding best friend).

This can be done in ${}^{12}C_9 \times {}^7C_3$ ways

$$= \frac{12!}{3!9!} \times \frac{7!}{3!4!} = \frac{12\times11\times10\times9!}{3\times2\times1\times9!} \times \frac{7\times6\times5\times4!}{3\times2\times1\times4!}$$

$$= 220\times35 = 7700$$

EXAMPLE 22

A candidate is required to answer 6 out of 10 questions which are divided into two groups, each containing 5 questions, and he is not permitted to attempt more than 4 from each group. In how many different ways can he make his selection?

Solution: The problem can be arranged as follows:

Option	First Group (5)	Second group (5)
(*i*)	4	2
(*ii*)	3	3
(*iii*)	2	4

(*i*) 4 questions can be selected from 5 of first group in ${}^5C_4 = {}^5C_1 = 5$ ways, and after they have been selected in any of these 5 ways, the 2 questions from 5 of the second group can be selected in ${}^5C_2 = 10$ ways.
Hence, the number of selections in option (*i*) = 5 × 10 = 50
Similarly in option (*ii*), number of selections = ${}^5C_3 \times {}^5C_3 = 10 \times 10 =$ 100 ways and in option (*iii*), number of selections = ${}^5C_2 \times {}^5C_4 = 10 \times 5 =$ 50 ways.
Therefore, the total number of ways in which a candidate may select 6 questions = 50 + 100 + 50 = 200.

EXAMPLE 23

How many words, each containing 2 vowels and 3 consonants can be formed with the letters of the word 'FACETIOUS'? [***C.U. B.Com. 1982***]

Solution: There are 5 vowels (A, E, I, O, U) and 4 consonants (F, C, T, S) in the word 'FACETIOUS' and all the letters are different.

2 vowels can be chosen from 5 vowels in ${}^5C_2$ ways and 3 consonants from 4 consonants in ${}^4C_3$ ways. Since each choice of 2 vowels can be associated with each choice of 3 consonants, the choice of 2 vowels and 3 consonants can be made in ${}^5C_2 \times {}^4C_3$ ways.

Again, each set of 5 letters can be arranged among themselves in 5! ways.

Therefore, total number of words framed = ${}^5C_2 \times {}^4C_3 \times 5!$

$= 10 \times 4 \times 120 = 4800.$

EXAMPLE 24

There are 8 points in a plane out of which 3 are collinear. How many straight lines can be formed by joining them? What will be the number of triangles formed?

Solution: By joining any two points we get a straight line. Therefore, the number of straight lines formed by joining 8 points in pairs = ${}^8C_2$

But out of 8 points 3 are collinear.

We cannot get ${}^8C_2$ straight lines joining any two of the three collinear points and instead we get only one straight line.

Therefore, the required number of straight lines is

${}^8C_2 - {}^3C_2 + 1 = 28 - 3 + 1 = 26$

By joining any three non-collinear points we get a triangle.

Therefore, the number of triangles formed by joining 8 points taken 3 at a time = ${}^8C_3$

But we cannot get ${}^3C_3$ triangles joining the three collinear points.

Therefore, the required number of triangles $= {}^8C_3 - {}^3C_3 = 56 - 1 = 55.$

EXAMPLE 25

From 6 bowlers, 2 wicketkeepers and 8 batsmen in how many ways a team of 11 players consisting of at least 4 bowlers, one wicketkeeper and at least 5 batsmen can be formed? [***C.U. B.Com. 1994, 2017(G)***]

Solution: The problem can be arranged as follows:

Option	Bowler (6)	Wicketkeeper (2)	Batsmen (8)	Total
(*i*)	4	1	6	11
(*ii*)	4	2	5	11
(*iii*)	5	1	5	11
(*iv*)	5	2	4	11
(*v*)	6	1	4	11

(*i*) 4 bowlers can be selected from 6 bowlers in ${}^6C_4$ ways, one wicketkeeper can be selected from 2 wicketkeepers in ${}^2C_1$ ways and 6 batsmen can be selected from 8 batsmen in ${}^8C_6$ ways.

Hence, the total number of ways of forming team with 4 bowlers, 1 wicket-keeper and 6 batsmen $= {}^6C_4 \times {}^2C_1 \times {}^8C_6$

$= 15 \times 2 \times 28 = 840.$

(*ii*) 4 bowlers can be selected from 6 bowlers in ${}^6C_4$ ways, 2 wicket-keepers can be selected from 2 wicketkeepers in ${}^2C_2$ ways and 5 batsmen can be selected in ${}^8C_5$ ways.

Hence, total number of ways $= {}^6C_4 \times {}^2C_2 \times {}^8C_5$

$= 15 \times 1 \times 56 = 840.$

(*iii*) 5 bowlers can be selected from 6 bowlers in ${}^6C_5$ ways, 1 wicketkeeper can be selected from 2 wicketkeepers in ${}^2C_1$ ways and 5 batsmen can be selected from 8 batsmen in ${}^8C_5$ ways.

Hence, total number of ways $= {}^6C_5 \times {}^2C_1 \times {}^8C_5$

$= 6 \times 2 \times 56 = 672.$

(*iv*) 5 bowlers can be selected from 6 bowlers in $^6C_5$ ways, 2 wicketkeepers can be selected from 2 wicketkeepers in $^2C_2$ ways and 4 batsmen can be selected from 8 batsmen = $^8C_4$ ways.
Hence, total number of ways = $^6C_5 \times {^2C_2} \times {^8C_4}$
$= 6 \times 1 \times 70 = 420.$

(*v*) Similarly, in case of option (*v*) total number of ways = $^6C_6 \times {^2C_1} \times {^8C_4}$
$= 1 \times 2 \times 70 = 140$

Therefore, total number of selection
= 840 + 840 + 672 + 420 + 140 = 2912.

EXAMPLE 26

In how many ways can a committee of 3 ladies and 4 gentlemen be nominated from a meeting consisting of 8 ladies and 7 gentlemen? What will be the number of ways if Mrs. X refuses to serve in a committee if Mr. Y is a member?

Solution: The committee of 3 ladies and 4 gentlemen can be formed out of 8 ladies and 7 gentlemen in $^8C_3 \times {^7C_4}$ ways = 56 × 35 = 1960.

If both Mrs. X and Mr. Y are members of the committee then the committee of 2 ladies and 3 gentlemen can be formed out of 7 ladies and 6 gentlemen in $^7C_2 \times {^6C_3}$ ways = 420 ways.

Therefore, the number of committees where Mrs. X and Mr. Y never serve together = 1960 – 420 = 1540.

EXAMPLE 27

In how many ways can a team of 5 persons be formed out of a total of 10 persons such that:

(*i*) two particular persons should be included in each team.
(*ii*) two particular persons should not be included in each team.

Solution:

(*i*) Two particular persons should be included in each team. Therefore, we have to select remaining 5 – 2 = 3 persons from 10 – 2 = 8 persons.
Hence, required number of ways = $^8C_3$

$$= \frac{8!}{3!5!} = \frac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 5!} = 56$$

(*ii*) Two particular persons should not be included in each team, i.e. we have to select 5 persons from 10 – 2 = 8 persons.
Hence, the required number of ways = $^8C_5$

$$= \frac{8!}{3!5!} = 56.$$

EXAMPLE 28

In a B.Com. examination paper on Business Studies 11 questions are set. In how many ways can you choose 6 questions to answer? If, however, Question No. 11 is made compulsory, in how many ways can you select to answer 6 questions in all?

Solution: 6 questions can be selected from 11 questions in $^{11}C_6$ ways

$$= \frac{11!}{5!6!} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6!}{5 \times 4 \times 3 \times 2 \times 1 \times 6!} = 462 \text{ ways.}$$

When the Question no. 11 is compulsory, then the remaining 5 questions are to be selected from the remaining 10 questions and this can be done in $^{10}C_5$ ways

$$= \frac{10!}{5!5!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5 \times 4 \times 3 \times 2 \times 1 \times 5!} = 252 \text{ ways.}$$

EXAMPLE 29

In a group of 6 boys and 4 girls, 4 children are to be selected. In how many different ways can they be selected such that at least one boy should be there?

Solution: In a group of 6 boys and 4 girls, 4 children are to be selected such that at least one boy should be there.

Hence, we have four choices as given below.

(*i*) 1 boy and 3 girls
(*ii*) 2 boys and 2 girls
(*iii*) 3 boys and 1 girl
(*iv*) 4 boys

(*i*) 1 boy out of 6 boys and 3 girls out of 4 girls can be selected in $^6C_1 \times {}^4C_3$ ways = 24 ways.

(*ii*) 2 boys out of 6 boys and 2 girls out of 4 girls can be selected in $^6C_2 \times {}^4C_2$ ways = 90 ways.

(*iii*) 3 boys out of 6 boys and 1 girl out of 4 girls can be selected in $^6C_3 \times {}^4C_1$ ways = 80 ways.

(*iv*) 4 boys can be selected out of 6 boys in $^6C_4$ ways = 15 ways.

Therefore, total number of ways = 24 + 90 + 80 + 15 = 209.

EXAMPLE 30

Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

Solution: Number of ways of selecting 3 consonants out of 7 and 2 vowels out of 4 = $^7C_3 \times {}^4C_2$

$$= \frac{7!}{3!4!} \times \frac{4!}{2!2!} = \frac{7 \times 6 \times 5 \times 4!}{3 \times 2 \times 1 \times 4!} \times \frac{4 \times 3 \times 2!}{2 \times 1 \times 2!}$$

$$= 35 \times 6 = 210.$$

It means that we can have 210 groups where each group contains total 5 letters (3 consonants and 2 vowels).

Number of ways of arranging 5 letters among themselves = $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

Hence, required number of ways = 210×120

= 25,200.

EXAMPLE 31

In a group of 15 boys there are 7 boy-scouts. In how many ways can 12 boys be selected so as to include (*i*) exactly 6 boy-scouts; (*ii*) at least 6 boy-scouts.

Solution:

(*i*) Out of 15 boys number of boy-scouts = 7.

Therefore, number of remaining boys, i.e. other than boy-scouts = 15 – 7 = 8.

12 boys are to be selected which includes exactly 6 boy-scouts.

Now, 6 boy-scouts can be selected out of 7 boy-scouts in $^7C_6$ ways and the remaining 6 boys can be selected out of remaining 8 boys in $^8C_6$ ways. As each way of selecting boy-scouts are associated with each way of selecting boys. So the total number of selections = $^7C_6 \times {^8C_6} = 7 \times 28 = 196$.

(*ii*) Possible options are as follows:

(*a*) 6 boy-scouts and 6 boys

(*b*) 7 boy-scouts and 5 boys

In case of (*a*) no. of selections = $^7C_6 \times {^8C_6}$ and in case of (*b*) no. of selections = $^7C_7 \times {^8C_5}$.

Therefore, the total number of selections

$$= {^7C_6} \times {^8C_6} + {^7C_7} \times {^8C_5}$$

$$= 7 \times 28 + 1 \times 56 = 196 + 56 = 252.$$

EXAMPLE 32

Find the number of combinations that can be made by taking 4 letters of the word 'COMBINATION'.

Solution: The word 'COMBINATION' consists of 11 letters of 8 different kinds

C, (O, O), M, B, (I, I), (N, N), A, T

Where two Os, two Is and two Ns are alike.

Combinations of 4 letters can be made as follows:

(*i*) All the four letters are different

(*ii*) 2 letters are alike, 2 are different

(*iii*) 2 letters are alike of one kind; 2 letters are alike of other kind.

(*i*) There are 8 different letters, so 4 different letters can be selected from 8 different letters in $^8C_4$ ways.

(*ii*) There are 3 pairs of alike letters [(O, O), (I, I), (N, N)]. So one pair can be selected in $^3C_1$ ways and the remaining 2 different letters can be selected from the remaining 7 different letters in $^7C_2$ ways.
Hence, the required number of combinations = $^3C_1 \times {}^7C_2$.

(*iii*) 2 pairs of alike letters can be selected from 3 pairs of alike letters in $^3C_2$ ways.
Therefore, the total number of combinations

$$= {}^8C_4 + {}^3C_1 \times {}^7C_2 + {}^3C_2$$
$$= 70 + 63 + 3 = 136.$$

EXAMPLE 33

At an election there are 7 candidates out of whom 3 members are to be selected and a voter is entitled to vote for any number of candidates not greater than the number to be elected. In how many ways may a voter choose to vote?

Solution: As per condition of the problem the voter may vote for one, two or three candidates out of the 7 candidates. As 7 candidates are contesting in the election, so a voter can cast one vote in $^7C_1$ ways, two votes in $^7C_2$ ways and three votes in $^7C_3$ ways.

Hence, the total number of ways in which a voter may choose to vote

$$= {}^7C_1 + {}^7C_2 + {}^7C_3 = 7 + 21 + 35 = 63.$$

EXAMPLE 34

A person has in his bag 15 coins of ₹10 each, 10 coins of ₹5 each, 5 coins of ₹2 each and 8 coins of ₹1 each. In how many different ways can he contribute to a charitable fund?

Solution: The person can contribute 15 coins of ₹10 each in (15 + 1) ways and for each of these ways he can contribute 10 coins of ₹5 each in (10 + 1) ways. In the similar way he can contribute 5 coins of ₹2 each and 8 coins of ₹1 each in (5 + 1) and (8 + 1) ways.

Hence, the number of ways in which the person can contribute to the charitable fund = (15 + 1)(10 + 1)(5 + 1)(8 + 1) ways.

But this includes the case in which the person does not contribute any coin at all.

Therefore, the required number of ways = (15 + 1)(10 + 1)(5 + 1)(8 + 1) – 1 = 9503

EXAMPLE 35

A student is to answer 8 out of 10 questions on an examination:

(*i*) How many choices has he?

(*ii*) How many if he must answer the first three questions?

(*iii*) How many if he must answer at least four of the first five questions?

[***C.A. Entrance, May 1979***]

Solution:

(*i*) 8 questions can be answered out of 10 questions in $^{10}C_8$ ways $= \dfrac{10!}{2!8!}$

$$= \frac{10 \times 9 \times 8!}{2 \times 1 \times 8!} = 45 \text{ ways.}$$

(*ii*) The first 3 questions are to be answered. So the remaining 5 questions to be answered out of remaining 7 questions. This can be done in $^{7}C_5$ ways

$$= \frac{7!}{2!5!} = \frac{7 \times 6 \times 5!}{2 \times 1 \times 5!} = 21 \text{ ways.}$$

(*iii*) Here we have the following options:

Options	**Group-A (say) (First 5 questions)**	**Group-B (say) (Balance of 5 questions)**
(*a*)	4	4
(*b*)	5	3

(*a*) 4 questions can be answered from 5 questions in Group-A in $^{5}C_4$ ways and 4 questions can be answered from 5 questions in Group-B in $^{5}C_4$ ways.

Therefore, the number of ways = $^{5}C_4 \times {}^{5}C_4 = 25$.

(*b*) 5 questions can be answered from 5 questions in Group-A in $^{5}C_5$ ways and 3 questions can be answered from 5 questions in Group-B in $^{5}C_3$ ways.

Therefore, the number of ways = $^{5}C_5 \times {}^{5}C_3 = 10$.

Therefore, the required number of ways = 25 + 10 = 35.

EXERCISE

A. SHORT TYPE

Permutations

1. Find the value of

(*i*) $^{7}P_1$ (*ii*) $^{10}P_2$ (*iii*) $^{n}P_0$ (*iv*) $^{5}P_5$ (*v*) $\dfrac{\lfloor 9}{\lfloor 4}$

[**Ans.** (*i*) 7, (*ii*) 90, (*iii*) 1, (*iv*) 120, (*v*) 15, 120]

2. (*i*) If $^{n}P_3 = 336$, find the value of n. [**Ans.** 8]

(*ii*) If $^{n}P_2 = 12$, find the value of n. [***C.U. B.Com. 2014(G)***] [**Ans.** 4]

(*iii*) If ${}^nP_5 : {}^nP_3 = 2 : 1$, then find n. [***C.U. B.Com. 2014(G), 2017(G)***]
[**Ans.** $n = 5$]

(*iv*) If ${}^{n+1}P_4 : {}^{n-1}P_3 = 72 : 5$, then find the value of n? [***C.U. B.Com. 2015(G)***]
[**Ans.** 8]

(*v*) If ${}^nP_6 = 30. {}^nP_4$, find the value of n. [***C.U. B.Com. 2015(H)***]
[**Ans.** 10]

3. In how many ways can 5 workers be appointed to 5 different posts? [***C.U. B.Com.'84***] [**Ans.** 120]
   [**Hints:** ${}^5P_5 = 120$]
4. In how many ways can 4 passengers be seated in 4 train compartments? [**Ans.** 24]
5. In how many ways 3 letters can be put in 3 different envelopes? [**Ans.** 6]
6. There are 4 launches plying between Howrah to Bagbazar. In how many ways can a man go from Howrah to Bagbazar and return by a different launch? [**Ans.** 12]
7. How many permutations can be done from the word DELHI taken together? [***C.U. B.Com. 2013(G)***] [**Ans.** 120]
8. There are 4 schools in a town. In how many ways can a man send his three children to the schools so that each of them may study in different schools? [**Ans.** 24]
9. In how many ways can two prizes be awarded to 10 students without giving both to the same boy? [**Ans.** 90]
10. A hall has 5 doors. In how many ways can a man enter into the room and go away by a different door? [**Ans.** 20]
11. How many different numbers can be formed using each of the 4 digits once with the digits 1, 2, 3, 4? [**Ans.** 24]
12. How many different numbers of four digits can be formed with the digits 3, 4, 7, 8, 9 (no figure being repeated in any number)? [**Ans.** 120]
13. How many different numbers of 3 digits can be formed with the digits 1, 2, 3, 4, 5 (no figure being repeated in any number)? [**Ans.** 60]
14. How many different words can be formed with the letters of the word MONDAY, no letter being repeated in any word? [**Ans.** 720]
    [**Hints:** ${}^6P_6$]
15. How many different words can be formed by the letters of the word ARTICLE? [**Ans.** 5040]
16. How many different permutations can be made by taking 5 of the letters of the word THURSDAY? [**Ans.** 6720]
17. (*i*) In how many ways can the letters of the word SALOON be arranged if the two O's do not come together? [***B.Com.' 82 (old)***] [**Ans.** 60]

(*ii*) How many different arrangements can be made with the letters of the word "GENTLEMEN"? [***C.U. B.Com. 2017(H)***]

[**Ans.** 30,240]

18. In how many ways can the letters of the word COTTON be arranged so that the two T's are not together? [**Ans.** 120]

[**Hints:** Total no. of permutations $= \frac{\lfloor 6}{\lfloor 2 \lfloor 2} = 180$, taking two T's as a single letter, no. of permutations $= \frac{\lfloor 5}{\lfloor 2} = 60$.

Therefore, reqd. no. of permutations in which the two T's will not be together $= 180 - 60 = 120$] [**Ans.** 120]

19. Find the total no. of words that can be formed with the letters of the word BALLOON taken all together? [**Ans.** 1260]

20. (*a*) In how many ways can 4 letters be posted to 3 letter boxes? [**Ans.** 81]

(*b*) In how many ways can 10 children sit in a merry-go-round relative to one another? [**Ans.** 362880]

21. In how many ways can 8 persons be arranged in a ring? [**Ans.** 5040]

22. In how many ways can 6 stones of different colours be strung together on a necklace? [**Ans.** 60]

23. In how many ways can 8 beads of different colours be strung into a ring?

[**Ans.** 2520]

24. In how many ways can 5 persons be seated at a round table? [**Ans.** 24]

25. In how many ways can 10 persons sit at a round table? [**Ans.** 362880]

Combinations

1. Find the value of:

(*i*) ${}^{10}C_0$ (*ii*) ${}^{15}C_1$ (*iii*) ${}^{10}C_6$ (*iv*) ${}^{6}C_6$

(*v*) ${}^{8}C_3$ (*vi*) ${}^{20}C_{17}$ (*vii*) ${}^{12}C_3$ (*viii*) ${}^{9}C_3 + {}^{12}C_8$

(*ix*) ${}^{9}C_4 + {}^{12}C_4$ (*x*) ${}^{10}C_3 + {}^{10}C_4$

[**Ans.** (*i*)1, (*ii*) 15, (*iii*) 210, (*iv*) 1, (*v*) 56, (*vi*) 1140, (*vii*) 220, (*viii*) 579, (*ix*) 621, (*x*) 330.]

2. If, ${}^{n}C_2 = 21$, find the value of *n*. [***C.U. B.Com.'94 '96***] [**Ans.** 7]

3. If ${}^{n}C_2 = 45$, find the value of *n*. [***C.U. B.Com. '98***] [**Ans.** 10]

4. If ${}^{n}C_3 = 120$, find the value of *n*. [***C.U. B.Com. 2011, 2015(H)***] [**Ans.** 8]

5. If ${}^{n}C_6 = {}^{n}C_8$, find the value of *n*. [***C.U. B.Com. '87***] [**Ans.** 14]

6. (*i*) If ${}^{11}C_x = {}^{11}C_y$ and $x \neq y$, then what is the value of $x + y$?

[***C.U. B.Com. '83***] [**Ans.** 11]

(*ii*) If ${}^{18}C_r = {}^{18}C_{r+2}$, find the value of ${}^{r}C_5$.
[***C.U. B.Com. 2000, 2012, 2016(G)***] [**Ans.** 56]

7. (*i*) If ${}^{18}C_r = {}^{18}C_{r+2}$, find the value of r. [***C.U. B.Com. '85; 89; 2012***] [**Ans.** 8]

(*ii*) If ${}^{15}C_r = {}^{15}C_{r+3}$, find the value of ${}^{8}C_r$. [***C.U. B.Com. 2017(H)***] [**Ans.** 28]

(*iii*) If ${}^{n}C_x = 56$ and ${}^{n}P_x = 336$, find n and x. [***C.U. B.Com. 2017(G)***] [**Ans.** $n = 8, x = 3$]

8. If ${}^{20}C_r = {}^{20}C_{2r+5}$, find the value of r. [***C.U. B.Com. '93***] [**Ans.** –5]

9. How many different committees consisting of 4 members can be formed out of 8 members? [***C.U. B.Com. '87***] [**Ans.** ${}^{8}C_4 = 70$]

10. How many different committees of 8 can be formed from 12 persons? [**Ans.** 495]

11. In how many ways can 5 questions be selected from 8 questions? [**Ans.** 56]

12. Six persons meet and shake hands. How many handshakes are there in all? [***C.U. B.Com. '95***] [**Ans.** 15]

13. (*a*) A man has 6 friends. In how many ways can he invite one or more of them at a time on dinner. [***N.B.U. B.Com. '98***] [**Ans.** $2^6 - 1 = 63$]

(*b*) A man has 7 friends. In how many ways can he invite one or more of them to a party? [***K.U. B.Com. '97***] [**Ans.** $2^7 - 1 = 127$]

14. In how many ways can a father take one or more of his 5 sons to a fair? [**Ans.** 31]

15. How many words (each word containing 2 vowels and 3 consonants) can be formed with the letters of the word FACETIOUS? [***C.U. B.Com. (New) '82]*** [**Ans.** 4800]

[**Hints:** ${}^{5}C_2 \times {}^{4}C_3 \times \underline{|5}$]

16. How many different committees of 6 members may be formed from 7 gentlemen and 5 ladies? [**Ans.** 924]

17. In an examination paper on Advanced Accounts, 10 questions are set. In how many different ways can an examinee choose 7 questions? [**Ans.** 120]

18. From 6 boys and 4 girls, 5 are to be selected for admission for a particular course. In how many ways can this be done if there must be exactly 2 girls? [**Ans.** 120]

[**Hints:** ${}^{4}C_2 \times {}^{6}C_3$]

19. How many straight lines can be obtained by joining eight points, no three of which are collinear? [**Ans.** 28]

20. A person has got 15 acquaintances out of whom 10 are relatives. In how many ways can he invite 9 guests so that 7 of them would be relatives? [**Ans.** 1200]

[**Hints:** ${}^{10}C_7 \times {}^{5}C_2$] [***C.U. B.Com. 1996, 2013(G)***]

B. SHORT ESSAY TYPE

Permutations

1. Three persons get into a railway carriage, where there are 8 seats. In how many ways can they seat themselves?

   [**Hints:** Since there are 8 vacant seats, the first man can choose any one of these 8 seats. There are thus 8 ways of filling the first seat, when that one is occupied 7 seats are left, therefore, the second man can occupy any one of the 7 seats. The last man can now seat himself in one of the remaining 6 seats.

   Therefore, no. of ways in which three persons can occupy 8 seats is $8 \times 7 \times 6 = 336$.]

2. There are 26 stations on a certain railway line. How many kinds of different single third-class tickets have to be printed, in order that it may be possible to travel from any station to another? [**Ans.** 650]

3. In how many ways can the letters of the word 'DAUGHTER' be arranged so that the vowels (*a*) may never be separated, (*b*) may occupy only odd positions, (*c*) are never together? [***C.U. B.Com. 2016(H)***]
   [**Ans.** (*a*) 4,320, (*b*) 2,880, (*c*) 36,000]

4. How many words can be formed of the letters in the word 'COSTING', the vowels being not separated? [**Ans.** 1,440]

5. How many words can be formed of the word 'FAILURE', the four vowels always coming together? [**Ans.** 576]

6. How many words can be formed of the letters in the word ARTICLE, so that the vowels may occupy only (*i*) the even positions; (*ii*) the odd positions?
   [**Ans.** (*i*) 144; (*ii*) 576]

7. In how many ways, can the letters of the word SANDESH be arranged so that the vowels occupy the even places? [**Ans.** 360]

8. In how many ways can the letters of the word ARRANGE be arranged so that two R's may not occur together? [**Ans.** 900]

9. In how many ways can the letters of the word BALLOON be arranged, so that the two L's do not come together? [**Ans.** 900]

10. In how may ways can the letters of the word 'BLOSSOM' be arranged if (*i*) the two S's are always together, (*ii*) the two S's are never together?
    [**Ans.** (*i*) 360; (*ii*) 900]

11. In how many ways can the letters of the word ASSISTANT, taken all together, be arranged? [**Ans.** 15,120]

12. In how many ways can the letters of the word MONDAY be arranged? How many of them begin with M? How many of them do not begin with M but end with Y? [***C.U. B.Com. '95***] [**Ans.** 720, 120, 96]

13. Out of the letters A, B, C, p, q, r how many different words can be formed if (*i*) the words always begin with a capital letters? (*ii*) the words always begin with a capital letter and end also with a capital letter? [***C.U. B.Com. '83***]
[**Ans.** 360, 140]

14. How many ways can the letters of the word 'TRIANGLE' be arranged? How many of these begin with T? How many of these begin with T and end with E?
[**Ans.** 40, 320, 5040, 720]

15. How many different permutations are there of all the letters of the word ALABAMA? Of these permutations how many contain the word LAMB?
[**Ans.** 210; 4]

16. In how many ways can 6 examination papers be arranged so that the best and worst papers are always separated? [**Ans.** 480]

    [**Hints:** If there is no restriction, the examination papers can be arranged in $\underline{|6}$ ways, taking the worst and best paper together, total no. of papers will be 5, which can be arranged in $\underline{|5}$ ways. But the two papers best and worst can be arranged among themselves in $\underline{|2}$ ways.

    Therefore, the no. of arrangements when these two papers are together

    $= \underline{|2} \times \underline{|5}$.

    Therefore, no. of arrangements in which the best and worst papers are always separated $= \underline{|6} - \underline{|2} \times \underline{|5} = 720 - 240 = 480$]

17. 6 papers are set in an examination, of which 2 are mathematical. In how many different orders can the papers be arranged so that (*i*) the two mathematical papers are together (*ii*) the two mathematical papers are not consecutive?
[**Ans.** 240; 480]

18. In how many ways can 8 oranges of different sizes be distributed among 8 boys of different ages so that the largest orange is always given to the youngest boy?
[**Ans.** 5040]

19. In how many of the permutations of 10 different things taken 4 at a time will one particular thing (*i*) always occur, (*ii*) never occur? [**Ans.** 2400, 720]

20. In how many ways can the words of the colours of the rainbow be arranged so that red and blue colours are always separated? [**Ans.** 3600]

21. Find the number of permutations of 8 things taken 3 at a time in which (*i*) a particular thing will never occur, (*ii*) a particular thing will always occur.
[**Ans.** 210, 42]

22. In how many ways can six men stand in a row, if two of them, A and B, (*i*) always stand together? (*ii*) never stand together? [**Ans.** 240; 480]

23. In how many ways can 15 Higher Secondary and 12 B.Sc. examination candidates be arranged in a line so that no two B.Sc. candidates may occupy consecutive positions? [**Ans.** $\underline{|15} \times {}^{6}P_{12}$]

24. There are 6 books in Economics, 3 in Mathematics and 2 in Accountancy. In how many ways can the books be arranged in a book-shelf so that books of the same subject will always be together? [**Ans.** 51,840]

25. In how many ways can 5 Commerce and 4 Science students be arranged in a row so that the Commerce and the Science students are placed alternately?
[***C.U. B.Com (Hons) '82***] [**Ans.** 2880]

26. Find the number of arrangements that can be made out of the letters of the following words.
    (*i*) ACCOUNTANT
    (*ii*) ASSASSINATION
    (*iii*) ENGINEERING
    (*iv*) MISSISSIPPI
    (*v*) MATHEMATICS.

[**Ans.** (*i*) 2,26,800 (*ii*) $\frac{\lfloor 13}{\lfloor 3 \lfloor 4 \lfloor 2 \lfloor 2}$ (*iii*) 2,77,200 (*iv*) 94,650 (*v*) 49,89,600]

27. How many numbers of six digits can be formed from the digits 1, 2, 3, 4, 5, 6; no digit being repeated? How many of them are not divisible by 5? [**Ans.** 600]

28. How many 4-digit numbers can be formed from the five digits 1, 2, 3, 4, 5 if no digit is repeated? [**Ans.** 60; 120]

29. How many odd numbers can be formed using each of the digits once: 1, 2, 3, 4, 5, 6? [***C.U. B.Com '93***] [**Ans.** 360]

30. How many different odd numbers of 5 digits can be formed with the digits 1, 2, 3, 4, 5, 6; the digits in any number being all different? [***C.U. B.Com. 2017(G)***]
[**Ans.** 360]

31. How many different even numbers can be formed with the digits 1, 2, 3, 4, 5; the digits in any number being all different? [**Ans.** 48]

32. How many even numbers greater than 300 can be formed with the digits 1, 2, 3, 4, 5; no repetitions being allowed? [**Ans.** 111]

33. How many even numbers of six digits can be formed of the digits 0, 1, 2, 3, 4, 5?
[**Ans.** 312]

34. How many odd numbers of six digits can be formed with the digits 0, 1, 2, 3, 4 and 5, each digit occurring only once? [**Ans.** 288]

35. How many (*i*) odd numbers, (*ii*) even numbers of five digits can be formed of the digits 0, 1, 2, 3, 4; each being used only once? [***C.U. B.Com. '92***]
[**Ans.** (*i*) 36; (*ii*) 60]

36. How many numbers of different digits each lying between 100 and 1000 can be formed with digits 2, 3, 4, 0, 8, 9 each being used only once?
[***C.U. B.Com. '90, 2017(H)***]
[**Ans.** 100]

[**Hints:** ${}^6P_3 - {}^5P_2$]

37. How many numbers greater than 4000 can be formed with the digits 2, 3, 4, 7, 8; no digit being repeated? [**Ans.** 192]

38. Find the numbers less than 1000 and divisible by 5 which can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; each digit occurring not more than once in each number. [**Ans.** 154]

39. Find the no. of numbers each less than 999 and divisible by 2 which can be formed with the digits 2, 3, 4, 5, 6, 7, no digit occurring more than once in any number. [**Ans.** 78]

40. How many numbers greater than 5000 can be formed with four of the digits 3, 4, 5, 6, 7 (no digit is to be repeated)? [**Ans.** 72]

41. A man has to post 5 letters and there are 4 letter boxes in the locality; in how many ways can he post the letters? [**Ans.** 1024]

42. In how many ways can six different coloured stones be stringed together on a necklace so that red and pink stones will not remain side by side? [**Ans.** 36]

43. In how many ways can 5 men and 2 ladies be arranged at a round table if the two ladies (*i*) sit together; (*ii*) are separated? [**Ans.** (*i*) 240 (*ii*) 480]

44. In how many ways can four prizes be awarded to 4 students when (*i*) each student is eligible for any number of prizes? (*ii*) each student is eligible for only one prize? [**Ans.** (*i*) 256; (*ii*) 24]

45. In how many ways can 5 prizes be given away to 4 boys when each boy is eligible for all the prizes? [**Ans.** 45]

46. A gentlemen has 6 friends to invite. In how many ways can he sent invitation cards to them if he has 3 servants to carry the cards? [**Ans.** $3^6 = 729$]

47. In how many cases can 5 letters be kept in 5 addressed envelopes by an illiterate so that no letter is placed in a proper envelope? [**Ans.** 44]

48. On behalf of a Company for its "great goals contest" of 1990 World Cup Soccer, 6 special goals were shown on the television. Lists of the goals placed in order of their 'greatness' were invited and five great prizes were declared for those lists tallying with the lists scheduled by the special judges to be the first, second, etc. How many minimum number of different lists will a competitor send so that no prizes may escape from him. [***C.U. B.Com. '91***] [**Ans.** 1236]

    [**Hints:** ${}^6P_1 + {}^6P_2 + {}^6P_3 + {}^6P_4 + {}^6P_5$]

49. Show that the letters of the word 'CALCUTTA' can be arranged in twice as many ways as the letters of the word 'AMERICA'.

50. There are 2 works each of 4 volumes and 3 works each of 2 volumes. In how many ways can all the books be arranged on a shelf so that volumes of the same work are not separated? [**Ans.** 552960]

    [**Hints:** $\lfloor 5 \times \lfloor 4 \times \lfloor 4 \times \lfloor 2 \times \lfloor 2 \times \lfloor 2$]

Combinations

1. A committee consists of 10 gentlemen and 8 ladies. How many different sub-committees can be formed consisting of 5 gentlemen and 3 ladies?
[***K.U. B.Com. '99***] [**Ans.** 14112]
2. Out of 9 Swaragists and 5 Ministerialists, how many different committees can be formed, each consisting of 6 Swaragists and 2 Ministerialists? [**Ans.** 840]
3. A Trade-Union Committee consists of 9 persons and has to elect a chairman, a secretary and a treasurer. How many various combinations can be there?
[**Ans.** 504]
4. At a meeting consisting of 10 government representatives and 15 representatives of private establishments it was decided to elect a sub-committee consisting of 3 government representatives and 5 private representatives. In how many ways can the election be made? [**Ans.** 3,60,360]
5. At an election, 8 candidates and 3 members are to be elected; a voter is entitled to vote for any number of candidates not greater than the number to be elected. How many ways may a voter cast his vote? [**Ans.** 92]
6. There are 7 gentlemen and 3 ladies contesting for 2 vacancies; an elector can vote for any number of candidates not exceeding the number of vacancies. In how many ways is it possible to vote? [**Ans.** 55]
7. A committee of 6 is to be formed from 7 Indians and 4 Pakistanis. In how many ways can this be done when the committee must contain at least 2 Pakistanis?
[**Ans.** 371]

   [**Hints:** (*i*) 2 Pakistanis out of 4 and 4 Indians out of 7; (*ii*) 3 Pakistanis out of 4 and 3 Indians out of 7; (*iii*) 4 Pakistanis out of 4 and 2 Indians out of 7]
8. In how many ways can a committee of 3 ladies and 4 gentlemen be appointed from a meeting consisting of 8 ladies and 7 gentlemen? What will be the number of ways if Mrs. X refuses to serve in a committee of Mr. Y is a member?
[***N.B.U. B.Com. '90***] [**Ans.** 1960; 1540]
9. A certain council consists of a chairman, 2 vice-chairman, and 12 other members. How many different committees consisting of 6 members can be formed, including always the chairman and only one vice-chairman?
[**Ans.** 990]
10. In how many ways can a committee of 5 be formed from 4 professors and 6 students so as to include at least 2 professors? [**Ans.** 186]
11. A committee of 7 is to be chosen from 13 students of whom 6 are science students and 7 are commerce students. In how many ways can the selection be made so as to retain a majority on the committee for commerce students? [**Ans.** 1057]
12. A committee of 3 experts is to be selected out of a panel of 7 persons, 3 of them are lawyers, 3 of them are chartered accountants (CA) and one is both a CA and

a lawyer. In how many ways can the committee be selected if it must have at least a CA and a lawyer? [**Ans.** 33]

13. In how many ways can a committee of 5 persons be selected out of 10 persons so that (*i*) youngest and eldest persons are always included (*ii*) if eldest person is included then youngest persons will be excluded? [**Ans.** 56; 196]

14. An executive committee of 6 is to be formed from 4 ladies and 7 gentlemen. In how many ways can this be formed when the committee contains (*a*) only 2 lady members (*b*) at least 2 lady members? [**Ans.** 210; 371]

15. There are 8 persons for a post. In how many ways can a selection of 4 be made out of them so that (*i*) 2 persons whose qualifications are below are excluded, (*ii*) 2 persons whose qualifications are above are included? [**Ans.** 15; 15]

16. Out of 16 men, in how many ways can a group of 7 men be selected so that, (*i*) particular 4 men will not come, (*ii*) particular 4 men will always come? [**Ans.** 792; 320]

17. Out of 7 commerce students and 3 science students, how many committees of 5 students can be formed so as to include at least one science students? [***C.U. B.Com. '93***] [**Ans.** 231]

18. A cricket team of 11 players is to be selected from two groups consisting of 6 and 8 players respectively. In how many ways can the selection be made on the supposition that the group of 6 shall contribute no fewer than 4 players? [**Ans.** 344]

19. Keeping a majority of batsmen in the side, in how many ways can a cricket team of 11 players be selected from 9 batsmen and 6 bowlers, so that there are at least 3 bowlers? [**Ans.** 1224]

20. 16 players are selected in preliminary in Indian side, to play first cricket test match against M.C.C. There are 2 wicket-keepers, 6 batsmen, 5 bowlers and 3 all rounders. In how many ways can a team of 11 be selected so that there is 1 wicket keeper, 5 batsmen, 3 bowlers and 2 all-rounders? [**Ans.** 360]

21. Among 20 members of a cricket club, there are 2 wicket keepers and 5 bowlers. In how many ways can an eleven be chosen so as to include only 1 of the wicket keepers and at least 3 bowlers? [**Ans.** 54, 054]

22. A candidate is required to answer 6 out of 12 questions which are divided into two groups each containing 6 questions and he is permitted to attempt not more than 4 from any group. In how many different ways can he make up his choice? (Necessary arguments are to be given). [***C.U. B.Com. (Hon's) '83***] [**Ans.** 850]

23. In an examination paper 10 questions are set. In how many different ways can you choose 6 questions to answer. If, however, no. 11 is made compulsory, in how many ways can you select to answer 6 questions in all? [**Ans.** 210; 126]

24. In an examination paper there are two groups each containing 4 questions. A candidate is required to attempt 5 questions, but not more than 3 questions from any group. In how many ways can 5 questions be selected? [**Ans.** 48]

25. A questions paper contains 12 questions, 7 in group A and 5 in group B and questions are arranged numbering from 1 to 12 in order. It is advised to answer question no. 4 and any 3 from group A; and question no. 8 and any 2 from group B. In how many ways can a candidate select 7 questions to answer? [**Ans.** 120]

26. A questions paper is divided into three groups, A, B, C which contains 4, 4 and 3 questions respectively. An examinee is required to answer 6 questions, 2 from A, 2 from B, 1 from group C and 1 from any group he likes. In how many ways can he afford to choose questions? [**Ans.** 252]

27. In an examination paper on Business Mathematics 11 questions are set. In how many different ways can you choose 6 questions to answer? If, however, question no. 1 is made compulsory, in how many ways can you select to answer 6 questions in all? [**Ans.** 462; 252]

28. The question paper of Cost Accounting and Income Tax contains 10 questions each. In how many ways can an examinee answer 6 questions taking at least 2 questions from each group? [**Ans.** 200]

29. Out of 17 consonants and 5 vowels, how many different words can be formed; each consisting of 3 consonants and 2 vowels? [**Ans.** 8,16,000]

30. Out of 10 different consonants and 5 different vowels, how many different words can be formed each consisting of 3 consonants and 2 vowels? [**Ans.** 14,400]

31. How many words of 3 consonants and 2 vowels can be formed, when there are 7 consonants and 4 vowels? [***C.U. B.Com. 2014(G)***] [**Ans.** 25,200]

32. Find the number of (*a*) combination and (*b*) permutation of the letters of the words (*i*) EXAMINATION, (*ii*) MATHEMATICS taken 4 at a time.
[**Ans.** (*i*) 136; 2454; (*ii*) 136; 2454]

33. Find the number of combinations and permutations of the letters of the word "EXPRESSION", taken 4 at a time. [**Ans.** 113; 2190]

34. How many words (each containing 3 consonants and 2 vowels) can be formed with the letters of the word "METHODICAL"? [**Ans.** 14,400]

35. Find the number of combination of the letters of the word "ALLITERATION" taken 4 at a time. Find also the corresponding number of permutations.
[**Ans.** 160; 1008]

36. Find the number of combinations that can be made by taking 4 letters of the word "COMBINATION". [**Ans.** 136]

37. There are 12 points in a plane, no three of which are in the same straight line with the exception of 5, which are all in same straight line, find the number (*i*) of straight lines (*ii*) of triangles which results from joining them. [**Ans.** 57; 210]

38. n points are in a space, no 3 of which are collinear. If the number of straight lines and triangles, with the given points only as vertices, obtained by joining them are equal find the value of n. [***C.U. B.Com. (Hon's) '85***] [**Ans.** 5]
39. How many diagonals can be drawn to a figure of 15 sides? [**Ans.** 90]

C. ADVANCED PROBLEMS

1. How many four digit numbers can be formed with the digits 1, 2, 5, 6, 7 no digit being repeated? How many of them are divisible by 5? [***C.U. B.Com. 2000***]
[**Ans.** 120; 24]
2. In a question paper there are six questions in group A and 4 questions in group B. In how many ways a candidate can select 6 questions taking at least 2 questions from each group with 1 question of group A being compulsory?
[***K.U. B.Com. '96***]
3. How many numbers each lying between 150 and 5000 can be formed with the digits 2, 4, 6, 7 and 8, if no digit occurs more than once in any number?
[***K.U. B.Com. '95***] [**Ans.** 180]
4. In how many ways can the letters of the word 'FRACTION' be arranged? How many of these begin with 'F'? How many of these do not begin with 'F' but end with 'N'? [***N.B.U. B.Com. '99***] [**Ans.** 40320, 5040, 4320]
5. How many three digits odd numbers can be formed by using the digits 1, 2, 3, 4, 5, 6 when (*i*) the repetition of digits is not allowed? (*ii*) the repetition of digits is allowed? [**Ans.** (*i*) 60, (*ii*) 108]
6. In Kolkata, a telephone number consists of 7 digits and none of these begins with 0. How many such telephone numbers are possible? [**Ans.** 9000000]
7. How many 4-digit numbers can be formed with digits 1, 2, 3, 4, 5, 6, 7 when the repetition of the digits is allowed? [**Ans.** 1296]
8. For the post of 5 clerks, there are 25 applicants, 2 posts are reserved for SC candidates and remaining for other, there are 7 SC candidates among the applicants. In how many ways can the selection be made? [**Hints:** ${}^7C_2 \times {}^{18}C_3$]
[**Ans.** 17136]
9. A man has 2 French, 3 German and 4 Spanish friends. (*i*) In how many ways can he invite one or more of them in his birthday party? What will be the number of ways if he has to invite (*ii*) at least one German friend and (*iii*) at least one friend of each country? [**Ans.** (*i*) 511, (*ii*) 448 (*iii*) 315]
10. Find the number of combinations and permutations of the letters of the word EXPRESSION, taken 4 at a time. [**Ans.** 113, 2190]

    [**Hints:** Three alternatives are possible (*i*) Two alike, two others alike. (*ii*) Two alike, two others different (*iii*) All four different.

Total no. of combination is $1 + {}^2C_1 \times {}^7C_2 \times {}^8C_4 = 113$.

Total no. of permutations is $1 \times \dfrac{\lfloor 4}{\lfloor 2 \lfloor 2} + 42 \times \dfrac{\lfloor 4}{\lfloor 2} + 70 \times \lfloor 4 = 2190$]

D. MULTIPLE CHOICE QUESTIONS (MCQs)

(i) Short Type

Permutations

Mark the correct alternative in each of the following:

1. If ${}^nP_2 = 20$, then find n. [***C.U. B.Com. 2017(H)***]
   (*a*) 6 (*c*) 4
   (*b*) 3 (*d*) 5 [**Ans.** (*d*)]
2. If ${}^{n-1}P_5 : {}^{n+1}P_5 = 11 : 24$, find n. [***C.U. B.Com. 2015(H)***]
   (*a*) 15 (*c*) 18
   (*b*) 12 (*d*) 20 [**Ans.** (*a*)]
3. How many permutations can be done from the word DELHI taken together? [***C.U. B.Com. 2013(G)***]
   (*a*) 130 (*c*) 120
   (*b*) 80 (*d*) 100 [**Ans.** (*c*)]
4. In how many ways can 5 workers be appointed to different posts? [***C.U. B.Com. 1984***]
   (*a*) 100 (*c*) 140
   (*b*) 120 (*d*) 110 [**Ans.** (*b*)]
5. There are 5 buses running between two towns Kolkata and Kalyani. In how many ways can a man go from Kolkata to Kalyani and return by a different bus? [***C.U. B.Com. 1990***]
   (*a*) 30 (*c*) 15
   (*b*) 35 (*d*) 20 [**Ans.** (*d*)]
6. In how many different ways can 4 prizes be distributed among 10 students so that none of the students get more than one prize? [***C.U. B.Com. 2014(H)***]
   (*a*) 5040 (*c*) 4280
   (*b*) 6080 (*d*) 5720 [**Ans.** (*a*)]
7. In how many ways can 11 persons be arranged in a row such that 3 particular persons should always be together?
   (*a*) 9!8! (*c*) 10!10!
   (*b*) 9!3! (*d*) 10!3! [**Ans.** (*b*)]
8. In how many ways can 10 software engineers and 10 civil engineers be seated around a round table so that they are positioned alternately?
   (*a*) 9!10! (*c*) 10!10!
   (*b*) 2×10! (*d*) 10!8! [**Ans.** (*a*)]

9. How many numbers between 100 and 1000 can be formed with the digits 2, 3, 4, 0, 8, 9 each digit occurring only once? [***C.U. B.Com. 1990, 2012, 2016(G)***]
   (*a*) 120 (*c*) 110
   (*b*) 90 (*d*) 100 [**Ans.** (*d*)]

10. In how many ways can 6 examination papers be arranged so that the best and worst papers may never come together?
   (*a*) 480 (*c*) 720
   (*b*) 520 (*d*) 460 [**Ans.** (*a*)]

11. In how many ways can 8 oranges of different sizes be distributed among 8 boys of different ages so that the largest orange is always given to the youngest boy? [***C.U. B.Com. 2014(H)***]
   (*a*) 4920 (*c*) 6020
   (*b*) 5040 (*d*) 4080 [**Ans.** (*b*)]

12. How many odd numbers of five digits can be formed of the digits 0, 1, 2, 3, 4? [***C.U. B.Com. 2014(G)***]
   (*a*) 42 (*c*) 40
   (*b*) 35 (*d*) 36 [**Ans.** (*d*)]

13. In how many different ways can the letters of the word 'SALOON' be arranged so that two O's do not come together? [***C.U. B.Com. 1982***]
   (*a*) 300 (*c*) 240
   (*b*) 280 (*d*) 250 [**Ans.** (*c*)]

14. In how many ways can the letters of the word 'ASSISTANT' taken all together be arranged? [***C.U. B.Com. 1984***]
   (*a*) 12600 (*c*) 20140
   (*b*) 15,120 (*d*) 16780 [**Ans.** (*b*)]

15. In how many ways can six men stand in a row, if two of them A and B always stand together? [***C.U. B.Com. 2013(H)***]
   (*a*) 240 (*c*) 220
   (*b*) 300 (*d*) 260 [**Ans.** (*a*)]

16. In how many ways can the letters of the word DAUGHTER be arranged so that the vowels may never be separated? [***C.U. B.Com. 2016(H)***]
   (*a*) 4760 (*c*) 4500
   (*b*) 3870 (*d*) 4320 [**Ans.** (*d*)]

17. How many 6 digit telephone numbers can be formed if each number starts with 35 and no digit appears more than once?
   (*a*) 1660 (*c*) 1680
   (*b*) 1440 (*d*) 1720 [**Ans.** (*c*)]

18. An event manager has 10 patterns of chairs and 8 patterns of tables. In how many ways can he make a pair of table and chair?
   (*a*) 80 (*c*) 78
   (*b*) 88 (*d*) 92 [**Ans.** (*a*)]

19. In how many different ways can 5 girls and 5 boys from a circle such that the boys and the girls alternate?
    (*a*) 3860 (*c*) 3020
    (*b*) 2880 (*d*) 2680 **[Ans.** (*b*)]

20. A child has 3 pockets and 4 coins. In how many ways can he put the coins in his pocket?
    (*a*) 72 (*c*) 76
    (*b*) 81 (*d*) 86 **[Ans.** (*b*)]

Combinations

Mark the correct alternative in each of the following:

1. If $^nC_2 = 28$, find n. [***C.U. B.Com. 2014(H)***]
   (*a*) 9 (*c*) 12
   (*b*) 8 (*d*) 10 **[Ans.** (*b*)]

2. If $^{18}C_r = {}^{18}C_{r+2}$, find $^rC_5$. [***C.U. B.Com. 2000, 2012, 2016(G)***]
   (*a*) 56 (*c*) 54
   (*b*) 60 (*d*) 58 **[Ans.** (*a*)]

3. Find the value of r, when $^{16}C_{r-1} = {}^{16}C_{r+1}$. [***C.U. B.Com. 2013(H)***]
   (*a*) 7 (*c*) 10
   (*b*) 9 (*d*) 8 **[Ans.** (*d*)]

4. How many different committees consisting of 4 members can be formed out of 8 members? [***C.U. B.Com. 2013(G), 2014(G)***]
   (*a*) 66 (*c*) 70
   (*b*) 68 (*d*) 74 **[Ans.** (*c*)]

5. In how many ways can a team of 11 cricketers be chosen from 9 batsmen and 6 bowlers to give a majority batsman if at least 4 bowlers are to be included?
   [***C.U. B.Com. 2014(H)***]
   (*a*) 1044 (*c*) 1140
   (*b*) 1024 (*d*) 1320 **[Ans.** (*a*)]

6. If $^nC_r = x.\,{}^nP_r$, what is the value of x ? [***C.U. B.Com. 2016(H)***]
   (*a*) $\frac{1}{r!}$ (*c*) $2r!$
   (*b*) $r!$ (*d*) $\frac{1}{2r}$ **[Ans.** (*a*)]

7. Six persons meet and shake hands. How many handshakes are there in all?
   [***C.U. B.Com. 1995***]
   (*a*) 12 (*c*) 18
   (*b*) 14 (*d*) 15 **[Ans.** (*d*)]

8. A man has 6 friends. In how many ways can he invite one or more of them in a dinner?
(*a*) 65 (*c*) 60
(*b*) 63 (*d*) 58 [**Ans.** (*b*)]

9. In how many ways can 3 mangoes be selected from 10 mangoes so as always to include a particular mango?
(*a*) 35 (*c*) 40
(*b*) 38 (*d*) 36 [**Ans.** (*d*)]

10. In how many ways can a committee of 5 be formed from 4 professors and 6 students so as to include at least 2 professors? [***C.U. B.Com. 1986, 2012***]
(*a*) 180 (*c*) 188
(*b*) 186 (*d*) 190 [**Ans.** (*b*)]

11. A question paper has two parts P and Q, each containing 10 questions. If a student needs to choose 8 from part P and 4 from part Q, in how many ways can he do that?
(*a*) 9610 (*c*) 9860
(*b*) 9450 (*d*) 8730 [**Ans.** (*b*)]

12. There are 10 points in a plane and 4 of them are collinear. Find the number of straight lines joining any two of them.
(*a*) 40 (*c*) 42
(*b*) 38 (*d*) 45 [**Ans.** (*a*)]

13. How many committees of five persons with a chairperson can be selected from 12 persons?
(*a*) 3780 (*c*) 3960
(*b*) 3860 (*d*) 3880 [**Ans.** (*c*)]

14. Determine the number of ways of choosing 5 cards out of a deck of 52 cards which include exactly one ace.
(*a*) 778320 (*c*) 753180
(*b*) 765270 (*d*) 689240 [**Ans.** (*a*)]

15. How many chords can be drawn through 21 points on a circle?
(*a*) 215 (*c*) 200
(*b*) 220 (*d*) 210 [**Ans.** (*d*)]

16. If 5 parallel straight lines are intersected by 4 parallel straight lines, then find the numbers of parallelograms thus formed.
(*a*) 60 (*c*) 64
(*b*) 56 (*d*) 62 [**Ans.** (*a*)]

17. How many five-letter words containing 3 vowels and 2 consonants can be formed using the letters of the word EQUATION so that the two consonants occur together?
(*a*) 1440 (*c*) 1690
(*b*) 1580 (*d*) 1450 [**Ans.** (*a*)]

18. There are 6 questions in a question paper. In how many ways can a student solve one or more questions?
    (*a*) 68 (*c*) 63
    (*b*) 65 (*d*) 60 [**Ans.** (*c*)]
19. In how many ways can 15 different things be divided into three groups containing 7, 5 and 3 things?
    (*a*) 240240 (*c*) 35070
    (*b*) 360360 (*d*) 372180 [**Ans.** (*b*)]
20. 3 gentlemen and 3 ladies are candidates for 2 vacancies. A voter has to vote for 2 candidates. In how many ways can one cast his vote?
    (*a*) 18 (*c*) 15
    (*b*) 20 (*d*) 25 [**Ans.** (*c*)]

(ii) Short Essay Type

Permutations

1. How many different words can be formed using all the letters of the word ALLAHABAD?
   A When vowels occupy the even positions
   B Both L do not occur together
   (*a*) 7650,200,4444 (*c*) 7890,120,650
   (*b*) 7560,60,1680 (*d*) None of these [**Ans.** (*d*)]
2. Without repetition, using digits 2, 3, 4, 5, 6, 8 and 0, how many numbers can be made which lie between 500 and 1000?
   (*a*) 147 (*c*) 70
   (*b*) 90 (*d*) 60 [**Ans.** (*b*)]
3. A bank has 6 digit account number with no repetition of digits within an account number. The first and last digit of the account number is fixed to be 4 and 7. How many such account numbers are possible?
   (*a*) 1680 (*c*) 5040
   (*b*) 10080 (*d*) 890 [**Ans.** (*a*)]
4. On a shelf, 2 books of Geography, 2 books of Social Science and 5 of Economics are to be arranged in such a way that the books of any subject are to be together. Find in how many ways can this be done?
   (*a*) 900 (*c*) 90
   (*b*) 2880 (*d*) 3864 [**Ans.** (*b*)]
5. A locker in bank has 3 digit lock. Mrinmoy forgot his password and trying all possible combinations. He took 6 seconds for each try. The problem was that each digit can be from 0 to 9. How much time will be needed to by Mrinmoy to try all the combinations?
   (*a*) 60 minutes (*c*) 100 minutes
   (*b*) 90 minutes (*d*) 120 minutes [**Ans.** (*c*)]

6. In how many different ways can the letters of the word 'GEOMETRY' be arranged so that the vowels always come together?
(*a*) 4320 (*c*) 40320
(*b*) 720 (*d*) 2160 [**Ans.** (*a*)]

7. In how many ways can the letters of the word ENCYCLOPAEDIA be arranged such that vowels only occupy the even positions?
(*a*) 128000 (*c*) 453600
(*b*) 635630 (*d*) 470500 [**Ans.** (*c*)]

8. How many 3-digit numbers can be formed from the digits 2, 3, 5, 6, 7 and 9, which are divisible by 5 and none of the digits is repeated?
(*a*) 10 (*c*) 5
(*b*) 20 (*d*) 15 [**Ans.** (*b*)]

9. The number of words from the letters of the word 'BHARAT' in which B and H will never come together, is
(*a*) 120 (*c*) 360
(*b*) 240 (*d*) none of these [**Ans.** (*b*)]

10. Suppose a licence plate contains two letters followed by three digits with the first digit not zero. How many different licence plates can be printed?
(*a*) 608400 (*c*) 608200
(*b*) 608600 (*d*) 606300 [**Ans.** (*a*)]

11. How many 6 digit telephone numbers can be formed if each number starts with 35 and no digit appears more than once?
(*a*) 360 (*c*) 1420
(*b*) 720 (*d*) 1680 [**Ans.** (*d*)]

12. Eight students should be accommodated in two 3-bed and one 2-bed rooms. In how many ways can they be accommodated?
(*a*) 720 (*c*) 660
(*b*) 560 (*d*) 480 [**Ans.** (*b*)]

13. In how many ways can you rearrange the word 'JUMBLE' such that the rearranged word starts with a vowel?
(*a*) 60 (*c*) 240
(*b*) 120 (*d*) 360 [**Ans.** (*c*)]

14. There are four bus routes between A and B; and three bus routes between B and C. A man can travel round-trip in number of ways by bus from A to C via B. If he does not want to use a bus route more than once, in how many ways can he make round trip?
(*a*) 14 (*c*) 72
(*b*) 19 (*d*) 144 [**Ans.** (*c*)]

15. The number of signals that can be sent by 6 flags of different colours taking one or more at a time is
(*a*) 21 (*c*) 63
(*b*) 1956 (*d*) 720 [**Ans.** (*b*)]

16. In how many different ways can four books A, B, C and D be arranged one above another in a vertical order such that the books A and B are never in continuous position?
    (*a*) 18 (*c*) 9
    (*b*) 14 (*d*) 12 [**Ans.** (*d*)]

17. Three dice (each having six faces with each face having one number from 1 to 6) are rolled. What is the number of possible outcomes such that at least one dice shows the number 2?
    (*a*) 36 (*c*) 91
    (*b*) 81 (*d*) 116 [**Ans.** (*c*)]

18. In how many different ways can six players be arranged in a line such that two of them Amit and Sumit are never together?
    (*a*) 480 (*c*) 240
    (*b*) 360 (*d*) 120 [**Ans.** (*a*)]

19. How many 3-digit even numbers greater than 500 can be formed out of the digits 6, 0, 3, 5 and 4?
    (*a*) 9 (*c*) 15
    (*b*) 12 (*d*) 18 [**Ans.** (*c*)]

20. How many five letter words (may not be meaningful) out of the first eight letters of the English alphabet can be formed with the vowels always included?
    (*a*) 1600 (*c*) 900
    (*b*) 2400 (*d*) 4800 [**Ans.** (*b*)]

Combinations

1. A team of six is to be formed by taking two from each of three groups of boys numbering 5, 6 and 7. How many such teams are possible?
   (*a*) 3150 (*c*) 3350
   (*b*) 3250 (*d*) 4500 [**Ans.** (*a*)]

2. A trekking group is to be formed having 6 members. They are to be selected from 3 girls, 4 boys and 5 teachers. In how many ways can the group be formed so that there are 3 teachers and 3 boys or 2 girls and 4 teachers?
   (*a*) 55 (*c*) 144
   (*b*) 90 (*d*) 27 [**Ans.** (*a*)]

3. 4 members from a group out of total 8 members
   (*i*) In how many ways it is possible to make the group if two particular members must be included.
   (*ii*) In how many ways it is possible to make the group if two particular members must not be included?
   (*a*) 360 and 360 (*c*) 15 and 15
   (*b*) 30 and 360 (*d*) 15 and 360 [**Ans.** (*c*)]

4. A box contains 3 black, 5 red and 2 green balls. In how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw?
   (*a*) 85 (*c*) 36
   (*b*) 48 (*d*) 29 [**Ans.** (*a*)]
5. From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done?
   (*a*) 642 (*c*) 756
   (*b*) 735 (*d*) 546 [**Ans.** (*c*)]
6. Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?
   (*a*) 1060 (*c*) 21,200
   (*b*) 25,200 (*d*) 220 [**Ans.** (*b*)]
7. A box contains 2 white balls, 3 black balls and 4 red balls. In how many ways can 3 balls be drawn from the box, if at least one black ball is to be included in the draw?
   (*a*) 64 (*c*) 42
   (*b*) 36 (*d*) 96 [**Ans.** (*a*)]
8. A bag contains 3 white balls, 2 black balls and 4 red balls. Three balls are drawn at random. Find out the number of ways of selecting the balls of different colours?
   (*a*) 46 (*c*) 62
   (*b*) 15 (*d*) 24 [**Ans.** (*d*)]
9. A question paper has two parts A and B, each containing 10 questions. If a student needs to choose 8 from part A and 4 from part B, in how many ways can he do that?
   (*a*) 1600 (*c*) 9450
   (*b*) 1200 (*d*) 6050 [**Ans.** (*c*)]
10. If how many ways can a team of 5 persons be formed out of a total of 10 persons such that two particular persons should not be included in any team?
    (*a*) 112 (*c*) 128
    (*b*) 56 (*d*) 28 [**Ans.** (*b*)]
11. If there are 9 horizontal lines and 9 vertical lines in a chess board, how many rectangles can be formed in the chess board?
    (*a*) 1296 (*c*) 930
    (*b*) 1026 (*d*) 74 [**Ans.** (*a*)]
12. Find the number of triangles that can be formed using 14 points in a plane such that 4 points are collinear?
    (*a*) 240 (*c*) 480
    (*b*) 360 (*d*) 120 [**Ans.** (*b*)]

13. There are 8 points in a plane out of which 3 are collinear. How many straight lines can be formed by joining them?
    (*a*) 18 (*c*) 26
    (*b*) 22 (*d*) 16 **[Ans.** (*c*)]
14. A box contains 20 balls. In how many ways can 8 balls be selected if each ball can be repeated any number of times?
    (*a*) ${}^{20}C_3$ (*c*) ${}^{20}C_7$
    (*b*) ${}^{27}C_3$ (*d*) ${}^{27}C_7$ **[Ans.** (*b*)]
15. In how many ways can 7 different balls be distributed in 5 different boxes if box 3 and box 5 can contain only one and two number of balls respectively and rest of the boxes can contain any number of balls?
    (*a*) 8505 (*c*) 10100
    (*b*) 12600 (*d*) 6400 **[Ans.** (*a*)]
16. 5 balls needs to be placed in 3 boxes. Each box can hold all the five balls. In how many ways can the balls be placed in the boxes if no box can be empty, all balls are identical but all boxes are different?
    (*a*) 8 (*c*) 4
    (*b*) 6 (*d*) 2 **[Ans.** (*b*)]
17. In how many ways can 30 identical toys be divided among 10 boys if each boy must get at least one toy?
    (*a*) ${}^{30}C_{10}$ (*c*) ${}^{29}C_9$
    (*b*) ${}^{29}C_{10}$ (*d*) ${}^{30}C_{10}$ **[Ans.** (*c*)]
18. Amaresh has 10 friends and he wants to invite 6 of them to a party. How many times will 3 particular friends always attend the party?
    (*a*) 125 (*c*) 110
    (*b*) 720 (*d*) 35 **[Ans.** (*d*)]
19. In how many ways can 10 engineers and 4 doctors be seated at a round table without any restriction?
    (*a*) 14! (*c*) 13!
    (*b*) ${}^{14}C_{10}$ (*d*) ${}^{13}C_{10}$ **[Ans.** (*c*)]
20. The Indian cricket team consists of 16 players. It includes 2 wicket keepers and 5 bowlers. In how many ways can a cricket eleven be selected if we have to select 1 wicket keeper and at least 4 bowlers?
    (*a*) 1092 (*c*) 1124
    (*b*) 1056 (*d*) 840 **[Ans.** (*a*)]

CHAPTER

2

Set Theory

SYLLABUS

Definition of Set, Presentation of Sets, Different Types of Sets—Null Set, Finite and Infinite Sets, Universal Set, Subset, Power Set etc.; Set Operations, Law of Algebra of Sets.

THEMATIC FOCUS

2.1 INTRODUCTION

The concept of modern mathematics started with sets. Set theory is an important branch of mathematics. It is the study of mathematical logic and its applications. Since its development in the 1870s by George Cantor and Richard Dedekind, set theory has become a foundational system of mathematics, and, therefore, its concepts are commonly applied in the study of mathematics. In every domain of mathematics we have to deal with sets such as the set of positive integers, the set of complex numbers, the set of points on a circle, the set of continuous functions, the set of integrable functions and so on. The object of set theory is to investigate the properties of sets from the most general point of view; generality is an essential aspect of the theory of set. Set theory basically deals with set and set operations.

2.2 DEFINITION OF A SET

Set is a collection of well-defined distinct objects. The word well-defined refers to a specific property which makes it easy to identify whether the given object belongs to the set or not. The word 'distinct' means that the objects of a set must be all different.

ILLUSTRATION 1

(*i*) Vowels of the English alphabets
(*ii*) Bunch of keys
(*iii*) Bouquet of flowers

OBSERVATIONS

1. A collection 'good books' is not a set, because the word 'good' is a relative term. What may appear good to one person may not be so to the other person. So, the objects are not well defined.
2. A group of 'young boys' is not a set, because the range of ages of young boys is not given and so it cannot be decided that which boy is to be considered young, i.e. the objects are not well defined.

2.3 ELEMENTS OF A SET

The different objects by which a set is made of are called elements or members of the set.

In Illustration 1, vowels, keys and flowers are the elements.

2.4 NOTATION OF A SET

The following conventions are used with sets.

⇒ Capital letters are used to denote sets.

⇒ Lowercase letters are used to denote elements of sets.

⇒ Curly braces { } denote a list of elements in a set.

If x is an element of set A, then we say $x \in A$ [x belongs to A or x is an element of the set A].

If x is not an element of set A, then we say $x \notin A$ [x does not belong to A or x is not an element of the set A]

ILLUSTRATION 2

Let V be the set of all vowels in the English alphabet. Then the set may be denoted as

$V = \{a, e, i, o, u\}$

Here, $a \in V$, $e \in V$, $i \in V$, $o \in V$ and $u \in V$ but $b \notin V$, $c \notin V$, $d \notin V$, etc.

2.5 REPRESENTATION OF A SET

A set can be represented by various methods. Two common methods used for representing set are:

1. Roster or Tabular method
2. Rule or Set-builder method

2.5.1 Roster or Tabular Method

In this method of representation, elements of the set are listed between two curly brackets{ } separated by commas.

ILLUSTRATION 3

(*i*) Let P is the set of first four natural numbers
Then, P = {1, 2, 3, 4}

(*ii*) Let Q is the set of all odd numbers less than 10.
Then, Q = {1, 3, 5, 7, 9}

(*iii*) Let R is the set of first four months of the year
Then, R = {January, February, March, April}

2.5.2 Rule or Set-builder Method

In this method of representation, elements of the set are described by their characteristics or property by using a symbol '*x*' or any other variable followed by a colon. The symbol ':' or '|' is used to denote such that and then we write the characteristics or property possessed by the elements of the set and enclose the whole description in braces. In this, the colon stands for 'such that' and braces stand for 'set of all'.

ILLUSTRATION 4

(*i*) Let E is a set of natural numbers less than 10.
Then, in Rule method, set E is writtten as: E = $\{x : x$ is natural numbers less than $10\}$
or
E = $\{x \mid x$ is natural numbers less than $10\}$

(*ii*) Let F is a set of odd numbers between 5 and 15. It can be written in the Rule method as
F = $\{x \mid x$ is an odd numbers, $5 < x < 15\}$
or
F = $\{x : x \in N, 5 < x < 15$ and N is an odd number$\}$

ILLUSTRATION 5

Represent the following set by using two methods.
The set of integers lying between 5 and 10.

Solution: Let A be the required set
Roster method: A = {6, 7, 8, 9}
Rule method: A = $\{x : x \in N, 5 < x < 10$ and N is an integer$\}$

OBSERVATIONS

(*i*) The change in order of writing the elements does not make any changes in the set.

(*ii*) If one or many elements of a set are repeated, the set remains the same.

2.6 TYPES OF SETS

2.6.1 Finite Set

A set which contains a definite number of elements is called a finite set.

ILLUSTRATION 6

(*i*) $A = \{2, 4, 6, 8, 10\}$
(*ii*) $B = \{1, 3, 5, 7, ..., 99\}$
(*iii*) $C = \{x : x \in N, x < 5\}$

2.6.2 Infinite Set

A set containing infinite number of elements is called an infinite set.

ILLUSTRATION 7

(*i*) $P = \{1, 2, 3, ...\}$
(*ii*) $Q = \{x : x \in N, x > 0\}$
(*iii*) $R = \{x : x \text{ is a prime number}\}$

2.6.3 Singleton Set

A set containing only one element is called a singleton set.

ILLUSTRATION 8

(*i*) $D = \{5\}$
(*ii*) $E = \{\Phi\}$
(*iii*) $F = \{x : x \text{ is an integer neither positive nor negative}\}$
(*iv*) $G = \{x : x \text{ is an even prime number}\}$

2.6.4 Null Set or Empty Set

A set having no element in it is called null set or empty set. A null set is denoted by the Greek letter Φ (pronounced as phi) or $\{\}$.

ILLUSTRATION 9

(*i*) $\Phi = \{x \mid x \text{ is an integer and } 4 < x < 5\}$
(*ii*) $\Phi = \{x : x \text{ is a perfect square of an integer and } 4 < x < 7\}$
(*iii*) $\Phi = \{x : x \text{ is a positive integer and } x < 0\}$

2.6.5 Equal Sets

Two sets A and B are said to be equal if both have same elements. Every element of A is an element of B and every element of B is an element of A. The symbol to denote an equal set is = .

ILLUSTRATION 10

(*i*) If A = {2, 3, 4, 5} and B = {5, 4, 3, 2} then A = B
(*ii*) If P = {a, b, c} and Q = {a, a, b, b, b, c, c} then P = Q

OBSERVATION

The order of elements or repetition of elements does not change the nature of set.

2.6.6 Equivalent Sets

Two sets A and B are said to be equivalent if the total number of elements of set A is equal to the total number of elements of set B. The symbol to denote equivalent set is ≡.

ILLUSTRATION 11

If A = {1, 2, 3, 4} and B = {p, q, r, s}, then A ≡ B.

OBSERVATIONS

(*i*) Equal sets are always equivalent, but equivalent sets may not be equal.
 (*a*) Let M = {a, b, c, d} and N = {d, c, b, a}
  Here, M = N and also M ≡ N
 (*b*) Let E = {p, q, r, s} and F = {i, j, k, l}
  Here, E ≡ F but E ≠ F
(*ii*) It is not essential that the elements of the two sets should be same.

2.6.7 Subset

A set A is a subset of a set B if every element of A is also an element of B.

Symbolically we write A ⊆ B (read as 'A is a subset of B' or 'A is contained in B')

ILLUSTRATION 12

(*i*) Let A = {2, 3, 4} and B = {1, 2, 3, 4}
 Clearly, $x \in A \Rightarrow x \in B$. Hence, A ⊆ B
(*ii*) If K = {5, 6, 7}, then the subsets of K are {5}, {6}, {7}, {5, 6}, {6, 7}, {5, 7}, {5, 6, 7}, and Φ.

OBSERVATIONS

(*i*) Every set is a subset of itself, i.e. $B \subseteq B$
(*ii*) Null set is a subset of every set.
(*iii*) Number of subsets of a set containing n elements is 2^n.

(a) Proper subset

A set A is said to be a proper subset of B if every element of the set A is also an element of the set B and there is at least one element in B which is not in A. Symbolically we write $A \subset B$ (read as 'A is a proper subset of B').

ILLUSTRATION 13

(*i*) If $A = \{2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5\}$, then $A \subset B$
(*ii*) If $Q = \{1, 3, 5\}$, then the proper subsets of Q are $\{1\}$, $\{3\}$, $\{5\}$, $\{1, 3\}$, $\{3, 5\}$, $\{1, 5\}$, $\{1, 3, 5\}$

OBSERVATIONS

(*i*) If $A \subseteq B$ and $A \neq B$ then A is a proper subset of B.
(*ii*) Number of proper subsets of a set containing n elements is $2^n - 1$.
(*iii*) $B \supset A$ means B is a superset of A.
(*iv*) A set is not a proper subset of itself.

(b) Improper subset

A set A is said to be a improper subset of B if A is a subset of B and B is also a subset of B, i.e. $A \subseteq B$ and $B \subseteq A$ and hence $A = B$.

ILLUSTRATION 14

If $A = \{4, 5, 6\}$ and $B = \{6, 5, 4\}$, then $A \subseteq B$ and $B \subseteq A$ and hence $A = B$.

Therefore, it is clear that every set A is an improper subset of itself, i.e. A is an improper subset of A.

(c) Properties of subsets

(*i*) If $A \subseteq B$ and $B \subseteq A$, then $A = B$
(*ii*) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

2.6.8 Power Set

The power set of a set A is the set of all possible subsets of the set A and is denoted by P(A).

ILLUSTRATION 15

$$A = \{r, s, t\}$$

All possible subsets of the set A

$$\{r\}, \{s\}, \{t\}, \{r, s\}, \{s, t\}, \{r, t\}, \{r, s, t\}, \Phi$$

Therefore, the power set of A is

$$P(A) = \{\{r\}, \{s\}, \{t\}, \{r, s\}, \{s, t\}, \{r, t\}, \{r, s, t\}, \Phi\}$$

OBSERVATION

Null set or empty set and set itself is also element or member of this set of subsets.

2.6.9 Universal Set

A universal set is the collection of all objects in a particular context or theory. All other sets in that framework constitute subsets of the universal set, which is denoted by U or S.

ILLUSTRATION 16

$A = \{a, b, c\}$, $B = \{b, c, d, e\}$, $C = \{d, e, f, g, h\}$
$U = \{a, b, c, d, e, f, g, h, i, j\}$
Clearly, all the sets A, B, and C are subsets of the set U.
Therefore, U is the Universal set of the sets A, B and C.

2.7 VENN DIAGRAM

Venn diagram is a diagram representing mathematical or logical sets pictorially as circles or closed curves within an enclosing rectangle (the Universal set). It shows all possible relations between a finite collection of different sets. John Venn, an English logician, invented this diagram. According to his name the diagram is known as 'Venn Diagram'. Generally, in Venn diagrams the universal set 'U' is represented by prints within a rectangle and its subsets are represented by points in closed curves (usually circles) within the rectangle. [See Figure 2.1]

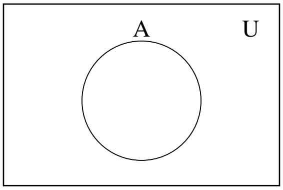


Figure 2.1

2.8 OVERLAPPING SETS

If the two sets A and B are not equal but they have some common elements, then they are called overlapping sets. [see Figure. 2.2]

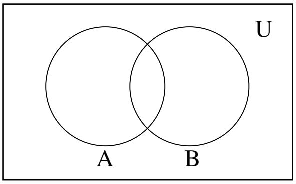


Figure 2.2

ILLUSTRATION 17

If A = {5, 7, 8, 9, 10}, B = {6, 8, 10} since both the sets have common elements 8 and 10 but they are not equal, so they are called overlapping sets.

2.9 DISJOINT SETS

Two sets A and B are said to be disjoint if they have no common elements. Two disjoint sets are represented by two non-intersecting circles. [see Figure. 2.3]

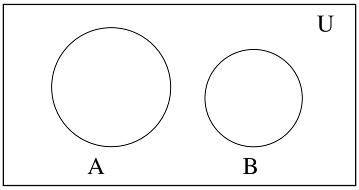


Figure 2.3

ILLUSTRATION 18

If A = {1, 2, 3} and B = {4, 5, 6}, then A and B are disjoint sets as they have no common elements.

2.10 CARDINAL NUMBER OF A SET

The number of distinct elements in a finite set A is called the cardinal number of A, and is denoted by $n(A)$.

ILLUSTRATION 19

(*i*) If A = $\{a, e, i, o, u\}$
Then, $n(A) = 5$

(*ii*) If B = {$x : x \in N, x < 7$}
Then, B = {1, 2, 3, 4, 5, 6}
Therefore, n(B) = 6

(*iii*) If C be the set of letters in the word ARITHMETIC,
i.e. C = {A, R, I, T, H, M, E, C}
Then, n(C) = 8

2.11 OPERATIONS ON SETS

When two or more sets combine together to form one set under the given conditions, then operations on sets are carried out.

2.11.1 Union of Sets

The union of two sets A and B is the set of all elements which belong either to A or to B, or to both A and B.

The union of A and B is denoted symbolically as A ∪ B (read as "A union B").

Thus, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

In Figure 2.4 the shaded region represents A ∪ B.

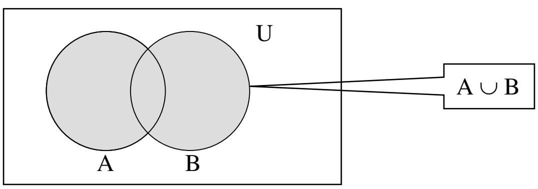


Figure 2.4

ILLUSTRATION 20

(*i*) If A = {2, 4, 6, 8} and B = {4, 6, 7, 9}
Then, A ∪ B = {2, 4, 6, 7, 8, 9}

(*ii*) If A = {a, b} B = {b, c, d} and C = {c, d, e, f}
Then, A ∪ B ∪ C = {a, b, c, d, e, f}

OBSERVATION

If A ⊂ B, then A ∪ B = B.
Also, A ∪ B = A, if B ⊂ A.

2.11.2 Intersection of Sets

The intersection of two sets A and B is the set of all elements that belong to both A and B.

The intersection of A and B is denoted symbolically as A ∩ B (read as "A intersection B").

Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

In Figure. 2.5 the shaded region represents A ∩ B

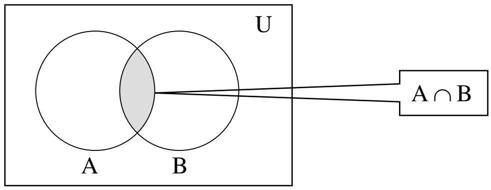


Figure 2.5

ILLUSTRATION 21

(*i*) If A = {5, 6, 7, 8} and B = {7, 8, 9, 10}
Then, A ∩ B = {7, 8}

(*ii*) If A = {*a, b, c*}, B = {*b, c, d*} and C = {*c, d, e*}
Then, A ∩ B ∩ C = {*c*}

OBSERVATION

If $A \subset B$ then $A \cap B = A$ and $A \cap B = B$, if $B \subset A$.

2.11.3 Difference of Sets

The difference of two sets A and B is the set of all elements which belong to A but do not belong to B.

The difference of A and B is denoted symbolically as A – B (read as "A difference B").

Thus, $A - B = \{x : x \in A \text{ and } x \notin B\}$

In Figure. 2.6 the shaded region represents A – B

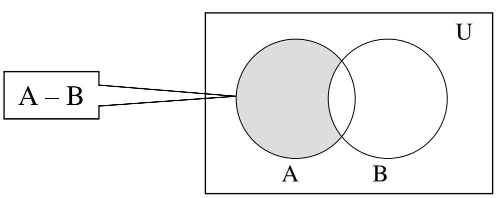


Figure 2.6

Similarly, the difference B – A is the set of all elements which belong to B but do not belong to A.

Thus, $B - A = \{x : x \in B \text{ and } x \notin A\}$

In Figure. 2.7 the shaded region represents B – A

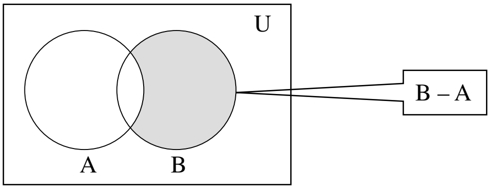


Figure 2.7

ILLUSTRATION 22

(*i*) If A = {*m*, *n*, *o*, *p*} and B = {*o*, *p*, *q*, *r*}
Then, A – B = {*m*, *n*} and B – A = {*q*, *r*}

(*ii*) If A = {1, 2, 3, 4, 5, 6, 7, 8} and B = {2, 4, 6, 8}
Then, A – B = {1, 3, 5, 7} and B – A = Φ

OBSERVATIONS

(*i*) A – B ≠ B – A

(*ii*) if B ⊂ A, then B – A = Φ

2.11.4 Symmetric Difference of Two Sets

The symmetric difference of two sets A and B is the set of all elements that belong to A or B but not both.

The symmetric difference of A and B is denoted symbolically as A Δ B (read as "A symmetric difference B").

Thus, A Δ B = (A – B) ∪ (B – A) = $\{x : x \in A$ or $x \in B, x \notin A \cap B\}$

In Figure. 2.8 the shaded region represents A Δ B.

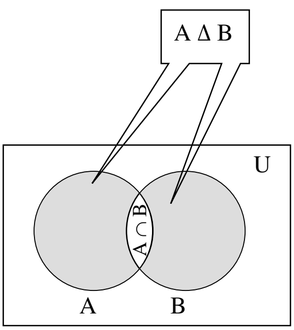


Figure 2.8

ILLUSTRATION 23

If A = {2, 4, 5, 6, 7, 9} and B = {2, 3, 4, 8, 9, 10}
Then, A – B = {5, 6, 7} and B – A = {3, 8, 10}
Therefore, A Δ B = {3, 5, 6, 7, 8, 10}

2.11.5 Complement of a Set

If A is a subset of the Universal set U, then complement of set A is the set of all elements which belong to U but do not belong to A.

The complement of set A is denoted symbolically as A′ or A^c or $\overline{A}$.

Thus, $A^c = \{x : x \in U , x \notin A\}$

In Figure. 2.9 the shaded region represents A^c.

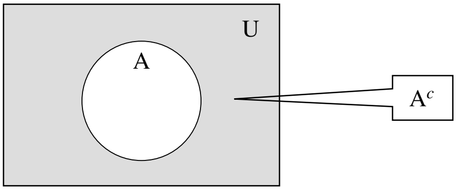


Figure 2.9

ILLUSTRATION 24

(*i*) If U = {1, 2, 3, 4, 5, 6, 7} and A = {2, 4, 6}, then A′ = {1, 3, 5, 7}

(*ii*) If U be the set of all the letters in the English alphabet and A be the set of all vowels in the English alphabet, then A^c is the set of all the consonants in the English alphabet.

2.12 LAWS OF ALGEBRA OF SETS

The algebra of sets defines the properties and laws of sets, the set-theoretic operations of union, intersection, and complementation and the relations of set equality and set inclusion. We shall state below some fundamental laws of algebra of sets. If A, B and C are any sets, then:

1. **Idempotent laws:**
   (*i*) $A \cup A = A$ (*ii*) $A \cap A = A$
2. **Identity laws:**
   (*i*) $A \cup U = U$ (*iii*) $A \cup \Phi = A$
   (*ii*) $A \cap U = A$ (*iv*) $A \cap \Phi = \Phi$
3. **Commutative laws:**
   (*i*) $A \cup B = B \cup A$ (*ii*) $A \cap B = B \cap A$
4. **Associative laws:**
   (*i*) $A \cup (B \cup C) = (A \cup B) \cup C$ (*ii*) $A \cap (B \cap C) = (A \cap B) \cap C$
5. **Distributive laws:**
   (*i*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (*ii*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6. Complement laws:

(*i*) $A \cup A' = U$ (*iii*) $(A')' = A$ (*v*) $\Phi' = U$

(*ii*) $A \cap A' = \Phi$ (*iv*) $U^c = \Phi$

[where A′ = complement of A]

7. De Morgan's laws:

(*i*) $(A \cup B)' = A' \cap B'$

(*ii*) $(A \cap B)' = A' \cup B'$

2.13 VERIFICATION OF ASSOCIATIVE, DISTRIBUTIVE AND DE MORGAN'S LAWS USING VENN DIAGRAM

Associative laws

By drawing Venn diagram verify that,

(*i*) $A \cup (B \cup C) = (A \cup B) \cup C$ (*ii*) $A \cap (B \cap C) = (A \cap B) \cap C$

Verification

Let A, B and C be three sets whose elements are represented by regions inside the circles. In each of the figures given below (See Figures 2.10 & 2.11), the shaded region represents the set given underneath the figure.

(*i*)

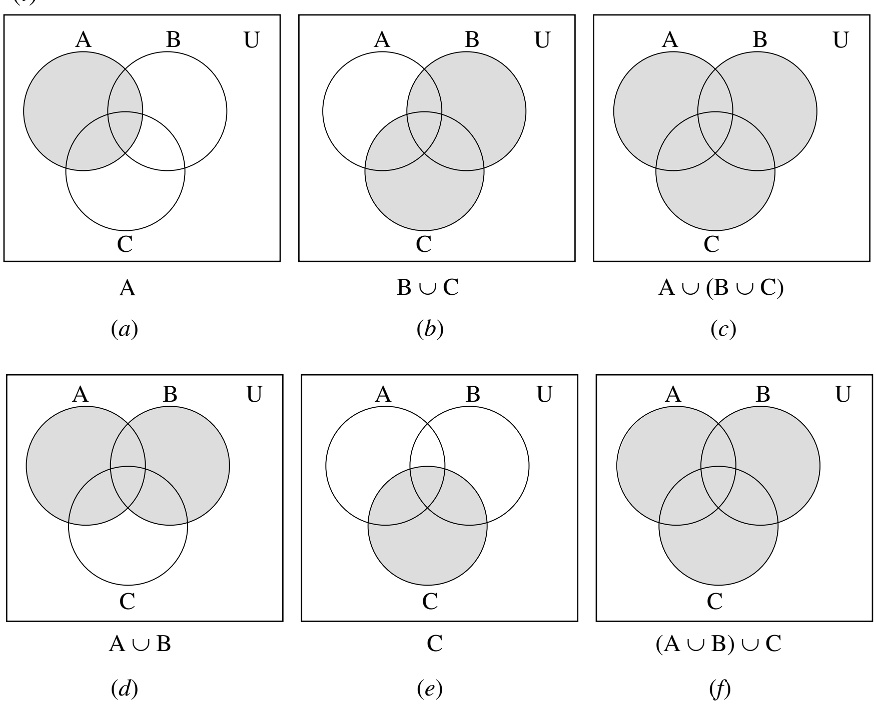


Figure 2.10

Shaded regions of diagrams (c) and (f) are the same.

Therefore, A ∪ (B ∪ C) = (A ∪ B) ∪ C

(*ii*)

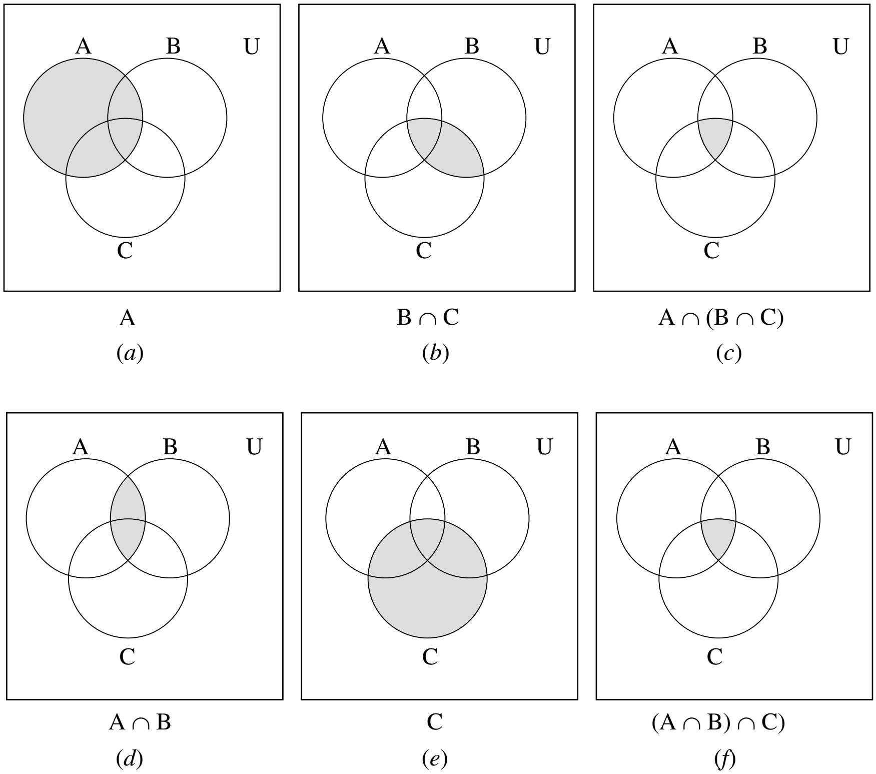


Figure 2.11

Shaded regions of the figures (*c*) and (*f*) are the same.

Therefore, A ∩ (B ∩ C) = (A ∩ B) ∩ C

Distributive laws

By drawing Venn diagram verify that,

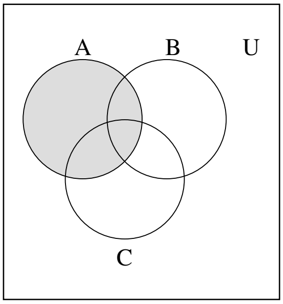
(*i*) A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

(*ii*) A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

Verification

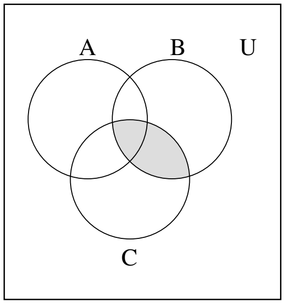
Let A, B and C be three sets whose elements are represented by regions inside the circles. In each of the figures given below (See Figures 2.12 & 2.13), the shaded region represents the set given underneath the figure.

(*i*)



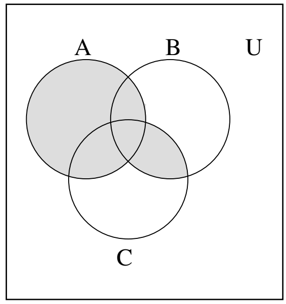
A

(*a*)



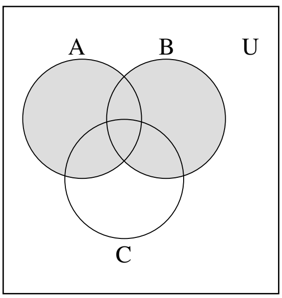
B ∩ C

(*b*)



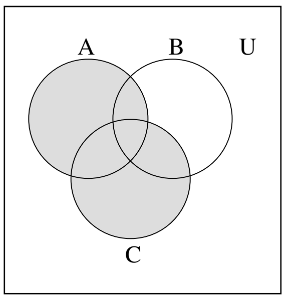
A ∪ (B ∩ C)

(*c*)



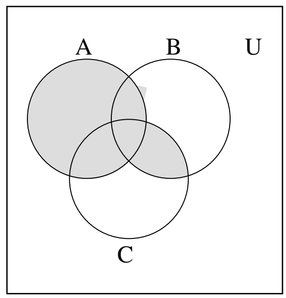
A ∪ B

(*d*)



A ∪ C

(*e*)



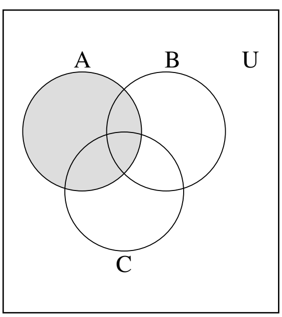
(A ∪ B) ∩ (A ∪ C)

(*f*)

Figure 2.12

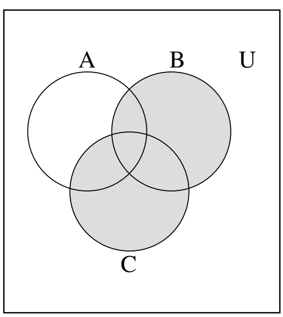
Shaded regions of the figures (c) and (f) are the same.
Therefore, A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

(*ii*)



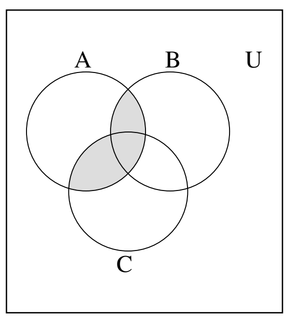
A

(*a*)



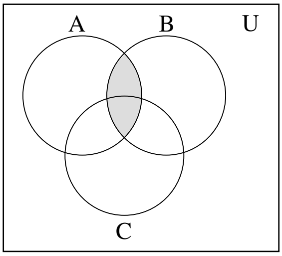
B ∪ C

(*b*)



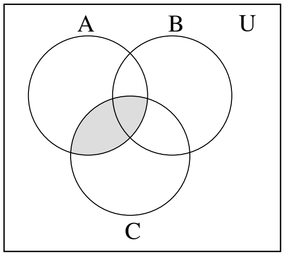
A ∩ (B ∪ C)

(*c*)



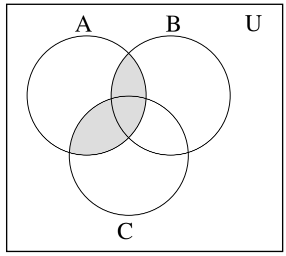
$A \cap B$

(*d*)



$A \cap C$

(*e*)



$(A \cap B) \cup (A \cap C)$

(*f*)

Figure 2.13

Shaded regions of diagrams (c) and (f) are the same.
Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan's laws

By drawing Venn diagram verify that,

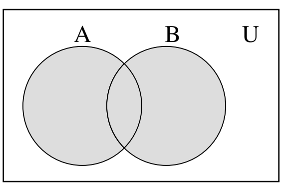
(*i*) $(A \cup B)' = A' \cap B'$

(*ii*) $(A \cap B)' = A' \cup B'$

Verification

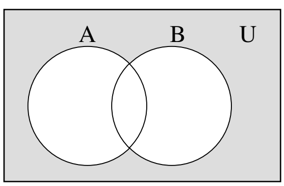
Let A and B be two sets whose elements are represented by regions inside the circles. In each of the figures given below (See Figures 2.14 & 2.15), the shaded region represents the set given underneath the figures.

(*i*)



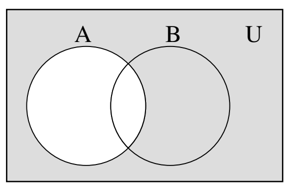
$A \cup B$

(*a*)



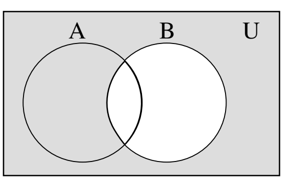
$(A \cup B)'$

(*b*)



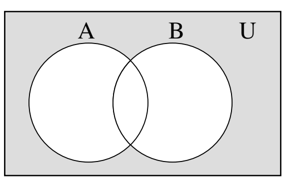
A'

(*c*)



B'

(*d*)



$A' \cap B'$

(*e*)

Figure 2.14

Shaded regions of the figures (b) and (e) are the same.
Therefore, $(A \cup B)' = A' \cap B'$

(*ii*)

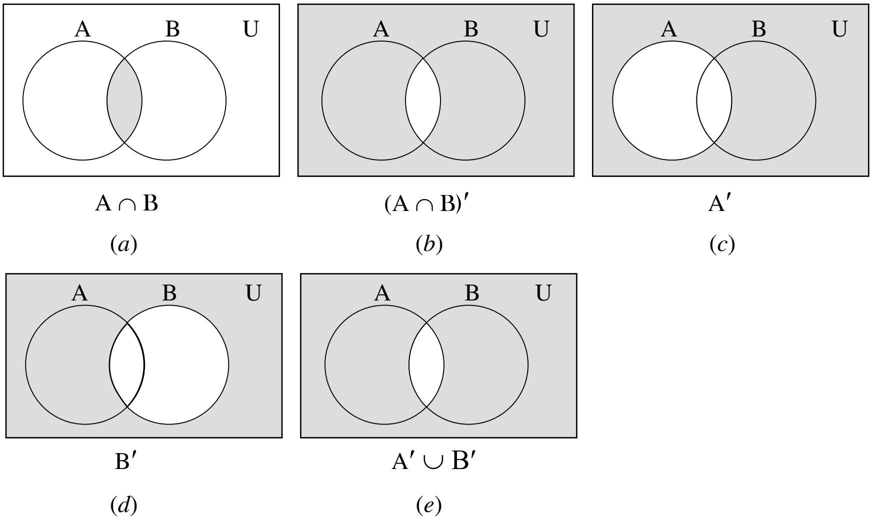


Figure 2.15

Shaded regions of figures (b) and (e) are the same.
Therefore, $(A \cap B)' = A' \cup B'$

2.14 PROOF OF ASSOCIATIVE, DISTRIBUTIVE AND DE MORGAN'S LAWS FROM DEFINITION

Associative laws

For any three sets A, B and C, prove that

(*i*) $A \cup (B \cup C) = (A \cup B) \cup C$
(*ii*) $A \cap (B \cap C) = (A \cap B) \cap C$

(*i*) Proof of $A \cup (B \cup C) = (A \cup B) \cup C$

Let x be an arbitrary element of $A \cup (B \cup C)$

Then, $x \in A \cup (B \cup C) \Rightarrow x \in A$ or $x \in (B \cup C)$

$\Rightarrow x \in A$ or $x \in B$ or $x \in C$

$\Rightarrow (x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow x \in (A \cup B)$ or $x \in C$

$\Rightarrow x \in (A \cup B) \cup C$

which shows that every element of $A \cup (B \cup C)$ is also an element of $(A \cup B) \cup C$

Therefore, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ ...(*i*)

Again, let y be an arbitrary element of $(A \cup B) \cup C$

Then, $y \in (A \cup B) \cup C \Rightarrow y \in (A \cup B)$ or $y \in C$

$\Rightarrow y \in A$ or $y \in B$ or $y \in C$

$\Rightarrow y \in A$ or ($y \in B$ or $y \in C$)

$\Rightarrow y \in A$ or $y \in (B \cup C)$

$\Rightarrow y \in A \cup (B \cup C)$

Therefore, $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ ...(*ii*)

From (*i*) and (*ii*), we can prove that

$A \cup (B \cup C) = (A \cup B) \cup C$ [From the properties of subsets, we know, if $A \subseteq B$ and $B \subseteq A$, then $A = B$]

(*ii*) Proof of $A \cap (B \cap C) = (A \cap B) \cap C$

Let x be an arbitrary element of $A \cap (B \cap C)$

Then, $x \in A \cap (B \cap C) \Rightarrow x \in A$ and $x \in (B \cap C)$

$\Rightarrow x \in A$ and $x \in B$ and $x \in C$

$\Rightarrow$ ($x \in A$ and $x \in B$) and $x \in C$

$\Rightarrow x \in (A \cap B)$ and $x \in C$

$\Rightarrow x \in (A \cap B) \cap C$

Therefore, $A \cap (B \cap C) \subseteq (A \cap B) \cap C$ ...(*i*)

Again, let y be an arbitrary element of $(A \cap B) \cap C$

Then, $y \in (A \cap B) \cap C \Rightarrow y \in (A \cap B)$ and $y \in C$

$\Rightarrow y \in A$ and $y \in B$ and $y \in C$

$\Rightarrow y \in A$ and ($y \in B$ and $y \in C$)

$\Rightarrow y \in A$ and $y \in (B \cap C)$

$\Rightarrow y \in A \cap (B \cap C)$

Therefore, $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ ...(*ii*)

From (*i*) and (*ii*) we can prove that

$A \cap (B \cap C) = (A \cap B) \cap C$

Distributive laws

For any three sets A, B and C, prove that

(*i*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(*ii*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(*i*) Proof of $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let x be an arbitrary element of $A \cup (B \cap C)$

Then, $x \in A \cup (B \cap C) \Rightarrow x \in A$ or $x \in (B \cap C)$

$\Rightarrow x \in A$ or ($x \in B$ and $x \in C$)

$\Rightarrow$ ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$)

$\Rightarrow (x \in (A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

Therefore, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ ...(*i*)

Again, let y be an arbitrary element of $(A \cup B) \cap (A \cup C)$

Then, $y \in (A \cup B) \cap (A \cup C) \Rightarrow y \in (A \cup B)$ and $y \in (A \cup C)$

$\Rightarrow (y \in A$ or $y \in B)$ and $(y \in A$ or $y \in C)$

$\Rightarrow y \in A$ or $(y \in B$ and $y \in C)$

$\Rightarrow y \in A$ or $y \in (B \cap C)$

$\Rightarrow y \in A \cup (B \cap C)$

Therefore, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ ...(*ii*)

From (*i*) and (*ii*) we can prove that

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(*ii*) Proof of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let x be an arbitrary element of $A \cap (B \cup C)$

Then, $x \in A \cap (B \cup C) \Rightarrow x \in A$ and $x \in (B \cup C)$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ ...(*i*)

Again, let y be an arbitrary element of $(A \cap B) \cup (A \cap C)$

Then, $y \in (A \cap B) \cup (A \cap C) \Rightarrow y \in (A \cap B)$ or $y \in (A \cap C)$

$\Rightarrow (y \in A$ and $y \in B)$ or $(y \in A$ and $y \in C)$

$\Rightarrow y \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow y \in A$ and $y \in (B \cup C)$

$\Rightarrow y \in A \cap (B \cup C)$

Therefore, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ ...(*ii*)

From (*i*) and (*ii*) we can prove that

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan's laws

For any two sets A and B, prove that

(*i*) $(A \cup B)' = A' \cap B'$

(*ii*) $(A \cap B)' = A' \cup B'$

(*i*) Proof of $(A \cup B)' = A' \cap B'$

Let x be an arbitrary element of $(A \cup B)'$

Then, $x \in (A \cup B)' \Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$
$\Rightarrow x \in (A' \cap B')$

Therefore, $(A \cup B)' \subseteq A' \cap B'$ ...(*i*)

Again, let y be an arbitrary element of $A' \cap B'$

Then, $y \in A' \cap B' \Rightarrow y \in A'$ and $y \in B'$
$\Rightarrow y \notin A$ and $y \notin B$
$\Rightarrow y \notin (A \cup B)$
$\Rightarrow y \in (A \cup B)'$

Therefore, $A' \cap B' \subseteq (A \cup B)'$ ...(*ii*)

From (*i*) and (*ii*) we can prove that

$(A \cup B)' = A' \cap B'$

(*ii*) Proof of $(A \cap B)' = A' \cup B'$

Let x be an arbitrary element of $(A \cap B)'$

Then, $x \in (A \cap B)' \Rightarrow x \notin (A \cap B)$
$\Rightarrow x \notin A$ or $x \notin B$
$\Rightarrow x \in A'$ or $x \in B'$
$\Rightarrow x \in (A' \cup B')$

Therefore, $(A \cap B)' \subseteq A' \cup B'$ ...(*i*)

Again, let y be an arbitrary element of $A' \cup B'$

Then, $y \in A' \cup B' \Rightarrow y \in A'$ or $y \in B'$
$\Rightarrow y \notin A$ or $y \notin B$
$\Rightarrow y \notin (A \cap B)$
$\Rightarrow y \in (A \cap B)'$

Therefore, $A' \cup B' \subseteq (A \cap B)'$ ...(*ii*)

From (*i*) and (*ii*) we can prove that

$(A \cap B)' = A' \cup B'$

2.15 CARTESIAN PRODUCT OF SETS

The Cartesian product of the two sets A and B is the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$. It is also called the product set of A and B. The Cartesian product of A and B is denoted symbolically as $A \times B$ (read as "A cross B").

Thus, $A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$

ILLUSTRATION 25

If $A = \{2, 3\}$ and $B = \{4, 5, 6\}$

Then, $A \times B = \{(2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$

$B \times A = \{(4, 2), (4, 3), (5, 2), (5, 3), (6, 2), (6, 3)\}$

OBSERVATIONS

(*i*) $A \times B \neq B \times A$, if A and B are equal sets
Then $A \times B = B \times A$

(*ii*) The number of elements in the sets $A \times B$ and $B \times A$ are equal

Theorem 1 For any three sets A, B and C, prove that,

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Proof

Let (x, y) be an arbitrary element of $A \times (B \cap C)$

Then, $(x, y) \in A \times (B \cap C) \Rightarrow x \in A$ and $y \in (B \cap C)$
$\Rightarrow x \in A$ and $(y \in B$ and $y \in C)$
$\Rightarrow (x \in A$ and $y \in B)$ and $x \in A$ and $y \in C$
$\Rightarrow (x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$
$\Rightarrow (x, y) \in (A \times B) \cap (A \times C)$

Therefore, $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ …(*i*)

Again, let (a, b) be can arbitrary element of $(A \times B) \cap (A \times C)$

Then, $(a, b) \in (A \times B) \cap (A \times C) \Rightarrow (a, b) \in (A \times B)$ and $(a, b) \in (A \times C)$
$\Rightarrow (a \in A$ and $b \in B)$ and $(a \in A$ and $b \in C)$
$\Rightarrow a \in A$ and $(b \in B$ and $b \in C)$
$\Rightarrow a \in A$ and $b \in B \cap C$
$\Rightarrow (a, b) \in A \times (B \cap C)$

Therefore, $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ …(*ii*)

From (*i*) and (*ii*) we can prove that

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

Theorem 2 For any three sets A, B and C, prove that

$$A \times (B - C) = (A \times B) - (A \times C)$$

Proof

Let (x, y) be an arbitrary element of $A \times (B - C)$

Then, $(x, y) \in A \times (B - C) \Rightarrow x \in A$ and $y \in (B - C)$
$\Rightarrow x \in A$ and $(y \in B$ and $y \notin C)$
$\Rightarrow (x \in A$ and $y \in B)$ and $(x \in A$ and $(y \notin C)$
$\Rightarrow (x, y) \in (A \times B)$ and $(x, y) \notin (A \times C)$
$\Rightarrow (x, y) \in (A \times B) - (A \times C)$

Therefore, $A \times (B - C) \subseteq (A \times B) - (A \times C)$ …(*i*)

Again, Let (a, b) be an arbitrary element of $(A \times B) - (A \times C)$

Then, $(a, b) \in (A \times B) - (A \times C) \Rightarrow (a, b) \in (A \times B)$ and $(a, b) \notin (A \times C)$
$\Rightarrow (a \in A$ and $b \in B)$ and $(a \in A$ and $b \notin C)$
$\Rightarrow a \in A$ and $(b \in B$ and $b \notin C)$

$\Rightarrow a \in A$ and $b \in (B - C)$

$\Rightarrow (a, b) \in A \times (B - C)$

Therefore, $(A \times B) - (A \times C) \subseteq A \times (B - C)$ ...(*ii*)

From (*i*) and (*ii*), we can prove that

$A \times (B - C) = (A \times B) - (A \times C)$

2.16 SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS

(*i*) If A is a finite set, then number of elements in set A is denoted by n (A).

(*ii*) If A, B and C are finite sets, and U be the finite Universal set, then

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
  [if A and B are disjoint sets, then $A \cap B = \Phi$ i.e. $n(A \cap B) = 0$]
- $n(A \cap B') = n(A - B) = n(A) - n(A \cap B)$
- $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$
- $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$
- $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$
- $n(A') = n(U) - n(A)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
- $n(A \cap B \cap C') = n(A \cap B) - n(A \cap B \cap C)$
- $n(A \cap B' \cap C') = n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$
- $n(A' \cap B' \cap C') = n(U) - n(A \cup B \cup C)$

ILLUSTRATIVE EXAMPLES

A. SHORT TYPE

EXAMPLE 1

State giving reason, whether the following objects form a set or not.

(*i*) The collection of difficult chapters in statistics.

(*ii*) The collection of all questions in this chapter.

Solution:

(*i*) The collection of difficult chapters in statistics is not a set, because a particular chapter considered difficult by one student might be considered simple by another.

(*ii*) The collection of all questions in this chapter is a set because if a question is given one can easily identify whether it is a question of this chapter or not.

EXAMPLE 2

Write the following sets in the roster form.

(*i*) A = {x : x is a letter in the word 'MATHEMATICS'}

(*ii*) B = {$x : x \in N$ and $2 < x < 8$}

Solution:

(*i*) The distinct letters in the word 'MATHEMATICS' are: M, A, T, H, E, I, C, S

So, the required set A = {M, A, T, H, E, I, C, S}

(*ii*) x is an integer and lies between 2 and 8.

Then, values of x are 3, 4, 5, 6, 7

So, the required set B = {3, 4, 5, 6, 7}

EXAMPLE 3

Write the following sets in the set-builder form:

(*i*) P = {1, 4, 9, 16, ...}

(*ii*) $Q = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ...\right\}$

Solution:

(*i*) We observe that elements in the given set are the square of all natural numbers. So, the set P in set-builder form is

$$P = \left\{x : x = n^2,\ n \in N\right\}$$

(*ii*) We observe that elements in the given set are the reciprocals of the squares of all natural numbers. So, the set Q in set-builder form is

$$Q = \left\{x : x = \frac{1}{n^2},\ n \in N\right\}$$

EXAMPLE 4

If a set R = {3, 6, 9, 10}. State whether the following statements are 'true' or 'false'. Justify your answer.

(*i*) $8 \in R$

(*ii*) $7 \notin R$

(*iii*) $10 \in A$

(*iv*) $4, 5, 7 \notin R$

Solution:

(*i*) False, because the element 8 does not belong to the given set R.

(*ii*) True, because the element 7 does not belongs to the given set R.

(*iii*) True, because the element 10 belongs to the given set R.

(*iv*) True, because the elements 4, 5 and 7 do not belong to the given set R.

EXAMPLE 5

Write the following statements in set-theoretic notations:

(*i*) 2 is an element of a set B

(*ii*) 5 does not belong to a set D

Solution: Set-theoretic notations of the given statements are:

(*i*) $2 \in B$ (*ii*) $5 \notin D$

EXAMPLE 6

Which of the following sets is a null set and why?

(*i*) $A = \{x : x \in N, 3 < x < 4\}$

(*ii*) $B = \{\Phi\}$

Solution:

(*i*) x is a natural number. Since there is no natural number between 3 and 4. So, the set A is a null set.

(*ii*) Here Φ is an element of set B. So, set B is not a null set.

EXAMPLE 7

Which of the following sets is finite and which is infinite?

(*i*) $G = \{x : x \in N$, and x is odd$\}$

(*ii*) $H = \{x : x \in N$, and $x^2 - 5x + 6 = 0\}$

Solution:

(*i*) $G = \{1, 3, 5, 7, 9, 11, \ldots\}$

Since, odd numbers are infinite in number, G is an infinite set.

(*ii*) $H = \{2, 3\}$, so, H is a finite set.

EXAMPLE 8

Which of the following sets are equal?

$A = \{x : x^2 + 8x + 15 = 0\}$

$B = \{-3, -5\}$

$C = \{2, -2, 3\}$

$D = \{2, 3\}$

Solution: Here the elements of the set A are −3 and −5 (These are the roots of the equation $x^2 + 8x + 15 = 0$)

Thus, $A = \{-3, -5\}$

Hence, $A = B$ [Since the elements of both the sets are same.]

EXAMPLE 9

If $A = \{1, 3, 5\}$, then write all the possible subsets of A.

Solution: We know that the number of subsets of any set = 2^n

Therefore, number of subsets of set A = $2^3 = 8$

Again, we know that null set Φ is a subset of every set.

The subset of A containing one element each – {1}, {3}, {5}

The subset of A containing two elements each – {1, 3}, {1, 5}, {3, 5}

The subset of A containing three elements – {1, 3, 5}

Therefore, number of all possible subsets of A are {1}, {3}, {5}, {1, 3}, {1, 5}, {3, 5}, {1, 3, 5}, Φ

EXAMPLE 10

If U = {1, 2, 3, 4, 5, 6, 7, 8} and A = {2, 4, 6}, find A^c

Solution: We observe that 1, 3, 5, 7, 8 are the only elements which do not belong to A.

Therefore, A^c = {1, 3, 5, 7, 8}

EXAMPLE 11

Find the cardinal number of a set from each of the following:

(*i*) M = $\{x : x \in N$, and $x^2 < 30\}$

(*iii*) N = $\{x : x$ is a factor of $20\}$

Solution:

(*i*) M = $\{x : x \in N$, and $x^2 < 30\}$

Then, M = {1, 2, 3, 4, 5}

Therefore, cardinal number of set M = 5, i.e. n(M) = 5

(*ii*) N = $\{x : x$ is a factor of $20\}$

Then, N = {1, 2, 4, 5, 10, 20}

Therefore, cardinal number of set N = 6, i.e. n(N) = 6

EXAMPLE 12

If A = {2, 4, 6, 8} and B = {2, 8, 3, 5}. Find the union of two sets A and B.

Solution: $A \cup B$ = {2, 3, 4, 5, 6, 8}

No element is repeated in the union of two sets.

The common elements 2, 8 are taken only once.

EXAMPLE 13

If P = {2, 4, 6, 8, 10} and Q = {1, 3, 8, 4, 6}. Find the intersection of two sets A and B.

Solution: Here, 4, 6 and 8 are the common elements in both the sets.

Therefore, $A \cap B$ = {4, 6, 8}

EXAMPLE 14

If R = {a, b, c, d, e, f} and S = {b, d, f, g}. Find the difference between the two sets:

(i) R and S
(ii) S and R

Solution:

(i) The elements a, c, e belong to R but not to S
Therefore, difference between R and S = R – S = {a, c, e}

(ii) The element g belongs to S but not to R
Therefore, difference between S and R = S – R = {g}

EXAMPLE 15

If U = {1, 3, 5, 7, 9, 11, 13}, then which of the following are subsets of U.
B = {2, 4}, A = {0}, C = {1, 9, 5, 13}, D = {5, 11, 1}, E = {13, 7, 9, 11, 5, 3, 1}, F = {2, 3, 4, 5}

Solution: Here, we observe that C, D and E have the elements which belong to U.
Therefore, C, D and E are the subsets of U.

EXAMPLE 16

Let P and Q be two finite sets such that $n(P) = 25$, $n(Q) = 30$ and $n(P \cup Q) = 40$, find $n(P \cap Q)$.

Solution: Using the formula $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$

Then, $n(P \cap Q) = n(P) + n(Q) - n(P \cup Q)$
$= 25 + 30 - 40 = 55 - 40 = 15$

EXAMPLE 17

In a group of 70 people, 30 like coffee and 45 like tea and each person likes at least one of the two drinks. How many like both coffee and tea?

Solution: Let C be the set of people who like coffee and T be the set of people who like tea.

Given that, $n(C \cup T) = 70$, $n(C) = 30$, $n(T) = 45$

Then, $n(C \cap T) = n(C) + n(T) - n(C \cup T)$
$= 30 + 45 - 70 = 5$

Therefore, 5 people like both tea and coffee.

EXAMPLE 18

Find the power set of the set {2, 4, 6} [***C.U. B.Com. 2009***]

Solution: Let A = {2, 4, 6}

All subsets of the set A are {2}, {4}, {6}, {2, 4}, {4, 6}, {2, 6}, {2, 4, 6}, Φ

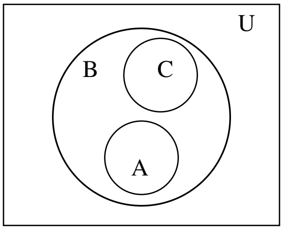
Therefore, power set of set A = P(A)

P(A) = {{2}, {4}, {6}, {2, 4}, {4, 6}, {2, 6}, {2, 4, 6}, Φ}

EXAMPLE 19

Draw a Venn diagram of three non-empty sets A, B, C such that $A \subset B$, $C \subset B$ and $A \cap C = \Phi$ [***C.U. B.Com. 1986***]

Solution:



EXAMPLE 20

Given A = {1, 2, 3, 4}, B = {3, 4, 5}, C = {1, 4, 5}

Verify the following statement:

$$A - (B \cup C) = (A - B) \cap (A - C)$$ [***C.U. B.Com. 1989***]

Solution:

$$B \cup C = \{x : x \in B \text{ or } x \in C\}$$
$$= \{1, 3, 4, 5\}$$
$$A = \{1, 2, 3, 4\}$$

Then, $A - (B \cup C) = \{x : x \in A \text{ and } x \notin B \cup C\}$

$= \{2\}$ …(*i*)

Again, $A - B = \{x : x \in A \text{ and } x \notin B\}$

$= \{1, 2\}$

and $A - C = \{x : x \in A \text{ and } x \notin C\}$

$= \{2, 3\}$

Then, $(A - B) \cap (A - C) = \{x : x \in (A - B) \text{ and } x \in (A - C)\}$

$= \{2\}$ … (*ii*)

From (*i*) and (*ii*), we can write

$$A - (B \cup C) = (A - B) \cap (A - C)$$

EXAMPLE 21

Write down the following statements in set-theoretic notations:

(*i*) 3 is an element of a set A (*ii*) 4 does not belong to set B (*iii*) C is a subset of D

(*iv*) P and Q are disjoint sets. [***C.U. B.Com.(H) 1988***]

Solution:

(*i*) $3 \in A$

(*ii*) $4 \notin B$

(*iii*) $C \subseteq D$

(*iv*) $P \cap Q = \Phi$

EXAMPLE 22

Which of the following sets is a null set Φ? Briefly say why?

(*i*) $A = \{x : x \text{ is } >1 \text{ and } x \text{ is } <1\}$

(*ii*) $B = \{x : x + 3 = 3\}$

(*iii*) $C = \{\Phi\}$ [***C.U. B.Com.(H) 1984***]

Solution:

(*i*) Here, x is >1 and x is <1. But x cannot assume a value which is simultaneously greater than 1 and less than 1. So, there is no element in set A satisfying the given condition. Therefore, A is a null set.

(*ii*) $x + 3 = 3$ or $x = 0$

Hence, the set B consists of one element 0 (zero). Therefore, B is a singleton set, it cannot be a null set.

(*iii*) Here, Φ is an element of the set C. It is a singleton set. Therefore, it cannot be a null set.

EXAMPLE 23

Let S = {1, 2, 3, 4, 5, 6} be the Universal set. Let $A \cup B = \{2, 3, 4\}$; find $A^c \cap B^c$ where A^c, B^c are complements of A and B respectively. Also show that $A \cup B$ and $A^c \cap B^c$ are disjoint sets. [***C.U. B.Com.(H) 1995***]

Solution: From De Morgan's law we know that,

$$\begin{aligned} A^c \cap B^c &= (A \cup B)^c = S - (A \cup B) \\ &= \{1, 2, 3, 4, 5, 6\} - \{2, 3, 4\} \\ &= \{1, 5, 6\} \end{aligned}$$

Now, $A \cup B = \{2, 3, 4\}$ and $A^c \cap B^c = \{1, 5, 6\}$

They have no common elements.

Therefore, $A \cup B$ and $A^c \cap B^c$ are disjoint sets.

EXAMPLE 24

Three sets A, B and C be such that A – B = {2, 4, 6}, and A – C = {2, 3, 5}; find $A - (B \cup C)$ and $A - (B \cap C)$. [***C.U. B.Com. 1997, 2013(G)***]

Solution: We know that,

$$\begin{aligned} A - (B \cup C) &= (A - B) \cap (A - C) \\ &= \{2, 4, 6) \cap (2, 3, 5\} = \{2\} \end{aligned}$$

and $A - (B \cap C) = (A - B) \cup (A - C)$
$= \{2, 4, 6\} \cup \{2, 3, 5\} = \{2, 3, 4, 5, 6\}$

EXAMPLE 25

(*i*) Find the H.C.F of 8 and 12
(*ii*) Find the L.C.M of 4 and 6

Solution:

(*i*) Let A be the set of factors of 8 and B be the set of factors of 12
then, $A = \{1, 2, 4, 8\}$ and $B = \{1, 2, 3, 4, 6, 12\}$
Now, $A \cap B = \{1, 2, 4\}$
Therefore, H.C.F = The highest element in $A \cap B = 4$

(*ii*) Let A be the set of multiples of 4 and B be the set of multiples of 6
then, $A = \{4, 8, 12, 16, 20, 24, \ldots\}$
$B = \{6, 12, 18, 24, 30, 36, \ldots\}$
Now, $A \cap B = \{12, 24, 36, 48, \ldots\}$
Therefore, L.C.M = the lowest element in $A \cap B = 12$

EXAMPLE 26

Given $A = \{1, 3\}$, $B = \{3, 5\}$ and $C = \{5, 10\}$, verify the following relations.

(*i*) $A \times B \neq B \times A$
(*ii*) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(*iii*) $A \times (B \cap C) = (A \times B) \cap (A \times C)$ [***C.U. B.Com. 1988***]

Solution: Given $A = \{1, 3\}$, $B = \{3, 5\}$ and $C = \{5, 10\}$

(*i*) $A \times B = \{(1, 3), (1, 5), (3. 3), (3, 5)\}$
$B \times A = \{(3, 1), (3, 3), (5, 1), (5, 3)\}$
Therefore, $A \times B \neq B \times A$ [Elements are not same]

(*ii*) $B \cup C = \{3, 5, 10\}$
$A \times (B \cup C) = \{(1, 3), (1, 5), (1, 10), (3, 3), (3, 5), (3, 10)\}$ ... (*i*)
$A \times B = \{(1, 3), (1, 5), (3, 3), (3, 5)\}$
$A \times C = \{(1, 5), (1, 10), (3, 5), (3, 10)\}$
Then, $(A \times B) \cup (A \times C) = \{(1, 3), (1, 5), (3, 3), (3, 5), (1, 10), (3, 10)\}$ … (*ii*)
From (*i*) and (*ii*) we can say,

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

(*iii*) $B \cap C = \{5\}$

$$A \times (B \cap C) = \{(1, 5), (3, 5)\} \quad \ldots(i)$$

Again, $A \times B = \{(1, 3), (1, 5), (3, 3), (3, 5)\}$
$A \times C = \{(1, 5), (1, 10), (3, 5), (3, 10)\}$

Then, $(A \times B) \cap (A \times C) = \{(1, 5), (3, 5)\}$ … (*ii*)

From (*i*) and (*ii*) we can say,

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

EXAMPLE 27

If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 5, 8\}$, $C = \{3, 4, 5, 6, 7\}$, find

(*i*) $A \cup (B \cap C)$

(*ii*) $A \cap (B \cup C)$ [***C.U. B.Com.(H) 2017***]

Solution: Given, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 5, 8\}$, $C = \{3, 4, 5, 6, 7\}$

(*i*) Now, $B \cap C$ = Common elements of B and C = $\{4, 5\}$

Therefore, $A \cup (B \cap C)$ = elements belong to A or $B \cap C$ = $\{1, 2, 3, 4, 5\}$

(*ii*) $B \cup C$ = elements belong to B or C

$= \{2, 3, 4, 5, 6, 7, 8\}$

Therefore, $A \cap (B \cup C)$ = common elements of A and $(B \cup C)$

$= \{2, 3, 4\}$

EXAMPLE 28

If $A = \{x : x$ is a natural number and $x \leq 6\}$, $B = \{x : x$ is the natural number and $3 \leq x \leq 8\}$, find $A - B$ and $A \cap B$. [***C.U. B.Com.(G) 2017***]

Solution: $A = \{1, 2, 3, 4, 5, 6\}$

$B = \{3, 4, 5, 6, 7, 8\}$

Therefore, $A - B = \{x : x \in A \text{ and } x \notin B\}$

$= \{1, 2\}$

and $A \cap B = \{x : x \in A \text{ and } x \in B\}$

$= \{3, 4, 5, 6\}$

B. SHORT ESSAY TYPE

EXAMPLE 29

Write the following sets in the roster form.

(*i*) $A = \{x : x \in N \text{ and } 2 \leq x \leq 6\}$

(*ii*) $B = \{x \mid x$ is a letter in the word 'ALGEBRA'$\}$

(*iii*) $C = \{x : x$ is a solution of the equation $x^2 + 2x - 15 = 0\}$

(*iv*) $D = \{x \mid x$ is a vowel in the word 'EQUATION'$\}$

(*v*) $E = \{x : x \in I \text{ and } x^2 < 22\}$

Solution:

(*i*) $A = \{2, 3, 4, 5, 6\}$

(*ii*) $B = \{A, L, G, E, B, R\}$

(*iii*) $x^2 + 2x - 15 = 0$

$\Rightarrow x^2 + 5x - 3x - 15 = 0 \Rightarrow x(x + 5) - 3(x + 5) = 0$

$\Rightarrow (x + 5)(x - 3) = 0 \Rightarrow x + 5 = 0$ or $x - 3 = 0$

$\Rightarrow x = -5, 3$

Therefore, C = {–5, 3}

(*iv*) Clearly, the word 'EQUATION' has the following vowels: A, E, I, O, U

Therefore, D = {A, E, I, O, U}

(*v*) The squares of integers less than 22 are: 0, ± 1, ± 2, ± 3, ± 4.

Therefore, E = [–4, –3, –2, –1, 0, 1, 2, 3, 4}

EXAMPLE 30

Write the following sets in the set-builder form.

(*i*) The set of even numbers between 6 and 15

(*ii*) The set of all odd natural numbers

(*iii*) The set of all letters in the word 'PARABOLA'

(*iv*) The set of reciprocals of natural numbers

(*v*) $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\ldots\ldots\ldots, \frac{9}{10}\right\}$

Solution:

(*i*) $\{x : x$ is an even number, $6 < x < 15\}$

(*ii*) $\{x : x$ is a natural number and $x = 2n + 1\}$ or $\{x : n \in N, x = 2n + 1\}$ or $\{2n + 1 : n \in N\}$

(*iii*) $\{x \mid x$ is a letter in the word 'PARABOLA'$\}$

(*iv*) $\left\{x \mid x = \frac{1}{n}, n \in N\right\}$

(*v*) $\left\{x : x = \frac{n}{n+1}, n \in N \text{ and } 1 \le x \le 9\right\}$

EXAMPLE 31

Which of the following sets is a null set?

(*i*) P = $\{x : x$ is an even prime number$\}$

(*ii*) Q = $\{x : x$ is a positive number less than zero$\}$

(*iii*) R = $\{x : x < x\}$

(*iv*) S = $\{x : 5 < x < 6, x \in N\}$

(*v*) T = $\{x : x^2 = 49$ and x is an odd integer$\}$

Solution:

(*i*) We know that 2 is the only even prime number. Therefore, P = {2}.

So, P is not a null set.

(*ii*) We know that there is no positive number which is less than zero.

So, Q is a null set.

(*iii*) There is no number which is less than itself.
So, R is a null set.

(*iv*) There is no natural number between 5 and 6.
So, S is a null set.

(*v*) $x^2 = 49$ or $x = \pm 7$, which are odd integers.
Therefore, T = {7, –7}. So, T is not a null set.

EXAMPLE 32

State with reasons whether each of the following statements is true or false.

(*i*) $4 \in \{4, 5, 6, 7\}$

(*ii*) $\{4\} \in \{4, 5, 6, 7]$

(*iii*) $\{5\} \subset \{4, 5, 6, 7\}$

(*iv*) $\{4, 5\} \subseteq \{4, 5, 6, 7\}$

(*v*) $\{4, 5, 6\} = \{5, 6, 7\}$

(*vi*) $\{4, 5, 6\} = \{4, 5, 5, 6, 4, 6, 5\}$

Solution:

(*i*) True, as 4 is an element of {4, 5, 6, 7}.

(*ii*) False, as {4} is not an element of {4, 5, 6, 7}, it is a singleton set.

(*iii*) True, as {5} is a proper subset of {4, 5, 6, 7}.

(*iv*) True, as each element of the set {4, 5} is also an element of the set {4, 5, 6, 7}. So, {4, 5} is a subset of {4, 5, 6, 7}

(*v*) False, as all elements of the sets are not same. 4 does not belong to {5, 6, 7} and 7 does not belong to {4, 5, 6}. So, the two sets are not equal.

(*vi*) True, as both the sets contain same elements. So, the two sets are equal.

EXAMPLE 33

Find the pairs of equal sets, from the following sets, if any, giving reasons.

$A = \{0\}$, $B = \{x : x > 15 \text{ and } x < 5\}$, $C = \{x : x - 5 = 0\}$,
$D = \{x : x^2 = 25\}$, $E = \{x : x \text{ is an integral positive root of equation } x^2 - 2x - 15 = 0\}$.

Solution:

$A = \{0\}$,
$B = \{x : x > 15 \text{ and } x < 5\} = \Phi$
$C = \{x : x - 5 = 0\} = \{5\}$
$D = \{x : x^2 = 25\} = \{5, -5\}$
$E = \{x : x \text{ is an integral positive root of equation } x^2 - 2x - 15 = 0\} = \{5\}$
Clearly C = E

EXAMPLE 34

State which of the following sets are finite and which are infinite?

(*i*) $I = \{x : x \in N \text{ and } x \text{ is prime}\}$

(*ii*) $J = \{x : x \in N \text{ and } x^2 = 25\}$

(*iii*) $K = \{x : x \in N \text{ and } x^2 \text{ is odd}\}$

(*iv*) L = $\{x : x \in N$ and $x^2 - 5x + 6 = 0\}$

(*v*) M = $\{x : x \in I$ and $x > -5\}$

Solution:

(*i*) I = $\{x : x \in N$ and x is prime$\} = \{2, 3, 5, 7, 11, \ldots\}$

As prime numbers are infinite in number, so I is an infinite set.

(*ii*) J = $\{x : x \in N$ and $x^2 = 25\}$

$x^2 = 25$ or $x = +5$ and -5, but $x \in N$, so J = $\{5\}$, which is a finite set.

(*iii*) K = $\{x : x \in N$ and x^2 is odd$\}$

$= \{\ldots, -5, -3, -1, 0, 1, 3, 5, \ldots\}$

Clearly, K is an infinite set.

(*iv*) L = $\{x : x \in N$ and $x^2 - 5x + 6 = 0\}$

$= \{2, 3\}$

So, L is a finite set.

(*v*) M = $\{x : x \in I$ and $x > -5\}$

$= \{-4, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Clearly, M is an infinite set.

EXAMPLE 35

Show the relationship among the following three sets in respect of subsets and supersets:

(*i*) N = $\{x : x$ is a positive integer$\}$

(*ii*) Z = $\{x : x$ is an integer$\}$

(*iii*) R = $\{x : x$ is a real number$\}$ [***C.U. B.Com.(H) 1985***]

Solution:

(*i*) N = $\{x : x$ is a positive integer$\}$

$= \{1, 2, 3, 4, 5, \ldots\}$

(*ii*) Z = $\{x : x$ is an integer$\} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

(*iii*) R = $\{x : x$ is a real number$\} = \{\ldots, -1, -.9, -.8, \ldots, 0, .1, .2, \ldots, 1, \ldots\}$ = {All rational, irrational and zero}

Therefore, N is a proper subset of Z (i.e. $N \subset Z$) and Z is a superset of N (i.e. $Z \supset N$)

Again, N is a proper subset of R (i.e. $N \subset R$) and R is a superset of N (i.e. $R \supset N$).

Finally, Z is a proper subset of R (i.e. $Z \subset R$) and R is a superset of Z (i.e. $R \supset Z$).

EXAMPLE 36

Show that:

(*i*) The complement of a universal set is an empty set.

(*ii*) A set and its complement are disjoint sets.

Solution:

(*i*) Let S be universal set.

S′ = The set of those elements which are not in S = empty set = Φ

Therefore, the complement of a universal set is an empty set.

(*ii*) Let A be any set

Then, A′ = set of those elements of S which are not in A.

Let $x \notin$ A, then x is an element of S not contained in A.

So, $x \in$ A′.

Therefore, A and A′ are disjoint sets.

EXAMPLE 37

Let A = {0, 2, 4, 6, 8}, B = {0, 1, 2, 3, 4} and C = {0, 3, 6, 9}.

Determine A ∪ B ∪ C and A ∩ B ∩ C.

Solution:

A ∪ B ∪ C = elements belongs to A or B or C

= {0, 1, 2, 3, 4, 6, 8, 9}

A ∩ B ∩ C = elements common to A, B and C

= {0}

EXAMPLE 38

Given A = {1, 2, 3}, B = {2, 4}, C = (2, 3, 5}

Find

(*i*) A ∩ B, A ∩ C and (A ∩ B) ∪ (A ∩ C)

(*ii*) B ∪ C and A ∩ (B ∪ C)

Hence, verify the result A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

[***C.U. B.Com.(H) 1987***]

Solution: Given, A = {1, 2, 3}, B = {2, 4}, C = (2, 3, 5}

(*i*) A ∩ B = {2}, A ∩ C = {2, 3}

(A ∩ B) ∪ (A ∩ C) = {2, 3} ...(*i*)

(*ii*) B ∪ C = {2, 3, 4, 5}

A ∩ (B ∪ C) = (2, 3} ...(*ii*)

From (*i*) and (*ii*) we can say

A ∩ (B ∪ C) = (A ∩ B) ∪ (B ∩ C)

EXAMPLE 39

If S = {a, b, c, d, e, f} be the universal set and A, B, C are three subsets of S, where A = {a, c, d, f}, B ∩ C = {a, b, f}, find (A ∪ B) ∩ (A ∪ C) and B′ ∪ C′.

[***C.U. B.Com.(H) 1983***]

Solution: Given, $S = \{a, b, c, d, e, f\}$, $A = \{a, c, d, f\}$
and $B \cap C = \{a, b, f\}$
Now, $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$

$$= \{a, c, d, f\} \cup \{a, b, f\}$$
$$= \{a, b, c, d, f\}$$

Again, $B' \cup C' = (B \cap C)' = S - (B \cap C)$

$$= \{a, b, c, d, e, f\} - \{a, b, f\}$$
$$= \{c, d, e\}$$

EXAMPLE 40

Three sets A, B, C be such that $A - B = \{2, 4, 6\}$ and $A - C = \{2, 3, 5\}$; find $A - (B \cup C)$ and $A - (B \cap C)$. [***C.U. B.Com.(H), 1997***]

Solution:

$$A - (B \cup C) = (A - B) \cap (A - C) = \{2, 4, 6\} \cap \{2, 3, 5\} = \{2\}$$
$$A - (B \cap C) = (A - B) \cup (A - C) = \{2, 4, 6\} \cup \{2, 3, 5\} = \{2, 3, 4, 5, 6\}$$

EXAMPLE 41

If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 3, 5, 7\}$, $B = \{1, 2, 4, 5, 7, 8\}$, verify that $(A \cup B)^c = A^c \cap B^c$.

Solution: Given $A = \{1, 3, 5, 7\}$, $B = \{1, 2, 4, 5, 7, 8\}$
Therefore, $A \cup B = \{1, 2, 3, 4, 5, 7, 8\}$
$(A \cup B)^c$ = The set of elements of U which are not in $A \cup B = \{6\}$ ...(*i*)
Again, $A^c = \{x : x \in U, x \notin A\} = \{2, 4, 6, 8\}$
and $B^c = \{x : x \in U, x \notin B\} = \{3, 6\}$

$$A^c \cap B^c = \{2, 4, 6, 8\} \cap \{3, 6\} = \{6\} \quad ...(ii)$$

From (*i*) and (*ii*) we can say

$$(A \cup B)^c = A^c \cap B^c$$

EXAMPLE 42

Distinguish among Φ, {Φ}, {O} and O.

Solution:

(*i*) Φ is null set having no element
(*ii*) {Φ}, Φ is an element, so it is a singleton set
(*vi*) {O}, O is an element, so it is a singleton set
(*vii*) 0 is only a number. It is neither a set nor an element of a set.

EXAMPLE 43

Given A = {1, 2, 3}, B = {1, 0}, and C = {2, 3}, then verify that:

(*i*) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(*iii*) $(A \times B) \cap (A \times C) = \Phi$

Solution:

(*i*) Given A = {1, 2, 3}, B = {1, 0} and C = {2, 3}

So, $A \cap B = \{1\}$

Now, $(A \cap B) \times C = \{1\} \times \{2, 3\}$

$$= \{(1, 2), (1, 3)\} \quad ...(i)$$

Again, $A \times C = \{1, 2, 3\} \times \{2, 3\}$

$$= \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$$

and $B \times C = \{1, 0\} \times \{2, 3\}$

$$= \{(1, 2), (1, 3), (0, 2), (0, 3)\}$$

So, $(A \times C) \cap (B \times C) = \{(1, 2), (1, 3)\}$ ...(*ii*)

From (*i*) and (*ii*) we can say

$$(A \cap B) \times C = (A \times C) \cap (B \times C)$$

(*ii*) $A \times B = \{1, 2, 3\} \times \{1, 0\}$

$$= \{(1, 1), (1, 0), (2, 1), (2, 0), (3, 1), (3, 0)\}$$

and $A \times C = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$

Therefore, $(A \times B) \cap (A \times C) = \Phi$ (verified)

EXAMPLE 44

Find the H.C.F of 21, 45, 105 using set theory. [***C.U. B.com. 2007***]

Solution: Let A, B, C be the set of factors of 21, 45 and 105 respectively.

A = {1, 3, 7, 21}, B = {1, 3, 5, 9, 15, 45}, C = {1, 3, 5, 7, 15, 21, 35, 105}

Therefore, $A \cap B \cap C = \{1, 3\}$

Therefore, H.C.F = the greatest element in $A \cap B \cap C = 3$.

EXAMPLE 45

Find the L.C.M of 6, 8, 12, using set theory.

Solution: Let P, Q, R be the set of multiples of 6, 8 and 12 respectively.

Therefore, P = {6, 12, 18, 24, 30, 36, 42, 48, …}

Q = {8, 16, 24, 32, 40, 48, …}

R = {12, 24, 36, 48, 60, 72, ...}

Therefore, $P \cap Q \cap R = \{24, 48, 72, \ldots\}$
Therefore, L.C.M = The lowest element of the set

$$P \cap Q \cap R = 24$$

EXAMPLE 46

Applying set operations, prove that, 3 + 4 = 7.

Solution: Let A and B be the two sets having 3 and 4 elements respectively.

$A = \{a, b, c\}$ and $B = \{q, r, s, t\}$

Therefore, $n(A) = 3$, $n(B) = 4$ and $A \cap B = \Phi$ i.e. A and B are disjoint sets.

$$A \cup B = \{a, b, c, q, r, s, t\}.$$

Therefore, $n(A \cup B) = 7$

We know that, $n(A \cup B) = n(A) + n(B)$ (In case of disjoint sets)

or $\quad 7 = 3 + 4$ (Proved)

EXAMPLE 47

If A and B are any two sets, then prove that:

(*i*) $A - B = A \cap B'$
(*ii*) $(A - B) \cup B = A \cup B$
(*iii*) $(A - B) \cap B = \Phi$
(*iv*) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

Solution:

(*i*) Let x be an arbitrary element of $A - B$.

