CIRCUIT THEORY

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He is an eminent writer and till now he has authored thirteen engineering books which are very popular among engineering students. He is known by name through his books in all engineering colleges in South India and some colleges in North India.

CIRCUIT THEORY

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Chennai

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Wife, **Mrs. C. Gnanaparanjothi (B.Sc, M.L.)**

Elder Son, **N. Bharath Raj**

Younger Son, **N. Vikram Raj**

CONTENT

CHAPTER 4 - AC SINGLE AND THREE-PHASE CIRCUITS............................ 4.1

PREFACE

The main objective of this book is to explore the basic concepts of Circuit Theory in a simple and easy-to-understand manner.

This text on Circuit Theory has been crafted and designed to meet students' requirements. Considering the highly mathematical nature of this subject, more emphasis has been given on the problem-solving methodology. Considerable effort has been made to elucidate mathematical derivations in a step-by-step manner. Exercise problems with varied difficulty levels are given in the text to help students get an intuitive grasp on the subject.

This book, with its lucid writing style and germane pedagogical features, will prove to be a master text for engineering students and practitioners.

Salient Features

The salient features of this book are:

- Proof of important concepts and theorems are clearly highlighted by shaded boxes
- Wherever required, problems are solved in multiple methods
- Additional explanations for solutions and proofs are provided in separate boxes
- Different types of fonts are used for text, proof and solved problems for better clarity
- Keywords are highlighted by bold and italic fonts
- Easy, concise and accurate study material
- Extremely precise edition where concepts are reinforced by pedagogy
- Demonstration of multiple techniques in problem solving–additional explanations and proofs highlighted
- Ample figures and examples to enhance students' understanding
- Practice through MCQ's
- Rich Pedagogy:
  - Solved Numerical Examples: 249
  - Short-answer Questions: 219
  - Figures: 1549
  - Practice Problems: 135
  - Review Questions (T/F): 109
  - MCQs: 145
  - Fill in the blanks: 109

Organization

This text is designed for an undergraduate course in Circuit Theory for engineering students. The book is organized into five chapters. The fundamental concepts, steady state analysis and transient state analysis are presented in a very easy and elaborative manner. Throughout the book, carefully chosen examples are presented so that the reader will have a clear understanding of the concepts discussed.

Chapter 1 starts with explanation of fundamental quantities involved in circuit theory, standard symbols and units used in circuit theory. The basic concepts of circuits are also presented in this chapter. The mesh and node analysis of circuits are discussed with special attention to dependent sources.

The concepts of series, parallel and star-delta network reduction are discussed in Chapter 2. The analysis of circuits using theorems are also presented in this Chapter.

The transient analysis of circuits are explained in Chapter 3 through Laplace transform. The analysis of single and three-phase circuits and measurement of power in three-phase circuits are presented in Chapter 4.

The concepts of resonance are discussed in detail in Chapter 5. The analysis of coupled circuits are also discussed.

The Laplace transform has been widely used in the analysis of Electric Circuits. Hence, an appendix on Laplace transform is included in this book. All the calculations in this book are performed using calculator in complex mode. An appendix is also included to help the readers to practice calculations in complex mode of calculator.

Since circuit theory is introduced as a course in the first year of engineering curriculum in most of the universities, this subject is considered tough by students entering into engineering courses. Hence, the author has taken special care in presenting the concepts in simple manner supported by carefully chosen solved problems.

Online Learning Center

The OLC of the book can be accessed at *http://www.mhhe.com/nagoorkani/ct/au*

The author hopes that that the teaching and student community will welcome the book. The readers can feel free to convey their criticism and suggestions to **kani@vsnl.com** for further improvement of the book.

A. Nagoor Kani

Publisher's Note

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Piracy-related issues may also be reported.

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A. Nagoor Kani

List of Symbols and Abbreviations

P	-	Active power
$\overline{Y}$	-	Admittance
AC, ac	-	Alternating current
A	-	Ampere
ω	-	Angular frequency
ω_r	-	Angular resonance frequency
S	-	Apparent Power
I_{ave}	-	Average value of current
V_{ave}	-	Average value of voltage
β	-	Bandwidth
B	-	Branch
C	-	Capacitance
X_C	-	Capacitive reactance
B_C	-	Capacitive susceptance
Q	-	Charge
k	-	Coefficient of coupling
j	-	Complex operator ($j = \sqrt{-1}$)
$\overline{S}$	-	Complex Power
G	-	Conductance
C	-	Coulomb
k_C	-	Critical coefficient of coupling
R_C	-	Critical resistance
I	-	Current
$i(0^+)$	-	Current at $t = 0^+$
$i(0^-)$	-	Current at $t = 0^-$
$i(\infty)$	-	Current at $t = \infty$
CC	-	Current Coil
$\bar{I}(j\omega), \bar{I}$	-	Current in frequency domain
I(s)	-	Current in Laplace domain
$i(t)$	-	Current in time domain

ζ	-	Damping ratio
E	-	DC source voltage
Δ	-	Determinant of matrix
DC, dc	-	Direct current
Y	-	Driving point admittance
Z	-	Driving point impedance
η_B	-	Efficiency of battery
W	-	Energy
R_{eq}	-	Equivalent resistance
F	-	Farad
ϕ	-	Flux
Ψ	-	Flux linkage
k_f	-	Form factor
π	-	Half period
H	-	Henry
Hz	-	Hertz
ω_h	-	Higher cut-off angular frequency
f_h	-	Higher cut-off frequency
j	-	Imaginary part
$\bar{Z}$	-	Impedance
θ	-	Impedance angle
L	-	Inductance
X_L	-	Inductive reactance
B_L	-	Inductive susceptance
e, $e(t)$	-	Instantaneous value of ac source voltage
q	-	Instantaneous value of charge
i, $i(t)$	-	Instantaneous value of current in time domain
i_C	-	Instantaneous value of current through capacitor
i_L	-	Instantaneous value of current through inductor
i_R	-	Instantaneous value of current through resistor
w	-	Instantaneous value of energy
p	-	Instantaneous value of power

v_C	-	Instantaneous value of voltage across capacitor
v_L	-	Instantaneous value of voltage across inductor
v_R	-	Instantaneous value of voltage across resistor
v, v(t)	-	Instantaneous value of voltage in time domain
J	-	Joule
K	-	Kelvin
kWh	-	kilowatt-hour
KCL	-	Kirchhoff's Current Law
KVL	-	Kirchhoff's Voltage Law
$\mathcal{L}$	-	Laplace operator
L	-	Links
I_L	-	Load Current
V_L	-	Load Voltage
R_L	-	Load Resistance
ω_l	-	Lower cut-off angular frequency
f_l	-	Lower cut-off frequency
Z	-	Magnitude of impedance
Y	-	Magnitude of admittance
I_m	-	Maximum value of current
V_m	-	Maximum value of voltage
m	-	Mesh
℧	-	Mho
M	-	Mutual inductance
ω_n	-	Natural frequency
$\bar{I}_N$	-	Neutral current
N	-	Neutral point
N	-	Nodes
Ω	-	Ohm
Ω-m	-	Ohm-metre
OC	-	Open circuit
k_p	-	Peak factor
ϕ	-	Phase difference between voltage and current

pf	-	Power factor
ϕ	-	Power factor angle
P	-	Power or Active power
PC	-	Pressure Coil
Q	-	Quality factor
Q_r	-	Quality factor at resonance
rad/s	-	Radians/second
X	-	Reactance
Q	-	Reactive Power
R	-	Resistance
ρ	-	Resistivity
f_r	-	Resonance frequency
s	-	Second
SC	-	Short circuit
S	-	Siemen
SDPT	-	Single Pole Double Throw
R_S	-	Source Resistance
B	-	Susceptance
T	-	Tesla
t	-	Time
τ	-	Time constant
V	-	Volt
VAR	-	Volt-Ampere-Reactive
V	-	Voltage
$v(0^+)$	-	Voltage at $t = 0^+$
$v(0^-)$	-	Voltage at $t = 0^-$
$v(\infty)$	-	Voltage at $t = \infty$
$\overline{V}(j\omega), \overline{V}$	-	Voltage in frequency domain
V(s)	-	Voltage in Laplace domain
W	-	Watt
W-h	-	Watt-hour
W-s	-	Watt-second
Wb	-	Weber/Weber-turn

Chapter 1

BASIC CIRCUIT ANALYSIS

1.1 Introduction to Circuits and Networks

1.1.1 Basic Phenomena

The energy associated with flow of electrons is called **electrical energy**. The flow of electrons is called **current**. The current can flow from one point to another point of an element only if there is a potential difference between these two points. The potential difference is called **voltage**.

When electric current is passed through a device or element, three phenomena have been observed. The three phenomena are,

(i) opposition to flow of current,

(ii) opposition to change in current or flux, and

(iii) opposition to change in voltage or charge.

The various effects of current like heating, arcing, induction, charging, etc., are due to the above phenomena. Therefore, three fundamental elements have been proposed which exhibit only one of the above phenomena when considered as an ideal element (of course, there is no ideal element in nature). These elements are resistor, inductor and capacitor.

1.1.2 Ideal Elements

The **ideal resistor** offers opposition only to the flow of current. The property of opposition to the flow of current is called **resistance** and it is denoted by R.

The **ideal inductor** offers opposition only to change in current (or flux). The property of opposition to change in current is called **inductance** and it is denoted by L.

The **ideal capacitor** offers opposition only to change in voltage (or charge). The property of opposition to change in voltage is called **capacitance** and it is denoted by C.

1.1.3 Electric Circuits

The behaviour of a device to electric current can be best understood if it is modelled using the fundamental elements R, L and C. For example, an incandescent lamp and a water heater can be modelled as ideal resistance. Transformers and motors can be modelled using resistance and inductance.

Practically, an **electric circuit** is a model of a device operated by electrical energy. The various concepts and methods used for analysing a circuit is called **circuit theory**. A typical circuit consists of sources of electrical energy and ideal elements R, L and C. The practical energy sources are batteries, generators (or alternators), rectifiers, transistors, op-amps, etc. The various elements of electric circuits are shown in Figs 1.1 and 1.2.

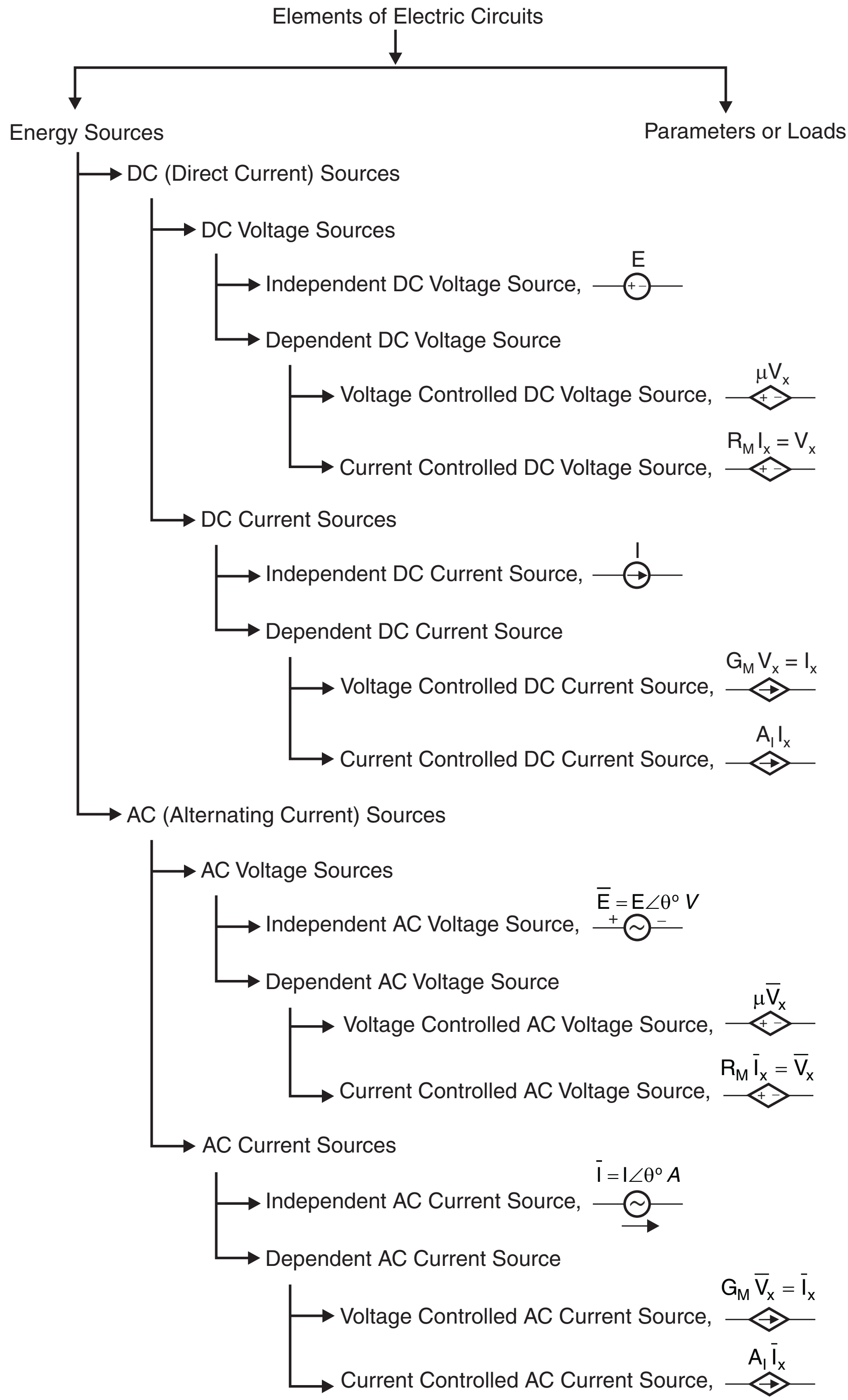


Fig. 1.1 : *Elements of electric circuits - Energy source.*

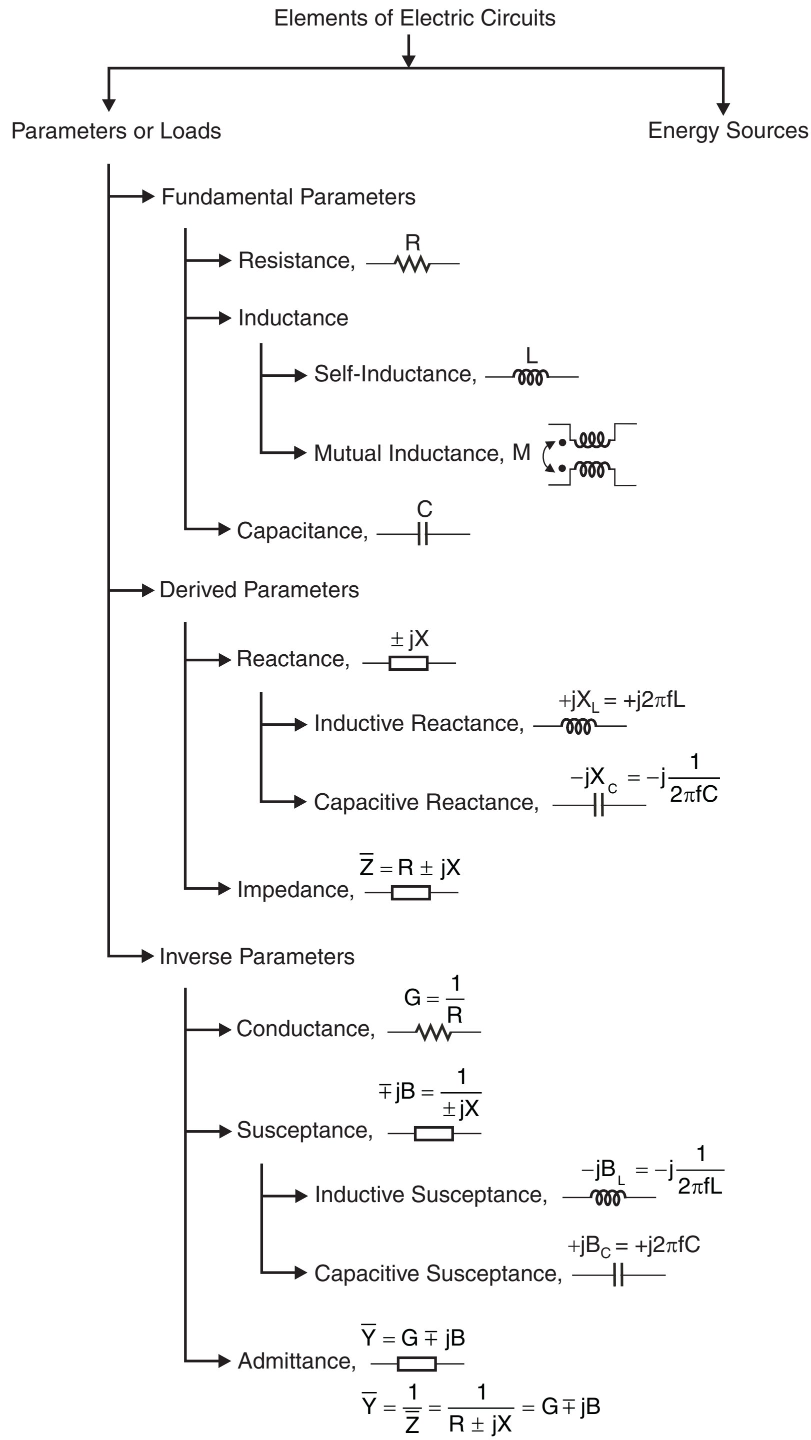


***Fig. 1.2 :** Elements of electric circuits - Parameters or loads.*

Elements which generate or amplify energy are called **active elements**. Therefore, energy sources are active elements. Elements which dissipate or store energy are called **passive elements**. Resistance dissipates energy in the form of heat, inductance stores energy in a magnetic field, and capacitance stores energy in an electric field. Therefore, resistance, inductance and capacitance are passive elements. If there is no active element in a circuit then the circuit is called a **passive circuit** or **network**.

Sources can be classified into independent and dependent sources. Batteries, generators and rectifiers are independent sources, which can directly generate electrical energy. Transistors and op-amps are dependent sources whose output energy depends on another independent source.

Practically, the sources of electrical energy used to supply electrical energy to various devices like lamps, fans, motors, etc., are called **loads**. The rate at which electrical energy is supplied is called **power**. Power in turn is the product of voltage and current.

Circuit analysis relies on the concept of **law of conservation of energy**, which states that energy can neither be created nor destroyed, but can be converted from one form to other. Therefore, the total energy/power in a circuit is zero.

1.1.4 Units

SI units are followed in this book. The SI units and their symbols for various quantities encountered in circuit theory are presented in Table 1.1. In engineering applications, large values are expressed with decimal multiples and small values are expressed with submultiples. The commonly used multiples and submultiples are listed in Table 1.2.

Table 1.1 : Units and Symbols

Quantity	Symbol for quantity	Unit	Unit symbol	Equivalent unit	Equivalent unit symbol
Charge	q, Q	Coulomb	C	-	-
Current	i, I	Ampere	A	Coulomb/second	C/s
Flux linkages	ψ	Weber-turn	Wb	-	-
Magnetic flux	ϕ	Weber	Wb	-	-
Energy	w, W	Joule	J	Newton-meter	N-m
Voltage	v, V	Volt	V	Joule/Coulomb	J/C
Power	p, P	Watt	W	Joule/second	J/s
Capacitance	C	Farad	F	Coulomb/Volt	C/V
Inductance	L, M	Henry	H	Weber/Ampere	Wb/A
Resistance	R	Ohm	Ω	Volt/Ampere	V/A
Conductance	G	Siemens	S	Ampere/Volt or mho	A/V *or* ℧

Table 1.1: Continued...

Quantity	Symbol for quantity	Unit	Unit symbol	Equivalent unit	Equivalent unit symbol
Time	t	Second	s	-	-
Frequency	f	Hertz	Hz	cycles/second	-
Angular frequency	ω	Radians/second	rad/s	-	-
Magnetic flux density	-	Tesla	T	Weber/ meter square	Wb/m^2
Temperature	-	Kelvin	^{o}K	-	-

Table 1.2 : Multiple and Submultiple used for Units

Multiplying factor	Prefix	Symbol
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h
10^{1}	deca	da

Multiplying factor	Prefix	Symbol
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

1.1.5 Definitions of Various Terms

The definitions of various terms that are associated with electrical energy like energy, power, current, voltage, etc., are presented in this section.

Energy : **Energy** is defined as the capacity to do work. It can also be defined as stored work. Energy may exist in many forms, such as electrical, mechanical, thermal, light, chemical, etc. It is measured in joules, which is denoted by J (or the unit of energy is joules).

In electrical engineering, one **joule** is defined as the energy required to transfer a power of one watt in one second to a load (or Energy = Power × Time). Therefore, $1J = 1\,W\text{-}s$.

In mechanical engineering, one joule is the energy required to move a mass of 1 kg through a distance of 1 m with a uniform acceleration of 1 m/s^2.

$$\text{Therefore, } 1\,J = 1\,N\text{-}m = 1\,kg\text{-}\frac{m}{s^2}\text{-}m$$

In thermal engineering, one joule is equal to a heat of 4.1855 (or 4.186) calories, and one **calorie** is the heat energy required to raise the temperature of 1 gram of water by $1^{o}C$.

$$\text{Therefore, } 1\,J = 4.1855\,calories$$

Power : **Power** is the rate at which work is done (or it is the rate of energy transfer). The unit of power is watt and denoted by W. If energy is transferred at the rate of one joule per second then one **watt** of power is generated.

An average value of power can be expressed as,

$$\text{Power, } \mathrm{P} = \frac{\text{Energy}}{\text{Time}} = \frac{\mathrm{W}}{\mathrm{t}} \qquad .....(1.1)$$

A time varying power can be expressed as,

$$\text{Instantaneous power, } p = \frac{\mathrm{d}w}{\mathrm{dt}} \qquad .....(1.2)$$

$$\text{Also, } p = \frac{\mathrm{d}w}{\mathrm{dt}} = \frac{\mathrm{d}w}{\mathrm{d}q} \times \frac{\mathrm{d}q}{\mathrm{dt}} = v\,i \qquad .....(1.3)$$

Hence, power is also given by the product of voltage and current.

Charge : **Charge** is the characteristic property of elementary particles of matter. The elementary particles are electrons, protons and neutrons. There are basically two types of charges in nature: positive charge and negative charge. The charge of an electron is called **negative charge**. The charge of a proton is called **positive charge**. Normally, a particle is neutral because it has equal number of electrons and protons. The particle is called charged, if some electrons are either added or removed from it. If electrons are added then the particle is called negatively charged. If electrons are removed then the particle is called positively charged.The unit used for measurement of charge is coulomb. One **coulomb** is defined as the charge which when placed in vacuum from an equal and similar charge at a distance of one metre repels it with a force of $9 \times 10^{9}\,N$. The charge of an electron is $1.602 \times 10^{-19}\,C$. Hence, $1/(1.602 \times 10^{-19}) = 6.24 \times 10^{18}$ electrons make up a charge of one coulomb.

Current : **Current** is defined as the rate of flow of electrons. It is measured in amperes. One **ampere** is the current flowing through a point if a charge of one coulomb crosses that point in one second. In SI units, one ampere is defined as that constant current in two infinite parallel conductors of negligible circular cross-section, one metre apart in vacuum, which produces a force between the conductors of 2×10^{-7} newton per metre length.

A steady current can be expressed as,

$$\text{Current, } \mathrm{I} = \frac{\text{Charge}}{\text{Time}} = \frac{\mathrm{Q}}{\mathrm{t}} \qquad .....(1.4)$$

A time varying current can be expressed as,

$$\text{Instantaneous current, } i = \frac{\mathrm{d}q}{\mathrm{dt}} \qquad .....(1.5)$$

where, Q = Charge flowing at a constant rate

t = Time

dq = Change in charge in a time of dt

dt = Time required to produce a change in charge dq.

Voltage : Every charge will have potential energy. The difference in potential energy between the charges is called **potential difference**. In electrical terminology, the potential difference is called **voltage**. Potential difference indicates the amount of work done to move a charge from one place to another. Voltage is expressed in volt. One **volt** is the potential difference between two points, when one joule of energy is utilised in transfering one coulomb of charge from one point to the other.

A steady voltage can be expressed as,

$$\text{Voltage, V} = \frac{\text{Energy}}{\text{Charge}} = \frac{\text{W}}{\text{Q}} \qquad .....(1.6)$$

A time varying voltage can be expressed as,

$$\text{Instantaneous voltage, } v = \frac{dw}{dq} \qquad .....(1.7)$$

$$\text{Also, } 1V = \frac{1\,J}{1\,C} = \frac{1\,J/s}{1\,C/s} = \frac{1\,W}{1\,A} \qquad .....(1.8)$$

$$\therefore \text{ Voltage, V} = \frac{\text{Power}}{\text{Current}} = \frac{\text{P}}{\text{I}} \qquad .....(1.9)$$

One **volt** is also defined as the difference in electric potential between two points along a conductor carrying a constant current of one ampere when the power dissipated between the two points is one watt.

1.1.6 Symbols used for Average, RMS and Maximum Values

The quantities like voltage, current, power and energy may be constant or varying with respect to time. For a time varying quantity we can define the value of the quantity as instantaneous, average, rms and maximum value. The symbols used for these values are listed in Table 1.3.

Table 1.3 : Symbols of DC and AC Variables

Quantity	DC	AC or Time varying				
		Instantaneous value	**Average value**	**Maximum value**	**RMS value**	**Phasors or Vectors**
Current	I	i	I_{ave}	I_m or I_p	I	$\bar{I}$
Voltage	V	v	V_{ave}	V_m or V_p	V	$\bar{V}$
Power	P	p	P	P_m	-	$\bar{S}$
Energy	W	w	W	W_m	-	-

1.1.7 Steady State Analysis and Transient Analysis

Circuit analysis can be classified into steady state analysis and transient analysis. The analysis of circuits during switching conditions is called **transient analysis**. During switching conditions, the current and voltage change from one value to the other. In purely resistive circuits this may not be a problem because the resistance will allow sudden change in voltage and current.

But in inductive circuits the current cannot change instantaneously and in capacitive circuits the voltage cannot change instantaneously. Hence, when the circuit is switched from one state to the other, the voltage and current cannot attain a steady value instantaneously in inductive or capacitive circuits. Therefore, during switching conditions there will be a small period during which the current and voltage will change from an initial value to a final steady value. The time from the instant of switching to the attainment of steady value is called **transient period**. Physically, the transient can be realised in switching of tubelights, fans, motors, etc.

In certain circuits the transient period is negligible and we may be interested only in steady value of the response. Therefore, steady state analysis is sufficient. The analysis of circuits under steady state (i.e., by neglecting the transient period) is called **steady state analysis**. Steady state analysis of circuits is discussed in this book in all chapters except Chapter 3.

In certain circuits the transient period is critical and we may require the response of the circuit during the transient period. Some practical examples where transient analysis is vital are starters, circuit breakers, relays, etc. Transient analysis of circuits is discussed in Chapter 3.

1.1.8 Assumptions in Circuit Theory

In circuit analysis the elements of the circuit are assumed to be linear, bilateral and lumped elements.

In **linear elements** the voltage-current characteristics are linear and the circuit consisting of linear elements is called **linear circuit** or **network**. The resistor, inductor and capacitor are linear elements. Some elements exhibit non-linear characteristics. For example, diodes and transistors have non-linear voltage-current characteristics, capacitance of a varactor diode is non-linear and inductance of an inductor with hystereris is non-linear. For analysis purpose, the non-linear characteristics can be linearised over a certain range of operation.

In a **bilateral element**, the relationship between voltage and current will be the same for two possible directions of current through the element. On the other hand, a **unilateral element** will have different voltage-current characteristics for the two possible directions of current through the element. The diode is an example of a unilateral element.

In practical devices like transmission lines, windings of motors, coils, etc., the parameters (R, L and C) are distributed in nature. But for analysis purpose we assume that the parameters are lumped (i.e., concentrated at one place). This approximation is valid only for low frequency operations and it is not valid in the microwave frequency range. All analysis in this book is based on the assumption that the elements are linear, bilateral and lumped elements.

1.2 Basic Concepts of Circuits and Networks

1.2.1 Basic Elements of Circuits

Circuits and Networks

An electric circuit consists of Resistors **(R)**, Inductors **(L)**, Capacitors **(C)**, voltage sources and/or current sources connected in a particular combination. When the sources are removed from a circuit, it is called a **network**.

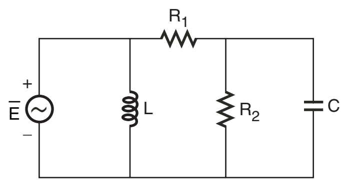


Fig. a : *Circuit.*

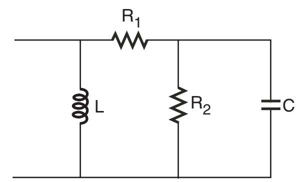


Fig. b : *Network.*

Fig. 1.3 : *Example of circuit and network.*

DC Circuits

The networks excited by dc sources are called **dc circuits**. In a dc source, the voltage and current do not change with time. Hence, the property of capacitance and inductance will not arise in steady state analysis of dc circuits.This chapter deals with steady state analysis of dc circuits. Therefore, in this chapter only resistive circuits are discussed.

Active and Passive Elements

The elements of a circuit can be classified into active elements and passive elements. The elements which can deliver energy are called **active elements**. The elements which consume energy either by absorbing or storing are called **passive elements**.

The active elements are voltage and current sources. The sources can be of different nature. The sources in which the current/voltage does not change with time are called **direct current sources** or in short **dc sources**. (But in dc sources, the current/voltage changes with load). The sources in which the current/voltage sinusoidally varies with time are called **sinusoidal sources** or **alternating current sources** or in short **ac sources**.

The passive elements of a circuit are resistors, inductors and capacitors, which exhibit the property of resistance, inductance and capacitance, respectively under ideal conditions. Resistance, inductance and capacitance are called **fundamental parameters** of a circuit. Practically, these parameters will be distributed in nature. For example, the resistance of a transmission line will exist throughout its length. But for circuit analysis, the parameters are considered as lumped.

The resistor absorbs energy (and the absorbed energy is converted into heat). The inductor and the capacitor store energy. When the power supply in the circuit is switched ON, the inductor and the capacitor store energy, and when the supply is switched OFF, the stored energy leaks away in the leakage path. (Hence, inductors and capacitors cannot be used as storage devices).

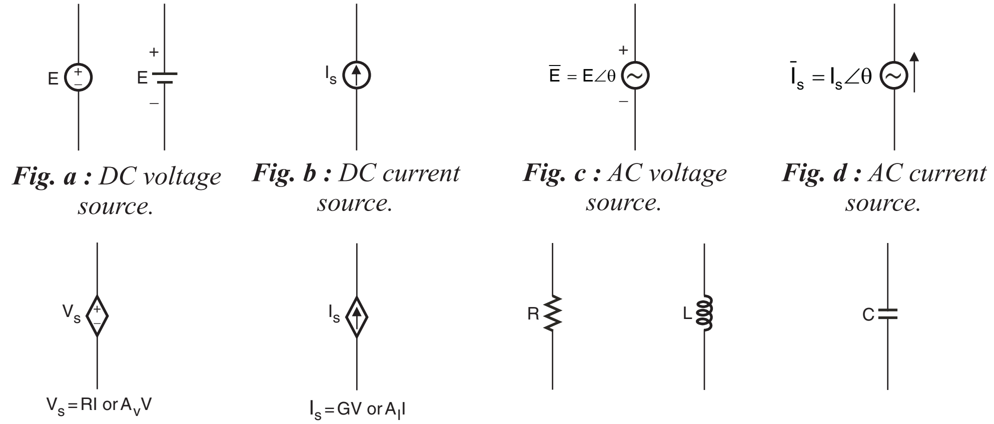


Fig. a : *DC voltage source.* ***Fig. b :*** *DC current source.* ***Fig. c :*** *AC voltage source.* ***Fig. d :*** *AC current source.*

Fig. e : *Dependent voltage source.* ***Fig. f :*** *Dependent current source.* ***Fig. g :*** *Resistance.* ***Fig. h :*** *Inductance.* ***Fig. i :*** *Capacitance.*

Fig. 1.4 : *Symbols of active and passive elements of circuits.*

Independent and Dependent Sources

Sources can be classified into independent and dependent sources. The electrical energy supplied by an independent source does not depend on another electrical source. Independent sources convert energy in some form into electrical energy. For example, a generator converts mechanical energy into electrical energy, a battery converts chemical energy into electrical energy, a solar cell converts light energy into electrical energy, a thermocouple converts heat energy into electrical energy, etc.

The electrical energy supplied by a dependent source depends on another source of electrical energy. For example, the output signal (energy) of a transistor or op-amp depends on the input signal (energy), where the input signal is another source of electrical energy.

In the circuit sense, the voltage/current of an **independent source** does not depend on voltage/ current in any part of the circuit. But the voltage/current of a **dependent source** depends on the voltage/current in some part of the same circuit.

1.2.2 Nodes, Branches and Closed Path

A typical circuit consists of lumped parameters, such as resistance, inductance, capacitance and sources of electrical energy like voltage and current sources connected through resistance-less wires.

In a circuit, the meeting point of two or more elements is called a **node**. If more than two elements meet at a node then it is called the **principal node**.

The path between any two nodes is called a **branch**. A branch may have one or more elements connected in series.

A **closed path** is a path which starts at a node and travels through some part of the circuit and arrives at the same node without crossing a node more than once.

The nodes, branches and closed paths of a typical circuit are shown in Fig. 1.5. The nodes of the circuit are the meeting points of the elements denoted as A, B, C, D, E and F. The nodes A, B, C and D are principal nodes because these nodes are meeting points of more than two elements.

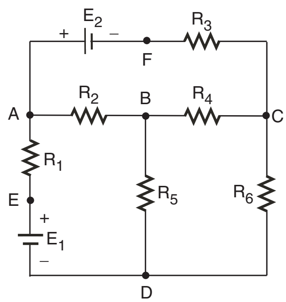


Fig. a : *Typical circuit.*

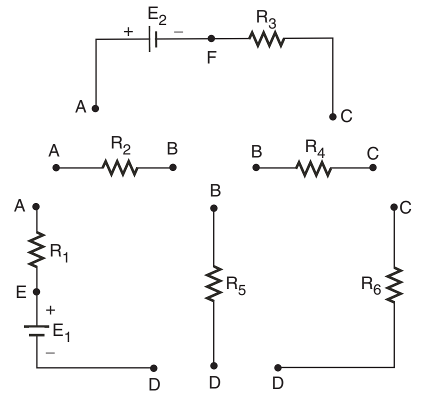


Fig. b : *Branches of the circuit in Fig. a.*

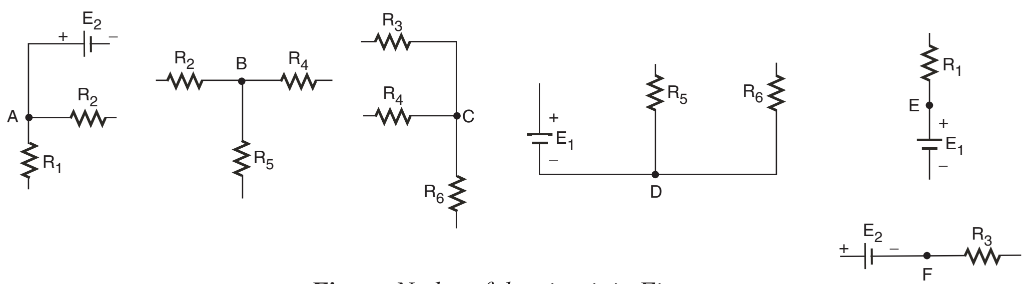


Fig. c : *Nodes of the circuit in Fig. a.*

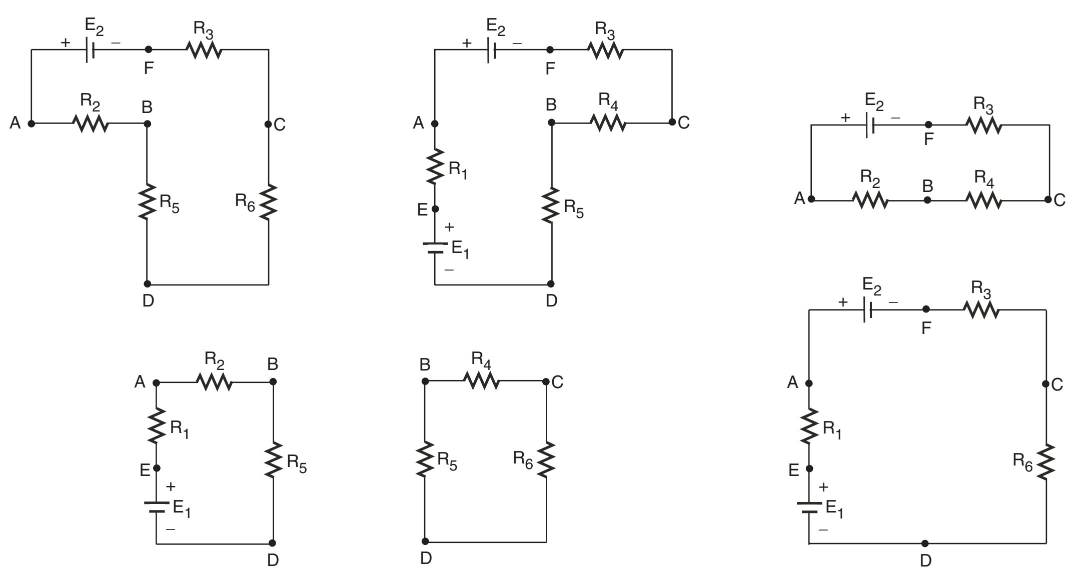


Fig. d : *Closed paths of the circuit in Fig. a.*

Fig. 1.5 : *A typical circuit and its branches, nodes and closed paths.*

1.2.3 Series, Parallel, Star and Delta Connections

The various types of connections that we may encounter in electric circuits are series, parallel, star and delta connections.

Series Connection

If two or more elements are connected such that the current through them is the same then the connection is called a **series connection**. In a circuit if the current in a path is the same then the elements in that path are said to be in series.

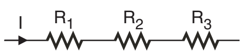


Fig. a : *Resistances in series.*

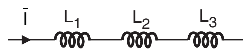


Fig. b : *Inductances in series.*

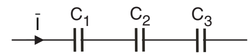


Fig. c : *Capacitances in series.*

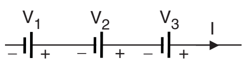


Fig. d : *Voltage sources in series.*

Fig. e : *Resistance and inductance in series.*

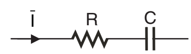


Fig. f : *Resistance and capacitance in series.*

Fig. g : *Resistance, inductance and capacitance in series.*

Fig. 1.6 : *Examples of series connected elements.*

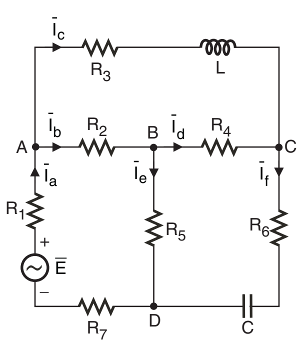


Fig. a : *Typical circuit.*

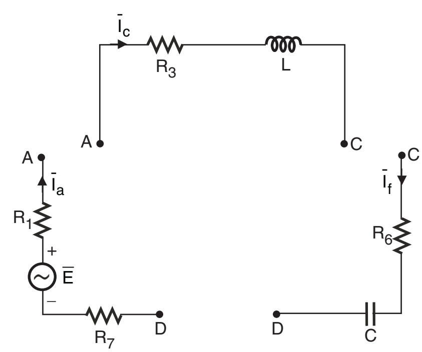


Fig. b : *Series paths in the circuit of Fig. a.*

Fig. 1.7 : *A typical circuit and its series paths.*

Parallel Connection

If two or more elements are connected such that the voltage across them is the same then the connection is called a **parallel connection**. In a circuit if the voltage across two or more paths is the same then, they are said to be in parallel.

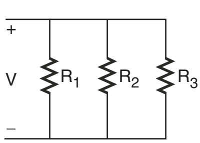


Fig. a : *Resistances in parallel.*

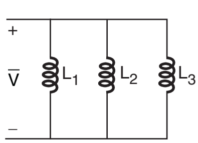


Fig. b : *Inductances in parallel.*

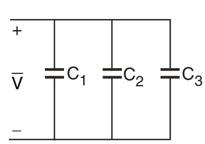


Fig. c : *Capacitances in parallel.*

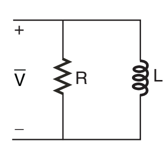


Fig. d : *R and L in parallel.*

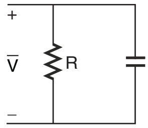


Fig. e : *R and C in parallel.*

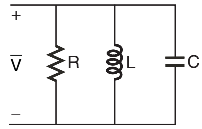


Fig. f : *R, L and C in parallel.*

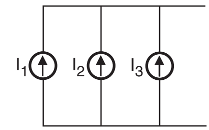


Fig. g : *Current sources in parallel.*

Fig. 1.8 : *Examples of parallel connected elements.*

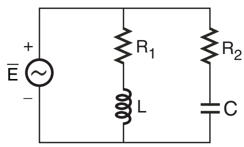


Fig. a : *The voltage source, series combination of R_1 and L and series combination of R_2 and C are in parallel.*

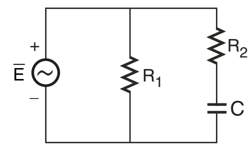


Fig. b : *The voltage source, Resistance R_1 and series combination of R_2 and C are in parallel.*

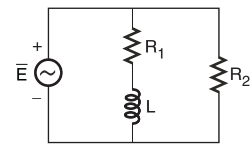


Fig. c : *The voltage source, series combination of R_1 and L and resistance R_2 are in parallel.*

Fig. 1.9 : *Simple circuits with parallel branches.*

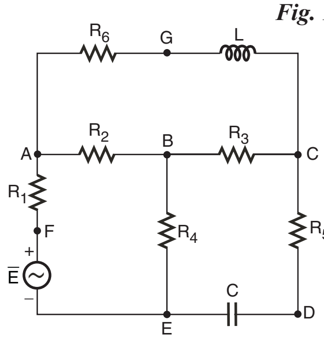


Fig. a : *A typical circuit.*

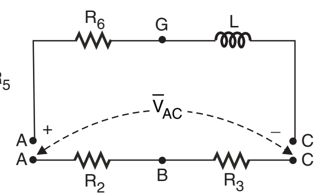


Fig. b : *The path AGC is parallel to the path ABC.*

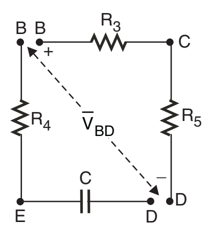


Fig. c : *The path BCD is parallel to the path BED.*

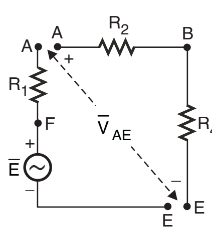


Fig. d : *The path ABE is parallel to the path AFE.*

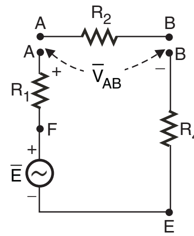


Fig. e : *The path AFEB is parallel to the resistance R_2.*

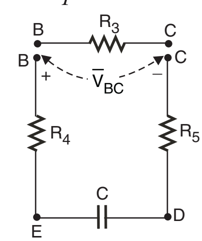


Fig. f : *The path BEDC is parallel to the resistance R_3.*

Fig. 1.10 : *A typical circuit and its parallel paths.*

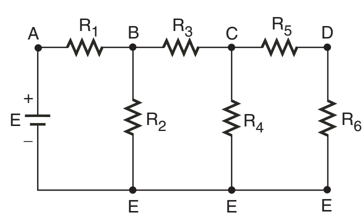


Fig. a : *A typical circuit.*

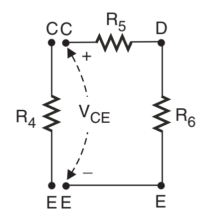


Fig. b : *R_4 in parallel with series combination of R_5 and R_6.*

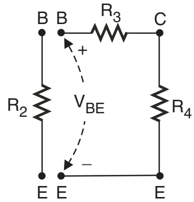


Fig. c : The path BCE is in parallel to resistance R_2.

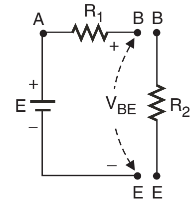


Fig. d : The path EAB is in parallel to resistance R_2.

Fig. 1.11 : *A typical circuit and its parallel paths.*

Star-Delta Connection

If three elements are connected to meet at a node then the three elements are said to be in a **star connection**. If three elements with a node in between any two elements are connected to form a closed path then they are said to be in a **delta connection**. The star connection is also called **T-connection** and delta connection is also called **Π-connection**.

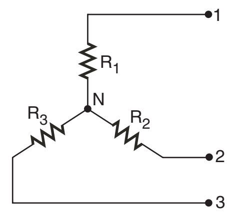


Fig. a : *Star connection.*

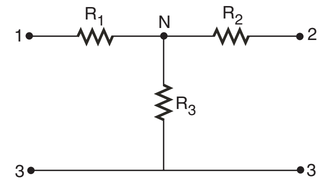


Fig. b : *T-connection.*

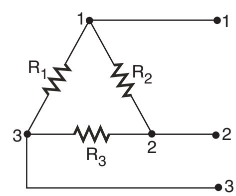


Fig. c : *Delta connection.*

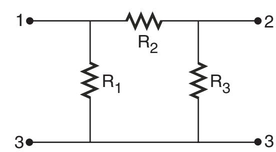


Fig. d : *Π-connection.*

Fig. 1.12 : *Basic star and delta connections.*

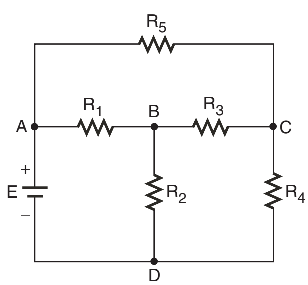


Fig. a : *A typical circuit.*

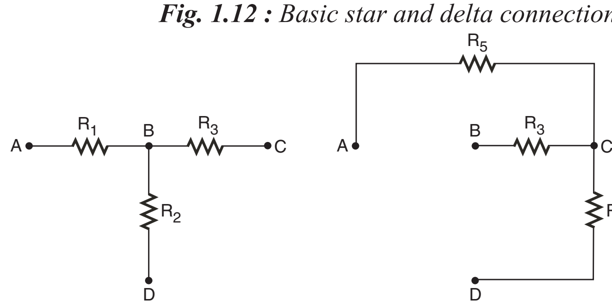


Fig. b : *Star connections in circuit of Fig. a.*

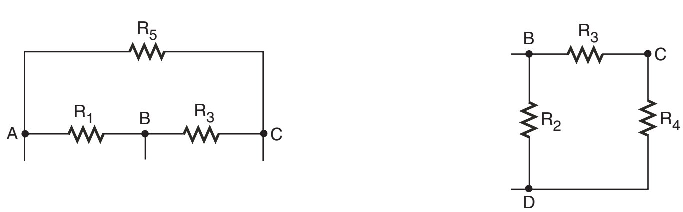


Fig. c : *Delta connections in circuit of Fig. a.*

Fig. 1.13 : *A typical circuit and its star and delta connections.*

1.2.4 Open Circuit and Short Circuit

In a circuit if there is an open path or path of infinite resistance between two nodes then that path is called an **open circuit** (OC). Since current can flow only in closed paths, the current in the open circuit will be zero.

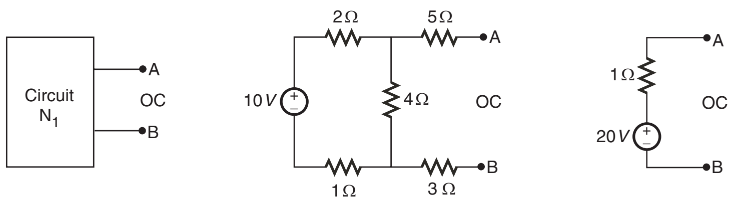


Fig. 1.14 : *Examples of open circuit (OC).*

While applying KVL to closed paths the open circuit can be included as an element of infinite resistance in the path because a voltage exists across the two open nodes of a circuit.

In a circuit if there is a closed path of zero resistance between two nodes then it is called **short circuit** (SC). Since the resistance of the short circuit is zero, the voltage across the short circuit is zero.

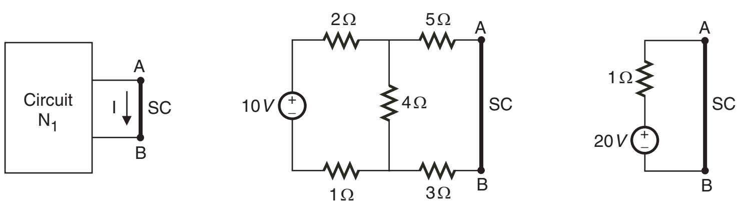


Fig. 1.15 : *Examples of short circuit (SC).*

In a circuit if there are elements parallel to a short circuit then they will not carry any current because the current will prefer the path of least resistance (or opposition) and so the entire current will flow through the short circuit. Hence, the elements parallel to a short-circuit need not be considered for analysis as shown in the example circuit of Fig. 1.16.

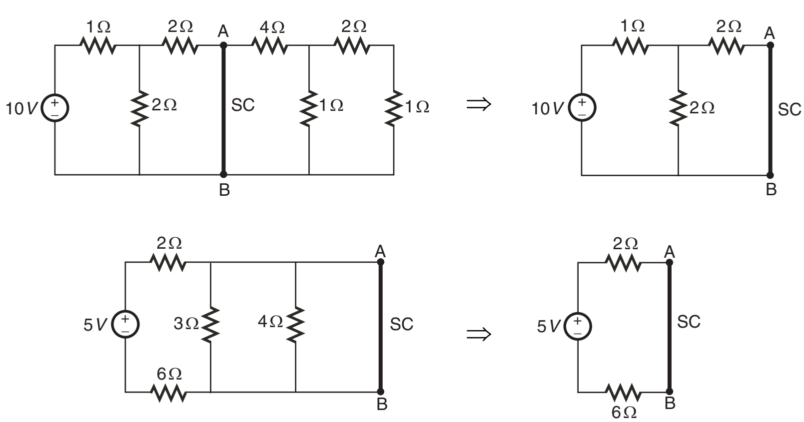


Fig. 1.16 : *Examples of short circuit.*

1.2.5 Sign Conventions

Every element of a circuit will have two terminals. When a circuit is excited (i.e., power supply is switched ON) a voltage is developed across the two terminals of the element such that one end is positive and the other end is negative, and a current flows through the element. When an element delivers energy, the current leaves the element from the positive terminal and when an element absorbs energy, the current enters at the positive terminal.

In a circuit, normally the sources deliver energy and the passive elements–resistance, inductance and capacitance absorb energy. Therefore, in a voltage/current source, when it delivers energy, the current leaves from the positive terminal. In the parameters R, L and C, the current enters at the positive terminal when they absorb energy.

Fig. a : *Voltage source delivering energy.*

Fig. b : *Current source delivering energy.*

Fig. c : *Resistance absorbing energy.*

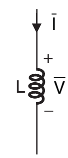


Fig. d : *Inductance absorbing energy.*

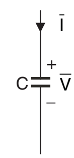


Fig. e : *Capacitance absorbing energy.*

Fig. 1.17 : *Sign conventions for sources when it delivers energy and parameters when they absorb energy.*

A chargeable battery is the best example for understanding the concept of energy delivery and absorption by sources. When the battery is connected to a load, it delivers energy. When the battery is charged, it absorbs energy. When a source absorbs energy, the current enters the source at the positive terminal, as shown in Fig. 1.18.