Then, $x \in (A - B) \Rightarrow x \in A$ and $x \notin B$

$\Rightarrow x \in A$ and $x \in B'$

$\Rightarrow x \in (A \cap B')$

Therefore, $A - B \subseteq A \cap B'$ ...(*i*)

Again, let y be an arbitrary element of $A \cap B'$.

Then, $y \in A \cap B' \Rightarrow y \in A$ and $y \in B'$

$\Rightarrow y \in A$ and $y \notin B$

$\Rightarrow y \in A - B$

Therefore, $A \cap B' \subseteq A - B$ ...(*ii*)

From (*i*) and (*ii*) we get $A - B = A \cap B'$

(*ii*) Let x be an arbitrary element of $(A - B) \cup B$.

Then, $x \in (A - B) \cup B \Rightarrow x \in (A - B)$ or $x \in B$

$\Rightarrow (x \in A$ and $x \notin B)$ or $x \in B$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \notin B$ or $x \in B)$

$\Rightarrow x \in A \cup B$

Therefore, $(A - B) \cup B \subseteq A \cup B$ ...(*i*)

Again, let y be an arbitrary element of $A \cup B$.

Then, $y \in (A \cup B) \Rightarrow y \in A \text{ or } y \in B$

$$\Rightarrow (y \in A \text{ or } y \in B) \text{ and } (y \notin B \text{ or } y \in B)$$
$$\Rightarrow (y \in A \text{ and } y \notin B) \text{ or } y \in B$$
$$\Rightarrow y \in (A - B) \cup B$$

Therefore, $A \cup B \subseteq (A - B) \cup B$ ...(*ii*)

From (*i*) and (*ii*) we get,

$$(A - B) \cup B = A \cup B$$

(*iii*) If possible let $(A - B) \cap B \neq \Phi$. Then, there exists at least one element x, (say), in $A - B \cap B$.

Now, $x \in (A - B) \cap B \Rightarrow x \in (A - B) \text{ and } x \in B$

$$\Rightarrow (x \in A \text{ and } x \notin B) \text{ and } x \in B$$
$$\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \in B)$$

But, $x \notin B$ and $x \in B$ is contradictory. It is not possible. So, our assumption is wrong.

Therefore, $(A - B) \cap B = \Phi$.

(*iv*) Let x be an arbitrary element of $(A - B) \cup (B - A)$.

Then, $x \in (A - B) \cup (B - A) \Rightarrow x \in A - B \text{ or } x \in B - A$

$$\Rightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)$$
$$\Rightarrow (x \in A \text{ or } x \notin B) \text{ and } (x \notin B \text{ or } x \notin A)$$
$$\Rightarrow x \in (A \cup B) \text{ and } x \notin (A \cap B)$$
$$\Rightarrow x \in (A \cup B) - (A \cap B)$$

Therefore, $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$ ...(*i*)

Again, let y be an arbitrary element of $(A \cup B) - (A \cap B)$.

Then, $y \in (A \cup B) - (A \cap B) \Rightarrow y \in (A \cup B) \text{ and } y \notin (A \cap B)$

$$\Rightarrow y \in A \text{ or } y \in B \text{ and } (y \notin A \text{ or } y \notin B)$$
$$\Rightarrow (y \in A \text{ and } y \notin B) \text{ or } (y \in B \text{ and } y \notin A)$$
$$\Rightarrow y \in (A - B) \text{ or } y \in (B - A)$$
$$\Rightarrow y \in (A - B) \cup (B - A)$$

Therefore, $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$ ...(*ii*)

From (*i*) and (*ii*) we get,

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

EXAMPLE 48

If A, B and C are any three sets, then prove that: $A - (B \cap C) = (A - B) \cup (A - C)$

[***C.U. B.Com. (H), 1988***]

Solution: Let x be an arbitrary element of A – (B ∩ C)

Then, $x \in A - (B \cap C) \Rightarrow x \in A$ and $x \notin (B \cap C)$

$$\Rightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C)$$

$$\Rightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C)$$

$$\Rightarrow x \in (A - B) \text{ or } x \in (A - C)$$

$$\Rightarrow x \in (A - B) \cup (A - C)$$

Therefore, $A - (B \cap C) \subseteq (A - B) \cup (A - C)$ ...(*i*)

Again, let y be an arbitrary element of (A – B) ∪ (A – C).

Then, $y \in (A - B) \cup (A - C) \Rightarrow y \in (A - B)$ or $y \in (A - C)$

$$\Rightarrow y \in A \text{ and } y \notin B \text{ or } y \in A \text{ and } y \notin C$$

$$\Rightarrow y \in A \text{ and } y \notin B \text{ or } y \notin C$$

$$\Rightarrow y \in A \text{ and } y \notin (B \cap C)$$

$$\Rightarrow y \in A - (B \cap C)$$

Therefore, $(A - B) \cup (A - C) \subseteq A - (B \cap C)$ ...(*ii*)

From (*i*) and (*ii*) we get,

$$A - (B \cap C) = (A - B) \cup (A - C)$$

EXAMPLE 49

Let A and B be two sets. Prove that: $(A - B) \cup B = A$ if and only if $B \subset A$.

Solution: $B \subset A$. We have to prove that $(A - B) \cup B = A$.

Now, $(A - B) \cup B = (A \cap B') \cup B$ [∵ $A - B = A \cap B'$]

$$= (A \cup B) \cap (B' \cup B)$$

$$= (A \cup B) \cap U$$

$$= A \cup B$$

$$= A$$ [∵ $B \subset A$]

EXAMPLE 50

For sets A, B and C using properties of sets, prove that:

(*i*) $A - (B \cup C) = (A - B) \cap (A - C)$

(*ii*) $(A \cap B) - C = (A - C) \cap (B - C)$

Solution:

(*i*) $A - (B \cup C) = A \cap (B \cup C)'$

$$= A \cap (B' \cap C')$$

$$= (A \cap B') \cap (A \cap C')$$

$$= (A - B) \cap (A - C)$$

(*ii*) $(A \cap B) - C = (A \cap B) \cap C'$

$$= (A \cap C') \cap (B \cap C')$$

$$= (A - C) \cap (B - C)$$

EXAMPLE 51

In a group of students, 50 students know Hindi, 25 know English and 12 know both. Each of the students knows either Hindi or English. How many students are there in the group?

Solution: Let H be the set of students who know Hindi

E be the set of students who know English

Then, $n(\text{H}) = 50$, $n(\text{E}) = 25$ and $n(\text{H} \cap \text{E})=12$

We know that, $n(\text{H} \cup \text{E}) = n(\text{H}) + n(\text{E}) - n(\text{H} \cap \text{E})$

$= 50 + 25 - 12$

$= 75 - 12 = 63$

Since each of the students knows either Hindi or English.

Therefore, the number of students in the group = $n(\text{H} \cup \text{E}) = 63$.

EXAMPLE 52

In a survey of 60 people, it was found that 21 people liked product A, 26 liked product B and 29 liked product C. If 14 people liked products A and B, 12 people liked products C and A, 14 people liked products B and C and 8 people liked all the three products, find

(*i*) how many people like at least one of the three products?

(*ii*) how many people do not like any of the three products?

(*iii*) how many people liked product C only ?

Solution: Let P, Q and R be the sets of people who like product A, B and C respectively.

Given, $n(\text{P}) = 21, n(\text{Q}) = 26,\ n(\text{R}) = 29, n(\text{P} \cap \text{Q}) = 14,\ n(\text{Q} \cap \text{R}) = 14, n(\text{P} \cap \text{R}) =$ $12 \text{ and } n(\text{P} \cap \text{Q} \cap \text{R}) = 8,$

We have to calculate:

(*i*) $n(\text{P} \cup \text{Q} \cup \text{R})$ (*ii*) $n(\text{P}' \cap \text{Q}' \cap \text{R}')$ (*iii*) $n(\text{P}' \cap \text{Q}' \cap \text{R})$

(*i*) $n(\text{P} \cup \text{Q} \cup \text{R}) = n(\text{P}) + n(\text{Q}) + n(\text{R}) - n(\text{P} \cap \text{Q}) - n(\text{Q} \cap \text{R}) - n(\text{P} \cap \text{R})$ $+ n(\text{P} \cap \text{Q} \cap \text{R})$

$= 21 + 26 + 29 - 14 - 14 - 12 + 8$

$= 84 - 40 = 44$

(*ii*) $n(\text{P}' \cap \text{Q}' \cap \text{R}') = n(\text{P} \cup \text{Q} \cup \text{R})' = N - n(\text{P} \cup \text{Q} \cup \text{R})$

$= 60 - 44 = 16$ [N = Total no. of people]

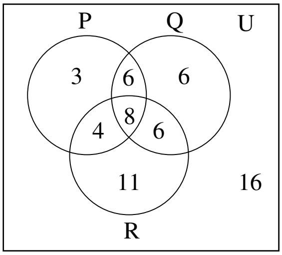
(*iii*) $\text{n}(\text{P}' \cap \text{Q}' \cap \text{R}) = n(\text{R}) - n(\text{P} \cap \text{R}) - n(\text{Q} \cap \text{R}) + n(\text{P} \cap \text{Q} \cap \text{R})$

$= 29 - 12 - 14 + 8$

$= 37 - 26 = 11$

*Alternatively***,** the problem can also be solved using Venn diagram.

The given Venn diagram represents the number of people of different regions.



From Venn diagram,

(*i*) The number of people who like at least one of the three products
= 3 + 6 + 8 + 4 + 6 + 6 + 11 = 44

(*ii*) the number of people who do not like any of the three products = 60 – 44 = 16

(*iii*) the number of people who like product C only
= 11

EXAMPLE 53

Out of 300 students in B.Com class of a school, 140 play Hockey, 115 play Football, 75 play Volleyball, 30 of these play Hockey and Football, 25 play Volleyball and Hockey, 25 play Football and Volleyball. Also, each student plays at least one of the three games. How many students play all the three games?

Solution: Let H, F and V be the sets of students who play Hockey, Football and Volleyball respectively.

Given N = Total number of students in B.Com class = 300, $n(\text{H}) = 140$, $n(\text{F}) = 115$, $n(\text{V}) = 75$, $n(\text{H} \cap \text{F}) = 30$, $n(\text{V} \cap \text{H}) = 25$, $n(\text{F} \cap \text{V}) = 25$.

Since, each student plays at least one of the three games, then, $n(\text{H} \cup \text{F} \cup \text{V}) = N = 300$

We have to calculate $n(\text{H} \cap \text{F} \cap \text{V})$

We know that,

$$n(\text{H} \cup \text{F} \cup \text{V}) = n(\text{H}) + n(\text{F}) + n(\text{V}) - n(\text{H} \cap \text{F}) - n(\text{V} \cap \text{H}) - n(\text{F} \cap \text{V}) + n(\text{H} \cap \text{F} \cap \text{V})$$

or $$300 = 140 + 115 + 75 - 30 - 25 - 25 + n(\text{H} \cap \text{F} \cap \text{V})$$

or $$300 = 250 + n(\text{H} \cap \text{F} \cap \text{V})$$

or $$n(\text{H} \cap \text{F} \cap \text{V}) = 50$$

Therefore, 50 students play all the three games.

EXAMPLE 54

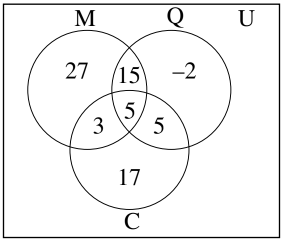
The production manager of Sen, Sarkar and Lahiri Company examined 100 items produced by the workers and furnished the following report to his boss:

Defect in measurement 50, defect in colouring 30, defect in quality 23, defect in quality and colouring 10, defect in measurement and colouring 8, defect in measurement and quality 20, and 5 are defective in all respect.

The manager was penalized for the report. Using appropriate result of set theory, explain the reason for penal measure. [***C.U. B.Com. 2002, 2006, 2011***]

Solution: Let M, C and Q be the sets of items which are defective in measurement, colouring and quality respectively.

Given $n(M) = 50$, $n(C) = 30$, $n(Q) = 23$, $n(Q \cap C) = 10$, $n(M \cap Q) = 20$, $n(M \cap C) = 8$, and $n(M \cap C \cap Q) = 5$



From the given Venn diagram, we observe that the number of items which are defective only in quality is – 2. But we know that the number of items cannot be negative. That means the information supplied by the manager was not correct. So, he was penalized.

EXAMPLE 55

It is known that in a group of people each person speaks at least one of the languages English, Hindi and Bengali; 31 speak English, 36 speak Hindi, 27 speak Bengali, 10 speak both English and Hindi, 9 both English and Bengali, 11 both Hindi and Bengali. Using a Venn diagram or otherwise, prove that the group contains at least 64 people and not more than 73 people.

[***C.U. B.Com.(H) 1983, 1990***]

Solution: Let E, H and B be the sets of people who can speak English, Hindi and Bengali respectively.

Given $n(E) = 31$, $n(H) = 36$, $n(B) = 27$, $n(E \cap H) = 10$, $n(E \cap B) = 9$, $n(H \cap B) = 11$.

Since each person of the group can speak at least one of the three languages. So, total number of people in the group = $n(E \cup H \cup B)$.

We have to calculate the least and the greatest values of $n(E \cup H \cup B)$.

We know that,

$$\begin{aligned} n(E \cup H \cup B) &= n(E) + n(H) + n(B) - n(E \cap H) - n(E \cap B) \\ &\quad - n(H \cap B) + n\,(E \cap H \cap B) \\ &= 31 + 36 + 27 - 10 - 9 - 11 + n\,(E \cap H \cap B) \\ &= 64 + n\,(E \cap H \cap B) \end{aligned}$$

As, least value of $n(E \cap H \cap B) = 0$

Therefore, the least value of $n(E \cup H \cup B) = 64$ i.e. the least number of people in the group = 64

Again, the greatest value of $n(E \cap H \cap B)$ = the minimum values of $n(E \cap H)$, $n(E \cap B)$ and $n(H \cap B)$

i.e. 10, 11 and 9, i.e. 9

Therefore, the greatest values of $n(E \cup H \cup B) = 64 + 9 = 73$ i.e. the greatest number of people in the group = 73.

EXERCISE

A. THEORETICAL

1. Define the following sets with examples:
   - (*i*) Null set
   - (*ii*) Finite and infinite sets
   - (*iii*) Singleton set
   - (*iv*) Universal set
   - (*v*) Equal sets
   - (*vi*) Equivalent sets
   - (*vii*) Subset and proper subset
   - (*viii*) Union of two sets
   - (*ix*) Intersection of two sets
   - (*x*) Complement of a set
   - (*xi*) Disjoint sets
   - (*xii*) Power set
2. Write short notes on:
   - (*i*) The three set operations (union, intersection and complementation) [***C.U. B.Com.(H), 1983***]
   - (*ii*) Ordered pair and Cartesian product. [***C.U. B.Com.(H), 1981***]
3. State De Morgan's laws of sets. Verify the laws in terms of Venn diagram. [***C.U. B.Com.(H), 1992***]
4. Define Universal set and subset. [***C.U. B.Com.(G), 2014***]
5. Define Null set with an example. [***C.U. B.Com.(H), 2014***]
6. Define subset with an example. [***C.U. B.Com.(G), 2015***]
7. Define a power set with an example. [***C.U. B.Com.(H), 2015, 2017***]
8. When are two sets A and B said to be disjoint sets? Give an example. [***C.U. B.Com.(G), 2016***]
9. Define equality of two sets. Can one of two equal sets be a proper subset of the other? [**Ans.** No]
10. Use a Venn diagram (or otherwise) prove that, for any two sets A, B

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

B. SHORT TYPE

1. Write the following sets in the roster form:
   - (*i*) $\{x : x \in N, x^2 < 20\}$ [**Ans.** $\{1, 2, 3, 4\}$]
   - (*ii*) $\{x : x \in N$ and x is a prime factor of 84$\}$ [**Ans.** $\{2, 3, 7\}$]
2. Write the following sets in the set-builder form:
   - (*i*) $\{1, 3, 5, 7, 9\}$ [**Ans.** $\{x : x$ is an odd natural number less than 10$\}$]
   - (*ii*) $\{2, 4, 8, 16\}$ [**Ans.** $\{x : x = 2^n, n \in N$ and, $n \le 4\}$]
3. If A = $\{6, 8, 10, 12\}$, then write which of the following statements are true.
   - (*i*) $6 \in A$
   - (*ii*) $8, 10 \in A$
   - (*iii*) $16 \notin A$
   - (*iv*) $\{6\} \in A$
   - (*v*) $\{4, 8\} \in A$

   [**Ans.** (*i*), (*ii*), (*iii*) are true]

4. Which of the following are examples of a null set?
   (*i*) $\{x : x^2 - 2 = 0$ and x is rational$\}$
   (*ii*) $\{x : x$ is whole numbers less then $0\}$
   (*iii*) $\{x : x \in N$ and $x^2 = 4\}$
   (*iv*) $\{x : x$ is an even prime number$\}$ [**Ans.** (*i*) and (*ii*)]

5. Which of the following sets are finite or infinite?
   (*i*) D = $\{x : x$ is the number of people living on earth$\}$
   (*ii*) W = $\{x : x$ is the time a person waits for a bus$\}$
   (*iii*) V = $\{x$ is an odd integer exceeding 889$\}$
   [***C.U. B.Com. 1985***]
   [**Ans.** D, V = infinite set and W = finite set]

6. State which of the following are null sets?
   (*i*) $\{x : 3x^2 - 4 = 0$, x is an integer$\}$
   (*ii*) $\{x : (x+3)(x+3) = 9, x$ is a real number$\}$
   (*iii*) $(A \cap B) - A$ [***C.U. B.Com, 1986***]
   [**Ans.** (*i*) & (*iii*) are null sets]

7. Write the following statements in set theoretic notations:
   (*i*) 3 is an element of a set A
   (*ii*) 4 does not belong to set B
   (*iii*) C is a subset of D
   (*iv*) P and Q are disjoint sets
   [***C.U. B.Com, 1988***] [**Ans.** (*i*) $3 \in A$ (*ii*) $4 \notin B$ (*iii*) $C \subseteq D$ (iv) $P \cap Q = \Phi$]

8. If A = $\{x, y, z\}$, then write the power set of A.
   [***C.U. B.Com, (G) 2015***]
   [**Ans.** P(A) = $\{\{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}, \{x, z\}, \{x, y, z\}, \Phi\}$]

9. If $a \in A$ and $a \in B$, does it follow $A \subseteq B$? Give reason.
   [***C.U. B.Com.(H), 1990***]
   [**Ans.** It is not always true, because other elements of set A may not belong to set B]

10. State which of the following statements is true or false:
    (*i*) $\{a\} \in \{a, b, c\}$
    (*ii*) $a \in \{a, b, c\}$
    (*iii*) $a \subset \{a, b, c\}$
    (*iv*) $a \not\subset \{a, b, c\}$
    [***C.U. B.Com, 1992***]
    [**Ans.** True: (*i*) & (*iv*) False: (*ii*) & (*iii*)]

11. Let A, B, C be three arbitrary events. Find expressions for the following events using the usual set theoretic notations:
    (*i*) Only A occurs;
    (*ii*) both A and B but not C occur;
    (*iii*) all three events occur;
    (*iv*) at least one event occurs;
    (*v*) at least two events occur.
    [***C.U.B.Com, 1985***] [Ans. (*i*) $A \cap B' \cap C'$ (*ii*) $A \cap B \cap C'$ (*iii*) $A \cap B \cap C$ (*iv*) $A \cup B \cup C$ (*v*) $(A \cap B) \cup (B \cap C) \cup (C \cap A)$]

12. Draw a Venn diagram of three non-empty sets A, B, C such that $A \subset B$, $C \subset B$, $A \cap C = \Phi$. [***C.U. B.Com. 1986***]

13. Given three arbitrary events A, B and C, express the following events in set theoretic notation.
    (*i*) At least one of A, B and C does not occur.
    (*ii*) Along with C either A or B occurs – but not both. [***C.U. B.Com. 1987***]
    [**Ans.** (*i*) $U - (A' \cup B' \cup C')$ (*ii*) $(C \cap A) \cup (C \cap B)$]

14. If A, B and C are three non-empty subsets of the Universal set S, draw a Venn diagram to illustrate the following case.
    $A \subset B, B \cap C \neq \Phi, A \cap C = \Phi$ and $C \not\subset B$. [***C.U.B.Com. 1989***]

15. List the sets $A \cup B, A \cap B, A \cap (B \cup C)$, given that $A = \{p, q, r, s\}$, $B = \{q, r, s, t\}$, $C = \{q, r, t\}$ [***C.U. B.Com. 1986***] [**Ans.** $\{p, q, r, s, t\}, \{q, r, s\}, \{q, r, s\}$

16. Given A = {1, 2, 3}, B = {2, 4}, C = {2, 3, 5}, D = {3, 4}. Find
    (*i*) $A \cap B, A \cap C$ and $(A \cap B) \cup (A \cap C)$
    (*ii*) $(B \cup C)$ and $A \cap (B \cup C)$
    Hence verify the result $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
    [***C.U. B.Com. 1988***]
    [**Ans.** $A \cap B = \{2\}$; $A \cap C = \{2, 3\}$; $(A \cap B) \cup (A \cap C) = \{2, 3\}$;
    $B \cup C = \{2, 3, 4, 5\}$; $A \cap (B \cup C) = \{2, 3\}$]

17. If A = {1, 2, 3, 4}, B = {2, 4, 5, 6}, C = {1, 3, 4, 6, 8} then find the set $A \cap (B \cup C)$.
    [***C.U. B.Com.(H) 2013***] [**Ans.** {1, 2, 3, 4}]

18. Let, S = {1, 2, 3, 4, 5, 6} be the universal set, set $A \cup B = \{2, 3, 4\}$. Find $A^c \cap B^c$ where the A^c, B^c are complements of A and B respectively.
    [**Ans.** $A^c \cap B^c = \{1, 5, 6\}$]

19. Let the sets A and B be given by A = {1, 2, 3, 4}, B = {2, 4, 6, 8, 10} and the universal set S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, Find $(A \cup B)'$ and $(A \cap B)'$.
    [***C.U. B.Com. 1991***] [**Ans.** {5, 7, 9}, {1, 3, 5, 6, 7, 8, 9, 10}]

20. For any two sets A and B, prove $(A \cup B)^c = A^c \cap B^c$
    [***C.U. B.Com.(H) 2014***]

21. For any two sets A and B, show that $(A \cap B)^c = A^c \cup B^c$
    [***C.U. B.Com.(G) 2015***]

22. If A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15} and B = {2, 4, 6, 8, 10, 12, 14, 16} find A – B and B – A. [Ans. {1, 3, 5, 7, 9, 11, 13, 15}, {10, 12, 14, 16}]
    [***C.U. B.Com. 2011***]

23. If A = {2, 3, 4, 5} and B = {1, 2, 3, 4}, Show that $B - A \neq A - B$
    [***C.U. B.Com. 2010***]

24. If A = {1, 2, 3, 4}, B = {3, 4, 5}, C = {1, 4, 5}, then verify $A - (B \cup C) = (A - B) \cap (A - C)$. [***C.U. B.Com. 2012***]

25. If three sets A, B, C be such that, A – B = {2, 4, 6}, A – C = {2, 3, 5}, then find $A - (B \cup C)$ and $A - (B \cap C)$.
    [***C.U. B.Com. 1997, 2013 (G), 2015(H)***] [**Ans.** {2}, {2, 3, 4, 5, 6}]

26. If the sets A = $\{p, q, r, s\}$, B = $\{r, s, t\}$, C = $\{p, s, t\}$, then prove that A – (B ∪ C) = (A – B) ∩ (A – C) [***C.U.B.Com.(G) 2016***]

27. Find A – (B ∪ C) and A – (B ∩ C) for the sets A = {1, 2, 3, 4}, B = {2, 4, 5, 6} and C = {3, 4, 6}. [***C.U. B.Com.(G) 2014***]
[**Ans.** {A – (B ∪ C) = {1}; A – (B ∩ C) = {1, 2, 3}]

28. If A = {1, 4}, B = {4, 5}, C = {5, 7}, find (A × B) ∪ (A × C)
[***C.U. B.Com.(H) 2016***]
[**Ans.** {(1, 4), (1, 5), (1, 7), (4, 4), (4, 5), (4, 7)}]

29. Given A = {1, 3}, B = {3, 5} and C = {5, 10} verify the following relations.
(*i*) A × B ≠ B × A
(*ii*) A × (B ∪ C) = (A × B) ∪ (A × C)
(*iii*) A × (B ∩ C) = (A × B) ∩ (A × C)
[***C.U. B.Com. 1988***]

30. If A = $\{x : x$ is an integer and $1 \le x \le 10\}$ and B = $\{x : x$ is a multiple of 3 and $5 \le x \le 30\}$, find A ∩ B.
[***C.U. B. Com.(H) 2016***] [**Ans.** {6, 3}]

31. Find sets A, B and C such that A ∩ B, B ∩ C and A ∩ C are non-empty sets and A ∩ B ∩ C = Φ.

32. If a set A has 4 elements and a set B has 6 elements what can be the minimum number of elements in the set (A ∪ B)? What conclusion can then be drawn about A, B? [***C.U. B.Com. 1995***]
[**Ans.** 6, A ⊂ B (A is proper subset of B)]

33. Which of the following sets are equal?
(*i*) A = {1, 2, 3}
(*ii*) $B = \{x \in R : x^2 - 2x + 1 = 0\}$
(*iii*) C = {1, 2, 2, 3}
(*iv*) $D = \{x \in R : x^3 - 6x^2 + 11x - 6 = 0\}$
[**Ans.** A = C = D]

34. From the sets given below, select equal sets and equivalent sets.
A = $\{0, a\}$, B = {1, 2, 3, 4}, C = {4, 8, 12}, D = {3, 1, 2, 4}, E = {1, 0}, F = {8, 4, 12}, G = {1, 5, 7, 11}, H = $\{a, b\}$
[**Ans.** Equal sets: B = D, C = F, Equivalent sets: A, E, H ; B, D, G ; C, F]

35. Are the following pair of sets equal? Give reason.
A = $\{x : x$ is a letter of the word "WOLF"}
B = $\{x : x$ is a letter of the word "FOLLOW"} [**Ans.** A = B]

36. If A and B are two sets such that n(A) = 45, n(A ∪ B) = 76 and n(A ∩ B) = 12, find n(B). [**Ans.** 43]

37. If A and B are two sets such that n(A) = 17, n(B) = 23 and n(A ∪ B) = 38, find n(A ∩ B) and n(A – B). [**Ans.** 2 and 15]

38. If A and B are two sets and U is the universal set such that $n(U) = 700$, $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$. Find $n(A' \cap B')$. [**Ans.** 300]
39. If A = {1, 2, 3, 4, 5, 6, 7, 8} and B = {1, 3, 5, 6, 7, 8, 9}, then find the value of $A \Delta B$. [**Ans.** {2, 4, 9}]
40. If A = {x : x is a natural number and $x \le 6$}, B = {x : x is the natural number and $3 \le x \le 8$}, find A – B and $A \cap B$.
[***C.U. B.Com.(G), 2017***] [**Ans.** {1, 2}, {3, 4, 5, 6}]

C. SHORT ESSAY TYPE

41. Write the following sets in roster form.
(*i*) The set of all letters in the word 'TRIGONOMETRY'.
(*ii*) The set of odd integers lying between –4 and 8.
(*iii*) The set of squares of integers.
(*iv*) $\{x : x \in Z, x^2 < 20\}$
(*v*) $\{x : x \in N \text{ and } 4x - 3 \le 15\}$
(*vi*) $\left\{x : x = \dfrac{n}{n^2+1} \text{ and } 1 \le n \le 3, \text{where } n \in N\right\}$

[**Ans.** (*i*) {T, R, I, G, O, N, M, E, Y} (*ii*) {–3, –1, 1, 3, 5, 7} (*iii*) {0, 1, 4, 9, 16, ...}
(*iv*) {–4, –3, –2, –1, 0, 1, 2, 3, 4} (*v*) {1, 2, 3, 4} (*vi*) $\left\{\dfrac{1}{2}, \dfrac{2}{5}, \dfrac{3}{10}\right\}$]

42. Write the following sets in set-builder form.
(*i*) {1, 4, 9, 16, ..., 100}
(*ii*) {5, 25, 125, 625}
(*iii*) {0}
(*iv*) $\left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \dfrac{1}{5}, - - -\right\}$
(*v*) {2, 5}
(*vi*) {1, 2, 5, 10}

[**Ans.** (*i*) $\{x^2 : x \in N, 1 \le x \le 10\}$ (*ii*) $\{5^n : n \in N, 1 \le n \le 4\}$ (*iii*) $\{x : x = 0\}$
(*iv*) $\left\{x : x = \dfrac{1}{n}, x \in N\right\}$ (*v*) {x : x is a prime number and a divisor of 10}
(*vi*) {x : x is a natural number and divisor of 10}]

43. Which of the following are examples of a null set?
(*i*) {x : x is a natural number, $x < 8$ and simultaneously $x > 12$}
(*ii*) {x : x is a point common to any two parallel lines}
(*iii*) $\{x : x \in N, x < 5 \text{ and } x > 8\}$
(*iv*) {x : x is the root of the equation $x^2 + 4x + 5 = 0$}
(*v*) {x : x is an odd integer and divisible by 2}
(*vi*) {x : x is an integer and $1 < x \le 2$}

[**Ans.** (*i*), (*ii*), (*iii*) and (*v*) are null sets]

44. Classify the following sets into finite set and infinite set. In case of finite sets, mention the cardinal number.
    (*i*) $\{x : x \in \text{N and } x \text{ is a factor of } 84\}$
    (*ii*) $\{x : x \in \text{W}, x \text{ is divisible by 4 and 9}\}$
    (*iii*) $\{x : x \in \text{W}, 3x - 7 \leq 8\}$
    (*iv*) $\{x : x = 5n, n \in \text{I and } x < 20\}$
    (*v*) $\left\{x : x = \dfrac{n}{n+1}, n \in \text{W and } n \leq 10\right\}$
    (*vi*) $\left\{x : x = \dfrac{2n}{n+3}, n \in \text{N and } 5 < n < 20\right\}$

[**Ans.** (*i*) finite; 12 (*ii*) infinite (*iii*) finite ; 6 (*iv*) infinite (*v*) finite; 11 (*vi*) finite; 14]

45. For any two sets A and B, prove without using Venn diagram, that $(A \cap B)' = A' \cup B'$ [Dash denotes complement] [***C.U. B.Com. 1999, 2008, 2011***]

46. Without using Venn diagram for any three sets A, B and C, prove that $A - (B \cup C) = (A - B) \cap (A - C)$. [***C.U. B.Com. 2004, 2011, 2013 (G)***]

47. For two non-empty sets A and B prove that, $A^c \cap B^c = (A \cup B)^c$ where A^c is the complement set of 'A'. [***C.U. B.Com. 2009, 2011***]

48. Prove for three sets of A, B, C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. [***C.U. B.Com.(G) 2014***]

49. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. [***C.U. B.Com. 1998, 2001, 2014 (H)***]

50. For any three sets A, B, C, prove that $A - (B \cap C) = (A - B) \cup (A - C)$ [***C.U. B.Com. (H), 2016***]

51. If A = {1, 2, 7} and B = {3, 5, 7} are the subsets of the universal set S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} then show that $(A \cup B)' = A' \cap B'$.
[***C.U. B.Com. 2005***] [**Ans.** $(A \cup B)' = A' \cap B'$ = {4, 6, 8, 9, 10}]

52. If $A \cup B = \{a, b, c, d\}$, $A \cap B = \{b, c\}$, $A \cup C = \{a, b, c, f\}$ and $A \cap C = \{a, b\}$, then find A, B and C. [***C.U. B.Com. 2005***]
[**Ans.** A = {*a*, *b*, *c*}, B = {*b*, *c*, *d*}, C = {*a*, *b*, *f*}]

53. Using set operations find the H.C.F of the numbers
    (*i*) 30, 105, 165 [***C.U. B.Com. 2005***] [**Ans.** 15]
    (*ii*) 21, 45 and 105 [***C.U. B.Com. 2007***] [**Ans.** 3]
    (*iii*) 15, 40, 105 [***C.U. B.Com. (H) 2016***] [**Ans.** 5]

54. Using set theory, find the L.C.M and H.C.F of 4, 6 and 8.
[***C.U. B.Com.(H), 2015***] [**Ans.** 24, 2]

55. If A = {1, 4}, B = {2, 3} and C = {3, 5}, then prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$
[***C.U. B.Com. 2006***]

56. Given A = {2, 3, 8}, B = {6, 4, 3}, find A × B. [*C.U. B.Com. 2001*]
[**Ans.** {(2, 6), (2, 4), (2, 3), (3, 6), (3, 4), (3, 3), (8, 6), (8, 4), (8, 3)}]
57. For any three sets A, B and C, prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$ [*C.U. B.Com.(H) 2015*]
58. If A = {1, 2, 3}, B {2, 3, 4}, S = {1, 3, 4}, T = {2, 4, 5}, verify that $(A \times B) \cap (S \times T) = (A \cap S) \times (B \cap T)$ [*C.A. Ent. June 1981*]
59. Applying set operations prove that, 4 + 5 = 9.
60. If L = {1, 2, 3, 4}, M = {3, 4, 5, 6} and N = {1, 3, 5}, then verify that $L - (M \cup N) = (L - M) \cap (L - N)$.
61. For any two sets A and B prove by using properties of sets that
    (*i*) $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$
    (*ii*) $(A \cup B) - A = B - A$
    (*iii*) $(A \cap B) \cup (A - B) = A$
62. For sets A, B and C using properties of sets, prove that $A - (B - C) = (A - B) \cup (A \cap C)$.
63. Let A, B and C be three sets such that $A \cup B = C$ and $A \cap B = \Phi$. Then, prove that $A = C - B$.
64. Let A, B and C be the sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Show that B = C.
65. Let A and B be sets, if $A \cap X = B \cap X = \Phi$ and $A \cup X = B \cup X$ for same set X, prove that A = B.

D. PRACTICAL

66. In a competition, a school awarded medals in different categories. 36 medals in dance, 12 medals in dramatics and 18 medals in music. If these medals went to total of 45 persons and only 4 persons got medals in all the three categories, how many received medals in exactly two of these categories? [**Ans.** 3]
67. In a group of 100 persons, 72 people can speak English and 43 can speak French. How many can speak English only? How many can speak French only and how many can speak both English and French? [**Ans.** 57, 28, 15]
68. In a class of 40 students, 15 like to play cricket and football and 20 like to play cricket. How many like to play football only but not cricket? [**Ans.** 20]
69. There is a group of 80 persons who can drive scooter or car or both. Out of these 35 can drive scooter and 60 can drive car. Find out how many can drive both scooter and car? How many can drive scooter only? How many can drive car only? [**Ans.** 15, 20, 45]
70. It was found that out of 45 girls, 10 joined singing but not dancing and 24 joined singing. How many joined dancing but not singing? How many joined both? [**Ans.** 21, 14]

71. In a class of 100 students, 55 students read History, 41 students read philosophy and 25 students read both the subjects. Find the number of students who study neither of the two subjects. [***C.U. B.Com. 2005***] [Ans. 29]

72. In an Engineering college 80 students get chance for Computer Science, 75 for Information Technology, 72 for Electronics, If 60 students get chance in 1^{st} and 2^{nd} , 50 in 2^{nd} and 3^{rd} , 40 in 1^{st} and 3^{rd} and 30 get chance in all 3 branches, how many seats are there in the Engineering college. [The college has only 3 disciplines.] [***C.U. B.Com. 2001***] [**Ans.** 107]

73. In a survey of 100 families, the number of families that read the most recent issues of various magazines were found to be : India Today 42, Sunday 30, New Delhi 28, India Today and Sunday 10, India Today and New Delhi 5, New Delhi and Sunday 8; all the three magazines 3. Find the number of families that read none of the magazines. [***C.U. B.Com 2000***] [**Ans.** 20]

74. In a survey of 150 students it was found that 40 students studied Economics, 50 students studied Mathematics, 60 students studied Accountancy and 15 students studied all three subjects. It was also found that 27 students studied Economics and Accountancy, 35 students studied Accountancy and Mathematics and 25 students studied Economics and Mathematics. Find the number who studied only Economics and the number who studied none of these subjects.
[***C.U. B.Com 1994***] [**Ans.** 3, 72]

75. In a group of people 31 speak English, 36 speak Hindi, 27 speak Bengali, 10 speak both English and Hindi, 9 both English and Bengali, 11 both Hindi and Bengali. Prove that the group contains at least 64 people.
[***C.U. B.Com. 1993***]

76. A company studies the product preferences of 300 consumers. It is found that 226 like product A, 51 like product B, 54 like product C, 21 like products A and B, 54 like products A and C, 39 like products B and C and 9 like all the three products. Prove that the study results are not correct. [Assume that each consumer likes at least one of the three products.] [***C.U. B.Com 1987***]
[**Ans.** $n(A \cup B \cup C) = 226 < 300$]

77. In a city three daily newspapers X, Y, Z are published; 65% of the citizens read X, 54% read Y and 45% read Z; 38% read X and Y; 32% read Y and Z; 28% read X and Z; 12% do not read any of these papers. If the total number of people in the city be 10,00,000, find the total number of citizens who read all the three newspapers. [You may use a Venn diagram of a standard formula for the enumeration of elements of sets].
[***C.U. B.Com. 1986***] [**Ans.** 2, 20, 000]

78. A school awarded 58 medals for Honesty, 20 for Punctuality and 25 for Obedience. If these medals were bagged by a total of 78 students and only 5 students got medals for all the three values, find the number of students who received medals for exactly two of the three values. [**Ans.** 15]

E. ADVANCED PROBLEMS

79. If U = $\{x : x \in N, x \leq 30\}$, A = $\{x : x$ is prime $< 5\}$, B = $\{x : x$ is a perfect square $\leq 10\}$ and C = $\{x : x$ is a perfect cube ≤ 30 , then verify the following results.
 (*i*) $(A \cup B)' = A' \cap B'$
 (*ii*) $(A \cap B)' = A' \cup B'$
 (*iii*) $(A \cap B) \cap C = A \cap (B \cap C)$
 (*iv*) $A' - B' = B - A$

80. If U = $\{x : x \in N, x \leq 10\}$, A = $\{x : x$ is prime$\}$ and B = $\{x : x$ is a factor of 24$\}$, verify the following results.
 (*i*) $A - B = A \cap B'$
 (*ii*) $(A \cup B)' = A' \cap B'$
 (*iii*) $(A \cap B)' = A' \cup B'$

81. Let x = the set of all letters in the word 'NEW DELHI' and Y = the set of all the letters in the word 'CHANDIGARH'. Find
 (*i*) $X \cup Y$ (*ii*) $X \cap Y$ (*iii*) $X - Y$

 Also verify that
 (*a*) $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$ and
 (*b*) $n(X - Y) = n(X \cup Y) - n(Y) = n(X) - n(X \cap Y)$

82. Find sets A, B and C such that $A \cap B$, $B \cap C$ and $A \cap C$ are non-empty sets and $A \cap B \cap C = \Phi$.

83. Let A and B be sets, if $A \cap X = B \cap X = \Phi$ and $A \cup X = B \cup X$ for same set X, prove that A = B.

84. For sets A, B and C using properties of sets, prove that:
 (*i*) $A - (B \cup C) = (A - B) \cap (A - C)$
 (*ii*) $(A \cap B) - C = (A - C) \cap (B - C)$

85. Let A and B be two sets and U be the universal set such as $n(A) = 25$, $n(B) = 28$ and $n(U) = 50$. Find (*i*) the greatest value of $n(A \cup B)$ (*ii*) the least value of $n(A \cap B)$ [**Ans.** 50, 3]

86. Out of 500 car owners investigated, 400 owned Maruti cars and 200 owned Hyundai cars; 50 owned both cars. Is this data correct? [**Ans.** No.]

87. A survey shows that 63% of the Indians like cheese, whereas 76% like apples. If x% of the Indians like both cheese and apples, find the value of x.
 [**Ans.** $39 \leq x \leq 63$]

88. From 50 students taking examination in Mathematics, Physics and Chemistry, 37 passed Mathematics, 24 Physics and 43 Chemistry. At least 19 passed Mathematics and Physics; at least 29 passed Mathematics and Chemistry and at least 20 passed Physics and Chemistry. If each student has passed in at least one of the subjects, find the largest number of students who could have passed in all the three subjects. [**Ans.** 14]

89. If $A \subset B$, then show that $C - B \subset C - A$

90. If A, B and C are three sets such that $A \cup B = C$ and $A \cap B = \Phi$, then show that $A = C - B$.

F. MULTIPLE CHOICE QUESTIONS (MCQs)

(i) Short Type

Mark the correct alternative in each of the following:

1. If A = {3, 5, 7, 9} and B = {2, 4, 6}, then
   (*a*) $4 \in A$ (*b*) $\{4\} \subset A$ (*c*) $B \subset A$ (*d*) $4 \in B$
   [**Ans.** (*d*)]

2. If $A \cap B = B$, then
   (*a*) $A \subseteq B$ (*b*) $B \subseteq A$ (*c*) $A = \Phi$ (*d*) $B = \Phi$
   [**Ans.** (*b*)]

3. For any set A, $(A')'$ is equal to
   (*a*) A' (*b*) A (*c*) Φ (*d*) U
   [**Ans.** (*b*)]

4. The number of subsets of a set containing *n* elements is
   (*a*) n (*b*) $2^n - 1$ (*c*) n^2 (*d*) 2^n
   [**Ans.** (*d*)]

5. The null set is represented by:
   (*a*) $\{\Phi\}$ (*b*) $\{0\}$ (*c*) Φ (*d*) 0
   [**Ans.** (*c*)]

6. If A = {2}, which of the following statement is correct?
   (*a*) A = 2 (*b*) $2 \subset A$ (*c*) $\{2\} \in A$ (*d*) $2 \in A$
   [**Ans.** (*d*)]

7. Which of the following is a correct statement?
   (*a*) $\Phi = 0$ (*b*) $\Phi = \{0\}$ (*c*) $\Phi = \{\Phi\}$ (*d*) $\Phi = \{\ \}$
   [**Ans.** (*d*)]

8. If $A \cup B = \Phi$, then:
   (*a*) $A = \Phi$ or $B = \Phi$ (*c*) $A = \Phi$ and $B \neq \Phi$
   (*b*) $A = \Phi$ and $B \neq \Phi$ (*d*) $A \neq \Phi$ and $B = \Phi$
   [**Ans.** (*b*)]

9. For any set A, $A \cap A$ is
   (*a*) A (*b*) Φ (*c*) U (*d*) A'
   [**Ans.** (*a*)]

10. If $A \cap B = \Phi$, then
    (*a*) $A - B = \Phi$ (*c*) $B - A = \Phi$
    (*b*) $A - B = A$ (*d*) None of these [**Ans.** (*b*)]

11. If U be the Universal set, then U^c =
    (*a*) U (*c*) Φ^c
    (*b*) Φ (*d*) None of these [**Ans.** (*b*)]

12. The number of subsets of the set {1, 2, 3} is:
    (*a*) 8 (*b*) 5 (*c*) 6 (*d*) 3
    [**Ans.** (*a*)]

13. Let A and B be two sets in the same universal set. Then, A – B is equal to
(*a*) $A \cap B$ (*b*) $A' \cap B$ (*c*) $A \cap B'$ (*d*) $A' \cap B'$
[**Ans.** (*c*)]

14. If three sets A, B, C be such that, A – B = {2, 4, 6}, A – C = {2, 3, 5}, then A – (B ∪ C) is
(*a*) {2, 3, 4, 5, 6} (*c*) {2, 4, 6}
(*b*) {2, 3, 5} (*d*) {2}
[***C.U.B.Com. (H) 2015***] [**Ans.** (*d*)]

15. Which of the following statement is false?
(*a*) $A - B = A \cap B'$ (*c*) $A - B = A - B'$
(*b*) $A - B = A - (A \cap B)$ (*d*) $A - B = (A \cup B) - B$
[**Ans.** (*c*)]

16. If A = {2, 3, 4, 5}, then the number of proper subsets of A is
(*a*) 16 (*b*) 15 (*c*) 17 (*d*) 18
[**Ans.** (*b*)]

17. Let A and B be two sets such that $n(A) = 8$, $n(B) = 7$, $n(A \cup B) = 13$. Then, $n(A \cap B)$ is equal to
(*a*) 20 (*b*) 28 (*c*) 15 (*d*) 2
[**Ans.** (*d*)]

18. In a set-builder method the null set is represented by
(*a*) Φ (*c*) { }
(*b*) $\{x : x \neq x\}$ (*d*) $\{x : x = x\}$
[**Ans.** (*b*)]

19. For any two sets A and B, $A \cap (A \cup B)'$ is equal to
(*a*) $A \cap B$ (*b*) Φ (*c*) B (*d*) A
[**Ans.** (*b*)]

20. Correct statement among $1 \subset \{1, 3, 4\}$, $\{1, 3\} \in \{1, 3, 4\}$ and $\{1, 4\} \subset \{1, 3, 4\}$ is:
(*a*) $1 \subset \{1, 3, 4\}$ (*c*) $\{1, 4\} \subset \{1, 3, 4\}$
(*b*) $\{1, 3\} \in \{1, 3, 4\}$ (*d*) None of these
[**Ans.** (*c*)]

21. The empty set is one which contains ________ element.
(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 0
[**Ans.** (*d*)]

22. The number of proper subsets of a set containing *n* element is
(*a*) 2^n (*b*) 2^{n-1} (*c*) $2^n - 1$ (*d*) n^2
[**Ans.** (*c*)]

23. If A = {1, 2, 3, 4, 5} and B = {2, 4, 6, 8, 10}, then $A \cap B$ is equal to
(*a*) {2, 4} (*c*) {4, 5, 6}
(*b*) {2, 4, 6} (*d*) {6, 8, 10}
[**Ans.** (*a*)]

24. If A = {2, 4} and B = {4, 5}, then A ∪ B is equal to
(*a*) {4} (*c*) {2, 4, 5}
(*b*) {2, 5} (*d*) None of these
[**Ans.** (*c*)]

25. If A = {x : x ∈ N, $x < 5$}, then the cardinal number of set A is equal to
(*a*) 5 (*b*) 6 (*c*) 4 (*d*) 3
[**Ans.** (*c*)]

26. If the number of elements in a set is 2, then number of subsets =
(*a*) 2 (*b*) 3 (*c*) 5 (*d*) 4
[**Ans.** (*d*)]

27. If the number of elements in a set is 3, then number of proper subsets =
(*a*) 8 (*b*) 7 (*c*) 6 (*d*) 5
[**Ans.** (*b*)]

28. Intersection of any set A with the null set is always ________ set.
(*a*) A (*b*) Φ (*c*) A′ (*d*) Φ′
[**Ans.** (*b*)]

29. A set and its complement are __________ sets.
(*a*) Equal (*c*) Disjoint
(*b*) Equivalent (*d*) Complement
[**Ans.** (*c*)]

30. The cardinal number of the set P = {x : x is a factor of 20} is
(*a*) 7 (*b*) 4 (*c*) 5 (*d*) 6
[**Ans.** (*d*)]

(ii) Short Essay Type

1. If the set A has p elements and B has q elements, then the number of elements in A × B is
(*a*) pq (*c*) p^2
(*b*) $p + q$ (*d*) $p + q + 1$ [**Ans.** (*a*)]

2. The smallest set A such that A∪ {1, 2} = {1, 2, 3, 5, 9} is
(*a*) {1, 2, 5, 9} (*c*) {2, 3, 5}
(*b*) {3, 5, 9} (*d*) None of these [**Ans.** (*b*)]

3. If A = {a, b], B = {c, d}, C= {d, e}, then { (a, c), (a, d), (a, e), (b, c), (b, d), (b, e)} is equal to
(*a*) A × (B ∩ C) (*c*) A × (B ∪ C)
(*b*) A ∩ (B ∪ C) (*d*) A ∪ (B ∩ C) [**Ans.** (*c*)]

4. Let A and B be two sets, then
(*a*) A ∩ B = A ∪ B (*c*) A ∪ B ⊆ A ∩ B
(*b*) A ∩ B ⊆ A ∪ B (*d*) None of these [**Ans.** (*b*)]

5. Let $A = \{(x, y) : y = e^x, x \in R\}$, $B = \{(x, y): y = e^{-x}, x \in R\}$, then
(*a*) A ∪ B = R^2 (*b*) A ∩ B ≠ Φ

(*c*) $A \cap B = \Phi$ (*d*) None of these [**Ans.** (*b*)]

6. If A, B, C be three sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then
(*a*) A = C (*c*) A = B
(*b*) B = C (*d*) A = B = C [**Ans.** (*b*)]

7. Two finite sets have n and m elements. The number of elements in the power set of first set is 48 more than the total number of elements in power set of the second set. Then the values of m and n are
(*a*) 6, 4 (*c*) 7, 4
(*b*) 6, 3 (*d*) 7, 6 [**Ans.** (*a*)]

8. In a class of 200 students, 70 played cricket, 60 played hockey and 80 played football. 30 played cricket and football, 30 played hockey and football, 40 played cricket and hockey. Find the maximum number of people playing all three games and also the minimum number of people playing at least one game.
(*a*) 60, 30 (*c*) 200, 100
(*b*) 30, 120 (*d*) 30, 110 [**Ans.** (*d*)]

9. In a recent survey conducted by cable T.V., among the people who watch DD, ZEE and STAR T.V., it is found that 80% of the people watched DD, 22% watched STAR T.V., and 15% watched ZEE. What is the maximum percentage of people, who can watch all the three channels?
(*a*) 15% (*c*) 9.5%
(*b*) 12.5% (*d*) 8.5% [**Ans.** (*a*)]

10. In a club, all the members are free to vote for one, two or three of the candidates. 20% of the members did not vote, 38% of the total members voted for at least 2 candidates. What % of the members voted for either 1 or 3 candidates, if 10% of the total members voted for all 3 candidates?
(*a*) 36% (*c*) 44%
(*b*) 40% (*d*) None of these [**Ans.** (*d*)]

11. In a competition, a school awarded medals in different categories. 36 medals in dance, 12 medals in dramatics and 18 medals in music. If these medals went to a total of 45 persons and only 4 persons got medals in all the three categories, how many received medals in exactly two of these categories?
(*a*) 2 (*c*) 3
(*b*) 4 (*d*) 5 [**Ans.** (*c*)]

12. In a group of 60 people, 27 like cold drinks and 42 like hot drinks and each person likes at least one of the two drinks. How many like both cold drinks and hot drinks?
(*a*) 7 (*c*) 6
(*b*) 9 (*d*) 8 [**Ans.** (*b*)]

13. If $n(A - B) = 18$, $n(A \cup B) = 70$ and $n(A \cap B) = 25$, then find $n(B)$
(*a*) 36 (*c*) 48
(*b*) 56 (*d*) 52 [**Ans.** (*d*)]

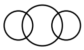
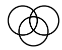
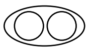
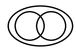
14. In a group, there were 115 people whose proofs of identity were being verified. Some had passport, some had voter ID and some had both. If 65 had passport and 30 had both, how many had voter ID only and not passport?

(*a*) 80 (*c*) 30
(*b*) 50 (*d*) None of these **[Ans.** (*d*)]

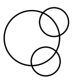
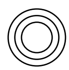
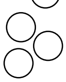
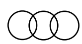
15. Of 20 adults, 5 belong to A, 7 belong to B, and 9 belong to C. If 2 belong to all three organisations and 3 belong to exactly 2 organisations, how many belong to none of these organizations?

(*a*) 6 (*c*) 7
(*b*) 8 (*d*) 5 **[Ans.** (*a*)]

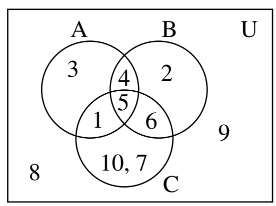
16. Which of the following diagrams indicates the best relation between Profit, Dividend and Bonus?

(*a*)  (*c*) 
(*b*)  (*d*)  **[Ans.** (*b*)]

17. Which of the following diagrams indicates the best relation between Hockey, Football and Cricket?

(*a*)  (*c*) 
(*b*)  (*d*)  **[Ans.** (*b*)]

18. From the adjoining Venn diagram, find the following sets

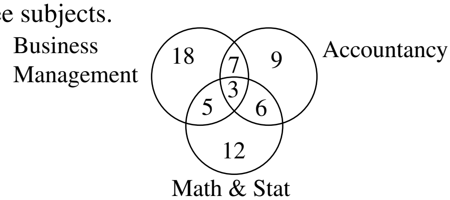


(*i*) A – B
(*a*) {3}
(*b*) {3, 1}
(*c*) {1, 3, 4, 5}
(*d*) {2, 6} **[Ans.** (*b*)]

(*ii*) (B ∪ C)′
(*a*) {3, 8, 9} (*c*) {3, 4, 5, 1}
(*b*) {8, 9} (*d*) {3, 4, 5, 1, 8, 9} **[Ans.** (*a*)]

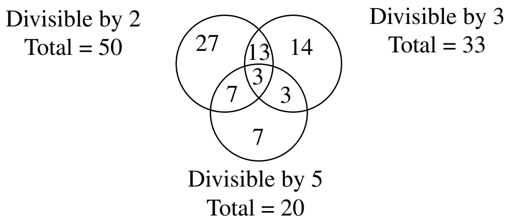
(*iii*) (A ∪ B) ∩ C
(*a*) {10, 7} (*c*) {1, 5, 6}
(*b*) {1, 2, 3, 4, 5, 6} (*d*) {1, 4, 5, 6} **[Ans.** (*c*)]

19. From the adjoining Venn diagram, find the number of students studying only one of the three subjects.



(*a*) 21 (*c*) 42
(*b*) 30 (*d*) 39 **[Ans.** (*d*)]

20. From the adjoining Venn diagram, find the number lies between 1 and 100 that are not divisible by 2, 3 and 5.



(*a*) 26
(*b*) 48
(*c*) 76
(*d*) 24

[**Ans.** (*d*)]

CHAPTER

3

Binomial Theorem

SYLLABUS

Statement of the Theorem for Positive Integral Index, General Term, Middle Term, Simple Properties of Binomial Coefficients

THEMATIC FOCUS

3.1 Introduction
3.2 Binomial Theorem for a Positive Integral Index
3.3 Properties of Binomial Expansion $(x + a)^n$
3.4 Applications of the Binomial Theorem
3.5 Some Particular Cases
3.6 Properties of the Binomial Coefficients
3.7 General Term
3.8 Middle Term
3.9 The 'r'th Term from the End
3.10 Equidistant Terms

3.1 INTRODUCTION

An expression having two terms, connected by + or – sign is called a **binomial expression**, e.g. $x + a$, $3x + 4y$, $5x - \frac{3}{4y}$, $\frac{7}{x} - \frac{5}{x^3}$, etc. The formula by which any power of binomial expression can be expanded in the form of a series is known as **binomial theorem**. The binomial theorem helps us to expand binomial expression to any given power without direct multiplication more easily and conveniently.

It can be explained by the following examples.

When we expand $(x + a)^2$ and $(x + a)^3$ by direct multiplication, we get

$$(x+a)^2 = (x+a)(x+a) = x^2 + 2xa + a^2$$

$$(x+a)^3 = (x+a)(x+a)^2 = (x+a)\left(x^2 + 2xa + a^2\right) = x^3 + 3x^2a + 3xa^2 + a^3$$

Here, we observe that the expression of $(x + a)^2$ is simple. We get the result by just multiplying $(x + a)$ by $(x + a)$. Expansion of $(x + a)^3$ is little tougher, here we get the result by multiplying $(x + a)$ by $(x + a)^2$. But when the expansion is raised to the power of ten or more, we find that this method becomes very tedious. Values of such binomial expressions can be obtained by the use of a general formula. Such a formula is known as binomial theorem. The theorem was invented by Sir Issac Newton in 1676.

3.2 BINOMIAL THEOREM FOR A POSITIVE INTEGRAL INDEX

3.2.1 Statement of the Theorem

Let us define 'x' as the first term, 'a' as the second term and 'n' as the exponent.

When n is a positive integer, then

$$(x+a)^n = x^n + {}^nC_1\, x^{n-1}\cdot a + {}^nC_2\, x^{n-2}a^2 + \cdots + {}^nC_r\, x^{n-r}\cdot a^r + \cdots + a^n$$

for all values of x and a.

[This formula is known as Binomial Theorem.]

Proof: We prove it by method of induction.

Step I: By direct multiplication, we get

$$(x+a)^2 = x^2 + 2xa + a^2 = x^2 + {}^2c_1 x^{2-1}\cdot a^1 + a^2$$

and $$(x+a)^3 = x^3 + 3x^2a + 3xa^2 + a^3 = x^3 + {}^3c_1 x^{3-1}\cdot a^1 + {}^3c_2 x^{3-2}\cdot a^2 + a^3$$

Thus, the theorem is true for $n = 2$ and 3.

Step II: Let us assume the theorem to be true for any definite positive integer m. Then, we get

$$(x+a)^m = x^m + {}^mC_1 x^{m-1}\cdot a^1 + {}^mC_2 x^{m-2}\cdot a^2 + \cdots + {}^mC_r x^{m-r}\cdot a^r + \cdots + a^m \quad ...(i)$$

Step III: Now multiplying both sides of (1) by $(x + a)$ we get,

$$(x+a)^{m+1} = (x+a)\begin{bmatrix} x^m + {}^mC_1 x^{m-1}\cdot a^1 + {}^mC_2 x^{m-2}\cdot a^2 \\ + \cdots + {}^mC_r x^{m-r}\cdot a^r + \cdots + a^m \end{bmatrix}$$

$$= \begin{pmatrix} x^{m+1} + {}^mC_1 x^m \cdot a + {}^mC_2 x^{m-1}\cdot a^2 + \cdots + {}^mC_r x^{m-r+1}\cdot a^r \\ + \cdots + {}^mC_{m-1}\cdot x^2\cdot a^{m-1} + x\cdot a^m \end{pmatrix}$$

$$+\begin{pmatrix} a\cdot x^m + {}^mC_1x^{m-1}\cdot a^2 + {}^mC_2x^{m-2}\cdot a^3 + \cdots + {}^mC_{r-1}\cdot x^{m-r+1}\cdot a^r \\ + \cdots + {}^mC_{m-1}\cdot x\cdot a^m + a^{m+1} \end{pmatrix}$$

$$= x^{m+1} + \left({}^mC_1 + 1\right)x^m\cdot a + \left({}^mC_2 + {}^mC_1\right)x^{m-1}\cdot a^2$$

$$+ \cdots + \left({}^mC_r + {}^mC_{r-1}\right)x^{m-r+1}\cdot a^r + \cdots + \left(1 + {}^mC_{m-1}\right)x\cdot a^m + a^{m+1}$$

$$= x^{m+1} + {}^{m+1}C_1x^m\cdot a + {}^{m+1}C_2x^{m-1}\cdot a^2 + \cdots + {}^{m+1}C_rx^{m+1-r}\cdot a^r$$

$$+ \cdots + {}^{m+1}C_mx\cdot a^m + a^{m+1} \quad .....(ii)$$

$$\Big[\because \quad {}^mC_r + {}^mC_{r-1} = {}^{m+1}C_r \qquad \text{for } r = 1, 2, 3, \cdots, m$$

i.e., $${}^mC_1 + 1 = {}^mC_1 + {}^mc_0 = {}^{m+1}C_1;\ {}^mC_2 + {}^mC_1$$

$$= {}^{m+1}C_2; \cdots 1 + {}^mC_{m-1} = {}^mC_m + {}^mC_{m-1} = {}^{m+1}C_m\Big]$$

Thus the expansion for $(x + a)^{m+1}$ is obtained from that of $(x + a)^m$ by replacing m by $m + 1$.

Now if the theorem is true for $n = m$, it is also true for $n = m + 1$.

Step IV: We have seen in Step-I that the theorem is true for $n = 3$, therefore it must be true for $n = 3 + 1 = 4$ by Step-III. Again, if the theorem is true for $n = 4$, it must be true for $n = 4 + 1 = 5$, and so on.

Therefore, the theorem is true for all positive integral values of n.

3.3 PROPERTIES OF BINOMIAL EXPANSION $(x + a)^n$

- The first term and last term of the expansion $(x + a)^n$ are x^n and a^n.
- The total number of terms in the expansion of $(x + a)^n$ is $n + 1$, i.e. one more than the exponent n.
- The sum of the exponents (indices) of 'x' and 'a' in any terms is equal to n.
- The exponent of 'x' decreases by 1 from 'n' to '0'.
- The exponent of 'a' increases by 1 from '0' to 'n'.
- The coefficients in the expansion follow a certain pattern known as Pascal's triangle.
- The coefficient of the second term and the second from the last term is n.
- The coefficients of the expansion are symmetric.

3.4 APPLICATIONS OF THE BINOMIAL THEOREM

The main application of the binomial theorem is to help us to expand a binomial expression to any given power without direct multiplication more easily and conveniently. Other than that, the binomial theorem is also used in:

(*a*) probability theory for probabilistic analyses;
(*b*) higher mathematics for solving problems in algebra, calculus, combinatorics and many other areas;
(*c*) scientific research for solving impossible equations (e.g. Einstein equations);
(*d*) forecast services such as weather forecast, disaster forecast, etc;
(*e*) in architecture to giving shape and determining the areas of infrastructure; and
(*f*) giving ranks to the candidate.

3.5 SOME PARTICULAR CASES

If n is a positive integer, then

$$(x+a)^n = {}^nC_0x^n \cdot a^0 + {}^nC_1x^{n-1} \cdot a^1 + {}^nC_2x^{n-2} \cdot a^2 + \cdots + {}^nC_rx^{n-r} \cdot a^r + \cdots + {}^nC_nx^0 \cdot a^n \quad ...(i)$$

In particular,

1. Replacing 'a' by $(-a)$ in (1), we get

$$(x-a)^n = {}^nC_0x^n \cdot a^0 - {}^nC_1x^{n-1} \cdot a^1 + {}^nC_2x^{n-2} \cdot a^2 + \cdots + (-1)^r \cdot {}^nC_rx^{n-r} \cdot a^r + \cdots + (-1)^n \cdot {}^nC_n \cdot x^0 \cdot a^n \quad ...(ii)$$

2. Adding (*i*) and (*ii*), we get

$$(x+a)^n + (x-a)^n = 2\left[{}^nC_0x^n \cdot a^0 + {}^nC_2x^{n-2} \cdot a^2 + {}^nC_4x^{n-4} \cdot a^4 + \cdots\right]$$
$$= 2[\text{sum of terms at odd places}]$$

3. Subtracting (*ii*) from (*i*), we get

$$(x+a)^n - (x-a)^n = 2\left[{}^nC_1x^{n-1} \cdot a^1 + {}^nC_3x^{n-3} \cdot a^3 + {}^nC_5x^{n-5} \cdot a^5 + \cdots\right]$$
$$= 2[\text{sum of terms at even places}]$$

4. Replacing x by 1 and a by x in (1), we get

$$(1+x)^n = {}^nC_0x^0 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_rx^r + \cdots + {}^nC_{n-1} \cdot x^{n-1} + {}^nC_nx^n$$

5. Replacing x by 1 and a by $(-x)$ in (1), we get

$$(1-x)^n = {}^nC_0x^0 - {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_{n-1}(-1)^{n-1} \cdot x^{n-1} + {}^nC_n(-1)^n \cdot x^n$$

3.6 PROPERTIES OF THE BINOMIAL COEFFICIENTS

The binomial coefficients have two important properties:

(*i*) The sum of all the binomial coefficients is 2^n

(*ii*) The sum of all the odd binomial coefficients is equal to the sum of all the even binomial coefficients and each in equal to 2^{n-1}.

Proof:

(*I*) In the binomial expression, we have

$$(x+a)^n = {}^nC_0x^n \cdot a^0 + {}^nC_1x^{n-1} \cdot a^1 + {}^nC_2x^{n-2} \cdot a^2 + \cdots + {}^nC_nx^0 \cdot a^n \quad \ldots(i)$$

where ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called binomial coefficients.

Putting $x = a = 1$ in equation no. (*i*), we get

$$ {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$$

Thus the sum of all the binomial coefficients is 2^n.

(*II*) Putting $x = 1$ and $a = -1$ in equation no. (*i*), we get

$$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - {}^nC_5 + \cdots$$

or $\quad {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots = {}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots$

Thus, the sum of all odd binomial coefficients = the sum of all even binomial coefficients.

Again, each of them $= \frac{1}{2} \times (\text{sum of them})$

$$= \frac{1}{2} \times \left({}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n \right)$$

$$= \frac{1}{2} \cdot 2^n = 2^{n-1}$$

i.e. $\quad {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots = {}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = 2^{n-1}$

3.7 GENERAL TERM

The $(r + 1)$th term in the expansion $(x + a)^n$ is called the General Term and is denoted by t_{r+1}.

From the expansion of $(x + a)^n$, we see that,

$$t_1 = \text{1st term} = t_{0+1} = {}^nC_0x^{n-0} \cdot a^0$$

$$t_2 = \text{2nd term} = t_{1+1} = {}^nC_1x^{n-1} \cdot a^1$$

$$t_3 = \text{3rd term} = t_{2+1} = {}^nC_2 \cdot x^{n-2} \cdot a$$

$$t_4 = \text{4th term} = t_{3+1} = {}^nC_3 x^{n-3} \cdot a^3$$

$$t_5 = \text{5th term} = t_{4+1} = {}^nC_4 x^{n-4} \cdot a^4$$

and in general,

$$t_{r+1} = (r+1)\text{th term} = {}^nC_r \cdot x^{n-r} \cdot a^r$$

NOTE

1. The general term in the expansion $(x-a)^n = (-1)^r \cdot {}^nC_r \cdot x^{n-r} \cdot a^r$
2. The general term in the expansion $(1+x)^n = {}^nC_r \cdot x^r$
3. The general term in the expansion $(1-x)^n = (-1)^r \cdot {}^nC_r \cdot x^r$
4. The coefficient of the general term in the expansion $(x+a)^n = {}^nC_r \cdot a^r$

ILLUSTRATION 1

Find the general term in the expansion of $\left(x^3 + \dfrac{1}{x^2}\right)^{12}$

Solution: General term $(t_{r+1}) = {}^{12}C_r \cdot \left(x^3\right)^{12-r} \cdot \left(\dfrac{1}{x^2}\right)^r$

$$= {}^{12}C_r x^{36-3r} \cdot \frac{1}{x^{2r}}$$

$$= {}^{12}C_r x^{36-5r}$$

ILLUSTRATION 2

Find the general term in the expansion of $\left(x - \dfrac{1}{x}\right)^8$.

Solution: General term $(t_{r+1}) = {}^8C_r\,(x)^{8-r} \cdot \left(-\dfrac{1}{x}\right)^r = {}^8C_r \cdot x^{8-r} \cdot \dfrac{(-1)^r}{x^r}$

$$= (-1)^r . {}^8C_r \cdot x^{8-2r}$$

3.8 MIDDLE TERM

The middle term of any expansion depends upon the value of index 'n'.

(*i*) If the value of index 'n' is even:

The total number of terms in the expansion of $(x + a)^n$ is $(n + 1)$ which is odd. So, there is only one middle term and it is $\left[\frac{(n+1)+1}{2}\right]$th term, i.e. $\left(\frac{n}{2}+1\right)$th term.

Therefore, the required middle term $= t_{\frac{n}{2}+1} = {}^nC_{\frac{n}{2}} \cdot (x)^{\frac{n}{2}} \cdot a^{\frac{n}{2}}$

(*ii*) If the value of index 'n' is odd:

The total number of terms in the expansion of $(x + a)^n$ is $(n + 1)$ which is even. So, there are two middle terms and they are $\left(\frac{n+1}{2}\right)$th term and $\left(\frac{n+1}{2}+1\right)$th, i.e. $\left(\frac{n+3}{2}\right)$th term.

Therefore, the first middle term

$$= t_{\frac{n+1}{2}} = t_{\frac{n-1}{2}+1} = {}^nC_{\frac{n-1}{2}} \cdot x^{n-\frac{n-1}{2}} \cdot a^{\frac{n-1}{2}}$$

$$= {}^nC_{\frac{n-1}{2}} \cdot x^{\frac{n+1}{2}} \cdot a^{\frac{n-1}{2}}$$

and the second middle term

$$= t_{\frac{n+3}{2}} = t_{\frac{n+1}{2}+1} = {}^nC_{\frac{n+1}{2}} \cdot x^{n-\frac{n+1}{2}} \cdot a^{\frac{n+1}{2}}$$

$$= {}^nC_{\frac{n+1}{2}} \cdot x^{\frac{n-1}{2}} \cdot a^{\frac{n+1}{2}}$$

ILLUSTRATION 3

Find the middle term in the expansion of $(x + a)^8$.

Solution: Here, the value of index 'n' = 8, which is even.

So the expansion has one middle term and it is $\left(\frac{8}{2}+1\right)$ th term, i.e. 5th term.

Therefore, the required middle term

$$= t_5 = t_{4+1} = {}^8C_4 \cdot (x)^{8-4} \cdot a^4$$

$$= \frac{8\times7\times6\times5}{4\times3\times2\times1} \cdot x^4 \cdot a^4 = 70x^4 \cdot a^4$$

ILLUSTRATION 4

Find the middle term in the expansion of $(x + a)^9$.

Solution: Here the value of index 'n' = 9, which is odd.

So, the expansion has two middle terms and they are $\left(\frac{9+1}{2}\right)$ th term, i.e. 5th term and (5 + 1), i.e. 6th term.

Therefore, the first middle term $= t_5 = t_{4+1} = {}^9C_4\, x^{9-4} \cdot a^4$

$$= \frac{9\times8\times7\times6}{4\times3\times2\times1} \cdot x^5 \cdot a^4$$

$$= 126\, x^5 \cdot a^4$$

and the second middle term $= t_6 = t_{5+1}$

$$= {}^9C_5 \cdot x^{9-5} \cdot a^5 = \frac{9\times8\times7\times6}{4\times3\times2\times1} \cdot x^4 \cdot a^5 = 126\, x^4 a^5.$$

3.9 THE '*r*th' TERM FROM THE END

The rth term from the end in the expansion $(x + a)^n$ is $(n - r + 2)$th term from the beginning.

3.10 EQUIDISTANT TERMS

Theorem : In the expansion of $(x + a)^n$, the coefficient of terms equidistant from the beginning and the end are equal.

Proof: The coefficient of $(r + 1)$th term from the beginning in the expansion $(x + a)^n = {}^nC_r$

And the coefficient of $(r + 1)$th term from the end in the expansion $(x + a)^n$ is $(n - r + 1)$th term from the beginning. [From 3.9]

So, the coefficient of $(r + 1)$th term from the end $= {}^nC_{n-r}$

But we know, ${}^nC_r = {}^nC_{n-r}$

Therefore, the coefficient of the $(r + 1)$th term from the beginning = the coefficient of the $(r + 1)$th term from the end.

Therefore, the coefficient of terms equidistant from the beginning and the end are equal.

ILLUSTRATIVE EXAMPLES

A. SHORT TYPE

EXAMPLE 1

Expand the binomial expression $(2x + 3)^4$.

Solution: $(2x + 3)^4 = (2x)^4 + {}^4C_1\,(2x)^3 \cdot 3 + {}^4C_2\,(2x)^2 \cdot 3^2 + {}^4C_3\,(2x) \cdot 3^3 + 3^4$

$$= 16x^4 + 4 \cdot 8x^3 \cdot 3 + \frac{4\times 3}{2\times 1} \cdot 4x^2 \cdot 9 + 4 \cdot 2x \cdot 27 + 81$$

$$= 16x^4 + 96x^3 + 216x^2 + 216x + 81.$$

EXAMPLE 2

Write the fifth term of $\left(x+\dfrac{1}{x}\right)^8$ in the most simplified form.

[***C.U.B.Com. 1983***]

Solution: General term $(t_{r+1}) = {}^8C_r\,(x)^{8-r}.\left(\dfrac{1}{x}\right)^r$

$$= {}^8C_r \cdot x^{8-r} \cdot \frac{1}{x^r} = {}^8C_r \,.\, x^{8-2r}$$

Putting $r = 4$

$$t_{4+1} = t_5 = {}^8C_4 \cdot x^{8-8} = \frac{8\times 7\times 6\times 5}{4\times 3\times 2\times 1} \cdot x^0 = 70 \text{ [As } x^0 = 1]$$

EXAMPLE 3

Determine the 11th term in the expansion of

$$\left(x-\frac{1}{x}\right)^{20}.$$

[***C.U.B.Com 2014***]

Solution: $t_{11} = t_{10+1} = {}^{20}C_{10}\,(x)^{20-10} \cdot \left(-\dfrac{1}{x}\right)^{10}$

$$= \frac{20!}{10!10!} \cdot x^{10} \cdot \frac{1}{x^{10}}$$

$$= \frac{20\times19\times18\times17\times16\times15\times14\times13\times12\times11}{10\times9\times8\times7\times6\times5\times4\times3\times2\times1} = 1{,}68{,}036$$

EXAMPLE 4

Find the general term in the binomial expansion of $\left(x+\dfrac{1}{x}\right)^{2n}$. Find the coefficient of x^n also. **[*C.U.B.Com. 1983*]**

Solution: General term of $\left(x+\dfrac{1}{x}\right)^{2n}$ is

$$t_{r+1} = {}^{2n}C_r\, x^{2n-r} \cdot \left(\frac{1}{x}\right)^r = {}^{2n}C_r \cdot x^{2n-r} \cdot \frac{1}{x^r}$$

$$= {}^{2n}C_r \cdot x^{2n-2r}$$

Now, $\quad x^{2n-2r} = x^n \quad$ or $\quad 2n-2r = n$

or $\quad n = 2r \qquad \therefore\ r = \dfrac{n}{2}$

Therefore, the required coefficient of $x^n = {}^{2n}C_{n/2}$

EXAMPLE 5

Find the middle term in the binomial expansion of $(x + 1)^6$.

Solution: Here, $n = 6$, which is even.

So the expansion has one middle term and it is $\left(\dfrac{6}{2}+1\right)$th term, i.e. 4th term
Therefore, the required middle term

$$= t_4 = t_{3+1} = {}^6C_3\, x^{6-3} \cdot 1^3 = \frac{6\times5\times4}{3\times2\times1} \cdot x^3$$

$$= 20x^3$$

EXAMPLE 6

Find the term independent of x in the expansion of $\left(x^2+\dfrac{1}{x}\right)^{12}$ and also its value.

[*C.U.B.Com. 1992*]

Solution: Let $(r + 1)$th term is independent of 'x', i.e. free from 'x' in the expansion $\left(x^2+\frac{1}{x}\right)^{12}$.

then, $t_{r+1} = {}^{12}C_r\,(x^2)^{12-r}\cdot\left(\frac{1}{x}\right)^r$

$$= {}^{12}C_r\,x^{24-2r}\cdot\frac{1}{x^r} = {}^{12}C_r\cdot x^{24-3r}$$

Now, $24 - 3r = 0$ (as the term is independent of 'x')

or $3r = 24$ or $r = 8$

Then $8 + 1 = 9$th term is independent of x and its value

$$= {}^{12}C_8 = \frac{12!}{8!4!} = \frac{12\times11\times10\times9}{4\times3\times2\times1} = 495.$$

EXAMPLE 7

Find the 4th term from the end in the expansion $\left(\frac{x^2}{2}-\frac{2}{x}\right)^5$.

Solution: Since rth term from the end in the expansion of $(x + a)^n$ is $(n - r + 2)$th term from the beginning.

Therefore, 4th term from the end is $(5 - 4 + 2)$, i.e. 3rd term from the beginning.

$$t_3 = t_{2+1} = {}^5C_2\left(\frac{x^2}{2}\right)^3\cdot\left(-\frac{2}{x}\right)^2 = \frac{5\times4}{2\times1}\cdot\frac{x^6}{8}\cdot\frac{4}{x^2} = 5x^4$$

EXAMPLE 8

Determine whether the expansion of $\left(x^2-\frac{2}{x}\right)^{10}$ will contain a term containing x^6?

Solution: Let $(r + 1)^{\text{th}}$ term contain x^6, then

$$t_{r+1} = {}^{10}C_r\,(x^2)^{10-r}\cdot\left(-\frac{2}{x}\right)^r = {}^{10}C_r x^{20-2r}.\frac{(-2)^r}{x^r}$$

$$= {}^{10}C_r x^{20-3r}\cdot(-2)^r = (-2)^r\cdot{}^{10}C_r x^{20-3r}$$

Now, $20 - 3r = 6$

or $3r = 14$

or $r = \frac{14}{3}$

As r is a fraction, so the given expansion cannot have a term containing x^6.

EXAMPLE 9

Expand $(x-3)^6$ in ascending powers of x, upto the term containing x^3.

Solution: $(x-3)^6 = (-3+x)^6$

$$= (-3)^6 + {}^6C_1(-3)^5 \cdot x + {}^6C_2(-3)^4 \cdot x^2 + {}^6C_3(-3)^3 \cdot x^3 + \cdots$$

$$= 729 + 6 \cdot (-243) \cdot x + 15 \cdot 81 \cdot x^2 + 20 \cdot (-27)\, x^3 + \ldots.$$

$$= 729 - 1458x + 1215x^2 - 540x^3 + \cdots$$

EXAMPLE 10

In the expansion of $(1 + x)^{10}$, the coefficient of $(2r + 1)^{\text{th}}$ term is equal to the coefficient of $(4r + 5)^{\text{th}}$ term. Find r

Solution: $t_{2r+1} = {}^{10}C_{2r}x^{2r}$

and $t_{4r+5} = t_{(4r+4)+1} = {}^{10}C_{4r+4} \cdot x^{4r+4}$

As per condition,

$${}^{10}C_{2r} = {}^{10}C_{4r+4}$$

or $\quad 2r + (4r + 4) = 10$

or $\quad 6r = 6$ or $r = 1$

B. SHORT ESSAY TYPE

EXAMPLE 11

Expand the following binomial expressions:

(*i*) $(3x + 7)^7$ (*ii*) $(5x - 2y)^5$ (*iii*) $(1 - x + x^2)^4$

Solution:

(*i*) $(3x + 7)^7 = (3x)^7 + {}^7C_1(3x)^6 \cdot 7 + {}^7C_2(3x)^5 \cdot 7^2$

$$+ {}^7C_3(3x)^4 \cdot 7^3 + {}^7C_4(3x)^3 \cdot 7^4 + {}^7C_5(3x)^2 \cdot 7^5$$

$$+ {}^7C_6(3x) \cdot 7^6 + 7^7$$

$$= 3^7 \cdot x^7 + 7 \cdot 3^6 \cdot x^6 \cdot 7 + \frac{7 \times 6}{2 \times 1} \cdot 3^5 \cdot x^5 \cdot 49 + \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \cdot 3^4 \cdot x^4 \cdot 7^3$$

$$+\frac{7\times6\times5}{3\times2\times1}\times3^3\cdot x^3\cdot7^4+\frac{7\times6}{2\times1}\cdot3^2\cdot x^2\cdot7^5+7\cdot3\cdot x\cdot7^6+7^7$$

$$=2187x^7+35{,}721x^6+2{,}50{,}047x^5+9{,}72{,}405x^4$$

$$+\ 22{,}68{,}945x^3+31{,}76{,}523x^2+24{,}70{,}629x+8{,}23{,}543$$

(*ii*) $(5x-2y)^5=(5x)^5+{}^5C_1(5x)^4\cdot(-2y)+{}^5C_2(5x)^3\cdot(-2y)^2$

$$+{}^5C_3(5x)^2\cdot(-2y)^3+{}^5C_4(5x)\cdot(-2y)^4+(-2y)^5$$

$$=3125x^5-5\cdot5^4\cdot x^2\cdot2y+\frac{5\times4}{2\times1}\cdot5^3\cdot x^3\cdot4y^2$$

$$-\frac{5\times4}{2\times1}\cdot5^2\cdot x^2\cdot8y^3+5\cdot5x\cdot16y^4-32y^5$$

$$=3125x^5-6250x^2\cdot y+5000x^3y^2$$

$$-\ 2000x^2y^3+400x\cdot y^4-32y^5$$

(*iii*) $(1-x+x^2)^4=\left\{(1-x)+x^2\right\}^4$

$$=(1-x)^4+{}^4C_1(1-x)^3\cdot x^2+{}^4C_2(1-x)^2\cdot(x^2)^2$$

$$+\ {}^4C_3(1-x)\cdot(x^2)^3+(x^2)^4$$

$$=(1-4x+6x^2-4x^3+x^4)+4x^2(1-3x+3x^2-x^3)$$

$$+\ 6x^4(1-2x+x^2)+4x^6(1-x)+x^8$$

$$=1-4x+6x^2-4x^3+x^4+4x^2-12x^3+12x^4$$

$$-\ 4x^5+6x^4-12x^5+6x^6+4x^6-4x^7+x^8$$

$$=1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8$$

EXAMPLE 12

Find 9th term in the expansion of $\left(2x^3-\frac{3}{x^2}\right)^{10}$.

Solution: General term = $t_{r+1}={}^{10}C_r(2x^3)^{10-r}.\left(\frac{-3}{x^2}\right)^r$

$$= {}^{10}C_r \cdot (2)^{10-r} \cdot x^{30-3r} \cdot \frac{(-3)^r}{x^{2r}}$$

$$= (2)^{10-r} \cdot (-3)^r \cdot {}^{10}c_r x^{30-5r}$$

Putting $r = 8$

$$t_{8+1} = t_9 = (2)^{10-8} \cdot (-3)^8 \cdot {}^{10}c_8 \cdot x^{30-40}$$

$$= 4.6561 \cdot \frac{10 \times 9}{2 \times 1} \cdot x^{-10}$$

$$= 11,80,980 \cdot \frac{1}{x^{10}}$$

EXAMPLE 13

In the expansion of $\left(x^3 + \frac{1}{x^2}\right)^{20}$, find the eleventh term.

Solution: General term = $t_{r+1} = {}^{20}C_r (x^3)^{20-r} \cdot \left(\frac{1}{x^2}\right)^r$

$$= {}^{20}C_r x^{60-3r} \cdot \frac{1}{x^{2r}}$$

$$= {}^{20}C_r x^{60-5r} \cdot$$

Putting $r = 10$,

$$t_{10+1} = t_{11} = {}^{20}C_{10} \cdot x^{60-50}$$

$$= {}^{20}C_{10} \cdot x^{10}$$

$$= \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11}{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \cdot x^{10}$$

$$= 1,68,036x^{10}$$

EXAMPLE 14

Find the coefficient of the term containing x^7 in the expression

$$\left(2x^2 + \frac{1}{4x}\right)^{11}$$ [***C.U.B.Com. 2013(G)***]

Solution: Let $(r + 1)$th term contains x^7

Now, $t_{r+1} = {}^{11}C_r (2x^2)^{11-r} \cdot \left(\frac{1}{4x}\right)^r = {}^{11}C_r\, 2^{11-r} \cdot x^{22-2r} \cdot \frac{1}{4^r . x^r}$

$$= 2^{11-r} \cdot \frac{1}{4^r} \cdot {}^{11}C_r x^{22-3r}$$

$$= 2^{11-r} \cdot \frac{1}{2^{2r}} \cdot {}^{11}C_r \cdot x^{22-3r} = 2^{11-3r} \cdot {}^{11}C_r \cdot x^{22-3r}$$

The term will contain x^7, if $22 - 3r = 7$

or $3r = 15$ or $r = 5$

Therefore, $(5 + 1)$th = 6th term contains x^7 and its coefficient

$$= 2^{11-15} \cdot {}^{11}c_5 = 2^{-4} \cdot \frac{11!}{6!5!}$$

$$= \frac{1}{2^4} \times \frac{11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= \frac{462}{16} = \frac{231}{8}.$$

EXAMPLE 15

In the expansion of $\left(5a^3 - \frac{2}{a^2}\right)^{13}$, find the coefficient of a^{-11}

[***C.U.B.Com. 2015(G)***]

Solution: Let $(r + 1)$th term contains a^{-11}

Now, $t_{r+1} = {}^{13}C_r (5a^3)^{13-r} \cdot \left(-\frac{2}{a^2}\right)^r$

$$= {}^{13}C_r \cdot 5^{13-r} \cdot a^{39-3r} \cdot \frac{(-2)^r}{a^{2r}}$$

$$= 5^{13-r} \cdot (-2)^r \cdot {}^{13}C_r \cdot a^{39-5r}$$

The term will contain a^{-11}, if $39 - 5r = -11$

Or $5r = 50$ or $r = 10$

$\therefore$ $(10 + 1)$th = 11th term contains a^{-11}

and its coefficient = $5^{13-10} \cdot (-2)^{10} \cdot {}^{13}C_{10}$

$$= 5^3 \cdot 2^{10} \cdot \frac{13 \times 12 \times 11}{3 \times 2 \times 1}$$

= 3,66,08,000

EXAMPLE 16

(i). Find the coefficient of x^3 in $(1+x+2x^2)(1-2x)^5$

Solution: $(1+x+2x^2)(1-2x)^5$

$$= (1+x+2x^2)\ [1^5 + {}^5C_1(-2x) + {}^5C_2(-2x)^2 + {}^5C_3(-2x)^3 + \cdots]$$

$$= (1+x+2x^2)\ [1 + 5\times -2x + 10\times 4x^2 + 10\times -8x^3 + \cdots]$$

$$= (1+x+2x^2)\ [1 - 10x + 40x^2 - 80x^3 + \cdots]$$

From the above multiplication of the expansion it is clear that the x^3 term in $(1+x+2x^2)(1-2x)^5$ is given by the sum of $1(-80x^3)$, $x(40x^2)$ and $2x^2(-10x)$.

So, the coefficient of x^3 is $-80 + 40 - 20 = -60$.

(ii). If the coefficient of $(r + 3)$th term in the expansion of $(1 + x)^{47}$ be the same as the coefficient of $(3r + 2)$th term, find these two terms.

[***C.U. B.Com. (G), 2017***]

Solution:

$$t_{r+3} = t_{(r+2)+1} = {}^{47}C_{r+2} \cdot x^{r+2}$$

$$t_{3r+2} = t_{(3r+1)+1} = {}^{47}C_{3r+1} \cdot x^{3r+1}$$

By the condition,

$$ {}^{47}C_{r+2} = {}^{47}C_{3r+1}$$

or $\quad (r + 2) + (3r + 1) = 47 \quad$ [If ${}^nc_p = {}^nc_q$ then $p + q = n$]

or $\quad 4r + 3 = 47$

or $\quad 4r = 44$

or $\quad r = 11$

Therefore, two terms are:

$$t_{r+3} = t_{11+3} = t_{14} = {}^{47}C_{13}x^{13}$$

and $\quad t_{3r+2} = t_{33+2} = t_{35} = {}^{47}C_{34}x^{34}$

EXAMPLE 17

Find the term independent of x in the expansion of $\left(x^2 - \frac{2}{x}\right)^6$.

Solution: Let $(r + 1)$th term is independent of x in the expansion of $\left(x^2 - \frac{2}{x}\right)^6$.

Now, $$t_{r+1} = {}^6C_r (x^2)^{6-r} \cdot \left(-\frac{2}{x}\right)^r$$

$$= {}^6C_r x^{12-2r} \cdot \frac{(-2)^r}{x^r} = (-2)^r \cdot {}^6C_r x^{12-3r}$$

Clearly, the $(r + 1)$th term will be independent of x if $12 - 3r = 0$ or $3r = 12$ or $r = 4$

Therefore, $(4 + 1)$th = 5th term is independent of 'x'

and its value = $(-2)^4 . {}^6C_4$

$$= 16 \cdot \frac{6 \times 5}{2 \times 1} = 240.$$

EXAMPLE 18

Find the term independent of x in the expansion of $\left(\frac{\sqrt{x}}{\sqrt{3}} + \frac{\sqrt{3}}{2x^2}\right)^{10}$

Solution: Let $(r + 1)$th term is independent of x in the expansion of $\left(\frac{\sqrt{x}}{\sqrt{3}} + \frac{\sqrt{3}}{2x^2}\right)^{10}$

Now, $$t_{r+1} = {}^{10}C_r \left(\frac{\sqrt{x}}{\sqrt{3}}\right)^{10-r} \cdot \left(\frac{\sqrt{3}}{2x^2}\right)^r$$

$$= {}^{10}C_r \left(\frac{x}{3}\right)^{\frac{10-r}{2}} \cdot \frac{3^{r/2}}{2^r . x^{2r}}$$

$$= {}^{10}C_r \frac{x^{\frac{10-r}{2} - 2r}}{2^r} \cdot 3^{\frac{r}{2} - \frac{10-r}{2}}$$

$$= {}^{10}C_r \frac{x^{\frac{10-5r}{2}} \cdot 3^{\frac{2r-10}{2}}}{2^r}$$

$$= \frac{3^{\frac{2r-10}{2}}}{2^r} . {}^{10}C_r \, x^{\frac{10-5r}{2}}$$

Since the term is independent of x, we have

$$\frac{10 - 5r}{2} = 0 \text{ or } 10 - 5r = 0 \text{ or } 5r = 10 \text{ or } r = 2$$

Therefore, (2 + 1) = 3rd term is independent of x and its value = $\dfrac{3^{\frac{4-10}{2}}}{2^2} \cdot {}^{10}C_2$

$$= \frac{3^{-3}}{2^2} \cdot {}^{10}C_2 = \frac{1}{27 \times 4} \times \frac{10 \times 9}{2 \times 1} = \frac{5}{12}$$

EXAMPLE 19

Find the middle term (terms) in the expansion of $\left(\dfrac{b}{x} + \dfrac{x}{b}\right)^9$.

Solution: Here, the value of index 'n' = 9, which is odd.

So, the expansion has two middle terms and they are $\left(\dfrac{9+1}{2}\right)$th term, i.e. 5th term and (5 + 1), i.e. 6th term.

Therefore, the first middle term = $t_5 = t_{4+1} = {}^9C_4 \left(\dfrac{b}{x}\right)^5 \cdot \left(\dfrac{x}{b}\right)^4$

$$= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \cdot \frac{b^5}{x^5} \cdot \frac{x^4}{b^4}$$

$$= \frac{126b}{x}$$

and the second middle term = $t_6 = t_{5+1} = {}^9C_5 \left(\dfrac{b}{x}\right)^4 \cdot \left(\dfrac{x}{b}\right)^5$

$$= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \cdot \frac{b^4}{x^4} \cdot \frac{x^5}{b^5}$$

$$= \frac{126x}{b}$$

EXAMPLE 20

Find the middle term in the expansion of $\left(2px - \dfrac{q}{x^2}\right)^{12}$

Solution: Here, the value of index = 12, which is even. So, the expansion has one middle term and it is $\left(\dfrac{12}{2} + 1\right)$th term, i.e. 7th term.

Therefore, the required middle term = $t_7 = t_{6+1} = {}^{12}C_6 (2px)^6 \cdot \left(-\dfrac{q}{x^2}\right)^6$

$$= \frac{12\times 11\times 10\times 9\times 8\times 7}{6\times 5\times 4\times 3\times 2\times 1}\cdot 64\cdot p^6\cdot x^6\cdot \frac{q^6}{x^{12}}$$

$$= 59136\frac{p^6\cdot q^6}{x^6}$$

EXAMPLE 21

Find the value of the terms independent of x in the expansion of $(1-x)^2\left(x+\frac{1}{x}\right)^6$.

Solution: $(1-x)^2\left(x+\frac{1}{x}\right)^6$.

$$= (1-2x+x^2)[x^6 + {}^6C_1 x^5\cdot\frac{1}{x} + {}^6C_2 x^4\cdot\frac{1}{x^2} + {}^6C_3 x^3\cdot\frac{1}{x^3}$$
$$+ {}^6C_4 x^2\cdot\frac{1}{x^4} + {}^6C_5 x\cdot\frac{1}{x^5} + \frac{1}{x^6}]$$

$$= (1-2x+x^2)[x^6 + {}^6C_1 x^4 + {}^6C_2 x^2 + {}^6C_3 + {}^6C_4\frac{1}{x^2} + {}^6C_5\frac{1}{x^4} + \frac{1}{x^6}]$$

From the above multiplication of the expansion it is clear that the value of terms independent of x is given by the sum of $1\cdot{}^6C_3$ and $x^2\cdot{}^6C_4\frac{1}{x^2}$.

Therefore, the value of the terms independent of x in the given expansion

$$= {}^6C_3 + {}^6C_4$$

$$= 20 + 15 = 35$$

EXAMPLE 22

If the 21st and 22nd terms in the expansion of $(1+x)^{44}$ are equal, find the value of x. **[*C.A. Entrance*]**

Solution: General term of the expansion $(1+x)^{44}$

$$t_{r+1} = {}^{44}C_r x^r$$

Therefore, $t_{21} = t_{20+1} = {}^{44}C_{20}x^{20}$ and $t_{22} = t_{21+1} = {}^{44}C_{21}x^{21}$

As per condition, $t_{21} = t_{22}$

or $$^{44}C_{20}x^{20} = {}^{44}C_{21}x^{21}$$

or $$\frac{x^{21}}{x^{20}} = \frac{^{44}C_{20}}{^{44}C_{21}}$$

or $$x = \frac{44!}{24!20!} \times \frac{23!21!}{44!}$$

$$= \frac{23! \times 21 \times 20!}{24 \times 23! \times 20!} = \frac{21}{24} = \frac{7}{8}$$

EXAMPLE 23

Using binomial theorem find the value of $(99)^4$

Solution: $(99)^4 = (100 - 1)^4$

$$= (100)^4 - {}^4C_1(100)^3 \cdot 1 + {}^4C_2(100)^2 \cdot 1^2 - {}^4C_3 100 \cdot 1^3 + 1^4$$

$= 100000000 - 4 \times 1000000 + 6 \times 10000 - 4 \times 100 + 1$

$= 100000000 - 4{,}00{,}00{,}00 + 60000 - 400 + 1$

$= 96059601.$

EXAMPLE 24

Find the value of $(1.05)^5$ upto 5 decimal places by using binomial theorem.

Solution: $$(1.05)^5 = (1 + 0.05)^5 = \left(1 + \frac{5}{100}\right)^5$$

$$= 1 + {}^5C_1\left(\frac{5}{100}\right) + {}^5C_2\left(\frac{5}{100}\right)^2 + {}^5C_3\left(\frac{5}{100}\right)^3$$

$$= + {}^5C_4\left(\frac{5}{100}\right)^4 + {}^5C_5\left(\frac{5}{100}\right)^5$$

$= 1 + 5 \times 0.05 + 10 \times 0.0025 + 10 \times 0.000125 + \ldots$

$= 1 + 0.25 + 0.025 + 0.00125$ (neglecting other terms)

$= 1.27625$ (correct upto 5 decimal places)

EXAMPLE 25

If the term independent of x in the expansion of $\left(\frac{k}{3}x^2 - \frac{3}{2x}\right)^9$ be 2268, find the value of k.

Solution: Let $(r+1)$th term in the given expansion is independent of *x*.

Now, $$t_{r+1} = {}^9C_r \left(\frac{k}{3}x^2\right)^{9-r} \cdot \left(-\frac{3}{2x}\right)^r$$

$$= {}^9C_r \left(\frac{k}{3}\right)^{9-r} \cdot x^{18-2r} \cdot \left(-\frac{3}{2}\right)^r \cdot \frac{1}{x^r}$$

$$= \left(\frac{k}{3}\right)^{9-r} \cdot \left(-\frac{3}{2}\right)^r \cdot {}^9C_r \cdot x^{18-3r}$$

As the term is independent of *x*, we have

$$18 - 3r = 0 \text{ or } 3r = 18 \quad \text{or } r = 6$$

Therefore, (6 + 1)th = 7th term is independent of *x* and its value

$$= \left(\frac{k}{3}\right)^{9-6} \cdot \left(-\frac{3}{2}\right)^6 \cdot {}^9C_6$$

$$= \left(\frac{k}{3}\right)^3 \times \frac{3^6}{2^6} \times \frac{9 \times 8 \times 7}{3 \times 2 \times 1}$$

$$= \frac{k^3}{3^3} \times \frac{3^6}{2^6} \times 84 = \frac{k^3 \times 27 \times 84}{64}$$

$$= \frac{k^3 \times 2268}{64}$$

As per condition, $\dfrac{k^3 \times 2268}{64} = 2268$

or $$k^3 = 64 = (4)^3$$

Therefore, $k = 4$

Therefore, the required value of *k* is 4.

EXAMPLE 26

Prove that, $2^{2n+2} - 3n - 4$ is divisible by 9 for all positive integer values of *n* greater then 1.

Solution: we have, $2^{2n+2} - 3n - 4$

$$= 2^{2(n+1)} - 3n - 4$$

$$= 4^{n+1} - 3n - 4$$

$$= (1+3)^{n+1} - 3n - 4$$

$$= 1 + {}^{n+1}C_1 3 + {}^{n+1}C_2 3^2 + {}^{n+1}C_3 3^3 + \cdots + {}^{n+1}C_{n+1} 3^{n+1} - 3n - 4$$

$$= 1 + (n+1)3 + \cdots {}^{n+1}C_2 3^2 + {}^{n+1}C_3 3^3 + \cdots + {}^{n+1}C_{n+1} 3^{n+1} - 3n - 4$$

$$= 1 + 3n + 3 + {}^{n+1}C_2 3^2 + {}^{n+1}C_3 3^3 + \cdots + 3^{n+1} - 3n - 4$$

$$= 3^2 \left[{}^{n+1}C_2 + {}^{n+1}C_3 3 + \cdots + 3^{n-1} \right]$$

From the above it is clear that $(2^{2n+2} - 3n - 4)$ is divisible by 3^2 or 9 for all positive integral values of n greater then 1. [Proved]

EXAMPLE 27

In the expansion of $(1 + x)^{12}$, the ratio of a certain coefficient to its proceeding coefficient is 6:7; find the values of the two coefficients.

Solution: Let, $\dfrac{\text{coefficient of } t_{r+1}}{\text{coefficient of } t_r} = \dfrac{6}{7}$

or $$\frac{{}^{12}C_r}{{}^{12}C_{r-1}} = \frac{6}{7}$$

or $$\frac{\dfrac{12!}{(12-r)!r!}}{\dfrac{12!}{(13-r)!(r-1)!}} = \frac{6}{7}$$

or $$\frac{12!}{(12-r)!r!} \times \frac{(13-r)!(r-1)!}{12!} = \frac{6}{7}$$

or $$\frac{(13-r) \times (12-r)! \times (r-1)!}{(12-r)!\, r \times (r-1)!} = \frac{6}{7}$$

or $$\frac{13-r}{r} = \frac{6}{7} \text{ or } 91 - 7r = 6r$$

or $13r = 91$ or $r = 7$.

Therefore, the value of 1st coefficient = $${}^{12}C_7 = \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1} = 792$$

and the value of 2nd coefficient = $${}^{12}C_6 = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 924$$

EXAMPLE 28

Prove that: $C_0 + 2C_1 + 3C_2 + \cdots + (n+1)C_n = (n+2)\cdot 2^{n-1}$

Solution: $C_0 + 2C_1 + 3C_2 + \cdots + (n+1)C_n$

$$= \left(C_0 + C_1 + C_2 + \cdots + C_n\right) + \left(C_1 + 2C_2 + \cdots + {}^nC_n\right)$$

$$= 2^n + \left\{n + 2\cdot\frac{n(n-1)}{2!} + 3\frac{n(n-1)(n-2)}{3!} + \cdots + n\right\}$$

$$= 2^n + n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \cdots + 1\right\}$$

$$= 2^n + n(1+1)^{n-1} = 2^n + n\cdot 2^{n-1}$$

$$= 2.2^{n-1} + n.2^{n-1}$$

$$= 2^{n-1}(n+2) = (n+2).2^{n-1}$$ [Proved]

EXAMPLE 29

Prove that: $\frac{C_0}{1} + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1} = \frac{1}{n+1}\left\{2^{n+1} - 1\right\}$ [***C.A. Entrance***]

Solution: $\frac{C_0}{1} + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1}$

$$= 1 + \frac{n}{2} + \frac{n(n-1)}{3!} + \cdots + \frac{1}{n+1}$$

$$= \frac{1}{n+1}\left[\left\{(n+1) + \frac{(n+1)\cdot n}{2!} + \frac{(n+1)\cdot n\cdot(n-1)}{3!} + \cdots + 1\right\}\right]$$

$$= \frac{1}{n+1}\left[\left\{1 + (n+1) + \frac{(n+1)n}{2!} + \frac{(n+1).n.(n-1)}{3!} + \cdots + 1\right\} - 1\right]$$

$$= \frac{1}{n+1}\left[(1+1)^{n+1} - 1\right] = \frac{1}{n+1}(2^{n+1} - 1)]$$ [Proved]

EXAMPLE 30

Prove that $C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = \frac{(2n)!}{(n!)^2}$

Solution: We know that, $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$ …(*i*)

Again, $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n$ ...(*ii*)

Multiplying both sides of equation no. (*i*) and (*ii*), we get

$$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + \cdots + C_nx^n) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_n)$$

clearly it is an identity. Hence, the coefficients of x^n in both sides are equal.

Now, the coefficient of x^n on the L.H.S = ${}^{2n}C_n = \dfrac{(2n)!}{(n!)^2}$

and the coefficients of x^n on the R.H.S.

$$= C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + C_3^{\,2} + \cdots + C_n^{\,2}$$

Therefore, $C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + C_3^{\,2} + \cdots + C_n^{\,2} = \dfrac{(2n)!}{(n!)^2}$ [Proved]

EXERCISE

A. SHORT TYPE

1. Write the fifth term of $\left(x+\dfrac{1}{x}\right)^3$ in the most simplified form.
   [***C.U.B. Com. '83***] [**Ans.** 70]
2. Find the 8th term in the expansion $(1 + x)^{11}$ in simplified form.
   [***C.U.B. Com. '96***] [**Ans.** $330x^7$]
3. Find the coefficient of x^5 in the expansion of $(x + 1)^8$. [**Ans.** 56]
4. Calculate the coefficient of x^8 in the expansion of $(1 + x)^8$. [**Ans.** 28]
5. Find the middle term in the binomial expansion of $(a + x)^{10}$.
   [***C.U.B. Com. '82***] [**Ans.** 6th term; $252a^5x^5$]
6. Find the middle term in the binomial expansion of $(x + 1)^8$. [**Ans.** 5th term]
7. Find the middle term in the binomial expansion of $\left(x-\dfrac{1}{x}\right)^{12}$. [**Ans.** 7th term]
8. In the expansion of $(x + 1)^{10}$, find (*i*) total number of terms and (*ii*) general term.
   [***C.U.B. Com. '98***] [**Ans.** (*i*) 11; (*ii*) ${}^{10}C_r x^{10-r}$]
9. If the number of terms in $(1 + x)^n$ be 11, find the 5th term and the value of n.
   [***C.U.B. Com. '99***] [**Ans.** $210x^4$; 10]
10. Find the term independent of x in the expansion of $\left(x+\dfrac{1}{x}\right)^{10}$.
    [**Ans.** 6th term; 252]

11. Find the coefficient of x^8 in the expansion of $(1 + x^2)^{10}$. [**Ans.** 5th term; 210]

12. Find the general term in the binomial expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$.

[***C.U. B. Com. '83***] [**Ans.** ${}^{2n}C_r x^{4n-3r}$]

13 Write the 4th term of $\left(x + \frac{1}{x}\right)^6$ in the simplified form.

[***C.U. B. Com. '86***] [**Ans.** 20]

14. Show that (when C_r means ${}^nC_r$) $C_0 + C_1 + C_2 + \cdots + C_n = 2^n$

[***C.U.B. Com. '91***]

[**Hints:** $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$.

Putting $x = 1$

$$2^n = C_0 + C_1 + C_2 + \cdots + C_n$$ (Proved)]

15. Show that (when C_r means ${}^nC_r$)

$$C_1 + C_2 + C_3 + \cdots + C_n = 2^n - 1.$$

B. SHORT ESSAY TYPE / PROBLEM TYPE

1. Expand the following binomials.

(*i*) $(2x + 3y)^5$

[**Ans.** $32x^2 + 240x^4 y + 720x^3 y^2 + 1080x^2 y^3 + 810xy^4 + 243y^5$]

[Hints: $(2x+3y)^5 = (2x)^5 + {}^5C_1 (2x)^{5-1}.3y$
$+ {}^5C_2 (2x)^{5-2}.(3y)^2 + {}^5C_3 (2x)^{5-3}.(3y)^3$
$+ {}^5C_4 (2x)^{5-4}.(3y)^4 + {}^5C_5 (2x)^{5-5}.(3y)^5$ etc.]

(*ii*) $(x + 3y)^6$

[**Ans.** $x^6 + 18x^5 y + 135x^4 y^2 + 540x^3 y^3 + 1215x^2 y^4 + 1458xy^5 + 729y^6$]

(*iii*) $\left(\frac{x}{3} + \frac{2}{y}\right)^6$

[**Ans.** $\frac{x^6}{729} + \frac{4x^5}{81y} + \frac{20}{27}\frac{x^4}{y^2} + \frac{160}{27}\frac{x^3}{y^2} + \frac{80}{3} + \frac{x^2}{y^4} + \frac{64x}{y^5} + \frac{64}{y^6}$]

(*iv*) $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$ [**Ans.** $\frac{64x^6}{729}-\frac{32x^4}{27}+\frac{20x^2}{3}-20+\frac{135}{4x^2}-\frac{243}{8x^4}+\frac{729}{64x^6}$]

(*v*) $(3x-2y)^4$ [**Ans.** $81x^4-216x^3y+216x^2y^2-96xy^2+16y^4$]

(*vi*) $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^4$ [**Ans.** $x^2-4x+6-\frac{4}{x}+\frac{1}{x^2}$]

2. Write in simplified form:

(*i*) 6th term in the expansion of $\left(2x-\frac{1}{3x^2}\right)^9$

[**Hints:** General term, $t_{r+1}={}^9C_r(2x)^{9-r}.(-\frac{1}{3x^2})^r$

putting $r = 5$, $t_6={}^9C_5(2x)^{9-5}.(-\frac{1}{3x^2})^5=-\frac{224}{27}.x^{-6}$]

(*ii*) 8th term of $\left(9x^2-\frac{1}{3x}\right)^{12}$ [**Ans.** $-21384x^3$]

(*iii*) 9th term of $\left(\frac{x}{2}+\frac{1}{x}\right)^{12}$ [**Ans.** $\frac{495}{16x^4}$]

(*iv*) 4th term of $\left(x^3+\frac{3}{x}\right)^8$ [**Ans.** $1512x^{12}$]

(*v*) 5th term of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^8$ [**Ans.** 1120]

(*vi*) 4th term of $\left(\frac{x}{a}-\frac{a}{x}\right)^{10}$ [**Ans.** $-\frac{120x^4}{a^4}$]

(*vii*) 5th term of $\left(\frac{3x}{4}+\frac{4}{3x}\right)^{12}$ [**Ans.** $\frac{40095}{256}x^4$]

(*viii*) 10th term of $\left(3x+\frac{2}{x}\right)^{12}$ [**Ans.** $2^{11}.3^3.55x^{-6}$]

3. Find the coefficient of:

   (*i*) x^{11} in the expansion of $(3x + 2x^2)^9$. [***C.U.B.Com. '91***]

   [Hints: $t_{r+1} = {}^9C_r(3x)^{9-r}.(2x^2)^r = {}^9C_r.3^{9-r}.2^r.x^{9+r}$

   The term will contain x^{11} if $9 + r = 11$ or $r = 2$

   $\therefore \quad t_3 = {}^9C_2.3^{9-2}.2^2.x^{9+2} = 314928x^{11}$.]

   (*ii*) x^{18} in the expansion of $(x^3 - 3x)^{10}$. [**Ans.** 153090]

   (*iii*) x^7 in the expansion of $\left(2x^2 + \frac{1}{4x}\right)^{11}$.

   [***C.U.B. Com '89***] [***K.U.B.Com '99***] [**Ans.** $\frac{231}{8}$]

   (*iv*) x^4 in the expansion of $\left(x^4 + \frac{1}{x^3}\right)^{15}$. [***C.U.B.Com.'86***] [**Ans.** 6435]

   (*v*) x^7 in the expansion of $\left(x^2 + \frac{1}{x}\right)^{11}$. [**Ans.** 462]

   (*vi*) x^4 in the expansion of $\left(2x^2 - \frac{1}{4x}\right)^{11}$. [**Ans.** $-\frac{231}{8}$]

   (*vii*) x^{-10} in the expansion of $\left(x^4 - \frac{1}{2x^3}\right)^{15}$. [**Ans.** 3003]

   (*viii*) x^{-11} in the expansion of $\left(x^2 - \frac{1}{x^3}\right)^{12}$.

   [***C.U.B.Com 2017 (H)***] [**Ans.** –792]

   (*ix*) x^{10} in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{20}$. [**Ans.** ${}^{20}C_{10}2^{10}$]

   (*x*) x^{-1} in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{10}$. [**Ans.** – 960]

4. Find the term independent of x in the expansion of:

   (*i*) $\left(x^2 + \frac{1}{x}\right)^{12}$ [***C.U.B.Com. '92, '98***]

[**Hints:** Suppose the $(r + 1)$th term to be independent of x.

Here $t_{r+1} = {}^{12}C_r (x^2)^{12-r}.(\frac{1}{x})^r = {}^{12}C_r x^{24-3r}$.

Now, this term will be independent of x, if $24 - 3r = 0$ or $r = 8$.

$\therefore$ the required term = ${}^{12}C_8 = {}^{12}C_4 = 495$]

(*ii*) $\left(3x^2 - \frac{1}{3x}\right)^9$. [***V.U.B.Com. '92***] [**Ans.** $t_7 = \frac{28}{9}$]

(*iii*) $\left(2x - \frac{1}{x^2}\right)^9$. [***C.U.B.Com. '84, '93***] [**Ans.** – 5376]

(*iv*) $\left(\frac{4}{3}x^2 - \frac{3}{2x}\right)^9$. [***C.U.B.Com.'82***] [**Ans.** $t_7 = 2268$]

(*v*) $\left(9x^2 - \frac{1}{3x}\right)^{12}$. [***C.U.B.Com '90, '95 2016 (G)***] [**Ans.** 495]

(*vi*) $\left(2x + \frac{1}{3x^2}\right)^9$. [**Ans.** $\frac{1792}{9}$]

(*vii*) $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$. [**Ans.** $\frac{7}{18}$]

(*viii*) $\left(x + \frac{1}{x}\right)^{12}$. [***C.U.B.Com. '96***] [**Ans.** 924]

(*ix*) $\left(x - \frac{2}{x^2}\right)^{15}$. [***C.U.B.Com. '97***] [**Ans.** – 96096]

(*x*) $\left(2x + \frac{1}{\sqrt{x}}\right)^{15}$. [***C.U.B. Com '99, 2014 (G)***] [**Ans.** 96096]

5. Find the middle terms in the expansion of:

(*i*) $\left(x + \frac{1}{x}\right)^{12}$

[**Hints:** Here $n = 12$ which is even and the number of terms is $12 + 1 = 13$ and there is only one middle term which is the $\left(\frac{12}{2} + 1\right)$th, i.e. the 7th term.

General term, $t_{r+1} = {}^{12}C_r x^{12-r}.(\frac{1}{x})^r$

$\therefore \ t_7 = {}^{12}C_6.x^{12-6}.(\frac{1}{x})^6 = {}^{12}C_6 = 924]$

(*ii*) $\left(3x - \frac{1}{2x}\right)^8$. [**Ans.** $\frac{2835}{8}$]

(*iii*) $\left(2x - \frac{1}{x}\right)^{12}$. [**Ans.** 59136]

(*iv*) $\left(x + \frac{1}{x}\right)^{21}$. [**Ans.** ${}^{21}C_{10}x; {}^{21}C_{10}\frac{1}{x}$]

(*v*) $\left(2x + \frac{1}{3x^2}\right)^9$. [**Ans.** $\frac{448}{9x^3}; \frac{224}{27x^6}$]

(*vi*) $\left(x - \frac{1}{x}\right)^{13}$. [**Ans.** $1716x, -1716x^{-1}$]

6. Evaluate (using binomial theorem):

(*i*) $(998)^{\frac{1}{3}}$ upto 3 places of decimals.

[**Hints:** $(998)^{\frac{1}{3}} = (1000-2)^{\frac{1}{3}} = 10\left(1 - \frac{2}{1000}\right)^{\frac{1}{3}}$

$$= 10(1-.002)^{\frac{1}{3}}$$

$$= 10\left\{1 + \frac{1}{3}(-0.002) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{\underline{|2}}(-0.002)^2 + \cdots\right\}$$

$$= 10\left\{1 - 0.000667) - \frac{1}{9} \times 0.000004 + \cdots\right\}$$

$$= 10 \times 0.999333 = 9.993]$$

(*ii*) $(999)^3$. [**Ans.** 99,70,02,999] (*iii*) $(1.02)^5$. (upto 3 decimal places) [**Ans.** 1.104]

(*iv*) $(1.03)^{10}$. (upto 5 decimal places) [**Ans.** 1.3439]

(*v*) $(0.99)^{15}$. (upto 4 decimal places) [**Ans.** 0.8601]

7. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^2$, prove that:

(*i*) $C_1 + 2C_2 + 3C_3 + \cdots + {}^nC_n = n.2^{n-1}$.

[**Hints:** $C_1 + 2C_2 + 3C_3 + \cdots + {}^nC_n$

$$= {}^nC_1 + 2.{}^nC_2 + 3.{}^nC_3 + \cdots + n.{}^nC_n$$

$$= n + 2\frac{n(n-1)}{\underline{|2}} + 3\frac{n(n-1)(n-2)}{\underline{|3}} + \cdots n.1$$

$$= n\left[1 + \frac{n-1}{\underline{|1}} + \frac{(n-1)(n-2)}{\underline{|2}} + \cdots 1\right]$$

$$= n\left[1 + {}^{n-1}C_1 + {}^{n-1}C_2 + \cdots {}^{n-1}C_{n-1}\right] \quad \left[1 = {}^{n-1}C_{n-1}\right]$$

$$= n(1+1)^{n-1} \qquad \left\{\begin{array}{l}\text{Putting } x = 1 \text{ in } (1+x)^{n-1} \\ = 1 + {}^{n-1}C_{n-1}x + {}^{n-1}C_2x^2 + \cdots + x^{n-1}\end{array}\right\}$$

$$= n \cdot 2^{n-1}]$$

(*ii*) $C_0 + 3C_1 + 5C_2 + \cdots + (2n+1)C_n = (n+1)2^n$.

(*iii*) $\dfrac{C_0}{1} + \dfrac{C_2}{3} + \dfrac{C_4}{5} + \dfrac{C_6}{7} + \cdots = \dfrac{2^n}{n+1}$

(*iv*) $C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots = 2^{n-1}$

(*v*) $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \cdots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

(*vi*) $\dfrac{C_1}{C_0} + 2.\dfrac{C_2}{C_1} + 3.\dfrac{C_3}{C_2} + \cdots + n.\dfrac{C_n}{C_{n-1}} = \dfrac{n(n+1)}{2}$

(*vii*) $C_0 - \dfrac{C_1}{2} + \dfrac{C_2}{3} \cdots + (-1)^n.\dfrac{C_n}{n+1} = \dfrac{1}{n+1}$

8. If the coefficient of x in the expansion of $\left(x^2 + \dfrac{k}{x}\right)^5$ is 270, find k.

[**Ans.** $k = 3$]

9. If the coefficient of x^2 and x^3 in the expansion of $(3 + kx)^9$ are equal, find the value of k. **[Ans.** $k = \frac{9}{7}$]

10. In the expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ the term free from x is 405. Find the value of k.

    [***C.U.B. Com. '91***] **[Ans.** ± 3]

11. If the middle term of the expansion $\left(\frac{k}{2} + 2\right)^8$ is 1120, find k. **[Ans.** $K = \pm 2$]

12. (*i*) Find the coefficient of (*i*) x^{15} in the expansion of $\left(2x^2 - \frac{1}{2x}\right)^{12}$.

    [***K.U.B.Com. '95***] **[Ans.** –140.80]

    (*ii*) x^6 in the expansion of $(1 + x)^8$ [***K.U.B.Com. '97***] **[Ans.** 28]

    (*iii*) x^{10} in the expansion of $\left(x - \frac{2}{x^2}\right)^{16}$. [***K.U.B.Com. '98***] **[Ans.** 480]

C. ADVANCED PROBLEMS

1. Find the coefficient of x in the expansion of $\left(1 - 2x^3 + 3x^5\right)\left(1 + \frac{1}{x}\right)^8$.

   [Ans. 154]

   [Hints:

   $$\left(1 - 2x^3 + 3x^5\right) \times \left(1 + {}^8C_1 \frac{1}{x} + {}^8C_2 \frac{1}{x^2} + {}^8C_3 \frac{1}{x^3} + {}^8C_4 \frac{1}{x^4} + \ldots\ldots + \frac{1}{x^8}\right)$$

   $\therefore$ Coefficient of $x = -2 \times {}^8C_2 + 3 \times {}^8C_4$ and x has no other coefficient in the product.

   $\therefore$ Required coefficient = –56 + 210 = 154]

2. Find the coefficient of x in the expansion of $\left(1 - 3x + x^2\right)\left(x - \frac{1}{x}\right)^{11}$.

   [Ans. – 110]

3. Find the coefficient of x^{15} in the expansion of $(x - x^2)^{10}$. **[Ans.** – 252]

4. Show that the middle term in the expansion of $\left(x + \frac{1}{x}\right)^{2n}$ is $\frac{1.3.5.7\ldots\ldots\ (2n-1)}{\underline{|n}} 2^n$.

5. Show that the middle term in the expansion of $\left(x-\frac{1}{x}\right)^{2n}$ is $\frac{1.3.5\cdots(2n-1)}{\lfloor n}.(-2)^n$.

6. Show that the coefficient of x^m and x^n are equal in the expansion of $(1+x)^{m+n}$.

7. Find the coefficient of x^{11} in the expansion of $(1-2x+3x^2)(1+x)^{11}$.

[***C.A. Nov. '82***] [**Ans.** 144]

8. In the expansion of $(1+x)^{12}$, the ratio of a certain coefficient to the preceding coefficeint is 6 : 7 ; find the values of the two coefficients.

[***I.C.W.A.I. Dec. '87***] [**Ans.** 792 and 924]

9. Prove that, $^2C_0+\frac{2^2}{2}C_1+\frac{2^3}{3}C_2+\cdots+\frac{2^{11}}{11}C_{10}=\frac{3^{11}-1}{11}$.

10. Apply the Binomial Theorem to find the value of $(.999)^4$ correct to 3 places of decimals. [**Ans.** 0.996]

11. Find the coefficient of x^r in the expansion of $\left(x-\frac{1}{x}\right)^n$, where each of n and r is a positive integer. [***C.U.B.Com(Hons.) '84***]

[**Ans.** $(-1)^{\frac{n-r}{2}}.{}^nC_{\frac{n-r}{2}}$]

12. Show that the coefficient of x in the expansion of $\left(x+\frac{1}{x}\right)^n$ is

$$\frac{n!}{\left(\frac{n+p}{2}\right)!\left(\frac{n-p}{2}\right)!}.$$

13. The coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^{2n}$ are in A.P., prove that $2n^2-9n+7=0$.

14. If the binomial coefficient of three successive terms in the binomial expansion of $(1+x)^n$, where n is a positive integer, are in the ratio 1 : 2 : 3, find the value of n.

[***I.C.W.A.I., June '79, '82***] [***C.A. May '83***] [**Ans.** $n = 14$]

15. If in the expansion of $(1+x)^{2n+1}$ the coefficients of x^r and x^{r+1} be equal, find r.

[**Ans.** $r = n$]

16. If the rth. term in the expansion of $(1+x)^{20}$ has its coefficient equal to that of $(r+4)$th term, find r. [**Ans.** $r = 9$]

17. In the expansion of $(x + a)^n$, if the sum of odd terms be P and sum of even terms be Q, then $P^2 - Q^2 = (x^2 - a^2)^n$.

[**Hints:** $(x+a)^n = \left(x^n + {}^nC_2 x^{n-2}a^2 + \cdots\right) + \left({}^nC_1 x^{n-1}.a + C_3 x^{n-3}.a^3 + \cdots\right)$

$= P + Q$

Similarly $(x - a)^n = P - Q$.]

18. If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots + C_nx^n$, prove that

(*i*) $C_0C_n + C_1C_{n-1} + C_2C_{n-2} + \cdots + C_nC_0 = \dfrac{\lfloor 2n}{(\lfloor n)^2}$

(*ii*) $C_1 - 2C_2 + 3C_3 - \cdots + (-1)^{n-1}.n.C_n = 0.$

(*iii*) $\dfrac{C_1}{2} + \dfrac{C_3}{4} + \dfrac{C_5}{6} + \cdots = \dfrac{2^n - 1}{n+1}.$

(*iv*) $2C_0 + C_1 + 2C_2 + C_3 + 2C_4 + C_5 + \cdots = 3.2^{n-1}$

[**Hints:** L.H.S. $= 2(C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + \cdots) - (C_1 + C_3 + C_5 + \cdots)$

$= 2.2^n - \dfrac{1}{2}.2^n = 3.2^{n-1}$]

(*v*) $C_0 + 2C_1 + 3C_2 + \cdots + (n+1)C_n = (n+2).2^{n-1}$

D. MULTIPLE CHOICE QUESTIONS (MCQs)

(i) Short Type

Mark the correct alternative in each of the following:

1. The number of terms in the expansion of $\left(2x + \dfrac{3}{x}\right)^7$ is:

(*a*) 7 (*c*) 9
(*b*) 8 (*d*) 6 [**Ans.** (*b*)]

2. The 8th term in the expansion of $(1 + x)^{11}$ is:

(*a*) $330x^7$ (*c*) $320x^7$
(*b*) $440x^8$ (*d*) $420x^6$ [**Ans.** (*a*)]

3. The 5th term in the expansion of $\left(\dfrac{x}{a} - \dfrac{a}{x}\right)^{10}$ is:

(*a*) $200\left(\dfrac{x}{a}\right)^9$ (*c*) $210\left(\dfrac{x}{a}\right)^2$

(*b*) $180\left(\dfrac{x}{a}\right)^3$ (*d*) $220\left(\dfrac{x}{a}\right)^5$ [**Ans.** (*c*)]

4. If the number of terms in $(1 + x)^n$ be 11, then the 5th term is:

   (*a*) $200x^5$ (*c*) $210x^5$
   (*b*) $180x^4$ (*d*) $210x^4$ **[Ans.** (*d*)]

5. If the number of terms in $\left(x+\frac{2}{x}\right)^n$ be 13, then the value of n is:

   (*a*) 14 (*c*) 10
   (*b*) 12 (*d*) 11 **[Ans.** (*b*)]

6. The number of terms in the expansion of $\left(1+\sqrt{2}x\right)^9+\left(1-\sqrt{2}x\right)^9$ is:

   (*a*) 5 (*c*) 9
   (*b*) 7 (*d*) 6 **[Ans.** (*a*)]

   [**Hints:** If n is odd, then the number of terms $=\frac{n+1}{2}$]

7. The number of terms in the expansion of $\left(2+3\sqrt{2}x\right)^{10}+\left(2-3\sqrt{2}x\right)^{10}$ is:

   (*a*) 5 (*c*) 4
   (*b*) 6 (*d*) 7 **[Ans.** (*b*)]

   [**Hints:** If n is even, then the number of terms $=\frac{n}{2}+1$]

8. The coefficient of x^{-17} in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$ is:

   (*a*) –1270 (*c*) 2500
   (*b*) –1365 (*d*) 1750 **[Ans.** (*b*)]

9. The coefficient of x^4 in the expansion of $\left(\frac{x}{2}-\frac{3}{x^2}\right)^{10}$ is:

   (*a*) $\frac{380}{290}$ (*c*) $\frac{505}{286}$
   (*b*) $\frac{425}{271}$ (*d*) $\frac{405}{256}$ [Ans. (d)]

10. The coefficient of x^{11} in the expansion of $(3x + 2x^2)^9$ is:

    (*a*) 314928 (*c*) 324715
    (*b*) 335028 (*d*) None of these **[Ans.** (*a*)]

11. The term independent of x in the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$ is:

    [***C.U.B.Com 1992***]

(*a*) 7 (*c*) 8
(*b*) 10 (*d*) 9 [**Ans.** (*d*)]

12. The term independent of x in the expansion of $\left(x-\frac{2}{x^2}\right)^{15}$ is:

(*a*) 5 (*c*) 7 [***C.U.B.Com 1997***]
(*b*) 6 (*d*) 8 [**Ans.** (*b*)]

13. Value of the term without x in the expansion of $\left(2x-\frac{1}{2x^2}\right)^{12}$ is:

(*a*) 7920 (*c*) 7860
(*b*) 7580 (*d*) None of these [**Ans.** (*a*)]

14. Value of the term independent of x in $\left(x^2+\frac{1}{x}\right)^{12}$ is: [***C.U.B.Com. 1998***]

(*a*) 515 (*c*) 495
(*b*) 480 (*d*) 505 [**Ans.** (*c*)]

15. The middle term in the expansion of $\left(\frac{2x^2}{3}+\frac{3}{2x^2}\right)^{10}$ is:

(*a*) 240 (*c*) 250
(*b*) 262 (*d*) 252 [**Ans.** (*d*)]

16. The middle term in the expansion of $\left(3x-\frac{1}{2x}\right)^{8}$ is:

(*a*) $\frac{2734}{7}$ (*c*) $\frac{2783}{5}$
(*b*) $\frac{2835}{8}$ (*d*) None of these [**Ans.** (*b*)]

17. The middle term of $(1 - 2x + x^2)^5$ is:

(*a*) $-252x^5$ (*c*) $275x^4$
(*b*) $-225x^5$ (*d*) $245x^4$ [**Ans.** (*a*)]

18. If rth term in the expansion of $\left(2x^2-\frac{1}{x}\right)^{12}$ is without x, then r is equal to:

(*a*) 7 (*c*) 9
(*b*) 10 (*d*) 8 [**Ans.** (*c*)]

19 If in the expansion of $(1 + x)^{20}$, the coefficients of rth and $(r + 4)^{\text{th}}$ terms are equal, then r is equal to:
(*a*) 9 (*c*) 10
(*b*) 8 (*d*) 12 [**Ans.** (*a*)]

20. If in the expansion of $(1 + x)^{15}$, the coefficients of $(2r + 3)$th and $(r - 1)$th terms are equal, then the value of r is:
    (*a*) 8 (*c*) 5
    (*b*) 6 (*d*) 7 [**Ans.** (*c*)]
21. If in the expansion of $(1 + y)^n$, the coefficients of 5th, 6th and 7th terms are in A.P., then n is equal to:
    (*a*) 6, 12 (*c*) 7, 21
    (*b*) 7, 14 (*d*) None of these [**Ans.** (*b*)]
22. If the fifth term of the expansion $\left(a^{2/3} + \frac{1}{a}\right)^n$ does not contain 'a', then n is equal to:
    (*a*) 10 (*c*) 9
    (*b*) 12 (*d*) 11 [**Ans.** (*a*)]
23. If $(1 + x)^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$, then the value of $c_0 + c_2 + c_4 + \ldots$ is:
    (*a*) 2^n (*c*) 2^{n+1}
    (*b*) 2^{n-1} (*d*) None of these [**Ans.** (*b*)]
24. If $(1 + x)^{10} = c_0 + c_1 x + c_2 x^2 + \cdots + c_{10} x^{10}$, then the value of $c_1 - 2c_2 + 3c_3 - 4c_4 + \ldots$ is:
    (*a*) 0 (*c*) 8
    (*b*) 5 (*d*) 7 [**Ans.** (*a*)]
25. If $(1 + x)^8 = c_0 + c_1 x + c_2 x^2 + \cdots + c_8 x^8$, then the value of $\frac{c_1}{2} + \frac{c_3}{4} + \frac{c_5}{6} + \ldots$ is:
    (*a*) $\frac{85}{3}$ (*c*) $\frac{86}{7}$
    (*b*) $\frac{78}{5}$ (*d*) $\frac{94}{3}$ [**Ans.** (*a*)]

(ii) Short Essay Type

1. 10th term in the expansion of $\left(2x^2 + \frac{1}{x^2}\right)^{25}$ is
    (*a*) $\frac{880}{x^2}$ (*c*) $\frac{1760}{x^2}$
    (*b*) $\frac{880}{x^3}$ (*d*) $\frac{1760}{x^3}$ [**Ans.** (*d*)]
2. If coefficients of x^7 and x^8 are equal in expression of $\left(2 + \frac{x}{3}\right)^n$, then $n =$
    (*a*) 54 (*c*) 56
    (*b*) 55 (*d*) 58 [**Ans.** (*b*)]

3. Coefficient of middle term in expansion of $\left(x-\frac{x^3}{5}\right)^8$ is

(*a*) $\frac{14}{625}$ (*c*) $\frac{70}{125}$

(*b*) $\frac{14}{125}$ (*d*) $\frac{70}{625}$ [**Ans.** (*d*)]

4. If middle term is kx^m in expansion of $\left(x+\frac{1}{x}\right)^{12}$, then $m =$

(*a*) 0 (*c*) –1

(*b*) 1 (*d*) –2 [**Ans.** (*d*)]

5. In the expansion of $(x + y)^{13}$ the coefficients of 3rd term and ______ th terms are equal.

(*a*) 8 (*c*) 12

(*b*) 11 (*d*) 13 [**Ans.** (*c*)]

6. 16th term and 17th term are equal in expansion of $(2 + x)^{40}$ then $x =$

(*a*) $\frac{17}{12}$ (*c*) $\frac{34}{13}$

(*b*) $\frac{17}{24}$ (*d*) $\frac{34}{15}$ [**Ans.** (*a*)]

7. If the ratio of coefficients of three consecutive terms in expansion of $(1 + x)^n$ is 1 : 7 : 42 then $n =$

(*a*) 35 (*c*) 55

(*b*) 45 (*d*) 65 [**Ans.** (*c*)]

8. Coefficient of x^{-3} in expansion of $\left(x-\frac{a}{x}\right)^{11}$ is ______

(*a*) $-792a^5$ (*c*) $-330a^7$

(*b*) $-792a^6$ (*d*) $-923a^7$ [**Ans.** (*c*)]

9. Coefficients of 5th, 6th and 7th terms are in A.P. for expansion of $(1 + x)^n$, then $n =$ ________

(*a*) –7 or 14 (*c*) 7 or 12

(*b*) –7 or 12 (*d*) 7 or 14 [**Ans.** (*d*)]

10. If r^{th} term in expansion of $\left(2x^3+\frac{5}{x^2}\right)^{10}$ is constant, then $r =$ _______

(*a*) 4 (*c*) 6

(*b*) 5 (*d*) 7 [**Ans.** (*d*)]

11. The two successive terms in the expansion of $(1 + x)^{24}$ whose coefficients are in the ratio 1 : 4 are
    (*a*) 3rd and 4th (*c*) 5th and 6th
    (*b*) 4th and 5th (*d*) 6th and 7th **[Ans.** (*c*)]

12. The coefficient of x^n in the expansion of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ are in the ratio
    (*a*) 1 : 2 (*c*) 3 : 1
    (*b*) 2 : 1 (*d*) 1 : 3 **[Ans.** (*b*)]

13. If A and B are coefficients of x^n in the expansions of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ respectively, then $\frac{A}{B}$ equals
    (*a*) 1 (*c*) $\frac{1}{2}$
    (*b*) 2 (*d*) $\frac{1}{n}$ **[Ans.** (*b*)]

14. The term independent of x and its value in the expansion of $\left(\frac{\sqrt{x}}{\sqrt{3}} + \frac{\sqrt{3}}{2x^2}\right)^{10}$ are
    (*a*) $3, \frac{5}{12}$ (*c*) $2, \frac{7}{9}$
    (*b*) $3, \frac{5}{6}$ (*d*) $2, \frac{3}{5}$ **[Ans.** (*a*)]

15. The coefficient of x^p and x^q (p and q are positive integers) in the expansion of $(1 + x)^{p+q}$ are
    (*a*) equal (*c*) reciprocal of eah other
    (*b*) equal with opposite signs (*d*) none of these **[Ans.** (*a*)]

16. The ratio of coefficient of x^{15} to the term independent of x in $\left(x^2 + \frac{2}{x}\right)^{15}$ is
    (*a*) 12 : 32 (*c*) 32 : 12
    (*b*) 1 : 32 (*d*) 32 : 1 **[Ans.** (*b*)]

17. If $(1 - x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$,then $a_0 + a_2 + a_4 + ... + a_{2n}$ equals
    (*a*) $\frac{3^n + 1}{2}$ (*c*) $\frac{1 - 3^n}{2}$
    (*b*) $\frac{3^n - 1}{2}$ (*d*) $3^n + \frac{1}{2}$ **[Ans.** (*a*)]

18. If $(1 + x)^n = c_0 + c_1x + c_2x^2 + ... + c_nx^n$, then the value of $c_0^2 + c_1^2 + c_2^2 + \cdots + c_n^2$ is

(*a*) 2^{n-1} (*c*) $(2n)!n!$

(*b*) $\dfrac{(2n)!}{(n!)^2}$ (*d*) $\dfrac{2^{n-1}}{n+1}$ [**Ans.** (*b*)]

19. If $(1 + x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots + c_nx^n$, then the value of $2c_0 + c_1 + 2c_2 + c_3 + 2c_4 + c_5 + \ldots$ is

(*a*) 2.2^n (*c*) 3.2^{n-1}

(*b*) 2.2^{n-1} (*d*) 3.3^{n-1} [**Ans.** (*c*)]

20. r and n are positive integers $r > 1$, $n > 2$ and coefficient of $(r + 2)^{\text{th}}$ term and $3r^{\text{th}}$ term in the expansion of $(1 + x)^{2n}$ are equal, then n equals

(*a*) $3r$ (*c*) $2r$

(*b*) $3r + 1$ (*d*) $2r + 1$ [**Ans.** (*c*)]

CHAPTER

4

Logarithm

SYLLABUS

Definition, Base and Index of Logarithm, General Properties of Logarithm, Common Problems

THEMATIC FOCUS

4.1 INTRODUCTION

Logarithms in mathematics were developed for making complicated calculations simple. The calculations which are not possible by using law of indices, can easily be solved by using laws of logarithms. For example, $3^x = 27$. By using law of indices we can easily calculate the value of x, as $3^x = 3^3$, then $x = 3$. But when $3^x = 10$, then how can you calculate the value of x? We can take another example, suppose you are asked to calculate the volume of a right circular cylinder when its radius $(r) = 0.231$ cm., height $(h) = 0.3175$ cm., then its volume (V) is given by: $V = \pi r^2 h = 3.146 \times (0.231)^2 \times 0.3175$. We can easily calculate this complicated calculation by using logarithm table. However, even having functions like multiplication, power, etc., in the calculator, logarithmic and exponential equations and functions are very common in mathematics. Logarithm is a computational tool to handle large numbers. By using this tool we can easily workout the tedious calculations of powers, roots and multiplications.

4.2 DEFINITION

If there are three quantities a, x and n, such that

a = positive real number but not unity ($a > 0$, but $a \neq 1$)

n = any real number ($n > 0$)

x = exponent or index of the power

are so related that $a^x = n$, then 'x' is said to be the logarithm of the number 'n' with respect to the base 'a' and written as:

$x = \log_a n$

(read as $x = \log n$ to the base 'a')

$x = \log_a n \Leftrightarrow a^x = n$

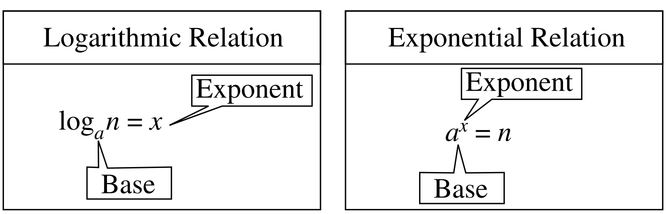
the symbol '$\Leftrightarrow$' means equivalent or the same)

From the definition it is clear that the logarithm of a number has no meaning if the base is not mentioned.

Hence, if $a^x = n$, then $x = \log_a n$

Conversely, if $x = \log_a n$ then $a^x = n$

Here, $a^x = n$ is called an exponential relation and $x = \log_a n$ is called a logarithmic relation



OBSERVATIONS

- The subscript 'a' in logarithmic relation becomes the base in exponential relation.
- The base 'x' in logarithmic relation becomes the exponent or superscript of 'a' in exponential relation.

EXAMPLES

Logarithmic Form	**Exponential Form**
$\log_6 36 = 2$	since $6^2 = 36$
$\log_2 8 = 3$	since $2^3 = 8$
$\log_9 1 = 0$	since $9^0 = 1$
$\log_{10} 0.01 = -2$	since $10^{-2} = 0.01$
$\log_5 0.008 = -3$	since $5^{-3} = 0.008$

OBSERVATIONS

From the above examples it is important to note that:

1. Logarithm is defined only positive real numbers. (as $a > 0$ but not unity, $n > 0$)
2. Logarithm of a positive real number may be negative, zero or positive.
3. Logarithm of a negative number is imaginary.
4. Logarithmic values of a given number are different for different bases.

4.3 PROPERTIES OF LOGARITHMS

1. Logarithm of 1 to any finite non-zero base is zero.

Proof: We know, $a^0 = 1$ ($a \neq 0$). Therefore, from the definition we have, $\log_a 1 = 0$.

2. Logarithm of a positive number to the same base is always 1.

Proof: Since $a^1 = a$. Therefore $\log_a a = 1$.

3. If $x = \log_a n$, then $a^{\log_a n} = n$

Proof: As $x = \log_a n$, therefore, $a^x = n$.

or $\quad a^{\log_a n} = n \quad$ [As $x = \log_a n$]

4.4 TYPES OF LOGARITHMS

In practice, however, two types of logarithms are used:

(*i*) **Natural or Napierian logarithm:** The logarithm of a number to the base '*e*' is called *Natural or Napierian logarithm* after the name of John Napier. Here, '*e*' is a mathematical constant and is an incommensurable number. It is known as Euler's number. It is also called Napier's constant. Its value is 2.7183 approximately. Natural logarithm is usually written as In. Thus 'In x' means $\log_e x$.

(*ii*) **Common logarithm:** The logarithm of a number to the base 10 is called *common logarithm.* This system was invented by Henry Briggs and this type of logarithm is used for mathematical calculations. If no base is given, the base is assumed to be 10. Logarithm tables also assume base 10.
Thus 'log 29' means $\log_{10} 29$.