Fig. a : *Voltage source absorbing energy.*

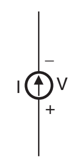


Fig. b : *Current source absorbing energy.*

Fig. 1.18 : *Sign conventions for sources when they absorb energy.*

The resistance always absorbs energy but the inductance and capacitance can deliver the stored energy temporarily. The inductance and capacitance store energy when the supply is switched ON and when the supply is switched OFF the stored energy is discharged in the available paths or leakage paths. When the inductance and capacitance discharge energy, the current leaves from the positive terminal as shown in Fig. 1.19.

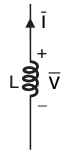


Fig. a : *Inductance discharging energy.*

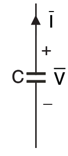


Fig. b : *Capacitance discharging energy.*

Fig. 1.19 : *Sign conventions for inductance and capacitance parameters when they discharge energy.*

1.2.6 Voltage and Current Sources

Voltage and current are two quantities that decide the energy supplied by the sources of electrical energy. Usually, the sources are operated by maintaining one of the two quantities as constant and by allowing the other quantity to vary depending on the load.

When voltage is maintained constant and current is allowed to vary then the source is called a **voltage source**. When current is maintained constant and voltage is allowed to vary then the source is called a **current source**.

1.2.7 Ideal and Practical Sources

In ideal conditions the voltage across an **ideal voltage source** should be constant for whatever current is delivered by the source. Similarly, the **ideal current source** should deliver a constant current for whatever voltage across its terminals.

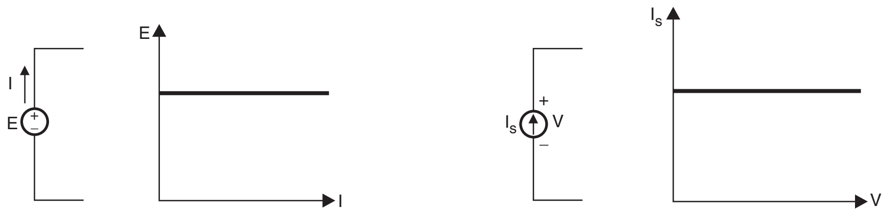


Fig. a : *Characteristics of an ideal voltage source.* ***Fig. b :*** *Characteristics of an ideal current source.*

Fig. 1.20 : *Characteristics of ideal sources.*

In reality, ideal conditions never exist (but for analysis purpose, the sources can be considered ideal). In practical voltage source, the voltage across the source decreases with increasing load current and the reduction in voltage is due to its **internal resistance**. In a practical current source, the current delivered by the source decreases with increasing load voltage and the reduction in current is due to its internal resistance.

Fig. a : *Characteristics of practical voltage source.* ***Fig. b :*** *Characteristics of practical current source.*

Fig. 1.21 : *Characteristics of practical sources.*

Let, E_s = Voltage across ideal source (or internal voltage of the source)

I_s = Current delivered by ideal source (or current generated by the source)

V = Voltage across the terminals of the source

I = Current delivered through the terminals of the source

R_s = Source resistance (or internal resistance).

A **practical voltage source** can be considered as a series combination of an ideal voltage source and a **source resistance**, R_s. The reduction in voltage across the terminals with increasing load current is due to the voltage drop in the source resistance. When the value of source resistance is zero the ideal condition is achieved in voltage sources. Hence, "*the source resistance for an ideal voltage source is zero*".

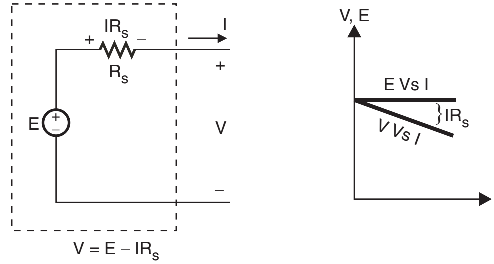


Fig. 1.22 : A practical dc voltage source.

A **practical current source** can be considered as a parallel combination of an ideal current source and a **source resistance**, R_s. The reduction in current delivered by the source is due to the current drawn by the parallel source resistance. When the value of source resistance is infinite the ideal condition is achieved in current sources. Hence, "*the source resistance for an ideal current source is infinite*".

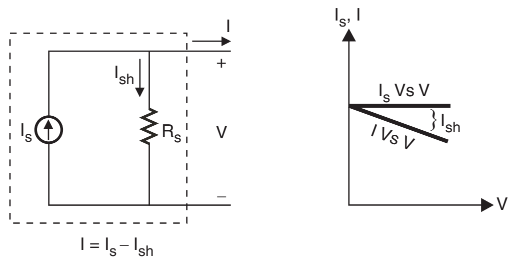


Fig. 1.23 : A practical dc current source.

1.2.8 DC Source Transformation

A practical voltage source can be converted into an equivalent practical current source and vice-versa, with the same terminal behaviour. In these conversions the current and voltage at the terminal of the equivalent source will be the same as that of the original source, so that the power delivered to a load connected at the terminals of original and equivalent source is the same.

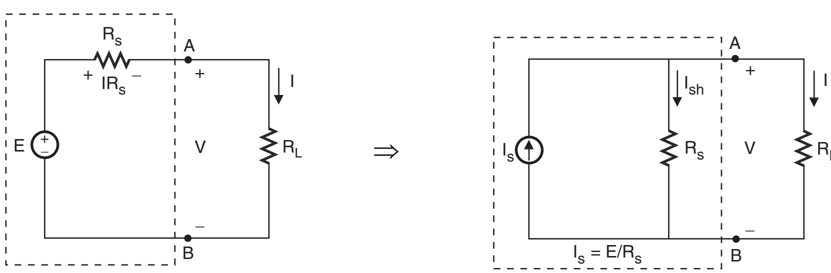


***Fig. a :** Voltage source.*

***Fig. b :** Equivalent current source of the voltage source in Fig. a.*

***Fig. 1.24 :** Conversion of voltage source to current source.*

A voltage source with series resistance can be converted into an equivalent current source with parallel resistance as shown in Fig. 1.24. Similarly, a current source with parallel resistance can be converted into an equivalent voltage source with series resistance as shown in Fig. 1.25. The proof for source conversions are presented in Chapter 2.

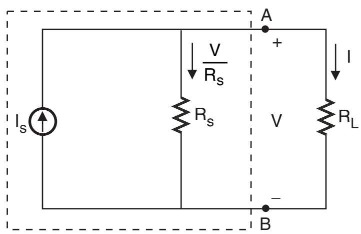


Fig. a : *Current source.*

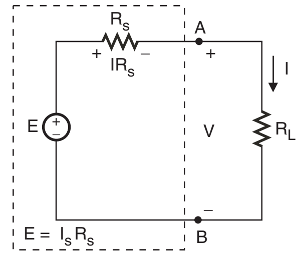


Fig. b : *Equivalent voltage source of the current source in Fig. a.*

Fig. 1.25 : *Conversion of current source to voltage source.*

1.2.9 Power and Energy

Power is the rate at which work is done or it is the rate of energy transfer.

Let, w = Instantaneous value of energy

q = Instantaneous value of charge.

Now, Instantaneous power, $p = \frac{dw}{dt} = \frac{dw}{dq} \times \frac{dq}{dt}$

We know that, $\frac{dw}{dq} = v$ and $\frac{dq}{dt} = i$

Refer equations (1.5) and (1.7).

$\therefore\ p = v\,i$

Therefore, power is the product of voltage and current. In circuits excited by dc sources, the voltage and current are constant and so the power is constant. This constant power is called **average power** or power and it is denoted by P.

$\therefore$ In dc circuits,

$$\boxed{\text{Power, P} = \text{VI}}$$

Power is the rate of work done and E**nergy** is the total work done. Hence, energy is given by the product of power and time. When time is expressed in second, the unit of energy is watt-second and when the time is expressed in hours, the unit of energy is watt-hour.

$$\boxed{\therefore\ \text{Energy, E} = \text{P t in } W\text{-}s \text{ or } W\text{-}h}$$

The larger unit of electrical energy is *kWh* and commercially one *kWh* of electrical energy is called one **unit**.

$$\boxed{\therefore\ \text{Energy, E} = \frac{\text{P t}}{1000 \times 3600} \text{ in } kWh}$$

1.3 Network Terminology

Topology is a branch of science which deals with the study of geometrical properties and special relations unaffected by continuous change of shape or size of figures. The concept of topology was first applied to networks by Kirchoff to study the relationship between the nodes and branches in a network.

A circuit or network can be drawn in different shapes and sizes by maintaining the relationship between the nodes and branches as shown in Fig. 1.26.

Therefore, "*the **network topology** is the study of the properties of the network which are unaffected when we stretch, twist or distort the size and shape of the network*". A network consists of interconnections of various elements. The physical arrangement of the elements and the length of wires used for connecting the elements may give rise to different types of layout for the circuits. As long as the relationship between the nodes and branches are maintained, the circuit response will be the same.

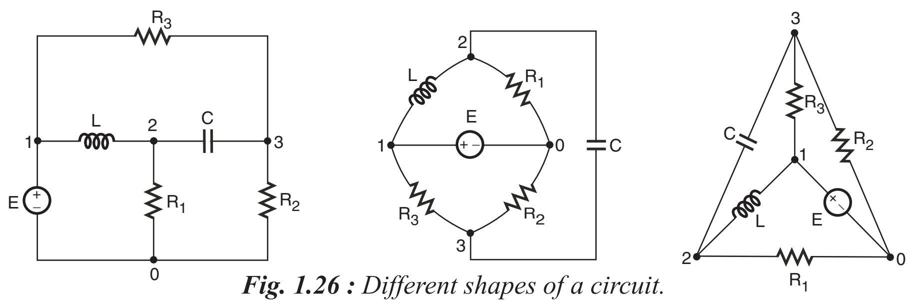


***Fig. 1.26 :** Different shapes of a circuit.*

1.3.1 Graph of a Network

The topological properties of networks are described by a graph. The **graph** of a network consists of nodes and branches of the network. In a network the branches have elements but in a graph the branches are drawn by lines. When arrows are placed on the branches of a graph it is called an **oriented graph**. The arrow indicates the direction of branch current and polarity of branch voltage.

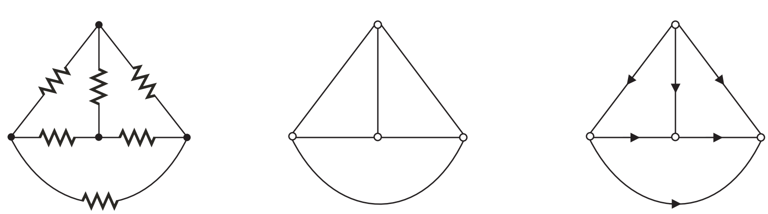


***Fig. a :** Network.* ***Fig. b :** Graph of network in Fig. a.* ***Fig. c :** Oriented graph of network in Fig. a.*

***Fig. 1.27 :** A typical network and its graph and oriented graph.*

A sequence of branches traversed while going from one node to another node is called a **path**. A graph is said to be a **connected graph** if there exists at least one path from each node of a graph to every other node of the graph.

To draw the graph of a circuit first redraw the circuit by replacing the sources by their internal impedances. The ideal voltage sources are replaced by short circuits and ideal current sources are replaced by open circuits. Now, the circuit becomes a network consisting of R, L and C elements only. Then represent the nodes of the network as small circles and the elements connected between the nodes as lines. The series connected elements are considered as a single branch. While drawing the graph of a network, the number of nodes and branches and the relationship between them has to be maintained. But the size and shape of graph and curvature of lines in the graph are not important.

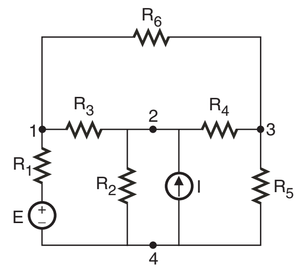


***Fig. a :** Typical circuit.*

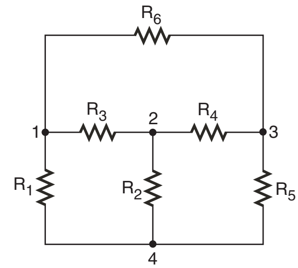


***Fig. b :** The circuit of Fig. a after replacing sources by their internal impedance.*

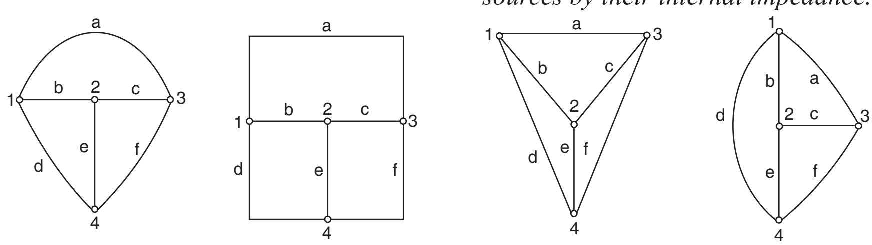


***Fig. c :** Various shapes of graphs for the circuit of Fig. a.*

***Fig. 1.28 :** A typical circuit and its different graphs.*

A typical circuit and its different graphs are shown in Fig. 1.28. In the graph, the nodes are represented by small circles and denoted by numerals 1, 2, 3 and 4. In the graph, the elements connected between the nodes are represented by lines. These lines are called branches and denoted by lower case letters a, b, c, d, e and f. This convention of denoting nodes by numerals and branches by lower case letters has been followed in this book.

1.3.2 Trees, Link, Twig and Cotree

When some of the branches in an original graph are removed, the resultant graph is called a **subgraph**. The **tree** is a subgraph which is obtained by removing some branches such that the subgraph includes all the nodes of the original graph, but does not have any closed paths. For a given graph, there may be more than one possible tree. Hence, a tree can be defined as any connected open set of branches which includes all nodes of a given graph. A tree of a graph with N nodes has the following properties:

- The tree contains all the nodes of the graph.
- The tree contains N – 1 branches.
- The tree does not have a closed path.

The branches removed to form a tree are called **links** or **chords**. By removing a link from a graph, one closed path can be eliminated. Alternatively, on adding a link to a tree one closed path is created. Hence, by adding the links one by one to a tree all closed paths can be created. Therefore, the number of closed paths in a graph is equal to the number of links.

The **cotree** is the complement of a tree. Hence, every tree has a cotree. The links connected to the nodes of a graph form a cotree. The branches of a tree are called **twigs** and the branches of a cotree are called **links**. A typical graph is shown in Fig. 1.29, and some possible trees of the graph and the cotree of each tree are shown in Table 1.4.

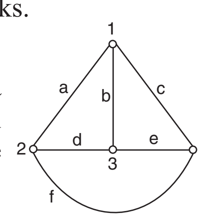


***Fig. 1.29 :** Graph.*

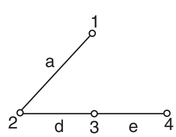
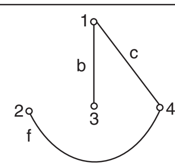
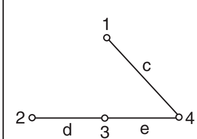
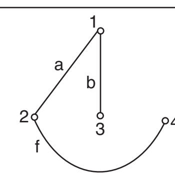
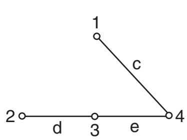
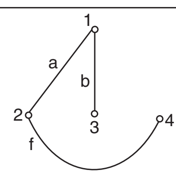
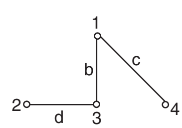
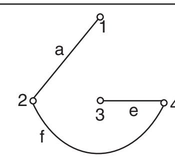
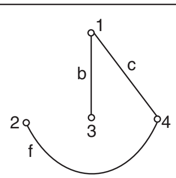
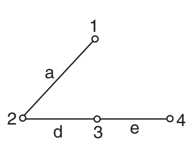
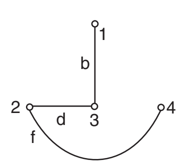
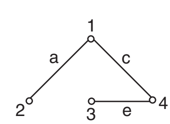
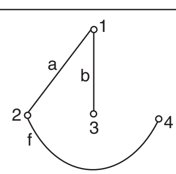
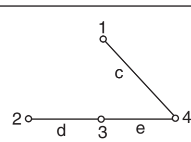
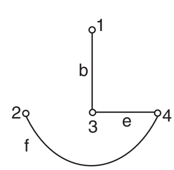
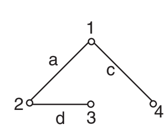
For most of the trees the cotree will also be in the form of a tree. But for some possible tree, the cotree may have closed paths and cotree may not be connected (i.e., all the nodes are not connected in a cotree).

A definite relationship exists between the number of nodes and branches in a tree. Any tree of the graph with B branches and N nodes will consist of N − 1 branches and the remaining branches are links.

Therefore, for a graph with B branches and N nodes, the number of links or chords is given by,

$$\boxed{\text{Link, L} = \text{B} - (\text{N} - 1) = \text{B} - \text{N} + 1}$$

Table 1.4 : The Trees and Cotrees of the Graph in Fig. 1.29

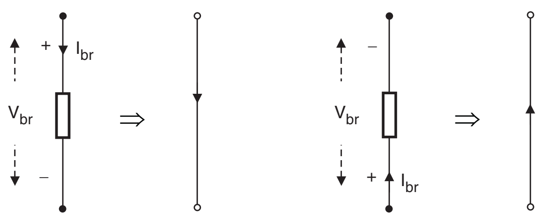
Tree	Cotree	Tree	Cotree
 Twig: [a, d, e]	 Link: [b, c, f]	 Twig: [c, d, e]	 Link: [a, b, f]
 Twig: [c, d, e]	 Link: [a, b, f]	 Twig: [b, c, d]	 Link: [a, e, f]
 Twig: [b, c, f]	 Link: [a, d, e]	 Twig: [b, d, f]	 Link: [a, c, e]
 Twig: [a, b, f]	 Link: [c, d, e]	 Twig: [b, e, f]	 Link: [a, c, d]

1.3.3 Network Variables

When a network is excited by connecting a source, every branch will have a current flowing through it and so a voltage will exist across the terminals of the branch. Hence, a graph (or network) with B branches will have B number of branch currents and B number of branch voltages. These branch currents and voltages are called **network variables**. The branch currents are called **current variables** and branch voltages are called **voltage variables** of the network.

An arrow is placed on the branch to indicate the direction of the branch current and polarity of the branch voltage. The arrow placed on the branch is called **reference** or **orientation**. In a branch, a single reference is used to represent both the directions of branch current and polarity of branch voltage.

The current-voltage relation of a branch is obtained by Ohm's law, by treating the branch as load. Hence, the set of references for the branches of a graph are called **load set reference**.



V_{br} = Branch voltage ; I_{br} = Branch current

Fig. 1.30 : *Orientation (or reference) of a branch.*

The conventional direction of branch current and polarity of branch voltage are shown in Fig. 1.30. In a network, branch current directions can be assumed arbitrarily and the polarity of branch voltages can be fixed as per Ohm's law, by treating the branches as loads. Alternatively, the polarity of branch voltages can be assumed arbitrarily and the direction of branch current can be fixed as per Ohm's law, by treating the branch as load.

1.3.4 Solution of Network Variables

In a network or a circuit we may be interested in the voltage and current in the various branches which is normally referred to as response. In a network if all the branch currents are known then the voltages can be obtained by Ohm's law. Alternatively, if the branch voltages are known then the currents can be obtained by Ohm's law.

Hence, in order to determine the response on current basis first we have to solve B number of branch currents and to determine the response on voltage basis first we have to solve B number of branch voltages.

For a unique solution of B number of variables, we have to form B number of equations involving the B variables and solve them. But in practice it can be shown that all the branch currents are not independent and so the independent current variables which are less than B, are sufficient to solve the currents. Similarly, all the branch voltages are not independent and so the independent voltage variables which are less than B, are sufficient to solve the voltages.

Independent Current Variables

In a network it can be proved that the branch currents of the links are **independent current variables**. When the links are removed in a network, all the closed paths are destroyed and so no current can flow in the network. The removal of a link is equal to making link current as zero. Therefore, when the link current are made zero, all the currents in the network become zero.

Hence, we can say that the branch currents depend on link currents. Therefore, "*the link currents are independent and branch currents are dependent*". In a network with N nodes and B branches we have B – N + 1 links. Therefore, in a network there will be B current variables in which B – N + 1 are independent current variables and the remaining N – 1 [i.e., B – (B – N + 1) = N – 1] currents are dependent current variables.

In order to determine the response of a network on current basis, it is sufficient if we form B – N + 1 equations involving independent current variables and solve them for a unique solution. Thereafter, the dependent current variables can be solved using the independent current variables.

Independent Voltage Variables

In a network it can be proved that the branch voltages of the twigs (or tree branches) are **independent voltage variables**. In a graph when all the twigs are short circuited, then all the nodes will be short circuited as well. Eventually, the voltages of the nodes become zero. Also the short circuiting of nodes will lead to short-circuiting of all the branches and so all the branch voltages will become zero.

Hence, we can say that, the branch voltages depend on twig voltages. Therefore, "*the twig voltages are independent and branch voltages are dependent*". In a network with N nodes and B branches we have N – 1 twigs. Therefore, in a network we have B voltage variables in which N – 1 are independent voltage variables and the remaining B – (N – 1) voltages are dependent voltage variables.

In order to determine the response of a network on voltage basis it is sufficient if we form N – 1 equations involving independent voltage variables and solve them for a unique solution. Thereafter, the dependent voltage variable can be solved using independent voltage variables.

1.4 Ohm's and Kirchhoff's Laws

The three fundamental laws that govern the electric circuit are Ohm's law, **K**irchhoff's **C**urrent **L**aw (KCL) and **K**irchhoff's **V**oltage **L**aw (KVL).

1.4.1 Ohm's Law

Ohm's law states that the potential difference (or voltage) across any two ends of a conductor is directly proportional to the current flowing between the two ends provided the temperature of the conductor remains constant.

The constant of proportionality is the resistance R of the conductor.

$$\therefore V \; \alpha \; I \quad \Rightarrow \quad V = I R \qquad \text{..... (1.10)}$$

From equation (1.10), we can say that when a current I flows through a resistance R, then the voltage V, across the resistance is given by the product of current and resistance.

1.4.2 Kirchhoff's Current Law (KCL) *(AU Dec'15, 2 Marks)*

Kirchhoff's Current Law states that the algebraic sum of currents at a node is zero.

$$\sum I = 0$$

Hence, we can say that current cannot stay at a point. While applying Kirchhoff's Current Law (KCL) to a node we have to assign polarity or sign (i.e., + or −) for the current entering and leaving that node. Let us assume that the currents entering the node are negative and currents leaving the node are positive.

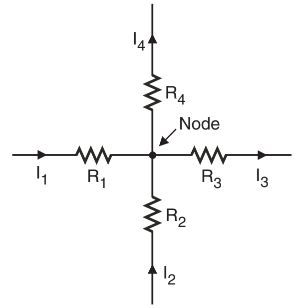


Fig. 1.31 : Currents in a node.

With reference to Fig. 1.31, we can say that currents I_1 and I_2 are entering the node and the currents I_3 and I_4 are leaving the node. Therefore, by Kirchhoff's Current Law we can write,

$$-I_1 - I_2 + I_3 + I_4 = 0$$

$$\therefore \; I_1 + I_2 = I_3 + I_4 \qquad \text{..... (1.11)}$$

From equation (1.11), we can say that, "*the sum of currents entering a node is equal to the sum of currents leaving that node*". This concept is easier to apply while solving problems using KCL.

1.4.3 Kirchhoff's Voltage Law (KVL)

Kirchhoff's Voltage Law states that the algebraic sum of voltages in a closed path is zero.

$$\sum V = 0$$

A closed path may have voltage rises and voltage falls when it is traversed or traced in a particular direction.While applying KVL to a closed path we have to assign polarity or sign (i.e., + or -) to voltage fall and rise. Let us assume voltage rise as positive and voltage fall as negative.

Consider the circuit shown in Fig. 1.32. Let us trace the circuit in the direction of current I. In the closed path ABCDEFGA, the voltage rise are E_1 and E_2 and voltage fall are IR_1, IR_2, IR_3, IR_4 and IR_5.

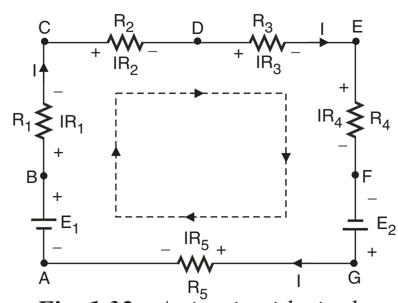


Fig. 1.32 : A circuit with single closed path.

Therefore, by KVL we can write,

$$E_1 + E_2 - IR_1 - IR_2 - IR_3 - IR_4 - IR_5 = 0$$

$$\therefore \; E_1 + E_2 = IR_1 + IR_2 + IR_3 + IR_4 + IR_5 \qquad \text{..... (1.12)}$$

From equation (1.12) we can say that, "*the sum of voltage rise in a closed path is equal to the sum of voltage fall in that closed path*". This concept is easier to apply while solving problems using KVL.

1.5 Resistive Elements

The devices that can be operated by electrical energy can be modelled by fundamental parameters R, L and C. In certain devices L and C are negligible and such devices can be modelled by pure resistance and so can be called resistive elements. Examples of such devices are incandescent lamp, water heater, ironbox and copper and aluminium wires.

(AU May'15, 2 Marks)

1.5.1 Resistance

Resistance is the property of an element (or matter) which opposes the flow of current (or electrons). The current carrying element is called a **conductor**. The resistance of a conductor (in the direction of current flow) is directly proportional to its length l and inversely proportional to the area of cross-section a.

$$\therefore \quad \text{Resistance, R} \;\alpha\; \frac{l}{a}$$

The proportionality constant is the **resistivity,** ρ of the material of the conductor.

$$\boxed{\therefore \; R = \frac{\rho l}{a}}$$

The unit of resistivity is ohm-metre(Ω-m). The resistivity of a material at a given temperature is constant. For example, the resistivity of copper is 1.72×10^{-8} Ω-m and that of aluminium is 2.69×10^{-8} Ω-m at 20^o C.

The resistance of a conductor is distributed throughout the length of the conductor. But for analysis purpose the resistance is assumed to be concentrated at one place, which is called lumped resistance. For connecting the lumped resistance to the other part of the circuit, resistance-less wires are connected to its ends as shown in Fig. 1.33. (Normally, the term resistance in circuit theory refers only to lumped resistance).

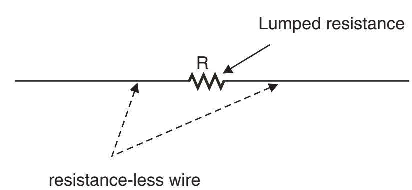


Fig. 1.33 : *A lumped resistance with resistance-less wires connected to its ends.*

1.5.2 Resistance Connected to DC Source

Consider a resistance R connected to dc source of voltage V volts as shown in Fig. 1.34. Since the resistance is connected across (or parallel to) the source, the voltage across the resistance is also V volts.

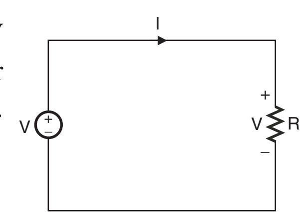


Fig. 1.34 : *Resistance connected to a dc source.*

By Ohm's law, the current through the resistance is given by,

$$I = \frac{V}{R} \quad \Rightarrow \quad V = IR \qquad \text{..... (1.13)}$$

Power in the resistance, $P = VI$ ..... (1.14)

Using equation (1.13), equation (1.14) can also be written as,

$$P = VI = V \times \frac{V}{R} = \frac{V^2}{R} \text{ and } P = VI = IR \times I = I^2 R$$

$$\boxed{\therefore \text{ Power, } P = VI} \quad \boxed{\text{or } P = \frac{V^2}{R}} \quad \boxed{\text{or } P = I^2 R}$$

1.5.3 Resistance in Series

Consider a circuit with series combination of two resistances R_1 and R_2 connected to a dc source of voltage V as shown in Fig.1.35(a). Let the current through the circuit be I.

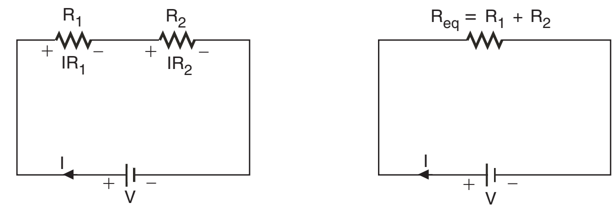


Fig. a : *Resistances in series.* ***Fig. b :*** *Equivalent circuit of Fig. a.*
Fig. 1.35 : *Resistances in series.*

It can be proved that the series-connected resistances R_1 and R_2 can be replaced by an equivalent resistance R_{eq} given by the sum of individual resistances R_1 and R_2 as shown in Fig. 1.35(b). The proof for resistance in series is presented in Chapter 2.

Voltage Division in Series Connected Resistances

Equations (1.15) and (1.16) given below, can be used to determine the voltages across series connected resistances shown in Fig. 1.36 in terms of total voltage across the series combination and the values of individual resistances. Hence, these equations are called **voltage division rule**. The proof for voltage division rule is presented in Chapter 2.

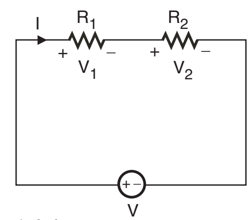


Fig. 1.36 : *Resistances in series.*

$$\boxed{V_1 = V \times \frac{R_1}{R_1 + R_2}} \quad .....(1.15)$$

$$\boxed{V_2 = V \times \frac{R_2}{R_1 + R_2}} \quad .....(1.16)$$

The following equation will be helpful to remember the voltage division rule.

In two series connected resistances,

$$\boxed{\text{Voltage across one of the resistance} = \frac{\begin{matrix}\text{Total voltage across} \\ \text{series combination}\end{matrix} \times \begin{matrix}\text{Value of the} \\ \text{resistance}\end{matrix}}{\text{Sum of the inidvidual resistances}}}$$

1.5.4 Resistance in Parallel

Consider a circuit with two resistances in parallel and connected to a dc source of voltage V as shown in Fig. 1.37(a). Let I be the current supplied by the source and I_1 and I_2 be the current through R_1 and R_2, respectively. Since the resistances are parallel to the source, the voltage across them will be the same.

It can be proved that the inverse of the equivalent resistance of parallel-connected resistances is equal to the sum of the inverse of individual resistances. The proof for resistance in parallel is presented in Chapter 2.

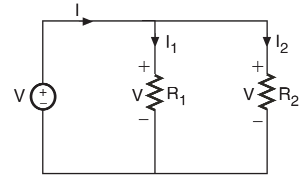


Fig. a : Resistances in parallel.

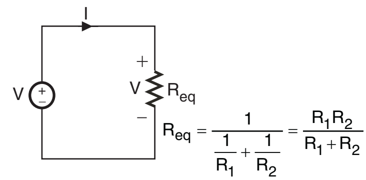


Fig. b : Equivalent circuit of Fig. a.

Fig. 1.37 : Resistances in parallel.

Current Division in Parallel Connected Resistances

Equations (1.17) and (1.18) given below, can be used to determine the currents in parallel connected resistances shown in Fig. 1.38 in terms of total current drawn by the parallel combination and the values of individual resistances. Hence, these equations are called **current division rule**. The proof for current division rule is presented in Chapter 2.

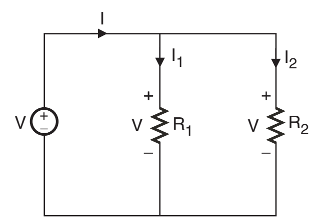


Fig. 1.38 : Resistances in parallel.

$$\boxed{I_1 = I \times \frac{R_2}{R_1 + R_2}} \qquad .....(1.17)$$

$$\boxed{I_2 = I \times \frac{R_1}{R_1 + R_2}} \qquad .....(1.18)$$

The following equation will be helpful to remember the current division rule.

In two parallel connected resistances,

$$\boxed{\text{Current through one of the resistance} = \frac{\begin{matrix}\text{Total current drawn by} \\ \text{parallel combination}\end{matrix} \times \begin{matrix}\text{Value of the} \\ \text{other resistance}\end{matrix}}{\text{Sum of the inidvidual resistances}}}$$

1.5.5 Analysis of Resistors in Series-Parallel Circuits

A typical circuit consists of a series-parallel connection of passive elements like resistance, inductance and capacitance and excited by voltage/current sources. The sources circulate current through all the elements of the circuit. Due to current flow, a voltage exists across each element of the circuit.

Basically, circuit analysis involves the solution of currents and voltages in various elements of a circuit. The currents and voltages can be solved using the three fundamental laws: Ohm's law, **K**irchhoff's **C**urrent **L**aw (KCL) and **K**irchhoff's **V**oltage **L**aw (KVL).

Now, the questions are: What will be the direction of current and polarity of voltage of an element in a circuit and how to find them ?

In practical cases we may come across circuits with single source or circuits with multiple sources. In circuits excited by a single source, with some experience it is possible to predict the direction of current because the current will leave from the positive end of the source and flow through available paths and then return to the negative end of the source.

But in circuits with multiple sources it will be difficult to find the direction of current through various elements. A common procedure to determine the current and voltage in various elements of a circuit is presented in the following section.

Procedure for Analysis of Circuits Using Fundamental Laws

1. Mark the nodes of the given circuit as A, B, C, D, etc. We can mark all the nodes including the meeting point of two elements.

2. Determine the number of branches in the given circuit. Attach a current to each branch of the circuit and arbitarily assume a direction for each branch current. Let, the branch currents be I_a, I_b, I_c, I_d, etc.

3. Write **K**irchhoff's **C**urrent **L**aw (KCL) equations at each principal node of the circuit. (Remember that a principal node is the meeting point of three or more elements.)

   The KCL equation is obtained by equating the sum of currents leaving the node to the sum of currents entering the node. Therefore, by KCL,

   Sum of currents entering the node = Sum of currents leaving the node.

4. Any circuit will have some independent currents and the remaining currents will depend on the independent currents. Hence, using the KCL equations, try to minimise the number of unknown currents by expressing some branch currents in terms of other branch currents. (The ultimate aim is to choose some independent currents and to express other currents in terms of independent currents.)

5. Let, the number of independent currents in the given circuit be M. Now we have to identify or choose M number of closed paths in the given circuit. For each closed path write a **K**irchhoff's **V**oltage **L**aw (KVL) equation.

   The KVL equation is formed by equating the sum of voltage fall in the closed path to the sum of voltage rise in that closed path. Therefore, by KVL,

   Sum of voltage fall in a closed path = Sum of voltage rise in a closed path.

6. The M number of KVL equations can be solved by any technique to find a unique solution for M independent branch currents.
7. From the knowledge of independent branch currents, determine the dependent branch currents.
8. Once the branch currents are known it is easy to find the voltage across the elements by Ohm's law. The voltage across an element is given by the product of resistance and current, i.e., by Ohm's law,

$$\text{Voltage} = \text{Resistance} \times \text{Current}$$

9. If we are interested in calculating the power of an element then the power can be calculated from the knowledge of voltage and current in the element.

   In purely resistive circuits excited by dc sources,

$$\text{Power} = (\text{Current})^2 \times \text{Resistance}$$

$$\text{or} \quad \text{Power} = \frac{(\text{Voltage})^2}{\text{Resistance}} \qquad \text{or} \quad \text{Power} = \text{Voltage} \times \text{Current}$$

Some Important Basic Concepts

1. When a source delivers energy, the current will leave from the positive end of the source and return to the negative end.
2. In series connected elements, the same current will flow.
3. In parallel connected elements, the voltage across them will be the same.
4. When a current flows through a resistance, the polarity of voltage across the resistance will be such that the current entering point is positive and the leaving point is negative as shown in Fig. 1.39.

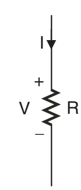


Fig. 1.39.

5. If the total voltage across two resistances R_1 and R_2 in series is V volts and V_1 and V_2 are the voltage across R_1 and R_2, respectively then,

$$V_1 = V \times \frac{R_1}{R_1 + R_2} \quad \text{and} \quad V_2 = V \times \frac{R_2}{R_1 + R_2}$$

6. If the total current through two resistances R_1 and R_2 in parallel is I amperes and I_1 and I_2 are the current through R_1 and R_2, respectively then,

$$I_1 = I \times \frac{R_2}{R_1 + R_2} \quad \text{and} \quad I_2 = I \times \frac{R_1}{R_1 + R_2}$$

1.5.6 Single Loop Circuit

A **single loop circuit** is one which has only one closed path. In a single loop circuit all the elements are connected in series and so current through all the elements will be the same. A single loop circuit can be analysed using **K**irchhoff's **V**oltage **L**aw (KVL) and Ohm's law.

A single loop circuit is shown in Fig. 1.40. Let, I be the current through the circuit. By Ohm's law, the voltage across a resistance is given by the product of resistance and current through the resistance.

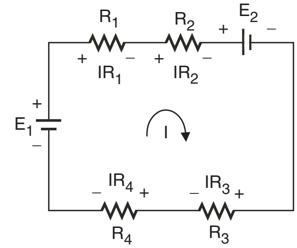


Fig. 1.40 : *Single loop circuit.*

Now, using KVL we can write,

$$IR_1 + IR_2 + E_2 + IR_3 + IR_4 = E_1 \Rightarrow I(R_1 + R_2 + R_3 + R_4) = E_1 - E_2$$

$$\therefore I = \frac{E_1 - E_2}{R_1 + R_2 + R_3 + R_4}$$

From the above equation, the current through the single loop circuit of Fig. 1.40, can be estimated. From the knowledge of current and resistance, the voltage across various elements can be estimated. From the knowledge of voltage and current, the power can be estimated.

1.5.7 Single Node Pair Circuit

A **single node pair circuit** is one which has only one independent node and a reference node. In a single node pair circuit all the elements are connected in parallel and so voltage across all the elements will be the same. A single node pair circuit can be analysed using **K**irchhoff's **C**urrent **L**aw (KCL) and Ohm's law.

A single node pair circuit is shown in Fig. 1.41. Let, V be the voltage of the independent node (node-1) with respect to the reference node.The voltage of the reference node is always zero. By Ohm's law, the current through the resistance is given by the ratio of voltage and resistance.

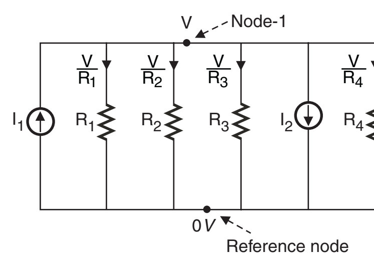


Fig. 1.41 : *Single node pair circuit.*

Now, using KCL we can write,

$$\frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} + I_2 + \frac{V}{R_4} = I_1 \Rightarrow V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) = I_1 - I_2$$

$$\therefore V = \frac{I_1 - I_2}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}}$$

From the above equation the voltage of the independent node (node-1) can be estimated. This voltage is the voltage across all the elements in the single node pair circuit. From the knowledge of voltage and resistance, the current through various resistances can be estimated. From the knowledge of voltage and current, the power in various elements can be estimated.

EXAMPLE 1.1

A 9 V Battery with internal resistance of 2 Ω is connected to a 16 Ω resistive load. Calculate **a)** power delivered to load, **b)** power loss in the battery and **c)** efficiency of the battery.

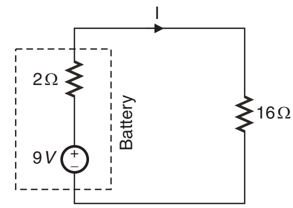


Fig. 1.

SOLUTION

The battery connected to resistive load can be represented by the circuit shown in Fig. 1.

Let, I = Current delivered by the battery.

Now, by Ohm's law,

$$I = \frac{9}{2 + 16} = 0.5\,A$$

Power delivered to load, $P_L = I^2 \times 16$

$$= 0.5^2 \times 16 = 4\,W$$

Power loss in the battery, $P_{LB} = I^2 \times 2$

$$= 0.5^2 \times 2 = 0.5\,W$$

$$\%\text{ Efficiency of battery, } \eta_B = \frac{\text{Load power}}{\text{Load power} + \text{Power loss}} \times 100$$

$$= \frac{P_L}{P_L + P_{LB}} \times 100 = \frac{4}{4 + 0.5} \times 100 = 88.9\%$$

EXAMPLE 1.2

An 8.4 A current generator with internal resistance of 200 Ω is connected to a 10 Ω resistive load. Calculate **a)** power delivered to load, **b)** power loss in the current generator and **c)** efficiency of the current generator.

SOLUTION

The current generator connected to resistive load can be represented by the circuit shown in Fig. 1.

Let, V_L = Voltage across the load.

Now, by Kirchhoff's current law,

$$\frac{V_L}{200} + \frac{V_L}{10} = 8.4 \Rightarrow V_L\left(\frac{1}{200} + \frac{1}{10}\right) = 8.4 \Rightarrow 0.105\,V_L = 8.4$$

$$\therefore V_L = \frac{8.4}{0.105} = 80\,V$$

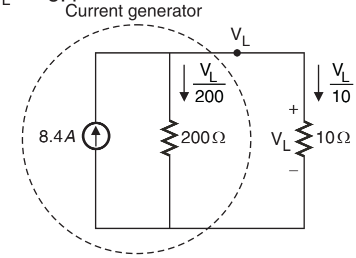


Fig. 1.

Power delivered to load, $P_L = \frac{V_L^2}{10}$

$$= \frac{80^2}{10} = 640\,W$$

Power loss in current generator, $P_{LCG} = \frac{V_L^2}{200} = \frac{80^2}{200} = 32\,W$

$$\%\text{ Efficiency of current generator, } \eta_{CG} = \frac{\text{Load power}}{\text{Load power} + \text{Power loss}} \times 100 = \frac{P_L}{P_L + P_{LCG}} \times 100$$

$$= \frac{640}{640 + 32} \times 100 = 95.2\%$$

EXAMPLE 1.3

Two batteries A and B with internal emf E_A and E_B and with internal resistance R_A and R_B are properly connected in parallel to supply a current of 160 *A* to a load resistance R_L. Given that, E_A = 120 *V*, $R_A = 0.15\,\Omega$, $R_B = 0.1\,\Omega$ and $I_B = 60\,A$. Calculate **a)** E_B and **b)** load power.

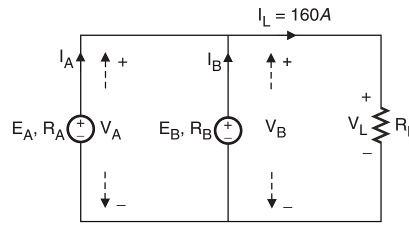


Fig. 1.

SOLUTION

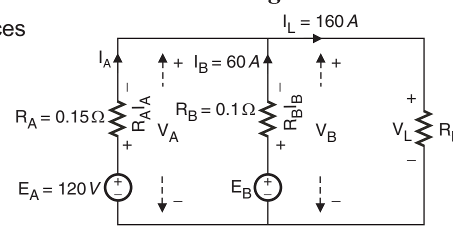


Fig. 2.

Let, E_A, E_B = Internal (or generated) voltage of sources

V_A, V_B = Terminal voltage of sources

I_A, I_B = Current supplied by the sources

I_L = Load current

V_L = Voltage across the load.

The sources are connected parallel to the load as shown in Fig. 1.

In Fig. 2, the sources are represented as ideal sources with source resistance connected in series with ideal source.

By KCL, we can write,

$$I_L = I_A + I_B$$

$$\therefore\ I_A = I_L - I_B$$

$$= 160 - 60 = 100\,A$$

By KVL, we can write,

$$E_A = R_A I_A + V_A$$

$$\therefore\ V_A = E_A - R_A I_A$$

$$= 120 - 0.15 \times 100$$

$$= 105\,V$$

Since the sources and load are in parallel,

$$V_A = V_B = V_L = 105\,V$$

By KVL, we can write, $E_B = R_B I_B + V_B$

$$= 0.1 \times 60 + 105 = 111\,V$$

Load power, $P_L = V_L I_L = 105 \times 160 = 16800\,W$

$$= \frac{16800}{1000}\,kW = 16.8\,kW$$

Also load resistance, $R_L = \frac{V_L}{I_L}$

$$= \frac{105}{160} = 0.65625\,\Omega$$

RESULT

$E_B = 111\,V$, $P_L = 16.8\,kW$, $R_L = 0.65625\,\Omega$

EXAMPLE 1.4 ***(AU Dec'15, 8 Marks)***

Determine the magnitude and direction of the current in the $2\,V$ battery in the circuit shown in Fig. 1.

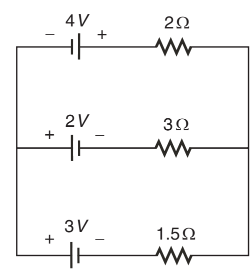


Fig. 1.

SOLUTION

Let us assume three branch currents I_a, I_b and I_c as shown in Fig. 2. The currents are assumed such that they leave from the positive terminal of the sources. The nodes in the circuit are denoted as A, B, C, D and E.

By KCL at node-A we get,

Current leaving node-A : I_a

Currents entering node-A : I_b, I_c

$$\therefore I_a = I_b + I_c \qquad .....(1)$$

With reference to Fig. 2, in the closed path ACBDA we get,

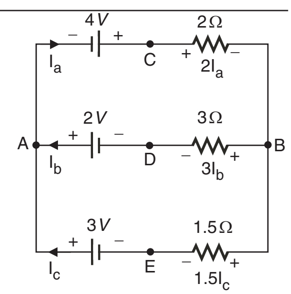


Fig. 2.

Voltage fall : $2I_a$, $3I_b$

Voltage rise : $4\,V$, $2\,V$

$$\therefore 2I_a + 3I_b = 4 + 2$$

Put, $I_a = I_b + I_c$

$$\therefore 2(I_b + I_c) + 3I_b = 6 \quad \Rightarrow \quad 5I_b + 2I_c = 6 \qquad .....(2)$$

With reference to Fig. 2, in the closed path ADBEA we get,

Voltage fall : $2\,V$, $1.5I_c$

Voltage rise : $3I_b$, $3\,V$

$$\therefore \quad 2 + 1.5I_c = 3I_b + 3$$

$$-3I_b + 1.5I_c = 3 - 2$$

$$-3I_b + 1.5I_c = 1 \qquad .....(3)$$

$$\begin{array}{lcrcrcr} \text{Equation (2)} \times 1.5 & \Rightarrow & 7.5I_b & + & 3I_c & = & 9 \\ \text{Equation (3)} \times (-2) & \Rightarrow & 6I_b & - & 3I_c & = & -2 \\ \hline \text{On adding,} & & 13.5I_b & & & = & 7 \end{array}$$

$$\therefore\ I_b = \frac{7}{13.5} = 0.5185\,A$$

Therefore, the current supplied by the $2\,V$ battery is $0.5185\,A$ in the direction B to A (Refer Fig. 2.).

The currents supplied by other sources can be estimated as shown below:

From equation (3), $I_c = \dfrac{1 + 3I_b}{1.5} = \dfrac{1 + 3 \times 0.5185}{1.5} = 1.7037\,A$

From equation (1), $I_a = I_b + I_c = 0.5185 + 1.7037 = 2.222\,A$

EXAMPLE 1.5

In the circuit shown in Fig. 1, the current in $5\,\Omega$ resistor is $5\,A$. Calculate the power consumed by the $5\,\Omega$ resistor. Also determine the current through $10\,\Omega$ resistance and the supply voltage E.

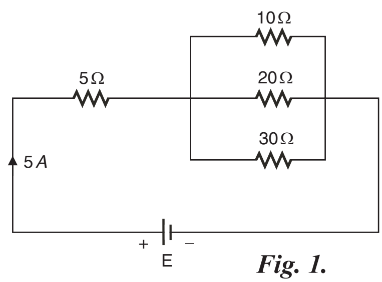


Fig. 1.

SOLUTION

Power consumed by $5\,\Omega$ resistor $=$ (Current)^2 $\times$ Resistance

$$= 5^2 \times 5 = 125\,W$$

The resistances $20\,\Omega$ and $30\,\Omega$ in parallel in Fig. 1, can be replaced by a single equivalent resistance as shown in Fig. 2. [Refer Chapter 2 for calculating equivalent resistance].

Let, I_s = Current supplied by source

I_1 = Current through $10\,\Omega$

I_2 = Current through $12\,\Omega$

V_1 = Voltage across $5\,\Omega$

V_2 = Voltage across $10\,\Omega$ and $12\,\Omega$ in parallel.

In Fig. 2, the current I_s divides into I_1 and I_2 and flows through parallel resistances $10\,\Omega$ and $12\,\Omega$. The currents I_1 and I_2 can be calculated by current division rule.

By current division rule,

$$I_1 = I_s \times \frac{12}{10 + 12} = 5 \times \frac{12}{10 + 12} = 2.7273\,A$$

By Ohm's law,

$$V_1 = 5 \times 5 = 25\,V$$

$$V_2 = 10 \times I_1 = 10 \times 2.7273 = 27.273\,V$$

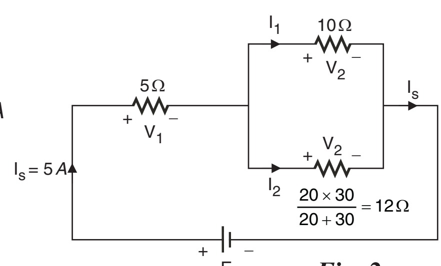


Fig. 2.

By KVL, we can write,

$$E = V_1 + V_2 = 25 + 27.273 = 52.273\,V$$

RESULT

Power consumed by 5Ω resistor = $125\,W$

Current through 10Ω resistor = $2.7273\,A$

Supply voltage, E = $52.273\,V$

EXAMPLE 1.6

In the circuit shown in Fig. 1, the voltage across 8Ω resistor is $20\,V$. What is the current through 12Ω resistor?. Also calculate the supply voltage.

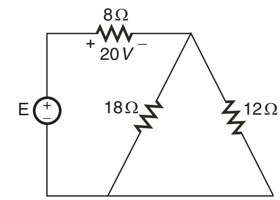


Fig. 1.

SOLUTION

Let, I_s be the current supplied by the source. The I_s divides into I_1 and I_2 and flows through parallel connected 18Ω and 12Ω resistances as shown in Fig. 2.

The current supplied by the source flows through 8Ω resistance. Since the voltage across 8Ω is known, the current I_s can be calculated by Ohm's law.

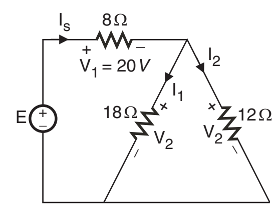


Fig. 2.

By Ohm's law,

$$I_s = \frac{20}{8} = 2.5\,A$$

By current division rule,

$$I_2 = I_s \times \frac{18}{18 + 12}$$

$$= 2.5 \times \frac{18}{18 + 12} = 1.5\,A$$

Let, V_1 = Voltage across 8Ω resistance.

V_2 = Voltage across parallel combination of 18Ω and 12Ω.

Given that, $V_1 = 20\,V$

By Ohm's law, $V_2 = 12 \times I_2 = 12 \times 1.5 = 18\,V$

With reference to Fig. 2, by KVL we can write,

$$E = V_1 + V_2 = 20 + 18 = 38\,V$$

RESULT

Current through 12Ω resistor = $1.5\,A$

Supply voltage, E = $38\,V$

EXAMPLE 1.7

In the circuit of Fig. 1, show that the power supplied by the current source is double of that supplied by the voltage source when $R = (10/3)\Omega$.

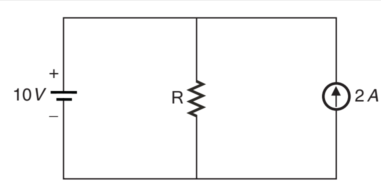


Fig. 1.

SOLUTION

Since the voltage source, current source and R are in parallel, the voltage across them will be the same, as shown in Fig. 2.

Let, I_1 = Current supplied by voltage source

I_2 = Current supplied by current source

I = Current through R.