4.5 GENERAL LAWS OF LOGARITHM

Law 1 Product Formula

$$\log_a (m \times n) = \log_a m + \log_a n$$

Proof: Let $\log_a m = x$ then, $a^x = m$

and $\log_a n = y$ then, $a^y = n$

Now, $a^x \cdot a^y = m \times n$

or $a^{x+y} = m \times n$

or $\log_a (m \times n) = x + y$ (from the definition of logarithm)

or $\log_a (m \times n) = \log_a m + \log_a n$ [Putting the values of x and y]

The above law can be extended in a similar way to the product of any number of factors i.e.

$$\log_a (m \times n \times p \times \ldots) = \log_a m + \log_a n + \log_a p + \ldots$$

Thus, the logarithm of the product of two or more positive factors is equal to the sum of their logarithms taken separately.

Law 2 Quotient Formula

$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Proof: Let $\log_a m = x$ then, $a^x = m$

and $\log_a n = y$ then, $a^y = n$

Now, $\frac{a^x}{a^y} = \frac{m}{n}$ or $a^{x-y} = \frac{m}{n}$

or $\log_a \left(\frac{m}{n}\right) = x - y$ (from the definition of logarithm)

or $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$ [Putting the values of x and y]

Thus, the logarithm of the quotient of two factors is equal to the difference of their logarithms.

Law 3 **Power Formula**

$$\log_a m^n = n \cdot \log_a m$$

Proof: Let $\log_a m = x$ then, $a^x = m$

or $(a^x)^n = (m)^n$

or $a^{nx} = (m)^n$

or $\log_a (m)^n = nx$ (from the definition if logarithm)

or $\log_a (m)^n = n. \log_a m$ (Putting the value of n)

Thus, the logarithm of a power of a factor is the product of the power and the logarithm of the factor.

Law 4 **Base Change Formula**

$$\log_a m = \log_b m \times \log_a b$$

Proof: Let $\log_a m = x$ then $a^x = m$

$\log_b m = y$ then $b^y = m$

and $\log_a b = z$ then $a^z = b$

Now, $a^x = m = b^y$ or $a^x = (a^z)^y$ (Putting the value of b)

or $a^x = a^{yz}$

or $x = yz$

or $\log_a m = \log_b m \times \log_a b$ [Putting the values of x, y and z]

Thus, the logarithm of a number m with respect to base 'a' is equal to the product of the logarithm of m with respect to the new base 'b' and the logarithm of b with respect to the old base 'a'.

Corollary 1: Putting $m = a$ in the law 4

$$\log_a m = \log_b m \times \log_a b$$

$$\log_a a = \log_b a \times \log_a b$$

or $\log_b a \times \log_a b = 1$ [as $\log_a a = 1$]

or $$\log_b a = \frac{1}{\log_a b}$$

Thus, the logarithm of a positive number 'a' with respect to a positive base 'b' ($\neq 1$) is equal to the reciprocal of logarithm 'b' with respect to the base 'a'.

Corollary 2: $\log_a m = \log_b m \times \log_a b$

or $$\log_b m = \frac{\log_a m}{\log_a b}$$

Thus, the logarithm of a positive number 'm' with respect to a positive base 'b' ($\neq 1$) is equal to the quotient of the logarithm of the number m and the logarithm of the number 'b' both with respect to any positive base a ($\neq 1$).

4.6 CHARACTERISTIC AND MANTISSA

The logarithm of a number has two parts, integral and decimal. The integral part of a common logarithm is called the *characteristic* and the decimal part is called the *mantissa*. The characteristic of the logarithm of a number may be zero or any integer (positive or negative) but its mantissa is always positive. The characteristic of the logarithm is determined by inspection, whereas mantissa is determined by logarithm table.

EXAMPLE

Suppose, log 21.43 = 1.3310, then characteristic is 1 and the mantissa is .3310 of the logarithm.

NOTE

Expression of decimal part of a negative decimal number into positive.

In order to express the decimal part of a negative decimal number as positive we are to subtract 1 from the integral part of the decimal number and then add 1 to its decimal part.

ILLUSTRATION 1

Express the decimal part of –3.2357 as positive.

Solution: $-3.2357 = -3 - 1 + 1 - 0.2357$

$$= -4 + (1 - 0.2357) = -4 + 0.7643$$

$$= \bar{4}.7643.$$

4.7 DETERMINATION OF CHARACTERISTIC

The number of characteristic of a logarithm depends upon the number of digits in the left of decimal point. In this respect we have two cases.

Case 1: *When the number of digits in the left of decimal point is greater than 1.*

Then count the digit, if the number of digits is K then the characteristic is $(K - 1)$.

ILLUSTRATION 2

Number (x):	512.37	25.325	7.8291
Number of digits in the left of decimal point (k):	3	2	1
Characteristic of log $x = (k–1)$:	3 – 1 = 2	2 – 1 = 1	1 – 1 = 0

Case 2: *When the number of digits in the left of decimal point is lying between 0 and 1*

Then count the number of zeroes appearing in the right side of the decimal point. If the number of zeroes is k, then the characteristic is – (k + 1). This is written as $\overline{(k+1)}$, read as (k + 1) bar.

ILLUSTRATION 3

Number (x):	0.8693	0.07231	0.00329
Number of zeroes in the right side of decimal point (k):	0	1	2
Characteristic of log x = $\overline{(k+1)}$:	$\overline{(0+1)} = \overline{1}$	$\overline{(1+1)} = \overline{2}$	$\overline{(2+1)} = \overline{3}$

4.8 DETERMINATION OF MANTISSA

Mantissa is a positive proper fraction. It is calculated with the help of a log table, given at the end of the book. The rules to calculate mantissa as well as the method to read logarithm table are as follows:

(*i*) Ignore the decimal point of a given number. For example, read 23.45 as 2345. After ignoring the decimal point, if the number of digits is not four, then make them four by inserting zeroes to the right side of a given number accordingly. Such as, read 30.5 as 3050 or, read 4.7 as 4700.

(*ii*) Take the first 2 digits of the number, i.e. 23 from 2345 and go down along the extreme left-hand column of the log table to the number 23.

(*iii*) Take the third digit of the number, i.e. 4 and read the intersection of horizontal row beginning with 23 and vertical column headed by 4, which gives the number 3692.

(*iv*) Now, take the fourth digit which is 5 and in a similar way read the intersection of horizontal row beginning with 23 and vertical column headed by 5 in the mean difference column, which gives the number 9.

(*v*) Then, by adding the number obtained in (*iv*) to the number obtained in (*iii*), we get 3692 + 9 = 3701.

(*vi*) Now, if we prefix the decimal to the number obtained by (*v*), the mantissa of the number 23.45 is .3701.

Log Table

											Mean Difference								
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16

After that, it is observed that the mantissa does not depend on the position of the decimal point in the number. It is same for the logarithms of all numbers with the same sequence of digits. In fact, at the time of determining mantissa by the log table, the decimal point of the number is neglected.

4.9 CALCULATION OF LOGARITHM OF A NUMBER

If x is a positive number then log x can be written as log x = characteristic of x + mantissa of x.

Therefore, the logarithm of a number is calculated by finding out its characteristic and mantissa, and adding them up.

ILLUSTRATION 4

Using log table find the value of

(*i*) log 752.3 (*iii*) log 6 (*v*) log 0.05234
(*ii*) log 2143 (*iv*) log 0.5632 (*vi*) log 0.004205

Solution:

	when $x > 1$			**when $0 < x < 1$**		
Number (x)	752.3	2143	6	0.5632	0.05234	0.004205
Number of digits in the left of decimal point	3	4	1	–	–	–
Number of zeroes in the right side of decimal point	–	–	–	0	1	2
Characteristic of log x	$3 - 1 = 2$	$4 - 1 = 3$	$1 - 1 = 0$	$\overline{(0 + 1)} = \bar{1}$	$\overline{(1 + 1)} = \bar{2}$	$\overline{(2 + 1)} = \bar{3}$
Mantissa of log x	0.8764	0.3310	0.7782	0.7507	0.7188	0.6237
Value of log x	2.8764	3.3310	0.7782	$\bar{1}.7507$	$\bar{2}.7188$	$\bar{3}.6237$

4.10 DETERMINATION OF MANTISSA OF A NUMBER CONSISTING OF MORE THAN FOUR DIGITS

Mantissa usually consists of a four-digit number. If the number of digits after ignoring the decimal point is more than four, say five digits, then to find mantissa by using the table, either.

(*i*) round off the last digit for rough calculations by adding 1 to the 4th digit if the 5th digit is equal to or greater than 5 or take the last digit is zero, in case the 5th digit is less than 5 or

(*ii*) apply the principle of proportional parts.

ILLUSTRATION 5

Find the log of 12344

Solution: This is a five-digit number, so the last, i.e. the fifth digit, will have to be rounded off. The fifth digit is 4, which is less than 5. So take the last digit is 0. Thus the mantissa of 12344 will be same as the mantissa 12340. Mantissa of 12340 is 0.0913 and characteristic is 4.

Therefore, log 12344 = 4.0913.

ILLUSTRATION 6

Find the log of 12346

Solution: This is a five-digit number, so the last digit, i.e. the fifth digit, will have to be rounded off. The fifth digit is 6 which is greater than 5. So the last digit 6 is to be rounded off as 1 and to be added to the second last number 4. Thus, the last digit becomes 4 + 1 = 5. Now, we have to find the mantissa for 1235, which is 0.0917.

Therefore, log 12346 = 4.0917

By applying principle of proportional parts

ILLUSTRATION 7

Find the log of 34567

Solution: This is a five-digit number, so its characteristic is 4.

Its mantissa is (using four figure log table)	0.5378
mean difference for 6	8
mean difference for 7	9
	0.53869

Thus, log 34567 = 4.53869

ILLUSTRATION 8

Find the log of 0.23457

Solution: The characteristic of log 0.23457 is $\bar{1}$.

Its mantissa is	0.3692
mean difference for 5	9
mean difference for 7	13
	0.37023

Thus, log 0.23457 = $\bar{1}.37023$

4.11 ANTILOGARITHM

Antilogarithm is the exact opposite of logarithm of a number. If log $n = x$, then n is called the antilogarithm of x and is written as n = antilog x. The table of antilogarithm is also given at the end of the book. The use of antilog table is similar to that of log table. Antilog tables are used for determining the inverse value of the mantissa. From the characteristic, the position of the decimal point can be determined.

After getting inverse value of the mantissa, consider the characteristic, i.e. integral part of the given number, for placing the decimal point in the number obtained from antilog table.

(a) If the value of characteristic is greater than zero

In this case, add 1 to the value of characteristic (say k) and you will get $(k + 1)$. Finally, prefix the decimal point after $(k + 1)$ digits in the number obtained from antilog table.

(b) If the value of characteristic is less than zero and negative

In this case, subtract 1 from the number (say k) under the bar sign and you will get $(k – 1)$. Attach $(k – 1)$ zeroes to the left-hand side of the number obtained from antilog table, and finally prefix the decimal point.

ILLUSTRATION 9

Find the numbers whose antilogarithms are:

(*i*) 1.4352 (*ii*) $\bar{2}.6347$

Solution:

(*i*) We first ignore the characteristic 1 and consider the decimal part 0.4352. Read the antilogarithm table in the same manner as logarithm table.

(*a*) Take the first two digits of the number, i.e. 43, from 4352 and go down along the extreme left-hand column of the antilog table to the number 43.

(*b*) Now, take the third digit of the number, i.e. 5, and read the intersection of horizontal row beginning with .43 and vertical column headed by 5 from the left, which given the number 2723.

(*c*) Again take the fourth digit, which is 2, and read the intersection of horizontal row beginning with .43 and vertical column headed by 2 in the mean difference column, which gives the number 1.

(*d*) Then, by adding the numbers obtained in (*c*) to the number obtained in (*b*), we get 2723 + 1 = 2724.

(*e*) Now, consider the value of characteristic i.e. 1 which is greater than zero. Thus, add 1 to it, and you will get 2.

Finally, prefix the decimal point after 2 digits in the number obtained in (*d*), i.e. 2724.

Therefore, the required number whose antilog is 1.4352 is 27.24.

Antilog table

											Mean Difference								
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6

(*ii*) Given log $n = \bar{2}.6347$

We first ignore the characteristic $\bar{2}$ and consider the decimal part 0.6347.

(*a*) Take the first two digits of the number, i.e. 63, from 6347 and go down along the extreme left-hand column of the antilog table to the number 63.

(*b*) Now, take the third digit of the number, i.e. 4, and read the intersection of horizontal row beginning with .63 and vertical column headed by 4 from the left, which gives the number 4305.

(*c*) Again, take the fourth digit, which is 7, and read the intersection of horizontal row beginning with .63 and vertical column headed by 7 in the mean difference column, which gives the number 7.

(*d*) Then, by adding the numbers obtained in (*c*) to the number obtained in (*b*), we get 4305 + 7 = 4312.

(*e*) Now, consider the value of characteristic, i.e. $\bar{2}$, which is less than zero and negative. Thus, subtract 1 from the number under the bar sign and you will get 2 – 1 = 1. Attach one zero to the left.

(*f*) Now, consider the value of characteristic, i.e. $\bar{2}$, which is less than zero and negative. Thus, subtract 1 from the number under the bar sign and you will get 2 – 1 = 1. Hence, attach one zero to the left-head side of the number 4312.

Finally, prefix the decimal point.

Therefore, the required number whose antilog is $\bar{2}.6347$ is 0.04312.

Antilog table

											Mean Difference								
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9

ILLUSTRATIVE EXAMPLES

A. SHORT TYPE

EXAMPLE 1

Change the following from exponential to logarithmic forms:

(*i*) $7^{-2} = \frac{1}{49}$; (*ii*) $\sqrt[4]{81} = 3$; (*iii*) $9^0 = 1$

Solution:

(*i*) $\log_7\left(\frac{1}{49}\right) = -2$; (*ii*) $(81)^{\frac{1}{4}} = 3$ or $\log_{81} 3 = \frac{1}{4}$; (*iii*) $\log_9 1 = 0$.

EXAMPLE 2

Change the following from logarithmic to exponential form:

(*i*) $\log_6 6 = 1$; (*ii*) $\log_4 64 = 3$; (*iii*) $\log_{\sqrt{3}} 27 = 6$

Solution:

(*i*) $6^1 = 6$; (*ii*) $4^3 = 64$; (*iii*) $(\sqrt{3})^6 = 27$.

EXAMPLE 3

Find the logarithms of:

(*i*) 1728 to the base $2\sqrt{3}$ (*ii*) 0.0001 to the base 0.01

Solution:

(*i*) Let x be required logarithm

then, $\log_{2\sqrt{3}} 1728 = x$

or $(2\sqrt{3})^x = 1728 = 2^6 \cdot 3^3$

or $(2\sqrt{3})^x = 2^6 \cdot (\sqrt{3})^6$

or $(2\sqrt{3})^x = (2\sqrt{3})^6$

or $x = 6$

Therefore, the required logarithm is 6.

(*ii*) Let y be the required logarithm

Then, $\log_{0.01} 0.0001 = y$

or $(0.01)^y = 0.0001$

or $(0.01)^y = (0.01)^2$

or $y = 2$

Therefore, the required logarithm = 2.

EXAMPLE 4

If logarithm of 5832 be 6, find the base.

Solution: Let x be the required base

Therefore, $\log_x 5832 = 6$

or $x^6 = 5832 = 3^6 \cdot 2^3$

or $x^6 = 3^6 \cdot (\sqrt{2})^6$

or $x^6 = (3\sqrt{2})^6$

or $x = 3\sqrt{2}$

Therefore, the required base is $3\sqrt{2}$.

EXAMPLE 5

Find the base if log 81 = 8

Solution: Let the base is x

Then, $\log_x 81 = 8$

or $x^8 = 81 = 3^4$

or $x^8 = (\sqrt{3})^8$

or $x = \sqrt{3}$

Therefore, the required base is $\sqrt{3}$.

EXAMPLE 6

If $\log_5 x = 3$, find the value of x.

Solution: $\log_5 x = 3$

or $x = 5^3 = 125$

Therefore, the required value of $x = 125$.

EXAMPLE 7

Find x when $\log_7 x = 4$.

Solution: $\log_7 x = 4$

or $x = 7^4 = 2401.$

EXAMPLE 8

If $2 + \log_{10} x = 2 \log_{10} y$, find x in terms of y.

Solution: $2 + \log_{10} x = 2 \log_{10} y$

or $2 \cdot \log_{10} 10 + \log_{10} x = 2 \log_{10} y$ [as $\log_{10} 10 = 1$]

or $\log_{10} 10^2 + \log_{10} x = \log_{10} y^2$

or $\log_{10} (10^2 \cdot x) = \log_{10} y^2$

or $10^2 \cdot x = y^2$

or $x = \dfrac{y^2}{10^2} = \dfrac{y^2}{100}$

Therefore, required x in terms of $y = \dfrac{y^2}{100}$.

EXAMPLE 9

Prove that, $\log_b a \times \log_c b \times \log_d c = \log_d a$

Solution: L.H.S $= \log_b a \times \log_c b \times \log_d c$

$= \log_c a \times \log_d c$ [as $\log_a m = \log_b m \times \log_a b$]

$= \log_d a$ (using the same law).

EXAMPLE 10

If $\log_5 (x^2 + x) - \log_5 x = 2$ then find the value of x.

Solution: $\log_5(x^2 + x) - \log_5 x = 2$

or $\log_5 \left(\dfrac{x^2 + x}{x}\right) = 2$

or $\log_5 \left\{ \dfrac{x(1+x)}{x} \right\} = 2$

or $\log_5(1 + x) = 2$

or $1 + x = 5^2 = 25$

or $x = 25 - 1 = 24$

Therefore, the required value of x is 24.

EXAMPLE 11

Show that, $\log_4 2 \times \log_2 3 = \log_4 5 \times \log_5 3$.

Solution: L.H.S $= \log_4 2 \times \log_2 3$

$= \log_4 3$ [as $\log_a m = \log_b m \times \log_a b$]

R.H.S $= \log_4 5 \times \log_5 3$

$= \log_4 3$ [using the same rule]

Therefore, L.H.S. = R.H.S.

EXAMPLE 12

Prove that $\dfrac{\log_a x}{\log_{ab} x} = 1 + \log_a b$.

Solution: L.H.S $= \dfrac{\log_a x}{\log_{ab} x}$

$= \log_a x \times \log_x ab$ [as $\log_b a = \dfrac{1}{\log_a b}$]

$= \log_a x \times (\log_x a + \log_x b)$

$= \log_a x \times \log_x a + \log_a x \times \log_x b$

$= 1 + \log_a b$ (Proved) [as $\log_b a \times \log_a b = 1$].

EXAMPLE 13

Prove that, $\log_2 \log_{\sqrt{2}} \log_3 81 = 2$.

Solution: L.H.S. $= \log_2 \log_{\sqrt{\ }} \log_3 81$

$= \log_2 \log_{\sqrt{2}} \cdot \log_3 3^4 = \log_2 \log_{\sqrt{2}} (4 \cdot \log_3 3)$

$= \log_2 \log_{\sqrt{2}} 4 = \log_2 \log_{\sqrt{2}} (\sqrt{2})^4$

$= \log_2 (4 \log_{\sqrt{2}} \sqrt{2}) = \log_2 4 = \log_2 2^2 = 2 \cdot \log_2 2$

$= 2$ (Proved) [as $\log_a a = 1$].

EXAMPLE 14

If $x^2 + y^2 = 6xy$, prove that $2 \log (x + y) = \log x + \log y + 3 \log 2$.

Solution: $x^2 + y^2 = 6xy$

or $x^2 + y^2 + 2xy = 8xy$ or $(x + y)^2 = 8xy$

or $\log (x + y)^2 = \log 8xy$ (taking log on both sides)

or $2 \cdot \log (x + y) = \log 8 + \log x + \log y$

or $2 \log (x + y) = \log 2^3 + \log x + \log y$

or $2 \log (x + y) = 3 \log 2 + \log x + \log y$ [Proved].

EXAMPLE 15

If $\log_{10} 8.75 = 0.9406$, find the value of $\log_{10} 875$.

Solution: $\log_{10} 8.75 = 0.9406$ [Given]

Now, $\log_{10} 875 = \log_{10}(8.75 \times 100) = \log_{10} 8.75 + \log_{10} 100$

$= 0.9406 + 2 \log_{10} 10$

$= 0.9406 + 2 = 2.9406.$

EXAMPLE 16

If $\log_{10} 2 = 0.3010$ and $\log_{10} 7 = 0.8451$, then find the value of $\log_{10} 2.8$.

Solution: $\log_{10} 2.8 = \log_{10}\left(\frac{28}{10}\right) = \log_{10} 28 - \log_{10} 10$

$= \log_{10}(7 \times 4) - 1$

$= \log_{10} (7 \times 2^2) - 1 = \log_{10} 7 + 2 \log_{10} 2 - 1$

$= 0.8451 + 2 \times 0.3010 - 1$

$= 0.8451 + 0.6020 - 1 = 0.4471$

Therefore, the required value of $\log_{10} 2.8$ is 0.4471.

EXAMPLE 17

Find the value of $\frac{1}{\log_3 60} + \frac{1}{\log_4 60} + \frac{1}{\log_5 60}$.

Solution: $\frac{1}{\log_3 60} + \frac{1}{\log_4 60} + \frac{1}{\log_5 60}$

$= \log_{60} 3 + \log_{60} 4 + \log_{60} 5$

$= \log_{60}(3 \times 4 \times 5) = \log_{60} 60 = 1.$

EXAMPLE 18

Solve: $\log_{10}(5x–2) = 2$.

Solution: $\log_{10}(5x - 2) = 2$

or $5x - 2 = 10^2 = 100$

or $5x = 100 + 2 = 102$

or $x = \frac{102}{5}$.

EXAMPLE 19

Find antilog of –0.4632.

Solution: $-0.4632 = -1 + 1 - 0.4632$ [the decimal part must be made positive]

$= -1 + (1 - 0.4632) = -1 + 0.5368$

$= \bar{1}.5368$

Therefore, antilog (-0.4632) = Antilog $(\bar{1}.5368)$

$= 0.3442$.

EXAMPLE 20

If $\log x = -2.0468$, find x.

Solution: $\log x = -2.0468$ [the decimal part must be made positive]

$= -2 - 0.0468$

$= -2 - 1 + 1 - 0.0468$

$= -3 + (1 - 0.0468)$

$= \bar{3} + 0.9532$

$= \bar{3}.9532$

Therefore, x = Antilog $(\bar{3}.9532)$

$= 0.008978$.

EXAMPLE 21

Find the value of $\sqrt{3}$ with the help of log table.

Solution: Let $x = \sqrt{3}$

or $\log x = \log\sqrt{3} = \log 3^{\frac{1}{2}} = \frac{1}{2}\log 3$

$= \frac{1}{2} \times 0.4771$ (from log table)

$= 0.23855 = 0.2386$ (approx)

or $\quad x = \text{Antilog } 0.2386$

$= 1.732$ (from antilog table)

Therefore, the required value of $\sqrt{3}$ is 1.732.

EXAMPLE 22

Simplify: 15.2×27.8 with the help of log table.

Solution: Let $x = 15.2 \times 27.8$

or $\quad \log x = \log (15.2 \times 27.8)$

or $\quad \log x = \log 15.2 + \log 27.8$

$= 1.1818 + 1.4440$

$= 2.6258$

or $\quad x = \text{Antilog } 2.6258$

$= 422.5.$

EXAMPLE 23

If $\log 2 = 0.3010$, then find the number of digits in 2^{64}.

Solution: $\log 2^{64} = 64 \log 2 = 64 \times 0.3010 = 19.264$

Since the characteristic of $\log 2^{64}$ is 19, therefore, required number of digits in $2^{64} = 19 + 1 = 20$.

EXAMPLE 24

Find the number of digits in 8^{10}.
(Given that $\log 2 = 0.3010$)

Solution: $\log 8^{10} = 10 \times \log 8 = 10 \times \log 2^3$

$= 30 \log 2 = 30 \times 0.3010 = 9.03$

Since the characteristic of $\log 8^{10}$ is 9, therefore, required number of digits in $8^{10} = 9 + 1 = 10$.

EXAMPLE 25

Find x if $\log_{49} \sqrt{7} = \frac{1}{4} x$.

Solution: $\log_{49} \sqrt{7} = \frac{1}{4} x$

or $\quad (49)^{\frac{x}{4}} = \sqrt{7}$

or $$(7^2)^{\frac{x}{4}} = 7^{\frac{1}{2}} \text{ or } 7^{\frac{x}{2}} = 7^{\frac{1}{2}}$$

or $$\frac{x}{2} = \frac{1}{2} \text{ or } 2x = 2 \text{ or } x = 1.$$

B. SHORT ESSAY TYPE

EXAMPLE 26

Simplify $\dfrac{\log\sqrt{27} + \log 8 - \log\sqrt{1000}}{\log 1.2}$.

Solution: $\dfrac{\log\sqrt{27} + \log 8 - \log\sqrt{1000}}{\log 1.2}$

$$= \frac{\log(27)^{\frac{1}{2}} + \log 2^3 - \log(1000)^{\frac{1}{2}}}{\log\frac{12}{10}}$$

$$= \frac{\frac{1}{2}\log 27 + 3\log 2 - \frac{1}{2}\log 1000}{\log 12 - \log 10}$$

$$= \frac{\frac{1}{2}\log 3^3 + 3\log 2 - \frac{1}{2}\log 10^3}{\log(3\times 2^2) - \log 10}$$

$$= \frac{\frac{3}{2}\log 3 + 3\log 2 - \frac{3}{2}\log 10}{\log 3 + 2\log 2 - \log 10}$$

$$= \frac{\frac{3}{2}(\log 3 + 2\log 2 - \log 10)}{(\log 3 + 2\log 2 - \log 10)} = \frac{3}{2}.$$

EXAMPLE 27

Show that, $7\log_{10}\frac{16}{15} + 5\log_{10}\frac{25}{24} + 3\log_{10}\frac{81}{80} = \log_{10} 2$. [***C.U. B.Com. 2011***]

Solution: L.H.S. = $7\log_{10}\frac{16}{15} + 5\log_{10}\frac{25}{24} + 3\log_{10}\frac{81}{80}$

$$= 7(\log_{10} 16 - \log_{10} 15) + 5(\log_{10} 25 - \log_{10} 24)$$

$$+ 3 (\log_{10} 81 - \log_{10} 80)$$
$$= 7\{\log_{10} 2^4 - \log_{10}(3 \times 5)\} + 5\{\log_{10} 5^2 - \log_{10}(2^3 \times 3)\}$$
$$+ 3\{\log_{10} 3^4 - \log_{10}(2^4 \times 5)\}$$
$$= 7(4 \log_{10} 2 - \log_{10} 3 - \log_{10} 5) + 5(2 \log_{10} 5 - 3 \log_{10} 2$$
$$- \log_{10} 3) + 3(4 \log_{10} 3 - 4 \log_{10} 2 - \log_{10} 5)$$
$$= 28 \log_{10} 2 - 7 \log_{10} 3 - 7 \log_{10} 5 + 10 \log_{10} 5 - 15 \log_{10} 2$$
$$- 5 \log_{10} 3 + 12 \log_{10} 3 - 12 \log_{10} 2 - 3 \log_{10} 5$$
$$= \log_{10} 2 = \text{R.H.S.}$$

EXAMPLE 28

Prove that $\log_3 \log_2 \log_{\sqrt{3}} 81 = 1$.

Solution: L.H.S. $= \log_3 \log_2 \log_{\sqrt{3}} 81$

$$= \log_3 \log_2 \log_{\sqrt{3}} (\sqrt{3})^8$$
$$= \log_3 \log_2 (8 \log_{\sqrt{3}} \sqrt{3}) = \log_3 \log_2 8 \quad [\text{as } \log_a a = 1]$$
$$= \log_3 \log_2 2^3 = \log_3 (3 \log_2 2)$$
$$= \log_3 3 = 1.$$

EXAMPLE 29

If $2 \log_4 x = 1 + \log_4 (x - 1)$, find the value of x.

Solution: $2 \log_4 x = 1 + \log_4 (x - 1)$

or $2 \log_4 x = \log_4 4 + \log_4(x - 1)$

or $2 \log_4 x^2 = \log_4\{4 \times (x - 1)\}$

or $\log_4 x^2 = \log_4(4x - 4)$

or $x^2 = 4x - 4$

or $x^2 - 4x + 4 = 0$

or $(x - 2)^2 = 0$

or $x - 2 = 0$

or $x = 2.$

EXAMPLE 30

If $\log_2 x + \log_4 x + \log_{16} x = \dfrac{21}{4}$, find the value of x.

Solution: $\log_2 x + \log_4 x + \log_{16} x = \dfrac{21}{4}$

or $$\frac{1}{\log_x 2}+\frac{1}{\log_x 4}+\frac{1}{\log_x 16}=\frac{21}{4}$$

or $$\frac{1}{\log_x 2}+\frac{1}{\log_x 2^2}+\frac{1}{\log_x 2^4}=\frac{21}{4}$$

or $$\frac{1}{\log_x 2}+\frac{1}{2\log_x 2}+\frac{1}{4\log_x 2}=\frac{21}{4}$$

or $$\frac{1}{\log_x 2}\left[1+\frac{1}{2}+\frac{1}{4}\right]=\frac{21}{4}$$

or $$\frac{1}{\log_x 2}\times\frac{4+2+1}{4}=\frac{21}{4}$$

or $$\frac{1}{\log_x 2}\times\frac{7}{4}=\frac{21}{4}$$

or $$\frac{1}{\log_x 2}=\frac{21}{4}\times\frac{4}{7}=3$$

or $$\log_x 2=\frac{1}{3}$$

or $$x^{\frac{1}{3}}=2$$

or $$\left(x^{\frac{1}{3}}\right)^3=2^3 \quad \text{[cubing both sides]}$$

or $$x=8.$$

EXAMPLE 31

Prove that, $\log_7 \sqrt{7\sqrt{7\sqrt{7\sqrt{7.....}}}}\,\infty = 1.$

Solution: Let $x=\sqrt{7\sqrt{7\sqrt{7\sqrt{7.....}}}}\,\infty$

or $$x^2 = 7\sqrt{7\sqrt{7\sqrt{7.....}}}\,\infty \quad \text{[squaring both sides]}$$

or $$x^2 = 7x$$

or $$x = 7 \quad [\text{as } x \neq 0]$$

Therefore, $\log_7 \sqrt{7\sqrt{7\sqrt{7\sqrt{7.....}}}}\,\infty = \log_7 7 = 1$ [Proved].

EXAMPLE 32

If, $x^2 + y^2 = 3xy$, show that $\log\left\{\frac{1}{\sqrt{5}}(x+y)\right\} = \frac{1}{2}(\log x + \log y)$

Solution: $x^2 + y^2 = 3xy$

or $$x^2 + y^2 + 2xy = 5xy$$

or $$(x + y)^2 = 5xy$$

or $$\frac{(x+y)^2}{5} = xy$$

or $$\frac{(x+y)^2}{(\sqrt{5})^2} = xy$$

or $$\left(\frac{x+y}{\sqrt{5}}\right)^2 = xy$$

or $$\log\left(\frac{x+y}{\sqrt{5}}\right)^2 = \log xy \text{ [taking logarithm on both sides]}$$

or $$2\log\left(\frac{x+y}{\sqrt{5}}\right) = \log x + \log y$$

or $$\log\left(\frac{x+y}{\sqrt{5}}\right) = \frac{1}{2}(\log x + \log y)$$

or $$\log\left\{\frac{1}{\sqrt{5}}(x+y)\right\} = \frac{1}{2}(\log x + \log y).$$

EXAMPLE 33

If $a = b^x$, $b = c^y$, $c = a^z$, than prove that $xyz = 1$.

Solution: $a = b^x$

or $\log_b a = x$

Similarly, $b = c^y$

or $\log_c b = y$

and $c = a^z$

or $\log_a c = z$

Now, $$xyz = \log_b a \times \log_c b \times \log_a c$$

$$= \log_c a \times \log_a c = 1 \quad [\text{as } \log_b a = \log_c a \times \log_b c].$$

EXAMPLE 34

If $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.47712$ and $\log_{10} 7 = 0.84510$, find the value of $\log_{10} 105$.

Solution: $\log_{10} 105 = \log_{10} (7 \times 5 \times 3)$

$$= \log_{10} 7 + \log_{10} 5 + \log_{10} 3$$

$$= \log_{10} 7 + \log_{10}\left(\frac{10}{2}\right) + \log_{10} 3$$

$$= \log_{10} 7 + \log_{10} 10 - \log_{10} 2 + \log_{10} 3$$

$$= 0.84510 + 1 - 0.30103 + 0.47712$$

$$= 2.02119.$$

EXAMPLE 35

If a, b, c are in G.P., prove that,

$$\log_x a + \log_x c = \frac{2}{\log_b x} \; [a, b, c, x > 0].$$

Solution: As, a, b, c are in G.P.

Therefore, $b^2 = ac$

Now, taking logarithm to the base x ($x > 0$) of both sides we get,

$$\log_x b^2 = \log_x ac \text{ or } 2 \cdot \log_x b = \log_x a + \log_x c$$

or $$2 \cdot \frac{1}{\log_b x} = \log_x a + \log_x c \text{ or } \log_x a + \log_x c = \frac{2}{\log_b x} \text{ [Proved].}$$

EXAMPLE 36

If $a^{2-x} \cdot b^{5x} = a^{x+3} \cdot b^{3x}$,

Prove that, $x \log\left(\frac{b}{a}\right) = \frac{1}{2} \log a.$

Solution: $a^{2-x} \cdot b^{5x} = a^{x+3} \cdot b^{3x}$

or $$\frac{a^{x+3}}{a^{2-x}} = \frac{b^{5x}}{b^{3x}}$$

or $$a^{x+3-2+x} = b^{5x-3x}$$

or $$a^{2x+1} = b^{2x}$$

or $$a^{2x} \cdot a = b^{2x}$$

or $$\frac{b^{2x}}{a^{2x}} = a$$

or $$\left(\frac{b}{a}\right)^{2x} = a$$

or $$\log\left(\frac{b}{a}\right)^{2x} = \log a \quad \text{[taking logarithm of both sides]}$$

or $$2x \log\left(\frac{b}{a}\right) = \log a$$

or $$x \cdot \log\left(\frac{b}{a}\right) = \frac{1}{2} \cdot \log a \text{ [Proved].}$$

EXAMPLE 37

If $y = a^{\frac{1}{1-\log_a x}}$ and $z = a^{\frac{1}{1-\log_a y}}$, show that, $x = a^{\frac{1}{1-\log_a z}}$.

Solution: Let $p = \log_a x$, $q = \log_a y$ and $r = \log_a z$

Then, $$y = a^{\frac{1}{1-p}} \qquad (i)$$

and $$z = a^{\frac{1}{1-q}} \qquad (ii)$$

Now, taking logarithm to the base a of both sides of equation no. (i) we get,

$$\log_a y = \log_a a^{\frac{1}{1-p}} = \frac{1}{1-p} \log_a a$$

or $$q = \frac{1}{1-p} \quad [\text{as } \log_a a = 1]$$

Again, taking logarithm to the base a of both sides of equation no (ii) we get,

$$\log_a z = \log_a a^{\frac{1}{1-q}} = \frac{1}{1-q} \cdot \log_a a$$

or $$r = \frac{1}{1-q}$$

or $$1 - q = \frac{1}{r}$$

or $$1 - \frac{1}{1-p} = \frac{1}{r}$$

or $$1 - \frac{1}{r} = \frac{1}{1-p}$$

or $$\frac{r-1}{r} = \frac{1}{1-p}$$

or $$1-p = \frac{r}{r-1}$$

or $$p = 1-\frac{r}{r-1}$$

or $$p = \frac{r-1-r}{r-1} = \frac{-1}{r-1} = \frac{-1}{-(1-r)}$$

or $$p = \frac{1}{1-r}$$

or $$\log_a x = \frac{1}{1-\log_a z}$$

or $$x = a^{\frac{1}{1-\log_a z}} \text{ (Proved).}$$

EXAMPLE 38

How many zeroes are there between the decimal point and the first significant figure in $(0.5)^{100}$? ***[I.C.W.A, June 1976]***

Solution: Let $x = (0.5)^{100}$

or
$$\begin{aligned}\log x &= 100 \log(0.5) = 100 \times (\bar{1}.6990)\\ &= 100 \times (-1 + 0.6990) = 100 \times (-0.3010)\\ &= -30.10 = -30 - 0.10\\ &= -30 - 1 + 1 - 0.10 = -31 + 0.90\\ &= \overline{31}.90\end{aligned}$$

The characteristic of $\log x = \overline{31}$, so number of zeroes between the decimal point and the first significant figure will be $(31-1) = 30$.

EXAMPLE 39

With the help of log tables, find the value of $\frac{61.92 \times 0.07046}{401.535}$ upto 4 decimal places.

Solution: Let $x = \frac{61.92 \times 0.07046}{401.535}$

or
$$\begin{aligned}\log x &= \log 61.92 + \log 0.07046 - \log 401.535\\ &= 1.7918 + \bar{2}.8480 - 2.6036\\ &= 1.7918 - 2 + 0.8480 - 2.6036\\ &= 2.6398 - 4.6036 = -1.9638\end{aligned}$$

$$= -1 - .9638 = -1 - 1 + 1 - .9638$$
$$= -2 + .0362 = \bar{2}.0362$$

or $$x = \text{Antilog } \bar{2}.0362$$
$$= 0.01086$$
$$= 0.0109.$$

EXAMPLE 40

Solve the equation $11^{4x-5} \times 3^{2x} = 5^{3-x} \div 7^{-x}$.

Solution: $11^{4x-5} \times 3^{2x} = 5^{3-x} \times 7^{x}$

or $$\log 11^{4x-5} + \log 3^{2x} = \log 5^{3-x} + \log 7^{x}$$
[taking logarithm of both sides]

or $$(4x - 5)\log 11 + 2x \cdot \log 3 = (3 - x)\log 5 + x \cdot \log 7$$

or $$4x \log 11 - 5 \log 11 + 2x \cdot \log 3 = 3 \log 5 - x \log 5 + x \log 7$$

or $$4x \log 11 + 2x \log 3 + x \log 5 - x \log 7 = 3 \log 5 - 5 \log 11$$

or $$x (4 \log 11 + 2 \log 3 + \log 5 - \log 7) = 3 \log 5 + 5 \log 11$$

or $$x(4 \times 1.0414 + 2 \times 0.4771 + 0.6990 - 0.8451) = 3 \times 0.6990 + 5 \times 1.0414$$

or $$x(4.1656 + 0.95542 + 0.6990 - 0.8451) = 2.0970 + 5.2070$$

or $$x \times 4.9737 = 7.304$$

or $$x = \frac{7.304}{4.9737}$$

or $$\log x = \log 7.304 - \log 4.9737$$
$$= .8635 - 0.6968 = 0.1667$$

or $$x = \text{Antilog } (0.1667) = 1.468.$$

EXAMPLE 41

Simplify: $6253 \left(1 + \frac{5}{100}\right)^8 \left(1 + \frac{5}{400}\right) - 6253.$

Solution: Let $x = 6253 \left(1 + \frac{5}{100}\right)^8 \left(1 + \frac{5}{400}\right)$

$$= 6253 \left(\frac{105}{100}\right)^8 \left(\frac{405}{400}\right)$$
$$= 6253(1.05)^8 (1.0125)$$

or $$\log x = \log 6253 + 8 \log 1.05 + \log 1.0125$$
$$= 3.7961 + 8 \times 0.0212 + 0.0054$$

$= 3.7961 + 0.1696 + 0.0054$

$= 3.9711$

or $\quad x = \text{Antilog } 3.9711 = 9356$

Hence, $\quad 6253\left(1+\frac{5}{100}\right)^8\left(1+\frac{5}{100}\right) - 6253$

$= 9356 - 6253 = 3103.$

EXERCISE

A. SHORT TYPE

1. Change the following from exponential to logarithmic forms:

   (*i*) $5^{-2} = \frac{1}{25}$; (*ii*) $\sqrt[5]{32} = 2$; (*iii*) $7^0 = 1$;

   [**Ans** (*i*) $\log_5\left(\frac{1}{25}\right) = -2$; (*ii*) $\log_{32} 2 = \frac{1}{5}$; (*iii*) $\log_7 1 = 0$]

2. Change the following from logarithmic to exponential form:

   (*i*) $\log_5 5 = 1$; (*ii*) $\log_5 625 = 4$; (*iii*) $\log_3\left(\frac{1}{81}\right) = -4$

   [**Ans.** (*i*) $5^1 = 5$; (*ii*) $5^4 = 625$; (*iii*) $3^{-4} = \frac{1}{81}$]

3. Find the logarithm of:

   (*i*) 1728 to the base $2\sqrt{2}$. [**Ans.** 6]

   (*ii*) 2025 to the base $3\sqrt{5}$. [**Ans.** 4]

   (*iii*) 125 to the base $5\sqrt{5}$. [**Ans.** 2]

   (*iv*) 324 to the base $3\sqrt{2}$. [**Ans.** 4]

4. (*i*) If log 9 = 2, find the base of logarithm. [***C.U. B.Com. '89***] [**Ans.** 3]

   (*ii*) If log 729 = 4, find the base of logarithm. [**Ans.** $3\sqrt{3}$]

   (*iii*) If log 8000 = 6, find the base of logarithm. [**Ans.** $2\sqrt{5}$]

   (*iv*) If log 1296 = 4, find the base of logarithm. [**Ans.** 6]

5. (*i*) If $\log_x \frac{1}{2} = \frac{1}{3}$, what will be the value of x?

   [***C.U. B.Com.'90***] [**Ans.** $\frac{1}{8}$]

(*ii*) If $\log_6 x = -3$, find the value of x? [***C.U. B.Com.'94***] [**Ans.** $\frac{1}{216}$]

(*iii*) If $\log_x 243 = 5$, find the value of x? [**Ans.** 3]

(*iv*) Find x when $\log_2 x = 3$. [***C.U. B.Com.'87***] [**Ans.** 8]

6. (*i*) Show that $\log_2 3 \times \log_3 2 = 1$ [***C.U. B.Com.'93***]

(*ii*) Show that $\log_b a \times \log_c b \times \log_a c = 1$.

(*iii*) Prove that $\log_a 6 = \log_a 1 \times \log_a 2 \times \log_a 3$. [***C.U. B.Com. '88***]

(*iv*) Prove that $\log (1 + 2 + 3) = \log 1 + \log 2 + \log 3$. [***C.U. B.Com. '97***]

(*v*) Prove that $\log_2 \log_{\sqrt{2}} \log_3 81 = 2$.

(*vi*) Prove that $\log_{\sqrt{2}} 16 \div \log_{\sqrt{3}} 9 = 2$. [***V.U. B.Com. '96***]

(*vii*) Prove that $\log_3 \log_3 27 = 1$. [***C.S. B.Com. '95***]

(*viii*) Prove that $\log_2 \log_2 \log_2 16 = 1$. [***C.U. B.Com. '99 2012, 2014(H)***]

7. (i) Taking $\log_{10} 2 = 0.3010$, find the value of $\log_{10} 8$.
[***C.U. B.Com. '86***] [**Ans.** 0.9030]

(*ii*) Find the value of $\log_5 125$ without using logarithmic tables.
[***C.U. B.Com. '87***] [**Ans.** 3]

8. By changing to an appropriate base find (without be help of log table the value of $\log_9 27$). [***C.U. B.Com. '84***] [**Ans.** $\frac{3}{2}$]

9. $\log_p x = a$, $\log_q x = b$, then prove that $\log_{\frac{p}{q}} x = \dfrac{ab}{a-b}$. [***C.U. B.Com. '98***]

10. The logarithm of a number to the base $\sqrt{2}$ is a; what is its logarithm to the base $2\sqrt{2}$? [**Ans.** $\frac{3}{2}$]

11. Find the number of digits in 3^{20}, having given $\log 3 = 0.4771$. [**Ans.** 10]

B. SHORT ESSAY TYPE/PROBLEM TYPE

1. If $\log_x 2 + \log_x 4 + \log_x 8 = 6$, find x. [***C.U. B.Com. 2017(H)***] [**Ans.** 2]
2. If a, b, c are any three consecutive positive integers, prove that $\log(1 + ac) = 2 \log b$.
3. If $a = \log_{24} 12$, $b = \log_{36} 24$ and $c = \log_{48} 36$, prove that $1 + abc = 2bc$.
4. If $\log_a bc = x, \log_b ca = y,$ and $\log_c ab = z,$ then show that $\dfrac{1}{1+x} + \dfrac{1}{1+y} + \dfrac{1}{1+z} = 1$
[***C.U. B.Com. '99, 2017(H)***]

5. If $\frac{\log a}{q-r} = \frac{\log b}{r-p} = \frac{\log c}{p-q}$, prove that $a^p \cdot b^q \cdot c^r = 1$.

6. If $\frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$, then prove that $xyz = 1$. **[*C.U. B.Com. '97*]**

7. If $\frac{\log x}{1} = \frac{\log y}{2} = \frac{\log z}{3}$, show that xyz are in G.P. **[*C.U. B.Com. '95*]**

8. Prove that $\log_3\left(\sqrt{3\sqrt{3\sqrt{3....\infty}}}\right) = 1$. **[*C.U. B.Com. 2017(G)*]**

9. (*i*) If $a^x = b^y = c^z$ and x, y, z be in G.P. then show that $\log_b a = \log_c b$.
   (*ii*) If $a^x = b^y = c^z$ and a, b, c be in G.P. then show that, $(x - y)\log a = (y - z)\log c$.

10. If a, b, c, are in G.P. and x, y, z be in A. P. show that
    $(y - z) \log a + (z - x)\log b + (z - y)\log c = 0$. **[*C.U. B.Com. '80, '96*]**

11. The first and the last term of a G.P. are a and k respectively. If the number of terms be n, prove sthat $n = 1 + \frac{\log k - \log a}{\log r}$, where r is the common ratio.
    [*C.U. B.Com. 2000*]

12. If a, b, c are in G.P. show that $\log_a x, \log_b x, \log_c x$ are in H.P.

13. If a, b, c be the pth, qth and rth term respectively of a G.P. then prove that $(q - r) \log a - (r - p) \log b + (p - q) \log c = 0$.

14. If $x^2 + y^2 = 6xy$, prove that $2 \log(x + y) = \log x + \log y + 3 \log 2$.

15. If $x^2 + y^2 = 11xy$, prove that $2 \log(x - y) = 2 \log 3 + \log x + \log y$.

16. If $a^2 + b^2 = 11ab$, show that $\log\left(\frac{a-b}{3}\right) = \frac{1}{2}(\log a + \log b)$.

17. If $a^2 + b^2 = 23ab$, prove that $\log\left(\frac{a+b}{5}\right) = \frac{1}{2}(\log a + \log b)$.

18. If $a^2 + b^2 = 7ab$, prove that $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}(\log a + \log b)$.
    [*C.A. Ent. 1990*]

19. If $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}(\log a + \log b)$, show that $\frac{a}{b} + \frac{b}{a} = 7$,
    [*C.U. B.Com. 2017(G)*]

20. Prove that

    (*i*) $\frac{1}{\log_{ab} abc} + \frac{1}{\log_{bc} abc} + \frac{1}{\log_{ca} abc} = 2$.

(*ii*) $x^{\log y-\log z}\times y^{\log z-\log x}\times z^{\log x-\log y}=1.$

[***C.U. B.Com. 2000***] [***K.U. B.Com. '98***]

(*iii*) $(yz)^{\log y-\log z}\times (zx)^{\log z-\log x}\times (xy)^{\log x-\log y}=1.$

(*iv*) $\dfrac{1}{\log_{\frac{p}{q}} x}+\dfrac{1}{\log_{\frac{q}{r}} x}+\dfrac{1}{\log_{\frac{r}{p}} x}=0.$

(*v*) $\dfrac{1}{\log_a bc+1}+\dfrac{1}{\log_b ca+1}+\dfrac{1}{\log_c ab+1}=1.$

21. Show that

(*i*) $23\log\dfrac{16}{15}+17\log\dfrac{25}{24}+10\log\dfrac{81}{80}=1$, (Base = 10) [***C.U. B.Com. '84***]

(*ii*) $7\log\dfrac{16}{15}+5\log\dfrac{25}{24}+3\log\dfrac{81}{80}=\log 2.$

(*iii*) $\log\dfrac{81}{8}-2\log\dfrac{3}{2}+3\log\dfrac{2}{3}+\log\dfrac{3}{4}=0.$

(*iv*) $\log\dfrac{11}{5}+\log\dfrac{490}{297}-2\log\dfrac{7}{9}=\log 2.$

(*v*) $\log\dfrac{75}{16}-2\log\dfrac{5}{9}+\log\dfrac{32}{243}=\log 2.$

22. Without using log tables, show that,

$$\frac{\log\sqrt{27}+\log\sqrt{8}-\log\sqrt{125}}{\log 6-\log 5}=\frac{3}{2}.$$ [***C.U. B.Com. '86***]

23. Find the value of $\dfrac{\log\sqrt{27}+\log\sqrt{8}+\log\sqrt{1000}}{\log 120}=\dfrac{2}{3}.$

[***C.U. B.Com. '98***] [**Ans.** $\frac{3}{2}$]

24. Using log table, find the value of

(*i*) $\dfrac{1}{(1.045)^{20}}$ [**Ans.** 0.415]

(*ii*) $\dfrac{1}{(1+0.12)^{20}}$ [***C.U. B.Com. '91***]

[**Hints:** Let A = $\dfrac{1}{(1+0.12)^{20}}$ or A = $(1.12)^{-20}$

or $\log A = -20 \log (1.12) = -20 \times 0.0492$

$= -0.984 = -1 + 1 - 0.984 = -1 + 0.016 = \bar{1}.016$

or $A = \text{Antilog } \bar{1}.016 = 0.1038$].

(*iii*) $\sqrt[7]{\dfrac{1}{1.235}}$. [**Ans.** 0.9703]

(*iv*) $(125)^{\frac{1}{10}} \times \dfrac{0.001834}{0.043160}$. [**Ans.** 0]

C. ADVANCED PROBLEM

1. Find the value of $\dfrac{2\log 6 + 6\log 2}{4\log 2 + \log 27 - \log 9}$. [**Ans.** 2]

2. Find the value of $\log_{10} 2 + 16 \log_{10} \dfrac{17}{15} + 12 \log_{10} \dfrac{25}{24} + 7\log_{10} \dfrac{81}{80}$. [**Ans.** 1]

3. If $a^{3-x} \cdot b^{5x} = a^{x+5} \cdot b^{3x}$, then prove that $x \log \dfrac{b}{a} = \log a$. [***C.A. Nov. '81***]

4. Prove that $(xy)^{\log\frac{x}{y}} \times (yz)^{\log\frac{y}{z}} \times (zx)^{\log\frac{z}{x}} = 1$.

5. Show that: $\dfrac{\log_a x}{\log_{ab} x} = 1 + \log_a b$.

6. Solve: $\log_{10} x - \log_{10} \sqrt{x} = \dfrac{2}{\log_{10} x}$. [***N.B.U. B.Com. '99***] [**Ans.** $x = 100$]

7. If $a^x = b^y = c^z = d^w$, show that,
$$\log_a(bcd) = x\left(\frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right).$$

8. If $x = 1 + \log_a bc$, $y = 1 + \log_b ca$, $z = 1 + \log_c ab$, prove that $xyz = xy + zy + zx$.

9. Prove that $2 \log(a + b) = 2 \log a + \log\left(1 + \dfrac{2b}{a} + \dfrac{b^2}{a^2}\right)$.

10. Given that $2 \log.(x^2.y) = 3 + \log x - \log y$ where x and y both are positive, express y in terms of x. If $x - y = 3$, find the value of x and y.

[**Ans.** $y = \dfrac{10}{x}$, $x = 5$, $y = 2$]

11. If $\dfrac{b-c}{\log a} = \dfrac{c-a}{\log b} = \dfrac{a-b}{\log c}$, show that $a^a . b^b . c^c = 1$.

12. If log 2 = 0.3010, log 3 = 0.4771 and log 7 = 0.8451, find log 29.4.
[*K.U. B.Com. '96*] [**Ans.** 1.4683]

13. Prove that $\log_3\log_2\log_2 256 = 1$. [*N.B.U. B.Com. '98*]

14. If $\log(x^3y^3) = a$ and $\log\left(\frac{x}{y}\right) = b$, find log x and log y in terms of a and b.
[*N.B.U. B.Com. '96*] $\left[\textbf{Ans. } \log x = \frac{a+3b}{5}; \log y = \frac{a-2b}{5}\right]$

15. If $x = \log_a bc$, $y = \log_b ca$ and $z = \log_c ab$, then show that,
$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1.$$
[*N.B.U. B.Com. '94*]

16. Prove that, $\log_5 \sqrt{5\sqrt{5\sqrt{5.....}}}\infty = 1$.

17. If $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$, find x. [**Ans.** 8]

18. Solve for t: $\log_t 2 \times \log_{\frac{t}{16}} 2 = \log_{\frac{t}{64}} 2$. [*I.C.W.A.I. Dec '87*]
[*B.U. B.Com. '88*] [**Ans.** t = 4 or 8]

19. Given $\log \frac{m+n}{7} = \frac{1}{2}(\log m + \log n)$, show that $\frac{m}{n} + \frac{n}{m} = 47$.

20. If $a^2 + c^2 = b^2$ then, find the value of $\frac{1}{\log_{b+c} a} + \frac{1}{\log_{b-c} a}$. [**Ans.** 2]

D. MULTIPLE CHOICE QUESTIONS (MCQs)

(i) Short Type

1. Logarithmic form of $5^0 = 1$ is
(*a*) $\log_1 5 = 0$
(*b*) $\log_5 1 = 0$
(*c*) $\log 5^0 = \log 1$
(*d*) None of these [**Ans.** (*b*)]

2. Logarithmic form of $3^{-2} = \frac{1}{9}$ is
(*a*) $\log_3\left(\frac{1}{9}\right) = -2$
(*b*) $\log_{\frac{1}{9}} 3 = -2$
(*c*) $\log 3^{-2} = \log \frac{1}{9}$
(*d*) None of these [**Ans.** (*a*)]

3. Logarithmic form of $\sqrt[4]{81} = 3$ is

(*a*) $\log \sqrt[4]{81} = \log 3$

(*b*) $\log_3 \sqrt[4]{81} = 1$

(*c*) $\log_{81} 3 = \frac{1}{4}$

(*d*) None of these **[Ans.** (*c*)]

4. Exponential form of $\log_{10} 10 = 1$ is

(*a*) $10^1 = 10$

(*b*) $10^{10} = 1$

(*c*) $1^{10} = 10$

(*d*) None of these **[Ans.** (*a*)]

5. Exponential form of $\log_2 16 = 4$ is

(*a*) $4^2 = 16$

(*b*) $2^4 = 16$

(*c*) $(16)^{\frac{1}{2}} = 4$

(*d*) None of these **[Ans.** (*a*)]

6. The logarithm of 125 to the base $5\sqrt{5}$ is

(*a*) 5

(*b*) 25

(*c*) 2

(*d*) None of these **[Ans.** (*c*)]

7. The logarithm of 729 to the base $3\sqrt{3}$ is

(*a*) 9

(*b*) 6

(*c*) 4

(*d*) $\sqrt{3}$ **[Ans.** (*c*)]

8. If log 25 = 2, then the base of logarithm is

(*a*) 3

(*b*) 4

(*c*) 2

(*d*) 5 **[Ans.** (*d*)]

9. If log 729 = 4, then the base of logarithm is

(*a*) $3\sqrt{3}$

(*b*) 9

(*c*) $9\sqrt{2}$

(*d*) $4\sqrt{3}$ **[Ans.** (*a*)]

10. When $\log_4 x = 3$, then the value of x is

(*a*) 16

(*b*) 12

(*c*) 64

(*d*) 9 **[Ans.** (*c*)]

11. If $\log x^{1/2} = \frac{1}{3}$, then the value of x is

(*a*) $\frac{1}{8}$

(*b*) $\frac{1}{6}$

(*c*) $\frac{1}{5}$

(*d*) $\frac{1}{4}$ **[Ans.** (*a*)]

12. If $\log_6 x = -3$, then the value of x is [***C.U. B.Com. 1994***]

(*a*) 216
(*b*) $\frac{1}{216}$
(*c*) $\frac{1}{108}$
(*d*) 108 **[Ans.** (*b*)]

13. If log 2 = 0.30103, then the value of log 2000 is [***C.U. B.Com. 2002***]
(*a*) 2.60106
(*b*) 3.30103
(*c*) 2.30103
(*d*) None of these **[Ans.** (*b*)]

14. If log 343 = 3, then the base of the logarithm is
(*a*) 6
(*b*) 5
(*c*) 7
(*d*) 4 **[Ans.** (*c*)]

15. The value of logarithm of $\log_5 125$ is
(*a*) 9
(*b*) 5^2
(*c*) 5
(*d*) 3 **[Ans.** (*d*)]

16. The value of logarithm of $\log_{\sqrt{7}} 49$ is
(*a*) 4
(*b*) 7
(*c*) 3
(*d*) 2 **[Ans.** (*a*)]

17. The value of $\log\left(\frac{9}{8}\right) - \log\left(\frac{27}{32}\right) + \log\left(\frac{3}{4}\right)$ is
(*a*) 8
(*b*) 0
(*c*) 6
(*d*) 4 **[Ans.** (*b*)]

18. The value of $\log 8 + \log \frac{1}{8}$ is
(*a*) 0
(*b*) 1
(*c*) 4
(*d*) 8 **[Ans.** (*a*)]

19. If $\log_{10} 2 = 0.301$, then the value of $\log_{10} 50$ is
(*a*) 2.333
(*b*) 1.301
(*c*) 1.699
(*d*) 1.602 **[Ans.** (*c*)]

20. If log 90 = 1.9542, then log 3 equals to
(*a*) 0.4771
(*b*) 0.4725
(*c*) 0.3922
(*d*) 0.5722 **[Ans.** (*a*)]

21. If $\log_8 x + \log_4 x + \log_2 x = 11$, then the value of x is
(*a*) 16
(*b*) 64
(*c*) 52
(*d*) 22 **[Ans.** (*b*)]

22. If log 2 = 0.3010 and log 3 = 0.4771, the value of $\log_5 512$ is
(*a*) 2.267
(*b*) 0.9542
(*c*) 3.876
(*d*) 3.0903 **[Ans.** (*c*)]

23. If $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, then the value of $\log_{10} 1.5$ is
(*a*) 0.2341
(*b*) 0.1726
(*c*) 0.1761
(*d*) 0.2762 **[Ans.** (*c*)]

24 If log 2 = 0.30103, the number of digits in 4^{50} is

(*a*) 30 (*c*) 32
(*b*) 31 (*d*) 50 [**Ans.** (*b*)]

25. If log 2 = 0.3010, then the number of digits in 2^{64} is

(*a*) 64 (*c*) 24
(*b*) 32 (*d*) 20 [**Ans.** (*d*)]

(ii) Short Essay Type

1. If log 2 = 0.3010 and log 3 = 0.4771, the value of $\log_5 512$ is
   (*a*) 3.876 (*c*) 2.967
   (*b*) 2.870 (*d*) 3.912 [**Ans.** (*a*)]
2. If log 27 = 1.431, then the value of log 9 is
   (*a*) 0.945 (*c*) 0.954
   (*b*) 0.934 (*d*) 0.958 [**Ans.** (*c*)]
3. If $\log_{10}5 + \log_{10}(5x + 1) = \log_{10}(x + 5) + 1$, then x is equal to:
   (*a*) 1 (*c*) 10
   (*b*) 5 (*d*) 3 [**Ans.** (*d*)]
4. If $\log_x\left(\frac{9}{16}\right) = -\frac{1}{2}$, then x is equal to:
   (*a*) $\frac{5}{9}$ (*c*) $\frac{81}{256}$
   (*b*) $\frac{256}{81}$ (*d*) $\frac{3}{4}$ [**Ans.** (*b*)]
5. If $\log\frac{a}{b} + \log\frac{b}{a} = \log(a + b)$, then:
   (*a*) $a + b = 1$ (*c*) $a = b$
   (*b*) $a - b = 1$ (*d*) $a^2 - b^2 = 1$ [**Ans.** (*a*)]
6. The value of x in the logarithmic equation $\log(x + 2) - \log(x - 1) = \log 2$ is
   (*a*) 2 (*c*) 4
   (*b*) 3 (*d*) 5 [**Ans.** (*c*)]
7. If $\log_x y = 100$ and $\log_3 x = 10$, then the value of y is:
   (*a*) 3^{10} (*c*) 3^{1000}
   (*b*) 3^{100} (*d*) 3^{10000} [**Ans.** (*c*)]
8. The value of $\log\frac{75}{16} - 2\log\frac{5}{9} + \log\frac{32}{243}$ is

(*a*) 1 (*c*) 2
(*b*) 1.2 (*d*) 2.02 [**Ans.** (*c*)]

9. The value of x satisfying $\log_{10}(2^x + x - 41) = x(1 - \log_{10}{}^5)$ is
(*a*) 41 (*c*) 36
(*b*) 39 (*d*) 43 [**Ans.** (*a*)]

10. The sum of two numbers a and b is $\sqrt{18}$ and their difference is $\sqrt{14}$. The value of $\log_b a$ is equal to
(*a*) 2 (*c*) –1
(*b*) 1 (*d*) $\frac{1}{2}$ [**Ans.** (*c*)]

11. If $x = \frac{\sqrt{10}+\sqrt{2}}{2}$ and $y = \frac{\sqrt{10}-\sqrt{2}}{2}$, then the value of $\log_2(x^2 + xy + y^2)$, is equal to
(*a*) 0 (*c*) 3
(*b*) 2 (*d*) 4 [**Ans.** (*c*)]

12. If 'x' and 'y' are real numbers, such that, $2 \log (2y - 3x) = \log x + \log y$, then $\frac{x}{y}$ is
(*a*) $\frac{3}{5}$ (*c*) $\frac{2}{3}$
(*b*) $\frac{5}{3}$ (*d*) $\frac{4}{9}$ [**Ans.** (*d*)]

13. The number of digits in 3^{3^3} is [Given that log 3 = 0.47712.]
(*a*) 12 (*c*) 14
(*b*) 13 (*d*) 15 [**Ans.** (*b*)]

14. If $\log_7 2 = m$, then $\log_{49} 28$ is equal to …
(*a*) $\frac{1}{1+2m}$ (*c*) $\frac{2m}{2m+1}$
(*b*) $\frac{1+2m}{2}$ (*d*) $\frac{2m+1}{2m}$ [**Ans.** (*b*)]

15. If $\frac{\log x}{l+m-2n} = \frac{\log y}{m+n-2l} = \frac{\log z}{n+l-2m}$, then xyz is equal to
(*a*) 0 (*c*) lmn
(*b*) 1 (*d*) 2 [**Ans.** (*b*)]

16. If $\log_3(30) = \frac{1}{a}$ and $\log_5(30) = \frac{1}{b}$ then the value of $3 \log_{30}(2)$ is
(*a*) $3(1 + a + b)$ (*c*) $3(1 - a - b)$
(*b*) $2(1 - a - b)$ (*d*) $3(1 + a - b)$ [**Ans.** (*c*)]

17. If $a^2 + b^2 = c^2$, then $\dfrac{1}{\log_{c+a}(b)} + \dfrac{1}{\log_{c-a}(b)} = ?$

(*a*) 1 (*c*) 4
(*b*) 2 (*d*) 8 [**Ans.** (*b*)]

18. If a, b, c be the pth, qth and rth terms of a GP, then the value of $(q - r)\log a + (r - p)\log b + (p - q)\log c$ is

(*a*) 0 (*c*) -1
(*b*) 1 (*d*) pqr [**Ans.** (*a*)]

19. If $\log_a(ab) = x$, then $\log_b(ab)$ is equal to

(*a*) $\dfrac{x}{x+1}$ (*c*) $\dfrac{x}{x-1}$
(*b*) $\dfrac{1}{x}$ (*d*) $\dfrac{x}{1-x}$ [**Ans.** (*c*)]

20. If p, q, r are in H.P. then $\log_e(p + r) + \log_e(p - 2q + r)$ is equal to

(*a*) $\log_e(p - r)$ (*c*) $3\log_e|p - r|$
(*b*) $2\log_e|p - r|$ (*d*) None of these [**Ans.** (*b*)]

21. If $\log_{10} x - \log_{10}\sqrt{x} = 2\log_x 10$, then the value of '$x$' is

(*a*) 185 (*c*) 190
(*b*) 200 (*d*) 100 [**Ans.** (*d*)]

22. If $\log_a(x^2) - 4 = 2\log_a 6$, where a > 0 and $x > 0$, then

(*a*) $x = a^6$ (*c*) $x = 2\log_a 6 + 4$
(*b*) $x = 6a^2$ (*d*) $x = \dfrac{6}{a}$ [**Ans.** (*b*)]

23. If $\log_{10} 2, \log_{10}(2^x + 1), \log_{10}(2^x + 3)$ are in A.P., then

(*a*) $x = 0$ (*c*) $x = \log_{10}2$
(*b*) $x = 1$ (*d*) $x = \dfrac{1}{2}\log_2 5$ [**Ans.** (*d*)]

24. If $\log_2 x + \log_2 y \geq 6$, then the least value of $x + y$ is

(*a*) 4 (*c*) 16
(*b*) 8 (*d*) 32 [**Ans.** (*c*)]

25. $7\log\left(\dfrac{16}{15}\right) + 5\log\left(\dfrac{25}{24}\right) + 3\log\left(\dfrac{81}{80}\right)$ is equal to

(*a*) 0 (*c*) log 2
(*b*) 1 (*d*) log 3 [**Ans.** (*c*)]

CHAPTER

5

Compound Interest and Annuities

SYLLABUS

Different Types of Interest Rates, Net Present Value, Types of Annuities, Continuous Compounding, Valuation of Simple Loans and Debentures, Problems Relating to Sinking Funds

THEMATIC FOCUS

COMPOUND INTEREST

5.1 INTRODUCTION

When money is lent, then the borrower has to pay some extra amount to the lender for the use of money. This extra amount is called **interest**. It is an expense for the person who borrows money and income for the person who lends money. Interest is charged on **Principal** amount at a certain **rate** for a specified period of **time**. For example, 10% per year, 3% per quarter or 2% per month etc. Various terms and their general representation are as follows:

(*a*) **Principal (*P*):** It means the amount of money that is originally borrowed from an individual or a financial institution. It does not include interest.

(*b*) **Number of years (*n*):** It means the time for which money is borrowed ('n' is expressed in number of periods).

(*c*) **Rate of interest (*r*):** The percentage of a sum of money that is charged for the use of the money for a particular period of time (generally one year).

(*d*) **Amount (*A*):** The sum of the Principal and interest together is called the amount

Amount = Principal + Interest

There are two types of interests:

(*i*) Simple interest, and (*ii*) Compound interest

Simple and compound interests are basic financial concepts. A thorough familiarity with them will help you make better decisions when taking a loan or making investments, which may save your money over the long term.

Simple interest: When interest is charged every year (or over a time period) on the amount originally lent (Principal amount) to the borrower, such an interest is called **simple interest**.

Here, year after year, even though the interest gets accumulated and is due to the lender, this accumulated interest in not taken into account for the purpose of calculating interest for later years. Simple interest is usually charged on short-term borrowings.

Simple interest formula: The simple interest I on a principal P for n years at an interest rate of r percent per year is given by the formula:

$$I = P.\ n.\ r$$

(Principal: P; No. of years: n; Rate of interest: r)

$$\text{Total Amount } (A) = P + I$$
$$= P + P.n.r = P\,(1 + n.r)$$

ILLUSTRATION 1

A loan of ₹ 10,000 has been issued for 6-years. Compute the amount to be repaid to the lender if simple interest is charged @ 5% per year.

Solution: Given P = ₹ 10,000, $r = 5\%$, $n = 6$

By putting the values of P, r and n into the simple interest formula:

$$\text{Simple interest} = P.\ n.\ r$$
$$= 10{,}000 \times 6 \times 0.05$$
$$= ₹\ 3{,}000$$

Therefore, at the end of sixth year, the amount of ₹ 13,000 (Principal ₹ 10,000 + Accumulated interest ₹ 3,000) will be repaid to the lender.

Compound interest: If the interest when due is added to the principal and thereafter earns interest, the interest (calculated in this way) is **compound interest**. The interest is computed on the Principal for the first period, on the principal and the first principal's interest for the second period, on the second principal and the second principal's interest for the third period, and so on, with the principal plus interest becoming a new principal at the end of each period. In the other words, the interest is reinvested to earn more interest. The interest may be compounded monthly, quarterly, half-yearly or annually. The interval of time between successive conversions of interest into principal is called the **interest period or conversion period**.

ILLUSTRATION 2

Suppose you have deposited ₹ 200 with a bank for three years at a rate of 5% per year compounded annually. The interest for the first year will be computed on ₹ 200 and you will have ₹ 210 (Principal ₹ 200 + interest ₹ 10) at the end of first year. The interest for the second year will be computed on ₹ 210 and at the end of second year you will have ₹ 220.50 (Principal ₹ 210 + interest ₹ 10.50). The interest for the third year will be computed on ₹ 220.50 and at the end of third year you will have ₹ 231.52 (Principal ₹ 220.50 + interest ₹ 11.02). Table 5.1 shows the computation for a 3-year period of investment.

Table 5.1: Computation for a 3-year period of investment

Years	Principal Amount (₹)	Rate of Interest	Interest	Compound Amount (₹)
1	200	5%	10	200 + 10 = 210
2	210	5%	10.50	210 + 10.50 = 220.50
3	220.50	5%	11.02	220.50 + 11.02 = 231.52

Under compound interest system, when interest is added to the principal amount, the resulting figure is known as **compound amount**. In Table 5.1, the compound amount at the end of each year has been computed in the last column. It is observed that the compound amount at the end of a year becomes the principal amount to compute the interest for the next year.

Compound Amount Formula

The above procedure of computing compound amount is lengthy and time consuming. We can compute compound amount for any number of periods by using the following formula :

$$A = P\left(1 + \frac{r}{m}\right)^{m.n}$$

Amount (A); Principal (P); Rate of interest (r); number of times per year, interest is compounded (m); time in years (n)

In the above formula, 'A' represents the compound amount (i.e. future value) after 'n' years compounded 'm' times a year at interest rate 'r' with starting amount (i.e. Principal) 'P'.

$$\text{Compound Interest } (I) = A - P$$
$$= P\left(1 + \frac{r}{m}\right)^{m.n} - P$$
$$= P\left\{\left(1 + \frac{r}{m}\right)^{mn} - 1\right\}$$

OBSERVATIONS

1. If 'r' be the rate of interest, 'P' be the principal amount and the interest be compounded at annually, half-yearly, quarterly or monthly, then the

formulae for calculating compound amount and compound interest are given in Table 5.2.

Table 5.2: Formulae for Calculating Compound Amount and Compound Interest

Conversion Period	Compound Amount (A)	Compound Interest (I) $= A - P$
(i) Annually	$A = P\,(1+r)^n$	$I = P\left\{(1+r)^n - 1\right\}$
(ii) Half-Yearly	$A = P\left(1+\frac{r}{2}\right)^{2n}$	$I = P\left\{\left(1+\frac{r}{2}\right)^{2n} - 1\right\}$
(iii) Quarterly	$A = P\left(1+\frac{r}{4}\right)^{4n}$	$I = P\left\{\left(1+\frac{r}{4}\right)^{4n} - 1\right\}$
(iv) Monthly	$A = P\left(1+\frac{r}{12}\right)^{12n}$	$I = P\left\{\left(1+\frac{r}{12}\right)^{12n} - 1\right\}$

2. **In logarithmic form**: $\log A = \log P + n \log (1 + r)$
3. **In case of depreciation of asset**:

$$A = P\,(1-r)^n$$

where, A = Scrap value, P = Original value of the assets, r = Rate of depreciation

ILLUSTRATION 3

Find the compound interest and the amount of ₹ 8750 at the rate of 6% p.a compounded quarterly for five years.

Solution: Here, $P = 8750$, $n = 5$, $r = 6/100 = 0.06$

Since, the interest is compounded quarterly, then, we can use the formula

$$A = P\left(1+\frac{r}{4}\right)^{4n}$$

$$= 8750\left(1+\frac{0.06}{4}\right)^{4\times 5} = 8750\,(1.015)^{20}$$

Therefore, $\log A = \log 8750 + 20 \log 1.015$

$= 3.9420 + 20 \times 0.0064 = 3.9420 + 0.128$

$= 4.07$

or A = antilog 4.07 = 11,750 (approx.)

Hence, the amount is ₹ 11,750 and

Compound interest = $A - P$ = 11,750 – 8750

= ₹ 3,000

ILLUSTRATION 4

A machine depreciated in value each year at 10% of its previous value and at the end of the fourth year its value was ₹ 1,31,220. Find its original value.

Solution: Here, A = ₹ 1,31,220, $r = \frac{10}{100} = 0.1$, $n = 4$

we are to calculate the value of P.

We know that, $A = P(1 - r)^n$

or 1,31,220 = $P\,(1 - 0.1)^4$

or 1,31,220 = $P\,(0.9)^4$

or $P = \frac{1,31,220}{(0.9)^4} = \frac{1,31,220}{0.6561}$

= ₹ 2,00,000

Therefore, the original value of the machine is ₹ 2,00,000.

OBSERVATION

Compound interest is greater than simple interest

Under simple interest system, the interest is computed only on principal amount whereas under compound interest system, the interest is computed on the combined total of the principal and accumulated interest. So, the compound interest is greater than simple interest.

ILLUSTRATION 5

Mr. A has deposited ₹ 6,000 in a saving account. Bank pays interest at a rate of 9% per year. Compute the amount of interest that will be earned over 5-year period:

(*i*) if the interest is simple

(*ii*) if the interest is compounded annually.

Solution:

(*i*) Simple interest = $P.n.r$

= 6,000 × 5 × 0.09

= ₹ 2700

(*ii*) Compound amount = $P\,(1 + r)^n$

= $6,000\,(1 + .09)^5$

$= 6{,}000\ (1.09)^5$
$= 6{,}000 \times 1.54$
$= ₹\ 9240$

Therefore compound interest $= A - P$
$= 9240 - 6000$
$= ₹\ 3{,}240$

Notice that compound interest is more than simple interest by ₹ 540 (₹ 3,240 – ₹ 2,700).

5.2 PRESENT VALUE OF A SINGLE AMOUNT

The value of money changes over time. The value of money received immediately is of greater value than equivalent sum to be received in future. Because if you have a certain amount in hand today you can invest it elsewhere and earn some interest on it.

The present value of an amount means today's value of amount to be received at a point of time in future.

Present value formula:

The present value of a single payment can be computed by using the following formula:

$$P \ \frac{A}{(1+r)^n}$$

Amount (A)
Number of periods (n)
Present value of the amount (P)
Rate of interest (r)

ILLUSTRATION 6

P Co. is expecting to receive ₹ 5,000 four years from now. Compute present value of this sum if the current market interest rate is 10% and the interest is compounded annually.

Solution: Here, $n = 4$, $r = 10\% = 0.1$ $A = ₹\ 5000$

By putting the values of 'n', 'r' and 'A' into the present value formula:

$$P = \frac{A}{(1+r)^n} = \frac{5000}{(1+0.1)^4} = \frac{5000}{(1-1)^4}$$

$$= \frac{5{,}000}{1.4641} = 5000 \times 0.683$$

$= ₹\ 3{,}415$

Therefore, the present value of ₹ 5000 is ₹ 3,415.

5.3 NOMINAL RATE AND EFFECTIVE RATE OF INTEREST

Nominal rate of interest: The rate of interest specified in case of compound interest, compounded a given number of times per year is called the **nominal rate of interest**.

Effective rate of interest: The rate of interest which, if compounded annually, would result in the same amount of interest is called the **effective rate of interest**.

From the above definitions, it can be said that, if interest is compounded annually then nominal rate of interest and effective rate of interest would be equal.

Relationship between nominal rate and effective rate of interest

If r_e represents effective rate of interest per unit per annum, r_n represents nominal rate of interest per unit per conversion period and m represents the number of conversion period in a year.

Then $$r_e = \left(1+\frac{r_n}{m}\right)^m - 1$$

ILLUSTRATION 7

Find the effective rate equivalent to nominal rate of 6% converted quarterly.

Solution: We know that $r_e = \left(1+\frac{r_n}{m}\right)^m - 1$

Here, $$r_n = \frac{6}{100} = 0.06,\ m = 4$$

Therefore, $$r_e = \left(1+\frac{0.06}{4}\right)^4 - 1$$

$$= (1 + 0.015)^4 - 1$$
$$= (1.015)^4 - 1$$
$$= 1.0613 - 1$$
$$= 0.0613$$

Therefore, the effective rate (percent) equivalent to nominal rate of 6% compounded quarterly = 0.0613 i.e. 6.13%.

ANNUITIES

5.4 INTRODUCTION

An **annuity** is simply a series of equal payments (or receipts) that occur at a regular interval. For instance, if any person deposits ₹ 10,000 in his savings bank account at the end of every year for a period of 10 years at 4% rate of interest, then the series of payments of ₹ 10,000 will be known as **annuity**. The payments can be different amounts, but must occur regularly, usually the time period is one year, so it is called an **annuity**. But the time period can be shorter, or even longer.

Understanding annuities is crucial for understanding loans and investments that require or yield periodic payments. For instance, how much of a mortgage can you afford if you can only pay ₹ 2,000 monthly? How much money will you have in your savings account if you deposit ₹ 1,500 at the beginning of each year for 20 years, and earn an annual interest rate of 4%, but it is compounded daily?

Various terms and their general representation are as follows:

1. **Payment Period or Payment Interval:** The length of time between two successive payments is called **payment period or payment interval**.
2. **Term:** The **term** of an annuity is the total time between the beginning of the first payment period and the end of the last one.
3. **Annual Rent:** The sum of payments made in one year is known as **annual rent**.
4. **Amount or Future Value:** The **amount or future value** of an annuity is the total amount that would be accumulated at the end of the term if each payment were invested at compound interest at the time of payment.
5. **Present Value or Capital Value:** The sum of present values of all payments is called **present value or capital value**.

5.5 TYPES OF ANNUITIES

Annuity Certain: An annuity in which payments begin and end at fixed dates is known as **annuity certain**.

Perpetual Annuity or Perpetuity: An annuity in which payments begin at a fixed date but continue forever is called **perpetual annuity or perpetuity**.

Contingent Annuity: An annuity whose payments continue for a period of time depending on events whose dates of occurrence cannot be accurately foretold is called **contingent annuity**. For example, the premiums on a life insurance policy constitute such an annuity.

Ordinary or Immediate Annuity: An annuity in which payments are made at the end of each period, is called **ordinary or immediate annuity**.

Annuity Due: An annuity in which payments are made at the beginning of each period it is called **annuity due**.

Deferred Annuity: An annuity that is to take effect after a certain time is known as **deferred annuity**.

Deferred Perpetuity: An annuity in which, the payment will commence after the lapse of deferred period and will then continue forever is called **deferred perpetuity**.

5.6 IMMEDIATE OR ORDINARY ANNUITY

5.6.1 Amount or Future Value of Immediate or Ordinary Annuity

If a fixed sum of money (P) is regularly invested at the end of a year for a certain period (n) of time, and the rate of interest payable on one rupee for one year is r, then the amount available (A) at the end of n years will be calculated by using the following formula:

$$A = P(1+r)^{n-1} + P(1+r)^{n-2} + \cdots + P(1+r) + P$$
$$= P + P(1+r) + \cdots + P(1+r)^{n-2} + P(1+r)^{n-1} \text{ [writing in reverse order]}$$
$$= P[1 + (1+r) + \cdots + (1+r)^{n-2} + (1+r)^{n-1}]$$
$$= P\left\{\frac{(1+r)^n - 1}{(1+r)-1}\right\} \text{ [as it is a G.P series with common ratio } (1+i)]$$
$$= P\left\{\frac{(1+r)^n - 1}{r}\right\} = \frac{P}{r}\left\{(1+r)^n - 1\right\}$$

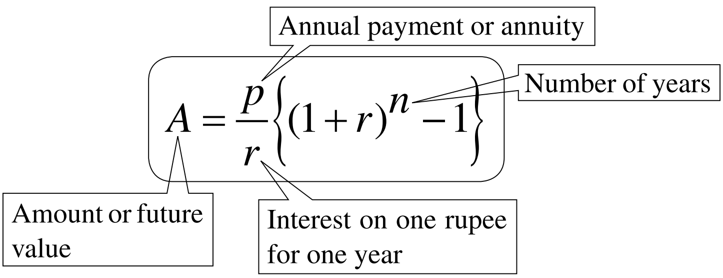


ILLUSTRATION 8

Mr. A deposits ₹ 5,000 at the end of each year for 5 years at 4% rate of interest. How much would he receive at the end of the 5th year?

Solution: Here , P = ₹ 5,000, $r = \frac{4}{100} = 0.04$ and $n = 5$

Therefore, $A = \frac{5000}{0.04}[(1.04)^5 - 1]$

$= 1{,}25{,}000\ (1.217 - 1)$

$= 1{,}25{,}000 \times 0.217$

= ₹ 27,125

5.6.2 Present Value of Immediate or Ordinary Annuity

Let P denote the annual payment of an immediate annuity, n is the number of years, r is the interest on one rupee for one year and V is the present value of annuity, then value of V will be calculated by using the following formula:

$$V = \frac{P}{(1+r)} + \frac{P}{(1+r)^2} + \frac{P}{(1+r)^3} + \cdots + \frac{P}{(1+r)^{n-1}} + \frac{P}{(1+r)^n}$$

$$= \frac{P}{(1+r)^n}\left[(1+r)^{n-1} + (1+r)^{n-2} + \cdots + (1+r)^2 + (1+r) + 1\right]$$

$$= \frac{p}{(1+r)^n}\left[1 + (1+r) + (1+r)^2 + \cdots + (1+r)^{n-2} + (1+r)^{n-1}\right]$$

[writing in reverse order]

$$= \frac{P}{(1+r)^n}\left[\frac{(1+r)^n - 1}{(1+r) - 1}\right]$$

[as it is a G.P. series with common ratio = (1 + r) >1 and 1st term = 1]

$$= \frac{P}{(1+r)^n}\left[\frac{(1+r)^n - 1}{r}\right] = \frac{P}{r}\left[\frac{(1+r)^n - 1}{(1+r)^n}\right]$$

$$= \frac{P}{r}\left[1 - \frac{P}{(1+r)^n}\right]$$

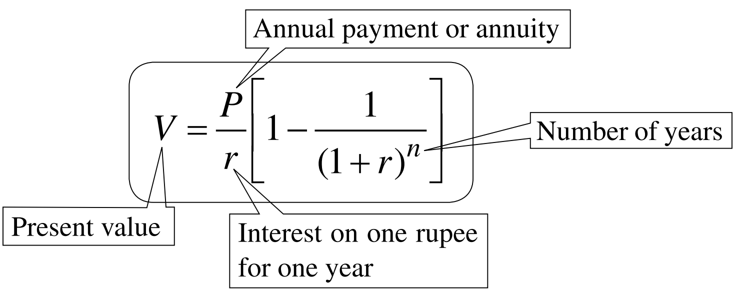


ILLUSTRATION 9

Calculate the present value of an annuity of ₹ 5,000 received annually for 5 years at a discount factor of 4%.

Solution: Here, P = ₹ 5,000, $r = \frac{4}{100} = 0.04$ and $n = 5$

Therefore, $V = \frac{p}{r}\left[1 - \frac{1}{(1+r)^n}\right]$

$$= \frac{5000}{0.04}\left[1 - \frac{1}{(1.04)^5}\right]$$

$$= 1,25,000\left(1-\frac{1}{1.217}\right)$$

= 1,25,000 (1 – 0.822)
= 1,25,000 × 0.178
= ₹ 22,250

5.7 DEFERRED ANNUITY

5.7.1 Amount of Deferred Annuity

The amount of a deferred annuity for *n* periods, deferred *m* periods is the value of the annuity at the end of its term. This will be calculated by using the following formula:

$$A = \frac{P}{r}\left[\frac{(1+r)^n - 1}{(1+r)^m}\right]$$

5.7.2 Present Value of Deferred Annuity

Present value of deferred annuity whose payments of instalments starts after *m* periods from the beginning and continue for *n* periods after the end of *m* periods will be calculated by using the following formula:

$$V = \frac{P}{r}\left[\frac{(1+r)^n - 1}{(1+r)^{m+n}}\right]$$

ILLUSTRATION 10

Mr. C borrows a certain sum of money at 5% p.a. compound interest and agrees to pay both the principal and the interest in 10 equal yearly instalments of ₹ 500 each. If the first instalment is to be paid at the end of 3 years from the date of borrowing and the other yearly instalments are paid regularly at the end of the subsequent years, find the sum borrowed by him.

Solution: Since the first instalment is to be paid at the end of 3 years as such the annuity, in this case, is deferred by 2 years. We have to calculate the present value of loan which is a deferred annuity.

$$V = \frac{P}{r}\left[\frac{(1+r)^n - 1}{(1+r)^{m+n}}\right]$$

$$= \frac{500}{0.05}\left[\frac{(1+0.05)^{10}-1}{(1+0.05)^{10+2}}\right]$$

$$= 10,000\left[\frac{(1.05)^{10}-1}{(1.05)^{12}}\right]$$

Here, P = ₹ 500
$i = 0.05$
$n = 10$
$m = 2$

Calculation:	
Let $x = (1.05)^{10}$	Let $y = (1.05)^{12}$
or log x = 10 log (1.05)	or log y = 12 log 1.05
= 10 × 0.0212 = 0.212	= 12 × 0.0212 = 0.2544
Therefore x = antilog (0.212) = 1.629	Therefore y = antilog (0.2544) = 1.797

Therefore, $V = 10,000\left[\frac{1.629-1}{1.797}\right] = 10,000 \times \frac{0.629}{1.797}$ = ₹3,500 (Approx.)

Therefore, the sum borrowed by Mr. C = ₹ 3,500.

5.8 ANNUITY DUE

5.8.1 Amount of an Annuity Due

Amount (A) of an annuity due at the end of n periods i.e. at the end of the term will be calculated by using the following formula:

$$A = \frac{P(1+r)}{r}\left[(1+r)^n - 1\right]$$

ILLUSTRATION 11

Mr. B deposits ₹ 2,000 at the beginning of each year for 4 years. How much do these accumulate at the end of 4th year? Assume the rate of interest is 4% p.a.

Solution: Here, P = ₹ 2000, $r = \frac{4}{100} = 0.04$ and $n = 4$

$$A = \frac{P(1+r)}{r}\left[(1+r)^n - 1\right]$$

$$= \frac{2000 \times 1.04}{0.04}\left[(1.04)^4 - 1\right]$$

= 52,000 (1.17 – 1) = 52,000 × 0.17

= ₹ 8,840

5.8.2 Present Value of an Annuity Due

The present value of an annuity due can be calculated by using the following formula:

$$V = \frac{P(1+r)}{r}\left[1-(1+r)^{-n}\right] = \frac{P(1+r)}{i}\left[1-\frac{1}{(1+r)^n}\right]$$

ILLUSTRATION 12

Calculate the present value of an annuity of ₹ 2,000 received at the beginning of each year for 4 years at a discount factor of 4%.

Solution: Here, P = ₹ 2,000, $i = 0.04$ and $n = 4$

$$\text{Therefore, } V = \frac{P(1+r)}{r}\left[1-\frac{1}{(1+r)^n}\right]$$

$$= \frac{2000\times 1.04}{0.04}\left[1-\frac{1}{(1.04)^4}\right]$$

$$= 52{,}000\left(1-\frac{1}{1.17}\right)$$

$$= 52{,}000 \times (1 - 0.85)$$

$$= 52{,}000 \times 0.15 = ₹\ 7{,}800$$

5.9 PRESENT VALUE OF PERPETUAL ANNUITY OR PERPETUITY

We know that the present value (V) of immediate annuity is given by:

$$V = \frac{P}{r}\left[1-\frac{1}{(1+r)^n}\right]$$

As per definition of perpetual annuity n tends to infinity and accordingly $\frac{1}{(1+r)^n} \to 0$, since $1 + r > 1$.

Therefore, $V = \frac{P}{r}$

ILLUSTRATION 13

Find the present value of perpetuity of ₹ 1000 at 5% p.a. compound interest.

Solution: Here, $P = ₹\ 1000,\ r = \frac{5}{100} = 0.05$

Therefore, $V = \frac{100}{0.05} = ₹\ 20{,}000$.

OBSERVATION

Net Present Value (NPV) = Sum of present value of cash inflows – Sum of present value of cash outflows

5.10 CONTINUOUS COMPOUNDING

Continuous compounding is an extreme case of compounding where the compounding period is infinitely small. This causes the interest to be compounded infinitely in any given period. In other words, the time interval between which interest is compounded is infinitely small causing the compounding periods per year to become infinite, which has the effect that interest is compounded continuously. For example, banks may pay interest continuously, and they call it daily compounding.

(*i*) The formula for calculating the amount (future value) based on continuous compounding is:

$$A = P.\ e.^{r \times n}$$

- Rate of interest
- Time in years
- Amount
- Principal
- Mathematical constant $e = 2.7183$

(*ii*) The formula for calculating interest in case of continuous compounding is:

$$I = P \times (e^{r.n} - 1) \quad [I = \text{Interest}]$$

5.10.1 Derivation of Continuous Compounding Formula

The continuous compounding formula can be found by first looking at the compound interest formula

$$A = P\left(1 + \frac{r}{q}\right)^{qn}$$

where q is the number of times compounded, n is number of years, and r is the rate of interest.

When q, or the number of times compounded is infinite, the formula can be rewritten as

$$A = P \times \left[\lim_{q \to \infty} \left(1 + \frac{r}{q}\right)^q \right]^n$$

The limit section in the middle of the formula can be shown as e^r, which leads to the formula

$$A = P.e^{r.n}$$

ILLUSTRATION 14

Find out the compound value of ₹ 2000, interest rate being 20 percent per annum compounded continuously for 5 years.

Solution: Here, P = ₹ 2,000, $r = \frac{20}{100} = 0.2$ and $n = 5$

Therefore $A = P.e^{r.n}$

$= 2000 \times e^{0.2 \times 5}$

$= 2000 \times e$

$= 2000 \times 2.7183$

= ₹ 5436.60

5.11 VALUATION OF SIMPLE LOAN

Loan is an amount collected from an outsider at an interest and repayable in a specified period (lump sum or in instalments). A loan, by definition, is an annuity, in that it consists of a series of future periodic payments. Payment of loan is known as amortization. A loan is amortized if both the principal and interest are paid by a sequence of equal periodic payments. The formula used to calculate periodic loan payments is exactly the same as the formula used to calculate payments on an ordinary annuity. The periodic payment needed to amortize a loan may be found, by solving the following formula.

$$P = \frac{Vr}{1 - (1 + r)^{-n}}$$

where V: Present value or original amount of loan; P: Periodic payment; r: Rate of interest; n: Number of periods

ILLUSTRATION 15

Mr. D borrows ₹ 1,000 for one year at 12% annual interest, compounded monthly. Find his monthly payment.

Solution: Here, V = ₹ 1,000, n = 12, and monthly interest rate (r) = $\frac{0.12}{12}$ = 0.01

Therefore, monthly payment (P) = $\frac{V \cdot r}{1-(1+r)^{-n}}$

$$= \frac{1000 \times 0.01}{1-(1+0.01)^{-12}} = \frac{10}{1-(1.01)^{-12}}$$

$$= \frac{10}{1-0.8882}$$

$$= \frac{10}{0.1118}$$

= ₹ 89.45

Let $x = (1.01)^{-12}$
$\log x = -12 \times \log 1.01$
$= -12 \times 0.0043$
$= -0.0516$
$= -1 + 1 - 0.0516$
$= -1 + 0.9484$
$= \bar{1}.9484$
$x = \text{antilog } \bar{1}.9484 = 0.8882$

The original loan amount is the present value of the future payments on the loan, much like the present value of an annuity. The original loan amount (V) may be calculated by solving the following formula.

$$V = \frac{P}{r}\left[1-\left(1+r\right)^{-n}\right].$$

OTHER FORMULAE

(*i*) Formula for calculating **principal** outstanding at the beginning of the t^{th} period = $= \frac{P}{r}\left[1-(1+r)^{-n+t-1}\right]$

(*ii*) formula for calculating **interest** contained in the t^{th} payment

$$= P\left[1-\left(1+r\right)^{-n+t-1}\right]$$

(*iii*) formula for calculating **principal** contained in t^{th} payment

$$= P - P\left[1-(1-r)^{-n+t-1}\right]$$

$$= P\left[1-1+(1+r)^{-n+t-1}\right]$$

$$= P.(1+r)^{-n+t-1}$$

(*iv*) Total interest paid = $np - V$

ILLUSTRATION 16

A loan of ₹ 20,000 is to be repaid by annual instalments of principal and interest over a period of 10 years. The rate of interest is 5% per annum compounded annually. Find

(*i*) the annual instalment;

(*ii*) the capital contained in the 6th instalment;

(*iii*) the principal repaid after 8 instalments have been paid.

Solution:

(*i*) We know that,

$$V = \frac{P}{r}\left[1-(1+r)^{-n}\right]$$

Here, V = ₹ 20,000, $r = 0.05$, $n = 10$,

P = Size of each instalment

$$\text{Therefore, } 20{,}000 = \frac{P}{0.05}\left[1-(1+0.05)^{-10}\right]$$

or $P\,[1 - (1.05)^{-10}] = 1000$

or $P\,(1 - 0.6138) = 1000$

or $P \times 0.3862 = 1000$

or $P = \dfrac{1000}{0.3862}$

= ₹ 2589.33

Therefore, the size of each annual instalment = ₹ 2,589.33.

> Let $x = (1.05)^{-10}$
>
> or $\log x = -10 \times \log(1.05)$
>
> $= -10 \times 0.0212$
>
> $= -0.212$
>
> $= -1 + 1 - 0.212$
>
> $= -1 + 0.788$
>
> $= \bar{1}.788$
>
> $x = \text{antilog } \bar{1}.788 = 0.6138$

(*ii*) The capital contained in the t^{th} instalment

$= P\,[(1 + r)^{-n+t-1}]$

Here, P = ₹ 2589.33, $r = 0.05$,

$-n + t - 1 = -10 + 6 - 1 = -5$

Therefore, the capital contained in the 6th instalment

$= 2589.33\,(1.05)^{-5}$

$= 2589.33 \times 0.7834$

= ₹ 2028.5

> Let, $x = (1.05)^{-5}$
>
> $\log x = -5 \log(1.05)$
>
> $= -5 \times 0.0212$
>
> $= -0.106$
>
> $= -1 + 1 - 0.106$
>
> $= \bar{1}.894$
>
> $x = \text{antilog } \bar{1}.894$
>
> $= 0.7834$

(*iii*) The principal repaid after 8 instalments have been paid

= Loan amount – Principal outstanding at the beginning of the 9th payment interval i.e. 9th period

$$= 20{,}000 - \frac{2589.33}{0.05} \times \left[1-(1.05)^{-10+9-1}\right]$$

$$= 20{,}000 - 51{,}786.6\left[1-(1.05)^{-2}\right]$$

$$= 20{,}000 - 51{,}786.6\left[1-\frac{1}{(1.05)^2}\right]$$

$$= 20{,}000 - 51{,}786.6\left[1-\frac{1}{1.1025}\right]$$

= 20,000 – 51,786.6 [1 – 0.907]
= 20,000 – 51786.6 × 0.093
= 20,000 – 4816.15
= ₹ 15,183.85

OBSERVATIONS

It is important to keep the rate per period and number of periods consistent with one another in the formula. If the loan payments are made monthly, then the rate per period needs to be adjusted to the monthly rate and the number of periods would be the number of months on the loan. If payments are quarterly, the terms of the loan payment formula would be adjusted accordingly.

5.12 VALUATION OF DEBENTURE/BOND

Meaning: A bond or debenture is an instrument of long-term debt issued by a company to the general public. These debts are normally repayable on a fixed date and pay a fixed rate of interest.

Valuation of bonds or debentures: The value of bonds or debentures is, generally determined through the capitalisation technique. The process of determination of the present value of a bond or debenture can be considered under two headings:

(*a*) Redeemable bond or debenture (i.e. definite maturity period).
(*b*) Irredeemable bond or debenture (i.e. no specified definite maturity period).

5.12.1 Present Value of a Redeemable Bond or Debenture

In case of redeemable bond or debenture, its present value can be determined by estimating its future cash flows, and then discounting the estimated future cash flows at an appropriate capitalization rate or discount rate. The estimated cash flows from the bond or debenture consists of the stream of future interest payments plus the principal repayment.

The following formula may be used to calculate the present value of the bond or debenture:

$$V = \left[\frac{I}{1+r} + \frac{I}{(1+r)^2} + \frac{I}{(1+r)^3} + \cdots + \frac{I}{(1+r)^n}\right] + \frac{M}{(1+r)^n}$$

$$= \left[\sum_{i=1}^{n} \frac{I}{(1+r)^i}\right] + \frac{M}{(1+r)^n}$$

$$= I\left[\frac{1-(1+r)^{-n}}{r}\right] + M(1+r)^{-n}$$

where, V = the present value of the bond or debenture
I = periodic interest payment
r = the capitalization rate or the discount rate
n = number of periods or payments
M = the maturity value of the bond or debenture

5.12.2 Present Value of a Perpetual or Irredeemable Bond or Debenture

In the case of irredeemable bond or debenture, its present value can be determined by simply discounting the stream of interest payments for the infinite period by an appropriate capitalization rate or discount rate.

The following formula may be used to calculate the present value of the bond or debenture.

$$V = \frac{I}{r}$$

where, V = the present value of the bond or debenture
I = periodic interest payment
r = the capitalization rate or the discount rate

ILLUSTRATION 17

Company A has issued a bond having face value of ₹ 1,00,000 carrying annual interest rate of 8% and maturing in 10 years. The market interest rate is 10%. Calculate the price of bond.

Solution: Here, $I = 1{,}00{,}000 \times \frac{8}{100} = 8{,}000$

$r = 10\% = 0.10$, $M = 1{,}00{,}000$ and $n = 10$ years

Therefore, price of bond (v) = $I\left[\frac{1-(1+r)^{-n}}{r}\right]+\frac{M}{(1+r)^{-n}}$

$$= 8,000\left[\frac{1-(1.1)^{-10}}{0.10}\right]+\frac{1,00,000}{(1.1)^{10}}$$

$$= 8,000\left[\frac{1-(1.1)^{-10}}{0.10}\right]+1,00,000\times(1.1)^{-10}$$

$$= 8,000\left[\frac{1-0.3855}{0.10}\right]+1,00,000\times 0.3855$$

$$= 8,000\left[\frac{0.6145}{0.10}\right]+38,550$$

= 8,000 × 6.145 + 38,500
= 49,160 + 38,500
= ₹ 87,660.

Calculation:
Let $x = (1.1)^{-10}$
$\log x = -10 \times \log 1.1$
$= -10 \times 0.0414$
$= -0.414$
$= -1 + 1 - 0.414$
$= \bar{1}.586$
$x = \text{antilog } \bar{1}.586$
$= 0.3855$

5.13 SINKING FUND

A sinking fund is an interest earning fund that is created in order to repay the large debt at the time of its maturity or to replace an asset, like machinery, plants, etc., on a future specified date by depositing a fixed amount periodically at compound rate of interest.

To calculate the liability 'A' to be redeemed after 'n' years by depositing a fixed amount 'P' periodically at an interest on one rupee for one year 'r', the following formula can be used:

$$A=\frac{P}{r}\left[(1+r)^{n}-1\right]$$

The above formula is same as the formula of amount of immediate annuity.

ILLUSTRATION 18

A sinking fund is created for the redemption of debentures of ₹ 1,20,000 at the end of 25 years. How much money should be provided out of profits each year for the sinking fund, if the investment can earn interest @ 4% p.a?

Solution: Let the sum of money to be deposited in sinking fund be ₹ P.

Here, A = ₹ 1,20,000, r = 0.04 and n = 25 years

We know that, $A = \frac{P}{r}\left[(1+r)^{n}-1\right]$

or $$1,20,000 = \frac{P}{0.04}\left[(1+0.04)^{25} - 1\right]$$

or $$4,800 = P\left[(1.04)^{25} - 1\right]$$

or $$4,800 = P(2.661 - 1)$$

or $$1.661 \times P = 4,800$$

or $$P = \frac{4,800}{1.661}$$

$$= ₹\ 2,890$$

Calculation:

Let $x = (1.04)^{25}$

or $\log x = 25 \log (1.04)$

$= 25 \times 0.0170$

$= 0.4250$

Therefore, $x = \text{antilog } 0.4250$

$= 2.661$

Therefore, the sum of money to be deposited in sinking fund every year out of profit is ₹ 2,890.

ILLUSTRATIVE EXAMPLES

COMPOUND INTEREST

A. SHORT TYPE

Finding the Simple Interest, Compound Interest and Rate of Interest

EXAMPLE 1

What is the simple interest on ₹ 20,000 at 18% p.a for a period of two years?

Solution: Simple Interest $(I) = P.n.r$

$$= 20,000 \times 2 \times \frac{18}{100}$$

$$= ₹\ 7,200.$$

EXAMPLE 2

What is the compound interest on ₹ 14,000 at 20% p.a. for a period of two years compounded annually?

Solution: Here, P = ₹ 14,000, r = 20% and n = 2 years

Therefore, $A = P(1 + r)^n$

$= 14,000 (1 + 0.2)^2$

$= 14,000 (1.2)^2$

$= 14,000 \times 1.44$

$= ₹\ 20,160$

Therefore, compound interest $= A - P =$ ₹ (20,160 – 14,000)

$= ₹\ 6,160.$

EXAMPLE 3

Ajay invested an amount of ₹ 8,000 in a fixed deposit scheme for 2 years at compound interest rate 5% p.a. compounded half-yearly. How much amount will Ajay get on maturity of the fixed deposit.

Solution: Here, P = ₹ 8,000, $n = 2$ years, $r = 0.05$ and $m = 2$ times a year.

we know, $A = P\ (1+\frac{r}{m})^{m.n}$

$$= 8000\left(1+\frac{0.05}{2}\right)^{2\times 2}$$

$$= 8000\ (1 + 0.025)^4$$
$$= 8000\ (1.025)^4$$
$$= 8000 \times 1.1038$$
= ₹ 8830.40

Therefore, Ajay will get ₹ 8,830.40 on maturity of the fixed deposit.

EXAMPLE 4

At what rate of compound interest per annum will a sum of ₹ 1200 become ₹ 1348.32 in 2 years?

Solution: Let the rate be r% p.a.

we know that, $A = P\ (1 + r)^n$

or $\quad 1348.32 = 1200\left(1+\frac{r}{100}\right)^2$

or $\quad \frac{1348.32}{1200} = \left(1+\frac{r}{100}\right)^2$

or $\quad \left(1+\frac{r}{100}\right)^2 = \frac{1348.32}{1200} = 1.1236$

or $\quad 1 + \frac{r}{100} = \sqrt{1.1236} = 1.06$

or $\quad \frac{r}{100} = 1.06 - 1 = 0.06$

or $\quad r = 0.06 \times 100 = 6$

Therefore, required rate of interest = 6%.

Finding the Period

EXAMPLE 5

In how many years a sum becomes 6 times itself at 10% p.a. simple interest?

Solution: Let P be the principal and A be the amount

Now, $A = 6P$

So, Interest $(I) = A - P = 6P - P = 5P$

We know that, $I = P.\ n.\ r$

or $5P = P.n.\ \dfrac{10}{100}$

or $\dfrac{n}{10} = \dfrac{5P}{P} = 5$

or $n = 50$

Therefore, required number of years = 50.

EXAMPLE 6

The compound interest on ₹ 30,000 at 7% p.a. is ₹ 4347. Calculate the required time.

Solution: Amount (A) = ₹ (30,000 + 4,347)

= ₹ 34,347

P = ₹ 30,000

$$r = \frac{7}{100} = 0.07$$

Let the time be n years

Then , $A = P\,(1 + r)^n$

or $34{,}347 = 30{,}000\,(1 + 0.07)^n$

or $(1.07)^n = \dfrac{34{,}347}{30{,}000} = 1.1449 = (1.07)^2$

or $n = 2$

Therefore, required number of years = 2.

Finding the Principal

EXAMPLE 7

Find the principal that amounts to ₹ 4913 in 3 years at 5% p.a. compound interest compounded annually.

Solution: Let the principal be ₹ P.

We know that, $A = P\,(1 + r)^n$

or $P = \dfrac{A}{(1+r)^n}$ $\qquad \left[\begin{aligned} \text{Here, } A &= 4913 \\ r &= 0.05 \\ n &= 3 \text{ years} \end{aligned}\right]$

or $P = \dfrac{4913}{(1+0.05)^3}$

$$= \frac{4913}{(1.05)^3} = \frac{4913}{1.1576} = 4244(\text{Approx})$$

Therefore, principal amount = ₹ 4,244.

Finding the Amount

EXAMPLE 8

Find the amount on ₹ 15,625 for 9 months at 16% p.a. compounded quarterly.

Solution: P = ₹ 15,625, n = 9 months = 3 quarters

$$r = 16\% \text{ p.a.} = \frac{16}{4}\% \text{ per quarter} = 4\% \text{ per quarter}$$

Amount (A) = $P(1 + r)^n$

$$= 15{,}625\left(1+\frac{4}{100}\right)^3$$

$$= 15{,}625\ (1 + 0.04)^3$$

$$= 15{,}625 \times (1.04)^3$$

$$= 15{,}625 \times 1.1248$$

$$= 17{,}575$$

Therefore, required amount = ₹ 17,575.

EXAMPLE 9

Find the amount for ₹ 6,000 at 10% p.a., compounded semi-annually for 2 years.

Solution: Semi-annually means we have to compound the interest for every six months.

Here, P = ₹ 6,000, $r = 10\% = \frac{10}{100} = 0.1$, n = 2 years, m = 2 times in a year.

$$\text{Amount } (A) = P\left(1+\frac{r}{m}\right)^{mn}$$

$$= 6{,}000\left(1+\frac{.1}{2}\right)^{2\times 2}$$

$$= 6{,}000\ (1.05)^4 = 6000 \times 1.2155$$

= ₹ 7,293.

B. SHORT ESSAY TYPE/PROBLEM TYPE

Finding the Simple Interest, Compound Interest and Rate of Interest

EXAMPLE 10

At what rate percent of compound interest, a sum of ₹ 2000 will amount to ₹ 2662 in 3 years?

Solution: Let the rate of compound interest be $r\%$

Here, P = ₹ 2000, A = ₹ 2662, n = 3 years

We know that, $A = P\left(1+\frac{r}{100}\right)^n$

or $\quad 2662 = 2000\left(1+\frac{r}{100}\right)^3$

or $\quad \left(1+\frac{r}{100}\right)^3 = \frac{2662}{2000} = 1.331$

or $\quad \left(1+\frac{r}{100}\right)^3 = (1.1)^3$

or $\quad 1+\frac{r}{100} = 1.1$

or $\quad \frac{r}{100} = 1.1 - 1 = 0.1$ or $r = 10$

Therefore, rate of compound interest = 10%.

EXAMPLE 11

Find the difference between compound interest and simple interest on ₹ 4000 for 1 year at 10% p.a., if the interest is compounded half-yearly.

Solution: Here, P = ₹ 4000, r = 10% = 0.1, n = 1 year, m = 2 times

Simple interest = $P.n.r$

$= 4000 \times 1 \times 0.1 =$ ₹ 400

$$A = P\left(1+\frac{r}{m}\right)^{mn}$$

$$= 4000\left(1+\frac{0.1}{2}\right)^{2\times1} = 4000\,(1 + .05)^2$$

$$= 4000\,(1.05)^2 = 4000 \times 1.1025 = 4{,}410$$

Compound interest = $A - P$

= 4,410 – 4000 = ₹ 410

Therefore, the difference between compound interest and simple interest = 410 – 400 = ₹ 10.

EXAMPLE 12

If compound interest on a certain sum for 2 years at 5% p.a. is ₹ 328, calculate the simple interest.

Solution: Let the principal amount be ₹ 100

$n = 2$ years, $r = 5\% = 0.05$

Simple interest $= P \times n \times r$

$= 100 \times 2 \times 0.05 = 10$

$A = P(1 + r)^n = 100(1 + .05)^2$

$= 100(1.05)^2 = 100 \times 1.1025 = 110.25.$

Compound interest $= A - P = 110.25 - 100$

= ₹ 10.25

If compound interest = 10.25 then simple interest = 10

If compound interest = 328 then simple interest $= \dfrac{10}{10.25} \times 328$

= ₹ 320.

EXAMPLE 13

A man invested ₹ 16,000 at compound interest for 3 years, interest compounded annually. If he got ₹ 18522 at the end of 3 years, what is rate of interest?

Solution: Let the rate of interest be r% p.a.

A = ₹ 18,522, P = ₹ 16,000, $n = 3$ years

We know that, $A = P\left(1 + \dfrac{r}{100}\right)^n$

or $\quad 18522 = 16000\left(1 + \dfrac{r}{100}\right)^3$

or $\quad \left(1 + \dfrac{r}{100}\right)^3 = \dfrac{18522}{16000} = 1.1576$

or $\quad \left(1 + \dfrac{r}{100}\right)^3 = (1.05)^3$

or $\quad 1 + \dfrac{r}{100} = 1.05$ or $\dfrac{r}{100} = 1.05 - 1 = 0.05$

or $\quad r = 0.05 \times 100 = 5$

Therefore, rate of interest = 5%.

Finding the Period

EXAMPLE 14

A certain sum of money invested at a certain rate of compound interest doubles in 6 years. In how many years will it become 8 times?

Solution: Let the principal amount be ₹ P.

This 'P' becomes '$2P$' in 6 years,

also, let it become '$8P$' in n years.

Now, $2P = P(1 + r)^6$

or $2 = (1 + r)^6$ ...(i)

Again, $8P = P(1 + r)^n$

or $(1 + r)^n = 8 = 2^3$

or $(1 + r)^n = \left\{(1+r)^6\right\}^3$ [From (i)]

or $(1 + r)^n = (1 + r)^{18}$

or $n = 18$

Therefore, the principal become 8 times in 18 years.

EXAMPLE 15

How many years will be required for a certain principal to get doubled at 8% compound interest per annum?

Solution: Let the principal be ₹ P

Therefore, amount = ₹ $2P$

$r = 8\% = 0.08$

number of years = n

Now, from $A = P(1 + i)^n$, we get,

$2P = P(1 + 0.08)^n$

or $2 = (1.08)^n$

or $\log 2 = n \log(1.08)$ [Taking logarithm on both sides]

or $n = \dfrac{\log 2}{\log 1.08} = \dfrac{0.3010}{0.0334} = 9.01$

Therefore, after 9 years (approx.) the principal will get doubled.

Finding the Principal

EXAMPLE 16

Find the principal for which the difference of simple interest and compound interest is ₹ 20 at 4% for 2 years.

Solution: Let the principal be ₹ P

Simple interest = $P.n.r$

$= P.2.0.04$

$= 0.08\,P$

Amount (A) = $P(1 + r)^n$

$= P(1 + 0.04)^2$

$= P(1.04)^2 = 1.0816P$

Compound interest = $A - P$

$= 1.0816P - P$

$= 0.0816P$

By the condition,

$0.0816P - 0.08P = 20$

or $0.0016P = 20$

or $P = \dfrac{20}{0.0016}$ = ₹ 12,500.

EXAMPLE 17

What sum will amount to ₹ 4298 in 5 years at 10% p.a. compound interest payable half-yearly?

Solution: Here, A = ₹ 4298, $n = 5$, $r = \dfrac{10}{100} = 0.1$, $m = 2$ times in a year

$$A = P\left(1 + \frac{r}{m}\right)^{mn}$$

or $4298 = P\left(1 + \dfrac{0.1}{2}\right)^{2\times5}$

or $4298 = P\,(1 + 0.05)^{10}$

or $4298 = P\,(1.05)^{10}$

or $\log 4298 = \log P + 10 \log 1.05$ (Taking logarithm on both sides)

or $3.6243 = \log P + 10 \times 0.0212$

or $\log P = 3.6243 - 0.212$

or $\log P = 3.4123$

or $P = \text{antilog } 3.4123 = 2584$

Therefore, required principal = ₹ 2584.

Finding the Amount

EXAMPLE 18

Compute the compounded value of an initial investment of ₹ 5000 at 12%, at the end of 3 years, if interest is compounded (*i*) annually, (*ii*) half-yearly, (*iii*) quarterly, (*iv*) monthly, and (*v*) after every 2 months.

Solution: Here, P = ₹ 5,000, $r = 12\% = 0.12$, $n = 3$ years

Now, (*i*) $A = P\,(1 + r)^n$

$= 5000\,(1 + 0.12)^3 = 5000\,(1.12)^3 = 5000 \times 1.405$

= ₹ 7,025

(*ii*) $m = 2$ times in a year

Therefore, $A = P\left(1 + \dfrac{r}{m}\right)^{mn} = P\left(1 + \dfrac{r}{2}\right)^{2.n}$

$$= 5000\left(1+\frac{0.12}{2}\right)^{2\times 3}$$

$= 5000\ (1.06)^6 = 5000 \times 1.419$

$=$ ₹ 7,095.

(*iii*) $m = 4$ times in a year.

$$A = P\left(1+\frac{r}{m}\right)^{mn}$$

$$= 5000\left(1+\frac{0.12}{4}\right)^{4\times 3}$$

$= 5000\ (1.03)^{12}$

$= 5000 \times 1.424$

$=$ ₹ 7120.

Calculation:

Let $x = (1.03)^{12}$

or $\log x = 12 \log 1.03$

$= 12 \times .0128$

$= 0.1536$

or $x = \text{antilog } 0.1536$

$= 1.424$

(*iv*) $m = 12$ times in a year

$$A = P\left(1+\frac{r}{m}\right)^{mn}$$

$$= 5000\left(1+\frac{0.12}{12}\right)^{12\times 3}$$

$= 5000\ (1 + 0.01)^{36}$

$= 5000\ (1.01)^{36}$

$= 5000 \times 1.429$

$=$ ₹ 7145.

Calculation:

Let $x = (1.01)^{36}$

or $\log x = 36 \times \log 1.01$

$= 36 \times .0043$

$= 0.1548$

or $x = \text{antilog } 0.1548$

$= 1.429$

(*v*) $m = 6$ times in a year

$$A = 5000\left(1+\frac{0.12}{6}\right)^{6\times 3}$$

$= 5000\ (1.02)^{18}$

$= 5000 \times 1.429$

$=$ ₹ 7,145.

Calculation:

Let $x = (1.02)^{18}$

or $\log x = 18 \times \log 1.02$

$= 18 \times .0086$

$= 0.1548$

or $x = \text{antilog } 0.1548$

$= 1.429$

Finding Present Value

EXAMPLE 19

A person invests money in a bank paying 6% interest compounded semi-annually. If the person expects to receive ₹ 8,000 in 6 years, what is the present value of the investment?

Solution: We know that, present value $(P) = \dfrac{A}{\left(1+\frac{r}{m}\right)^{mn}}$

Here, A = ₹ 8,000 , r = 6% = 0.06, n = 6 years, m = 2 times in a year

$$\text{Therefore, } P = \frac{8000}{\left(1+\frac{0.06}{2}\right)^{2\times 6}} = \frac{8,000}{(1+0.03)^{12}}$$

$$= \frac{8,000}{(1.03)^{12}}$$

$$= \frac{8,000}{1.424}$$

= ₹ 5618 (Approx.)

Therefore, the present value = ₹5,618

Calculation:

Let $x = (1.03)^{12}$

or $\log x = 12 \times \log 1.03$

$= 12 \times .0128$

$= 0.1536$

$x = \text{antilog } 0.1536$

$= 1.424$

EXAMPLE 20

Find the present value of ₹ 10,000 due in 12 years at 6% p.a. compound interest.

[***C.U. B.Com. 2006***]

Solution: Here, A = ₹ 10,000, n = 12 years, r = 6 % = 0.06

Let present value be ₹ P

$$\text{Then, } P = \frac{A}{(1+i)^n}$$

$$= \frac{10,000}{(1+0.06)^{12}} = \frac{10,000}{(1.06)^{12}}$$

$= 10,000\,(1.06)^{-12}$

$= 10,000 \times 0.4971$

= ₹ 4,971

Therefore, The present value is ₹ 4971.

Calculation:

Let $x = (1.06)^{-12}$

or $\log x = -12 \times \log 1.06$

or $\log x = -12 \times 0.0253$

$= -0.3036$

$= -1 + 1 - 0.3036$

$= \bar{1}.6964$

or $x = \text{antilog } \bar{1}.6964$

$= 0.4971$

Finding Depreciation

EXAMPLE 21

A machine depreciated 10% p.a. for first two years and then 7% p.a. for the next three years, depreciation being calculated on the diminishing value. If the value of the machine be ₹ 10,000 initially, find the average rate of depreciation and the depreciated value of the machine at the end of the fifth year.

[***C.U. B.Com. 2007***]

Solution: In the first two years the machine depreciates at 10% p.a.

Let A be the depreciated value of the machine at the end of two years,

then $A = 10,000\,(1 - 0.1)^2 = 10,000 \times (0.9)^2$

$= 10,000 \times 0.81 = 8,100$

In the next three years the machine depreciates at 7% p.a.

Let B be the depreciated value of the machine at the end of these three years, then $B = 8{,}100\ (1 - 0.07)^3 = 8{,}100 \times (0.93)^3$

$= 8{,}100 \times 0.804357$

$= 6{,}515.3 = 6515$ (Approx)

Therefore, the depreciated value of the machine at the end of 5 years is ₹ 6,515.

Now, the original (principal) value (P) of the machine is ₹ 10,000 and its depreciated value (A) at the end of the fifth year is ₹ 6,515.

then, $A = P\ (1 - r)^n$

or $6515 = 10{,}000\ (1 - r)^5$

or $\log 6515 = \log 10{,}000 + 5 \log (1 - r)$ [Taking logarithm on both sides]

or $3.8140 = 4 + 5 \log (1 - r)$

or $5 \log (1 - r) = 3.8140 - 4$

or $\log (1 - r) = \dfrac{3.8140 - 4}{5} = -0.0372$

$= -1 + 1 - 0.0372$

$= \bar{1}.9628$

or $1 - r = \text{antilog } \bar{1}.9628 = 0.9179$

or $r = 1 - 0.9179 = 0.0821$ i.e. 8.21%

Therefore, the average rate of depreciation is 8.21% p.a.

EXAMPLE 22

A machine, the life of which is 5 years costs ₹ 12,000. Calculate the scrap value of the end of its life, depreciation on the reducing balance being charged at 25% p.a. [***C.U. B.Com. 2011***]

Solution: Let scrap value be ₹ A.

Given, span of life (n) = 5 years, original value (P) = ₹ 12,000, rate of depreciation (r) = 25% = $\dfrac{25}{100} = 0.25$

Now, $A = P\ (1 - r)^n$

$= 12{,}000\ (1 - 0.25)^5$

$= 12{,}000\ (0.75)^5$

or $\log A = \log 12{,}000 + 5 \log 0.75$ [Taking logarithm on both sides]

or $= 4.0792 + 5 \times (\bar{1}.8751)$

$= 4.0792 + 5\ (-1 + 0.8751)$

$= 4.0792 - 5 + 4.3755$

$= 3.4547$

or $A = \text{antilog } 3.4547 = 2849$

Therefore, the scrap value of the machine at the end of its life is ₹ 2,849.

EXAMPLE 23

A machine depreciates at the rate of 10% of its value at the beginning of a year. The machine was purchased for ₹ 5,810 and the scrap value realised when sold was ₹ 2250. Find the number of years the machine was used.

Solution: Here, P = ₹ 5,810, $A = 2250$, $r = 10\% = 0.10$

We have to calculate the value of 'n'

We know that, $A = P(1 - r)^n$

Then, $2250 = 5810(1 - 0.1)^n$

or $2250 = 5810(0.9)^n$

or $\log 2250 = \log 5810 + n. \log 0.9$ [Taking logarithm on both sides]

or $3.3522 = 3.7642 + n(\bar{1}.9542)$

or $3.3522 = 3.7642 + n(-1 + 0.9542)$

or $3.3522 = 3.7642 = n(-0.0458)$

or $-0.4120 = -0.0458\, n$

or $n = \dfrac{0.4120}{0.0458} = 9\,(\text{Approx})$

Therefore, the machine was used for 9 years.

Finding Effective Rate

EXAMPLE 24

A man wants to invest ₹ 10,000 for 5 years. He may invest the amount at 10% p.a. compound interest accruing at the end of each quarter of the year or he may invest it at 10.25% p.a. compound interest accruing at the end of each year. Which investment will give him better return?

Solution: In the first case,

$r = 10\% = 0.1, m = 4$

Hence, effective rate of interest $(r_e) = \left(1 + \dfrac{r}{m}\right)^m - 1$

$$= \left(1 + \frac{0.1}{4}\right)^4 - 1 = (1 + 0.025)^4 - 1$$

$$= (1.025)^4 - 1 = 0.1038 \text{ i.e. } 10.38\%$$

In the second case,

Interest is compounded annually.

Therefore, nominal rate of interest is equal to effective rate of interest.

Hence, $r_e = 10.25\%$

Since 10.38 > 10.25, the first investment will give him better return.

Miscellaneous

EXAMPLE 25

A certain sum is to be divided between *A* and *B* so that after 5 years the amount received by A is equal to the amount received by *B* after 7 years. The rate of interest is 10%, interest compounded annually. Find the ratio of amounts invested by them.

Solution: Let the sum (principal) received by *A* and *B* are *x* and *y* respectively.

Amount received by *A* after 5 years

$$= x\left(1+\frac{10}{100}\right)^5 = x\,(1 + .1)^5 = x\,(1.1)^5$$

Amount received by *B* after 7 years

$$= y\left(1+\frac{10}{100}\right)^7 = y\,(1 + .1)^7 = y\,(1.1)^7$$

By the condition,

$$x\,(1.1)^5 = y\,(1.1)^7$$

or $$x = \frac{y(1.1)^7}{(1.1)^5} = y(1.1)^2$$

or $$x = 1.21\,y$$

or $$\frac{x}{y} = 1.21 = \frac{121}{100}$$

Therefore, required ratio = 121 : 100.

EXAMPLE 26

A man divided a sum of ₹ 18,750 between his two sons of age 10 years and 13 years respectively in such a way that each would receive the same amount at 3% p.a. compound interest when he attains the age of 30 years. Find the original share of the younger son. [***C.U. B.Com 2010***]

Solution: Let the original share of the younger son was ₹ *x*.

Hence, the original share of the elder son was ₹ (18,750 – *x*)

Younger son will attain 30 years after (30 – 10) = 20 years

Elder son will attain 30 years after (30 – 13) = 17 years

As per condition, the amount of ₹ *x* after 20 years at 3% p.a. compound interest = the amount of ₹ (18,750 – *x*) after 17 years at 3 % p.a. compound interest

Hence, $$x\,(1 + 0.03)^{20} = (18,750 - x)\,(1 + 0.03)^{17}$$

or $$x\,(1.03)^{20} = (18,750 - x)\,(1.03)^{17}$$

or $x(1.03)^3 = 18,750 - x$

or $x \times 1.092727 = 18,750 - x$

or $x \times 1.092727 + x = 18,750$

or $2.092727\,x = 18,750$

or $x = \dfrac{18750}{2.092727} =$ ₹ 8959.60

Therefore, original share of younger son = ₹ 8959.60.

EXAMPLE 27

A man can buy a flat either for ₹ 1,00,000 cash, or for ₹ 50,000 down and ₹ 60,000 After one year. If the money is worth 10% per year compounded half- yearly. Which plan should he choose? **[*C.U. B.Com 2012*]**

Solution: In the second plan, the purchase price of the flat = ₹ 50,000 (down payment) + the present value of ₹ 60,000 paid after one year.

Present value of ₹ 60,000 at 10% p.a. compound interest compounded half-yearly payable after one year

$$P = A\left(1+\frac{r}{2}\right)^{-2n}$$

$$= 60,000\left(1+\frac{.1}{2}\right)^{-2\times 1} = 60,000\,(1+0.05)^{-2}$$

$$= 60,000\,(1.05)^{-2}$$

$$= \frac{60,000}{(1.05)^2} = \frac{60,000}{1.1025} = 54,422 \text{ (Approx.)}$$

Therefore, purchase price of the flat in the second plan = 50,000 + 54,422 = ₹ 1,04,422

which is greater than the purchase price (₹ 1,00,000) in the first plan.

Hence, the man should select the first plan.

EXAMPLE 28

If the population of a town increases every year by 2% of total population at the beginning of that year, in how many years will the total increase of population be 40% [Given log 14 = 1.1461 and log 1.02 = 0.0086]

Solution: Let the population at the beginning (P) = 100
Again, after n years, let the population increase by 40%

Therefore, after n years, the population(A) = 140

Rate of increase of population = 2%

$$\text{i.e. } r = \frac{2}{100} = 0.02$$

We know that, $A = P(1 + r)^n$

then, $140 = 100(1 + 0.02)^n$

or $140 = 100(1.02)^n$

or $\log 140 = \log 100 + n \log 1.02$ [Taking logarithm both sides]

or $2.1462 = 2 + n(0.0086)$

or $n(0.0086) = 2.1462 - 2 = 0.1462$

or $n = \dfrac{0.1462}{0.0086} = 17$ (Approx.)

Therefore, the population increases by 40% in 17 years.

ANNUITIES

A. SHORT ESSAY TYPE/PROBLEM TYPE

Finding the Amount of Annuity

EXAMPLE 1

A person deposits ₹ 5000 at the end of each year for 5 years at 4% rate of interest. How much would he receive at the end of the 5th year?

Solution: Here, $P = ₹\ 5000$, $r = \dfrac{4}{100} = 0.04$, $n = 5$

We know that, $A = \dfrac{P}{r}\left[(1+r)^n - 1\right]$

$$= \frac{5000}{0.04}\left[(1+0.04)^5 - 1\right]$$

$$= 1{,}25{,}000\left[(1.04)^5 - 1\right]$$

$$= 1{,}25{,}000(1.217 - 1)$$

$$= 1{,}25{,}000 \times 0.217$$

$$= ₹\ 27{,}125$$

Therefore, the person will receive ₹ 27,125 at the end of the 5th year.

EXAMPLE 2

Rahim deposits ₹ 500 at the end of every year for 6 years at 6% interest. Determine Rahim's money value at the end of 6 years.

Solution: Here, $P = ₹\ 500$, $r = \dfrac{6}{100} = 0.06$, $n = 6$ years

Therefore , $A = \dfrac{500}{0.06}\left[(1+.06)^6 - 1\right]$

$$= 8{,}333\left[(1.06)^6 - 1\right]$$

$= 8{,}333(1.419 - 1)$
$= 8{,}333 \times 0.419$
$= ₹\ 3491.53$

Therefore, Rahim's money value at the end of 6 years = ₹ 3491.53

Calculation:
Let $x = (1.06)^6$
or $\log x = 6 \log 1.06$
$= 6 \times 0.0253$
$= 0.1518$
$x = \text{antilog } 0.1518$
$= 1.419$

EXAMPLE 3

Kanan deposits ₹ 2,500 at the beginning of every year for 5 years in a savings bank account at 4% compound interest. What is his money value at the end of 5 years ?

Solution: Here, $P = ₹\ 2500$, $r = 0.04$, $n = 5$

$$\text{Therefore, } A = \frac{2500}{0.04}\left[(1+0.04)^5 - 1\right](1+0.04)$$

$$= \frac{2500(1.04)}{0.04}\left[(1.04)^5 - 1\right]$$

$= 65{,}000\ (1.218 - 1)$
$= 65{,}000 \times 0.218$
$= ₹\ 14{,}170$

Therefore, Kanan's money value at the end of 5 years = ₹ 14,170.

Calculation:
Let $x = (1.04)^5$
$\log x = 5 \log (1.04)$
$= 5 \times 0.0170$
$= 0.085$
or $x = \text{antilog } 0.085$
$= 1.218$

EXAMPLE 4

On 1.1.2010, Mr. *X* purchases a deferred annuity of ₹ 500 for 20 years at 5% p.a. compound interest, the first payment is to be received on 1.1.2017. Find the amount that he will have to pay.

Solution: Here, $P = ₹\ 500$, $n = 20$, $m = 7$, $r = \frac{5}{100} = 0.05$

$$\text{We know that, } V = \frac{P}{r}\left[\frac{(1+r)^n - 1}{(1+r)^{m+n}}\right]$$

$$= \frac{500}{0.05}\left[\frac{(1+0.05)^{20} - 1}{(1+0.05)^{27}}\right]$$

$$= \frac{500}{0.05}\left[\frac{(1.05)^{20} - 1}{(1.05)^{27}}\right]$$

$$= \frac{500}{0.05}\left[\frac{2.655 - 1}{3.736}\right]$$

$$= 10{,}000 \times \frac{1.655}{3.736}$$

$= 10{,}000 \times 0.442987$

$=$ ₹ 4429.87

Therefore, the person will have to pay ₹ 4429.87 to purchase the annuity.

Calculation:	
Let $x = (1.05)^{20}$	Let $y = (1.05)^{27}$
or $\log x = 20 \log 1.05$	$\log y = 27 \log 1.05$
$= 20 \times 0.0212$	$= 27 \times 0.0212$
$= 0.424$	$= 0.5724$
Therefore, $x = \text{antilog } 0.424$	$y = \text{antilog } 0.5724$
$= 2.655$	$= 3.736$

Finding the Present Value

EXAMPLE 5

Mr. Rahul Roy wishes to determine the present value of the annuity consisting of cash flows of ₹ 40,000 p.a. for 6 years. The rate of interest he can earn from his investment is 10 percent.

Solution: Here, $P =$ ₹ 40,000, $r = \frac{10}{100} = 0.1$, $n = 6$ years

We know that, $V = \frac{P}{r}\left[1 - \frac{1}{(1+r)^n}\right]$

$$= \frac{40{,}000}{0.1}\left[1 - \frac{1}{(1+0.1)^6}\right]$$

$$= 4{,}00{,}000\left[1 - \frac{1}{(1.1)^6}\right]$$

$$= 4{,}00{,}000\left[1 - \frac{1}{1.772}\right]$$

$$= 4{,}00{,}000\left[\frac{1.772 - 1}{1.772}\right]$$

$$= 4{,}00{,}000 \times \frac{0.772}{1.772}$$

$= 4{,}00{,}000 \times 0.435666$

$=$ ₹ 1,74,266.

Calculation:

Let $x = (1.1)^6$

$\log x = 6 \log (1.1)$

$= 6 \times 0.0414$

$= 0.2484$

$x = \text{antilog } 0.2484$

$= 1.772.$

EXAMPLE 6

Find the present value of the annuity of ₹ 2000 for 10 years at 5% compound of interest p.a., the payment being made at the beginning of each year.

Solution: It is a problem relates to annuity due, as each instalment is paid at the beginning of a year.

We know that, $V = \frac{P(1+r)}{r}\left[1-(1+r)^{-n}\right]$

Here, $P =$ ₹ 2000, $r = \frac{5}{100} = 0.05$, $n = 10$ years

Therefore, $V = \frac{2000(1+0.05)}{0.05}\left[1-\left(1+0.05\right)^{-10}\right]$

$= \frac{2000 \times 1.05}{0.05} \times \left[1-\left(1.05\right)^{-10}\right]$

= 42,000 (1 – 0.6138)

= 42,000 (0.3862)

= ₹ 16,220.40

Therefore, required present value = ₹ 16,220.40.

Calculation:

Let $x = (1.05)^{-10}$

$\log x = -10. \log 1.05$

$= -10 \times 0.0212$

$= -0.212$

$= -1 + 1 - 0.212$

$= -1 + 0.788$

$= \bar{1}.788$

$x = \text{antilog } \bar{1}.788$

$= 0.6138.$

EXAMPLE 7

Mr. D. Podder borrows a certain sum of money at 12% p.a. compound interest and agrees to pay both principal and interest in 10 equal yearly instalments of ₹ 2400 each. If the first instalment is to be paid at the end of 5 years from the date of borrowing and other yearly instalments are paid regularly at the end of the subsequent years, find the sum borrowed by him.

Solution: It is a problem of deferred annuity, as the first instalment is deferred by 4 years.

Here, $P =$ ₹ 2400, $r = \frac{12}{100} = 0.12$, $n = 10$ years and $m = 4$

We know that, $V = \frac{P}{r}\left\{\frac{(1+r)^n - 1}{(1+r)^{m+n}}\right\}$

$= \frac{2400}{0.12}\left\{\frac{(1.12)^{10} - 1}{(1.12)^{14}}\right\}$

$= 20{,}000 \times \frac{3.105 - 1}{4.884}$

$$= 20{,}000 \times \frac{2.105}{4.884}$$
$$= 20{,}000 \times 0.43099$$
$$= ₹\ 8619.80$$

Therefore, Mr. Podder borrowed ₹ 8619.80 (Approx).

Calculation:	
Let $x = (1.12)^{10}$	$y = (1.12)^{14}$
or $\log x = 10 \log 1.12$	Therefore, $\log y = 14 \times \log 1.12$
$= 10 \times 0.0492$	$= 14 \times 0.0492$
$= 0.492$	$= 0.6888$
or $x = \text{antilog } 0.492$	$y = \text{antilog } 0.6888$
$= 3.105$	$= 4.884.$

Finding the Instalment

EXAMPLE 8

Mr. B. Roy borrows ₹ 50,000 at 4% compound interest and agrees to pay both the principal and interest in 10 equal instalments at the end of each year. Find the amount of each instalment.

Solution: Here, $V = ₹\ 50{,}000$, $r = \frac{4}{100} = 0.04$, $n = 10$ years.

We know that, $V = \frac{P}{r}\left[1-(1+r)^{-n}\right]$

or $50{,}000 = \frac{P}{0.04}\left[1-(1.04)^{-10}\right]$

or $2{,}000 = P\,(1 - 0.6761)$

or $2{,}000 = P \times 0.3239$

or $P = \frac{2{,}000}{0.3239}$

$= ₹\ 6174.75$

Therefore, the amount of each instalment = ₹ 6174.75.

Calculation:

Let $x = (1.04)^{-10}$

$\log x = -10 \log (1.04)$

$= -10 \times 0.0170$

$= -0.170$

$= -1 + 1 - 0.170$

$= -1 + 0.830$

$= \bar{1}.830$

Therefore, $x = \text{antilog } \bar{1}.830$

$= 0.6761$

EXAMPLE 9

A government constructed housing flat costs ₹ 2,50,000, 50% is to be paid at the time of possession and the balance, reckoning compound interest at 9% p.a. is to be paid in 12 equal annual instalments. Find the amount each such instalment. [Given: $\frac{1}{(1.09)^{12}} = 0.3558$]

Solution: Amount to be paid at the time of possession = 50% of ₹ 2,50,000 i.e. ₹ 1,25,000.

Therefore, balance ₹ 1,25,000 to be paid in 12 equal instalments.

We know that, $V = \frac{P}{r}\left[1-(1+r)^{-n}\right]$

Here, V = ₹ 1,25,000, $r = \frac{9}{100} = 0.09$, $n = 12$

Hence, $1{,}25{,}000 = \frac{P}{0.09}\left[1-(1+0.09)^{-12}\right]$

or $1{,}25{,}000 = \frac{P}{0.09}\left[1-\frac{1}{(1.09)^{12}}\right]$

or $11{,}250 = P\,(1 - 0.3558)$

or $0.6442\,P = 11{,}250$

or $P = \frac{11{,}250}{0.6442}$ = ₹ 17,463.52

Therefore, the amount of each instalment = ₹ 17,463.52.

EXAMPLE 10

A sum of ₹ 2522 borrowed from a money lender at 5% p.a. compounded annually. If this amount is to be paid back in three equal annual instalment, find the annual instalment. [Given $(1.05)^{-3} = 0.8638$]

Solution: Let each instalment be ₹ P

Here, V = ₹ 2522, $r = \frac{5}{100} = 0.05$, $n = 3$ years

Hence, $V = \frac{P}{r}\left[1-(1+r)^{-n}\right]$

$2{,}522 = \frac{P}{0.05}\left[1-(1+0.05)^{-3}\right]$

or $2{,}522 = \frac{P}{0.05}\left[1-(1.05)^{-3}\right]$

or $126.1 = P\,(1 - 0.8638)$

or $0.1362\,P = 126.1$

or $P = \frac{126.1}{0.1362}$

= ₹ 925.84

Therefore, the amount of each instalment is ₹ 925.84.

Finding the Period

EXAMPLE 11

In how many years will an annuity of ₹ 1600 amount to ₹ 16,256 at 4% p.a. compound interest.

Solution: Here, $A =$ ₹ 16,256 , $P =$ ₹ 1600, $r = \frac{4}{100} = 0.04$

We know that, $A = \frac{P}{r}\left[(1+r)^n - 1\right]$

Hence, $16{,}256 = \frac{1600}{0.04}\left[(1+0.04)^n - 1\right]$

or $650.24 = 1600\left[(1.04)^n - 1\right]$

or $(1.04)^n - 1 = \frac{650.24}{1600} = 0.4064$

or $(1.04)^n = 1 + 0.4064 = 1.4064$

or $n \log(1.04) = \log(1.4064)$ [Taking logarithm both sides]

or $n \times 0.0170 = 0.1481$

or $n = \frac{0.1481}{0.0170} = 8.71$

$= 9$ years (approx.)

Perpetual Annuity

EXAMPLE 12

What is the present value of perpetual annuity of ₹ 2500 a year at $2\frac{1}{2}$ % p.a.?

Solution: Here, $P =$ ₹ 2500, $r = \frac{2\frac{1}{2}}{100} = \frac{5}{200} = 0.025$

We know that, present value of perpetual annuity is:

$$V = \frac{P}{r} = \frac{2500}{0.025} = ₹\ 1{,}00{,}000$$

Therefore, the required present value is ₹ 1,00,000.

Endowment Fund

EXAMPLE 13

A person desires to create an endowment fund to provide for a prize of ₹ 300 every year. If the fund can be invested at 10% p.a. compound interest, find the amount of endowment.

Solution: Here the endowment is to continue forever.

$P = ₹\ 300, r = 0.10$

$$V = \frac{P}{r} = \frac{300}{0.10} = ₹\ 3{,}000.$$

EXAMPLE 14

For endowing an annual scholarship of ₹ 15,000, Mr A. Paul wishes to make three equal annual contributions. The first award of the scholarship is to be made three years after the last of his three contributions. What would be the value of each contribution, assuming interest at 6% p.a. compounded annually?

Solution: The first scholarship is payable at the end of 5^{th} year and then it continued forever. Thus, we have a perpetuity of ₹ 15,000 deferred by 4 years.

Let P be the annual contribution, then present value of endowment

$$V = P + \frac{P}{r}\left[1-(1+r)^{-n}\right]$$

Since the last two payments are made in two annual payments after 1^{st} instalment, then, $n = 2$ and $r = \frac{6}{100} = 0.06$

$$V = P + \frac{P}{0.06}\left[1-(1.06)^{-2}\right]$$

$$= P + \frac{P}{0.06}\left[1-\frac{1}{(1.06)^{-2}}\right]$$

$$= P + \frac{P}{0.06}\left[1-\frac{1}{1.1236}\right]$$

$$= P + \frac{P}{0.06}[1-0.8899]$$

$$= P + \frac{P}{0.06}\times 0.11$$

$$= P\left(1+\frac{0.11}{0.06}\right)$$

$$= P\,(1 + 1.83)$$

$$= 2.83\,P$$

For deferred value of perpetuity

$$V = \frac{15000}{0.06}\times\frac{1}{(1.06)^4} \qquad \left[\text{As } V = \frac{P}{r}\cdot\frac{1}{(1+r)^m}\right]$$

$$= \frac{15000}{0.06}\times\frac{1}{1.2645}$$

$= 198019.80 = 1,98,020$ (Approx.)

Now, $2.83\ P = 1,98,020$

or $P = \dfrac{1,98,020}{2.83} = 69,972$ (Approx.)

Therefore, the value of each contribution is ₹ 69,972.

Continuous Compounding

EXAMPLE 15

A person deposits ₹ 10,000 in a bank which pays an interest of 8% p.a. compounded continuously. How much amount will be accumulated in his account after 5 years?

Solution: Here, $P =$ ₹ 20,000, $r = \dfrac{8}{100} = 0.08$, $n = 5$ years

We know that, $A = p.e^{rn}$

or $A = 10,000 \cdot e^{0.08 \times 5}$

$= 10,000 \times e^{0.4}$

or $\log A = \log 10,000 + 0.4 \log e$ [Taking logarithm both sides]

or $\log A = \log (10)^4 + 0.4 \times \log 2.71828$ [As $e = 2.71828$]

$= 4 \log 10 + 0.4 \times 0.4343$

$= 4 + 0.17372$

$= 4.17372$

$A = \text{antilog } 4.17372 = 14,910$ (approx.)

Therefore, the amount that will be accumulated in the account = ₹ 14,910 (approx.)

Valuation of Loan

EXAMPLE 16

A loan of ₹ 5,000 is to be paid in 6 equal annual payments, interest being at 8% p.a. compound interest and first payment being made after a year. Find the amount of annual instalment.

Solution: Here, $V =$ ₹ 5000, $r = 0.08$, $n = 6$ years

We know that, $V = \dfrac{P}{r}\left[1-(1+r)^{-n}\right]$

or $P = \dfrac{v.r}{1-(1+r)^{-n}}$

$= \dfrac{5000 \times 0.08}{1-(1+0.08)^{-6}}$

$= \dfrac{400}{1-(1.08)^{-6}}$

Calculation:

Let $x = (1.08)^{-6}$

or $\log x = -6 \log (1.08)$

$= -6 \times 0.0334$

$= -0.2004$

$= -1 + 1 - 0.2004$

$= \bar{1}.7996$

Therefore, $x = \text{antilog } \bar{1}.7996$

$= 0.6304$

$$= \frac{400}{1-0.6304}$$

$$= \frac{400}{0.3696}$$

$$= 1{,}082.25$$

Therefore, the amount of annual instalment is ₹ 1,082.25.

Valuation of Debenture/Bond

EXAMPLE 17

Bablu invested in a bond whose value is ₹ 20,000 and matures in 15 years. The interest rate is 6% semi-annually and required rate of return is 13%. Find out the current price of the bond.

Solution: Here, $I = 20{,}000 \times \frac{6}{100} = 1200$

$r = 13\% = 0.13$, $m =$ ₹ 20,000 and $n = 15$ years.

Therefore, price of bond $(V) = \frac{I}{2}\left[\frac{1-\left(1+\frac{r}{2}\right)^{-2n}}{\frac{r}{2}}\right] + \frac{m}{\left(1+\frac{r}{2}\right)^{2n}}$

$$= \frac{1200}{2}\left[\frac{1-\left(1+\frac{0.13}{2}\right)^{-2\times15}}{\frac{0.13}{2}}\right] + \frac{20{,}000}{\left(1+\frac{0.13}{2}\right)^{2\times15}}$$

$$= 600\left[\frac{1-(1+0.065)^{-30}}{0.065}\right] + \frac{20{,}000}{(1+0.065)^{30}}$$

$$= 600\left[\frac{1-(1.065)^{-30}}{0.065}\right] + \frac{20{,}000}{(1.065)^{30}}$$

$$= 600\left[\frac{1-\frac{1}{6.592}}{0.065}\right] + \frac{20{,}000}{6.592}$$

$$= 600\left[\frac{\frac{6.592-1}{6.592}}{0.065}\right] + \frac{20{,}000}{6.592}$$

Calculation:

Let $x = (1.065)^{30}$

or $\log x = 30 \log 1.065$

$= 30 \times 0.0273$

$= 0.819$

Therefore, $x = \text{antilog } 0.819$

$= 6.592$

$$= 600\left[\frac{5.592}{6.592\times 0.065}\right]+\frac{20,000}{6.592}$$

$$= 600\left[\frac{5.592}{0.42848}\right]+\frac{20,000}{6.592}$$

$= 600 \times 13.05 + 3033.98$

$= 7830 + 3033.98$

$= 10,863.98 =$ ₹ 10864 (Approx.)

Sinking Fund

EXAMPLE 18

A company has ₹ 20,00,000, 6% debentures outstanding today. The company has to redeem the debentures after 5 years and establishes a sinking fund to provide funds for redemption. Sinking Fund investments earn interest @10% p.a. The investments are made at the end of each year. What annual payment must the firm make to ensure that the needed ₹ 20,00,000 is available on the designated date?

Solution: Let the amount to be invested at the end of each year in sinking fund be ₹ P.

Here, $A =$ ₹ 20,00,000, $r = 0.10$ and $n = 5$ years

We know that, $A = \frac{P}{r}\left[(1+r)^n - 1\right]$

or $\quad 20,00,000 = \frac{P}{0.10}\left[(1+0.10)^5 - 1\right]$

or $\quad 20,00,000 = \frac{P}{0.10}\left[(1.1)^5 - 1\right]$

or $\quad 2,00,000 = P\,(1.611 - 1)$

or $\quad 2,00,000 = 0.611\,P$

or $\quad P = \frac{2,00,000}{0.611}$

$= $ ₹ 3,27,332.24

Calculation:
Let $x = (1.1)^5$
$\log x = 5 \log 1.1$
$= 5 \times .0414$
$= 0.207$
Therefore, $x =$ antilog 0.207
$= 1.611$

Therefore, the company has to invest ₹ 3,27,332.24 at the end of each year to get accumulated amount of ₹ 20,00,000.

EXAMPLE 19

A sinking fund is created for replacing some machinery for ₹ 54,000 after 25 years. The scrap value of the machine at the end of the period is ₹ 4,000. How much should be set aside from profit each year for the sinking fund when the rate of compound interest is 5%?

Solution: Replacement cost of the machine after 25 years = ₹ 54,000

Less: Scrap value after 25 years = ₹ 4,000

Amount Required ₹ 50,000

Let, the required sum to be set aside each year for replacement of the machine be ₹ P.

Here, A = ₹ 50,000, n = 25 years, r = 0.05

Now, $A = \frac{P}{r}\left[(1+r)^n - 1\right]$

or $50{,}000 = \frac{P}{0.05}\left[(1.05)^{25} - 1\right]$

or $2500 = P(3.388 - 1)$

or $2.388\,P = 2500$

or $P = \frac{2500}{2.388} = 1046.90$

Calculation:

Let $x = (1.05)^{25}$

$\log x = 25 \log (1.05)$

$= 25 \times 0.0212$

$= 0.5300$

Therefore, x = antilog 0.5300

$= 3.388$

Therefore, the required sum to be set aside every year for sinking fund is ₹ 1046.90.

Finding Cash Down Price

EXAMPLE 20

A truck is purchased on instalment basis, such that ₹ 10,000 is to be paid on the signing of the contract and five yearly instalments of ₹ 5000 each payable at the end of 1^{st}, 2^{nd}, 3^{rd}, 4^{th} and 5^{th} years. If interest is charged at 10% p.a. what would be the cash down price?

Solution: Here, P = ₹ 5000, $r = \frac{10}{100} = 0.1$, n = 5 years

Now, $V = \frac{P}{r}\left[1 - (1+r)^{-n}\right]$

$= \frac{5000}{0.1}\left[1 - (1.1)^{-5}\right]$

$= 50{,}000 \left[\frac{(1.1)^5 - 1}{(1.1)^5}\right]$

$= 50{,}000 \left[\frac{1.61051 - 1}{1.61051}\right]$

$= 50{,}000 \times \frac{0.61051}{1.61051}$

$= 50{,}000 \times 0.379079$

$= 18953.95$

Therefore, required each down price of the truck = ₹ 18953.95 + ₹ 10,000 = ₹ 28953.95

Selecting the Plan

EXAMPLE 21

Which is better: an annuity of ₹ 150 to last for 10 years or a perpetuity of ₹ 79.20 p.a. to commence 7 years hence, the rate of interest being 5% ?

Solution: For comparison between two plans we have to compute present values. The annuity having greater present value is better than the other.

Present value of first annuity

Here, P = ₹ 3000, $r = 0.05$, $n = 10$ years

$$V = \frac{P}{r}\left[1-(1+r)^{-n}\right]$$

$$= \frac{150}{0.05}\left[1-(1+0.05)^{-10}\right]$$

$$= 3000\left[1-(1.05)^{-10}\right]$$

$$= 3000\,(1 - 0.6138)$$

$$= 3000 \times 0.3862$$

= ₹ 1158.6

Calculation:

Let $x = (1.05)^{-10}$

$\log x = -10 \log (1.05)$

$= -10 \times 0.0212$

$= -0.212$

$= -1 + 1 - 0.212$

$= \bar{1}.788$

Therefore, $x = \text{antilog } \bar{1}.788$

$= 0.6138$

Present value of second annuity

It is a deferred perpetuity which commences after 7 years.

Hence, $V = \dfrac{P}{r}.\dfrac{1}{(1+r)^m}$

Hence, P = ₹ 79.2, $r = 0.05$, $m = 7$

Therefore, $V = \dfrac{79.2}{0.05}.\dfrac{1}{(1+0.05)^7} = \dfrac{79.2}{0.05.(1.05)^7}$

$$= 79.2 \times 20 \times (1.05)^{-7}$$

$$= 1584 \times (1.05)^{-7}$$

$$= 1584 \times 0.7106$$

= ₹ 1125.59

Calculation:

Let $x = (1.05)^{-7}$

or $\log x = -7 \log (1.05)$

$= -7 \times 0.0212$

$= -0.1484$

$= -1 + 1 - 0.1484$

$= \bar{1}.8516$

Therefore, $x = \text{antilog } \bar{1}.8516$

$= 0.7106$

It is clear from the above valuation that the present value of the first plan is greater than that of the second plan.

Therefore, the first annuity is better than the second deferred perpetuity.

Miscellaneous Problems

EXAMPLE 22

PQR company estimates it will have to replace a piece of equipment at a cost of ₹ 8,00,000 in 5 years. To do this a sinking fund is established by making equal monthly payments into an account paying 6.6% compounded interest monthly. How much should each payment be?

Solution: Here, A = ₹ 8,00,000, $r = 6.6\% = \frac{6.6}{100} = 0.066$, $n = 5$ years, $m = 12$ times in a year

Now, $$A = \frac{P}{\frac{r}{m}}\left[\left(1+\frac{r}{m}\right)^{m.n} - 1\right]$$

or $$8{,}00{,}000 = \frac{P}{\frac{0.066}{12}}\left[\left(1+\frac{0.066}{12}\right)^{5\times 12} - 1\right]$$

or $$8{,}00{,}000 \times \frac{0.066}{12} = P\,[(1 + 0.0055)^{60} - 1]$$

or $4400 = P\,[(1.0055)^{60} - 1]$

or $4400 = P\,(1.374 - 1)$

or $0.374\,P = 4400$

or $$P = \frac{4400}{0.374} = 11{,}764.70$$

Therefore, amount of each payment is ₹ 11,764.70.

Calculation:

Let $x = (1.0055)^{60}$

or $\log x = 60 \times \log (1.0055)$

$= 60 \times 0.0023$

$= 0.138$

Therefore, x = antilog 0.138

$= 1.374$

EXAMPLE 23

Mr. H. P. Samaddar retires at the age of 60 and earns a pension of ₹ 60,000 a year. He wants to commute one-fourth of his pension to ready money. If the expectation of life at this age be 15 years, find the amount he will receive when money is worth 9% p.a. compound. (It is assumed that pension for a year is due at the end of the year.)

Solution: He commutes $\frac{1}{4}$ of the pension $= \frac{1}{4}\times 60{,}000$ = ₹ 15,000

Given, P = ₹ 15,000, $r = \frac{9}{100}$ 0.09, $n = 15$ years

Then, $$V = \frac{P}{r}\left[1-(1+r)^{-n}\right]$$

$$= \frac{15{,}000}{0.09}\left[1-(1+0.09)^{-15}\right]$$

$$= 1,66,667\left[1-(1.09)^{-15}\right]$$
$$= 1,66,667\ (1 - 0.2748)$$
$$= 1,66,667 \times 0.7252$$
$$= 1,20,867 \text{ (Approx.)}$$

Therefore, Mr. Samaddar will receive ₹ 1,20,867 on commuting one-fourth of his pension.

Calculation:

Let $x = (1.09)^{-15}$

or $\log x = -15 \log 1.09$
$= -15 \times 0.0374$
$= -0.561$
$= -1 + 1 - 0.561$
$= \bar{1}.439$

$x = \text{antilog } \bar{1}.439$
$= 0.2748$

EXAMPLE 24

A sinking fund is created for the redemption of debentures of ₹ 1,00,000 at the end of 25 years. How much money should be provided out of profits each year for the sinking fund, if the investment can earn interest @4% p.a.?

Solution: Let the amount to be deposited in sinking fund be ₹ P

Here, A = ₹ 1,00,000, $r = 4\% = 0.04$ and $n = 25$ years

We know that, $A = \dfrac{P}{r}\left[(1+r)^n - 1\right]$

or $1,00,000 = \dfrac{P}{0.04}\left[(1+0.04)^{25} - 1\right]$

or $4,000 = P\,[(1.04)^{25} - 1]$

or $4,000 = P\,(2.661 - 1)$

or $1.661\,P = 4000$

or $P = \dfrac{4000}{1.661} = 2408.19$

Therefore the required amount to be deposited in sinking fund every year out of profit is ₹ 2408.19

Calculation:

Let $x = (1.04)^{25}$

or $\log x = 25 \log 1.04$
$= 25 \times 0.0170$
$= 0.4250$

Therefore, $x = \text{antilog } 0.4250$
$= 2.661$

EXAMPLE 25

A company borrows ₹ 10,000 on condition to repay it with compound interest at 5% p.a. by annual instalments of ₹ 1000 each. In how many years will the debt be paid off?

Solution: Here, V = ₹ 10,000, P = ₹ 1000, $r = 0.05$

We know that, $V = \dfrac{P}{r}\left[1-(1+r)^{-n}\right]$

or $10,000 = \dfrac{1000}{0.05}\left[1-(1.05)^{-n}\right]$

or $$500 = 1000\left[1 - \frac{1}{(1.05)^n}\right]$$

or $$1 = 2\left[1 - \frac{1}{(1.05)^n}\right]$$

or $$1 = 2 - \frac{2}{(1.05)^n}$$

or $\dfrac{2}{(1.05)^n} = 1$ or $(1.05)^n = 2$

or $n.\log(1.05) = \log 2$ [Taking logarithm both sides]

or $n\,(0.0212) = 0.3010$

or $n = \dfrac{0.3010}{0.0212} = 14.2$

Therefore, the debt will be paid off in 14.2 years.

EXERCISE

COMPOUND INTEREST

A. SHORT TYPE

Finding the Simple and Compound Interest and Rate of Interest

1. The simple interest on ₹ 956 for $2\frac{1}{2}$ years is ₹ 119.50. Find the rate of interest per annum.

   [**Hints:** Here P = ₹ 956, $n = 2\frac{1}{2} = 2.5$ years

   I = ₹ 119.50

   We know that, $I = \dfrac{P.nr}{100}$ or $119.50 = \dfrac{956 \times 2.5 \times r}{100}$

   or $r = \dfrac{119.50 \times 100}{956 \times 2.5} = 5$

   Therefore, Rate of interest = 5% p.a.]

2. At what rate of simple interest will ₹ 2000 amount to ₹ 2,220 in 2 years?

   [**Ans.** 5.5% p.a.]

3. What will be the compound interest on ₹ 1000 for 20 years at 5% p.a.?

   [Here P = ₹ 1000, n = 20 yrs, $i = \dfrac{5}{100} = 0.05$

Therefore, $A = P(1 + r)^n = 1000 (1.05)^{20}$ or $\log A = \log 1000 + 20 \log 1.05$

or $\log A = 3 + 20 \times 0.0212 = 3.424$

Therefore, A = antilog (3.424) = ₹ 2655

C.I = $A - P$ = ₹ (2655 – 1000) = ₹ 1655.

4. What will be the compound interest, interest being compounded annually, on ₹ 3000 for 2 years at 10% p.a.
5. Find the compound interest on ₹ 1000 for 3 years at 10% p.a.? [**Ans.** ₹ 331]
6. If interest is compounded at the end of each year, find the compound interest on ₹ 10,000 at 5% p.a. for 2 years. [***C.U. B.Com. 2010***] [**Ans.** ₹ 1,025]

Finding the Period

7. In what time will ₹ 860 amount to ₹ 1943.60 at 7% p.a. simple interest.

   [**Hints:** Here P = ₹ 860, A = ₹ 1943.60, $r = \frac{7}{100} = 0.07$

   We know that, $A = P(1 + nr)$ or $1943.60 = 860 (I + n \times 0.07)$

   or $n = \frac{1083.60}{60.2} = 18$ yrs.]
8. In what time will ₹ 2000 amount to ₹ 2600 at 5% interest?

   [**Ans.** 6 years]
9. How many years will it require for ₹ 2 to amount to ₹ 15 at 5% p.a. compounded annually?

   [**Hints:** Here A = ₹ 15, P = ₹ 2, $r = \frac{5}{100} = 0.05$

   Therefore, $A = P(1 + r)^n$ or $15 = 2 (1.05)^n$ or $\log 15 = \log 2 + n. \log 1.05$

   or $n = \frac{\log 15 - \log 2}{\log 1.05} = \frac{1.1761 - 0.3010}{0.0212} = 41.3$ yrs.]
10. In what time will a sum of money double itself @ 10% p.a. compound interest, payable half-yearly? [***C.U. B.Com. '85***] [**Ans.** 7.1 yrs.]

    [Given: log 2 = 0.3010, log 1.05 = 0.0212]
11. In what time will a sum of money double itself at 4% p.a. compound interest payable half-yearly? [Given: log 2 = 0.3010, log 1.02 = 0.0086]

    [**Ans.** 17.5 yrs.]
12. In what time will a sum of money double itself at 5% p.a. compound interest?

    [Given: log 2 = 0.3010, log 105 = 2.0212] [***C.U. B.Com. '89***] [**Ans.** 14.198 yrs.]

Finding the Principal

13. What principal will amount to ₹ 370 in 6 years at 8% p.a. simple interest?

[**Hint:** Here A = ₹ 370, n = 6 years, $i = \frac{8}{100} = 0.08$,

We know that $A = P(1 + nr)$ or $370 = P(I + 6 \times .08)$

or $P = \frac{370}{1.48}$ = ₹ 250]

14. What principal will amount to ₹ 720.32 in 4 years at 3% p.a. compound interest?

    [**Hint:** A = ₹ 720.32, n = 4 yrs., r = .03

    now $A = P(I + r)^n$ or $720.32 = P(1 + .03)^4$

    or log 720.32 = log P + 4 log 1.03

    or 2.857 = log P + 4 × 0.0128 or log P = 2.806

    or P = antilog 2.806 = ₹ 640]

15. What sum will amount to ₹ 10,000 in 18 years at 4% p.a. compound interest?
    [Given: log 104 = 2.0170, log 493.629 = 2.6934] [**Ans.** ₹ 4936.29]

16. What sum of money will amount to ₹ 1000 in 12 years at $4\frac{1}{2}\%$ p.a. compound interest? [**Ans.** ₹ 589.90]

17. What sum will amount to ₹ 1000 in 2 years at 5% p.a. compound interest payable half-yearly. [**Ans.** ₹ 906]

18. Find the present value of ₹ 5000 due in 15 years at 5½% p.a. compound interest? [**Ans.** ₹ 2239.85]

Finding the Amount and Compound Interest

19. Find the amount and compound interest for ₹ 3500 for two years if the rate of interest is 4% p.a. compounded half-yearly. [**Ans.** ₹ 3788.40]

20. Find the amount and the compound interest on ₹ 10,000 for 4 years, if interest is 5% p.a. payable yearly? [**Ans.** ₹ 12,156.26; ₹ 2,156.26]

21. Find the amount of ₹ 300 at the rate of interest 2% p.a. compounded annually for four years. Also calculate the interest. [**Ans.** ₹ 324.70; ₹ 24.70]

B. SHORT ESSAY TYPE/PROBLEM TYPE

Finding the Simple and Compound Interest and Rate of Interest

1. Mr. S Sen borrowed ₹ 6000 from a money lender but he could not pay any amount in period of 4 years. Accordingly the money lender demanded now ₹ 7500 from him. What rate per cent p.a. compound interest did the latter require for lending his money? [**Ans.** 5.7%]

2. Purchasing of National Saving Certificate makes the investment double itself in 6 years. Find the rate of interest accrued, if compounded half-yearly.
   [***C.U. B.Com. '92***] [**Ans.** 11.8%]

3. The difference between the simple and compound interest on a certain sum for 3 years at 5% p.a. is ₹ 228.75. Find the compound interest on the sum for 2 years at 5% p.a. [**Ans.** ₹3,075]

Finding the Period

4. Find the number of years in which a sum of money will be triple itself at 4% p.a. compound interest. [**Ans.** 28.06]
5. In what time will the C.I. on ₹ 1200 be ₹ 124.60 at 8% p.a. payable quarterly? [**Ans.** 1.25 yrs.]

Finding the Principal

6. Mr S. Roy intends to invest a sum of money which will amount ₹ 5000 in 10 years at 4% p.a. compound interest . What amount should he invest? [**Ans.** ₹ 3381]
7. The difference between simple and compound interest on a sum put out for 5 years at 3% was ₹ 46.80. Find the sum. [**Ans.** ₹ 5200]
8. The difference between the simple and compound interest on a sum put out for 2 years at 5% was ₹6.90. Find the sum. [**Ans.** ₹ 2300]
9. What sum of money invested at 5% p.a. payable half-yearly for 2 years will amount ₹ 1000? [**Ans.** ₹ 906.15]
10. A sum of money invested at compound interest amounts to ₹ 10,816 at the end of the second year and to ₹ 11,248.64 at the end of the third year. Find the rate of interest and the sum invested? [***C.U. B.Com. '83***] [**Ans.** 4%; ₹ 10,000]
11. A sum of money at compound interest, calculated yearly, amounts to ₹ 1102.50 and ₹ 1157.625 in 2 years and 3 years respectively. Find the sum and the rate of interest. [**Ans.** ₹ 1000; 5%]
12. A man wishes to have ₹ 2500 available in a bank account when his daughter's first year college expenses begin. How much must he deposit now at 3.5% compounded annually, if the girl to start college six years hence from now? [**Ans.** ₹ 2,034]
13. A sum of money is lent at 8% p.a. compound interest. If the interest for the second year exceeds that for the first year by ₹ 32. Find the original principal. [**Ans:** ₹ 5,000]
14. Mr. Brown was given the choice of two payment plans on a piece of property. He may pay ₹ 10,000 at the end of 4 years, or ₹ 12,000 at the end of 9 years. Assuming money can be invested annually at 4 % per year converted annually. What plan should Mr. Brown choose ? [***I C. W. A. I. June, '83***] [**Ans.** 2^{nd} plan]

Finding the Amount and Compound Interest

15. A man deposits ₹ 5,000 in a Savings Bank which pays compound interest at the rate of $4\frac{1}{2}\%$ for the first two years and then at the rate of 5% for the next three years. Find his amount after five years. [*C. U. B.Com. '81*]
[**Ans.** ₹ 6319.90]

16. C. P. Mukherjee left ₹ 20,000 to be divided between his son Amit and daughter Arati. Arati's share was to amount to a certain sum of money at the end of 5 years and Amit's share was to amount to an equal sum at the end of 7 years. Calculate the sum of money each should receive if the interest rate was 4% p.a. (Compound) [**Ans.** ₹ 12,640 (approx)]

17. Find the interest and the amount of ₹ 7350 at the rate of 6% p.a compounded quarterly for five years. [**Ans.** ₹ 2520; ₹ 9870]

18. When a boy is born, ₹ 500 is placed to his credit in an account that pays (*i*) 6 % compounded annually, (*ii*) 6% compounded quarterly, (*iii*) 6% compounded monthly. If the account is not disturbed, what amount will there be to his credit on his twentieth birth-day? [*I. C. W. A. I. Dec. '79*]
[**Ans.** (*i*) ₹ 1,603; (*ii*) ₹ 1626, (*iii*) ₹ 1596]

Finding Present Value

19. Find the present value of ₹ 10,000 due in 12 years at 6% p.a. compound interest. [Given log 1.06 = 0.0253, log 4971 = 3.6964]
[*C. U. B.Com. 2006, 2017(G)*][**Ans.** ₹ 4971]

20. Find the present value of ₹ 5000 due in 15 years at $5\frac{1}{2}\%$ p.a. compound interest.
[**Ans.** ₹ 2244 (approx.)]

21. Find the present value of ₹ 450 due in four years at 2% p.a. compounded half yearly. [**Ans.** ₹ 415.80]

Regarding Depreciation

22. A machine depreciates 10 % p.a. for first two years and the 7% p.a. for the next three years, depreciation being calculated on the diminishing value. If the value of the machine be ₹ 10,000 initially, find the average rate of depreciation and the depreciated value of the machine at the end of the fifth year.
[*C. U. B.Com. '74*] [**Ans.** 6.968%]

23. A machine depreciates at the rate of 10% of its value at the beginning of a year. The machine was purchased for Ra. 5,810 and the scrap value realised when sold was ₹ 2,250. Find the number of years the machine was used.
[*C. U. B.Com.(H) 2015*] [**Ans.** 9 years]

24. A machine, the life of which is estimated to be 10 years, costs ₹ 10,000. Calculate its scrap value at the end of its life, depreciation on reducing instalment system being charged @ 10% p.a. [**Ans.** ₹ 3,488]

25. A machine the life of which is 20 years costs ₹ 12000. Calculate the scrap value at the end of its life, depreciation on the reducing balance being charged at 25% p.a. [**Ans.** ₹ 38.12]

26. A machine is depreciated at the rate of 10% on reducing balance. The original cost was ₹ 10,000 and the ultimate scrap value was ₹ 3,750. Find the effective life of the machine. [***C. U. B.Com. (Hons.) '89, 2013***]
[**Ans.** 9.3 years]

Effective Rate

27. A sum of ₹ 50 is put for 4 years at 5% interest p.a. compound interest correct to paise. Find also the effective rate of interest p.a. [**Ans.** 5.3 yrs.]

[**Hints:** Given: P = ₹ 50, $r = \dfrac{5}{100} = 0.05$, $n = 4$ yrs.

$$\therefore A = P\left(I + \frac{r}{4}\right)^{4n} = 50\left(I + \frac{.05}{4}\right)^{4\times4}$$

or $\log A = \log 50 + 16 \log (1.0125)$ or $A = 61.46$

C.I = 61.46 – 50 = ₹ 11.46

$$\text{Now effective rate } = 100\left\{\left(I + \frac{.05}{4}\right)^{4} - 1\right\} = 100\left\{(1.0125)^4 - 1\right\}$$

= 5.3 $\therefore$ Effective rate = 5.3%]

28. Find the effective rate of interest corresponding to the nominal rate of 6% p.a, payable quarterly. [**Ans.** 6.12%]

C. ADVANCED PROBLEMS

29. A man left ₹ 18,000 with the direction that it should be divided in such a way that his 3 sons aged 9, 12 and 15 years should receive the same amount when they reached the age of 25. If the rate of interest is 3½% p.a, what should each son receive when he is 25 years old? [**Ans.** ₹ 9,341(approx.)]

30. A man can buy a flat for ₹ 1,00,000 cash, or for ₹ 50,000 down and ₹ 60,000 at the end year. If money is worth 10% per year compounded half-yearly which plan should he choose? [***C. U. B.Com. '95***]
[**Ans.** 1st case of payment is desirable]

31. A man left for his three sons aged 10, 12 and 14 years ₹ 10,000, ₹ 8,000 and ₹ 6,000 respectively. The money is invested in 3%, 6% and 10% C. I. respec-

tively. They will receive the amount when each of them attains the age of 21 years. Find using five figure log-table, hoe much each would recevive.
[*C. U. B.Com. '67*] [**Ans.** ₹ 13,843, ₹ 13,617, ₹ 11,691]

32. A man left ₹ 20,000 for his son and daughter. The amount that the daughter will receive at the end of 5 years, the same amount the son will receive after 7 years. Assuming the rate of compound interest at 4%, find the present share of each.
[**Ans.** ₹ 12,640]

33. An air pump used to extract air from vessel removes one tenth of the air at each stroke. Find the fraction of the original volume of air left after the 11^{th} stroke.
[*C. U. B.Com. '73*] [**Ans.** .3134]

34. If the population of a town increases every year by 1.8% of the population at the beginning of that year, in how many years will the total increase population be 30% ? [*B. U. B.Com. (Hons.) '87*] [**Ans.** 15 years]

35. The population of a country is 85,000. Each year it increases by 5% of the population of that year. What will be the population of the country after 5 years ?
[**Ans.** 1,06,300]

36. Mr. Prasenjit Sarkar can buy a flat for ₹ 80,000 cash or for ₹ 40,000 down and ₹ 50,000 at the end of the year . If money is worth 10% per year, compounded half-yearly, which plan should he choose? [**Ans.** First offer]

37. If the population of a town increases every year by 5% of the population at the beginning of that year, in how many years will the population be doubled?
[**Ans.** 14.2 years]

38. A man borrowed a certain amount of money at the rate of 3% p.a. simple interest. The man lent the same amount at the rate of 5% compound interest per annum. If at the end of three years, the man made a profit of ₹ 541, Find the sum borrowed. [**Ans.** ₹ 8,000]

39. In what time will a sum of money double itself at 5% p.a. compound interest?
[Given log 2 = 0.3010, log 1.05 = 0.0212] [*C. U. B.Com.(G) 2014, 2016*]
[**Ans.** 14.2 years]

40. The population in a town increases every year by 2% of the population of the town at the beginning of the year . In how many years will the total increase of population be 40%? [Given: $\log_{10}{}^{14} = 1.146$, $\log_{10}{}^{102} = 2.0086$]
[*C. U. B.Com.(G) 2013*] [**Ans.** 16.99 years]

ANNUITIES

A. SHORT ESSAY TYPE/PROBLEM TYPE

Finding the Amount of Annuity

1. Find the amount of an immediate annuity of ₹ 100 p.a. left unpaid for 10 years, allowing 5% p.a. compound interest.

[**Hints:** Here P = ₹ 100, n = 10, $r = \frac{5}{100} = 0.05$

Therefore, $A = \frac{P}{r}\left\{(1+r)^n - 1\right\}$

$= \frac{100}{0.05}\left\{(1+0.05)^{10} - 1\right\}$

$= \frac{100}{0.05}(1.629 - 1)$

$= \frac{100}{0.05} \times .629$

= ₹ 1,258]

Let $x = (1.05)^{10}$
or $\log x = 10 \log 1.05$
$= 10 \times 0.0212$
$= 0.212$
or $x = \text{antilog } 0.212$
$= 1.629$

2. Find the amount of an annuity of ₹ 100 in 20 years allowing compound interest $4\frac{1}{2}$ %. [Given log 1.045 = 0.0191163 and log 24.117 = 1.3823260]. [**Ans.** ₹ 3,137.12]
3. A man decides to deposit ₹ 300 at the end of each year in a bank which pays 3% p.a. (compound interest) if the instalments are allowed to accumulate, what will be the total accumulation at the end of 15 years? [**Ans.** ₹ 5560]
4. A man deposits ₹30 at the end of each year in a bank which pays 2.5% p.a. compound interest. Find the amount of his accumulation at the end of 20 years. Given, log 1025 = 3.0107239 and log 1.6386 = 0.214478. [**Ans.** ₹ 766.32]
5. A man decides to deposit ₹ 500 at the end of each year in a bank which pays 8% p.a. compound interest. If the instalments are allowed to accumulate, what will be the total accumulation at the end of 7 years? [**Ans.** ₹ 4,456.25]
6. If the present value of an annuity for 10 years at 6% p.a. compound interest is ₹ 15,000, find the annuity. [***C. U. B.Com. '88***] [**Ans.** ₹ 2163.46]

Finding the Present Value

7. Calculate the present value of an annuity of ₹ 5,000 for 10 years if the interest be 5% p.a.

[**Hints:** Here P = ₹ 5,000; n = 10;

$i = \frac{5}{100} = 0.05$

Therefore, $V = \frac{P}{r}\left\{1 - (1+r)^{-n}\right\}$

$= \frac{5000}{0.05}\left\{1 - (1.05)^{-10}\right\}$

= 1,00,000 (1 – 0.6138)

= 1,00,000 × 0.3862

= ₹ 38,620.

Let $x = (1.05)^{-10}$
or, $\log x = -10 \times \log 1.05$
$= -10 \times 0.0212$
$= -0.212$
$= \bar{1} + (1 - 0.212)$
$= \bar{1}.788$
or, $x = \text{antilog } \bar{1}.788 = 0.6138$

8. Find the present value of an annuity of ₹ 300 per annum for 5 years at 4%.
[**Ans.** ₹ 1333.50]

9. What is the present value of an annuity certain of ₹ 500 for 15 years, reckoning compound interest of $4\frac{1}{2}$ % p.a.? [Given $(1.045)^{15} = 1.933528$]
[**Ans.** ₹ 5,369.77]

10. A professor retires at the age of 60 years. He will get the pension of ₹ 42,000 a year paid in half yearly instalments for the rest of his life. Reckoning his expectation of his life to be 15 years and that interest is at 10% p.a. payable half-yearly, what single sum is equivalent to his pension.
[***C. U. B.Com. '96, 2006, 2012***]
[**Ans.** ₹ 3,22,812]

[**Hints:** A= annual rent = ₹ 42,000, n = 15, i = 0.10

Now, $V = \frac{P}{r}\left\{1-\left(1+\frac{r}{2}\right)^{-2n}\right\} = \frac{42,000}{0.1}\left\{1-\left(1+\frac{0.1}{0}\right)^{-30}\right\}$]

11. A man retires at the age of 60 years and his employer gives him a pension of ₹ 1200 a year paid in half-yearly instalments for the rest of his life. Reckoning his expectation of life to be 13 years and that interest is at 4% p.a. payable half-yearly, what single sum is equivalent to his pension. [***C. U. B.Com. '54***]
[**Ans.** ₹ 12,075]

Valuation of Loan

12. A man borrows ₹ 30,000 at 14 % compound interest and agrees to pay both the principal and the interest in 15 equal annual instalments at the end of each year. Find the amount of these instalments. [**Ans.** ₹ 4884.29]

13. S. Roy borrow ₹ 20,000 at 4% C.I. and agrees to play both the principal and interest in 10 equal instalments at the end of each year. Find the amount of these instalments. [**Ans.** ₹ 2,470]

[**Hints:** Here V = ₹ 20,000; r = 0.4; n = 10; P = ?]

14. S. Roy borrows ₹ 20,000 at 4% compound interest and agrees to pay both the principal and the interest in 10 equal annual instalments at the end of each year. Find the amount of these instalments. [***C. U. B.Com. '72***]
[**Ans.** ₹ 2,469.90]

15. A loan of ₹ 10,000 is to be repaid in 30 equal annual instalments of ₹ P. Find P; if the compound interest is charged at the rate of 4% p.a. (Annuity is an annuity immediate).

[Given: $(1.04)^{30} = 3.2434$] [***C. U. B.Com. '82] [V. U. '92***]
[**Ans.** ₹ 578.40]

Valuation of Debenture/Bond

16. ABC Company has issued a bond having face value of ₹ 50,000 carrying annual interest rate of 12% and maturing in 8 years. The market interest rate is 15%. Calculate the price of bond.

    [Given: $(1.15)^{-8} = 0.3269$] [**Ans.** ₹ 43,269]

17. The face value of the bond is ₹ 1000 (maturity value ₹ 1000). The bond has a 10% coupon rate (annual interest rate) payable semi-annually and the yield to maturity (market rate of return) is 9%. The bond matures in 5 years period from now. What is the value of the bond? [**Ans.** ₹ 1040 (approx.)]

Finding the Instalment

18. What sum should be invested every year at 8% p.a. compound interest for 10 years to replace plant and machinery, which is expected to cost then 20% more than its present cost of ₹ 50,000? [*C. U. B.Com. '86*] [**Ans.** ₹ 4145.07]

    [**Hints:** A = ₹ (50,000 + 10,000) = ₹ 60,000; $r = 0.08$; $n = 10$, $P = ?$]

19. A government constructed housing flat costs ₹ 1,36,000; 40% is to be paid at the time of possession and the balance, reckoning compound interest @ 9% p.a., is to be paid in 12 equal annual instalments. Find the amount of each such instalment.

    [Given: $(1.09)^{-12} = 0.3558$] [*C. U. B.Com. '84*] [**Ans.** ₹ 11,400]

    [**Hints:** $V = 1{,}36{,}000 - 1{,}36{,}000 \times \dfrac{40}{100}$ = ₹ 81,600; $n = 12$; $r = 0.06$; $P = ?$]

20. A man wishes to buy a house valued at ₹ 50,000. He is prepared to pay ₹ 20,000 now and the balance in 10 equal annual instalments. If the interest is calculated at 8% p.a. What should he pay annually?

    [Given: $\dfrac{1}{(1.08)^{10}} = 0.4634$] [*C. U. B.Com '87*] [**Ans.** ₹ 4472.61]

21. A man wishes to buy a Government constructed housing flat which costs ₹ 1,00,000. He has to pay 20% now and the balance reckoning compound interest @ 12% p.a., by 20 equal annual instalments. Find the amount of each such instalment.

    [Given: $(1.12)^{-20} = 0.1037$] [*C. U. B.Com. '91*] [**Ans.** ₹ 10,710.70]

22. A TV set is purchased on instalment basis such that 60% of its cash down price is to paid on signing of the contract and the balance in six equal instalments payable half-yearly @ 10% p.a. compound interest (Compounded half-yearly). If the cash down price of the set is ₹ 4,200, find the value of each instalment.

    [Given: $(1.05)^{-6} = 0.7461$] [*C. U. B.Com.(Hons.) '86*] [**Ans.** ₹ 330.84]

23. A man purchased a house valued at ₹ 3,00,000. He paid ₹ 2,00,000 at the time of purchase and agreed to pay the balance with interest at 12% p.a. compounded

half-yearly in 20 equal half-yearly instalments. If the first instalment is paid after the end of six months from the date of purchase, find the amount of each instalment.

[Given: log 10.6 = 1.0253 and log 31.19 = 1.494] [***C. U. B. Com. '94, '97***]

[**Ans.** ₹ 8719.66]

24. The life time of a machine is 12 years. The cost price of the machine is ₹ 1,00,000. The estimate scrap value and the increase in the cost of machine after 12 years are ₹ 30,000 and 20% respectively. Find the amount of each equal annual instalment to be deposited at 12% interest p.a. compounded annually.

[Given: log 1.12 = 0.0864 and log 10.88 = 1.0368]

[***C. U. B. Com. '93***] [**Ans.** ₹ 3731.86]

Finding the Period

25. A firm borrows ₹ 1000 on condition to repay it with compound interest at 4% p.a. by annual instalments of ₹ 100 each. In how many years will the debt be paid off ? [***N. B. U. B. Com. '84***] [**Ans.** 13.1 years]

26. A company borrows ₹ 10,000 on condition to repay it with compound interest at 5% p.a. by annual instalments of ₹ 1000 each. In how many years will the debt the paid off?

[Hints: V = ₹ 10,000; P = ₹ 1000; $r = .05$, $n = ?$]

[**Ans.** 14.2 years]

27. A man buys an old piano for ₹ 500.00, agreeing to pay ₹ 100 down and the balance in equal monthly instalment of ₹ 20 with interest at 6%. How long will it take him to complete payment?

[***I.C.W.A.I. Dec. '79***] [**Ans.** 22 months (approx.)]

28. Suppose that a company has a commitment for redemption of debentures of ₹ 5,73,000 at the end of a certain number of years and find it necessary to put ₹ 10,000 at the end of every year in a sinking fund for this purpose. The rate of compound interest is 10% p.a. When is the redumption due?

[Given: $(1.1)^{20} = 6.73$] [**Ans.** 20 years]

Perpetual Annuity

29. What is the value (i.e., present value) of perpetual annuity of ₹ 1000 p.a. at 4%

[**Hints:** P = ₹ 1000; $r = .04$; $V = \frac{P}{r} = \frac{1000}{0.04}$ = ₹ 25,000]

30. What is the value of perpetual annuity of ₹ 825 a year at $6\frac{1}{2}$% p.a.?

[**Ans.** ₹ 13,200]

31. Find the value of a perpetuity of ₹ 500 at $2\frac{1}{2}$% p.a.? [**Ans.** ₹ 20,000]

32. What perpetuity can be purchased by investing ₹ 5500 at $2\frac{3}{4}\%$ p.a. C.I.? [**Ans.** ₹ 151]

Endowment Fund

33. A person desires to create an endowment fund to provide for a prize of ₹ 300 every year. If the fund can be invested at 10% p.a. compound interest, find the amount of endowment. [**Ans.** ₹ 3,000]

34. A person desires to endow a bed in a hospital the cost of which is ₹ 5000 per year. If the money is worth 4% p.a., how much sums should he provide to create the endowment fund.

    [**Hints:** P = ₹ 5000; $r = 0.04$; $V = \frac{P}{r} = \frac{5000}{0.04}$ = ₹ 1,25,000]

Continuous Compounding

35. A National Savings Certificate costs ₹ 15 and realises ₹ 20 after 10 years. Find the rate of interest involved when it is added continuously.

    [**Ans.** 2.88%]

36. A person deposits ₹ 6,000 in a bank which pays an interest of 9% per annum compounded continuously. How much amount will be in his account after 10 years? [**Ans.** ₹ 15,170]

37. If interest is compounded continuously at an annual rate of 6%, how long will it take for a principal to double? [**Ans.** 11.5 years]

38. Find out the compound value of ₹ 5000, interest rate being 10% p.a. compounded continuously for 6 years. [Given $e^{0.6} = 1.8221$]

    [**Ans.** ₹ 9110.50]

39. What is an investment's doubling time, to the nearest 10,000ths of a year, if it earns 5% interest compounded continuously? [**Ans.** 13.8629 years]

    [**Hints:** $P.e^{0.05.n} = 2P$, or $e^{0.05.n} = 2$]

40. The population of Kolkata city was 45,75,876 in the year 2000, and was 47,67,952 in the year 2008. Predict the population of Kolkata city in the year 2020. [Given: $\log e = 0.4343$; $\log 1.042 = 0.0179$, $e^{0\cdot 1} = 1.1052$]

    [**Hints:** $A_n = 45{,}75{,}876 \cdot e^{r\cdot n}$; $A_8 = 4575876 \cdot e^{r\cdot 8} = 47{,}67{,}952$; Therefore, $r = 0.005$] [**Ans.** 50,57,258]

41. Suppose that a population of bacteria doubles every 43 minutes. If a bacteria population initially consists of 50 bacteria, how many will there be after 6 hours?

    [**Hints:** $A_n = 50 \cdot e^{r\cdot n}$; $A_{43} = 50 \cdot e^{r\cdot 43} = 100$ or $e^{r\cdot 43} = 2$ or $r = 0.0161197$ $A_{360} = 50 \cdot e^{r\cdot 360} = 16{,}566.1325$.] [**Ans.** 16,566.1325]

Sinking Fund

42. A machine costs a company ₹ 65,000 and its effective life is estimated to be 25 years. A sinking fund is created for replacing the machine at the end of its life time, when its scrap realises a sum of ₹ 2500 only. Calculate what amount should be provided every year from the profits earned for the sinking fund, if it accumulates $3\frac{1}{2}\%$ p.a. compound.
[Given that $(1.035)^{25} = 2.358$] [*C.U. B.Com. '83*] [**Ans.** ₹ 1612]

43. A machine costs a company ₹ 80,000 and its effective life is estimated to be 25 years. A sinking fund is created for replacing the machine at the end of its life time, when its scrap realises a sum of ₹ 5000 only. Calculate what amount should be provided every year, out of profits, for the sinking fund, if it accumulates of $11\frac{1}{2}\%$ p.a. compound. [**Ans.** ₹ 607.39]
[**Hints:** A = required sum to be invested = ₹ 80,000 – 5,000 = 75,000, $n = 25$; $i = 0.115$; calculate the amount to be provided each year (P)]

44. A sinking fund is created for the redumption of debentures of ₹ 1,00,000 at the end of 25 years. How much money should be provided out of profits each year for the sinking fund, if the investment can earn interest 4% p.a.
[**Ans.** ₹ 2401.19]

45. A sinking fund is created for replacing some machinery worth ₹ 54,000 after 25 years. The scrap value of the machine at the end of the period is ₹ 4000. How much should be set aside from profit each year for the sinking fund, when the rate of C.I. is 5%? [*C.U. B.Com. '74*] [**Ans.** ₹ 1046.90]

Finding Cash Down Price

46. A wagon is purchased on instalment basis, such that ₹ 5000 is to be paid on the signing of the contact and four-yearly instalments of ₹ 3000 each payable at the end of the first, second, third and fourth years. If interest is charged at 5% p.a. what would be the cash down price? [*C.U. B.Com. '64*]
[**Ans.** ₹ 15644]

47. A motor cycle is purchased on instalment basis, such that ₹ 3400 is to be paid on the signing of the contract and four yearly instalments of ₹ 2,400 each payable at the end of the first, second, third and fourth years. If the interest is charged at 8% p.a. what would be the cash down price?
[Given: $(1.08)^4 = 1.36$] [*C.U. B.Com.(Hons) '83*]
[**Ans.** ₹ 11,341.18]
[**Hints:** P = ₹ 2400; $n = 4$; $i = 0.08$, Now calculate present value, then cash down price = ₹ 3400 + ₹ 7941.18 (P.V) = ₹ 11341.18]

Selecting the Plan

48. A man can buy a flat for ₹ 1,00,000 cash, or for ₹ 50,000 cash down and ₹ 60,000 at the end of one year. If money is worth 10% per year compounded half-yearly, which plan should he choose? [***C.U. B.Com. '95***]

[**Ans.** 1st plan]

B. MISCELLANEOUS PROBLEMS

49. A person retires at the age of 60. He is entitled to get a pension of ₹ 200 per month payable at the end of every six months. He is expected to live upto the age of 70. If the rate of interest be 12% p.a payable half yearly, what single sum he is entitled to get at the time of his retirement equivalent to his pension?

    [Given: $(1.06)^{20} = 0.3119$] [***C. U. B.Com. Hons. '84***]

    [**Ans.** ₹ 13,762]

50. The cost price of a machine is ₹ 80,000. The estimated scrap value of the machine at the end of its life time of 10 years is ₹ 12,000. Find the amount of each equal annual instalment to be deposited at 9% p.a. compound interest annually just sufficient to meet the cost of a new machine after 10 years assuming an increase of 40% of the price of the machine then. The first instalment is to be paid of the end of the first year.

    [Given: $(1.09)^{10} = 2.368$] [***C. U. B.Com. Hons. '85***]

    [**Ans.** ₹ 6578.35]

51. The cost price of a flat is ₹ 1,36,000. If 40% of the price is to be paid at the time of possession and the balance is to be paid in 12 equal annual instalments reckoning compound interest at the rate of 9% p.a. find the amount of each such instalment.

    [Given $\dfrac{1}{(1.09)^{12}} = 0.3558$] [***K. U. B.Com. '95***]

    [**Ans.** ₹ 11,400]

52. If the present value of an annuity for 10 years at a compound interest of 6% p.a. is ₹ 15,000, find the annuity. [***K. U. B.Com. '97, 99***]

    [**Ans.** 2163.46]

53. Find the present value of an annuity due to ₹ 700 p.a. payable at the beginning of each year for 2 years at 6% p.a. compounded annually.

    [Given $(1.06)^{-1} = 0.943$] [***C. U. B.Com.(G) 2013, 2016***]

    [**Ans.** ₹ 1,283.37 (Approx.)]

54. Mr. X borrows ₹ 20,000 at 4% compound interest and agrees to pay both the principal and interest in 10 equal instalments at the end of each year. Find the amount of these instalments. [***C. U. B.Com.(G) 2014, 2015***]

    [***K. U. B.Com. '98***] [**Ans.** ₹ 2,470]

55. A machine costs the company ₹ 97,000 and its effective life is estimated to be 12 years. If the scrap realises ₹ 2,000 only, what amount should be retained out of profits at the end of each year to accumulate at compound interest at 5% p.a. to purchase a new machine at original cost? [***N.B.U. B.Com. '98***]

[**Ans.** ₹ 5960 (Approx.)]

56. A man decides to deposit ₹ 10,000 at the end of each year in a bank which pays 10% p.a. compound interest. If the instalments are allowed to accumulate what will be the total accumulation at the end of 9 years?

[Given $(1.1)^9 = 2.2583$] [***C. U. B.Com. 2001***]

[**Ans.** ₹ 1,25,830]

57. A Moped is purchased on instalments basis such that ₹ 3400 is to be paid on the signing of the contract and four yearly instalments of ₹ 2400 each payable at the end of the first, second, third and fourth years. If the interest is charged at 3% p.a. what would be the cash down price?

[Given that $(1.08)^4 = 1.36$] [***C. U. B.Com.(H) 2014***]

[**Ans.** ₹ 11,341]

58. (*a*) How much would you have to invest in an account earning 8% interest compounded continuously, for it to be worth ₹10,00,000 in 30 years? [Given: $e^{2\cdot4} = 11.023$]

[**Hints:** $A = P \cdot e^{r \cdot n} = 10{,}00{,}000$ or $P = \frac{10{,}00{,}000}{e^{2\cdot4}} =$ ₹ 90717.95]

[**Ans.** ₹ 90717.95]

(*b*) At what interest rate (compounded continuously) would ₹ 3000 grow to ₹ 3,00,000 in 25 years? [Given $\log e = 0.4343$] [**Ans.** 18.42%]

C. MULTIPLE CHOICE QUESTIONS (MCQs)

(i) Short Type

Compound Interest

Mark the correct alternative in each of the following:

1. Simple interest on ₹ 20,000 at 18% p.a. for a period of two years is
   (*a*) ₹ 7200 (*c*) ₹ 6800
   (*b*) ₹ 8400 (*d*) ₹ 5400 [**Ans.** (*a*)]

2. A sum of money will yield a simple interest of ₹ 7200 in four years at 12% p.a. The sum is
   (*a*) ₹ 16,000 (*c*) ₹ 14000
   (*b*) ₹ 15,000 (*d*) ₹ 13000 [**Ans.** (*b*)]

3. A sum becomes six times itself at 20% p.a. simple interest in
   (*a*) 26 years (*c*) 25 years
   (*b*) 23 years (*d*) 24 years [**Ans.** (*c*)]

4. A sum of money amounts to ₹ 5800 in three years at 15% p.a. simple interest. The sum is
   (*a*) ₹ 3700 (*c*) ₹ 3500
   (*b*) ₹ 4200 (*d*) ₹ 4000 [**Ans.** (*d*)]
5. Compound interest on ₹ 1400 at 20% p.a. for a period of two years compounded annually is
   (*a*) ₹ 575 (*c*) ₹ 600
   (*b*) ₹ 616 (*d*) ₹ 585 [**Ans.** (*b*)]
6. A sum of money becomes ₹ 6690 after three years and ₹ 10,035 after 6 years on compound interest. The sum is
   (*a*) ₹ 4445 (*c*) ₹ 4460
   (*b*) ₹ 4400 (*d*) ₹ 4520 [**Ans.** (*c*)]
7. ₹ 1600 at 10% per annum compound interest compounded half yearly amounts to ₹ 1944.81 in
   (*a*) 2 yrs (*c*) 4 yrs
   (*b*) 3 yrs (*d*) 5 yrs [**Ans.** (*a*)]
8. The difference between simple and compound interests compounded annually on a certain sum of money for 2 years at 4% per annum is Re. 1 . The sum is
   (*a*) ₹ 750 (*c*) ₹ 720
   (*b*) ₹ 625 (*d*) ₹ 670 [**Ans.** (*b*)]
9. The difference between simple interest and compound interest on a sum for 2 years at 8%, when the interest is compounded annually is ₹ 16. If the interest was compounded half-yearly, the difference in two interests would be nearly
   (*a*) ₹ 21.85 (*c*) ₹ 16.80
   (*b*) ₹ 16 (*d*) ₹ 24.64 [**Ans.** (*d*)]
10. There is 60% increase in an amount in 6 years at simple interest. The compound interest of ₹ 12,000 after 3 years at the same rate is
    (*a*) ₹ 3120 (*c*) ₹ 3972
    (*b*) ₹ 2970 (*d*) ₹ 6240 [**Ans.** (*c*)]
11. The compound interest on ₹ 30,000 at 7% p.a. is ₹ 4347. The period (in years) is:
    (*a*) 2 (*c*) 3
    (*b*) 21 (*d*) 4 [**Ans.** (*a*)]
12. The rate of compound interest per annum will a sum of ₹ 1200 become ₹ 1348.32 in 2 years is
    (*a*) 6% (*c*) 7%
    (*b*) 6.5% (*d*) 7.5% [**Ans.** (*a*)]
13. The least number of complete years in which a sum of money put out at 20% compound interest will be more than double is
    (*a*) 3 (*c*) 5
    (*b*) 4 (*d*) 6 [**Ans.** (*b*)]

14. The effective annual rate of interest corresponding to a nominal rate of 6% per annum payable half-yearly is
(*a*) 6.08% (*c*) 6.09%
(*b*) 6.07% (*d*) 6.06% [**Ans.** (*c*)]

15. The difference between simple and compound interest on ₹ 1200 for one year at 10% per annum reckoned half-yearly is
(*a*) ₹ 2.50 (*c*) ₹ 3.50
(*b*) ₹ 3 (*d*) ₹ 4 [**Ans.** (*b*)]

16. The difference between compound interest and simple interest on an amount of ₹ 15,000 for 2 years is ₹ 96. The rate of interest per annum is
(*a*) 8 (*c*) 12
(*b*) 10 (*d*) 14 [**Ans.** (*a*)]

17. The value *k* of a machine depreciates every year at the rate of 10% on its value at the beginning of that year. If the present value of the machine is ₹ 729, its worth 3 years ago was:
(*a*) ₹ 800.15 (*c*) ₹ 750.17
(*b*) ₹ 1000 (*d*) ₹ 947.10 [**Ans.** (*b*)]

18. A sum of ₹ 550 was taken as loan. This is to be repaid in two equal annual instalments. If the rate of interest be 20% compounded annually, then the value of each instalment is
(*a*) ₹ 396 (*c*) ₹ 350
(*b*) ₹ 421 (*d*) ₹ 360 [**Ans.** (*d*)]

19. A loan was repaid in two annual instalments of ₹ 112 each. If the rate of interest be 10% per annum compounded annually, the sum borrowed was
(*a*) ₹ 217.50 (*c*) ₹ 210
(*b*) ₹ 290 (*d*) ₹ 200 [**Ans.** (*c*)]

20. A sum of money placed at compound interest doubles itself in 5 years. It will amount to eight times itself in
(*a*) 20 years (*c*) 10 years
(*b*) 15 years (*d*) 12 years [**Ans.** (*b*)]

21. If ₹ 7500 are borrowed at compound interest at the rate of 4% per annum, the the amount to be paid after 2 years is:
(*a*) ₹ 7800 (*c*) ₹ 8082
(*b*) ₹ 8100 (*d*) ₹ 8112 [**Ans.** (*d*)]

22. The difference of compound interest on ₹ 800 for 1 year at 20% per annum when compounded half yearly and quarterly is:
(*a*) Nil (*c*) ₹ 4.40
(*b*) ₹ 6.60 (*d*) ₹ 2.50 [**Ans.** (*c*)]

23. Simple interest on a sum at 4% per annum for 2 years is ₹ 80. The compound interest on the same sum for the same period is
(*a*) ₹ 81.60 (*c*) ₹ 160
(*b*) ₹ 1081.60 (*d*) ₹ 181.60 [**Ans.** (*a*)]

24. The principal amount which earns ₹ 132 as compound interest for the second year at 10% p.a. is
    (*a*) ₹ 1320 (*c*) ₹ 1000
    (*b*) ₹ 1200 (*d*) ₹ 1480 [**Ans.** (*b*)]

25. The present value of ₹ 10,000 due in 12 years at 6% p.a. compound interest is [***C.U. B.Com. 2006***]
    (*a*) ₹ 4872 (*c*) ₹ 4729.31
    (*b*) ₹ 4531.7 (*d*) ₹ 4971 [**Ans.** (*d*)]

26. A sum of money triple itself at 6% interest p.a. compounded half-yearly in
    (*a*) 18.6 years (*c*) 16.8 years
    (*b*) 17.25 years (*d*) 18 years [**Ans.** (*a*)]

27. The present value of ₹ 450 due in four years at 2% p.a. compounded half yearly is :
    (*a*) ₹ 412 (*c*) ₹ 415.80
    (*b*) ₹ 502.70 (*d*) ₹ 517.32 [**Ans.** (*c*)]

28. A sum of money double itself at 5% p.a. compound interest in
    [Given log 2 = 0.3010, log 105 = 2.0212] [***CU. B.Com. 1989, 2003***]
    (*a*) 15.2 years (*c*) 13.82 years
    (*b*) 14.2 years (*d*) 15.7 years [**Ans.** (*b*)]

29. A machine depreciated in value each year at 10% of its previous value and at the end of fourth year its value was ₹ 1,31,220. Its original value is
    (*a*) ₹ 2,50,000 (*c*) ₹ 2,20,000
    (*b*) ₹2,00,000 (*d*) ₹ 1,80,000 [**Ans.** (*b*)]

30. An asset costing ₹ 1500 will depreciate to a scrap value of ₹ 120 in 10 years. The rate of depreciation is
    (*a*) 20.19% (*c*) 22.33%
    (*b*) 23.22% (*d*) 21% [**Ans.** (*c*)]

Annuities

1. An Annuity is a(n)
   (*a*) level stream of perpetual cash flows.
   (*b*) level stream of cash flows occurring for a fixed period of time.
   (*c*) increasing stream of perpetual cash flows.
   (*d*) decreasing stream of cash flows occurring for a fixed period of time.
   [**Ans.** (*b*)]

2. Amount of money today which is equal to series of payments in future
   (*a*) nominal value of annuity (*c*) present value of annuity
   (*b*) sinking value of annuity (*d*) future value of annuity
   [**Ans.** (*c*)]

3. Process of loan repayment by instalment payments is classified as
   (*a*) amortizing a loan
   (*b*) depreciation a loan
   (*c*) appreciation of loan
   (*d*) appreciation of investment
   [**Ans.** (*a*)]

4. Formula used for annuity A as $\frac{P}{r}\left\{1-(1+r)^{-n}\right\}$ used to calculate
   (*a*) future value of annuity
   (*b*) nominal value of annuity
   (*c*) sinking value of annuity
   (*d*) present value of annuity
   [**Ans.** (*d*)]

5. If the nominal rate of interest is 10% per annum and there is quarterly compounding, the effective rate of interest will be :
   (*a*) 10% p.a.
   (*b*) 10.10% p. a.
   (*c*) 10.25% p.a.
   (*d*) 10.38% p.a. [**Ans.** (*d*)]

6. Relationship between annual nominal rate of interest and annual effective rate of interest, if frequency of compounding is greater than one :
   (*a*) effective rate > nominal rate
   (*b*) effective rate < nominal rate
   (*c*) effective rate = nominal rate
   (*d*) none of the above [**Ans.** (*a*)]

7. Mr X takes a loan of ₹ 50,000 from HDFC bank. The rate of interest is 10% p.a. The first instalment will be paid at the end of year 5. The amount of equal annual instalments if Mr X whishes repay the amount in five instalments is
   (*a*) ₹ 19500
   (*b*) ₹ 19400
   (*c*) ₹ 19310
   (*d*) None of the above [**Ans.** (*c*)]

8. Assumption in calculating annuity is that every payment is
   (*a*) equal
   (*b*) different
   (*c*) nominal
   (*d*) marginal [**Ans.** (*a*)]

9. Accumulated series of deposits as future sum money is classified as :
   (*a*) marginal fund
   (*b*) nominal fund
   (*c*) sinking fund
   (*d*) annuity fund [**Ans.** (*c*)]

10. Series of payments made periodically is classified as
    (*a*) annuity
    (*b*) effective payment
    (*c*) marginal payment
    (*d*) nominal payment [**Ans.** (*a*)]

11. If nominal rate of return is 10% p.a. and annual effective rate of interest is 10.25% p.a., the frequency of compounding is
    (*a*) 1
    (*b*) 2
    (*c*) 3
    (*d*) none of the above [**Ans.** (*b*)]

12. Present value tables for annuity cannot be straight away applied to varied stream of cash flows.
    (*a*) True
    (*b*) False [**Ans.** (*a*)]

13. How is an annuity due defined?
    (*a*) A stream of cash flows occurring for less than one year
    (*b*) An annuity stream of payments that are disbursed rather than receiver.

(*c*) An annuity stream of payments that are received rather than disbursed
(*d*) A set of equal cash flows occurring at the end of each period
(*e*) A set of equal cash flows occurring at the beginning of each period

[**Ans.** (*e*)]

14. Which of the following is a series of constant cash flows that occur at the end of each period for some fixed number of periods?
(*a*) Ordinary annuity
(*b*) Annuity due
(*c*) Perpetuity
(*d*) None of the given options [**Ans.** (*a*)]

15. Which of the following is a special case of annuity, where the stream of cash flows continues forever ?
(*a*) Ordinary annuity (*c*) Annuity due
(*b*) Special annuity (*d*) Perpetuity [**Ans.** (*d*)]

16. The process of determining the present value of a payment or a stream of payments that is to be received in the future is known as:
(*a*) Discounting
(*b*) Compounding
(*c*) Factorization
(*d*) None of the given options [**Ans.** (*a*)]

17. In which of the following type of annuity, cash flows occur at the beginning of each period ?
(*a*) Ordinary annuity
(*b*) Annuity due
(*c*) Perpetuity
(*d*) None of the given options [**Ans.** (*a*)]

18. Securities future value is ₹ 10,00,000 and present value of securities is ₹ 5,00,000 with an interest rate of 4.5% 'N' will be
(*a*) 16.7473 years (*c*) 15.7473 years
(*b*) 0.0304 months (*d*) 0.7575 years [**Ans.** (*c*)]

19. The value of a perpetual annuity of ₹ 7500 a year at 9% p.a. is
(*a*) ₹ 82,750 (*c*) ₹ 74,444.44
(*b*) ₹ 83,333.33 (*d*) ₹ 85,222.22 [**Ans.** (*b*)]

20. A person desires to endow a bed in a hospital, the cost of which is ₹ 5000 per year. If the money is worth 4% p.a., required sums to provide to create the endowment fund is
(*a*) ₹ 1,50,000 (*c*) ₹ 1,25,000
(*b*) ₹ 1,75,000 (*d*) ₹ 1,60,000 [**Ans.** (*c*)]

21. If the present value of an annuity for 10 years at 6% p.a. compound interest is ₹ 15,000, then annuity is: [***C.U. B.Com. 1988***]
(*a*) ₹ 2175.67 (*c*) ₹ 2721.63
(*b*) ₹ 2200.57 (*d*) ₹ 2163.46 [**Ans.** (*d*)]

22. If the interest is compounded continuously, the required rate at which the principal would double itself in 8 years is
(*a*) 8.66% (*c*) 8.12%
(*b*) 7.95% (*d*) 7.83% [**Ans.** (*a*)]

23. How long will it take ₹ 4000 to amount to ₹ 7000 if it is invested at 7% compounded continuously?
(*a*) 8 years (*c*) 9 years
(*b*) 7 years (*d*) 8.5 years [**Ans.** (*a*)]

24. The effective rate equivalent to nominal rate of 6% converted quarterly is
(*a*) 5.93% (*c*) 5.23%
(*b*) 6.73% (*d*) 6.13% [**Ans.** (*d*)]

25. The present value of ₹ 4000 due after 6 years if the interest rate is 7% compounded semi-annually is:
(*a*) ₹ 2680 (*c*) ₹ 2770
(*b*) ₹ 2710 (*d*) ₹ 2560 [**Ans.** (*b*)]

26. In how many years will an annuity of ₹ 400 amount to ₹ 4064 at 3% p.a. compound interest?
(*a*) 8 years (*c*) 9 years
(*b*) 8.5 years (*d*) 9.5 years [**Ans.** (*c*)]

27. The amount of an annuity of ₹ 700 in 6 years allowing compound interest at 4%, the payment being made half-yearly is:
(*a*) ₹ 9397.50 (*c*) ₹ 9597.63
(*b*) ₹ 8987.75 (*d*) ₹ 9773.27 [**Ans.** (*a*)]

28. The amount of perpetual annuity of ₹ 50 at 5% compound interest per annum is
(*a*) ₹ 1200 (*c*) ₹ 1000
(*b*) ₹ 1100 (*d*) ₹ 1250 [**Ans.** (*c*)]

29. If the present value of an annuity for 10 years at 6% p.a. compound interest is ₹ 2500, then the annuity is:
(*a*) ₹ 350 (*c*) ₹ 330
(*b*) ₹ 340 (*d*) ₹ 320 [**Ans.** (*b*)]

30. An annuity stream where the payments occur forever is called a(n):
(*a*) Annuity due (*c*) Perpetuity
(*b*) Indemnity (*d*) Ordinary annuity [**Ans.** (*c*)]

(ii) Short Essay Type

Compound Interest

1. Mr. A invested amount of ₹ 8,000 in a fixed deposit for 2 years at compound interest rate of 5% p.a. How much A will get on the maturity of the fixed deposit.
(*a*) ₹ 8,750 (*c*) ₹ 8,520
(*b*) ₹ 8,820 (*d*) ₹ 8,650 [**Ans.** (*b*)]

2. Simple interest on a certain sum of money for 3 years at 8% p.a. is half the compound interest on ₹ 4,000 for 2 years at 10% p.a. The sum placed on simple interest is

   (*a*) ₹ 1,750 (*c*) ₹ 1,650
   (*b*) ₹ 2,250 (*d*) ₹ 1,850 [**Ans.** (*a*)]

3. If the simple interest on a sum of money for 2 years at 5% p.a. is ₹ 50, what will be the compound interest on same value.

   (*a*) ₹ 51.50 (*c*) ₹ 51.25
   (*b*) ₹ 51.00 (*d*) ₹ 51.75 [**Ans.** (*c*)]

4. On a sum of money, simple interest for 2 years is ₹ 660 and compound interest is ₹ 696.30, the rate of interest being the same in both cases. The rate of interest is

   (*a*) 10% (*c*) 13%
   (*b*) 12% (*d*) 11% [**Ans.** (*d*)]

5. Divide the total sum of ₹ 3,364 between Ram and Shyam in such a way so that Ram's share at the end of 5 years may equal to Shyam's share at the end of 7 years with compound interest rate at 5%. Then Ram's and Shyam's share are

   (*a*) 1,564 and 1,800 (*c*) 1,764 and 1,600
   (*b*) 1,864 and 1,500 (*d*) 1,664 and 1,700 [**Ans.** (*c*)]

6. A man saves ₹ 200 at the end of each year and lends the money at 5% compound interest. How much will it become at the end of 3 years?

   (*a*) ₹ 662.02 (*c*) ₹ 660.02
   (*b*) ₹ 661.02 (*d*) ₹ 661.03 [**Ans.** (*a*)]

7. A bank offers 5% compound interest calculated on half-yearly basis. A customer deposits ₹ 1,600 each on 1st January and 1st July of a year. At the end of the year, the amount he would have gained by way of interest is

   (*a*) ₹ 122 (*c*) ₹ 125
   (*b*) ₹ 120 (*d*) ₹ 121 [**Ans.** (*d*)]

8. The difference between simple interest and compound on ₹ 1,200 for one year at 10% p.a. reckoned half yearly is:

   (*a*) ₹ 3 (*c*) ₹ 3.5
   (*b*) ₹ 4 (*d*) ₹ 3.75 [**Ans.** (*a*)]

9. The difference between compound interest and simple interest on an amount of ₹ 15,000 for 2 years is ₹ 96. What is the rate of interest p.a.?

   (*a*) 10% (*c*) 9%
   (*b*) 7% (*d*) 8% [**Ans.** (*d*)]

10. Soumen had invested same amount of sums at simple as well as compound interest. The time period of both the sums was 2 years and rate of interest too was same 4% p.a. At the end, he found a difference of ₹ 50 in both the interests received. What were the sums invested?

   (*a*) ₹ 30,250 (*c*) ₹ 31,250
   (*b*) ₹ 31,000 (*d*) ₹ 32,250 [**Ans.** (*c*)]

11. Amit has ₹ 1301 with him. He divided it amongst his sons Ajit and Asit and asked them to invest it at 4% rate of interest compounded annually. It was seen that Ajit and Asit got same amount after 17 and 19 years respectively. How much did Amit give to Asit?
    (*a*) ₹ 615 (*c*) ₹ 715
    (*b*) ₹ 625 (*d*) ₹ 725 [**Ans.** (*b*)]
12. Suresh gave his friend ₹ 11,000. This loan was to be repaid in 3 yearly instalments with a rate of interest of 20% compounded annually. How much would be the value of each instalment?
    (*a*) ₹ 5,221.97 (*c*) ₹ 4,627.69
    (*b*) ₹ 3,667.65 (*d*) ₹ 5,122.79 [**Ans.** (*a*)]
13. A town has population of 50,000 in 2015. In one year i.e. by 2016 it increased by 25%. Next year i.e. in 2017, it decreased by 30%. The next year in 2018 there was an increase of 40%. What is the population at end of 2018?
    (*a*) 63,150 (*c*) 65,150
    (*b*) 62,250 (*d*) 61,250 [**Ans.** (*d*)]
14. In 3 years by compound interest, a sum becomes ₹ 900. But in 4 years by compound interest, the same sum becomes ₹ 1000. What is the sum and the rate of interest?
    (*a*) ₹ 656.10, 11.11% (*c*) ₹ 676.20, 12.15%
    (*b*) ₹ 666.20, 11.25% (*d*) ₹ 686.20, 12.00% [**Ans.** (*a*)]
15. The compound interest on a certain sum for 2 years at 10% p.a. is ₹ 525. The simple interest on the same sum for double the time at half the rate present per annum is:
    (*a*) ₹ 800 (*c*) ₹ 500
    (*b*) ₹ 700 (*d*) ₹ 600 [**Ans.** (*c*)]

Annnuities

1. Mr. X takes a loan of ₹ 50,000 from HDFC Bank. The rate of interest is 10% p.a. The first instalment will be paid at the end of year 5. The amount of equal annual instalments if Mr. X wishes to repay the amount in 5 instalments is
   (*a*) ₹ 19,420 (*c*) ₹ 19,510
   (*b*) ₹ 19,310 (*d*) ₹ 19,370 [**Ans.** (*b*)]
2. A deposit of ₹ 100 is placed into a college fund at the beginning of every month for 10 years. The fund earns 9% annual interest, compounded monthly, and paid at the end of the month. The amount in the account right after the last deposit is
   (*a*) ₹ 19365.63 (*c*) ₹ 19351.43
   (*b*) ₹ 19452.13 (*d*) ₹ 19551.73 [**Ans.** (*c*)]
3. Tarun decides to set aside ₹ 50 at the end of each month for his child's college education. If the child were to be born today, how much will be available for

its college education when she/he turns 19 years old? Assume on interest rate of 5% compounded monthly.

(*a*) ₹ 14,460.20 (*c*) ₹ 16,560.20
(*b*) ₹ 15,360.10 (*d*) ₹ 17,460.10 [**Ans.** (*d*)]

4. Ranjit asks you to help him determine the appropriate price to pay for an annuity offering a retirement income of ₹ 1,000 a month for 10 years. Assume the interest rate is 6% compounded monthly. The appropriate price is

(*a*) ₹ 90,073.45 (*c*) ₹ 90,170.15
(*b*) ₹ 90,063.25 (*d*) ₹ 90,083.35 [**Ans.** (*a*)]

5. Amiya wants to deposit ₹ 300 into a fund at the beginning of each month. If he can earn 10% compounded interest monthly, how much amount will be there in the fund at the end of 6 years?

(*a*) ₹ 29,579 (*c*) ₹ 30,579
(*b*) ₹ 27,379 (*d*) ₹ 29,679 [**Ans.** (*d*)]

6. The monthly rent on an apartment is ₹ 950 per month payable at the beginning of each month. If the current interest is 12% compounded monthly, what single payment 12 months in advance would be equal to a year's rent?

(*a*) ₹ 10,801.50 (*c*) ₹ 10,706.70
(*b*) ₹ 10,607.80 (*d*) ₹ 10,501.70 [**Ans.** (*a*)]

7. A certain amount was invested on Jan. 1, 2017 such that it generated a periodic payment of ₹ 1,000 at the beginning of each month of the calendar year 2017. The interest rate on the investment was 13.2%. The original investment is

(*a*) ₹ 10,506.27 (*c*) ₹ 11,307.32
(*b*) ₹ 11,206.23 (*d*) ₹ 11,507.23 [**Ans.** (*c*)]

8. A man purchases a refrigerator by making a cash payment of ₹ 5,000. In addition he has to pay ₹ 1,000 at the end of each six month for 4 years. If money is worth 10% compounded semi-annually the cash-price for refrigerator is

(*a*) ₹ 11,326 (*c*) ₹ 11,466
(*b*) ₹ 11,556 (*d*) ₹ 12,456 [**Ans.** (*c*)]

9. A person sets up a sinking fund in order to collect ₹ 50,00,000 after 19 years for his child education in AIIMS, New Delhi. The amount he set aside into a an account paying 5% p.a. compound half yearly is

(*a*) ₹ 1,85,100.80 (*c*) ₹ 1,90,700.85
(*b*) ₹ 1,80,200.70 (*d*) ₹ 1,75,900.70 [**Ans.** (*a*)]

10. Dipika plans on retiring on her 60th birthday. She wants to put the same amount of funds aside each year for the next twenty years-starting next year, so that she will be able to withdraw ₹ 50,000 per year for 20 years once she retires, with the first withdrawl on her 61st birthday. Dipika is 20 years old today. How much must she set aside each year for her retirement, if she can earn 10% on her funds?

(*a*) ₹ 1,205.25 (*c*) ₹ 1,107.35
(*b*) ₹ 1,104.75 (*d*) ₹ 1,206.75 [**Ans.** (*b*)]

11. A woman will need an amount of ₹ 2,000 to go on vacations with her husband at the end of each year for 10 years. For this purpose she wants to invest some money in a saving bank but does not know the exact amount of money to invest. What amount does she require to invest now to receive an income of ₹ 2,000 at the end of each year for 10 years if the interest rate is 15%.
    (*a*) ₹ 10,250 (*c*) ₹ 10,038
    (*b*) ₹ 10,120 (*d*) ₹ 10,048 [**Ans.** (*c*)]

12. Kabir is an athlete who believes that his playing career will last 7 year. He deposits ₹ 22,000 at the end of each year for 7 years in an account paying 6% compounded annually. How much will he have on deposit after 7 year?
    (*a*) ₹ 1,90,062.32 (*c*) ₹ 1,70,462.13
    (*b*) ₹ 1,80,115.37 (*d*) ₹ 1,84,664.43 [**Ans.** (*d*)]

13. Abinash has decided to deposit ₹ 200 at the end of each month in an account that pays interest of 4.8% compounded monthly for retirement in 20 year. How much will be in the account at that time?
    (*a*) ₹ 82,235.05 (*c*) ₹ 80,225.12
    (*b*) ₹ 80,335.01 (*d*) ₹ 82,230.15 [**Ans.** (*b*)]

14. Suppose that a court settlement results in a ₹ 7,50,000 award. If this is invested at 9% compounded semi-annually, how much will it provide at the beginning of each half year for a period of 7 years?
    (*a*) ₹ 70,205.97 (*c*) ₹ 70,515.25
    (*b*) ₹ 70,325.85 (*d*) ₹ 70,410.77 [**Ans.** (*a*)]

15. A deferred annuity is purchased that will pay ₹ 10,000 per quarter for 15 years after being deferred for 5 years. If money is worth 6% compounded quarterly, what is the present value of this annuity?
    (*a*) ₹ 2,90,286.15 (*c*) ₹ 2,92,386.85
    (*b*) ₹ 2,91,372.25 (*d*) ₹ 2,92,426.35 [**Ans.** (*c*)]

CHAPTER

6

Correlation and Association

SYLLABUS

Bivariate Data, Scatter Diagram, Pearson's Correlation Coefficient, Spearman's Rank Correlation, Measures of Association of Attributes

THEMATIC FOCUS

6.1 BIVARIATE DATA

Univariate, bivariate and multivariate are the various types of data that are based on the number of variables. We have studied a number of methods for displaying and summarizing a single measurement variable which is known as **univariate data**. For example, mean, median and mode marks of a group of students, standard deviation of wages of a group of employees, skewness of weights of a group of men, etc. **Multivariate data** is the data in which analyses are based on more than two variables per observation. Usually multivariate data is used for explanatory purposes. When we conduct a study that examines the relationship between two variables, then we are working with **bivariate data**. Suppose we conducted a study to see if there was a relationship between the height and weight of each male student in a mathematics class. Since we are working with two variables (height and weight), we would be working with bivariate data. The study of bivariate data provides tools, techniques and methods for the purpose of analysis and inference of bivariate data distribution.

Bivariate data consists of a set of pairs of observations relating to the simultaneous measurement of two variables of x and y, i.e. (x, y). Hence different pairs of observations are like (x_1, y_1), (x_2, y_2), $\ldots(x_n, y_n)$. On the basis of the above analysis, bivariate data can be defined as follows:

Bivariate data: Bivariate data are the data in which analyses are based on two variables per observation simultaneously.

6.2 CORRELATION

It is a statistical method which helps to find whether two variables are related and to what extent they are related. Correlation is considered as the sympathetic movement of two or more variables. Whenever two variables are so related that any change in the value of one variable may yield change in the value of the other variables, then the variables are said to be correlated.

6.2.1 Definition of Correlation

A statistical tool used to measure the relationship between two or more variables such that the movement in one variable is accompanied by the movement of another is called **correlation**.

6.2.2 Types of Correlation

Three important types of correlation classified on the basis of movement, number and the ratio of change between the variables are as follows:

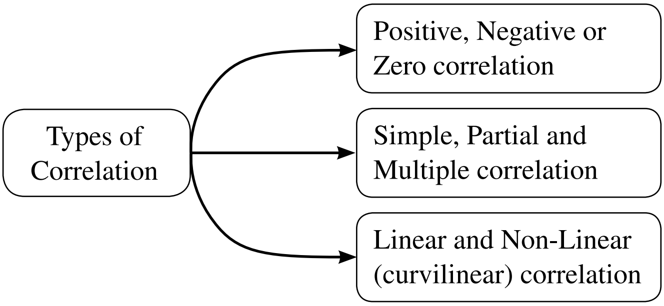


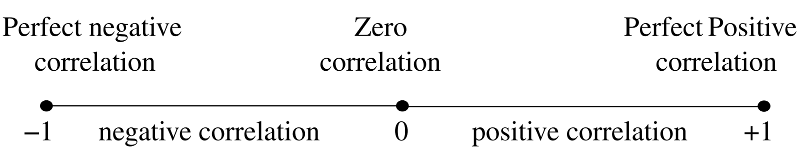
Figure 6.1 Types of correlation

1. *Positive, Negative, or Zero Correlation*

Whether the correlation between the variables is positive or negative depends on its direction of change. The correlation is **positive** when both the variables move in the same direction, i.e. when the increase in one variable (x) is followed by a corresponding increase in the other variable (y) and vice versa. The positive correlations range from 0 to +1. For example, when price increases, supply also increases; when price decreases, supply decreases. Price and supply are two variables, which are positively correlated.

On the other hand, the increase in one variable (x) results in a corresponding decrease in the other variable (y), i.e. when both the variables move in the opposite direction, the correlation is said to be **negative** correlation. The negative correlation ranges from 0 to −1. For example, when price increases, demand decreases; when price decreases, demand increases. So price and demand are negatively correlated.

Zero correlation means no relationship between the two variables x and y; i.e. the change in one variable (x) is not associated with the change in the other variable (y). For example, body weight and intelligence, shoe size and monthly salary; etc. The zero correlation is the mid-point of the range −1 to +1.



2. *Simple, Partial and Multiple Correlation*

The correlation is simple, partial or multiple depends on the number of variables studied. If there are only two variables under study, the correlation is said to be **simple**. For example, the correlation between price and demand is simple. When one variable is related to a number of other variables, the correlation is not simple. It is **multiple** if there is one variable on one side and a set of variables on the other side. For example, relationship between the yield of wheat per acre with the amount of both rainfall and fertilizer used together is a multiple correlation.

Whereas, in the case of a **partial correlation** we study more than two variables, but consider only two among them that would be influencing each other, such that the effect of the other influencing variable in kept constant. Such as, in the above example, if we study the relationship between the yield and fertilizers used during the periods when certain average temperature existed, then it is a problem of partial correlation.

3. Linear and Non-linear (Curvilinear) Correlation

Whether the correlation between the variables is linear or non-linear depends on the constancy of ratio of change between the variables. When the change in one variable results in the constant change in the other variable, we say the correlation is **linear**. When there is a linear correlation, the points plotted will be in a straight line.

EXAMPLE

Consider the variables with the following values:

x:	10	20	30	40	50
y:	30	60	90	120	150

Here, there is a linear relationship between the variables. There is a ratio 1 : 3 at all points. Also, if we plot them they will be in a straight line.

The correlation is called non-linear or curvilinear when the amount of change in one variable is not in a constant ratio to the change in the other variable.

EXAMPLE

Consider the variables with the following values:

x	10	20	30	40	50
y	10	35	70	85	110

Here, there is a non-linear relationship between the variables. The ratio between the variables is not fixed for all points. The graphical representation of the two variables will be a curved line. Such a relationship between the two variables is termed as the curvilinear correlation.

6.3 COEFFICIENT OF CORRELATION

To measure the degree of association or relationship between two variables quantitatively, an index of relationship is used and is termed as **coefficient of**

correlation. It is a numerical index that tells us to what extent the two variables are related and to what extent the variations in one variable changes with the variations in the other. It measures only the degree of linear association between two variables. The values of the correlation coefficient can vary from –1 to +1.

6.4 METHODS OF COMPUTING COEFFICIENT OF CORRELATION

In case of ungrouped data of bivariate distribution, the following three methods are used to compute the value of coefficient of correlation:

1. Scatter diagram
2. Pearson's product moment coefficient of correlation
3. Spearman's Rank coefficient of correlation

6.4.1 Scatter Diagram

A scatter diagram is a graphic picture of simple bivariate data. Suppose two related variables x and y of n pairs of observations have the values (x_1, y_1), (x_2, y_2), ... (x_n, y_n). These points are plotted on a rectangular co-ordinate system putting the independent variable on the x-axis and the dependent variable on the y-axis in the form of dots (.), i.e. for each pair of x, y values we put a dot, and thus obtain a set of points in the xy plane for n pairs of values. This diagrammatic representation (i.e. the diagram of dots) of bivariate data is known as **scatter diagram**. The scatter diagram is known by many names, such as scatter plot, scatter graph, and correlation chart.

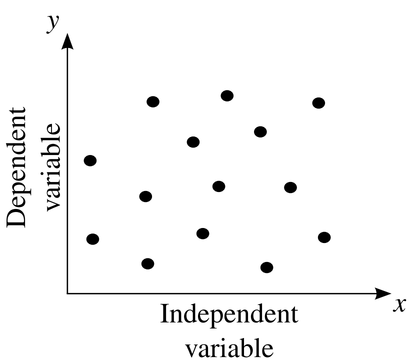


Figure 6.2 Scatter diagram

The scatter diagram is used to find the correlation between two variables. This diagram helps to determine how closely the two variables are related. After determining the correlation between the variables, you can then predict the behaviour of the dependent variable based on the measure of the independent variable. This diagram is very useful when one variable is easy to measure and the other is not.

TYPES OF SCATTER DIAGRAM

The scatter diagram can be classified into two main types. Again each type is subdivided under different categories.

TYPE OF SCATTER DIAGRAM

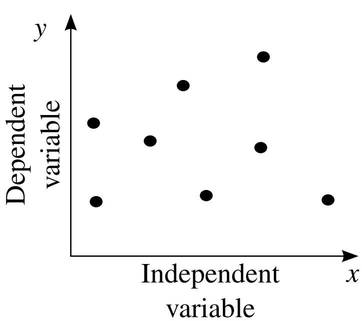
First type based on type of correlation

- Scatter diagram with no correlation
- Scatter diagram with moderate correlation
- Scatter diagram with strong correlation

Second type based on slope or trend

- Scatter diagram with strong positive correlation
- Scatter diagram with weak positive correlation
- Scatter diagram with strong negative correlation
- Scatter diagram with weak negative correlation.
- Scatter diagram with weakest (or no) correlation

Table 6.1 First type based on type of correlation

Category	**Interpretation**
Scatter diagram with no correlation 	Data points are spread so randomly that we cannot draw any line through them. There is no relation between two variables.

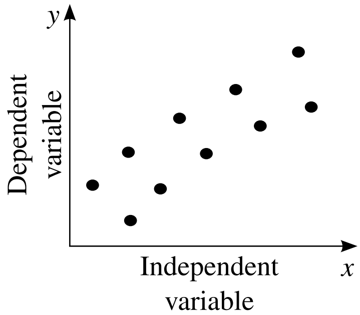
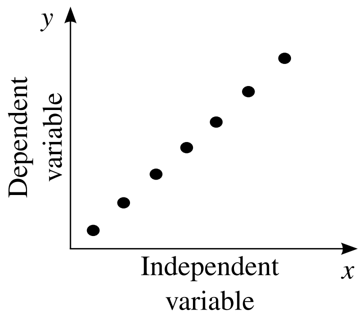
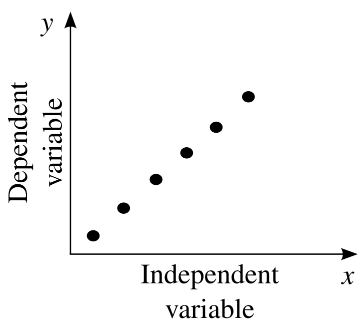
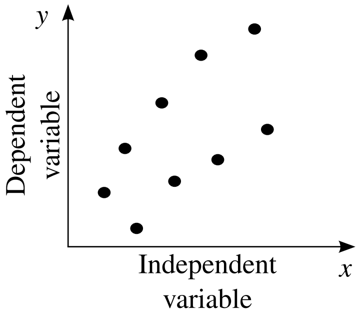
Scatter diagram with moderate correlation 	Data points are little closer together and we can feel that some kind of relation exists between the variables.
Scatter diagram with strong correlation 	Data points are grouped very close to each other such that we can draw a line by following their pattern.

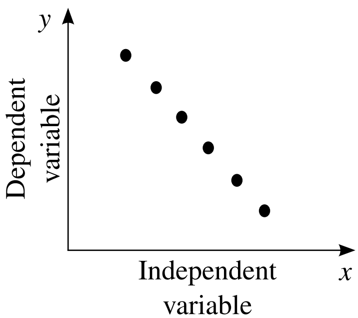
Table 6.2 Second type based on the slope or trend

Category	Interpretation
Scatter diagram with strong positive correlation 	This type of diagram is also known as scatter diagram with positive slant. In positive slant, the correlation will be positive. We can say that the scope of straight line drawn along the data points will go up. The pattern will resemble the straight line.

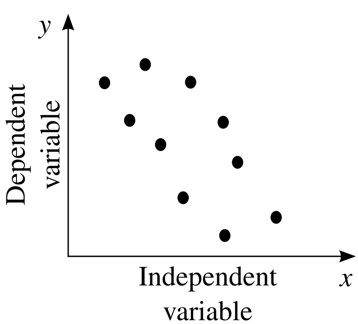
<table>
<tr>
<td>
Scatter diagram with weak positive correlation



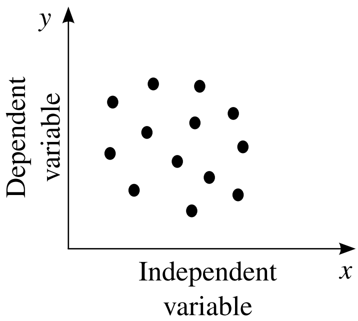
</td>
<td>Here as the value of x increases the value of y will also tend to increase, but the pattern will not closely resemble a straight line.</td>
</tr>
<tr>
<td>
Scatter diagram with strong negative correlation



</td>
<td>This type of diagram is also known as scatter diagram with negative slant. In negative slant, the correlation will be negative, i.e. as the value of x increases, the value of y will decrease. The slope of a straight line drawn along the data points will go down.</td>
</tr>
<tr>
<td>
Scatter diagram with weak negative correlation



</td>
<td>Here as the value of x increases the value of y will tend to decrease, but the pattern will not be as well defined.</td>
</tr>
</table>

Scatter diagram with no correlation	In this diagram no path is formed, i.e. all the points are scattered around without any system. Then there is no association between the variables. It might just be a blob of points with no visible trend.
	

6.4.2 Pearson's Product Moment Coefficient of Correlation

The coefficient of correlation, *r*, is often called the "Pearson *r*" after Pofessor Karl Pearson who developed the product-moment method, following the earlier work of Gallon and Bravais. Of the several methods of measuring correlation, the Karl Pearson's method is most widely used in practice. The Pearson's coefficient of correlation is denoted by the symbol *r*.

Computation of r for ungrouped data

Here, using the formula for computation of *r* depends on "where from the deviations are taken". In different situations deviations can be taken either from actual mean or from zero or for assumed mean. Type of formula conveniently applied for the calculation of coefficient correlation depends upon mean value (either in fraction or whole).

A. When deviations are taken from actual mean

The formula of *r* when deviations are taken from actual means of the two distributions *X* and *Y*.

$$r_{XY} = \frac{\Sigma xy}{n.\sigma_x.\sigma_y} \tag{6.1}$$

where,
r_{XY} = correlation between *X* and *Y*
x = deviations from actual mean of *X* series = $X - \overline{X}$
y = deviations from actual mean of *Y* series = $Y - \overline{Y}$
Σxy = sum of all the products of deviations (*X* and *Y*)
σ_x = standard deviation of the distribution of *X* series
σ_y = standard deviation of the distribution of *Y* series

If we write $\sigma_x = \sqrt{\dfrac{\Sigma x^2}{N}}$, and $\sigma_y = \sqrt{\dfrac{\Sigma y^2}{N}}$, the N's cancel and the formula (6.1) becomes

$$r_{xy} = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \times \Sigma y^2}} \tag{6.2}$$

Where x and y are deviations from the actual means and Σx^2 and Σy^2 are the sums of squared deviations in x and y taken from the two means.

This formula is suitable in the following cases:

(*i*) To find out correlation between short, ungrouped series
(*ii*) When mean values of both the variables are not in fraction
(*iii*) When deviations are to be taken from actual means of the two distributions

Steps to compute

Step 1: Determine the two means $\overline{X}$ and $\overline{Y}$

Step 2: Take the deviations of X series from $\overline{X}$ and Y series from $\overline{Y}$ and denote these deviations by x and y respectively. Check them by finding algebraic sums, which should be zero.

Step 3: Square these deviations and find out Σx^2 and Σy^2.

Step 4: Multiply the deviations x and y and find Σxy.

Step 5: Substitute the values of Σxy, Σx^2 and Σy^2 in the above formula.

ILLUSTRATION 1

When deviations are taken from actual mean

Calculate Karl Pearson's coefficient of correlation from the following data:

X :	96	34	70	94	46
Y :	90	80	40	90	50

Solution: **Calculation of coefficient of correlation**

X	Y	$x = X - \overline{X}$	$y = Y - \overline{Y}$	x^2	y^2	$x \cdot y$
96	90	28	20	784	400	560
34	80	–34	10	1156	100	–340
70	40	2	–30	4	900	–60
94	90	26	20	676	400	520
46	50	–22	–20	484	400	440
$\Sigma X = 340$	$\Sigma Y = 350$	$\Sigma x = 0$	$\Sigma y = 0$	$\Sigma x^2 = 3104$	$\Sigma y^2 = 2200$	$\Sigma xy = 1120$

$$\overline{X} = \frac{\Sigma x}{n} = \frac{340}{5} = 68 \qquad \sigma_x = \sqrt{\frac{\Sigma x^2}{n}} = \sqrt{\frac{3104}{5}} = \sqrt{620.8} = 24.91$$

$$\overline{Y} = \frac{\Sigma y}{n} = \frac{350}{5} = 70 \qquad \sigma_y = \sqrt{\frac{\Sigma y^2}{n}} = \sqrt{\frac{2200}{5}} = \sqrt{440} = 20.98$$

Appling formula (6.1) $r_{XY} = \frac{\Sigma xy}{n \cdot \sigma_x \cdot \sigma_y} = \frac{1120}{5 \times 24{\cdot}91 \times 20{\cdot}98}$

$$= \frac{1120}{2613} = 0.428$$

$$= 0.43 \text{ (Approx)}$$

Applying formula (6.2) $r_{xy} = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \times \Sigma y^2}} = \frac{1120}{\sqrt{3104 \times 2200}}$

$$= \frac{1120}{2613} = 0.428$$

$$= 0.43 \text{ (Approx)}$$

B. When deviations are taken from zero, i.e. calculation of *r* from original/raw scores (Direct Method)

It is an another method with ungrouped data, which does not require the use of deviations. It deals entirely with original scores when values of x and y are small. The formula may look forbidding but is really easy to apply.

The formula in such a case is:

$$r_{xy} = \frac{n{\cdot}\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{\left[n{\cdot}\Sigma x^2 - (\Sigma x)^2\right]\left[n{\cdot}\Sigma y^2 - \Sigma y^2\right]}} \tag{6.3}$$

This formula is suitable in the following cases:

(*i*) When mean values are in fractions
(*ii*) To compute *r* from direct raw scores
(*iii*) When data are small ungrouped
(*iv*) When a good calculating machine is available

Steps to Compute

Step 1: Square all *x* and *y* measurements
Step 2: Multiply *x* and *y* for every pair of scores

Step 3: Find out $\sum x$, $\sum y$, $\sum x^2$, $\sum y^2$ and $\sum xy$ (i.e. sum of all columns)
Step 4: Substitute the values of Σx, Σy, Σxy, Σx^2 and Σy^2 in the above formula.

ILLUSTRATION 2

Calculation of r from original score (Direct method)

Find the coefficient of correlation of the following data:

x :	9	5	8	6	7	4	5	6
y :	7	4	6	8	9	5	2	4

Solution: **Computation of coefficient of correlation**

x	y	x^2	y^2	xy
9	7	81	49	63
5	4	25	16	20
8	6	64	36	48
6	8	36	64	48
7	9	49	81	63
4	5	16	25	20
5	2	25	4	10
6	4	36	16	24
$\Sigma x = 50$	$\Sigma y = 45$	$\Sigma x^2 = 332$	$\Sigma y^2 = 291$	$\Sigma xy = 296$

Here $n = 8$

$$r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{\left[n\Sigma x^2 - (\Sigma x)^2\right]\left[n\Sigma y^2 - (\Sigma y)^2\right]}}$$

$$= \frac{8 \times 296 - 50 \times 45}{\sqrt{\left[8.332 - (50)^2\right]\left[8.291 - (45)^2\right]}}$$

$$= \frac{2368 - 2250}{\sqrt{(2656 - 2500)(2328 - 2025)}}$$

$$= \frac{118}{\sqrt{156 \times 303}} = \frac{118}{\sqrt{47268}}$$

$$= \frac{118}{217.41} = 0.54.$$

C. **When deviations are taken from assumed mean**

If the actual means are in fractions, then the calculation of correlation using formula 6.2 would involve too many calculations. To avoid difficult calculations, deviations are taken from assumed means. The following formula will be used in this case.

$$r_{xy} = \frac{n\Sigma d_x d_y - \Sigma d_x \cdot \Sigma d_y}{\sqrt{n\Sigma d_x^2 - \left(\Sigma d_x\right)^2} \cdot \sqrt{n\Sigma d_y^2 - \left(\Sigma d_y\right)^2}} \quad (6.4)$$

where, d_x = deviations from assumed mean in x-series (i.e. $X - A$)

d_y = derivations from assumed mean in y-series (i.e. $Y - B$)

This formula is suitable when

(*i*) actual means are in fractions and the multiplication and squaring of these values is often a tedious task,

(*ii*) we want to avoid fractions.

Steps to compute

Step 1: Find the actual mean of x and y series.

Step 2: Choose assumed mean of both x and y.

Step 3: Find the derivations of x series from assumed mean and denote these derivations by d_x and obtain the total, i.e. Σd_x. Similarly, in the case of y series find Σd_y.

Step 4: Square d_x and obtain the total Σd_x^2. Similarly find Σd_y^2

Step 5: Multiply d_x and d_y and obtain the total, i.e. $\Sigma d_x d_y$

Step 6: Substitute the values of $\Sigma d_x d_y$, Σd_x, Σd_y, Σd_x^2 and Σd_y^2 in the formula given above.

ILLUSTRATION 3

When derivations are taken from assumed mean

Find Karl Pearson's coefficient of correlation between the values of x and y given below:

Subject :	A	B	C	D	E	F	G	H	I	J	K	L
X :	50	54	56	59	60	62	61	65	67	71	71	74
Y :	22	25	34	28	26	30	32	30	28	34	36	40

Solution: **Computation of coefficient of correlation**

Subject	x	y	$d_x = X - A$	$d_y = Y - B$	d_x^2	d_y^2	$d_x \cdot d_y$
A	50	22	–10	–8	100	64	80
B	54	25	–6	–5	36	25	30
C	56	34	–4	4	16	16	–16
D	59	28	–1	–2	1	4	2
E	60	26	0	–4	0	16	0
F	62	30	2	0	4	0	0
G	61	32	1	2	1	4	2
H	65	30	5	0	25	0	0
I	67	28	7	–2	49	4	–14
J	71	34	11	4	121	16	44
K	71	36	11	6	121	36	66
L	74	40	14	10	196	100	140
	$\sum x = 750$	$\sum y = 365$	$\sum d_x = 30$	$\sum d_y = 5$	$\sum d_x^2 =$ 670	$\sum d_y^2 =$ 285	$\sum d_x d_y =$ 334

Actual Mean

$$\bar{x} = \frac{\Sigma x}{n} = \frac{750}{12} = 62.5$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{365}{12} = 30.42$$

Assumed Mean

$A = 60, B = 30$

$$r_{xy} = \frac{n\Sigma d_x d_y - \Sigma d_x \cdot \Sigma d_y}{\sqrt{n\Sigma d_x^2 - (\Sigma d_x)^2}\sqrt{n\Sigma d_y^2 - (\Sigma d_y)^2}}$$

$$= \frac{12 \times 334 - 30 \times 5}{\sqrt{12 \times 670 - (30)^2}\sqrt{12 \times 285 - (5)^2}}$$

$$= \frac{4008 - 150}{\sqrt{8040 - 900}\sqrt{3420 - 25}}$$

$$= \frac{3858}{\sqrt{7140}\sqrt{3395}} = \frac{3858}{84.5 \times 58.3}$$

$$= \frac{3858}{4926.35} = 0.78$$

D. Coefficient of correlation by covariance method

If $(X_1,Y_1),(X_2,Y_2)..........,(X_n,Y_n)$ be n pairs of observations on two variables x and y, then covariance of x and y is defined as

$$\text{cov}(X,Y)=\frac{1}{n}\Sigma\left(X-\overline{X}\right)\left(Y-\overline{Y}\right)$$

Where, $\overline{X}$ = mean of X-series and $\overline{Y}$ = mean of Y-series. Now coefficient of correlation (r) is defined by

$$r=\frac{\text{cov}(X,Y)}{\sigma_X.\sigma_Y} \quad \text{[where, } \sigma X = \text{S.d of } X\text{-series] and } \sigma Y = \text{S.d of } Y\text{-series]} \qquad (6.5)$$

$$=\frac{\frac{1}{n}\Sigma\left(X-\overline{X}\right)\left(Y-\overline{Y}\right)}{\sqrt{\frac{\Sigma\left(X-\overline{X}\right)^2}{n}}\sqrt{\frac{\Sigma\left(Y-\overline{Y}\right)^2}{n}}}$$

$$=\frac{\frac{1}{n}\Sigma xy}{\sqrt{\frac{\Sigma x^2}{n}}\sqrt{\frac{\Sigma y^2}{n}}} \quad \text{where } x=X-\overline{X} \text{ and } y=Y-\overline{Y}$$

$$=\frac{\Sigma xy}{\sqrt{\Sigma x^2}\sqrt{\Sigma y^2}}$$

$$=\frac{\Sigma xy}{\sqrt{\Sigma x^2\times\Sigma y^2}}$$

[Using this formula, illustrations have been solved.]

Note: $\text{Cov}(X,Y)=\frac{1}{n}\Sigma\left(X-\overline{X}\right)\left(Y-\overline{Y}\right)=\frac{1}{n}\Sigma\left(XY-X\overline{Y}\right)-\left(\overline{X}Y+\overline{XY}\right)$

$$=\frac{1}{n}\left[\Sigma XY-\overline{Y}\Sigma X-\overline{X}\,\Sigma Y+\Sigma\overline{X}\cdot\overline{Y}\right]$$

$$=\frac{\Sigma XY}{n}-\overline{Y}\frac{\Sigma X}{n}-\overline{X}\frac{\Sigma Y}{n}+\frac{\Sigma\overline{X}.\overline{Y}}{n}$$

$$=\frac{\Sigma XY}{n}-\overline{Y}\cdot\overline{X}-\overline{X}\cdot\overline{Y}+\overline{XY}$$

$$=\frac{\Sigma XY}{n}-\overline{X}.\overline{Y}=\frac{\Sigma XY}{n}-\frac{\Sigma X}{n}.\frac{\Sigma Y}{n}$$

[This form is generally used for calculation.]

Uses of Product Moment *r*

Correlation is one of the most widely used analytical procedures in the field of Educational and Psychological Measurement and Evaluation.

It is useful in:

(*i*) measuring the degree of association (or relationship) between two variables.
(*ii*) estimating the value of the dependent variable on the basis of the value of the independent variable.
(*iii*) validating a test; e.g. a group intelligence test.
(*iv*) determining the degree of objectivity of a test.
(*v*) determining the reliability and validity of the test.
(*vi*) factor analysis technique for determining the factor loading of the underlying variables in human abilities.
(*vii*) determining the role of various correlates to a certain ability.
(*viii*) educational and vocational guidance and decision making.
(*ix*) determining regression coefficients provided standard deviation of two variables are known.

Advantages and Disadvantages of Product Moment *r*

(*i*) *Advantages:*
(*a*) The coefficient of correlation indicates both the directions (i.e. whether the correlation is positive or negative) as well as degree of correlation.
(*b*) The values of *r* lie between –1 and +1. Absence of correlation is denoted by zero.
(*c*) It takes into account all items of the variable and is thus based on a suitable measure of variation.

(*ii*) *Disadvantages*:
(*a*) It always assumes always the linear relationship between the variables regardless of the facts that assumption is correct or not.
(*b*) It takes more time to compute as compared to other methods.
(*c*) It is not easy to interpret the significance of coefficient of correlation. Very often it is misinterpreted.
(*d*) It is effected by the extreme items.
(*e*) If the data are not reasonably homogeneous the coefficient of correlation may give a misleading picture.

Assumptions of Product Moment *r*

(*i*) **Normal distribution:** The variables from which we want to calculate the correlation should be normally distributed. The assumption can be laid from random sampling.
(*ii*) **Linearity:** The product-moment correlation can be shown in straight line which is known as linear correlation.

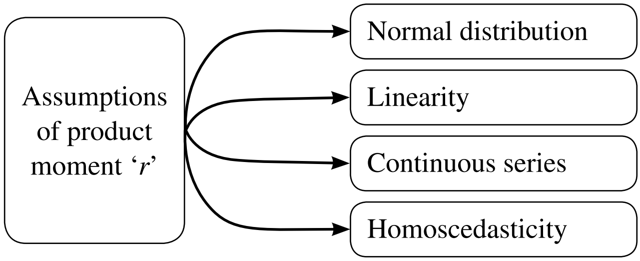


Figure 6.3 Assumptions of Product Moments 'r'

(*iii*) **Continuous series:** Measurement of variables on continuous series

(*iv*) **Homoscedasticity:** It must satisfy the condition of homoscedasticity (equal variability).

Properties of Product Moment *r*

(*i*) The value of coefficient of correlation '*r*' remains unchanged when a constant is added or subtracted on or from one or both variables.

(*ii*) The coefficient of correlation '*r*' is a pure number, i.e. it has no unit.

(*iii*) The value of the coefficient of correlation '*r*' remains unaltered when one or both sets of variate values are multiplied or divided by some constant, i.e. *r* is independent of change of origin and change of scale of observations.

(*iv*) The value of the coefficient of correlation '*r*' lies between −1 to +1.

(*v*) If the two variables are independent of each other, then the values of '*r*' is zero.

Theorem 1 The value of the coefficient of correlation is independent of change of origin and change of scale of observations.

Proof

Let (x_1, y_1), (y_2, y_2), ..., (x_n, y_n) be n pairs of observations of the two variables x and y. Let us change the origin of x and y to a and b and the scales to c and d respectively.

Then, let $u_i = \dfrac{x_i - a}{c}$ and $v_i = \dfrac{y_i - b}{d}$; $(i = 1, 2, 3, ..., n)$

where, a, b, c, d, are constants and c, d being positive.

Now, $x_i = a + cu_i$ and $y_i = b + dv_i$

$$\therefore \quad \overline{x} = a + c\overline{u} \text{ and } \overline{y} = b + d\overline{v}$$

$$\text{and } \sigma_x = c.\sigma_u \text{ and } \sigma_y = d.\sigma_v$$

$$\text{Hence, } \operatorname{cov}(x, y) = \sum(x_i - \overline{x})(y_i - \overline{y})$$

$$= \sum(a + cu_i - a - c\overline{u})(b + dv_i - b - d\overline{v})$$

$$= \sum cd(u_i - \overline{u})(v_i - \overline{v})$$
$$= cd \sum (u_i - \overline{u})(v_i - \overline{v})$$
$$= c.d.\ \text{cov}(u, v)$$

$$\therefore \quad r_{xy} = \frac{\text{cov}(x, y)}{\sigma_x \cdot \sigma_y} = \frac{c \cdot d \cdot \text{cov}(u, v)}{c \cdot \sigma_u \cdot d \cdot \sigma_v} = \frac{\text{cov}(u, v)}{\sigma_u \cdot \sigma_v} = r_{uv}$$

That is, coefficient of correlation of x and y = coefficient of correlation of u and v. Therefore, coefficient of correlation is independent of change of origin and change of scale of observations.

Theorem 2 The value of the coefficient of correlation 'r' lies between -1 and $+1$, i.e. $-1 \le r \le 1$.

Proof

Let (x_1, y_1), (x_2, y_2) ,……., (x_n, y_n) be n pairs of observations of the two variables x and y.

$$\text{Let, } u_i = \frac{x_i - \overline{x}}{\sigma_x} \text{ and } v_i = \frac{y_i - \overline{y}}{\sigma_y}; \quad (i = 1, 2, 3, \ldots\ldots, n)$$

Where, $\overline{x}$ and $\overline{y}$ are the means and σ_x and σ_y are the standard deviations of x and y respectively.

$$\text{Now, } \sum u_i^2 = \sum \left(\frac{x_i - \overline{x}}{\sigma_x} \right)^2 = \frac{\sum (x_i - \overline{x})^2}{\sigma_x^2} = \frac{n.6x^2}{\sigma_x^2} = n$$

$$\text{Similarly, } \sum v_i^2 = n$$

$$\text{Also, } \sum u_i v_i = \sum \left(\frac{x_i - \overline{x}}{\sigma_x} \right) \left(\frac{y_i - \overline{y}}{\sigma_y} \right) = \frac{\sum (x_i - \overline{x})(y_i - \overline{y})}{\sigma_x . \sigma_y}$$

$$= \frac{n.\text{cov}(x, y)}{\sigma_x \cdot \sigma_y} = n \cdot r \quad \left[\text{as } r = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y} \right]$$

$$\text{Now, } \sum (u_i + v_i)^2 \ge 0$$
$$\text{Or} \quad \sum (u_i^2 + v_i^2 + 2u_i v_i) \ge 0$$
$$\text{Or} \quad \sum u_i^2 + \sum v_i^2 + 2 \sum u_i v_i \ge 0$$
$$\text{Or} \quad n + n + 2nr \ge 0$$
$$\text{Or} \quad 2n + 2nr \ge 0$$
$$\text{Or} \quad 2n(1 + r) \ge 0$$
$$\text{Or} \quad 1 + r \ge 0$$
$$\text{Or} \quad r \ge -1, \text{ i.e. } -1 \le r \qquad (i)$$

$$\text{Again, } \sum (u_i - v_i)^2 \ge 0$$

Or, $\sum(u_i^2 + v_i^2 - 2u_i v_i) \geq 0$

Or. $\sum u_i^2 + \sum v_i^2 - 2\sum u_i v_i \geq 0$

Or. $n + n - 2nr \geq 0$

Or. $2n - 2nr \geq 0$

Or. $2n(1 - r) \geq 0$

Or. $1 - r \geq 0$

Or. $1 \geq r$, i.e. $r \leq 1$ (*ii*)

From (*i*) and (*ii*) we can write $-1 \leq r \leq 1$

6.4.3 Spearman's Rank Correlation Coefficient

There are some attributes in Education and Psychology (such as intelligence, leadership, honesty, morality, character, etc.) where quantitative measurement is not possible. Again, there are some problems when data exhibits some relationship but not linear correlation, and cannot be described by Pearson's correlation coefficient. In such cases, the objects or individuals may be ranked and arranged in order of merit or proficiency on two variables when these two sets of ranks have agreement between them, we measure the degree of relationship by rank correlation. For example, the evaluation of a group of students on the basis of leadership ability, the ordering of women in a beauty contest, students ranked in order of preference or the pictures may be ranked according to their asthetic values. Spearman has developed a formula called rank correlation coefficient to measure the extent or degree of correlation between two sets of ranks.

The formula to calculate the rank correlation coefficient is.

$$R = 1 - \frac{6\sum d^2}{n^3 - n} \qquad (6.6)$$

Difference between paired ranks (in each case)

Number of pairs of observations

Rank correlation coefficient

Steps to compute

Step 1: Create a table by listing two given data sets.

Step 2: Assign ranks to the data sets (if it is not given). Ranking is achieved by giving the ranking '1' to the biggest number in a column, '2' to the second biggest value and so on. The smallest value in the column will get the lowest ranking. This should be done for both sets of measurements.

Step 3: In case of equal ranks or tie in ranks, tied scores are given the mean (average) rank. For example, in a set of observations three observations have similar values of 20 each and they possess 4^{th}, 5^{th} and 6^{th} rank in the group. Instead of assigning the 4^{th}, 5^{th} and 6^{th} ranks, the average of the three ranks i.e. $\frac{4+5+6}{3} = 5$ is to be assigned to each of them.

Step 4: Find differences of the ranks (d).

Step 5: Square these differences (d^2) to remove negative values and then sum them (Σd^2).

Step 6: Calculate the coefficient (R) using the above formula. The answer will always be between +1.0 (a perfect positive correlation) and −1.0 (a perfect negative correlation).

While solving the rank correlation coefficient one may come across the following types of problems:

(*a*) Where actual ranks are given
(*b*) Where actual ranks are not given
(*c*) Equal ranks or tie in ranks

ILLUSTRATION 4

A. *Where actual ranks are given*

Ten competitors in a beauty contest are ranked by two judges in the following order. Calculate Spearman's rank correlation coefficient.

Competitors	:	A	B	C	D	E	F	G	H	I	J
1^{st} Judge	:	1	6	5	10	3	2	4	9	7	8
2^{nd} Judge	:	6	4	9	8	1	2	3	10	5	7

[*C.U. B.Com. (G) 2016*]

Solution: **Calculation of Spearman's Rank correlation coefficient**

Competitors	1^{st} Judge (r_1)	2^{nd} Judge (r_2)	$d = r_1 - r_2$	d^2
A	1	6	−5	25
B	6	4	2	4
C	5	9	−4	16
D	10	8	2	4
E	3	1	2	4
F	2	2	0	0
G	4	3	1	1
H	9	10	−1	1
I	7	5	2	4
J	8	7	1	1
n =10			$\Sigma d = 0$	$\Sigma d^2 = 60$

$$R = 1 - \frac{6\sum d^2}{n^3 - n}$$

$$= 1 - \frac{6 \times 60}{10^3 - 10}$$

$$= 1 - \frac{360}{990} = 1 - 0.36 = 0.64 \text{ (Approx.)}$$

ILLUSTRATION 5

B. *Where actual ranks are not given*

In a contest, two judges assessed the performance of the eight candidates A, B, C, D, E, F, G and H as shown in the data following:

	A	B	C	D	E	F	G	H
1st Judge :	52	67	40	72	55	48	60	43
2nd Judge :	52	51	43	54	56	40	60	49

Find the rank correlation of the above data applying Spearman's method.

[*C.U. B.Com. 1992, 2014 (H)*]

Solution: **Calculation of Rank correlation coefficient**

Candidates	**1st Judge**	**2nd Judge**	**Rank in 1st Judge's assessment (r_1)**	**Rank in 2nd Judge's assessment (r_2)**	$d = r_1 - r_2$	d^2
A	52	52	5	4	1	1
B	67	51	2	5	–3	9
C	40	43	8	7	1	1
D	72	54	1	3	–2	4
E	55	56	4	2	2	4
F	48	40	6	8	–2	4
G	60	60	3	1	2	4
H	43	49	7	6	1	1
$n = 8$					$\Sigma d = 0$	$\Sigma d^2 = 28$

$$R = 1 - \frac{6\sum d^2}{n^3 - n}$$

$$= 1 - \frac{6 \times 28}{8^3 - 8}$$

$$= 1 - \frac{168}{504} = 1 - 0.33 = 0.67.$$

C. Equal ranks or tie in ranks

The formula to calculate the rank correlation coefficient when there is a tie in the ranks is:

$$R = 1 - \frac{6\left\{\sum d^2 + \sum\left(\frac{t^3 - t}{12}\right)\right\}}{n^3 - n} \qquad (6.7)$$

where t = number of items whose ranks are common.

ILLUSTRATION 6

The following data give the scores of 10 students on two trials of test with a gap of 2 weeks i.e. Trial I and Trial II. Compute the correlation between the scores of two trials by rank difference method:

Student :	A	B	C	D	E	F	G	H	I	J
Trial I :	10	15	11	14	16	20	10	8	7	9
Trial II :	16	16	24	18	22	24	14	10	12	14

Solution: **Computation of rank correlation coefficient**

Student	**Trial-I (X)**	**Trial-II (Y)**	**Rank on Trial-I (r_1)**	**Rank on Trial-II (r_2)**	$d = r_1 - r_2$	d^2
A	10	16	6.5	5.5	1	1.00
B	15	16	3	5.5	–2.5	6.25
C	11	24	5	1.5	3.5	12.25
D	14	18	4	4	0	0
E	16	22	2	3	–1	1.00
F	20	24	1	1.5	–0.5	0.25
G	10	14	6.5	7.5	–1	1.00
H	8	10	9	10	–1	1.00
I	7	12	10	9	1	1.00
J	9	14	8	7.5	0.5	0.25
$n = 10$					$\Sigma d = 0$	$\Sigma d^2 = 24$

In the above table Columns 2 and 3 show that more than one students are getting the same scores. In column 2 students A and G are getting the same score viz. 10 and they possess 6th and 7th rank in the group. So the rank of A and G will be the average of the two ranks, i.e. $\frac{6+7}{2} = \frac{13}{2} = 6.5$.

In column 3 in respect of scores on Trial II, ties occur at three places. Students C and F have the same score and hence are assigned the average rank of $\frac{1+2}{2}=\frac{3}{2}=1.5$. Students A and B have rank position 5 and 6, hence are assigned $5.5\left(\frac{5+6}{2}\right)$ rank each. Similarly students G and J have been assigned $7.5\left(\frac{7+8}{2}\right)$ rank each.

In Trial-I, $t = 2$

In Trial-II, $t = 2, 2$ and 2 [t = number of items whose ranks are common]

Therefore,

$$\sum\left(\frac{t^3-t}{12}\right)=\frac{2^3-2}{12}+\frac{2^3-2}{12}+\frac{2^3-2}{12}+\frac{2^3-2}{12}$$

$$=\frac{6}{12}+\frac{6}{12}+\frac{6}{12}+\frac{6}{12}$$

$$=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$$

$$=2$$

Therefore, $$R=1-\frac{6\left\{\sum d^2+\sum\left(\frac{t^3-t}{12}\right)\right\}}{n^3-n}$$

$$=1-\frac{6(24+2)}{1000-10}$$

$$=1-\frac{6\times 26}{990}=1-\frac{156}{990}$$

$$=1-0.157$$

$$=0.843\,(\text{Approx.})$$

Advantages and Disadvantages of Rank Correlation Method

(i) *Advantages*:

- Computation of 'R' under this method is quicker and easier than computation of 'r' by the Pearson's Product Moment Method.
- It is quiet easy to interpret 'R'.
- It is an acceptable method if data are available only in ordinal form or number of paired variable lies between 5 and 30 with a minimum or a few ties in ranks.

(ii) *Disadvantages*

- If the number of observations are more, giving ranks to them becomes a tedious job.

- When the interval data are converted into rank-ordered data the information about the size of the score differences is lost; e.g. in Illustration-6, if D in Trial-II gets scores from 18 up to 21, his rank remain only 4.

6.5 ASSOCIATION OF ATTRIBUTES

6.5.1 Introduction

So far we have discussed the statistics of variables of which quantitative measurement is possible, e.g. heights and weights of students for a class. We have used various statistical techniques like measures of central tendency, dispersion, skewness in the analysis of such variable and correlation to measure the relationship between two or more variables. But there are certain characteristics like blindness, deafness, fairness, honesty, etc. that cannot be measured quantitatively, we can only study the presence or absence of certain quality in a group of individuals. These types of qualitative characteristics are known as **attributes**.

Hence, **attribute** refers to the quality of a characteristic. The theory of attributes deals with qualitative types of characteristics that are calculated by using quantitative measurements. Therefore, the attribute needs slightly different kinds of statistical treatments, which the variables do not get. In the theory of attributes, the researcher puts more emphasis on quality (rather than on quantity). Since the statistical techniques deal with quantitative measurements, qualitative data is converted into quantitative data in the theory of attributes.

For example, there are 4 students and their heights in inches are 52, 54, 70 and 72. Here height is a characteristic and the figures 52, 54, 70 and 72 are the values of a variable. These figures are the result of measurements. We know that the measurements generate a continuous variable. Thus the variable for heights is a continuous variable. Suppose we select 4 bolts from a certain lot and inspect them. The lot contains good as well as defective bolts. The sample may contain 0, 1, 2, 3, 4 defective bolts. The values 0, 1, 2, 3 and 4 are the values of a discrete variable. Out of 4 students whose heights are given above, 2 are tall with heights of 70 and 72 inches and 2 are short with heights of 52 and 54 inches. When we use the words tall and short, any variable is not under consideration. We do not make any measurements. We only see who is tall and who is short. Here level of height tall or short is not a variable, it is called an attribute. Out of 4 bolts 2 are good and 2 are defective. Here also any variable is not under consideration. We only count the defective bolts and good bolts. We examine whether the quality of being defective is present in a bolt or not. The status of bolt is an attribute with two outcomes, good and defective. Thus an attribute is a quality and the data is collected to see how many objects possess the quality of being defective and how many elements do not possess this quality. The data on the attribute is the result of recording the presence or absence of a certain quality (attribute) in the individuals.

6.5.2 Classification and Notations

The population in the theory of attributes is divided into two classes, namely the negative class and the positive class. The positive class signifies that the attribute is present in that particular item under study, and this class in the theory of attributes is represented by capital letters such as A, B, etc. The negative class signifies that the attribute is not present in that particular item under study, and this class in the theory of attributes is represented by the Greek letter 'α' (alpha) and 'β' (beta), etc.

Thus 'α' = not *A* and 'β' = not B. For example, if *A* represents boys, the 'α' would represents girls. Similarly, if *B* represents tall, then 'β' would denotes short. The combination of the different attributes is denoted by (AB), (Aβ), (α, β) and (α, β). Thus, in this example (AB) would mean number of tall boys and (αβ) short girls. The number of observations that have been allocated in the attributes is known as the **class frequencies**. These class frequencies are symbolically denoted by bracketing the attribute terminologies like (AB), (αβ), etc. Thus, if the number of tall boys is 20, the frequency of class (AB) is 20.

6.5.3 Order of Classes and Class Frequencies

A class represented by the '*n*' attribute refers to the class that has the *n*th order and its frequency as a frequency of the *n*th order. *A* class having one attribute is known as the class by the first order and a class having two attributes as class of the second order, and so on. Thus when there are two attributes *A* and *B* only and *N* is denoted by the total number of observations (called the frequency), then *N* is the frequency of the zero order (A), (B), (α), (β) are the frequencies of the first order (AB), (Aβ), (αB), (αβ) are the frequencies of the second order.

In such a case, classes of the second order would also be known as the classes of the ultimate order. If *N* is counted as a class, then the total number of classes is always equal to 3^n, *n* represents the number of attributes. But the total number of classes of the ultimate order is always equal to 2^n.

If two attributes *A* and *B* are taken into consideration, then the total number of classes will be $3^2 = 9$ (A, α, B, β, AB, Aβ, αB, αβ and *N*) but the total number of classes of the ultimate order would be $2^2 = 4$ (i.e. AB, Aβ, αB, αβ).

The contingency table of order (2 × 2) for two attributes *A* and *B* can be shown as given below:

	A	**α**	**Total**
B	(AB)	(αB)	(B)
β	(Aβ)	(αβ)	(β)
Total	(A)	(α)	*N*

6.5.4 Relationship between the Class Frequencies

The frequency of a lower class can always be expressed in terms of class frequencies of the higher order. Thus we can write:

N = (A) + (α) = (B) + (β)

(A) = (AB) + (Aβ)

(α) = (αB) + (αβ)

(B) = (AB) + (αB)

(β) = (Aβ) + (αβ)

6.6 CONSISTENCY OF DATA

Consistency of a set of class frequencies may be defined as the property that none of these is negative and so the data are said to be consistent. It means there is a consistency between the different frequencies and there is no conflict in any way. Otherwise, the data for class frequencies are said to be 'inconsistent' i.e. if any class frequency is negative, we say that the data is inconsistent.

ILLUSTRATION 7

From the following cases find out whether the data are consistent or not:

(i) N = 1000, (A) = 600, (AB) = 50 and (B) = 500

(ii) N = 60, (A) = 51, (B) = 32 and (AB) = 25

Solution:

(i) Given: N = 100, (A) = 600, (AB) = 50 and (B) = 500

Substituting the values in the following table:

	A	**α**	**Total**
B	50	450	500
β	550	–50	500
Total	600	400	1000

As (αβ) is negative, then the given data is inconsistent.

(ii) Given: N = 60, (A) = 51, (B) = 32 and (AB) = 25

Substituting the values in the following table:

	A	**α**	**Total**
B	25	7	32
β	26	2	28
Total	51	9	60

As all frequencies are positive, then we can conclude that the given data is consistent.

6.7 ASSOCIATION

The statistical term association is defined as a relationship between two random variables which makes them statistically dependent. It refers to rather a general relationship without specifics of the relationship being mentioned, and it is not necessary to be a causal relationship.

6.7.1 Difference between Association and Correlation

Association and correlation are two methods of explaining a relationship between two statistical variables. There are some differences between correlation and association, which are as follows:

(*i*) Correlation is used to measure the degree of relationship between two variables, whereas association is used to measure the degree of relationship between two attributes.

(*ii*) Association is a concept, but correlation is a measure of association and mathematical tools are provided to measure the magnitude of the correlations.

(*iii*) Association refers to a more generalised term and correlation can be considered as a special case of association, where the relationship between the variables is linear in nature.

6.8 METHODS OF STUDYING ASSOCIATION

The following methods may be used to study whether two attributes are associated or not:

(*a*) Comparison of observed and expected frequencies method
(*b*) Proportion method
(*c*) Yule's coefficient of association

6.8.1 Comparison of Observed and Expected Frequencies Method

Under this method, the actual observation is compared with the expectation. If the actual observation is more than the expectation, the attributes are said to be positively associated; if the actual observation is less than the expectation, the attributes are said to be negatively associated and if the actual observation is equal to the expectation, the attributes are said to be independent.

Symbolically, two attributes A and B are

(*i*) positively associated

$$\text{if} \quad (\text{AB}) = \frac{(\text{A})\times(\text{B})}{N}$$

[actual observation] [expectation]

(*ii*) negatively associated

If $(AB) < \frac{(A)\times(B)}{N}$

[actual observation] [expectation]

(*iii*) independent

if $(AB) = \frac{(A)\times(B)}{N}$

[actual observation] [expectation]

It is also true in case of attributes α and B; α and β and A and B.

ILLUSTRATION 8

As from the following data find out whether attributes (i) (AB), (ii) (Aβ), (iii) (αB) and (iv) (αβ) are independent, associated or disassociated. Given N = 80, (A) = 32, (B) = 64, (AB) = 24.

Solution:

(i) Expectation of $(AB) = \frac{(A)\times(B)}{N} = \frac{32\times 64}{80} = 25.6$

The actual observation (AB) = 24

As actual observation (24) is less than the expectation (25.6), then, attributes are negatively associated or disassociated.

(ii) In order to find out the nature of association between the attributes (Aβ), (αB) and (αβ) we shall have to determine the unknown values by preparing a nine square table:

	A	**α**	**Total**
B	24	40	64
β	8	8	16
Total	32	48	80

From the above table we get, (Aβ) = 8, (αB) = 40, (αβ) = 8, (α) = 48, (β) = 16

Expectation of $(A\beta) = \frac{(A)\times(\beta)}{N} = \frac{32\times 16}{80} = 6.4$

The actual observation (Aβ) = 8

As the actual observation (8) is more than the expectation (6.4), then the attributes *A* and β are positively associated.

(iii) Expectation of $(\alpha B) = \frac{(\alpha)\times(B)}{N} = \frac{48\times 64}{80} = 38.4$

The actual observation (αB) = 40

As the actual observation (40) is more than the expectation (38.4), then the attributes α and *B* are positively associated.

(iv) Expectation of (αβ) = $\frac{(\alpha)\times(\beta)}{N} = \frac{48\times 16}{80} = 9.6$

The actual observation (αβ) = 8

As the actual observation (8) is less than the expectation (9.6), then the attributes α and β are negatively associated or disassociated.

[This method can not measure the degree of association between the attributes. It can only measure the nature of association.]

6.8.2 Proportion Method

If there is no relationship of any kind between two attributes, we define them independent and we find the same proportion of A's amongst the B's as amongst the β's. for example, if blindness and dumbness are not associated, then the proportion of blind people amongst the dumb and amongst the not-dumbs must be equal.

Symbolically, two attributes *A* and *B* are

(*i*) positively associated if $\frac{(\mathrm{AB})}{(\mathrm{B})} > \frac{(\mathrm{A}\beta)}{(\beta)}$

(*ii*) negatively associated if $\frac{(\mathrm{AB})}{(\mathrm{B})} < \frac{(\mathrm{A}\beta)}{(\beta)}$

(*iii*) independent if $\frac{(\mathrm{AB})}{(\mathrm{B})} = \frac{(\mathrm{A}\beta)}{(\beta)}$

If the relation (iii) holds good, the corresponding relations

$$\frac{(\alpha\mathrm{B})}{(\mathrm{B})} = \frac{(\alpha\beta)}{(\beta)};\ \frac{(\mathrm{AB})}{(\mathrm{A})} = \frac{(\alpha\beta)}{(\alpha)};\ \frac{(\mathrm{AB})}{(\mathrm{A})} = \frac{(\alpha\beta)}{(\alpha)}$$

must also hold true.

ILLUSTRATION 9

In a population of 250 people the number of married is 100. Out of 75 people who are unemployed 30 belonged to the married group. It is required to find out whether the attributes marriage and unemployed are independent, positively associated or negatively associated.

Solution: Let A denote married people.

∴ α represents unmarried people.

Let B devote number of unemployed people

∴ β would denote employed people.

We are given the total number of people, i.e. $N = 250$.

(A) = 100, (B) = 75 and (AB), i.e. the number of married people who are unemployed = 30.

∴ Proportion of married people who are unemployed:

i.e., $\frac{(AB)}{(A)} = \frac{30}{100} = 0.3$ or 30%

and proportion of unmarried people who are unemployed:

i.e., $\frac{(\alpha B)}{(\alpha)} = \frac{45}{150} = 0.3$ or 30%

Nine square table

	A	**α**	**Total**
B	30	45	75
β	70	105	175
Total	100	150	250

As the two proportions are the same, then we conclude that the attributes, marriage and unemployed, are independent.

[Like previous method, this method also can not measure the degree of association. It can only measure the nature of association.]

6.8.3 Yule's Coefficient of Association

It is the most popular and superior method of studying association of attributes. This method not only measures the nature of association, but also the degree or extent to which the two attributes are associated. To find out the degree of association between two attributes Yule has defined coefficient of association Q by the formula,

$$Q = \frac{(AB)(\alpha\beta) - (A\beta)(\alpha B)}{(AB)(\alpha\beta) + (A\beta)(\alpha B)} \tag{6.8}$$

This is a relative measure of association. The absolute measure of association is given by

$$(AB)(\alpha\beta) - (A\beta)(\alpha B) = 0$$

$$\text{or } (AB)(\alpha\beta) = (A\beta)(\alpha B)$$

The coefficient of association varies between +1 and −1. When the value of Q is +1 there is complete association, i.e. perfect positive association between the attributes, because in that case the second term in both the numerator and denominator of the value of Q is Zero. When the value of Q is −1 there is perfect dissociation, i.e. perfect negative association between the attributes, because in that case the first term in both the numerator and denominator is zero. When the value of Q is zero, the two attributes are independent, i.e. there is no association.

ILLUSTRATION 10

Find the Association between Literacy and Unemployment from the following figures:

Total adults	10,000
Literates	1,290
Unemployed	1,390
Literate unemployed	820

Comment on the results.

Solution: Let A denote literates

∴ α will denote illiterates

Let B denote unemployed

∴ β will denote employed

Given: $(A) = 1290$, $(B) = 1390$, $(AB) = 820$, $N = 10000$

By putting the known values in the nine square table, we can find out the unknown values.

	A	**α**	**Total**
B	820	570	1,390
β	470	8140	8,610
Total	1290	8710	10,000

$(\alpha B) = (B) - (AB)$

$= 1390 - 820 = 570$

$(A\beta) = (B) - (AB)$

$= 1290 - 820 = 470$

$(\alpha) = N - (A)$

$= 10000 - 1290 = 8710$

$(\alpha\beta) = (\alpha) - (\alpha \mathrm{B})$

$= 8710 - 570 = 8140$

$$Q = \frac{(AB)(\alpha\beta) - (A\beta)(\alpha B)}{(AB)(\alpha\beta) + (A\beta)(\alpha B)}$$

$$= \frac{(820 \times 8140) - (470 \times 570)}{(820 \times 8140) + (470 \times 570)}$$

$$= \frac{6674800 - 267900}{6674800 + 267900} = \frac{6406900}{6942700} = 0.923$$

Therefore, there is a high degree of positive association between literacy and unemployment.

ILLUSTRATIVE EXAMPLES

A. SHORT TYPE

Correlation

EXAMPLE 1

If $x = X - \overline{X}$ and $y = Y - \overline{Y}$, and $\Sigma x^2 = 10$, $\Sigma y^2 = 24$, $\Sigma xy = 12$, then find the coefficient of correlation between the two variables x and y. [***C.U. B.Com. 1992***]

Solution: Here, deviation are taken from actual mean, therefore,

$$r_{xy} = \frac{\Sigma xy}{\sqrt{\sum x^2 \times \sum y^2}} = \frac{12}{\sqrt{10 \times 24}} = \frac{12}{\sqrt{240}} = \frac{12}{15.49}$$

$= 0.77$ (approx.)

EXAMPLE 2

Find the correlation coefficient, if $\sigma_x^2 = 6.25$, $\sigma_y^2 = 4$, cov$(x, y) = 0.9$.

[***C.U. B.Com. 2000, 2012, 2015(G)***]

Solution: Given: $\sigma_x^2 = 6.25$ or $\sigma_x = \sqrt{6.25} = 2.5$

$\sigma_y^2 = 4$ or $\sigma_y = \sqrt{4} = 2$

and cov$(x, y) = 0.9$

$$\text{Now, } r = \frac{\text{cov}(x, y)}{\sigma_x . \sigma_y} = \frac{0.9}{2.5 \times 2} = \frac{0.9}{5} = 0.18.$$

EXAMPLE 3

If $r_{xy} = 0.6$, find the value of r_{uv} when

$u = 2x - 3$, $v = -3y + z$. [***C.U. B.Com. 2014(H)***]

Solution: We know that the value of the coefficient of correlation is independent of change of origin and change of scale of observations.

Therefore, if $u = 2x - 3$ and $v = -3y + 2$, then

$$r_{uv} = -r_{xy} \text{ [as coefficient of } x \text{ and } y \text{ are of opposite signs]}$$
$$= -0.6.$$

EXAMPLE 4

If $r_{xy} = 0.6$, $\text{cov}(x, y) = 12$, $\sigma_x^2 = 25$, find the value of σ_y.

[***C.U. B.Com. 2014 (H)***]

Solution: we know that,

$$r_{xy} = \frac{\text{cov}(x,y)}{\sigma_x.\sigma_y}$$

or, $0.6 = \frac{12}{5.\sigma_y}$ [as $\sigma_x^2 = 25$ then $\sigma_x = 5$]

or, $3.\sigma_y = 12$

or, $\sigma_y = \frac{12}{3} = 4.$

EXAMPLE 5

If $r_{xy} = 0.6$, $\sigma_x = 4$ and $\sigma_y = 5$, then
Find $\text{cov}(x, y)$. [***C.U. B.Com. 2015 (H)***]

Solution: We know that,

$$r_{xy} = \frac{\text{cov}(x,y)}{\sigma_x.\sigma_y}$$

or, $0.6 = \frac{\text{cov}(x,y)}{4 \times 5}$

or, $\text{cov}(x,y) = 0.6 \times 20 = 12.$

EXAMPLE 6

If $x = X - \overline{X}$, $y = Y - \overline{Y}$, where $\overline{X}$, $\overline{Y}$ being respectively arithmetic means of X and Y, and if $\Sigma x^2 = 60$, $\Sigma xy = 57$, $r = 0.95$ and variance of $y = 6\frac{2}{3}$, find 'n'.

[***C.U. B.Com. 2009***]

Solution: Here, deviations are taken from actual mean, therefore,

$$r_{xy} = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \times \Sigma y^2}}$$

$$\text{or, } 0.95 = \frac{57}{\sqrt{60 \times \Sigma y^2}}$$

$$\text{or, } (0.95)^2 = \frac{(57)^2}{60 \times \Sigma y^2} \text{ [squaring both sides]}$$

or, $0.9025 \times 60 \times \Sigma y^2 = 3249$
or, $54.15 \times \Sigma y^2 = 3249$

$$\text{or, } \Sigma y^2 = \frac{3249}{54.15} = 60$$

Given that, variance of y is $6\frac{2}{3}$, i.e $\frac{20}{3}$

$$\text{Therefore, } \sigma_y^2 = \frac{20}{3}$$

$$\text{or, } \frac{\Sigma(y - \overline{y})^2}{n} = \frac{20}{3}$$

$$\text{or, } \frac{\Sigma y^2}{n} = \frac{20}{3} \quad [\text{as } y = Y - \overline{Y}]$$

$$\text{or, } \frac{60}{n} = \frac{20}{3}$$

$$\text{or, } 20n = 180 \text{ or } n = \frac{180}{20} = 9$$

Therefore, the required value of n is 9.

Rank Correlation

EXAMPLE 7

Find R (Spearman Rank correlation coefficient) when $\Sigma d^2 = 30$ and $n = 10$
[***C.U. B.Com. 2001, 2012, 2014 (G), 2016 (G), 2016(H)***]

Solution: We know that,

$$R = 1 - \frac{6\Sigma d^2}{n^3 - n}$$

$$= 1 - \frac{6 \times 30}{10^3 - 10} \quad (\text{as } \Sigma d^2 = 30 \text{ and } n = 10)$$

$$= 1 - \frac{180}{990} = 1 - 0.18$$

$$= 0.82$$

Therefore, Spearman's Rank correlation coefficient is 0.82.

Association of Attributes

EXAMPLE 8

From the following data show that attributes α and β are independent.