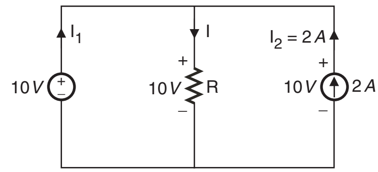


Fig. 2.

By Ohm's law,

$$I = \frac{\text{Voltage across resistance}}{\text{Resistance}} = \frac{10}{10/3} = 3\,A$$

By KCL, we can write, $I = I_1 + I_2$

Given that, $I_2 = 2A$

$$\therefore \; I_1 = I - I_2 = 3 - 2 = 1A$$

Now, Power supplied by 10 *V* source $= 10 \times I_1 = 10 \times 1 = 10\,W$

Power supplied by 2 *A* source $= 10 \times I_2 = 10 \times 2 = 20\,W$

From the above results it is clear that the power supplied by the current source is 20 *W*, which is double the power supplied by the voltage source.

EXAMPLE 1.8

Find the power dissipated in each resistor in the circuit of Fig. 1.

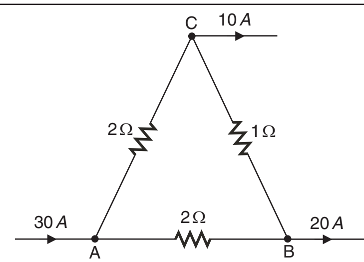


Fig. 1.

SOLUTION

The power dissipated in the resistors can be calculated from the knowledge of current through the resistors. Let us denote the current through the resistors as I_a, I_b and I_c as shown in Fig. 2.

In the closed path ACBA using KVL, we can write,

$$2I_a + I_b = 2I_c \;\Rightarrow\; I_a + \frac{I_b}{2} = I_c$$

$$\therefore \; I_c = I_a + 0.5I_b \qquad \text{..... (1)}$$

At node-A, by KCL, we get,

$$I_a + I_c = 30 \qquad \text{..... (2)}$$

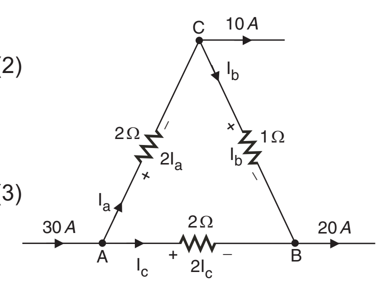


Fig. 2.

On substituting for I_c from equation (1) in equation (2), we get,

$$I_a + I_a + 0.5I_b = 30 \;\Rightarrow\; 2I_a + 0.5I_b = 30 \qquad \text{..... (3)}$$

At node-C, by KCL, we get,

$$I_b + 10 = I_a \;\Rightarrow\; I_a - I_b = 10 \qquad \text{..... (4)}$$

Equation (3) × 1 $\Rightarrow$ $2I_a + 0.5I_b = 30$

Equation (4) × 0.5 $\Rightarrow$ $0.5I_a - 0.5I_b = 5$

On adding $2.5I_a = 35$

$$\therefore \; I_a = \frac{35}{2.5} = 14\,A$$

From equation (4) we get, $I_b = I_a - 10 = 14 - 10 = 4A$

From equation (1) we get, $I_c = I_a + 0.5I_b = 14 + 0.5 \times 4 = 16A$

We know , Power consumed by a resistor $= (\text{Current})^2 \times$ Resistance

Power consumed by 2Ω resistor between nodes A and C $= I_a^2 \times 2 = 14^2 \times 2 = 392\,W$

Power consumed by 2Ω resistor between nodes A and B $= I_c^2 \times 2 = 16^2 \times 2 = 512\,W$

Power consumed by 1Ω resistor between nodes B and C $= I_b^2 \times 1 = 4^2 \times 1 = 16\,W$

EXAMPLE 1.9

In the circuit shown in Fig. 1, find **a)** the total current drawn from the battery, **b)** voltage across $2\,\Omega$ resistor and **c)** current passing through the $5\,\Omega$ resistor.

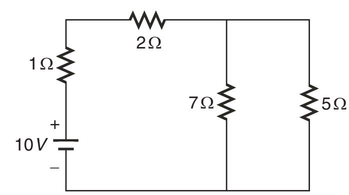


Fig. 1.

SOLUTION

Let, I_T be the total current supplied by the source. This current flows through $1\,\Omega$ and $2\,\Omega$ in series and then it divides into I_1 and I_2 and flows through parallel combination of $7\,\Omega$ and $5\,\Omega$ as shown in Fig. 2.

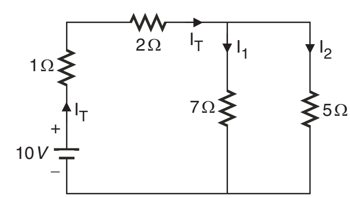


Fig. 2.

The parallel combination of $7\,\Omega$ and $5\,\Omega$ can be reduced to a single equivalent resistance as shown in Fig. 3. (Refer Chapter 2 for calculating equivalent resistance.)

By Ohm's law, we can write,

$$I_T = \frac{10}{1 + 2 + 2.9167} = 1.6901\,A$$

$\therefore$ Voltage across 2 Ω resistor $= I_T \times 2 = 1.6901 \times 2$

$$= 3.3802\,V$$

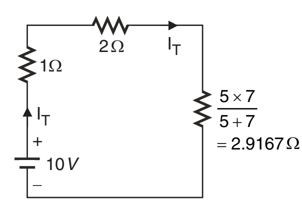


Fig. 3.

By current division rule,

$$\text{Current through } 5\,\Omega \text{ resistor}, I_2 = I_T \times \frac{7}{5 + 7}$$

$$= 1.6901 \times \frac{7}{5 + 7} = 0.9859\,A$$

RESULT

a) Total current supplied by the source, $I_T = 1.6901\,A$

b) Voltage across $2\,\Omega$ resistor $= 3.3802\,V$

c) Current through $5\,\Omega$ resistor $= 0.9859\,A$

EXAMPLE 1.10

Calculate the current in all the elements of the circuit shown in Fig. 1.

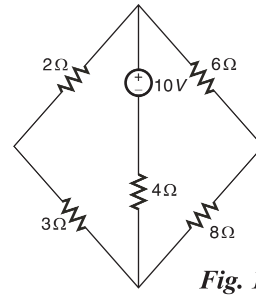


Fig. 1.

SOLUTION

Since the circuit has only one source. The direction of current through the elements can be found as shown in Fig. 2.

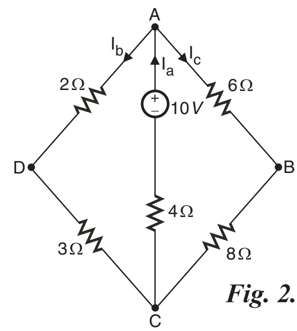


Fig. 2.

By applying KCL at node-A we get,

Currents leaving node-A : I_b, I_c

Current entering node-A : I_a

$$\therefore \quad I_b + I_c = I_a$$

$$\therefore \quad I_c = I_a - I_b \qquad ..... (1)$$

From equations (1) we can say that current I_c can be expressed in terms of I_a and I_b. Hence, the circuit has only two independent currents, I_a and I_b and they can be solved by writing two KVL equations in the closed paths ABCA and ADCA.

With reference to Fig. 3, in the closed path ADCA we get,

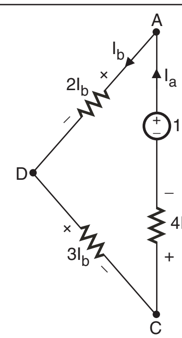


Fig. 3.

Voltage fall : $2I_b$, $3I_b$, $4I_a$

Voltage rise : 10 *V*

$$\therefore \; 2I_b + 3I_b + 4I_a = 10$$

$$4I_a + 5I_b = 10 \qquad ..... (2)$$

With reference to Fig. 4, in the closed path ABCA we get,

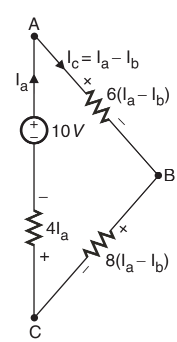


Fig. 4.

Voltage fall : $6(I_a - I_b)$, $8(I_a - I_b)$, $4I_a$

Voltage rise : 10 *V*

$$\therefore \; 6(I_a - I_b) + 8(I_a - I_b) + 4I_a = 10$$

$$14(I_a - I_b) + 4I_a = 10$$

$$14I_a - 14I_b + 4I_a = 10$$

$$18I_a - 14I_b = 10 \qquad ..... (3)$$

Equation (2) × 14 $\Rightarrow$ $56I_a + 70I_b = 140$

Equation (3) × 5 $\Rightarrow$ $90I_a - 70I_b = 50$

On adding $146I_a = 190$

$$\therefore I_a = \frac{190}{146} = 1.3014\,A$$

From equation (2), we get, $I_b = \dfrac{10 - 4I_a}{5}$

$$= \frac{10 - 4 \times 1.3014}{5} = 0.9589\,A$$

From equation (1), we get, $I_c = I_a - I_b = 1.3014 - 0.9589 = 0.3425\,A$

RESULT

The current through the elements (i.e., branch currents) are,

$$I_a = 1.3014\,A \quad ; \quad I_b = 0.9589\,A \quad ; \quad I_c = 0.3425\,A$$

EXAMPLE 1.11 *(AU June'14, 8 Marks)*

Determine the current in all the resistors of the circuit shown in Fig 1.

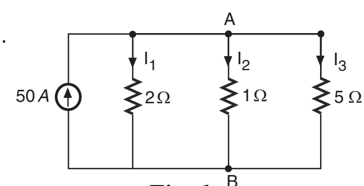


Fig. 1.

SOLUTION

Let, voltage at node-A be V_A

Now, by Ohm's law,

$$I_1 = \frac{V_A}{2} \quad ; \quad I_2 = \frac{V_A}{1} \quad ; \quad I_3 = \frac{V_A}{5}$$

By KCL, at node-A,

$$I_1 + I_2 + I_3 = 50 \quad \Rightarrow \quad \frac{V_A}{2} + \frac{V_A}{1} + \frac{V_A}{5} = 50 \quad \Rightarrow \quad 0.5V_A + V_A + 0.2\,V_A = 50$$

$$\therefore 1.7\,V_A = 50 \quad \Rightarrow \quad V_A = \frac{50}{1.7} = 29.4118\,V$$

$$\therefore I_1 = \frac{V_A}{2} = \frac{29.4118}{2} = 14.7059\,A$$

$$I_2 = \frac{V_A}{1} = \frac{29.4118}{1} = 29.4118\,A$$

$$I_3 = \frac{V_A}{5} = \frac{29.4118}{5} = 5.8823\,A$$

RESULT

The current in the resistors are,

$$I_1 = 14.7059\,A \quad ; \quad I_2 = 29.4118\,A \quad ; \quad I_3 = 5.8823\,A$$

1.6 Mesh Current Method of Analysis for DC and AC Circuits

Mesh analysis is a useful technique to solve the currents in various elements of a circuit. Mesh analysis is preferred, if the circuit is excited by voltage sources, and the current through various elements is unknown. Mesh analysis can also be extended to circuits excited by both voltage and current sources and to circuits excited by both independent and dependent sources.

In a circuit, each branch will have a current through it. Hence, the number of currents in a circuit is equal to the number of branches. In a circuit some of the currents will be independent and the remaining currents depend on independent currents. The number of independent currents in a circuit is given by number of links in the graph of the circuit. (Refer Section 1.3.2.)

In mesh analysis, independent currents are solved by writing Kirchhoff's Voltage Law (KVL) equations for various meshes in a circuit. If the graph of a circuit has B branches and N nodes then the number of links, L is given by L = B − N + 1. Hence, L number of meshes are chosen in a given circuit. "***Mesh** is defined as a closed path which does not contain any other loops within it*". Let us denote the number of meshes by m. In a circuit the number of meshes, m is equal to links, L.

For each mesh, an independent current is assigned called **mesh current** and for each mesh, an equation is formed using Kirchhoff's Voltage Law. The equation is formed by equating the sum of voltage rise to sum of voltage drop in a mesh. These m number of mesh equations are arranged in a matrix form and mesh currents are solved by Cramer's rule. A simple procedure to form mesh basis matrix equation directly from circuit by inspection without forming KVL equations is also discussed in this chapter.

Mesh analysis is applicable to planar circuits. "*A circuit is said to be a **planar circuit** if it can be drawn on a plane surface without crossovers*".

1.6.1 Mesh Analysis of Resistive Circuits Excited by DC Sources

A circuit with B branches will have B number of currents and in this some currents are independent and the remaining currents depend on independent currents. The number of independent currents m is given by m = B − N + 1, where N is the number of nodes.

In order to solve the independent currents of a circuit we have to choose m meshes (or closed paths) in the circuit. For each mesh we have to attach a current called mesh current. The mesh currents are the independent currents of the circuit. Let, I_1, I_2, I_3, ...,I_m be mesh currents.

For each mesh, a KVL equation is formed by equating the sum of voltage rise to sum of voltage fall in the mesh. Since there are m meshes we can form m equations.

In resistive circuits excited by dc sources, the voltages and currents are real (i.e., they are not complex). For resistive circuits, the m number of equations can be arranged in the matrix form as shown in equation (1.19), which is called mesh basis matrix equation. The formation of mesh basis matrix equation from the KVL equations is explained in some of the solved problems ahead.

The mesh basis matrix equation (1.19), can be written in a simplified form as shown in equation (1.20).

Note : *The bold faced letters represent matrices.*

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} & \cdots & R_{1m} \\ R_{21} & R_{22} & R_{23} & \cdots & R_{2m} \\ R_{31} & R_{32} & R_{33} & \cdots & R_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ R_{m1} & R_{m2} & R_{m3} & \cdots & R_{mm} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_m \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ \vdots \\ E_{mm} \end{bmatrix} \qquad \text{..... (1.19)}$$

$$\mathbf{R\,I = E} \qquad \text{..... (1.20)}$$

where, $\mathbf{R}$ = Resistance matrix of order m × m

$\mathbf{I}$ = Mesh current matrix of order m × 1

$\mathbf{E}$ = Source voltage matrix of order m × 1

m = Number of meshes.

In equation (1.19), the elements of resistance matrix and source voltage matrix can be determined from the given circuit. Hence, the unknowns are mesh currents, which have to be solved by any standard technique.

Alternatively, equation (1.19) can be formed directly from the circuit by inspection without writing KVL equations. A procedure to form mesh basis matrix equation by inspection is given below:

Procedure to Form Mesh Basis Matrix Equation by Inspection

Consider the mesh basis matrix equation shown below for a circuit with three meshes.

Let, I_1, I_2, I_3 be the mesh currents.

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \qquad \text{..... (1.21)}$$

The elements of equation (1.21) for circuits with independent sources are,

R_{11} = Sum of resistances in mesh-1

R_{22} = Sum of resistances in mesh-2

R_{33} = Sum of resistances in mesh-3

$R_{12} = R_{21}$ = Sum of resistances common between mesh-1 and mesh-2

$R_{13} = R_{31}$ = Sum of resistances common between mesh-1 and mesh-3

$R_{23} = R_{32}$ = Sum of resistances common between mesh-2 and mesh-3

E_{11} = Sum of voltage sources in mesh-1

E_{22} = Sum of voltage sources in mesh-2

E_{33} = Sum of voltage sources in mesh-3.

The resistances R_{11}, R_{22}, R_{33} are called **self-resistance** of mesh-1, mesh-2, mesh-3, respectively.

The resistances R_{12}, R_{13}, R_{21}, R_{23}, R_{31}, R_{32} are called **mutual-resistance** between meshes.

The formation of the elements of resistance matrix and source voltage matrix are explained below:

i) The self-resistance R_{jj} is given by the sum of all the resistances in the j[th] mesh. The self-resistances will be always positive.

ii) The mutual-resistance R_{jk} is given by the sum of all the resistances common between mesh-j and mesh-k.

The common resistance R_{jk} is positive if the mesh currents I_j and I_k flow in the same direction through the common resistance as shown in Fig. 1.42 and it is negative if the mesh currents I_j and I_k flow in the opposite direction through the common resistance as shown in Fig. 1.43.

In a circuit with only independent sources (reciprocal circuit), $R_{jk} = R_{kj}$.

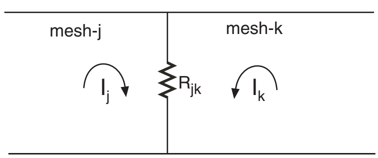


Fig. 1.42 : *Example for positive R_{jk}.*

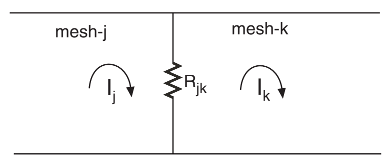


Fig. 1.43 : *Example for negative R_{jk}.*

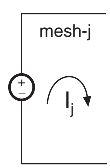


Fig. 1.44 : *Example for positive source voltage.*

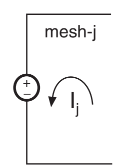


Fig. 1.45 : *Example for negative source voltage.*

iii) The source voltage matrix element E_{jj} is given by the sum of all the voltage sources in the j[th] mesh. A source voltage is positive if it is a rise in voltage in the direction of mesh current as shown in Fig. 1.44. A source voltage is negative if it is a fall or drop in voltage in the direction of mesh current as shown in Fig. 1.45.

Note : *In a circuit with both independent and dependent sources (non-reciprocal circuit) $R_{jk} \neq R_{kj}$*

Solution of Mesh Currents

In the mesh basis matrix equation [i.e., equation (1.19)], the unknowns are mesh currents $I_1, I_2, I_3 \ldots I_m$. The mesh currents can be obtained by premultiplying equation (1.19), by the inverse of resistance matrix.

Consider equation (1.20),

$$\mathbf{R\,I = E}$$

On premultiplying both sides by $\mathbf{R}^{-1}$, we get,

$$\mathbf{R}^{-1}\mathbf{R}\,\mathbf{I} = \mathbf{R}^{-1}\mathbf{E}$$

$$\mathbf{U}\,\mathbf{I} = \mathbf{R}^{-1}\mathbf{E} \qquad \boxed{\mathbf{R}^{-1}\mathbf{R} = \mathbf{U} = \text{Unit matrix}}$$

$$\therefore\ \mathbf{I} = \mathbf{R}^{-1}\mathbf{E} \qquad \text{..... (1.22)} \qquad \boxed{\mathbf{UI} = \mathbf{I}}$$

Equation (1.22) will be the solution for mesh currents. Equation (1.22) can be solved by **Cramer's rule**, by which the k^{th} mesh current I_k is given by equation (1.23).

$$\boxed{I_k = \frac{\Delta_{1k}}{\Delta}E_{11} + \frac{\Delta_{2k}}{\Delta}E_{22} + \frac{\Delta_{3k}}{\Delta}E_{33} + \ldots\ldots + \frac{\Delta_{mk}}{\Delta}E_{mm} = \frac{1}{\Delta}\sum_{j=1}^{m}\Delta_{jk}E_{jj}} \qquad \text{..... (1.23)}$$

where,

Δ_{jk} = Cofactor of R_{jk}

E_{jj} = Sum of voltage sources in mesh-j

Δ = Determinant of resistance matrix.

Proof for Cramer's Rule

Consider equation (1.22), for a circuit with three meshes.

$$\boldsymbol{I} = \boldsymbol{R}^{-1}\boldsymbol{E} \Rightarrow \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}^{-1} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \qquad \text{..... (1.24)}$$

We know that,

$$\boldsymbol{R}^{-1} = \frac{\text{Adjoint of } \boldsymbol{R}}{\text{Determinant of } \boldsymbol{R}} = \frac{\text{Transpose of } \boldsymbol{R}_{cof}}{\text{Determinant of } \boldsymbol{R}} = \frac{\boldsymbol{R}_{cof}^{T}}{\Delta}$$

where, Δ = Determinant of R

R_{cof} = Cofactor matrix (matrix formed by cofactor of elements of R matrix).

Let, Δ_{11} = Cofactor of R_{11}

Δ_{12} = Cofactor of R_{12}

and in general, Δ_{jk} = Cofactor of R_{jk}

$$\therefore\ \boldsymbol{R}_{cof} = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} \end{bmatrix} \xrightarrow{\text{Transpose}} \boldsymbol{R}_{cof}^{T} = \begin{bmatrix} \Delta_{11} & \Delta_{21} & \Delta_{31} \\ \Delta_{12} & \Delta_{22} & \Delta_{32} \\ \Delta_{13} & \Delta_{23} & \Delta_{33} \end{bmatrix}$$

$$\therefore\ \boldsymbol{R}^{-1} = \frac{\boldsymbol{R}_{cof}^{T}}{\Delta} = \frac{1}{\Delta}\begin{bmatrix} \Delta_{11} & \Delta_{21} & \Delta_{31} \\ \Delta_{12} & \Delta_{22} & \Delta_{32} \\ \Delta_{13} & \Delta_{23} & \Delta_{33} \end{bmatrix} \qquad \text{..... (1.25)}$$

On substituting for R^{-1} from equation (1.25) in equation (1.24), we get,

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} \Delta_{11} & \Delta_{21} & \Delta_{31} \\ \Delta_{12} & \Delta_{22} & \Delta_{32} \\ \Delta_{13} & \Delta_{23} & \Delta_{33} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix}$$

On multiplying the matrices on the right-hand side of the above equation and equating to the terms on the left-hand side we get,

$$I_1 = \frac{\Delta_{11}}{\Delta}E_{11} + \frac{\Delta_{21}}{\Delta}E_{22} + \frac{\Delta_{31}}{\Delta}E_{33}$$

$$I_2 = \frac{\Delta_{12}}{\Delta} E_{11} + \frac{\Delta_{22}}{\Delta} E_{22} + \frac{\Delta_{32}}{\Delta} E_{33}$$

$$I_3 = \frac{\Delta_{13}}{\Delta} E_{11} + \frac{\Delta_{23}}{\Delta} E_{22} + \frac{\Delta_{33}}{\Delta} E_{33}$$

The above equations can be used to form a general equation for mesh current. In general, the k^{th} mesh current of a circuit with m meshes is given by,

$$I_k = \frac{\Delta_{1k}}{\Delta} E_{11} + \frac{\Delta_{2k}}{\Delta} E_{22} + \frac{\Delta_{3k}}{\Delta} E_{33} + ...+ \frac{\Delta_{mk}}{\Delta} E_{mm} = \frac{1}{\Delta}\sum_{j=1}^{m} \Delta_{jk} E_{jj}$$

Short-cut Procedure for Cramer's Rule

A short-cut procedure exists for Cramer's rule which is shown below:

Let us consider a circuit with three mesh. The mesh basis matrix equation for a three mesh circuit is,

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix}$$

Let us define three determinants as shown below:

$$\Delta_1 = \begin{vmatrix} E_{11} & R_{12} & R_{13} \\ E_{22} & R_{22} & R_{23} \\ E_{33} & R_{32} & R_{33} \end{vmatrix} ; \qquad \Delta_2 = \begin{vmatrix} R_{11} & E_{11} & R_{13} \\ R_{21} & E_{22} & R_{23} \\ R_{31} & E_{33} & R_{33} \end{vmatrix} ; \qquad \Delta_3 = \begin{vmatrix} R_{11} & R_{12} & E_{11} \\ R_{21} & R_{22} & E_{22} \\ R_{31} & R_{32} & E_{33} \end{vmatrix}$$

Here, Δ_1 = Determinant of resistance matrix after replacing the first column of resistance matrix by source voltage column matrix.

Δ_2 = Determinant of resistance matrix after replacing the second column of resistance matrix by source voltage column matrix.

Δ_3 = Determinant of resistance matrix after replacing the third column of resistance matrix by source voltage column matrix.

Let, Δ = Determinant of resistance matrix

$$\Delta = \begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{vmatrix}$$

Now mesh currents I_1, I_2 and I_3 are given by,

$$\boxed{I_1 = \frac{\Delta_1}{\Delta}} \quad ; \quad \boxed{I_2 = \frac{\Delta_2}{\Delta}} \quad ; \quad \boxed{I_3 = \frac{\Delta_3}{\Delta}}$$

Cross-Check

The equation for mesh currents obtained by short-cut procedure is the same as equation (1.23), and verified as shown below:

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{1}{\Delta}\begin{vmatrix} E_{11} & R_{12} & R_{13} \\ E_{22} & R_{22} & R_{23} \\ E_{33} & R_{32} & R_{33} \end{vmatrix}$$

Expanding along first column

$$= \frac{1}{\Delta}[E_{11}\ \Delta_{11} + E_{22}\ \Delta_{21} + E_{33}\ \Delta_{31}] = \frac{\Delta_{11}}{\Delta}E_{11} + \frac{\Delta_{21}}{\Delta}E_{22} + \frac{\Delta_{31}}{\Delta}E_{33}$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{1}{\Delta}\begin{vmatrix} R_{11} & E_{11} & R_{13} \\ R_{21} & E_{22} & R_{23} \\ R_{31} & E_{33} & R_{33} \end{vmatrix}$$

Expanding along second column

$$= \frac{1}{\Delta}[E_{11}\ \Delta_{12} + E_{22}\ \Delta_{22} + E_{33}\ \Delta_{32}] = \frac{\Delta_{12}}{\Delta}E_{11} + \frac{\Delta_{22}}{\Delta}E_{22} + \frac{\Delta_{32}}{\Delta}E_{33}$$

$$I_3 = \frac{\Delta_3}{\Delta} = \frac{1}{\Delta}\begin{vmatrix} R_{11} & R_{12} & E_{11} \\ R_{21} & R_{22} & E_{22} \\ R_{31} & R_{32} & E_{33} \end{vmatrix}$$

Expanding along third column

$$= \frac{1}{\Delta}[E_{11}\ \Delta_{13} + E_{22}\ \Delta_{23} + E_{33}\ \Delta_{33}] = \frac{\Delta_{13}}{\Delta}E_{11} + \frac{\Delta_{23}}{\Delta}E_{22} + \frac{\Delta_{33}}{\Delta}E_{33}$$

Various Steps to Obtain the Solution of Mesh Currents and Branch Currents in a Circuit

***Step 1* :** Draw the graph of the circuit.

***Step 2* :** Determine the branches B and nodes N. The number of mesh currents m is given by $m = B - N + 1$.

***Step 3* :** Select m number of meshes of the circuit and attach a mesh current to each mesh.

***Step 4* :** In the given circuit choose arbitrary direction for branch and mesh currents. Let us denote mesh currents by I_1, I_2, I_3,....., and branch currents by I_a, I_b, I_c, I_d, I_e,...... Write the relationship between mesh and branch currents.

Preferably, the directions of mesh currents are chosen in the same orientation. For example, the direction of all the mesh currents can be chosen clockwise (alternatively, the direction of all the mesh currents can be chosen anticlockwise). When all the mesh currents are chosen in the same orientation, all the mutual-resistances (R_{jk}) will be negative.

***Step 5* :** Form the mesh basis matrix equation by inspection and solve the mesh currents using Cramer's rule. For a circuit with three meshes, the mesh basis matrix equation and solution of mesh currents using Cramer's rule are given below:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix}$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{1}{\Delta}\begin{vmatrix} E_{11} & R_{12} & R_{13} \\ E_{22} & R_{22} & R_{23} \\ E_{33} & R_{32} & R_{33} \end{vmatrix}$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{1}{\Delta}\begin{vmatrix} R_{11} & E_{11} & R_{13} \\ R_{21} & E_{22} & R_{23} \\ R_{31} & E_{33} & R_{33} \end{vmatrix}$$

$$I_3 = \frac{\Delta_3}{\Delta} = \frac{1}{\Delta}\begin{vmatrix} R_{11} & R_{12} & E_{11} \\ R_{21} & R_{22} & E_{22} \\ R_{31} & R_{32} & E_{33} \end{vmatrix}$$

Step 6 : Solve the branch currents using the relationship between branch and mesh currents.

> ***Note :*** *1. After solving the branch currents if any of the current is found to be negative, then the actual direction is opposite to that of the assumed direction. If interested we can draw the circuit by indicating the actual direction of current.*
>
> *2. If the directions of the current are already given in the circuit, then we have to solve for the given direction of the current.*

EXAMPLE 1.12

Solve the currents in various branches of the circuit shown in Fig. 1, by mesh analysis.

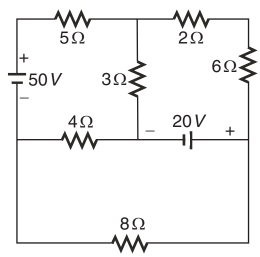


Fig. 1.

SOLUTION

The graph of the given circuit is shown in Fig. 2. It has six branches and four nodes. Hence, the number of meshes m in the circuit is, $m = B - N + 1 = 6 - 4 + 1 = 3$.

The circuit has six currents (corresponding to six branches) and in this three currents are independent (corresponding to three meshes).

Let us assume three mesh currents I_1, I_2 and I_3 as shown in Fig. 2. The directions of the current are chosen arbitrarily. The circuit with chosen mesh currents is shown in Fig. 3.

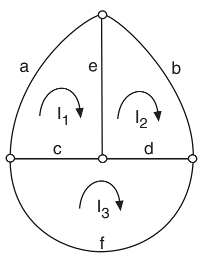


Fig. 2.

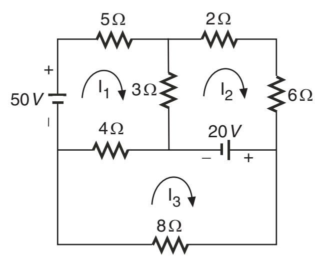


Fig. 3.

Method I : Formation of mesh basis equation by applying KVL

In this method, the mesh equations are formed using Kirchhoff's Voltage Law. The mesh equation for a mesh is formed by equating the sum of voltage fall to the sum of voltage rise. The voltage rise and fall are determined by tracing the circuit in the direction of the mesh current.

With reference to Fig. 4, the mesh equation for mesh-1 is formed as shown below :

Voltage fall : $5I_1, 3I_1, 4I_1$

Voltage rise : $50, 3I_2, 4I_3$

$$\therefore \quad 5I_1 + 3I_1 + 4I_1 = 50 + 3I_2 + 4I_3$$

$$12I_1 - 3I_2 - 4I_3 = 50 \qquad ..... (1)$$

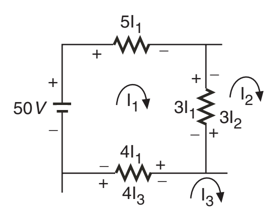


Fig. 4.

With reference to Fig. 5, the mesh equation for mesh-2 is formed as shown below :

Voltage fall : $3I_2, 2I_2, 6I_2, 20$

Voltage rise : $3I_1$

$$\therefore \quad 3I_2 + 2I_2 + 6I_2 + 20 = 3I_1$$

$$-3I_1 + 11I_2 = -20 \qquad ..... (2)$$

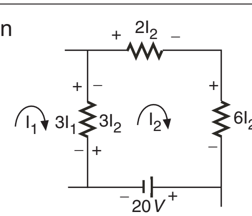


Fig. 5.

With reference to Fig. 6, the mesh equation for mesh-3 is formed as shown below :

Voltage fall : $4I_3, 8I_3$

Voltage rise : $20, 4I_1$

$$\therefore\ 4I_3 + 8I_3 = 20 + 4I_1$$

$$-4I_1 + 12I_3 = 20 \qquad \text{..... (3)}$$

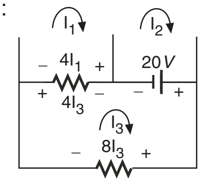


Fig. 6.

Equations (1), (2) and (3) are the mesh equations of the circuit shown in Fig. 3. The mesh equations are summarised here for convenience.

$$12I_1 - 3I_2 - 4I_3 = 50$$
$$-3I_1 + 11I_2 = -20$$
$$-4I_1 + 12I_3 = 20$$

The mesh equations can be arranged in the matrix form as shown below and then solved by Cramer's rule.

$$\begin{bmatrix} 12 & -3 & -4 \\ -3 & 11 & 0 \\ -4 & 0 & 12 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 50 \\ -20 \\ 20 \end{bmatrix} \qquad \text{..... (4)}$$

Method II : Formation of mesh basis matrix equation by inspection

In this method, the mesh basis matrix equation is formed directly from the circuit shown in Fig. 3 by inspection. The circuit has three meshes. The general form of mesh basis matrix equation for three mesh circuit is shown in equation (5).

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \qquad \text{..... (5)}$$

The elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 5 + 3 + 4 = 12$	$R_{12} = R_{21} = -3$	$E_{11} = 50$
$R_{22} = 3 + 2 + 6 = 11$	$R_{13} = R_{31} = -4$	$E_{22} = -20$
$R_{33} = 4 + 8 = 12$	$R_{23} = R_{32} = 0$	$E_{33} = 20$

On substituting the above terms in equation (5), we get equation (6) and the solution of equation (6) will give the mesh currents.

$$\begin{bmatrix} 12 & -3 & -4 \\ -3 & 11 & 0 \\ -4 & 0 & 12 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 50 \\ -20 \\ 20 \end{bmatrix} \qquad \text{..... (6)}$$

Solution of mesh currents

It is observed that the mesh basis matrix equation obtained in method I and II are the same. In equation (6), the unknown are I_1, I_2 and I_3. In order to solve I_1, I_2 and I_3, let us define four determinants Δ, Δ_1, Δ_2 and Δ_3 as shown below:

$$\Delta = \begin{vmatrix} 12 & -3 & -4 \\ -3 & 11 & 0 \\ -4 & 0 & 12 \end{vmatrix};\quad \Delta_1 = \begin{vmatrix} 50 & -3 & -4 \\ -20 & 11 & 0 \\ 20 & 0 & 12 \end{vmatrix};\quad \Delta_2 = \begin{vmatrix} 12 & 50 & -4 \\ -3 & -20 & 0 \\ -4 & 20 & 12 \end{vmatrix};\quad \Delta_3 = \begin{vmatrix} 12 & -3 & 50 \\ -3 & 11 & -20 \\ -4 & 0 & 20 \end{vmatrix}$$

The determinants are evaluated by expanding along first row and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 12 & -3 & -4 \\ -3 & 11 & 0 \\ -4 & 0 & 12 \end{vmatrix} = 12\times[11\times12-0]-(-3)\times[-3\times12-0]+(-4)\times[0-(-4)\times11]$$

$$= 1584 - 108 - 176 = 1300$$

$$\Delta_1 = \begin{vmatrix} 50 & -3 & -4 \\ -20 & 11 & 0 \\ 20 & 0 & 12 \end{vmatrix} = 50\times[11\times12-0]-(-3)\times[-20\times12-0]+(-4)\times[0-20\times11]$$

$$= 6600 - 720 + 880 = 6760$$

$$\Delta_2 = \begin{vmatrix} 12 & 50 & -4 \\ -3 & -20 & 0 \\ -4 & 20 & 12 \end{vmatrix} = 12\times[-20\times12-0]-50\times[-3\times12-0] +(-4)\times[-3\times20-(-4)\times(-20)]$$

$$= -2880 + 1800 + 560 = -520$$

$$\Delta_3 = \begin{vmatrix} 12 & -3 & 50 \\ -3 & 11 & -20 \\ -4 & 0 & 20 \end{vmatrix} = 12\times[11\times20-0]-(-3)\times[-3\times20-(-4)\times(-20)]+50\times[0-(-4)\times11]$$

$$= 2640 - 420 + 2200 = 4420$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{6760}{1300} = 5.2\,A$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{-520}{1300} = -0.4\,A$$

$$I_3 = \frac{\Delta_3}{\Delta} = \frac{4420}{1300} = 3.4\,A$$

Here, the mesh current I_2 is negative. Hence, the actual direction of I_2 is opposite to that of assumed direction. Since there are six branches in the given circuit, we can assume six currents I_a, I_b, I_c, I_d, I_e and I_f as shown in Fig. 7. The direction of branch currents are chosen such that they are all positive. The relation between mesh and branch currents can be obtained from Fig. 7 and the branch currents are evaluated as shown below:

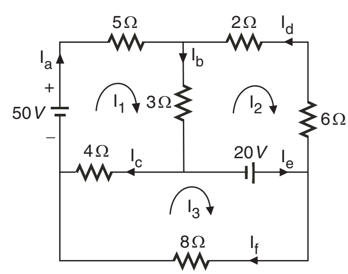


Fig. 7.

$$I_a = I_1 = 5.2\,A$$

$$I_b = I_1 - I_2 = 5.2 - (-0.4) = 5.6\,A$$

$$I_c = I_1 - I_3 = 5.2 - 3.4 = 1.8\,A$$

$$I_d = -I_2 = -(-0.4) = 0.4\,A$$

$$I_e = I_3 - I_2 = 3.4 - (-0.4) = 3.8\,A$$

$$I_f = I_3 = 3.4\,A$$

EXAMPLE 1.13

Determine the currents in various elements of the bridge circuit shown in Fig. 1, using mesh analysis.

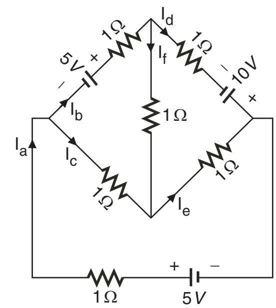


Fig. 1.

SOLUTION

The graph of the given circuit is shown in Fig. 2.

It has 6 branches and 4 nodes.

Hence, the number of meshes m in the circuit is,

m = B – N + 1 = 6 – 4 + 1 = 3.

The circuit has 6 currents (corresponding to six branches) and in this 3 currents are independent (corresponding to three meshes).

Let us assume three mesh currents as shown in Fig. 2. The direction of the current are chosen arbitrarily. The circuit with chosen mesh currents is shown in Fig. 3.

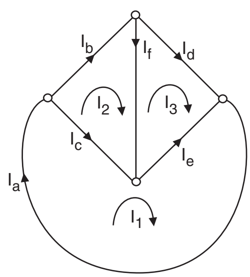


Fig. 2.

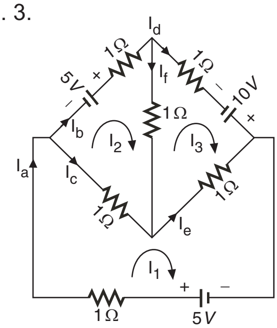


Fig. 3.

Method I : Formation of mesh basis equation by applying KVL

In this method, the mesh equations are formed using Kirchhoff's Voltage Law.

The mesh equation for a mesh is formed by equating the sum of voltage fall to the sum of voltage rise.

The voltage rise and fall are determined by tracing the circuit in the direction of the mesh current.

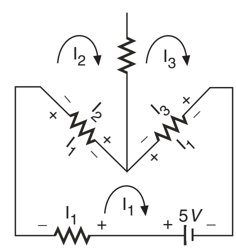


Fig. 4.

With reference to Fig. 4, the mesh equation for mesh-1 is formed as shown below:

Voltage fall : I_1, I_1, I_1

Voltage rise : I_2, I_3 , 5

$\therefore\ I_1 + I_1 + I_1 = I_2 + I_3 + 5$

$3I_1 - I_2 - I_3 = 5$ .....(1)

With reference to Fig. 5, the mesh equation for mesh-2 is formed as shown below:

Voltage fall : I_2, I_2, I_2

Voltage rise : I_1, I_3 , 5

$\therefore\ I_2 + I_2 + I_2 = I_1 + I_3 + 5$

$-I_1 + 3I_2 - I_3 = 5$ ..... (2)

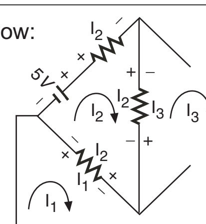


Fig. 5.

With reference to Fig. 6, the mesh equation for mesh-3 is formed as shown below:

Voltage fall : I_3, I_3, I_3

Voltage rise : I_1, I_2 , 10

$\therefore \quad I_3 + I_3 + I_3 = I_1 + I_2 + 10$

$-I_1 - I_2 + 3I_3 = 10$ ..... (3)

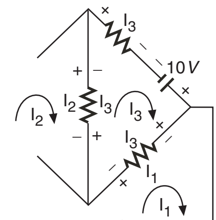


Fig. 6.

Equations (1), (2) and (3) are the mesh equations of the circuit shown in Fig. 3. The mesh equations are summarised here for convenience.

$$3I_1 - I_2 - I_3 = 5$$
$$-I_1 + 3I_2 - I_3 = 5$$
$$-I_1 - I_2 + 3I_3 = 10$$

The mesh equations can be arranged in the matrix form as shown below and then solved by Cramer's rule.

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix} \qquad ..... (4)$$

Method II : Formation of mesh basis equation by inspection

In this method, the mesh basis matrix equation is formed directly from the circuit shown in Fig.3 by inspection. The circuit has three meshes. The general form of mesh basis matrix equation for three mesh circuit is shown in equation (5).

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \qquad ..... (5)$$

The elements of the resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 1 + 1 + 1 = 3$	$R_{12} = R_{21} = -1$	$E_{11} = 5$
$R_{22} = 1 + 1 + 1 = 3$	$R_{13} = R_{31} = -1$	$E_{22} = 5$
$R_{33} = 1 + 1 + 1 = 3$	$R_{23} = R_{32} = -1$	$E_{33} = 10$

On substituting the above terms in equation (5), we get,

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix} \qquad ..... (6)$$

Solution of mesh currents

It is observed that the mesh basis matrix equation obtained in method I and II are the same. In equation (6) the unknowns are I_1, I_2 and I_3. In order to solve I_1, I_2 and I_3, let us define four determinants Δ, Δ_1, Δ_2 and Δ_3 as shown below:

$$\Delta = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix}; \quad \Delta_1 = \begin{vmatrix} 5 & -1 & -1 \\ 5 & 3 & -1 \\ 10 & -1 & 3 \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} 3 & 5 & -1 \\ -1 & 5 & -1 \\ -1 & 10 & 3 \end{vmatrix}; \quad \Delta_3 = \begin{vmatrix} 3 & -1 & 5 \\ -1 & 3 & 5 \\ -1 & -1 & 10 \end{vmatrix}$$

The determinants are evaluated by expanding along first row and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 3\times[3\times3-(-1)\times(-1)]-(-1)\times[-1\times3-(-1)\times(-1)]$$

$$+(-1)\times[-1\times(-1)-(-1)\times3]$$

$$= 24 - 4 - 4 = 16$$

$$\Delta_1 = \begin{vmatrix} 5 & -1 & -1 \\ 5 & 3 & -1 \\ 10 & -1 & 3 \end{vmatrix} = 5\times[3\times3-(-1)\times(-1)]-(-1)\times[5\times3-10\times(-1)]+(-1)\times[5\times(-1)-10\times3]$$

$$= 40 + 25 + 35 = 100$$

$$\Delta_2 = \begin{vmatrix} 3 & 5 & -1 \\ -1 & 5 & -1 \\ -1 & 10 & 3 \end{vmatrix} = 3\times[5\times3-10\times(-1)]-5\times[-1\times3-(-1)\times(-1)]+(-1)\times[-1\times10-(-1)\times5]$$

$$= 75 + 20 + 5 = 100$$

$$\Delta_3 = \begin{vmatrix} 3 & -1 & 5 \\ -1 & 3 & 5 \\ -1 & -1 & 10 \end{vmatrix} = 3\times[3\times10-(-1)\times5]-(-1)\times[-1\times10-(-1)\times5]+5\times[-1\times(-1)-(-1)\times3]$$

$$= 105 - 5 + 20 = 120$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{100}{16} = 6.25\,A$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{100}{16} = 6.25\,A$$

$$I_3 = \frac{\Delta_3}{\Delta} = \frac{120}{16} = 7.5\,A$$

The relation between mesh and branch currents can be obtained from Fig. 3 and branch currents are evaluated as shown below:

$I_a = I_1 = 6.25\,A$

$I_b = I_2 = 6.25\,A$

$I_c = I_1 - I_2 = 6.25 - 6.25 = 0$

$I_d = I_3 = 7.5\,A$

$I_e = I_1 - I_3 = 6.25 - 7.5 = -1.25\,A$

$I_f = I_2 - I_3 = 6.25 - 7.5 = -1.25\,A$

EXAMPLE 1.14

(AU Dec'14, 16 Marks)

In the circuit shown in Fig.1, find **(a)** mesh currents in the circuit, **(b)** current supplied by the battery and **(c)** potential difference between terminals B and D.

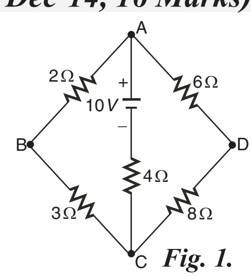


Fig. 1.

SOLUTION

Since the given circuit has only one source, it is possible to predict the exact directions of the current.

The current will start from the positive end of the supply and when it enters node-A, it will divide into two parts. These two currents will again meet at node-C and enter the negative end of the supply through 4 Ω resistor.

The circuit has three branch currents and in this two are independent. Hence, we can take two mesh currents.

The actual directions of mesh and branch currents are shown in Fig. 2. Using the circuit shown in Fig. 2, the mesh basis matrix equation is formed as shown below:

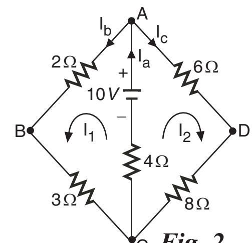


Fig. 2.

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \end{bmatrix} \qquad ..... (1)$$

The elements of the resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 2 + 4 + 3 = 9$	$R_{12} = R_{21} = 4$	$E_{11} = 10$
$R_{22} = 4 + 6 + 8 = 18$		$E_{22} = 10$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 9 & 4 \\ 4 & 18 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \qquad ..... (2)$$

In equation (2), the unknowns are I_1 and I_2. In order to solve I_1 and I_2, let us define three determinants Δ, Δ_1 and Δ_2 as shown below:

$$\Delta = \begin{vmatrix} 9 & 4 \\ 4 & 18 \end{vmatrix}; \qquad \Delta_1 = \begin{vmatrix} 10 & 4 \\ 10 & 18 \end{vmatrix}; \qquad \Delta_2 = \begin{vmatrix} 9 & 10 \\ 4 & 10 \end{vmatrix}$$

The determinants are evaluated by expanding along first row and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 9 & 4 \\ 4 & 18 \end{vmatrix} = 9 \times 18 - 4 \times 4 = 146$$

$$\Delta_1 = \begin{vmatrix} 10 & 4 \\ 10 & 18 \end{vmatrix} = 10 \times 18 - 10 \times 4 = 140$$

$$\Delta_2 = \begin{vmatrix} 9 & 10 \\ 4 & 10 \end{vmatrix} = 9 \times 10 - 4 \times 10 = 50$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{140}{146} = 0.9589\,A$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{50}{146} = 0.3425\,A$$

b) To find the battery current

With reference to Fig. 2, the battery current is given by, $I_a = I_1 + I_2$

$\therefore$ Battery current, $I_a = I_1 + I_2 = 0.9589 + 0.3425 = 1.3014\,A$

c) To find potential difference between the termianals ''B'' and ''D''

The given circuit is redrawn as shown in Fig. 3. With reference to Fig. 3, using KVL , we can write,

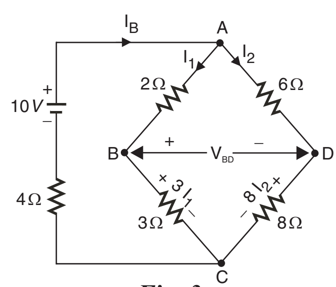


Fig. 3.

$$V_{BD} + 8I_2 = 3I_1 \qquad \Rightarrow \qquad V_{BD} = 3I_1 - 8I_2$$

$$\therefore V_{BD} = 3 \times 0.9589 - 8 \times 0.3425 = 0.1367\,V$$

EXAMPLE 1.15

(AU May'15, 8 Marks)

Use branch currents in the network shown in Fig. 1 to find the current supplied by the 60 V source. Solve the circuit by the mesh current method.

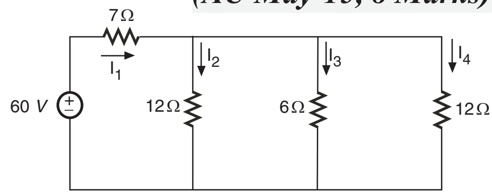


Fig. 1.

SOLUTION

The direction of the mesh currents are chosen to match the given branch currents as shown in Fig. 2. With reference to Fig.2, the mesh basis matrix equation is formed as shown below:

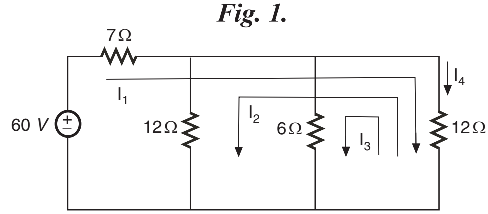


Fig. 2.

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \quad ..... (1)$$

Now, $I_4 = I_1 - I_2 - I_3$

The elements of the resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 7 + 12 = 19$	$R_{12} = R_{21} = -12$	$E_{11} = 60$
$R_{22} = 12 + 12 = 24$	$R_{13} = R_{31} = -12$	$E_{22} = 0$
$R_{33} = 6 + 12 = 18$	$R_{23} = R_{32} = +12$	$E_{33} = 0$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 19 & -12 & -12 \\ -12 & 24 & 12 \\ -12 & 12 & 18 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 60 \\ 0 \\ 0 \end{bmatrix} \quad ..... (2)$$

In equation (2), the unknowns are I_1, I_2 and I_3. In order to solve I_1, I_2 and I_3, let us define four determinants Δ, Δ_1, Δ_2 and Δ_3 as shown below:

$$\Delta = \begin{vmatrix} 19 & -12 & -12 \\ -12 & 24 & 12 \\ -12 & 12 & 18 \end{vmatrix}; \quad \Delta_1 = \begin{vmatrix} 60 & -12 & -12 \\ 0 & 24 & 12 \\ 0 & 12 & 18 \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} 19 & 60 & -12 \\ -12 & 0 & 12 \\ -12 & 0 & 18 \end{vmatrix}; \quad \Delta_3 = \begin{vmatrix} 19 & -12 & 60 \\ -12 & 24 & 0 \\ -12 & 12 & 0 \end{vmatrix}$$

The determinants are evaluated by expanding along first row and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 19 & -12 & -12 \\ -12 & 24 & 12 \\ -12 & 12 & 18 \end{vmatrix} = 19 \times [24 \times 18 - 12^2] - (-12) \times [-12 \times 18 - (-12) \times 12]$$

$$+ (-12) \times [-12 \times 12 - (-12) \times 24]$$

$$= 5472 - 864 - 1728 = 2880$$

$$\Delta_1 = \begin{vmatrix} 60 & -12 & -12 \\ 0 & 24 & 12 \\ 0 & 12 & 18 \end{vmatrix} = 60 \times [24 \times 18 - 12^2] = 17280$$

$$\Delta_2 = \begin{vmatrix} 19 & 60 & -12 \\ -12 & 0 & 12 \\ -12 & 0 & 18 \end{vmatrix} = -60 \times [-12 \times 18 - (-12) \times 12] = 4320$$

$$\Delta_3 = \begin{vmatrix} 19 & -12 & 60 \\ -12 & 24 & 0 \\ -12 & 12 & 0 \end{vmatrix} = 60 \times [-12 \times 12 - (-12) \times 24] = 8640$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{17280}{2880} = 6\,A$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{4320}{2880} = 1.5\,A$$

$$I_3 = \frac{\Delta_3}{\Delta} = \frac{8640}{2880} = 3\,A$$

$$I_4 = I_1 - I_2 - I_3 = 6 - 1.5 - 3 = 1.5\,A$$

Current supplied by 60 V source = I_1 = 6 A

EXAMPLE 1.16

Solve the mesh currents shown in Fig. 1.

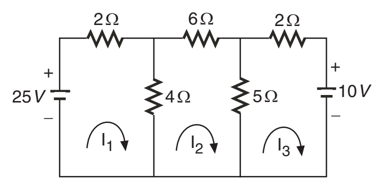


Fig. 1.

SOLUTION

The mesh currents and their direction are given in the problem and so we need not assume the currents. Using the circuit shown in Fig. 1, the mesh basis matrix equation is formed as shown below:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \quad \text{..... (1)}$$

The elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 2 + 4 = 6$	$R_{12} = R_{21} = -4$	$E_{11} = 25$
$R_{22} = 4 + 6 + 5 = 15$	$R_{13} = R_{31} = 0$	$E_{22} = 0$
$R_{33} = 5 + 2 = 7$	$R_{23} = R_{32} = -5$	$E_{33} = -10$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 6 & -4 & 0 \\ -4 & 15 & -5 \\ 0 & -5 & 7 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 25 \\ 0 \\ -10 \end{bmatrix} \quad \text{..... (2)}$$

In equation (2), the unknowns are I_1, I_2 and I_3. In order to solve I_1, I_2 and I_3, let us define four determinants Δ, Δ_1, Δ_2 and Δ_3 as shown below:

$$\Delta = \begin{vmatrix} 6 & -4 & 0 \\ -4 & 15 & -5 \\ 0 & -5 & 7 \end{vmatrix}; \quad \Delta_1 = \begin{vmatrix} 25 & -4 & 0 \\ 0 & 15 & -5 \\ -10 & -5 & 7 \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} 6 & 25 & 0 \\ -4 & 0 & -5 \\ 0 & -10 & 7 \end{vmatrix}; \quad \Delta_3 = \begin{vmatrix} 6 & -4 & 25 \\ -4 & 15 & 0 \\ 0 & -5 & -10 \end{vmatrix}$$

The determinants are evaluated by expanding along first row and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 6 & -4 & 0 \\ -4 & 15 & -5 \\ 0 & -5 & 7 \end{vmatrix} = 6\times[15\times7-(-5)\times(-5)]-(-4)\times[-4\times7-0]+0$$

$$= 480 - 112 = 368$$

$$\Delta_1 = \begin{vmatrix} 25 & -4 & 0 \\ 0 & 15 & -5 \\ -10 & -5 & 7 \end{vmatrix} = 25\times[15\times7-(-5)\times(-5)]-(-4)\times[0-(-10)\times(-5)]+0$$

$$= 2000 - 200 = 1800$$

$$\Delta_2 = \begin{vmatrix} 6 & 25 & 0 \\ -4 & 0 & -5 \\ 0 & -10 & 7 \end{vmatrix} = 6\times[0-(-10)\times(-5)]-25\times[-4\times7-0]+0$$

$$= -300 + 700 = 400$$

$$\Delta_3 = \begin{vmatrix} 6 & -4 & 25 \\ -4 & 15 & 0 \\ 0 & -5 & -10 \end{vmatrix} = 6\times[15\times(-10)-0]-(-4)\times[-4\times(-10)-0]+25\times[-4\times(-5)-0]$$

$$= -900 + 160 + 500 = -240$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{1800}{368} = 4.8913\ A$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{400}{368} = 1.0870\ A$$

$$I_3 = \frac{\Delta_3}{\Delta} = \frac{-240}{368} = -0.6522\ A$$

EXAMPLE 1.17 ***(AU June'16, 8 Marks)***

In the circuit shown in Fig. 1, find I_L by mesh analysis.

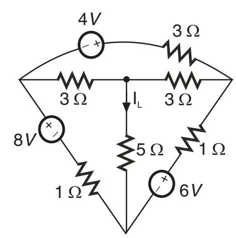


Fig. 1.

SOLUTION

Let us choose mesh currents as shown in Fig. 2, the mesh basis matrix equation is formed as shown below:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \quad \text{..... (1)}$$

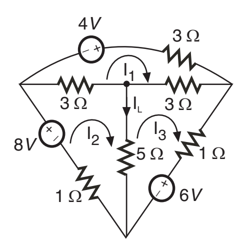


Fig. 2.

The elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 3 + 3 + 3 = 9$	$R_{12} = R_{21} = -3$	$E_{11} = 4$
$R_{22} = 3 + 5 + 1 = 9$	$R_{13} = R_{31} = -3$	$E_{22} = 8$
$R_{33} = 5 + 3 + 1 = 9$	$R_{23} = R_{32} = -5$	$E_{33} = -6$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 9 & -3 & -3 \\ -3 & 9 & -5 \\ -3 & -5 & 9 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ -6 \end{bmatrix} \quad \text{..... (2)}$$

Here, $I_L = I_2 - I_3$.

In order to solve the mesh currents I_2 and I_3, let us define three determinants Δ, Δ_2 and Δ_3 as shown below:

$$\Delta = \begin{vmatrix} 9 & -3 & -3 \\ -3 & 9 & -5 \\ -3 & -5 & 9 \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} 9 & 4 & -3 \\ -3 & 8 & -5 \\ -3 & -6 & 9 \end{vmatrix}; \quad \Delta_3 = \begin{vmatrix} 9 & -3 & 4 \\ -3 & 9 & 8 \\ -3 & -5 & -6 \end{vmatrix}$$

The determinants are evaluated by expanding along first row and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 9 & -3 & -3 \\ -3 & 9 & -5 \\ -3 & -5 & 9 \end{vmatrix} = 9 \times [9^2 - (-5)^2] - (-3) \times [-3 \times 9 - (-3) \times (-5)]$$
$$+ (-3) \times [-3 \times (-5) - (-3) \times 9]$$
$$= 504 - 126 - 126 = 252$$

$$\Delta_2 = \begin{vmatrix} 9 & 4 & -3 \\ -3 & 8 & -5 \\ -3 & -6 & 9 \end{vmatrix} = 9 \times [8 \times 9 - (-6) \times (-5)] - 4 \times [-3 \times 9 - (-3) \times (-5)]$$
$$+ (-3) \times [-3 \times (-6) - (-3) \times 8]$$
$$= 378 + 168 - 126 = 420$$

$$\Delta_3 = \begin{vmatrix} 9 & -3 & 4 \\ -3 & 9 & 8 \\ -3 & -5 & -6 \end{vmatrix} = 9 \times [9 \times (-6) - (-5)] \times 8] - (-3) \times [-3 \times (-6) - (-3) \times 8]$$
$$+ 4 \times [-3 \times (-5) - (-3) \times 9]$$
$$= -126 + 126 + 168 = 168$$

$$I_L = I_2 - I_3 = \frac{\Delta_2}{\Delta} - \frac{\Delta_3}{\Delta} = \frac{\Delta_2 - \Delta_3}{\Delta} = \frac{420 - 168}{252} = 1\,A$$

EXAMPLE 1.18

In the circuit shown in Fig. 1, find E such that $I_2 = 0$.

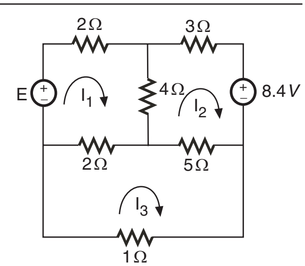


Fig. 1.

SOLUTION

In mesh analysis, when the solution of mesh current is obtained by Cramer's rule, the mesh current I_2 is given by,

$$I_2 = \frac{\Delta_2}{\Delta} \qquad \text{..... (1)}$$

In equation (1), if $I_2 = 0$, then $\Delta_2 = 0$. Therefore, in order to find the value of E, we can form the mesh basis matrix equation. Then form the determinant Δ_2 and equate the determinant to zero.

Using Fig. 1, the mesh basis matrix equation is formed by inspection as shown below:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \qquad \text{..... (2)}$$

The elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 2 + 4 + 2 = 8$	$R_{12} = R_{21} = -4$	$E_{11} = E$
$R_{22} = 4 + 3 + 5 = 12$	$R_{13} = R_{31} = -2$	$E_{22} = -8.4$
$R_{33} = 2 + 5 + 1 = 8$	$R_{23} = R_{32} = -5$	$E_{33} = 0$

On substituting the above terms in equation (2), we get,

$$\begin{bmatrix} 8 & -4 & -2 \\ -4 & 12 & -5 \\ -2 & -5 & 8 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E \\ -8.4 \\ 0 \end{bmatrix} \quad \text{..... (3)}$$

In equation (3) by Cramer's rule, the unknown current I_2 is given by,

$$I_2 = \frac{\Delta_2}{\Delta}, \text{ where } \Delta_2 = \begin{vmatrix} 8 & E & -2 \\ -4 & -8.4 & -5 \\ -2 & 0 & 8 \end{vmatrix}$$

On expanding Δ_2, we get,

$$\Delta_2 = \begin{vmatrix} 8 & E & -2 \\ -4 & -8.4 & -5 \\ -2 & 0 & 8 \end{vmatrix} = 8\times[-8.4\times 8 - 0] - E\times[-4\times 8 - (-2)\times(-5)] + (-2)\times[0 - (-2)\times(-8.4)]$$

$$= -537.6 + 42E + 33.6 = -504 + 42E$$

On equating $\Delta_2 = 0$, we get,

$$-504 + 42E = 0$$

$$\therefore \; 42\,E = 504$$

$$E = \frac{504}{42} = 12\,V$$

EXAMPLE 1.19

Solve the current in 12 Ω resistor by mesh analysis.

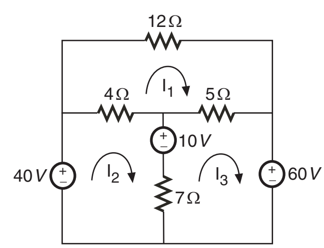


Fig. 1.

SOLUTION

The mesh currents and their directions are given in the problem and so we need not assume the currents. Using the circuit shown in Fig. 1 the mesh basis matrix equation is formed as shown below:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \quad \text{..... (1)}$$

The elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 12 + 4 + 5 = 21$	$R_{12} = R_{21} = -4$	$E_{11} = 0$
$R_{22} = 4 + 7 = 11$	$R_{13} = R_{31} = -5$	$E_{22} = 40 - 10 = 30\,V$
$R_{33} = 7 + 5 = 12$	$R_{23} = R_{32} = -7$	$E_{33} = 10 - 60 = -50\,V$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 21 & -4 & -5 \\ -4 & 11 & -7 \\ -5 & -7 & 12 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 30 \\ -50 \end{bmatrix} \quad \text{..... (2)}$$

The current through 12Ω resistance is I_1. To solve the current I_1 by Cramer's rule let us define the determinants Δ and Δ_1 as shown below:

$$\Delta = \begin{vmatrix} 21 & -4 & -5 \\ -4 & 11 & -7 \\ -5 & -7 & 12 \end{vmatrix}; \qquad \Delta_1 = \begin{vmatrix} 0 & -4 & -5 \\ 30 & 11 & -7 \\ -50 & -7 & 12 \end{vmatrix}$$

The determinants are evaluated by expanding along first row and the mesh current I_1 is solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 21 & -4 & -5 \\ -4 & 11 & -7 \\ -5 & -7 & 12 \end{vmatrix} = 21\times[11\times12-(-7)\times(-7)]-(-4)\times[-4\times12-(-5)\times(-7)]$$
$$+(-5)\times[-4\times(-7)-(-5)\times11]$$
$$= 1743 - 332 - 415 = 996$$

$$\Delta_1 = \begin{vmatrix} 0 & -4 & -5 \\ 30 & 11 & -7 \\ -50 & -7 & 12 \end{vmatrix} = 0-(-4)\times[30\times12-(-50)\times(-7)]+(-5)\times[30\times(-7)-(-50)\times11]$$
$$= 40 - 1700 = -1660$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{-1660}{996} = -1.6667\ A$$

EXAMPLE 1.20

Solve the mesh currents in the circuit shown in Fig. 1.

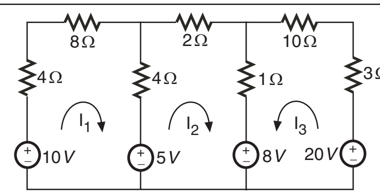


Fig. 1.

SOLUTION

With reference to Fig.1, the mesh basis matrix equation is formed as shown below:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \qquad ..... (1)$$

Note : *Here, the directions of the mesh currents are given in the problem itself.*

The elements of the resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 4 + 8 + 4 = 16$	$R_{12} = R_{21} = -4$	$E_{11} = 10 - 5 = 5$
$R_{22} = 4 + 2 + 1 = 7$	$R_{13} = R_{31} = 0$	$E_{22} = 5 - 8 = -3$
$R_{33} = 1 + 10 + 3 = 14$	$R_{23} = R_{32} = 1$	$E_{33} = 20 - 8 = 12$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 16 & -4 & 0 \\ -4 & 7 & 1 \\ 0 & 1 & 14 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 12 \end{bmatrix} \qquad ..... (2)$$

In equation (2), the unknowns are I_1, I_2 and I_3. In order to solve I_1, I_2 and I_3, let us define four determinants Δ, Δ_1, Δ_2 and Δ_3 as shown below:

$$\Delta = \begin{vmatrix} 16 & -4 & 0 \\ -4 & 7 & 1 \\ 0 & 1 & 14 \end{vmatrix}; \quad \Delta_1 = \begin{vmatrix} 5 & -4 & 0 \\ -3 & 7 & 1 \\ 12 & 1 & 14 \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} 16 & 5 & 0 \\ -4 & -3 & 1 \\ 0 & 12 & 14 \end{vmatrix}; \quad \Delta_3 = \begin{vmatrix} 16 & -4 & 5 \\ -4 & 7 & -3 \\ 0 & 1 & 12 \end{vmatrix}$$

The determinants are evaluated by expanding along first row and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 16 & -4 & 0 \\ -4 & 7 & 1 \\ 0 & 1 & 14 \end{vmatrix} = 16\times[7\times14 - 1\times1] - (-4)\times[-4\times14 - 0] + 0$$

$$= 1552 - 224 = 1328$$

$$\Delta_1 = \begin{vmatrix} 5 & -4 & 0 \\ -3 & 7 & 1 \\ 12 & 1 & 14 \end{vmatrix} = 5\times[7\times14 - 1\times1] - (-4)\times[-3\times14 - 12\times1] + 0$$

$$= 485 - 216 = 269$$

$$\Delta_2 = \begin{vmatrix} 16 & 5 & 0 \\ -4 & -3 & 1 \\ 0 & 12 & 14 \end{vmatrix} = 16\times[-3\times14 - 12\times1] - 5\times[-4\times14 - 0] + 0$$

$$= -864 + 280 = -584$$

$$\Delta_3 = \begin{vmatrix} 16 & -4 & 5 \\ -4 & 7 & -3 \\ 0 & 1 & 12 \end{vmatrix} = 16\times[7\times12 - 1\times(-3)] - (-4)\times[-4\times12 - 0] + 5\times[-4\times1 - 0]$$

$$= 1392 - 192 - 20 = 1180$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{269}{1328} = 0.2026\,A$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{-584}{1328} = -0.4398\,A$$

$$I_3 = \frac{\Delta_3}{\Delta} = \frac{1180}{1328} = 0.8886\,A$$

EXAMPLE 1.21

(AU Dec'15, 8 Marks)

Determine the power dissipation in the 4 Ω resistor of the circuit shownin Fig. 1.

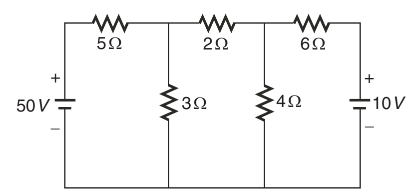


Fig. 1.

SOLUTION

The graph of the given circuit is shown in Fig. 2. It has five branches and three nodes. Hence, the number of meshes m in the circuit is, m = B − N + 1 = 5 − 3 + 1 = 3.

The circuit has five currents (corresponding to five branches) and in this three currents are independent (corresponding to three meshes). Let us assume three mesh currents I_1, I_2 and I_3 as shown in Fig. 2.

The direction of the currents are chosen arbitrarily. The circuit with chosen mesh currents is shown in Fig. 3.

Now, the current through 4 Ω resistor is $(I_2 - I_3)$ in the direction shown in Fig. 3.

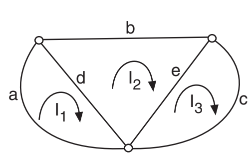


Fig. 2.

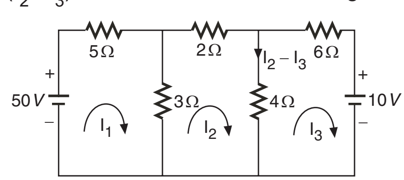


Fig. 3.

$\therefore$ Power dissipated in 4Ω resistor $= |I_2 - I_3|^2 \times 4$

Using the circuit shown in Fig. 3, the mesh basis matrix equation is formed as shown below :

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \qquad .....(1)$$

The elements of resistance matrix and source voltage matrix are formed as shown below :

$R_{11} = 5 + 3 = 8$	$R_{12} = R_{21} = -3$	$E_{11} = 50$
$R_{22} = 3 + 2 + 4 = 9$	$R_{13} = R_{31} = 0$	$E_{22} = 0$
$R_{33} = 4 + 6 = 10$	$R_{23} = R_{32} = -4$	$E_{33} = -10$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 8 & -3 & 0 \\ -3 & 9 & -4 \\ 0 & -4 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ -10 \end{bmatrix} \qquad ..... (2)$$

Here, we have to solve the mesh currents I_2 and I_3. In order to solve I_2 and I_3, let us define three determinants Δ, Δ_2 and Δ_3 as shown below:

$$\Delta = \begin{vmatrix} 8 & -3 & 0 \\ -3 & 9 & -4 \\ 0 & -4 & 10 \end{vmatrix}; \qquad \Delta_2 = \begin{vmatrix} 8 & 50 & 0 \\ -3 & 0 & -4 \\ 0 & -10 & 10 \end{vmatrix}; \qquad \Delta_3 = \begin{vmatrix} 8 & -3 & 50 \\ -3 & 9 & 0 \\ 0 & -4 & -10 \end{vmatrix}$$

The determinants are evaluated by expanding along first row and then the currents I_2 and I_3 are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 8 & -3 & 0 \\ -3 & 9 & -4 \\ 0 & -4 & 10 \end{vmatrix} = 8 \times [9 \times 10 - (-4) \times (-4)] - (-3) \times [-3 \times 10 - 0] + 0$$
$$= 592 - 90 = 502$$

$$\Delta_2 = \begin{vmatrix} 8 & 50 & 0 \\ -3 & 0 & -4 \\ 0 & -10 & 10 \end{vmatrix} = 8 \times [0 - (-10) \times (-4)] - 50 \times [-3 \times 10 - 0] + 0$$
$$= -320 + 1500 = 1180$$

$$\Delta_3 = \begin{vmatrix} 8 & -3 & 50 \\ -3 & 9 & 0 \\ 0 & -4 & -10 \end{vmatrix} = 8 \times [9 \times (-10) - 0] - (-3) \times [-3 \times (-10) - 0] + 50 \times [-3 \times (-4) - 0]$$
$$= -720 + 90 + 600 = -30$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{1180}{502} = 2.3506\,A$$

$$I_3 = \frac{\Delta_3}{\Delta} = \frac{-30}{502} = -0.0598\,A$$

$\therefore$ Power dissipated in 4 Ω resistor $= |I_2 - I_3|^2 \times 4 = |2.3506 - (-0.0598)|^2 \times 4$

$$= |2.4104|^2 \times 4 = 2.4104^2 \times 4 = 23.2401W$$

EXAMPLE 1.22

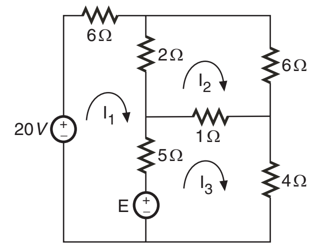


Fig. 1.

Determine the voltage E which causes the current I_1 to be zero for the circuit shown in Fig. 1.

SOLUTION

In mesh analysis, when the solution of mesh currents is obtained by Cramer's rule, the mesh current I_1 is given by,

$$I_1 = \frac{\Delta_1}{\Delta} \quad \text{..... (1)}$$

In equation (1), if $I_1 = 0$, then $\Delta_1 = 0$. In order to find the value of E, we can form the mesh basis matrix equation. Then form the determinant Δ_1 and equate the determinant to zero.

Using the circuit shown in Fig. 1, the mesh basis matrix equation is formed by inspection.

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \quad \text{..... (2)}$$

The elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 6 + 2 + 5 = 13$ | $R_{12} = R_{21} = -2$ | $E_{11} = 20 - E$

$R_{22} = 2 + 6 + 1 = 9$ | $R_{13} = R_{31} = -5$ | $E_{22} = 0$

$R_{33} = 5 + 1 + 4 = 10$ | $R_{23} = R_{32} = -1$ | $E_{33} = E$

On substituting the above terms in equation (2), we get,

$$\begin{bmatrix} 13 & -2 & -5 \\ -2 & 9 & -1 \\ -5 & -1 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 20-E \\ 0 \\ E \end{bmatrix}$$

$$\text{Now,} \quad \Delta_1 = \begin{vmatrix} 20-E & -2 & -5 \\ 0 & 9 & -1 \\ E & -1 & 10 \end{vmatrix} \quad \text{..... (3)}$$

On expanding Δ_1 along column-1, we get,

$$\Delta_1 = (20-E) \times [9 \times 10 - (-1) \times (-1)] - 0 + E \times [-2 \times (-1) - 9 \times (-5)]$$

$$= (20 - E) \times 89 + 47\,E = 1780 - 89\,E + 47\,E = 1780 - 42\,E$$

On equating Δ_1 to zero, we get,

$$0 = 1780 - 42\,E$$

$$\therefore \quad 42E = 1780 \quad \Rightarrow \quad E = \frac{1780}{42} = 42.381\,V$$

EXAMPLE 1.23

For the circuit shown in Fig. 1, find **(a)** the power delivered to 4 Ω resistor using mesh analysis and **(b)** to what voltage should the 80 *V* battery be changed so that no power is delivered to the 4 Ω resistor?

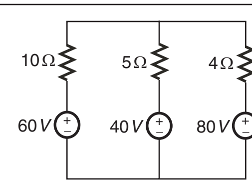


Fig. 1.

SOLUTION

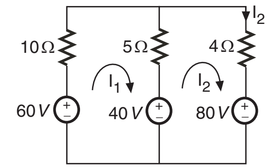


Fig. 2.

Let us assume two mesh currents I_1 and I_2 as shown in Fig. 2. The direction of the currents are chosen as clockwise. Now, the current through 4 Ω resistor is I_2.

$$\therefore \quad \text{Power delivered to } 4\,\Omega \text{ resistor } = |I_2|^2 \times 4$$

a) To find the power delivered to 4Ω resistor

Using the circuit shown in Fig. 2, the mesh basis matrix equation is formed as shown below:

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \end{bmatrix} \quad \text{..... (1)}$$

The elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 10 + 5 = 15$	$R_{12} = R_{21} = -5$	$E_{11} = 60 - 40 = 20$
$R_{22} = 5 + 4 = 9$		$E_{22} = 40 - 80 = -40$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 15 & -5 \\ -5 & 9 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 20 \\ -40 \end{bmatrix}$$

Let us solve I_2 by Cramer's rule.

Now, $I_2 = \dfrac{\Delta_2}{\Delta}$

$$\text{where,} \quad \Delta = \begin{vmatrix} 15 & -5 \\ -5 & 9 \end{vmatrix} = 15 \times 9 - (-5) \times (-5) = 110$$

$$\Delta_2 = \begin{vmatrix} 15 & 20 \\ -5 & -40 \end{vmatrix} = 15 \times (-40) - (-5) \times 20 = -500$$

$$\therefore \quad I_2 = \frac{\Delta_2}{\Delta} = \frac{-500}{110} = -4.5455\,A$$

$$\therefore \quad \text{Power delivered to } 4\,\Omega \text{ resistor } = |I_2|^2 \times 4 = |-4.5455|^2 \times 4$$
$$= 4.5455^2 \times 4 = 82.6463\,W$$

b) To find the change in voltage in 80 *V* source such that power delivered to 4Ω resistor is zero

Let us take the new value of 80 *V* source as E.

Now in equation (1), E_{22} is given by, $E_{22} = 40 - E$. Equation (1) for case (b) is given below:

$$\begin{bmatrix} 15 & -5 \\ -5 & 9 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 40 - E \end{bmatrix} \quad \text{..... (2)}$$

In equation (2), by Cramer's rule, I_2 is given by $I_2 = \Delta_2 / \Delta$. If power delivered to 4 Ω is zero, then I_2 should be zero. For I_2 to be zero, the determinant Δ_2 should be zero.

$$\text{Now,} \quad \Delta_2 = \begin{vmatrix} 15 & 20 \\ -5 & 40 - E \end{vmatrix} = 15 \times (40 - E) - (-5) \times 20$$
$$= 600 - 15E + 100 = 700 - 15E$$

On equating Δ_2 to zero, we get,

$$0 = 700 - 15E$$

$$15E = 700$$

$$E = \frac{700}{15} = 46.6667\,V$$

$\therefore$ The value of 80 *V* should be reduced to 46.6667 *V* to make the power delivered to 4 Ω resistor as zero.

EXAMPLE 1.24

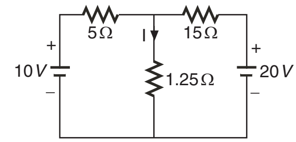


Fig. 1.

In the circuit shown in Fig. 1, find the current I by mesh method and the power supplied by each battery to the 1.25 Ω resistor.

SOLUTION

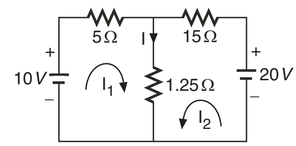


Fig. 2.

Let us assume two mesh currents as shown in Fig. 2. Now the current I is given by the sum of I_1 and I_2.

Using the circuit shown in Fig. 1, the mesh basis matrix equation is formed as shown below:

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \end{bmatrix} \quad .....(1)$$

The elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 5 + 1.25 = 6.25$	$R_{12} = R_{21} = 1.25$	$E_{11} = 10$
$R_{22} = 15 + 1.25 = 16.25$		$E_{22} = 20$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 6.25 & 1.25 \\ 1.25 & 16.25 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} \quad .....(2)$$

In equation (2), the unknowns are I_1 and I_2. In order to solve I_1 and I_2, let us define three determinants Δ, Δ_1 and Δ_2 as shown below and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 6.25 & 1.25 \\ 1.25 & 16.25 \end{vmatrix} = 6.25 \times 16.25 - 1.25 \times 1.25 = 100$$

$$\Delta_1 = \begin{vmatrix} 10 & 1.25 \\ 20 & 16.25 \end{vmatrix} = 10 \times 16.25 - 20 \times 1.25 = 137.5$$

$$\Delta_2 = \begin{vmatrix} 6.25 & 10 \\ 1.25 & 20 \end{vmatrix} = 6.25 \times 20 - 1.25 \times 10 = 112.5$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{137.5}{100} = 1.375\,A$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{112.5}{100} = 1.125\,A$$

$$\therefore \quad I = I_1 + I_2 = 1.375 + 1.125 = 2.5\,A$$

Let P_{10} and P_{20} be the power delivered by 10 *V* and 20 *V* sources.

Now, $P_{10} = 10 \times I_1 = 10 \times 1.375 = 13.75\,W$

$P_{20} = 20 \times I_2 = 20 \times 1.125 = 22.5\,W$

Let P_5 and P_{15} be the power consumed by 5 Ω and 15 Ω resistances, respectively.

Now, $P_5 = I_1^2 \times 5 = 1.375^2 \times 5 = 9.4531\ W$

$P_{15} = I_2^2 \times 15 = 1.125^2 \times 15 = 18.9844\ W$

Let P_{L10} and P_{L20} be the power delivered to load (i.e., to 1.25 Ω resistor) by the 10 *V* and 20 *V* sources, respectively.

Now, $P_{L10} = P_{10} - P_5 = 13.75 - 9.4531 = 4.2969\ W$

$P_{L20} = P_{20} - P_{15} = 22.5 - 18.9844 = 3.5156\ W$

Cross-Check

Power consumed by 1.25 Ω resistance, $P_L = I^2 \times 1.25 = 2.5^2 \times 1.25 = 7.8125\ W$

Also, $P_L = P_{L10} + P_{L20} = 4.2969 + 3.5156 = 7.8125\ W$

EXAMPLE 1.25 *(AU June'16, 16 Marks)*

In the circuit shown in Fig. 1, find voltage across 5 Ω resistor using source transformation technique and verify the results using mesh analysis.

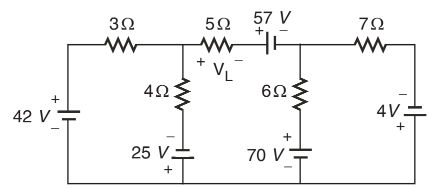


Fig. 1.

SOLUTION

Method 1: Source Transformation Technique

Let, V_L = Voltage across 5Ω resistor.

The voltage sources in Fig. 2 are converted to current sources as shown in Fig. 3.

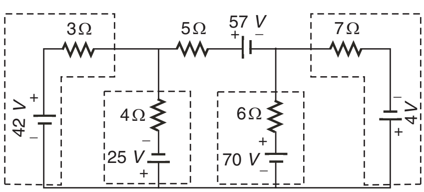


Fig. 2.

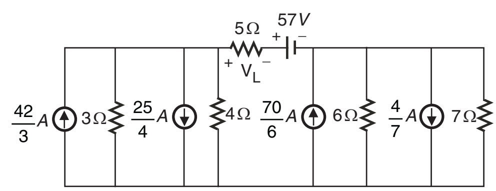


Fig. 3.

The parallel current sources in Fig. 4 are converted to a single equivalent current source in Fig. 5. Similarly the parallel resistances in Fig. 4 are converted to a single equivalent resistance in Fig. 5.

The current sources in Fig. 5 are converted to voltage sources as shown in Fig. 6.

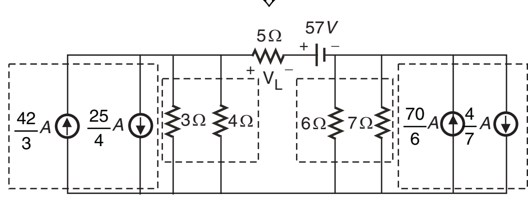


Fig. 4.

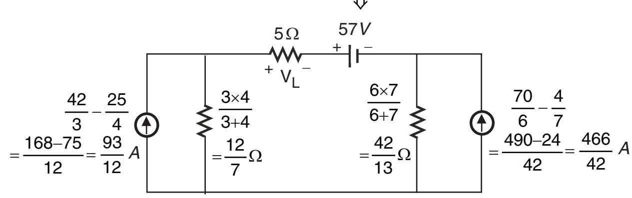


Fig. 5.

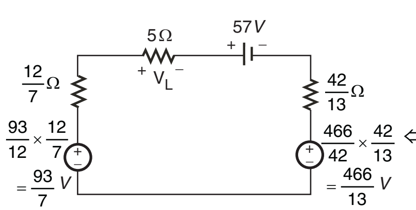


Fig. 6.

The circuit of Fig. 6 is redrawn as shown in Fig. 7.

With reference to Fig. 7 by voltage division rule, we can write,

$$V_L = -\frac{7240}{91} \times \frac{5}{5+\frac{450}{91}}$$

$$= -\frac{7240}{91} \times \frac{5}{\frac{5 \times 91 + 450}{91}}$$

$$= -\frac{7240 \times 5}{5 \times 91 + 450} = -40\,V$$

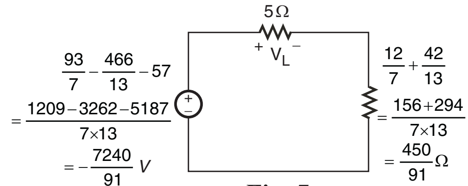


Fig. 7.

Method 2: Mesh Analysis

Let, V_L = Voltage across 5Ω resistor.

Let us choose mesh currents as shown in Fig. 8.

Now, $V_L = 5I_2$

For the circuit of Fig. 8, the mesh basis matrix equation is formed as shown below:

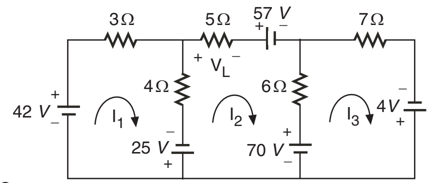


Fig. 8.

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \qquad ..... (1)$$

The elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 3 + 4 = 7$	$R_{12} = R_{21} = -4$	$E_{11} = 42 + 25 = 67$
$R_{22} = 4 + 5 + 6 = 15$	$R_{13} = R_{31} = 0$	$E_{22} = -25 - 57 - 70 = -152$
$R_{33} = 6 + 7 = 13$	$R_{23} = R_{32} = -6$	$E_{33} = 70 + 4 = 74$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 7 & -4 & 0 \\ -4 & 15 & -6 \\ 0 & -6 & 13 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 67 \\ -152 \\ 74 \end{bmatrix} \qquad ..... (2)$$

In order to solve I_2 by Cramer's rule, let us define two determinant Δ and Δ_2 as shown below:

$$\Delta = \begin{vmatrix} 7 & -4 & 0 \\ -4 & 15 & -6 \\ 0 & -6 & 13 \end{vmatrix}; \qquad \Delta_2 = \begin{vmatrix} 7 & 67 & 0 \\ -4 & -152 & -6 \\ 0 & 74 & 13 \end{vmatrix}$$

The determinants are evaluated by expanding along first row and the mesh current I_2 is solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 7 & -4 & 0 \\ -4 & 15 & -6 \\ 0 & -6 & 13 \end{vmatrix} = 7 \times [15 \times 13 - (-6)^2] - (-4) \times [-4 \times 13 - 0] + 0 = 1113 - 208 = 905$$

$$\Delta_2 = \begin{vmatrix} 7 & 67 & 0 \\ -4 & -152 & -6 \\ 0 & 74 & 13 \end{vmatrix} = 7 \times [-152 \times 13 - 74 \times (-6)] - 67 \times [-4 \times 13 - 0] + 0 = -10724 + 3484 = -7240$$

$$\therefore I_2 = \frac{\Delta_2}{\Delta} = \frac{-7240}{905} = -8$$

$$\therefore V_L = 5I_2 = 5 \times (-8) = -40\,V$$

1.6.2 Mesh Analysis of Circuits Excited by Both Voltage and Current Sources

The mesh analysis can be extended to circuits excited by both voltage and current sources. In such circuits if each current source has a parallel impedance then it can be converted into an equivalent voltage source with series impedance. After conversion, the circuit will have only voltage sources and so the procedure for obtaining mesh basis matrix equation by inspection and its solution discussed in Sections 1.6.1 and 1.6.4 can be directly applied to these circuits.

In certain circuits excited by both voltage and current sources, the current source may not have a parallel resistance. In this situation the current source cannot be converted into a voltage source. In this case the value of each current source is related to mesh currents and one of the mesh currents can be expressed in terms of the source current and other mesh currents. The remaining mesh currents can be solved by writing Kirchhoff's Voltage Law (KVL) equations.

Alternatively, the mesh basis matrix equation can be formed directly by inspection, by taking the voltage of the current sources as unknown and relating the value of each current source to mesh currents. Here for each current source one mesh current is eliminated by expressing the mesh current in terms of the source current and other mesh currents. While forming the mesh basis matrix equation, the voltage of current sources should be entered in the source matrix.

Now in the matrix equation some mesh currents will be eliminated and an equal number of unknown source voltages will be introduced. Thus, the number of unknowns will remain the same as the number of meshes m. On multiplying the mesh basis matrix equation, we get m equations which can be solved to give a unique solution for unknown currents.

1.6.3 Supermesh Analysis

In circuits excited by both voltage and current sources, if a current source lies common to two meshes then the common current source can be removed for analysis purpose and the resultant two meshes can be considered as one single mesh called **supermesh**. In order to solve the two mesh currents of a supermesh, two equations are required. One of the equations is the KVL equation of the supermesh and the other equation is obtained by equating the source current to the sum or difference of the mesh currents (depending on the direction of the mesh currents). An example of formation of supermesh is shown in Fig. 1.46. Also, Example 1.27 is solved using the supermesh analysis technique.

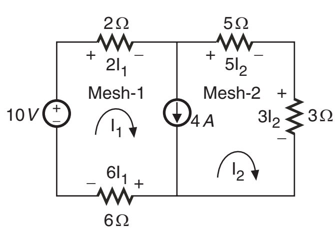


Fig. a : Two mesh circuit with mesh currents in same orientation.

⇒

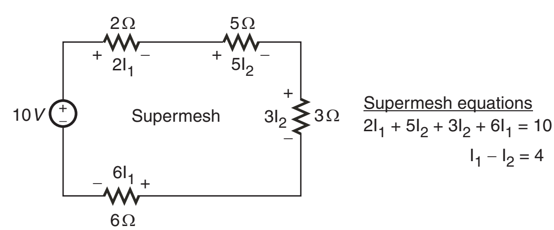


Fig. b : Supermesh of circuit shown in Fig. a and its equations.

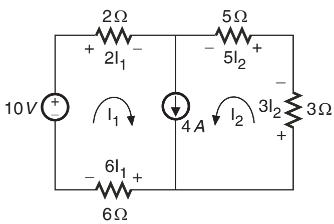
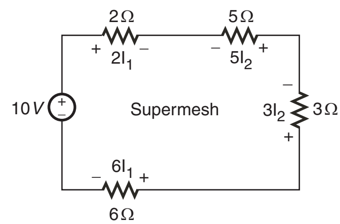


Fig. c : Two mesh circuit with mesh currents in opposite orientation.

⇒



Supermesh equations

$2I_1 + 6I_1 = 5I_2 + 3I_2 + 10$

$I_1 + I_2 = 4$

Fig. d : Supermesh of circuit shown in Fig. c and its equations.

***Fig. 1.46 :** Examples of formation of supermesh.*

EXAMPLE 1.26

Find the voltage between A and B of the circuit shown in Fig. 1, using mesh analysis.

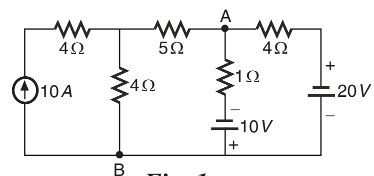


Fig. 1.

SOLUTION

The graph of the given circuit is shown in Fig. 2. It has five branches and three nodes. Hence, the number of meshes m in the circuit is m = B − N + 1 = 5 − 3 + 1 = 3.

The circuit has five currents (corresponding to five branches) and in this three currents are independent (corresponding to three meshes). Let us assume three mesh currents as shown in Figs 2 and 3.

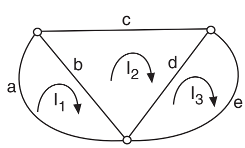


Fig. 2.

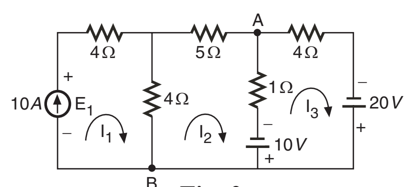


Fig. 3.

The directions of mesh currents are chosen arbitrarily. Here, one of the mesh has 10*A* current source which cannot be converted to a voltage source because the source does not have parallel impedance. Hence, we can take this current as a known mesh current, but the voltage across the source E_1 is unknown. Therefore, the number of unknowns remain as three (i.e., unknowns are E_1, I_2 and I_3) and so we can write three mesh equations using KVL (corresponding to three meshes) and a unique solution is obtained by solving the three equations. The mesh equations can be obtained by two methods.

> ***Note :*** *In solutions of simultaneous equations a unique solution can be obtained only if the number of unknowns are equal to the number of equations.*

Method I : Formation of mesh equations by applying KVL

In this method the mesh equations are formed using Kirchhoff's Voltage Law. The mesh equation for a mesh is formed by equating the sum of voltage fall to the sum of voltage rise. The voltage rise and fall are determined by tracing the circuit in the direction of the mesh current.

With reference to Fig. 4, the mesh equation for mesh-2 is formed as shown below:

Voltage fall : $4I_2, 5I_2, I_2$

Voltage rise : $4I_1, I_3, 10\,V$

$$\therefore\ 4I_2 + 5I_2 + I_2 = 4I_1 + I_3 + 10$$

$$-4I_1 + 10I_2 - I_3 = 10 \qquad I_1 = 10A$$

$$-4\times 10 + 10I_2 - I_3 = 10 \quad \Rightarrow \quad 10I_2 - I_3 = 50 \qquad \text{..... (1)}$$

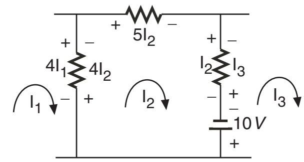


Fig. 4.

With reference to Fig. 5, the mesh equation for mesh-3 is formed as shown below:

Voltage fall : $I_3, 4I_3, 10\text{ V}$

Voltage rise : $I_2, 20\text{ V}$

$$\therefore\ I_3 + 4I_3 + 10 = I_2 + 20 \ \Rightarrow\ -I_2 + 5I_3 = 10 \qquad \text{..... (2)}$$

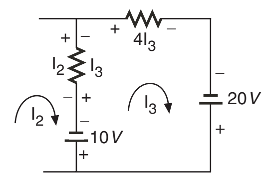


Fig. 5.

Mesh equations (1) and (2) are sufficient for solving I_2 and I_3.

Method II : Formation of mesh equations by inspection

In this method the mesh basis matrix equations is formed by inspection using the circuit shown in Fig. 3.

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \qquad \text{..... (3)}$$

$R_{11} = 4 + 4 = 8$	$R_{12} = R_{21} = -4$	$E_{11} = E_1$	$I_1 = 10$
$R_{22} = 4 + 5 + 1 = 10$	$R_{13} = R_{31} = 0$	$E_{22} = 10$	
$R_{33} = 1 + 4 = 5$	$R_{23} = R_{32} = -1$	$E_{33} = -10 + 20 = 10$	

On substituting the above terms in equation (3), we get,

$$\begin{bmatrix} 8 & -4 & 0 \\ -4 & 10 & -1 \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} 10 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_1 \\ 10 \\ 10 \end{bmatrix} \qquad \text{..... (4)}$$

On multiplying the matrices on left-hand side of equation (4), and equating to terms on right-hand side, we get the following equations:

From row - 2, $\quad -40 + 10I_2 - I_3 = 10 \quad \Rightarrow \quad 10I_2 - I_3 = 50$ ..... (5)

From row - 3, $\quad -I_2 + 5I_3 = 10$ ..... (6)

Solution of mesh currents

It is observed that the mesh equations obtained by both the methods are the same.

From equation (5), we get,

$$I_3 = 10I_2 - 50 \qquad \text{..... (7)}$$

On substituting for I_3 from equation (7) in equation (6), we get,

$$-I_2 + 5(10 I_2 - 50) = 10 \quad \Rightarrow \quad 49I_2 - 250 = 10$$

$$\therefore \quad I_2 = \frac{260}{49} = 5.3061\,A$$

$$I_3 = 10I_2 - 50 = 10 \times 5.3061 - 50 = 3.061\,A$$

To find voltage across A and B

Let us denote the meeting point of 4 Ω and 5 Ω as node-C and the meeting point of 1 Ω and 10 *V* source as node-D as shown in Fig. 6.

There are two shortest path to find the voltage across A and B. They are closed path ABCA and ABDA. Let voltage across A and B be denoted as V_{AB}.

With reference to Fig. 6, in path-ABCA by KVL we can write,

$$V_{AB} + 4I_2 + 5I_2 = 4I_1$$

$$\therefore \quad V_{AB} = 4I_1 - 9I_2 = 4 \times 10 - 9 \times 5.3061 = -7.7549\,V$$

In path-ABDA by KVL we can write,

$$V_{AB} + 10 + I_3 = I_2$$

$$\therefore V_{AB} = I_2 - I_3 - 10 = 5.3061 - 3.061 - 10 = -7.7549\,V$$

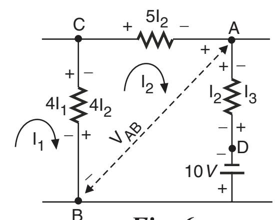


Fig. 6.

EXAMPLE 1.27

In the circuit shown in Fig. 1, find the current supplied by the voltage source and the voltage across the current source by mesh analysis.

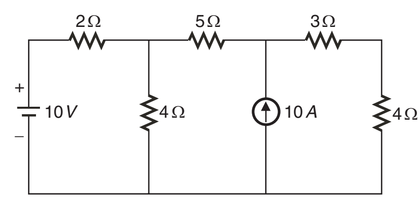


Fig. 1.

SOLUTION

Let us assume three mesh currents as shown in Fig. 2. The current delivered by the voltage source is I_1. Let the voltage across the current source be E with current leaving point as positive. Also, the voltage across various elements of the circuit are shown in Fig. 2.

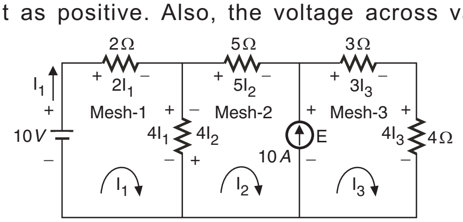


Fig. 2.

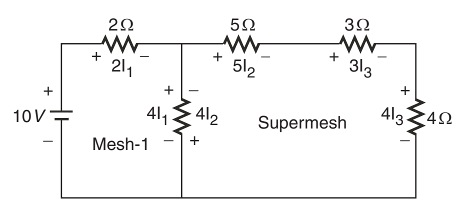


Fig. 3.

With reference to Fig. 2, the relation between mesh currents I_2 and I_3 is

$$I_3 - I_2 = 10 \quad \Rightarrow \quad I_3 = 10 + I_2 \qquad ..... (1)$$

Let us combine mesh-2 and mesh-3 and form a supermesh as shown in Fig. 3. The KVL equation for the supermesh is formed as shown ahead.

$$4I_2 + 5I_2 + 3I_3 + 4I_3 = 4I_1 \quad \Rightarrow \quad 4I_1 = 9I_2 + 7I_3$$

$$\therefore\ 4I_1 = 9I_2 + 7(10 + I_2) \qquad \boxed{\text{Using equation (1)}}$$

$$\therefore\ 4I_1 = 16I_2 + 70 \quad \Rightarrow \quad 16I_2 = 4I_1 - 70$$

$$\therefore\ I_2 = \frac{4}{16}I_1 - \frac{70}{16} \quad \Rightarrow \quad I_2 = 0.25I_1 - 4.375 \qquad ..... (2)$$

The KVL equation for the mesh-1 is formed as shown below:

$$2I_1 + 4I_1 = 10 + 4I_2$$

$$\therefore\ 6I_1 = 10 + 4(0.25I_1 - 4.375) \qquad \boxed{\text{Using equation (2)}}$$

$$\therefore\ 6I_1 = 10 + I_1 - 17.5 \quad \Rightarrow \quad 5I_1 = -7.5 \quad \Rightarrow \quad I_1 = \frac{-7.5}{5} - 1.5\,A$$

$$\therefore\ I_2 = 0.25I_1 - 4.375$$

$$= 0.25(-1.5) - 4.375 = -4.75\,A$$

$$\therefore\ I_3 = 10 + I_2 = 10 - 4.75 = 5.25\,A$$

With reference to Fig. 2 by KVL,

$$E = 3I_3 + 4I_3 = 7I_3 = 7 \times 5.25 = 36.75\,V$$

$\therefore$ Current supplied by the voltage source, $I_1 = -1.5\,A$

Voltage across the current source, $E = 36.75\,V$

EXAMPLE 1.28

(AU June'14, 8 Marks)

Find out the current in each branch of the circuit shown in Fig 1.

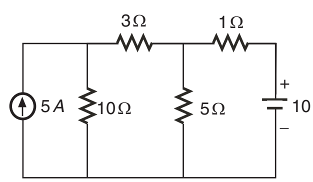


Fig. 1.

SOLUTION

Let us assume the four branch currents are I_a, I_b, I_c and I_d as shown in Fig. 2. The current source of Fig.2 can be represented by an equivalent voltage source of value 50 V with a source resistance of 10 Ω in series as shown in Fig. 3. Let us assume two mesh currents I_1 and I_2 as shown in Fig. 3.

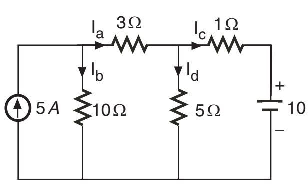


Fig. 2.

$\Rightarrow$

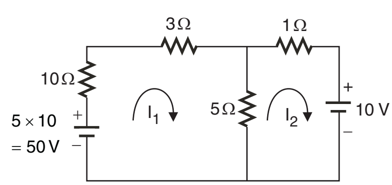


Fig. 3.

Using the circuit shown in Fig. 3, the mesh basis matrix equation is formed as shown below :

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \end{bmatrix} \qquad ..... (1)$$

The elements of the resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 10 + 3 + 5 = 18$	$R_{12} = R_{21} = -5$	$E_{11} = 50$
$R_{22} = 5 + 1 = 6$		$E_{22} = -10$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 18 & -5 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 50 \\ -10 \end{bmatrix} \quad .....(2)$$

In equation (2), the unknowns are I_1 and I_2. In order to solve I_1 and I_2, let us define three determinants Δ, Δ_1 and Δ_2 as shown below:

$$\Delta = \begin{vmatrix} 18 & -5 \\ -5 & 6 \end{vmatrix}; \qquad \Delta_1 = \begin{vmatrix} 50 & -5 \\ -10 & 6 \end{vmatrix}; \qquad \Delta_2 = \begin{vmatrix} 18 & 50 \\ -5 & -10 \end{vmatrix}$$

The determinants are evaluated as shown below and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 18 & -5 \\ -5 & 6 \end{vmatrix} = 18 \times 6 - (-5)^2 = 108 - 25 = 83$$

$$\Delta_1 = \begin{vmatrix} 50 & -5 \\ -10 & 6 \end{vmatrix} = 50 \times 6 - (-10 \times -5) = 300 - 50 = 250$$

$$\Delta_2 = \begin{vmatrix} 18 & 50 \\ -5 & -10 \end{vmatrix} = 18 \times (-10) - (-5 \times 50) = -180 + 250 = 70$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{250}{83} = 3.012\,A$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{70}{83} = 0.8434\,A$$

The branch currents are,

$$I_a = I_1 = 3.012\ A$$

$$I_b = 5 - I_a = 5 - 3.012\,A = 1.988\,A$$

$$I_c = I_2 = 0.8434\,A$$

$$I_d = I_1 - I_2 = 3.012 - 0.8434 = 2.1686\,A$$

EXAMPLE 1.29 ***(AU June'14, 8 Marks)***

Determine the current in each mesh of the circuit shown in Fig. 1.

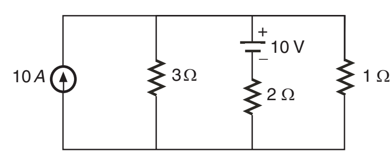


Fig. 1.

SOLUTION

Let, voltage across 10 A curret source be V_S and I_1 , I_2 and I_3 be mesh currets as shown in Fig. 2. Here $I_1 = 10\,A.$

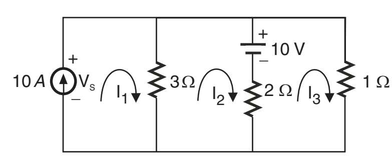


Fig. 2.

Using the circuit shown in Fig. 2, the mesh basis matrix equation is formed as shown below:

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \quad ..... (1)$$

The elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 3$	$R_{12} = R_{21} = -3$	$E_{11} = V_S$	
$R_{22} = 3 + 2 = 15$	$R_{13} = R_{31} = 0$	$E_{22} = -10$	$I_1 = 10\,A$
$R_{33} = 2 + 1 = 7$	$R_{23} = R_{32} = -2$	$E_{33} = 10$	

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 3 & -3 & 0 \\ -3 & 5 & -2 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_s \\ -10 \\ 10 \end{bmatrix} \qquad .....(2)$$

From row-2, we get,

$$-30 + 5I_2 - 2I_3 = -10$$

$$\therefore\ 5I_2 - 2I_3 = 20 \qquad .....(1)$$

From row-3, we get,

$$-2I_2 + 3I_3 = 10 \qquad .....(2)$$

Equation (1) × 3 $\Rightarrow$ $15I_2 - 6I_3 = 60$

Equation (2) × 2 $\Rightarrow$ $-4I_2 + 6I_3 = 20$

Add $11I_2 = 80$

$$\therefore\ I_2 = \frac{80}{11} = 7.2727\ A$$

From equation (2), $I_3 = \dfrac{10 + 2I_2}{3} = \dfrac{10 + 2 \times 7.2727}{3} = 8.1818\ A$

The mesh currents I_1 , I_2, and I_3 are given by,

$I_1 = 10\ A$

$I_2 = 7.2727\ A$

$I_3 = 8.1818\ A$

1.6.4 Mesh Analysis of Circuits Excited by AC Sources (Mesh Analysis of Reactive Circuits)

The reactive circuits consist of resistances, inductive and capacitive reactances. Therefore, the voltage and current of reactive circuits are complex (i.e., they have both real and imaginary components). In general, the elements of these circuits are referred to as impedances.