(α) = 125, (β) = 45, (αβ) = 25 and N = 225

Solution: Expectation of $(\alpha\beta) = \frac{(\alpha)\times(\beta)}{N} = \frac{125\times 45}{225} = 25$

The actual observation (αβ) = 25

As the actual observation (25) is equal to the expectation (25), then the attributes α and β are independent.

EXAMPLE 9

From the following case find out whether the data are consistent or not:

(A) = 100, (B) = 150, (AB) = 140, N = 500

Solution: We know that,

$(A\beta) = (A) - (AB) = 100 - 140 = -40$

As (AB) is negative, then the given data is inconsistent.

EXAMPLE 10

Using proportion method, determine the nature of association between A and B:

(AB) = 30, (A) = 80, (αB) = 20, (α) = 120

Solution: $\frac{(AB)}{(A)} = \frac{30}{80} = 0.375$

and $\frac{(\alpha B)}{\alpha} = \frac{20}{120} = 0.167$

Since, $\frac{(AB)}{(A)} > \frac{(\alpha B)}{\alpha}$ the attributes A and B are positively associated.

B. SHORT ESSAY TYPE

Correlation

When Deviations are Taken from Actual Mean

EXAMPLE 11

Find the coefficient of correlation from the following data:

x	:	3	5	7	8	9	15	16
y	:	15	18	22	24	19	25	31

Solution: **Calculation of coefficient of correlation**

X	Y	$x = X - \bar{X}$	$y = Y - \bar{Y}$	x^2	y^2	xy
3	15	–6	–7	36	49	42
5	18	–4	–4	16	16	16
7	22	–2	0	4	0	0
8	24	–1	2	1	4	–2
9	19	0	–3	0	9	0
15	25	6	3	36	9	18
16	31	7	9	49	81	63
$\Sigma X=63$	$\Sigma Y = 154$	$\Sigma x = 0$	$\Sigma y = 0$	$\Sigma x^2 = 142$	$\Sigma y^2 = 168$	$\Sigma xy = 137$

$$\bar{X} = \frac{\Sigma X}{n} = \frac{63}{7} = 9$$

$$\bar{Y} = \frac{\Sigma Y}{n} = \frac{154}{7} = 22$$

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \times \Sigma y^2}} = \frac{137}{\sqrt{142 \times 168}} = \frac{137}{\sqrt{23856}}$$

$$= \frac{137}{\sqrt{154.45}} = 0.887$$

When Deviations are Taken from Zero (Raw Score)

EXAMPLE 12

Calculate ‘r’ by using Pearson product moment correlation coefficient formula of the following data:

x :	1	2	3	4	5	6
y :	6	4	3	5	4	2

Solution: **Calculation of coefficient of correlation**

x	y	x^2	y^2	xy
1	6	1	36	6
2	4	4	16	8
3	3	9	9	9
4	5	16	25	20
5	4	25	16	20
6	2	36	4	12
$\Sigma x = 21$	$\Sigma y = 24$	$\Sigma x^2 = 91$	$\Sigma y^2 = 106$	$\Sigma xy = 75$

Here $n = 6$

$$r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{[n\Sigma x^2 - (\Sigma x)^2][n\Sigma y^2 - (\Sigma y)^2]}}$$

$$= \frac{6 \times 75 - 21 \times 24}{\sqrt{[6 \times 91 - (21)^2[6 \times 106 - (24)^2]}}$$

$$= \frac{450 - 504}{\sqrt{(546 - 441)(636 - 576)}}$$

$$= \frac{-54}{\sqrt{105 \times 60}} = \frac{-54}{\sqrt{6300}}$$

$$= \frac{-54}{79.37} = -0.68$$

When Deviations are Taken from Assumed Mean

EXAMPLE 13

Calculate the coefficient of correlation for the ages of husband and wife:

Age of husband	:	23	27	28	29	30	31	33	35	36	39
Age of wife	:	18	22	23	24	25	26	28	29	30	32

Solution: **Calculation of coefficient of correlation**

Age of husband (x)	Age of wife (y)	$d_x = x - A$	$d_y = y - B$	d_x^2	d_y^2	$d_x.d_y$
23	18	–8	–8	64	64	64
27	22	–4	–4	16	16	16
28	23	–3	–3	9	9	9
29	24	–2	–2	4	4	4
30	25	–1	–1	1	1	1
31	26	0	0	0	0	0
33	28	2	2	4	4	4
35	29	4	3	16	9	12
36	30	5	4	25	16	20
39	32	8	6	64	36	48
$\Sigma x = 311$	$\Sigma y = 257$	$\Sigma d_x = 1$	$\Sigma d_y = -3$	$\Sigma d_x^2 = 203$	$\Sigma d_y^2 = 159$	$\Sigma d_x d_y = 178$

Actual mean

$$\bar{x} = \frac{\Sigma x}{n} = \frac{311}{10} = 31.1$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{257}{10} = 25.7$$

Assumed mean

$A = 31, B = 26$

$$r_{xy} = \frac{n.\sum d_x.d_y - \sum d_x.\sum d_y}{\sqrt{n.\sum d_x^{\,2} - (\sum d_x)^2}\sqrt{n.\sum d_y^{\,2} - (\sum d_y)^2}}$$

$$= \frac{10 \times 178 - 1 \times -3}{\sqrt{10 \times 203 - (1)^2}\sqrt{10 \times 159 - (-3)^2}}$$

$$= \frac{1780 + 3}{\sqrt{2030 - 1}\sqrt{1590 - 9}}$$

$$= \frac{1783}{\sqrt{2029}\sqrt{1581}} = \frac{1783}{45.04 \times 39.76}$$

$$= \frac{1783}{1790.79} = 0.995$$

Co-variance method

EXAMPLE 14

Find correlation coefficient from the following data:

$\sum x = 56,\ \sum y = 40,\ \sum x^2 = 524,\ \sum y^2 = 256,\ \sum xy = 364,\ n = 8.$

Solution:

$$\text{cov}(x, y) = \frac{\sum xy}{n} - \frac{\sum x}{n}.\frac{\sum y}{n}$$

$$= \frac{364}{8} - \frac{56}{8}.\frac{40}{8}$$

$$= 45.5 - 7 \times 5$$

$$= 45.5 - 35$$

$$= 10.5$$

$$\sigma_x = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2} = \sqrt{\frac{524}{8} - \left(\frac{56}{8}\right)^2}$$

$$= \sqrt{65.5 - 49} = \sqrt{16.5} = 4.06$$

$$\sigma_y = \sqrt{\frac{\sum y^2}{n} - \left(\frac{\sum y}{n}\right)^2} = \sqrt{\frac{256}{8} - \left(\frac{40}{8}\right)^2}$$

$$= \sqrt{32 - 25} = \sqrt{7} = 2.645$$

$$\therefore \quad r_{xy} = \frac{\text{cov}(x, y)}{\sigma_x.\sigma_y} = \frac{10.5}{4.06 \times 2.645}$$

$$= \frac{10.5}{10.74} = 0.977.$$

EXAMPLE 15

In order to find the correlation coefficient between two variables x and y from 12 pairs of observations, the following calculation were made:

$\Sigma x = 30$, $\Sigma y = 5$, $\Sigma x^2 = 670$, $\Sigma y^2 = 285$, and $\Sigma xy = -334$.

On subsequent verification, it was found that the pair ($x = 11$, $y = 4$) was copied wrongly, the correct value being ($x = 10$, $y = 14$). Find the correct value of correlation coefficient. [***C.U. B.Com. 2001, 2011***]

Solution: Corrected $\Sigma x = 30 - 11 + 10 = 29$

Corrected $\Sigma y = 5 - 4 + 14 = 15$

Corrected $\Sigma x^2 = 670 - 11^2 + 10^2 = 649$

Corrected $\Sigma y^2 = 285 - 4^2 + 14^2 = 465$

Corrected $\Sigma xy = -334 - 11 \times 4 + 10 \times 14$

$= -334 - 44 + 140 = -238$

$n = 12$ (given).

Putting the corrected different values in the formula, we get

$$r = \frac{n\sum xy - \sum x \times \sum y}{\sqrt{[n\sum x^2 - (\sum x)^2][n\sum y^2 - (\sum y)^2]}}$$

$$= \frac{12 \times -238 - 29 \times 15}{\sqrt{[12 \times 649 - (29)^2][12 \times 465 - (15)^2]}}$$

$$= \frac{-2856 - 435}{\sqrt{(7788 - 841)(5580 - 225)}} = \frac{-3291}{\sqrt{6947 \times 5355}}$$

$$= \frac{-3291}{\sqrt{6947}\sqrt{5355}} = \frac{-3291}{83.35 \times 73.18} = \frac{-3291}{6099.55}$$

$$= -0.5395 .$$

EXAMPLE 16

A computer while calculating the correlation coefficient between two variables x and y from 25 pairs of observations obtained the following result: $n = 25$, $\Sigma x = 125$, $\Sigma y = 100$, $\Sigma x^2 = 650$, $\Sigma y^2 = 460$ and $\Sigma xy = 508$. It was, however, discovered at the time of checking that two pairs of observations were not correctly copied. They were taken as (6,14) and (8,6), while the correct values were (8,12) and (6,8). Prove that the correct value of the correlation coefficient should be $\frac{2}{3}$.

[***C.U. B.Com. 1990, 2000***]

Solution: Corrected $\Sigma x = 125 - (6 + 8) + (8 + 6) = 125$

Corrected $\Sigma y = 100 - (14 + 6) + (12 + 8) = 100$

Corrected $\Sigma x^2 = 650 - (6^2 + 8^2) + (8^2 + 6^2) = 650$

Corrected $\Sigma y^2 = 460 - (14^2 + 6^2) + (12^2 + 8^2)$

$= 460 - 232 + 208 = 436$

$$\text{Corrected } \Sigma xy = 508 - (6 \times 14 + 8 \times 6) + (8 \times 12 + 6 \times 8)$$
$$= 508 - 132 + 144 = 520$$

Therefore, correct value of the correlation coefficient

$$r = \frac{n\sum xy - \sum x \times \sum y}{\sqrt{n\sum x^2 - (\sum x)^2}\sqrt{n\sum y^2 - (\sum y)^2}}$$

$$= \frac{25 \times 520 - 125 \times 100}{\sqrt{25 \times 650 - (125)^2}\sqrt{25 \times 436 - (100)^2}}$$

$$= \frac{13000 - 12500}{\sqrt{16250 - 15625}\sqrt{10900 - 10000}} = \frac{500}{\sqrt{625}\sqrt{900}} = \frac{500}{25 \times 30} = \frac{500}{750} = \frac{2}{3}$$

EXAMPLE 17

In studying a sets of pairs of related variates, the following results are obtained:
$N = 100, \sum x_i = 57$, $\sum y_i = 43$, $\Sigma(x_i - \bar{x})^2 = 169$, $\Sigma(y_i - \bar{y})^2 = 64$, $\Sigma(x_i - \bar{x})(y_i - \bar{y}) = 101$.

Find the correlation coefficient. [***C.U. B.Com. 1984***]

Solution: From the definition of 'r'. we get

$$r = \frac{\text{cov}(x, y)}{\sigma_x . \sigma_y}$$

$$= \frac{\frac{1}{n}\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}}\sqrt{\frac{\Sigma(y_i - \bar{y})^2}{n}}}$$

$$= \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\Sigma(x_i - \bar{x})^2}\sqrt{\Sigma(y_i - \bar{y})^2}} = \frac{101}{\sqrt{169}\sqrt{64}}$$

$$= \frac{101}{13 \times 8} = \frac{101}{104} = 0.97.$$

EXAMPLE 18

Find Karl Pearson's coefficient of correlation between age and the playing habits of the people from the following information:

Age group (years)	No. of people	No. of players
15 and less then 20	200	150
20 and less then 25	270	162
25 and less then 30	340	170
30 and less then 35	360	180
35 and less then 40	400	180
40 and less then 45	300	120

Also mention what does your calculated r indicate.

Solution: We take the mid value of age group as x and the percentage of players in respective groups as y. We get the following values of x and y.

X :	17.5	22.5	27.5	32.5	37.5	42.5
Y :	75	60	50	50	45	40

Calculation of coefficient of correlation

X	Y	$x = \frac{X - 27.5}{5}$	$y = \frac{Y - 50}{5}$	x^2	y^2	xy
17.5	75	–2	5	4	25	–10
22.5	60	–1	2	1	4	–2
27.5	50	0	0	0	0	0
32.5	50	1	0	1	0	0
37.5	45	2	–1	4	1	–2
42.5	40	3	–2	9	4	–6
—	—	$\Sigma x = 3$	$\Sigma y = 4$	$\Sigma x^2 = 19$	$\Sigma y^2 = 34$	$\Sigma xy = -20$

$$r = \frac{n\sum xy - \sum x \times \sum y}{\sqrt{n\sum x^2 - (\sum x)^2}\sqrt{n\sum y^2 - (\sum y)^2}}$$

$$= \frac{6 \times -20 - 3 \times 4}{\sqrt{6 \times 19 - (3)^2}\sqrt{6 \times 34 - (4)^2}} = \frac{-120 - 12}{\sqrt{114 - 9}\sqrt{204 - 16}}$$

$$= \frac{-132}{\sqrt{105}\sqrt{188}} = \frac{-132}{10.25 \times 13.71} = \frac{-132}{140.53}$$

$= -0.939 = -0.94$ (Approx.)

The value of 'r' indicates that there is a negative relationship between age and playing habits of people, i.e. with the increase in age the playing habit of people decreases sharply.

Rank Correlation

EXAMPLE 19

Ten students got the following marks in Economics and Statistics:

Students :	1	2	3	4	5	6	7	8	9	10
Marks in Statistics :	78	36	98	25	75	82	90	62	65	39
Marks in Economics :	84	51	91	60	68	62	86	58	53	47

Find the rank correlation coefficient. [***C.U. B.Com. 2010***]

Solution: **Calculation of rank correlation coefficient**

Students	Marks in Statistics (x)	Marks in Economics (y)	Rank in Statistics (r_1)	Rank in Economics (r_2)	$d=r_1-r_2$	d^2
1	78	84	4	3	1	1
2	36	51	9	9	0	0
3	98	91	1	1	0	0
4	25	60	10	6	4	16
5	75	68	5	4	1	1
6	82	62	3	5	−2	4
7	90	86	2	2	0	0
8	62	58	7	7	0	0
9	65	53	6	8	−2	4
10	39	47	8	10	−2	4
$n=10$					$\Sigma d = 0$	$\Sigma d^2 = 30$

$$R = 1 - \frac{6\sum d^2}{n^3 - n}$$

$$= 1 - \frac{6 \times 30}{10^3 - 10} = 1 - \frac{180}{990}$$

$$= 1 - 0.18 = 0.82$$

EXAMPLE 20

The coefficient of Rank correlation of the marks obtained by 10 students in English and Mathematics was found to be 0.5. It was later found that the difference in ranks in the two subjects obtained by one of the students was wrongly token as 3 instead of 7. Find the correct Rank correlation coefficient.

[***C.U. B.Com. 2013(H), 2015(H)***]

Solution: Given $R = 0.5$ and $n = 10$

We know that,

$$R = 1 - \frac{6\sum d^2}{n^3 - n}$$

or, $0.5 = 1 - \frac{6\sum d^2}{10^3 - 10}$ or $\frac{6\sum d^2}{990} = 1 - 0.5$

or, $6\sum d^2 = 990 \times 0.5 = 495$

or, $\sum d^2 = \frac{495}{6} = 82.5$

By question, it is the incorrect value of Σd^2,

the corrected value of $\Sigma d^2 = 82.5 - 3^2 + 7^2$

$$= 82.5 - 9 + 49 = 122.5$$

Therefore, corrected rank correlation coefficient (R_1)

$$R_1 = 1 - \frac{6 \times \text{corrected} \sum d^2}{n^3 - n} = 1 - \frac{6 \times 122.5}{10^3 - 10}$$

$$= 1 - \frac{735}{990} = 1 - 0.74 = 0.26 \text{ (Approx.)}$$

Association of Attributes

EXAMPLE 21

A survey was conducted in respect of marital status and success in examinations. Out of 2000 persons who appeared for an examination, 80% of them were boys, and the rest were girls. Among 300 married boys, 140 were successful, while 1,100 boys were successful among unmarried boys. In respect of 100 married girls 40 were successful, while 200 unmarried girls were successful. Construct two separate nine-square tables and discuss the association between marital status and passing of examination.

Solution: Let A denote married boys

$\therefore$ α would denote unmarried boys

Let B denote those who were successful

$\therefore$ β would denote those who were unsuccessful

In respect of boys, the following information is given N = 1,600, (A) = 300, (AB) = 140, (αB) = 1,100

For finding missing values let us prepare a nine-square table

	A	**α**	**Total**
B	140	1100	1240
β	160	200	360
Total	300	1300	1600

$$\text{Expectation of } (AB) = \frac{(A) \times (B)}{N}$$

$$= \frac{300 \times 1240}{1600}$$

$$= 232.5$$

The actual observation (AB) = 140

As the actual observation (140) is less than the expectation (232.5).

Then, the attributes marriage and success in the examination are negatively associated.

In respect of girls, the following information is given:
$N = 400$, $(AB) = 40$, $(A) = 100$, $(\alpha B) = 200$
For finding missing values let us prepare a nine-square table

	A	**α**	**Total**
B	40	200	240
β	60	100	160
Total	100	300	400

$$\text{Expectation of } (AB) = \frac{(A) \times (B)}{N}$$
$$= \frac{100 \times 240}{400}$$
$$= 60$$

The actual observation $(AB) = 40$

As the actual observation (40) is less than the expectation (60), then, the attributes marriage and success in the examination are negatively associated.

EXAMPLE 22

Out of 3,000 unskilled workers of a factory, 2000 come from rural areas and out of 1200 skilled workers, 300 come from rural areas. Determine the association between skill and residence by the method of proportions.

Solution: Let A denote skilled workers
∴ α would denote unskilled workers
Let B denote workers from rural areas
∴ β would denote workers from urban areas

Given: $(A) = 1200$, $(\alpha) = 3000$, (B), $= 2000$, $(\alpha B) = 2000$, and $(AB) = 300$

$$\text{Now, } \frac{(AB)}{(A)} = \frac{300}{1200} = 0.25$$

$$\text{and } \frac{(\alpha B)}{(\alpha)} = \frac{2000}{3000} = 0.67$$

As $\frac{(AB)}{(A)} < \frac{(\alpha B)}{(\alpha)}$, then there is negative association between skill and residence.

EXAMPLE 23

A teacher examined 280 students in Economics and Auditing and found that 160 failed in Economics, 140 failed in Auditing and 80 failed in both the subjects. Is there any association between failure in Economics and Auditing?

Solution: Let A denote students who failed in Economics
∴ α would denote students who passed in Economics
Let B denote students who failed in Auditing
∴ ß would denote students who passed in Auditing

Putting the given information in a nine-square table we get

	A	**α**	**Total**
B	80	60	140
β	80	60	140
Total	160	120	280

Now, $Q = \dfrac{(AB)(\alpha\beta) - (A\beta)(\alpha B)}{(AB)(\alpha\beta) + (A\beta)(\alpha B)}$

$$= \frac{(80 \times 60) - (80 \times 60)}{(80 \times 60) + (80 \times 60)}$$

$$= \frac{4800 - 4800}{4800 + 4800} = 0$$

As Yule's coefficient of association is zero, then there is no association between failure in Economics and Auditing.

EXAMPLE 24

In an experiment on immunization of cattle from tuberculosis, the following results were obtained:

	Died or affected	Unaffected
Inoculated	12	26
Not inoculated	16	6

Examine the effect of vaccine in controlling susceptibility to tuberculosis.

Solution:

Let A denote inoculation
and B denote died or affected

For finding missing value let us prepare the following nine square table

	A	**α**	**Total**
B	12	26	38
β	26	6	32
Total	38	32	70

Yule's coefficient of association

$$Q = \frac{(AB)(\alpha\beta) - (A\beta)(\alpha B)}{(AB)(\alpha\beta) + (A\beta)(\alpha B)}$$

$$= \frac{12 \times 6 - 26 \times 16}{12 \times 6 + 26 \times 16}$$

$$= \frac{72 - 416}{72 + 416} = \frac{-344}{488}$$

$$= -0.7$$

As Yule's coefficient of association in negative, then there is a negative association between vaccine and susceptibility to tuberculosis, i.e. the vaccine prevents the attack of tuberculosis to a very considerable extent.

EXAMPLE 25

The male population of a certain state in India is 331 lakhs. The number of literate males is 66 lakhs; the number of male criminals is 33 thousands, and the number of literate male criminals is 6 thousands. Do you find any association between literacy and criminality?

Solution: Let A denote literate
$\therefore$ B denote criminals

Then in lakhs, $(A) = 66$, $(N) = 331$, $(B) = 0.33$ and $(AB) = 0.06$

Now, Expectation of $(AB) = \dfrac{(A) \times (B)}{N}$

$$= \frac{66 \times 0.33}{331} = 0.066$$

The actual observation $(AB) = 0.06$

As the actual observation is less than the expectation.

Then literacy and criminality are negatively associated, i.e literacy checks criminality.

EXERCISE

A. THEORY

1. Define correlation. What is meant by positive and negative correlation? [***C.U. B.Com. 1982***]
2. What do you mean by the coefficient of correlation? Show that its value lies between −1 and +1. [***V.U. B.Com. 1997***]

3. What do you mean by bivariate data?
4. Define scatter diagram. Give the scatter diagram when the product moment correlation coefficient *r* is zero and ±1. [***C.U. B.Com. 1997***]
5. Write short notes on:
   (*a*) coefficient of correlation [***C.U. B.Com. 1982***]
   (*b*) bivariate data
   (*c*) Rank correlation coefficient
   (*d*) Scatter diagram
6. Define Rank correlation. Write Spearman's formula for rank correlation coefficient *R*. What are the limits of *R*? Interpret in case when *R* assumes the minimum value. [***C.U. B.Com. 1983***]
7. What do you mean by "Association of Attributes". State the different methods by which it is measured.
8. Distinguish between concept of association and correlation.
9. Define 'Association' and 'Disassociation' between two attributes.
10. When are two attributes said to be
    (*i*) positively associated
    (*ii*) negatively associated
    (*iii*) independent

B. PRACTICAL/SHORT TYPE

Correlation

1. Give the expression for the measurement of correlation coefficient.
   [**Ans.** $\dfrac{\text{Cov}(x, y)}{\sigma_x \cdot \sigma_y}$]
2. Given cov(X, Y) = −30, $\sigma_x = 6$, $\sigma_y = 10$; find the coefficient of correlation between *X* and Y. [***C.U. B.Com. 1985***] [**Ans.** – 0.5]
3. Find the correlation coefficient from the following data:
   $\sigma_x^2 = 2.25$; $\sigma_y^2 = 1$ and cov(x, y) = 0.9. [***C.U. B.Com. 1988***] [**Ans.** + 0.6]
4. If covariance between *X* and *Y* variables is 10 and the variances of *X* and *Y* are respectively 16 and 9, find the coefficient of correlation. [**Ans.** 0.833]
5. Karl Pearson's coefficient of correlation between two variables *X* and *Y* is 0.46, their covariance is +3.68. If the variance of *X* is 16, find the standard deviation of Y. [**Ans.** 2]
6. Coefficient of correlation between two variables *X* and *Y* is 0.48. Their co-variance is 36. The variance of *X* is 16, find standard deviation of *Y* series. [**Ans.** 18.7]

7. If r = 0.6, cov(X,Y) = 12 and σ_x =5, find the value of σ_y
[*C.U. B.Com. 1982*] [**Ans.** 4]

8. Karl Pearson's coefficient of correlation between two variables *x* and *y* is 0.52, their covariance is +78. If the variance of *x* is 16, find the standard deviation of *y*. [*C.U. B.Com. 1983*] [**Ans.** 5]

9. If r = 0.4, cov(x, y) =10 and σ_y = 5, then find the value of σ_x
[*C.U. B.Com. 1995*] [**Ans.** 5]

10. (i) If r = 6, cov(x, y) = 12 and σ_y = 5, find the value of σ_x.
[*C.U. B.Com. 1998*] [*V.U. B.Com. '89, '94, '98*]

    (ii) Find the correlation coefficient if $\sigma_x^2 = 6.25$, $\sigma_y^2 = 4$, cov(x, y) = 0.9
[*C.U. B.Com. 2000*] [**Ans.** (i) 4; (ii) 0.18]

11. In studying a set of pairs of related variates, the following results are obtained. N = 10, $\Sigma x_i = 5.8$, $\Sigma y_i = 4.3$, $\Sigma(x_i - \bar{x})^2 = 144$, $\Sigma(y_i - \bar{y})^2 = 49$; $\Sigma(x_i - \bar{x})(y_i - \bar{y}) = 77$. Find the correlation coefficient. [**Ans.** 0.92]

12. Karl Pearson's coefficient of correlation between two variables *x* and *y* is 0.28, their covariance is +7.6. If the variance of *x* is 9, find the standard deviation of y-series. [**Ans.** 9.048]

13. if $x = X - \bar{X}$, $y = Y - \bar{Y}$, and $\Sigma x^2 = 25$, $\Sigma y^2 = 64$, $\Sigma xy = 18$, then find the value of correlation coefficient between *x* and y. [**Ans.** 0.45]

14. If $x = X - \bar{X}$, $y = Y - \bar{Y}$, and $\Sigma x^2 = 10$, $\Sigma y^2 = 24$, $\Sigma xy = 12$, then find the coefficient correlation between the two variables *x* and y.
[*C.U. B.Com. 1992*] [**Ans.** 0.77]

15. Find the correlation coefficient between *x* and *y* when variance of *x* = 2.25, variance of *y* is 1 and covariance of *x* and *y* = 0.9.
[*C.U. B.Com. 1999*] [**Ans.** 0.60]

16. Given r = 0.8, $\Sigma xy = 80$, $\sigma_x = 2$, $\Sigma y^2 = 100$, where $x = X - \bar{X}$ and $y = Y - \bar{Y}$; find the number of items. [**Ans.** n = 25]

17. If the coefficient of correlation is zero, what should be the nature of correlation?
[**Ans.** no relation]

18. If the correlation is perfect then what is the value of r? [**Ans.** 1]

Rank Correlation

19. If $\Sigma D^2 = 33$ and N = 10; find the value of the coefficient of rank correlation, where D represents the difference between the ranks of two series and N is the number of pairs of observations. [*C.U. B.Com. 1985*] [**Ans.** +0.8]

20. Write the formula of Edward Spearman for the determination of rank correlation coefficient.
[*C.U. B.Com. 1997*] [**Ans.** $R = 1 - \frac{6\sum d^2}{N(N^2-1)}$]

21. If $\sum D^2 = 45$ and $N = 10$; find the value of the coefficient of rank correlation, where D represents the difference between the ranks of two series and N is the number of pairs of observations. [**Ans.** 0.73]

Association of Attributes

22. Test for consistency of the data given below:
    (i) (A) = 100, (B) = 150, (AB) = 60, N = 500 [**Ans.** Consistent]
    (ii) (A) = 300, (B) = 240, (AB) = 60, N = 400 [**Ans.** Inconsistent]
23. Find if *A* and *B* are independent, positively associated or negatively associated from the data given below:
    (A) = 470, (B) = 620, (AB) = 320, N = 1000 [**Ans.** Positively associated]
24. Find whether *A* and *B* are independent in the following case:
    (AB) = 128, (αB) = 384, (A) = 152, (α) = 456 [**Ans.** Independent]
25. From the following data compute Yule's coefficient of association.
    (AB) = 96, (Aβ) = 224, (αβ) = 336, (αB) = 144 [**Ans.** 0]

C. SHORT ESSAY/PROBLEM TYPE

Correlation

1. Given the bivariate data:

x :	1	5	3	2	1	1	7	3
y :	6	1	0	0	1	2	1	5

   Calculate Karl Pearsons correlation coefficient. [**Ans.** –0.29]

2. Calculate the coefficient of correlation from the following data:

x :	1	2	3	4	5
y :	3	2	5	4	6

   [***C.U. B.Com. 1984, '87***] [***K.U. B.Com. '99***] [**Ans.** 0.8]

3. Find the coefficient correlation from the following data:

x :	1	2	3	4	5
y :	6	8	11	8	12

   [***C.U. B.Com. 1991, '95***] [***K.U. B.Com. '98***] [**Ans.** 0.77]

4. Find out the coefficient of correlation between X and Y.

X :	2	2	4	5	5
Y :	6	3	2	6	4

   [**Ans.** 0.04]

5. Calculate the correlation coefficient from the following data:

x :	1	3	5	7	9	11
y :	4	3	6	10	2	8

   [**Ans.** 0.36]

6. Compute the correlation coefficient between corresponding values of X and Y in the following table:

X :	2	4	5	6	8	11
Y :	18	12	10	8	7	5

[*C.U. B.Com. (Hons.) '90*] [**Ans.** –0.92]

7. Calculate the coefficient correlation of the following data:

X :	2	5	7	10	19	17
Y :	26	29	26	30	34	35

[**Ans.** 0.92]

8. Calculate the coefficient correlation of the following data:

X :	63	60	67	61	69	70
Y :	61	65	64	63	68	63

[*C.U. B.Com. 1988*] [**Ans.** 0.32]

9. Find the coefficient of correlation form the following data:

x :	10	12	13	16	17	20	25
y :	19	22	24	27	29	33	37

[*C.U. B.Com. 1989*] [**Ans.** 0.99]

10. What is the scatter diagram? Calculate the correlation coefficient of the following data:

x :	11	12	13	14	18	15
y :	13	12	15	14	12	11

[*C.U. B.Com. 1997*] [**Ans.** 0.34]

11. Find the correlation coefficient between x and y from the following data:

x :	45	55	56	60	65	68	70
y :	56	50	48	62	64	65	65

[*C.U. B.Com. 1999*] [**Ans.** 0.74]

12. Calculate Karl Pearson's correlation coefficient of the following paired data:

x :	28	37	40	38	35	33	40	32	34	33
y :	23	32	33	34	30	26	29	31	34	38

[**Ans.** 0.406]

13. Calculate Karl Pearson's coefficient of correlation from the following data:

Roll No. of students :	1	2	3	4	5
Marks in Accountancy :	48	35	17	23	47
Marks in Statistics :	45	20	40	25	45

[**Ans.** 0.429]

14. Compute the Karl Pearson's correlation coefficient between the corresponding values of X and Y in the following table:

X :	3	5	7	8	9	15	16
Y :	15	18	22	24	19	25	31

[*C.U. B.Com. '80*] [**Ans.** 0.89]

15. Obtain the correlation coefficient from the following:

X :	6	2	10	4	8
Y :	9	11	5	8	7

[*D. S. W. '77*] [**Ans.** –0.92]

16. The following table gives the relative values of two variables:

X :	42	44	58	50	90	88	60
Y :	56	40	50	52	60	70	58

Calculate Karl Pearson's coefficient of correlation. [**Ans.** 0.717]

17. In studying a set of pairs of related variates, the following results are obtained:

$N = 100, \Sigma x_i = 57, \Sigma y_i = 43, \Sigma(x_i - \bar{x})^2 = 169, \Sigma(y_i - \bar{y})^2 = 64,$
$\Sigma(x_i - \bar{x})(y_i - \bar{y}) = 101$. Find the correlation coefficient.

[***C.U. B.Com. (Hons), '84***] [**Ans.** 0.97]

18. Calculate correlation coefficient from the following results:

$n = 100, \Sigma x = 140, \Sigma y = 150, \Sigma(x-10)^2 = 180, \Sigma(y-15)^2 = 215,$
$\Sigma(x-10)(y-15) = 60.$

[***C. A. May '85***] [**Ans.** 0.915]

19. Given $\sum_{i=1}^{11}(x_i - \bar{x})^2 = 100, \sum_{i=1}^{11}(y_i - \bar{y})^2 = 990;$

$\sum_{i=1}^{11}(x_i - \bar{x})(y_i - \bar{y}) = 330.$

Find the value of correlation coefficient. [**Ans.** 1]

20. Calculate correlation coefficient from the following data:

$N = 10, \Sigma x = 90, \Sigma y = 100, \Sigma(x-5)^2 = 170, \Sigma(y-10)^2 = 250$

$\Sigma(x-5)(y-10) = 40$ [**Ans.** 0.8]

21. Calculate r from the following given results:

$n = 10; \Sigma x = 125; \Sigma x^2 = 1585; \Sigma y = 80; \Sigma y^2 = 650; \Sigma xy = 1007.$ [**Ans.** 0.47]

22. Given $\Sigma x = 56, \Sigma y = 40, \Sigma x^2 = 524, \Sigma y^2 = 256, \Sigma xy = 364, n = 8.$

Find the correlation coefficient. [**Ans.** 0.98]

23. From the following data, find the correlation coefficient.

$\sum_{i=1}^{25} X_i = 125, \sum_{i=1}^{25} Y_i = 100, \sum_{i=1}^{25} X^2_i = 650, \sum_{i=1}^{25} Y^2_i = 460$, and $\sum_{i=1}^{25} X_i Y_i = 508,$

[**Ans.** 0.207]

24. From the data given below find the number of items, i.e., n. $r = 0.5, \Sigma xy = 120,$ $\sigma_y = 8, \Sigma_x^2 = 90.$

[**Ans.** 10]

[Hints: $r = \dfrac{\Sigma xy}{\sqrt{\Sigma x^2 \cdot \Sigma y^2}}; \sigma_y = \sqrt{\dfrac{\Sigma y^2}{n}}$ or $\Sigma y^2 = n.\sigma_y^2$]

25. Given $r = 0.8, \Sigma xy = 60, \sigma_y = 2.5$ and $\Sigma x^2 = 90$, find the number of items. x and y are deviations from arithmetic mean. [**Ans.** $n = 10$]

26. Find the number of items from the following data:
$r = 0.95$, $\Sigma xy = 57$, $\Sigma x^2 = 60$, $\sigma_y^2 = 6\frac{2}{3}$.
Where x and y denote deviations of the variates X and Y from their respective means. [**Ans.** 9]

27. In order to find the correlation coefficient between two variables X and Y from 20 pairs of observations, the following calculation were made: $\Sigma X = 120$, $\Sigma Y = 80$, $\Sigma X^2 = 1440$, $\Sigma Y^2 = 650$, $\Sigma XY = 886$. It was found later on that the pair ($X = 10$, $Y = 5$) was copied wrongly, instead of the correct value ($X = 11$, $Y = 4$). Find the corrected value of correlation coefficient. [**Ans.** 0.821]

28. In order to find correlation coefficient between two variables X and Y from 12 pairs of observation, the following results were made: $\Sigma X = 30$, $\Sigma Y = 5$, $\Sigma X^2 = 670$, $\Sigma Y^2 = 285$, $\Sigma XY = 334$. On subsequent verification it was found that pair ($X = 11$, $Y = 4$) was copied wrongly, the correct value being ($X = 10$, $Y = 14$). Find the correct value of correlation coefficient. [**Ans.** 0.775]

29. In order to find the correlation coefficient between two variables x and y from 25 pairs of observations, the following calculations were made:

$n = 25$, $\Sigma x = 125$ $\Sigma x^2 = 650$

$\Sigma y = 100$, $\Sigma y^2 = 460$ $\Sigma xy = 508$

Subsequently is was found that the pairs ($x = 6$, $y = 8$) was copied wrongly the correct value being ($x = 8$, $y = 6$). Find the correct value of correlation coefficient. [***C.U. B.Com. 1990, 2000***] [**Ans.** 0.256]

30. Coefficient of correlation between X and Y for 20 items is 0.3; mean of X is 15 and that of Y is 20, standard deviations are 4 and 5 respectively. At the time of calculation one time 27 has wrongly been taken as 17 in case of X series and 35 instead of 30 in case of y series. Find the correct coefficient of correlation. [**Ans.** 0.504]

[**Hints**: correct $\overline{X} = 15.5$, $\overline{Y} = 19.75$, $\sum X^2 = 5260$, $\sigma_x = 4.77$,

$\sum Y^2 = 8175$, $\sigma_Y = 4.32$, $\sum XY = 207.875$]

31. While calculating the coefficient of correlation between two variables x and y, the following results were obtained; $x = 25$, $\Sigma x = 125$, $\Sigma y = 100$, $\Sigma x^2 = 650$, $\Sigma y^2 = 460$, $\Sigma xy = 508$. It was however later discovered at the time of checking that two pairs of observations (x, y) were copied (6, 14) and (8, 6), while the correct values were (8, 12) and (6, 8) respectively. Determine the correct value of the coefficient of correlation.
[***I.C.W.A. June '77, '78***] [**Ans.** 0.67]

Rank Correlation

32. Define Rank Correlation. Write the Spearman's formula for determining rank correlation coefficient R. For two series, we have $\sum D^2 = 30$ and $N = 10$; find R.

(D is the difference of ranks of the two series. N = number of individuals in each series) [*C.U. B.Com. 1993*] [**Ans.** 0.818]

33. Two ladies were asked to rank 7 different types of lipsticks. The ranks given by them are given below:

Lipsticks	A	B	C	D	E	F	G
Neelu	2	1	4	3	5	7	6
Neena	1	3	2	4	5	6	7

Calculate Spearman's rank correlation coefficient. [**Ans.** 0.786]

34. Ten competitors in a beauty contest are ranked by two judges in the following order. Calculate Spearman's rank correlation coefficient.

Competitors :	A	B	C	D	E	F	G	H	I	J
1^{st} Judge :	1	6	5	10	3	2	4	9	7	8
2^{nd} Judge :	6	4	9	78	1	2	3	10	5	7

[*C.U. B.Com. 1996*] [**Ans.** 0.64]

35. In a contest judges assessed the performances of eight candidates as shown below:

Candidates :	A	B	C	D	E	F	G	H
1^{st} Judge :	5	3	2	1	4	7	8	6
2^{nd} Judge :	4	2	1	3	5	8	6	7

Using Spearman's formula obtain the rank correlation coefficient.

[*C.U. B.Com. 1998*] [**Ans.** 0.83]

36. The rankings of ten students in Statistics and Economics are as follows:

Statistics :	3	5	8	4	7	10	2	1	6	9
Economics :	6	4	9	8	1	2	3	10	5	7

What is the coefficient of rank correlation? [**Ans.** –0.3]

37. Twelve pictures submitted in a competition were ranked by two judges with results as shown below:

Pictures :	1	2	3	4	5	6	7	8	9	10	11	12
Rank by 1^{st} Judge :	5	9	6	7	1	3	4	12	2	11	10	8
Rank by 2^{nd} Judge :	5	8	9	11	3	1	2	10	4	12	7	6

Calculate the rank correlation coefficient. [**Ans.** 0.791]

38. Find the Rank correlation coefficient of the following data of marks obtained by 10 students in Mathematics and Statistics:

Student (Roll No.) :	1	2	3	4	5	6	7	8	9	10
Marks in Mathematics :	80	38	95	30	74	84	91	60	66	40
Marks in Statistics :	85	50	92	58	70	65	88	56	52	46

[*C.U. B.Com. 1986*] [**Ans.** 0.84]

39. In a certain examination 10 students obtained the following marks in Mathematics and Physics. Find the Spearman's rank correlation coefficient:

Student (Roll no)	:	1	2	3	4	5	6	7	8	9	10
Marks in Mathematics	:	90	30	82	45	32	65	40	88	73	66
Marks in Physics	:	85	42	75	68	45	63	60	90	62	58

[***C.U. B.Com. 1993***] [**Ans.** 0.84]

40. In a contest, two judges assessed the performances of eight candidates A, B, C, D, E, F, G and H as shown in the following table:

		A	B	C	D	E	F	G	H
First Judge	:	52	67	40	72	55	48	60	43
Second Judge	:	52	51	43	54	56	40	60	49

Find the rank correlation of the above data applying Spearman's method.

[***C.U. B.Com. 1992***] [**Ans.** 0.67]

41. Compute Rank correlation from the following table of Index Numbers of supply (X) and price (Y):

X	:	115	134	120	130	124	128
Y	:	130	132	128	130	127	125

[**Ans.** 0.33]

42. The following are the marks obtained by a group of 10 students in Economics and Statistics:

Students	:	1	2	3	4	5	6	7	8	9	10
Marks in Economics	:	8	36	98	25	75	82	90	62	65	39
Marks in Statistics	:	84	51	91	60	68	62	86	58	53	47

[**Ans.** 0.58]

Equal Ranks

43. Calculate the rank coefficient of correlation of the following data:

X	:	80	78	75	75	68	67	60	59
Y	:	12	13	14	14	14	16	15	17

[**Ans.** –0.929]

44. In the following table are recorded data showing the test scores made by 10 salesmen on an intelligence test and their weekly sales:

Salesmen	:	1	2	3	4	5	6	7	8	9	10
Test scores	:	50	70	50	60	80	50	90	50	60	60
Sales ('000Rs)	:	25	60	45	50	45	20	55	30	45	30

Calculate the rank correlation coefficient between intelligence and efficiency in salesmanship. [**Ans.** 0.70]

45. Calculate the rank coefficient of correlation of the following data;

X :	29	32	45	67	56	36	45
Y :	56	40	52	68	40	72	40

[**Ans.** –0.036]

46. The students obtained the following marks in Mathematics and Statistics. Calculate the rank correlation coefficient.

Roll No.	1	2	3	4	5	6	7	8	9	10
Marks in Mathematics:	78	52	48	68	52	25	90	52	48	69
Marks in Statistics:	68	42	60	58	42	30	78	42	58	61

[**Ans.** 0.71]

47. The coefficient of rank correlation between marks in Statistics and marks in Accountancy obtained by a certain group of students is 0.8. If the sum of the squares of the differences in ranks is given to be 33, find the number of students in the group.
[***Bombay University, B.Com***] [**Ans.** 10]

Hints: $0.8 = 1 - \dfrac{6 \times 33}{N^3 - N}$ or, $N^3 - N - 990 = 0$

or, $N^3 - (10)^3 - (N-10) = 0$ or, $(N-10)\left(N^2 + 99 + 10N\right) = 0$ Now proceed]

48. The coefficient of rank correlation of the marks obtained by 10 students in Mathematics and Statistics was found to be 0.5. It was then detected that the difference in ranks in the two subjects for one particular student was wrongly taken to be 3 in place of 7. What should be the correct rank correlation coefficient? [***C.U. B.Com. (Hons.) '83***] [**Ans.** 0.26]

49. The coefficient of rank correlation between the marks in Statistics and Mathematics obtained by a certain group of students is $\frac{2}{3}$ and the sum of the squares of the differences in ranks is 55. Find the number of students in the group.

[**Ans.** 10]

50. The coefficient of rank correlation of marks obtained by 10 students in Statistics and Accountancy was found to be 0.2. It was later discovered that the difference in ranks in the two subjects obtained by one of the students was wrongly taken as 9 instead of 7. Find the correct value of the coefficient of rank correlation. [***D.U. B.Com. (Hons.) '92***] [**Ans.** 0.394]

Association of Attributes

51. The following information in case of two attributes is given below:

(AB) = 400, (A) = 800, N = 2500 , (B) = 1600

Name the remaining classes and find out their frequencies.

[**Ans.** (Aβ)= 400, (αB) = 1200, (αβ) = 500, (α) = 1700 and (β) = 900]

52. Test the consistency of the data given below:

    (i) (AB) = 200, (A) = 300, (α) = 200, (B) = 250, (αβ) = 150, (N) = 500
    (ii) (AB) = 250, (A) = 150, (αB) = 1000, (αβ) = 600, (β) = 500, (N) = 1750

    [**Ans.** (i) consistent (ii) inconsistent]

53. Explain from the data given below whether *A* and *B* are independent, positively associated or negatively associated:

    (i) N = 10000, (A) = 500, (B) = 6000, (AB) = 3150

    (ii) (A) = 150, (α) = 250, (B) = 200, (αB) = 125

    [**Ans.** (i) positively associated (ii) independent]

54. From the following ultimate class frequencies find the frequencies of the positive and negative classes and the total number of observations.

    (AB) = 100, (αB) = 80, (Aβ) = 50, (αβ) = 40

    [**Ans.** N = 270, (A) = 150, (B) = 180, (α) = 120, (β) = 90]

55. 88 residents of an Indian city, who were interviewed during a sample survey, are classified below according to their smoking and tea drinking habits. Calculate Yule's coefficient of association and comment on its value.

	Smokers	Non-smokers
Tea Drinkers	40	33
Non-tea Drinkers	3	12

[**Ans.** Q = 0.658]

56. Calculate Yule's coefficient of association between marriage and failure of students from the following data pertaining to 500 students:

	Passed	Failed	Total
Married	90	65	155
Unmarried	260	110	370

[**Ans.** Q = +0.261]

57. The following table is reproduced from a memoir written by Karl Pearson:

		Eye colour in son	
		Not light	Light
Eye colour in father	Not light	230	148
	Light	151	471

Discuss whether the colour of the son's eye is associated with that of the father.

[**Ans.** Q = 0.65]

58. The following table gives the condition of home and the health condition of a child in 180 homes. Is there any association between the two?

		Condition of home		
		Clean	Not clean	Total
Health condition of child	Good	70	30	100
	Bad	20	60	80
	Total	90	90	180

[**Ans.** Q = 0.75]

59. From the following data show that the attributes α and β are independent;

(α) = 60, (β) = 125, (αβ) = 20, N = 375 [*C.U. B.Com. 2014(G)*]

60. 176 people in an industrial complex who were interviewed during a sample survey are classified according to habits of watching TV and reading books. Calculate Yule's coefficient of association and comment on its value.

	Book reader	Non-reader
T.V. Watcher	80	66
Non-TV	6	24

[*C.U. B.Com. 2016(H)*] [**Ans.** Q = 0.658, positively associated]

Miscellaneous

61. Find rank correlation coefficient from the following marks obtained by 8 students in two subjects Mathematics and Accountancy.

Student :	A	B	C	D	E	F	G	H
Mathematics :	50	76	52	72	80	44	67	35
Accountancy :	58	63	68	44	64	65	46	49

[*K.U.B.Com. '95*] [**Ans.** – 0.048]

62. The following table gives the distribution of production and also the relativity defective items among them, according to size-groups. Find the correlation coefficient between size and defect in quality and its probable error.

Size-group	**No. of items**	**No. of defective items**
15–16	200	150
16–17	270	162
17–18	340	170
18–19	360	180
19–20	400	180
20–21	300	114

[*D.U. B.Com. (Hons.) '89*] [**Ans.** –0.95; 0.027 Approx.]

[Hints: Size of group = X, % of defective items = Y.

X	**Y**	$d_x = x - 18.5$	$d_y = y - 50$	d_x^2	d_y^2	$d_x.d_y$
15.5	75	−3	+25	9	625	−75
16.5	60	−2	+10	4	100	−20
17.5	50	−1	0	1	0	0
18.5	50	0	0	0	0	0
19.5	45	+1	−5	1	25	−5
20.5	38	+2	−12	4	144	−24
		$\sum d_x = -3$	$\sum d_y = +18$	$\sum d_x^2 = 19$	$\sum d_y^2 = 894$	$\sum d_x d_y = -124$

Now calculate the value of 'r'.

And Probable error = $0.675\left(\frac{1-r^2}{\sqrt{N}}\right)$]

63. Given the following information:

$r = 0.8,\ \Sigma xy = 60,\ \sigma_y = 2.5,\ \Sigma x^2 = 90.$

Find the number of items. [*D.U. B.Com. (Hons.) '93*] [**Ans.** 10]

$$\left[\textbf{Hints}: r = \frac{\Sigma xy}{\sqrt{\Sigma x^2}\sqrt{\Sigma y^2}}, \sigma_y = \sqrt{\frac{\Sigma y^2}{N}} \text{ or, } \Sigma y^2 = 6.25N\right]$$

64. Find the correlation between the age and playing habit of the following students:

Age (yrs.) :	15	16	17	18	19	20
No. of Students :	250	200	150	120	100	80
Regular Players :	200	150	90	48	30	12

[*D.U. B.Com. (Hons.) '96*] [**Ans.** –0.9684]

65. The coefficient of rank correlation for certain data is found to be 0.6. If the sum of the squares of the differences is given to 66, find the number of items in the group. [**Ans.** 10]

65. 300 people of German and French nationalities were interviewed for finding their preference for music of their own language. The following facts were gathered out of 100 German nationals, 60 liked music of their own language, whereas 70 French nationals out of 200 liked German music. Out of 100 French nationals 55 liked music of their own language and 35 German nationals out of 200 Germans liked French music.

Using coefficient of association, state whether Germans prefer their own music in comparison with Frenchmen. [*C.A. May 1982*]

[**Ans.** Coefficient of association between German music and German national is 0.472 and between French music and French national is 0.704. Therefore, Frenchmen prefer their own music in comparison with Germans]

D. MULTIPLE CHOICE QUESTIONS (MCQs)

(i) Short Type

Correlation

Mark the correct alternative in each of the following:

1. If cov$(x,y) = -30$, $\sigma_x = 6$ and $\sigma_y = 10$, then the coefficient of correlation between x and y will be

(*a*) –0.75 (*b*) –0.50 (*c*) –0.45 (*d*) –0.35

[*C.U. B.Com. 1985*] [**Ans.** (*b*)]

2. If $\sigma_x^2 = 6.25$, $\sigma_y^2 = 4$ and $\text{cov}(x,y) = 0.9$, then the coefficient of correlation will be [***C.U. B.Com. 2000, 2012, 2015 (G)***] [**Ans.** (*a*)]

   (*a*) 0.18 (*b*) 0.17 (*c*) 0.19 (*d*) 0.20

3. Given $r_{xy} = 0.8$, if $u = x + 5$ and $v = y - 5$, then the value of r_{uv} will be

   (*a*) 0.85 (*b*) 0.7 (*c*) 0.75 (*d*) 0.8

   [***C.U. B.Com. 2011, 2014 (G)***] [**Ans.** (*d*)]

4. The correlation coefficient of two variables x and y is 0.6 and their covariance is 12. If the standard deviation of x be 5, the standard deviation of y will be

   (*a*) 2 (*b*) 5 (*c*) 4 (*d*) 3

   [***C.U. B.Com. 2011***] [**Ans.** (*c*)]

5. If $r_{xy} = 0.6$, $4 = 2x - 3$ and $v = -3y + 2$, then the value of r_{uv} will be

   (*a*) –0.55 (*b*) –0.6 (*c*) –0.5 (*d*) –0.45

   [***C.U. B.Com. 2014 (H)***] [**Ans.** (*b*)]

6. If $r = 0.4$, $\text{cov}(x, y) = 10$ and $\sigma_y = 5$, then the value of σ_x will be

   (*a*) 5 (*b*) 4 (*c*) 6 (*d*) 3

   [***C.U. B.Com. 2014 (G)***] [**Ans.** (*a*)]

7. Karl Pearson's coefficient of correlation between two variables x and y is 0.46, their covariance is 3.68; if the variance of x is 16, then the standard deviation of y will be

   (*a*) 3 (*b*) 4 (*c*) 2 (*d*) 5

   [***C.U. B.Com. 2013 (H)***] [**Ans.** (*c*)]

8. If $r_{xy} = 0.6$, $\sigma_x = 4$ and $\sigma_y = 5$, then the value of $\text{cov}(x,y)$ will be

   (*a*) 9 (*b*) 10 (*c*) 11 (*d*) 12

   [***C.U. B.Com. 2015 (H)***] [**Ans.** (*d*)]

9. If $x = X - \overline{X}$ and $y = Y - \overline{Y}$, and $\sum x^2 = 10$, $\sum y^2 = 24$, $\sum xy = 12$, then the coefficient of correlation between the two variables x and y will be

   (*a*) 0.77 (*b*) 0.82 (*c*) 0.85 (*d*) 0.71

   [***C.U. B.Com. 1992***] [**Ans.** (*a*)]

10. The correlation coefficient between x and y is 0.5 and $u = 2x + 11$ and $v = 3y + 7$, then the correlation coefficient between u and v will be

    (*a*) 0.3 (*b*) 0.5 (*c*) 0.6 (*d*) 0.4

    [***C.U. B.Com. 2008***] [**Ans.** (*b*)]

11. If $x = X - \overline{X}$, $y = Y - \overline{Y}$, where $\overline{X}$, $\overline{Y}$ being respectively arithmetic means of X and Y and if $\sum x^2 = 60$, $\sum xy = 57$, $r = 0.95$ and variance of $y = 6\frac{2}{3}$, then the value of n will be

    (*a*) 6 (*b*) 7 (*c*) 9 (*d*) 8

    [***C.U. B.Com. 2009***] [**Ans.** (*c*)]

12. From the following data the Karl Pearson coefficient of correlation is

x	:	6	8	10	7	10	7
y	:	12	10	8	12	8	10

(*a*) 0.97 (*c*) –0.93
(*b*) 0.85 (*d*) 0.65 [**Ans.** (*c*)]

13. From the following data the Karl Pearson coefficient of correlation is

x	:	11	15	15	12	15	10
y	:	18	13	11	15	11	16

(*a*) –0.99 (*c*) 0.80
(*b*) –0.89 (*d*) –0.50 [**Ans.** (*b*)]

14. Number of observations $n = 10$, mean of x =22, mean of $y = 15$, sum of squared deviations of x from mean value = 148, sum of squared deviations of y from mean value = 168, sum of multiplication of deviation of x and $y = 124$.

From the above details the coefficient of correlation will be
(*a*) 0.79 (*c*) 0.65
(*b*) 0.87 (*d*) 0.43 [**Ans.** A]

15. Sum of deviations of x from mean value = 8, sum of squared deviation of y from mean value = 54. Sum of multiplication of deviation of x and $y = 32$. Sum of squared deviations of x from mean value = 60.

From the above details the coefficient of correlation will be
(*a*) 0.58 (*c*) 0.61
(*b*) 0.56 (*d*) 0.47 [**Ans.** (*b*)]

16. If the coefficient of correlation is 0.8, the coefficient of determination will be
(*a*) 0.98 (*c*) 0.66
(*b*) 0.64 (*d*) 0.54 [**Ans.** (*b*)]

[**Hints**: coefficient of determination = (coefficient of correlation)^2]

17. If the coefficient of determination is 0.49, what is the coefficient of correlation.
(*a*) 0.7 (*c*) 0.9
(*b*) 0.8 (*d*) 0.6 [**Ans.** (*a*)]

18. If the coefficient of correlation between x and y is $\frac{2}{3}$ and the standard deviations of x is 3 and standard deviation of y is 4, the covariance between x and y will be
(*a*) 3 (*b*) 6 (*c*) 7 (*d*) 8 [**Ans.** (*d*)]

19. If the correlation is perfect then what is the value of r?
(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4 [**Ans.** (*a*)]

20. If there is no relation between two variables, then what will be the value of coefficient of correlation of these two variables?
(*a*) 1 (*b*) 0 (*c*) –1 (*d*) 2 [**Ans.** (*b*)]

21. Given $r = 0.8$, $\Sigma xy = 80$, $\sigma_x = 2$, $\Sigma y^2 = 100$, where $x = X - \overline{X}$ and $y = Y - \overline{Y}$; the number of items will be
(*a*) 12 (*c*) 20
(*b*) 15 (*d*) 25 [**Ans.** (*d*)]

22. If the two variables are independent of each other, then value of '*r*' is
(*a*) 1 (*b*) 2 (*c*) 0 (*d*) 0.5 [**Ans.** (*c*)]

23. The value of the coefficient of correlation lies between
(*a*) ±1 (*b*) ±2 (*c*) ±3 (*d*) ±1.5 [**Ans.** (*a*)]

24. What is the covariance if the coefficient of correlation between *x* and *y* is 0.87 and the variance of *x* and *y* are 36 and 25 respectively
(*a*) 18.25 (*b*) 26.10 (*c*) 19.25 (*d*) 21.6 [**Ans.** (*b*)]

25. When we conduct a study that examines the relationship between two variables, then we are working with
(*a*) Univariate data (*c*) Bivariate data
(*b*) Multivariate data (*d*) None of there [**Ans.** (*c*)]

Rank Correlation

26. If $\Sigma d^2 = 33$ and $n = 10$, where d represents the difference between the rank of two series and *n* is the number of pairs of observations, then the value of the coefficient of rank correlations will be [***C.U. B.Com. 1985***]
(*a*) 0.68 (*c*) 0.75
(*b*) 0.7 (*d*) 0.8 [**Ans.** (*d*)]

27. The value of R (Spearman Rank correction coefficient) when $\Sigma d^2 = 30$ and $n = 10$, is
(*a*) 0.75 (*c*) 0.82
(*b*) 0.65 (*d*) 0.9
[***C.U. B.Com. 2001, 2012, 2014 (G), 2016 (G), 2016 (H)***] [**Ans.** (*c*)]

28. The following are the ranks of 10 students in English and Maths

Sr. no.	:	1	2	3	4	5	6	7	8	9	10
Rank in Math	:	9	6	4	5	10	3	1	7	2	8
Rank in English	:	8	9	3	6	7	1	2	5	5	10

The coefficient of rank correlation between the marks in Maths and English is
(*a*) 0.61 (*c*) 0.59
(*b*) 0.769 (*d*) 0.79 [**Ans.** (*b*)]

29. In case of tie in ranks, the rank of the tied scores are calculated by using
(*a*) Mean (*c*) Mode
(*b*) Median (*d*) S.D [**Ans.** (*a*)]

30. Differences of the ranks (d) are squared to remove
(*a*) deviations (*c*) positive values
(*b*) negative values (*d*) none of there [**Ans.** (*b*)]

Association of Attributes

31. A qualitative characteristic is called
    (*a*) constant
    (*b*) variable
    (*c*) attribute
    (*d*) association [**Ans.** (*c*))

32. If an attribute has two classes, it is called
    (*a*) Trichotomy
    (*b*) Dichotomy
    (*c*) Simple classification
    (*d*) Manifold classification [**Ans.** (*b*)]

33. With two attributes *A* and *B*, the total number of ultimate frequencies is
    (*a*) Two
    (*b*) Four
    (*c*) Six
    (*d*) Nine [**Ans.** (*b*)]

34. If $(AB) = \frac{(A)(B)}{N}$, the two attributes *A* and *B* are
    (*a*) Independent
    (*b*) Dependent
    (*c*) Correlated
    (*d*) Quantitative [**Ans.** (*a*)]

35. If the class frequency (AB) = 0, the value of Q is equal to
    (*a*) 0
    (*b*) 1
    (*c*) −1
    (*d*) 0 to 1 [**Ans.** (*c*)]

36. If for two attributes the class frequencies are (AB) (αβ) = (Aβ) (αB), then Q is equal to:
    (*a*) 0
    (*b*) −1
    (*c*) +1
    (*d*) α [**Ans.** (*a*)]

37. If two attributes *A* and *B* are independent, then the coefficient of association is:
    (*a*) −1
    (*b*) +1
    (*c*) 0
    (*d*) 0.5 [**Ans.** (*c*)]

38. If $(AB) < \frac{(A)(B)}{N}$, the association between two attributes *A* and *B* is;
    (*a*) Positive
    (*b*) Zero
    (*c*) Symmetrical
    (*d*) Negative [**Ans.** (*d*)]

39. If (AB) (αβ) > (Aβ) (αB), then *A* and *B* are said to be
    (*a*) Negatively associated
    (*b*) Positively associated
    (*c*) Independent
    (*d*) Difficult to tell [**Ans.** (*b*)]

40. If two attributes *A* and *B* have perfect positive association, the value of coefficient of association is equal to
    (*a*) 0
    (*b*) −1
    (*c*) (r − 1) (c − 1)
    (*d*) +1 [**Ans.** (*d*)]

(ii) Short Essay Type

Correlation

1.

No. of study hours:	2	4	6	8	10
No. of sleeping hours:	10	9	8	7	6

The correlation coefficient between the number of study hours and the number of sleeping hours of different students is

(*a*) 0.97 (c) –1

(b) –0.89 (d) +1 [**Ans.** (*c*)]

2.

	X series	Y series
Number of Items	15	15
Arithmetic Mean	25	18
Sum of square deviations	136	138

The summation of the products of the deviations of X and Y series from their arithmetic means = 122.

From the above data, the coefficient of correlation between X and Y will be:

(*a*) 0.97 (*c*) –0.89

(*b*) 0.89 (*d*) –0.97 [**Ans.** (*b*)]

3.

X:	8	6	4	3	4
Y:	9	7	4	4	6

From the above data the Karl Pearson's coefficient of correlation is

(*a*) –0.87 (*c*) –0.92

(*b*) 0.89 (*d*) 0.94 [**Ans.** (*d*)]

4.

Height (in meter):	1.60	1.64	1.71
Weight (in kg):	53	57	60

Covariance for the above data is

(*a*) 0.126 (*c*) 0.141

(*b*) 0.135 (*d*) 0.153 [**Ans.** (*a*)]

5. N = 25, $\sum x = 125$, $\sum y = 100$, $\sum x^2 = 650$, $\sum y^2 = 436$, $\sum xy = 520$
Correlation coefficient from the above data is

(*a*) 0.75 (*c*) –0.667

(*b*) 0.72 (*d*) –0.59 [**Ans.** (*c*)]

6. If $n = 10$, $\sum(x - \bar{x})^2 = 144$, $\sum(y - \bar{y})^2 = 49$ and $\sum(x - \bar{x})(y - \bar{y}) = 77$, then the value of r is

(*a*) –0.91 (*c*) 0.92

(*b*) –0.95 (*d*) 0.95 [**Ans.** (*c*)]

7. If $r = 0.8$, $\sum xy = 80$, $\sigma_x = \sigma_y = 2$, then the number of pairs of n (where $x = X - \overline{X}$ and $y = Y - \overline{Y}$) is

(*a*) 24 (*c*) 26
(*b*) 25 (*d*) 27 [**Ans.** (*b*)]

8. If $2u + 5x = 17$, $5v - 2y = 11$ and cov $(x, y) = 3$, the cov(u, v) is

(*a*) –3 (*c*) –2
(*b*) –4 (*d*) –5 [**Ans.** (*a*)]

9. If $r = 0.8$, $\sum xy = 60$, $\sigma_y = 2.5$, and $\sum x^2 = 9$ then the number of items (x and y are deviations from the arithmetic means) is

(*a*) 7 (*c*) 9
(*b*) 8 (*d*) 10 [**Ans.** (*d*)]

10. If $r_{xy} = 0.6$, find r_{uv} where

(*i*) $u = 3x + 5, v = 4y - 3$ (*ii*) $u = 3x + 5, v = -4y + 3$

[***C.U. B. Com. 1987***]

(*a*) 0.7, –0.7 (*c*) 0.6, –0.6
(*b*) 0.5, –0.5 (*d*) 0.8, –0.8 [**Ans.** (*c*)]

11. For 10 pairs of observations of x and y, the correlation coefficient is 0.7. Here, $\bar{x} = 15$, $\bar{y} = 18$, $\sigma_x = 4$, $\sigma_y = 5$. Later it is found that the pair ($x = 10$, $y = 8$) is wrongly copied. If it is omitted, the correlation coefficient of the remaining 9 pairs of observations is

(*a*) –0.71 (*c*) 0.69
(*b*) –0.63 (*d*) 0.62 [**Ans.** (*d*)]

12. The correlation coefficient and covariance of two variables x and y are respectively 0.28 and 7.6. If the variance of x is 9, the standard deviation of y is

[***V.U. B. Com. '94***]

(*a*) 7 (*c*) 9
(*b*) 8 (*d*) 10 [**Ans.** (*c*)]

13. In a question on correlation, the value of r is 0.917 and its probable error is 0.034. Then the value of N is

(*a*) 10 (*c*) 12
(*b*) 11 (*d*) 13 [**Ans.** (*a*)]

[**Hints:** P.E. = $0.6745 \times \dfrac{1-r^2}{\sqrt{N}}$]

14. In a question on correlation the value of r is 0.64 and its P.E. = 0.1312. The value of N is

(*a*) 7 (*c*) 9
(*b*) 8 (*d*) 10 [**Ans.** (c)]

15. $r = 0.5$, $\sum xy = 120$, $\sigma_y = 8$, $\sum x^2 = 90$

From the data given above, the number of items, i.e., n is

(*a*) 9 (*c*) 11
(*b*) 10 (*d*) 12 [**Ans.** (*b*)]

Rank Correlation

16. The rankings of ten students in Statistics and Economics are as follows:

Statistics:	3	5	8	4	7	10	2	1	6	9
Economics:	6	4	9	8	1	2	3	10	5	7

The coefficient of rank correlation is
(*a*) –0.2 (*c*) +0.3
(*b*) +0.2 (*d*) –0.3 [**Ans.** (*d*)]

17. The coefficient of rank correlation between marks in Statistics and marks in Accountancy obtained by a certain group of students is 0.8. If the sum of the squares of the differences in ranks is given to be 33, then the number of students in the group is
(*a*) 9 (*c*) 11
(*b*) 10 (*d*) 12 [**Ans.** (*b*)]

18. The coefficient of rank correlation of the marks obtained by 10 students in Statistics and Accountancy was found to be 0.8. It was later discovered that the difference in ranks in the two subjects obtained by one of the students was wrongly taken as 7 instead of 9. The correct coefficient of rank correlation is
(*a*) 0.509 (*c*) 0.606
(*b*) 0.512 (*d*) 0.612 [**Ans.** (*c*)]

19. The coefficient of the rank correlation between debenture prices and share prices is found to be 0.143. If the sum of squares of the differences in ranks is given to be 48, then the value of N is
(*a*) 6 (*c*) 8
(*b*) 7 (*d*) 9 [**Ans.** (*b*)]

20. From the following data, Spearman's rank correlation is

X:	10	12	8	15	20	25	40
Y:	15	10	6	25	16	12	8

(*a*) 0.14 (*c*) 0.12
(*b*) 0.13 (*d*) 0.15 [**Ans.** (*a*)]

Association of Attributes

21. Given N = 2,000, (A) = 1,500, (B) = 100, (AB) = 350. Are the data consistent?
(*a*) Yes (b) No [**Ans.** (*b*)]

22. Given N = 280, (A) = 250, (B) = 85, (AB) = 35. Are the data consistent?
(*a*) Yes (*c*) Cannot say [**Ans.** (*b*)]
(*b*) No

23. In a sample of 1,000 individuals, 100 possess the attribute A and 300 possess attribute B. If A and B are independent, how many individual possess A and B?
(*a*) 28 (*c*) 30
(*b*) 25 (*d*) 35 [**Ans.** (*c*)]

24. In a report of consumer preference, it was given that out of 500 persons surveyed, 400 preferred variety A, 380 preferred variety B and 270 liked both A and B. Are the data consistent?
    (*a*) Yes (*c*) Cannot say **[Ans.** (*b*)]
    (*b*) No

25. Given N = 1,482, (A) = 368, (B) = 343 and (AB) = 35, Then Yule's coefficient of association is
    (*a*) 0.58 (*c*) 0.63
    (*b*) –0.55 (*d*) –0.57 [**Ans.** (*d*)]

26. Total adults = 10,000, Literates = 1,290, Unemployed = 1,390, Literate unemployed = 820. The association between literacy and unemployment from the above figures is
    (*a*) 0.923 (*c*) 0.792
    (*b*) 0.891 (*d*) 0.956 [**Ans.** (*a*)]

27. From the following data find out the nature of ($\alpha\beta$).
    N = 100, (A) = 40, (B) = 80, (AB) = 30
    (*a*) Independent
    (*b*) Disassociated
    (*c*) Positively associated
    (*d*) Negatively associated [**Ans.** (*b*)]

28. Find if, A and B are independent, dependent, positively associated or negatively associated from the data given below:
    (A) = 470, (B) = 620, (AB) = 320, N = 1,000
    (*a*) Independent
    (*b*) Disassociated
    (*c*) Positively associated
    (*d*) Negatively associated [**Ans.** (*c*)]

29. Given N = 800, (A) = 470, (β) = 450 and (AB) = 230. Then Yule's coefficient of association is
    (*a*) 0.215 (*c*) 0.273
    (*b*) 0.253 (*d*) 0.262 [**Ans.** (*b*)]

30. Given (AB) = 100, (αB) = 80, (Aβ) = 50, ($\alpha\beta$) = 40. The total number of observations are
    (*a*) 320 (*c*) 300
    (*b*) 250 (*d*) 270 [**Ans.** (*d*)]

CHAPTER

7

Regression Analysis

SYLLABUS

Least Squares Method, Simple Regression Lines, Properties of Regression, Identification of Regression Lines

THEMATIC FOCUS

7.1 REGRESSION

Regression is a statistical measure used in finance, investment and other disciplines to determine the strength of the relationship between one dependent variable (usually denoted by *y*) and one or more independent variables. It also can be used to predict the future value of one variable based on the values of others. It helps investment and financial managers to value assets, understand the relationship between variables, such as commodity prices, stocks of businesses dealing in those commodities and to predict sales based on weather, previews sales, GDP growth or other conditions.

7.2 REGRESSION ANALYSIS

Regression analysis is a statistical tool which investigates the relationship between variables in the form of mathematical equations. It is the most useful form of analysis as it studies the variables individually and determines their significance with greater accuracy. Say, for example, you want to find out the effect of price-increase on the customer's demand for fixing selling price of a product or you want to study the relationship between salaries and qualifications on the job performance of an employee. These studies will naturally involve a lot of co-related variables that will individually have an effect on the dependent variable. These complex questions can be easily answered with the help of regression analysis. In short, a good regression analysis needs sound reasoning and proper interpretation of data for highly accurate predictions, forecast and solutions.

7.3 TYPES OF REGRESSION TECHNIQUES

Regression techniques basically involve the assembling of data on the variables under study and estimating the quantitative effect of a variable on another. There are different kinds of regression techniques such as:

7.3.1 Linear Regression

This is a simple and easy to use method that models the relationship between a scalar dependent variable y and one or more explanatory (independent) variables denoted as x. Usually, more than one independent variable influences the dependent variable. When one independent variable is used in a regression, it is called a **Simple Linear Regression**; when two or more independent variables are used, it is called a **Multiple Linear Regression**. This method uses linear predictor functions for data modelling wherein unknown parameters are estimated from the data. Since linear models are linearly dependent on unknown parameters, they are easier to fit than non-linear models and lead to easier determination of statistical parameters. Multiple regression analysis introduces several additional complexities but may produce more realistic results then simple regression analysis.

7.3.2 Least Squares Method

The typical procedure for finding the line of best fit is called the least-squares method. This method is a commonly used method to solve for linear regression equations. It's best suited for data fitting applications such as fitting a straight line on to the points i.e. closeness to all the points in a scatter diagram etc. The criteria for what we mean by 'closeness' is called the least square principle. Recall the discussion on variance, where we learnt how the variance squares the deviations around the mean. In regression we will square the deviations around the regression line instead of around the mean. The best fit regression line that has the smallest value for the squared deviations around it, the least squared deviations. That's essentially the whole idea of least squares. Therefore, the least square method is based upon the principle that the sum of the squared residuals (residual is nothing but the difference between the actual or observed y value and y value calculated by the linear regression line equation) should be made as small as possible so the regression line has the least error. This method can be used for linear as well as non-linear regression depending on the nature of the residuals and equations.

7.3.3 Non-linear Regression

When the relationship between variable are represented by curved lines, then it is called non-linear regression. The non-linear regression analysis uses the method of successive approximations. Here, the data are modeled by a function, which is a non-linear combination of model parameters and depends on one or more explanatory (independent) variables. Therefore, in non-linear regression too, the models could be based on simple or multiple regressions. This method takes into account the nature of relationship between the variables and tries to find some kind of transformation in them so that the relationship can be expressed easily as a straight line.

7.4 REGRESSION LINE

In statistics, a regression line is a line that best describes the behavior of a set of data. It is the best fit straight line that we can draw through or between the points on the scatter diagram. Obviously a straight line cannot connect all the dots because then you would have to bounce up and down and up and down from one dot to the next, and it would not be a straight line. So we want to be able to draw a single line that comes as close to all the dots as possible. **Regression line** is the line that best fits the data, such that the overall distance from the line to the points (variable values) plotted on a graph is the smallest. In other words, the line which is used to minimize the squared deviations of predictions is called **Regression line**. It is also known as the line of 'best fit'.

Consider the scatter diagram given below. One possible line of best fit has been drawn on the diagram. Some of the points lie above the line and some lie below it. The vertical distance each point is above or below the line has been added to the diagram. These distances are called deviations or errors – they are symbolized as $d_1, d_2, \ldots, d_n$. When drawing in a regression line, the aim is to make the line fit the points as closely as possible. We do this by making the total of the squares of the deviations as small as possible, i.e. we minimize Σd_i^2. If a line of best fit is found using this principle, it is called the **least-squares regression line**.

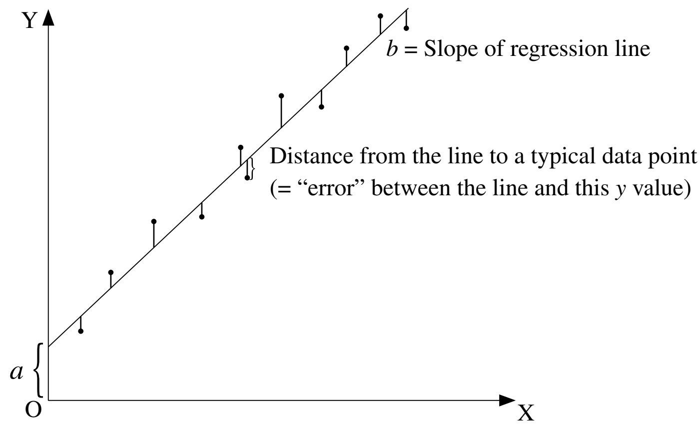


Figure 7.1 Regression line

The regression line formula is like the following:

$$y = a + bx + e$$

The multiple regression formula look like this:

$$y = a + b_1x_1 + b_2x_2 + b_3x_3 + \cdots + b_nx_n + e$$

where,

y = the value of the dependent variable (y), what is being predicted or explained

a or alpha = a constant; intercept of regression line; equals the value of y when the value of $x = 0$

b or beta = the coefficient of x; the slope of the regression line; how much y changes for each one-unit change in x.

x = the value of the independent variable (x), what is predicting or explaining the value of y

e = the error term; the error in predicting the value of y, given the value of x (it is not displayed in most regression equations).

A simple equation for the regression line is

$$y = a + bx$$

There are as many number of regression lines as variables. Suppose we take two variables, say x and y, then there will be two regression lines:

(1) Regression line of *y* on *x*: This give the most probable values of y from the given values of x.

(2) Regression line of *x* on *y*: This gives the most probable values of x from the given values of y.

The algebraic expression of these regression lines is called as **Regression Equations**. There will be two regression equations for the two regression lines.

(i) Regression equation of y on x : $y = a + bx$

(ii) Regression equation of x on y : $x = a + by$

7.5 DERIVATION OF THE REGRESSION EQUATIONS

(1) Regression Equation of *y* on *x*

We know that the regression equation of y on x by method of least squares from observation is $y = a + bx$ ...(1)

Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be n pair of observations of the two variables x and y.

Now to find the values of a and b, we apply the method of least squares and solve the following two normal equations:

$$\Sigma y = na + b\Sigma x \quad \ldots(2)$$

and

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad \ldots(3)$$

Now, dividing (2) by n, we get

$$\frac{\Sigma y}{n} = a + b\frac{\Sigma x}{n} \text{ or } \bar{y} = a + b\bar{x} \quad \ldots(4)$$

Subtracting (4) from (1) we get $y - \bar{y} = b(x - \bar{x})$ ...(5)

Again multiplying (2) by Σx and (3) by n we get

$$(\Sigma x)(\Sigma y) = n\,a\,(\Sigma x) + (\Sigma x)^2$$

and,

$$n\,\Sigma xy = n\,a\,(\Sigma x) + n\,b(\Sigma x^2)$$

Now,

$$(\Sigma x)(\Sigma y) - n\Sigma xy = b(\Sigma x)^2 - nb\left(\Sigma x^2\right)(\text{by subtracting})$$

or

$$b\left[(\Sigma x)^2 - n\left(\Sigma x^2\right)\right] = (\Sigma x)(\Sigma y) - n\Sigma xy$$

or

$$b = \frac{(\Sigma x)(\Sigma y) - n\Sigma xy}{(\Sigma x)^2 - n\left(\Sigma x^2\right)} = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{n\left(\Sigma x^2\right) - (\Sigma x)^2} \text{ [Changing sign]}$$

$$= \frac{\dfrac{\Sigma xy}{n} - \dfrac{\Sigma x}{n}\cdot\dfrac{\Sigma y}{n}}{\dfrac{\Sigma x^2}{n} - \left(\dfrac{\Sigma x}{n}\right)^2} = \frac{\text{cov}(x, y)}{\sigma_x^{\,2}} \quad \ldots(6)$$

Replacing b by b_{yx} in (5), the regression equation of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

where, b_{yx} = regression coefficient of y on x

$$= \frac{\text{cov}(x, y)}{\sigma_x^{\,2}} \qquad \dots(7)$$

Again we know $r = \dfrac{\text{cov}(x, y)}{\sigma_x . \sigma_y}$ or $\text{cov}(x, y) = r.\sigma_x.\sigma_y$

Then from (7), $b_{yx} = \dfrac{r.\sigma_x.\sigma_y}{\sigma_x^{\,2}} = r.\dfrac{\sigma_y}{\sigma_x}$

(2) Regression Equation of *x* on *y*

The regression equation of x on y by method of least squares from observation is

$$x = a + by \qquad \dots(8)$$

Now to find the values of a and b, we are to apply the method of least squares and solve the following two normal equations:

$$\Sigma x = n\,a + b\,\Sigma y \qquad \dots(9)$$

and $$\Sigma xy = a\,\Sigma y = b\,\Sigma y^2 \qquad \dots(10)$$

solving (9) and (10) for a and b and proceeding in the same way as before, we get the regression equation of x on y as $x - \bar{x} = b(y - \bar{y})$. ...(11)

where, $b = \dfrac{\dfrac{\Sigma xy}{n} - \dfrac{\Sigma x}{n}.\dfrac{\Sigma y}{n}}{\dfrac{\Sigma y^2}{n} - \left(\dfrac{\Sigma y}{n}\right)^2} = \dfrac{\text{cov}(x, y)}{\sigma_y^{\,2}}$

Replacing b by b_{xy} in (*ii*), the regression equation of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

where, b_{xy} = regression coefficient of x on y

$$= \frac{\text{cov}(x, y)}{\sigma_y^{\,2}} = r.\frac{\sigma_x}{\sigma_y}$$

Theorem 1 Show that the correlation coefficient is the geometric mean of regression coefficients

Proof: We know $b_{yx} = r.\dfrac{\sigma_y}{\sigma_x}$ and $b_{xy} = r.\dfrac{\sigma_x}{\sigma_y}$

Now multiplying we get

$$b_{yx} \times b_{xy} = r.\frac{\sigma_y}{\sigma_x} \times r.\frac{\sigma_x}{\sigma_y}$$

$$= r^2$$

or $$r = \pm\sqrt{b_{yx} \times b_{xy}}$$

It is clear from the relationship that r is the geometric mean between the two regression coefficients.

This relation clearly shows that b_{yx} and b_{xy} must have the same sign otherwise r^2 will be negative which is impossible.

(*i*) If b_{yx} and b_{xy} are both positive, then $r = +\sqrt{b_{yx} \times b_{xy}}$

(*ii*) If b_{yx} and b_{xy} are both negative, then $r = -\sqrt{b_{yx} \times b_{xy}}$

7.6 REGRESSION COEFFICIENTS

There are two regression coefficient:

(*i*) Regression coefficient of y on x

$$b_{yx} = \frac{\frac{\Sigma xy}{n} - \frac{\Sigma x}{n} \cdot \frac{\Sigma y}{n}}{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

(*ii*) Regression coefficient of x on y

$$b_{xy} = \frac{\frac{\Sigma xy}{n} - \frac{\Sigma x}{n} \cdot \frac{\Sigma y}{n}}{\frac{\Sigma y^2}{n} - \left(\frac{\Sigma y}{n}\right)^2}$$

Since σ_x, σ_y and r are independent of the change of origin, b_{yx} and b_{xy} are also independent of the change of origin. Thus the formulae for actual computation of b_{yx} and b_{xy} are given below.

(*a*) When deviations are taken from assumed mean of x and y.

That is, $U = x - A$ and $V = y - B$, then

$$b_{yx} = \frac{\frac{\Sigma uv}{n} - \frac{\Sigma u}{n} \cdot \frac{\Sigma v}{n}}{\frac{\Sigma u^2}{n} - \left(\frac{\Sigma u}{n}\right)^2} \text{ and } b_{xy} = \frac{\frac{\Sigma uv}{n} - \frac{\Sigma u}{n} \cdot \frac{\Sigma v}{n}}{\frac{\Sigma v^2}{n} - \left(\frac{\Sigma v}{n}\right)^2}$$

(*b*) When deviations are taken from arithmetic mean of x and y.

That is, $U = x - \bar{x}$ and $V = y - \bar{y}$, then $\Sigma U = 0$ and $\Sigma V = 0$

Hence, $b_{yx} = \dfrac{\Sigma uv}{\Sigma u^2}$ and $b_{xy} = \dfrac{\Sigma uv}{\Sigma v^2}$

7.7 PROPERTIES OF REGRESSION LINES (INCLUDING CORRELATION COEFFICIENT AND REGRESSION COEFFICIENTS)

(1) The two regression lines $y - \bar{y} = b_{yx}(x - \bar{x})$ and $x - \bar{x} = b_{xy}(y - \bar{y})$ are satisfied when $x = \bar{x}$ and $y = \bar{y}$, it implies that the two lines of regression intersect at the point $(\bar{x}, \bar{y})$.

(2) Putting $b_{yx} = r.\frac{\sigma_y}{\sigma_x}$ in the regression equation of y on x

We get, $y - \bar{y} = r.\frac{\sigma_y}{\sigma_x}(x - \bar{x})$

or $$\frac{y - \bar{y}}{\sigma_y} = r.\frac{x - \bar{x}}{\sigma_x}$$

Again, putting $b_{xy} = r.\frac{\sigma_x}{\sigma_y}$ in the regression equation of x on y we get

$x - \bar{x} = r.\frac{\sigma_x}{\sigma_y}(y - \bar{y})$

or $$\frac{x - \bar{x}}{\sigma_x} = r.\frac{y - \bar{y}}{\sigma_y}$$

(*a*) Two lines of regression are different. They coincide, i.e. become identical when $r = -1$ or 1 or in other words, there is a perfect negative or positive correlation between the two variables under discussion.

(*b*) The two lines of regression are perpendicular to each other when $r = 0$ (i.e. parallel to the x-axis and y-axis)

(3) The slope of the regression line of y on x is b_{yx} and slope of the regression line of x on y is $\frac{1}{b_{xy}}$.

(4) The correlation between the variables depend on the distance between two regression lines, such as the nearer the regression lines to each other the higher is the degree of correlation, and the further the regression lines to each other the lesser is the degree of correlation.

(5) The correlation coefficient is the **geometric mean** of two regression coefficients. Symbolically , it can be expressed as: $r = \sqrt{b_{yx} \times b_{xy}}$.

(6) The value of the coefficient of correlation cannot exceed unity, i.e. 1. Therefore, if one of the regression coefficients is greater than unity, the other must be less than unity.

(7) The coefficient of correlation will have the same sign as that of the regression coefficients, such as if the regression coefficients have a positive sign, then '*r*' will be positive or vice versa.

(8) The sign of both the regression coefficients will be same, i.e. they will be either positive or negative. Thus, it is not possible that one regression coefficient is negative while the other is positive.

(9) Arithmetic mean of the regression coefficients is greater than the correlation coefficient r, provided $r > 0$. Symbolically, it can be represented as $\frac{b_{xy}+b_{yx}}{2} > r$.

(10) The regression coefficients are independent of the change of origin, but not of the scale. By origin, we mean that there will be no effect on the regression coefficients if any constant is subtracted from the value of x and y. By scale, we mean that if the value of x and y is either multiplied or divided by some constant, then the regression coefficients will also change.

Proof: Let $u = \frac{x-a}{b}$ and $v = \frac{y-c}{d}$, i.e. $x = a + bu$

and $$y = c + dv \text{ or } \bar{x} = a + b\bar{u} \text{ and } \bar{y} = c + d\bar{v}$$

We know, $\text{cov}(x,y) = \frac{1}{n}\Sigma(x-\bar{x})(y-\bar{y})$

$$= \frac{1}{n}\Sigma(a+bu-a-b\bar{u})(c+dv-c-d\bar{v})$$

$$= \frac{1}{n}\Sigma b(u-\bar{u}).d(v-\bar{v})$$

$$= bd.\frac{1}{n}\Sigma(u-\bar{u})(v-\bar{v})$$

$$= b.d.\text{cov}(u,v)$$

$$\sigma_x^{\ 2} = \frac{1}{n}\Sigma(x-\bar{x})^2 = \frac{1}{n}\Sigma(a+bu-a-b\bar{u})^2$$

$$= \frac{1}{n}\Sigma(bu-b\bar{u})^2 = b^2.\frac{1}{n}\Sigma(u-\bar{u})^2$$

$$= b^2.\sigma_u^{\ 2}$$

and $$\sigma_y^{\ 2} = \frac{1}{n}\Sigma(y-\bar{y})^2 = \frac{1}{n}\Sigma(c+dv-c-d\bar{v})^2$$

$$= \frac{1}{n}\Sigma(dv-d\bar{v})^2 = d^2\,\frac{1}{n}\Sigma(v-\bar{v})^2$$

$$= d^2.\sigma_v^{\ 2}$$

Therefore,
$$b_{yx} = \frac{\text{cov}(x, y)}{\sigma_x^{\ 2}} = \frac{bd\,\text{cov}(u, v)}{b^2 . \sigma_u^{\ 2}}$$
$$= \frac{d}{b} . \frac{\text{cov}(u, v)}{\sigma_u^{\ 2}} = \frac{d}{b} b_{vu}$$

Similarly,
$$b_{xy} = \frac{b}{d} . b_{uv}$$

Thus the regression coefficients are independent of *a* and *c* but dependent of *b* and *d*, i.e. independent of the change of origin but not of the scale.

ILLUSTRATION 1

If $u = -5x + 3$ and $v = 7y + 2$ and the regression coefficient of x on y is 0.7, find the regression coefficient of u on v.

Solution: $u = -5x + 3$

or $5x = -4 + 3$

or $x = -\frac{1}{5} u + \frac{3}{5}$

It is in the form $x = a + bu$, where $b = -\frac{1}{5}$

Again, $v = 7y + 2$ or $7y = v - 2$

or $y = \frac{1}{7} v - \frac{2}{7}$

It is in the form $y = c + dv$, where $d = \frac{1}{7}$

Therefore, regression coefficient of u on v

$$b_{uv} = \frac{d}{b} . b_{xy} = \frac{\frac{1}{7}}{-\frac{1}{5}} \times 0.75$$
$$= -\frac{5}{7} \times 0.7 = -0.5$$

7.8 IDENTIFICATION OF REGRESSION LINES

Suppose the two regression equations are

(*i*) $a_1 x + b_1 y + c_1 = 0$ and (*ii*) $a_2 x + b_2 y + c_2 = 0$

Out of these two equations which of the two is the equation of x on y; is not given. In such a case the following steps should be followed:

(1) Assume any one of the equations [suppose equation (i)] as the regression equations of x on y and calculate slope, i.e. $b_{xy}\left(\text{here } b_{xy} = -\dfrac{a_1}{b_1}\right)$.

(2) Calculate slope of another equation [here equation (ii)]

That is, $b_{yx}\left(\text{here } b_{yx} = -\dfrac{b_2}{a_2}\right)$

(3) Multiply b_{xy} and b_{yx}, i.e. find $r^2.\left(\text{here } \dfrac{a_1b_2}{a_2b_1}\right)$

(i) If $r^2 < 1\left(\text{here } \dfrac{a_1b_2}{a_2b_1} < 1\right)$ then our assumption is correct, i.e. $a_1x + b_1y + c_1 = 0$ in the regression equation of x on y and $a_2x + b_2y + c_2 = 0$ is the regression equation of y on x.

(ii) If $r^2 > 1\left(\mathit{here}\dfrac{a_1b_2}{a_2b_1} > 1\right)$ then our assumption is wrong. Therefore, $a_1x + b_1y + c_1 = 0$ is equation of y on x and $a_2x + b_2y + c_2 = 0$ is equation of x on y. Calculate regression coefficients of the changed equations following the same procedure.

NOTE

If one of the regression coefficient is greater than unity, the other must be less than unity. [Property No. 6]

ILLUSTRATION 2

Two lines of regression are $4x - 5y + 30 = 0$ and $20x - 9y = 56$. Calculate coefficient of correlation between x and y.

Solution: For calculating the value of coefficient of correlation (r), we will have to find out the regression coefficients. But we do not know which of the two regression equations is the equation of x on y.

We assume first equation as the regression equation of x on y

$$4x - 5y + 30 = 0$$

or $$4x = -30 + 5y$$

or $$x = -\frac{30}{4} + \frac{5}{4}y \text{ or } b_{xy} = \frac{5}{4}$$

From 2nd equation we get

$$20x - 9y = 56$$

or $$-9y = 56 - 20x$$

or $$y = \frac{56}{-9} - \frac{20}{-9}x$$

$$= -\frac{56}{9} + \frac{20}{9}x$$

or $$b_{yx} = \frac{20}{9}$$

Now $r^2 = b_{xy} \times b_{yx} = \frac{5}{4} \times \frac{20}{9} = \frac{100}{36} = 2.78 > 1$

Since $r^2 > 1$, our assumption is wrong.

Hence, the first equation is equation of y on x

then, $$4x - 5y + 30 = 0$$

or $$-5y = -30 - 4x$$

or $$y = \frac{30}{5} + \frac{4}{5}x$$

or $$b_{yx} = \frac{4}{5}$$

Second equation in equation of x on y

$$20x - 9y = 56$$

or $$20x = 56 + 9y$$

or $$x = \frac{56}{20} + \frac{9}{20}y$$

or $$b_{xy} = \frac{9}{20}$$

then, $r^2 = b_{yx} \times b_{xy} = \frac{4}{5} \times \frac{9}{20} = \frac{36}{100}$

$$= 0.36$$

or $$r = \sqrt{0.36} = 0.6$$

[As b_{xy} and b_{yx} both are positive, then r will also be positive]

Therefore, the required value of coefficient of correlation is 0.6

7.9 USES OF REGRESSION

1. Regression is used in Economics, Commerce and other disciplines to determine the strength of relationship between one dependent variable and one or more independent variables.
2. Regression is used to estimate the future value of one variable based on the values of others.
3. Regression lines are used to study the relative variation of the two variables.
4. Regression lines are very useful for forecasting procedures. By using the equation obtained from the regression line an analyst can forecast future behaviour of the dependent variable by inputting different values for the independent ones.
5. Regression analysis helps investment and financial manager to perform valuations for many different securities.
6. We calculate coefficient of correlation with the help of regression coefficients. The square of correlation coefficient (r^2) is called the **coefficient of determination** which measure the degree of association of correlation that exists between the two variables.

7.10 DIFFERENCE BETWEEN CORRELATION AND REGRESSION

Both correlation and regression can be said as the tools used in statistics that actually deals through two or more than two variables. Even though both identify with some topic, there exist contrasts between these two methods. The main difference is that correlation finds out the degree of relationship between two variables, while regression explains the nature of relationship. Other differences between these methods are given below:

1. The correlation term is used when (*i*) both variables are random variables, and (*ii*) the end goal is simply to find a number that expresses the relation between the variables.

   The regression term is used when (*i*) one of the variables is a fixed variable, and (*ii*) the end goal is to use the measure of relation to predict or estimate values of the random variable based on values of the fixed variable.
2. In correlation, there exists no distinction amongst explanatory (independent) and dependent variable that shows correlation amongst x and y is as like as y and x.

   In regression, there exists the distinction amongst explanatory and dependent variable that shows the regression of y on x is is not quite the same as x on y.
3. Correlation coefficient is independent of the change of scale and origin. Regression coefficients are independent of the change of origin, but not of the scale.

4. Correlation never fit in a line which passes through the points of data. Regression finds the finest line which estimates the behavior of y from x.
5. Correlation does not help us in ascertaining whether one variable is the cause and the other the effect. But regression helps us in studying the cause and effect relationship between the two variables.

7.11 EXPLANATION OF HAVING TWO REGRESSION LINES

Regression line is obtained by minimizing the sum of squares of overall distances (i.e. vertical and horizontal distances) from the line to the points (variable values) plotted on a graph. One line cannot serve both the purposes, i.e. minimize the sum of squares of vertical distances as well as horizonal distances. So it is essential to have two regression lines. The line which is used to minimize the sum of squares of the vertical, distances is known as the regression line of x on y. In this case it is assumed that the values of x are exactly known but the values of y are subject to error. The line which is used to minimize the sum of squares of horizontal distances is known as the regression line of x on y. In this case it is assumed that the values of x are exactly known but the values of x are subject to error. The first line is used to obtain best estimates of y for given values of x, and the second line is used to obtain best estimates of x for given values of y.

ILLUSTRATIVE EXAMPLES

A. SHORT TYPE

EXAMPLE 1

Find the regression equation of y on x from the following values:

$\bar{x} = 10, \bar{y} = 15$ and $b_{yx} = 2.50$ [***C.U. B.Com. 2016***]

Solution: We know that the regression equation of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

Putting the values of $\bar{x}$, $\bar{y}$ and b_{yx} we get

$$y - 15 = 2.50\,(x - 10)$$

or $y - 15 = 2.50x - 25$

or $y = 2.50x - 25 + 15$

or $y = 2.50x - 10$

EXAMPLE 2

If two regression equations are $8x - 10y + 66 = 0$ and $40x - 18y = 214$, find the average values of x and y. [***C.U. B.Com. 2010, 2016 (H)***]

Solution: We know that two regression equations intersect at a point $(\bar{x}, \bar{y})$. Hence, for finding the average values of x and y, we are to solve two regression equations.

$$8x - 10y + 66 = 0 \quad \ldots(i)$$

$$40x - 18y - 214 = 0 \quad \ldots(ii)$$

Multiplying equation (i) by 5 and then subtracting from equation (ii) we get

$$\begin{array}{l} 40x - 18y - 214 = 0 \\ 40x - 50y + 330 = 0 \\ - \quad + \quad - \\ \hline \quad 32y - 544 = 0 \end{array}$$

or $$32y = 544 \text{ or } y = \frac{544}{32} = 17$$

Putting the value of y in equation (i) we get

$$8x - 170 + 66 = 0$$

or $$8x - 104 = 0 \text{ or } 8x = 104 \text{ or } x = \frac{104}{8} = 13$$

Hence, $$\bar{x} = 13 \text{ and } \bar{y} = 17.$$

EXAMPLE 3

Using the following regression coefficient, find the value of correlation coefficient r where $b_{yx} = –0.6$ and $b_{xy} = –1.35$. [***C.U. B.Com. 2011***]

Solution: We know that, $r = \sqrt{b_{yx} \times b_{xy}}$

$$= \sqrt{-0.6 \times -1.35}$$

$$= \sqrt{0.81} = 0.9$$

As b_{xy} and b_{yx} both are negative, therefore, value of r will be –0.9.

EXAMPLE 4

If $\bar{x} = 36$, $\bar{y} = 85$, $\sigma_x = 11$, $\sigma_y = 8$ and correlation coefficient between x and y = 0.66, then find the two regression equations. [***C.U. B.Com. 2013 (G)***]

Solution: Given that, $\bar{x} = 36$, $\bar{y} = 85$, $\sigma_x = 11$, $\sigma_y = 8$ and $r_{xy} = 0.66$

Regression equation of x on y:

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

or $$x - 36 = 0.9075(y - 85) \quad \left[bxy = r_{xy}\frac{\sigma_x}{\sigma_y} = 0.66 \times \frac{11}{8} = 0.9075\right]$$

or $$x - 36 = 0.9075y - 77.1375$$

or $$x = 0.9075y - 77.1375 + 36$$

or $$x = 0.9075y - 41.1375$$

Regression equation of y on x:

$$y - \overline{y} = b_{yx}(x - \overline{x})$$

or $$y - 85 = 0.48(x - 36) \quad [b_{yx} = r_{xy} \cdot \frac{\sigma_y}{\sigma_x} = 0.66 \times \frac{8}{11} = 0.48]$$

or $$y - 85 = 0.48x - 17.28$$

or $$y = 0.48x - 17.28 + 85$$

or $$y = 0.48x + 67.72$$

EXAMPLE 5

If $x + 2y = 5$ and $2x + 3y = 8$ be two regression lines then find r_{xy}.

[***C.U. B.Com. 2014 (H)***]

Solution: Assume the regression equation of y on x is

$$x + 2y = 5$$

or $$2y = 5 - x$$

or $$y = \frac{5}{2} - \frac{1}{2}.x$$

Therefore, $$b_{yx} = -\frac{1}{2}$$

The regression equation of x on y is

$$2x + 3y = 8$$

or $$2x = 8 - 3y$$

or $$x = 4 - \frac{3}{2}.y$$

Therefore, $$b_{xy} = -\frac{3}{2}$$

Now, $$r^2 = b_{yx} \times b_{xy} = -\frac{1}{2} \times -\frac{3}{2} = \frac{3}{4} < 1$$

Therefore, our assumption is correct.

Hence, $$r^2 = \frac{3}{4}$$

or $$r = \sqrt{\frac{3}{4}} = \sqrt{0.75} = 0.87$$

As b_{xy} and b_{yx} both are negative, therefore, the value of r will be –0.87.

EXAMPLE 6

If $\sigma_x = 10$, $\sigma_y = 12$, $b_{xy} = -0.8$, find the value of r_{xy}. [***C.U. B.Com. 2015 (G)***]

Solution: We know, $b_{xy} = r.\frac{\sigma x}{\sigma y}$

or $$-0.8 = r_{xy}.\frac{10}{12} \text{ or } r_{xy} = -0.8 \times \frac{12}{10} = -0.96$$

EXAMPLE 7

The regression equation of y on x is $3x - 5y = 13$ and the regression equation of x on y is $2x - y = 7$. Estimate the value of x when $y = 10$. [***C.U. B.Com. 1996***]

Solution: We are to estimate the value of x for a given value of y

So, regression equation of x on y is to be selected

That is $2x - y = 7$

or $2x - 10 = 7$ [Putting the value of y]

or $2x = 17$

or $x = \frac{17}{2} = 8.5$.

EXAMPLE 8

If $r_{xy} = 0.6$, $\sigma_y = 4$ and $b_{yx} = 0.48$, find the value of σ_x. [***C.U. B.Com. 2003***]

Solution: We know that,

$$b_{yx} = r_{xy}.\frac{\sigma_y}{\sigma_x}$$

or $$0.48 = 0.6 \times \frac{4}{\sigma_x}$$

or $0.48\ \sigma_x = 2.4$

or $$\sigma_x = \frac{2.4}{0.48} = 5$$

Therefore, required value of σ_x is 5.

B. SHORT ESSAY TYPE

EXAMPLE 9

Find the two regression equations from the following table:

X	1	2	3	4	5
Y	2	3	4	5	6

If $X = 2.5$, what will be the value of Y? [***B.U. B.Com (H), 1994***]

Solution: **Calculation of regression equations**

X	Y	$x = X - \bar{X}$	$y = Y - \bar{Y}$	x^2	y^2	xy
1	2	–2	–2	4	4	4
2	3	–1	–1	1	1	1
3	5	0	1	0	1	0
4	4	1	0	1	0	0
5	6	2	2	4	4	4
$\Sigma X = 15$	$\Sigma Y = 20$	$\Sigma x = 0$	$\Sigma y = 0$	$\Sigma x^2 = 10$	$\Sigma y^2 = 10$	$\Sigma xy = 9$

$$\bar{X} = \frac{\Sigma X}{n} = \frac{15}{5} = 3,\ \bar{Y} = \frac{\Sigma Y}{n} = \frac{20}{5} = 4 \text{ (both are integers).}$$

Regression coefficient of Y on X (b_{YX}) $= \dfrac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\Sigma(X - \bar{X})^2}$

$$= \frac{\Sigma xy}{\Sigma x^2} = \frac{9}{10} = 0.9$$

Regression coefficient of X on Y (b_{XY}) $= \dfrac{\Sigma(X - \bar{X})(Y - \bar{Y})}{\Sigma(Y - \bar{Y})^2} = \dfrac{\Sigma xy}{\Sigma y^2}$

$$= \frac{9}{10} = 0.9$$

Therefore,

Regression equation of Y on X is

$$Y - \bar{Y} = b_{YX}(X - \bar{X})$$

or $$Y - 4 = 0.9(X - 3) \text{ or } Y - 4 = 0.9X - 2.7$$

or $$Y = 0.9X - 2.7 + 4 \text{ or } Y = 0.9X + 1.3$$

If $X = 2.5$, then $Y = 0.9 \times 2.5 + 1.3 = 2.25 + 1.3 = 3.55$

Regression equation of x on y is

$$X - \bar{X} = b_{XY}(Y - \bar{Y})$$

or $$X - 3 = 0.9(Y - 4) \text{ or } X - 3 = 0.9Y - 3.6$$

or $$X = 0.94 - 3.6 + 3 \text{ or } X = 0.9Y - 0.6$$

EXAMPLE 10

Find the two linear regression equations from the following:

X	12	23	37	46	57	76	82
Y	36	42	57	64	68	82	95

[***C.U. B.Com 2016 (G)***]

Solution: **Calculation of regression equations**

X	Y	$x = X - a$	$y = Y - b$	x^2	y^2	xy
12	36	–35	–27	1225	729	945
23	42	–24	–21	576	441	504
37	57	–10	–6	100	36	60
46	64	–1	1	1	1	–1
57	68	10	5	100	25	50
76	82	29	19	841	361	551
80	95	33	32	1089	1024	1056
$\Sigma X = 331$	$\Sigma Y = 444$	$\Sigma x = 2$	$\Sigma y = 3$	$\Sigma x^2 = 3932$	$\Sigma y^2 = 2617$	$\Sigma xy = 3165$

Actual mean: $\overline{X} = \dfrac{\Sigma X}{n} = \dfrac{331}{7} = 47.29$

$$\overline{Y} = \frac{\Sigma Y}{n} = \frac{444}{7} = 63.43$$

$\overline{X}$ and $\overline{Y}$ are not integers, hence deviations are to be taken from assumed mean or calculation may be done directly (without taking any deviation).

Assumed mean: $A = 47$, $B = 63$

Since the regression coefficients are independent of origin,

Hence,

Regression coefficient of X on Y $(b_{XY}) = b_{xy} = \dfrac{n\Sigma xy - \Sigma x.\Sigma y}{n.\Sigma y^2 - (\Sigma y)^2}$

$$= \frac{7 \times 3165 - 2 \times 3}{7 \times 2617 - 9} = \frac{22155 - 6}{18319 - 9} = \frac{22149}{18310} = 1.21$$

and regression coefficient of Y on X $(b_{YX}) = b_{yx} = \dfrac{n\Sigma xy - \Sigma x.\Sigma y}{n.\Sigma x^2 - (\Sigma x)^2}$

$$= \frac{7 \times 3165 - 2 \times 3}{7 \times 3932 - 4} = \frac{22155 - 6}{27524 - 4} - \frac{22149}{27520} = 0.8$$

Therefore,

Regression equation of X on Y is

$$X - \overline{X} = b_{XY}(Y - \overline{Y})$$

or $\quad X - 47.29 = 1.21(Y - 63.43)$

or $\quad X - 47.29 = 1.21Y - 76.75$

or $\quad X = 1.21Y - 76.75 + 47.29$

or $\quad X = 1.21Y - 29.46$

Regression equation of Y on X is

$$Y - \overline{Y} = b_{YX}(X - \overline{X})$$

or $$Y - 63.43 = 0.8(X - 47.29)$$

or $$Y - 63.43 = 0.8X - 37.83$$

or $$Y = 0.8X - 37.83 + 63.43$$

or $$Y = 0.8X + 25.60$$

EXAMPLE 11

With the help of a suitable regression line estimate the value of x when $y = 22$ by using the following data:

x:	4	5	8	9	11
y:	16	10	8	7	6

[***C.U. B.Com. 2011***]

Solution: We are to estimate the value of x for a given value of y; therefore, we are to find the regression equation of x on y.

Calculation of regression equation of x on y

x	y	y^2	xy
4	16	256	64
5	10	100	50
8	8	64	64
9	7	49	63
11	6	36	66
$\Sigma x = 37$	$\Sigma y = 47$	$\Sigma y^2 = 505$	$\Sigma xy = 307$

Here, $$\overline{x} = \frac{\Sigma x}{n} = \frac{37}{5} = 7.4 \text{ and } \overline{y} = \frac{\Sigma y}{n} = \frac{47}{5} = 9.4$$

and $$b_{xy} = \frac{n\Sigma xy - \Sigma x.\Sigma y}{n.\Sigma y^2 - (\Sigma y)^2} = \frac{5 \times 307 - 37 \times 47}{5 \times 505 - (47)^2}$$

$$= \frac{1535 - 1739}{2525 - 2209} = \frac{-204}{316} = -0.646 \text{ (Approx.)}$$

Therefore, the regression equation of x on y is.

$$x - \overline{x} = b_{xy}(y - \overline{y}) \text{ or } x - 7.4 = -0.646(y - 9.4)$$

or $$x - 7.4 = -0.646y + 6.0724$$

or $$x = -0.646y + 6.0724 + 7.4$$

or $$x = -0.646y + 13.4724$$

Putting $y = 22$ in the above equation we get

$$\begin{aligned} x &= -0.646 \times 22 + 13.4724 \\ &= -14.212 + 13.4724 \\ &= -0.7396 \\ &= -0.74 \text{(Approx)} \end{aligned}$$

Therefore, the required estimated value of x is –0.74 when $y = 22$.

EXAMPLE 12

You are given the following data:

	x	y
A.M	36	85
S.D.	11	8

and correlation coefficient between x and y is 0.66

(*i*) Find the two regression equations

(*ii*) Estimate the value of x when $y = 33.5$

(*iii*) Estimate the value of y when $x = 73.3$ [***C.U. B.Com. 2000***]

Solution: Given that, $\bar{x} = 36$, $\bar{y} = 85$, $\sigma_x = 11$, $\sigma_y = 8$ and $r = 0.66$

$$b_{yx} = r.\frac{\sigma_y}{\sigma_x} = 0.66 \times \frac{8}{11} = 0.48$$

$$b_{xy} = r.\frac{\sigma_x}{\sigma_y} = 0.66 \times \frac{11}{8} = 0.91$$

(*i*) Therefore, the regression equation of y on x is,

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

or $$y - 85 = 0.48(x - 36)$$

or $$y - 85 = 0.48x - 17.28$$

or $$y = 0.48x - 17.28 + 85$$

or $$y = 0.48x + 67.72$$

and the regression equation of x on y is

$$x - \bar{x} = b_{xy}(y = \bar{y})$$

or $$x - 36 = 0.91(y - 85)$$

or $$x - 36 = 0.91y - 77.35$$

or $$x = 0.91y - 77.35 + 36$$

or $$x = 0.91y - 41.35$$

(*ii*) To estimate the value of *x* for a given value of *y* we are to use regression equation of *x* on *y*.

That is, $x = 0.91y - 41.35$

or $$x = 0.91 \times (-33.5) - 41.35 \text{ (putting } y = 33.5)$$

or $$x = 30.485 - 41.35$$
$$= -10.865$$

Therefore, the estimated value of *x* is –10.865 when *y* = 33.5

(*iii*) To estimate the value of *y* for a given value of *x* we are to use regression equation of *y* on *x*

That is, $y = 0.48 + 67.72$

or $$y = 0.48 \times 73.3 + 67.72 \text{ [puting } x = 73.3]$$
$$= 35.184 + 67.72 = 102.904$$

Therefore, the estimated value of *y* is 102.904 when *x* = 73.3.

EXAMPLE 13

The lines of regression of *y* on *x* and *x* on *y* are respectively *y* = *x* + 5 and 16*x* = 9*y* – 94. Find the variance of *x* if the variance of *y* is 16. Also, find the covariance of *x* and *y*.

Solution: The regression equation of *y* on *x* is *y* = *x* + 5

Therefore, the regression coefficient of *y* on *x* (b_{yx}) = 1

Again, the regression equation of *x* on *y* is

$$16x = 9y - 94$$

or $$x = \frac{9}{16}y - \frac{94}{16}$$

Therefore, the regression coefficient of *x* on *y* (b_{xy}) = $\frac{9}{16}$

We know that, $r = \sqrt{b_{yx} \times b_{xy}}$

$$= \sqrt{1 \times \frac{9}{16}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$$

Given that, variance of *y* = 16

That is, $\sigma_y^2 = 16$ or $\sigma_y = 4$

Now, $$b_{xy} = r.\frac{\sigma_x}{\sigma_y}$$

or $$\frac{9}{16} = \frac{3}{4}.\frac{\sigma_x}{4}$$

or $$\frac{9}{16} \times \frac{16}{3} = \sigma_x \text{ or } \sigma_x = 3$$

Therefore, variance of $x = \sigma_x^2 = (3)^2 = 9$

Again, $$b_{xy} = \frac{\text{cov}(x, y)}{\sigma_y^2}$$

or $$\frac{9}{16} = \frac{\text{cov}(x, y)}{16}$$

or $$\text{cov}(x, y) = 9$$

EXAMPLE 14

Find the regression equation of x on y from the following data and find the estimated value of x, when $y = 6$.

$$\Sigma x = 24, \Sigma y = 44, \Sigma xy = 306, \Sigma x^2 = 164, \Sigma y^2 = 574, n = 4$$

[***C.U. B.Com. 2008***]

Solution: Given: $\Sigma x = 24$, $\Sigma y = 44$, $\Sigma xy = 306$, $\Sigma x^2 = 164$, $\Sigma y^2 = 574$, $n = 4$

Now, $$\bar{x} = \frac{\Sigma x}{n} = \frac{24}{4} = 6, \bar{y} = \frac{\Sigma y}{n} = \frac{44}{4} = 11$$

Regression coefficient of x on y (b_{xy}) $$= \frac{n\Sigma xy - \Sigma x.\Sigma y}{n.\Sigma y^2 - (\Sigma y)^2}$$

$$= \frac{4 \times 306 - 24 \times 44}{4 \times 574 - (44)^2} = \frac{1224 - 1056}{2296 - 1936}$$

$$= \frac{168}{360} = 0.47$$

Therefore, the regression equation of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y}) \text{ or } x - 6 = 0.47(y - 11)$$

or $$x - 6 = 0.47y - 5.17 \text{ or } x = 0.47y - 5.17 + 6 \text{ or } x = 0.47y + 0.83$$

when $$y = 6$$

Then, the estimated value of x is

$$x = 0.47 \times 6 + 0.83$$
$$= 2.82 + 0.83 = 3.65$$

EXAMPLE 15

If $u = 2x - 3$ and $v = \frac{1}{3}y + 1.5$, find the values of regression coefficients b_{uv} and b_{vu} when $b_{xy} = 0.5$ and $b_{yx} = 1.4$

Solution: $u = 2x - 3$

or $2x = u + 3$

or $x = \frac{3}{2} + \frac{1}{2}u$

It is in the form $x = a + bu$ $\left(\text{when } u = \frac{x-a}{b}\right)$

where $a = \frac{3}{2}$ and $b = \frac{1}{2}$

again, $v = \frac{1}{3}y + 1.5$

or $\frac{1}{3}y = -1.5 + v$

or $y = -4.5 + 3v$

It is in the form $y = c + dv$ $\left(\text{when } v = \frac{y-c}{d}\right)$

where, $c = -4.5$ and $d = 3$

Now, $b_{xy} = \frac{\text{cov}(x, y)}{\sigma_y^2} = \frac{bd\,\text{cov}(u, v)}{d^2.\sigma_v^2} = \frac{b}{d} \cdot \frac{\text{cov}(u, v)}{\sigma_v^2} = \frac{b}{d} \cdot b_{uv}$

or $0.5 = \frac{\frac{1}{2}}{3} b_{uv}$ or $b_{uv} = 1.5 \times 2 = 3$

Similarly, $b_{yx} = \frac{d}{b}.b_{vu}$

or $b_{vu} = \frac{b}{d}.b_{yx} = \frac{\frac{1}{2}}{3} \times 1.4 = \frac{1.4}{6} = 0.23$

Therefore, the required values of b_{uv} and b_{vu} are 3 and 0.23 respectively.

EXAMPLE 16

For the variables x and y, variance of $x = 12$, regression equations are: $x + 2y = 5$ and $2x + 3y = 8$

Find the following:

(*i*) The average values of x and y

(*ii*) Correlation coefficient between x and y

(*iii*) Standard deviation of y. [***C.U. B.Com. 2004***]

Solution:

(*i*) We know that two regression equations intersect at a point $(\bar{x}, \bar{y})$. Hence, for finding $\bar{x}$ and $\bar{y}$, we are to solve two regression equations.

$$x + 2y = 5 \qquad \ldots(i)$$
$$2x + 3y = 8 \qquad \ldots(ii)$$

Multiplying equation (*i*) by 2 and then subtracting from equation (*ii*) we get

$$\begin{array}{r} 2x + 3y = 8 \\ \underset{-}{2x} \underset{-}{+} 4y = \underset{-}{10} \\ \hline -y = -2 \end{array}$$

or $$y = 2$$

Putting the value of y in equation (*i*) we get

$$x + 2 \times 2 = 5$$

or $$x + 4 = 5 \text{ or } x = 5 - 4 = 1$$

Hence, $$\bar{x} = 1 \text{ and } \bar{y} = 2$$

(*ii*) For finding correlation coefficient (r) we are to find the values of b_{yx} and b_{xy}. Out of the two equations, which one is meant for y on x is not given. Let us assume that equation (*i*) is meant for y on x and equation (*ii*) is for x on y.

From (*i*), $2y = -x + 5$ or $y = -\frac{1}{2}x + \frac{5}{2}$

or $$b_{yx} = -\frac{1}{2}$$

and from (*ii*), $2x = -3y + 8$

or $$x = -\frac{3}{2}y + 4$$

or $$b_{xy} = -\frac{3}{2}$$

Now, $$b_{yx} \times b_{xy} = -\frac{1}{2} \times -\frac{3}{2} = \frac{3}{4} < 1.$$

So our assumption is correct.

Therefore, $$r^2 = b_{yx} \times b_{xy} = \frac{3}{4}$$

or $$r = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} = \frac{1.732}{2} = 0.866$$

As b_{xy} and b_{yx} both are negative, then value of r will be –0.866.

(*iii*) Given, variance of $x = 12$

That is, $\sigma_x^2 = 12$

or $\sigma_x = \sqrt{12} = 3.46$

Now, $b_{yx} = r.\dfrac{\sigma_y}{\sigma_x}$

or $-\dfrac{1}{2} = -0.866.\dfrac{\sigma_y}{3.46}$

or $\sigma_y = \dfrac{3.46}{2 \times 0.866} = \dfrac{3.46}{1.732}$

$= 1.99 = 2$ (approx.)

Therefore, standard deviation of y is 2.

EXAMPLE 17

In order to find regression coefficients between two variables x and y from 5 pairs of observations the following results are given:

$\Sigma x = 30, \Sigma y = 40, \Sigma x^2 = 220, \Sigma y^2 = 340, \Sigma xy = 214.$ Later it was found that one particular set of observations namely $x = 4$, $y = 8$ was wrongly taken. The correct value being $x = 2$, $y = 6$, Find the corrected values of the regression coefficients and hence find the equations.

Solution: Here $n = 5$

For determining correct value we have to subtract the wrong observation and then add the correct observations.

Corrected $\Sigma x = 30 - 4 + 2 = 28$

Corrected $\Sigma y = 40 - 8 + 6 = 38$

Corrected $\Sigma x^2 = 220 - 4^2 + 2^2 = 220 - 16 + 4 = 208$

Corrected $\Sigma y^2 = 340 - 8^2 + 6^2 = 340 - 64 + 36 = 312$

Corrected $\Sigma xy = 214 - (4 \times 8) + (2 \times 6) = 214 - 32 + 12 = 194$

Corrected $b_{yx} = \dfrac{5 \times 194 - 28 \times 38}{5 \times 208 - (28)^2} = \dfrac{970 - 1064}{1040 - 784} = \dfrac{-94}{256} = -0.37$

Corrected $b_{xy} = \dfrac{5 \times 194 - 28 \times 38}{5 \times 312 - (38)^2} = \dfrac{970 - 1064}{1560 - 1444} = \dfrac{-94}{116} = -0.81$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{28}{5} = 5.6, \quad \bar{y} = \frac{\Sigma y}{n} = \frac{38}{5} = 7.6$$

Regression equation of y on x:

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

or $$y - 7.6 = -0.37(x - 5.6)$$

or $$y = -0.37x + 2.072 + 7.6$$

or $$y = -0.37x + 9.672$$

Regression equation of x on y:

$$x - \overline{x} = b_{xy}(y - \overline{y})$$

or $$x - 5.6 = -0.81(y - 7.6)$$

or $$x - 5.6 = -0.81y + 6.156$$

or $$x = -0.81y + 6.156 + 5.6$$

or $$x = -0.81y - 11.756$$

EXAMPLE 18

The correlation coefficient (r) = 0.60, variance of x and y are respectively 2.25 and 4.00; $\overline{x} = 10$, $\overline{y} = 20$. From the above data find the regression equations. Find the estimated value of y when $x = 25$. [***C.U. B.Com 2006(old)***]

Solution: Given that $r = 0.60; \sigma_x^2 = 2.25$ or $\sigma_x = \sqrt{2.25} = 1.5$;

$$\sigma_y^2 = 4 \text{ or } \sigma_y = \sqrt{4} = 2; \overline{x} = 10, \overline{y} = 20$$

Regression equation of x on y:

$$x - \overline{x} = b_{xy}(y - \overline{y}) \text{ or } x - \overline{x} = r \cdot \frac{\sigma_x}{\sigma_y}(y - \overline{y})$$

or $$x - 10 = 0.6.\frac{1.5}{2}(y - 20) \text{ or } x - 10 = 0.45\,(y - 20)$$

or $$x - 10 = 0.45y - 9 \text{ or } x = 0.45y - 9 + 10$$

or $$x = 0.45y + 1$$

Regression equation of y on x:

$$y - \overline{y} = b_{yx}(x - \overline{x}) \text{ or } y - \overline{y} = r \cdot \frac{\sigma_y}{\sigma_x}(x - \overline{x})$$

or $$y - 20 = 0.6.\frac{2}{1.5}(x - 10) \text{ or } y - 20 = 0.8(x - 10)$$

or $$y = 0.8x - 8 + 20 \text{ or } y = 0.8x + 12$$

when $$x = 25$$

Then $$y = 0.8 \times 25 + 12 = 20 + 12 = 32$$

EXERCISE

A. THEORY

1. Briefly explain the concept of regression.
2. What are the regression coefficients?
3. Why are there two regression lines?
4. Show that correlation coefficient is a geometric mean of regression coefficients. [*C.U. B.Com. 1991*]
5. If two regression lines coincide, determine the value of the correlation coefficient between x and y and mention when they will be perpendicular to each other.
6. Obtain the equations of the two lines of regression for a bivariate distribution. [*C.U.B.Com. 1985, 1987*]
7. What are the differences between correlation and regression?
8. Mention the important properties of regression lines.
9. What are the uses of regression?
10. Write short notes on:
    (*i*) Regression
    (*ii*) Regression analysis
    (*iii*) Regression coefficients [*C.U. B.Com. 1982*]
    (*iv*) Regression equations [*C.U. B.Com. 1983*]

B. SHORT TYPE

1. Find the regression equation of y on x from the following values:
   (*i*) $\bar{x} = 15, \bar{y} = 20, b_{yx} = 3.5$ [**Ans.** $y = 3.5x - 32.5$]
   (*ii*) $\bar{x} = 10, \bar{y} = 15, b_{yx} = 2.5$ [**Ans.** $2y = 5x - 20$]
2. Find the regression equation of x on y from the following values:
   (*i*) $\bar{x} = 15, \bar{y} = 10$ and $b_{xy} = 2.5$ [**Ans.** $x = 2.5y - 10$]
   (*ii*) $\bar{x} = 6, \bar{y} = 1$ and $b_{xy} = -0.4$ [**Ans.** $x = -0.4y + 6.4$]
3. Find the equations of the lines of regression using the following data:

$$\bar{x} = 4, \bar{y} = 5, b_{xy} = 0.39 \text{ and } b_{xy} = 0.69$$

[*C.U. B.com. 1982*] [**Ans.** $y = 0.39x + 3.44; x = 0.69y + 0.55$]

4. Find the regression equation of y on x from the following values:
   $\bar{x} = 10, \bar{y} = 15$ and $b_{xy} = 2.50$ [*C.U. B.Com. 1988*] [**Ans.** $y = 2.5x - 10$]

5. If $\overline{x} = 4, \overline{y} = 5, b_{yx} = 0.35, b_{xy} = 0.65$ then find the equation of the regression lines.

   [***C.U. B.Com. 1991***] $\left[\textbf{Ans. } x = 0.65y + 0.75; y = 0.35x + 3.6\right]$

6. If $\overline{x} = 3, \overline{y} = 4, b_{xy} = b_{yx} = 0.9$, then find the equation of the regression lines.

   $\left[\textbf{Ans. } y = 0.9x + 1.3; x = 0.9y - 0.6\right]$

7. Estimate the correlation coefficient between x and y from the following two regression lines: $y = 1.5x + 2.3$, $x = 0.4y + 1.8$ [**Ans.** 0.77]

8. Find the value of the correlation coefficient r when $b_{yx} = -0.4$ and $b_{xy} = -0.9$. (Here b_{yx} and b_{xy} are the regression coefficients of the two regression lines).

   [***C.U. B.Com 1983, 86, 89***] [**Ans.** –0.6]

9. The regression coefficient of y on x and x on y are 1.2 and 0.3 respectively. Find the coefficient of correlation. [***C.U. B.Com. 1994***] [**Ans.** +0.6]

10. The regression coefficient of y on x and x on y are –1.2 and –0.3 respectively. Find the coefficient of correlation.

    [***C.U. B.Com. 1997***] [**Ans.** –0.6]

11 If the regression coefficient are 0.8 and 0.6, what would be the value of the coefficient of correlation. [**Ans.** 0.69]

12. Find the mean values of x and y when the regression equations are $3x - 2y = 4.5$ and $2x - y = 3.5$ [**Ans.** $\overline{x} = 2.5, \overline{y} = 1.5$]

13. Find $\overline{x}$ and $\overline{y}$ if the regression equations are $5x - 7y = 15$ and $4x - 15y = -35$.

    [**Ans.** $\overline{x} = 10, \overline{y} = 5$]

14. Find $\overline{x}$ and $\overline{y}$; if the regression equations are $5x - 2y - 4 = 0$ and $4x - 7y + 13 = 0$

    [**Ans.** $\overline{x} = 2, \overline{y} = 3$]

15. If the two regression equations are $8x - 10y + 66 = 0$ and $40x - 18y = 214$, find the average values of x and y.

    [***C.U. B.Com. 2010, 2016(H)***] [**Ans.** $\overline{x} = 13, \overline{y} = 17$]

16. The regression equation of y on $3x - 5y = 13$ and the regression equation of x on y is $2x - y = 7$. Estimate the value of x when $y = 10$.

    [***C.U. B.Com. 1996***] [**Ans.** 8.5]

17. The regression equation of y on x is $4x - 3y = 11$ and the regression equation of x on y is $5x - 2y = 9$

    Estimate (*i*) the value of x when $y = 3$ (*ii*) the value of y when $x = 5$

    [**Ans.** (*i*) 3, (*ii*) 3]

18. The regression equation of y on x is $15x - 4y = 14$ and the regression equation of x on y is $7x + 2y = 11$. Estimate (*i*) the value of x when $y = 2$ (*ii*) the value of y when $x = 4$. [**Ans.** (*i*) $x = 1$ (*ii*) $y = 11.5$]

19. If $\sigma_x = 10$, $\sigma_y = 12$, $b_{xy} = -0.8$ find the value of r.

    . [***C.U. B.Com. 1998***] [**Ans.** –0.96]

20. If $\sigma_x = 4.5$, $\sigma_y = 13$ and $b_{yx} = 1.04$, then find the value of r. [**Ans.** 0.36]

21. If $r = \pm 1$, will the two regression lines be perpendicular to each other? [***V.U. B.Com. 1988***] [**Ans.** No]

22. Find the regression coefficients for the following data:

$$n = 10, \sigma_x = 12, \sigma_y = 8 \text{ and } \Sigma(x - \bar{x})(y - \bar{y}) = 250$$ [**Ans.** 0.26]

C. SHORT ESSAY / PROBLEM TYPE

1. Obtain two lines of regression from the following data:

***x*:**	4	5	6	8	11
***y*:**	12	10	8	7	5

[**Ans.** $y = -0.93x + 14.72$; $x = -0.98y + 15.03$]

2. From the following data find the two regression equations:

***x*:**	1	2	3	4	5
***y*:**	2	3	5	4	6

Predict the value of y when $x = 2.5$.
[***C.U. B.Com. 1983, 89***] [**Ans.** $y = 0.9x + 1.3$; $x = .9y - .6$; 3.55]

3. Find the linear regression equation of Y on X for the data:

***x*:**	1	2	3	4	5
***y*:**	3	2	5	4	6

[***C.U. B.Com. 1985***] [**Ans.** $y = 0.8x + 1.6$]

4. Find the linear regression equation of Y on X for the following data:

***X*:**	1	2	3	4	5
***Y*:**	6	8	11	8	12

Find also the most probable value of Y when $X = 2.5$.
[***C.U. B.Com. 1992***] [***K.U. B.Com. '97, '99***] [**Ans.** $Y = 1.2X + 5.4$;8.4]

5. Find the linear regression equation of y on x from the following data:

***x*:**	1	2	3	4	5
***y*:**	3	3	5	4	6

[***C.U. B.Com. 1994***] [**Ans.** $y = 0.8x + 1.6$]

6. From the following data find the two regression equations:

Age (Years):	1	3	4	5	7
Weight (Kg.)	3	5	8	12	17

What will be the most probable weight of a baby at the age of 8 years?

[***C.U. B.Com. 1996***] [**Ans.** $Y = 2.4X - 0.8$; $X = 0.39Y + 0.49$;18.8 kg.]

7. (*i*) What do you mean by Regression ?
   (*ii*) Find the two regression equations from the following data:

***x*:**	6	2	10	4	8	12	14	16
***y*:**	9	11	5	8	7	11	16	18

[***C.U. B.Com. 1998***] [**Ans.** $x = 0.67y + 1.88; y = 0.55x + 5.675.$]

8. From the following data obtain the two regression equations and calculate the correlation coefficient:

***x*:**	1	2	3	4	5	6	7	8	9
***y*:**	9	8	10	12	11	13	14	16	15

Estimate the value of y which should correspond on an average to $x = 6.2$

[**Ans.** $x = 0.95y - 6.4; y = 0.95x + 7.25; y = 13.14, r = 0.95$]

9. Find the equation of the line of regression of Y on X from the following data:

Husband's age (*X*)	25	27	29	33	26	32	35	31	35	30
Wife's age (*Y*)	20	24	24	29	21	26	28	30	29	28

[***C.U. B.Com. 1988***] [**Ans.** 0.319]

10. Given the bivariate data:

***X*:**	1	5	3	2	1	1	7	3
***Y*:**	6	1	0	0	1	2	1	5

(*a*) Fit the regression line of Y on X and hence predict Y if $X = 10$.
(*b*) Fit the regression line of X on Y and hence predict X, if $Y = 2.5$.

[**Ans.** $(a)\ Y = 2.86 - 0.30X;\ -0.14\ \ (b) X = 3.43 - 0.284Y; 2.73$]

11. Write the equation of the line of regression of Y on X.

***X*:**	78	89	97	69	59	79	68	61
***Y*:**	125	137	156	112	107	136	123	108

[**Ans.** $Y = 1.22X + 34$]

12. The grade of 9 students at the college test (X) and at the University examination (Y) are as follows:

***X*:**	77	50	71	72	81	94	96	99	67
***Y*:**	82	66	78	84	47	85	99	99	68

Find a linear regression equation for the data and then estimate the grade of the University examination for a student who received 85 in the college test but was sick at the time of University examination.

[***C.U. B.Com. (Hons) '83***] [**Ans.** $Y = 0.615X + 30.36;\ 82.635$]

13. By using the following data, find out the two lines of regression and from them compute the Karl Pearson's coefficient of correlation:

$$\Sigma x = 250; \Sigma y = 300; \Sigma xy = 7900; \Sigma x^2 = 6500; \Sigma y^2 = 10,000 \text{ and } N = 10$$

$\left[\textbf{Ans. } x = 0.4y + 13; y = 1.6x - 10; r = 0.8\right]$

14. Find the regression line of y on x from the following results:

$$N = 10, \Sigma x = 350, \Sigma y = 310, \Sigma (x-35)^2 = 162$$

$$\Sigma (y-31)^2 = 222, \Sigma (x-35)(y-31) = 92$$

[**Ans.** $y = 0.568x - 19.88$]

15. Given the following data:

$$\bar{x} = 36, \bar{y} = 85, \sigma_x = 11, \sigma_y = 8, r = 0.66$$

Find the two regression equations and estimate the value of x when $y = 75$.

$\left[\textbf{Ans. } x = 0.9075y - 41.1375; y = 0.48x + 67.72; x = 26.925\right]$

16. Find the regression equation of x on y from the following data :

$$n = 10, \Sigma x = 30, \Sigma y = 90, \Sigma x^2 = 110, \Sigma y^2 = 858, \Sigma xy = 294 \cdot$$

Find the estimated value of x, when $y = 8$. [**Ans.** $x = 0.5y - 1.5$; $x = 2.5$]

17. You are given the following data:

	x	*y*
Arithmetic Mean:	20	25
Standard Deviation:	5	4

Correlation coefficient between X and Y is 0.5.

Find the two regression equations.

$$\left[\text{Given: } b_{yx} = r.\frac{\sigma_y}{\sigma_x} \text{ and } b_{xy} = r\frac{\sigma_x}{\sigma_y}\right]$$

[***C.U. B.Com. 1986***] $\left[\textbf{Ans. } y = 0.48x + 15.4; x = 0.75y + 1.25\right]$

18. The following data are given for marks in English (x) and Mathematics (y) in a certain examination:

	English	**Mathematics**
Mean Marks	39.5	47.5
S. D. Marks	10.8	16.8

Coefficient of correlation between marks in English and Mathematics = +0.40. Find the two regression equations.

[***C.U. B.Com. 1990***] $\left[\textbf{Ans. } x = 0.25y + 27.63; y = 0.62x + 23.01\right]$

19. The following results were obtained from the record of age (x) and blood pressure (y) of a group of 10 women:

	x	y
Mean	53	142
Variance	130	165

$$\Sigma(x-\bar{x})(y-\bar{y}) = 1{,}220$$

Find the regression equation of y on x and use it to estimate the blood pressure of a woman of age 45.

[*C.U. B.Com. 1993*] [**Ans.** $y = 0.94x + 92.18; 134.48$]

20. Given

	x **series**	y **series**
Mean	18	100
Standard Deviation	14	20

Coefficient of correlation between X and Y is + 0.8. Find out the most probable value of y if x is 70 and most probable value of x if y is 90.

[**Ans.** $y = 79.48 + 1.14x, y = 159.28; x = 0.56y - 38, x = 12.4$]

21. Given the following data find what will be the probable yield when the rainfall is 29".

	Rainfall	**Production**
$\bar{x}$	25"	40 units per acre
σ	3"	6 units

r between rainfall and production = 0.8 [**Ans.** 50 units per acre]

22. In a correlation study the following values are obtained:

	x	y
Mean	65	67
Standard Deviation	2.5	3.5

Coefficient of correlation 0.8.

Find the two regression equations that are associated with the above values.

[**Ans.** $x = 0.5714y + 26.72; y = 1.12x - 5.8$]

23. For the variables x and y the equations of regression lines of y on x and x on y respectively are $4x - 5y + 33 = 0$ and $20x - 9y = 107$. What is the correlation coefficient. If variance of x is 9 find the standard deviation of y. Also find $\bar{x}, \bar{y}$ [*C.U. B.Com. 1999*] [**Ans.** + 0.6; 4; $\bar{x} = 13, \bar{y} = 17$]

24. Two random variables have the least square regression lines with equations $3x + 2y = 26$ and $6x + y = 31$.

Find the mean values and the coefficient of correlation between x and y.

[**Ans.** $\bar{x} = 4$, $\bar{y} = 7$, $r = -0.5$]

25. For certain x and y series which are correlated, the two lines of regression are $5x - 6y + 90 = 0$ and $15x - 8y - 130 = 0$. Find the means of the two series and the correlation coefficient. [**Ans.** $\bar{x} = 30$, $\bar{y} = 40$, $r = 0.667$]

26. Find out σ_y and r from the following data: $3x = y$, $4y = 3x$ and $\sigma_x = 2$.

[**Ans.** $\sigma_y = 3$; $r = 0.5$]

27. The equations of two lines of regression obtained in a correlation analysis are as following:

$$2x = 8 - 3y \text{ and } 2y = 5 - x$$

obtain the value of the correlation coefficient. [**Ans.** – 0.866]

28. Two lines of regression are given by $x + 2y = 5$ and $x + 3y = 9$ and $\sigma^2_x = 12$. Calculate the values of $\bar{x}$, $\bar{y}$, σ^2_y and r. [**Ans.** $\bar{x} = 1$, $\bar{y} = 2$, $\sigma_y^2 = 4$, $r = -0.87$]

29. The lines of regression of y on x and x on y are respectively $y = x + 5$ and $16x = 9y - 94$. Find the variance of x if the variance of y is 16. Also find the covariance of x and y. [**Ans.** $\sigma_x^2 = 9$; cov $(x, y) = 9$]

30. From the data given below find:
   (*a*) The two regression equations.
   (*b*) The coefficient of correlation between the marks Economics and Statistics.
   (*c*) The most likely marks in Statistics when marks in Economics are 30.

Marks in Economies	25	28	35	32	31	36	29	38	34	42
Marks in Statistics	43	46	49	41	36	32	31	30	33	39

[***D.U. '82***] $\left[\textbf{Ans.}\ (a)\ y = 59.1 - 0.66x; x = 40.7 - 0.23y\ (b)\ r = 0.389\ (c)\ y = 39.3\right]$

31. Calculate the (*i*) two regression coefficient, (*ii*) coefficients of correlation, and (*iii*) the two regression equations from the following informations:

$$N = 10, \Sigma x = 350, \Sigma y = 310, \Sigma(x-35)^2 = 162, \Sigma(y-31)^2 = 222,$$
$$\Sigma(x-35)(y-31) = 92$$

[***D.U. B.Com. '92***]

$\left[\textbf{Ans.}\ b_{xy} = 0.41,\ b_{yx} = 0.57,\ r = 0.24, x = 22.15 + 0.41y, y = 11.12 + 0.57x\right]$

32. In studying a set of pairs of related variates a statistician has completed the preliminary calculations. The results are as follows:

$$n = 16, \Sigma x = 749, \Sigma y = 77.90, \Sigma x^2 = 42.177, \Sigma y^2 = 454.81, \Sigma xy = 3{,}156.80$$

Compute the linear regression equation of x on y.

[***C.U. B.Com. (Hons)'81***] [**Ans.** $x = 78.4 - 6.49y$]

33. Find the regression equation of x on y from the following data:

$$\sigma_x = 24, \Sigma y = 44, \Sigma xy = 306, \Sigma x^2 = 164, \Sigma y^2 = 574, N = 4$$

[**Ans.** $x = 0.467y + 0.863$, $x = 3.7$]

34. Find the two regression equations from the following data:

$$n = 20, \Sigma x = 80, \Sigma y = 40, \Sigma xy = 480, \Sigma x^2 = 1680, \Sigma y^2 = 320$$

[**Ans.** $y = 0.235x + 1.06$, $x = 1.33y + 1.334$]

35. Calculate the regression coefficients from the following informations:

$$\Sigma x = 50, \Sigma y = 30, \Sigma xy = 1000, \Sigma x^2 = 3000, \Sigma y^2 = 1800, N = 10$$

[***C.A. (F) Nov.'95***] [**Ans.** $b_{yx} = 0.31$; $b_{xy} = 0.50$]

36. To study the relationship between expenditure on accommodation, (₹ x) and Expenditure on Food and Entertainment, (₹ y), an enquiry into 50 families gave the following results.

$$\sum x = 8500, \sum y = 9600, \sigma_x = 60, \sigma_y = 20, r = 0.6.$$

Estimate the Expenditure on Food and Entertainment when Expenditure on accommodation is ₹ 200. [***C.A. (Inter) Nov. '79***] [**Ans.** ₹ 198]

37. For a bivariate data, you are given the following information: $\Sigma(x - 58) = 46$, $\Sigma(x - 58)^2 = 3086$, $\Sigma(y - 58) = 9$, $\Sigma(y - 58)^2 = 483$, $\Sigma(x - 58)(y - 58) = 1905$ Number of pairs of observations = 7.

You are required to determine (*i*) the two regression equations and (*ii*) the coefficient of correlation between x and y series.

[***D.U. B.Com. (Hons.) '88***] [**Ans.** $y = 0.37x + 35.399$; $x = 2.20y - 65.87$; 0.902]

D. MISCELLANEOUS

38. You are given the following data:

	X	**Y**
A.M.	36	85
S.D.	11	8

If correlation coefficient between X and Y is 0.66 find the two regression equations. Also find the value of X when $Y = 33.5$ and that of Y when $X = 73.3$ from the two regression lines.

[**Ans.** $X = 0.9075Y - 41.1375$; $Y = 0.48X + 67.72$; -10.73625; 102.904]

39. Find the means of X and Y variables and the coefficient of correlation between them from the following two regression equations:

$$2y - x - 50 = 0; 3y - 2x - 10 = 0$$

[***D.U. B.Com. (Hons.) '83***] [**Ans.** $\bar{x} = 130$, $\bar{y} = 90$, 0.866]

40. The following data about the sales and advertisement expenditure of a firm is given below:

	Sales (*in crores of* ₹)	***Advertisement expenditure*** (*in crores of* ₹)
Mean	40	6
Standard deviations	10	1.5

Coefficient of correlation = $r = 0.9$

(*i*) Estimate the likely sales for a proposed advertisement expenditure of ₹ 10 crores.