The general mesh basis matrix equation for reactive circuit is

$$\mathbf{Z\,I = E} \qquad ..... (1.26)$$

where, $\mathbf{Z}$ = Impedance matrix of order m × m

$\mathbf{I}$ = Mesh current matrix of order m × 1

$\mathbf{E}$ = Source voltage matrix of order m × 1

m = Number of meshes

Equation (1.26) can be expanded as shown in equation (1.27).

$$\begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} & \bar{Z}_{13} & \cdots & \bar{Z}_{1m} \\ \bar{Z}_{21} & \bar{Z}_{22} & \bar{Z}_{23} & \cdots & \bar{Z}_{2m} \\ \bar{Z}_{31} & \bar{Z}_{32} & \bar{Z}_{33} & \cdots & \bar{Z}_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ \bar{Z}_{m1} & \bar{Z}_{m2} & \bar{Z}_{m3} & \cdots & \bar{Z}_{mm} \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \\ \vdots \\ \bar{I}_m \end{bmatrix} = \begin{bmatrix} \bar{E}_{11} \\ \bar{E}_{22} \\ \bar{E}_{33} \\ \vdots \\ \bar{E}_{mm} \end{bmatrix} \qquad ..... (1.27)$$

Note : The over bar is used to denote complex quantities.

The formation of mesh basis matrix equation and the solution of mesh and branch currents are similar to that of resistive circuits except that the solution of currents involves complex arithmetic.

Therefore, the k[th] mesh current of a reactive circuit with m meshes is given by,

$$\bar{I}_k = \frac{1}{\Delta}\sum_{j=1}^{m}\Delta_{jk}\bar{E}_{jj} \quad \text{..... (1.28)}$$

where, Δ_{jk} = Cofactor of $\bar{Z}_{jk}$.

$\bar{E}_{jj}$ = Sum of voltage sources in j^{th} mesh.

Δ = Determinant of impedance matrix.

***Note :** Refer equation (1.23).*

Instead of using above equation for solution of mesh currents, a short-cut for Cramer's rule can be followed.

Consider the mesh basis matrix equation for three mesh circuit consisting of reactive elements.

$$\begin{bmatrix}\bar{Z}_{11} & \bar{Z}_{12} & \bar{Z}_{13}\\ \bar{Z}_{21} & \bar{Z}_{22} & \bar{Z}_{23}\\ \bar{Z}_{31} & \bar{Z}_{32} & \bar{Z}_{33}\end{bmatrix}\begin{bmatrix}\bar{I}_1\\ \bar{I}_2\\ \bar{I}_3\end{bmatrix} = \begin{bmatrix}\bar{E}_{11}\\ \bar{E}_{22}\\ \bar{E}_{33}\end{bmatrix}$$

Let us define the four determinants as

$$\Delta = \begin{vmatrix}\bar{Z}_{11} & \bar{Z}_{12} & \bar{Z}_{13}\\ \bar{Z}_{21} & \bar{Z}_{22} & \bar{Z}_{23}\\ \bar{Z}_{31} & \bar{Z}_{32} & \bar{Z}_{33}\end{vmatrix}; \quad \Delta_1 = \begin{vmatrix}\bar{E}_{11} & \bar{Z}_{12} & \bar{Z}_{13}\\ \bar{E}_{22} & \bar{Z}_{22} & \bar{Z}_{23}\\ \bar{E}_{33} & \bar{Z}_{32} & \bar{Z}_{33}\end{vmatrix}; \quad \Delta_2 = \begin{vmatrix}\bar{Z}_{11} & \bar{E}_{11} & \bar{Z}_{13}\\ \bar{Z}_{21} & \bar{E}_{22} & \bar{Z}_{23}\\ \bar{Z}_{31} & \bar{E}_{33} & \bar{Z}_{33}\end{vmatrix}; \quad \Delta_3 = \begin{vmatrix}\bar{Z}_{11} & \bar{Z}_{12} & \bar{E}_{11}\\ \bar{Z}_{21} & \bar{Z}_{22} & \bar{E}_{22}\\ \bar{Z}_{31} & \bar{Z}_{32} & \bar{E}_{33}\end{vmatrix}$$

Here, Δ = Determinant of impedance matrix

Δ_1 = Determinant of impedance matrix after replacing the first column of impedance matrix by source voltage column matrix

Δ_2 = Determinant of impedance matrix after replacing the second column of impedance matrix by source voltage column matrix

Δ_3 = Determinant of impedance matrix after replacing the third column of impedance matrix by source voltage column matrix.

Now mesh currents $\bar{I}_1$, $\bar{I}_2$ and $\bar{I}_3$ are given by,

$$\bar{I}_1 = \frac{\Delta_1}{\Delta}$$

$$\bar{I}_2 = \frac{\Delta_2}{\Delta}$$

$$\bar{I}_3 = \frac{\Delta_3}{\Delta}$$

EXAMPLE 1.30 ***(AU Dec'16, 12 Marks)***

In the circuit shown in Fig. 1, find $\bar{I}_2$ and voltage drop across 1 Ω resistor.

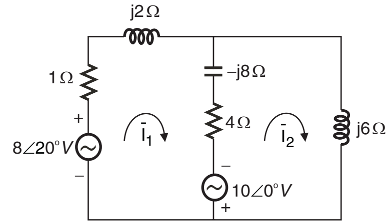


Fig. 1.

SOLUTION

With reference to Fig. 1, the mesh basis matrix equation is formed by inspection.

$$\begin{bmatrix}\bar{Z}_{11} & \bar{Z}_{12}\\ \bar{Z}_{21} & \bar{Z}_{22}\end{bmatrix}\begin{bmatrix}\bar{I}_1\\ \bar{I}_2\end{bmatrix} = \begin{bmatrix}\bar{E}_{11}\\ \bar{E}_{22}\end{bmatrix} \quad ..... (1)$$

$\bar{Z}_{11} = 1 + j2 - j8 + 4 = 5 - j6$

$\bar{Z}_{12} = \bar{Z}_{21} = -(-j8 + 4) = -4 + j8$

$\bar{Z}_{22} = 4 - j8 + j6 = 4 - j2$

$\bar{E}_{11} = 8\angle 20^\circ + 10\angle 0^\circ = 7.5175 + j2.7362 + 10$

$= 17.5175 + j2.7362$

$\bar{E}_{22} = -10\angle 0^\circ = -10$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix}5-j6 & -4+j8\\ -4+j8 & 4-j2\end{bmatrix}\begin{bmatrix}\bar{I}_1\\ \bar{I}_2\end{bmatrix} = \begin{bmatrix}17.5175+j2.7362\\ -10\end{bmatrix} \quad ..... (2)$$

To solve the unknowns (i.e., mesh currents) of equation (2) by Cramer's rule, we can define three determinants Δ, Δ_1 and Δ_2 as shown below:

$$\Delta = \begin{vmatrix}5-j6 & -4+j8\\ -4+j8 & 4-j2\end{vmatrix} \quad ; \quad \Delta_1 = \begin{vmatrix}17.5175+j2.7362 & -4+j8\\ -10 & 4-j2\end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix}5-j6 & 17.5175+j2.7362\\ -4+j8 & -10\end{vmatrix}$$

The determinants are evaluated by expanding along first row and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix}5-j6 & -4+j8\\ -4+j8 & 4-j2\end{vmatrix} = (5-j6)\times(4-j2)-(-4+j8)^2$$
$$= 56 + j30$$

$$\Delta_1 = \begin{vmatrix}17.5175+j2.7362 & -4+j8\\ -10 & 4-j2\end{vmatrix} = [(17.5175+j2.7362)\times(4-j2)]-[(-10)\times(-4+j8)]$$
$$= 35.5424 + j55.9098$$

$$\Delta_2 = \begin{vmatrix}5-j6 & 17.5175+j2.7362\\ -4+j8 & -10\end{vmatrix} = (5-j6)\times(-10)-(-4+j8)\times(17.5175+j2.7362)$$
$$= 41.9596 - j69.1952$$

$$\bar{I}_1 = \frac{\Delta_1}{\Delta} = \frac{35.5424+j55.9098}{56+j30} = 0.9087 + j0.5116 = 1.0428\angle 29.4^\circ\ A$$

$$\bar{I}_2 = \frac{\Delta_2}{\Delta} = \frac{41.9596-j69.1952}{56+j30} = 0.0679 - j1.272 = 1.2738\angle -86.9^\circ\ A$$

Let, $\bar{V}_1$ = Voltage drop across 1Ω resistor.

Now, $\bar{V}_1 = \bar{I}_1 \times 1 = \bar{I}_1 = 1.0428\angle 29.4^\circ\ V$

EXAMPLE 1.31

Solve the currents in various branches of the circuit shown in Fig. 1, using mesh analysis.

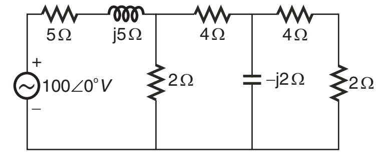


Fig. 1.

SOLUTION

The graph of the given circuit is shown in Fig. 2. It has five branches and three nodes. Hence, the number of meshes m in the circuit is m = B − N + 1 = 5 − 3 + 1 = 3.

The circuit has five currents (corresponding to five branches) and in this three currents are independent (corresponding to three meshes).

Let us assume the mesh currents $\bar{I}_1$, $\bar{I}_2$ and $\bar{I}_3$ and the branch currents $\bar{I}_a, \bar{I}_b, \bar{I}_c, \bar{I}_d$ and $\bar{I}_e$ as shown in Figs 2 and 3.

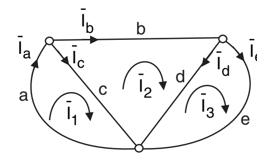


Fig. 2.

The directions of the currents are chosen arbitrarily. With reference to Fig. 3, the mesh basis matrix equation is formed as shown below:

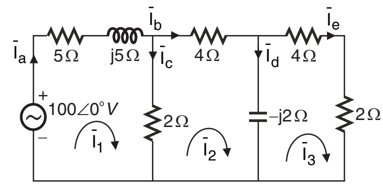


Fig. 3.

$$\begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} & \bar{Z}_{13} \\ \bar{Z}_{21} & \bar{Z}_{22} & \bar{Z}_{23} \\ \bar{Z}_{31} & \bar{Z}_{32} & \bar{Z}_{33} \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} \bar{E}_{11} \\ \bar{E}_{22} \\ \bar{E}_{33} \end{bmatrix} \qquad \text{..... (1)}$$

$$\bar{Z}_{11} = 5 + j5 + 2 = 7 + j5 \qquad \bar{Z}_{12} = \bar{Z}_{21} = -2 \qquad \bar{E}_{11} = 100\angle 0^\circ = 100$$

$$\bar{Z}_{22} = 2 + 4 - j2 = 6 - j2 \qquad \bar{Z}_{13} = \bar{Z}_{31} = 0 \qquad \bar{E}_{22} = 0$$

$$\bar{Z}_{33} = -j2 + 4 + 2 = 6 - j2 \qquad \bar{Z}_{23} = \bar{Z}_{32} = -(-j2) = j2 \qquad \bar{E}_{33} = 0$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 7+j5 & -2 & 0 \\ -2 & 6-j2 & j2 \\ 0 & j2 & 6-j2 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} \qquad \text{..... (2)}$$

In equation (2), the unknowns are $\bar{I}_1$, $\bar{I}_2$ and $\bar{I}_3$. In order to solve $\bar{I}_1$, $\bar{I}_2$ and $\bar{I}_3$ by Cramer's rule, let us define four determinants Δ, Δ_1, Δ_2 and Δ_3 as shown below:

$$\Delta = \begin{vmatrix} 7+j5 & -2 & 0 \\ -2 & 6-j2 & j2 \\ 0 & j2 & 6-j2 \end{vmatrix}; \qquad \Delta_1 = \begin{vmatrix} 100 & -2 & 0 \\ 0 & 6-j2 & j2 \\ 0 & j2 & 6-j2 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} 7+j5 & 100 & 0 \\ -2 & 0 & j2 \\ 0 & 0 & 6-j2 \end{vmatrix}; \qquad \Delta_3 = \begin{vmatrix} 7+j5 & -2 & 100 \\ -2 & 6-j2 & 0 \\ 0 & j2 & 0 \end{vmatrix}$$

The determinants are evaluated by expanding along first row and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 7+j5 & -2 & 0 \\ -2 & 6-j2 & j2 \\ 0 & j2 & 6-j2 \end{vmatrix} = (7+j5)\times[(6-j2)\times(6-j2) - j2\times j2]$$

$$-(-2)\times[-2\times(6-j2) - 0] + 0$$

$$= (7 + j5) \times [36 - j24] + 2 \times [-12 + j4] = 348 + j20$$

***Note** : All calculations are performed using the calculator in complex mode.*

$$\Delta_1 = \begin{vmatrix} 100 & -2 & 0 \\ 0 & 6-j2 & j2 \\ 0 & j2 & 6-j2 \end{vmatrix} = 100\times[(6-j2)^2-(j2)^2]-(-2)\times[0-0]+0$$

$$= 100\times(36-j24) = 3600-j2400$$

$$\Delta_2 = \begin{vmatrix} 7+j5 & 100 & 0 \\ -2 & 0 & j2 \\ 0 & 0 & 6-j2 \end{vmatrix} = (7+j5)\times[0-0]-100\times[-2\times(6-j2)-0]+0$$

$$= -100\times[-12+j4] = 1200-j400$$

$$\Delta_3 = \begin{vmatrix} 7+j5 & -2 & 100 \\ -2 & 6-j2 & 0 \\ 0 & j2 & 0 \end{vmatrix} = (7+j5)\times[0-0]-(-2)\times[0-0]+100\times[-2\times j2-0]$$

$$= 100\times(-j4) = -j400$$

$$\bar{I}_1 = \frac{\Delta_1}{\Delta} = \frac{3600 - j2400}{348+j20} = 9.9157-j7.4664 = 12.412\angle-37^\circ\ A$$

$$\bar{I}_2 = \frac{\Delta_2}{\Delta} = \frac{1200 - j400}{348+j20} = 3.3711-j1.3432 = 3.629\angle-21.7^\circ\ A$$

$$\bar{I}_3 = \frac{\Delta_3}{\Delta} = \frac{-j400}{348+j20} = -0.0658-j1.1456 = 1.147\angle-93.3^\circ\ A$$

With reference to Fig. 3, the following relations between mesh and branch currents are obtained. Now the branch currents are evaluated using the mesh currents $\bar{I}_1$, $\bar{I}_2$ and $\bar{I}_3$.

$$\bar{I}_a = \bar{I}_1 = 12.412\angle-37^\circ\ A$$

$$\bar{I}_b = \bar{I}_2 = 3.629\angle-21.7^\circ\ A$$

$$\bar{I}_c = \bar{I}_1-\bar{I}_2 = 9.9157-j7.4664-(3.3711-j1.3432)$$
$$= 6.5446-j6.1232 = 8.962\angle-43.1^\circ\ A$$

$$\bar{I}_d = \bar{I}_2-\bar{I}_3 = 3.3711-j1.3432-(-0.0658-j1.1456)$$
$$= 3.4369-j0.1976 = 3.443\angle-3.3^\circ\ A$$

$$\bar{I}_e = \bar{I}_3 = 1.147\angle-93.3^\circ\ A$$

EXAMPLE 1.32

In the circuit shown in Fig. 1, find the mesh currents.

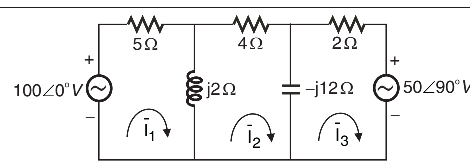


Fig. 1.

SOLUTION

With reference to Fig. 1, the mesh basis matrix equation is formed by inspection as shown below:

$$\begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} & \bar{Z}_{13} \\ \bar{Z}_{21} & \bar{Z}_{22} & \bar{Z}_{23} \\ \bar{Z}_{31} & \bar{Z}_{32} & \bar{Z}_{33} \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} \bar{E}_{11} \\ \bar{E}_{22} \\ \bar{E}_{33} \end{bmatrix} \quad \text{..... (1)}$$

$\bar{Z}_{11} = 5 + j2$	$\bar{Z}_{12} = \bar{Z}_{21} = -j2$	$\bar{E}_{11} = 100\angle0^\circ = 100$
$\bar{Z}_{22} = j2 + 4 - j12 = 4 - j10$	$\bar{Z}_{13} = \bar{Z}_{31} = 0$	$\bar{E}_{22} = 0$
$\bar{Z}_{33} = -j12 + 2 = 2 - j12$	$\bar{Z}_{23} = \bar{Z}_{32} = -(-j12) = j12$	$\bar{E}_{33} = -(50\angle90^\circ) = -j50$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 5+j2 & -j2 & 0 \\ -j2 & 4-j10 & j12 \\ 0 & j12 & 2-j12 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ -j50 \end{bmatrix} \qquad \text{..... (2)}$$

To solve the unknowns (i.e., mesh currents) of equation (2) by Cramer's rule, we can define four determinants Δ, Δ_1, Δ_2 and Δ_3 as shown below:

$$\Delta = \begin{vmatrix} 5+j2 & -j2 & 0 \\ -j2 & 4-j10 & j12 \\ 0 & j12 & 2-j12 \end{vmatrix} \quad ; \quad \Delta_1 = \begin{vmatrix} 100 & -j2 & 0 \\ 0 & 4-j10 & j12 \\ -j50 & j12 & 2-j12 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} 5+j2 & 100 & 0 \\ -j2 & 0 & j12 \\ 0 & -j50 & 2-j12 \end{vmatrix} \quad ; \quad \Delta_3 = \begin{vmatrix} 5+j2 & -j2 & 100 \\ -j2 & 4-j10 & 0 \\ 0 & j12 & -j50 \end{vmatrix}$$

The determinants are evaluated by expanding along first row and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 5+j2 & -j2 & 0 \\ -j2 & 4-j10 & j12 \\ 0 & j12 & 2-j12 \end{vmatrix} = (5+j2)\times[(4-j10)\times(2-j12)-j12\times j12]-(-j2)\times[-j2\times(2-j12)-0]+0$$

$$= 296 - j276 + 8 - j48$$

$$= 304 - j324$$

$$\Delta_1 = \begin{vmatrix} 100 & -j2 & 0 \\ 0 & 4-j10 & j12 \\ -j50 & j12 & 2-j12 \end{vmatrix} = 100\times[(4-j10)\times(2-j12)-(j12)^2]-(-j2)\times[0-(-j50)\times j12]+0$$

$$= 3200 - j6800 - j1200 = 3200 - j8000$$

$$\Delta_2 = \begin{vmatrix} 5+j2 & 100 & 0 \\ -j2 & 0 & j12 \\ 0 & -j50 & 2-j12 \end{vmatrix} = (5+j2)\times[0-(-j50)\times j12]-100\times[-j2\times(2-j12)-0]+0$$

$$= -3000 - j1200 + 2400 + j400$$

$$= -600 - j800$$

$$\Delta_3 = \begin{vmatrix} 5+j2 & -j2 & 100 \\ -j2 & 4-j10 & 0 \\ 0 & j12 & -j50 \end{vmatrix} = (5+j2)\times[(4-j10)\times(-j50)]-0]-(-j2)\times[-j2\times(-j50)-0] + 100\times[-j2\times j12-0]$$

$$= -2100 - j2000 - j200 + 2400$$

$$= 300 - j2200$$

$$\bar{I}_1 = \frac{\Delta_1}{\Delta} = \frac{3200-j8000}{304-j324} = 18.0595 - j7.0682 = 19.393\angle{-21.4^\circ}\,A$$

$$\bar{I}_2 = \frac{\Delta_2}{\Delta} = \frac{-600-j800}{304-j324} = 0.3891 - j2.2169 = 2.251\angle{-80^\circ}\,A$$

$$\bar{I}_3 = \frac{\Delta_3}{\Delta} = \frac{300-j2200}{304-j324} = 4.0731 - j2.8958 = 4.998\angle{-35.4^\circ}\,A$$

EXAMPLE 1.33 *(AU May'17, 8 Marks)*

In the circuit shown in Fig. 1, find $\bar{I}_o$ using mesh analysis.

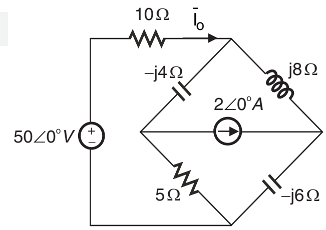


Fig. 1.

SOLUTION

Let us assume three mesh currents $\bar{I}_1$, $\bar{I}_2$ and $\bar{I}_3$ as shown in Fig. 2. With reference to Fig. 2, the mesh basis matrix equation is formed by inspection as shown below:

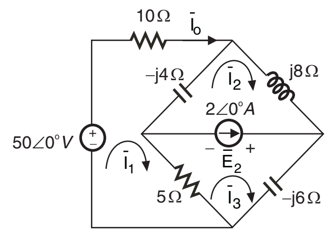


Fig. 2.

$$\begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} & \bar{Z}_{13} \\ \bar{Z}_{21} & \bar{Z}_{22} & \bar{Z}_{23} \\ \bar{Z}_{31} & \bar{Z}_{32} & \bar{Z}_{33} \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} \bar{E}_{11} \\ \bar{E}_{22} \\ \bar{E}_{33} \end{bmatrix} \qquad \text{..... (1)}$$

$\bar{Z}_{11} = 10 - j4 + 5 = 15 - j4$	$\bar{Z}_{12} = \bar{Z}_{21} = -(-j4) = j4$	$\bar{E}_{11} = 50\angle 0^\circ = 50$
$\bar{Z}_{22} = j8 - j4 = j4$	$\bar{Z}_{13} = \bar{Z}_{31} = -5$	$\bar{E}_{22} = -\bar{E}_2$
$\bar{Z}_{33} = 5 - j6$	$\bar{Z}_{23} = \bar{Z}_{32} = 0$	$\bar{E}_{33} = \bar{E}_2$

Here, $\bar{I}_2 = -2\angle 0^\circ = -2$ and $\bar{I}_3 = 2\angle 0^\circ = 2$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 15-j4 & j4 & -5 \\ j4 & j4 & 0 \\ -5 & 0 & 5-j6 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 50 \\ -\bar{E}_2 \\ \bar{E}_2 \end{bmatrix} \qquad \text{.....(2)}$$

From row-1 we get

$$(15 - j4)\bar{I}_1 + j4(-2) - 5(2) = 50$$

$$(15 - j4)\bar{I}_1 = 50 + 10 + j8$$

$$\therefore\ \bar{I}_1 = \frac{60 + j8}{15 - j4} = 3.6017 + j1.4938 = 3.8992\angle 22.5^\circ\ A$$

$$\bar{I}_0 = 3.8992\angle 22.5^\circ\ A$$

EXAMPLE 1.34

In the circuit shown in Fig. 1, find $\bar{E}_2$ such that the current in $(1 + j1)\,\Omega$ branch is zero.

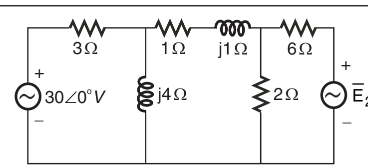


Fig. 1.

SOLUTION

Let us assume three mesh currents $\bar{I}_1$, $\bar{I}_2$ and $\bar{I}_3$ as shown in Fig. 2. With reference to Fig. 2, the mesh basis matrix equation is formed by inspection as shown below:

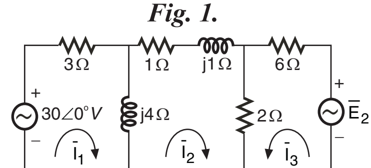


Fig. 2.

$$\begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} & \bar{Z}_{13} \\ \bar{Z}_{21} & \bar{Z}_{22} & \bar{Z}_{23} \\ \bar{Z}_{31} & \bar{Z}_{32} & \bar{Z}_{33} \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} \bar{E}_{11} \\ \bar{E}_{22} \\ \bar{E}_{33} \end{bmatrix} \quad ..... (1)$$

$\bar{Z}_{11} = 3 + j4$	$\bar{Z}_{12} = \bar{Z}_{21} = -j4$	$\bar{E}_{11} = 30\angle 0^\circ = 30$
$\bar{Z}_{22} = j4 + 1 + j1 + 2 = 3 + j5$	$\bar{Z}_{13} = \bar{Z}_{31} = 0$	$\bar{E}_{22} = 0$
$\bar{Z}_{33} = 2 + 6 = 8$	$\bar{Z}_{23} = \bar{Z}_{32} = 2$	$\bar{E}_{33} = \bar{E}_2$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 3+j4 & -j4 & 0 \\ -j4 & 3+j5 & 2 \\ 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} 30 \\ 0 \\ \bar{E}_2 \end{bmatrix} \quad ..... (2)$$

It is given that the current through $(1 + j1)\,\Omega$ impedance is zero and so the mesh current $\bar{I}_2$ is zero. When the mesh currents are solved by Cramer's rule, $\bar{I}_2$ is given by Δ_2 / Δ. For $\bar{I}_2$ to be zero, the determinant Δ_2 should be zero. Therefore, the value of $\bar{E}_2$ can be obtained by equating Δ_2 to zero.

$$\Delta_2 = \begin{vmatrix} 3+j4 & 30 & 0 \\ -j4 & 0 & 2 \\ 0 & \bar{E}_2 & 8 \end{vmatrix} = (3+j4) \times [0 - 2 \times \bar{E}_2] - 30 \times [-j4 \times 8 - 0] + 0$$

$$= -\bar{E}_2(6 + j8) + j960$$

Put $\Delta_2 = 0$, $\therefore\ 0 = -\bar{E}_2(6 + j8) + j960$

$$\bar{E}_2(6 + j8) = j960$$

$$\therefore \quad \bar{E}_2 = \frac{j960}{6+j8} = 76.8 + j57.6 = 96\angle 36.9^\circ\ V$$

The value of voltage source, $\bar{E}_2 = 96\angle 36.9^\circ\ V$

1.6.5 Mesh Analysis of Circuits Excited by Independent and Dependent Sources

Mesh analysis can be extended to circuits excited by both dependent and independent sources. When a circuit has a dependent source, the dependent variable should be related to mesh currents and then the dependent source should be treated as a source while forming the mesh basis matrix equation.

If a dependent source depends on a voltage V_x in some part of a circuit then the voltage V_x should be expressed in terms of mesh currents. If a dependent source depends on a current I_x in some part of a circuit then the current I_x should be expressed in terms of mesh currents.

Circuits with Dependent Voltage Source

When a circuit has a dependent voltage source then express the value of the source in terms of mesh currents. While forming the mesh basis matrix equation, enter the value of the dependent source at the appropriate location in the source matrix on the right-hand side.

Now, some of the terms in the source matrix on the right-hand side will be a function of mesh currents and so they can be transferred to the left-hand side with the opposite sign. Then the mesh basis matrix equation can be solved using Cramer's rule. This procedure is explained below with an example.

Consider a circuit with three meshes and a dependent voltage source in mesh-2. Let the mesh basis matrix equation without considering the dependent voltage source be as shown in equation (1.29).

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \qquad ..... (1.29)$$

Let the value of the dependent source in mesh-2, when expressed in terms of the mesh currents, be $2I_1 - 2I_3$. Let the voltage of the dependent source be such that it is a rise in voltage in the direction of mesh current I_2. Hence, the value of the dependent source $2I_1 - 2I_3$ is added as a positive quantity to the element in the second row of the source matrix as shown in equation (1.30).

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} + 2\,I_1 - 2\,I_3 \\ E_{33} \end{bmatrix} \qquad ..... (1.30)$$

From row-2 of equation (1.30), we get,

$$R_{21}I_1 + R_{22}I_2 + R_{23}I_3 = E_{22} + 2I_1 - 2I_3$$

$$R_{21}I_1 - 2I_1 + R_{22}I_2 + R_{23}I_3 + 2I_3 = E_{22}$$

$$\therefore (R_{21} - 2)I_1 + R_{22}I_2 + (R_{23} + 2)I_3 = E_{22} \qquad ..... (1.31)$$

Using equation (1.31), equation (1.30) can be written as shown in equation (1.32).

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} - 2 & R_{22} & R_{23} + 2 \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \qquad ..... (1.32)$$

In equation (1.30), the terms $2I_1$ and $-2I_3$ on the right-hand side are functions of mesh currents I_1 and I_3, respectively. In equation (1.32), these two terms are transferred to the left-hand side with the opposite sign. Now equation (1.32) can be solved by Cramer's rule.

Circuits with Dependent Current Source

When a circuit has a dependent current source then express the value of the source in terms of mesh currents. If the dependent current source has parallel impedance then it can be converted into a dependent voltage source with series impedance and the analysis can be proceeded as explained in Section 1.6.5.

If the dependent current source does not have parallel impedance then it cannot be converted into a voltage source. In this case the value of the current source is related to mesh currents. Then for each current source one mesh current is eliminated by expressing the mesh current in terms of the source current and other mesh currents. The mesh basis matrix equation can be formed by inspection, by taking voltage across the dependent current source as unknown. While forming the mesh basis matrix equation, the voltage of current sources should be entered in the source matrix.

Now in the matrix equation some mesh currents will be eliminated and an equal number of unknown source voltages will be introduced. Thus, the number of unknowns will remain the same as the number of meshes m. On multiplying the mesh basis matrix equation we get m number of equations which can be solved to give a unique solution for unknowns and hence mesh currents.

EXAMPLE 1.35

Solve the mesh currents of the circuit shown in Fig. 1.

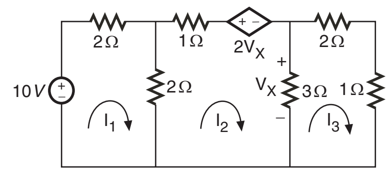


Fig. 1.

SOLUTION

The given circuit has three meshes. The general form of mesh basis matrix equation for three-mesh circuit is shown in equation (1).

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \qquad .....(1)$$

With reference to Fig. 1, the elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 2 + 2 = 4$	$R_{12} = R_{21} = -2$	$E_{11} = 10$
$R_{22} = 2 + 1 + 3 = 6$	$R_{13} = R_{31} = 0$	$E_{22} = -2V_x$
$R_{33} = 3 + 2 + 1 = 6$	$R_{23} = R_{32} = -3$	$E_{33} = 0$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 6 & -3 \\ 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -2V_x \\ 0 \end{bmatrix} \qquad ..... (2)$$

The value of dependent voltage source "$-2V_x$" should be expressed in terms of mesh currents.

With reference to Fig. 1 we can write,

$$V_x = 3(I_2 - I_3)$$

$$\therefore\ -2V_x = -2 \times 3(I_2 - I_3) = -6I_2 + 6I_3 \qquad ..... (3)$$

Using equation (3), equation (2) can be written as shown in equation (4).

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 6 & -3 \\ 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -6I_2 + 6I_3 \\ 0 \end{bmatrix} \qquad ..... (4)$$

In equation (4), the terms on the right-hand side which are a function of mesh currents are transferred to the left-hand side with the opposite sign as shown in equation (5).

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 6+6 & -3-6 \\ 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \qquad ..... (5)$$

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 12 & -9 \\ 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \qquad ..... (6)$$

In equation (6), the unknowns are I_1, I_2 and I_3. In order to solve I_1, I_2 and I_3, let us define the four determinants Δ, Δ_1, Δ_2 and Δ_3 as shown below:

$$\Delta = \begin{vmatrix} 4 & -2 & 0 \\ -2 & 12 & -9 \\ 0 & -3 & 6 \end{vmatrix}; \quad \Delta_1 = \begin{vmatrix} 10 & -2 & 0 \\ 0 & 12 & -9 \\ 0 & -3 & 6 \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} 4 & 10 & 0 \\ -2 & 0 & -9 \\ 0 & 0 & 6 \end{vmatrix}; \quad \Delta_3 = \begin{vmatrix} 4 & -2 & 10 \\ -2 & 12 & 0 \\ 0 & -3 & 0 \end{vmatrix}$$

The determinants are evaluated by expanding along first row and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 4 & -2 & 0 \\ -2 & 12 & -9 \\ 0 & -3 & 6 \end{vmatrix} = 4\times[12\times6-(-3)\times(-9)]-(-2)\times[-2\times6-0]+0$$

$$= 180-24 = 156$$

$$\Delta_1 = \begin{vmatrix} 10 & -2 & 0 \\ 0 & 12 & -9 \\ 0 & -3 & 6 \end{vmatrix} = 10\times[12\times6-(-3)\times(-9)]-(-2)\times[0-0]+0 = 450$$

$$\Delta_2 = \begin{vmatrix} 4 & 10 & 0 \\ -2 & 0 & -9 \\ 0 & 0 & 6 \end{vmatrix} = 4\times[0-0]-10\times[-2\times6-0]+0 = 120$$

$$\Delta_3 = \begin{vmatrix} 4 & -2 & 10 \\ -2 & 12 & 0 \\ 0 & -3 & 0 \end{vmatrix} = 4\times[0-0]-(-2)\times[0-0]+10\times[-2\times(-3)-0] = 60$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{450}{156} = 2.8846\,A$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{120}{156} = 0.7692\,A$$

$$I_3 = \frac{\Delta_3}{\Delta} = \frac{60}{156} = 0.3846\,A$$

EXAMPLE 1.36

Determine the current I_L in the circuit shown in Fig. 1, using mesh analysis.

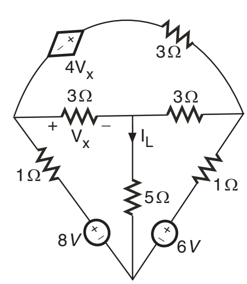


Fig. 1.

SOLUTION

The graph of the given circuit is shown in Fig. 2. It has six branches and four nodes. Hence, the number of meshes m in the circuit is, m = B − N + 1 = 6 − 4 + 1 = 3.

Let us assume three mesh currents I_1, I_2 and I_3 as shown in Fig. 3.

Now, the current, $I_L = I_1 - I_2$

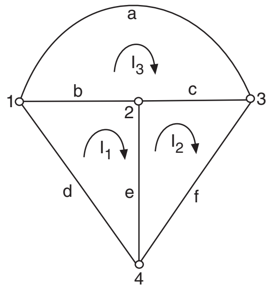


Fig. 2.

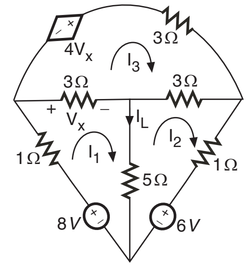


Fig. 3.

The general mesh basis matrix equation for three mesh circuit is shown in equation (1).

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \quad ..... (1)$$

With reference to Fig. 3, the elements of resistance matrix and source voltage matrix are formed as shown below :

$R_{11} = 1 + 3 + 5 = 9$	$R_{12} = R_{21} = -5$	$E_{11} = 8$
$R_{22} = 5 + 3 + 1 = 9$	$R_{13} = R_{31} = -3$	$E_{22} = -6$
$R_{33} = 3 + 3 + 3 = 9$	$R_{23} = R_{32} = -3$	$E_{33} = 4V_x$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 9 & -5 & -3 \\ -5 & 9 & -3 \\ -3 & -3 & 9 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ 4\,V_x \end{bmatrix} \quad ..... (2)$$

Let us express the value of dependent sources in terms of mesh currents. With reference to Fig. 3, we can write,

$$V_x = 3(I_1 - I_3)$$

$$\therefore \; 4V_x = 4 \times 3(I_1 - I_3) = 12I_1 - 12I_3 \quad ..... (3)$$

On substituting for $4V_x$ from equation (3), in equation (2) we get,

$$\begin{bmatrix} 9 & -5 & -3 \\ -5 & 9 & -3 \\ -3 & -3 & 9 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ 12\,I_1 - 12\,I_3 \end{bmatrix} \quad ..... (4)$$

In equation (4), the terms on the right-hand side which are a function of mesh currents are transferred to the left-hand side with the opposite sign as shown in equation (5).

$$\begin{bmatrix} 9 & -5 & -3 \\ -5 & 9 & -3 \\ -3-12 & -3 & 9+12 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ 0 \end{bmatrix} \quad .....(5)$$

$$\begin{bmatrix} 9 & -5 & -3 \\ -5 & 9 & -3 \\ -15 & -3 & 21 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ 0 \end{bmatrix} \quad .....(6)$$

In equation (6), the unknowns are I_1, I_2 and I_3. In order to solve I_1 and I_2 let us define three determinants Δ, Δ_1 and Δ_2 as

$$\Delta = \begin{vmatrix} 9 & -5 & -3 \\ -5 & 9 & -3 \\ -15 & -3 & 21 \end{vmatrix}; \quad \Delta_1 = \begin{vmatrix} 8 & -5 & -3 \\ -6 & 9 & -3 \\ 0 & -3 & 21 \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} 9 & 8 & -3 \\ -5 & -6 & -3 \\ -15 & 0 & 21 \end{vmatrix}$$

The determinants are evaluated by expanding along the first row and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 9 & -5 & -3 \\ -5 & 9 & -3 \\ -15 & -3 & 21 \end{vmatrix} = 9\times[9\times 21-(-3)^2]-(-5)\times[-5\times 21-(-15)\times(-3)]$$

$$+(-3)\times[-5\times(-3)-(-15)\times 9]$$

$$= 1620 - 750 - 450 = 420$$

$$\Delta_1 = \begin{vmatrix} 8 & -5 & -3 \\ -6 & 9 & -3 \\ 0 & -3 & 21 \end{vmatrix} = 8\times[9\times 21-(-3)^2]-(-5)\times[-6\times 21-0]+(-3)\times[-6\times(-3)-0]$$

$$= 1440 - 630 - 54 = 756$$

$$\Delta_2 = \begin{vmatrix} 9 & 8 & -3 \\ -5 & -6 & -3 \\ -15 & 0 & 21 \end{vmatrix} = 9\times[-6\times 21-0]-8\times[-5\times 21-(-15)\times(-3)]$$

$$+(-3)\times[0-(-15)\times(-6)]$$

$$= -1134 + 1200 + 270 = 336$$

$$\therefore\ I_L = I_1 - I_2 = \frac{\Delta_1}{\Delta} - \frac{\Delta_2}{\Delta} = \frac{\Delta_1 - \Delta_2}{\Delta} = \frac{756-336}{420} = 1\,A$$

EXAMPLE 1.37 *(AU Dec'16, 8 Marks)*

Determine the current I_o in the circuit shown in Fig. 1, using mesh analysis.

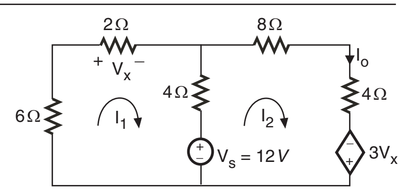


Fig. 1.

SOLUTION

The given circuit has two meshes. The general form of mesh basis matrix equation for two-mesh circuit is shown in equation (1).

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \end{bmatrix} \quad .....(1)$$

With reference to Fig. 1, the elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 6 + 2 + 4 = 12$ | $E_{11} = -12$

$R_{12} = R_{21} = -4$ | $E_{22} = 3V_x + 12$

$R_{22} = 4 + 8 + 4 = 16$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 12 & -4 \\ -4 & 16 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -12 \\ 3V_x + 12 \end{bmatrix} \quad ..... (2)$$

The value of dependent voltage source $3V_x$ should be expressed in terms of mesh currents.

With reference to Fig. 1 we can write,

$$V_x = 2I_1$$

$$\therefore\ 3V_x = 3 \times 2I_1 = 6I_1 \qquad ..... (3)$$

Using equation (3), equation (2) can be written as shown in equation (4).

$$\begin{bmatrix} 12 & -4 \\ -4 & 16 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -12 \\ 6I_1 + 12 \end{bmatrix} \qquad ..... (4)$$

In equation (4), the terms on the right-hand side which are a function of mesh currents are transferred to the left-hand side with the opposite sign as shown in equation (5).

$$\begin{bmatrix} 12 & -4 \\ -4-6 & 16 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -12 \\ 12 \end{bmatrix} \qquad ..... (5)$$

$$\begin{bmatrix} 12 & -4 \\ -10 & 16 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -12 \\ 12 \end{bmatrix} \qquad ..... (6)$$

To determine current I_o

From Fig. 1, we get, $I_o = I_2$

In order to solve I_2, let us define the two determinants Δ and Δ_2 as shown below:

$$\Delta = \begin{vmatrix} 12 & -4 \\ -10 & 16 \end{vmatrix} \ ; \quad \Delta_2 = \begin{vmatrix} 12 & -12 \\ -10 & 12 \end{vmatrix}$$

The determinants are evaluated by expanding along the first row and the mesh currents are solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 12 & -4 \\ -10 & 16 \end{vmatrix} = 12 \times 16 - (-4) \times (-10) = 152$$

$$\Delta_2 = \begin{vmatrix} 12 & -12 \\ -10 & 12 \end{vmatrix} = 12^2 - (-12) \times (-10) = 24$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{24}{152} = 0.1579\ A$$

$$\therefore\ I_o = I_2 = 0.1579\ A$$

EXAMPLE 1.38

In the circuit of Fig. 1, determine the power delivered to the 4 Ω resistor using mesh analysis.

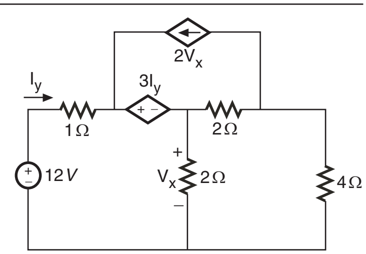


Fig. 1.

SOLUTION

The graph of the given circuit is shown in Fig. 2. It has six branches and four nodes. Hence, the number of meshes m in the circuit is, m = B − N + 1 = 6 − 4 + 1 = 3.

Let us assume three mesh currents I_1, I_2 and I_3 as shown in Fig. 3. Now the current through 4 Ω resistor is I_3.

$\therefore$ Power delivered to the $4\,\Omega$ resistor = $|I_3|^2 \times 4$

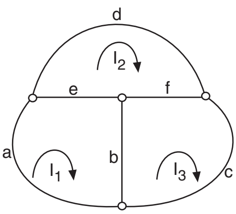


Fig. 2.

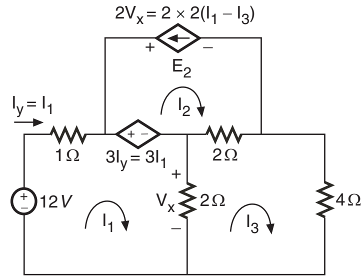


Fig. 3.

The general mesh basis matrix equation for three mesh circuit is shown in equation (1).

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \qquad \text{..... (1)}$$

Let the voltage across dependent current source be E_2 as shown in Fig. 3. With reference to Fig. 3, the elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 1 + 2 = 3$	$R_{12} = R_{21} = 0$	$E_{11} = 12 - 3I_y$
$R_{22} = 2$	$R_{13} = R_{31} = -2$	$E_{22} = 3I_y - E_2$
$R_{33} = 2 + 2 + 4 = 8$	$R_{23} = R_{32} = -2$	$E_{33} = 0$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 3 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 12 - 3\,I_y \\ 3\,I_y - E_2 \\ 0 \end{bmatrix} \qquad \text{..... (2)}$$

Let us express the value of dependent sources in terms of mesh currents. With reference to Fig. 3, we can write,

$$I_y = I_1 \quad \Rightarrow \quad 3I_y = 3I_1 \qquad \text{..... (3)}$$

$$V_x = 2(I_1 - I_3) \quad \Rightarrow \quad 2V_x = 2 \times 2(I_1 - I_3) = 4I_1 - 4I_3$$

In mesh-2, $I_2 = -2V_x$

$$\therefore \quad I_2 = -(4I_1 - 4I_3) = -4I_1 + 4I_3 \qquad \text{..... (4)}$$

Using equations (3) and (4), equation (2) can be written as shown in equation (5).

$$\begin{bmatrix} 3 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} I_1 \\ -4\,I_1 + 4\,I_3 \\ I_3 \end{bmatrix} = \begin{bmatrix} 12 - 3\,I_1 \\ 3\,I_1 - E_2 \\ 0 \end{bmatrix} \qquad \text{..... (5)}$$

From row-3 of equation (5), we get,

$$-2I_1 - 2(-4I_1 + 4I_3) + 8I_3 = 0$$

$$\therefore \quad 6I_1 = 0 \quad \Rightarrow \quad I_1 = 0$$

From row-1 of equation (5), we get,

$$3I_1 - 2I_3 = 12 - 3I_1$$

$$\therefore \quad -2I_3 = 12 \qquad \boxed{\text{Put, } I_1 = 0}$$

$$\therefore \quad I_3 = \frac{12}{-2} = -6\,A$$

Power delivered to the $4\,\Omega$ resistor $= |I_3|^2 \times 4 = |-6|^2 \times 4 = 6^2 \times 4 = 144\,W$

EXAMPLE 1.39

Determine the voltage V_L in the circuit shown in Fig. 1, using mesh analysis.

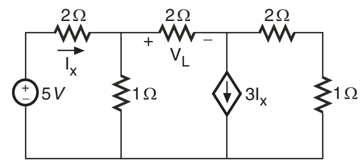


Fig. 1.

SOLUTION

The graph of the given circuit is shown in Fig. 2. It has five branches and three nodes. Hence, the number of meshes m in the circuit is, $m = B - N + 1 = 5 - 3 + 1 = 3$. Let us assume three mesh currents I_1, I_2 and I_3 as shown in Fig. 3.

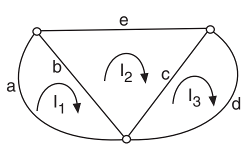


Fig. 2.

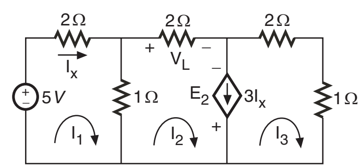


Fig. 3.

Now, the voltage, $V_L = 2I_2$

The general mesh basis matrix equation for three mesh circuit is shown in equation (1).

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \qquad \text{..... (1)}$$

Let the voltage across dependent source be E_2 as shown in Fig. 3.

With reference to Fig. 3, the elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 2 + 1 = 3$	$R_{12} = R_{21} = -1$	$E_{11} = 5$
$R_{22} = 1 + 2 = 3$	$R_{13} = R_{31} = 0$	$E_{22} = E_2$
$R_{33} = 2 + 1 = 3$	$R_{23} = R_{32} = 0$	$E_{33} = -E_2$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 5 \\ E_2 \\ -E_2 \end{bmatrix} \qquad \text{..... (2)}$$

With reference to Fig. 3, we can write,

$$I_x = I_1 \quad \Rightarrow \quad 3I_x = 3I_1$$

Also, $\quad 3I_x = I_2 - I_3 \quad \Rightarrow \quad 3I_1 = I_2 - I_3 \quad \Rightarrow \quad I_3 = -3I_1 + I_2 \qquad$ ..... (3)

On substituting for I_3 from equation (3) in equation (2), we get,

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ -3I_1 + I_2 \end{bmatrix} = \begin{bmatrix} 5 \\ E_2 \\ -E_2 \end{bmatrix} \quad ..... (4)$$

From row-2 of equation (4), we get,

$$-I_1 + 3I_2 = E_2 \quad ..... (5)$$

From row-3 of equation (4), we get,

$$3(-3I_1 + I_2) = -E_2 \quad \Rightarrow \quad -9I_1 + 3I_2 = -E_2 \quad ..... (6)$$

On adding equations (5) and (6), we get,

$$\therefore \quad -10I_1 + 6I_2 = 0 \quad \Rightarrow \quad I_1 = \frac{6}{10}I_2 \quad ..... (7)$$

From row-1 of equation (4), we get,

$$3I_1 - I_2 = 5$$

$$3 \times \frac{6}{10} I_2 - I_2 = 5$$ Using equation (7)

$$\therefore \quad 0.8I_2 = 5 \quad \Rightarrow \quad I_2 = \frac{5}{0.8} = 6.25\,A$$

$$\therefore \quad V_L = 2I_2 = 2 \times 6.25 = 12.5\,V$$

EXAMPLE 1.40

(AU May'17, 10 Marks)

Determine the voltage V_x and current I_x as shown in Fig. 1, using mesh analysis.

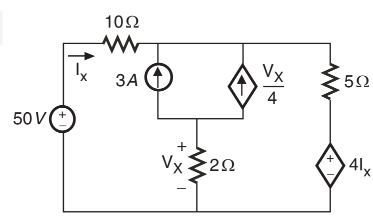


Fig. 1.

SOLUTION

The graph of the given circuit is shown in Fig. 2. It has six branches and four nodes. Hence, the number of meshes m in the circuit is m = B − N + 1 = 6 − 4 + 1 = 3. Let us assume three mesh currents I_1, I_2 and I_3 as shown in Fig. 3.

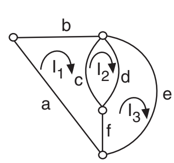


Fig. 2.

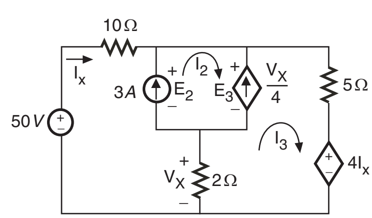


Fig. 3.

The general mesh basis matrix equation for three mesh circuit is shown in equation (1).

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \quad ..... (1)$$

Let the voltage across indepent current source be E_2 and voltage across dependent current source be E_3.