(*ii*) What should be the advertisement expenditure if the firm proposes a sales target of 60 crores of rupees? [***D.U.B.Com. (Hons.) '85***]

[**Ans.** (*i*) X (sales) = $6Y$ (Adv.exp) + 4, ₹ 64 crores
(*ii*) $Y = 0.135\,X + 0.60$, ₹8.70 crores]

41. The following data relate to marks in Advanced Accounts and Business Statistics in B.Com.(Hons.) 1st year examination of a particular year in Delhi University:

Mean marks in Advanced Accounts = 30

Mean Marks in Business Statistics = 35

Standard Deviation of Marks in Advanced Accounts = 10

Standard Deviation of Marks in Business Statistics = 7

Coefficient of correlation between the marks of Advanced Accounts and Business Statistics = 0.8

Form the two regression lines calculate the expected marks in Advanced Accounts if the marks secured by a student in Business Statistics are 40.

[***D.U. B.Com. (Hons.) '87***] [**Ans.** Marks in Advanced Account = X, Marks in Business Statistics = Y; $Y = 0.5X + 18.2$; $X = \frac{8}{7}Y - 10$; 36 Marks]

42. In order to find the correlation coefficient between two variables x and y from 12 pairs of observations, the following calculations were made: $\Sigma x = 30$; $\Sigma x^2 = 670$; $\Sigma y = 5$; $\Sigma y^2 = 285$; $\Sigma xy = 344$ on subsequent verification it was discovered that the pair ($x = 11$, $y = 4$) was copied wrongly, the correct values being $x = 10$, $y = 14$. After making necessary corrections, find: (*a*) the two regression coefficients. (*b*) the two regression equations (*c*) the correlation coefficient.

[***D.U. B.Com. (Hons.) '90***]

[**Ans.** (*a*) 0.898, 0.694 (*b*) $x = 0.898y + 1.294$, $y = 0.694x - 0.427$ (*c*) 0.79]

43. A panel of judges A and B graded seven debators and independently awarded the following marks:

Debator:	1	2	3	4	5	6	7
Marks by *A*:	40	34	28	30	44	38	31
Marks by *B*:	32	39	26	30	38	34	28

An eighth debtor was awarded 36 marks by judge A while judge B was not present.

If judge B was also present, how many marks would you expect him to award to eighth debator assuming same degree of relationship exists in judgment?

[***D.U. B.Com. (H) '93***] [**Ans.** 33 (Approx).]

[**Hints:** $y = 0.587x + 11.885$. Putting $x = 36$ we get $y = 33.017$ or 33 Approx.]

44. The line of regression of marks in Statistics (x) on marks in Accountancy (y) for a class of 50 students is $3y - 5x + 180 = 0$. Average marks in Accountancy is 44 and variance of marks in Statistics is $\frac{9}{16}$ th of variance of marks in Accountancy. Find: (*i*) the average marks in statistics; and (*ii*) coefficient of correlation between marks in Statistics and marks in Accountancy.

[***D. U.B.Com (H) '94***] [**Ans.** (*i*) 62.4 marks (*ii*) 0.8]

E. MULTIPLE CHOICE QUESTIONS (MCQs)

(i) Short Type

Mark the correct alterative in each of the following:

1. If $bxy = -\frac{3}{20}, byx = -\frac{3}{5}$, then the value of r_{xy} is [***C.U. B.Com. 2013 (G)***]

(*a*) –0.25 (*c*) –0.4

(*b*) –0.3 (*d*) $-\frac{4}{5}$ **[Ans.** (*b*)**]**

2. If regression coefficient of y on x is –0.4 and regression coefficient of x on y is –0.9, the correlation coefficient between x and y is

[***C.U. B.Com. 2013 (H)***]

(*a*) –0.6 (*c*) –0.45

(*b*) –0.5 (*d*) –0.7 [**Ans.** (*a*)]

3. Two lines of regression are given by $3x - 2y = 5$, $2x - y = 4$, the value of $\bar{x}$ and $\bar{y}$ are [***C.U. B.Com. 2012***]

(*a*) (3 and 2) (*c*) (4 and 3)

(*b*) (5 and 4) (*d*) (3 and 1) [**Ans.** (*a*)]

4. If $\sigma_x = 10$, $\sigma_y = 12$, $b_{xy} = -0.8$, the value of r_{xy} is

(*a*) 0.75 (*c*) –0.96

(*b*) 0.89 (*d*) –0.98 [**Ans.** (*c*)]

5. $\sigma_x = 36, \bar{y} = 30, \bar{x} = 36, r = 0.8, \sigma_y = 32$
   The coefficient of regression of *x* and *y* from the above information is
   (*a*) 0.48 (*c*) 0.40
   (*b*) 0.55 (*d*) 0.90 [**Ans.** (*d*)]

6. In simple linear regression, the numbers of unknown constants are:
   (*a*) one (*c*) three
   (*b*) two (*d*) four [**Ans.** (*b*)]

7. If the value of any regression coefficient is zero, then two variables are.
   (*a*) dependent (*c*) qualitative
   (*b*) independent (*d*) correlated [**Ans.** (*b*)]

8. In the regression equation $y = a + bx$, the y is called
   (*a*) dependent variable (*c*) qualitative variable
   (*b*) independent variable (*d*) none of the above [**Ans.** (*a*)]

9. When b_{xy} is positive, then b_{yx} will be:
   (*a*) zero (*c*) negative
   (*b*) one (*d*) positive [**Ans**. (*d*)]

10. Regression coefficient is independent of
    (*a*) units of measurement (*c*) both (a) and (b)
    (*b*) scale and origin (*d*) none of these [**Ans.** (*c*)]

11. When the two regression lines are parallel to each other, then their slopes are:
    (*a*) zero (*c*) same
    (*b*) different (*d*) positive [**Ans.** (*c*)]

12. In the regression equation $y = a + bx$, *a* is called
    (*a*) *x*-intercept (*c*) dependent variable
    (*b*) *y*-intercept (*d*) none of the above [**Ans.** (*b*)]

13. The regression equations always passes through:
    (*a*) (x, y) (*c*) $(\bar{x}, \bar{y})$
    (*b*) (a, b) (*d*) $(\bar{x}, y)$ [**Ans.** (*c*)]

14. The straight line graph of the linear equation $y = a + bx$, slope will be downward if:
    (*a*) $b < 0$ (*c*) $b = 0$
    (*b*) $b > 0$ (*d*) $b \neq 0$ [**Ans.** (*a*)]

15. If $y = -10x$ and $x = -0.1y$, then *r* is equal to
    (*a*) 10 (*c*) 1
    (*b*) 0.1 (*d*) –1 [**Ans.** (*d*)]

16. If $r_{xy} = 1$, then
    (*a*) $b_{yx} = b_{xy}$ (*c*) $b_{yx} < b_{xy}$
    (*b*) $b_{yx} > b_{xy}$ (*d*) $b_{yx} \cdot b_{xy} = 1$ [**Ans.** (*d*)]

17. If $r_{xy} > 0$, then b_{yx} and b_{xy} are both
    (*a*) 0 (*c*) > 0
    (*b*) < 0 (*d*) < 1 [**Ans.** (*c*)]

18. If $b_{xy} = 0.20$ and $r_{xy} = 0.50$, then b_{yx} is equal to
(*a*) 0.20 (*c*) 0.50
(*b*) 0.25 (*d*) 0.125 [**Ans.** (*d*)]

19. If $b_{xy} = -0.5$ and $r_{xy} = -1$, then b_{yx} is equal to
(*a*) –1 (*c*) –0.5
(*b*) –2 (*d*) 0.5 [**Ans.** (*b*)]

20. If $\sigma_x = 10$, $\sigma_y = 12$ and $b_{xy} = -0.8$, then the value of r is
(*a*) 0.92 (*c*) –0.96
(*b*) –0.86 (*d*) 0.89
[***V.U. B.Com 1992***] [**Ans.** (*c*)]

(ii) Short Essay Type

21. $x = 0.64y + 19.10$; $y = x + 5.25$
The regression coefficient b_{xy} from the above details is
(*a*) 0.85 (*c*) 0.98
(*b*) 0.64 (*d*) 1 [**Ans.** (*b*)]

22. $9x = 5y + 9.10$; $y = 3x - 7$
The regression coefficient b_{xy} from the above details is
(*a*) $\frac{5}{9}$ (*c*) 1.08
(*b*) $\frac{9}{5}$ (*d*) 2.3 [**Ans.** (*a*)]

23. $x = \frac{7}{3}y + 28.10$; $y = 1.5x + 10$
The regression coefficient b_{yx} from the above details is
(*a*) 2.9 (*c*) 3/77
(*b*) 1.5 (*d*) 7/3 [**Ans.** (*b*)]

24. If the regression coefficient b_{xy} is 2.5, the value of a in the given equation $2x = ay + 12.6$ is
(*a*) 4 (*c*) 3.32
(*b*) 2.5 (*d*) 5.0 [**Ans.** (*d*)]

25. If the regression coefficient b_{yx} is 3.0, the value of b in given equation $2y = bx +$ 18 is
(*a*) 2.5 (*c*) 6.0
(*b*) 1.5 (*d*) 4.0 [**Ans.** (*c*)]

26. From the regression equations $2x - 8y + 60 = 0$ and $40x - 18y - 220 = 0$, the value of b_{xy} and b_{yx} are
(*a*) $\left(\frac{9}{20}, \frac{1}{4}\right)$ (*b*) $\left(\frac{19}{20}, \frac{2}{5}\right)$

(*c*) $\left(\frac{8}{20}, \frac{4}{5}\right)$ (*d*) $\left(\frac{15}{20}, \frac{1}{5}\right)$ **[Ans.** (*a*)]

27. From the regression equations $8x - 10y + 66 = 0$ and $40x - 18y - 214 = 0$, the value of mean x and mean y are
(*a*) (19,21) (*c*) (11,15)
(*b*) (13,17) (*d*) (16,19) **[Ans.** (*b*)]

28. If $r_{xy} = 0.5$, $\sigma_x = 1.5$, $\sigma_y = 3.0$, $\bar{x} = 10$ and $\bar{y} = 8$, the regression line of x on y is
(*a*) $x = 0.25y + 8$ (*c*) $x = 1.54y - 10.6$
(*b*) $x = 0.56y + 9$ (*d*) $x = 0.34y - 8$ **[Ans.** (*a*)]

29. If $r_{xy} = 0.6$, $\sigma_x = 4$, $\sigma_y = 1.33$, $\bar{x} = 15$, and $\bar{y} = 10$, the regression line y on x is
(*a*) $y = 2.6x - 14$ (*c*) $y = 1.5x - 10$
(*b*) $y = 1.88x - 15$ (*d*) $y = 0.2x + 7$ **[Ans.** (*d*)]

30. Average rainfall in W.B. = 40 cm, standard deviation of rainfall = 3 cm, mean of paddy yield = 800 quintal, standard deviation of paddy production = 10 quintal, correlation = 0.6, the estimate of production of paddy in 2017 corresponding to the estimate of 72 cm rainfall is
(*a*) 978 quintal (*c*) 753.84 quintal
(*b*) 640.9 quintal (*d*) 773 quintal **[Ans.** (*c*)]

31. If $n = 10$, $\sigma_x = 15$, $\sigma_y = 10$ and $\Sigma(x - \bar{x})(y - \bar{y}) = 980$, then b_{yx} is
(*a*) 0.435 (*c*) 0.413
(*b*) 0.529 (*d*) 0.517 **[Ans.** (*a*)]

32. The regression equation of y on x and x on y are $y = x + 5$ and $16x = 9y - 94$ respectively. If the variance of y is 16, the standard deviation of x is

[*V.U. B.Com 1990*]

(*a*) 2 (*c*) 4
(*b*) 3 (*d*) 5 **[Ans.** (*b*)]

33. If $\bar{x} = 5, \bar{y} = 12, \sigma_x = 3, \sigma_y = 4$ and $r = 0.75$, the regression equation of x on y is
(*a*) $x = 0.42y + 1.15$ (*c*) $x = 0.45y + 2.32$
(*b*) $x = 0.5y - 2.32$ (*d*) $x = 0.56y - 1.75$ **[Ans.** (*d*)]

34. The regression equation of x on y from the following data $b_{yx} = -1.2$, $b_{xy} = -0.3, \bar{x} = 10, \bar{y} = 12$ is
(*a*) $x = -0.3y + 13.6$ (*c*) $x = 0.29y + 11.25$
(*b*) $x = 0.42y - 12.3$ (*d*) $x = -0.23y + 12.7$ **[Ans.** (*a*)]

35. Two lines of regression are given by $x + 2y = 5$ and $2x + 3y = 8$. The values of $\bar{x}$ and $\bar{y}$ are [*V.U. B.Com.'97*]
(*a*) 3, 4 (*c*) 1, 2
(*b*) 2, 3 (*d*) 2, 5 **[Ans.** (*c*)]

36. The regression coefficient of y on x for the following information:
$\Sigma x = 50, \Sigma y = 30, \Sigma xy = 1000, \Sigma x^2 = 3000, \Sigma y^2 = 1800, \text{N} = 10$ is

(*a*) 0.35 (*c*) 0.41
(*b*) 0.31 (*d*) 0.45 [**Ans.** (*b*)]

37. The regression coefficient of y on x is 4 and that of x on y is 1/9. The value of r is 2/3. If standard deviation of y is 12, then the standard deviation of x is
(*a*) 2 (*c*) 4
(*b*) 3 (*d*) 5 [**Ans.** (*a*)]

38. The regression equation of x and y from the following data is:
$\Sigma x = 24, \Sigma y = 44, \Sigma xy = 306, \Sigma x^2 = 164, \Sigma y^2 = 574, n = 4$
(*a*) $x = 0.52y + 0.29$ (*c*) $x = 0.467y + 0.86$
(*b*) $x = 0.37y + 1.73$ (*d*) $x = 0.49y - 3.23$ [**Ans.** (*c*)]

39. The value of σ_y for the following data $3x - y, 4y = 3x$ and $\sigma_x = 2$ is
(*a*) 6 (*c*) 4
(*b*) 5 (*d*) 3 [**Ans.** (*d*)]

40. Given $b_{xy} = 0.85, b_{yx} = 0.89$ and the standard deviation of $x = 6$. Then the value of σ_y is
(*a*) 5.69 (*c*) 6.25
(*b*) 6.14 (*d*) 5.78 [**Ans.** (*b*)]

CHAPTER 8

Index Numbers

SYLLABUS

Meaning and Types of Index Numbers, Problems of Constructing Index Numbers, Construction of Price and Quantity Indices, Test of Adequacy, Errors in Index Numbers, Chain Base Index Numbers; Base Shifting, Splicing, Deflating, Consumer Price Index and its Uses

THEMATIC FOCUS

8.1 INTRODUCTION

Economic activities have a tendency to change and it occurs due to the change in an economic variable or a group of economic variables. It is necessary to measure such change of economic activities with the help of a statistical device.

Index number is such a statistical device which measures percentage changes of an economic variable or variables to derive a conclusion about the change of the economic activities over a specified time. For example, we want to measure the change in the average retail price of tea in 2017 with that in 2010 or we want to measure the retail price of Darjeeling tea with that of Assam tea. We may also like to know the increase in the yield of paddy in India during 2017 as compared to 2010. We can easily measure such changes by expressing the price or quantity of a given period as percentage of the price or quantity in the base period (taken as 100), i.e. the period with which comparison is made. These percentages which measure changes in the prices or quantities at a specified time with that of the base period is known as **index number**. Suppose, average retail price of tea per kg. in 2010 and 2017 are ₹ 120 and ₹ 150 respectively. Then index number of tea for the year 2017 is 125 $\left(\frac{150}{120}\times 100\right)$, taking base year as 2010, means there is an increase of 25% in the average retail price of tea as compared to the corresponding figure for the year 2010.

The problems arise when we want to measure changes in a complex phenomenon like agricultural production or industrial production, wages, export, import, cost of living, etc. This involves measuring changes in the prices or quantities of many commodities. Since the prices or quantities of different commodities change by different degrees or units, they can not be measured directly. For example, changes in agricultural production in a country are not capable of direct measurement, but it is possible to study relative changes in agricultural production by studying the changes in the values of some such variables which affect agricultural production, and which are capable of direct measurement. So, we have to consider all items of production for measuring the combined changes in a group related variables and each item may have undergone a different fractional increase (or even a decrease). Index number provides such composite measurement which may be defined as a device for combining the changes or variations that have come in groups of related variables over a period of time, with a view to obtain a figure that represents the 'net' result of the change in the constitute variables.

In reality, Index numbers are used to feel the pulse of the economy and they reveal the inflationary or deflationary tendencies. Hence, index numbers are described as barometers of economic activity. If one wants to have an idea as to what is happening in an economy, he should check the important indicators, like the index numbers of industrial production, agricultural production, business activity, etc.

8.1.1 Meaning of Index Number

An index number is a statistical measure designed to show changes in a variables or a group of related variables with respect to time, geographic location or other characteristics such as income, profession, etc.

8.2 TYPES OF INDEX NUMBERS

There are three types of index numbers which are commonly used.

8.2.1 Price Index Numbers

Price index numbers measure the relative changes in prices of a commodity between two periods. It is basically the ratio of the price of a certain number of commodities in the present year as against base year. The Whole Price Index (WPI), Consumer Price Index (CPI) are some of the popularly used price indices.

8.2.2 Quantity Index Numbers

Quantity Index Numbers measure the changes in the physical quantity of goods produced, consumed or sold of an item or a group of items from one period to another. The index of industrial production is the popularly used quantity index.

8.2.3 Value Index Numbers

The value index combines price and quantity changes to present a more spatial comparison. It is also known as a combination index. These pertain to compare changes in the monetary value of imports, exports, production or consumption of commodities. It has limited use as it can not distinguish the effects of price and quantity separately.

8.3 CHARACTERISTICS OF INDEX NUMBERS

1. Index Numbers are Specialized Averages

An index number is an average with a difference. Averages can be used to compare only those series which are expressed in the same units. Whereas index number is obtained as a result of an average of all items which are expressed in different units.

2. Index Numbers Measure Changes that are not Directly Measurable

An index number is used for measuring the magnitude of changes in phenomenon that are not directly measurable. The cost of living, business activity in a country are not directly measurable but it is possible to study relative changes in these activities by measuring the changes in the values of variables/factors which effect these activities.

3. Index Numbers are Expressed as Percentage

An index number is calculated as a ratio of the current value to a base value and expressed in terms of percentages to show the extent of relative change. The index number for the base year is always taken as 100.

4. Expressed in Number

Index number can be expressed in number only.

5. Universal Utility

Changes in the quantity of agricultural production, industrial production, imports and exports can also measured through index numbers.

8.4 USES OF INDEX NUMBERS

- **Helps in policy formulation:** Index numbers measure trends of various phenomena and on the basis of these trends and tendencies the government can formulate different policies, such as determining the rates of dearness allowance, price policies, etc.
- **Helps in studying trends:** Index numbers reveal a general trend of the phenomenon under study such as trend of exports, imports, national income, etc.
- **Helps in determining purchasing power of money:** Index numbers measure the purchasing power of money and determine the real wages.
- **Helps in deflating various values:** Index numbers play a vital role in adjusting the original data to reflect reality. For example, nominal income can be transformed into real income by using income deflators. Index numbers are also helpful in deflating national income on the basis of constant prices.

- **Helps in measuring effectiveness:** Index numbers measure the effectiveness of teaching system in the field of health. It can be used to show the general health condition of the people and to indicate the adequacy of hospital facilities.

8.5 PROBLEMS OF CONSTRUCTING INDEX NUMBERS

Before constructing index numbers, a careful thought must be given to the following problems:

- **The purpose of the index:** At the very outset the purpose of constructing the index must be clearly decided. There cannot be any all-purpose index. Every index is of limited and particular use. Thus, a price index that is intended to measure consumers' prices must not include wholesale prices. Therefore, it is important to be clear about the purpose of the index number before its construction.
- **Selection of the items:** The second problem in the construction of index numbers is the selection of the items. The items which are related to and are relevant with the purpose for which the index is constructed should be included.
- **Selection of price quotation:** After the items have been selected, the next problem is to obtain price quotations for these items. It is a well-known fact that prices of many items vary from place to place and even from shop to shop in the same market. It is neither possible nor necessary to collect prices of the items from all markets in the country where it is dealt with, we should take a sample of the markets. A selection must be made to represent places and persons. These places should be well known for trading these items.
- **Selection of the base year:** The base year is defined as that year with reference to which the price changes in other years are compared and expressed as percentages. It is therefore necessary that

  (*i*) the base year should be normal, and

  (*ii*) it should not be too far in the past.

  There are two methods for selecting the base year. One method is the selection of a certain year as a base year, known as fixed base method. While the other is chain base method, where relatives of each year are calculated on the basis of the prices of the preceding year. The chain base index numbers are called Link Relatives.
- **Selection of the average:** As index numbers are themselves specialised averages, it has to be decided first as to which average should be used for their construction. The choice lies between arithmetic mean and geometric mean alone as other measures of averages are rarely used. Theoretically, geometric mean is the best for this purpose. But, in practice, arithmetic mean is used because it is easier to follow.
- **Selection of the weights:** Generally, all the items included in the construction of index numbers are not equally important. Therefore, if the index numbers are to be representative, proper weights should be assigned to various items in relation to their relative importance. A more important

item will get more weight. Because of a rise in the prices of essential items the poor consumer is more affected than others. Weights should be unbiased and be rationally and not arbitrarily selected.

- **Selection of an appropriate formula:** A large number of formulae have been devised for constructing the index. The problem is that of selecting the most appropriate formula. The choice of the formula would depend on the purpose of the index as well as on the data available.

8.6 COMMON NOTATIONS

The following notations will be used to derive the various mathematical formulae of index number.

p_o = Price per unit in the base year (denoted by suffix *o*)
p_n = Price per unit in the current year (denoted by suffix *n*)
q_o = Quantity in the base year (denoted by suffix *o*)
q_n = Quantity in the current year (denoted by suffix *n*)
P_{on} = Price index number for the current or given year *n* with respect to the base year *o*
Q_{on} = Quantity index number for the current or given year *n* with respect to the base year *o*
I_{on} = Index number for the current or given year *n* with respect to the base year *o*
I_{no} = Index number for the current or given year *o* with respect to the base year *n*

8.7 METHODS OF CONSTRUCTING INDEX NUMBERS

Several methods have been considered for constructing index numbers. Principal methods are illustrated in Chart 8.1.

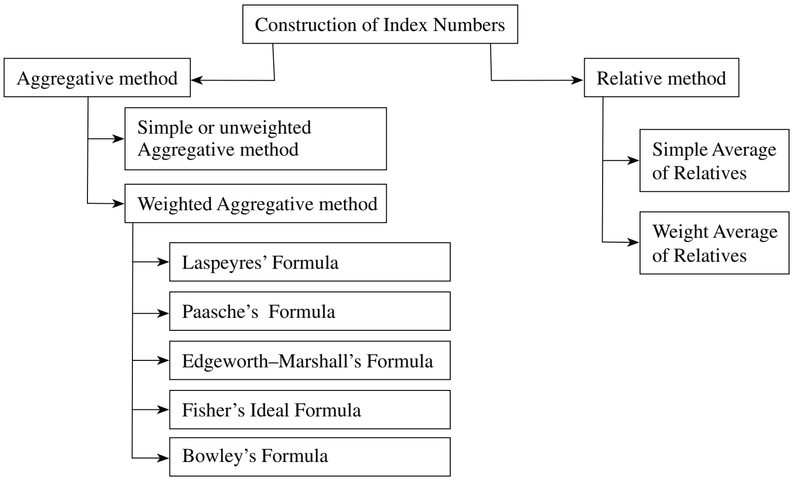


Chart 8.1 Methods for Construction of Index Numbers

In this section we shall be mainly interested in Price Index Numbers showing changes with respect to time, although methods stated above can be applied to other cases. In general, the present level of prices is compared with the level of prices in the past. The present period is called the current period or year and some period in the past is called the base period or year.

8.7.1 Aggregative Method

In aggregative method, the aggregate (average) price of all items in a given year is expressed as a percentage of the aggregate (average) price of all items in the base year, giving the index number. Hence,

$$\text{Price Index Number} = \frac{\text{Aggregate price in the given or current year}}{\text{Aggregate price in the base year}} \times 100$$

(*i*) **Simple or unweighted aggregative method:** Under this method the total of the current year prices for various items is divided by the total of the base year and multiplying the result by 100.

Symbolically,

$$I_{on} = \frac{\Sigma p_n}{\Sigma p_o} \times 100 \qquad \ldots (8.1)$$

ILLUSTRATION 1

Construct the price index number for 2017, taking the year 2010 as the base year by using simple aggregative method:

Commodity	Price in the year 2010	Price in the year 2017
A	30	40
B	25	30
C	35	50
D	60	80
E	50	75

Solution: **Calculation of price index number by simple aggregative method**

Commodity	Price in 2010 (₹) (p_o)	Price in 2017 (₹) (p_n)
A	30	40
B	25	30
C	35	50
D	60	80
E	50	75
Total	$\Sigma p_o = 200$	$\Sigma p_n = 275$

$$\text{Price Index Number} = \frac{\Sigma p_n}{\Sigma p_o} \times 100 = \frac{275}{200} \times 100 = 137.5$$

Therefore, the price index for the year 2017, taking 2010 as base year, is 137.5, showing that there is an increase of 37.5% in the prices in 2017 as against 2010.

(*ii*) **Weighted aggregative method:** This method is same as simple aggregative method with the only difference that the weights are assigned to the various items included in the index.
Symbolically,

$$I_{on} = \frac{\Sigma p_n w}{\Sigma p_o w} \times 100 \text{ [where } w \text{ represents the 'weight']} \qquad \text{...(8.2)}$$

The importance of the price of a commodity in the overall picture described by an index number is generally determined by its quantity produced, consumed, marketed or sold. We, therefore, use the **quantities** in the base year or given year or the average of several years as weights. There are various formulae of assigning weights to an index. The more important ones are as follows:

(*a*) **Laspeyres' Formula:** In this formula base year quantities (q_o) are used as weights. Therefore, substituting w by q_o in equation 8.2 we get Laspeyres' price index formula.

$$\text{Laspeyres' Price Index } (I_{on}) = \frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times 100 \qquad \text{...(8.3)}$$

(*b*) **Paasche's Formula:** In this formula current year quantities (q_n) are used as weights. Therefore, substituting w by q_n in equation 8.2 we get Paasche's Price Index Formula.

$$\text{Paasche's Price Index } (I_{on}) = \frac{\Sigma p_n q_n}{\Sigma p_o q_n} \times 100 \qquad \text{...(8.4)}$$

(*c*) **Edgeworth–Marshall's Formula:** In this formula the average of the quantities of the base year and the current year are used as weights. Therefore, substituting w by $\left(\frac{q_o + q_n}{2}\right)$ in equation 8.2 we get Edgeworth–Marshall's Price Index Formula.

Edgeworth–Marshall's Price Index

$$(I_{on}) = \frac{\Sigma p_n\left(\dfrac{q_o + q_n}{2}\right)}{\Sigma p_o\left(\dfrac{q_o + q_n}{2}\right)} \times 100 = \frac{\Sigma p_n q_o + \Sigma p_n q_n}{\Sigma p_o q_o + \Sigma p_o q_n} \times 100 \qquad ...(8.5)$$

(*d*) **Fisher's Ideal Formula:** This formula is the geometric mean of Laspeyres' formula and Paasche's formula and is given by:

Fisher's Ideal Price Index

$$(I_{on}) = \sqrt{\text{Laspeyres' Price Index} \times \text{Paasche's Price Index}}$$

$$= \sqrt{\frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times \frac{\Sigma p_n q_n}{\Sigma p_o q_n}} \times 100 \qquad ...(8.6)$$

Fisher's formula is known as 'Ideal' because of the following reasons:

(*i*) It is based on the geometric mean which is theoretically considered the best average for the construction of index numbers.

(*ii*) Both the current year and base year prices and quantities are taken into account by this index.

(*iii*) It satisfies both the time reversal test and factor reversal test, i.e. the tests of adequacy of index numbers.

(*iv*) It neutralizes the upward bias of Laspeyres' index and downward bias of Paasche's index to a great extent. In fact, Fisher's ideal index is free from any bias.

(*e*) **Bowley's Formula:** This formula is the arithmetic mean of Laspeyres' formula and Paasche's formula and is given by:

Bowley's Price Index $(I_{on}) = \frac{1}{2}$ [Laspeyres' Price Index + Paasche's Price Index]

$$= \frac{1}{2}\left[\frac{\Sigma p_n q_o}{\Sigma p_o q_o} + \frac{\Sigma p_n q_n}{\Sigma p_o q_n}\right] \times 100 \qquad ...(8.7)$$

ILLUSTRATION 2

Find the index number by using weighted aggregative method from the following data:

Commodities	Base Price (2014)	Current Price (2017)	Weight
Rice	36	54	10
Dal	30	50	3
Fish	130	155	2
Potato	40	35	4
Oil	110	110	5

Solution: **Calculation of Price Index Number by weighted aggregative method**

Commodities	Base Price (2014) (p_o)	Current Price (2017) (p_n)	Weight (w)	$p_o w$	$p_n w$
Rice	36	54	10	360	540
Dal	30	50	3	90	150
Fish	130	155	2	260	310
Potato	40	35	4	160	140
Oil	110	110	5	550	550
Total				$\Sigma p_o w =$ 1420	$\Sigma p_n w =$ 1690

$$\text{Price Index Number} = \frac{\Sigma\, p_n w}{\Sigma\, p_o w} \times 100 = \frac{1690}{1420} \times 100 = 119.01$$

Therefore, the price index for the year 2017, taking 2014 as base year is 119.01.

ILLUSTRATION 3

Using (*i*) Laspeyres' (*ii*) Paasche's (*iii*) Edgeworth-Marshall (*iv*) Fisher (*v*) Bowley's formulae, find the index number from the following data:

Commodity	Price per unit		Unit	
	Base Year	Current Year	Base Year	Current Year
A	4	10	50	40
B	3	8	10	8
C	2	4	5	4

Solution: **Calculation of Price Index Numbers**

Commodity	Base year price (p_o)	Current year price (p_n)	Base year qty. (q_o)	Current year qty. (q_n)	$p_o q_o$	$p_n q_o$	$p_o q_n$	$p_n q_n$
A	4	10	50	40	200	500	160	400
B	3	8	10	8	30	80	24	64
C	2	4	5	4	10	20	8	16
Total	–	–	–	–	240	600	192	480

(*i*) Laspeyres' Price Index Number = $\frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times 100 = \frac{600}{240} \times 100 = 250$

(*ii*) Paasche's Price Index Number = $\frac{\Sigma p_n q_n}{\Sigma p_o q_n} \times 100 = \frac{480}{192} \times 100 = 250$

(*iii*) Edgeworth-Marshall's Price Index Number

$$= \frac{\Sigma p_n q_o + \Sigma p_n q_n}{\Sigma p_o q_o + \Sigma p_o q_n} \times 100 = \frac{600 + 480}{240 + 192} \times 100 = \frac{1080}{432} \times 100 = 250$$

(*iv*) Fisher's Price Index Number = $\sqrt{\text{Laspeyres' Index} \times \text{Paasche's Index}}$

$$= \sqrt{250 \times 250} = 250$$

(*v*) Bowley's Price Index Number = $\frac{1}{2}$ [Laspeyres' Index + Paasche's Index]

$$= \frac{1}{2}\,[250 + 250] = 250$$

8.7.2 Relative Method

Price Relative

The price of each item in the current year is expressed as a percentage of price in base year. This is called price relative and expressed as the following formula:

$$\text{Price Relative} = \frac{\text{Price in the given or current year}}{\text{Price in the base year}} \times 100 = \frac{P_n}{p_o} \times 100$$

(*i*) **Simple Average of Relatives Method:** In this method, the price relatives for all items is calculated and then averaged to get the index number. Thus,

(*a*) If Arithmetic Mean (A.M.) is taken as average, then we have simple

A.M. of Relative Index Number (I_{on}) = $\frac{\Sigma\left(\frac{p_n}{p_o} \times 100\right)}{N}$ [where N is the number of items]

(*b*) If Geometric Mean (G.M.) is taken as average, then we have Simple

G.M. of Relative Index Number (I_{on}) = $\sqrt[N]{(\text{Product of Price Relatives})}$

(*ii*) **Weighted Average of Relatives Method** In this method, either current year values ($p_n.q_n$) or base year values ($p_o q_o$) are used as weight. The index number for the current year is calculated by dividing the sum of the products of the current year's price relatives and base year or current year values by the total of the weights. Thus,

(*a*) If Arithmetic Mean (A.M.) is taken as average, then we have weighted A.M. of Relative Index Number (I_{on}) =

$$\frac{\Sigma\left(\frac{p_n}{p_o}\times 100\right).V}{\Sigma V} \text{ or } \frac{\Sigma P.V}{\Sigma V}$$

where V = value weights and P = $\frac{p_n}{p_o} \times 100$

Taking $V = p_o q_o$, Laspeyres' Index Number and $V = p_o q_n$, Paasche's Index Number can be derived from above formula.

(*b*) If Geometric Mean (G.M.) is taken as average, then we have weighted G.M. of Relative Index Number (I_{on}) = antilog $\left[\frac{\Sigma(\log P).V}{\Sigma V}\right]$

where P = $\frac{p_n}{p_o} \times 100$ for every item, and V = value weight.

ILLUSTRATION 4

Compute price index using simple average and weighted average of price relatives by applying (a) A.M. and (b) G.M.

Commodity	**Base year price (₹)**	**Current year price (₹)**	**Base year quantity**
P	52	71	2
Q	12	16	4
R	30	27	3
S	5	6	1

Solution: **Calculation of Price Index Numbers**

Commodity	**Base year price (p_o)**	**Current year price (p_n)**	**Base year qty. (q_o)**	$V = p_o q_o$	$P = \frac{p_n}{p_o} \times 100$	**P.V.**	**log P**	**V.log P**
P	52	71	2	104	136.54	14200.16	2.1352	222.0608
Q	12	16	4	48	133.33	6399.84	2.1250	102.0000
R	30	27	3	90	90.00	8100.00	1.9542	175.8780
S	5	6	1	5	120.00	600.00	2.0792	10.3960
Total	–	–	–	247	479.87	29300.00	8.2936	510.3348

(*i*) Simple A.M. of Price Relative Index Number

$$= \frac{\Sigma\left(\frac{p_n}{p_o}\times 100\right)}{N} = \frac{479.87}{4} = 119.9675$$

(*ii*) Simple G.M. of Price Relative Index Number

$$= \sqrt[n]{\text{Product of Price Relatives}}$$

$$= \sqrt[4]{136.54\times 133.33\times 90\times 120}$$

Let I be the Index Number

Therefore, $\quad I = \sqrt[4]{136.54\times 133.33\times 90\times 120}$

$$= (136.54 \times 133.33 \times 90 \times 120)^{\frac{1}{4}}$$

or $\quad \log I = \frac{1}{4}$ (log 136.54 + log 133.33 + log 90 + log 120)

$$= \frac{1}{4}\times \Sigma \log P$$

$$= \frac{1}{4}\times 8.2936$$

$$= 2.0734$$

Therefore, $\quad$ I = antilog 2.0734

= 118.4

Therefore, Price Index Number = 118.4

(*iii*) Weighted A.M. of Price Relative Index Number

$$= \frac{\Sigma\left(\frac{p_n}{p_o}\times 100\right)V}{\Sigma V} = \frac{\Sigma PV}{\Sigma V} = \frac{29300}{247} = 118.62$$

(*iv*) Weighted G.M. of Price Relative Index Number

$$= \text{antilog}\left[\frac{\Sigma(\log P).V}{\Sigma V}\right] = \text{antilog}\left(\frac{510.3348}{247}\right) = \text{antilog }(2.066)$$

= 116.4.

8.8 QUANTITY INDEX NUMBERS

Price index numbers measure the changes in the price of certain goods. Quantity index numbers, on the other hand, measure the changes in the volume of productions, construction or employment over a period of years. Methods of construction of quantity index number are similar to those involved in price index numbers. Quantity index numbers can be obtained easily by changing *p* to *q* and *q* to *p* in the various formulae of price index numbers.

Method	Price Index Number (P_{on})	Quantity Index Number (Q_{on})
I. Aggregative		
A. Simple aggregative	$\frac{\Sigma p_n}{\Sigma p_o} \times 100$	$\frac{\Sigma q_n}{\Sigma q_o} \times 100$
B. Weighted aggregative	$\frac{\Sigma p_n w}{\Sigma p_o w} \times 100$	$\frac{\Sigma q_n w}{\Sigma q_o w} \times 100$
(*i*) Laspeyres' Formula	$\frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times 100$	$\frac{\Sigma q_n p_o}{\Sigma q_o p_o} \times 100$
(*ii*) Paasche's Formula	$\frac{\Sigma p_n q_n}{\Sigma p_o q_n} \times 100$	$\frac{\Sigma q_n p_n}{\Sigma q_o p_n} \times 100$
(*iii*) Fisher's Formula	$\sqrt{\frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times \frac{\Sigma p_n q_n}{\Sigma p_o q_n}} \times 100$	$\sqrt{\frac{\Sigma q_n p_o}{\Sigma q_o p_o} \times \frac{\Sigma q_n p_n}{\Sigma q_o p_n}} \times 100$
(*iv*) Edgeworth-Marshall's Formula	$\frac{\Sigma p_n (q_o + q_n)}{\Sigma p_o (q_o + q_n)} \times 100$	$\frac{\Sigma q_n (p_o + p_n)}{\Sigma q_o (p_o + p_n)} \times 100$
(*v*) Bowely's Formula	$\frac{1}{2}\left[\frac{\Sigma p_n q_o}{\Sigma p_o q_o} + \frac{\Sigma p_n q_n}{\Sigma p_o q_n}\right] \times 100$	$\frac{1}{2}\left[\frac{\Sigma q_n p_o}{\Sigma q_o p_o} + \frac{\Sigma q_n p_n}{\Sigma q_o p_n}\right] \times 100$
II Relative		
A. Simple A.M. of Relative Index	$\frac{\Sigma\left(\frac{p_n}{p_o} \times 100\right)}{N}$	$\frac{\Sigma\left(\frac{q_n}{q_o} \times 100\right)}{N}$
B. Weighted A.M. of Relative Index	$\frac{\Sigma\left(\frac{p_n}{p_o} \times 100\right) w}{\Sigma w}$	$\frac{\Sigma\left(\frac{q_n}{q_o} \times 100\right) w}{\Sigma w}$

ILLUSTRATION 5

Calculate Quantity Index Numbers from the following data using (a) Laspeyres' Formula (b) Paasche's Formula and (c) Fisher's Formula.

Items	Base year Price (₹) (p_o)	Base year Quantity (kg.) (q_o)	Current year Price (₹) (p_n)	Current year Quantity (kg.) (q_n)
P	6	52	11	58
Q	4	102	5	122
R	5	62	7	62
S	12	32	15	26
T	8	42	11	38

Solution: **Calculation of Quantity Index Numbers**

Items	p_o	q_o	p_n	q_n	p_oq_o	p_nq_n	p_nq_o	p_oq_n
P	6	52	11	58	312	638	572	348
Q	4	102	5	122	408	610	510	488
R	5	62	7	62	310	434	434	310
S	12	32	15	26	384	390	480	312
T	8	42	11	38	336	418	462	304
Total	–	–	–	–	1750	2490	2458	1762

(*a*) Laspeyres' Quantity Index Number

$$(Q_{on}) = \frac{\Sigma q_n p_o}{\Sigma q_o p_o} \times 100 = \frac{1762}{1750} \times 100 = 100.69$$

(*b*) Paasche's Quantity Index Number

$$(Q_{on}) = \frac{\Sigma q_n p_n}{\Sigma q_o p_n} \times 100 = \frac{2490}{2458} \times 100 = 101.3$$

(*c*) Fisher's Quantity Index Number

$$(Q_{on}) = \sqrt{\frac{\Sigma q_n p_o}{\Sigma q_o p_o} \times \frac{\Sigma q_n p_n}{\Sigma q_o p_n}} \times 100$$

$$= \sqrt{\frac{1762}{1750} \times \frac{2490}{2458}} \times 100 = 100.99 = 101 \text{ (Approx.)}$$

8.9 TESTS OF ADEQUACY OF INDEX NUMBER

We have discussed several methods for constructing simple and weighted index numbers. The question arises which method of index number is the most suitable in a given situation. However, some tests have been suggested to determine the adequacy of a method of index number. These tests are:

8.9.1 Time Reversal Test

According to Fisher the formula for calculating the index number should be such that it will give the same ratio between one point of comparison and the other, no matter which of the two is taken as base. This means that the index number should work both backwards as well as forwards. When the data for any two years are treated by the same method, but with the base reversed, the two index numbers should be reciprocals of each other and hence, their product should be unity. Mathematically, the index number (I_{on}) for the year *n* with respect to the base year *o* is reciprocal of the index number (I_{no}) for the year *o* with respect to the base year *n*. Symbolically,

$$I_{on} = \frac{1}{I_{no}} \text{ or } I_{on} \times I_{no} = 1 \text{ (omitting the factor 100 from each index)}$$

The index number formula which obey the above relation is said to satisfy the Time Reversal Test. This test is satisfied by

(*i*) Simple aggregative formula
(*ii*) Simple geometric mean of price relatives formula
(*iii*) Weighted geometric mean of price relatives formula
(*iv*) Edgeworth–Marshall's formula
(*v*) Fisher's ideal index formula

8.9.2 Factor Reversal Test

Another test proposed by Fisher is known as Factor Reversal Test. In the words of Fisher, just as each formula should permit the interchange of the two times without giving inconsistent results, similarly it should permit interchanging the prices and quantities without giving inconsistent results which means two results multiplied together should give the true value ratio. The test says that the change in price multiplied by change in quantity should be equal to total change in value. In other words, an index number formula is said to satisfy factor reversal test if the product of the price index (P_{on}) and quantity index (Q_{on}) equals the total value ratio of the current year and the base year. Symbolically,

$$P_{on} \times Q_{on} = \frac{\Sigma p_n q_n}{\Sigma p_o q_o} \text{ where } \frac{\Sigma p_n q_n}{\Sigma p_o q_o} = \text{value ratio}$$

This test is satisfied only by Fisher's ideal index number formula.

8.9.3 Circular Test

Another test of adequacy applied in index number studies is the circular test. This test was suggested by Westerguard and C.M. Walsch. It is an extension of time reversal test for more than two years and based on the shift ability of the base. Accordingly, the index should work in a circular fashion. Thus, if I_{o1} be the index number for the year 1 with respect to the base year o, I_{12} be the index number for the year 2 with respect to the base year 1 and so on, $I_{n-1,\, n}$, the index number for the year n with respect to the base year $n - 1$ and I_{no} the index number for the year o with respect to the base year n then the circular test is satisfied if

$$I_{01} \times I_{12} \times I_{23} \times \ldots\ldots\ldots\ldots\ldots\ldots \times I_{(n-1),\, n} \times I_{no} = 1$$

This test is satisfied by

(*i*) Simple aggregative index number formula
(*ii*) Simple G.M. of relatives formula

ILLUSTRATION 6

Show that neither Laspeyres' formula nor Paasche's formula obeys Time Reversal and Factor Reversal Tests of Index Numbers.

[***C.U. B.Com. 2012, 2014, 2015***]

Proof:

1. Time Reversal Test:

(*a*) Laspeyres' index number formula $(I_{on}) = \frac{\Sigma p_n q_o}{\Sigma p_o q_o}$ (omitting the factor 100)

Interchanging '*o*' and '*n*' we have,

$$I_{no} = \frac{\Sigma p_o q_n}{\Sigma p_n q_n}$$

Now, $$I_{on} \times I_{no} = \frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times \frac{\Sigma p_o q_n}{\Sigma p_n q_n} \neq 1$$

thus, Laspeyres' formula does not obey time reversal test.

(*b*) Paasche's index number formula $(I_{on}) = \dfrac{\Sigma p_n q_n}{\Sigma p_o q_n}$ (omitting the factor 100)

Interchanging '*o*' and '*n*' we have,

$$I_{no} = \frac{\Sigma p_o q_o}{\Sigma p_n q_o}$$

Then, $$I_{on} \times I_{no} = \frac{\Sigma p_n q_n}{\Sigma p_o q_n} \times \frac{\Sigma p_o q_o}{\Sigma p_n q_o} \neq 1$$

thus, Paasche's formula also does not obey time reversal test.

2. Factor Reversal Test:

(*a*) Laspeyres' price index number formula

$$(P_{on}) = \frac{\Sigma p_n q_o}{\Sigma p_o q_o} \text{ (omitting the factor 100)}$$

Interchanging '*p*' and '*q*', we get Laspeyres' Quantity index formula

$$(Q_{on}) = \frac{\Sigma q_n p_o}{\Sigma q_o p_o} = \frac{\Sigma p_o q_n}{\Sigma p_o q_o}$$

Then, $$P_{on} \times Q_{on} = \frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times \frac{\Sigma p_o q_n}{\Sigma p_o q_o} \neq \frac{\Sigma p_n q_n}{\Sigma p_o q_o} \text{ (value ratio)}$$

Thus, Laspeyres' formula does not obey factor reversal test.

(*b*) Paasche's Price Index formula $(P_{on}) = \dfrac{\Sigma p_n q_n}{\Sigma p_o q_n}$ (omitting the factor 100)

Interchanging '*p*' and '*q*', we get,

Paasche's Quantity Index formula $(Q_{on}) = \dfrac{\Sigma q_n p_n}{\Sigma q_o p_n} = \dfrac{\Sigma p_n q_n}{\Sigma p_n q_o}$

Then, $$P_{on} \times Q_{on} = \frac{\Sigma p_n q_n}{\Sigma p_o q_n} \times \frac{\Sigma p_n q_n}{\Sigma p_n q_o} \neq \frac{\Sigma p_n q_n}{\Sigma p_o q_o} \text{ (value ratio)}$$

Thus, Paasche's formula also does not obey factor reversal test.

ILLUSTRATION 7

Prove that Fisher's Ideal Index number satisfies both time reversal test and factor reversal test. [***C.U. B.Com. 2009, 2013 (H), 2014 (G)***]

Proof:

1. **Time Reversal Test:**

   Fisher's Ideal Index Number Formula

$$(I_{on}) = \sqrt{\frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times \frac{\Sigma p_n q_n}{\Sigma p_o q_n}} \text{ (omitting the factor 100)}$$

   Interchanging 'o' and 'n', we have

$$I_{no} = \sqrt{\frac{\Sigma p_o q_n}{\Sigma p_n q_n} \times \frac{\Sigma p_o q_o}{\Sigma p_n q_o}}$$

   Then,

$$I_{on} \times I_{no} = \sqrt{\frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times \frac{\Sigma p_n q_n}{\Sigma p_o q_n}} \times \sqrt{\frac{\Sigma p_o q_n}{\Sigma p_n q_n} \times \frac{\Sigma p_o q_o}{\Sigma p_n q_o}}$$

$$= \sqrt{\frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times \frac{\Sigma p_n q_n}{\Sigma p_o q_n} \times \frac{\Sigma p_o q_n}{\Sigma p_n q_n} \times \frac{\Sigma p_o q_o}{\Sigma p_n q_o}}$$

$$= \sqrt{1} = 1$$

   Thus, Fisher's Ideal Index Number satisfies time reversal test.

2. **Factor Reversal Test:**

   Fisher's Price Index Number formula

$$P_{on} = \sqrt{\frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times \frac{\Sigma p_n q_n}{\Sigma p_o q_n}} \text{ (omitting the factor 100)}$$

   Interchanging 'p' and 'q', we have

   Fisher's Quantity Index Number formula

$$Q_{on} = \sqrt{\frac{\Sigma q_n p_o}{\Sigma q_o p_o} \times \frac{\Sigma q_n p_n}{\Sigma q_o p_n}} = \sqrt{\frac{\Sigma p_o q_n}{\Sigma p_o q_o} \times \frac{\Sigma p_n q_n}{\Sigma p_n q_o}}$$

   Then,

$$P_{on} \times Q_{on} = \sqrt{\frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times \frac{\Sigma p_n q_n}{\Sigma p_o q_n} \times \frac{\Sigma p_o q_n}{\Sigma p_o q_o} \times \frac{\Sigma p_n q_n}{\Sigma p_n q_o}}$$

$$= \sqrt{\frac{\Sigma p_n q_n \times \Sigma p_n q_n}{\Sigma p_o q_o \times \Sigma p_o q_o}} = \frac{\Sigma p_n q_n}{\Sigma p_o q_o}$$

   Thus, Fisher's Ideal Index Number satisfies factor reversal test.

ILLUSTRATION 8

Prove that the simple aggregative formula satisfies Circular Test.

Proof: We know,

Simple aggregative index number for the year 1 with base year 0

$$I_{01} = \frac{\Sigma p_1}{\Sigma p_0} \text{ (omitting the factor 100)}$$

Simple aggregative index number for the year 2 with base year 1

$$I_{12} = \frac{\Sigma p_2}{\Sigma p_1} \text{ (omitting the factor 100)}$$

Simple aggregative index number for the year 3 with base year 2

$$I_{23} = \frac{\Sigma p_3}{\Sigma p_2} \text{ (omitting the factor 100)}$$

and so on,

Simple aggregative index number for the year n with base year $(n - 1)$

$$I_{(n-1),\, n} = \frac{\Sigma p_n}{\Sigma p_{n-1}} \text{ (omitting the factor 100)}$$

and,

Simple aggregative index number for the year 0 with base year n

$$I_{no} = \frac{\Sigma p_o}{\Sigma p_n} \text{ (omitting the factor 100)}$$

Clearly, $I_{01} \times I_{12} \times I_{23} \times \ldots\ldots\ldots \times I_{(n-1),\, n} \times I_{no}$

$$= \frac{\Sigma p_1}{\Sigma p_0} \times \frac{\Sigma p_2}{\Sigma p_1} \times \frac{\Sigma p_3}{\Sigma p_2} \times \ldots\ldots \times \frac{\Sigma p_n}{\Sigma p_{n-1}} \times \frac{\Sigma p_o}{\Sigma p_n} = 1$$

Thus, simple aggregative formula satisfies circular test.

8.10 ERRORS IN INDEX NUMBERS

Errors may crop up in the construction of index numbers due to faulty selection of items, inadequate information about price quotation and due to lack of representative character of price quotation, etc. Index numbers are mainly affected by the following three types of errors:

8.10.1 Formula Error

Different formulae are used for the calculation of index numbers and as such we get different values of an index number. There is no specific formula which can be considered the most suitable formula for the construction of an index number. Each formula used introduces an error which can never be eliminated. This error is known as formula error.

8.10.2 Homogeneity Error

For the construction of index numbers same commodities are considered at the base year and at the current year. But in reality, due to the change of taste, consumption habit, obsolescence, commodities consumed during the long period between base year and current year may not be the same. Hence, the homogeneity in the composition of the commodities can not be strictly maintained and as such are not comparable. This introduces an error called homogeneity error.

8.10.3 Sampling Error

All items not considered for the construction of index number. It is done, only on the basis of sample items. As many items are left out during the construction process, the calculated index number do not represent the actual changes in the phenomenon. The error thus occurred due to the consideration of sample items is known as sampling error.

8.11 CHAIN BASE INDEX NUMBERS

The various formulae discussed so far assume that the base period remains the same throughout the series of the index. This method though convenient has certain limitations. The index of a given year on a given fixed base year is not affected by changes in the prices or the quantities in all the intermediate years between two periods or years. New items may have to be included and old ones may have to be deleted in order to make the index more representative. In such cases it may be desirable to use the **chain base method**. To construct index numbers by chain base method, a series of index numbers are computed for each year with preceding year as the base. These index numbers are known as **link relatives or link indices**.

Therefore, link index for nth year = Index number of nth year with $(n-1)$th year as base.

Let $p_o, p_1, p_2, p_3, \ldots$ represent prices during the successive years, denoted by 0,1,2,3,... Then, the link indices of each year will be as under

$$I'_{01} = \frac{p_1}{p_0} \times 100,\ I'_{12} = \frac{p_2}{p_1} \times 100,\ I'_{23} = \frac{p_3}{p_2} \times 100, \ldots$$

These link indices or link relatives when multiplied successively known as the chaining process give link to a common base. The products thus obtained are expressed as percentage and give the required index number. This index number is called **chain base index number**. For example, the chain base index number for the nth year with 0th year as base is given by (expressed as percentages)

$$I_{on} = I'_{01} \times I'_{12} \times I'_{23} \times \ldots \times I'_{(n-2)(n-1)} \times I'_{(n-1),n}$$

$$= I'_{0,(n-1)} \times I'_{(n-1),n}$$

= [chain base index number for $(n-1)$th year with 0th year as base] × [link index of the nth year with $(n-1)$th year as base]

(Here I′ is used for link index and I is used for chain base index)

thus,

Chain base index for any year

$$= \frac{\text{(chain base index of previous year)} \times \text{(link index of the current year)}}{100}$$

i.e.

$$I_{01} = I'_{01},\ I_{02} = I_{01} \times I'_{12}$$

$$I_{03} = I_{02} \times I'_{23} \text{ and so on.}$$

Therefore, the steps of chain base index are:

(*i*) Express the figures of each year as a percentage of the preceding year to obtain link relatives or link indices.

(*ii*) The link relatives are chained together by successive multiplication to get a chain index.

Advantages:

(1) In this method the price relatives of a year can be compared with the price levels of the immediately preceding year. Businesses mostly interested in comparing this time period rather than comparing rates related to the distant past will utilize this method.

(2) In this method it is possible to include new items in an index number or to delete old items which are no longer important which is not possible with the fixed base method.

(3) The effects of all intermediate years between two periods are taken into consideration.

(4) The weights of the different items can be adjusted frequently.

(5) Index numbers computed by this method are free to a great extent from seasonal variation than those obtained by the other methods.

Disadvantages:

(1) Under this method comparisons cannot be made over a long period because the long range comparisons of chained percentages are not strictly valid.

(2) This method requires tedious calculations and hence time consuming.

(3) The significance of index numbers obtained by this method is difficult to understand.

ILLUSTRATION 9

Construct Index Numbers by chain base method from the following data of wholesale prices:

Year:	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
Prices:	75	50	65	60	72	70	69	75	84	80

Solution: **Computation of chain base index numbers**

Year	Prices	Link relatives	Chain base index numbers
2005	75	100	100
2006	50	$\frac{50}{75}\times 100 = 66.67$	$\frac{66.67\times 100}{100} = 66.67$
2007	65	$\frac{65}{50}\times 100 = 130$	$\frac{130\times 66.67}{100} = 86.67$
2008	60	$\frac{60}{65}\times 100 = 92.31$	$\frac{92.31\times 86.67}{100} = 80$
2009	72	$\frac{72}{60}\times 100 = 120$	$\frac{120\times 80}{100} = 96$
2010	70	$\frac{70}{72}\times 100 = 97.22$	$\frac{92.22\times 96}{100} = 93.33$
2011	69	$\frac{69}{70}\times 100 = 98.57$	$\frac{98.57\times 93.33}{100} = 92$
2012	75	$\frac{75}{69}\times 100 = 108.69$	$\frac{108.69\times 92}{100} = 100$
2013	84	$\frac{84}{75}\times 100 = 112$	$\frac{112\times 100}{100} = 112$
2014	80	$\frac{80}{84}\times 100 = 95.24$	$\frac{95.24\times 112}{100} = 106.67$

8.12 BASE SHIFTING, SPLICING AND DEFLATING

Base Shifting: Sometimes, it becomes necessary to change the base year of a series of index numbers from one period to another for the comparison. This change of reference base period is usually referred to as

"Shifting the base". Shifting of base is generally required due to the following reasons:

(*i*) The base year is too old to compare the current year and we want to choose a recent year as new base to make the index numbers more useful.

(*ii*) If different series of index numbers are based on different base years and they are to be compared from each other. In this case all the series are expressed with respect to a common base.

Under these circumstances it is necessary to recompute all index numbers using new base year. Such computation of index numbers using new base year is to divide index number in each year by the index number corresponding to the new base year and then to express the result as percentages.

Symbolically,

Index number (with new base year)

$$= \frac{\text{Old index number for the current year}}{\text{Old index number for the new base year}} \times 100$$

[$\dfrac{100}{\text{Old index number for the new base year}}$ is known as multiplying factor]

ILLUSTRATION 10

The following table represents the price index number of a commodity with base 2008. Shift the base to 2015

Year:	2012	2013	2014	2015	2016
Index number:	120	145	165	175	190

Solution: **Shifting of base from 2008 to 2015**

Year	Index Number (Base – 2008)	Index Number (Base – 2015)
2012	120	$\frac{120}{175} \times 100 = 68.57$
2013	145	$\frac{145}{175} \times 100 = 82.86$
2014	165	$\frac{165}{175} \times 100 = 94.28$
2015	175	$\frac{175}{175} \times 100 = 100$
2016	190	$\frac{190}{175} \times 100 = 108.57$

[In this calculation $\frac{100}{175}$ is the multiplying factor.]

Splicing: Sometimes, a series of index numbers with an old base is discontinued and a new series of index numbers is constructed by taking some recent year as base. In this situation index numbers of these two series are not comparable because both are based on different years. In order to make them comparable it is necessary to convert these series of index numbers of different bases into a continuous series of index numbers of a common base.

The statistical procedure which convert an old index number with a new one is called **Splicing**. Splicing arises only, if the following conditions are satisfied:

(*i*) If the old and the new series of index numbers have been constructed with the same items;

(*ii*) If the old and the new series of index numbers have different base year; and

(*iii*) If the old and the new series of index numbers have at least one overlapping year.

Types of Splicing: Splicing are of two types:

(*i*) **Forward Splicing:** When the new indices are to be spliced with the base year of the old indices, then it is called forward splicing. The following formula is used for computation:

Spliced Index Number (from new to old)

$$= \frac{\text{New Index Number for the current year} \times \text{Old Index Number for the new base year}}{100}$$

$$\left[\frac{\text{Old Index Number for the new base year}}{100} \text{ is known as multiplying factor}\right]$$

(*ii*) **Backward splicing:** When the old indices are to be spliced with the base year of the new indices, then it is called backward splicing. The following formula is used for computation:

Spliced index number (from old to new)

$$= \frac{\text{Old Index Number for the current year}}{\text{Old Index Number for the new base year}} \times 100$$

$$\left[\frac{100}{\text{Old Index Number for the new base year}} \text{ is known as multiplying factor.}\right]$$

ILLUSTRATION 11

Two series of price index numbers are given below. Splice them on the new base 2012 and also on the old base 2006.

Year	2008	2009	2010	2011	2012	2013	2014	2015	2016
Series A (Base – 2006)	130	145	161	169	185				
Series B (Base – 2012)					100	125	140	155	175

Solution: **Splicing of two series of Index Numbers**

Year	**Series A Index Numbers (Base – 2006)**	**Series B Index Numbers (Base – 2012)**	**Spliced Index Numbers**	
			Base Year -2006	**Base Year -2012**
2008	130		130	$\frac{130 \times 100}{185} = 70.27$
2009	145		145	$\frac{145 \times 100}{185} = 78.38$
2010	161		161	$\frac{161 \times 100}{185} = 87.03$
2011	169		169	$\frac{169 \times 100}{185} = 91.35$
2012	185	100	185	100
2013		125	$\frac{125 \times 185}{100} = 231.25$	125
2014		140	$\frac{140 \times 185}{100} = 259.00$	140
2015		155	$\frac{155 \times 185}{100} = 286.75$	155
2016		175	$\frac{175 \times 185}{100} = 323.75$	175

Deflating: Deflating means making allowances for the changes in the purchasing power of money due to change in general price level. A rise in price level means a reduction in the purchasing power of money. The process of adjusting a series of wages or income according to current price changes to find out the level of real wages or income is called **deflating.** The following formula is used to find out real wages or income.

$$\text{Real wages or income} = \frac{\text{Money wages or income}}{\text{Price Index or Consumer Price Index}} \times 100$$

The adjusted figures of wages or income can be converted into index numbers for the purposes of comparison. It is known as Real wage (or income) indices.

Such indices show changes in the purchasing power of the money wages (or income) of the workers. Technically, it is known as deflating the index number.

Real wage (or income) Index Number or deflated Index Number

$$= \frac{\text{Real wage (or income) of current year}}{\text{Real wage (or income) of base year}} \times 100$$

$$= \frac{\text{Index of Money wage (or income)}}{\text{Price Index or Consumer Price Index}} \times 100$$

ILLUSTRATION 12

The following table gives the annual income of a worker and the general price index numbers during 2008–2016. Calculate (i) Real Income and (ii) Real Income Index Number with 2008 as base.

Year	2008	2009	2010	2011	2012	2013	2014	2015	2016
Income (₹)	7200	8400	10,000	11,000	12,000	12,800	13600	14400	15000
General Price Index Number	100	120	150	160	250	320	450	530	600

Solution: **Calculations of (i) Real Income and (ii) Real Income Index Number**

Year	**Income (₹) (x)**	**General Price Index Number (I)**	**Real Income** $\left(y = \frac{x}{I} \times 100\right)$	**Real Income Index Number** $= \left(\frac{y}{7200} \times 100\right)$ **(Base 2008 = 100)**
2008	7200	100	$\frac{7200}{100} \times 100 = 7200.00$	100
2009	8400	120	$\frac{8400}{120} \times 100 = 7000.00$	97.22
2010	10,000	150	$\frac{10,000}{150} \times 100 = 6666.67$	92.59
2011	11,000	160	$\frac{11,000}{160} \times 100 = 6875.00$	95.49

2012	12,000	250	$\frac{12000}{250}\times 100 = 4800.00$	66.67
2013	12,800	320	$\frac{12800}{320}\times 100 = 4000.00$	55.55
2014	13,600	450	$\frac{13600}{450}\times 100 = 3022.22$	41.97
2015	14,400	530	$\frac{14400}{530}\times 100 = 2716.98$	37.73
2016	15,000	600	$\frac{15000}{600}\times 100 = 2500.00$	34.72

8.13 CONSUMER PRICE INDEX OR COST OF LIVING INDEX

Different classes of people in a society consume different types of commodities and if the same type of commodities then in different proportions. A change in the level of prices affect different classes of people in different manners. General index numbers fail to give an exact idea of the effect of the change in the general price level on the cost of living of different classes of people. This necessitates the construction of special purpose index numbers known as **Consumer Price Index Numbers**. These indices are designed to measure the effect of changes in the price of group of commodities and services on the purchasing power of a particular class of society during any given period with reference to some fixed base. The group of commodities and services will contain items like Food, House Rent, Education, Clothing, Fuel and Light, Miscellaneous like Transport, Washing, Newspaper, etc. Consumer Price Index Numbers are also called cost of living index numbers or Retail price index numbers.

8.13.1 Construction of Consumer Price Index Numbers

The following steps are involved in the construction of consumer price index numbers.

(1) **Class of people:** The first step in the construction of consumer price index is that the class of people should be defined clearly. As far as possible a homogeneous group of persons (e.g. industrial workers, officers, school teachers, etc. residing in a particular well-defined area) regarding their income and consumption pattern are considered. It is therefore necessary to specify the class of people and locality where they reside.

(2) **Family budget inquiry and allocation of weights:** The next step is to conduct a family budget inquiry of the category of people concerned. For this purpose some families are selected at random and inquiries are conducted to determine the goods and services to be included in the construction of index numbers. The inquiry includes questions on family size, income, the quality and quantity of resources consumed and money spent on them under various headings, such as clothing and footwear, fuel and lighting, housing, etc. and the weights are assigned in proportions to the expenditure on different items. This step has many practiced problems as no two families have the same income and consumption pattern.

(3) **Collection of consumer prices:** The next step is to collect data on the retail prices of the selected commodities for the current period and the base period from the locality where the people reside or from where they make their purchase. Collection of retail prices is not a easy task as the prices vary from place to place and from shop to shop.

Finally, consumer price index numbers are computed by using suitable index number formulae.

8.13.2 Methods of Construction

Following two methods are used to construct consumer price index numbers:

8.13.2.1 Aggregative Expenditure Method

In this method, the prices of commodities of current year as well as base year are multiplied by the quantities consumed in the base year. The aggregate expenditure of current year is divided by the aggregate expenditure of the base year and the quotient is multiplied by 100. Symbolically,

$$\text{Consumer Price Index } (P_{on}) = \frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times 100$$

where,

$\Sigma\, p_n q_o$ = Aggregative expenditure of the current year

$\Sigma\, p_o q_o$ = Aggregative expenditure of the base year.

This method is same as Laspeyres' method.

8.13.2.2 Family Budget Method

In this method, the price relatives for each commodity are obtained and these price relatives are multiplied by the value weights for each item and the product is divided by the total of weights. Symbolically,

$$\text{Consumer Price Index} = \frac{\Sigma\left(\frac{p_n}{p_o}\right) p_o q_o}{\Sigma\, p_o q_o} \times 100$$

$$= \frac{\Sigma\, I \times W}{\Sigma\, W}$$

where, Price Relative (I) = $\frac{p_n}{p_o} \times 100$, W = $p_o q_o$

8.13.3 Applications and Uses

(*i*) Consumer price index numbers are primarily used for the calculation of dearness allowance (D.A.) to maintain the same standard of living as in the base year.

(*ii*) These index numbers are used for formulation of wage policy, general economic policy, taxation at government level.

(*iii*) Purchasing power of money can be measured with the help of consumer price index numbers.

$$\text{Purchasing power of money} = \frac{1}{\text{Consumer Price Index Number}}$$

(*iv*) Real wage (or income) can be measured by dividing the Actual wage (or income) received during a period by the corresponding consumer price index number of that period.

$$\text{Real wage (or income)} = \frac{\text{Actual wage (or income)}}{\text{Consumer Price Index Number}} \times 100$$

ILLUSTRATION 13

Compute the cost of living index number using both the Aggregate Expenditure Method and Family Budget Method, from the following information:

Commodity	Unit consumption in base year	Price in base year	Price in current year
Wheat	200	1.00	1.20
Rice	50	3.00	3.50
Pulses	50	4.00	5.00
Ghee	20	20.00	30.00
Sugar	40	2.50	5.00
Oil	50	10.00	15.00
Fuel	60	2.00	2.50
Clothing	40	15.00	18.00

[***Calicut University, B.Com. 1987***]

Solution: **Calculation of cost of living index**

Commodity	q_o	p_o	p_n	$p_n q_o$	$w = p_n q_o$	$I = \frac{p_n}{p_o} \times 100$	**I.W**
Wheat	200	1.00	1.20	240	200	120	24,000.00
Rice	50	3.00	3.50	175	150	116.67	17,500.50
Pulses	50	4.00	5.00	250	200	125	25,000.00
Ghee	20	20.00	30.00	600	400	150	60,000.00
Sugar	40	2.50	5.00	200	100	200	20,000.00
Oil	50	10.00	15.00	750	500	150	75,000.00
Fuel	60	2.00	2.50	150	120	125	15,000.00
Clothing	40	15.00	18.00	720	600	120	72,000.00
Total				3085	2270		3,08,500.50

Cost of Living Index Number (CLI):

(*a*) Under Aggregate Expenditure Method

$$\text{CLI} = \frac{\Sigma\, p_n q_o}{\Sigma\, p_o q_o} \times 100 = \frac{3085}{2270} \times 100 = 135.9$$

(*b*) Under Family Budget Method

$$\text{CLI} = \frac{\Sigma\, \text{IW}}{\Sigma\, \text{W}} = \frac{308500.50}{2270} = 135.9$$

ILLUSTRATIVE EXAMPLES

GROUP A: SHORT ESSAY TYPE

EXAMPLE 1

Prepare simple aggregative price index number from the following data:

Commodity	**Rate unit**	**Price (2008) (₹)**	**Price (2017) (₹)**
Wheat	per 10 kg	200	280
Rice	per 10 kg	400	500
Pulses	per 10 kg	500	700
Sugar	per kg	28	40
Oil	per litre	80	100

Solution: **Calculation of simple aggregative price index number**

Commodity	Rate unit	Price (2008) (₹)	Price (2017) (₹)
Wheat	per 10 kg	200	280
Rice	per 10 kg	400	500
Pulses	per 10 kg	500	700
Sugar	per kg	28	40
Oil	per litre	80	100
Total		1208	1620

Simple Aggregative Price Index Number

$$= \frac{\Sigma p_n}{\Sigma p_o} \times 100 = \frac{1620}{1208} \times 100 = 134.1$$

EXAMPLE 2

Find the price index number by using weighted aggregative method from the following data:

Commodity	Price per unit (₹)		Weight
	Base year (2015)	Current year (2017)	
A	32	40	80
B	80	120	50
C	2	2	10
D	10	11	40
E	4	3	20

Solution: **Calculation of weighted aggregative price index number**

Commodity	Base year price (p_o)	Current year price (p_n)	Weight (w)	$p_o.w$	$p_o.w$
A	32	40	80	2560	3200
B	80	120	50	4000	6000
C	2	2	10	20	20
D	10	11	40	400	440
E	4	3	20	80	60
Total				7060	9720

Weighted Aggregative Price Index Number

$$= \frac{\Sigma p_n w}{\Sigma p_o w} \times 100 = \frac{9720}{7060} \times 100 = 137.68$$

EXAMPLE 3

Compute the index number for the years 2011, 2012, 2013 and 2014, taking 2010 as base year, from the following data:

Year	2010	2011	2012	2013	2014
Price	120	144	168	204	216

Solution: **Calculation of Price relative index numbers for different years**

Year	Price relative index number
2010	$\frac{120}{120} \times 100 = 100$
2011	$\frac{144}{120} \times 100 = 120$
2012	$\frac{168}{120} \times 100 = 140$
2013	$\frac{204}{120} \times 100 = 170$
2014	$\frac{216}{120} \times 100 = 180$

EXAMPLE 4

Find price index number by the method of relatives using arithmetic mean from the following data:

Commodities:	Wheat	Milk	Fish	Sugar
Base Price :	5	8	25	6
Current Price:	7	10	32	12

[***C.U. B.Com. 2015 (H)***]

Solution: **Calculation of Price Index Number**

Commodities	Base Price (p_o)	Current Price (p_n)	$\frac{P_n}{P_o} \times 100$
Wheat	5	7	140
Milk	8	10	125
Fish	25	32	128
Sugar	6	12	200
Total			593

Simple Arithmetic Mean of Price Relative

$$\text{Index Number} = \frac{\Sigma\left(\frac{p_n}{p_o}\times 100\right)}{n} = \frac{593}{4} \quad (n = \text{number of commodities} = 4)$$

$= 148.25$

EXAMPLE 5

Using the data given below calculate the Laspeyres' Price Index Number for the year 2014 with the year 2011 as base year:

Commodities	Price (₹)		Quantity (kg)	
	2011	2014	2011	2014
A	4	5	95	120
B	60	70	118	130
C	35	40	50	70

[*C.U. B.Com. 2015 (G)*]

Solution: **Calculation of Price Index Number**

Commodities	Price (₹)		Quantity (kg)		p_nq_o	p_oq_o
	2011 (p_o)	2014 (p_n)	2011 (q_o)	2014 (q_n)		
A	4	5	95	120	475	380
B	60	70	118	130	8260	7080
C	35	40	50	70	2000	1750
Total					10735	9210

$$\text{Laspeyres' Price Index Number} = \frac{\Sigma p_nq_o}{\Sigma p_oq_o}\times 100$$

$$= \frac{10735}{9210}\times 100$$

$$= 116.56$$

EXAMPLE 6

Using the data given below calculate the Price Index Number for the year 2013 by Paasche's formula with the year 2010 as base:

Commodity	Price per unit (₹)		Quantity ('000 kg)	
	2010	2013	2010	2013
Rice	9.3	14.5	100	90
Wheat	6.4	13.7	11	19
Pulse	5.1	12.7	5	3

[*C.U. B.Com. 2014 (H)*]

Solution: **Calculation of Price Index Number**

Commodity	Price per unit (₹)		Quantity ('000 kg)		p_nq_n	p_oq_n
	2010 (p_o)	2013 (p_n)	2010 (q_o)	2013 (q_n)		
Rice	9.3	14.5	100	90	1305	837
Wheat	6.4	13.7	11	19	260.3	121.6
Pulse	5.1	12.7	5	3	38.1	15.3
Total					1603.4	973.9

$$\text{Paasche's Price Index Number} = \frac{\Sigma p_nq_n}{\Sigma p_oq_n} \times 100$$

$$= \frac{1603.4}{973.9} \times 100$$

$$= 164.64$$

EXAMPLE 7

Calculate the Fisher's Ideal Index Number from the following:

Commodities	2005		2010	
	Quantity (kg)	Price (₹)	Quantity (kg)	Price (₹)
Wheat	10	100	6	110
Rice	15	150	18	170
Cloth	50	5	30	4

[***C.U. B.Com. 2015 (G)***]

Solution: **Calculation of Price Index Number**

Commodities	2005		2010		p_nq_o	p_oq_o	p_nq_n	p_oq_n
	Quantity (kg) (q_o)	Price (₹) (p_o)	Quantity (kg) (q_n)	Price (₹) (p_n)				
Wheat	10	100	6	110	1100	1000	660	600
Rice	15	150	18	170	2550	2250	3060	2700
Cloth	50	5	30	4	200	250	120	150
Total					3850	3500	3840	3450

$$\text{Fisher's Price Index Number} = \sqrt{\frac{\Sigma p_nq_o}{\Sigma p_oq_o} \times \frac{\Sigma p_nq_n}{\Sigma p_oq_n}} \times 100$$

$$= \sqrt{\frac{3850}{3500} \times \frac{3840}{3450}} \times 100 = \sqrt{1.1 \times 1.113} \times 100$$

$$= \sqrt{1.2243} \times 100 = 1.1064 \times 100 = 110.64$$

EXAMPLE 8

Using the data given below calculate the price index number for the year 1988 by (i) Laspeyres' formula, (ii) Paasche's formula and (iii) Fisher's formula with the year 1979 as base:

Commodity	**Price (₹)**		**Quantity ('000 kg)**	
	1979	**1988**	**1979**	**1988**
Rice	9.3	4.5	100	90
Wheat	6.4	3.7	11	10
Pulses	5.1	2.7	5	3

[*C.U. B.Com. 1990, 2012, 2014 (G)*]

Solution: **Calculation of Price Index Number**

Commodity	**Price (₹)**		**Quantity**		p_nq_o	p_oq_o	p_nq_n	p_oq_n
	1979 (p_o)	**1988** (p_n)	**1979** (q_o)	**1988** (q_n)				
Rice	9.3	4.5	100	90	450	930	405	837
Wheat	6.4	3.7	11	10	40.7	70.4	37	64
Pulses	5.1	2.7	5	3	13.5	25.5	8.1	15.3
Total					504.2	1025.9	450.1	916.3

(*i*) Laspeyres' Price Index Number $= \frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times 100 = \frac{504.2}{1025.9} \times 100$

$= 0.49147 \times 100 = 49.147$

(*ii*) Paasche's Price Index Number $= \frac{\Sigma p_n q_n}{\Sigma p_o q_n} \times 100 = \frac{450.1}{916.3} \times 100$

$= 0.49121 \times 100 = 49.121$

(*iii*) Fisher's Price Index Number $= \sqrt{\frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times \frac{\Sigma p_n q_n}{\Sigma p_o q_n}} \times 100$

$$= \sqrt{\frac{504.2}{1025.9} \times \frac{450.1}{916.3}} \times 100$$

$$= \sqrt{0.49147 \times 0.49121} \times 100$$

$$= \sqrt{0.241415} \times 100$$

$$= 0.49134 \times 100 = 49.134$$

EXAMPLE 9

Calculate the quantity index for 2015 from the following data using Laspeyres' formula with 2015 as the base year:

Commodities	2015		2016	
	Price	**Quantity**	**Price**	**Quantity**
A	1	6	5	4
B	3	5	8	5
C	4	8	10	6

[*C.U. B.Com. 2016 (G)*]

Solution: **Calculation of Quantity Index Number**

Commodities	2015 (Base Year)		2016 (Current Year)		$q_n p_o$	$q_o p_o$
	Price (p_o)	**Quantity** (q_o)	**Price** (p_n)	**Quantity** (q_n)		
A	1	6	5	4	4	6
B	3	5	8	5	15	15
C	4	8	10	6	24	32
Total					43	53

$$\text{Laspeyres' Quantity Index Number} = \frac{\Sigma q_n p_o}{\Sigma q_o p_o} \times 100$$

$$= \frac{43}{53} \times 100$$

$$= 0.81132 \times 100$$

$$= 81.132$$

EXAMPLE 10

Using (i) Laspeyres' (ii) Paasche's (iii) Marshall-Edgeworth and (iv) Fisher's formula, find the Quantity Index Number from the following data:

Commodity	Price per unit		Quantity (in relevant units)	
	Base year	**Current year**	**Base year**	**Current year**
A	4	12	60	50
B	3	10	20	12
C	2	6	10	6

[*C.U. B.Com. (H) 1992*]

Solution: Let base year's price = p_o

current year's price = p_n

base year's quantity = q_o

current year's quantity = q_n

Calculation of Quantity Index Number

Commodity	p_o	p_n	q_o	q_n	$q_o p_o$	$q_o p_n$	$q_n p_o$	$q_n p_n$
A	4	12	60	50	240	720	200	600
B	3	10	20	12	60	200	36	120
C	2	6	10	6	20	60	12	36
Total					320	980	248	756

(*i*) Laspeyres' Quantity Index Number $= \dfrac{\Sigma q_n p_o}{\Sigma q_o p_o} \times 100 = \dfrac{248}{320} \times 100 = 77.5$

(*ii*) Paasche's Quantity Index Number $= \dfrac{\Sigma q_n p_n}{\Sigma q_o p_n} \times 100 = \dfrac{756}{980} \times 100 = 77.14$

(*iii*) Marshall–Edgeworth Quantity Index Number $= \dfrac{\Sigma q_n (p_o + p_n)}{\Sigma q_o (p_o + p_n)} \times 100$

$$= \frac{\Sigma q_n p_o + \Sigma q_n p_n}{\Sigma q_o p_o + \Sigma q_o p_n} \times 100 = \frac{248 + 756}{320 + 980} \times 100$$

$$= \frac{1004}{1300} \times 100 = 77.23$$

(*vi*) Fisher's Quantity Index Number $= \sqrt{\dfrac{\Sigma q_n p_o}{\Sigma q_o p_o} \times \dfrac{\Sigma q_n p_n}{\Sigma q_o p_n}} \times 100$

$$= \sqrt{\frac{248}{320} \times \frac{756}{980}} \times 100 = \sqrt{0.775 \times 0.771} \times 100$$

$$= \sqrt{0.597525} \times 100 = 0.773 \times 100 = 77.3$$

EXAMPLE 11

Calculate the Price index number for the year 2011 with 2001 as base year using Laspeyres' or Paasche's formula, whichever will be applicable, on the basis of following data:

Commodity	Price (₹)		Money value ('000 ₹) 2011
	2001	2011	
P	22	30	240
Q	16	18	72
R	20	25	150
S	8	12	36

(Here money value means total value of commodity which is obtained by price × quantity)

Solution: Let P_o, P_n be the prices for the base year and current year respectively and q_n be the quantity for the current year.

Here, Money value (v) for the current year is given, i.e. $p_n \cdot q_n$ is given. $[v = p_n \cdot q_n]$

So, we can calculate current year's quantity by using the formula $\frac{p_n q_n}{p_n}$. But as the base year's quantity is not available, we can not use Laspeyres' formula. Hence, we have to find the price index number by using Paasche's formula.

Calculation of Price Index Number

Commodity	p_o	p_n	$v = p_n q_n$	$q_n = \frac{v}{p_n}$	$p_o q_n$
P	22	30	240	8	176
Q	16	18	72	4	64
R	20	25	150	6	120
S	8	12	36	3	24
Total			498		384

$$\text{Paasche's Price Index Number} = \frac{\Sigma p_n q_n}{\Sigma p_o q_n} \times 100$$

$$= \frac{498}{384} \times 100$$

$$= 129.69.$$

EXAMPLE 12

With the help of the following data, show that Fisher's formula satisfies the Time Reversal Test:

Commodities	2011		2012	
	Quantity (kg)	**Price (₹)**	**Quantity (kg)**	**Price (₹)**
Rice	50	32	50	30
Wheat	35	30	40	25
Sugar	55	16	50	18

[***C.U. B.Com. 2013 (G), 2014 (G)***]

Solution: Let p_o, p_n be the prices and q_o, q_n be the quantities for the base year (2011) and current year (2012) respectively.

Calculation of Fisher's Price Index

Commodities	q_o	p_o	q_n	p_n	p_nq_o	p_oq_o	p_nq_n	p_oq_n
Rice	50	32	50	30	1500	1600	1500	1600
Wheat	35	30	40	25	875	1050	1000	1200
Sugar	55	16	50	18	990	880	900	800
Total					3365	3530	3400	3600

Fisher's Price Index of current year with respect to base year

$$(P_{on}) = \sqrt{\frac{\Sigma p_nq_o}{\Sigma p_oq_o} \times \frac{\Sigma p_nq_n}{\Sigma p_oq_n}} \quad \text{[omitting the factor 100]}$$

$$= \sqrt{\frac{3365}{3530} \times \frac{3400}{3600}}$$

Fisher's Price Index of base year with respect to current year

$$(P_{no}) = \sqrt{\frac{\Sigma p_oq_n}{\Sigma p_nq_n} \times \frac{\Sigma p_oq_o}{\Sigma p_nq_o}} \quad \text{[omitting the factor 100]}$$

$$= \sqrt{\frac{3600}{3400} \times \frac{3530}{3365}}$$

Now,

$$P_{on} \times P_{no} = \sqrt{\frac{3365}{3530} \times \frac{3400}{3600} \times \frac{3600}{3400} \times \frac{3530}{3365}}$$

$$= \sqrt{1} = 1$$

Hence, Fisher's Price Index satisfies Time Reversal Test.

EXAMPLE 13

Using the following data verify that Paasche's formula for index does not satisfy Factor Reversal Test:

Commodity	2005		2008	
	Quantity	Price (₹)	Quantity	Price (₹)
x	50	32	50	30
y	35	30	40	25
z	55	16	50	18

[***C.U. B.Com. 2010, 2016(H)***]

Solution: Let p_o, p_n be the prices and q_o, q_n be the quantities for the base year (2005) and current year (2008) respectively.

Calculation of Paasche's Index Number

Commodity	q_o	p_o	q_n	p_n	p_oq_n	p_oq_o	p_nq_n	p_nq_o
x	50	32	50	30	1600	1600	1500	1500
y	35	30	40	25	1200	1050	1000	875
z	55	16	50	18	800	880	900	990
Total					3600	3530	3400	3365

Paasche's Price Index $(P_{on}) = \dfrac{\Sigma\, p_nq_n}{\Sigma\, p_oq_n}$ [omitting the factor 100]

$$= \frac{3400}{3600}$$

Paasche's Quantity Index $(Q_{on}) = \dfrac{\Sigma\, q_np_n}{\Sigma\, q_op_n}$ [omitting the factor 100]

$$= \frac{3400}{3365}$$

$\therefore$ $$P_{on} \times Q_{on} = \frac{3400}{3600} \times \frac{3400}{3365} = 0.94 \times 1.01$$

$$= 0.9494$$

Now, value ratio $= \dfrac{\Sigma\, p_nq_n}{\Sigma\, p_oq_o} = \dfrac{3400}{3530} = 0.9632$

Therefore, $$P_{on} \times Q_{on} \neq \frac{\Sigma\, p_nq_n}{\Sigma\, p_oq_o}$$

Hence, Paasche's Index formula does not satisfy Factor Reversal Test.

EXAMPLE 14

Compute Fisher's index number on the basis of the following data:

Commodity	Base year		Current year	
	Price (₹)	Expenditure (₹)	Price (₹)	Expenditure (₹)
A	5	25	10	60
B	1	10	2	24
C	4	16	8	40
D	2	40	5	75

Also apply Factor Reversal Test to the above index number.

***Solution*:** We know that, Expenditure = Price × Quantity

Therefore, $$\text{Quantity} = \frac{\text{Expenditure}}{\text{Price}}$$

Calculation of Fisher's indices

Commodity	Base year		Current year		$q_o = \frac{c}{b}$	$q_n = \frac{e}{d}$	$p_o q_n$	$p_n q_o$
	Price (p_o)	Expenditure ($p_o q_o$)	Price (p_n)	Expenditure ($p_n q_n$)				
a	**b**	**c**	**d**	**e**	**f**	**g**	**h**	**i**
A	5	25	10	60	5	6	30	50
B	1	10	2	24	10	12	12	20
C	4	16	8	40	4	5	20	32
D	2	40	5	75	20	15	30	100
Total		91		199			92	202

$$\text{Fisher's Price Index Number} = \sqrt{\frac{\Sigma p_n q_o}{\Sigma p_o q_o} \times \frac{\Sigma p_n q_n}{\Sigma p_o q_n}} \times 100$$

$$= \sqrt{\frac{202}{91} \times \frac{199}{92}} \times 100$$

$$= \sqrt{\frac{40198}{8372}} \times 100 = \sqrt{4.8015} \times 100$$

$$= 2.1912 \times 100 = 219.12$$

$$\text{Fisher's Quantity Index Number} = \sqrt{\frac{\Sigma p_o q_n}{\Sigma p_o q_o} \times \frac{\Sigma p_n q_n}{\Sigma p_n q_o}} \times 100$$

$$= \sqrt{\frac{92}{91} \times \frac{199}{202}} \times 100 = \sqrt{\frac{18308}{18382}} \times 100 = 0.998 \times 100 = 99.8$$

Now, $$P_{on} \times Q_{on} = \sqrt{\frac{202}{91} \times \frac{199}{92}} \times \sqrt{\frac{92}{91} \times \frac{199}{202}} \quad \text{(omitting the factor 100)}$$

$$= \sqrt{\frac{202}{91} \times \frac{199}{92} \times \frac{92}{91} \times \frac{199}{202}} = \frac{199}{91} = \frac{\Sigma p_n q_n}{\Sigma p_o q_o}$$

Hence, Fisher's Ideal Index Number satisfies the Factor Reversal Test.

EXAMPLE 15

From the following data, calculate the cost of living index numbers:

Group	Weight (W)	Index numbers (Base 2004 – 05 = 100)
Food	50	241
Clothing	2	221
Fuel etc.	3	204
Rent etc.	16	256
Miscellaneous	29	179

[*C.U. B.Com. 2016 (G)*]

Solution: **Calculation of Cost of Living Index Number**

Group	Weight (W)	Index number (I)	I.W
Food	50	241	12050
Clothing	2	221	442
Fuel etc.	3	204	612
Rent etc.	16	256	4096
Miscellaneous	29	179	5191
Total	100		22391

$$\text{Cost of Living Index} = \frac{\Sigma IW}{\Sigma W} = \frac{22391}{100} = 223.91$$

EXAMPLE 16

Compare the general cost of living indices for the two years 2006 and 2012 from the following table:

Group	Weight	Group Indices	
		2006	2012
Food	71	370	380
Clothing	3	423	504
House Rent	7	110	116
Miscellaneous	10	279	283

[*C.U. B.Com. 2013 (G)*]

Solution: **Calculation of cost of living indices**

Group	Weight	Group Indices		I_1W	I_2W
		2006 (I_1)	2012 (I_2)		
Food	71	370	380	26270	26980
Clothing	3	423	504	1269	1512
House Rent	7	110	116	770	812
Miscellaneous	10	279	283	2790	2830
Total	91			31099	32134

$$\text{Cost of living index for the year 2006} = \frac{\Sigma I_1 W}{\Sigma W} = \frac{31099}{91} = 341.75$$

$$\text{Cost of Living index for the year 2012} = \frac{\Sigma I_2 W}{\Sigma W} = \frac{32134}{91} = 353.12$$

EXAMPLE 17

Following information relating to the workers in a town are given:

Items	Consumer's Price Index in 2011 (base year = 2000)	Percentage Expenditure on the items
Food	225	52
Clothing	175	8
Fuel	155	10
House Rent	250	14
Miscellaneous	150	16

The average wage per month in 2000 was ₹ 12,000. What should be the average wage per month in 2011 in that town so that the standard of living in 2011 remains same as that in 2000? [*C.U. B.Com. 2013 (H)*]

Solution: Calculation of the cost of living index number for the year 2011

Items	Consumer's Price Index in 2011 (*I*) (base year = 2000)	% Expenditure on the items (*W*)	*I.W*
Food	225	52	11700
Clothing	175	8	1400
Fuel	155	10	1550
House Rent	250	14	3500
Miscellaneous	150	16	2400
Total		100	20550

Cost of living index number for the year 2011 $= \dfrac{\Sigma IW}{\Sigma W} = \dfrac{20550}{100} = 205.5$

Therefore, the average wage per month in 2011 was ₹ 205.5 with respect to the base 2000 when the wage per month was ₹ 100.

Average wage per month

2000	2011
100	205.5
12,000	$\dfrac{205.5}{100} \times 12{,}000 = ₹ 24{,}660$

So, to maintain the same standard of living as in 2000, the average wage per month of the worker should be ₹ 24,660 in 2011.

EXAMPLE 18

Determine the weight for the food group with cost of living index number for 2014 with 2010 as base is 175 from:

Group	% increase in expenditure	Weight
Food	95	–
Clothing	90	12
Fuel etc.	20	18
Rent etc.	50	20
Miscellaneous	70	10

[***C.U. B.Com. 2015(H)***]

Solution: Let weight for the food group be 'x'.

Group	% increase in expenditure	Current Index (I)	Weight (W)	$I.W.$
Food	95	195	x	$195x$
Clothing	90	190	12	2280
Fuel etc.	20	120	18	2160
Rent etc.	50	150	20	3000
Miscellaneous	70	170	10	1700
Total			$60 + x$	$9140 + 195x$

$$\text{Cost of living index} = \frac{\Sigma IW}{\Sigma W}$$

or $\quad 175 = \dfrac{9140 + 195x}{60 + x}$

or $\quad 10500 + 175x = 9140 + 195x$

or $\quad 195\,x - 175x = 10500 - 9140$

or $\quad 20x = 1360$

or $\quad x = \dfrac{1360}{20} = 58$

Therefore, the weight for the food group is 58.

EXAMPLE 19

Consumer price index number goes up from 110 to 200 and the salary of a worker is also raised from ₹ 325 to ₹ 500. Does the worker really gain, and if so, by how much in real terms? [***C.U. B.Com. 1984, 2012***]

Solution: Consumer price index number goes up from 110 to 200. To maintain the same standard of living, the salary of the worker should be proportional to consumer price index number.

Consumer price index number	**Salary**
110	325
200	$\frac{325}{110} \times 200 =$ ₹ 590.91

But his salary was raised from ₹ 325 to ₹ 500. So, in real terms, he does not gain. He should get an additional amount of ₹(590.91 – 500) = ₹ 90.91 to maintain same standard of living as before.

EXAMPLE 20

The net salary of an employee was ₹ 3,000 in the year 2000. The Consumer Price Index Number in the year 2011 is 250 with 2000 as base year. Calculate the dearness allowance to be paid to the employee if he has to be rightly compensated.

[***C.U. B.Com. 2011***]

Solution: Consumer price index number in the year 2011 is 250 with 2000 as base year. To maintain the same standard of living, salary should be proportional to consumer price index number.

Consumer price index number	**Salary**
100	3000
250	$\frac{3000}{100} \times 250 = 7500$

Therefore, the dearness allowance of ₹ (7500 – 3000) = ₹ 4500 to be paid to the employee if he has to be rightly compensated.

EXAMPLE 21

For the year 2004, the following table gives the cost of living index numbers for different groups together with their respective weights (1991 as base year):

Group	**Group index**	**Weight**
Food	425	62
Clothes	475	4
Fuel	300	6
House rent	400	12
Misc.	250	16

Obtain the overall cost of living index number. Suppose a person was earning ₹ 6000 in 1991, what should be his earning in 2004 if his standard of living is as level as in 1991? [***C.U. B.Com. 2008***]

Solution: **Computation of overall cost of living index number**

Group	**Group Index (*I*)**	**Weight (*W*)**	**I.W.**
Food	425	62	26350
Clothes	475	4	1900
Fuel	300	6	1800
House Rent	400	12	4800
Misc.	250	16	4000
Total		100	38850

The overall cost of living index number = $\frac{\Sigma IW}{\Sigma W} = \frac{38850}{100} = 388.5$

The above cost of living index number indicates that a person who is getting ₹ 100 in 1991 should receive ₹ 388.5 in 2004 in order to maintain same standard of living. As per question the person was earning ₹ 6000 in 1991. Hence, his earning in 2004 should be as follows:

Cost of living Index number	**Earnings**
100	6000
388.5	$\frac{6000}{100} \times 388.5 =$ ₹ 23,310

Therefore, his earning in 2004 should be ₹ 23,310.

EXAMPLE 22

The following table gives the annual income of a person and the general price index number for five years:

Year	2010	2011	2012	2013	2014
Income (₹)	1800	2100	2500	2750	3000
General Price Index number	100	104	115	160	280

Determine the real income of the person.

Solution:

$$\text{Real Income} = \frac{\text{Actual Income}}{\text{Price Index}} \times 100$$

Calculation of Real Income

Year	Income (₹)	Index number	Real income (₹)
2010	1800	100	$\frac{1800}{100} \times 100 = 1800.00$
2011	2100	104	$\frac{2100}{104} \times 100 = 2019.23$
2012	2500	115	$\frac{2500}{115} \times 100 = 2173.91$
2013	2750	160	$\frac{2750}{160} \times 100 = 1718.75$
2014	3000	280	$\frac{3000}{280} \times 100 = 1071.43$

EXAMPLE 23

When the cost of an item was increased by 50% a man maintaining his former scale of consumption said that the rise in price of the item had increased his cost of living by 5%. What percent of his cost of living was due to buying the item before the rise of price?

Solution: Let x be the required percentage

Group	weight (w)	% increase (i)	i.w.
Items of increase	x	50	$50x$
Remaining items	$100 - x$	0	0
Total	100		$50x$

Increase of cost of living index = 5

Now, $$5 = \frac{50x}{100} \text{ or } 50x = 500 \text{ or, } x = 10.$$

Therefore, 10% of his cost of living was due to buying the item before the rise of price.

EXAMPLE 24

A price index number series was started in 2006 as base. By 2010 it rose by 25%. The link relative for 2011 was 95. In this year a new series was started. This new series rose by 15 points by next year. But during next four years the rise was not rapid. During 2016 the price level was only 5% higher than 2014 and in 2014 it was 8% higher than 2012. Splice the two series and calculate the index numbers for the various years by shifting the base to 2012. [***D.U. M.Com. 1983***]

Solution: **Splicing of Index Numbers**

Year	Index Number (2006 = 100)	Index Number (2011 = 100)	Old Series spliced to new
2006	100	–	$100 \times \frac{100}{118.75} = 84.21$
2010	125	–	$125 \times \frac{100}{118.75} = 105.26$
2011	$\left(125 \times \frac{95}{100}\right) = 118.75$	100	100
2012	–	115	115
2014	–	$\left(115 \times \frac{108}{100}\right)$ 124.2	124.20
2016	–	$\left(124.2 \times \frac{105}{100}\right)$ 130.41	130.41

Shifting base to 2012

Year	Index Number
2006	$84.21 \times \frac{100}{115} = 73.23$
2010	$105.26 \times \frac{100}{115} = 91.53$
2011	$100 \times \frac{100}{115} = 86.96$
2012	$115 \times \frac{100}{115} = 100.00$
2014	$124.20 \times \frac{100}{115} = 108.00$
2016	$130.41 \times \frac{100}{115} = 113.40$

EXAMPLE 25

From the data given below, construct a cost of living index number by using family budget method for 2016 with 2006 an base year:

Commodity	A	B	C	D	E	F
Qty in units in 2006	50	25	10	20	30	40
Price per unit in 2006 (₹):	10	5	8	7	9	6
Price per unit in 2016 (₹):	6	4	3	8	10	12

[***Osmania University, B.Com. 1986***]

Solution: **Calculation of cost of living index by using family budget method**

Commodity	Qty in 2006 (q_o)	Price in 2006 (p_o)	Price in 2016 (p_n)	$I = \frac{P_n}{P_o} \times 100$	$W = p_o q_o$	I.W.
A	50	10	6	60.00	500	30,000.00
B	25	5	4	80.00	125	10,000.00
C	10	8	3	37.50	80	3,000.00
D	20	7	8	114.28	140	15,999.20
E	30	9	10	111.11	270	29,999.70
F	40	6	12	200.00	240	48,000.00
Total					1355	1,36,998.90

EXAMPLE 26

Construct chain index numbers (Base 1992 = 100) for the year 1993–97.

Year	1993	1994	1995	1996	1997
Link Index	103	98	105	112	108

[***C.U. B.Com. 1999***]

Solution: **Calculation of chain index numbers**

Year	Link Index	Chain Index Number
1992	100	100
1993	103	$\frac{103 \times 100}{100} = 103$
1994	98	$\frac{98 \times 103}{100} = 100.94$
1995	105	$\frac{105 \times 100.94}{100} = 105.987$
1996	112	$\frac{112 \times 105.987}{100} = 118.71$
1997	108	$\frac{108 \times 118.71}{100} = 128.21$

EXAMPLE 27

In the following series of index numbers, shift the base from 1970 to 1973.

Year	1970	1971	1972	1973	1974	1975	1976	1977
Index	100	105	110	125	135	180	195	205

[***Lucknow University, 1982***]

Solution: **Shifting of base from 1970 to 1973**

Year	**Index Number (Base 1970 = 100)**	**Index Number (Base 1973 = 100)**
1970	100	$\frac{100}{125} \times 100 = 80$
1971	105	$\frac{100}{125} \times 105 = 84$
1972	110	$\frac{100}{125} \times 110 = 88$
1973	125	$\frac{100}{125} \times 125 = 100$
1974	135	$\frac{100}{125} \times 135 = 108$
1975	180	$\frac{100}{125} \times 180 = 144$
1976	195	$\frac{100}{125} \times 195 = 156$
1977	205	$\frac{100}{125} \times 205 = 164$

[In this calculation $\frac{100}{125}$ is the multiplying factor.]

EXAMPLE 28

Assume that an index number is 100 in 2008, it rises 3% in 2009, falls 1% in 2010 and rises 2% in 2011 and 3% in 2012; rise and fall begin with respect to the previous year. Calculate the index for five years, using 2012 as the base year.

Solution: **Shifting of base from 2008 to 2012**

Year	**Index Number (Base 2008 = 100)**	**Index Number (Base 2012 = 100)**
2008	100	$\frac{100}{107} \times 100 = 93$
2009	$\frac{103}{100} \times 100 = 103$	$\frac{103}{107} \times 100 = 96$
2010	$\frac{99}{100} \times 103 = 102$	$\frac{102}{107} \times 100 = 95$
2011	$\frac{102}{100} \times 102 = 104$	$\frac{104}{107} \times 100 = 97$
2012	$\frac{103}{100} \times 104 = 107$	100

EXAMPLE 29

Splice the following two series of index numbers with (i) base 2000 and (ii) base 2005:

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
Series A	100	120	130	140	160	200	–	–	–	–	–
Series B	–	–	–	–	–	100	105	90	120	140	160

Solution: **Splicing of two series of index numbers**

Year	**Series A Index Number**	**Series B Index Number**	**Spliced Index Numbers**	
	Base 2000 = 100	**Base 2005 = 100**	**Base 2000 = 100**	**Base 2005 = 100**
2000	100	–	100	$\frac{100}{200}\times 100 = 50$
2001	120	–	120	$\frac{100}{200}\times 120 = 60$
2002	130	–	130	$\frac{100}{200}\times 130 = 65$
2003	140	–	140	$\frac{100}{200}\times 140 = 70$
2004	160	–	160	$\frac{100}{200}\times 160 = 80$
2005	200	100	200	$\frac{100}{200}\times 200 = 100$
2006	–	105	$\frac{200}{100}\times 105 = 210$	105
2007	–	90	$\frac{200}{100}\times 90 = 180$	90
2008	–	120	$\frac{200}{100}\times 120 = 240$	120
2009	–	140	$\frac{200}{100}\times 140 = 280$	140
2010	–	160	$\frac{200}{100}\times 160 = 320$	160

EXAMPLE 30

Deflate the per capita income shown in the following table on the basis of the rise in the cost of living index.

Year	2005	2006	2007	2008	2009	2010	2011	2012
CLI	100	110	120	130	150	200	250	350
Per capita income (₹)	65	70	75	80	90	100	110	130

Solution: **Deflating per capita income middle**

Year	**CLI (2005 = 100)**	**Per capita income (₹)**	**Real income (₹)**
2005	100	65	65.00
2006	110	70	$\frac{70}{110}\times 100 = 63.64$
2007	120	75	$\frac{75}{120}\times 100 = 62.50$
2008	130	80	$\frac{80}{130}\times 100 = 61.54$
2009	150	90	$\frac{90}{150}\times 100 = 60.00$
2010	200	100	$\frac{100}{200}\times 100 = 50.00$
2011	250	110	$\frac{110}{250}\times 100 = 44.00$
2012	350	130	$\frac{130}{350}\times 100 = 37.14$

EXAMPLE 31

Calculate the chain base index numbers chained to 2011 from the average price of following three commodities:

Commodity	**2011**	**2012**	**2013**	**2014**	**2015**
Wheat	4	6	8	10	12
Rice	16	20	24	30	36
Sugar	8	10	16	20	24

[***Kurukshetra University, B.Com. 1985***]

Solution: **Calculation of chain indices**

Commodity	**Relatives based on preceding year**				
	2011	**2012**	**2013**	**2014**	**2015**
Wheat	100	$\frac{6}{4}\times100 = 150$	$\frac{8}{6}\times100 = 133.33$	$\frac{10}{8}\times100 = 125$	$\frac{12}{10}\times100 = 120$
Rice	100	$\frac{20}{16}\times100 = 125$	$\frac{24}{20}\times100 = 120.00$	$\frac{23}{24}\times100 =125$	$\frac{36}{30}\times100 = 120$
Sugar	100	$\frac{10}{8}\times100 = 125$	$\frac{16}{10}\times100 = 160.00$	$\frac{20}{16}\times100 = 125$	$\frac{24}{20}\times100 = 120$
Total of link relatives	300	400	413.33	375	360
Average of link relatives	100	133.33	137.78	125	120
Chain Index (2011 = 100)	100	$\frac{133.33\times100}{100}$ = 133.33	$\frac{137.78\times133.33}{100}$ = 183.70	$\frac{125\times183.70}{100}$ = 229.63	$\frac{120\times229.63}{100}$ = 275.55

$$\text{Cost of living index} = \frac{\Sigma IW}{\Sigma W} = \frac{136998.90}{1355} = 101.11 \text{ (Approx.)}$$

EXERCISE

A. THEORY

1. Define an index number and its uses. [***C.U. B.Com. 1978***]
2. What are errors in index numbers?
3. Discuss the various problems in constructing index numbers. [***C.U. B.Com. 1976***]
4. What are the characteristics of index numbers?
5. State different types of index numbers.
6. Explain the different methods of constructing index numbers.
7. What are the tests prescribed for a good index number? Describe the index number which satisfies these tests.
8. Describe time reversal test and factor reversal test of index numbers. Show that neither Laspeyres' price index nor Paasche's price index satisfies either test. [***C.U. B.Com. (H), 1986***]

9. Verify that Fisher's Ideal Index satisfies factor reversal test and also time reversal test. [***C.U. B.Com. (H), 1984***]

10, Describe the chain base method of construction of index numbers. How does it compare with the fixed base method? [***C.U. B.Com. (H), 1990***]

11. What do you mean by Price Relative? Write two formulae for price index number using price relatives.

12. What do you mean a link index? Discuss the relative merits and demerits of chain base and fixed base index numbers. [***C.U. M.Com. 79,81***]

13. What is meant by cost of living index number? How is it constructed? Mention its application and uses.

14. Briefly describe the various steps involved in constructing the cost of living index number.

15. What is meant by (*i*) base shifting (*ii*) splicing, and (*iii*) deflating of index numbers? Explain with illustrations.

16. What is the need for deflating the index numbers? Illustrate your answer with the help of an example.

17. Why is Fisher's formula known as 'Ideal'? Give reasons.

18. (*a*) Discuss the importance and uses of weights in the construction of index numbers.
    (*b*) How do you choose the base year for construction of index numbers? [***C.U. B.Com. (H), 1991***]

19. Write the formulae for the following Price Index Numbers and also mention the weights used in them
    (*i*) Laspeyres' index,
    (*ii*) Paasche's index,
    (*iii*) Marshall-Edgeworth's index.

20. Write short notes on
    (*i*) Price Index Number
    (*ii*) Quantity Index Number
    (*iii*) Chain base method
    (*iv*) Consumer Price Index Number
    (*v*) Uses of Index Numbers [***C.U. B.Com. (H), 84,88***]
    (*vi*) Test of adequacy of index numbers [***C.U. B.Com. (H),90,92***]
    (*vii*) Role of family budget survey in the construction of cost of living index number [***C.U. B.Com. (H),87***]
    (*viii*) Base shifting
    (*ix*) Splicing
    (*x*) Deflating

B. SHORT TYPE

1. If Laspeyres' and Paasche's price index numbers are 125.6 and 154.3 respectively, find Fisher's ideal price index number.

   [***C.U. B.Com. 2008***] [**Ans.** 139.21 (Approx.)]

2. The prices of a commodity in the year 1975 and 1980 were 25 and 30 respectively. Find the price relatives (*i*) taking 1975 as base year; (*ii*) taking 1980 as base year. Verify the truth of Time Reversal Test.

   [***C.U. B.Com. 1984***] [**Ans.** (*i*) 120; (*ii*) 83.33; True]

3. The price index of rice of 1993 with 1973 as the base year is 175. Find the percentage increase in the general price level of rice in 1993 over 1973.

   [***C.U. B.Com. 1994***] [**Ans.** 75%]

4. The prices of mustard oil per quintal in 2000 was ₹ 300, while in 2008 it was ₹ 420. Find the index number for 2008 with 2000 as base year.

   [***C.U. B.Com. 1993***] [**Ans.** 140]

5. The price of rice per quintal was ₹ 380, ₹ 415 and ₹ 502 in the year 1980, 1985 and 1990 respectively. Find the index number of the price of rice of 1990 with base year 1980. [***C.U. B.Com. 1983***] [**Ans.** 132.1]

6. During a certain period the cost of living index number goes up from 110 to 200 and the salary of a worker is also raised from ₹ 325 to ₹ 500. Does the worker really gain? [***C.U. B.Com. 1985***] [**Ans.** No]

7. If Laspeyres' and Paasche's Price Index Numbers are 147.5 and 154.3, find Bowley's Price Index Number. [**Ans.** 150.9]

8. Net monthly income of an employee was ₹ 2000 in 2010. The consumer price index number in 2015 is 220 with 2010 as base year. Calculate the additional dearness allowance to be paid to the employee if he has to be rightly compensated.

   [**Ans.** ₹ 2400]

9. Find the price index number from the following data using simple aggregative method.

Commodity:	**A**	**B**	**C**
Price in the year 2012:	6	5	7
Price in the year 2017:	8	6	10

   [**Ans.** 133.33]

10. (*a*) Using simple average of Price Relative Method find the price index for 2016, taking 2010 as base year from the following data:

Commodity	**A**	**B**	**C**
Price (in 2010) per unit	5	10	20
Price (in 2016) per unit	8	12	30

   [**Ans.** 143.33]

(*b*) Consumer price index number for the year 2017 was 313 with 2010 as the base year. The average monthly wages in 2017 of the workers in a factory was ₹ 160. Calculate real wage. [**Ans.** ₹ 51.12]

C. SHORT ESSAY TYPE

11. Find the simple aggregative index number for each of the following:

(*i*) For the year 2016 with 2000 as base year

Commodity	Price in 2000	Price in 2016
A	100	125
B	55	75
C	10	15
D	105	125
E	12.5	12.5

[**Ans.** 124.78]

(*ii*)

Commodity	Base Price	Current Price
Rice	36	42
Wheat	30	35
Fish	42	38
Pulse	25	33

[**Ans.** 111.28]

(*iii*) For the years 2011 and 2012 taking 2009 as base year

Commodity	A	B	C	D	E	F
Price in 2009	10	25	40	30	25	100
Price in 2011	12	30	50	30	25	110
Price in 2012	15	30	60	40	30	120

[**Ans.** 11.74; 128.26]

12. (*i*) Calculate Price Index from the following data using weighted aggregative formula.

Commodity	Unit	Weight	Price per unit	
			Base period	Current period
A	Quintal	14	90	120
B	Kilogram	20	10	17
C	Dozen	35	40	60
D	Litre	15	50	95

[***C.U. B.Com. 1993***] [**Ans.** 153.6] (*ii*) Find by weighted aggregative method, the index number from the following data:

Commodity	Base price	Current price	Weight
Rice	32	50	8
Wheat	25	25	6
Oil	90	100	7
Fish	120	140	3
Potato	35	40	5

[***N.B.U. B.Com. 1969***] [**Ans.** 199.03]

(*iii*) Using weighted aggregative method find the price index number.

Commodity	A	B	C	D	E
Unit	quintal	kg	dozen	litre	pound
Base Price	80	10	40	50	12
Current Price	110	15	56	95	18
Weight	14	20	35	15	16

[**Ans.** 150.55]

13. The average price of mustard oil per quintal in the years 2012 to 2016 are given. Find the index numbers for all the years taking 2014 as the base year.

Year	2012	2013	2014	2015	2016
Price	295	275	300	225	250

[**Ans.** 88.3, 91.7, 100, 75, 83.3]

14. Find the index number by the (*i*) unweighted and (*ii*) weighted aggregative methods using the data given below:

Commodity	Base Price	Current Price	Weight
Rice	36	54	10
Pulse	30	50	3
Fish	130	155	2
Potato	40	35	4
Oil	110	110	5

[***B.U. B.Com. 1969***] [**Ans.** 116.8; 119]

15. Using Arithmetic Mean method, find the index number from the following data:

Commodity	Rice	Wheat	Fish	Potato	Coal
Base year price ₹	30	22	54	20	15
Current year price ₹	35	25	64	25	18

[***C.U. B.Com. 1976***] [**Ans.** 118.76]

16. Find the Arithmetic Mean method, the price index number from the following data:

Commodity	Base Price	Current Price
Wheat	5	7
Milk	8	10
Fish	25	32
Sugar	6	12

[***C.U. B.Com. 1996***] [**Ans.** 148.25]

17. The prices of a number of commodities are given below in the current year (1975) and base year (1970). Using geometric mean calculate a price index based on price relatives.

Commodities	A	B	C	D	E	F
Base price (1970)	45	60	20	50	85	120
Current price (1975)	55	70	30	75	90	130

[***B.U. B.Com. 1976***] [**Ans.** 124.3]

18. Find the Price Index by relative method using (*a*) Arithmetic Mean, and (*b*) Geometric Mean from the following data:

Commodity	Base Price	Current Price
A	30	48
B	22	44
C	16	40

[**Ans.** (*a*) 203; (*b*) 200]

19. From the following data using (*i*) A.M. (*ii*) G.M. method regarding price relative find index number:

Commodity	A	B	C	D	E
Base price	25	20	30	12	90
Current price	30	22	33	15	99

[**Ans.** (*i*) 115; (*ii*) 114.8]

20. Construct Laspeyres' price index number from the following data:

Commodities	Base Price (₹)	Year Quantity (in suitable units)	Current Price (₹)	Year Quantity (in suitable units)
A	2	8	4	6
B	5	10	6	5
C	4	14	5	10
D	2	19	2	13

[***C.U. B.Com. 1988***] [**Ans.** 125]

21. Using the data given below calculate the Price Index Number for the year 2013 by Paasche's formula with the year 2010 as base:

Commodity	Price (₹) per unit		Quantity ('000 kg.)	
	2010	2013	2010	2013
Rice	9.3	14.5	100	90
Wheat	6.4	13.7	11	19
Pulse	5.1	12.7	5	3

[***C.U. B.Com. 2014 (H)***] [**Ans.** 164.64]

22. Calculate price index number for 2012 with 2000 as base year from the following data by using Fisher's method:

Commodities	2000		2012	
	Quantity (kg)	Price (₹)	Quantity (kg.)	Price (₹)
A	5	2.00	7	4.50
B	7	2.50	10	3.20
C	6	8.00	6	4.50

[***C.U. B.Com. 2013(G)***] [**Ans.** 99.53]

23. Using the data given below calculate price index number for 2016 with 2010 as base by (*i*) Laspeyres', (*ii*) Paasche's, (*iii*) Fisher's, (*iv*) Bowley's and (*v*) Marshall–Edgeworth formulae.

Commodity	Price (₹)		Quantity (kg)	
	2016	2010	2016	2010
A	30	20	82	74
B	40	50	140	125
C	60	70	33	40

[**Ans.** (*i*) 91.36; (*ii*) 91.69; (*iii*) 91.44; (*iv*) 91.52; (*v*) 91.53]

24. Taking 2015 as base year find the index number by using a suitable formula:

Commodity	Wheat		Rice		Sugar	
Year	Price	Quantity	Price	Quantity	Price	Quantity
2015	14	15	20	5	4	10
2016	24	12	27	4	7	8

[**Ans.** Fisher's formula: 161.4]

25. Using appropriate formula find the price index number from the following data:

Commodity	2010		2014
	Price (₹)	Quantity	Price (₹)
A	6	50	9
B	2	100	3
C	4	60	6
D	10	30	4

[**Ans.** Laspeyres's index number; 147]

26. Calculate the price index number for the year 1998 with 1996 as base using Laspeyres' or Paasche's formula, whichever will be applicable, on the basis of the following data:

Commodity	Price (in ₹)		Money Value (₹'000)
	1996	1998	1996
P	12.50	14.00	112.50
Q	10.50	12.00	126.00
R	15.00	14.00	105.00
S	9.40	11.20	47.00

[Here money value means total value of a commodity]

[**Ans.** Laspeyres' formula: 109]

27. Using appropriate formula find the price index from the following data:

Commodity	Price (₹)		Quantity
	2010	2015	2015
x	8	12	5
y	10	11	6
z	7	8	5

[**Ans.** Paasche's formula: 123]

28. Calculate the price index number for 2016 with 2006 as base on the basis of following data:

Commodity	Price (₹)		Money Value
	2006	2016	2016
P	56	62.5	250
Q	50	48	300
R	28	30	210
S	22.4	18.8	94
T	20	16	80

[**Ans.** Paasche's index: 98.9]

29. Find 'x' from the following data:

Commodities	Base Year		Current Year	
	Price (₹)	Quantity	Price (₹)	Quantity
A	1	10	2	5
B	1	5	x	2

Given that the ratio between Laspeyres' and Paasche's index number is 28 : 27.

[***C.U. B.Com. 2003, 2012, 2016 (H)***] [**Ans.** $x = 4$]

30. Find the quantity index number using (*i*) Laspeyres' (*ii*) Paasche's formula from the given table:

Commodity	**A**		**B**		**C**	
Year	**Price**	**Quantity**	**Price**	**Quantity**	**Price**	**Quantity**
2012	5	10	8	6	6	3
2013	4	12	7	7	5	4

[**Ans.** 120.68; 120.62]

31. Using Fisher's Ideal Formula, compute quantity index number for 2014 with 2012 as base year, given the following information:

Year	**Commodity A**		**Commodity B**		**Commodity C**	
	Price per unit (₹)	**Expenditure (₹)**	**Price per unit (₹)**	**Expenditure (₹)**	**Price per unit (₹)**	**Expenditure (₹)**
2012	5	50	8	48	6	18
2013	4	48	7	49	5	20

[**Ans.** 120.6]

32. Prepare price and quantity index numbers for 2012 with 2001 as base year from the following data by using (*i*) Laspeyres', (*ii*) Paasche's and (*iii*) Fisher's method:

Commodity	**Unit**	**2001**		**2012**	
		Quantity	**Price (₹)**	**Quantity**	**Price (₹)**
A	kg	5	2.00	7	4.50
B	quintal	7	2.50	10	3.20
C	dozen	6	8.00	6	4.50
D	kg	2	1.00	9	1.80

[***C.U. B.Com. (H), 1981***]

[**Ans.** Price Index No. (*i*) 97.42; (*ii*) 111.15; (*iii*) 104.06 Quantity Index No. (*i*) 123.87; (*ii*) 141.32; (*iii*) 132.31]

33. Using the following data, compute Fisher's ideal price and quantity index numbers for the current year.

Items	**Base year**		**Current year**	
	Price (₹/kg)	**Quantity (kg)**	**Price (₹/kg)**	**Quantity (kg)**
A	8	6	12	4
B	10	8	12	8
C	14	4	18	4
D	4	6	2	10
E	10	10	14	8

[**Ans.** 124.01; 91.11]

34. Using the following data verify that Paasche's formula for index does not satisfy factor reversal test:

Commodity	Base year		Current year	
	Price (₹)	Quantity (kg)	Price (₹)	Quantity (kg)
x	4	10	6	15
y	6	15	4	20
z	8	5	10	4

[***C.U. B.Com. 1995, 2016 (G)***]

35. Using the following data, show that Fisher's Ideal formula satisfies the Factor Reversal Test:

Commodity	Price (₹) per unit		Quantity	
	Base year	Current year	Base year	Current year
A	6	10	50	56
B	2	2	100	120
C	4	6	60	60
D	10	12	30	24
E	8	12	40	36

[***C.U. B.Com. 2011, 2014 (H)***]

36. From the following data, show that Fisher's Price Index Number formula satisfies Time Reversal Test:

Commodities	Base year		Current year	
	Price (₹)	Quantity (kg)	Price (₹)	Quantity (kg)
A	8	6	12	5
B	10	5	11	6
C	7	8	8	5

[***C.U. B.Com. 2013 (H)***]

37. Calculate from the following data the Fisher's Ideal Index and show how it satisfies the Time Reversal test and Factor Reversal test.

Item	Price (₹)		Quantity	
	2015	2016	2015	2016
A	8	20	50	60
B	2	6	15	10
C	1	2	20	25
D	2	5	10	8
E	1	5	40	30

[**Ans.** 266.61]

38. Construct (*a*) fixed base, and (*b*) chain base index numbers from the following data relating to production of electricity.

Year	2012	2013	2014	2015	2016	2017
Production	25	27	30	24	28	29

[**Ans.** 100, 108, 120, 96, 112, 116]

39. Prepare fixed base index numbers from the chain base index numbers given below:

Year	2009	2010	2011	2012	2013	2014
Chain Index	94	104	104	93	103	102

[**Ans.** 94, 97.76, 101.67, 94.55, 97.39, 99.34]

40. Compute the chain index number with 2003 prices as base from the following table giving the average wholesale prices of the commodities A, B and C for the years 2004–2008.

Commodity	**Average wholesale price (₹)**				
	2004	**2005**	**2006**	**2007**	**2008**
A	20	16	28	35	21
B	25	30	24	36	45
C	20	25	30	24	30

[**Ans.** 100, 108.33, 135.41, 160.23, 165.57]

41. The following are index numbers of prices (2009 = 100):

Year	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
Index	100	120	180	207	243	270	300	360	400	420

Shift the base from 2009 to 2015 and recast the index numbers.

[**Ans**. 33, 40, 60, 69, 81, 90, 100, 120, 133, 140]

42. The following are index numbers of wholesale prices of a commodity based on 2008

Year	2008	2009	2010	2011	2012	2013	2014
Index	100	108	120	150	210	225	240

Prepare new index numbers taking 2010 as base.

[**Ans.** 83.33, 90, 100, 125, 175.5, 187.5, 200]

43. Splice the following two series of index numbers with (*a*) base 2000 and (*b*) 2005:

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
Series A	100	120	135	160	178	200	–	–	–	–	–
Series B	–	–	–	–	–	100	110	98	115	125	150

[**Ans.** (*a*) 100, 120, 135, 160, 178, 200, 220, 196, 230, 250, 300
(*b*) 50, 60, 67.5, 80, 89, 100, 110, 98, 115, 125, 150]

44. Monthly wages average in different years is as follows:

Year	2007	2008	2009	2010	2011	2012	2013
Wages (₹)	200	240	350	360	360	380	400
Price index	100	150	200	220	230	250	250

[**Ans**. 100, 80, 87.5, 82, 78.5, 76, 80]

45. Given the following data:

Year	**Weekly take-home Pay (wages) (₹)**	**Consumer price index**
2008	109.50	112.8
2009	112.20	118.2
2010	116.40	127.4
2011	125.08	138.2
2012	135.40	143.5
2013	138.10	149.8

(*i*) What was the real average weekly wage for each year?

(*ii*) In which year did the employees have the greatest buying power?

[**Ans.** (*i*) 97.07, 94.92, 91.37, 90.51, 94.36, 92.19; (*ii*) 2008]

46. Construct the cost of living index number from the following data:

Group	Food	Fuel & Light	Clothing	House Rent	Miscellaneous
Weight	47	7	8	14	24
Group index	247	293	289	100	236

[**Ans.** 230.36]

47. Find the general cost of living index of 2015 from the following table.

Class	Food	Dress	Rent	Fuel	Miscellaneous
Class index	241	221	256	204	179
Weight	50	2	16	3	29

[***C.U. B.Com. 1991***] [**Ans.** 223.91]

48. From the table of group index numbers and group expenditures given below calculate the cost of living index number:

Group	**Index number**	**% of total expenditure**
Food	428	45
Clothing	250	15
Fuel & light	220	8
House rent	125	20
Others	175	12

[**Ans.** 293.7]

49. The group indices and the corresponding weights for the cost of living index number of the working class in an industrial city for the years 1990 and 1992 are given below.

Group	**Weight**	**Group Index**	
		1990	**1992**
Food	60	120	150
Clothing	5	150	190
Fuel	10	130	160
House rent	15	110	130
Miscellaneous	10	120	140

Compare the cost of living indices for the two years 1990 and 1992. If a worker getting ₹ 2000 in 1990 and ₹ 2300 in 1992, state how much extra allowance should be given to him to maintain his standard of living at 1990 level.

[**Ans.** ₹ 162.81]

50. For the year 2004, the following table gives the cost of living index numbers for different groups together with their prospective weights (1991 as base year):

Group	**Group Index**	**Weight**
Food	425	62
Clothes	475	4
Fuel	300	6
House rent	400	12
Misc.	250	16

Obtain the overall cost of living index number. Suppose a person was earning ₹ 6000 in 1991, what should be his earning in 2004 if his standard of living is at the same level as in 1991?

[***C.U. B.Com. 2008***] [**Ans.** 388.5, ₹ 23,310]

51. The cost of living index uses the following weights:

Food – 40, rent –15, clothing – 20, fuel – 10, miscellaneous – 15.

During the period 1992–2002, the cost of living index number rose from 100 to 205.65. Over the same period the percentage rise in prices were:

Rent – 60, Clothing – 180, Fuel – 75, Miscellaneous – 165. What was the percentage change in the price of food?

[***C.U. B.Com. 2007***] [**Ans.** 71% increase]

52. A textile worker in the city of Bombay earns ₹ 350 per month. The cost of living index for a particular month is given as ₹ 136. Using the following data find out the amounts he spent on house rent and clothing.

Group	Food	Clothing	House rent	Fuel	Misc.
Expenditure	140	?	?	56	63
Group index	180	150	100	110	80

[**Ans.** ₹ 42 and ₹ 49]

53. The following data show the cost of living indices for the groups food, clothing, fuel and light, house rent and miscellaneous, with their respective weights, for middle-class people of Kolkata in 2007. Obtain the general cost of living index number. Mr. X was getting as salary ₹ 250 in 1989 and ₹ 429 in 2007. State how much he ought to have received as extra allowance in 2007 to maintain his standard of living at 1989 level.

Group	Base 1989 = 100	
	Group Index	**Group Weight**
Food	411.8	61.4
Clothing	544.8	4.5
Fuel & Light	388.0	6.5
House rent	116.9	8.9
Miscellaneous	284.5	18.7

[***C.U. B.Com. 1987***] [**Ans.** ₹ 486.50]

54. In 2015 for working class people wheat was selling at an average price of ₹ 320 per 20 kg, cloth at ₹ 40 per metre, house rent ₹ 600 per house and other items at ₹ 200 per unit. By 2016 cost of wheat rose by ₹ 80 per 20 kg, house rent by ₹ 300 per house and other items doubled in price. The working class cost of living index for the year 2016 (with 2015 as base) was 160. By how much did the price of the cloth rise during the period? [**Ans.** ₹ 66]

55. Construct the consumer price index number for 2016 on the basis of 2000 from the following data using family budget method:

Items	Price in 2000	Price in 2016	Weights
	(₹)	(₹)	
Food	200	280	30
Rent	100	200	20
Clothing	150	120	20
Fuel & Lighting	50	100	10
Miscellaneous	100	200	20

[**Ans.** 158]

56. When the cost of rice was increased by 60%, a person, who maintained his former consumption scale, said that the rise had increased his cost of living by 7%. What percentage of his cost of living was due to buying rice before the change in price? [***C.U. B.Com. 2016(H)***] [**Ans.** 11.67%]

57. The group index numbers for the current year as compared with a fixed base period were respectively 410, 150, 343, 248 and 285. Calculate the consumer price index number for the current year. Mr. X was getting ₹ 240 in the base period and ₹ 430 in the current year. State how much he ought to have received as extra allowance to maintain his former standard of living.

[**Ans.** 325, ₹ 350]

D. MULTIPLE CHOICE QUESTIONS (MCQs)

(i) Short Type

1. Index number can be used for
   (*a*) fixed prices
   (*b*) constant prices
   (*c*) forecasting
   (*d*) different prices [**Ans.** (*c*)]
2. The base period for construction of index number is always taken as
   (*a*) 0
   (*b*) 100
   (*c*) 200
   (*d*) 1 [**Ans.** (*b*)]
3. If all values are of equal importance, the index numbers are called
   (*a*) composite
   (*b*) value index
   (*c*) weighted
   (*d*) unweighted [**Ans.** (*d*)]
4. Index numbers are expressed in
   (*a*) percentage
   (*b*) ratios
   (*c*) squares
   (*d*) combinations [**Ans.** (*a*)]
5. An index number is called a simple index when it is computed from
   (*a*) single variable
   (*b*) bi-variable
   (*c*) multiple variables
   (*d*) none of these [**Ans.** (*a*)]
6. The ratio of a sum of prices in current period to the sum of prices in the base period, expressed as a percentage is called
   (*a*) simple price index number
   (*b*) quantity index number
   (*c*) simple aggregative price index number
   (*d*) weighted aggregative price index number [**Ans.** (*c*)]
7. When the price of a year is divided by the price of a particular year we get;
   (*a*) link relative
   (*b*) simple relative
   (*c*) both A and B
   (*d*) none of these [**Ans.** (*b*)]
8. A number that measures relative change in a single variable with respect to a base is called:
   (*a*) composite index number
   (b) good index number
   (*c*) simple index number
   (*d*) quantity index number
   [**Ans.** (*c*)]
9. A number that measures an average relative change in a group of related variables with respect to a base is called

(*a*) composite index number
(*b*) simple index number
(*c*) quantity index number
(*d*) price index number
[**Ans.** (*a*)]

10. When relative change is measured for a fixed period, it is called
(*a*) chain base method
(*b*) fixed base method
(*c*) simple aggregative method
(*d*) cost of living index method
[**Ans.** (*b*)]

11. An index number constructed to measure the relative change in the price of an item or a group of items is called
(*a*) quantity index number
(*b*) price index number
(*c*) volume index number
(*d*) none of these [**Ans.** (*b*)]

12. When the base year values are used as weights, the weighted average of relatives price index number is the same as the
(*a*) Laspeyres' index
(*b*) Paasche's index
(*c*) Simple aggregative index
(*d*) Quantity index [**Ans.** (*a*)]

13. While computing a weighted index, the current period quantities are used in the
(*a*) Fisher's ideal method
(*b*) Laspeyers' method
(*c*) Paasche's method
(*d*) Marshall-Edgeworth method
[**Ans.** (*c*)]

14. Fisher's ideal index number is the geometric mean of the
(*a*) Laspeyres' and Marshall-Edgeworth indices
(*b*) Paasche's and Marshall-Edgeworth indices
(*c*) Laspeyres' and Paasche's indices
(*d*) None of the above [**Ans.** (*c*)]

15 Price relatives are a percentage ratio of current year price and
(*a*) base year quantity
(*b*) current year quantity
(*c*) pervious year quantity
(*d*) base year quantity [**Ans.** (*d*)]

16. An index that measures the average relative change in group of variables keeping in view the relative importance of the variables is called:
(*a*) simple index number
(*b*) weighted index number
(*c*) composite index number
(*d*) price index number [**Ans.** (*b*)]

17. In chain base method, the base period is
(*a*) fixed
(*b*) not fixed
(*c*) constant
(*d*) zero [**Ans.** (*b*)]

18. The chain base indices are not suitable for
(*a*) ratios
(*b*) percentages
(*c*) short range comparisons
(*d*) long range comparisons
[**Ans.** (*d*)]

19. Chain base indices are free from
(*a*) seasonal variations
(*b*) ratios
(*c*) errors
(*d*) percentages [**Ans.** (*a*)]

20. When the price of a year is divided by the price of the preceding year, we get
    (*a*) value index
    (*b*) link relative
    (*c*) simple relative
    (*d*) none of these [**Ans.** (*b*)]
21. The most appropriate average in averaging the price relatives is:
    (*a*) arithmetic mean
    (*b*) geometric mean
    (*c*) harmonic mean
    (*d*) median [**Ans.** (*b*)]
22. Chain process is used to make comparisons of price index numbers in:
    (*a*) price relative
    (*b*) simple relative
    (*c*) link relative
    (*d*) none of the above [**Ans.** (*c*)]
23. Consumer's price index is also known as
    (*a*) cost of living index
    (*b*) wholesale price index number
    (*c*) sensitive index number
    (*d*) composite index number
    [**Ans.** (*a*)]
24. Purchasing power of money can be accessed through:
    (*a*) volume index
    (*b*) simple index
    (*c*) Fisher's index
    (*d*) consumer price index
    [**Ans.** (*d*)]
25. Cost of living index are obtained by
    (*a*) Paasche's formula
    (*b*) Fisher's ideal formula
    (*c*) Marshall-Edgeworth formula
    (*d*) family budget method formula
    [**Ans.** (*d*)]
26. The aggregative expenditure method and family budget method for constructing cost of living index number always give
    (*a*) different results
    (*b*) same results
    (*c*) approximate results
    (*d*) none of these [**Ans.** (*b*)]
27. What type of index number can help the government to formulate its price policies and to take appropriate economic measures to control prices:
    (*a*) consumer's price index
    (*b*) wholesale price index
    (*c*) quantity index
    (*d*) none of these [**Ans.** (*a*)]
28. The price level of a country in a certain year has increased 55% over the base period. The index number is
    (*a*) 55
    (*b*) 155
    (*c*) 255
    (*d*) none of these [**Ans.** (*b*)]
29. If the prices of all commodities in a place have increased by 1.25 times in comparison to the base period prices, then the index number of prices for the place is now:
    (*a*) 100
    (*b*) 125
    (*c*) 225
    (*d*) none of these [**Ans.** (*c*)]
30. Laspeyres' index = 110, Paasche's index = 108, then Fisher's ideal index is equal to:
    (*a*) 110
    (*b*) 108
    (*c*) 100
    (*d*) 109 [**Ans.** (*d*)]

31. Index number calculated by Fisher's formula is ideal because it satisfies:
    (*a*) circular test
    (*b*) factor reversal test
    (*c*) time reversal test
    (*d*) all of the above [**Ans.** (*d*)]

32. The test which is not obeyed by any of the weighted index numbers unless the weights are constant:
    (*a*) circular test
    (*b*) time reversal test
    (*c*) factor reversal test
    (*d*) none of these [**Ans.** (*a*)]

33. Which of the following measures changes in retail price of the commodities?
    (*a*) wholesale price index
    (*b*) weighted index
    (*c*) consumer price index
    (*d*) none of these [**Ans.** (*c*)]

34. Cost of living index numbers are also used to find real wage by the process of
    (*a*) base shifting
    (*b*) splicing of index number
    (*c*) deflating of index number
    (*d*) none of the above [**Ans.** (*c*)]

35. Consumer price index can be constructed by the average of:
    (*a*) stable price
    (*b*) price relative
    (*c*) increasing price
    (*d*) decreasing price [**Ans.** (*b*)]

36. Index numbers are called the barometer of
    (*a*) economy
    (*b*) data calculation
    (*c*) statistical observations
    (*d*) none of these [**Ans.** (*a*)]

37. An index that is designed to measure changes in quantities over time is known as the
    (*a*) Laspeyeres' index
    (*b*) Time index
    (*c*) Paasche's index
    (*d*) Quantity index [**Ans.** (*d*)]

38. A composite price index where the prices of the items in the composite are weighted by their relative importance is known as the
    (*a*) Simple aggregate price index
    (*b*) Price relative
    (*c*) Weighted aggregate price index
    (*d*) Consumer price index [**Ans.** (*b*)]

39. A weighted aggregate price index where the weight for each item is its current period quantity is called the
    (*a*) Consumer price index
    (*b*) Laspeyres' index
    (*c*) Paasche's index
    (*d*) Price relative [**Ans.** (*c*)]

40. If the wholesale price index for first week is 200 and for second week is 250 then rate of inflations is.
    (*a*) 200
    (*b*) 250
    (*c*) 300
    (*d*) 400 [**Ans.** (*a*)]

(ii) Short Essay Type

41.

Commodity:	A	B	C	D
Base Price (₹):	36	30	42	25
Current Price (₹):	42	35	38	33

The Price Index number by the method of simple aggregative from the above data is

(*a*) 115.23 (*c*) 111.28
(*b*) 112.32 (*d*) 110.35 [**Ans.** (*c*)]

42.

Commodity:	P	Q	R	S
Price per unit:				
2015	80	50	90	30
2017	95	60	100	45

Price Index number from the above data by simple aggregative method taking prices of 2015 as base is

(*a*) 115 (*c*) 125
(*b*) 120 (*d*) 130 [**Ans.** (*b*)]

43.

Commodities	M	N	O	P	Q
Price (2007) (₹):	100	40	10	60	90
Price (2017) (₹):	140	60	20	70	100

Price Index number for 2017 taking 2007 as the base year from the above data by simple aggregative method is

(*a*) 130 (*c*) 135
(*b*) 120 (*d*) 140 [**Ans.** (*a*)]

44.

Year:	2012	2013	2014	2015	2016	2017
Price (₹):	10	14	16	20	22	24

Taking 2012 as base year the index number of the year 2017 is

(*a*) 210 (*c*) 230
(*b*) 220 (*d*) 240 [**Ans.** (*d*)]

45.

Commodity:	A	B
Price (2005) (₹):	6	30
Price (2015) (₹):	12	45

The Price Index by relative method using Arithmetic mean from the above data is

(*a*) 182 (*c*) 175
(*b*) 170 (*d*) 164 [**Ans.** (*c*)]

46.

Commodity:	X	Y	Z
Base Price (₹):	30	22	16
Current Price (₹):	48	44	40

The Price Index by relative method using Arithmetic mean from the above data is

(*a*) 200 (*c*) 205
(*b*) 203 (*d*) 208 [**Ans.** (*b*)]

47.

Commodity:	A	B	C
Current year price (₹):	5	3	2
Base year price (₹):	4	2	1
Weight:	60	50	30

The weighted price index number for the above data is

(*a*) 140 (*c*) 150
(*b*) 130 (*d*) 120 [**Ans.** (*c*)]

48.

Commodity:	A	B	C	D	E
Price (2010) (₹):	100	80	160	220	40
Price (2015) (₹):	140	120	180	240	40

The index number for the year 2015 taking 2010 as the base year from the above data by simple average of price relative method is

(*a*) 122.32 (*c*) 115.35
(*b*) 112.23 (*d*) 117.43 [**Ans.** (*a*)]

49.

Commodity:	A	B	C
Price (2002) (₹):	8	10	7
Price (2017) (₹):	12	11	8
Quantity (2017):	5	6	5

Using Paasche's Index number formula the price index from the above data is

(*a*) 115 (*c*) 123
(*b*) 120 (*d*) 125 [**Ans.** (*c*)]

50.

Commodity:	A	B	C
Price (2005) (₹):	8	12	10
Quantity (2005):	5	4	6
Price (2015) (₹):	12	16	12

Using Laspeyres' Index number formula the price index from the above data is

(*a*) 132.43 (*c*) 135.24
(*b*) 130.31 (*d*) 139.13 [**Ans.** (*a*)]

51. Given that $\Sigma p_0 q_0 = 96, \Sigma p_n q_0 = 115, \Sigma p_0 q_n = 126$ and $\Sigma p_n q_n = 155$. The Fisher's Ideal Price Index Number is
    (*a*) 125.32 (*c*) 127.35
    (*b*) 121.24 (*d*) 123.41 [**Ans.** (*b*)]

52.

Commodity:	A	B	C
Base year Price:	4	3	2
Current year Quantity:	50	12	6
Base year Quantity:	60	20	10

Using Laspeyres' formula, the Quantity Index Number from the above data is
(*a*) 75.2 (*c*) 78.6
(*b*) 69.3 (*d*) 77.5 [**Ans.** (*d*)]

53.

Commodity:	A	B	C
Current year Price:	10	9	4
Current year quantity:	40	2	2
Base year quantity:	50	10	5

Using Paasche's formula, the Quantity Index Number from the above data is
(*a*) 62.72 (*c*) 65.23
(*b*) 69.83 (*d*) 68.21 [**Ans.** (*b*)]

54. Given that $\Sigma p_0 q_0 = 77.5, \Sigma p_n q_0 = 75.5, \Sigma p_0 q_n = 96$ and $\Sigma p_n q_n = 106.7$. The Fisher's Quantity Index Number is
    (*a*) 128.31 (*c*) 132.31
    (*b*) 130.23 (*d*) 125.21 [**Ans.** (*c*)]

55. Net monthly income of an employee was ₹3000 in 2010. The consumer price index number in 2015 is 250 with 2010 as base year. The additional dearness allowance to be paid to the employee if he has to be rightly compensated is
    (*a*) ₹4,800 (*c*) ₹4,500
    (*b*) ₹5,000 (*d*) ₹5,200 [**Ans.** (*c*)]

56. With the year 2010 as the base, the C.L.I in 2017 stood at 250. Mr. A was getting a monthly salary of ₹500 in 2010 and ₹750 in 2017. How much should Mr. A have received as extra allowances in 2017 to maintain his standard of living as in 2010?
    (*a*) ₹525 (*c*) ₹450
    (*b*) ₹550 (*d*) ₹500 [**Ans.** (*d*)]

57. The net monthly salary of an employee was ₹3,000 in 2012. If the Consumer Price Index number in 2017 is 250 (2012 is the base year), the expected salary of that employee in 2017 is
    (*a*) ₹7,000 (*c*) ₹8,000
    (*b*) ₹7,500 (*d*) ₹8,500 [**Ans.** (*b*)]

58. Find the general cost of living index of 2017 from the following data:

Group:	Food	Clothing	Fuel	House Rent	Miscellaneous
Group index:	428	240	200	125	170
Weight:	45	15	8	20	12

(*a*) 300 (*c*) 280
(*b*) 290 (*d*) 270 [**Ans.** (*b*)]

59.

Expenses:	Food	Fuel	Clothing	Rent	Miscellaneous
Price (₹) 2017:	1500	250	750	300	400
Price (₹) 2008:	1400	200	500	200	250

The cost of living index during the year 2017 as compared with 2008 is
(*a*) 133.12 (*c*) 130.23
(*b*) 135.25 (*d*) 134.49 [**Ans.** (*d*)]

60. The Consumer Price Index for April 2015 was 125. The food price index was 120 and other items index (excluding food) was 135. What percentage of total weight of the index is given to food?
(*a*) 66.67% (*c*) 64.44%
(*b*) 65.52% (*d*) 63.33% [**Ans.** (*a*)]

CHAPTER

9

Time Series Analysis

SYLLABUS

Causes of Variation in Time Series Data, Components of Time Series, Additive and Multiplicative Models, Determination of Trend by Semi Average, Moving Average and Least Squares (of Linear, Quadratic and Exponential Trend) Methods; Computation of Seasonal Indices by Simple Average, Ratio-to-Moving Average, Ratio-to-Trend and Link Relative Methods; Simple Forecasting Through Time Series Data

THEMATIC FOCUS

9.1 INTRODUCTION

Estimation for the future is an important task of economists and businessmen. An economist has to estimate the likely population of the country in the coming year so that proper planning can be carried out regarding supply of food, creation of jobs, provision for medical facilities, housing, etc. A businessman has to estimate likely sales in the coming year so that he can adjust production by avoiding the possibility of unsold stock or inadequate production to meet the demand. The first step to be followed to estimate for the future is to gather the related information from the past. Quantitative values of the information are usually recorded over equal time intervals: daily, weekly, monthly, quarterly, half-yearly, yearly or any other time measure. When quantitative data are arranged in the order of their occurrence, the resulting statistical series is called a time series.

9.2 TIME SERIES

According to **Croxton and Cowden**, "A time series consists of data arranged chronologically." According to **Kenny and Keeping**, "A set of data depending on the time is called time series." According to **Patterson**, "A time series consists of statistical data which are collected, recorded or observed over successive increments." It is clear from the above definitions that time series is a series of values of variables, the values of which vary according to the passage of time. In such type of variables, the time factor plays an important role in affecting the variables to a marked extent. Therefore, time series is a sequence of data points, typically consisting of successive measurements made over a time interval. Examples of time series are annual birth rate figures for the country, monthly sales of products of a company, weekly wholesale price of wheat, daily industrial production in India, hourly temperature recorded by the meteorological office in a day, etc.

9.3 TIME SERIES ANALYSIS

Time series data fluctuate with time due to effect of various factors, such as increase in population, change of customs, habits and tastes of people, development of new technology, changes in weather conditions, economic depression or recession, etc., which are always changing with time. The joint action of these factors changes in the value of time series data. It is not possible to find out the causes of fluctuations over time through observations only. A careful and indepth analysis of time series data is required. We thus have to observe the evolution of the time series and draw some conclusions about its past behavior that would allow us to infer something about its probable future behavior and apply same statistical methods for this purpose. Time series analysis comprises methods for analyzing time series data to extract meaningful statistics and other characteristics of time series data. It focuses on comparing values of a single time series or multiple dependent time series at different points in time. We use time series analysis for the following purposes:

- Financial Market Analysis
- Budget Analysis
- Inventory Management
- Marketing and Sales Forecasting
- Economic Forecasting
- Census Analysis

9.4 OBJECTIVES OF TIME SERIES ANALYSIS

The objectives of time series analysis are:

(*i*) To identify the factors which cause fluctuations in the values of the time series.

(*ii*) To isolate and measure the effects of various components.

(*iii*) To forecast or predict future values of the time series variable.

9.5 UTILITY OF TIME SERIES ANALYSIS

Analysis of time series has a lot of utilities for various fields of human interest, viz. business, economics, sociology, politics, administration, etc. It also plays a significant role in the fields of physical, and natural sciences. Some of its utilities are given below:

(*i*) **It helps in studying the past behavior:** Time series analysis helps in studying the past behavior of the variable under study. The analysis discloses the type and nature of variation that have taken place in the variable over a period of time.

(*ii*) **It helps in forecasting:** Time series analysis helps in forecasting the future value of a variable after a certain period, which is very essential for a businessman in planning the future operations and in the formulation of policies.

(*iii*) **It helps in the evaluation of current accomplishments:** Evaluation of the actual performances with reference to the predetermined targets is highly necessary to judge the efficiency. Time series analysis helps us in comparing the actual performance with that which was expected on the basis of past behavior of the data and if there are any variations it can be analysed and necessary corrections can be made in subsequent forecasting.

(*iv*) **It helps in making comparative studies:** Comparative study of data relating to two, or more periods, regions, or industries reveals a lot of valuable information which guide a management in taking the proper course of action for the future. Time series analysis provides a scientific basis for making comparisons by studying and isolating the effects of various components and thereby knowing their behaviour. It is also possible to make geo-graphical or regional comparisons amongst data collected on the basis of time.

9.6 COMPONENTS OF TIME SERIES

The factors that are responsible for bringing about changes in a time series are called components of time series. The following are the important components of time series.

(*i*) **Secular trend:** Secular Trend is the main component of a time series which results from long-term effects of socio-economic and political factors. The tendency of the data to increase or decrease over a period of years is known as Secular Trend. It can be linear or non-linear. Most of the series in economics and business show an upward tendency, for example production of different commodities, population, volume of bank deposits, amount of currency in circulation, etc. But all time series do not show an upward tendency. A declining tendency is noticed in case of death rate. Therefore, the tendency of the data to increase or decrease over a period of years is known as Secular Trend also called as long term trend. The graphs of the trends are as follows:

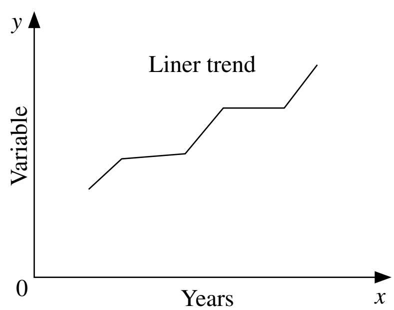


Fig 9.1 (a)

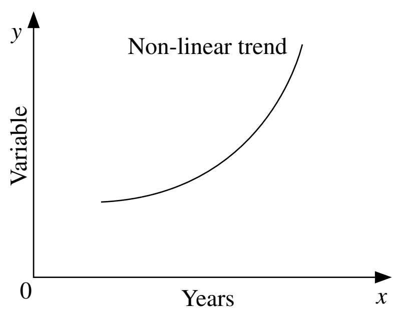


Fig. 9.1 (b)

(*ii*) **Seasonal variations:** These are short term movements occurring in data due to seasonal factors. The two principal factors liable for seasonal changes are the climate or weather and customs. Nearly every type of business activity is influenced by seasonal variations. It may be a greater degree or a lesser degree depends upon the characteristics of season. There is a greater demand for dress materials just prior to Puja festival and relatively low demand after Puja festival, for woollen clothes in winter rather than summer period. According to **Belterson**, "The seasonal variations in a time series are repetitive, recurrent pattern of change which occurs within a year or a short period of time period."

(*iii*) **Cyclical fluctuations:** The third characteristic movement in economic time series is cyclical fluctuation. A typical business cycle consists of a period of prosperity followed by periods of recession, depression, and then recovery with no fixed duration of the cycle. Cyclical variations are influenced by activities in the previous years. The period when the movement of business activities is downward is known as 'a period of recession'. Similarly when in an economy the series of related prices, wages, production, demand etc., undergo rapid increase, it is known as a 'prosperity'. It will be clear from the following diagram:

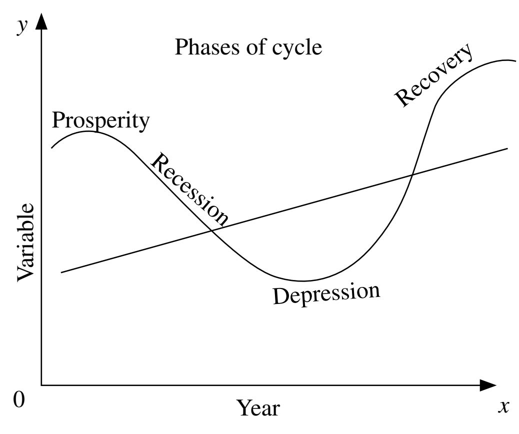


Fig. 9.2 Long term trend with cyclical fluctuations

(*iv*) **Irregular movements:** It is the residual time series after the trend, cycle and the seasonal components have been removed. In many situations, the value of a variable may be completely unpredictable, changing in a random manner. These fluctuations are the result of such unforeseen and unpredictable forces that operate in absolutely erratic and irregular manner. There is no definite pattern and there is no regular period of time of their occurrence. They are normally short-term variations caused by non-recurring factors such as floods, draughts, wars, earthquakes, fires, pestilence etc. Irregular movements are also known as 'episodic' variations and include all variations other than those accounted for by trend, seasonal and cyclical variations.

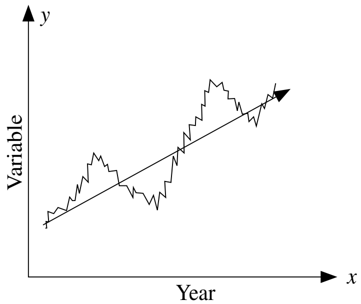


Fig. 9.3 Long-term Trend with Cyclical and Seasonal Variations/ Movements

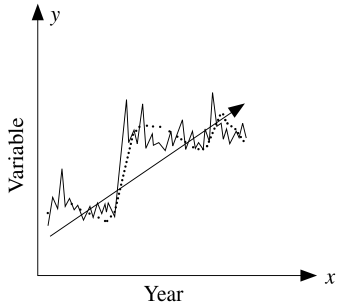


Fig 9.4 Long-term Trend with Cyclical, Seasonal and Random Variations/Movements

9.7 TIME SERIES MODELS

A time series may not be affected by all its four components. Some of these components may affect a few time series, while the other series may be effected by all of them. Hence, in time series analysis, these effects are isolated.

The process of breaking the series into its component parts is called the **decomposition of time series**. In the decomposition of a time series into its components the following two models are generally used:

9.7.1 Additive Model

During the development of additive models there is an implicit assumption that the different components affect the time series additively. Symbolically,

$$Y = T + S + C + I$$

where Y refers to original data, T refers to trend, S refers to seasonal variations, C refers to cyclical fluctuations, and I refers to irregular movements. The additive model assumes that all the four components of time series are independent of one another. For this model, the components are expressed in the same units.

9.7.2 Multiplicative Model

The most commonly used model in the decomposition of time series is multiplicative model. In this model, it is assumed that the four components have multiplicative relationship a symbolically,

$$Y = T \times S \times C \times I$$

In this model, it is assumed that the factors influencing the components are dependent and they can affect one another. This is a much more sensible and practical model for real-life situations since, in most situations, factors are interdependent. For the multiplicative model, the trend has the same units as the '*Y*' values and three other components are considered to be unitless, thus acting as indices. Whenever the model to be used is not specified, then we use the multiplicative.

9.8 MEASUREMENT OF TREND

The following three methods are commonly used for measuring trends:

(*i*) Semi-Average Method
(*ii*) Moving Average Method
(*iii*) Method of Least Squares

9.8.1 Semi-average Method

This method can be used if a straight line trend is to be obtained. Since two points are necessary to obtain a straight line, it is obvious that we may select two representative prints and connect them by a straight line. The given data are divided into two parts, preferably with the same number of years. For example, if we are given data from 2009 to 2016, that is over a period of 8 years, the two equal parts will be 4 years each from 2009 to 2012 and from 2013 to 2016. In case of odd number of years, two equal parts can be made simply by omitting the middle year. For example, if data are given for 11 years from 2007 to 2017, then the middle year 2012 will be omitted. After the data have been divided into two parts, we have to calculate the average (arithmetic mean) of time series values for each part separately. These averages are called **semi-averages**. We thus get two points. Each point is plotted at the mid-point of the class interval covered by the respective part and then the two points are joined by a straight line which gives the required trend line. The line can be extended downwards or upwards to get intermediate values or to predict future values.

9.8.1.A Merits and Limitations of the Method

Merits

(*i*) This method is simple and easy to understand.
(*ii*) This method is an objective method because anyone applying this to a given data would get identical trend value.
(*iii*) This method is flexible because it is permissible to select representative periods to determine the two points.

Limitations

(*i*) This method is used only when the trend is linear or almost linear.
(*ii*) This is only a crude method of measuring trend, since we do not know whether the effects of other components are completely eliminated or not.
(*iii*) Semi-averages are affected by extreme values and hence the trend line may not be reliable.

ILLUSTRATION 1

Measure the trend by the method of semi-averages by using the table given below. Also write the equation of the trend line with origin at 2008–09.

Years	Value (₹)
2008–09	18600
2009–10	22600
2010–11	38100
2011–12	40900
2012–13	41400
2013–14	40100
2014–15	46600
2015–16	60700
2016–17	57200
2017–18	53400

Solution: **Calculation of trend values**

Years	Values (₹)	Semi-totals	Semi-average	Trend values
2008–09	18600			28464 – 3856 = 24608
2009–10	22600			32320 – 3856 = 28464
2010–11	38100	1,61,600	32,320	32320
2011–12	40900			32320 + 3856 = 36176
2012–13	41400			36176 + 3856 = 40032
2013–14	40100			40032 + 3856 = 43888
2014–15	46600			43888 + 3856 = 47744
2015–16	60700	2,58,000	51, 600	51600
2016–17	57200			51600 + 3856 = 55456
2017–18	53400			55456 + 3856 = 59312

Trend for 2010–11 = 32,320
Trend for 2015–16 = 51,600
Increase in trend in 5 years = 19280
Increase in trend in 1 year = 3856

The trend for one year is 3856. This is called the slope of the trend line and is denoted by b. Thus $b = 38356$. The trend for 2011–12 is calculated by adding 3856 to 32320 and similar calculations are done for the subsequent years. The trend for 2009–10 is less than the trend for 2010–11. Thus the trend for 2009–10 is 32320–3856 = 28464. The trend for the year 2008–09 = 24608. This is called the intercept because 2008 – 09 is the origin. The intercept is the value of y when $x = 0$. The intercept is denoted by a. The equation of the trend line is

$$y = a + bx = 24608 + 3856 \ (2008 - 2009 = 0)$$

where y shows the trend values. This equation can be used to calculate the trend values of the time series. It can also be used for forecasting the future values of the variable.

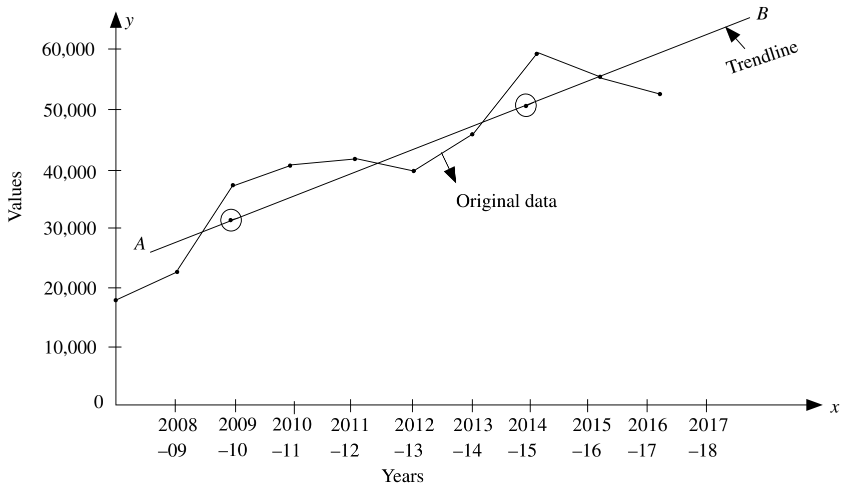


Fig. 9.5

Semi-averages are plotted on the graph paper against the mid-year of the two parts. The required trend line *AB* is the straight line obtained by joining the plotted two points and has been shown in Figure 9.5.

9.8.2 Moving Average Method

Moving average method is a very effective method of smoothing a time series. This consists in averaging out seasonal and other short term fluctuations from the series so that we are left with only the trend in the series. The number of items taken for averaging will be the number required to cover the period over which the fluctuations occur. Suppose that there are n time periods denoted by $t_1, t_2, t_3, \ldots, t_n$ and the corresponding values of the y variable are $y_1, y_2, y_3, \ldots\ldots y_n$. First of all we have to decide the period of the moving averages. For a short time series we use a period of 3 or 4 values, and for a long time series the period may be 7, 10 or more. For a quarterly time series we always calculate averages taking 4-quarters at a time, and in a monthly time series, 12 monthly moving averages are calculated. Suppose the given time series is in years and we have decided to calculate 3-year moving averages. The moving averages denoted by A_1, A_2, A_3, … , A_{n-2} are calculated as under:

Years (t)	**Variable (y)**	**3-year moving totals**	**3-Year moving averages**
t_1	y_1	–	–
t_2	y_2	$y_1 + y_2 + y_3$	$\frac{y_1 + y_2 + y_3}{3} = A_1$
t_3	y_3	$y_2 + y_3 + y_4$	$\frac{y_2 + y_3 + y_4}{3} = A_2$
t_4	y_4	⋮	⋮
⋮	⋮	⋮	⋮
t_{n-2}	y_{n-2}	⋮	⋮
t_{n-1}	y_{n-1}	$y_{n-2} + y_{n-1} + y_n$	$\frac{y_{n-2} + y_{n-1} + y_n}{3} = A_{n-2}$
t_n	y_n	–	–

The average of the first 3 values is A_1 and is placed against the middle year of t_1, t_2 and t_3 i.e. t_2. We leave the first value y_1 and calculate the average for the next three values. The average is A_2 and is placed against the middle year of t_2, t_3 and t_4 i.e. t_3. Similar process is carried out to calculate the remaining moving averages.

When the period of moving averages is even, say 4-year moving averages, this can be calculated as under.

Years (t)	**Variable (y)**	**4-year moving averages (not centered)**	**4-year moving averages (centered)**
t_1	y_1	–	–
t_2	y_2	–	–
		$\frac{y_1 + y_2 + y_3 + y_4}{4} = a_1$	
t_3	y_3		$\frac{a_1 + a_2}{2} = A_1$
		$\frac{y_2 + y_3 + y_4 + y_5}{4} = a_2$	
t_4	y_4		$\frac{a_2 + a_3}{2} = A_2$
		$\frac{y_3 + y_4 + y_5 + y_6}{4} = a_3$	
t_5	y_5		$\frac{a_3 + a_4}{2} = A_3$
		⋮	⋮

The first average is a_1, which is placed against the middle of t_2 and t_3 which is not centred but it should be placed corresponding to the year. Similarly, the second average a_2 is placed against the middle of t_3 and t_4 and so on. So the two averages a_1 and a_2 are further averaged to get an average of $\frac{a_1 + a_2}{2} = A_1$, which refers to the center of t_3 and is placed against t_3. This process is known as centering of moving averages. The process continuous until the end of the series to get 4-years moving averages centred.

9.8.2.A Merits and Limitations of the Method

Merits

(*i*) This method is quite simple to understand and use.

(*ii*) This method is flexible for the reason that if some more figures are added to the data, the entire calculations are not altered. It only provides additional trend values.

(*iii*) If the period of moving average is equal to the period of cyclical fluctuations, the fluctuations automatically disappear.

(*iv*) This method reduces the short term seasonal and irregular movements to a great extent.

(*v*) The pattern i.e. the nature or shape of the trend curve is determined by the data. It does not depend on the desire of the drawer.

(*vi*) This method is also used for measuring seasonal variations, cyclical fluctuations and irregular movements.

Limitations

(*i*) This method is appropriate only when the trend is linear. If the trend is not linear, computed moving average may deviate considerably from the actual trend.

(*ii*) It is not possible to have a trend value for each and every year. Trend values for same periods at the beginning and at the end can not be determined.

(*iii*) As moving average is not represented by a mathematical function, the leading objectives of trend analysis, like forecasting, remains unfulfilled.

(*iv*) There is no hard and fast rule for the selection of a period of moving average and one has to use his own judgement.

(*v*) This method may generate cycles or other movements which were not present in the original data.

ILLUSTRATION 2

Using moving average method, calculate 3-yearly moving averages from the following data:

Year:	2005	2006	2007	2008	2009	2010	2011	2012	2013
Percentage of insured people:	11.3	13.0	14.5	16.6	18.7	19.3	21.4	24.1	27.2

[***C.U. 2014 (G)***]

Solution: **Calculation of 3-yearly moving averages**

Year	Percentage of insured people	3-Yearly moving total	3-yearly moving average (Trend values)
2005	11.3	–	–
2006	13.0	38.8	12.93
2007	14.5	44.1	14.70
2008	16.6	49.8	16.60
2009	18.7	54.6	18.20
2010	19.3	59.4	19.80
2011	21.4	64.8	21.60
2012	24.1	72.7	24.23
2013	27.2	–	–

ILLUSTRATION 3

Find the 4-yearly moving averages by moving average method from the following table:

Year:	2000	’01	’02	’03	’04	’05	’06	’07	’08	’09	’10
Production (kg):	506	620	1036	673	588	696	1116	738	663	773	1189

[***C.U. 2013(G), 2014(H)***]

First method

Solution: **Calculation of 4-yearly moving averages**

Year	Production (kg)	4-yearly moving total	4-yearly moving average (not centered)	4-yearly moving average (centered)
(a)	(b)	(c)	(d) = c ÷ 4	(e)
2000	506	–	–	–
’01	620	–	–	–
		2835	708.75	
’02	1036			719
		2917	729.25	
’03	673			738.75
		2993	748.25	
’04	588			758.25
		3073	768.25	
’05	696			776.375
		3138	784.50	
’06	1116			739.875
		3213	803.25	
’07	738			812.875
		3290	822.50	
’08	663			831.625
		3363	840.75	
’09	773	–	–	–
’10	1189	–	–	–

Second method

Year	Production (kg)	4-yearly moving total (not centered)	2-yearly moving total (centered)	4-yearly moving average (centered)
(a)	(b)	(c)	(d)	(e) = (d) ÷ 8
2000	506	–	–	–
’01	620	–	–	–
		2835		
’02	1036		5752	719
		2917		
’03	673		5910	738.75
		2993		
’04	588		6066	758.25
		3037		
’05	696		6211	776.375
		3138		
’06	1116		6351	793.875
		3213		
’07	738		6503	812.875
		3290		
’08	663		6653	831.625
		3363		
’09	773	–	–	–
’10	1189	–	–	–

ILLUSTRATION 4

For the following table find the trend by the method of moving average, assuming that a 5-yearly cycle is present with weights 1, 2, 2, 2, 1 respectively.

Year:	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013
Sales (In lacs ₹):	2	6	1	5	3	7	2	6	4	8	3

[*C.U. B.Com. 2014(H)*]

Solution: Calculation of 5-yearly weighted moving averages

Year	Sales (in lacs ₹)	5-Yearly weighted moving total	5-Yearly weighted moving average
(a)	(b)	(c)	(d) = (c) ÷ 8
2003	2	–	–
2004	6	–	–
2005	1	2 × 1 + 6 × 2 + 1 × 2 + 5 × 2 + 3 × 1 = 29	3.625
2006	5	6 × 1 + 1 × 2 + 5 × 2 + 3 × 2 + 7 × 1 = 31	3.875
2007	3	1 × 1 + 5 × 2 + 3 × 2 + 7 × 2 + 2 × 1 = 33	4.125
2008	7	5 × 1 + 3 × 2 + 7 × 2 + 2 × 2 + 6 × 1 = 35	4.375
2009	2	3 × 1 + 7 × 2 + 2 × 2 + 6 × 2 + 4 × 1 = 37	4.625

Year	Sales (in lacs ₹)	5-Yearly weighted moving total	5-Yearly weighted moving average
(a)	(b)	(c)	(d) = (c) ÷ 8
2010	6	$7 \times 1 + 2 \times 2 + 6 \times 2 + 4 \times 2 + 8 \times 1 = 39$	4.875
2011	4	$2 \times 1 + 6 \times 2 + 4 \times 2 + 8 \times 2 + 3 \times 1 = 41$	5.125
2012	8	–	–
2013	3	–	–

Here, total weight = 1 + 2 + 2 + 2 + 1 = 8

9.8.3 Method of Least Squares

When different data are plotted on the graph paper, then all the points do not lie on a single curve. A large number of curves can be drawn between the points. It is necessary to decide which line best describes the data. The principle of least squares helps in determining the line that best describes the data. This principle consists in finding the best fitting curve to the time series data as that curve, from all possible curves, for which: (*i*) the sum of the vertical deviations of the actual (observed) values from the fitting curve is zero, and, (*ii*) the sum of the squared vertical deviations is minimum, i.e. no other curve would have a smaller sum of squared deviations. The method which is used to find the best fitting curve to the data is called **least square method**. It is the best and most objective method of determining trend in time series. This method may be used either to fit linear trend or non-linear trend.

9.8.3.A Linear Trend

When the points on the graph paper follow a straight line pattern, then the trend is called **linear trend** and is represented by the equation:

$$y = a + bt$$

where a and b are two constants determined by the method of least squares from the normal equations:

$$\Sigma y = na + b. \Sigma t \qquad \ldots(i)$$

$$\Sigma yt = a\Sigma t + b\Sigma t^2 \qquad \ldots(ii)$$

when n is the number of years, normal equations are obtained by multiplying $y = a + bt$, by the coefficients of a and b, i.e. by 1 and t throughout and then summing up.

9.8.3.B Non-linear Trend

Two types of non-linear trend are generally considered. They are (*a*) Parabolic Trend and (*b*) Exponential Trend.

(*i*) **Parabolic Trend:** The parabolic trend equation is taken as, $y = a + bt + ct^2$ where a, b and c are three constants determined from the normal equations:

$$\Sigma y = na + b\Sigma t + c\Sigma t^2 \quad \ldots(iii)$$

$$\Sigma ty = a\Sigma t + b\Sigma t^2 + c\Sigma t^3 \quad \ldots(iv)$$

$$\Sigma t^2 y = a\Sigma t^2 + b\Sigma t^3 + c\Sigma t^4 \quad \ldots(v)$$

(*ii*) **Exponential Trend:** The exponential trend is taken as,

$$y = a.b^t$$

taking logarithm on both sides, we have

$$\log y = \log a + t.\log b$$

We now put, $\log y = y$, $\log a = A$ and $\log b = B$

Thus we get,

$$y = \text{A} + \text{B}.t$$

which is in linear form and therefore, A and B can be obtained from the normal equations:

$$\Sigma y = n\text{A} + \text{B}.\Sigma t \text{ and } \Sigma ty = \text{A}\Sigma t + \text{B}\Sigma t^2$$

9.8.3.C Choice of Origin and Unit of 't'

The values are collected in regular interval of time in time series. In order to make the calculation simpler to a great extent the origin and unit of t should be choiced properly.

When the number of years is odd: When the number of years is odd then the middle of the time span is to be taken as origin and the interval (i.e. the gap between two consecutive times of collection of data) is to be taken as scale. For example, if five values of y are supplied corresponding to the years 2005, 2008, 2011, 2014, 2017, then we select the mid-year 2011 as origin and

$$t = \frac{\text{year} - 2011}{3}$$

In this case Σt and Σt^3 become zero and as a result the calculations become very simple.

When the number of years is even: When the number of years is even then the middle of time span is taken as origin and half of the interval is taken as scale.

For example, if six values of y are supplied corresponding to the years 2002, 2005, 2008, 2011, 2014 and 2017, then we select the middle of the years 2008 and 2011 as origin and

$$t = \frac{\text{year} - 2009.5}{1.5}$$

In this case also Σt and Σt^3 become zero.

9.8.3.D Merits and Limitations of the Method

Merits

(*i*) This method is completely free from personal bias of the analyst as it is a mathematical method of measuring trend and hence very objective.

(*ii*) Under this method the trend values for the entire time series can be obtained.

(*iii*) This method gives the line of best fit because from this line sum of positive and negative deviations is zero and the total of the squares of these deviations is minimum.

(*iv*) The mathematical curves fitted to the data is most suitable for forecasting purposes.

(*v*) This method provides us with a rate of growth per period, i.e. with this rate of growth, we can easily determine the value for any past or previous year by the process of successive addition, or deduction from the trend values of the origin of t.

Limitations

(*i*) The calculations for this method are more difficult than in the other methods.

(*ii*) This method is very much rigid in the sense that an addition of some more values or some corresponding additional years changes the entire calculations of the trend equation.

(*iii*) Under this method future predictions are based only on trend values and ignore completely the cyclical, seasonal and irregular components of the series for the purpose.

(*iv*) This method is unsuitable for a series in which the differences between the successive observations are not found to be constant or nearly so. This method is also inappropriate for both very short and very long series.

(*v*) Under this method great care is needed for the determination of the type of trend curve to be fitted, i.e. linear, parabolic, exponential, or any other more complicated curve. An erratic selection of the type of curve may lead to fallacious conclusions.

ILLUSTRATION 5

Fit a linear trend equation to the following data:

Year	2008	2009	2010	2011	2012	2013	2014
Production ('000 tons)	17	23	30	39	43	51	58

Hence, estimate the production for the year 2015. [***C.U. 2015 (H)***]

Solution: Let $y = a + bt$ be the equation of straight line. Normal equations are

$$\Sigma y = na + b\Sigma t \quad \ldots(i)$$

and
$$\Sigma yt = a\Sigma t + b\Sigma t^2 \quad \ldots(ii)$$

Here, n = number of years = 7, which is odd, so we take the middle year 2011 as origin and t unit = 1 year a and b are constants, whose values are to be calculated by using equation (i) and (ii).

Fitting of Straight Line Trend

Year	Production ('000 Tons) (y)	t = year – 2011	t^2	yt
2008	17	–3	9	–51
2009	23	–2	4	–46
2010	30	–1	1	–30
2011	39	0	0	0
2012	43	1	1	43
2013	51	2	4	102
2014	58	3	9	174
Total	261	0	28	192

From (i), $261 = 7a + b.0$

or $7a = 261 \text{ or } a = \dfrac{261}{7} = 37.29$

From (ii), $192 = a.0 + b.\ 28$

or $28b = 192 \text{ or } b = \dfrac{192}{28} = 6.86$

Therefore, the trend equation is

$$y = 37.29 + 6.86t$$

For 2015, $t = 2015 - 2011 = 4$

Thus, the estimated production in 2015 = $37.29 + 6.86 \times 4$

$= 37.29 + 27.44 = 64.73$ ('000 tons).

ILLUSTRATION 6

Fit a straight line trend by the method of least squares to the following data:

Year	1980	1981	1982	1983	1984	1985
Sales (in tons)	210	225	275	220	240	235

Find also the trend values and estimate the sales in 1987.

[***C.U. 1989, 1992, 2014 (H)***]

Solution: Here the number of years (n) is 6, i.e. even. We take the middle of the years 1982 and 1983 as origin and t unit = $\frac{1}{2}$ year . Then the values of t corresponding to the years 1982 and 1983 will be –1 and 1 and other values of t are calculated accordingly

Let $y = a + bt$ be the equation of straight line.

Here a and b are constant, whose values are to be calculated by using the following two normal equations:

$$\Sigma y = na + b.\Sigma t \quad \ldots(i)$$

and

$$\Sigma yt = a\Sigma t + b\Sigma t^2 \quad \ldots(ii)$$

Fitting of straight line trend

Year	Sales (in tones) (y)	$t = \frac{\text{year} - 1982.5}{.5}$	t^2	yt	Trend values
1980	210	–5	25	–1050	225.97
1981	225	–3	9	–675	229.15
1982	275	–1	1	–275	232.53
1983	220	+1	1	220	235.81
1984	240	+3	9	720	239.19
1985	235	+5	25	1175	242.37
Total	1405	0	70	115	–

From (i), $1405 = 6a + b \times 0$

or $6a = 1405$ or $a = \frac{1405}{6} = 234.17$

From (ii), $115 = a \times 0 + b \times 70$

or $70b = 115$ or $b = \frac{115}{70} = 1.64$

Therefore, the trend equation is

$$y = 234.17 + 1.64t \quad \ldots(iii)$$

The trend values are calculated by substituting the values of t in equation (iii) and are shown in the table.

For the year 1987, $t = \frac{1987 - 1982.5}{.5} = \frac{4.5}{.5} = 9$

Thus, the estimate for sales in 1987 is

$$y = 234.17 + 1.64 \times 9 = 234.17 + 14.76$$

$$= 248.93 \text{ (tonnes)}.$$

ILLUSTRATION 7

Production of certain commodity is given below:

Year	2009	2010	2011	2012	2013
Production ('000 tonnes)	7	9	10	7	5

Fit a parabolic curve of second degree to the production.
(Assume the year of origin : 2011)

Solution: Let $y = a + bt + ct^2$ be the equation of the parabolic curve.

t unit = 1 year, origin = The middle year 2011.

n = number of years = 5

The values of constants a, b, c can be obtained with the following simultaneous equations:

$$\Sigma y = na + b\Sigma t + c\Sigma t^2 \quad \ldots(i)$$
$$\Sigma ty = a\Sigma t + b\Sigma t^2 + c\Sigma t^3 \quad \ldots(ii)$$
$$\Sigma t^2y = a\Sigma t^2 + b\Sigma t^3 + c\Sigma t^4 \quad \ldots(iii)$$

Fitting of Parabolic Trend

Year	**Production ('000 tonnes) (y)**	**t = year – 2011**	**t^2**	**t^3**	**t^4**	**ty**	**t^2y**
2009	7	–2	4	–8	16	–14	28
2010	9	–1	1	–1	1	–9	9
2011	10	0	0	0	0	0	0
2012	7	1	1	1	1	7	7
2013	5	2	4	8	16	10	20
Total	38	0	10	0	34	–6	64

By substituting the values in the simultaneous equations we get

From (i), $38 = 5a + b \times 0 + 10c$ or $38 = 5a + 10c$ ...(iv)

From (ii), $-6 = a \times 0 + 10b + c \times 0$ or $-6 = 10b$ or $b = -0.6$

From (iii), $64 = 10a + b \times 0 + 34c$ or $64 = 10a + 34c$ ...(v)

The values of 'a' and 'c' can be obtained from equation (iv) and (v).

$$5a + 10c = 38 \quad \ldots(iv)$$
$$10a + 34c = 64 \quad \ldots(v)$$

Multiplying equation (iv) by 2 and subtracting from equation (v) we get

$$\begin{aligned} 10a + 34c &= 64 \\ \underset{-}{10a} + \underset{-}{20c} &= \underset{-}{76} \\ \hline 14c &= -12 \end{aligned}$$

or $$c = \frac{-12}{14} = -0.857$$

Putting the value of c in equation (iv) we get

$$5a + 10 \times -0.857 = 38$$

or $$5a - 8.57 = 38 \text{ or } 5a = 38 + 8.57$$

or $$5a = 46.57 \text{ or } a = \frac{46.57}{5} = 9.31$$

Therefore, 2nd degree parabolic equation is

$$y = 9.31 + (-0.6)t + (-0.857)t^2$$

or $$y = 9.31 - 0.6t - 0.857t^2$$

ILLUSTRATION 8

Fit an exponential curve to the following data by the method of least squares:

Years	2010	2011	2012	2013	2014	2015	2016
Sales (₹ in lakhs)	32	47	65	92	132	190	275

Solution: Since the number of years (n) = 5, which is odd, the middle year 2013 is taken as origin (i.e. $t = 0$) and t unit = 1 year

Let $y = a.b^t$ be the equation of the exponential curve where y represents sales.

Taking logarithm on both sides we get

$$\log y = \log a + t.\log b$$

Putting $\log y = Y$, $\log a = \text{A}$ and $\log b = \text{B}$ we get

$$y = \text{A} + \text{B}t$$

Fitting of Exponential Trend

Years	**Sales (₹ in lakhs) (*y*)**	***t***	***y* = log y**	***t*²**	***ty***
2010	32	–3	1.5051	9	–4.5153
2011	47	–2	1.6721	4	–3.3442
2012	65	–1	1.8129	1	–1.8129
2013	92	0	1.9638	0	0
2014	132	1	2.1206	1	2.1206
2015	190	2	2.2788	4	4.5576
2016	275	3	2.4393	9	7.3179
Total		0	13.7926	28	4.3237

The normal equations are

$$\Sigma y = nA + B\Sigma t$$
$$\Sigma ty = A\Sigma t + B\Sigma t^2$$

Substituting the values from the table in normal equations, we get,

$$13.7926 = 7A + B.0$$

or $7A = 13.7926 \text{ or } A = \dfrac{13.7926}{7}$

or $A = 1.97$

or $\log a = 1.97 \text{ or } a = \text{antilog } (1.97)$

or $a = 93.33$

and, $4.3237 = A \times 0 + B.\ 28$

or $28B = 4.3237 \text{ or } B = \dfrac{4.3237}{28} = 0.1544$

or $\log b = 0.1544 \text{ or } b = \text{antilog } (0.1544)$

or $b = 1.427$

Thus, the equation of the exponential curve is

$y = 93.33 \times (1.427)^t$ with origin at the year 2013 and t unit = 1 year.

9.8.4 Measurement of Seasonal Variations

Seasonal variations are those short-term fluctuations which occur regularly and periodically within a period of less than one year such as days, weeks, months or quarters. Climate and weather conditions, social customes and traditions and habits of people at different parts of the year are the main factors responsible for seasonal fluctuations. The purpose of studying seasonal variations is to determine the effect of seasonal swings on the value of a given phenomenon, to forecast short-term fluctuations and to eliminate them for the study of cyclical and irregular variation. Organisations affected by seasonal variations need to identify and measure this seasonality to help with planning for temporary increases or decreases in labour requirements, inventory, training, periodic maintenance and so forth. Apart from these considerations, the organisations need to know if the variations they have experienced have been more or less expected given the usual seasonal variations. It is important that these variations be measured accurately for three seasons. First, the investigator wants to eliminate seasonal variations from the data he is studying. Second, a precise knowledge of the seasonal pattern aid in planning future operations. Lastly, complete knowledge of seasonal variations is of use to those who are trying to remove the cause of seasonal or are attempting to mitigate the problem by diversification, off setting opposing seasonal patterns or some other means.

Seasonal variation is measured in terms of an index called seasonal index. It is an average that can be used to compare an actual observation relative to what it would be if there were no seasonal variations. The measures that are generally employed to compute such seasonal index are:

1. Method of simple averages
2. Moving average method
3. Ratio to trend method
4. Link relative method

9.8.4.A Method of Simple Averages

This is the simplest method of measuring seasonal index. This method is used only when the given time series data does not contain trend or cyclical fluctuations to any appreciable extent. The following steps are necessary for calculating the seasonal index.

(*i*) Arrange the given data by years, months or quarters.

(*ii*) Calculate totals for each period (yearly, monthly or quarterly).

(*iii*) Calculate average for each period (For quarterly data $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ and $\bar{x}_4$).

(*iv*) Calculateoverallorgrandaverage $\left(\dfrac{\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \bar{x}_4}{4} = \bar{X} \text{ for quarterly data}\right)$

(*v*) Calculate seasonal index by using the following formula

(*a*) For the additive model

$$\text{Seasonal Index (variation)} = \text{Average} - \text{grand average}$$

(*b*) For the multiplicative method

$$\text{Seasonal Index (Variation)} = \frac{\text{Average}}{\text{Grand average}} \times 100$$

ILLUSTRATION 9

Calculate the seasonal index from the following data using simple average method:

Year/Quarter	Q_1	Q_2	Q_3	Q_4
2011	72	68	80	70
2012	76	70	82	74
2013	74	66	84	80
2014	76	74	84	78
2015	78	74	86	82

[***C.U. 2016 (H)***]

Solution: **Calculation of Seasonal Index**

Year \ Quarter	Q_1	Q_2	Q_3	Q_4	Total
2011	72	68	80	70	–
2012	76	70	82	74	–
2013	74	66	84	80	–
2014	76	74	84	78	–
2015	78	74	86	82	–
Total	376	352	416	384	1528
Average	75.2	70.4	83.2	76.8	305.6
Grand average*	–	–	–	–	76.4
Using additive model					
Seasonal Index (Average – Grand average)	–1.2	–6.0	6.8	0.4	0
Using Multiplicative model					
Seasonal Index $\left(\frac{\text{Average}}{\text{Grand average}} \times 100\right)$	98.43	92.15	108.9	100.52	400

$$\text{*Grand average} = \frac{305.6}{4} = 76.4$$

9.8.4.B Moving Average Method

This method is one of the simplest and widely used devices for measuring seasonal variation which takes the trend into consideration. The steps to compute seasonal variation are as follows:

(*i*) Arrange the given data by years, months or quarters.
(*ii*) Compute the trend values by the method of moving averages.
(*iii*) Calculate Deviation from trend (in ease of additive model) or ratio to moving average (in case of multiplicative model).
(*iv*) Arrange the deviations from trend or ratio to moving average in a separate table and use the simple average method to obtain seasonal variations.

ILLUSTRATION 10

Estimate the seasonal indices by the method of moving averages from the table given below by **additive model**.

Total production of paper ('000 tons)

Year / Quarter	I	II	III	IV
2001	35	39	36	40
2002	42	34	28	32
2003	36	38	34	41

[*C.U. B.Com. 2007*]

Solution: **Calculation of moving averages and deviations from trend**

Year / Quarter	Production ('000 tons)	4-Quarter moving total (not centered)	2-item moving total (centered)	4-quarter moving average	Deviation from trend
(a)	(b)	(c)	(d)	(e) = (d) ÷ 8	(f) = (b) – (e)
2001 I	35	–	–	–	–
II	39	–	–	–	–
		150			
III	36		307	38.375	–2.375
		157			
IV	40		309	38.625	1.375
		152			
2002 I	42	144	296	37.000	5.000
II	34	136	280	35.000	–1.000
III	28	130	266	33.250	–5.250
IV	32	134	264	33.000	–1.000
2003 I	36	140	274	34.250	1.750
II	38	149	289	36.125	1.875
III	34	–	–	–	–
IV	41	–	–	–	–

Calculation of Seasonal Indices

Year \ Quarter	Deviations from trend I	II	III	IV	Total
2001	–	–	–2.375	1.375	–
2002	5.000	–1.000	–5.250	–1.000	–
2003	1.750	1.875	–	–	–
Total (Quarter wise)	6.750	0.875	–7.625	0.375	0.375
Average (Total ÷ 2)	3.375	0.4375	–3.8125	0.1875	0.1875
Grand Average *	–	–	–	–	0.046875
Seasonal Index (Average – Grand average)	3.328125	0.390625	–3.859375	0.140625	0

$$\text{*Grand average} = \frac{0.1875}{4} = 0.046875$$

ILLUSTRATION 11

Estimate the seasonal indices by the method of moving averages from the following information:

Year / Quarter	Sales (in crore ₹)			
	I	II	III	IV
2006	28	37	41	34
2007	32	40	43	39
2008	39	45	48	42

[*C.U. B.Com. 2009*]

Solution: **Calculation of moving averages and ratios to moving averages**

Year / Quarter	Sales (in crore ₹)	4-Quarter moving total (not centered)	2-item moving total (centered)	4-quarter moving average	Ratio to moving average (in %)
(a)	(b)	(c)	(d)	(e) = (d) ÷ 8	(f) = (b) / (e) × 100
2006 I	28	–	–	–	–
II	37	–	–	–	–
III	41	140	284	35.500	115.49
IV	34	144	291	36.375	93.47
2007 I	32	147	296	37.000	86.49
II	40	149	303	37.875	105.61
III	43	154	315	39.375	109.21
IV	39	161	327	40.875	95.41
2008 I	39	166	337	42.125	92.58
II	45	171	345	43.125	104.35
III	48	174	–	–	–
IV	42	–	–	–	–
		–			

Calculation of seasonal indices (Using multiplicative model)

Quarter / Year	Ratio to moving average (%)				
	I	II	III	IV	Total
2006	–	–	115.49	93.47	–
2007	86.49	105.61	109.21	95.41	–
2008	92.58	104.35	–	–	–
Total (Quarter-wise)	179.07	209.96	224.70	188.88	–
Average (Total ÷ 2)	89.535	104.98	112.35	94.44	401.305
Grand Average *	–	–	–	–	100.32625

Year \ Quarter	Ratio to moving average (%)				
	I	II	III	IV	Total
Seasonal Index $\left(\frac{\text{Average}}{\text{Grand average}} \times 100\right)$	89.25	104.64	111.98	94.13	400.00

$$*\text{Grand average} = \frac{401.305}{4} = 100.32625$$

9.8.4.C Ratio to Trend Method

This method is similar to ratio-to-moving average method. The only difference is the way of obtaining the trend values. In the ratio-to-moving average method, the trend values are obtained by the method of moving averages, whereas in the ratio-to-trend method, the corresponding trend is obtained by the method of least squares. These trend values are eliminated from the given series by expressing the original data as percentage of the corresponding trend values. In this method, the multiplicative model is always considered. As in the moving average method, these percentages are arranged by quarters and the average trend-ratio for each quarter is found out. Then average of these averages for each quarter, i.e. grand average, is calculated. The percentage of each quarter average to the grand average will be the seasonal indices. Same procedure is to be followed for monthly or weekly data.

9.8.4.D Link Relative Method

This method was developed by Prof. Karl Pearson, hence this method is also known as the Pearson method of seasonal variation. When this method is adopted the following steps are taken to calculate the seasonal variation indices:

(*i*) Calculate the link relatives of all the seasonal data by the following formula,

$$\text{Link Relative (LR) of the current season} = \frac{\text{Data of current season}}{\text{Data of the immediately preceding season}} \times 100$$

(The link relative for the first and foremost season cannot be obtained)

(*ii*) Arrange all the link relatives thus obtained season wise (year, month, week or quarter) and find the average of the link relatives for each season either by mean or by median.

(*iii*) Convert each of the averages of the link relatives into chain relatives (CR) by the following formula,

$$\text{Chain relative of the current season} = \frac{\begin{array}{c}\text{Average of link}\\ \text{relative of the}\\ \text{current season}\end{array} \times \begin{array}{c}\text{Chain relative}\\ \text{of the immediately}\\ \text{preceding season}\end{array}}{100}$$

(The initial chain relative of the foremost season is taken as 100)

(*iv*) Calculate the revised chain relative of the foremost season on the basis of the chain relative or the last season by the following formula.

$$\text{Revised chain relative of the foremost season} = \frac{\begin{array}{c}\text{Average line}\\ \text{relative of the}\\ \text{foremost season}\end{array} \times \begin{array}{c}\text{Chair relative}\\ \text{of the last}\\ \text{season}\end{array}}{100}$$

(*v*) Determine the constants correcting the chain relatives by

$$d = \frac{\begin{array}{c}\text{Revised chain relative}\\ \text{of the foremost season}\end{array} - \begin{array}{c}\text{Initial chain relative of}\\ \text{the foremost season}\end{array}}{\text{Number of seasons in a year}}$$

Where d = difference between the two chain relatives of the foremost season that is used as the correcting factor.

(*vi*) Subtract d, $2d$, $3d$ and $4d$ (and so on) respectively from the chain relatives of the 2nd , 3rd, 4th, 5th and so on seasons to find the preliminary indices of the seasonal variations.

(*vii*) Express the preliminary indices as the percentage of their average to get the required seasonal indices. The total of these seasonal indices will be 400.

9.9 BUSINESS FORECASTING

The future is always uncertain. So, there is need of an organized system of forecasting in a business. Business forecasting is a method or a technique for estimating future aspects of a business or the operation. It is a method for translating past data or experience into estimates of the future. It helps the management to cope with the uncertainty of the future and to take short-term and long-term decisions.

Thus, scientific business forecasting involves:

(*i*) Analysis of the past economic conditions and

(*ii*) Analysis of the present economic conditions so as to predict the future course of events accurately.

In the words of **Allen**, "Forecasting is a systematic attempt to probe the future by inference from known facts. The purpose is to provide management with information on which it can base planning decisions."

C.E. Sulton observes, "Business forecasting is the calculation of probable events, to provide against the future. It, therefore, involves a 'look ahead' in business and an idea of predetermination of events and their financial implications as in the case of budgeting."

According to **John G. Glover**, "Business forecasting is the research procedure to discover those economic, social and financial influences governing business activity, so as to predict or estimate current and future trends or forces which may have a bearing on company policies or future financial, production and marketing operations."

The essence of all the above definitions is that business forecasting is a technique to analysis the economic, social and financial forces affecting the business with an object of predicting future events on the basis of past and present information.

9.9.1 Methods of Business Forecasting

There are a number of different methods that a business can use for effective financial planning. All the methods fall into one of two overarching approaches: qualitative and quantitative.

Qualitative Method: Qualitative method has generally been applied in predicting the short-term success of companies, products and services, but meets limitations due to its reliance on opinion over measurable data. This method includes:

(*i*) **Executive opinions:** In this method, the expert opinions of key personnel of various departments, such as production, sales, purchasing and operations, are gathered to arrive at future predictions. Based on their expectations, the management team makes revisions in the resulting forecast.

(*ii*) **Delphi method:** Under this method a series of questionnaires are prepared and answered by a group of experts, who are kept separate from each other. Once the results of the first questionnaire are compiled, a second questionnaire is prepared based on the results of the first. This second document is again presented to the experts, who are then asked to revaluate their responses to the first questionnaire. This process continues until the researchers have a narrow shortlist of opinions.

(*iii*) **Scenario writing:** In this method, on the basis of diverse starting criteria the forecaster generates different outcomes. The management team decides on the most likely outcome from the numerous scenarios presented.

(*iv*) **Consumer surveys:** Business enterprises often conduct market surveys of consumers. The data are collected through personal interviews, telephonic conversations or survey questionnaires, and extensive statistical analysis is conducted to generate forecasts.

(*v*) **Sales Force Polling:** Some business enterprises believe that salespersons are closely attached with the consumers, so they could provide significant insight regarding customers' behaviour. Hence, estimates are derived based on the average of sales force polling.

Quantitative Method: Quantitative methods are concerned solely with data and avoid the fickleness of the people underlying the numbers. They also try to predict where variables like sales, gross domestic product, housing prices and so on, will be in the long term, measured in months or years. This method includes.

(*i*) **Time series methods:** Time series methods make forecasts based solely on historical patterns in the data. This method uses time as an independent variable to produce demand. In a time series, measurements are taken at successive points or over successive periods. The measurements may be taken every hour, day, week, month, or year, or at any other regular (or irregular) interval. Time series methods are one of the simplest methods to deploy and can be quite accurate, particularly over the short term. Some techniques that fall within this method are simple averaging, exponential smoothing and trend projection. Smoothing methods are appropriate when a time series displays no significant effects of trend, cyclical, or seasonal components. In such a case, the goal is to smooth out the irregular component of the time series by using an averaging process. The moving average method is the most widely used smoothing technique. Trend projection method uses the underlying long-term trend of time series of data to forecast its future value.

(*ii*) **Causal forecasting methods:** Causal methods use the cause and effect relationship between the variable whose future values are being forecasted and other related variables or factors (such as changes in consumers' disposable incomes, the interest rate, the level of consumer confidence, and unemployment levels). The widely known causal method is called regression analysis, a statistical technique used to develop mathematical model showing how a set of variables is related. The mathematical relationship can be used to generate forecasts.

(*iii*) **Performa Financial Statements:** Performa statements use sales figures and costs from the previous two or three years after excluding certain one-time costs. This method is mainly used in mergers and acquisitions, as well as in cases where a new company is forming and statements are needed to request capital from investors.

9.10 STEPS OF BUSINESS FORECASTING

The following steps are taken into consideration in the process of forecasting. These are also known as elements of forecasting:

(*i*) **Developing the basis:** The first step involved in forecasting is developing the basis of systematic investigation of economic situation, position of industry and products. The future estimates of sales and general business operations have to be based on the results of such investigation.

(*ii*) **Estimating future business operations:** On the basis of information or data collected through investigation, future business operations are estimated. The quantitative estimates for future scale of operations are made on the basis of certain assumptions.

(*iii*) **Regulating forecast:** The next step is regulating forecasts. To determine any deviation the forecasts are compared with actual results. The reasons for this variations are ascertained so that corrective action is taken in future.

(*iv*) **Reviewing the Forecasting Process:** If any deviation is found between forecast and actual performance then improvements can be made in the process of forecasting. This type of review of forecasting process will improve forecasts in future.

9.11 LIMITATIONS OF BUSINESS FORECASTING

Business forecasting is very useful for businesses, as it allows them to plan production, financing and so on. Inspite of many advantages, some people consider business forecasting "as an unnecessary mental gymnastics and reject it as a sheer waste of time, money and energy." The reasons behind it are as follows:

(*i*) Forecasting is made on the basis of historical or old data and there is no guarantee that the conditions in the past will persist into the future.

(*ii*) Forecasting is influenced by the pessimistic or optimistic attitude of the forecaster. Guesswork in forecasts cannot be eliminated. Despite all precautious, an element of error is bound to creep in the forecasts.

(*iii*) Forecasts involve constant monitoring and revision with the changed circumstances.

(*iv*) Forecasting is a conceptual knot. In a worst case scenario, management becomes a slave to historical data and trends rather than worrying about what the business is doing now.

(*v*) Forecasts can't integrate their own impact. By having forecasts, accurate or inaccurate, the actions of businesses are influenced by a factor that can't be included as a variable.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1

Using 3-year moving average method determine the trend and short-term fluctuations for the following data:

Year :	2011	2012	2013	2014	2015	2016	2017
Value :	21	34	45	28	40	57	73

Solution: **Calculation of 3-year moving averages**

Year	Value	3-year moving total	3-year moving average	Short term fluctuations
(a)	(b)	(c)	(d) = (c) ÷ 3	(e) = (b) – (d)
2011	21	–	–	–
2012	34	100	33.33	0.67
2013	45	107	35.67	9.33
2014	28	113	37.67	-9.67
2015	40	125	41.67	-1.67
2016	57	170	56.67	0.33
2017	73	–	–	–

EXAMPLE 2

Construct 5-yearly moving averages of the number of students studying in a college shown below:

Year	No. of students
1951	332
1952	317
1953	357
1954	392
1955	402
1956	405
1957	410
1958	427
1959	405
1960	431

[*C.U. B.Com. 2000*]

Solution: **Calculation of 5-yearly moving average**

Year	No. of students	5-yearly moving total	5-yearly moving average
(a)	(b)	(c)	(d) = (c) ÷ 5
1951	332	–	–
1952	317	–	–
1953	357	1800	360
1954	392	1873	374.6
1955	402	1966	393.2
1956	405	2036	407.2

Year	No. of students	5-yearly moving total	5-yearly moving average
(a)	(b)	(c)	(d) = (c) ÷ 5
1957	410	2049	409.8
1958	427	2078	415.6
1959	405	–	–
1960	431	–	–

EXAMPLE 3

For the following series of observations verify that the 4-year centered moving average is equivalent to a 5-year moving average with weights 1, 2, 2, 2, 1 respectively:

Year:	'02	'03	'04	'05	'06	'07	'08	'09	'10	'11	'12
Sales (₹ '000):	2	6	4	5	3	7	2	6	4	8	3

[*C.U. B.Com. 2013(H)*]

Solution: **Calculation of 4-year moving average**

Year	Sales (₹ '000)	4-year moving total (not centered)	2-item moving total (centered)	4-year moving average
(a)	(b)	(c)	(d)	(e) = (d) ÷ 8
'02	2	–	–	–
'03	6	–	–	–
		17		
'04	4		35	4.375
		18		
'05	5		37	4.625
		19		
'06	3		36	4.500
		17		
'07	7		35	4.375
		18		
'08	2		37	4.625
		19		
'09	6		39	4.875
		20		
'10	4		41	5.125
		21		
'11	8	–	–	–
'12	3	–	–	–

Calculation of 5-year weighted moving average

Year	Sales (₹ '000)	5-year weighted moving total	5-year weighted moving average
(a)	(b)	(c)	(d) = (c) ÷ 8
'02	2	–	–
'03	6	–	–
'04	4	$2 \times 1 + 6 \times 2 + 4 \times 2 + 5 \times 2 + 3 \times 1 = 35$	4.375
'05	5	$6 \times 1 + 4 \times 2 + 5 \times 2 + 3 \times 2 + 7 \times 1 = 37$	4.625
'06	3	$4 \times 1 + 5 \times 2 + 3 \times 2 + 7 \times 2 + 2 \times 1 = 36$	4.500
'07	7	$5 \times 1 + 3 \times 2 + 7 \times 2 + 2 \times 2 + 6 \times 1 = 35$	4.375

Year	Sales (₹ '000)	5-year weighted moving total	5-year weighted moving average
(a)	(b)	(c)	(d) = (c) ÷ 8
'08	2	$3 \times 1 + 7 \times 2 + 2 \times 2 + 6 \times 2 + 4 \times 1 = 37$	4.625
'09	6	$7 \times 1 + 2 \times 2 + 6 \times 2 + 4 \times 2 + 8 \times 1 = 39$	4.875
'10	4	$2 \times 1 + 6 \times 2 + 4 \times 2 + 8 \times 2 + 3 \times 1 = 41$	5.125
'11	8	–	–
'12	3	–	–

From the last column of the above two tables we see that 4-year centered moving average is equivalent to a 5-year moving average with weights 1, 2, 2, 2, 1 respectively.

EXAMPLE 4

Find the quarterly trend values from the following data by the Moving Average Method, using an appropriate period:

Quarter / year	Quarterly output ('000 tons)		
	2001	2002	2003
I	52	59	57
II	54	63	61
III	67	75	72
IV	55	65	60

[*C.U. B.Com. 2004*]

Solution: We shall have to determine trend values by the method of moving averages of period 4 quarters.

Calculation of quarterly trend values by moving average method

Year and Quarter	Output ('000 tons)	4-Quarter moving total (not centered)	2-item moving total (centered)	4-quarter moving average (or trend values) (centered)
2001 I	52	–	–	–
II	54	–	–	–
		228		
III	67		463	57.875
		235		
IV	55		479	59.875
		244		
2002 I	59		496	62.000
		252		
II	63		514	64.250
		262		
III	75		522	65.250
		260		
IV	65		518	64.750
		258		
2003 I	57		513	64.125
		255		
II	61		505	63.125
		250		
III	72	–	–	–
IV	60	–	–	–

EXAMPLE 5

You are given the annual profit figures for a certain firm for the years 2006 to 2012. Fit a least square straight line trend to the data and estimate the expected profit for the year 2015:

Year	2006	2007	2008	2009	2010	2011	2012
Profit (₹ lakhs)	60	72	75	78	80	85	95

[***C.U. B.Com. 2013(H)***]

Solution: Let $y = a + bt$ be the equation of straight line.

Here, a and b are constant, whose values are to be calculated by using the following two normal equations:

$$\Sigma y = n.a + b.\ \Sigma t \quad \ldots(i)$$

and

$$\Sigma yt = a\Sigma t + b\Sigma t^2 \quad \ldots(ii)$$

Here the number of years (n) is 7, which is odd, so we take the middle year 2009 as origin and t unit = 1 year. And y is the expected profit.

Fitting of straight line trend

Year	**Profit (₹ lakhs) (y)**	**t = year – 2009**	**t^2**	***yt***
2006	60	–3	9	–180
2007	72	–2	4	–144
2008	75	–1	1	–75
2009	78	0	0	0
2010	80	1	1	80
2011	85	2	4	170
2012	95	3	9	285
Total	545	0	28	136

From (i), $\quad 545 = 7a + b \times 0$ or $7a = 545$ or $a = \dfrac{545}{7} = 77.86$

From (ii), $\quad 136 = a \times 0 + b.28$ or $28b = 136$ or $b = \dfrac{136}{28} = 4.86$

Therefore, the trend equation is

$$y = 77.86 + 4.86t$$

For 2015, $\quad t = 2015 – 2009 = 6$

Hence, the expected profit for the year 2015

$$= 77.86 + 4.86 \times 6 = 77.86 + 29.16$$
$$= ₹\ 107.02$$

EXAMPLE 6

Obtain the trend values for the following series by fitting a straight line trend equation.

Year	Goods carried ('00000 metric tons)
2009–10	1480
2010–11	1576
2011–12	1620
2012–13	1840
2013–14	1920
2014–15	1950
2015–16	2000
2016–17	2020

[***C.U. B.Com (H)***]

Solution: Let, $y = a + bt$ be the equation of the straight line.

Here, a and b are constant, whose values are to be calculated by using the following two normal equations;

$$\Sigma y = n.a + b.\ \Sigma t \quad \ldots(i)$$

and

$$\Sigma yt = a\Sigma t + b\Sigma t^2 \quad \ldots(ii)$$

Here, number of years(n) is 8, which is even, so we take the middle of the years 2012–13 and 2013–14 as origin and t unit = 6 months and y is goods carried ('00000 metric tons).

Calculation of straight line trend and trend values.

Year	Goods carried ('00000 metric tons)	t	t^2	yt	Trend values ('00000 metric tons)
2009–10	1480	–7	49	–10360	1510.32
2010–11	1576	–5	25	–7880	1593.30
2011–12	1620	–3	9	–4860	1676.28
2012–13	1840	–1	1	–1840	1759.26
2013–14	1920	1	1	1920	1842.24
2014–15	1950	3	9	5850	1925.22
2015–16	2000	5	25	10000	2008.20
2016–17	2020	7	49	14140	2091.18
Total	14,406	0	168	6970	–

From (i), $14406 = 8a + b \times 0$

or $8a = 14406$ or $a = \dfrac{14406}{8} = 1880.75$

From (ii), $6970 = a \times 0 + b \times 168$

or. $168b = 6970$ or $b = \dfrac{6970}{168} = 41.49$

Therefore, the straight line trend equation is,

$$y = 1800.75 + 41.49t \qquad \ldots(iii)$$

Trend values of goods carried are computed by putting $t = -7, -5, -3, -1, 1, 3, 5, 7$ successively in equation (iii).

EXAMPLE 7

Fit a quadratic trend equation by the method of least squares from the following data and estimate the trend value for the year 2004:

Year	1997	1998	1999	2000	2001	2002
Price (₹)	250	207	228	240	281	392

[***C.U. B.Com. 2004***]

Solution: Let $y = a + bt + ct^2$ be the equation of quadratic trend.

Here, a, b and c are constant, whose values are to be calculated by using the following three normal equations:

$$\Sigma y = na + b\Sigma t + c\Sigma t^2 \qquad \ldots(i)$$

$$\Sigma yt = a\Sigma t + b\Sigma t^2 + c\Sigma t^3 \qquad \ldots(ii)$$

$$\Sigma yt^2 = a\Sigma t^2 + b\Sigma t^3 + c\Sigma t^4 \qquad \ldots(iii)$$

Here, the number of years (n) is 6, which is even, so we take the middle of the years 1999 and 2000 on origin and t unit = 6 months and y is the price.

Fitting of quadratic trend equation

Year	**Price (₹) (y)**	$t = \dfrac{\textbf{year} - \textbf{1999.5}}{\textbf{.5}}$	t^2	t^3	t^4	yt	yt^2
1997	250	–5	25	–125	625	–1250	6250
1998	207	–3	9	–27	81	–621	1863
1999	228	–1	1	–1	1	–228	228
2000	240	1	1	1	1	240	240
2001	281	3	9	27	81	843	2529
2002	392	5	25	125	625	1960	9800
Total	1598	0	70	0	1414	944	20910

From equation (i), $1598 = 6a + b \times 0 + c \times 70$

or $$6a + 70c = 1598 \quad \ldots(iv)$$

From equation (ii), $944 = a \times 0 + b \times 70 + c \times 0$

or $$70b = 944 \text{ or } b = \frac{944}{70} = 13.486$$

From equation (iii), $20910 = a \times 70 + b \times 0 + c \times 1414$

or $$70a + 1414c = 20910 \quad \ldots(v)$$

The values of 'a' and 'c' can be obtained from equation (iv) and equation (v),

$$6a + 70c = 1598 \quad \ldots(\text{iv})$$

$$70a + 1414c = 20910 \quad \ldots(\text{v})$$

Multiplying equation (iv) by 70 and equation (v) by 6, we get

$$\begin{array}{r} 420a + 4900c = 111860 \\ 420a + 8484c = 125460 \\ - \quad\quad - \quad\quad\quad - \\ \hline -3584c = -13600 \end{array} \quad \text{(By subtracting)}$$

or $$c = \frac{13600}{3584} = 3.7946$$

Putting the value of 'c' in equation (iv), we get

$$6a + 70 \times 3.7946 = 1598$$

or $$6a + 265.622 = 1598$$

or $$6a = 1598 - 265.622 = 1332.378$$

or $$a = \frac{1332.378}{6} = 222.063$$

Hence, the quadratic trend equation is

$$y = 222.063 + 13.486t + 3.7946t^2$$

For the year 2004, $t = \dfrac{2004 - 1999.5}{.5} = 9$

Therefore, the trend value for the year 2004

$$= 222.063 + 13.486 \times 9 + 3.7946\,(9)^2$$

$$= 222.063 + 121.374 + 307.3626$$

$$= ₹\ 650.7996$$

EXAMPLE 8

Fit an exponential trend $y = ab^t$ to the following data by the method of least squares and find the trend value for the year 2017.

Year (x)	2011	2012	2013	2014	2015
Production ('000 tons)	132	142	157	170	191

Solution: Here, the number of years (n) is 5, which is odd, so we take the middle year 2013 as origin and t unit = 1 year.

The exponential trend $y = a.b^t$ is transformed into a linear form by taking logarithm on both sides. Thus,

log y = log a + t. log b or $y = A + B.t$ where y = log y, A = log a and B = log b. The constants A and B are to be determined by solving the normal equations

$$\Sigma y = nA + b\Sigma t \text{ and } \Sigma yt = A\Sigma t + B\Sigma t^2$$

Fitting Exponential Trend

x	y	$y = \log y$	$t = x - 2013$	t^2	yt
2011	132	2.1206	–2	4	–4.2412
2012	142	2.1523	–1	1	–2.1523
2013	157	2.1859	0	0	0
2014	170	2.2304	1	1	2.2304
2015	191	2.2810	2	4	4.5620
Total		10.9802	0	10	0.3989

From 1st normal equation, we get

$$10.9802 = 5A + 0 \times B \text{ or } 5A = 10.9802 \text{ or } A = 2.19604$$

or $\log a = 2.19604$ or $a = \text{antilog}(2.19604) = 157.01$

From 2nd normal equation we get,

$$0.3989 = 0 \times A + 10B \text{ or, } 10B = 0.3989$$

or $B = 0.03989$, or $\log b = 0.03989$

or $b = \text{antilog}(0.03989) = 1.0962$

Therefore, the equation for exponential trend is

$$y = 157.01\,(1.0962)^t \text{ with origin at 2013 and } t \text{ unit} = 1 \text{ year.}$$

For the year 2017, $t = 2017 - 2013 = 4$

Putting the value of t in the equation $Y = A + Bt$

we get $y = 2.19604 + 0.03989 \times 4 = 2.35560$

or $\log y = 2.35560$ or $y = \text{antilog}(2.35560) = 226.8$

Thus, the trend value for the year 2017 is 226.8 ('000 tons).

EXAMPLE 9

Compute the seasonal index for the following data using simple average method:

Year / Quarter	Q_1	Q_2	Q_3	Q_4
2010	75	60	54	59
2011	86	65	63	80
2012	90	72	66	85
2013	100	78	72	93

Solution: Assuming that trend and cyclical fluctuations are absent in the given data.

Calculation of seasonal index

Year / Quarter	Q_1	Q_2	Q_3	Q_4	Total
2010	75	60	54	59	–
2011	86	65	63	80	–
2012	90	72	66	85	–
2013	100	78	72	93	–
Total	351	275	255	317	–
Average	87.75	68.75	63.75	79.25	299.50
Grand average*	–	–	–	–	74.875
Using additive model					
Seasonal Index (Average – Grand average)	12.875	–6.125	–11.125	4.375	0
Using multiplicative model					
Seasonal Index $\left(\frac{\text{Average}}{\text{Grand average}} \times 100\right)$	117.20	91.82	85.14	105.84	400

$$\text{*Grand average} = \frac{299.50}{4} = 74.875$$

EXAMPLE 10

Using additive model compute seasonal variations of the following by the method of moving average and obtain deseasonalised data for the four quarters of 2013.

Years	Quarterly output to paper in million tons			
	I	II	III	IV
2011	37	38	37	40
2012	41	34	25	31
2013	35	37	35	41

[*C.U. B.Com. 1993*]

Solution: Since quarterly figures are given, a centered moving average with a period of four quarters is necessary.

Calculation of moving average and deviations from trend

Years and Quarters	Quarterly output (million tons)	4-Quarter moving total (not centered)	2-item moving total (centered)	4-quarter moving average (centered)	Deviations from trend (million tons)
(a)	(b)	(c)	(d)	(e) = (d) ÷ 8	(f) = (b) – (e)
2011 I	37	–	–	–	–
II	38	–	–	–	–
III	37	152	308	38.5	–1.5
IV	40	156	308	38.5	1.5
2012 I	41	152	292	36.5	4.5
II	34	140	271	33.9	0.1
III	25	131	256	32.0	–7.0
IV	31	125	253	31.6	–0.6
2013 I	35	128	266	33.2	1.8
II	37	138	286	35.7	1.3
III	35	148	–	–	–
IV	41	– –	–	–	–

Calculation of seasonal variations

Years \ Quarters	Deviations from trend (million tons) I	II	III	IV	Total
2011	–	–	–1.5	1.5	–
2012	4.5	0.1	–7.0	–0.6	–
2013	1.8	1.3	–	–	–
Total	6.3	1.4	–8.5	0.9	–
Average	3.15	0.7	–4.25	0.45	0.05
Grand average*	–	–	–	–	0.0125
Seasonal variation (Average – Grand average)	3.1375	0.6875	–4.2625	0.4375	0

$$\text{* Grand average} = \frac{0.05}{4} = 0.0125$$

Deseasonalised data for 2013:

For 1st quarter 35 – 3.1375 = 31.8625

For 2nd quarter 37 – 0.6875 = 36.3125

For 3rd quarter 35 + 4.2625 = 39.2625

For 4th quarter 41 – 0.4375 = 40.5625

EXAMPLE 11

Using multiplicative model, find the seasonal indices by the method of moving average from the following series of observations:

Year / Quarter	Sales ('000 ₹)		
	2009	**2010**	**2011**
I	101	106	110
II	93	96	101
III	79	83	88
IV	98	103	106

Solution: **Calculation of moving average and ratio to moving average**

Year and Quarter	Sales ('000 ₹)	4-Quarter moving total (not centered)	2-item moving total (centered)	4-quarter moving average (centered)	Ratio to moving average
(a)	**(b)**	**(c)**	**(d)**	**(e) = (d) ÷ 8**	**(f) = (b) – (e) × 100**
2009 I	101	–	–	–	–
II	93	–	–	–	–
III	79	371	747	93.38	84.6
IV	98	376	755	94.38	103.8
2010 I	106	379	762	95.25	111.3
II	96	383	771	96.38	99.6
III	83	388	780	97.50	85.1
IV	103	392	789	98.62	104.4
2011 I	110	397	799	99.88	110.1
II	101	402	807	100.88	100.1
III	88	405 –	–	–	–
IV	106	–	–	–	–

Calculations of seasonal Index

Year \ Quarter	Ratio to moving average				
	I	**II**	**III**	**IV**	**Total**
2009	–	–	84.6	103.8	–
2010	111.3	99.6	85.1	104.4	–
2011	110.1	100.1	–	–	–
Total	221.4	199.7	169.7	208.2	–
Average	110.7	99.8	84.8	104.1	399.4
Grand average*	–	–	–	–	99.85

Quarter / Year	Ratio to moving average				
	I	**II**	**III**	**IV**	**Total**
Seasonal index $\left(\frac{\text{Average}}{\text{Grand average}} \times 100\right)$	111	100	85	104	400

$$\text{*Grand average} = \frac{399.4}{4} = 99.85\ .$$

EXAMPLE 12

Calculate seasonal indices by the method of link relatives from the data given below:

Year	Quarterly output ('000 tons) of coal for 4 years			
	I	**II**	**III**	**IV**
2008	65	58	56	61
2009	58	63	63	67
2010	70	59	56	52
2011	60	55	51	58

Solution: **Calculation of average link relatives**

Quarter / Year	Link Relatives			
	Q_1	Q_2	Q_3	Q_4
2008	–	89.23	96.55	108.93
2009	111.48	92.65	100.00	106.35
2010	104.48	84.29	94.92	92.86
2011	115.38	91.67	92.73	113.73
Total	331.34	357.84	384.20	421.87
Average	110.45	89.46	96.05	105.47

The Link Relative (L.R.) for the 1st quarter Q_1 of the 1st year 2008 cannot be determined. For the other three quarters of 2008,

$$\text{L.R. for } Q_2 = \frac{58}{65} \times 100 = 89.23$$

$$\text{L.R. for } Q_3 = \frac{56}{58} \times 100 = 96.55$$

$$\text{L.R. for } Q_4 = \frac{61}{56} \times 100 = 108.93$$

Similarly, the other link relatives are determined.

From the average link relatives obtained in the last row of the above table, we now find chain relatives (C.R.), taking 100 as the chain relative for Q_1.

$$\text{C.R. for } Q_1 = 100$$

$$\text{C.R. for } Q_2 = \left(\frac{89.46 \times 100}{100}\right) = 89.46$$

$$\text{C.R. for } Q_3 = \left(\frac{96.05 \times 89.46}{100}\right) = 85.93$$

$$\text{C.R. for } Q_4 = \left(\frac{105.47 \times 85.93}{100}\right) = 90.63$$

$$\text{Revised C.R. for } Q_1 = \left(\frac{110.45 \times 90.63}{100}\right) = 100.10$$

Therefore, $$d = \frac{100.10 - 100}{4} = 0.025$$

The preliminary indices are respectively 100, 89.46 – 0.025, 85.93 – 2 × 0.025, 90.63 – 3 × 0.025, i.e. 100, 89.435, 85.88, 90.555, or 100, 89.44, 85.88, 90.56

Average of the preliminary indices = $\dfrac{100 + 89.44 + 85.88 + 90.56}{4} = 91.47$

The required seasonal indices are

For 1st quarter = $\dfrac{100}{91.47} \times 100 = 109.33$, For 2nd quarter = $\dfrac{89.44}{91.47} \times 100 = 97.78$

For 3rd Quarter = $\dfrac{85.88}{91.47} \times 100 = 93.89$, For 4th quarter = $\dfrac{90.56}{91.4} \times 100 = 99.00$.

EXERCISE

A. THEORETICAL

(a) Short Essay Type

1. What do you mean by "time series"? Define the components of time series. [***C.U. B.Com. 2011, 2014(G), 2014(H)***]
2. Explain different components of time series with examples. [***C.U. B.Com. 2012***]
3. Write four points of advantages and disadvantages for the calculation of moving average in time series. [***C.U. B.Com. 2013(G), 2016(H)***]
4. Define Time series. Explain with examples the difference between the secular trend and seasonal variation. [***C.U. B.Com. 2013(H)***]

5. What do you mean by seasonal variation? Explain with a few examples the utility of such a study. [***C.U. B.Com. 2015(G)***]
6. Explain the uses of time series with examples. [***C.U. B.Com. 2015(H)***]
7. What are the component of a time series? With which component of a time series would you mainly associate each of the following:
   (*i*) A need for increased wheat production due to constant increase in population. [**Ans.** Long-term trend]
   (*ii*) A fire in a factory delaying production by four week. [**Ans.** Irregular movement]
   (*iii*) An after-puja sale in a department store [**Ans.** Seasonal variation]
   (*iv*) An era of prosperity [**Ans.** Cyclical]
8. Explain the object and utility of time series analysis. [***C.U. B.Com. 2000***]
9. Define the line of best fit. [***C.U. B.Com. 1989***]
10. Explain what is meant by secular trend in time series analysis. Briefly mention the important types of forces which influence an economic time series [***C.U. B.Com. 1985***]
11. What is meant by business forecast?

(b) Essay Type

1. What do you mean by a time series? What are the components of a time series? What are the merits of the moving average method? [***C.U. B.Com. 1999***]
2. Point out the similarities and dissimilarities between the seasonal and the cyclical components of a time series. [***C.U. B.Com. 1988***]
3. What do you mean by seasonal fluctuations in time series? Give an example. What are the major uses of seasonal indices in time series analysis? [***C.U. B.Com. 1990***]
4. Explain the nature of cyclical variations in a time series. How do seasonal variations differ from them? Give an outline of the moving average method of measuring seasonal variations.
5. Briefly describe Link Relative Methods for finding the seasonal variations.
6. What steps are followed in business forecasting process?
7. Describe the limitations of business forecasting.

B. PRACTICAL

(a) Short Essay Type

1. Given the numbers 1, 0, –1, 0, 1, 0, –1, 0, 1; find the moving average of order four. [**Ans.** 0, 0, 0, 0, 0]
2. Given the numbers 2, 6, 1, 5, 3, 7, 2. Obtain moving average of order 3. [***C.U. B.Com. 1984***] [**Ans.** Nil, 3, 4, 3, 5, 4, Nil]

3. Given the numbers 2, 6, 1, 5, 3, 7, 2; Write the weighted moving average of period 3, the weights being 1, 4,1.

[***I.C.W.A June 1984***] [**Ans.** 4.5, 2.5, 4.0, 4.0, 5.5]

4. Find the trend for the following series using a three-year moving average:

Year	1	2	3	4	5	6	7
Value	2	4	5	7	8	10	13

[***C.U., B.Com. 83***] [**Ans.** Nil, 3.67, 5.33, 6.67, 8.33, 10.33, Nil]

5. Find the trend for the following series using a three-year weighted moving average with weights 1, 2, 1.

Year	1	2	3	4	5	6	7
Values	2	4	5	7	8	10	13

[**Ans.** Nil, 3.75, 5.25, 6.75, 8.25, 10.25, Nil]

(b) Essay Type

6. The sale of a commodity in tonnes varied from January, 2016 to December 2016 in the following manner:

280	300	280	280	770	240
230	230	220	200	210	200

Fit a trend line by the method of semi averages.

[**Ans.** Averages to be plotted 275; 215]

7. Draw a trend line by the semi average method using the following data:

Year	2010	2011	2012	2013	2014	2015
Production of steel (in '000 tonnes)	30	38	42	50	48	56

8. Apply the semi average method and obtain trend values for all the years from the following data :

Year	2011	2012	2013	2014	2015	2016	2017	2018
Value	45	43	47	49	46	45	42	43

[**Ans.** 46.75, 46.25, 45.75, 45.25, 44.75, 44.25, 43.75, 43.25]

9. Students reading in a college in different years are as follows:

Calculate the five yearly moving averages of the data:

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
Student	332	317	357	392	402	405	410	427	405	431

[**Ans.** 360, 374.6, 393.2, 407.2, 409.8, 415.6]

10. Calculate trend value from the following data relating to the production of tea in India by the Moving Average Method, on the assumption of a four-yearly cycle:

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
Production (Million lbs)	464	515	518	467	502	540	557	571	586	612

[***C.U. B.Com. 2011, 2014 (a)***] [**Ans.** 495.75, 503.625, 511.625, 529.50, 553, 572.50]

11. Determine trend by the method of moving averages from the following figures of quarterly production of a commodity :

Quarter / Year	**Production (in thousand tons)**		
	2015	**2016**	**2017**
I	115	119	149
II	180	189	209
III	108	149	179
IV	99	119	145

[**Ans.** 126.0, 127.6, 133.9, 141.5, 147.5, 147.8, 154.0, 160.2, 167.2, (Take 4-quarter period)]

12. Fit a straight line trend equation by the method of least square:

Years	2009	2010	2011	2012	2013
Values	83	94	99	97	104

[***C.U. B.Com. 2014 (G)***] [**Ans.** $y = 95.4 + 4.5\ t$]

13. You are given the annual profit figures for a certain firm for the years 2005 to 2011. Fit a straight line trend to the data and estimate the expected profit for the year 2012:

Year	2005	2006	2007	2008	2009	2010	2011
Profit (Lakhs)	30	36	38	32	40	42	48

[***C.U. B.Com. 2013 (G)***] [**Ans.** $y = 38 + 2.43\ t$]

14. Fit a least square trend line to the following data:

Year	2004	2005	2006	2007	2008	2009
Decimal ('000 units)	8	12	15	17	22	24

Hence find estimated decimal for 2010.

[**Ans.** $y = 16.33 + 1.6\ t$ (taking demand as '000 units); Demand for the year 2010 = 27.53 ('000 units)] [***C.U. B.Com. 2010, 2016 (H)***]

15. Fit a straight line trend equation by the method of least squares from the following data and estimate the profit for the year 2015:

Year	1975	1980	1985	1990	1995	2000	2005	2010
Profit (₹ lakh)	10	13	15	20	22	28	33	40

[***C.U. B.Com. 2016 (G)***] [**Ans.** $y = 22.625 + 0.42\ t$; $y = 41.525$ lakh]

16. Fit a least square trend line to the following data:

Year	1980	1985	1990	1995	2000	2005
Volume of sale (in suitable units)	12	15	17	22	24	30

Estimate the volume of sales for 2010. [***C.U. B.Com. 1988, 2000***]

[**Ans.** $y = 20 + 1.743\ t$; origin = middle of 1990 & 1995; t Unit = $2\frac{1}{2}$ yrs.; 32.20 units]

17. Fit a straight line by using the method of least square from the data given below and also compute trend values.

Year	2010	2011	2012	2013	2014	2015	2016	2017
No. of sheep (in lakhs)	56	55	57	47	42	38	35	32

[**Ans.** $y = 44.5 - 1.86\ t$, origin middle of 2013 & 2014; t unit = $\frac{1}{2}$ yr; 57.52, 53.8, 50.08, 46.36, 42.64, 38.92, 35.20, 31.48]

18. Fit a parabolic curve to the following data:

Year	2011	2012	2013	2014	2015
Value	3	2	3	6	11

[**Ans.** $y = 3 + 2t + t^2$]

19. Fit an equation of the type $y = a + bt + ct^2$ to the following data:

Year	2014	2015	2016	2017	2018
Purchase ('000)	1	5	10	22	38

[**Ans.** $y = 10.78 + 9.1t + 2.21t^2$; Origin 2016]

20. Fit an equation of the type $y = a + bt + ct^2$ to the following data:

Year	2011	2012	2013	2014	2015	2016
Sales (in million tons)	100	105	115	100	112	118

[**Ans.** $y = 107.66 + 1.37t + 0.06\ t^2$; origin = middle of 2013 & 2014]

21. Fit a parabolic curve of second degree to the data given below and estimate the value of 2017 and comment on it:

Year	2011	2012	2013	2014	2015
Sales (in '000 ₹)	10	12	13	10	8

[***I.C.W.A. (Final) Dec. 1981***] [**Ans.** $y = 12.314 - 0.6t - 0.857t^2$; trend values: 10.086, 12.057, 12.314, 10.857, 7.686; –3.798 ('000 ₹); a negative number in sales, so this curve is inadmissible in general]

22. Fit an exponential trend of the form $y = a.b^t$ to the data giving the population figures below:

Census year	1951	1961	1971	1981	1991	2001	2011
Population (in crores)	25.0	25.1	27.9	31.9	36.1	43.9	54.7

Estimate the population figures for the year 2021 using the fitted exponential trend equation. [***D.U. M.Com.***] [**Ans.** $y = 33.56\ (1.142)^t$; 157.15 crore]

23. The following table gives the profits of a concern for five years ending 2016:

Years	2012	2013	2014	2015	2016
Profit (in ₹ '000)	1.6	4.5	13.6	40.2	125.0

Fit an equation of the type $y = a.b^t$ [***D.U. Eco (H)***] [**Ans.** $y = 13.79\ (2.982)^t$]

24. Compute the average seasonal movements by the method of quarterly total (average) for the following series of observation:

Quarters / Year	**Total production of paper (thousand tons)**			
	I	**II**	**III**	**IV**
2011	37	38	37	40
2012	41	34	25	31
2013	25	37	35	41

[**Ans.** 1.75, 0.42, -3.59, 1.42, using additive model]

25. Construct indices of seasonal variations from the following time series data on consumption of cold drinks, which contains only seasonal and irregular variations.

Quarters / Year	**Consumption of cold drinks ('000 bottles)**			
	I	**II**	**III**	**IV**
2011	90	75	87	70
2012	75	80	78	75
2013	80	75	75	72
2014	85	82	80	81

[***C.U. M.Com. '75; B.Com. (H) '81***]

[**Ans.** 105, 99, 101, 95, using multiplicative model]

26. Calculate the seasonal indices in the case of the following quarterly data in certain units. You may use any method you think appropriate:

Year	Q_1	Q_2	Q_3	Q_4
2010	39	21	52	81
2011	45	23	63	76
2012	44	26	69	75
2013	53	23	64	84

[***C.U. M.Com***] [**Ans.** 86.4, 44.4, 118.4, 150.8, using multiplicative model]

27. Compute the average seasonal movements by the method of averages for the following data:

Total production of steel (in million tonnes)

Year	**Quarter**			
	I	**II**	**III**	**IV**
2012	33	30	31	30
2013	24	18	27	34
2014	34	28	30	29
2015	30	25	29	31

Also obtain the deseasonalised data.

[**Ans.** 1.31, –3.69, 0.31, 2.06, using additive model]

28. Obtain seasonal indices from the following data using moving average method:

Year / Season	**Output in thousand units**				
	2014	**2015**	**2016**	**2017**	**2018**
Summer	31	42	49	47	51
Rain	39	44	53	51	54
Winter	45	57	65	62	66

[**Ans.** –4.49, –2.27, 6.76, in thousand units]

29. Obtain seasonal fluctuations from the following time series data:

Quarters / Year	**Quarterly output of coal for 4 years**			
	I	**II**	**III**	**IV**
2014	65	58	56	61
2015	68	63	63	67
2016	70	59	56	52
2017	60	55	51	58

[**Ans.** 5.33, –1.33, –3.00, –1.00, using additive model]

30. Calculate seasonal indices by the ratio to moving average method from the following data:

Wheat prices (in ₹ Per quintal)

Quarters / Year	**2012**	**2013**	**2014**	**2015**
Q_1	75	86	90	100
Q_2	60	65	72	78
Q_3	54	63	66	72
Q_4	59	80	85	93

[**Ans.** 122.366, 92.426, 84.694, 100.514]

31. Calculate the seasonal indices by the ratio to moving average method from the following data:

Quarters / Year	I	II	III	IV
2015	68	62	61	63
2016	65	58	66	61
2017	68	63	63	67

[**Ans.** 105.10, 95.69, 99.34, 99.87]

32. Calculate the seasonal indices by the method of link relatives from the following data:

Quarter	**Year**			
	2014	**2015**	**2016**	**2017**
I	75	86	90	100
II	60	65	72	78
III	54	63	66	72
IV	59	80	82	93

[***C.U. Eco (H) 1980***] [**Ans.** 124.2, 93.5, 82.5, 99.8]

33. Deseasonalise the following production data by the method of moving average:

Quarterly output ('000 tons)

Year / Quarter	2010	2011	2012	2013
I	30	49	35	75
II	49	60	62	79
III	50	61	60	65
IV	35	20	25	70

[**Ans.** 28.28, 38.70, 39.28, 57.74, 47.28, 39.70, 50.28, 42.74, 33.28, 51.70, 49.28, 47.74, 73.28, 68.70, 54.28, 92.74]

34. The following table gives the annual sales (in ₹ '000) of commodity

Year	**Sales**	**Year**	**Sales**
2007	710	2013	644
2008	705	2014	783
2009	680	2015	781
2010	687	2016	805
2011	757	2017	872
2012	629		

Determine the trend by calculating the 5-yearly moving averages.

[**Ans.** Nil, Nil, 709.8, 691.6, 679.4, 700.0, 718.8, 728.4, 777.0, Nil, Nil]

35. The following table gives the annual sales (in ’000 ₹) of a commodity.

Year	2010	2011	2012	2013	2014	2015
Sales	75	68	70	78	80	88

Obtain the trend by (*i*) semi-average method (*ii*) Fitting a straight line using least square method. [***I.C.W.A. Dec ‘96***] [**Ans.** (*i*) 1st part: 71; 2nd part: 82; (*ii*) $y = 76.5 + 1.56t$; Origin: mid of 2012 & 2013, t unit = $\frac{1}{2}$ yr.]

36. Production figures of a flour factory (in ’000 tons) are given below:

Year	2000	2001	2002	2003	2004	2005	2006
Production	12	10	14	11	13	15	16

(*i*) Fit a straight line trend to the data
(*ii*) Estimate the production for 2007, 2009 and 2010
(*iii*) Find the slope of the straight line trend.
(*iv*) Do the figures show a rising trend or a falling trend?
(*v*) What does the difference between the given figures and trend values indicate?

[**Ans.** (*i*) $y = 0.75t + 13$, origin 2003; (*ii*) 16, 17.50 and 18.25 in ’000 tons; (*iii*) 0.75 (in ’000 tons); (*iv*) rising trend as slope is positive, (*v*) trend values: 10.75, 11.50, 12.25, 13, 13.75, 14.50, 15.25; the difference indicate the elimination of trend values by additive model]

C. MULTIPLE CHOICE QUESTIONS (MCQs)

(i) Short Type

1. An orderly set of data arranged in accordance with their time of occurrence is called
(*a*) Arithmetic series (*c*) Harmonic series
(*b*) Geometric series (*d*) Time series [**Ans.** (*d*)]

2. The secular trend is measured by the method of semi averages when:
(*a*) Time series consists of even number of values
(*b*) Time series based on yearly values
(*c*) Trend is linear
(*d*) None of these [**Ans.** (*c*)]

3. The method of making average is used to find the:
(*a*) Secular trend (*c*) Cyclical fluctuation
(*b*) Seasonal variation (*d*) Irregular movement [**Ans.** (*a*)]

4. In a straight line equation $y = a + bt$; b is the
(*a*) x-intercept (*c*) Trend
(*b*) y-intercept (*d*) Slope [**Ans.** (*d*)]

5. Value of b in the trend line $y = a + bt$ is
   (*a*) Both negative and positive (*c*) Always negative
   (*b*) Always positive (*d*) Always zero [**Ans.** (*a*)]
6. In semi average method, we decide the data into :
   (*a*) Two parts (*c*) Three parts
   (*b*) Two equal parts (*d*) Difficult to tell [**Ans.** (*b*)]
7. Moving average method is used for measurement of trend when:
   (*a*) Trend is linear (*c*) Trend is curvilinear
   (*b*) Trend is non-linear (*d*) None of these [**Ans.** (*a*)]
8. The long term trend of a time series graph appears to be:
   (*a*) Straight line
   (*b*) Upward
   (*c*) Downward
   (*d*) Parabolic curve or third degree curve [**Ans.** (*d*)]
9. The most commonly used mathematical method for measuring the trend is
   (*a*) Moving average method (*c*) Method of least squares
   (*b*) Semi average method (*d*) None of these [**Ans.** (*c*)]
10. In moving average method, we cannot find the trend values of same
    (*a*) Middle periods
    (*b*) End periods
    (*c*) Starting periods
    (*d*) Between extreme periods. [**Ans.** (*d*)]
11. The best fitting trend is one which the sum of square of residuals is:
    (*a*) Maximum (*c*) Least
    (*b*) Negative (*d*) Zero [**Ans.** (*c*)]
12. Moving average
    (*a*) Smooth out the time series
    (*b*) Give the trend in a straight line
    (*c*) Measure the seasonal variations
    (*d*) None of these [**Ans.** (*a*)]
13. In time series seasonal variations can occur within a period of
    (*a*) Four years (*c*) One year
    (*b*) Three years (*d*) Five years [**Ans.** (*c*)]
14. A complete cycle passes through
    (*a*) Two stages (*c*) Four stages
    (*b*) Three stages (*d*) Difficult to tell [**Ans.** (*c*)]
15. A complete cycle consists of a period of
    (*a*) Prosperity and depression
    (*b*) Prosperity and recession
    (*c*) Prosperity and recovery
    (*d*) Recession and recovery [**Ans.** (*b*)]

16. Most frequently used mathematical model of a time series is
    (*a*) Additive model (*c*) Mixed model
    (*b*) Multiplicative model (*d*) Regression model [**Ans.** (*b*)]

17. A pattern that is repeated throughout a time series and has a recurrence period of at most one year is called
    (*a*) Cyclical variation (*c*) Seasonal variation
    (*b*) Irregular variation (*d*) Long-term variation
    [**Ans.** (*c*)]

18. Increase in the number of patients in the hospital due to heat stroke is
    (*a*) Secular trend (*c*) Seasonal variation
    (*b*) Irregular movement (*d*) Cyclical functions [**Ans.** (*c*)]

19. Indicate which of the following is an example of seasonal variations?
    (*a*) The sale of air condition increases during summer
    (*b*) Death rate decreased due to advancement in science
    (*c*) Recovery in business
    (*d*) Sudden causes by wars [**Ans.** (*a*)]

20. The fire in a factory is an example of
    (*a*) Secular trend (*c*) Cyclical fluctuation
    (*b*) Seasonal variation (*d*) Irregular movements
    [**Ans.** (*d*)]

21. Damages due to floods, droughts, strikes, fires are:
    (*a*) Trend (*c*) Cyclical
    (*b*) Seasonal (*d*) Irregular [**Ans.** (*d*)]

22. In general pattern of increase or decrease in economic or social phenomena is shown by:
    (*a*) Seasonal trend (*c*) Secular trend
    (*b*) Cyclical trend (*d*) Irregular trend [**Ans.** (*c*)]

23. Depression in business is
    (*a*) Secular trend (*c*) Seasonal
    (*b*) Cyclical (*d*) Irregular [**Ans.** (*b*)]

24. The difference between the actual value of the time series and the forecasted value is called:
    (*a*) Residual (*c*) Sum of squares of residual
    (*b*) Sum of variation (*d*) All of the above [**Ans.** (*a*)]

25. If we first subtract the trend value. (*T*) for each quarter from the original value (Y), then average the values for a given quarter over successive years, then for short-term data we get:
    (*a*) Unseasonal data
    (*b*) Deseasonalised data
    (*c*) Cyclical component
    (*d*) Seasonal component [**Ans.** (*d*)]

26. Deseasonalised data involves
    (*a*) Subtracting the trend component (*T*) from the original data (*Y*)
    (*b*) Adding the seasonal component (*S*) to the original data (*Y*)
    (*c*) Adding the trend component (*T*) to the original data (*Y*)
    (*d*) Subtracting the seasonal component (*S*) from the original data (*Y*) [**Ans.** (*d*)]

27. We can use the regression line for past data for forecasting future data. We are then using the line which
    (*a*) Maximizes the sum of deviations of past data from the line
    (*b*) Minimizes the sum of squared deviations of past data from the line
    (*c*) Minimizes the sum of deviations of past data from the line
    (*d*) Maximizes the sum of squared deviations of past data from the line
    [**Ans.** (*b*)]

28. Which of the following is not a forecasting technique?
    (*a*) Judgemental (*c*) Time horizon
    (*b*) Time series (*d*) Associative [**Ans.** (*c*)]

29. In which of the following forecasting techniques, data obtained from past experience is analysed?
    (*a*) Judgmental forecast (*c*) Associative model
    (*b*) Time series forecast (*d*) All of the above [**Ans.** (*b*)]

30. Delphi method is used for
    (*a*) Judgemental forecast (*c*) Associative model
    (*b*) Time series forecast (*d*) All of the above [**Ans.** (*a*)]

31. When we use an approach which implies that the forecast for the next time period should take into account the observed error in the earlier forecast for the current time period, then we are using:
    (*a*) Exponential smoothing (*c*) Regression analysis
    (*b*) Decision tree analysis (*d*) Time series analysis
    [**Ans.** (*a*)]

32. Which of the following is a major problem for forecasting, especially when using regression analysis?
    (*a*) The past cannot be known
    (*b*) The future is not entirely certain
    (*c*) The future exactly follows the patterns of the past
    (*d*) The future may not follow the patterns of the past [**Ans.** (*a*)]

33. Operations generated forecasts often do not have to do with
    (*a*) Inventory requirements (*c*) Time requirements
    (*b*) Resource needs (*d*) Sales [**Ans.** (*d*)]

34. Which of the following is not true for forecasting?
    (*a*) Forecasts are rarely perfect
    (*b*) The underlying casual system will remain same in the future.
    (*c*) Forecast for group of items is accurate then individual item.
    (*d*) Short-range forecasts are less accurate than long-range forecasts. [**Ans.** (*d*)]

(ii) Short Essay Type

35. Given the numbers 2, 6, 1, 5, 3, 7, 2; moving average of order 3 is [***C.U. B.Com (H) 1984***]

(*a*) 3, 5, 4, 3, 4
(*b*) 3, 4, 3, 5, 4
(*c*) 2, 3, 4, 5, 3
(*d*) 3, 2, 3, 4, 2 [**Ans.** (*b*)]

36.

Year	1	2	3	4	5	6	7
Value	2	4	5	7	8	10	13

The trend values for the above series using a three– year moving average are [***C.U. B.Com (H) 1983***]

(*a*) 3.56, 4.32, 5.23, 7.33, 8.67
(*b*) 3.33, 5.33, 6.33, 7.33, 8.33
(*c*) 3.67, 5.33, 6.67, 8.33, 10.33
(*d*) 3.67, 5.67, 6.67, 8.67, 10.67 [**Ans.** (*c*)]

37. Given the numbers 2, 6, 1, 5, 3, 7, 2; The weighted moving average of period 3, the weights being 1, 4, 1 are [***I.C.W.A June 1984***]

(*a*) 4.5, 2.5, 4.0, 4.0, 5.5
(*b*) 4.5, 3.5, 2.5, 3.5, 5.5
(*c*) 4.25, 3.5, 3.75, 4.5, 5.5
(*d*) 4.5, 3.25, 5.25, 4.75, 3.5 [**Ans.** (*a*)]

38.

Quarter	1^{st}	2^{nd}	3^{rd}	4^{th}
Sales (₹ '000)	23.7	25.2	21.4	65.4
Seasonal Index	0.78	1.24	0.50	1.48

Deseasonalise values of the above sales data using multiplicative model are

(*a*) 32.4, 22.3, 42.5, 44.3
(*b*) 30.4, 20.3, 42.8, 44.2
(*c*) 20.4, 30.4, 41.4, 42.4
(*d*) 30.4, 40.4, 20.3, 42.3 [**Ans.** (*b*)]

39.

Quarter	I	II	III	IV
Sales (₹ '00)	25.8	28.2	26.5	38.6
Seasonal index	82	98	85	135

Deseasonalise values of the above profit data using multiplicative model are

(*a*) 31.26, 29.56, 31.16, 29.56
(*b*) 32.16, 28.26, 30.18, 29.16
(*c*) 30.26, 28.16, 32.16, 28.56
(*d*) 31.46, 28.76, 31.18, 28.59 [**Ans.** (*d*)]

40.

Quarter	I	II	III	IV
Production (in '000 tons)	47.3	60.2	58.5	52.0
Seasonal Index	120	85	130	65

Deseasonalise values of the above data using multiplicative model are
(*a*) 38.35, 72.45, 70, 80
(*b*) 39.42, 70.82, 45, 80
(*c*) 37, 40, 60, 75
(*d*) 36.12, 65.72, 50.32, 70.22 [**Ans.** (*b*)]

41. Assuming the trend line (annual total equation) concerning the output (in tons) of a certain commodity is $y = 2400 + 360x$ with origin at October 2007. The Seasonal Index for April is 90. The seasonally adjusted output for April, 2011 is
(*a*) 1,300 tons (*c*) 1,314 tons
(*b*) 1,320 tons (*d*) 1,215 tons [**Ans.** (*c*)]
[**Hints:** The Monthly trend line equation is $y = 200 + 30t$. For April 2011, $t = 42$]

42. The following table gives the normal weight of a baby during the first six month of life

Age (in months)	0	2	3	5	6
Weight (in lbs)	5	7	8	10	12

Fit a straight line trend to the above data.
(*a*) $y = 4.806 + 1.123t$ (*c*) $y = 3.72 + 2.43t$
(*b*) $y = 4.35 + 2.21t$ (*d*) $y = 3.69 + 1.32t$ [**Ans.** (*a*)]

43.

Year	1	2	3	4	5	6	7
Value	2	4	5	7	8	10	13

From the above series of observation, 3-yearly weighted moving averages with weight 1, 2, 1 respectively are [***I.C.W.A. Dec. 1990***]
(*a*) 3.75, 4.75, 5.5, 6.25, 8.35
(*b*) 3.75, 5.25, 6.75, 8.25, 10.25
(*c*) 3.57, 5.67, 2.77, 6.56, 9.26
(*d*) 3.65, 5.35, 6.35, 8.45, 10.35 [**Ans.** (*b*)]

44.

Year	1	2	3	4	5
Number	28	38	46	40	56

A straight line by using the method of least square from the data given above is
(*a*) $y = 37.5 + 2.7t$ (*c*) $y = 41.6 + 5.8t$
(*b*) $y = 42.5 + 4.3t$ (*d*) $y = 38.9 + 5.2t$ [**Ans.** (*c*)]

45. In a time series forecasting problem, if the seasonal indices for quarters 1, 2, and 3 are 0.80, 0.90 and 0.95 respectively. What can you say about the seasonal index of quarter 4?
(*a*) It will be equal to 1
(*b*) It will be less than 1
(*c*) It will be greater than 1
(*d*) Seasonality does not exist [**Ans.** (*c*)]
[**Hints:** The seasonal indices must sum to 4, since there are 4 quarters]

46. 10, 14, 18, 22, 26, 30, 34, 38, 42, 46

5 yearly moving averages from the data given above are

(*a*) 18, 22, 26, 30, 34, 38, 42
(*b*) 18, 20, 25, 28, 32, 36, 45
(*c*) 20, 24, 28, 32, 36, 40, 44
(*d*) 20, 26, 30, 35, 38, 40, 42 [**Ans.** (*a*)]

47. 37.4, 31.1, 38.7, 39.5, 47.9, 42.6

4 yearly moving averages from the above data are

(*a*) 38.79, 42.28 (*c*) 38.34, 46.29
(*b*) 37.34, 40.27 (*d*) 37.99, 40.74 [**Ans.** (*d*)]

48. **Total Production of paper (100 tonnes)**

Year	Quarter			
	I	II	III	IV
2015	7	6	5	9
2016	9	5	4	6
2017	8	4	3	3

Using additive model, seasonal variations by the method of averages for the above data are

(*a*) 2.52, 1.25, –1.24, –2.53
(*b*) 2.25, –0.75, –1.75, 0.25
(*c*) 2.35, –1.15, 0.52, –1.98
(*d*) –2.15, –1.12, 2.17, 1.10 [**Ans.** (*b*)]

49.

Year	Quarter			
	I	II	III	IV
2014	8	4	3	3
2015	9	5	4	6
2016	7	6	5	9

Using multiplicative model seasonal indices by the method of averages for the above data are

(*a*) 135.23, 87.29, 73.17, 106.32
(*b*) 140.23, 85.36, 70.28, 99.32
(*c*) 139.13, 86.96, 69.56, 104.35
(*d*) 136.23, 90.26, 68.13, 105.25 [**Ans.** (*c*)]

50.

Year	2013	2014	2015	2016	2017
Value	11.3	13.0	9.7	10.6	10.7

A suitable straight line trend to the above data by the method of least squares is

(*a*) $y = 23 + 0.25t$ (*c*) $y = 22 + 0.12t$
(*b*) $y = 25 + 1.7t$ (*d*) $y = 21 + 0.1t$ [**Ans.** (*d*)]

CHAPTER 10

Probability Theory

SYLLABUS

Meaning of Probability; Different Definitions of Probability; Conditional Probability; Compound Probability; Independent Events, Simple Problems

THEMATIC FOCUS

10.1 INTRODUCTION

In our everyday life the 'probability' or 'chance' is a commonly used term. Sometimes, we use it to say 'it will probably rain today', 'Your estimation is probably correct', 'The result of the 1st year B.Com. Examination will probably be published next week. All these terms, possibility and probability convey the same meaning. But in statistics probability has a certain special connotation, unlike in the layman's view. Probability theory had its start in the 17th century, when two French mathematicians, **Blaise Pascal and Pierre de Fermat** carried out a correspondence discussing mathematical problems dealing with games of chances. In 1954 **Antoine Gornband** took an interest in this area. Thereafter many experts in statistics tried to develop the idea by their own expertise. The theory of probability was initially restricted to the games of chances by tossing coins, throwing dice, etc. Its importance has enormously increased in recent years. Today it is applied in almost all disciplines – physics, chemistry, biology, psychology, economics, education, business, industry, engineering, etc. The theory of probability is the fundamental basis of the subject of statistics. Sometimes statistical analysis becomes paralyzed without the theorem of probability.

To quote **Levin,** 'We live in a world in which we are unable to forecast the future with completed certainty. Our need to cope with uncertainty leads us to the study and use of probability theory. In fact the role played by probability in modern science is that of a substitute for certainty.'

10.2 MEANING OF PROBABILITY

Garrett has defined the term probability as "Probability of a given event is defined as the expected frequency of occurrence of the event among events like sort." The general meaning of the word probability is likelihood (chance). The likelihood of occurrence of the events of an experiment are different i.e. in a certain experiment the degree of occurrence varies from event to event. The probability theory provides a means of getting an idea of the likelihood of occurrence of different events resulting from a random experiment in terms of quantitative measures ranging between zero and one, i.e. zero for an event which cannot occur (i.e. impossible event) and one for an event certain to occur.

For example:

The probability that Mr. A will die one day is equal to unity, as it is absolutely certain that he will die someday. Symbolically, $p = 1$ [p denotes the probability]. The probability that the sky will fall is 0, as the sky will never fall, it is an impossible event, symbolically, $p = 0$.

But in actual practice, there are many cases which lie between absolute certainty and absolute impossibility, i.e. they have probability between 1 and 0, and by convention it is expressed in decimals.

10.3 IMPORTANCE OF PROBABILITY

The concept of probability is of great importance in our day-to-day life.

(*i*) The probability theory is very helpful for making predictions. In research investigations, estimations and predictions play an important role. With the help of statistical methods, estimations can be made for further analyses.

(*ii*) It is an important decision-making tool for the businessmen, policymakers, Government officials and other organisations.

(*iii*) It is concerned with the planning and controlling and with the occurrence of accidents of all kinds.

(*iv*) The concept of probability is not only applied in business and commercial lines, but it is also applied to all scientific investigation and everyday life.

(*v*) It is one of the inseparable tools for all types of formal studies that involve uncertainty.

10.4 SOME IMPORTANT TERMS USED IN PROBABILITY THEORY

Before setting the definitions of probability we must be acquainted with certain terms which are frequently used in the probability theory.

(*i*) Random experiment: An experiment is a planned operation carried out under controlled conditions. If the result is not predetermined and depend on chance, then the experiment is said to be a random experiment.

For example: When a coin in tossed, either the head or the tail will fall upwards when it lands. But the result of any toss cannot be predicted in advance, and is said to 'depend on chance'. Thus, 'tossing a coin' is a random experiment. Similarly, 'throwing a dice' and 'drawing a card' from a well-shuffled pack of 52 cards are also random experiments.

(*ii*) Trial: Any particular performance of a random experiment is known as a trial.

For example: Each roll of a dice or toss of a coin is a trial.

(*iii*) Outcome: An outcome is the result of a single trial of an experiment.

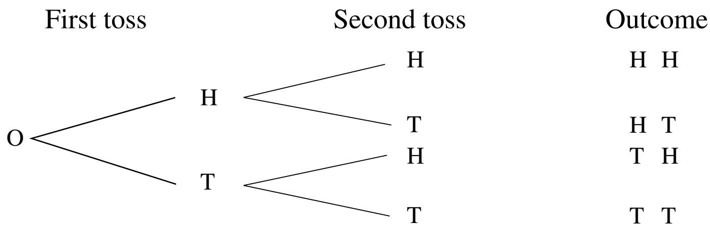
For example: If a fair coin is tossed, when the coin comes to rest, it can show a tail or a head, each of which is an outcome.

(*iv*) Sample space: The sample space of a random experiment is the set of all possible outcomes. Three ways to represent a sample space are: (*a*) to list the possible outcomes, (*b*) to create a tree diagram, or (*c*) to create a Venn diagram. The uppercase letter S or Ω is used to denote sample space.

For example: In the random experiment of tossing two coins at the same time.

(*i*) there are four possible outcomes – HH, HT, TH and TT. In sample space it is represented by $S = \{HH, HT, TH, TT\}$, where H = head and T = tail. Each outcome is known as a sample point.

(*ii*) **Using tree diagram**



(*iii*) **Using Venn diagram**

Let A = tails on the first coin

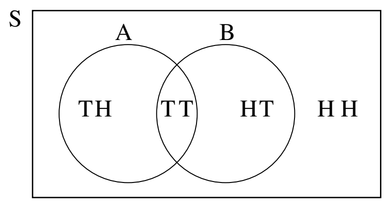
B = tails on the second coin

Then A = {TT, TH} and B = {TT}

A or B = {TH, TT, HT}

The outcome HH is in neither A nor B.

The Venn diagram is as follows:



(*v*) **Event:** A set of one or more outcomes of a random experiment is called an event. Therefore, an event is always a subset of the corresponding sample-space. Uppercase letters like A and B represent events. Events are of two types – (i) Elementary (simple) event and (ii) compound (composite) event.

- ***Elementary event:*** An event consisting of a single outcome is known as elementary event.
- ***Compound event:*** An event consisting of more than a single possible outcome, i.e. two or more outcomes is known as compound event.

For example: In the experiment of tossing two coins, the event both heads is an elementary event, as it corresponds a single outcome (HH) and so also the event 'both tails'; but the event 'one head and one tail' is a compound event consisting of two outcomes (HT) and (TH).

In throwing two dice the event of getting 'a sum of 2' is an elementary event as it corresponds to a single outcome (1, 1); but the event of getting 'a sum of 8' is a compound event as it corresponds to 5 outcomes (2, 6), (3, 5), (4, 4), (5, 3) and (6, 2).

(*vi*) **Mutually exclusive events:** Events are said to be mutually exclusive when they do not occur simultaneously. Among the events, if one event will remain present in a trial, the other events will not take place. In other words, occurrence of one precludes the occurrence of all the others.

For example: If a girl is beautiful, she cannot be ugly. If we take another events live dead and alive, it can be said that a person may be either alive or dead at a point of time. But lie cannot be both alive and dead simultaneously. If a coin is tossed either the head will appear or tail will appear. But both cannot appear at the same time. It refers that in tossing a coin the occurrence of head and tail comes under mutually exclusive events.

For two mutually exclusive events A and B, $A \cap B = \Phi$(Phi).

(*vii*) Mutually exhaustive events: Events are said to be mutually exhaustive if at least one of them must necessarily occur.

For example: In case of tossing a coin the two events 'head' and 'tail' are mutually exhaustive as one of the events must necessarily occur.

In case of throwing a dice the events 'odd number' and 'even number' are mutually exhaustive, as in the throw of a dice either odd or even number must necessarily occur.

(*viii*) Equally likely events: Events are said be equally likely, when there is equal chance of occurring, i.e. one event does not occur more often than the others.

For example: If an *unbiased* coin or dice is thrown, each face may be expected to occur in equal numbers in the long run. If a coin or dice is *biased* then each face is not expected to appear equally.

(*ix*) Complementary events: The event 'E' and the event 'not E' are called complementary events of the latter event. If C occurs, its complement is E which does not occur.

Complement of an event E is denoted by E' or $\bar{E}$ or E^c.

For example: When a coin is tossed, getting 'head' and getting 'tail' are complementary events of each other.

When a dice is thrown, getting 'even face' and 'odd face' are complementary events of each other.

(*x*) Independent and Dependent Events: Two or more events are said to be independent when the occurrence of one trial does not affect the other. It indicates the fact that if trials are made one by one, one trial is not affected by the other trial. One trial never describes anything about the other trials.

For Example: If a coin is tossed several times, then one trial is not affected by the other. The first trial never describes anything what event will come in the second trial. So the second trial is completely independent to that of the first trial.

Two or more events are said to be dependent when the occurrence and non-occurrence of one event in a trial may affect the occurrence of the other trials.

For Example: If a card is drawn from a pack of playing cards and is not replaced, then in the second trial probability will be affected.

10.5 DIFFERENT DEFINITIONS OF PROBABILITY

There are three important definitions of probability:

(*i*) Classical (Mathematical or Priori) definition
(*ii*) Frequency (Statistical or empirical) definition
(*iii*) Axiomatic (Set theoretic) definition

10.5.1 Classical Definition

If in a random experiment there are n (finite) mutually exclusive, exhaustive and equally likely outcomes, out of which m of them are favourable to a particular event A, then the probability of occurrence of the event A is denoted by P(A) and is defined as $P(A) = \frac{m}{n}$.

i.e. $$\text{Probability of an event} = \frac{\text{Number of outcomes favourable to the event}}{\text{Total number of mutually exclusive, exhaustive and equally likely outcomes of the random experiment}}$$

This definition can be applied in a situation in which all possible outcomes and the favourable outcomes in the event A can be counted. It is one of the oldest and simplest definition of probability. It originated in 18^{th} century and has been given by a French mathematician named "**Laplace**". The classical definition is also called the priori definition of probability. The word priori is from prior, and is used because the definition is based on the previous knowledge that the outcomes are equally likely.

- **Some deductions**

(*a*) If A be a certain event, then P(A) = 1
(*b*) If A be an impossible event, then P(A) = 0
(*c*) Since both the number of favourable and total outcomes are always positive, so $P(A) \geq 0$
(*d*) As number of favourable outcomes cannot exceed the total number of outcomes, so $0 \leq m \leq n$, or $0 \leq \frac{m}{n} \leq 1$ or $0 \leq P(A) \leq 1$.
Thus, the probability of any event lies between 0 and 1.
(*e*) If A′ denotes not occurring of A, then P(A) + P(A′) = 1, i.e. P(A′) = 1 – P(A). If P(A) = p and P(A′) = q, then $p + q = 1$.

- **Limitations of classical definition**

(1) Classical definition is only confined with games of chance, i.e. tossing coin, throwing dice, drawing cards, etc.
(2) This definition is not applicable if the outcomes are not mutually exclusive, exhaustive and equally likely.

(3) The definition fails when the total number of possible outcomes is very large or infinite.
(4) There is a fallacy within the definition, that before setting up the definition of probability how could we know that the events are equally probable, i.e. the probability of their occurrence are equal.

ILLUSTRATION 1

Throw a dice. What is the probability that an even number will appear?

Solution: The six numbers on a dice are 1, 2, 3, 4, 5, 6

Therefore, total number of outcomes = 6

On the dice, 2, 4 and 6 are the even numbers.

Therefore, number of favourable outcomes = 3

So, the probability that an even number will fall = $\frac{3}{6} = \frac{1}{2}$

ILLUSTRATION 2

In a bag, there are 3 blue and 4 red marbles. Take out any two marbles out of the bag. What is the probability that the two marbles are blue-coloured?

Solution: The bag contains 7 marbles (3 blue and 4 red marbles), 2 marbles out of 7 marbles can be taken out in ${}^7C_2$ ways, i.e. 21 ways.

(Recall that r things can be selected out of n different things in ${}^nC_r$ ways from combinations).

Number of blue coloured marbles in the bag = 3

2 blue out of 3 blue marbles can be taken out in ${}^3C_2$ ways, i.e. 3 ways.

Therefore, the probability = $\frac{3}{21} = \frac{1}{7}$.

10.5.2 Frequency Definition

If a random experiment is repeated a large number of times, say n times, under identical conditions and if an event A is observed to occur m times, then the probability of the event A is defined as the limit of the relative frequency $\frac{m}{n}$ as n tends to infinitely. Symbolically, we write

$$P(A) = \underset{n\to\infty}{\text{Limit}} \frac{m}{n}$$

The definition assumes that as n increases indefinitely, the ratio $\frac{m}{n}$ tends to become stable at the numerical value P(A). Theoretically, probability can never be obtained from the above limit. However, in practice a close estimate of P(A) based on a large number of observations can be obtained. For practical convenience, the estimate of P(A) can be written as if it were actually P(A), and the relative frequency definition of probability way be expressed as: $P(A) = \frac{m}{n}$.

As its name suggests, the relative frequency definition relates to the relative frequency with which an event occurs in the long run. In situations where we can say that an experiment has been repeated a very large number of times, the relative frequency definition can be applied.

As such, this definition is very useful in those practical situations where the classical definition cannot be applied. This type of probability is also called empirical probability as it is based on empirical evidence, i.e. on observational data. It can also be called statistical probability for it is this very probability that forms the basis of mathematical statistics.

Let us try to understand this concept by means of an experiment, that is by tossing a coin.

No one can tell which way a coin will fall but we expect the proportion of heads and tails after a large no. of tosses to be nearly equal. An experiment to demonstrate this point was performed by **Kerrich** in Denmark in 1946. He tossed a coin 10,000 times, and obtained altogether 5067 heads and 4933 tails. The behavior of the proportion of heads throughout the experiment is shown as in the following figure:

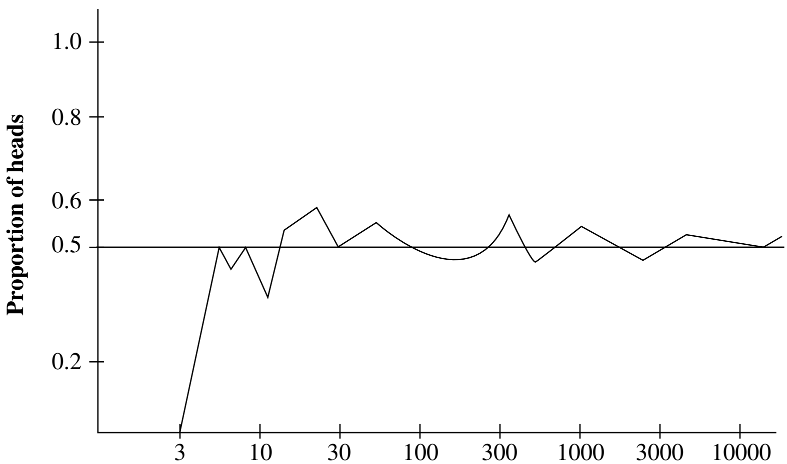


Figure 10.1 Number of Tosses

As you can see, the curve fluctuates widely at first, but begins to settle down to a more or less stable value as the number of spins increases. It seems reasonable to suppose that the fluctuations would continue to diminish if the experiment were continued indefinitely, and the proportion of heads would cluster more and more closely about a limiting value which would be very near, if not exactly, one-half. This hypothetical limiting value is the (Statistical) probability of heads.

- **Limitations of the frequency definition**

(*i*) In this definition for calculation of probability a large number of experiments are to be performed under identical conditions.

(*ii*) The limiting value is only a mathematical concept, this definition can never be obtained in practice.

(*iii*) The experimental conditions may not remain the same when a random experiment is repeated a large number of times and the relative frequency $\frac{m}{n}$ may not attain a unique limiting value when $n \to \infty$.

10.5.3 Axiomatic Definition

The Russian mathematician **Andrei N. Kolmogrov,** proposed a definition of probability in 1933 based on a set of axioms, which is the one that we keep on using nowadays.

If S be a sample space with sample points $A_1, A_2 \ldots A_n$. To each sample point, we assign a real number, denoted by the symbol $P(A_i)$, and called the probability of A_i, that must satisfy the following basic axioms;

Axiom 1: The probability of any event A_i is positive or zero. Namely $P(A_i) \geq 0$.

Axiom 2: The probability of the sure event is 1. Namely $P(S) = 1$.

Axiom 3: The probability of the union of any set of two by two incompatible events is the sum of the probabilities of the events. That is, if we have, for example, events A, B and these are two by two incompatible, then $P(A \cup B) = P(A) + P(B)$

Similarly, extending the result to n mutually exclusive events x_1, x_2, x_3, x_y and so on,

$$P(x_1 \cup x_2 \cup x_3 \cup x_y \cup \ldots) = P(x_1) + P(x_2) + P(x_3) + P(x_4) + \ldots$$

NOTE

In mathematics, an axiom is a result that is accepted without the need for proof. In this case, we say that this is the axiomatic definition of probability because we define probability as a function that satisfies these three axioms. Also, we might have chosen different axioms, and then probability would be another thing.

ILLUSTRATION 3

In a presidential election, there are four candidates A, B, C and D. Based on polling analysis, it is estimated that A has a 20 per cent chance of winning the election, while B has a 40 per cent chance of winning. What is the probability that A or B wins the election?

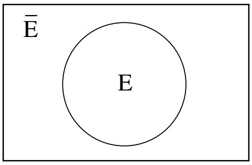
Solution: Here the events {A wins}, {B wins}, {C wins} and {D wins} are disjoint since more than one of them cannot occur at the same time. For example, if A wins then B cannot win. From the third axiom of probability, the probability of the union of two disjoint events is the summation of individual probabilities. Therefore,

$$\begin{aligned} \text{P(A wins or B wins)} &= \text{P(\{A wins\}} \cup \text{\{B wins\})} \\ &= \text{P(\{A wins\}} + \text{P\{B wins\})} \\ &= 0.2 + 0.4 = 0.6 \end{aligned}$$

10.6 PROBABILITY OF EVENT 'NOT E'

If E is any event, then P(not E) = 1 – P(E)

Proof: Let S be the sample space of a random experiment and E be any event associated with the random experiment, then 'not E' (also denoted by $\bar{E}$, E' or E^c) is also an event such that $E \cap \bar{E} = \Phi$ i.e., E and $\bar{E}$ are mutually disjoint and $E \cup \bar{E} = S$



Now, $P(E \cup \bar{E}) = P(E) + P(\bar{E})$ and $P(E \cup \bar{E}) = P(S) = 1$

or $P(E) + P(\bar{E}) = 1$

or $P(\bar{E}) = 1 - P(E)$

or P(not E) = 1 – P(E)

10.7 ODDS IN FAVOUR OF AND ODDS AGAINST AN EVENT

Let S be the sample space of a random experiment and all the outcomes are equally likely. If an event E can happen in 'a' ways and fail in 'b' ways, then odds in favour of E are $a : b$, odds against E are $b : a$, and probability of E is $P(E) = \dfrac{a}{a+b}$

If the probability of event E is given to be p, then odds in favour of event E = $\dfrac{p}{1-p}$, i.e. $p : (1-p)$

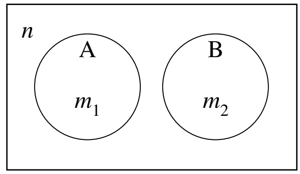
and odds against event E = $\dfrac{1-p}{p}$, i.e. $(1-p) : p$

10.8 THEOREM OF TOTAL PROBABILITY

Theorem 1 If A and B be two mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

i.e., the probability that either A or B occurs is the sum of the probabilities of the events A and B. [***B.U. B.Com. (H) 2007***]

Proof: Since the events A and B are mutually exclusive, they can be represented by two disjoint sets.



Let there be n mutually exclusive, exhaustive and equally likely outcomes, out of which m_1 and m_2 outcomes are favourable to the event A and B respectively. So $P(A) = \frac{m_1}{n}$, $P(B) = \frac{m_2}{n}$

As by notation $A \cup B$ means the occurrence of the event either A or B or both, so the favourable outcomes to the event $A \cup B$ is $m_1 + m_2$.

Hence, $P(A \cup B)$ = probability that either A or B occurs

$$= \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B)$$

i.e., $P(A \cup B) = P(A) + P(B)$

The above theorem can be extended to any number of mutually exclusive events.

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots P(A_n)$$

Corollary 1: If the events $A_1, A_2, \ldots A_n$ are mutually exclusive and exhaustive, then

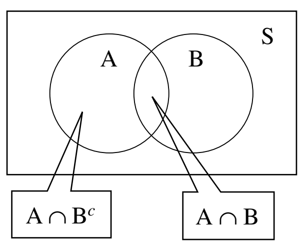
$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \ldots P(A_n) = 1$$

Corollary 2: The event A and its complement, i.e. A and A^c are exhaustive and mutually exclusive.

$$P(A \cup A^c) = 1 \text{ or } P(A) + P(A^c) = 1 \text{ or } P(A) = 1 - P(A^c) \text{ or } P(A^c) = 1 - P(A).$$

Corollary 3. For any two events A and B, $A \cap B^c$ and $A \cap B$ are mutually exclusive.

Also we have,



$$A = (A \cap B^c) \cup (A \cap B)$$

or $P(A) = P(A \cap B^c) + P(A \cap B)$

or $P(A \cap B^c) = P(A) - P(A \cap B)$

Theorem 2 If A and B be any two events (not necessarily mutually exclusive), then $P(A \cup B)$, i.e., the probability that at least one of the two events A and B occurs is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

[***C.U. B.Com(H) '94, '99, 2002; B.U. B.Com(H) 2008; V.U. B.Com.(H) 2008***]

Proof: Let $n(\text{S})$ be the total number of sample points in a finite sample space. S of a random experiment and $n(\text{A} \cup \text{B})$ the number of sample points in $(\text{A} \cup \text{B})$. Hence by the definition of probability.

$$\text{P}(\text{A} \cup \text{B}) = \frac{n(\text{A} \cup \text{B})}{n(\text{S})} = \frac{n(\text{A}) + n(\text{B}) - n(\text{A} \cap \text{B})}{n(\text{S})}$$

[From set theory we know $n(\text{A} \cup \text{B}) = n(\text{A}) + n(\text{B}) - n(\text{A} \cap \text{B})$]

$$= \frac{n(\text{A})}{n(\text{S})} + \frac{n(\text{B})}{n(\text{S})} - \frac{n(\text{A} \cap \text{B})}{n(\text{S})} = \text{P}(\text{A}) + \text{P}(\text{B}) - \text{P}(\text{A} \cap \text{B})$$

Corollary: For there events A, B, C not necessaring mutnally execlusive, the poobability of occurrence of at least one of them is given by

$$\text{P}(\text{A} \cup \text{B} \cup \text{C}) = \text{P}(\text{A}) + \text{P}(\text{B}) + \text{P}(\text{C}) - \text{P}(\text{A} \cap \text{B}) - \text{P}(\text{A} \cap \text{C}) - \text{P}(\text{B} \cap \text{C}) + \text{P}(\text{A} \cap \text{B} \cap \text{C})$$

Proof: $\text{P}(\text{A} \cup \text{B} \cup \text{C}) = \text{P}[(\text{A} \cup \text{B}) \cup \text{C}] = \text{P}(\text{A} \cup \text{B}) + \text{P}(\text{C}) - \text{P}[(\text{A} \cup \text{B}) \cap \text{C}]$

$= [\text{P}(\text{A}) + \text{P}(\text{B}) - \text{P}(\text{A} \cap \text{B})] + \text{P}(\text{C}) - \text{P}[(\text{A} \cap \text{C}) \cup (\text{B} \cap \text{C})]$

[By distributive law]

$= \text{P}(\text{A}) + \text{P}(\text{B}) - \text{P}(\text{A} \cap \text{B}) + \text{P}(\text{C}) - [\text{P}(\text{A} \cap \text{C}) + \text{P}(\text{B} \cap \text{C}) - \text{P}\{(\text{A} \cap \text{C}) \cap (\text{B} \cap \text{C})\}]$

$= \text{P}(\text{A}) + \text{P}(\text{B}) - \text{P}(\text{A} \cap \text{B}) + \text{P}(\text{C}) - \text{P}(\text{A} \cap \text{C}) - \text{P}(\text{B} \cap \text{C}) + \text{P}(\text{A} \cap \text{B} \cap \text{C})$

$= \text{P}(\text{A}) + \text{P}(\text{B}) + \text{P}(\text{C}) - \text{P}(\text{A} \cap \text{B}) - \text{P}(\text{A} \cap \text{C}) - \text{P}(\text{B} \cap \text{C}) + \text{P}(\text{A} \cap \text{B} \cap \text{C})$

ILLUSTRATION 4

From a set of 17 balls marked 1, 2, 3, …, 16, 17 one is drawn at random. (a) What is the chance that its number is a multiple of 3 or of 7? (b) What is the probability that its number is an even number greater than 9?

[***C.U. B.Com (H) 2000***]

Solution: Let S be the required sample space then S = {1, 2, 3, …, 16, 17} therefore, $n(\text{S}) = 17$

(*a*) Let A be the event that the number of the ball is a multiple of 3 and B be the event that the number of the ball is a multiple of 7
then A = {3, 6, 9, 12, 15} therefore, $n(\text{A}) = 5$
and B = {7, 14} therefore, $n(\text{B}) = 2$
Also $\text{A} \cap \text{B} = \Phi$
Therefore, $\text{P}(\text{A}) = \frac{n(\text{A})}{n(\text{S})} = \frac{5}{17}$ and $\text{P}(\text{B}) = \frac{n(\text{B})}{n(\text{S})} = \frac{2}{17}$
Since A and B are mutually exclusive events, by the theorem of total probability, we have

$$P(A \cup B) = P(A) + P(B) = \frac{5}{17} + \frac{2}{17} = \frac{7}{17}$$

Hence, the required probability that its number is a multiple of 3 or of 7 is $P(A \cup B) = \frac{7}{17}$

(*b*) Let C be the event that the number of the ball is an even number greater than 9.

then C = {10, 12, 14, 16}, therefore, $n(C) = 4$

Hence, the required probability that its number is an even number greater than 9 = $\frac{4}{17}$

ILLUSTRATION 5

A card is drawn from a well-shuffled pack of 52 cards. What is the probability of the card being either black or an ace?

Solution: A well-shuffled pack contains 52 cards, i.e. 52 sample points.

Let S be the required sample space

then $n(S) = 52$

Let A be the event that the card is black and B be the event that the card is an ace.

Since a pack contains 26 black cards and 4 ace cards out of which 2 black aces.

Then, $n(A) = 26$, $n(B) = 4$ and $n(A \cap B) = 2$

Therefore, $P(A) = \frac{n(A)}{n(S)} = \frac{26}{52}$, $P(B) = \frac{n(B)}{n(S)} = \frac{4}{52}$

and $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{2}{52}$

Hence, the probability that the card drawn is either black or an ace is $P(A \cup B)$. Since the events A and B are not mutually exclusive, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{58} = \frac{7}{13}$$

10.9 CONDITIONAL PROBABILITY

Of two events A and B, if A occurs when B has already occurred, then the probability of A with respect to B is termed as **conditional probability** of A with respect to B. It is denoted by P(A/B) and defined by $P(A/B) = \frac{P(A \cap B)}{P(B)}$ provided $P(B) > 0$.

Similarly, by definition $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$ provided $P(A) > 0$

Hence, $P(A \cap B) = P(A) \cdot P(B/A)$

This relation is known as multiplication Theorem of Probability. The vertical bar '/' is read as 'given' or such that.

ILLUSTRATION 6

Soumi took two tests. The probability of her passing both tests is 0.6. The probability of her passing the first test is 0.8. What is the probability of her passing the second test given that she has passed the first test?

Solution: Let T_1 be the event of passing first test and T_2 be the event of passing second test

Given that, $P(T_1 \cap T_2) = 0.6$ and $P(T_1) = 0.8$

We are to calculate $P(T_2/T_1)$, the probability of passing the second test given that first test has already passed.

From the definition of conditional probability, we have

$$P(T_2 / T_1) = \frac{P(T_1 \cap T_2)}{P(T_1)} = \frac{0.6}{0.8} = 0.75$$

10.10 COMPOUND PROBABILITY

Probability of occurrence of any two events A and B simultaneously is the product of the unconditional probability of the occurrence of B and the probability of occurrence of the event A under the condition that B has already occurred. Symbolically,

$$P(A \cap B) = P(B) \cdot P(A/B)$$

Proof: Let there be n mutually exclusive, exhaustive, and equally likely events, out of which m cases are favourable to the event B. Out of these m cases, the event A has occurred m_1 times.

Therefore, $P(A \cap B) = \dfrac{m_1}{n}$, $P(B) = \dfrac{m}{n}$, $P(A/B) = \dfrac{m_1}{m}$

Thus, $P(A \cap B) = \dfrac{m_1}{n} = \dfrac{m}{n} \times \dfrac{m_1}{m} = P(B) \cdot P(A/B)$

Corollaries

1. $P(A \cap B) = P(A) \cdot P(B/A)$
2. $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$ or $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$

3. $P(A \cap B \cap C) = P(A \cap B). P(C/A \cap B)$
$= P(A) \cdot P(B/A) \cdot P(C/A \cap B)$

ILLUSTRATION 7

An urn has 6 black and 4 white marbles. Two of them are drawn without replacement. Find the probability of getting a black and a white marble.

Solution: Let S be the sample space. Then $n(S) = 10$

x be the event of drawing a black marble, $n(x) = 6$

y be the event of drawing a white marble, $n(y) = 4$

Now, $P(x) = \frac{6}{10} = \frac{3}{5}$, $P(y) = \frac{4}{10} = \frac{2}{5}$

$P(x/y)$ = Probability of getting white marble after drawing black = $\frac{4}{9}$

Since the events are not dependent, we have

$$P(x \cap y) = P(x). P(y/x) = \frac{3}{5} \times \frac{4}{9} = \frac{12}{45} = \frac{4}{15}$$

10.11 PROBABILITY OF INDEPENDENT EVENTS

In probability theory, the independent events are very common. As the name suggests, the independent events refer to those two or more events which do not depend on one another.

• Definition

Two events A and B in a probability space S are defined to be independent events, if the occurrence of one of them does not influence the occurrence of the other. Such two events may occur in two different experiments conducted simultaneously or the same event in dependent trials of an experiment where the outcome of one of the trials is not influenced by the outcomes of other trials. Two events, A and B are independent if the probability of A and B, that is, the probability of their intersection is equal to the product of their separate probability.

So, $P(A \cap B) = P(A) \cdot P(B)$

This result can be generalized in the following form:

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1) \cdot P(A_2) \cdot P(A_3) \ldots P(A_n)$$

• Union of Independent Events

If A and B are two independent events, then

$$P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$$

- **Conditional Probability of Independent Events**

If A and B are independent events, then the conditional probability P(B/A) is the same as P(B) as the occurrence of event A does not influence the occurrence of B.

If A and B are independent events, then P(A/B) = P(A) and P(B/A) = P(B)

10.12 COMPOUND INDEPENDENT EVENTS

An event has a one or more possible outcome from an experiment. An event consisting of more than one events is called a compound event. For example the combined trial of rolling a dice and tossing a coin, find the probability of getting a two/head outcome is said to be a compound event. In this event, there are two results for the desired outcome. Also, the probability of getting a two as outcome is completely independent of probability of getting a head. Such compound events are referred to as compound independent events.

The type of compound events having results which show no dependency on each other are known as independent compound events. On the other hand, if compound events have dependent results, they are said to be conditional.

While calculating the probability of compound events, it is necessary to determine whether the events are independent or not.

ILLUSTRATION 8

A coin is tossed and a six-sided dice is rolled. Find the probability of getting a head on the coin and a 6 on the dice.

Solution: These two events (the coin and dice) are independent events because the flipping of the coin does not affect the rolling of the dice. The events are independent of each other.

Let's find the probability of each independent event.

P(Heads) = $\frac{1}{2}$ [As there is only one head on a coin and there are two total outcomes (heads or tails)]

P(6) = $\frac{1}{6}$ [As there is only one 6 on a dice and there are 6 total outcomes on a dice (1, 2, 3, 4, 5, 6)]

Now we need to find the probability of tossing a head on the coin and rolling a 6 on the dice. So, we need to combine both events. Now, let's apply the rule of independent events.

$$P(\text{heads and a 6}) = P(\text{heads}) \times P(6) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

Therefore, the probability of flipping a head on the coin and rolling a 6 on the dice is $\frac{1}{12}$.

Corollary: $P(A \cap B) = P(A). P(B)$ is accepted as the mathematical definition of two independent events A and B.

Let	$P(A \cap B) = P(A) \cdot P(B)$
Again,	$P(A \cap B) = P(A) \cdot P(B/A)$
or	$P(A) \cdot P(B) = P(A) \cdot P(B/A)$
Therefore,	$P(B) = P(B/A)$

which shows that A and B are independent. So, A and B are said to be independent if and only if $P(A \cap B) = P(A). P(B)$.

- **Difference between mutually exclusive events and independent events**

For two mutually exclusive events, the occurrence of one guarantees the non-occurrence of the other. But for two independent events, the occurrence of one does not guarantee the occurrence or the non-occurrence of the other.

10.13 BAYES' THEOREM

The concept of conditional probability can be extended to "revise" probabilities based on new information and to determine the probability that a particular effect was due to a specific cause. The procedure for revising these probabilities is known as Bayes' theorem. This rule is named after the British mathematician Rev. Thomas Bayes (1701 – 1761). As per this rule –

If there be n mutually exclusive and exhaustive events $A_1, A_2, \ldots A_n$ and $P(A_1), P(A_2), \ldots P(A_n)$ are known and for a particular event A, which can occur if any one of A_i occurs, and $P(B/A_1), P(B/A_2), \ldots, P(B/A_n)$ are also known, then

$$P(A_i/B) = \frac{P(A_i) \cdot P(B / A_i)}{\sum_{i=1}^{n} P(A_i) \cdot P(B / A_i)}$$

where $P(B_1/A)$, $i = 1, 2, \ldots, n$ are the probabilities of B under the condition that A_i has actually occurred.