With reference to Fig. 3, the elements of resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 10 + 2 = 12$	$R_{12} = R_{21} = 0$	$E_{11} = 50 - E_2$
$R_{22} = 0$	$R_{13} = R_{31} = -2$	$E_{22} = E_2 - E_3$
$R_{33} = 5 + 2 = 7$	$R_{23} = R_{32} = 0$	$E_{33} = E_3 - 4I_x$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 12 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 7 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 50 - E_2 \\ E_2 - E_3 \\ E_3 - 4I_x \end{bmatrix} \qquad \text{..... (2)}$$

With reference to Fig. 3, we can write the following equations:

$I_2 - I_1 = 3 \quad \Rightarrow \quad I_2 = 3 + I_1$ .....(3)

$V_x = 2(I_1 - I_3) \quad \Rightarrow \quad V_x = 2I_1 - 2I_3$ .....(4)

$I_3 - I_2 = \dfrac{V_x}{4} \quad \Rightarrow \quad I_3 - I_2 = \dfrac{2I_1 - 2I_3}{4}$ $\boxed{\text{Using equation (4)}}$

$\therefore \; I_3 - I_2 = 0.5I_1 - 0.5I_3 \quad \Rightarrow \quad 1.5I_3 = 0.5I_1 + I_2 \quad \Rightarrow \quad I_3 = \dfrac{0.5}{1.5} I_1 + \dfrac{I_2}{1.5} = 0.3333I_1 + 0.6667I_2$ .....(5)

$I_x = I_1 \quad \Rightarrow \quad \therefore \; 4I_x = 4I_1$ .....(6)

On substituting equations (3), (5) and (6) in equation (2) we get,

$$\begin{bmatrix} 12 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 7 \end{bmatrix} \begin{bmatrix} I_1 \\ 3 + I_1 \\ 0.3333I_1 + 0.6667I_2 \end{bmatrix} = \begin{bmatrix} 50 - E_2 \\ E_2 - E_3 \\ E_3 - 4I_1 \end{bmatrix} \qquad \text{..... (7)}$$

From row-1 of equation (7), we get,

$$12I_1 - 2(0.3333I_1 + 0.6667I_2) = 50 - E_2 \qquad \text{..... (8)}$$

From row-2 of equation (7), we get,

$$0 = E_2 - E_3 \qquad \text{..... (9)}$$

From row-3 of equation (7), we get,

$$-2I_1 + 7(0.3333I_1 + 0.6667I_2) = E_3 - 4I_1 \qquad \text{..... (10)}$$

On adding equation (8), (9) and (10) we get,

$$10I_1 + 5(0.3333I_1 + 0.6667I_2) = 50 - 4I_1$$

$$10I_1 + 4I_1 + 1.6665I_1 + 3.3335I_2 = 50$$

$$15.6665I_1 + 3.3335\,(3 + I_1) = 50 \qquad \boxed{\text{Using equation (3)}}$$

$$19I_1 = 50 - 10.0005$$

$$\therefore \; I_1 = \frac{50 - 10}{19} = 2.1053\,A$$

$$\therefore \; I_2 = 3 + I_1 = 3 + 2.1053 = 5.1053\,A$$

$$I_3 = 0.3333I_1 + 0.6667I_2 = 0.3333 \times 2.1053 + 0.6667 \times 5.1053$$
$$= 4.1054\ A$$

$$\therefore\ I_x = I_1 = 2.1053\ A$$

$$V_x = 2I_1 - 2I_3 = 2 \times 2.1053 - 2 \times 4.1054 = -4.0002\ V = -4\ V$$

Alternate method

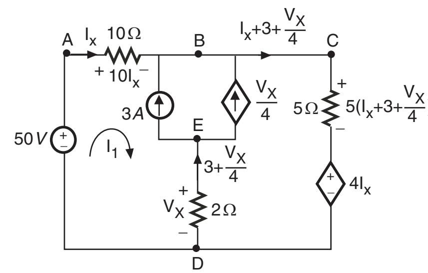


Fig. 4.

With reference to Fig. 4, by KCL at node E, we get,

$$V_x = -2\left(3 + \frac{V_x}{4}\right)$$

$$V_x = -6 - 0.5V_x$$

$$1.5V_x = -6$$

$$\therefore\ V_x = \frac{-6}{1.5} = -4\,V$$

With reference to Fig. 5, by KVL in the closed path ABCDA we get,

$$10I_x + 5\left(I_x + 3 + \frac{V_x}{4}\right) + 4I_x = 50$$

$$10I_x + 5I_x + 15 + 5\frac{V_x}{4} + 4I_x = 50$$

$$19I_x + \frac{5 \times (-4)}{4} = 50 - 15$$

$$\therefore\ 19I_x = 50 - 15 + 5$$

$$\therefore\ I_x = \frac{40}{19} = 2.1053\ A$$

1.7 Node Voltage Method of Analysis for DC and AC Circuits

Node analysis is a useful technique to solve the voltage across various elements of a circuit. Node analysis is preferred when the circuit is excited by current sources and the voltage across various elements are unknown. Node analysis can also be extended to circuits excited by both voltage and current sources and to circuits excited by both independent and dependent sources.

In a circuit each branch will have a voltage across it. Hence, the number of voltages in the circuit are equal to the number of branches. In a circuit some of the voltages will be independent and the remaining voltages depend on the independent voltages. The number of independent voltages in a circuit can be determined from the graph of the circuit. It is given by the branches of the tree (or twigs) of the graph. The voltages of the tree branches are same as the node voltages. (Refer Section 1.3.4.)

In nodal analysis, the independent voltages are solved by writing **K**irchhoff's **C**urrent **L**aw (KCL) equations for various nodes in the circuit. A tree of the graph with N nodes will have N – 1 branches or twigs. Hence, the number of independent voltages n = N – 1. The nodes of the circuit are same as nodes of the graph and so a circuit will also have N number of nodes. "*A **node** is meeting point of two or more elements. When more than two elements meet at a point then the* *node is called* ***principal node***". The voltage of a node can be expressed only with reference to another node. Hence, one of the nodes is chosen as the **reference node** and the node voltages are expressed with respect to the reference node.

For each node except the reference node, a voltage is assigned called **node voltage**. The voltage of the reference node is always zero. Using KCL, an equation is formed for each node by equating the sum of currents leaving the node to the sum of currents entering the node. These equations are arranged in the form of a matrix and node voltages are solved by Cramer's rule. A simple procedure to form a node basis matrix equation directly from the circuit by inspection without forming KCL equation is also discussed in Chapter 1, Section 1.7.1.

1.7.1 Node Analysis of Resistive Circuits Excited by DC Sources

A circuit with N nodes and B branches will have N – 1 independent voltages and B – (N – 1) dependent voltages which depend on independent voltages. Let us denote the number of independent voltages by n, where, n = N – 1.

In order to solve the independent voltages of a circuit we have to identify the N nodes of the circuit and choose one of the node as the reference node. For each node, except the reference, we have to attach a voltage called node voltage. The node voltages are the independent voltages of the circuit. Let V_1, V_2, V_3, ..., V_n be the node voltages.

For each node except the reference node, a KCL equation is formed by equating the sum of currents leaving the node to the sum of currents entering the node. Since there are n independent nodes, we can form n equations.

In resistive circuits excited by dc sources, the voltages and currents are real (i.e., they are not complex). For resistive circuits, the n equations can be arranged in a matrix form as shown in equation (1.33), which is called the node basis matrix equation. The formation of the node basis matrix equation from KCL equations is explained in some of the solved problems.

The node basis matrix equation (1.33), can be written as shown in equation (1.34).

Note *: The bold faced letter represent matrices.*

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} & \cdots & G_{1n} \\ G_{21} & G_{22} & G_{23} & \cdots & G_{2n} \\ G_{31} & G_{32} & G_{33} & \cdots & G_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ G_{n1} & G_{n2} & G_{n3} & \cdots & G_{nn} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \\ \vdots \\ I_{nn} \end{bmatrix} \qquad ..... (1.33)$$

$$\mathbf{G}\,\mathbf{V} = \mathbf{I} \qquad ..... (1.34)$$

where, **G** = Conductance matrix of order n × n

V = Node voltage matrix of order n × 1

I = Source current matrix of order n × 1

n = Number of nodes except reference node.

In equation (1.33), the elements of conductance matrix and source current matrix can be determined from the given circuit. Hence, the unknowns are node voltages, which has to be solved by any standard technique.

Alternatively, equation (1.33) can be formed directly from the circuit by inspection without writing KCL equations. A procedure to form node basis matrix equation by inspection is given below:

Procedure to Form Node Basis Matrix Equation by Inspection

Consider the node basis matrix equation shown below for a circuit with three nodes excluding the reference node.

Let V_1, V_2, V_3 be the node voltages.

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \qquad \text{..... (1.35)}$$

The elements of equation (1.35) for circuits with independent sources are

G_{11} = Sum of conductances connected to node-1

G_{22} = Sum of conductances connected to node-2

G_{33} = Sum of conductances connected to node-3

$G_{12} = G_{21}$ = Sum of conductances connected between node-1 and node-2

$G_{13} = G_{31}$ = Sum of conductances connected between node-1 and node-3

$G_{23} = G_{32}$ = Sum of conductances connected between node-2 and node-3

I_{11} = Sum of current sources connected to node-1

I_{22} = Sum of current sources connected to node-2

I_{33} = Sum of current sources connected to node-3

The conductances G_{11}, G_{22}, G_{33} are called **self-conductance** of node-1, node-2 and node-3, respectively.

The conductances G_{12}, G_{13}, G_{21}, G_{23}, G_{31}, G_{32} are called **mutual-conductance** between nodes.

The formation of the elements of conductance matrix and source current matrix are explained below:

i) The self-conductance G_{jj} is given by the sum of all the conductances connected to the j^{th} node. The self-conductances will always be positive.

ii) The mutual-conductance G_{jk} is given by negative of sum of all the conductances connected between node-j and node-k.

In a circuit with only independent sources (Reciprocal network), $G_{jk} = G_{kj}$.

iii) The source current matrix element I_{jj} is given by the sum of all the current sources connected to the j^{th} node. A current source is positive if it drives current towards a node as shown in Fig. 1.47, and it is negative if it drives current away from the node as shown in Fig. 1.48.

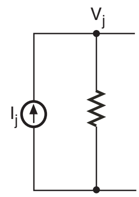


Fig. 1.47 : *Example for positive current source.*

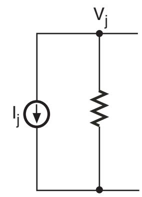


Fig. 1.48 : *Example for negative current source.*

Note : *In a circuit with both independent and dependent sources (non-reciprocal circuit)* $G_{jk} \neq G_{kj}$

Solution of Node Voltages

In the node basis matrix equation [i.e., equation (1.33)] the unknowns are node voltages $V_1, V_2, V_3 \ldots V_n$. The node voltages can be obtained by premultiplying equation (1.33), by the inverse of conductance matrix.

Consider equation (1.34),

$$\mathbf{G\,V = I}$$

On premultiplying both sides by $\mathbf{G}^{-1}$, we get,

$$\mathbf{G^{-1}G\,V = G^{-1}I}$$

$$\mathbf{U\,V = G^{-1}I} \qquad \boxed{\mathbf{G^{-1}G = U} = \text{Unit matrix}}$$

$$\therefore\ \mathbf{V = G^{-1}I} \qquad .....(1.36) \qquad \boxed{\mathbf{U\,V = V}}$$

Equation (1.36), will be the solution for node voltages. Equation (1.36), can be solved by **Cramer's rule**, by which the k^{th} node voltage V_k is given by equation (1.37).

$$\boxed{V_k = \frac{\Delta'_{1k}}{\Delta'} I_{11} + \frac{\Delta'_{2k}}{\Delta'} I_{22} + \frac{\Delta'_{3k}}{\Delta'} I_{33} + \ldots\ldots + \frac{\Delta'_{nk}}{\Delta'} I_{nn} = \frac{1}{\Delta'} \sum_{j=1}^{n} \Delta'_{jk} I_{jj}} \qquad ..... (1.37)$$

where, Δ'_{jk} = Cofactor of G_{jk}

I_{jj} = Sum of current sources connected to node-j

Δ' = Determinant of conductance matrix

Proof for Cramer's Rule

Consider equation (1.36), for a circuit with three nodes excluding reference node.

$$\mathbf{V} = \mathbf{G}^{-1}\mathbf{I} \quad \Rightarrow \quad \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix}^{-1} \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \qquad .....(1.38)$$

We know that,

$$\mathbf{G}^{-1} = \frac{\text{Adjoint of } \mathbf{G}}{\text{Determinant of } \mathbf{G}} = \frac{\text{Transpose of } \mathbf{G}_{cof}}{\text{Determinant of } \mathbf{G}} = \frac{\mathbf{G}^T_{cof}}{\Delta'}$$

where, Δ' = *Determinant of* $\mathbf{G}$.

$\mathbf{G}_{cof}$ = *Cofactor matrix (matrix formed by cofactor of elements of* $\mathbf{G}$ *matrix).*

Let, Δ'_{11} = *Cofactor of* G_{11}

Δ'_{12} = *Cofactor of* G_{12}

and in general, Δ'_{jk} = *Cofactor of* G_{jk}

$$\therefore \quad \mathbf{G}_{cof} = \begin{bmatrix} \Delta'_{11} & \Delta'_{12} & \Delta'_{13} \\ \Delta'_{21} & \Delta'_{22} & \Delta'_{23} \\ \Delta'_{31} & \Delta'_{32} & \Delta'_{33} \end{bmatrix} \xrightarrow{\text{Transpose}} \mathbf{G}^T_{cof} = \begin{bmatrix} \Delta'_{11} & \Delta'_{21} & \Delta'_{31} \\ \Delta'_{12} & \Delta'_{22} & \Delta'_{32} \\ \Delta'_{13} & \Delta'_{23} & \Delta'_{33} \end{bmatrix}$$

$$\therefore \mathbf{G}^{-1} = \frac{\mathbf{G}^T_{cof}}{\Delta'} = \frac{1}{\Delta'} \begin{bmatrix} \Delta'_{11} & \Delta'_{21} & \Delta'_{31} \\ \Delta'_{12} & \Delta'_{22} & \Delta'_{32} \\ \Delta'_{13} & \Delta'_{23} & \Delta'_{33} \end{bmatrix} \qquad .....(1.39)$$

On substituting for $\mathbf{G}^{-1}$ *from equation (1.39) in equation (1.38), we get,*

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \frac{1}{\Delta'} \begin{bmatrix} \Delta'_{11} & \Delta'_{21} & \Delta'_{31} \\ \Delta'_{12} & \Delta'_{22} & \Delta'_{32} \\ \Delta'_{13} & \Delta'_{23} & \Delta'_{33} \end{bmatrix} \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix}$$

On multiplying the matrices on the right-hand side of the above equation and equating to the terms on the left-hand side we get,

$$V_1 = \frac{\Delta'_{11}}{\Delta'} I_{11} + \frac{\Delta'_{21}}{\Delta'} I_{22} + \frac{\Delta'_{31}}{\Delta'} I_{33}$$

$$V_2 = \frac{\Delta'_{12}}{\Delta'} I_{11} + \frac{\Delta'_{22}}{\Delta'} I_{22} + \frac{\Delta'_{32}}{\Delta'} I_{33}$$

$$V_3 = \frac{\Delta'_{13}}{\Delta'} I_{11} + \frac{\Delta'_{23}}{\Delta'} I_{22} + \frac{\Delta'_{33}}{\Delta'} I_{33}$$

The above equations can be used to form a general equation for node voltage. In general, the k^{th} *node voltage of a circuit with n nodes excluding reference is given by,*

$$V_k = \frac{\Delta'_{1k}}{\Delta'} I_{11} + \frac{\Delta'_{2k}}{\Delta'} I_{22} + \frac{\Delta'_{3k}}{\Delta'} I_{33} + ... + \frac{\Delta'_{nk}}{\Delta'} I_{nn} = \frac{1}{\Delta'} \sum_{j=1}^{n} \Delta'_{jk} I_{jj}$$

Short-cut Procedure for Cramer's Rule

A short-cut procedure for Cramer's rule is explained below:

Let us consider a circuit with three nodes excluding reference. The node basis matrix equation for this case is,

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix}$$

Let us define three determinants as shown below:

$$\Delta'_1 = \begin{vmatrix} I_{11} & G_{12} & G_{13} \\ I_{22} & G_{22} & G_{23} \\ I_{33} & G_{32} & G_{33} \end{vmatrix}$$

$$\Delta'_2 = \begin{vmatrix} G_{11} & I_{11} & G_{13} \\ G_{21} & I_{22} & G_{23} \\ G_{31} & I_{33} & G_{33} \end{vmatrix}$$

$$\Delta'_3 = \begin{vmatrix} G_{11} & G_{12} & I_{11} \\ G_{21} & G_{22} & I_{22} \\ G_{31} & G_{32} & I_{33} \end{vmatrix}$$

Here,

Δ'_1 = Determinant of conductance matrix after replacing the first column of conductance matrix by source current column matrix

Δ'_2 = Determinant of conductance matrix after replacing the second column of conductance matrix by source current column matrix

Δ'_3 = Determinant of conductance matrix after replacing the third column of conductance matrix by source current column matrix.

Let, Δ' = Determinant of conductance matrix

$$\Delta' = \begin{vmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{vmatrix}$$

Now, node voltages V_1, V_2 and V_3 are given by,

$$V_1 = \frac{\Delta'_1}{\Delta'}$$

$$V_2 = \frac{\Delta'_2}{\Delta'}$$

$$V_3 = \frac{\Delta'_3}{\Delta'}$$

Cross-Check

The equation for node voltages obtained by short-cut procedure is the same as equation (1.37), and verified as shown below:

$$V_1 = \frac{\Delta'_1}{\Delta'} = \frac{1}{\Delta'}\begin{vmatrix} I_{11} & G_{12} & G_{13} \\ I_{22} & G_{22} & G_{23} \\ I_{33} & G_{32} & G_{33} \end{vmatrix} \qquad \boxed{\text{Expanding along first column}}$$

$$= \frac{1}{\Delta'}[I_{11}\ \Delta'_{11} + I_{22}\ \Delta'_{21} + I_{33}\ \Delta'_{31}]$$

$$= \frac{\Delta'_{11}}{\Delta'}I_{11} + \frac{\Delta'_{21}}{\Delta'}I_{22} + \frac{\Delta'_{31}}{\Delta'}I_{33}$$

$$V_2 = \frac{\Delta'_2}{\Delta'} = \frac{1}{\Delta'}\begin{vmatrix} G_{11} & I_{11} & G_{13} \\ G_{21} & I_{22} & G_{23} \\ G_{31} & I_{33} & G_{33} \end{vmatrix} \qquad \boxed{\text{Expanding along second column}}$$

$$= \frac{1}{\Delta'}[I_{11}\ \Delta'_{12} + I_{22}\ \Delta'_{22} + I_{33}\ \Delta'_{32}]$$

$$= \frac{\Delta'_{12}}{\Delta}I_{11} + \frac{\Delta'_{22}}{\Delta'}I_{22} + \frac{\Delta'_{32}}{\Delta'}I_{33}$$

$$V_3 = \frac{\Delta'_3}{\Delta'} = \frac{1}{\Delta'}\begin{vmatrix} G_{11} & G_{12} & I_{11} \\ G_{21} & G_{22} & I_{22} \\ G_{31} & G_{32} & I_{33} \end{vmatrix} \qquad \boxed{\text{Expanding along third column}}$$

$$= \frac{1}{\Delta'}[I_{11}\ \Delta'_{13} + I_{22}\ \Delta'_{23} + I_{33}\ \Delta'_{33}]$$

$$= \frac{\Delta'_{13}}{\Delta'}I_{11} + \frac{\Delta'_{23}}{\Delta'}I_{22} + \frac{\Delta'_{33}}{\Delta'}I_{33}$$

Various Steps to Obtain the Solution of Node Voltages and Branch Voltages in a Circuit

Step 1 **:** Draw the graph of the circuit.

Step 2 **:** Determine the branches B and nodes N. The number of node voltages n is given by $n = N - 1$.

Step 3 **:** Choose one of the nodes as reference. Let us denote the reference node as 0 (zero) and other nodes as 1, 2, 3, ..., n.

Step 4 **:** Let us denote the node voltages as V_1, V_2, V_3,....., and the branch voltages as V_a,V_b, V_c, V_d, V_e,...... Write the relationship between node and branch voltages.

Step 5 **:** Form the node basis matrix equation by inspection and solve the node voltages using Cramer's rule. For a circuit with three nodes excluding the reference, the node basis matrix equation and solution of node voltages using Cramer's rule are given below:

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix}$$

$$V_1 = \frac{\Delta'_1}{\Delta'} = \frac{1}{\Delta'}\begin{vmatrix} I_{11} & G_{12} & G_{13} \\ I_{22} & G_{22} & G_{23} \\ I_{33} & G_{32} & G_{33} \end{vmatrix}$$

$$V_2 = \frac{\Delta'_2}{\Delta'} = \frac{1}{\Delta'}\begin{vmatrix} G_{11} & I_{11} & G_{13} \\ G_{21} & I_{22} & G_{23} \\ G_{31} & I_{33} & G_{33} \end{vmatrix}$$

$$V_3 = \frac{\Delta'_3}{\Delta'} = \frac{1}{\Delta'}\begin{vmatrix} G_{11} & G_{12} & I_{11} \\ G_{21} & G_{22} & I_{22} \\ G_{31} & G_{32} & I_{33} \end{vmatrix}$$

Step 6 : Solve the branch voltages using the relationship between branch and node voltages.

> ***Note :*** *After solving the node voltages if any of the voltage is found to be negative then that node has a potential lesser than the reference node.*

EXAMPLE 1.41

Write and solve the node voltage equations for the circuit shown in Fig. 1.

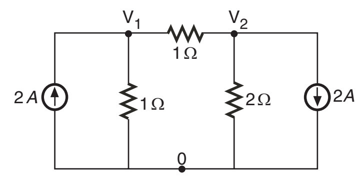


Fig. 1.

SOLUTION

With reference to Fig. 2, the node equation for node-1 is formed as shown below:

Currents leaving node-1 : $\frac{V_1 - V_2}{1}, \frac{V_1}{1}$

Current entering node-1 : 2 A

$$\therefore \frac{V_1 - V_2}{1} + \frac{V_1}{1} = 2$$

$$V_1 - V_2 + V_1 = 2$$

$$2V_1 - V_2 = 2 \qquad \text{..... (1)}$$

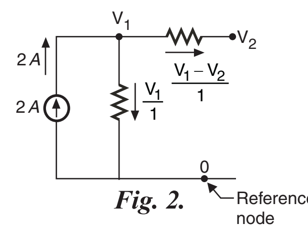


Fig. 2.

With reference to Fig. 3, the node equation for node-2 is formed as shown below:

Currents leaving node-2 : $\frac{V_2 - V_1}{1}, \frac{V_2}{2}$, 2 A

Current entering node-2 : Nil

$$\therefore \frac{V_2 - V_1}{1} + \frac{V_2}{2} + 2 = 0$$

$$V_2 - V_1 + 0.5V_2 = -2$$

$$-V_1 + 1.5V_2 = -2 \qquad \text{..... (2)}$$

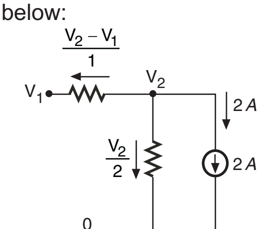


Fig. 3.

Equations (1) and (2) are the node equations of the circuit, which are summarised below for convenience.

$$2V_1 - V_2 = 2 \qquad \text{..... (1)}$$

$$-V_1 + 1.5V_2 = -2 \qquad \text{..... (2)}$$

Equation (1) × 1 $\Rightarrow$ $2V_1 - V_2 = 2$

Equation (2) × 2 $\Rightarrow$ $-2V_1 + 3V_2 = -4$

On adding $2V_2 = -2$

$$\therefore \ V_2 = -\frac{2}{2} = -1V$$

From equation (2), we get,

$$V_1 = 1.5V_2 + 2$$
$$= 1.5 \times (-1) + 2 = 0.5V$$

The node voltages are,

$$V_1 = 0.5V$$
$$V_2 = -1V$$

EXAMPLE 1.42 *(AU Dec'16, 8 Marks)*

Write and solve the node voltage equations for the circuit shown in Fig. 1.

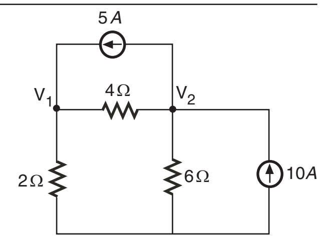


Fig. 1.

SOLUTION

With reference to Fig. 2, the node equation for node-1 is formed as shown below:

Currents leaving node-1 : $\frac{V_1 - V_2}{4}, \frac{V_1}{2}$

Current entering node-1 : $5A$

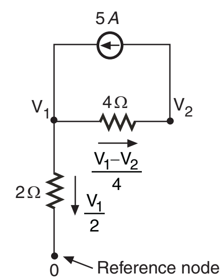


Fig. 2.

$$\therefore \ \frac{V_1 - V_2}{4} + \frac{V_1}{2} = 5$$

$$0.25V_1 - 0.25V_2 + 0.5V_1 = 5$$

$$0.75V_1 - 0.25V_2 = 5 \quad .....(1)$$

With reference to Fig. 3, the node equation for node-2 is formed as shown below:

Currents leaving node-2 : $\frac{V_2 - V_1}{4}, \frac{V_2}{6}, 5A$

Current entering node-2 : $10A$

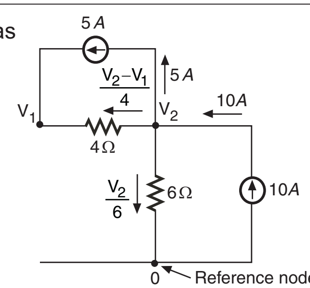


Fig. 3.

$$\therefore \ \frac{V_2 - V_1}{4} + \frac{V_2}{6} + 5 = 10$$

$$0.25V_2 - 0.25V_1 + 0.1667V_2 = 10 - 5$$

$$-0.25V_1 + 0.416V_2 = 5 \quad .....(2)$$

Equations (1) and (2) are the node equations of the circuit, which are summarised below for convenience.

$$0.75V_1 - 0.25V_2 = 5 \quad ..... (1)$$

$$-0.25V_1 + 0.416V_2 = 5 \quad ..... (2)$$

Equation (1) × 1 ⇒ $0.75V_1 - 0.25V_2 = 5$

Equation (2) × 3 ⇒ $-0.75V_1 + 1.25V_2 = 15$

On adding $V_2 = 20$

$$\therefore \; V_2 = 20\,V$$

From equation (1), we get,

$$V_1 = \frac{5 + 0.25V_2}{0.75}$$

$$= \frac{5 + 0.25 \times 20}{0.75} = 13.3333\,V$$

EXAMPLE 1.43

Determine the voltages across various elements of the circuit shown in Fig. 1, using the node method.

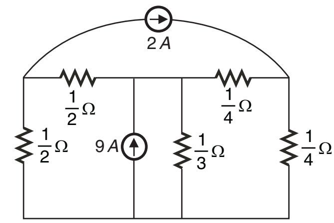


Fig. 1.

SOLUTION

The graph of the given circuit is shown in Fig. 2. It has seven branches and four nodes.

Let us choose one of the node as reference as shown in Fig. 2. Let the voltages of other three nodes be V_1, V_2 and V_3.

The reference node is denoted by 0 to indicate that its voltage is zero volt. The circuit with chosen node voltages is shown in Fig. 3.

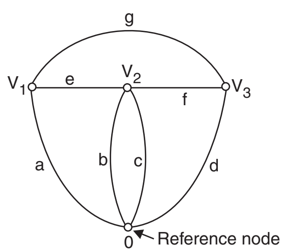


Fig. 2.

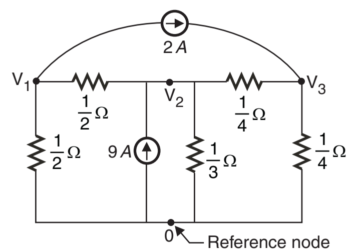


Fig. 3.

Method I : Formation of node basis matrix equation by applying KCL

In this method, the node equations are formed using Kirchhoff's Current Law. The node equation for a node is formed by equating the sum of currents leaving that node to the sum of currents entering that node.

While writing the node equation for a node it is assumed that all the resistances connected to that node will draw current from that node. Hence, the current in the resistances will always leave the node.

With reference to Fig. 4, the node equation for node-1 is formed as shown below:

Currents leaving node-1 : $\frac{V_1}{1/2}, \frac{V_1 - V_2}{1/2}, 2\,A$

Current entering node-1 : Nil

$$\therefore \; \frac{V_1}{1/2} + \frac{V_1 - V_2}{1/2} + 2 = 0$$

$$2V_1 + 2V_1 - 2V_2 + 2 = 0$$

$$4V_1 - 2V_2 = -2 \qquad \text{..... (1)}$$

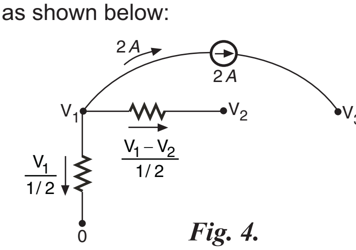


Fig. 4.

With reference to Fig. 5, the node equation for node-2 is formed as shown below:

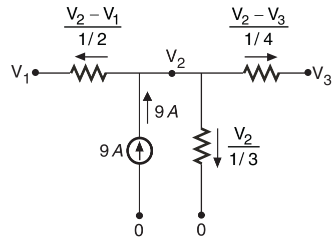


Fig. 5.

Currents leaving node-2 : $\frac{V_2 - V_1}{1/2}, \frac{V_2 - V_3}{1/4}, \frac{V_2}{1/3}$

Current entering node-2 : $9\,A$

$$\therefore \quad \frac{V_2 - V_1}{1/2} + \frac{V_2 - V_3}{1/4} + \frac{V_2}{1/3} = 9$$

$$2V_2 - 2V_1 + 4V_2 - 4V_3 + 3V_2 = 9$$

$$-2V_1 + 9V_2 - 4V_3 = 9 \qquad \text{..... (2)}$$

With reference to Fig. 6, the node equation for node-3 is formed as shown below:

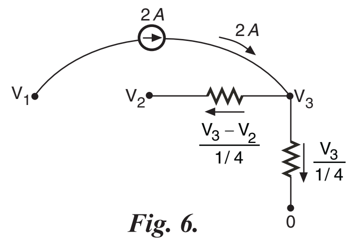


Fig. 6.

Currents leaving node-3 : $\frac{V_3 - V_2}{1/4}, \frac{V_3}{1/4}$

Current entering node-3 : $2\,A$

$$\therefore \quad \frac{V_3 - V_2}{1/4} + \frac{V_3}{1/4} = 2$$

$$4V_3 - 4V_2 + 4V_3 = 2$$

$$-4V_2 + 8V_3 = 2 \qquad \text{..... (3)}$$

Equations (1), (2) and (3) are node equations of the circuit shown in Fig. 3. The node equations are summarised below for convenience.

$$4V_1 - 2V_2 = -2$$

$$-2V_1 + 9V_2 - 4V_3 = 9$$

$$-4V_2 + 8V_3 = 2$$

The node equations can be arranged in a matrix form as shown below and then solved by Cramer's rule.

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 9 & -4 \\ 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ 2 \end{bmatrix} \qquad \text{..... (4)}$$

Method II : Formation of node basis matrix equation by inspection

In this method, the node basis matrix equation is formed by inspection using the circuit shown in Fig. 3. The general node basis matrix equation for a circuit with three nodes excluding the reference is shown in equation (5).

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \qquad \text{..... (5)}$$

The elements of conductance matrix and source current matrix are formed as shown below:

$G_{11} = 2 + 2 = 4$	$G_{12} = G_{21} = -2$	$I_{11} = -2$
$G_{22} = 2 + 3 + 4 = 9$	$G_{13} = G_{31} = 0$	$I_{22} = 9$
$G_{33} = 4 + 4 = 8$	$G_{23} = G_{32} = -4$	$I_{33} = 2$

On substituting the above terms in equation (5), we get,

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 9 & -4 \\ 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ 2 \end{bmatrix} \qquad ..... (6)$$

Solution of node voltages

It is observed that the node basis matrix equations obtained in methods-I and II are the same. In equation (6) the unknowns are V_1, V_2 and V_3. In order to solve V_1, V_2 and V_3, let us define four determinants Δ', Δ'_1, Δ'_2 and Δ'_3 as shown below:

$$\Delta' = \begin{vmatrix} 4 & -2 & 0 \\ -2 & 9 & -4 \\ 0 & -4 & 8 \end{vmatrix}; \quad \Delta'_1 = \begin{vmatrix} -2 & -2 & 0 \\ 9 & 9 & -4 \\ 2 & -4 & 8 \end{vmatrix}; \quad \Delta'_2 = \begin{vmatrix} 4 & -2 & 0 \\ -2 & 9 & -4 \\ 0 & 2 & 8 \end{vmatrix}; \quad \Delta'_3 = \begin{vmatrix} 4 & -2 & -2 \\ -2 & 9 & 9 \\ 0 & -4 & 2 \end{vmatrix}$$

The determinants are evaluated by expanding along first row and the node voltages are solved by Cramer's rule.

$$\Delta' = \begin{vmatrix} 4 & -2 & 0 \\ -2 & 9 & -4 \\ 0 & -4 & 8 \end{vmatrix} = 4 \times [9 \times 8 - (-4)^2] - (-2) \times [-2 \times 8 - 0] + 0 = 224 - 32 = 192$$

$$\Delta'_1 = \begin{vmatrix} -2 & -2 & 0 \\ 9 & 9 & -4 \\ 2 & -4 & 8 \end{vmatrix} = -2 \times [9 \times 8 - (-4)^2] - (-2) \times [9 \times 8 - 2 \times (-4)] + 0 = -112 + 160 = 48$$

$$\Delta'_2 = \begin{vmatrix} 4 & -2 & 0 \\ -2 & 9 & -4 \\ 0 & 2 & 8 \end{vmatrix} = 4 \times [9 \times 8 - 2 \times (-4)] - (-2) \times [-2 \times 8 - 0] + 0 = 320 - 32 = 288$$

$$\Delta'_3 = \begin{vmatrix} 4 & -2 & -2 \\ -2 & 9 & 9 \\ 0 & -4 & 2 \end{vmatrix} = 4 \times [9 \times 2 - (-4) \times 9] - (-2) \times [-2 \times 2 - 0] + (-2) \times [-2 \times (-4) - 0] = 216 - 8 - 16 = 192$$

$$V_1 = \frac{\Delta'_1}{\Delta'} = \frac{48}{192} = 0.25\,V$$

$$V_2 = \frac{\Delta'_2}{\Delta'} = \frac{288}{192} = 1.5\,V$$

$$V_3 = \frac{\Delta'_3}{\Delta'} = \frac{192}{192} = 1\,V$$

To solve branch voltages

The given circuit has seven branches. Let us denote the branch voltages as V_a, V_b, V_c, V_d, V_e, V_f and V_g as shown in Fig. 7. The sign of branch voltages are chosen such that they are all positive. The relation between the branch and node voltages are obtained using the circuit shown in Fig. 7 and the branch voltages are solved as shown below:

$V_a = V_1 = 0.25\,V$

$V_b = V_2 = 1.5\,V$

$V_c = V_2 = 1.5\,V$

$V_d = V_3 = 1\,V$

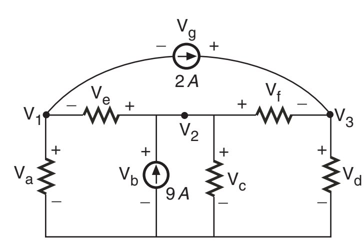


Fig. 7.

$$V_e = V_2 - V_1 = 1.5 - 0.25 = 1.25\,V$$

$$V_f = V_2 - V_3 = 1.5 - 1 = 0.5\,V$$

$$V_g = V_3 - V_1 = 1 - 0.25 = 0.75\,V$$

Note : *The branch voltages are voltages across various elements in the circuit.*

EXAMPLE 1.44

In the network shown in Fig. 1, find the current through the $2\,\Omega$ resistor, using the node method.

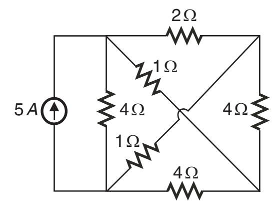


Fig. 1.

SOLUTION

The given circuit is redrawn as shown in Fig. 2. The graph of the circuit is shown in Fig. 3. It has seven branches and four nodes. Let us choose one of the nodes as reference as shown in Fig. 3. Let the voltages of other three nodes be V_1, V_2 and V_3. The reference node is denoted by 0.

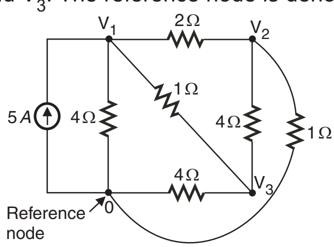


Fig. 2.

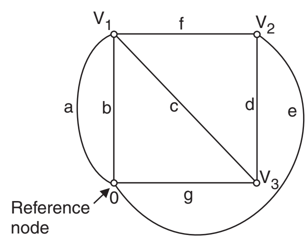


Fig. 3.

In the $2\,\Omega$ resistor, the current will flow from node-1 to node-2 if $V_1 > V_2$, and when $V_2 > V_1$, the current will flow from node-2 to node-1. Let the current through $2\,\Omega$ resistance be I_x.

If $V_1 > V_2$, then $I_x = \dfrac{V_1 - V_2}{2}$.

If $V_2 > V_1$, then $I_x = \dfrac{V_2 - V_1}{2}$.

In both the cases it is enough if we solve the node voltages V_1 and V_2.

The node basis matrix equation is formed by inspection using the circuit shown in Fig. 2. The general node basis matrix equation for a circuit with three nodes excluding the reference is shown in equation (1).

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \qquad \text{..... (1)}$$

The elements of conductance matrix and source current matrix are formed as shown below :

$$G_{11} = \frac{1}{4} + \frac{1}{1} + \frac{1}{2} = 1.75 \qquad G_{12} = G_{21} = -\frac{1}{2} = -0.5 \qquad I_{11} = 5$$

$$G_{22} = \frac{1}{2} + \frac{1}{4} + \frac{1}{1} = 1.75 \qquad G_{13} = G_{31} = -\frac{1}{1} = -1 \qquad I_{22} = 0$$

$$G_{33} = \frac{1}{4} + \frac{1}{1} + \frac{1}{4} = 1.5 \qquad G_{23} = G_{32} = -\frac{1}{4} = -0.25 \qquad I_{33} = 0$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 1.75 & -0.5 & -1 \\ -0.5 & 1.75 & -0.25 \\ -1 & -0.25 & 1.5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

In order to solve the node voltages V_1 and V_2, let us define three determinants Δ', Δ'_1 and Δ'_2 as shown below:

$$\Delta' = \begin{vmatrix} 1.75 & -0.5 & -1 \\ -0.5 & 1.75 & -0.25 \\ -1 & -0.25 & 1.5 \end{vmatrix}; \quad \Delta'_1 = \begin{vmatrix} 5 & -0.5 & -1 \\ 0 & 1.75 & -0.25 \\ 0 & -0.25 & 1.5 \end{vmatrix}; \quad \Delta'_2 = \begin{vmatrix} 1.75 & 5 & -1 \\ -0.5 & 0 & -0.25 \\ -1 & 0 & 1.5 \end{vmatrix}$$

The determinants are evaluated by expanding along the first row and the node voltages are solved by Cramer's rule.

$$\Delta' = \begin{vmatrix} 1.75 & -0.5 & -1 \\ -0.5 & 1.75 & -0.25 \\ -1 & -0.25 & 1.5 \end{vmatrix} = 1.75 \times [1.75 \times 1.5 - (-0.25)^2] - (-0.5) \times [-0.5 \times 1.5 - (-1) \times (-0.25)] + (-1) \times [-0.5 \times (-0.25) - (-1) \times 1.75]$$

$$= 4.4844 - 0.5 - 1.875 = 2.1094$$

$$\Delta'_1 = \begin{vmatrix} 5 & -0.5 & -1 \\ 0 & 1.75 & -0.25 \\ 0 & -0.25 & 1.5 \end{vmatrix} = 5 \times [1.75 \times 1.5 - (-0.25)^2] - 0 + 0 = 12.8125$$

$$\Delta'_2 = \begin{vmatrix} 1.75 & 5 & -1 \\ -0.5 & 0 & -0.25 \\ -1 & 0 & 1.5 \end{vmatrix} = 0 - 5 \times [-0.5 \times 1.5 - (-1) \times (-0.25)] + 0 = 5$$

$$\therefore \quad V_1 = \frac{\Delta'_1}{\Delta'} = \frac{12.8125}{2.1094} = 6.074\,V$$

$$V_2 = \frac{\Delta'_2}{\Delta'} = \frac{5}{2.1094} = 2.3703\,V$$

$$\therefore \quad I_x = \frac{V_1 - V_2}{2} = \frac{6.074 - 2.3703}{2} = 1.8519\,A$$

Since $V_1 > V_2$, the direction of current I_x is from node-1 to node-2.

EXAMPLE 1.45

In the circuit shown in Fig. 1, find the potential difference between A and D.

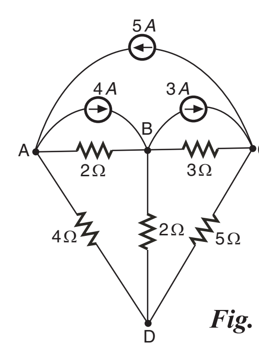


Fig. 1.

SOLUTION

The given circuit has four nodes. Let us choose the node-D as reference node and so the voltage of node-D is zero volt. All other node voltages are expressed with respect to reference node. Let the voltages of node A, B and C be V_1, V_2 and V_3, respectively. Now the voltage between A and D is V_1.

The node basis matrix equation is formed by inspection using the circuit shown in Fig. 2. The general node basis matrix equation for a circuit with three nodes excluding the reference is shown in equation (1).

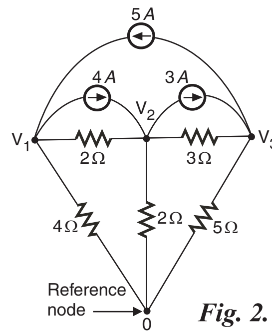


Fig. 2.

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \quad \text{..... (1)}$$

The elements of conductance matrix and source current matrix are formed as shown below:

$$G_{11} = \frac{1}{2} + \frac{1}{4} = 0.75 \qquad G_{12} = G_{21} = -\frac{1}{2} = -0.5 \qquad I_{11} = 5 - 4 = 1$$

$$G_{22} = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = 1.33 \qquad G_{13} = G_{31} = 0 \qquad I_{22} = 4 - 3 = 1$$

$$G_{33} = \frac{1}{3} + \frac{1}{5} = 0.53 \qquad G_{23} = G_{32} = -\frac{1}{3} = -0.33 \qquad I_{33} = 3 - 5 = -2$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 0.75 & -0.5 & 0 \\ -0.5 & 1.33 & -0.33 \\ 0 & -0.33 & 0.53 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \text{..... (2)}$$

In order to solve the node voltage V_1, let us define two determinants Δ' and Δ'_1 as shown below:

$$\Delta' = \begin{vmatrix} 0.75 & -0.5 & 0 \\ -0.5 & 1.33 & -0.33 \\ 0 & -0.33 & 0.53 \end{vmatrix} ; \qquad \Delta'_1 = \begin{vmatrix} 1 & -0.5 & 0 \\ 1 & 1.33 & -0.33 \\ -2 & -0.33 & 0.53 \end{vmatrix}$$

The determinants are evaluated by expanding along the first row and the node voltage V_1 is solved by Cramer's rule.

$$\Delta' = \begin{vmatrix} 0.75 & -0.5 & 0 \\ -0.5 & 1.33 & -0.33 \\ 0 & -0.33 & 0.53 \end{vmatrix} \begin{aligned} &= 0.75 \times [1.33 \times 0.53 - (-0.33)^2] - (-0.5) \times [-0.5 \times 0.53 - 0] + 0 \\ &= 0.447 - 0.1325 = 0.3145 \end{aligned}$$

$$\Delta'_1 = \begin{vmatrix} 1 & -0.5 & 0 \\ 1 & 1.33 & -0.33 \\ -2 & -0.33 & 0.53 \end{vmatrix} \begin{aligned} &= 1 \times [1.33 \times 0.53 - (-0.33)^2] - (-0.5) \times [1 \times 0.53 - (-2) \times (-0.33)] + 0 \\ &= 0.596 - 0.065 = 0.531 \end{aligned}$$

$$\therefore \; V_1 = \frac{\Delta'_1}{\Delta'} = \frac{0.531}{0.3145} = 1.6884\,V$$

Voltage between node A and D = $V_{AD} = V_1 = 1.6884\,V$

EXAMPLE 1.46

In the circuit shown in Fig. 1, determine the power supplied by the current sources.

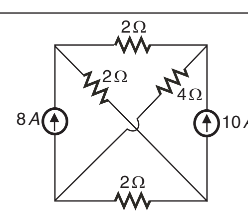


Fig. 1.

SOLUTION

The given circuit has four nodes. Let us choose one of the node voltage as reference node, which is indicated by 0. Let the voltages of the other three nodes be V_1, V_2 and V_3 as shown in Fig. 2.

The power supplied by $8\,A$ source in watts = $V_1 \times 8$

The power supplied by $10\,A$ source in watts = $(V_2 - V_3) \times 10$

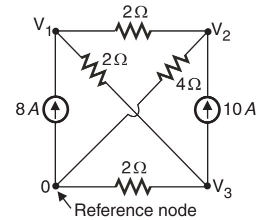


Fig. 2.

The node basis matrix equation is formed by inspection using the circuit shown in Fig. 2. The general node basis matrix equation for a circuit with three nodes excluding the reference is shown in equation (1).

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \quad \text{..... (1)}$$

The elements of conductance matrix and source current matrix are formed as shown below:

$$G_{11} = \frac{1}{2} + \frac{1}{2} = 1 \qquad G_{12} = G_{21} = -\frac{1}{2} = -0.5 \qquad I_{11} = 8$$

$$G_{22} = \frac{1}{2} + \frac{1}{4} = 0.75 \qquad G_{13} = G_{31} = -\frac{1}{2} = -0.5 \qquad I_{22} = 10$$

$$G_{33} = \frac{1}{2} + \frac{1}{2} = 1 \qquad G_{23} = G_{32} = 0 \qquad I_{33} = -10$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.75 & 0 \\ -0.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ -10 \end{bmatrix} \quad \text{..... (2)}$$

In equation (2), the unknowns are V_1, V_2 and V_3. In order to solve V_1, V_2 and V_3, let us define four determinants, Δ', Δ'_1, Δ'_2 and Δ'_3 as shown below:

$$\Delta' = \begin{vmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.75 & 0 \\ -0.5 & 0 & 1 \end{vmatrix} \; ; \qquad \Delta'_1 = \begin{vmatrix} 8 & -0.5 & -0.5 \\ 10 & 0.75 & 0 \\ -10 & 0 & 1 \end{vmatrix}$$

$$\Delta'_2 = \begin{vmatrix} 1 & 8 & -0.5 \\ -0.5 & 10 & 0 \\ -0.5 & -10 & 1 \end{vmatrix} \; ; \qquad \Delta'_3 = \begin{vmatrix} 1 & -0.5 & 8 \\ -0.5 & 0.75 & 10 \\ -0.5 & 0 & -10 \end{vmatrix}$$

The determinants are evaluated by expanding along the first row and the node voltages are solved by Cramer's rule.

$$\Delta' = \begin{vmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.75 & 0 \\ -0.5 & 0 & 1 \end{vmatrix} = 1 \times [0.75 \times 1 - 0] - (-0.5) \times [-0.5 \times 1 - 0] + (-0.5) \times [0 - (-0.5) \times 0.75]$$

$$= 0.75 - 0.25 - 0.1875 = 0.3125$$

$$\Delta'_1 = \begin{vmatrix} 8 & -0.5 & -0.5 \\ 10 & 0.75 & 0 \\ -10 & 0 & 1 \end{vmatrix} = 8 \times [0.75 \times 1 - 0] - (-0.5) \times [10 \times 1 - 0] + (-0.5) \times [0 - (-10) \times 0.75]$$

$$= 6 + 5 - 3.75 = 7.25$$

$$\Delta'_2 = \begin{vmatrix} 1 & 8 & -0.5 \\ -0.5 & 10 & 0 \\ -0.5 & -10 & 1 \end{vmatrix} = 1 \times [10 \times 1 - 0] - 8 \times [-0.5 \times 1 - 0] + (-0.5) \times [-0.5 \times (-10) - (-0.5) \times 10]$$

$$= 10 + 4 - 5 = 9$$

$$\Delta'_3 = \begin{vmatrix} 1 & -0.5 & 8 \\ -0.5 & 0.75 & 10 \\ -0.5 & 0 & -10 \end{vmatrix} = \begin{matrix} 1 \times [0.75 \times (-10) - 0] - (-0.5) \times [-0.5 \times (-10) - (-0.5) \times 10] \\ + 8 \times [0 - (-0.5) \times 0.75] \end{matrix}$$

$$= -7.5 + 5 + 3 = 0.5$$

$$V_1 = \frac{\Delta'_1}{\Delta'} = \frac{7.25}{0.3125} = 23.2\,V$$

$$V_2 = \frac{\Delta'_2}{\Delta'} = \frac{9}{0.3125} = 28.8\,V$$

$$V_3 = \frac{\Delta'_3}{\Delta'} = \frac{0.5}{0.3125} = 1.6\,V$$

Power supplied by 8 A current source = $V_1 \times 8 = 23.2 \times 8 = 185.6\,W$

Power supplied by 10 A current source = $(V_2 - V_3) \times 10 = (28.8 - 1.6) \times 10 = 272\,W$

EXAMPLE 1.47

Find the power in the 4 Ω resistor of the circuit shown in Fig. 1, using the node method.

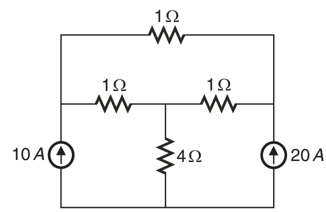


Fig. 1.

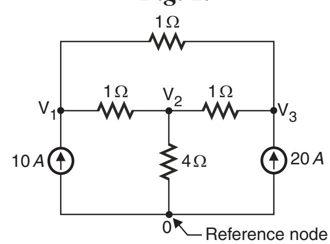


Fig. 2.

SOLUTION

To estimate the power in the 4 Ω resistor, first we have to determine the voltage across 4 Ω resistor. The given circuit has four nodes. Let us choose one of the node as reference node and it is indicated by 0. Let the voltages of other three nodes be V_1, V_2 and V_3 as shown in Fig. 2. Now the voltage across 4 Ω resistor is V_2 volts.

$\therefore$ Power in the 4 Ω resistor in watts $= \dfrac{V_2^2}{4}$.

The node basis matrix equation is formed by inspection using the circuit shown in Fig. 2. The general node basis matrix equation for a circuit with three nodes excluding the reference is shown in equation (1).

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \qquad .....(1)$$

The elements of conductance matrix and source current matrix are formed as shown below:

$$G_{11} = \frac{1}{1} + \frac{1}{1} = 2 \qquad G_{12} = G_{21} = -\frac{1}{1} = -1 \qquad I_{11} = 10$$

$$G_{22} = \frac{1}{1} + \frac{1}{1} + \frac{1}{4} = 2.25 \qquad G_{13} = G_{31} = -\frac{1}{1} = -1 \qquad I_{22} = 0$$

$$G_{33} = \frac{1}{1} + \frac{1}{1} = 2 \qquad G_{23} = G_{32} = -\frac{1}{1} = -1 \qquad I_{33} = 20$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2.25 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 20 \end{bmatrix} \quad \text{..... (2)}$$

In order to solve the node voltage V_2, let us define two determinants Δ' and Δ'_2 as shown below:

$$\Delta' = \begin{vmatrix} 2 & -1 & -1 \\ -1 & 2.25 & -1 \\ -1 & -1 & 2 \end{vmatrix} \quad ; \quad \Delta'_2 = \begin{vmatrix} 2 & 10 & -1 \\ -1 & 0 & -1 \\ -1 & 20 & 2 \end{vmatrix}$$

The determinants are evaluated by expanding along the first row and the node voltage V_2 is solved by Cramer's rule.

$$\Delta' = \begin{vmatrix} 2 & -1 & -1 \\ -1 & 2.25 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 2\times[2.25\times2-(-1)^2]-(-1)\times[-1\times2-(-1)^2]+(-1)\times[(-1)^2-(-1)\times2.25]$$

$$= 7-3-3.25 = 0.75$$

$$\Delta'_2 = \begin{vmatrix} 2 & 10 & -1 \\ -1 & 0 & -1 \\ -1 & 20 & 2 \end{vmatrix} = 2\times[0-20\times(-1)]-10\times[-1\times2-(-1)^2]+(-1)\times[-1\times20-0]$$

$$= 40 + 30 + 20 = 90$$

$$\therefore \ V_2 = \frac{\Delta'_2}{\Delta'} = \frac{90}{0.75} = 120\,V$$

Power in the 4Ω resistor $= \dfrac{V_2^2}{4} = \dfrac{120^2}{4} = 3600\,W$

EXAMPLE 1.48 ***(AU May'17, 6 Marks)***

In the circuit shown in Fig. 1, write mesh equations by inspection and solve V_x and I_x. Verify the result by node analysis.

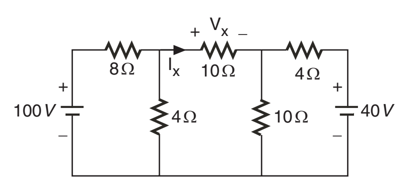


Fig. 1.

SOLUTION

Method 1: Mesh Analysis

The mesh currents and their direction are given in Fig 2. The circuit has three meshes. The general form of mesh basis matrix equation for three mesh circuit is shown in equation (1).

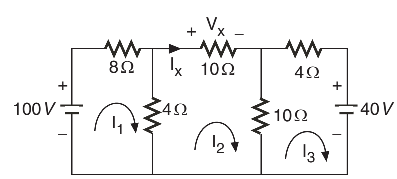


Fig. 2.

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{bmatrix} \quad \text{..... (1)}$$

The elements of the resistance matrix and source voltage matrix are formed as shown below:

$R_{11} = 8 + 4 = 12$	$R_{12} = R_{21} = -4$	$E_{11} = 100$
$R_{22} = 4 + 10 + 10 = 24$	$R_{13} = R_{31} = 0$	$E_{22} = 0$
$R_{33} = 10 + 4 = 14$	$R_{23} = R_{32} = -10$	$E_{33} = -40$

On substituting the above terms in equation (2), we get mesh equation,

$$\begin{bmatrix} 12 & -4 & 0 \\ -4 & 24 & -10 \\ 0 & -10 & 14 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ -40 \end{bmatrix} \qquad .....(2)$$

Here, $I_x = I_2$ and $V_x = 10I_2$

In order to solve I_2, two determinants Δ and Δ_2 are defined as shown below and evaluated by expanding along the first row and I_2 is solved by Cramer's rule.

$$\Delta = \begin{vmatrix} 12 & -4 & 0 \\ -4 & 24 & -10 \\ 0 & -10 & 14 \end{vmatrix} = 12 \times [24 \times 14 - (-10)^2] - (-4) \times [-4 \times 14 - 0] + 0 = 2608$$

$$\Delta_2 = \begin{vmatrix} 12 & 100 & 0 \\ -4 & 0 & -10 \\ 0 & -40 & 14 \end{vmatrix} = 12 \times [0 - (-40) \times (-10)] - 100[-4 \times 14 - 0] + 0 = 800$$

$$\therefore\ I_x = I_2 = \frac{\Delta_2}{\Delta} = \frac{800}{2608} = 0.30675\ A$$

$$V_x = 10I_x = 10 \times 0.30675 = 3.0675\ V$$

Method 2: Node Analysis

Let us convert the voltage sources in Fig. 1 into current sources as shown below and the circuit is redrawn with current sources in Fig. 7.

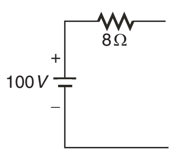


Fig. 3.

⇒

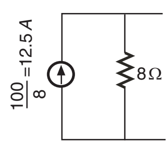


Fig. 4.

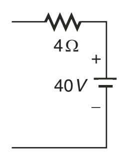


Fig.5.

⇒

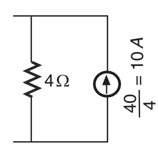


Fig. 6.

The node basis matrix equation is formed by inspection using the circuit shown in Fig. 7. The general node basis matrix equation for a circuit with two nodes excluding the reference is shown in equation (3).

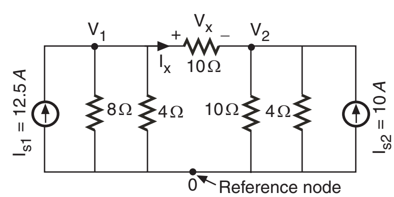


Fig. 7.

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \end{bmatrix} \qquad ..... (3)$$

The elements of conductance matrix and source current matrix are formed as shown below:

$$G_{11} = \frac{1}{8} + \frac{1}{4} + \frac{1}{10} = 0.475 \qquad I_{11} = 12.5$$

$$G_{12} = G_{21} = -\frac{1}{10} = -0.1 \qquad I_{22} = 10$$

$$G_{22} = \frac{1}{10} + \frac{1}{10} + \frac{1}{4} = 0.45$$

On substituting the above terms in equation (3), we get the node equation,

$$\begin{bmatrix} 0.475 & -0.1 \\ -0.1 & 0.45 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 12.5 \\ 10 \end{bmatrix} \qquad ..... (4)$$

In order to solve the node voltages V_1 and V_2, let us define three determinants Δ', Δ'_1, and Δ'_2 as shown below and evaluated by expanding along first row and node voltages are solved by Cramer's rule.

$$\Delta' = \begin{vmatrix} 0.475 & -0.1 \\ -0.1 & 0.45 \end{vmatrix} = 0.475 \times 0.45 - (-0.1)^2 = 0.20375$$

$$\Delta'_1 = \begin{vmatrix} 12.5 & -0.1 \\ 10 & 0.45 \end{vmatrix} = 12.5 \times 0.45 - 10 \times (-0.1) = 6.625$$

$$\Delta'_2 = \begin{vmatrix} 0.475 & 12.5 \\ -0.1 & 10 \end{vmatrix} = 0.475 \times 10 - (-0.1) \times 12.5 = 6$$

$$\therefore\ V_x = V_1 - V_2 = \frac{\Delta'_1}{\Delta} - \frac{\Delta'_2}{\Delta} = \frac{\Delta'_1 - \Delta'_2}{\Delta'} = \frac{6.625 - 6}{0.20375} = 3.0675\,V$$

$$I_x = \frac{V_x}{10} = \frac{3.0675}{10} = 0.30675\,A$$

1.7.2 Node Analysis of Circuits Excited by Both Voltage and Current Sources

Node analysis can be extended to circuits excited by both voltage and current sources. In such circuits if each voltage source has a series impedance then they can be converted into an equivalent current source with parallel impedance. After conversion, the circuit will have only current sources and so the procedure for obtaining node basis matrix equation by inspection and its solution discussed in Sections 1.7.1 and 1.7.4 can be directly applied to these circuits.

In circuits excited by both voltage and current sources, the voltage source may not have series resistance. In this situation the voltage source cannot be converted into a current source. In this case, the value of each voltage source is related to node voltages and for each voltage source one of the node voltages can be expressed in terms of source voltage and other node voltages. The remaining node voltages can be solved by writing Kirchhoff's Current Law equations.

Alternatively, the node basis matrix equation can be formed directly by inspection by taking the current delivered by the voltage sources as unknown and relating the value of each voltage source to node voltages. Here for each voltage source one node voltage is eliminated by expressing the node voltage in terms of the source voltage and other node voltages. While forming the node basis matrix equation, the current of the voltage sources should be entered in the source matrix.

Now in the matrix equation some node voltages will be eliminated and an equal number of unknown source currents will be introduced. Thus, the number of unknowns will remain the same as n where n is number of nodes in the circuit except the reference node. On multiplying the node basis matrix equation we get n equations which can be solved to give a unique solution for unknowns.

1.7.3 Supernode Analysis

In circuits excited by both voltage and current sources, if a voltage source is connected between two nodes then the voltage source can be short-circuited for analysis purpose and the shorted two nodes can be considered as one single node called **supernode**. In order to solve the two node voltages of a supernode, two equations are required. One of the equations is the KCL equation of the supernode and the other equation is obtained by equating the source voltage to the difference of the node voltages. An example of formation of a supernode is shown in Fig. 1.49. Also, Example 1.57 is solved using supernode analysis technique.

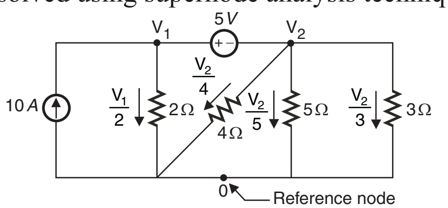
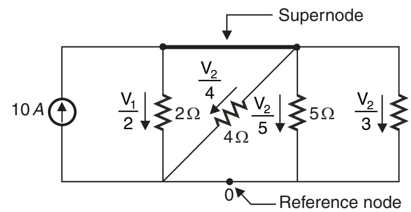


***Fig. a :** Circuit with two independent nodes.*

$\Rightarrow$



$$\frac{V_1}{2} + \frac{V_2}{4} + \frac{V_2}{5} + \frac{V_2}{3} = 10$$

$$V_1 - V_2 = 5$$

***Fig. b :** Supernode of the circuit shown in Fig. a and its equations.*

***Fig. 1.49 :** Example of formation of a supernode.*

EXAMPLE 1.49

In the circuit shown in Fig. 1, find the voltage across the 40 Ω resistor and the power supplied by 5 *A* source, using node analysis.

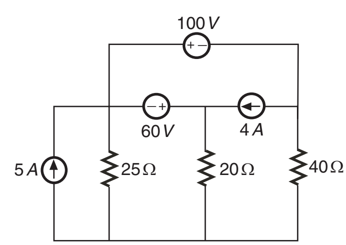


Fig. 1.

SOLUTION

The given circuit has four nodes. In this one of the nodes is chosen as reference. Let the voltages of the other three nodes be V_1, V_2 and V_3 as shown in Fig. 2. Here, the voltage sources do not have a resistance in series and so they cannot be converted into a current source. Let I_{s1} and I_{s2} be the currents supplied by 100 *V* and 60 *V* sources, respectively.

With reference to Fig. 2, we can write,

$$V_1 - V_3 = 100\,V \quad \text{and} \quad V_2 - V_1 = 60\,V$$

From the above equations we can say that node voltages V_2 and V_3 can be expressed in terms of V_1. Now the number of unknowns in the circuit are three and they are V_1, I_{s1} and I_{s2}. Therefore, we can write three node equations using KCL (corresponding to three nodes) and a unique solution for unknowns can be obtained by solving the three equations.

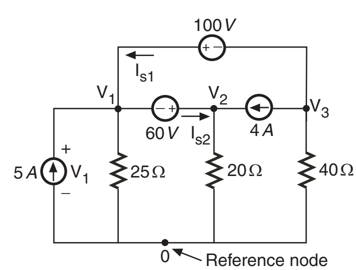


Fig. 2.

The node basis matrix equation is formed by inspection using the circuit shown in Fig. 2.

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \quad .....(1)$$

$$G_{11} = \frac{1}{25} = 0.04 \qquad G_{12} = G_{21} = 0 \qquad I_{11} = 5 + I_{s1} - I_{s2}$$

$$G_{22} = \frac{1}{20} = 0.05 \qquad G_{13} = G_{31} = 0 \qquad I_{22} = 4 + I_{s2}$$

$$G_{33} = \frac{1}{40} = 0.025 \qquad G_{23} = G_{32} = 0 \qquad I_{33} = -4 - I_{s1}$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.025 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 5 + I_{s1} - I_{s2} \\ 4 + I_{s2} \\ -4 - I_{s1} \end{bmatrix} \quad .....(2)$$

With reference to Fig. 2, the following relations can be obtained between node voltages:

$$V_2 - V_1 = 60 \qquad\qquad V_1 - V_3 = 100$$

$$\therefore \; V_2 = 60 + V_1 \quad ..... (3) \qquad\qquad \therefore \; V_3 = V_1 - 100 \quad ..... (4)$$

Using equations (3) and (4), equation (2) can be written as shown in equation (5).

$$\begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.025 \end{bmatrix} \begin{bmatrix} V_1 \\ 60 + V_1 \\ V_1 - 100 \end{bmatrix} = \begin{bmatrix} 5 + I_{s1} - I_{s2} \\ 4 + I_{s2} \\ -4 - I_{s1} \end{bmatrix} \quad .....(5)$$

On multiplying the matrices on the left-hand side of equation (5) and equating to the terms on the right-hand side we get the following three equations:

$$0.04\,V_1 = 5 + I_{s1} - I_{s2} \quad .....(6)$$

$$0.05\,(60 + V_1) = 4 + I_{s2} \quad .....(7)$$

$$0.025\,(V_1 - 100) = -4 - I_{s1} \quad .....(8)$$

On adding the above three equations, we get,

$$0.04\,V_1 + 0.05(60 + V_1) + 0.025\,(V_1 - 100) = 5$$

$$0.04\,V_1 + 3 + 0.05V_1 + 0.025V_1 - 2.5 = 5$$

$$0.115V_1 + 0.5 = 5$$

$$V_1 = \frac{5 - 0.5}{0.115} = 39.13\,V$$

$$\therefore \; V_1 = 39.13\,V$$

From equation (3), $V_2 = 60 + V_1 = 60 + 39.13 = 99.13\,V$

From equation (4), $V_3 = V_1 - 100 = 39.13 - 100 = -60.87\,V$

With reference to Fig. 2, the voltage across the $40\,\Omega$ resistor is V_3.

$\therefore$ Voltage across the $40\,\Omega$ resistor = $V_3 = -60.87\,V$

The negative voltage across the $40\,\Omega$ resistor indicates that the current through the $40\,\Omega$ is flowing towards the node. (Remember that while forming node equations it is assumed that the currents through resistances are leaving the node.)

If we are interested in positive voltage across the $40\,\Omega$ resistor then the polarity of voltage across the $40\,\Omega$ resistor is assumed as shown in Fig. 3.

Now, $V_{40} = -V_3 = -(-60.87)$

$= 60.87\,V$

With reference to Fig. 2, we can say that the voltage across $5\,A$ current source is V_1.

$\therefore$ Power supplied by $5\,A$ source = $V_1 \times 5 = 39.13 \times 5 = 195.65\,W$

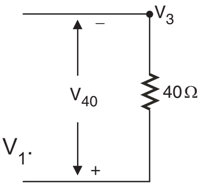


Fig. 3.

EXAMPLE 1.50

In the circuit shown in Fig. 1, find the value of E using node analysis which will make the voltage across $10\,\Omega$ resistance as zero.

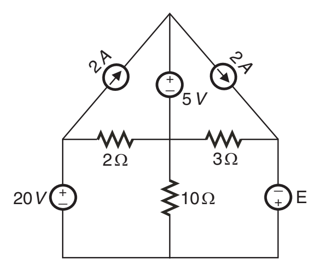


Fig. 1.

SOLUTION

The given circuit has five nodes. In this one of the nodes is chosen as reference. Let the voltages of other four nodes be V_1, V_2, V_3 and V_4 as shown in Fig. 2. Here, the voltage sources do not have series resistances and so they cannot be converted into a current source. Let us treat the currents supplied by the voltage sources as unknown quantities, and the values of voltage sources can be related to node voltages. Let I_{s1}, I_{s4} and I_{s3} be the currents supplied by the $20\,V$, $5\,V$ and E volt sources, respectively.

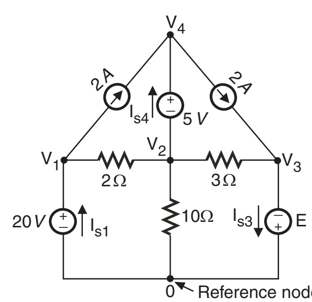


Fig. 2.

The node basis matrix equation of the given circuit is formed by inspection using the circuit shown in Fig. 2.

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \\ I_{44} \end{bmatrix} \qquad .....(1)$$

$$G_{11} = \frac{1}{2}$$
$$G_{22} = \frac{1}{2} + \frac{1}{10} + \frac{1}{3} = \frac{15+3+10}{30} = \frac{28}{30} = \frac{14}{15}$$
$$G_{33} = \frac{1}{3}$$
$$G_{44} = 0$$

$$G_{12} = G_{21} = -\frac{1}{2}$$
$$G_{13} = G_{31} = 0$$
$$G_{14} = G_{41} = 0$$
$$G_{23} = G_{32} = -\frac{1}{3}$$
$$G_{24} = G_{42} = 0$$
$$G_{34} = G_{43} = 0$$

$$I_{11} = I_{s1} - 2$$
$$I_{22} = -I_{s4}$$
$$I_{33} = -I_{s3} + 2$$
$$I_{44} = 2 + I_{s4} - 2 = I_{s4}$$

$$V_1 = 20$$
$$V_2 = 0 \text{ (given)}$$
$$V_4 - V_2 = 5 \quad \therefore V_4 = 5$$
$$V_3 = -E$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{14}{15} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 20 \\ 0 \\ -E \\ 5 \end{bmatrix} = \begin{bmatrix} I_{s1} - 2 \\ -I_{s4} \\ -I_{s3} + 2 \\ I_{s4} \end{bmatrix} \qquad .....(2)$$

From row-4 of equation (2), we get, $I_{s4} = 0$ ..... (3)

From row-2 of equation (2), we get,

$$\left(-\frac{1}{2} \times 20\right) + \left(\frac{14}{15} \times 0\right) + \left(-\frac{1}{3} \times (-E)\right) + (0 \times 5) = -I_{s4}$$

$$-10 + 0 + \frac{E}{3} + 0 = -I_{s4}$$

$$\therefore \frac{E}{3} = -I_{s4} + 10 \qquad .....(4)$$

From equation (3), we know that, $I_{s4} = 0$,

$$\therefore \frac{E}{3} = 0 + 10$$
$$E = 10 \times 3 = 30\,V$$

EXAMPLE 1.51

In the circuit shown in Fig. 1, determine the current delivered by 24 *V* source using node analysis.

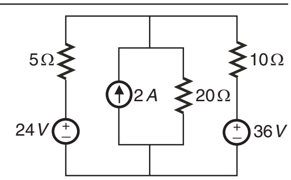


Fig. 1.

SOLUTION

The given circuit has only two principal nodes. Let us choose one of the principal nodes as reference and the voltage of the other principal node as V_1, as shown in Fig. 2. Let us take the voltage at the meeting point of 5 Ω and 24 *V* source as V_2 and the voltage at the meeting point of 10 Ω and 36 *V* as V_3. With reference to Fig. 2, we can say that,

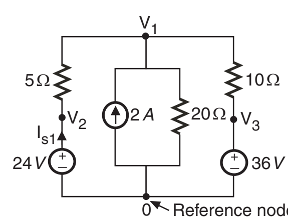


Fig. 2.

$V_2 = 24\,V$ and $V_3 = 36\,V$

With reference to Fig. 3, the node equation for node-1 can be written as shown below:

Currents leaving node-1 : $\frac{V_1 - V_2}{5}, \frac{V_1 - V_3}{10}, \frac{V_1}{20}$

Current entering node-1 : $2\,A$

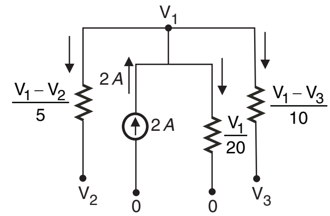


Fig. 3.

$$\therefore \quad \frac{V_1 - V_2}{5} + \frac{V_1 - V_3}{10} + \frac{V_1}{20} = 2$$

$$\frac{V_1}{5} - \frac{V_2}{5} + \frac{V_1}{10} - \frac{V_3}{10} + \frac{V_1}{20} = 2$$

$$\therefore \quad \left(\frac{1}{5} + \frac{1}{10} + \frac{1}{20}\right) V_1 = 2 + \frac{V_2}{5} + \frac{V_3}{10}$$

Put, $V_2 = 24$ and $V_3 = 36$

$$\therefore \quad \left(\frac{1}{5} + \frac{1}{10} + \frac{1}{20}\right) V_1 = 2 + \frac{24}{5} + \frac{36}{10}$$

$$(0.2 + 0.1 + 0.05)V_1 = 2 + 4.8 + 3.6$$

$$0.35V_1 = 10.4$$

$$\therefore \quad V_1 = \frac{10.4}{0.35} = 29.7143\,V$$

Let, I_{s1} be the current delivered by 24 V source as shown in Fig. 2.

$$I_{s1} = \frac{V_2 - V_1}{5} = \frac{24 - 29.7143}{5} = -1.1429\,A$$

Since the current delivered by 24 V source is negative, we can say that it absorbs power instead of delivering power.

EXAMPLE 1.52

In the circuit shown in Fig. 1, solve the voltages across various elements using node method and determine the power in each element of the circuit.

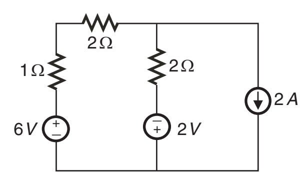


Fig. 1.

SOLUTION

The given circuit has five nodes and in this only two nodes are principal nodes. Let us choose one of the nodes as the reference node, which is indicated by 0. The voltage of the reference node is zero volt. Let us choose three other nodes and assign node voltages V_1, V_2 and V_3 as shown in Fig. 2. Let the current delivered by 2 V and 6 V sources be I_{s1} and I_{s2}, respectively. With reference to Fig. 2, the following relation can be obtained for node voltages:

$$V_1 = 2\,V \quad ; \qquad V_2 - V_1 = 6\,V$$

$$\therefore \quad V_2 = 6 + V_1 = 6 + 2 = 8\,V$$

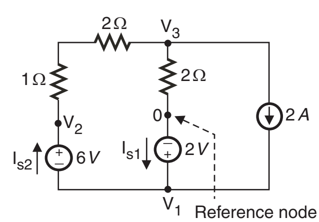


Fig. 2.

In the circuit shown in Fig. 2, the voltage V_1 and V_2 are known quantities, but the currents I_{s1} and I_{s2} are unknown quantities. Hence, the total number of unknowns are three (i.e., V_3, I_{s1} and I_{s2}) and so three node equations can be formed and they can be solved to give a unique solution.

Using Fig. 2, the node basis matrix equation is formed by inspection as shown below:

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \qquad .....(1)$$

$$G_{11} = 0$$
$$G_{22} = \frac{1}{1+2} = \frac{1}{3}$$
$$G_{33} = \frac{1}{1+2}+\frac{1}{2} = \frac{1}{3}+\frac{1}{2} = \frac{5}{6}$$

$$G_{12} = G_{21} = 0$$
$$G_{13} = G_{31} = 0$$
$$G_{23} = G_{32} = \frac{-1}{1+2} = -\frac{1}{3}$$

$$I_{11} = I_{s1}+2-I_{s2}$$
$$I_{22} = I_{s2}$$
$$I_{33} = -2$$

$$V_1 = 2$$
$$V_2 - V_1 = 6$$
$$\therefore V_2 = 6+V_1 = 6+2 = 8$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{s1}+2-I_{s2} \\ I_{s2} \\ -2 \end{bmatrix} \qquad ..... (2)$$

The node equations are obtained by multiplying the matrices on the left-hand side and equating to the terms on right-hand side.

From row-1 we get, $0 = I_{s1} + 2 - I_{s2} \Rightarrow I_{s1} = -2 + I_{s2}$ ..... (3)

From row-2 we get, $\frac{8}{3}-\frac{1}{3}V_3 = I_{s2} \Rightarrow I_{s2} = \frac{8-V_3}{3}$ .....(4)

From row-3 we get, $-\frac{8}{3}+\frac{5}{6}V_3 = -2$ .....(5)

From equation (5), we can write,

$$V_3 = \frac{6}{5}\times\left(-2+\frac{8}{3}\right) = \frac{6}{5}\left(\frac{-6+8}{3}\right) = \frac{12}{15} = 0.8\,V$$

On substituting, $V_3 = 0.8$ in equation (4), we get,

$$I_{s2} = \frac{8-V_3}{3} = \frac{8-0.8}{3} = 2.4\,A$$

On substituting, $I_{s2} = 2.4\,A$ in equation (3), we get,

$$I_{s1} = -2 + I_{s2} = -2 + 2.4 = 0.4\,A$$

In Fig. 2 it can be observed that the current through series combination of 1 Ω and 2 Ω is I_{s2} and the current through the 2 Ω resistance in series with 2 *V* source is I_{s1}. Now, the voltage across the resistances are given by the product of current and resistance.

Let the voltage across the resistances be V_a, V_b and V_c and the voltage across 2 *A* source be E_2 as shown in Fig. 3.

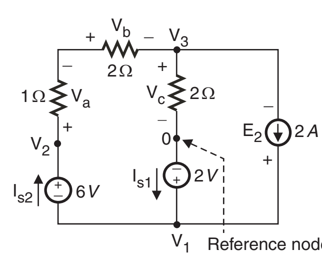


Fig. 3.

Now, $V_a = 1 \times I_{s2} = 1 \times 2.4 = 2.4\,V$

$V_b = 2 \times I_{s2} = 2 \times 2.4 = 4.8\,V$

$V_c = 2 \times I_{s1} = 2 \times 0.4 = 0.8\,V$

$E_2 = V_1 - V_3 = 2 - 0.8 = 1.2\,V$

Estimation of power in each element

In dc circuits, the power in an element is given by the product of voltage and current in that element. The resistances always absorb power. The sources can either deliver power or absorb power. In a source if the current leaves at the positive end of the source then it delivers power.

Power consumed by the 1 Ω resistor $= V_a \times I_{s2} = 2.4 \times 2.4 = 5.76\,W$

Power consumed by the 2 Ω resistor $= V_b \times I_{s2} = 4.8 \times 2.4 = 11.52\,W$

Power consumed by the 2 Ω resistor in series with 2 V source $= V_c \times I_{s1} = 0.8 \times 0.4 = 0.32\,W$

Power delivered by 6 V source $= 6 \times I_{s2} = 6 \times 2.4 = 14.4\,W$

Power delivered by 2 V source $= 2 \times I_{s1} = 2 \times 0.4 = 0.8\,W$

Power delivered by 2 A source $= E_2 \times 2 = 1.2 \times 2 = 2.4\,W$

> ***Note :*** *It is observed that the sum of power delivered (14.4 + 0.8 + 2.4 = 17.6 W) is equal to the sum of power consumed (5.76 + 11.52 + 0.32 = 17.6 W).*

EXAMPLE 1.53

Use nodal analysis to determine the values of voltages at various nodes in the circuit shown in Fig. 1.

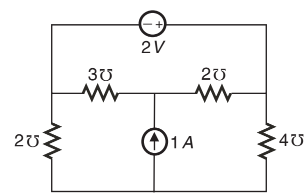


Fig. 1.

SOLUTION

The given circuit has four nodes. In this one of the node is chosen as reference node and it is indicated by 0. The voltage of the reference node is zero. Let the voltages of the other three nodes be V_1, V_2 and V_3 with respect to the reference node, as shown in Fig. 2. The voltage source in the circuit does not have series resistance and so it cannot be converted into a current source. Let I_{s3} be the current supplied by the 2 V source. With reference to Fig. 2, we can write,

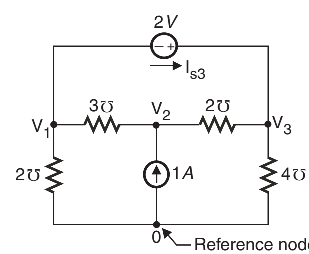


Fig. 2.

$$V_3 - V_1 = 2\,V \quad \Rightarrow \quad V_3 = 2 + V_1$$

From the above equation we can say that the node voltage V_3 can be expressed in terms of V_1. Now, the number of unknowns in the circuit are three and they are V_1, V_2 and I_{s3}. Therefore, we can write three node equations using KCL (corresponding to three nodes) and a unique solution for unknowns can be obtained by solving the three equations.

The node basis matrix equation for the circuit shown in Fig. 2 is obtained by inspection as shown below:

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \qquad .....(1)$$

$$\begin{aligned} G_{11} &= 3+2 = 5 \\ G_{22} &= 3+2 = 5 \\ G_{33} &= 2+4 = 6 \end{aligned} \quad \Bigg| \quad \begin{aligned} G_{12} &= G_{21} = -3 \\ G_{13} &= G_{31} = 0 \\ G_{23} &= G_{32} = -2 \end{aligned} \quad \Bigg| \quad \begin{aligned} I_{11} &= -I_{s3} \\ I_{22} &= 1 \\ I_{33} &= I_{s3} \end{aligned} \quad \Bigg| \quad \begin{aligned} V_3 - V_1 &= 2 \\ \therefore\; V_3 &= 2 + V_1 \end{aligned}$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 5 & -3 & 0 \\ -3 & 5 & -2 \\ 0 & -2 & 6 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ 2+V_1 \end{bmatrix} = \begin{bmatrix} -I_{s3} \\ 1 \\ I_{s3} \end{bmatrix} \qquad .....(2)$$

The node equations of the circuit are obtained by multiplying the matrices on the left-hand side of equation (2) and equating to the terms on the right-hand side.

From row-1, we get,

$$5V_1 - 3V_2 = -I_{s3} \qquad ..... (3)$$

From row-2, we get,

$$-3V_1 + 5V_2 - 2 \times (2 + V_1) = 1 \Rightarrow -3V_1 + 5V_2 - 4 - 2V_1 = 1 \Rightarrow -5V_1 + 5V_2 = 5 \qquad ..... (4)$$

From row-3, we get,

$$-2V_2 + 6 \times (2 + V_1) = I_{s3} \Rightarrow -2V_2 + 12 + 6V_1 = I_{s3} \Rightarrow 6V_1 - 2V_2 = I_{s3} - 12 \qquad ..... (5)$$

On adding equations (3), (4) and (5), we get,

$$5V_1 - 3V_2 - 5V_1 + 5V_2 + 6V_1 - 2V_2 = -I_{s3} + 5 + I_{s3} - 12 \Rightarrow 6V_1 = -7$$

$$\therefore \; V_1 = \frac{-7}{6} = -1.1667\,V$$

From equation (4), we get, $V_2 = \dfrac{5 + 5V_1}{5} = \dfrac{5 + 5 \times (-1.1667)}{5} = -0.1667\,V$

With reference to Fig. 2, we can write,

$$V_3 - V_1 = 2$$

$$\therefore \; V_3 = 2 + V_1 = 2 + (-1.1667) = 0.8333\,V$$

The node voltages are,

$$V_1 = -1.1667\,V; \quad V_2 = -0.1667\,V \quad \text{and} \quad V_3 = 0.8333\,V$$

EXAMPLE 1.54

(AU June'14, 8 Marks)

Use nodal analysis to determine the values of voltages at various nodes in the circuit shown in Fig. 1.

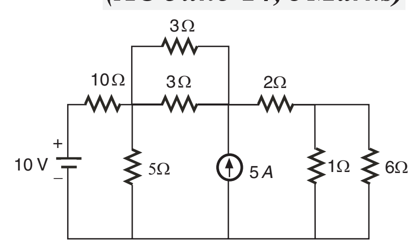


Fig. 1.

SOLUTION

The given circuit has five nodes. In this one of the nodes is chosen as the reference. Let the voltages of the other four nodes be V_1, V_2, V_3 and V_4 as shown in Fig.2. Here, V_4 = 10 *V*. Let us convert the 10 *V* voltage source in series with 10 Ω resistance into equivalent current source as shown in Fig. 3.

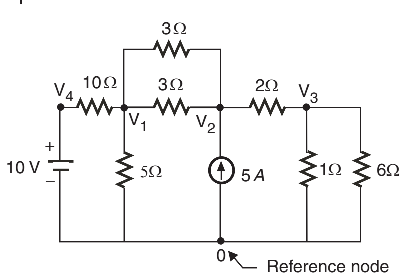


Fig. 2.

⇒

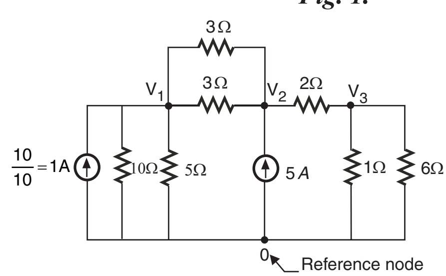


Fig. 3.

The node basis matrix equation is formed by inspection using the circuit shown in Fig. 3.

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \qquad .....(1)$$

With reference to Fig. 3, the elements of conductance matrix and source current matrix are obtained as shown below :

$$G_{11} = \frac{1}{10} + \frac{1}{5} + \frac{1}{3} + \frac{1}{3} = 0.9967 \qquad G_{12} = G_{21} = -\left(\frac{1}{3} + \frac{1}{3}\right) = -0.6667 \qquad I_{11} = 1$$

$$G_{22} = \frac{1}{3} + \frac{1}{3} + \frac{1}{2} = 1.1667 \qquad G_{13} = G_{31} = 0 \qquad I_{22} = 5$$

$$G_{33} = \frac{1}{2} + \frac{1}{1} + \frac{1}{6} = 1.6667 \qquad G_{23} = G_{32} = -\frac{1}{2} = -0.5 \qquad I_{33} = 0$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 0.9667 & -0.6667 & 0 \\ -0.6667 & 1.1667 & -0.5 \\ 0 & -0.5 & 1.6667 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$$

To solve the node voltages V_1, V_2 and V_3, let us define four determinants Δ', Δ'_1, Δ'_2 and Δ'_3 as shown below:

$$\Delta' = \begin{vmatrix} 0.9667 & -0.6667 & 0 \\ -0.6667 & 1.1667 & -0.5 \\ 0 & -0.5 & 1.1667 \end{vmatrix} \qquad \Delta'_1 = \begin{vmatrix} 1 & -0.6667 & 0 \\ 5 & 1.1667 & -0.5 \\ 0 & -0.5 & 1.1667 \end{vmatrix}$$

$$\Delta'_2 = \begin{vmatrix} 0.9667 & 1 & 0 \\ -0.6667 & 5 & -0.5 \\ 0 & 0 & 1.1667 \end{vmatrix} \qquad \Delta'_3 = \begin{vmatrix} 0.9667 & -0.6667 & 1 \\ -0.6667 & 1.1667 & 5 \\ 0 & -0.5 & 0 \end{vmatrix}$$

The determinants are evaluated by expanding along the first row and node voltages are solved by Cramer's rule.

$$\Delta' = \begin{vmatrix} 0.9667 & -0.6667 & 0 \\ -0.6667 & 1.1667 & -0.5 \\ 0 & -0.5 & 1.6667 \end{vmatrix} = 0.9667 \times [1.1667 \times 1.6667 - (-0.5)^2] +$$

$$0.6667 \times [-0.6667 \times 1.6667 - 0]$$

$$= 0.8972$$

$$\Delta'_1 = \begin{vmatrix} 1 & -0.6667 & 0 \\ 5 & 1.1667 & -0.5 \\ 0 & -0.5 & 1.6667 \end{vmatrix} = 1 \times \left[1.1667 \times 1.6667 - (-0.5)^2\right] + 0.6667 \times [5 \times 1.6667 - 0]$$

$$= 7.2505$$

$$\Delta'_2 = \begin{vmatrix} 0.9667 & 1 & 0 \\ -0.6667 & 5 & -0.5 \\ 0 & 0 & 1.6667 \end{vmatrix} = 0.9667 \times [5 \times 1.6667 - 0] - 1 \times [-0.6667 \times 1.6667 - 0]$$

$$= 9.1672$$

$$\Delta'_3 = \begin{vmatrix} 0.9667 & -0.6667 & 1 \\ -0.6667 & 1.1667 & 5 \\ 0 & -0.5 & 0 \end{vmatrix} = 0.9667 \times [0 + 0.5 \times 5] + 0.6667 \times [0 - 0] + 1 \times [0.6667 \times 0.5 - 0]$$

$$= 2.7501$$

$$V_1 = \frac{\Delta_1'}{\Delta} = \frac{7.2505}{0.8972} = 8.0813\,V$$

$$V_2 = \frac{\Delta_2'}{\Delta'} = \frac{9.1672}{0.8972} = 10.2176\,V$$

$$V_3 = \frac{\Delta_3'}{\Delta'} = \frac{2.7501}{0.8972} = 3.0652\,V$$

The node voltages are,

$V_1 = 8.0813\,V$; $V_2 = 10.2176\,V$; $V_3 = 3.0652\,V$ and $V_4 = 10\,V$

EXAMPLE 1.55 ***(AU Dec'14, 12 Marks)***

Determine the node voltages and the currents across all the resistors of the circuit shown in Fig. 1, using node method.

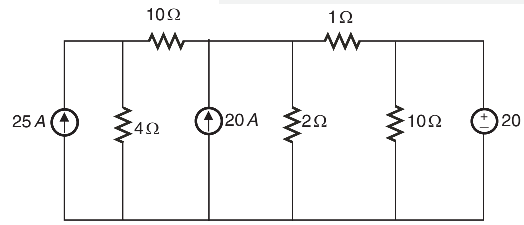


Fig. 1.

SOLUTION

Solution of node voltages

The given circuit has four nodes.In this one of the nodes is chosen as the reference node, which is indicated by 0. The voltage of the reference node is zero volt. Let us choose three other nodes and assign node voltages V_1, V_2 and V_3 as shown in Fig. 2. Let the current delivered by 20 V source be I_s.

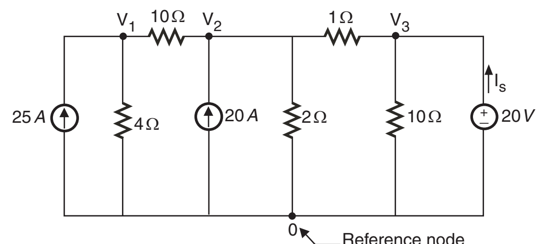


Fig. 2.

With reference to Fig. 2, we get,

$V_3 = 20\,V$

From the above equation we can say that the node voltage V_3 is a known quantity. Now, the number of unknowns in the circuit are three and they are V_1, V_2 and I_s. Therefore, we can write three node equations using KCL (corresponding to three nodes) and a unique solution for unknowns can be obtained by solving the three equations.

The node basis matrix equation for the circuit shown in Fig. 2 is obtained by inspection as shown below:

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \quad .....(1)$$

$$G_{11} = \frac{1}{4} + \frac{1}{10} = 0.35 \qquad G_{12} = G_{21} = -\frac{1}{10} = -0.1$$

$$G_{22} = \frac{1}{10} + \frac{1}{2} + \frac{1}{1} = 1.6 \qquad G_{13} = G_{31} = 0 \qquad V_3 = 10\,V$$

$$G_{33} = \frac{1}{1} + \frac{1}{10} = 1.1 \qquad G_{23} = G_{32} = -\frac{1}{1} = -1$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 0.35 & -0.1 & 0 \\ -0.1 & 1.6 & -1 \\ 0 & -1 & 1.1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ 20 \end{bmatrix} = \begin{bmatrix} 25 \\ 20 \\ I_s \end{bmatrix} \quad .....(2)$$

The node equations of the circuit are obtained by multiplying the matrices on the left-hand side of equation (2) and equating to the terms on the right-hand side.

From row-1, we get,

$$0.35V_1 - 0.1\,V_2 = 25 \qquad .....(3)$$

From row-2, we get,

$$-0.1V_1 + 1.6\,V_2 - 20 = 20 \quad \Rightarrow \quad -0.1\,V_1 + 1.6\,V_2 = 40 \qquad .....(4)$$

On multiplying equation (3) by 16, we get,

$$5.6\,V_1 - 1.6\,V_2 = 400 \qquad .....(5)$$

On adding equations (4) and (5), we get,

$$-0.1\,V_1 + 1.6\,V_2 + 5.6\,V_1 - 1.6\,V_2 = 40 + 400 \quad \Rightarrow \quad 5.5\,V_1 = 440$$

$$\therefore\ V_1 = \frac{440}{5.5} = 80\,V$$

From equation (2), we get, $V_2 = \dfrac{40 + 0.1V_1}{1.6} = \dfrac{40 + 0.1 \times 80}{1.6} = 30\,V$

The node voltages are,

$$V_1 = 80\,V \quad ; \quad V_2 = 30\,V \quad \text{and} \quad V_3 = 20\,V$$

To solve branch voltages and currents

The given circuit has five resistance branches. Let us denote the resistance branch voltages as V_a, V_b, V_c, V_d and V_e and resistance branch currents as I_a, I_b, I_c, I_d and I_e, as shown in Fig. 3. The signs of branch voltages and currents are chosen such that they are all positive. The branch voltages and currents are solved as shown below:

$$V_a = V_1 = 80\,V$$

$$V_b = V_1 - V_2 = 80 - 30 = 50\,V$$

$$V_c = V_2 = 30\,V$$

$$V_d = V_2 - V_3 = 30 - 20 = 10\,V$$

$$V_e = V_3 = 20\,V$$

$$I_a = \frac{V_a}{4} = \frac{80}{4} = 20\,A$$

$$I_b = \frac{V_b}{10} = \frac{50}{10} = 5\,A$$

$$I_c = \frac{V_c}{2} = \frac{30}{2} = 15\,A$$

$$I_d = \frac{V_d}{1} = \frac{10}{1} = 10\,A$$

$$I_e = \frac{V_e}{10} = \frac{20}{10} = 2\,A$$

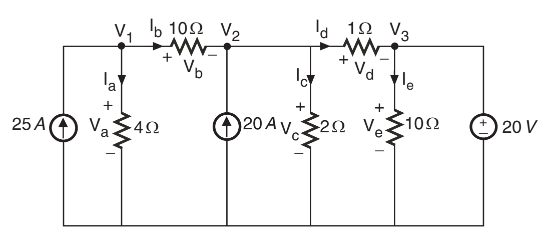


Fig. 3.

EXAMPLE 1.56 *(AU May'15, 16 Marks)*

Use nodal analysis to solve the circuit shown in Fig.1.

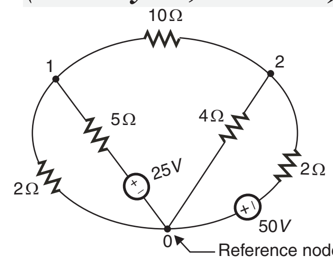


Fig. 1.

SOLUTION

Let the node voltages be V_1 and V_2. Let us convert the 25 *V* voltage source in series with 5 Ω resistance into an equivalent current source as shown in Fig. 2.Similarly convert the 50 *V* voltage source in series with 2 Ω resistance into an equivalent current source as shown in Fig. 2.The node basis matrix equation of the given circuit is formed by inspection using the circuit as shown in Fig. 2.

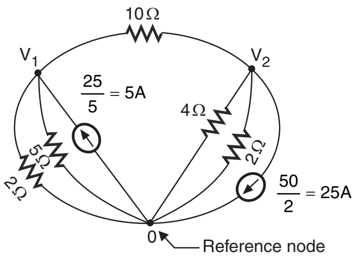


Fig. 2.

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \end{bmatrix} \quad .....(1)$$

With reference to Fig. 2, the elements of conductance matrix and source current matrix are obtained as shown below :

$$G_{11} = \frac{1}{2}+\frac{1}{5}+\frac{1}{10} = 0.8 \qquad G_{12} = G_{21} = -\frac{1}{10} = -0.1 \qquad I_{11} = 5$$

$$G_{22} = \frac{1}{10}+\frac{1}{4}+\frac{1}{2} = 0.85 \qquad I_{22} = -25$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 0.8 & -0.1 \\ -0.1 & 0.85 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -25 \end{bmatrix}$$

In order to solve the node voltages V_1 and V_2, let us define three determinants Δ', Δ'_1 and Δ'_2 as shown below:

$$\Delta' = \begin{vmatrix} 0.8 & -0.1 \\ -0.1 & 0.85 \end{vmatrix}; \quad \Delta'_1 = \begin{vmatrix} 5 & -0.1 \\ -25 & 0.85 \end{vmatrix}; \quad \Delta'_2 = \begin{vmatrix} 0.8 & 5 \\ -0.1 & -25 \end{vmatrix}$$

The determinants are evaluated by expanding along the first row and the node voltages are solved by Cramer's rule.

$$\Delta' = \begin{vmatrix} 0.8 & -0.1 \\ -0.1 & 0.85 \end{vmatrix} = 0.8 \times 0.85 - (-0.1)^2 = 0.67$$

$$\Delta'_1 = \begin{vmatrix} 5 & -0.1 \\ -25 & 0.85 \end{vmatrix} = 5 \times 0.85 - (-25) \times (-0.1) = 1.75$$

$$\Delta'_2 = \begin{vmatrix} 0.8 & 5 \\ -0.1 & -25 \end{vmatrix} = 0.8 \times (-25) - (-0.1) \times 5 = -19.5$$

$$V_1 = \frac{\Delta'_1}{\Delta} = \frac{1.75}{0.67} = 2.6119\,V$$

$$V_2 = \frac{\Delta'_2}{\Delta} = \frac{-19.5}{0.67} = -29.1045\,V$$

EXAMPLE 1.57

Determine the node voltages and, hence, the power supplied by a $5\,A$ source into the circuit shown in Fig. 1, using supernode analysis technique.

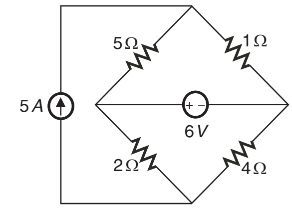


Fig. 1.

SOLUTION

Let us choose the reference node 0 and three node voltages V_1, V_2 and V_3 as shown in Fig. 2. Now, the voltage across $5\,A$ source is V_1 and the power delivered by $5\,A$ source is $V_1 \times 5$ watts

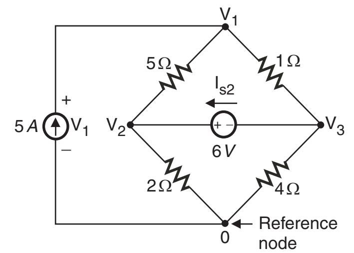


Fig. 2.

With reference to Fig. 2, the relation between node voltages V_2 and V_3 is,

$$V_2 - V_3 = 6 \quad \Rightarrow \quad V_3 = V_2 - 6 \qquad .....(1)$$

Let us short-circuit node-2 and node-3 to form a supernode as shown in Fig. 3.

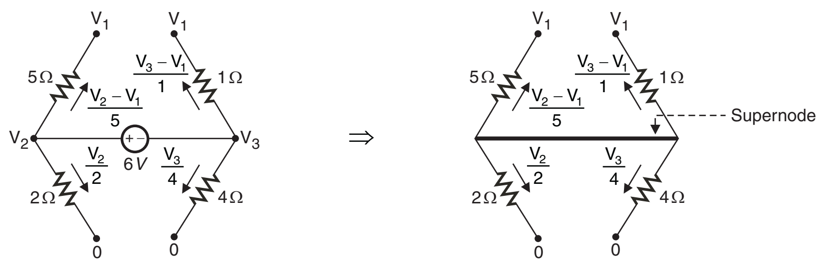


Fig. 3 : *Formation of a supernode.*

The KCL equation of the supernode is formed as shown below:

$$\frac{V_2 - V_1}{5} + \frac{V_3 - V_1}{1} + \frac{V_2}{2} + \frac{V_3}{4} = 0$$

$$0.2V_2 - 0.2V_1 + V_3 - V_1 + 0.5V_2 + 0.25V_3 = 0$$

$$-1.2V_1 + 0.7V_2 + 1.25V_3 = 0$$

$$-1.2V_1 + 0.7V_2 + 1.25\,(V_2 - 6) = 0 \qquad \boxed{\text{Using equation (1)}}$$

$$-1.2V_1 + 1.95V_2 - 7.5 = 0$$

$$\therefore \quad V_2 = \frac{7.5 + 1.2\,V_1}{1.95} = \frac{7.5}{1.95} + \frac{1.2}{1.95}V_1 = 3.8462 + 0.6154\,V_1 \qquad .....(2)$$

With reference to Fig. 4, the KCL equation of node-1 is formed as shown below:

$$\frac{V_1 - V_2}{5} + \frac{V_1 - V_3}{1} = 5$$

$$0.2V_1 - 0.2V_2 + V_1 - V_3 = 5$$

$$1.2V_1 - 0.2V_2 - (V_2 - 6) = 5$$

$$1.2V_1 - 1.2V_2 = 5 - 6 \qquad \boxed{\text{Using equation (1)}}$$

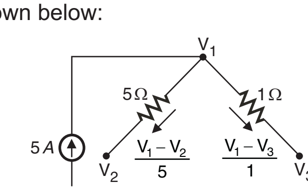


Fig. 4.

$$1.2V_1 - 1.2(3.8462 + 0.6154V_1) = -1 \qquad \boxed{\text{Using equation (2)}}$$

$$1.2V_1 - 4.6154 - 0.7385V_1 = -1$$

$$0.4615V_1 = 3.6154$$

$$\therefore\ V_1 = \frac{3.6154}{0.4615} = 7.834\ V$$

From equation (2), $V_2 = 3.8462 + 0.6154\ V_1 = 3.8462 + 0.6154 \times 7.834 = 8.6672\ V$

From equation (3), $V_3 = V_2 - 6 = 8.6672 - 6 = 2.6672\ V$

Power supplied by $5\,A$ source $= V_1 \times 5 = 7.834 \times 5 = 39.17\ W$

1.7.4 Node Analysis of Circuits Excited by AC Sources (Nodal Analysis of Reactive Circuits)

Reactive circuits consist of resistances and inductive and capacitive reactances. Therefore, the voltages and currents of reactive circuits are complex (i.e., they have both real and imaginary components). In general, the elements of a circuit are referred to as impedances. In node analysis, the admittance (which is the inverse of impedance) is more convenient.

The general node basis matrix equation for reactive circuit is,

$$\mathbf{Y\,V} = \mathbf{I} \qquad \text{..... (1.40)}$$

where, $\mathbf{Y}$ = Admittance matrix of order n × n

$\mathbf{V}$ = Node voltage matrix of order n × 1

$\mathbf{I}$ = Source current matrix of order n × 1

n = Number of nodes except the reference node.

Equation (1.40) can be expanded as shown in equation (1.41).

$$\begin{bmatrix} \overline{Y}_{11} & \overline{Y}_{12} & \overline{Y}_{13} & \cdots & \overline{Y}_{1n} \\ \overline{Y}_{21} & \overline{Y}_{22} & \overline{Y}_{23} & \cdots & \overline{Y}_{2n} \\ \overline{Y}_{31} & \overline{Y}_{32} & \overline{Y}_{33} & \cdots & \overline{Y}_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \overline{Y}_{n1} & \overline{Y}_{n2} & \overline{Y}_{n3} & \cdots & \overline{Y}_{nn} \end{bmatrix} \begin{bmatrix} \overline{V}_1 \\ \overline{V}_2 \\ \overline{V}_3 \\ \vdots \\ \overline{V}_n \end{bmatrix} = \begin{bmatrix} \overline{I}_{11} \\ \overline{I}_{22} \\ \overline{I}_{33} \\ \vdots \\ \overline{I}_{nn} \end{bmatrix} \qquad \text{..... (1.41)}$$

***Note :** The over bar is used to denote complex quantities.*

The formation of node basis matrix equation and the solution of node and branch voltages are similar to that of resistive circuits except that the solution of voltages involves complex arithmetic.

Therefore, the k^{th} node voltage of a reactive circuit with n nodes excluding reference is given by,

$$\overline{V}_k = \frac{1}{\Delta'} \sum_{j=1}^{n} \Delta'_{jk}\ \overline{I}_{jj} \qquad \text{..... (1.42)}$$

***Note :** Refer equation (1.38)*

where, Δ'_{jk} = Cofactor of $\overline{Y}_{jk}$

$\overline{I}_{jj}$ = Sum of current sources connected to j^{th} node

Δ' = Determinant of admittance matrix.

Instead of using equation (1.42) for solution of node voltages, the short-cut procedure for Cramer's rule can be followed.