Proof: By the law of conditional probability

$$P(B/A_i) = \frac{P(A_i \cap B)}{P(A_i)} \text{or, } P(A_i \cap B) = P(A_i) \cdot P(B/A_i)$$

Similarly, $P(B \cap A_i) = P(B) \cdot P(A_i/B)$

Now since $P(A_i \cap B) = P(B \cap A_i)$, it follows that

$$P(A_i) \cdot P(B/A_i) = P(B) \cdot P(A_i/B)$$

or

$$P(A_i/B) = \frac{P(A_i) \cdot P(B / A_i)}{P(B)} \quad \ldots(1)$$

Since the events $A_1, A_2, \ldots, A_n$ are mutually exclusive and $P(B) \neq 0$

Therefore, $P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$

$$= P(A_1) \cdot P(B/A_1) + P(A_2). P(B/A_2) + \cdots + P(A_n) \cdot P(B/A_n)$$

$$= \sum_{i=1}^{n} P(A_i) \cdot P(B / A_i)$$

From (1), $P(A_i/B) = \dfrac{P(A_i) \cdot P(B / A_i)}{\sum_{i=1}^{n} P(A_i) \cdot P(B / A_i)}$

ILLUSTRATION 9

Two urns A and B contain 6 white and 4 black balls; 5 white and 7 black balls respectively. If two balls are transferred from A to B, finally one black ball is drawn from the 2nd urn B, find the probability that the balls transferred from the box A were one white and one black.

Solution: Let us assume the following events:

B_1 = transferred balls are both white

B_2 = transferred balls are one white and one black

B_3 = transferred balls are both black

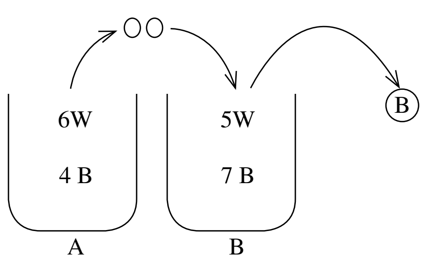
E = ultimately ball drawn from the urn B is black.

It is required to find the probability $P(B_2/E)$

Here, $P(B_1) = \dfrac{^6C_2}{^{10}C_2} = \dfrac{15}{45} = \dfrac{1}{3}$

$P(B_2) = \dfrac{^6C_1 \times {^4C_1}}{^{10}C_2} = \dfrac{24}{45} = \dfrac{8}{15}$

$P(B_3) = \dfrac{^4C_2}{^{10}C_2} = \dfrac{6}{45} = \dfrac{2}{15}$



Again, $P(E/B_1)$ = Probability that the ultimate draw from the urn B is black assuming the transferred balls are both white $= \dfrac{7}{14} = \dfrac{1}{2}$

$P(E/B_2)$ = Probability that the ultimate draw from the urn B is black assuming the transferred balls are one white and one black $= \dfrac{8}{14} = \dfrac{4}{7}$

$P(E/B_3)$ = Probability that the ultimate draw from the urn B is black assuming the transferred balls are both black $= \dfrac{9}{14}$

So by Bayes' theorem, we have

$$P(B_2 / E) = \frac{P(B_2) \cdot P(E / B_2)}{P(B_1) \cdot P(E / B_1) + P(B_2) \cdot P(E / B_2) + P(B_3) \cdot P(E / B_3)}$$

$$= \frac{\frac{8}{15} \cdot \frac{4}{7}}{\frac{1}{3} \cdot \frac{1}{2} + \frac{8}{15} \cdot \frac{4}{7} + \frac{2}{15} \cdot \frac{9}{14}} = \frac{64}{117}.$$

ILLUSTRATIVE EXAMPLES

A. SHORT TYPE

EXAMPLE 1

Two coins are tossed together. Describe the sample space.

Solution: Each coin may come up heads (H) or tails (T). To describe the outcome of two coins, we use ordered pairs. Thus the sample space is

$$S = \{HH, HT, TH, TT\}$$

EXAMPLE 2

From a group of 2 boys and 3 girls, two children are selected. Find the sample space associated to this random experiment.

Solution: Let the two boys be taken as B_1 and B_2 and three girls be taken as G_1, G_2 and G_3. Clearly, there are 5 children, out of which two children can be chosen in ${}^5C_2$ ways. So, there are ${}^5C_2 = 10$ elementary events associated to this experiment and are given by:

B_1B_2, B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3, G_1G_2, G_1G_3 and G_2G_3

Thus, the same space S associated to this random experiment is

$$S = \{B_1B_2, B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3, G_1G_2, G_1G_3, G_2G_3\}$$

EXAMPLE 3

A dice is rolled. Let D be the event 'dice shows 5' and E be the event 'dice shows odd number; Are the events D and E mutually exclusive?

Solution: When a dice is rolled, sample space S = {1, 2, 3, 4, 5, 6}

Event D = {5} and Event E {1, 3, 5}

Here: $D \cap E = \{5\} \neq \Phi$

i.e. events D and E are not mutually exclusive.

EXAMPLE 4

A pair of dice is rolled. What is the event that the sum is greater than 8.

Solution: The sum is greater than 8, i.e., the sum is 9, 10, 11 or 12 so the events associated to this experiments are

(3, 6), (4, 5), (5, 4), (6, 3), (4, 6), (5, 5), (6, 4), (5, 6), (6, 5), (6, 6)

EXAMPLE 5

A coin is tossed two times. Consider the following events:
A: 'no head appears', B: 'exactly one head appears'
Are the events A and B mutually exhaustive?

Solution: The sample space of the experiment is

S = {HH, HT, TH, TT}

A = {TT}, B = {HT, TH}

Now $A \cup B = \{HT, TH, TT\} \neq S$

So, A and B are not mutually exhaustive events.

EXAMPLE 6

Three coins are tossed once. Find the probability of getting at most two heads.

Solution: Let S be the sample space associated with the random experiment of tossing three coins. Then

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Clearly there are 8 elements in S.

Therefore, total number of elementary events = 8

At most two heads can be obtained in any one of the following ways:

HHT, THH, HTH, HTT, THT, TTH, TTT

Therefore, favourable number of elementary events = 7

Hence, required probability = $\frac{7}{8}$.

EXAMPLE 7

A dice is thrown. Find the probability of getting a prime number.

Solution: When a dice is rolled, sample space S = {1, 2, 3, 4, 5, 6} clearly, there are 6 elements in S.

Therefore, total number of elementary events = 6

A prime number in obtained, if we get any one of 2, 3, 5 as an outcome.

So, favourable number of elementary events = 3

Hence, required probability = $\frac{3}{6} = \frac{1}{2}$.

EXAMPLE 8

What is the probability that a number selected from the numbers 1, 2, 3, …, 25 is prime number, when each of the given numbers is equally likely to be selected?

Solution: Let S be the sample space and A be the event "selecting a prime number"

Then, S = {1, 2, 3, …, 25} and A = {2, 3, 5, 7, 11, 13, 17, 19, 23}

Therefore, total number of elementary events = 25 and, favourable number of elementary events = 9

Hence, required probability = $\frac{9}{25}$.

EXAMPLE 9

One card is drawn from a pack of 52 cards, each of the 52 cards being equally likely to be drawn. Find the probability that the card drawn is red and a king.

Solution: Out of 52 cards, one card can be drawn in ${}^{52}C_1 = 52$ ways

Therefore, total number of elementary events = 52

There are 2 cards which are red and king, i.e. red kings

Therefore, favourable number of elementary events = ${}^{2}C_1 = 2$

Hence, required probability = $\frac{2}{52} = \frac{1}{26}$.

EXAMPLE 10

A single letter is selected at random from the word 'PROBABILITY'. What is the probability that it is a vowel?

Solution: There are 11 letters in the word 'PROBABILITY', out of which 4 are vowels and 7 are consonants.

Therefore, total number of elementary events = 11

and favourable number of elementary events = 4

Hence, required probability = $\frac{4}{11}$.

EXAMPLE 11

Two dice are thrown simultaneously. Find the probability of getting a total of at least 4.

Solution: When two dice are thrown simultaneously, the total number of outcomes = 6 × 6 = 36. All the outcomes are equally likely. Let A be the event of getting a total of at least 4, i.e. total ≥ 4, then $\overline{A}$ denotes the events of getting a total of less than 4, i.e. a total of 2, 3.

Therefore, $\overline{A} = \{(1, 1), (1, 2), (2, 1)\}$

The number of outcomes favourable to the event $\overline{A} = 3$

Therefore, $P(\overline{A}) = \frac{3}{36} = \frac{1}{12}$

Then, $P(A) = 1 - P(\overline{A}) = 1 - \frac{1}{12} = \frac{11}{12}$.

EXAMPLE 12

A ball is drawn from a bag containing 3 white and 5 black balls. What are the odds against drawing a white ball?

Solution: Since there are 3 white out of a total of 8 balls, the probability of drawing a white ball is $\frac{3}{8}$

Therefore, the odds against drawing a white ball are (1 – *p*): p, i.e.

$$\left(1 - \frac{3}{8}\right) : \frac{3}{8} = \frac{5}{8} : \frac{3}{8} = 5 : 3$$

EXAMPLE 13

Given $P(A) = \frac{2}{5}$ and $P(B) = \frac{1}{5}$. Find $P(A \cup B)$, if A and B are mutually exclusive events.

Solution: Since A and B are mutually exclusive events

Therefore, $P(A \cup B) = P(A) + P(B) = \frac{2}{5} + \frac{1}{5} = \frac{3}{5}$

EXAMPLE 14

What is the probability of drawing either a shade or an ace from a pack of 52 cards? [***C.U. B.Com 2001***]

Solution: Let A be the event of drawing a spade and B be the event of drawing an ace.

There are 13 spade cards, 4 ace cards and one spade and ace card in a pack of 52 cards

Therefore, $P(A) = \frac{13}{52}$, $P(B) = \frac{4}{52}$ and $P(A \cap B) = \frac{1}{52}$

Thus the probability of drawing either a spade or an ace from a pack of 52 cards = $P(A \cup B)$

Therefore, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}.$$

EXAMPLE 15

A pair of fair dice is thrown. Find the probability that the sum is 10 or greater if 5 appears on the first dice. [***C.U. B.Com. 1985***]

Solution: When a pair of fair dice is thrown, then total number of outcomes = $6 \times 6 = 36$.

Let A be the event that the sum is 10 or greater if 5 appears on the first dice.

Therefore, A = {(5, 5), (5, 6)}

The number of outcomes favourable to the event A = 2

Therefore, $P(A) = \frac{2}{36} = \frac{1}{18}$.

EXAMPLE 16

Given: $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$, find P(A/B) and P(B/A). Are the two events A and B independent? [***C.U. B.Com. 1986***]

Solution: We know that,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

or $$\frac{3}{4} = \frac{3}{8} + \frac{5}{8} - P(A \cap B)$$

or $$P(A \cap B) = \frac{3}{8} + \frac{5}{8} - \frac{3}{4} = \frac{3+5-6}{8} = \frac{2}{8} = \frac{1}{4}$$

Therefore, $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{5}{8}} = \frac{1}{4} \times \frac{8}{5} = \frac{2}{5}$$

and $$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{3}{8}} = \frac{1}{4} \times \frac{8}{3} = \frac{2}{3}$$

Again, $$P(A).\,P(B) = \frac{3}{8} \times \frac{5}{8} = \frac{15}{64} \neq P(A \cap B)$$

So, the two events A and B are not independent.

EXAMPLE 17

Events E and F are such that P (not E or not F) = 0.25. State whether E and F are mutually exclusive.

Solution: We have, P(not E or not F) = 0.25

i.e. $P(\overline{E} \cup \overline{F}) = 0.25$ or $P(\overline{E \cap F}) = 0.25$

or $1 - P(E \cap F) = 0.25$ or $P(E \cap F) = 1 - 0.25 = 0.75 \neq 0$

Hence, E and F are not mutually exclusive.

EXAMPLE 18

What is the probability that a non-leap year, selected at random, will have 53 Tuesdays or Wednesdays?

Solution: A non-leap year has 365 days, 364 days make 52 weeks and 1 day is left. In each week, there is one Tuesday and one Wednesday.

So, we have to find the probability that the remaining day is Tuesday or Wednesday.

Out of 7 days, the number of outcomes favourable to the event 'Tuesday or Wednesday' is 2.

Therefore, required probability = $\frac{2}{7}$.

EXAMPLE 19

What is the probability that a randomly chosen 2-digit positive integer is a multiple of 3?

Solution: 2-digit positive integers are 10, 11, 12, …, 99.

Therefore, total number of 2-digit positive integers = 90

and the number of multiples of 3 = 12, 15, …, 99

= 30

Therefore, required probability = $\frac{30}{90} = \frac{1}{3}$.

EXAMPLE 20

A hockey match is played from 4 p.m. to 6 p.m. A man arrives late for the match. What is the probability that he misses the only goal of the match which is scored at the 30th minute of the match?

Solution: Total duration of the match = 2 hours = 120 minutes. The man sees the goal only if he arrives within first 30 minutes. Let A be the event that the man sees the goal.

So, $P(A) = \frac{30 \text{ minutes}}{120 \text{ minutes}} = \frac{1}{4}$.

Therefore, the probability that the man misses the goal = $P(\overline{A}) = 1 - P(A)$

$$= 1 - \frac{1}{4} = \frac{3}{4}.$$

B. SHORT ESSAY TYPE

EXAMPLE 21

Three fair coins are tossed. Construct the sample space of the outcomes of the random experiment. Find the probability of (i) at least one head (ii) exactly one tail. [***C.U. B.Com. 1985***]

Solution: When three fair coins are tossed the total number of outcomes are = $2^3 = 8$, which are as follows:

S = {HHH, HHT, HTH, HTT, TTH, THT, THH}

Favourable outcomes of at least one head = 7

{HHH, HHT, HTH, HTT, TTT, TTH, THT, THH}

Outcomes of exactly one tail = 3 {HHT, THH, HTH}

(*i*) Probability of getting at least one head = $\frac{7}{8}$

(*ii*) Probability of getting exactly one tail = $\frac{3}{8}$

EXAMPLE 22

Find the chance of throwing at least 8 in a single cast with two dice.

[***C.U. B.Com. 1991***]

Solution: When a single cast of two dice is occurred then total number of outcomes = $6^2 = 36$, which are as follows:

$$S = \begin{Bmatrix} (1, 1) & (2, 1) & (3, 1) & (4, 1) & (5, 1) & (6, 1) \\ (1, 2) & (2, 2) & (3, 2) & (4, 2) & (5, 2) & (6, 2) \\ (1, 3) & (2, 3) & (3, 3) & (4, 3) & (5, 3) & (6, 3) \\ (1, 4) & (2, 4) & (3, 4) & (4, 4) & (5, 4) & (6, 4) \\ (1, 5) & (2, 5) & (3, 5) & (4, 5) & (5, 5) & (6, 5) \\ (1, 6) & (2, 6) & (3, 6) & (4, 6) & (5, 6) & (6, 6) \end{Bmatrix}$$

Favourable outcomes of getting at least 8 = 15

{(2, 6), (3, 5), (3, 6), (4, 4), (4, 5), (4, 6), (5, 3), (5, 4), (5, 5), (5, 6), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

Therefore, required probability of setting at least 8 = $\frac{15}{36} = \frac{5}{12}$

EXAMPLE 23

A pair of unbiased dice is thrown. If the two numbers appearing be different, then find the probabilities that (i) the sum is six and (ii) the sum is 5 or less.

[***C.U. B.Com. 1995***]

Solution: If a pair of unbiased dice is thrown, then total number of outcomes = $6^2 = 36$, i.e. n(S) = 36.

Let A be the event that sum is six and B be the event that the sum is 5 or less, then

A = {(1, 5), (2, 4), (4, 2), (5, 1)}, therefore, n(A) = 4

B = {(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 1), (3, 2), (4, 1)}, therefore, n(B) = 8

(*i*) The probability that the sum is six = P(A) = $\dfrac{n(A)}{n(S)} = \dfrac{4}{36} = \dfrac{1}{9}$

(*ii*) The probability that the sum is 5 or less = P(B) = $\dfrac{n(B)}{n(S)} = \dfrac{8}{36} = \dfrac{2}{9}$

EXAMPLE 24

Find the probability that a leap year chosen at random has 53 Sundays.

[***C.U. B.Com. 1993***]

Solution: A leap year means 366 days or (52 × 7 + 2) days contains 52 complete weeks, i.e. 52 Sundays and 2 more consecutive days are as follows:

(Sunday, Monday), (Monday, Tuesday), (Tuesday, Wednesday), (Wednesday, Thursday), (Thursday, Friday), (Friday, Saturday), (Saturday, Sunday).

Therefore, total number of outcomes = 7

Out of which two pairs contain Sunday (Sunday, Monday) and (Saturday, Sunday).

Therefore, number of favourable outcomes = 2

Hence, the probability of getting 53 Sundays in a leap year = $\dfrac{2}{7}$.

EXAMPLE 25

Find the probability that a leap year, selected at random, will contain 53 Sundays or 53 Mondays?

Solution: In a leap year there are 366 days, i.e. 52 full weeks and 2 more consecutive days. In each week there is one Sunday and one Monday. Therefore, we have to find the probability of having a Sunday or a Monday out of the remaining 2 days.

Now the 2 days can be as follows;

(Sunday, Monday), (Monday, Tuesday), (Tuesday, Wednesday), (Wednesday, Thursday), (Thursday, Friday), (Friday, Saturday), (Saturday, Sunday).

In these 7 pairs, Sunday or Monday occurs 3 times (Sunday, Monday), (Monday, Tuesday), (Saturday, Sunday)

Let E be the event containing 'Sunday or Monday', then the number of outcomes favourable to the event E = 3

Therefore, $P(E) = \frac{3}{7}$.

EXAMPLE 26

From a pack of 52 cards, two are drawn at random; find the chance that one is a king and the other a queen.

Solution: From a pack of 52 cards a draw of 2 cards can be made in ${}^{52}C_2$ ways. A king can be drawn in ${}^4C_1$ ways and similarly a queen in ${}^4C_1$ ways. Since each of the former can be associated with each of the latter, a king and a queen can be drawn in ${}^4C_1 \times {}^4C_1$ ways.

Therefore, the required chance $= \dfrac{{}^4C_1 \times {}^4C_1}{{}^{52}C_2} = \dfrac{4 \times 4}{52 \times 51}$

$$= \frac{8}{663} = 0.12$$

EXAMPLE 27

A bag contains 4 white and 3 black balls. It 4 balls are drawn at random, what is the probability that 2 are white and 2 are black?

Solution: From a bag containing 4 white and 3 black balls or 7 balls, four balls are drawn at random.

Then total number of outcomes $= {}^7C_4 = \dfrac{7!}{4!3!} = 35$

Now, favourable outcomes of 2 white balls $= {}^4C_2 = \dfrac{4!}{2!2!} = 6$

Favourable outcomes of 2 black balls $= {}^3C_2 = \dfrac{3!}{2!1!} = 3$

Therefore, probability of getting 2 white and 2 black balls $= \dfrac{6 \times 3}{35} = \dfrac{18}{35}$.

EXAMPLE 28

In a lot of 20 articles, 5 are defective, 4 articles are drawn at random from the lot. Find the probability that exactly 2 of the drawn articles are defective.

[***C.U. B.Com 1998***]

Solution: Out of 20 articles, 5 are defective

Therefore, non-defective articles = 20 – 5 = 15.

4 articles are drawn at random from the lot

So, total number of ways $= {}^{20}C_4 = \dfrac{20!}{16!4!}$

$$= \frac{20 \times 19 \times 18 \times 17 \times 16!}{16! \times 4 \times 3 \times 2 \times 1} = 323$$

Out of 4 articles drawn, 2 are defective, i.e. remaining 2 are non-defective.

Favourable number of ways = ${}^{15}C_2 \times {}^5C_2$

$$= \frac{15!}{13!2!} \times \frac{5!}{2!3!}$$

$$= \frac{15 \times 14 \times 13!}{13! \times 2 \times 1} \times \frac{5 \times 4 \times 3!}{3! \times 2 \times 1} = 70$$

Hence, required probability = $\frac{70}{323} = 0.2167$

EXAMPLE 29

10 balls are distributed at random among three boxes. What is the probability that the first box will contain 3 balls? [***C.U. B.Com. 1998***]

Solution: The first ball can be distributed at random into 3 boxes in 3 different ways and for each such ways the second ball can also be distributed into these three boxes in 3 different ways. Therefore, 2 balls can be distributed into the given 3 boxes in $3 \times 3 = 3^2$ ways. Similarly, We can say that 10 balls can be distributed at random into the given boxes in 3^{10} ways. Therefore, total number of ways = 3^{10}.

The first box will contain 3 balls. Out of 10 balls 3 balls can be distributed in first box in ${}^{10}C_3$ ways; for each such way one ball from the remaining 7 balls can be distributed into the remaining 2 boxes in 2 different ways. Therefore, 7 balls can be distributed into the 2 boxes in 2^7 ways.

Therefore, No. of favourable cases = ${}^{10}C_3 \times 2^7$

Thus, required probability = $\frac{{}^{10}C_3 \times 2^7}{3^{10}} = \frac{\frac{10!}{7!3!} \times 128}{59049}$

$$= \frac{15360}{59049} = 0.26$$

EXAMPLE 30

Four persons are chosen at random from a group containing 3 men, 2 women and 4 children. Find the probability that exactly two of them are children.

Solution: Out of 9 persons, 4 can be chosen in ${}^9C_4$ ways. Two children can be chosen from 4 children in ${}^4C_2$ ways. Remaining two persons can be chosen from 3 men and 2 women, i.e. from 5 persons in ${}^5C_2$ ways. Hence, the number of ways of choosing four persons such that exactly two of them are children = ${}^4C_2 \times {}^5C_2$.

Therefore, the required probability = $\dfrac{^4C_2 \times {}^5C_2}{^9C_4}$

$$= \frac{\frac{4\times 3}{2\times 1}\times\frac{5\times 4}{2\times 1}}{\frac{9\times 8\times 7\times 6}{4\times 3\times 2\times 1}} = \frac{6\times 10}{126} = \frac{10}{21}$$

EXAMPLE 31

In a family there are 3 children. Find the probability that all of them will have different birth days.

Solution: The birthday of the first child may fall on any one of the 365 days, the second child's birthday may fall on anyone of the 365 days and similarly for the third child.

So, the total number of all possible ways in which the 3 children can have their birthdays

$$= 365 \times 365 \times 365$$

For the number of favourable cases, we see that the first child can have his birthday on any one of the 365 days and then the second child has a different birthday on any one of remaining 364 days in the year, and similarly, the third child can have a different birthday on any of the remaining 363 days.

Therefore, the number of cases favourable to the event A ('different birthdays') is 365 × 364 × 363.

Hence, the required probability that all the 3 children will have different birthdays.

$$= \frac{365\times 364\times 363}{365\times 365\times 365} = 0.99179 = 0.99 \text{ (Approx.)}$$

EXAMPLE 32

A five-digit number is formed by the digits 1, 2, 3, 4, 5 without repetition. Find the probability that the number is divisible by 4.

Solution: Total number of five-digit numbers formed by the digits 1, 2, 3, 4, 5 is 5!

So, the total number of outcomes = 5! = 120.

We know that a number is divisible by 4 if the number formed by last two digits is divisible by 4. Therefore, last two digits can be 12, 24, 32, 52 that is, last two digits can be filled in 4 ways. But corresponding to each of these ways there are 3! = 6 ways of filling the remaining three places.

Therefore, the total number of five-digit numbers formed by the digits 1, 2, 3, 4, 5 and divisible by 4 is 4 × 6 = 24.

Therefore, favourable number of outcomes = 24.

So, required probability = $\dfrac{24}{120} = \dfrac{1}{5}$.

EXAMPLE 33

A word consists of 9 letters: 5 consonants and 4 vowels. Three letters are chosen at random. What is the probability that more than one vowel will be selected?

Solution: Three letters can be chosen out of 9 letters in ${}^9C_3$ ways

Therefore, total number of outcomes = ${}^9C_3$

More than one vowels can be chosen in one of the following ways:

(i) 2 vowels and one consonant or, (ii) 3 vowels

so, favourable number of outcomes = ${}^4C_2 \times {}^5C_1 + {}^4C_3$

Hence, required probability = $\dfrac{{}^4C_2 \times {}^5C_1 + {}^4C_3}{{}^9C_3} = \dfrac{17}{42}$.

EXAMPLE 34

There are 4 letters and 4 addressed envelopes. Find the probability that all the letters are not dispatched in the right envelopes.

Solution: Four letters can be put in four addressed envelopes in 4! ways.

Therefore, total number of outcomes = 4!

All four letters can be put in correct envelopes in exactly one way.

Therefore, the probability that all four letters are put in correct envelopes = $\dfrac{1}{4!}$

Hence, required probability = $1 - \dfrac{1}{4!} = \dfrac{23}{24}$.

EXAMPLE 35

A bag contains 5 white and 4 black balls. One ball is drawn from the bag and replaced and then a second ball is drawn. What is the chance that the two balls drawn are of different colours? [***C.U. B.Com. 1989***]

Solution: Let W be the set of white balls and B be the set of black balls. From a bag containing 9 balls (5 white and 4 black balls), one ball is drawn, then total number of outcomes = ${}^9C_1 = 9$

The ball will be white, then favourable outcomes = ${}^5C_1 = 5$

The ball will be black, then favourable outcomes = ${}^4C_1 = 4$

If the first draw is made, for a white ball,

probability = $\dfrac{5}{9}$ and for a black ball, probability = $\dfrac{4}{9}$

As the ball is replaced before the second draw, so the probability for white and black ball is the same.

Two draws are made for two different colours. So, the total probability = $\dfrac{5}{9} \times \dfrac{4}{9} = \dfrac{20}{81}$ (as two events are independent)

Therefore, the chance of drawing balls of different colours = $\frac{20}{81}$.

EXAMPLE 36

Each of the two identical bags contain 5 white and 5 red balls. One ball is transferred at random without notice from the second bag to the first bag. Then one ball is drawn from the first bag. Find the probability that the ball drawn is a red ball. [***C.U. B.Com. 1991***]

Solution: The ball transferred from the second bag to the first bag, may be white or may be red.

If that ball be white then the bag contains {(5 + 1) or 6 white + 5 red} or 11 balls.

One ball is drawn, then the total number of outcomes = ${}^{11}C_1 = 11$

Therefore, favourable outcomes of getting a red ball = ${}^{5}C_1 = 5$

Hence, probability of getting a red ball = $\frac{5}{11}$

If that ball be red, then total number of outcomes will be same.

Favourable outcomes of that ball is red (5 + 1) = ${}^{6}C_1 = 6$

Therefore, probability of getting a red ball = $\frac{6}{11}$

Therefore, total probability of getting a red ball = $\frac{5}{11} \times \frac{6}{11} = \frac{30}{121}$.

EXAMPLE 37

The probability that a person stopping at a petrol pump will get his tyres checked is 0.12; the probability that he will get his oil checked is 0.29, and the probability that he will get both checked is 0.07.

(*i*) What is the probability that a person stopping at this pump will have neither his tyres checked nor oil checked.

(*ii*) Find the probability that a person who has his oil checked, will also have his tyres checked.

Solution: Let A be the event of tyres checked

B be the event of oil checked

Given that, $P(A) = 0.12$, $P(B) = 0.29$ and $P(A \cap B) = 0.07$

Therefore, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.12 + 0.29 - 0.07 = 0.34$

(*i*) The probability of getting neither oil checked nor tyres checked

$$= P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - 0.34 = 0.66$$

(*ii*) Probability of getting tyres checked, when oil has already been checked $= P(A/B)$ [This relates to conditional probability]

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.07}{0.29} = 0.24$$

EXAMPLE 38

$P(A/B) = 0.75$, $P(B/A) = 0.6$ and $P(A) = 0.4$. Evaluate $P(A/\overline{B})$.

[***C.U. B.Com. 1989***]

Solution: Given that $P(A/B) = 0.75$, $P(B/A) = 0.6$ and $P(A) = 0.4$

We are to calculate $P(A/\overline{B})$

We know that, $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$

or $$P(A \cap B) = P(A) \cdot P(B/A) = 0.4 \times 0.6 = 0.24$$

Again, $$P(B) = \frac{P(A \cap B)}{P(A / B)} = \frac{0.24}{0.75} = \frac{8}{25} = 0.32$$

$$P(\overline{B}) = 1 - P(B) = 1 - 0.32 = 0.68$$

We know that,

$$P(A \cap \overline{B}) = P(A) - P(A \cap B) = 0.04 - 0.24 = 0.16$$

Therefore, $$P(A/\overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})} = \frac{0.16}{0.68} = \frac{4}{17} = 0.2353.$$

EXAMPLE 39

A fair dice is thrown, write the sample space. What is the chance that either an odd number or a number greater than 4 will turn up.

Solution: When a fair dice is thrown, then total number of outcomes = 6 (1, 2, 3, 4, 5, 6) or $n(S) = 6$

Let A be the event of getting an odd number and B be the event of getting a number greater than 4.

Favourable outcomes:

$$A = \{1, 3, 5\} \quad \text{or} \quad n(A) = 3$$
$$B = \{5, 6\} \quad \text{or} \quad n(B) = 2$$
$$A \cap B = \{5\} \quad \text{or} \quad n(A \cap B) = 1$$

Therefore, probability of getting an odd number = $P(A) = \dfrac{3}{6} = \dfrac{1}{2}$

Probability of getting a number greater than 4 = $P(B) = \dfrac{2}{6} = \dfrac{1}{3}$

Probability of getting an odd number and a number greater than 4 = $P(A \cap B) = \dfrac{1}{6}$

We are to calculate the probability of getting either an odd number or a number greater than 4 = P(A ∪ B)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{3+2-1}{6} = \frac{4}{6} = \frac{2}{3}$$

Therefore, the chances of getting either an odd number or the number greater than 4 is $\frac{2}{3}$.

EXAMPLE 40

The chance that in a Management Accountant's job an applicant has a post-graduate degree is 30% and that he has had some work experience as Accountant is 70% and that he has both is 20%. Out of 400 applicants how many applicants would have either a post-graduate degree or some professional experience or both? [***C.U. B.Com. 1986]***

Solution: Let M be the event having post-graduate degree

E be the event having some work experience

Given that, P(M) = 0.30, P(E) = 0.70 and P(M ∩ E) = 0.20

We are to calculate P(M ∪ E)

Now, P(M ∪ E) = P(M) + P(E) – P(M ∩ E) = 0.30 + 0.70 – 0.20

= 1.00 – 0.20 = 0.80

Therefore, out of 400 applicants, number of applicants having either post-graduate degree or some professional experience or both = 0.80 × 400 = 320.

EXAMPLE 41

A committee of 5 members is to be formed from 6 gentlemen and 4 ladices. Find the probability that the ladies may be majority in the committee.

[***C.U. B.Com 1996***]

Solution: The problem can be arranged as under

Case No.	No. of gentlemen (6)	No. of ladies (4)
1	2	3
2	1	4

5 members are to be selected out of 10 members

So, the total number of cases = $^{10}C_5 = \frac{10!}{5!5!}$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5! \times 5 \times 4 \times 3 \times 2 \times 1} = 252$$

Case 1: 2 gentlemen can be selected out of 6 gentlemen in ${}^6C_2$ ways and 3 ladies can be selected out of 4 ladies in ${}^4C_3$ ways.

Therefore, favourable number of cases = ${}^6C_2 \times {}^4C_3$

$$= \frac{6!}{4!2!} \times \frac{4!}{3!1!} = \frac{6\times5\times4!}{4!\times2\times1} \times \frac{4\times3!}{3!} = 15\times4 = 60$$

Hence, probability = $\frac{60}{252} = \frac{5}{21}$

Case 2: 1 gentleman can be selected out of 6 gentlemen and 4 ladies can be selected out of 4 ladies in ${}^6C_1 \times {}^4C_4$ ways.

Therefore, favourable number of cases = ${}^6C_1 \times {}^4C_4 = 6 \times 1 = 6$

Hence, probability = $\frac{6}{252} = \frac{1}{42}$

Therefore, required probability = $\frac{5}{21} + \frac{1}{42} = \frac{10+1}{42} = \frac{11}{42}$.

EXAMPLE 42

40% of the students in a class are girls. If 60% and 70%, boys and the girls respectively, of the class pass a certain test, what is the probability that a randomly selected student from the class will have passed the test?

[***C.U. B.Com. 1990***]

Solution: Let G be the event of girl student
B be the event of boy student
and A be the event of students who passed the test

Now, P(G) = probability of selecting girl student = $\frac{40}{100} = \frac{4}{10}$

P(B) = probability of selecting boy student = $\frac{60}{100} = \frac{6}{10}$

P(A/G) = probability of passing the test from the girl student = $\frac{70}{100} = \frac{7}{10}$

Similarly P(A/B) = $\frac{60}{100} = \frac{6}{10}$

Therefore, P(A) = Probability of passing the test
= P(G). P(A/G) + P(B). P(A/B)

$$= \frac{4}{10}\times\frac{7}{10} + \frac{6}{10}\times\frac{6}{10} = \frac{28}{100} + \frac{36}{100} = \frac{34}{100} = 0.64$$

EXAMPLE 43

A bag contains 5 white and 4 black balls. A ball is drawn at random from the bag and put into another bag which contains 3 white and 7 black balls. A ball is drawn randomly from the second bag. What is the probability that it is white.

[***C.U. B.Com. 1997***]

Solution: Let A = event of transferring a white ball from the first bag

B = event of transferring a black ball from the first bag

C = event of drawing a white ball from the second beg

To draw a white ball from the second bag, either

(*i*) a white ball is transferred from the first bag to the second bag and a white ball is drawn from it; or

(*ii*) a black ball is transferred from the first bag to the second bag and a white ball is drawn from it.

That is, we should have either (A and C) or (B and C), i.e. either $(A \cap C)$ or $(B \cap C)$. Now since the events $A \cap C$ and $B \cap C$ are mutually exclusive, the required probability is $P[(A \cap C) \text{ or } (B \cap C)] = P[(A \cap C) \cup (B \cap C)]$

$$= P(A \cap C) + P(B \cap C)$$
$$= P(A) \cdot P(C/A) + P(B) \cdot P(C/B)$$

Now, $P(A) = \dfrac{5}{5+4} = \dfrac{5}{9}$, $P(B) = \dfrac{4}{5+4} = \dfrac{4}{9}$

$P(C/A) = \dfrac{3}{11}$ (Since a white ball is transferred from 1st bag to the 2nd bag)

$P(C/B) = \dfrac{3}{11}$ (Since a black ball is transferred from 1st bag to the 2nd bag)

Therefore, required probability = $\dfrac{5}{9}.\dfrac{4}{11}+\dfrac{4}{9}.\dfrac{3}{11}=\dfrac{20}{99}+\dfrac{12}{99}=\dfrac{32}{99}$

EXAMPLE 44

A_1, A_2, and A_3 are three events, show that the simultaneous occurrence of the events is $P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2/A_1) \cdot P(A_3/A_1A_2)$. State under which condition $P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$. [***C.U. B.Com. 1992***]

Solution: Let $n(S)$ be the total number of sample points in a finite sample space S of a random experiment of which there are $n(A_1)$ sample points in A_1 and $n(A_1 \cap A_2)$ sample points in $(A_1 \cap A_2)$.

Therefore, $P(A_1 \cap A_2) = \dfrac{n(A_1 \cap A_2)}{n(S)}$ ...(1)

Since, $P(A_1) > 0$, i.e. $n(A_1) > 0$,

Equation (1) can be written in the form

$$P(A_1 \cap A_2) = \frac{n(A_1 \cap A_2)}{n(A_1)} \times \frac{n(A_1)}{n(S)} \quad \ldots(2)$$

But, $\frac{n(A_1 \cap A_2)}{n(A_1)} = P(A_2/A_1)$ and $\frac{n(A_1)}{n(S)} = P(A_1)$

Hence, $P(A_1 \cap A_2) = P(A_1) \cdot P(A_2/A_1)$

Again, since $P(A_1) > 0$ and $P(A_2/A_1) > 0$, we have from (2)

$P(A_2 \cap A_1) > 0$, and hence

$$P(A_1 \cap A_2 \cap A_3) = P\{(A_1 \cap A_2) \cap A_3\} = P(A_1 \cap A_2).\ P(A_3 / A_1 \cap A_2)$$
$$= P(A_1) \cdot P(A_2/A_1) \cdot P(A_3 / A_1 \cap A_2)$$

When the events A_1, A_2 and A_3 are independent, we have

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$$

EXAMPLE 45

A and B are independent events with P(A) = 0.3, P(B) = 0.5. Evaluate $P(A \cup B)$.

[***C.U. B.Com. 1989***]

Solution: Given that P(A) = 0.3, P(B) = 0.5

P(A) and P(B) are two independent events

Therefore, $P(A \cap B) = P(A) \cdot P(B) = 0.3 \times 0.5 = 0.15$

We are to calculate $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.3 + 0.5 - 0.15$

$= 0.8 - 0.15 = 0.65$

Therefore, $P(A \cup B) = 0.65$.

EXAMPLE 46

The probabilities of solving a problem by three students A, B, C are $\frac{2}{7}, \frac{3}{8}$, and $\frac{1}{2}$ respectively. If all of them try independently, find the probability that the problem is solved. Find also the probability that the problem could not be solved.

[***C.U. B.Com. 1992***]

Solution: Here given that,

Probability of solving a problem by A = P(A)= $\frac{2}{7}$

Probability of solving a problem by B = P(B) = $\frac{3}{8}$

Probability of solving a problem by C = P(C) = $\frac{1}{2}$

We are to calculate P(A ∪ B ∪ C) = P(A) + P(B) + P(C) – P(A ∩ B) – P(A ∩ C) – P(B ∩ C) + P(A ∩ B ∩ C)

$$= \frac{2}{7}+\frac{3}{8}+\frac{1}{2}-\left(\frac{2}{7}\times\frac{3}{8}\right)-\left(\frac{2}{8}\times\frac{1}{2}\right)-\left(\frac{3}{8}\times\frac{1}{2}\right)+\left(\frac{2}{7}\times\frac{3}{8}\times\frac{1}{2}\right)$$

(as three students try to solve the problem independently)

$$= \frac{2}{7}+\frac{3}{8}+\frac{1}{2}-\left(\frac{3}{28}+\frac{1}{7}+\frac{3}{16}\right)+\frac{3}{56}$$

$$= \frac{16+21+28+3}{56}-\frac{12+16+21}{112}=\frac{68}{56}-\frac{49}{112}$$

$$= \frac{136-49}{112}-\frac{87}{112}$$

Hence, probability of the problem could not be solved

$$= 1-\frac{87}{112}=\frac{112-87}{112}=\frac{25}{112}$$

Therefore, probability of the problem could be solved is $\frac{87}{112}$ and the problem could not be solved is $\frac{25}{112}$.

EXAMPLE 47

An unbiased coin is tossed twice in succession. Write the sample space. Show that the events 'first toss is a head' and the event 'second toss is a head' are an independent event. ***[C.U. B.Com. 1995]***

Solution: When a coin is tossed twice in succession, the result (getting a 'head') of the first tossing does not affect the result (getting a 'tail') of the second tossing. Hence, the two events are independent.

Sample space S = {HH, HT, TH, TT}

A = event that the first toss is a head = {HH, HT}

B = event that the second toss is a head = {HH, TH}

Therefore, A ∩ B = {HH}

Then, $P(A) = \frac{2}{4} = \frac{1}{2}$, $P(B) = \frac{2}{4}=\frac{1}{2}$ and $P(A \cap B) = \frac{1}{4}$

Now, $P(A).\ P(B) = \frac{1}{2}.\frac{1}{2}=\frac{1}{4}= P(A \cap B)$

Therefore, the event 'first toss is a head' and the event 'second toss is a head' are independent.

EXAMPLE 48

Box A contains 8 bulbs of which 5 are non-defective and Box B contains 5 bulbs of which 2 are defective. A bulb is drawn at random from each box. What is the probability that (i) both the bulbs are non-defective and (ii) one is defective and one is non-defective? [***C.U. B.Com. 1995***]

Solution: Let occurrence of a non-defective bulb be a success

In box A, $P(A) = \frac{5}{8}$ (non-defective)

In box B, $P(\overline{B}) = \frac{2}{5}$ (defective)

Therefore, probability of non-defective bulb in box B, $P(B) = 1 - \frac{2}{5} = \frac{3}{5}$

(*i*) When both the bulbs are non-defective,
then probability $= P(A \cap B) = P(A) \cdot P(B)$ [as A and B are independent]

$$= \frac{5}{8} \cdot \frac{3}{5} = \frac{3}{8}$$

(*ii*) When one is defective, and one is non-defective,
then probability $= P(A \cap \overline{B}) + P(\overline{A} \cap B)$

$$= P(A) \cdot P(\overline{B}) + P(\overline{A}) \cdot P(B)$$

$$= \frac{5}{8} \cdot \frac{2}{5} + \left(1 - \frac{5}{8}\right) \cdot \frac{3}{5} = \frac{1}{4} + \frac{3}{8} \cdot \frac{3}{5}$$

$$= \frac{1}{4} + \frac{9}{40} = \frac{19}{40}.$$

EXAMPLE 49

If A and B two independent events, prove that $\overline{A}$ and $\overline{B}$ are also independent where $\overline{A}$ and $\overline{B}$ have usual meaning. [***C.U. B.Com. 1994***]

Solution: Given that A and B are two independent events

By De-Morgan's Law

$$P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B)$$
$$= 1 - \{P(A) + P(B) - P(A \cap B)\}$$
$$= 1 - P(A) - P(B) + P(A \cap B)$$
$$= 1 - P(A) - P(B) + P(A) \cdot P(B)$$ [as A and B are two independent events]
$$= \{1 - P(A)\} - P(B)\{1 - P(A)\}$$
$$= \{1 - P(A)\}\{1 - P(B)\} = P(A') \cdot P(B')$$

As $P(A' \cap B') = P(A') \cdot P(B')$

Therefore, A' and B' are two independent events.

EXAMPLE 50

If A and B events with $P(A + B) = \frac{3}{4}$, $P(\overline{A}) = \frac{2}{3}$ and $P(AB) = \frac{1}{4}$ then find (i) P(A) (ii) P(B) and (iii) $P(\overline{AB})$. [***C.U. B.Com. 1995***]

Solution: $P(A + B) = P(A \cup B) = \frac{3}{4}$ and $P(AB) = P(A \cap B) = \frac{1}{4}$

(*i*) $P(A) = 1 - P(\overline{A}) = 1 - \frac{2}{3} = \frac{1}{3}$

(*ii*) $P(A \cup B) = P(A) + P(B) - P(AB)$

or $\frac{3}{4} = \frac{1}{3} + P(B) - \frac{1}{4}$

or, $P(B) = \frac{3}{4} + \frac{1}{4} - \frac{1}{3} = 1 - \frac{1}{3} = \frac{2}{3}$

(*iii*) $P(\overline{AB}) = 1 - P(AB) = 1 - \frac{1}{4} = \frac{3}{4}$.

EXAMPLE 51

In a given race, the odds in favour of four horses A, B, C and D are 1:3, 1:4, 1:5 and 1:6 respectively. Assuming that dead heat is impossible, find the chance that one of them wins the race.

Solution: Let E_1, E_2, E_3 and E_4 be the events of winning the horses A, B, C and D respectively. Since the odd of winning in favour of the horses A, B, C and D are 1:3, 1:4, 1:5 and 1:6 respectively,

Therefore, $P(E_1) = \frac{1}{4}$, $P(E_2) = \frac{1}{5}$, $P(E_3) = \frac{1}{6}$ and $P(E_4) = \frac{1}{7}$

As only one of the horses can win the race, the events E_1, E_2, E_3 and E_4 are mutually exclusive.

The probability that one of them wins the race

$$= P(E_1 \cup E_2 \cup E_3 \cup E_4) = P(E_1) + P(E_2) + P(E_3) + P(E_4)$$

$$= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$$

$$= \frac{105 + 84 + 70 + 60}{420} = \frac{319}{420}.$$

EXAMPLE 52

One purse contains 1 dollar and ₹3, second purse contains 2 dollars and ₹4 and third purse contains 3 dollar and ₹1. If a coin is taken out of one of the purses selected at random, find the chance that it is a dollar. [***C.U. B.Com.(H) 2000***]

Solution: Let A, B, C be the events of selecting the first, second and third purses respectively. Let D be the event that a coin drawn from the selected purse is a dollar.

Then $P(A) = \frac{1}{3}, P(B) = \frac{1}{3}, P(C) = \frac{1}{3}, P(D/A) = \frac{1}{4}, P(D/B) = \frac{2}{6} = \frac{1}{3}, P(D/C) = \frac{3}{4}$.

Probability of selecting 1 dollar from the first purse

$$= P(A \cap D) = P(A) \cdot P(D/A) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$$

Probability of selecting 1 dollar from the second purse

$$= P(B \cap D) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

and probability of selecting 1 dollar from the third purse

$$= P(C \cap D) = \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$$

Therefore, by the theorem of total probability, the required probability

$$= P(A \cap D) + P(B \cap D) + P(C \cap D) = \frac{1}{12} + \frac{1}{9} + \frac{1}{4} = \frac{3+4+9}{36} = \frac{16}{36} = \frac{4}{9}.$$

EXAMPLE 53

The four possible outcomes $e_i (i = 1, 2, 3, 4)$ of an experiment are equally likely. Suppose the events A, B, C are defined as follows:

$A = \{e_1, e_2\}$, $B = \{e_2, e_3\}$, $C = \{e_3, e_4\}$. Prove that A is independent of B and B is independent of C. Does this imply that A is independent of C? Give reasons.

[***C.U. B.Com. 2000***]

Solution: Here sample space $S = \{e_1, e_2, e_3, e_4\}$

Therefore, $n(S) = 4$

Again, $A = \{e_1, e_2\}$, $B = \{e_2, e_3\}$, $C = \{e_3, e_4\}$.

Therefore, $n(A) = n(B) = n(C) = 2$.

Hence, $P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2}$

Now, $A \cap B = \{e_2\}$, $B \cap C = \{e_3\}$, $A \cap C = \Phi$ (null set)

$n(A \cap B) = 1$, $n(B \cap C) = 1$ and $n(A \cap C) = 0$

Now, $P(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(B)$

Therefore, A and B are independent events

Again, $P(B \cap C) = \frac{1}{4} = \frac{1}{2} . \frac{1}{2} = P(B) \cdot P(C)$

Therefore, B and C are independent events

Now, $P(A \cap C) = \frac{0}{4} = 0$ and $P(A) \cdot P(C) = \frac{1}{2} . \frac{1}{2} = \frac{1}{4}$

Therefore, $P(A \cap C) \neq P(A) \cdot P(C)$.

Hence, A and C are not independent, they are dependent events.

EXAMPLE 54

In a group of 14 males and 6 females, 8 and 3 of the males and the females respectively are aged above 40. What is the probability that a person selected at random from the group is aged above 40, given that the selected person is a female? [***C.U. B.Com. 1990***]

Solution: Let M be the event of selecting male, F be the event of selecting female and A be the event of selecting person aged above 40.

Then P(M) = Probability of selecting male = $\frac{14}{20}$

P(F) = Probability of selecting female = $\frac{6}{20}$

P(A/M) = Probability of selecting person aged above 40 from males = $\frac{8}{14}$

Similarly, P(A/F) = $\frac{3}{6}$

By Bayes' theorem we get,

$$P(F/A) = \frac{P(F).P(A/F)}{P(M).P(A/M)+P(F).P(A/F)}$$

$$= \frac{\frac{6}{20}\times\frac{3}{6}}{\frac{14}{20}\times\frac{8}{14}+\frac{6}{20}\times\frac{3}{6}} = \frac{\frac{3}{20}}{\frac{2}{5}+\frac{3}{20}} = \frac{\frac{3}{20}}{\frac{8+3}{20}}$$

$$= \frac{\frac{3}{20}}{\frac{11}{20}} = \frac{3}{11}.$$

EXAMPLE 55

Three identical boxes B_1, B_2, B_3 contain respectively 4 white and 3 red balls, 3 white and 7 red balls, and 2 white and 3 red balls. A box is chosen at random and a ball is drawn out of it. If the ball is found to be white, what is the probability that Box B_2 was selected?

Solution: Let A be the event that the drawn ball is white.

We have to find the probability of selecting B_2 box, given that A has happened, i.e. $P(B_2/A)$.

As the 3 boxes are identical in appearance

Therefore, $P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$

Also, $P(A/B_1)$ = Probability of getting a white ball, assuming that Box B_1 was selected = $\frac{4}{7}$

Similarly, $P(A/B_2) = \frac{3}{10}$, $P(A/B_3) = \frac{2}{5}$

The problem can be arranged as under

Box (B_i)	Probability of selecting boxes $P(B_i)$	$P(A/B_i)$	$P(B_i).\ P(A/B_i)$
B_1	$\frac{1}{3}$	$\frac{4}{7}$	$\frac{4}{21}$
B_2	$\frac{1}{3}$	$\frac{3}{10}$	$\frac{1}{10}$
B_3	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{2}{15}$
Total	1		$\frac{89}{210}$

$\Sigma P(B_i) = 1$, as B_i's are mutually exclusive and exhaustive. Using Bayes' formula

$$P(B_2/A) = \frac{P(B_2).P(A/B_2)}{\sum P(B_i).P(A/B_i)} = \frac{\frac{1}{10}}{\frac{89}{210}} = \frac{1}{10} \times \frac{210}{89} = \frac{21}{89}.$$

EXERCISE

THEORETICAL PROBLEMS

1. Define with examples:
   - (*i*) Random experiment [***C.U. B.Com (H) 1983, 1993***]
   - (*ii*) Trial
   - (*iii*) Outcome
   - (*iv*) Sample space [***C.U. B.Com (H) 1983***]
   - (*v*) Event [***C.U. B.Com (H) 1983***]
   - (*vi*) Mutually exclusive events [***B.U. B.Com 2008, C.U. B.Com (H) 1979, 1980, 1984, 1989, 1992, V.U. B.Com (H) 2008, 11***]
   - (*vii*) Mutually exhaustive events [***C.U. B.Com (H) 1980, '84***]
   - (*viii*) Equally likely events [***C.U. B.Com (H) '84]***
   - (*ix*) Complementary events [***C.U. M.Com. 1979***]
   - (*x*) Independent events [***B.U. B.Com(H) 2008; V.U. B.Com (H) 2008, 11; C.U. B.Com (H) 1989, 1992***]
   - (*xi*) Dependent events

2. Write the classical definition of probability and point out its limitations. ***[C.U. B.Com 1992, B.U. B.Com (H) 2007, V.U. B.Com (H) 2010]***
3. State the frequency definition of probability and point out its limitations.
4. State the axiomatic definition of probability.
5. What do you mean by "odds in favour of and odds against an event"?
6. State and prove the theorem of total probability for two mutually exclusive events. How is the result modified when the events are not mutually exclusive? ***[C.U. B.Com (H) 1999; V.U. B.Com (H) 2010]***
7. State and prove the theorem of total probability. [***C.U. B.Com (H) 1996, 1999***]
8. State and prove the "Addition theorem" of probability for two mutual exclusive events. [***C.U. B.Com (H) 1980, 1991***]
9. Define conditional probability of event A, given that event B has occurred. [***C.U. B.Com (H) 1985***]
10. What is meant by compound event in probability? State and prove the theorem of compound probability. [***C.U. B.Com (H) '81, '90, '95, '97***]
11. When are two events mutually exclusive? When are they mutually independent? Let A and B be any two events. The probability of A and the probability of B are each positive. Show that A and B cannot be independent if they are mutually exclusive. [***C.U. B.Com (H) 1991***]
12. (*a*) Write short explanatory notes (with illustrations wherever necessary) on sample space; elementary event and composite event [***C.U. B.Com (H) 1986***]
    (*b*) What do you mean by compound Independent Events?
13. What is the difference between mutually exclusive events and independent events?
14. State Bayes' theorem of conditional probability.
15. State the importance of probability.
16. Show that for two non-mutually exclusive events A and B
    $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [***C.U. B.Com. (H) 1992***]
17. Let S be a sample space and let A be any event in the field of events F. State three axioms to define a probability function P(A) on the field F. Using these axioms prove that,
    (*i*) $P(A) + P(\overline{A}) = 1$
    (*ii*) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for only two events A and B in F. [***C.U. B.Com(H) 1982, 1994, 1996***]
18. If A and B are two independent events, prove that the events (*i*) A^c and B^c (*ii*) A^c and B (*iii*) A and B^c are also independent. [***C.U. B.Com. (H) 1981***]
19. If E is any event then show that P(not E) = 1 – P(E).
20. A_1, A_2 and A_3 are three events show that the simultaneous occurrence of the events is $P(A_1 \cap A_2 \cap A_3) = P(A_1).\ P(A_2/A_1),\ P(A_3/A_1 \cap A_2)$

State under which condition

$P(A_1 \cap A_2 \cap A_3) = P(A_1).\ P(A_2).\ P(A_3)$ [*C.U. B.Com. (H) 1992*]

PRACTICAL PROBLEMS

A. SHORT TYPE

21. If a coin is tossed two times, describe the sample space associated to this experiment. [**Ans.** S = {HH, HT, TH, TT}]

22. Three coins are tossed simultaneously. Describe the sample space. [**Ans.** HHH, HHT, HTH, HTT, THH, THT, TTH, TTT]

23. Describe the sample space when a coin in tossed thrice and number of heads is recorded. [**Ans.** {0, 1, 2, 3}]

24. A card is drawn from a deck of playing cards, and its suit is noted. Describe the sample space. [**Ans.** Spade, Heart, Diamond, Club]

25. A card is drawn from a pack of 52 cards, and its colour is noted. Write the sample space for this experiment. [**Ans.** {Red, Black}]

26. If A and B are two events, then when A and B are said to be (*i*) mutually exclusive and (*ii*) exhaustive? [**Ans.** (i) $A \cap B = \Phi$ (ii) $A \cup B = S$]

27. A coin is tossed thrice. If E denotes the 'number of heads is odd' and F denotes the 'number of tails is odd', then find the cases favourable to the event $E \cap F$. [**Ans.** Zero as $E \cap F = \Phi$]

28. A box contains 1 white and 3 identical black balls. Two balls are drawn at random in succession without replacement. Write the sample space for this experiment. [**Ans.** {WB, BW, BB}]

29. In a simultaneous toss of two coins, find the probability of getting exactly one head. [**Ans.** $\frac{1}{2}$]

30. A dice is thrown. Find the probability of getting a number multiple of 2 or 3. [**Ans.** $\frac{2}{3}$]

31. If odds in favour of an event be 2:3, find the probability of occurrence of this event. [**Ans.** $\frac{2}{5}$]

32. If odds against an event be 7:9, find the probability of non-occurrence of this event. [**Ans.** $\frac{7}{16}$]

33. A and B are two mutually exclusive events of an experiment. If $P(\overline{A}) = 0.65$, $P(A \cup B) = 0.65$ and $P(B) = p$, find the value of p. [**Ans.** 0.30]

34. The probability of two events A and B are 0.25 and 0.50 respectively. The probability of their simultaneous occurrence is 0.14. Find the probability that neither A nor B occurs. [**Ans.** 0.39]

35. There are four men and six women on the city council. If one council member is selected, then what is the probability that it is a woman? [**Ans.** $\frac{3}{5}$]

36. A card is drawn from an ordinary pack and a gambler bets that it is a spade or an ace. What are the odds against his winning the bet? [**Ans.** 9:4]

37. While shuffling a pack of 52 cards, 2 cards are accidently dropped. Find the probability that the missing cards are of different colours. [**Ans.** $\frac{26}{51}$]

[**Hints:** $\frac{^{26}C_1 \times {}^{26}C_1}{^{56}C_2}$]

38. If the letters of the word ALGORITHM are arranged at random in a row, then what is the probability that the letters GOR must remain as a unit? [**Ans.** $\frac{1}{72}$]

39. If A and B are mutually exclusive events, then show that $P(A) \le P(\overline{B})$.
[Hints: $P(A \cup B) \le 1 \Rightarrow P(A) + P(B) \le 1 \Rightarrow P(A) \le 1 - P(B) \Rightarrow P(A) \le P(\overline{B})$]
[As A and B are mutually exclusive events]

40. The probability of an event A occurring is 0.5 and of B is 0.3. If A and B are mutually exclusive events, then find the probability of neither A nor B occurring. [**Ans.** 0.2]

41. A fair dice is thrown. Write the sample space. What is the chance that either an odd number or a number greater than 4 will turn up? [***C.U. B.Com 1986***] [**Ans.** $\frac{2}{3}$]

42. If $P(A) = \frac{2}{11}$ and $P(B) = \frac{6}{7}$, what is the probability that at least one of the two independent events A and B will occur? [**Ans.** $\frac{68}{77}$]

43. Given that $P(A) = \frac{2}{3}$, $P(B) = \frac{3}{5}$ and $P(A \cup B) = \frac{13}{15}$. Are A and B independent? [**Ans.** yes]

44. Calculate P(A/B) if $P(A) = \frac{3}{4}$, $P(B) = \frac{3}{5}$ and $P(B/A) = \frac{18}{25}$ [**Ans.** $\frac{9}{10}$]

45. If $P(A) = \frac{2}{5}$, $P(B) = \frac{1}{3}$ and $P(A \cup B) = \frac{1}{2}$ then find the values P(A/B) and P(B/A). [***C.U. B.Com. 2005***] [**Ans.** $P(A/B) = \frac{7}{10}$, $P(B/A) = \frac{7}{12}$]

46. Given: $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$, find P(A/B) and P(B/A). Are the two events A and B independent? [***C.U. B.Com 1986***]

    [**Ans.** $P(A/B) = \frac{2}{3}$, $P(B/A) = \frac{2}{3}$ and A & B are not independent]

47. Two unbiased dice are thrown together. Find the probability of obtaining 2 in both the dice.

    [***C.U. B.Com 2005***] [**Ans.** $\frac{1}{36}$]

48. One number is chosen from the numbers 1 to 21. Find the probability that it may be a prime number.

    [**Ans.** $\frac{8}{21}$]

49. From a pack of 52 cards one card is drawn at random. Find the probability that the card is a king.

    [**Ans.** $\frac{1}{13}$]

50. A bag contains 3 white and 5 black balls. What is the chance that a ball drawn at random will be black?

    [Ans. $\frac{5}{8}$]

B. SHORT ESSAY TYPE

51. In a simultaneous toss of two coins, find the probability of getting:
    (*i*) 2 heads
    (*ii*) exactly one head
    (*iii*) exactly one tail
    (*iv*) exactly two tails
    (*v*) no tail. [**Ans.** (*i*) $\frac{1}{4}$ (*ii*) $\frac{1}{2}$ (*iii*) $\frac{1}{2}$ (*iv*) $\frac{1}{4}$ (*v*) $\frac{1}{4}$]

52. Two dice are thrown simultaneously, find the probability of getting:
    (*i*) the sum as a prime number
    (*ii*) a total of at least 10
    (*iii*) same number on both dice
    (*iv*) a multiple of 3 as the sum. [**Ans.** (*i*) $\frac{5}{12}$ (*ii*) $\frac{1}{6}$ (*iii*) $\frac{1}{6}$ (*iv*) $\frac{1}{3}$]

53. Two dice are cast. Find the probability of getting the sum of the numbers (*i*) to be 7, (*ii*) not to be 8 or 10.

    [**Ans.** (*i*) $\frac{1}{6}$ (*ii*) $\frac{7}{9}$]

54. From a pack of 52 cards one card is drawn at random. Find the probability that the card is (*i*) a red one, (*ii*) a diamond and (*iii*) a king.

[**Ans.** (*i*) $\frac{1}{2}$ (*ii*) $\frac{1}{4}$ (*iii*) $\frac{1}{13}$]

55. A card is drawn from a deck of 52 cards. Find the probability of getting a king or a heart or a red card.

[**Ans.** $\frac{7}{13}$]

56. If one card is drawn at random from a well-shuffled pack of 52 cards, find the chance that the card is (*i*) a diamond, (*ii*) not a diamond, (*iii*) an ace, (*iv*) neither a spade nor a heart. ***[C.U. B.Com (H) 1998]***

[**Ans.** (*i*) $\frac{1}{4}$ (*ii*) $\frac{3}{4}$ (*iii*) $\frac{1}{13}$ (*iv*) $\frac{1}{2}$]

57. Three coins are tossed. Write the sample space. What is the probability of getting (*a*) exactly two heads; (*b*) at least two heads.

[**Ans.** (*a*) $\frac{3}{8}$, (*b*) $\frac{1}{2}$]

58. Three dice are thrown together. Find the probability of getting a total of at least 6.

[**Ans.** $\frac{103}{108}$]

59. Find the probability that a leap year, selected at random, will contain 53 Tuesdays or 53 Wednesdays?

[**Ans.** $\frac{2}{7}$]

60. A bag contains 8 red and 5 white balls. Three balls are drawn at random. Find the probability that (*i*) all the three balls are white, (*ii*) all the three balls are red, (*iii*) one ball is red and two balls are white.

[**Ans.** (*i*) $\frac{5}{143}$ (*ii*) $\frac{28}{143}$ (*iii*) $\frac{40}{143}$]

61. A box contains 10 red marbles, 20 blue marbles and 30 green marbles. Five marbles are drawn from the box, What is the probability that (*i*) all will be blue? (*ii*) at least one will be green?

[**Ans.** (*i*) $\frac{^{20}C_5}{^{60}C_5}$ (*ii*) $1-\frac{^{30}C_5}{^{60}C_5}$]

62. In a pack of 10 calculators, 3 are known to be defective. If 2 are selected at random from the pack, what is the probability that at least one is defective.

[**Ans.** $\frac{8}{15}$]

63. A bag contains 4 white balls and 3 black balls. Two balls are drawn at random one after another without replacement. Find the probability that both the balls drawn are black.

[**Ans.** $\frac{1}{7}$]

64. A box contains 9 red and 6 white balls. Two balls are drawn at random one after another. Find the probability that both the balls are white, when the first drawn ball is not replaced before the second drawing.

[***C.U. B.Com. 2005***] [**Ans.** $\frac{1}{7}$]

65. 8 men in a company of 25 are graduates. If 3 men are picked out of the 25 at random, what is the probability that (*i*) they are all graduates, (*ii*) at least one is graduate?

[***C.U. B.Com. (H) 2013***] [**Ans.** $\frac{14}{575}, \frac{81}{115}$]

66. A box contains 9 red balls, 5 blue balls and 6 green balls. Three balls are drawn from the box at random. Find the probability that (*i*) all will be blue (*ii*) at least one will be green.

[**Ans.** (*i*) $\frac{1}{114}$ (*ii*) $\frac{194}{285}$]

67. The letters of the word EQUATION are arranged in a row. Find the probability that (*i*) all vowels are together, (*ii*) the arrangement starts with a vowel and ends with a consonant.

[**Ans.** (*i*) $\frac{1}{14}$ (*ii*) $\frac{15}{56}$]

68. A letter is chosen at random from the word 'ASSASSINATION'. Find the probability that letter is (*i*) a vowel (*ii*) a consonant.

[**Ans.** (*i*) $\frac{6}{13}$ (*ii*) $\frac{7}{13}$]

69. Find the probability that in a random arrangement of the letters of the word 'UNIVERSITY' the two I's come together.

[**Ans.** $\frac{1}{5}$]

70. Tickets numbers from 1 to 20 are mixed up together and then a ticket is drawn at random. What is the probability that the ticket has a number which is a multiple of 3 or 7?

[**Ans.** $\frac{2}{5}$]

71. 20 cards are numbered from 1 to 20. One card is drawn at random. What is the probability that the number on the cards is: (*i*) a multiple of 4? (*ii*) greater than 12 (*iii*) divisible by 5 (*iv*) not a multiple of 6?

[**Ans.** (*i*) $\frac{1}{4}$ (*ii*) $\frac{2}{5}$ (*iii*) $\frac{1}{5}$ (*iv*) $\frac{17}{20}$]

72. What are the odds in favour of getting a spade if a card is drawn from a well-shuffled deck of cards? What are the odds in favour of getting a king?

[**Ans.** 1:3, 1:12]

73. One number is chosen from numbers 1 to 200. Find the probability that it is divisible by 4 or 6?

[**Ans.** $\frac{67}{200}$]

74. The probability that a person will get an electric contract is $\frac{2}{5}$ and the probability that he will not get plumbing contract is $\frac{4}{7}$. If the probability of getting at least one contract is $\frac{2}{3}$, what is the probability that he will get both? [**Ans.** $\frac{17}{105}$]

75. In a town of 6000 people 1200 are over 50 years old and 2000 are female. It is known that 30% of the females are over 50 years. What is the probability that a random chosen individual from the town is either a female or over 50 years?

[**Ans.** $\frac{13}{30}$]

76. In a class of 60 students 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that: (*i*) the student opted for NCC or NSS (*ii*) the student has opted neither NCC nor NSS (*iii*) the student has opted NSS but not NCC.

[**Ans.** (*i*) $\frac{19}{30}$ (*ii*) $\frac{11}{30}$ (*iii*) $\frac{2}{15}$]

77. The probability that a student will pass the final examination in both English and Hindi is 0.5 and the probability of passing neither is 0.1. If the probability of passing the English examination is 0.75. What is the probability of passing the Hindi examination? [**Ans.** 0.65]

78. In supplies of 3 components, viz. base, neck and switch, for an electric lamp, the percentages of defectives on a day were 5, 20 and 10 respectively. An assembled lamp is considered defective if at least one of the 3 components is defective. If components are selected randomly, what is the probability that an assembled lamp would be defective? [**Ans.** 0.316]

79. Two sets of candidates are competing for the positions in the Board of Directors of a company. The probabilities that the first and second sets will win are 0.6 and 0.4 respectively. If the first set wins, the probability of introducing a new product is 0.8, and the corresponding probability, if the second set wins, is 0.3. What is the probability that the new product will be introduced?

[**Ans.** 0.60]

80. A bag contains 8 red balls and 5 white balls. Two successive draws of 3 balls are made without replacement. Find the probability that the first drawing will give 3 white balls and the second, 3 red balls.

[**Ans.** $\frac{7}{429}$]

81. Boxes 1 and 2 contain 4 white, 3 red and 3 blue balls; and 5 white, 4 red and 3 blue balls respectively. If one ball is drawn at random from each box, what is the probability that both the balls are of the same colour?

[***C.U. B.Com (H) 2000; V.U. B.Com. (H) 2011***] [**Ans.** $\frac{41}{120}$]

82. A, B, C and D are four mutually exclusive and exhaustive events. If the odds against the events B, C, D are respectively 7:2, 7:5 and 13:5, find the odds in favour of the event A. [**Ans.** 1:11]

83. There are two bags. The first contains 2 red and 1 white balls, whereas the second bag has only 1 red and 2 white balls. One ball is taken out at random from the first bag and is put in the second bag. Then a ball is chosen at random from the second bag. What is the probability that this last ball is red? [Ans. $\frac{5}{12}$]

84. There are three men aged 60, 65 and 70 years. The probability to live 5 years more is 0.8 for a 60-year-old, 0.6 for a 65-year-old and 0.3 for a 70-year-old person. Find the probability that at least two of the three persons will remain alive 5 years hence. [***C.U. B.Com (H) 1981***] [**Ans.** 0.612]

85. Two letters are drawn at random from the word HOME. Write the sample space. Now find the probability that,
   (*i*) both the letters are vowels
   (*ii*) at least one is a vowel
   (*iii*) one of the letters chosen should be 'M' [***C.U. B.Com (Hon.) 1984***]

   [**Ans.** : S{(H,O), (H,M), (H,E), (O,M), (O,E), (M,E)}; (*i*) $\frac{1}{6}$ (*ii*) $\frac{5}{6}$ (*iii*) $\frac{1}{2}$]

86. (*a*) 20 dates are named at random, what is the probability that 5 of them will be Sundays? [***C.U. B.Com. (H) 1999***] [**Ans.** $15.504 \times \frac{6^{15}}{7^{20}}$]

   (*b*) State and prove the theorem of total probability. Two fair dice are thrown simultaneously. The two scores are then multiplied together. Calculate the probability that the product is (*i*) 12 and (*ii*) even. [***C.U. B.Com. (H) 1997***] [**Ans.** (i) $\frac{1}{7}$ (ii) $\frac{3}{4}$]

87. From a set of 17 balls marked 1, 2, 3, …, 16, 17; one ball is drawn at random. What is the chance that its number is either a multiple of 3 or of 7? [***C.U. B.Com. (H) 2000***] [**Ans.** $\frac{7}{17}$]

88. A and B are two events, not mutually exclusive, connected with a random experiment E. If $P(A) = \frac{1}{4}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = \frac{1}{2}$, find the values of the following probabilities; (*i*) $P(A \cap B)$, (*ii*) $P(A \cap B^c)$, (*iii*) $P(A^c \cup B^c)$ where c stands for the complement. [***C.U. B.Com. (H) 1980***]

   [**Ans.** (*i*) $\frac{3}{20}$, (*ii*) $\frac{1}{10}$, (*iii*) $\frac{17}{20}$]

89. Let S be a sample space and let A be any event in the field of events F. State three axioms to define a probability function P(A) on the field F. Using

these axioms prove that (*i*) $P(A) + P(\overline{A}) = 1$, (*ii*) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, for any two events A and B in F. [***C.U. B.Com. (H) 1982***]

90. (*a*) A and B are independent events with $P(A) = 0.3$, $P(B) = 0.5$. Evaluate $P(A \cup B)$. [C.U. B.Com. (H) 1988] [Ans. 0.65]

(*b*) $P(A/B) = 0.75$, $P(B/A) = 0.6$ and $P(A) = 0.4$, Evaluate $P(A/\overline{B})$.

[***C.U. B.Com. (H) 1989***] [**Ans.** 0.2353]

91. If A and B be events with $P(A + B) = \frac{3}{4}$, $P(\overline{A})\ \frac{2}{3}$ and $P(AB) = \frac{1}{4}$, then find (*i*) P(A), (*ii*) P(B) and (*iii*) $P(\overline{AB})$.

[***C.U. B.Com. (H) 1995***] [**Ans.** (*i*) $\frac{1}{3}$ (*ii*) $\frac{2}{3}$ (*iii*) $\frac{3}{4}$]

92. $P(A) = \frac{2}{3}$, $P(B) = \frac{1}{2}$, $P(A \cup B) = 1$. Find the value of P(A/B), $P(A/B^c)$. $P(A^c \cap B^c)$.

[***C.U. B.Com. (H) 2003***] [**Ans.** $\frac{1}{3}$, 1, 0]

93. (*i*) If $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(A \cup B) = \frac{5}{12}$, find $P(A \cap B)$, $P(\overline{A} \cap \overline{B})$.

[**Ans.** $\frac{1}{6}$, $\frac{7}{12}$]

(*ii*) If $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{5}$, find P(A/B) and $P(\overline{A}/\overline{B})$.

[**Ans.** $\frac{3}{5}$, $\frac{11}{20}$]

94. (*i*) If A and B are two events with $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$, find P(A/B) and P(B/A). Are A and B independent?

[**Ans.** $\frac{2}{5}$, $\frac{2}{3}$, No]

(*ii*) If $P(\overline{A} \cup \overline{B}) = \frac{5}{6}$, $P(A) = \frac{1}{2}$ and $P(\overline{B}) = \frac{2}{3}$, are the events A and B independent? [**Ans.** yes]

95. A, B and C are three mutually exclusive and exhaustive events. Find P(B), if $\frac{1}{3}P(C) = \frac{1}{2}P(A) = P(B)$

[***B.U. B.Com.(H) 2007; V.U. B.Com. (H) 2007***] [**Ans.** $\frac{1}{6}$]

96. A candidate is selected for interview for three posts. For the first post there are 3 candidates, for the second there are 4, and for the third there are 2, what is the chance of his getting at least one post?

[**Ans.** $\frac{3}{4}$]

97. A bag contains 4 red and 3 blue balls. Two drawings of 2 balls are made. Find

the probability of drawing 2 red balls first and the 2 blue balls second.

(*i*) If the balls are returned to the bag after the first draw.

(*ii*) If the balls are not returned after the first draw.

[**Ans.** (*i*) $\frac{2}{49}$ (*ii*) $\frac{3}{35}$]

98. In a bolt factory, machines X, Y and Z manufacture respectively 20, 35 and 45 per cent of the total output. Of their output, 8, 6 and 5 per cent respectively are defective bolts. One bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured in the machine Z?

[**Ans.** $\frac{45}{119}$]

99. There are two identical boxes containing respectively 4 white and 3 red balls, and 3 white and 7 red balls. A box is chosen at random and a ball is drawn from it. If the ball is white, what is the probability that it is from the first box?

[***C.U. B.Com. 2002***] [**Ans.** $\frac{21}{61}$]

100. The chance that doctor A will diagnose disease B correctly is 60%. The chance that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of doctor A, who had disease B died. What is the chance that his disease was correctly diagnosed?

[**Ans.** $\frac{6}{13}$]

C. MULTIPLE CHOICE QUESTIONS (MCQs)

(i) Short Type

1. When a coin is tossed, what is the probability of getting head.

(*a*) 1 (*b*) 2 (*c*) $\frac{1}{2}$ (*d*) 0

[**Ans.** (*c*)]

2. When a coin is tossed, what is the probability of getting tail.

(*a*) 1 (*b*) 2 (*c*) $\frac{1}{2}$ (*d*) 0

[**Ans.** (*c*)]

3. Two unbiased coins are tossed. What is the probability of getting at most one tail?

(*a*) $\frac{1}{2}$ (*b*) $\frac{1}{3}$ (*c*) $\frac{3}{2}$ (*d*) $\frac{3}{4}$

[**Ans.** (*d*)]

4. Three unbiased coins are tossed, what is the probability of getting at least two heads?

(*a*) $\frac{1}{3}$ (*b*) $\frac{1}{6}$ (*c*) $\frac{1}{2}$ (*d*) $\frac{1}{8}$

[**Ans.** (*c*)]

5. In a throw of dice what is the probability of getting a number greater than 4.

(*a*) $\frac{1}{2}$ (*b*) $\frac{1}{3}$ (*c*) $\frac{1}{5}$ (*d*) $\frac{1}{6}$

[**Ans.** (*b*)]

6. What is the probability of getting a sum 8 from two throws of dice.

(*a*) $\frac{7}{36}$ (*b*) $\frac{5}{36}$ (*c*) $\frac{11}{36}$ (*d*) $\frac{1}{6}$

[**Ans.** (*b*)]

7. Two dice are thrown simultaneously. What is the probability of getting two numbers whose product is even?

(*a*) $\frac{3}{4}$ (*b*) $\frac{1}{4}$ (*c*) $\frac{7}{4}$ (*d*) $\frac{1}{2}$

[**Ans.** (*a*)]

8. When two coins are tossed simultaneously, what are the chances of getting at least one tail?

(*a*) $\frac{3}{4}$ (*b*) $\frac{1}{5}$ (*c*) $\frac{4}{5}$ (*d*) $\frac{1}{4}$

[**Ans.** (*a*)]

9. Three unbiased coins are tossed. What is the probability of getting at least 2 tails?

(*a*) 0.75 (*b*) 0.5 (*c*) 0.25 (*d*) 0.2

[**Ans.** (*b*)]

10. Two dice are tossed simultaneously. Find the probability that the total is a prime number.

(*a*) $\frac{7}{9}$ (*b*) $\frac{5}{12}$ (*c*) $\frac{1}{6}$ (*d*) $\frac{5}{9}$

[**Ans.** (*b*)]

11. A dice is rolled twice. What is the probability of getting sum 9?

(*a*) $\frac{2}{3}$ (*b*) $\frac{1}{3}$ (*c*) $\frac{1}{9}$ (*d*) $\frac{3}{9}$

[**Ans.** (*c*)]

12. Three coins are tossed. What is the probability of getting at most two tails?

(*a*) $\frac{7}{8}$ (*b*) $\frac{1}{8}$ (*c*) $\frac{2}{8}$ (*d*) $\frac{4}{8}$

[**Ans.** (*a*)]

13. A card is drawn from a pack of 52 cards. The probability of getting a queen of club or a king of heart is

(*a*) $\frac{1}{13}$ (*b*) $\frac{2}{13}$ (*c*) $\frac{1}{26}$ (*d*) $\frac{1}{52}$

[**Ans.** (*c*)]

14. From a pack of 52 cards, 1 card is drawn at random. Find the probability of a face card drawn.

(*a*) $\frac{3}{13}$ (*c*) $\frac{1}{4}$

(*b*) $\frac{1}{52}$ (*d*) None of above

[**Ans.** (*a*)]

15. There is a pack of 52 cards and Rohan draws two cards together. What is the probability that one is spade and one is heart?

(*a*) $\frac{11}{103}$ (*b*) $\frac{13}{102}$ (*c*) $\frac{11}{104}$ (*d*) $\frac{11}{102}$

[**Ans.** (*b*)]

16. What is probability of drawing two clubs from a well shuffled pack of 52 cards?

(*a*) $\frac{13}{51}$ (*b*) $\frac{1}{17}$ (*c*) $\frac{1}{26}$ (*d*) $\frac{13}{17}$

[**Ans.** (*b*)]

17. A card is drawn from a pack of 52 cards. The probability of getting a queen of club or a king of heart is:

(*a*) $\frac{1}{13}$ (*b*) $\frac{2}{13}$ (*c*) $\frac{1}{26}$ (*d*) $\frac{1}{52}$

[**Ans.** (*c*)]

18. One card is drawn at random from a pack of 52 cards. What is the probability that the card drawn is a face card (Jack, Queen and King only)?

(*a*) $\frac{1}{13}$ (*b*) $\frac{3}{13}$ (*c*) $\frac{1}{4}$ (*d*) $\frac{9}{52}$

[**Ans.** (*b*)]

19. What will be the possibility of drawing a jack or a spade from a well shuffled standard deck of 52 playing cards?

(*a*) $\frac{4}{13}$ (*b*) $\frac{1}{26}$ (*c*) $\frac{1}{13}$ (*d*) $\frac{17}{52}$

[**Ans.** (*a*)]

20. From a pack of 52 cards, two cards are drawn together at random. What is the probability of both the cards being kings?

(*a*) $\frac{1}{15}$ (*b*) $\frac{25}{57}$ (*c*) $\frac{35}{256}$ (*d*) $\frac{1}{221}$

[**Ans.** (*d*)]

21. Two cards are drawn together from a pack of 52 cards. The probability that one is a spade and one is a heart, is.

(*a*) $\frac{3}{20}$ (*b*) $\frac{29}{34}$ (*c*) $\frac{47}{100}$ (*d*) $\frac{13}{102}$

[**Ans.** (*d*)]

22. On rolling a dice 2 times, the sum of 2 numbers that appear on the uppermost face is 8. What is the probability that the first throw of dice yields 4?

(*a*) $\frac{2}{36}$ (*b*) $\frac{1}{36}$ (*c*) $\frac{1}{6}$ (*d*) $\frac{1}{5}$

[**Ans.** (*b*)]

23. A bag contains 6 black and 8 white balls. One ball is drawn at random. What is the probability that the ball drawn is white?

(*a*) $\frac{3}{4}$ (*b*) $\frac{4}{7}$ (*c*) $\frac{1}{8}$ (*d*) $\frac{3}{7}$

[**Ans.** (*b*)]

24. A box contains 5 green, 4 yellow and 3 white balls. Three balls are drawn at random. What is the probability that they are not of same colour.

(*a*) $\frac{52}{55}$ (*b*) $\frac{3}{55}$ (*c*) $\frac{41}{44}$ (*d*) $\frac{3}{44}$

[**Ans.** (*c*)]

25. A bag contain 10 black and 20 white balls, One ball is drawn at random. What is the probability that ball is white

(*a*) 1 (*b*) $\frac{2}{3}$ (*c*) $\frac{1}{3}$ (*d*) $\frac{4}{3}$

[**Ans.** (*b*)]

(ii) Short Essay Type

26. In a box, there are 8 red, 7 blue and 6 green balls. One ball is picked up randomly. What is the probability that it is neither blue nor green?

(*a*) $\frac{2}{3}$ (*b*) $\frac{8}{21}$ (*c*) $\frac{3}{7}$ (*d*) $\frac{9}{22}$

[**Ans.** (*a*)]

27. A box has 6 black, 4 red, 2 white and 3 blue shirts. What is the probability that 2 red shirts and 1 blue shirt get chosen during a random selection of 3 shirts from the box?

(*a*) $\frac{18}{455}$ (*b*) $\frac{7}{15}$ (*c*) $\frac{7}{435}$ (*d*) $\frac{7}{2730}$

[**Ans.** (*a*)]

28. A beg contains 2 red, 3 green and 2 blue balls. Two balls are drawn at random. What is the probability that none of the balls drawn is blue?

(*a*) $\frac{10}{21}$ (*c*) $\frac{2}{7}$

(*b*) $\frac{11}{21}$ (*d*) $\frac{5}{7}$ [**Ans.** (*a*)]

29. A pot has 2 white, 6 black, 4 grey and 8 green balls. If one ball is picked randomly from the pot, what is the probability of it being black or green?

(*a*) $\frac{3}{4}$ (*b*) $\frac{7}{10}$ (*c*) $\frac{4}{3}$ (*d*) $\frac{1}{10}$

[**Ans.** (*b*)]

30. A box has 6 black, 4 red, 2 white and 3 blue shirts. When 2 shirts are picked randomly, what is the probability that either both are white or both are blue?

(*a*) $\frac{4}{105}$ (*b*) $\frac{1}{35}$ (*c*) $\frac{1}{105}$ (*d*) $\frac{1}{15}$

[**Ans.** (*a*)]

31. A bag contains 4 white, 5 red and 6 blue balls. Three balls are drawn at random from the bag. The probability that all of them are red, is:

(*a*) $\frac{1}{22}$ (*b*) $\frac{3}{22}$ (*c*) $\frac{2}{91}$ (*d*) $\frac{2}{77}$

[**Ans.** (*c*)]

32. A box has 6 black, 4 red, 2 white and 3 blue shirts. Find the probability of drawing 2 black shirts if they are picked randomly?

(*a*) $\frac{1}{8}$ (*b*) $\frac{2}{15}$ (*c*) $\frac{6}{15}$ (*d*) $\frac{1}{7}$

[**Ans.** (*d*)]

33. In a box, there are 8 red, 7 blue and 6 green balls. One ball is picked up randomly. What is the probability that it is neither red nor green?

(*a*) $\frac{1}{3}$ (*c*) $\frac{7}{19}$ (*e*) $\frac{9}{21}$

(*b*) $\frac{3}{4}$ (*d*) $\frac{8}{21}$

[**Ans.** (*a*)]

34. A box has 10 black and 10 white balls. What is the probability of getting two balls of the same color?

(*a*) $\frac{10}{19}$ (*b*) $\frac{9}{38}$ (*c*) $\frac{9}{19}$ (*d*) $\frac{5}{38}$

[**Ans.** (*c*)]

35. Tickets numbered 1 to 20 are mixed up and then a ticket is drawn at random. What is the probability that the ticket drawn has a number which is a multiple of 3 or 5?

(*a*) $\frac{1}{2}$ (*b*) $\frac{2}{5}$ (*c*) $\frac{8}{15}$ (*d*) $\frac{9}{20}$

[**Ans.** (*d*)]

36. Tickets numbered 1 to 50 are mixed and one ticket is drawn at random. Find the probability that the ticket drawn has a number which is a multiple of 4 or 7?

(*a*) $\frac{9}{25}$

(*b*) $\frac{9}{50}$

(*c*) $\frac{18}{25}$

(*d*) None of these **[Ans.** (*a*)**]**

37. A class consists of 15 girls and 10 boys. Three students are to be randomly selected. Find the probability that one boy and two girls are picked.

(*a*) $\frac{1}{50}$ (*b*) $\frac{3}{25}$ (*c*) $\frac{21}{46}$ (*d*) $\frac{25}{122}$

[Ans. (*c*)**]**

38. In a class, there are 15 boys and 10 girls. Three students are selected at random. The probability that 1 girl and 2 boys are selected, is:

(*a*) $\frac{21}{46}$ (*b*) $\frac{25}{117}$ (*c*) $\frac{1}{50}$ (*d*) $\frac{3}{25}$

[Ans. (*a*)**]**

39. Two friends A and B apply for a job in the same company. The chances of A getting selected is 2/5 and that of B is 4/7. What is the probability that both of them get selected?

(*a*) $\frac{8}{35}$

(*b*) $\frac{34}{35}$

(*c*) $\frac{27}{35}$

(*d*) None of these **[Ans.** (*a*)**]**

40. What are the chances that no two boys are sitting together for a photograph if there are 5 girls and 2 boys?

(*a*) $\frac{1}{21}$ (*b*) $\frac{4}{7}$ (*c*) $\frac{2}{7}$ (*d*) $\frac{5}{7}$

[Ans. (*d*)**]**

41. What is the possibility of having 53 Thursdays in a non-leap year?

(*a*) $\frac{6}{7}$ (*b*) $\frac{1}{7}$ (*c*) $\frac{1}{365}$ (*d*) $\frac{53}{365}$

[Ans. (*b*)**]**

42. In a set of 30 game cards, 17 are white and rest are green. 4 white and 5 green are marked IMPORTANT. If a card is chosen randomly from this set, what is the possibility of choosing a green card or an 'IMPORTANT' card?

(*a*) $\frac{13}{30}$ (*b*) $\frac{22}{30}$ (*c*) $\frac{17}{30}$ (*d*) $\frac{9}{13}$

[Ans. (*c*)**]**

43. In a lottery, there are 10 prizes and 25 blanks. A lottery is drawn at random. What is the probability of getting a prize?

(*a*) $\frac{1}{10}$ (*b*) $\frac{2}{5}$ (*c*) $\frac{2}{7}$ (*d*) $\frac{5}{7}$

[**Ans.** (*c*)]

44. A box contains 20 electric bulbs, out of which 4 are defective. Two bulbs are chosen at random from this box. The probability that at least one of these is defective is

(*a*) $\frac{7}{19}$ (*b*) $\frac{6}{19}$ (*c*) $\frac{5}{19}$ (*c*) $\frac{4}{19}$

[**Ans.** (*a*)]

46. A speaks truth in 75% of cases and B in 80% of cases. In what percentage of cases are they likely to contradict each other, narrating the same incident.

(*a*) 30% (*b*) 35% (*c*) 40% (*d*) 45%

[**Ans.** (*b*)]

46. A box has 6 black, 4 red, 2 white and 3 blue shirts. What is probability of picking at least 1 red shirt in 4 shirts that are randomly picked?

(*a*) $\frac{4}{15}$ (*b*) $\frac{24}{455}$ (*c*) $\frac{69}{91}$ (*d*) $\frac{22}{91}$

[**Ans.** (*c*)]

47. A box has 5 black and 3 green shirts. One shirt is picked randomly and put in another box. The second box has 3 black and 5 green shirts. Now a shirt is picked from the second box. What is the probability of it being a black shirt?

(*a*) $\frac{4}{9}$ (*b*) $\frac{29}{72}$ (*c*) $\frac{8}{72}$ (*d*) $\frac{3}{16}$

[**Ans.** (*b*)]

48. In a drawer there are 4 white socks, 3 blue socks and 5 grey socks. Two socks are picked randomly. What is the possibility that both the socks are of the same color?

(*a*) $\frac{4}{11}$ (*b*) 1 (*c*) $\frac{2}{33}$ (*d*) $\frac{19}{66}$

[**Ans.** (*d*)]

49. In a drawer there are 5 black socks and 3 green socks. Two socks are picked randomly one after the other without replacement. What is the possibility that both the socks are black?

(*a*) $\frac{5}{14}$ (*b*) $\frac{5}{8}$ (*c*) $\frac{3}{8}$ (*d*) $\frac{5}{16}$

[**Ans.** (*a*)]

50. There are 2 pots. One pot has 5 red and 3 green marbles. Other has 4 red and 2 green marbles. What is the probability of drawing a red marble?

(*a*) $\frac{9}{14}$ (*b*) $\frac{31}{48}$ (*c*) 1 (*d*) $\frac{1}{2}$

[**Ans.** (*b*)]

51. A card is drawn from a well shuffled pack of playing cards. The probability that it is either diamond or a king is

(*a*) $\frac{16}{13}$ (*b*) $\frac{15}{52}$ (*c*) $\frac{4}{13}$ (*d*) $\frac{5}{26}$

[**Ans.** (*c*)]

52. Probability that a man will be alive 25 years hence is 0.3 and the probability that his wife will be alive 25 years hence is 0.4. The probability that 25 years hence only the man will be alive is

(*a*) 0.18 (*b*) 0.15 (*c*) 0.12 (*d*) 0.16

[**Ans.** (*a*)]

53. A piece of electronic equipment has two essential parts A and B. In the past, part A failed 30% of the times, part B failed 20% of the times and both failed simultaneously 5% of the times. Assuming that both parts must operate to enable the equipment to function, the probability that the equipment will function is

(*a*) 0.27 (*b*) 0.52 (*c*) 0.45 (*d*) 0.55

[**Ans.** (*d*)]

54. If $P(A) = \frac{1}{4}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = \frac{1}{2}$, find $P(A^c \cup B^c)$, where A and B are two non mutually exclusive events connected with a random experiment E and A^c is the complement event of A.

(*a*) 0.81 (*b*) 0.85 (*c*) 0.79 (*d*) 0.65

[**Ans.** (*b*)]

55. If P(A) = 0.3, P(B) = 0.2 and P(C) = 0.1 and A, B, C are independent events, the probability of occurrence of atleast one of the three events A, B, C is

(*a*) 0.392 (*b*) 0.457 (*c*) 0.496 (*d*) 0.38

[**Ans.** (*c*)]

56. Out of all the 2-digit integers between 1 and 200, a 2-digit number has to be selected at random. The probability that the selected number is not divisible by 7 is

(*a*) $\frac{77}{90}$ (*b*) $\frac{66}{90}$ (*c*) $\frac{55}{90}$ (*d*) $\frac{44}{90}$

[**Ans.** (*a*)]

57. A box of nine golf gloves contains two left-handed and seven right-handed gloves. If three gloves are selected without replacement, the probability that all of them are left-handed is

(*a*) $\frac{49}{81}$ (*b*) $\frac{7}{18}$ (*c*) 1 (*d*) 0

[**Ans.** (*d*)]

58. In a certain college, the students engage in sports in the following proportion Football (F): 60% of all students, Basketball (B): 50% of all students, Both Football and Basketball : 30% of all students. If a student is selected at random the probability that he will play neither sports is

(*a*) 0.25 (*c*) 0.32
(*b*) 0.20 (*d*) 0.18 [**Ans.** (*b*)]

59. The result of an examination given to a class on three papers A, B and C are 40% failed in paper A, 30% failed in B, 25% failed in paper C, 15% failed in paper A and B both, 12% failed in B and C both, 10% failed in A and C both, 3% failed in A, B and C. The probability of a randomly selected candidates passing in all three papers is

(*a*) 0.53 (*c*) 0.39
(*b*) 0.42 (*d*) 0.36 [**Ans.** (*c*)]

60. A doctor is to visit a patient. The probability that he will come by car, taxi, scooter or by other means of transport are 0.3, 0.2, 0.1 and 0.4. The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3}$ and $\frac{1}{2}$, if he comes by car, taxi and scooter. But if he comes by other means of transport he will not be late. When he arrives he is late. Therefore, the probability that he comes by car is

(*a*) $\frac{1}{2}$ (*c*) $\frac{1}{4}$
(*b*) $\frac{1}{3}$ (*d*) $\frac{2}{7}$ [**Ans.** (*a*)]

Solution of C.U. Question Paper–2017 (New Syllabus)

APPENDIX

A

FOR HONOURS CANDIDATES

MODULE–I

Group–A

1. Answer the following questions:

   (*a*) If A = {1, 2}, then find the power set of A.

Solution: Power set of A:

$P(\text{A}) = [\{1\}, \{2\}, \{1, 2\}, \Phi\ ]$

(*b*) If $^{15}C_r = {}^{15}C_r + 3$, then find $^{8}C_r$

Solution: Given that, $^{15}C_r = {}^{15}C_{r+3}$

or $r + r + 3 = 15$ [$\because$ when $^{n}c_p = {}^{n}c_q$ then $p + q = n$]

or $2r + 3 = 15$

or $2r = 15 - 3 = 12$ or $r = \dfrac{12}{2} = 6$

$\therefore$ $^{8}C_r = {}^{8}C_6 = \dfrac{8!}{6!\,2!} = \dfrac{8 \times 7 \times 6!}{6! \times 2 \times 1} = 28$

or

If $^{n}p_2 = 20$, then find n.

Solution: Given, $^{n}p_2 = 20$

or $\dfrac{n!}{(n-2)!} = 20$ or $\dfrac{n(n-1)\,(n-2)!}{(n-2)!} = 20$

or $n(n-1) = 5 \times (5-1)$

or $n = 5$.

(*c*) How many different arrangements can be made with letters of the word "GENTLEMEN"?

Solution: The word 'GENTLEMEN' contains 9 letters of which E occurs 3 times, N occurs 2 times and the rest are all different.

$\therefore$ the required number of words

$$= \frac{9!}{3! \times 2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3! \times 2 \times 1}$$

$$= 30,240$$

(*d*) If A = {1, 2, 3,4}, B = {2, 4, 5, 8}, C = {3, 4, 5, 6, 7}, find $A \cup (B \cap C)$

or

Find $A \cap (B \cup C)$ for the above sets.

Solution: A = {1, 2, 3, 4}, B = {2, 4, 5, 8}, C = {3, 4, 5, 6, 7} $B \cap C$ = {4, 5}

$\therefore$ $A \cup (B \cap C)$ = {1, 2 , 3, 4, 5}

or

$B \cup C$ = {2, 3, 4, 5, 6, 7, 8}

$A \cap (B \cup C)$ = {2, 3, 4}

Group–B

2. Answer the following questions:

(*a*) How many numbers of different digits lying between 100 and 1000 can be formed with the digits 2, 3, 4, 0, 8, 9 each being used only once?

Solution: The numbers lying between 100 and 1000 must be the numbers of 3 digits and zero cannot be the first digit.

$\therefore$ the total number of permutations of the given 6 digits taken 3 at a time

$$= {}^6p_3 = \frac{6!}{3!} = 6 \times 5 \times 4 = 120$$

The first digit of some of these 120 numbers is zero. To obtain these numbers whose first digit is zero, we put zero in the first place and remaining two places can be filled in by the remaining 5 digits in ${}^5p_2$ ways $= \frac{5!}{3!} = 5 \times 4 = 20$ ways.

Hence, there are 20 numbers whose first digit is zero.

$\therefore$ the required number of permutations = 120 – 20 = 100.

or

Show that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

Solution: L.H.S $= {}^nC_r + {}^nC_{r-1}$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$= \frac{n!}{r \times (r-1)!\ (n-r)!} + \frac{n!}{(r-1)!\ (n-r+1).(n-r)!}$$

$$= \frac{n!}{(r-1)!\ (n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!\ (n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$$

$$= \frac{n!\,(n+1)}{r.(r-1)!\ (n-r+1).(n-r)!} = \frac{(n+1)!}{r!(n+1-r)!} = {}^{n+1}C_r$$

(*b*) If $x = \log_a(bc)$, $y = \log_b(ca)$ and $z = \log_c(ab)$, show that $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1}$

Solution: $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1}$

$$= \frac{1}{\log_a(bc) + \log_a a} + \frac{1}{\log_b(ca) + \log_b b} + \frac{1}{\log_c(ab) + \log_c c}$$

$$= \frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_c(abc)}$$

$$= \log_{abc} a + \log_{abc} b + \log_{abc} c$$

$$= \log_{abc} abc = 1$$

or

Solve for x if $\log_x 2 + \log_x 4 + \log_x 8 = 6$

Solution: $\log_x 2 + \log_x 4 + \log_x 8 = 6$

or $\log_x 2 + \log_x 2^2 + \log_x 2^3 = 6$

or $\log_x{}^2 + 2\log_x{}^2 + 3\log_x{}^2 = 6$

or $6\log_x{}^2 = 6$ or $\log_x{}^2 = 1$

or $x^1 = 2$ or $x = 2$

(*c*) Find the coefficient of x^{-11} in the expansion of $\left(x^2 - \frac{1}{x^3}\right)^{12}$.

Solution: Suppose that the $(r + 1)$th term contains x^{-11} in the expansion of $\left(x^2 - \frac{1}{x^3}\right)^{12}$.

Now, the $(r + 1)$th term of the expansion

$$= {}^{12}C_r\ (x^2)^{12-r} \cdot \left(-\frac{1}{x^3}\right)^r = {}^{12}C_r x^{24-2r} \cdot \frac{(-1)^r}{x^{3r}}$$

$$= {}^{12}C_r.\ (-1)^r.\ x^{24-5r}$$

Since, the $(r + 1)$th term contains x^{-11}, we must have,

$$24 - 5r = -11 \quad \text{or} \quad 5r = 35 \text{ or } r = 7$$

Therefore, (7 + 1)th, i.e 8th term contains x^{-11} and its coefficient

$$= {}^{12}C_7\,(-1)^7 = (-1).\frac{12!}{7!5!} = (-1).\frac{12 \times 11 \times 10 \times 9 \times 8 \times 7!}{7! \times 5 \times 4 \times 3 \times 2 \times 1} = -792$$

(*d*) Prove that for any two sets A and B, $(A \cup B)^c = A^c \cap B^c$, where A^c denotes the complement of the set A.

Solution: Let x be an arbitrary element of $(A \cup B)^c$.

Therefore, $x \in (A \cup B)^c \Rightarrow x \notin (A \cup B)$

$$\Rightarrow x \notin A \text{ and } x \notin B$$

$$\Rightarrow x \in A^c \text{ and } x \in B^c$$

$$\Rightarrow x \in (A^c \cap B^c)$$

$\therefore \quad (A \cup B)^c \subseteq A^c \cap B^c \quad \ldots(i)$

Again, let y be an arbitrary element of $A^c \cap B^c$

Therefore, $y \in A^c \cap B^c \Rightarrow y \in A^c \text{and } y \in B^c$

$$\Rightarrow y \notin A \text{ and } y \notin B$$

$$\Rightarrow y \notin (A \cup B)$$

$$\Rightarrow y \in (A \cup B)^c$$

therefore $\quad A^c \cap B^c \subseteq (A \cup B)^c \ \ldots (ii)$

From (*i*) and (*ii*) we can write

$$(A \cup B)^c = A^c \cap B^c$$

or

Each student in a class of 40, plays at least one of the games hockey, football and cricket: 16 play hockey, 20 football, 26 cricket, 5 play hockey and football, 14 football and cricket and 2 hockey, football and cricket. Find the number of students who play hockey and cricket but not football.

Solution: Let H, F and C denote the set of students who play hockey, football and cricket respectively.

Then by problem, we have

$n(H) = 16$, $n(F) = 20$, $n(C) = 26$, $n(H \cap F) = 5$, $n(F \cap C) = 14$, $n(H \cap F \cap C) = 2$.

It is also given that each student of the class plays at least one of the three games. Therefore, total number of students in the class = $n(H \cup F \cup C) = 40$

Now we have,

$n(H \cup F \cup C) = n(H) + n(F) + n(C) - n(H \cap F) - n(H \cap C) - n(F \cap C) + n(H \cap F \cap C)$

or $40 = 16 + 20 + 26 - 5 - n(H \cap C) - 14 + 2$

or $n(H \cap C) = 5$

$\therefore$ Number of students who play hockey and cricket but not football = $n(H \cap C \cap F') = n(H \cap C) - n(H \cap C \cap F) = 5 - 2 = 3$

Group – C

3. (*a*) The original value of a machine is ₹ 24,537. Each year it depreciates by 18% of its value at the beginning of the year. In what time will the value of the machine be one tenth of the original value? [It is given that log (0.82) = –0.0862]

Solution: Let after *n* years the value of the machine be one tenth of the original value. Now, to find the value of *n* we use the following formula:

$$A = P(1 - i)^n$$

A = Value of the machine after *n* years = $\frac{1}{10} \times 24537$ = ₹ 2453.7

P = Original value of the machine = ₹ 24,537

i = rate of depreciation = 18% = $\frac{18}{100} = 0.18$

Putting these values we get,

$2453.7 = 24{,}537\,(1 - 0.18)^n$

or $(0.82)^n = \frac{1}{10}$ or $\log(0.82)^n = \log\left(\frac{1}{10}\right)$

or $n\log(0.82) = \log 1 - \log 10$

or $n \times -0.0862 = 0 - 1$

or $-n \times 0.862 = -1$ or $n = \frac{1}{0.0862} = 11.6$

Therefore, the required time = 11.6 years.

(*b*) A man buys a house of ₹ 60,00,000 on condition that he will pay ₹ 30,00,000 cash down and the balance in 10 equal annual instalments, the first to be paid one year after the date of purchase. Calculate the amount of each instalment, compound interest being computed @ 5% p.a. [Given $(1.05)^{-10} = 0.6139$]

Solution: Cash down price = ₹ 30,00,000

Balance amount to be paid in 10 equal annual instalments = (60,00,000 – 30,00,000) = ₹ 30,00,000

Let ₹ A be the required amount of each instalment.

We know that the present value

$$P = \frac{A}{i}\left[1-(1+i)^{-n}\right]$$

Here, P = ₹ 30,00,000, i = 5% = 0.05, n = 10

Putting these values, we get

$$30,00,000 = \frac{A}{0.05}\left[1-(1+0.05)^{-10}\right]$$

or $$1,50,000 = A\left[1-(1.05)^{-10}\right]$$

or $$1,50,000 = A[1-0.6139]$$

or $$1,50,000 = A \times 0.3861$$

or $$A = \frac{1,50,000}{0.3861}$$ = ₹ 3,88,500 (approx.)

Therefore, the required amount of each instalment = ₹ 3,88,500.

MODULE–II

Group–A

5. Answer the following questions:
   (*b*) Karl Pearson's coefficient of correlation between two variables x and y is 0.28, their covariance is 7.6. If the variance of x is 9, find the standard deviation of y.

Solution: Given that $r_{xy} = 0.28$, $\text{cov}(x,y) = 7.6$ and $\sigma_x^2 = 9$ or $\sigma_x = 3$

We know that, $$r_{xy} = \frac{\text{cov}(x,y)}{\sigma_x . \sigma_y}$$

or $$0.28 = \frac{7.6}{3.\sigma_y} \quad \text{or} \quad \sigma_y = \frac{7.6}{3 \times 0.28}$$

or $$\sigma_y = \frac{7.6}{2.34} = 3.25 \text{ (approx.)}$$

Therefore, standard deviation of $y(\sigma_y) = 3.25$.

(*c*) If $\bar{x}$ = 6, $\bar{y}$ = 7, b_{yx} = 0.45 and b_{xy} = 0.65, then find both regression equations.

Solution:

Regression equations of x and y:

$$x - \bar{x} = b_{xy}\,(y - \bar{y})$$

or $x - 6 = 0.65\ (y - 7)$ or $x - 6 = 0.65y - 4.55$

or $x = 0.65y - 4.55 + 6$

or $x = 0.65y + 1.45$

Regression equation of y on x:

$$y - \bar{y} = b_{xy}\ (x - \bar{x})$$

or $y - 7 = 0.45\ (x - 6)$ or $y - 7 = 0.45x - 2.70$

or $y = 0.45 \times -2.70 + 7$

or $y = 0.45x + 4.3$

or

If $b_{xy} = -0.4$ and $b_{yx} = -0.9$, find r_{xy}

Solution: We know that, $r = \sqrt{b_{xy} \times b_{yx}}$

$$= \sqrt{-0.4 \times -0.9} = \sqrt{0.36}$$

$$= 0.6$$

Since, b_{xy} and b_{yx} both are negative, therefore, $r = -0.6$.

Group – B

6. Answer the following questions:
   (*a*) Find the Price Index Number by the method of arithmetic mean of price relatives from the following.

Commodity	**Base Price**	**Current Price**
Wheat	5	7
Milk	8	10
Fish	25	32
Sugar	6	12

Solution: Calculation of Price Index Number by the method of arithmetic mean of price relatives.

Commodity	**Base Price (p_0)**	**Current Price (p_n)**	$\frac{p_n}{p_o} \times 100$
Wheat	5	7	140
Milk	8	10	125
Fish	25	32	128
Sugar	6	12	200
Total	–	–	593

$$\text{Price Index Number } (I_{on}) = \frac{\sum\left(\frac{p_n}{p_o}\times 100\right)}{N} = \frac{593}{4}$$

$$= 148.25$$

or

Find the general cost of living index of 2016 from the following table:

Class	**Food**	**Clothing**	**House Rent**	**Fuel**	**Miscellaneous**
Group Index	620	575	325	255	280
Weight	30	20	25	15	10

Solution: **Calculation of cost of living index**

Class	**Group Index (I)**	**Weight (W)**	**I.W.**
Food	620	30	18600
Clothing	575	20	11500
House Rent	325	25	8125
Fuel	255	15	3825
Miscellaneous	280	10	2800
Total		100	44,850

$$\text{Cost of living index} = \frac{\Sigma IW}{\Sigma W} = \frac{44850}{100} = 448.5$$

(*b*) From the following data, find an appropriate regression equation and predict the value of y for $x = 2.5$:

x	1	2	3	4	5	7	10
y	2	2	5	4	6	9	12

Solution: In order to predict the value of y we have to find out the regression equation of y on x.

Calculation of regression equation of y on x

x	y	x^2	xy
1	2	1	2
2	2	4	4
3	5	9	15
4	4	16	16
5	6	25	30
7	9	49	63
10	12	100	120
32	**40**	**204**	**250**

$$\bar{x} = \frac{\Sigma x}{n} = \frac{32}{7} = 4.57 \quad \text{(Here } n = 7\text{)}$$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{40}{7} = 5.71$$

$$b_{yx} = \frac{\frac{\Sigma xy}{n} - \frac{\Sigma x}{n}.\frac{\Sigma y}{n}}{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2} = \frac{\frac{250}{7} - \frac{32}{7}.\frac{40}{7}}{\frac{204}{7} - \left(\frac{32}{7}\right)^2}$$

$$= \frac{35.71 - 4.57 \times 5.71}{29.14 - (4.57)^2} = \frac{35.71 - 26.095}{29.14 - 20.885}$$

$$= \frac{9.615}{8.255} = 1.165.$$

Therefore, Regression equation of y on x

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

or $\quad y - 5.71 = 1.165\ (x - 4.57)$

or $\quad y - 5.71 = 1.165x - 5.32$

or $\quad y = 1.165x - 5.32 + 5.71$

or $\quad y = 1.165x + 0.39$

When $x = 2.5$

Then, $y = 1.165 \times 2.5 + 0.39$

$= 2.9125 + 0.39 = 3.3025$

or

The coefficient of rank correlation of the marks obtained by 10 students in Mathematics and Statistics was found to be 0.8. It was then detected that the difference in ranks in the two subjects for one particular student was wrongly taken to be in place of 7. What should be the correct rank correction coefficient?

Solution: We know that coefficient of rank correlation

$$R = 1 - \frac{6\Sigma d^2}{n^3 - n} \quad [d = \text{difference in ranks and } n = \text{no. of students}]$$

Given that $R = 0.8$, $n = 10$

Putting these values we get

$$0.8 = 1 - \frac{6\Sigma d^2}{10^3 - 10}$$

or $\dfrac{6\Sigma d^2}{1000-10} = 1-0.8$

or $\dfrac{6\Sigma d^2}{990} = 0.2$ or $6\Sigma d^2 = 198$

or $\Sigma d^2 = 33$

Correct $\Sigma d^2 = 33 - 3^2 + 7^2 = 33 - 9 + 49$

$= 33 + 40 = 73$

Therefore, correct rank correlation coefficient

$$R = 1 - \frac{6 \times 73}{1000-10} = 1 - \frac{438}{990}$$

$= 1 - 0.44 = 0.56.$

(*c*) Compute the seasonal index quarterly average for the following data:

Year	**1st quarter**	**2nd quarter**	**3rd quarter**	**4th quarter**
2010	75	60	54	59
2011	86	65	63	80
2012	90	72	66	85
2013	100	78	72	93

Solution: **Calculation of seasonal index**

Year	**1st quarter**	**2nd quarter**	**3rd quarter**	**4th quarter**	**Total**
2010	75	60	54	59	–
2011	86	65	63	80	–
2012	90	72	66	85	–
2013	100	78	72	93	–
Total	351	275	255	317	1198
Average	87.75	68.75	63.75	79.25	299.5
Grand Average	–	–	–	–	74.875
Seasonal Index $\left(\dfrac{\text{Average}}{\text{Grand Average}} \times 100\right)$	117.2	91.82	85.14	105.84	400.00

Group – C

7. (a) Construct 5-yearly moving averages of the number of students studying in a college shown below:

Year	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
No. of students	332	317	357	392	402	405	410	427	405	431	467	483

Solution: **Calculation of 5-yearly moving averages**

Year	No. of students	5-year moving total	5-year moving average
2004	332	–	–
2005	317	–	–
2006	357	1800	360.0
2007	392	1873	374.6
2008	402	1966	393.2
2009	405	2036	407.2
2010	410	2049	409.8
2011	427	2078	415.6
2012	405	2140	428.0
2013	431	2213	442.6
2014	467	–	–
2015	483	–	–

or

Fit a straight line trend equation by the method of least squares and estimate the trend values:

Year	2008	2009	2010	2011	2012	2013	2014	2015	2016
Values	117	123	130	139	143	151	158	163	170

Hence estimate the production for the year 2017.

Solution: Let $y = a + bt$ be the equation of the straight line trend. Here the number of years is odd, so mid-year 2012 is taken as origin and one year as unit.

Fitting a straight line trend

Year	Values (y)	t = year – 2012	t^2	$y.t$
2008	117	–4	16	–468
2009	123	–3	9	–369
2010	130	–2	4	–260
2011	139	–1	1	–139

Year	Values (y)	t = year – 2012	t^2	$y.t$
2012	143	0	0	0
2013	151	1	1	151
2014	158	2	4	316
2015	163	3	9	489
2016	170	4	16	680
Total	$\Sigma y = 1294$	$\Sigma t = 0$	$\Sigma t^2 = 60$	$\Sigma yt = 400$

Normal equations are $\Sigma y = na + b\Sigma t$ … (i)

and $\Sigma yt = a\Sigma t + b\Sigma t^2$ … (ii)

From (i), $1294 = 9a + b.0$ or $9a = 1294$ or $a = 143.78$

From (ii), $400 = a.0 + b.60$ or $60b = 400$ or $b = 6.67$

Therefore, the trend equation is

$y = 143.78 + 6.67.t$ with origin at 2012 and t unit = 1 year.

The value of t for the year 2017 will be 5. Hence the estimate of production for the year 2017 is

$$y = 143.78 + 6.67 \times 5 = 143.78 + 33.35 = 177.13$$

8. (b) Using the following data, verify that Paasche's formula does not satisfy Factor Reversal Test:

Commodity	Base year		Current year	
	Price	Quantity	Price	Quantity
X	4	10	6	15
Y	6	15	4	20
Z	8	5	10	4

Solution: Let p_0, p_n denote prices in the base year and current year and q_0, q_n the quantities in the base year and current year respectively.

Commodity	p_0	q_0	p_n	q_n	p_0q_0	p_nq_0	p_0q_n	p_nq_n
X	4	10	6	15	40	60	60	90
Y	6	15	4	20	90	60	120	80
Z	8	5	10	4	40	50	32	40
					170	170	212	210

$$\Sigma p_0q_0 = 170, \Sigma p_nq_0 = 170, \Sigma p_0q_n = 212, \Sigma p_nq_n = 210$$

Omitting the factor 100 from each index,

$$P_{0n} = \frac{\Sigma p_nq_n}{\Sigma p_0q_n} = \frac{210}{212} = 0.99$$

$$Q_{on} = \frac{\Sigma p_nq_n}{\Sigma p_oq_n} = \frac{210}{170} = 1.235$$

Value ratio = $\dfrac{\sum p_n q_n}{\sum p_0 q_0} = \dfrac{210}{170} = 1.235$

Clearly, $P_{0n} \times Q_{0n} \neq \dfrac{\sum p_n q_n}{\sum p_0 q_0}$

That is the product of the price index and quantity index is not equal to the value ratio of the current period and base period.

Therefore, Paasche's index formula does not satisfy Factor Reversal Test.

PART–III (HONS.) – 2017

Group – C

3. Answer the following equations:

(*a*) Four men in a company of 10 employees are engineers. If 2 men are selected at random, then find the probability that exactly one of them will be an engineer.

Solution: Out of 10 employees four are engineers.

Therefore, other than engineers = 10 – 4 = 6

2 men are selected at random

So, total number of cases = ${}^{10}C_2 = \dfrac{10\times 9}{2\times 1} = 45$

Out of 2 men selected at random, one will be an engineer, so, second man will be other than engineer.

Therefore, the number of favourable cases = ${}^{6}C_1 \times {}^{4}C_1$

$= 6 \times 4 = 24$

Therefore, the probability that exactly one will be an engineer = $\dfrac{24}{45} = \dfrac{8}{15}$

or

A problem of Mathematics is given to three students A, B and C whose chances of solving it are $\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}$ respectively. What is the chance that the problem will be solved?

Solution: Let X, Y and Z denote the respective event that A, B and C can solve the given problem.

Then we have $P(X) = \dfrac{1}{2}$, $P(Y) = \dfrac{1}{3}$ and $P(Z) = \dfrac{1}{4}$

Therefore, $P(X^c) = 1 - P(X) = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

$P(Y^c) = 1 - P(Y) = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

$$P(Z^c) = 1 - P(Z) = 1 - \frac{1}{4} = \frac{3}{4}$$

The probability that the problem will not be solved

$= P\left(X^c \cap Y^c \cap Z^c\right) = P\left(X^c\right).P\left(Y^c\right).P\left(Z^c\right)$ (As the events are independent)

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$$

Therefore, the probability that the problem will be solved =

$$P\left(X \cup Y \cup Z\right) = 1 - P\left(X^c \cap Y^c \cap Z^c\right)$$

$$= 1 - \frac{1}{4} = \frac{3}{4}.$$

FOR ALL OTHER CATEGORIES OF CANDIDATES

MODULE – I

Group – A

1. Answer the following questions:

   (*a*) If ${}^nP_5 : {}^nP_3 = 2 : 1$, find the value of n.

Solution: ${}^nP_5 : {}^nP_3 = 2 : 1$

or $\quad \frac{n!}{(n-5)!} : \frac{n!}{(n-3)!} = 2 : 1$

or $\quad \frac{n!}{(n-5)!} \times \frac{(n-3)(n-4)(n-5)!}{n!} = \frac{2}{1}$

or $\quad (n-3)(n-4) = 2.1 = (5-3)(5-4)$

or $\quad n = 5$

or

If ${}^nC_x = 56$ and ${}^nP_x = 336$, find n and x.

Solution: ${}^nC_x = 56$

or $\quad \frac{n!}{x!(n-x)!} = 56 \quad \ldots (i)$

${}^nP_x = 336$

or $\quad \frac{n!}{(n-x)!} = 336 \quad \ldots (ii)$

Dividing equation (*ii*) by equation (*i*), we get

$$\frac{n!}{(n-x)!} \div \frac{n!}{x!(n-x)!} = 336 \div 56$$

or $$\frac{n!}{(n-x)!} \times \frac{x!(n-x)!}{n!} = \frac{336}{56}$$

or $$x! = 6 = 3.2.1 = 3!$$

Therefore, $x = 3$

Putting the value of 'x' in equation (ii) we get

$$\frac{n!}{(n-x)!} = 336 \quad \text{or} \quad \frac{n(n-1)(n-2)(n-3)!}{(n-3)!} = 336$$

or $$n(n-1)(n-2) = 8.7.6$$
$$= 8(8-1)(8-2)$$

Therefore, $n = 8$

(b) Find the number of different odd numbers of 5 digits that can be formed with the digits 1, 2, 3, 4, 5, 6 without repetition.

Solution: A number is odd if its unit place is occupied by an odd number. In this case there are 3 odd numbers namely 1, 3, 5. Thus, the unit place can be filled in $^3P_1$ ways = 3 ways.

After filling the unit place, the remaining 4 places can be filled by the remaining 5 numbers in $^5P_4$ ways, i.e. $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

Therefore, required odd numbers = $3 \times 120 = 360$.

(c) Find the power set of {2, 3, 5}

Solution: Let A = {2, 3, 5}

Subsets of set A = {2}, {3}, {5}, {2, 3}, {3, 5}, {2, 5}, {2, 3, 5} Φ

Therefore, power set of set A

P(A) = [{2}, {3}, {5}, {2, 3}, {3, 5}, {2, 5}, {2, 3, 5}, Φ]

or

Given A = {1, 2, 3} and B = {4, 6, 8}. Find A × B

Solution: A = {1, 2, 3} and B = {4, 6, 8}

Therefore, A × B = {(1, 4), (1, 6), (1, 8), (2, 4), (2, 6), (2, 8), (3, 4), (3, 6), (3, 8)}

(e) If A = {x : x is a natural number and $x \le 6$},
B = {x : x is the natural number and $3 \le x \le 8$},

Find A – B and A ∩ B.

Solution: A = {x : x is a natural number and $x \le 6$}
= {1, 2, 3, 4, 5, 6}

B = {x : x is the natural number and $3 \le x \le 8$}
= {3, 4, 5, 6, 7, 8}

Therefore, A – B = {1, 2}
$A \cap B$ = {3, 4, 5, 6}.

Group – B

2. Answer the following questions:
   (*a*) From 6 bowlers, 2 wicket keepers and 8 batsmen; in how many ways can a team of 11 players consisting of at least 4 bowlers, 1 wicket keeper and at least 5 batsmen be formed?

Solution: The problem can be arranged as under.

Situation	Bowlers (6)	Wicket keepers (2)	Batsmen (8)	Total
(*i*)	4	1	6	11
(*ii*)	5	1	5	11

Therefore, total number of ways of selection

$$= {}^6C_4 \times {}^2C_1 \times {}^8C_6 + {}^6C_5 \times {}^2C_1 \times {}^8C_5$$

$$= \frac{6\times5}{2\times1}\times2\times\frac{8\times7}{2\times1}+6\times2\times\frac{8\times7\times6}{3\times2\times1}$$

= 840 + 672 = 1512 ways.

or

A committee of 7 is to be chosen from 13 students of whom 6 are science students and 7 are commerce students. In how many ways can the selection be made so as to retain a majority in the committee for commerce students?

Solution: The problem can be arranged as under.

Situation	Science students (6)	Commerce students (7)	Total
(*i*)	3	4	7
(*ii*)	2	5	7
(*iii*)	1	6	7

Therefore, total number of ways of selection

$$= {}^6C_3 \times {}^7C_4 + {}^6C_2 \times {}^7C_5 + {}^6C_1 \times {}^7C_6$$

$$= \frac{6\times5\times4}{3\times2\times1}\times\frac{7\times6\times5}{3\times2\times1}+\frac{6\times5}{2\times1}\times\frac{7\times6}{2\times1}+6\times7$$

= 700 + 315 + 42 = 1057

(*b*) Prove that, $\log_3\left(\sqrt{3\sqrt{3\sqrt{3\text{..........}\infty}}}\right) = 1$

Solution: Let $x = \sqrt{3\sqrt{3\sqrt{3.........\infty}}}$ $\quad \therefore \quad x^2 = 3\sqrt{3\sqrt{3.....\infty}}$

[Squaring both sides]

or $\quad x^2 = 3x$ or $x^2 - 3x = 0$

or $\quad x(x-3) = 0$

or $\quad x - 3 = 0$ (as $x \neq 0$)

or $\quad x = 3$

$\therefore \quad \log_3\left(\sqrt{3\sqrt{3\sqrt{3........\infty}}}\right) = \log_3 3 = 1$

Or

If $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}(\log a + \log b)$, show that $\frac{a}{b} + \frac{b}{a} = 7$.

Solution: $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}.(\log a + \log b)$

or $\quad \log\left(\frac{a+b}{3}\right) = \frac{1}{2}\log ab$

or $\quad \log\left(\frac{a+b}{3}\right) = \log ab^{\frac{1}{2}} = \log\sqrt{ab}$

or $\quad \frac{a+b}{3} = \sqrt{ab}$

or $\quad a + b = 3\sqrt{ab}$

or $\quad (a+b)^2 = 9ab$ [Squaring both sides]

or $\quad a^2 + b^2 + 2ab = 9ab$

or $\quad a^2 + b^2 = 7ab$

or $\quad \frac{a^2}{ab} + \frac{b^2}{ab} = \frac{7ab}{ab}$ [Dividing both sides by *ab*]

or $\quad \frac{a}{b} + \frac{b}{a} = 7$

(*c*) If the coefficient of $(r+3)^{th}$ term in the expansion of $(1+x)^{47}$ be the same as the coefficient of $(3r+2)^{th}$ term, find these two terms.

Solution:

$(r+3)^{th}$ term in the expansion of $(1+x)^{47}$

$t_{r+3} = t_{(r+2)+1}$

$= {}^{47}C_{r+2} \cdot x^{r+2}$

$(3r + 2)^{th}$ term in the expansion of $(1 + x)^{47}$

$$t_{3r+2} = t_{(3r+1)+1} = {}^{47}C_{3r+1}x^{3r+1}$$

Therefore, the coefficients of $(r + 3)^{\text{th}}$ term and $(3r + 2)^{\text{th}}$ term are respectively ${}^{47}C_{r+2}$ and ${}^{47}C_{3r+1}$

Now, ${}^{47}C_{r+2} = {}^{47}C_{3r+1}$ (according to the problem)

or $(r + 2) + (3r + 1) = 47$

or $4r + 3 = 47$ or $4r = 44$ or $r = 11$

(*d*) In a class of 100 students, 55 students read History, 41 students read English and 25 students read both the subjects. Find the number of students who study neither of the subjects.

Solution: Let H and E be the set of students who read History and English respectively.

Then, $n(\text{H}) = 55$, $n(\text{E}) = 41$, $n(\text{H} \cap \text{E}) = 25$ and $\text{N} = 100$

We are to calculate $\text{n}(\text{H}' \cap \text{E}')$

We know, $n(\text{H} \cup \text{E}) = n(\text{H}) + n(\text{E}) - n(\text{H} \cap \text{E})$

$= 55 + 41 - 25$

$= 96 - 25 = 71$

Now, $n(\text{H}' \cap \text{E}') = n(\text{H} \cup \text{E})' = 100 - n(\text{H} \cup \text{E})$

$= 100 - 71 = 29$

or

For any three sets A, B, C prove that

$$\text{A} \times (\text{B} \cap \text{C}) = (\text{A} \times \text{B}) \cap (\text{A} \times \text{C})$$

Proof: Let (a, b) be an arbitrary element of $\text{A} \times (\text{B} \cap \text{C})$

Therefore, $(a, b) \in \text{A} \times (\text{A} \cap \text{C})$

$\Rightarrow a \in \text{A}$ and $b \in (\text{B} \cap \text{C})$

$\Rightarrow a \in \text{A}$ and $\{b \in \text{B}$ and $b \in c\}$

$\Rightarrow \{a \in \text{A}$ and $b \in \text{B}\}$ and $\{a \in \text{A}$ and $b \in c\}$

$\Rightarrow (a, b) \in \text{A} \times \text{B}$ and $(a, b) \in \text{A} \times \text{C}$

$\Rightarrow (a, b) \in (\text{A} \times \text{B}) \cap (\text{A} \times \text{C})$

Therefore, $\text{A} \times (\text{B} \cap \text{C}) \subseteq (\text{A} \times \text{B}) \cap (\text{A} \times \text{C})$ … (*i*)

Again, let (c, d) be an arbitrary element of $(\text{A} \times \text{B}) \cap (\text{A} \times \text{C})$

Therefore, $(c, d) \in (\text{A} \times \text{B}) \cap (\text{A} \times \text{C})$

$\Rightarrow (c, d) \in (\text{A} \times \text{B})$ and $(c, d) \in (\text{A} \times \text{C})$

$\Rightarrow \{\text{C} \in \text{A}$ and $d \in \text{B}\}$ and $\{\text{C} \in \text{A}$ and $d \in c\}$

$\Rightarrow \text{C} \in \text{A}$ and $\{d \in \text{B}$ and $d \in c\}$

$\Rightarrow \text{C} \in \text{A}$ and $d \in (\text{B} \cap \text{C})$

$\Rightarrow (c, d) \in \text{A} \times (\text{B} \cap \text{C})$

Therefore, $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ … (*ii*)

From (*i*) and (*ii*) we can write

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Group – C

3. (*a*) Find the present value of ₹ 10,000 due in 12 years at 6% p.a. compound interest. [Given $(1.06)^{12} = 2.012$]

Solution: Here, A = ₹ 10,000, $n = 12$, $i = 0.06$

Present value $(P) = A(1 + i)^{-n}$

$$= 10{,}000 \cdot (1 + 0.06)^{-12}$$
$$= 10{,}000 \cdot (1.06)^{-12}$$
$$= \frac{10{,}000}{(1.06)^{12}} = \frac{10{,}000}{2.012}$$
$$= 4970 \text{ (Approx.)}$$

Hence, the present value = ₹ 4,970.

(*b*) A man wishes to buy a house valued at ₹ 50,00,000. He is prepared to pay ₹ 20,00,000 now and the balance in 10 equal instalments. If the interest is calculated at 8% p.a. compound annually, what should he pay annually?

$$\left[\textit{Given} \frac{1}{(1.08)^{10}} = 0.4634\right]$$

Solution:

Cash down price = ₹ 20,00,000

Balance amount to be paid in 10 equal instalments = ₹ (50,00,000 – 20,00,000) = ₹ 30,00,000.

Let ₹ A be the required amount of each instalment.

We know that the present value

$$P = \frac{A}{i}\left[1 - (1 + i)^{-n}\right]$$

Here, P = ₹ 30,00,000, $i = 8\% = 0.08$, $n = 10$

Putting these values, we get

$$30{,}00{,}000 = \frac{A}{0.08}\left[1 - (1 + 0.08)^{-10}\right]$$

or $$2{,}40{,}000 = A\left[1 - (1.08)^{-10}\right]$$

or $$2{,}40{,}000 = A(1 - 0.4634)$$

or $\quad 2,40,000 = A \times 0.5366$

or $\quad A = \dfrac{2,40,000}{0.5366} =$ ₹ 4,47,260.53

MODULE – II

Group – A

5. Answer the following question:
   (*c*) Karl Pearson's coefficient of correlation between two variables *x* and *y* is 0.52, their covariance is 7.8. If the variance of *x* is 16, find the S.D. of *y*.

Solution: Given that, $r_{xy} = 0.52, \operatorname{cov}(x, y) = 7.8$

$$\sigma_x^2 = 16 \text{ or } \sigma_x = 4$$

We are to calculate the value of σ_y

we know that, $r_{xy} = \dfrac{\operatorname{cov}(x, y)}{\sigma_x \cdot \sigma_y}$

or $\quad 0.52 = \dfrac{7.8}{4 \cdot \sigma_y} \quad$ or $\quad \sigma_y = \dfrac{7.8}{2.08} = 3.75$.

   (*d*) If the two regression coefficients are $b_{xy} = -0.4$ and $b_{yx} = -0.9$, find the value of correlation coefficient r_{xy}.

Solution: We know that, $r = \sqrt{b_{xy} \times b_{yx}}$

$$= \sqrt{-0.4 \times -0.9} = \sqrt{0.36}$$
$$= 0.6$$

Since, b_{xy} and b_{yx} are negative,

Therefore, $r = -0.6$

Group – B

6. Answer the following questions:
   (*b*) Calculate the correlation coefficient of the following data;

x	63	60	67	70	61	69
y	61	65	64	63	63	68

Solution: **Calculation of correlation coefficient**

x	y	$U = x - \bar{x}$	$V = x - \bar{x}$	u^2	v^2	$u.v$
63	61	–2	–3	4	9	6
60	65	–5	1	25	1	–5

x	y	$U = x - \bar{x}$	$V = x - \bar{x}$	u^2	v^2	$u.v$
67	64	2	0	4	0	0
70	63	5	–1	25	1	–5
61	63	–4	–1	16	1	4
69	68	4	4	16	16	16
390	**384**	**0**	**0**	**90**	**28**	**16**

$$\bar{x} = \frac{\Sigma x}{n} = \frac{390}{6} = 65$$

$$\bar{y} = \frac{\Sigma y}{x} = \frac{384}{6} = 64$$

$$r_{xy} = r_{uv} = \frac{\Sigma uv}{\sqrt{\Sigma u^2}\sqrt{\Sigma v^2}} = \frac{16}{\sqrt{90}\sqrt{28}}$$

$$= \frac{16}{\sqrt{2520}} = \frac{16}{50.2} = 0.32$$

or

You are given that variance of x = 36. The regression equations are $60x - 27y = 321$ are $12x - 15y + 99 = 0$. Find the correlation coefficient between the variables; the average values of x and y; the S.D. of y.

Solution: Given that $\sigma_x^2 = 36$ or $\sigma_x = 6$

Let $60x - 27y = 321$ be the regression equation of x on y and $12x - 15y + 99 = 0$ be the regression equation of y on x.

$$60x - 27y = 321$$

or $$60x = 27y + 321$$

or $$x = \frac{27}{60}y + \frac{321}{60}$$

$\therefore$ $$b_{xy} = \frac{27}{60} = \frac{9}{20}$$

Again, $12x - 15y + 99 = 0$

or $$15y = 12x + 99$$

or $$y = \frac{12}{15}x + \frac{99}{15}$$

Therefore, $b_{yx} = \frac{12}{15} = \frac{4}{5}$

Now, $r^2 = b_{xy} \times b_{yx} = \frac{9}{20} \times \frac{4}{5} = \frac{9}{25}$

or $\quad r = \sqrt{\frac{9}{25}} = \frac{3}{5} = 0.6 < 1$

Since the value of r is less than unity, our assumption is correct.

Therefore, $r = 0.6$

$$60x - 27y = 321 \ \ldots (i)$$
$$12x - 15y = -99 \ \ldots (ii)$$

Multiplying equation (*ii*) by 5 and then subtracting from equation (*i*) we get

$$\begin{array}{rrrrr} 60x & - & 27y & = & 321 \\ 60x & - & 75y & = & -495 \\ - & & + & & + \\ \hline & & 48y & = & 816 \end{array}$$

or $\quad y = \frac{816}{48} = 17$

Putting the value of 'y' in equation (*ii*) we get

$12x - 15 \times 17 = -99$

or $\quad 12x - 255 = -99 \quad$ or $\quad 12x = 255 - 99 = 156$

or $\quad x = \frac{156}{12} = 13$

Hence, $\bar{x} = 13$ and $\bar{y} = 17$

We know that, $b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}$

or $\quad \frac{4}{5} = 0.6 \cdot \frac{\sigma_y}{6}$ or $\cdot \frac{4.8}{0.6} = \sigma_y$

or $\quad \sigma_y = 8$

Therefore, S.D. of $y(\sigma_y) = 8$

(*e*) Construct seasonal indices from the following time series data:

Year/quarter	I	II	III	IV
2014	90	75	87	70
2015	75	80	78	75
2016	80	75	75	72

Solution: **Calculation of seasonal indices**

Year/quarter	I	II	III	IV	Total
2014	90	75	87	70	–
2015	75	80	78	75	–
2016	80	75	75	72	–
Total	245	230	240	217	932
Average	81.7	76.67	80	72.33	310.7

Year/quarter	I	II	III	IV	Total
Grand average	–	–	–	–	77.675
Seasonal indices $\frac{\text{Average}}{\text{Grand Average}} \times 100$	105.18	98.71	103	93.11	400

Group –C

7. (*a*) Fit a least squares trend line to the following data:

Year	2008	2009	2010	2011	2012	2013	2014
Average production per month ('0000 tons)	20	22	21	24	25	23	28

Hence, find the average production per month in the year 2017.

Solution: Let $y = a + bt$ be the equation of the straight line trend. Here the number of years is odd, so mid year 2011 is taken as origin and one year as unit.

Fitting a straight line trend

Year	**Average production per month ('0000 tons) (*y*)**	***t* = year – 2011**	t^2	***y.t***
2008	20	–3	9	–60
2009	22	–2	4	–44
2010	21	–1	1	–21
2011	24	0	0	0
2012	25	1	1	25
2013	23	2	4	46
2014	28	3	9	84
Total	$\sum y = 163$	$\sum t = 0$	$\sum t^2 = 28$	$\sum yt = 30$

Normal equations are:

$$\sum y = na + b\sum t \quad \ldots (i)$$

and $$\sum yt = a\sum t + b\sum t^2 \quad \ldots (ii)$$

From (*i*), $163 = 7a + b.\,0$ or $7a = 163$ or $a = 23.28$

From (*ii*), $30 = a.\,0 + b.\,28$ or $28b = 30$ or $b = 1.07$

Therefore, the trend equation is

$y = 23.28 + 1.07t$ with origin at 2011 and t unit = 1 year.

The value of t for the year 2017 will be 6. Hence, the average production ('0000 tons) per month for the year 2017

$$y = 23.28 + 1.07 \times 6$$
$$= 23.28 + 6.42 = 29.7$$

or

For the following series of observations, verify that the 4-year centred moving averages are equivalent to a 5-year weighted moving average with weights 1, 2, 2, 2, 1 respectively:

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
Sale (₹ '0000)	2	6	1	5	3	7	2	6	4	8	3

Solution: **Calculation of 4-year centred moving average**

Year	Sales (₹ '0000)	4-year moving total (not centered)	2-item moving total (centered)	4-year moving average
2006	2	–	–	–
2007	6	–	–	–
		14		
2008	1		29	3.625
		15		
2009	5		31	3.875
		16		
2010	3		33	4.125
		17		
2011	7		35	4.375
		18		
2012	2		37	4.625
		19		
2013	6		39	4.875
		20		
2014	4		41	5.125
		21		
2015	8	–	–	–
2016	3	–	–	–

Calculation of 5-year weighted moving average

Year(*a*)	Sales (*b*) (₹ '0000)	5 – year weighted moving total (*c*)	5–year weighted moving average (*d* = *c* / 8)
2006	2	–	–
2007	6	–	–
2008	1	$2 \times 1 + 6 \times 2 + 1 \times 2 + 5 \times 2 + 3 \times 1 = 29$	3.625
2009	5	$6 \times 1 + 1 \times 2 + 5 \times 2 + 3 \times 2 + 7 \times 1 = 31$	3.875
2010	3	$1 \times 1 + 5 \times 2 + 3 \times 2 + 7 \times 2 + 2 \times 1 = 33$	4.125
2011	7	$5 \times 1 + 3 \times 2 + 7 \times 2 + 2 \times 2 + 6 \times 1 = 35$	4.375
2012	2	$3 \times 1 + 7 \times 2 + 2 \times 2 + 6 \times 2 + 4 \times 1 = 37$	4.625

Year(*a*)	**Sales (*b*) (₹ '0000)**	**5 – year weighted moving total (*c*)**	**5–year weighted moving average (*d* = *c* / 8)**
2013	6	$7 \times 1 + 2 \times 2 + 6 \times 2 + 4 \times 2 + 8 \times 1 = 39$	4.875
2014	4	$2 \times 1 + 6 \times 2 + 4 \times 2 + 8 \times 2 + 3 \times 1 = 41$	5.125
2015	8	–	–
2016	3	–	–

From the last column of the above two tables it is clear that 4-year centered moving averages are equivalent to a 5-year weighted moving average with weights 1, 2, 2, 2, 1 respectively.

8. (*a*) From the following data, prove that Fisher's Ideal formula satisfies both time Reversal and Factor Reversal Tests of index number:

Commodity	**Base year**		**Current year**	
	Price	**Quantity**	**Price**	**Quantity**
A	4	20	6	18
B	5	15	6	12
C	2	30	3	30
D	3	25	5	28

Solution: Let p_0, p_n denote prices in the base year and current year and q_0, q_n be the quantities in the base year and current year respectively.

Commodity	p_0	q_0	p_n	q_n	p_0q_0	p_nq_0	p_0q_n	p_nq_n
A	4	20	6	18	80	120	72	108
B	5	15	6	12	75	90	60	72
C	2	30	3	30	60	90	60	90
D	3	25	5	28	75	125	84	140
Total	–	–	–	–	**290**	**425**	**276**	**410**

$\Sigma p_0q_0 = 290$, $\Sigma p_nq_0 = 425$, $\Sigma p_0q_n = 276$, $\Sigma p_nq_n = 410$

Fisher's Ideal index number (omitting the factor 100) is

$$I_{0n} = \sqrt{\frac{\sum p_nq_0}{\sum p_0q_0} \times \frac{\sum p_nq_n}{\sum p_0q_n}} = \sqrt{\frac{425}{290} \times \frac{410}{276}}$$

Interchanging '0' and '*n*', we get

$$I_{n0} = \sqrt{\frac{\sum p_0q_n}{\sum p_nq_n} \times \frac{\sum p_0q_0}{\sum p_nq_0}}$$

$$= \sqrt{\frac{276}{410} \times \frac{290}{425}}$$

Now, $I_{0n} \times I_{n0} = \sqrt{\frac{425}{290} \times \frac{410}{276}} \times \sqrt{\frac{276}{410} \times \frac{290}{425}}$

$$= \sqrt{\frac{425}{290} \times \frac{410}{276} \times \frac{276}{410} \times \frac{290}{425}} = \sqrt{1} = 1$$

Hence, Fisher's Ideal formula satisfies Time Reversal Test.

Again, Fisher's Price Index Number (omitting the factor 100) is

$$P_{0n} = \sqrt{\frac{\Sigma p_n q_0}{\Sigma p_0 q_0} \times \frac{\Sigma p_n q_n}{\Sigma p_0 q_n}} = \sqrt{\frac{425}{290} \times \frac{410}{276}}$$

Fisher's Quantity Index Number (omitting the factor 100)

$$Q_{0n} = \sqrt{\frac{\Sigma q_n p_0}{\Sigma q_0 p_0} \times \frac{\Sigma q_n p_n}{\Sigma q_0 p_n}} = \sqrt{\frac{276}{290} \times \frac{410}{425}}$$

Then, $P_{0n} \times Q_{0n} = \sqrt{\frac{425}{290} \times \frac{410}{276}} \times \sqrt{\frac{276}{290} \times \frac{410}{425}}$

$$= \sqrt{\frac{425}{290} \times \frac{410}{276} \times \frac{276}{290} \times \frac{410}{425}}$$

$$= \sqrt{\frac{410 \times 410}{290 \times 290}} = \frac{410}{290}$$

Now, value ratio = $\frac{\Sigma p_n q_n}{\Sigma p_0 q_0} = \frac{410}{290}$

Since, $P_{0n} \times Q_{0n}$ = value ratio.

Therefore, Fisher's Ideal Formula satisfies the Factor Reversal Test.

APPENDIX

B

Log Tables

Logarithms

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	14	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	4	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10

Logarithms

											Mean Differences								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9653	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Antilogarithms

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

Antilogarithms

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20