Consider the node basis matrix equation for a circuit with three nodes except the reference node.

$$\begin{bmatrix} \bar{Y}_{11} & \bar{Y}_{12} & \bar{Y}_{13} \\ \bar{Y}_{21} & \bar{Y}_{22} & \bar{Y}_{23} \\ \bar{Y}_{31} & \bar{Y}_{32} & \bar{Y}_{33} \end{bmatrix} \begin{bmatrix} \bar{V}_1 \\ \bar{V}_2 \\ \bar{V}_3 \end{bmatrix} = \begin{bmatrix} \bar{I}_{11} \\ \bar{I}_{22} \\ \bar{I}_{33} \end{bmatrix}$$

Let us define four determinants as shown below:

$$\Delta' = \begin{vmatrix} \bar{Y}_{11} & \bar{Y}_{12} & \bar{Y}_{13} \\ \bar{Y}_{21} & \bar{Y}_{22} & \bar{Y}_{23} \\ \bar{Y}_{31} & \bar{Y}_{32} & \bar{Y}_{33} \end{vmatrix}; \qquad \Delta'_1 = \begin{vmatrix} \bar{I}_{11} & \bar{Y}_{12} & \bar{Y}_{13} \\ \bar{I}_{22} & \bar{Y}_{22} & \bar{Y}_{23} \\ \bar{I}_{33} & \bar{Y}_{32} & \bar{Y}_{33} \end{vmatrix}$$

$$\Delta'_2 = \begin{vmatrix} \bar{Y}_{11} & \bar{I}_{11} & \bar{Y}_{13} \\ \bar{Y}_{21} & \bar{I}_{22} & \bar{Y}_{23} \\ \bar{Y}_{31} & \bar{I}_{33} & \bar{Y}_{33} \end{vmatrix}; \qquad \Delta'_3 = \begin{vmatrix} \bar{Y}_{11} & \bar{Y}_{12} & \bar{I}_{11} \\ \bar{Y}_{21} & \bar{Y}_{22} & \bar{I}_{22} \\ \bar{Y}_{31} & \bar{Y}_{32} & \bar{I}_{33} \end{vmatrix}$$

Here, Δ' = Determinant of admittance matrix

Δ'_1 = Determinant of admittance matrix after replacing the first column of admittance matrix by source current column matrix

Δ'_2 = Determinant of admittance matrix after replacing the second column of admittance matrix by source current column matrix

Δ'_3 = Determinant of admittance matrix after replacing the third column of admittance matrix by source current column matrix.

Now, node voltages $\bar{V}_1$, $\bar{V}_2$ and $\bar{V}_3$ are given by,

$$\bar{V}_1 = \frac{\Delta'_1}{\Delta'}$$

$$\bar{V}_2 = \frac{\Delta'_2}{\Delta'}$$

$$\bar{V}_3 = \frac{\Delta'_3}{\Delta'}$$

EXAMPLE 1.58

Determine the power consumed by the 10 Ω resistor in the circuit shown in Fig. 1, using nodal analysis.

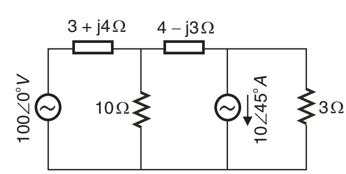
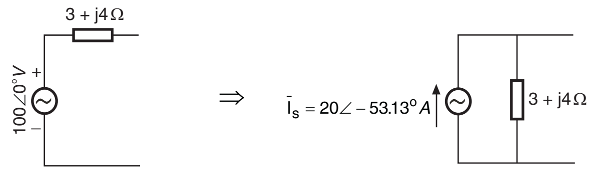


Fig. 1.

SOLUTION

Let us convert the $100\angle 0^\circ\ V$ voltage source in series with 3 + j4 Ω impedance into a current source $\bar{I}_S$ in parallel with 3 + j4 Ω impedance.



$$\bar{I}_S = \frac{100\angle 0^\circ}{3+j4} = \frac{100}{3+j4} = 12 - j16\ A = 20\angle -53.13^\circ\ A$$

The modified circuit is shown in Fig. 2. The circuit of Fig. 2 has three nodes. Let us choose one of the nodes as the reference node, which is denoted as 0. The voltage of the reference node is zero. Let the voltages of the other two nodes be $\bar{V}_1$ and $\bar{V}_2$ with respect to the reference node, as shown in Fig. 2.

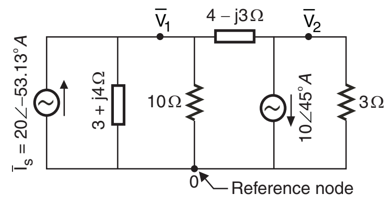


Fig. 2.

The node basis matrix equation of the circuit of Fig. 2 is formed by inspection as shown below:

$$\begin{bmatrix} \bar{Y}_{11} & \bar{Y}_{12} \\ \bar{Y}_{21} & \bar{Y}_{22} \end{bmatrix} \begin{bmatrix} \bar{V}_1 \\ \bar{V}_2 \end{bmatrix} = \begin{bmatrix} \bar{I}_{11} \\ \bar{I}_{22} \end{bmatrix} \qquad ..... (1)$$

$$\bar{Y}_{11} = \frac{1}{3+j4} + \frac{1}{4-j3} + \frac{1}{10} = 0.38 - j0.04$$

$$\bar{Y}_{22} = \frac{1}{4-j3} + \frac{1}{3} = 0.493 + j0.12$$

$$\bar{Y}_{12} = \bar{Y}_{21} = -\left(\frac{1}{4-j3}\right) = -0.16 - j0.12$$

$$\bar{I}_{11} = 20\angle -53.13^\circ = 20\cos(-53.13) + j20\sin(-53.13) = 12 - j16$$

$$\bar{I}_{22} = -[10\angle 45^\circ] = -(10\cos 45^\circ + j10\sin 45^\circ) = -7.071 - j7.071$$

Note : *All calculations are performed using the calculator in complex mode.*

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 0.38 - j0.04 & -0.16 - j0.12 \\ -0.16 - j0.12 & 0.493 + j0.12 \end{bmatrix} \begin{bmatrix} \bar{V}_1 \\ \bar{V}_2 \end{bmatrix} = \begin{bmatrix} 12 - j16 \\ -7.071 - j7.071 \end{bmatrix} \qquad ..... (2)$$

To determine the power consumed by the 10 Ω resistance, it is sufficient if we calculate $\bar{V}_1$ in equation (2). In order to solve $\bar{V}_1$, let us define two determinants Δ' and Δ'_1 as shown below:

$$\Delta' = \begin{vmatrix} 0.38 - j0.04 & -0.16 - j0.12 \\ -0.16 - j0.12 & 0.493 + j0.12 \end{vmatrix}; \qquad \Delta'_1 = \begin{vmatrix} 12 - j16 & -0.16 - j0.12 \\ -7.071 - j7.071 & 0.493 + j0.12 \end{vmatrix}$$

Now, the voltage $\bar{V}_1$ is given by, $\bar{V}_1 = \dfrac{\Delta'_1}{\Delta'}$.

$$\Delta' = \begin{vmatrix} 0.38 - j0.04 & -0.16 - j0.12 \\ -0.16 - j0.12 & 0.493 + j0.12 \end{vmatrix} = [(0.38 - j0.04) \times (0.493 + j0.12)] - [-0.16 - j0.12]^2$$

$$= 0.1809 - j0.0125$$

$$\Delta'_1 = \begin{vmatrix} 12-j16 & -0.16-j0.12 \\ -7.071-j7.071 & 0.493+j0.12 \end{vmatrix} = [(12-j16) \times (0.493 + j0.12)] - [(-7.071-j7.071)\times(-0.16-j0.12)]$$

$$= 7.5532 - j8.4279$$

$$\therefore\ \overline{V}_1 = \frac{\Delta'_1}{\Delta'} = \frac{7.5532-j8.4279}{0.1809-j0.0125} = 44.759-j43.496 = 62.412\angle-44.2^\circ\, V$$

$$\text{Power consumed by the } 10\,\Omega \text{ resistor} = \frac{|\overline{V}_1|^2}{10} = \frac{62.412^2}{10} = 389.5\, W$$

EXAMPLE 1.59

Find the voltages across various elements in the circuit shown in Fig. 1, using node method.

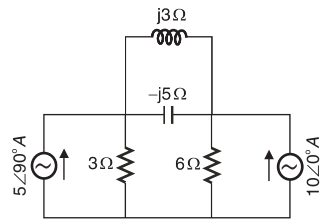


Fig. 1.

SOLUTION

The graph of the given circuit is shown in Fig. 2. It has six branches and three nodes. Hence, the circuit will have six voltages corresponding to six branches. The branch voltages depend on the node voltages. In node analysis, the voltage of one of the nodes is chosen as the reference and it is equal to zero volt. In the circuit of Fig. 3, the reference node is denoted as 0. The voltages of the other two nodes are denoted as $\overline{V}_1$ and $\overline{V}_2$.

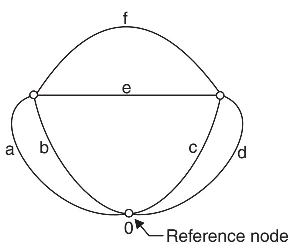


Fig. 2.

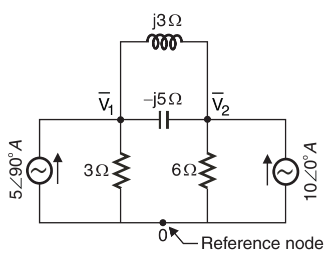


Fig. 3.

The node basis matrix equation of the circuit shown in Fig. 3 is obtained by inspection as shown below:

$$\begin{bmatrix} \overline{Y}_{11} & \overline{Y}_{12} \\ \overline{Y}_{21} & \overline{Y}_{22} \end{bmatrix} \begin{bmatrix} \overline{V}_1 \\ \overline{V}_2 \end{bmatrix} = \begin{bmatrix} \bar{I}_{11} \\ \bar{I}_{22} \end{bmatrix} \quad \text{..... (1)}$$

$$\overline{Y}_{11} = \frac{1}{3} + \frac{1}{j3} + \frac{1}{-j5} = 0.333 - j0.133 \qquad \bar{I}_{11} = 5\angle 90^\circ = j5$$

$$\overline{Y}_{22} = \frac{1}{j3} + \frac{1}{-j5} + \frac{1}{6} = 0.167 - j0.133 \qquad \bar{I}_{22} = 10\angle 0^\circ = 10$$

$$\overline{Y}_{12} = \overline{Y}_{21} = -\left(\frac{1}{j3} + \frac{1}{-j5}\right) = j0.133$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 0.333-j0.133 & j0.133 \\ j0.133 & 0.167-j0.133 \end{bmatrix} \begin{bmatrix} \overline{V}_1 \\ \overline{V}_2 \end{bmatrix} = \begin{bmatrix} j5 \\ 10 \end{bmatrix} \quad \text{.....(2)}$$

To solve the node voltages by Cramer's rule, let us define three determinants Δ', Δ'_1 and Δ'_2 as shown below:

$$\Delta' = \begin{vmatrix} 0.333 - j0.133 & j0.133 \\ j0.133 & 0.167 - j0.133 \end{vmatrix} ; \ \Delta'_1 = \begin{vmatrix} j5 & j0.133 \\ 10 & 0.167 - j0.133 \end{vmatrix} ; \ \Delta'_2 = \begin{vmatrix} 0.333 - j0.133 & j5 \\ j0.133 & 10 \end{vmatrix}$$

Now, the node voltages are given by, $\overline{V}_1 = \dfrac{\Delta'_1}{\Delta}$ and $\overline{V}_2 = \dfrac{\Delta'_2}{\Delta}$.

$$\Delta' = \begin{vmatrix} 0.333 - j0.133 & j0.133 \\ j0.133 & 0.167 - j0.133 \end{vmatrix} = [(0.333 - j0.133) \times (0.167 - j0.133)] - [j0.133]^2$$

$$= 0.0556 - j0.0665$$

$$\Delta'_1 = \begin{vmatrix} j5 & j0.133 \\ 10 & 0.167 - j0.133 \end{vmatrix} = [j5 \times (0.167 - j0.133)] - [10 \times j0.133]$$

$$= 0.665 - j0.495$$

$$\Delta'_2 = \begin{vmatrix} 0.333 - j0.133 & j5 \\ j0.133 & 10 \end{vmatrix} = [(0.333 - j0.133) \times 10] - [j0.133 \times j5]$$

$$= 3.995 - j1.33$$

$$\therefore \ \overline{V}_1 = \frac{\Delta'_1}{\Delta'} = \frac{0.665 - j0.495}{0.0556 - j0.0665} = 9.302 + j2.2227 = 9.564\angle 13.4^\circ \ V$$

$$\overline{V}_2 = \frac{\Delta'_2}{\Delta'} = \frac{3.995 - j1.33}{0.0556 - j0.0665} = 41.3339 + j25.5163 = 48.575\angle 31.7^\circ \ V$$

To find branch voltages

The branch voltages are denoted by $\overline{V}_a, \overline{V}_b, \overline{V}_c, \overline{V}_d, \overline{V}_e$ and $\overline{V}_f$, as shown in Fig. 4. The polarites of branch voltages are chosen arbitrarily. The branch voltages depend on the node voltages. The relation between branch and node voltages are obtained with reference to Fig. 4 as shown below:

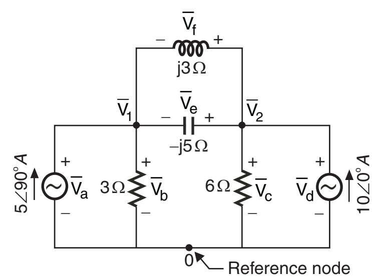


Fig. 4.

$$\overline{V}_a = \overline{V}_b = \overline{V}_1 = 9.564\angle 13.4^\circ \ V$$

$$\overline{V}_c = \overline{V}_d = \overline{V}_2 = 48.575\angle 31.7^\circ \ V$$

$$\overline{V}_e = \overline{V}_f = \overline{V}_2 - \overline{V}_1$$

$$= (41.3339 + j25.5163) - (9.302 + j2.2227)$$

$$= 32.0319 + j23.2936 = 39.606\angle 36^\circ \ V$$

1.7.5 Node Analysis of Circuits Excited by Independent and Dependent Sources

Node analysis can be extended to circuits excited by both dependent and independent sources. When a circuit has a dependent source, the dependent variable should be related to node voltages and then the dependent source should be treated as a source while forming the node basis matrix equation.

If a dependent source depends on a voltage V_x in some part of a circuit then the voltage V_x should be expressed in terms of node voltages. If a dependent source depends on a current I in some part of a circuit then the current I_x should be expressed in terms of node voltages.

Circuits with Dependent Current Source

When a circuit has a dependent current source then express the value of the source in terms of node voltages. While forming the node basis matrix equation, enter the value of the dependent source at the appropriate location in the source matrix on the right-hand side.

Now some of the terms in the source matrix on the right-hand side will be a function of node voltages and so they can be transferred to the left-hand side with the opposite sign. Then the node basis matrix equation can be solved by Cramer's rule. This procedure is explained here with an example.

Consider a circuit with three nodes excluding the reference node and with a dependent current source between node-2 and the reference node. Let the node basis matrix equation without considering the dependent current source be as shown in equation (1.43).

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \qquad .....(1.43)$$

Let the value of the dependent current source when expressed in terms of node voltages be $3\,V_1 - 3V_3$. Let the dependent current source drive the current towards node-2. Hence, the value of the dependent source $3V_1 - 3V_3$ is added as a positive quantity to the element in the second row of the source matrix as shown in equation (1.44).

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} + 3\,V_1 - 3V_3 \\ I_{33} \end{bmatrix} \qquad .....(1.44)$$

From row-2 of equation (1.44), we get,

$$G_{21}V_1 + G_{22}V_2 + G_{23}V_3 = I_{22} + 3\,V_1 - 3V_3$$

$$G_{21}V_1 - 3\,V_1 + G_{22}V_2 + G_{23}V_3 + 3V_3 = I_{22}$$

$$\therefore\ (G_{21} - 3)\,V_1 + G_{22}V_2 + (G_{23} + 3)V_3 = I_{22} \qquad .....\ (1.45)$$

Using equation (1.45), equation (1.44) can be written as shown in equation (1.46).

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} - 3 & G_{22} & G_{23} + 3 \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \qquad .....\ (1.46)$$

In equation (1.44), the terms $3\,V_1$ and $-3V_3$ on the right-hand side are a function of node voltages V_1 and V_3, respectively. In equation (1.46) these two terms are transferred to the left-hand side with the opposite sign. Now equation (1.46) can be solved by Cramer's rule.

Circuits with Dependent Voltage Source

When a circuit has a dependent voltage source then express the value of the source in terms of node voltages. If the dependent voltage source has a series impedance then it can be converted into a dependent current source with parallel impedance and the analysis can be proceeded .

If the dependent voltage source does not have series impedance then it cannot be converted into a current source. In this case the value of the voltage source is related to the node voltages. Then for each voltage source one node voltage is eliminated by expressing the node voltage in terms of the source voltage and other node voltages. The node basis matrix equation can be formed by inspection by taking the current delivered by the dependent voltage source as unknown. While forming the node basis matrix equation, the current of the voltage sources should be entered in the source matrix.

Now in the matrix equation some node voltages will be eliminated and an equal number of unknown source currents will be introduced. Thus, the number of unknowns will remain the same as n, where n is the number of nodes in the circuit except the reference node. On multiplying the node basis matrix equation, we get n number of equations which can be solved to give a unique solution for unknowns and hence the node voltage.

EXAMPLE 1.60

Determine the node voltages of the circuit shown in Fig. 1.

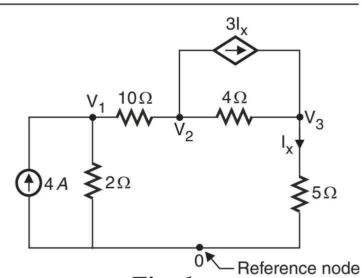


Fig. 1.

SOLUTION

The given circuit has three nodes excluding the reference. The general node basis matrix equation of a circuit with three nodes excluding the reference is shown in equation (1).

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \quad .....(1)$$

With reference to Fig. 1, the elements of conductance matrix and source current matrix are obtained as shown below :

$G_{11} = \frac{1}{2}+\frac{1}{10} = 0.6$	$G_{12} = G_{21} = -\frac{1}{10} = -0.1$	$I_{11} = 4$
$G_{22} = \frac{1}{10}+\frac{1}{4} = 0.35$	$G_{13} = G_{31} = 0$	$I_{22} = -3I_x$
$G_{33} = \frac{1}{4}+\frac{1}{5} = 0.45$	$G_{23} = G_{32} = -\frac{1}{4} = -0.25$	$I_{33} = 3I_x$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 0.6 & -0.1 & 0 \\ -0.1 & 0.35 & -0.25 \\ 0 & -0.25 & 0.45 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3I_x \\ 3I_x \end{bmatrix} \quad ..... (2)$$

Let us express the value of dependent current source in terms of node voltages. With reference to Fig. 1, we can write,

$$I_x = \frac{V_3}{5} \; ; \qquad \therefore \; 3I_x = 3 \times \frac{V_3}{5} = 0.6V_3 \qquad .....(3)$$

On substituting for $3I_x$ from equation (3) in equation (2), we get,

$$\begin{bmatrix} 0.6 & -0.1 & 0 \\ -0.1 & 0.35 & -0.25 \\ 0 & -0.25 & 0.45 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -0.6V_3 \\ 0.6V_3 \end{bmatrix} \qquad .....(4)$$

The terms $-0.6V_3$ and $+0.6V_3$ in the source matrix on the right-hand side of equation (4) can be transferred to the left-hand side with the opposite sign as shown in equation (5).

$$\begin{bmatrix} 0.6 & -0.1 & 0 \\ -0.1 & 0.35 & -0.25+0.6 \\ 0 & -0.25 & 0.45-0.6 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \qquad ..... (5)$$

$$\begin{bmatrix} 0.6 & -0.1 & 0 \\ -0.1 & 0.35 & 0.35 \\ 0 & -0.25 & -0.15 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \qquad ..... (6)$$

To solve the node voltages V_1, V_2 and V_3, let us define four determinants Δ', Δ'_1, Δ'_2 and Δ'_3 as shown below:

$$\Delta' = \begin{vmatrix} 0.6 & -0.1 & 0 \\ -0.1 & 0.35 & 0.35 \\ 0 & -0.25 & -0.15 \end{vmatrix} \qquad \Delta'_1 = \begin{vmatrix} 4 & -0.1 & 0 \\ 0 & 0.35 & 0.35 \\ 0 & -0.25 & -0.15 \end{vmatrix}$$

$$\Delta'_2 = \begin{vmatrix} 0.6 & 4 & 0 \\ -0.1 & 0 & 0.35 \\ 0 & 0 & -0.15 \end{vmatrix} \qquad \Delta'_3 = \begin{vmatrix} 0.6 & -0.1 & 4 \\ -0.1 & 0.35 & 0 \\ 0 & -0.25 & 0 \end{vmatrix}$$

The determinants are evaluated by expanding along the first row and node voltages are solved by Cramer's rule.

$$\Delta' = \begin{vmatrix} 0.6 & -0.1 & 0 \\ -0.1 & 0.35 & 0.35 \\ 0 & -0.25 & -0.15 \end{vmatrix} = 0.6 \times [0.35 \times (-0.15) - (-0.25) \times 0.35]$$

$$- (-0.1) \times [-0.1 \times (-0.15) - 0] + 0$$

$$= 0.021 + 0.0015 = 0.0225$$

$$\Delta'_1 = \begin{vmatrix} 4 & -0.1 & 0 \\ 0 & 0.35 & 0.35 \\ 0 & -0.25 & -0.15 \end{vmatrix} = 4 \times [0.35 \times (-0.15) - (-0.25) \times 0.35] - 0 + 0 = 0.14$$

$$\Delta'_2 = \begin{vmatrix} 0.6 & 4 & 0 \\ -0.1 & 0 & 0.35 \\ 0 & 0 & -0.15 \end{vmatrix} = 0 - 4 \times [-0.1 \times (-0.15) - 0] + 0 = -0.06$$

$$\Delta'_3 = \begin{vmatrix} 0.6 & -0.1 & 4 \\ -0.1 & 0.35 & 0 \\ 0 & -0.25 & 0 \end{vmatrix} = 0 - 0 + 4 \times [-0.1 \times (-0.25) - 0] = 0.1$$

Now, the node voltages are,

$$V_1 = \frac{\Delta'_1}{\Delta'} = \frac{0.14}{0.0225} = 6.2222\,V$$

$$V_2 = \frac{\Delta'_2}{\Delta'} = \frac{-0.06}{0.0225} = -2.6667\,V$$

$$V_3 = \frac{\Delta'_3}{\Delta'} = \frac{0.1}{0.0225} = 4.4444\,V$$

EXAMPLE 1.61

Determine the power delivered to the $10\,\Omega$ resistor in the circuit shown in Fig. 1.

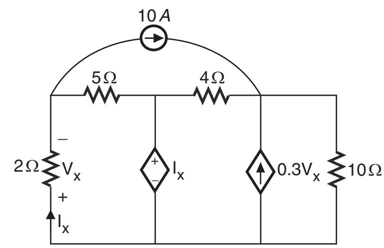


Fig. 1.

SOLUTION

The given circuit has four nodes. Let us choose one of the nodes as reference. Let the voltage of the other three nodes be V_1, V_2 and V_3 as shown in Fig. 1. Let I_{s2} be the current delivered by the dependent voltage source.

Now, power delivered to $10\,\Omega$ resistor $= \dfrac{V_3^2}{10}$

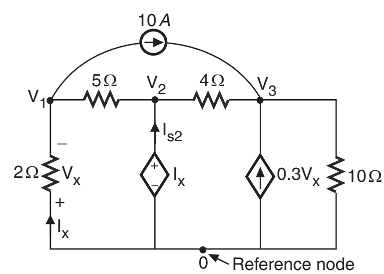


Fig. 2.

The general node basis matrix equation of a circuit with three nodes excluding the reference is given by equation (1).

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \qquad .....(1)$$

With reference to Fig. 1, the elements of conductance matrix and source current matrix can be formed as shown below:

$$G_{11} = \frac{1}{2}+\frac{1}{5} = 0.7 \qquad G_{12} = G_{21} = -\frac{1}{5} = -0.2 \qquad I_{11} = -10$$

$$G_{22} = \frac{1}{5}+\frac{1}{4} = 0.45 \qquad G_{13} = G_{31} = 0 \qquad I_{22} = I_{s2}$$

$$G_{33} = \frac{1}{4}+\frac{1}{10} = 0.35 \qquad G_{23} = G_{32} = -\frac{1}{4} = -0.25 \qquad I_{33} = 10+0.3V_x$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 0.7 & -0.2 & 0 \\ -0.2 & 0.45 & -0.25 \\ 0 & -0.25 & 0.35 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -10 \\ I_{s2} \\ 10+0.3V_x \end{bmatrix} \qquad ..... (2)$$

Let us express the value of dependent sources in terms of node voltages. With reference to Fig. 2, we can write,

$$V_x = -V_1 \quad \Rightarrow \quad 0.3V_x = -0.3V_1 \qquad ..... (3)$$

$$I_x = -\frac{V_1}{2} = -0.5\,V_1 \qquad ..... (4)$$

Also, with reference to Fig. 2, we get,

$$V_2 = I_x \qquad ..... (5)$$

From equations (4) and (5), we can write,

$$V_2 = -0.5V_1 \qquad ..... (6)$$

Using equations (3) and (6), equation (2) can be written as shown in equation (7).

$$\begin{bmatrix} 0.7 & -0.2 & 0 \\ -0.2 & 0.45 & -0.25 \\ 0 & -0.25 & 0.35 \end{bmatrix} \begin{bmatrix} V_1 \\ -0.5V_1 \\ V_3 \end{bmatrix} = \begin{bmatrix} -10 \\ I_{s2} \\ 10-0.3V_1 \end{bmatrix} \qquad ..... (7)$$

From row-1 of equation (5), we get,

$$0.7V_1 - 0.2 \times (-0.5V_1) = -10 \quad \Rightarrow \quad 0.8V_1 = -10 \quad \Rightarrow \quad V_1 = \frac{-10}{0.8} = -12.5\,V$$

From row-3 of equation (5), we get,

$$-0.25 \times (-0.5V_1) + 0.35V_3 = 10 - 0.3V_1$$

$$0.125V_1 + 0.35V_3 = 10 - 0.3V_1$$

$$0.35V_3 = 10 - 0.3V_1 - 0.125V_1$$

$$\therefore \; V_3 = \frac{10 - 0.425V_1}{0.35} = \frac{10 - 0.425 \times (-12.5)}{0.35} = 43.75\,V$$

$$\therefore \; \text{Power delivered to } 10\,\Omega \text{ resistor} = \frac{V_3^2}{10} = \frac{(43.75)^2}{10} = 191.40625\,W$$

EXAMPLE 1.62 *(AU Dec'15, 16 Marks)*

Determine the voltage V_x in the circuit shown in Fig. 1, using node analysis.

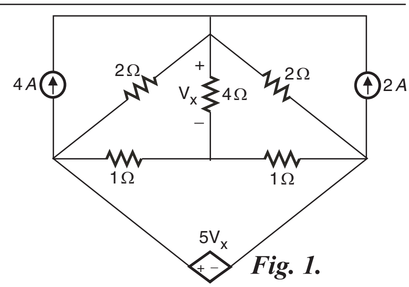


Fig. 1.

SOLUTION

The given circuit has four nodes. Let us choose one of the nodes as reference. Let voltages of the other nodes with respect to the reference be V_1, V_2 and V_3 as shown in Fig. 2. Let I_{s2} be the current delivered by the dependent voltage source. Now, the voltage $V_x = V_1$.

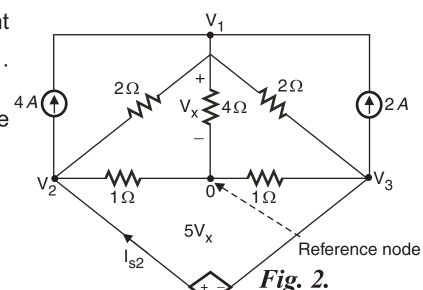


Fig. 2.

The general node basis matrix equation of a circuit with three nodes excluding the reference is given by equation (1).

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \end{bmatrix} \qquad .....(1)$$

With reference to Fig. 2, the elements of conductance matrix and source current matrix can be formed as shown below :

$$G_{11} = \frac{1}{2}+\frac{1}{4}+\frac{1}{2} = 1.25 \qquad G_{12} = G_{21} = -\frac{1}{2} = -0.5 \qquad I_{11} = 4+2 = 6$$

$$G_{22} = \frac{1}{2}+\frac{1}{1} = 1.5 \qquad G_{13} = G_{31} = -\frac{1}{2} = -0.5 \qquad I_{22} = I_{s2}-4$$

$$G_{33} = \frac{1}{2}+\frac{1}{1} = 1.5 \qquad G_{23} = G_{32} = 0 \qquad I_{33} = -I_{s2}-2$$

On substituting the above terms in equation (1), we get,

$$\begin{bmatrix} 1.25 & -0.5 & -0.5 \\ -0.5 & 1.5 & 0 \\ -0.5 & 0 & 1.5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 6 \\ I_{s2}-4 \\ -I_{s2}-2 \end{bmatrix} \quad ..... (2)$$

With reference to Fig. 2, we can write,

$$V_x = V_1 \quad ..... (3)$$

$$V_2 - V_3 = 5V_x \quad ..... (4)$$

From equation (4), we get,

$$V_2 = 5V_x + V_3$$

$$\therefore \quad V_2 = 5V_1 + V_3 \qquad \boxed{\text{Using equation (3)}} \quad ..... (5)$$

On substituting for V_2 from equation (5) in equation (2), we get,

$$\begin{bmatrix} 1.25 & -0.5 & -0.5 \\ -0.5 & 1.5 & 0 \\ -0.5 & 0 & 1.5 \end{bmatrix} \begin{bmatrix} V_1 \\ 5V_1+V_3 \\ V_3 \end{bmatrix} = \begin{bmatrix} 6 \\ I_{s2}-4 \\ -I_{s2}-2 \end{bmatrix} \quad ..... (6)$$

From row-1 of equation (6), we get,

$$1.25V_1 - 0.5(5V_1 + V_3) - 0.5V_3 = 6$$

$$\therefore \quad -1.25V_1 - V_3 = 6 \quad \Rightarrow \quad V_3 = -1.25V_1 - 6 \quad ..... (7)$$

From row-2 of equation (6), we get,

$$-0.5V_1 + 1.5(5V_1 + V_3) = I_{s2} - 4$$

$$7V_1 + 1.5V_3 = I_{s2} - 4 \quad ..... (8)$$

From row-3 of equation (6), we get,

$$-0.5V_1 + 1.5V_3 = -I_{s2} - 2 \quad ..... (9)$$

On adding equations (8) and (9), we get,

$$7V_1 + 1.5V_3 - 0.5V_1 + 1.5V_3 = I_{s2} - 4 - I_{s2} - 2$$

$$6.5V_1 + 3V_3 = -6$$

$$6.5V_1 + 3(-1.25V_1 - 6) = -6 \qquad \boxed{\text{Using equation (7)}}$$

$$\therefore \quad 2.75V_1 - 18 = -6 \quad \Rightarrow \quad V_1 = \frac{-6+18}{2.75} = 4.3636\,V$$

Since, $V_x = V_1$, $V_x = 4.3636\,V$

1.8 Summary of Important Concepts

1. An electric circuit consists of parameters - R, L and C and sources connected in a particular combination.
2. A circuit without sources is called a network.
3. The sources in which the current/voltage does not change with time are called direct current sources.
4. The networks excited by dc sources are called dc circuits.
5. The elements which can deliver energy are called active elements.
6. The elements which consume energy either by absorbing or storing are called passive elements.
7. Resistance, inductance and capacitance are called fundamental parameters.
8. The voltage/current of an independent source does not depend on voltage/current in any part of the circuit.
9. The voltage/current of a dependent source depends on the voltage/current in some part of the same circuit.
10. A node is the meeting point of two or more elements.
11. The principal node is the meeting point of more than two elements.
12. The path between any two nodes is called a branch.
13. A path that starts and ends at a same node after travelling through some part of a circuit is called a closed path.
14. The connection of two or more elements in which the same current flows is called series connection.
15. The connection of two or more elements such that same voltage exists across them is called parallel connection.
16. When three elements meet at a node, then they are said to be in star or T-connection.
17. When three elements are connected to form a closed path with a node in between any two elements then they are said to be in delta or Π-connection.
18. An open path or path of infinite resistance between two nodes is called an open circuit.
19. A closed path of zero resistance between two nodes is called a short circuit.
20. When current leaves an element from the positive terminal, it delivers energy.
21. When current enters an element at the positive terminal, it absorbs energy.
22. Network topology is the study of properties of a network which are unaffected when we stretch, twist or distort the size and shape of the network.
23. A graph describes the topological properties of a network and consists of nodes and branches of a network.
24. When arrows are placed on the branches of a graph, it is called an oriented graph.

25. A tree is a subgraph obtained by removing some branches of a graph such that all nodes of the graph are included without a closed path.
26. A tree has N nodes and N – 1 branches.
27. The branches removed to form a tree are called links or chords and the branches of a tree are called twigs.
28. In a graph with B branches and N nodes, the number of links is, $L = B - N + 1$.
29. The cotree is the complement of a tree and it is obtained by connecting the links to the nodes of a graph.
30. The branch currents and voltages are called network variables.
31. The arrow placed on a branch is called reference or orientation.
32. In a branch, a single orientation is used to represent current and voltage direction.
33. When reference is placed on a branch by treating it as load then the reference is called load set reference.
34. In a graph, link currents are independent current variables and twig currents are dependent current variables.
35. In a graph, twig voltages are independent voltage variables and link voltages are dependent voltage variables.
36. A single-loop circuit is one which has only one closed path.
37. A single node pair circuit is one which has only one independent node and a reference node.
38. Ohm's law states that voltage across a conductor is directly proportional to the current through it.
39. **K**irchhoff's **C**urrent **L**aw (KCL) states that the algebraic sum of currents in a node is zero. By KCL, the sum of currents entering a node = Sum of currents leaving a node.
40. **K**irchhoff's **V**oltage **L**aw (KVL) states that the algebraic sum of voltages in a closed path is zero. By KVL, the sum of voltage raise in a closed path = Sum of voltage fall in a closed path.
41. In a source, when voltage is constant and current varies with load then it is called a voltage source.
42. In a source, when current is constant and voltage varies with load then it is called a current source.
43. In an ideal voltage source the source resistance is zero.
44. In an ideal current source the source resistance is infinite.
45. A voltage source with internal resistance R_s can be represented by an ideal voltage source in series with an external resistance of value R_s.
46. A current source with internal resistance R_s can be represented by an ideal current source in parallel with an external resistance of value R_s.

47. A voltage source E in series with resistance R_s can be converted into an equivalent current source I_s ($I_s = E/R_s$) parallel with resistance R_s.

48. A current source I_s in parallel with resistance R_s can be converted into an equivalent voltage source E ($E = I_s R_s$) in series with resistance R_s.

49 When a source voltage depends on a voltage in some part of the same circuit then the source is called **V**oltage **C**ontrolled **V**oltage **S**ource (VCVS).

50. When a source voltage depends on a current in some part of the same circuit then the source is called **C**urrent **C**ontrolled **V**oltage **S**ource (CCVS).

51. When a source current depends on a voltage in some part of a same circuit then the source is called **V**oltage **C**ontrolled **C**urrent **S**ource (VCCS).

52. When a source current depends on a current in some part of a same circuit then the source is called **C**urrent **C**ontrolled **C**urrent **S**ource (CCCS).

53. Power is the rate of work done and energy is the total work done.

54 Commercially, one *kilowatt-hour* of electrical energy is called one *unit*.

55. Resistance is the property of an element by which it opposes the flow of current.

56. The voltage-current relation in a resistance is governed by Ohm's law. If V and I are voltage and current in a resistance is R then, V = IR.

57. Current division rule: If I is the total current through two parallel connected resistances R_1 and R_2 then the currents I_1 and I_2 flowing through R_1 and R_2 are,

$$I_1 = I \times \frac{R_2}{R_1 + R_2} \quad \text{and} \quad I_2 = I \times \frac{R_1}{R_1 + R_2}$$

58. Voltage division rule: If V is the total voltage across two series connected resistances R_1 and R_2 then the voltages V_1 and V_2 across R_1 and R_2 are,

$$V_1 = V \times \frac{R_1}{R_1 + R_2} \quad \text{and} \quad V_2 = V \times \frac{R_2}{R_1 + R_2}$$

59. Mesh is defined as a closed path which does not contain any other loops within it.

60. Mesh analysis is used to solve independent current variables of a circuit.

61. The number of current variables in a circuit is equal to the number of branches.

62. The number of independent currents in a circuit is given by the number of links in the graph of a circuit.

63. The number of links L in a circuit with B branches and N nodes is given by, L = B – N + 1.

64. In mesh analysis, the independent currents are solved by writing KVL equations for various meshes of a circuit.

65. The mesh basis matrix equation for a resistive circuit is,

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} & \cdots & R_{1m} \\ R_{21} & R_{22} & R_{23} & \cdots & R_{2m} \\ R_{31} & R_{32} & R_{33} & \cdots & R_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_{m1} & R_{m2} & R_{m3} & \cdots & R_{mm} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_m \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ \vdots \\ E_{mm} \end{bmatrix}$$

66. The mesh currents are solved by Cramer's rule.

67. The k^{th} mesh current I_k by Cramer's rule is,

$$I_k = \frac{1}{\Delta} \sum_{j=1}^{m} \Delta_{jk} \; E_{jj}$$

where, m = Number of meshes in the circuit.

Δ_{jk} = Cofactor of R_{jk}.

E_{jj} = Sum of voltage sources in mesh-j.

Δ = Determinant of resistance matrix.

68. For a circuit with three meshes, the mesh currents by Cramer's rule are,

$$I_1 = \frac{\Delta_{11}}{\Delta} E_{11} + \frac{\Delta_{21}}{\Delta} E_{22} + \frac{\Delta_{31}}{\Delta} E_{33}$$

$$I_2 = \frac{\Delta_{12}}{\Delta} E_{11} + \frac{\Delta_{22}}{\Delta} E_{22} + \frac{\Delta_{32}}{\Delta} E_{33}$$

$$I_3 = \frac{\Delta_{13}}{\Delta} E_{11} + \frac{\Delta_{23}}{\Delta} E_{22} + \frac{\Delta_{33}}{\Delta} E_{33}$$

69. The mesh currents for a circuit with three meshes using the short-cut procedure for Cramer's rule are,

$$I_1 = \frac{\Delta_1}{\Delta} \; ; \quad I_2 = \frac{\Delta_2}{\Delta} \; ; \quad I_3 = \frac{\Delta_3}{\Delta}$$

where, $\Delta = \begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{vmatrix}$; $\Delta_1 = \begin{vmatrix} E_{11} & R_{12} & R_{13} \\ E_{22} & R_{22} & R_{23} \\ E_{33} & R_{32} & R_{33} \end{vmatrix}$; $\Delta_2 = \begin{vmatrix} R_{11} & E_{11} & R_{13} \\ R_{21} & E_{22} & R_{23} \\ R_{31} & E_{33} & R_{33} \end{vmatrix}$; $\Delta_3 = \begin{vmatrix} R_{11} & R_{12} & E_{11} \\ R_{21} & R_{22} & E_{22} \\ R_{31} & R_{32} & E_{33} \end{vmatrix}$

70. When a current source lies common to two meshes then the common current source can be removed for analysis purpose and the resultant two meshes can be considered as one single mesh called supermesh.

71. Node analysis is used to solve the independent voltage variables of a circuit.

72. The number of voltage variables in a circuit is equal to the number of branches.

73. The number of independent voltages in a circuit is given by the number of twigs (or tree branches) in the graph of a cirucit.

74 The number of twigs (or tree branches) n in a circuit with N nodes is given by, $n = N - 1$.

75. In node analysis, the independent voltages are solved by writing KCL equations for the various nodes of a circuit.

76. In node analysis, one of the nodes is chosen as the reference node and its voltage is considered as zero and all other node voltages are solved with respect to the reference node.

77. The node basis matrix equation for a resistive circuit is,

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} & \cdots & G_{1n} \\ G_{21} & G_{22} & G_{23} & \cdots & G_{2n} \\ G_{31} & G_{32} & G_{33} & \cdots & G_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ G_{n1} & G_{n2} & G_{n3} & \cdots & G_{nn} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \\ \vdots \\ I_{nn} \end{bmatrix}$$

78. The node voltages are solved by Cramer's rule.

79. The k^{th} node voltage V_k by Cramer's rule is,

$$V_k = \frac{1}{\Delta'} \sum_{j=1}^{n} \Delta'_{jk}\, I_{jj}$$

where, n = Number of independent nodes in a circuit

Δ'_{jk} = Cofactor of G_{jk}

I_{jj} = Sum of current sources connected to node-j

Δ' = Determinant of conductance matrix.

80. For circuit with three nodes excluding the reference node, the node voltages by Cramer's rule are,

$$V_1 = \frac{\Delta'_{11}}{\Delta'} I_{11} + \frac{\Delta'_{21}}{\Delta'} I_{22} + \frac{\Delta'_{31}}{\Delta'} I_{33}$$

$$V_2 = \frac{\Delta'_{12}}{\Delta'} I_{11} + \frac{\Delta'_{22}}{\Delta'} I_{22} + \frac{\Delta'_{32}}{\Delta'} I_{33}$$

$$V_3 = \frac{\Delta'_{13}}{\Delta'} I_{11} + \frac{\Delta'_{23}}{\Delta'} I_{22} + \frac{\Delta'_{33}}{\Delta'} I_{33}$$

81. The node voltages of a circuit with three nodes excluding the reference node using the short-cut procedure for Cramer's rule are,

$$V_1 = \frac{\Delta'_1}{\Delta'}$$

$$V_2 = \frac{\Delta'_2}{\Delta'}$$

$$V_3 = \frac{\Delta'_3}{\Delta'}$$

where, $\Delta' = \begin{vmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{vmatrix}$; $\Delta'_1 = \begin{vmatrix} I_{11} & G_{12} & G_{13} \\ I_{22} & G_{22} & G_{23} \\ I_{33} & G_{32} & G_{33} \end{vmatrix}$;

$\Delta'_2 = \begin{vmatrix} G_{11} & I_{11} & G_{13} \\ G_{21} & I_{22} & G_{23} \\ G_{31} & I_{33} & G_{33} \end{vmatrix}$; $\Delta'_3 = \begin{vmatrix} G_{11} & G_{12} & I_{11} \\ G_{21} & G_{22} & I_{22} \\ G_{31} & G_{32} & I_{33} \end{vmatrix}$

82. When a voltage source is connected between two nodes it can be short-circuited for analysis purpose and the short-circuited two nodes can be considered as one single node called supernode.

1.9 Short-answer Questions

Q1.1 What is the difference between a circuit and a network?

A network will not have any independent sources whereas a circuit will have independent sources. When independent sources are connected to a network it becomes a circuit.

Q1.2 Define active and passive elements.

The elements which can deliver energy are called active elements. The elements which consume energy either by absorbing or storing are called passive elements.

Q1.3 List the active and passive elements of an electric circuit.

The active elements of electric circuits are voltage source and current source. (Both independent and dependent source.) The passive elements of electric circuits are resistor, inductor and capacitor.

Q1.4 Define the dependent source of a circuit.

If the electrical energy supplied by a source depends on the voltage or current in some other part of the same circuit then it is called a dependent source.

Q1.5 What is node and principal node?

In a circuit the meeting point of two or more elements is called a node. If more than two elements meet at a node then the meeting point is called the principal node.

Q1.6 Define the branch of a circuit.

The path between any two nodes in a circuit is called a branch.

Q1.7 Define series and parallel connection.

If two or more elements are connected such that the current through them is the same then the connection is called a series connection.

If two or more elements are connected such that the voltage across them is the same then the connection is called a parallel connection.

Q1.8 What is meant by open circuit and short circuit?

A path of infinite resistance between any two nodes is called an open circuit. The current through an open circuit is zero.

A path of zero resistance between any two nodes is called a short circuit. The voltage across a short circuit is zero.

Q1.9 Define Ohm's law.

Ohm's law states that the potential difference (or voltage) across any two ends of a conductor is directly proportional to the current flowing between the two ends provided the temperature of the conductor remains constant.

Q1.10 What are the limitations of Ohm's law? ***(AU Dec'16, 2 Marks)***

(i) Ohm's law cannot be applied for non-linear elements.

(ii) Ohm's law cannot be applied to a unilateral network which has diodes and transistors.

Q1.11 Define Kirchhoff's laws.

1. Kirchhoff's current law states that the algebraic sum of currents in a node is zero.
2. Kirchhoff's voltage law states that the algebraic sum of voltages in a closed path is zero.

Q1.12 Draw the characteristics of an ideal and a practical voltage source.

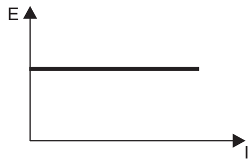


Fig. Q1.12.1 : *Characteristic of an ideal voltage source.*

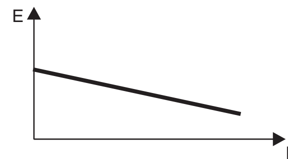


Fig. Q1.12.2 : *Characteristic of a practical voltage source.*

Q1.13 Draw the characteristics of an ideal and a practical current source.

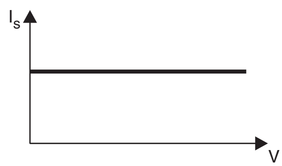


Fig. Q1.13.1 : *Characteristic of an ideal current source.*

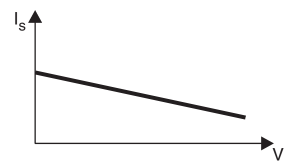


Fig. Q1.13.2 : *Characteristic of a practical current source.*

Q1.14 A 10 A current source has a source resistance of 100 Ω. What will be the equivalent voltage source?

Solution

The current source can be converted into an equivalent voltage source as shown in Fig. Q1.14.1 below:

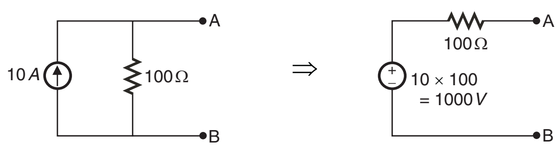


Fig. Q1.14.1 : *Current source to voltage source conversion.*

Q1.15 Convert the voltage source shown in Fig. Q1.15.1 into current source.

Solution

Given that, $E = 20\,V$, $R_s = 5\Omega$

Now, $I_s = \dfrac{E}{R_s} = \dfrac{20}{5} = 4\,A$

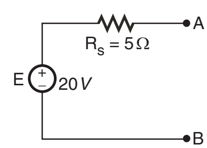


Fig. Q1.15.1.

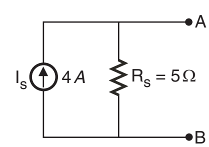


Fig. Q1.15.2.

The voltage source of Fig. Q1.15.1 can be represented by an equivalent current source of value 4 *A* with a source resistance of 5 Ω in parallel, as shown in Fig. Q1.15.2.

Q1.16 A 2 kW, 220 V water heater is used to heat a water tank for 45 minutes. What will be the number of units of energy consumed?

Solution

$$\text{Energy consumed} = \text{Power} \times \text{Time}$$

$$= 2\,kW \times \frac{45}{60}\text{ Hours} = 1.5\,kWh = 1.5\ units$$

Q1.17 ***An electrical appliance consumes 1.2 kWh in 30 minutes at 120 V. What is the current drawn by the appliance?*** **(AU Dec'14, 2 Marks)**

Solution

Energy consumed in 30 minutes = 1.2 *kWh*

Energy consumed in 1 hour (60 minutes) = $1.2 \times \frac{60}{30} = 2.4\ kWh$

$\therefore$ Power rating of the device, P = 2.4 *kW* = 2400 *W*

$\therefore$ Current, $I = \frac{P}{V} = \frac{2400}{120} = 20\ A$

Q1.18 ***What is a graph?***

A graph is a topological description of a network and consists of nodes and branches.

Q1.19 ***Define tree, link and cotree.***

Tree:

A tree is a subgraph and can be defined as a connected open set of branches which includes all nodes of a given graph.

Link:

The branches removed from the graph of a network to form a tree are called links.

Cotree:

The cotree is the complement of a tree.

Q1.20 ***What are network variables?***

Branch currents and voltages are called network variables. Branch currents are called current variables and branch voltages are called voltage variables.

Q1.21 ***Determine the value of current I in the circuit shown in Fig. Q1.21.***

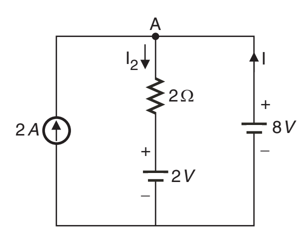


Fig. Q1.21.

Solution

The voltage at node-A = 8 *V*

Current through 2 Ω resistance, $I_2 = \frac{8-2}{2} = 3\ A$

By KCL, $2 + I = I_2 \Rightarrow I = I_2 - 2 = 3 - 2 = 1\ A$

Q1.22 ***Obtain the current in each branch of the network shown in Fig. Q1.22.1. using Kirchhoff's Current Law.***

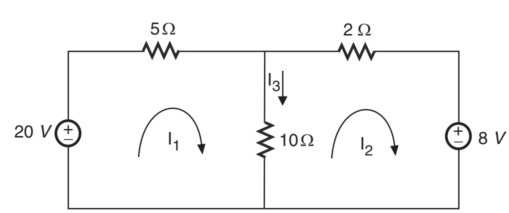


Fig. Q1.22.1.

Solution **(AU May'15, 2 Marks)**

Let V_A be the voltage at node-A.

By, KCL at node-A we get,

$$I_1 - I_2 - I_3 = 0 \quad \Rightarrow \quad \frac{20 - V_A}{5} - \frac{V_A - 8}{2} - \frac{V_A}{10} = 0$$

$$\therefore \frac{20}{5} - \frac{V_A}{5} - \frac{V_A}{2} + \frac{8}{2} - \frac{V_A}{10} = 0 \quad \Rightarrow \quad 4 - 0.2\,V_A - 0.5V_A + 4 - 0.1V_A = 0$$

$$8 - 0.8V_A = 0 \quad \Rightarrow \quad 0.8V_A = 8$$

$$\therefore V_A = \frac{8}{0.8} = 10\,V$$

$$I_1 = \frac{20 - V_A}{5} = \frac{20 - 10}{5} = 2\,A$$

$$I_2 = \frac{V_A - 8}{2} = \frac{10 - 8}{2} = 1\,A$$

$$I_3 = \frac{V_A}{10} = \frac{10}{10} = 1\,A$$

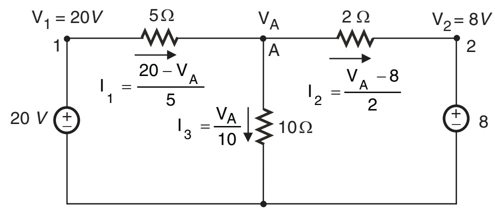


Fig. Q1.22.2.

Q1.23 ***Find the voltage across AB in the circuit shown in Fig. Q1.23.1. (AU June'16 & May'17, 8 Marks)***

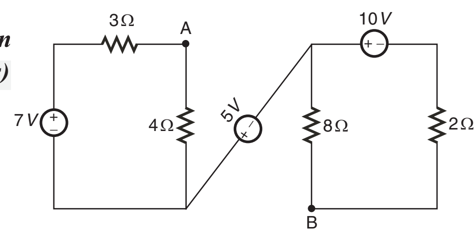


Fig. Q1.23.1.

Solution

With reference to Fig. Q1.23.2, by voltage division rule,

Voltage across 4 Ω resistor, $V_2 = \dfrac{7 \times 4}{4 + 3} = 4\,V$

Voltage across 8 Ω resistor, $V_3 = 10 \times \dfrac{8}{8 + 2} = 8\,V$

On tracing from A to B, as shown in Fig. Q1.23.2.

$$V_{AB} = V_2 + 5 + V_3 = 4 + 5 + 8 = 17\,V$$

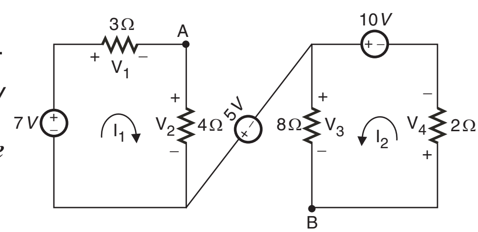


Fig. Q1.23.2.

Q1.24 ***Determine the currents I_1, I_2, I_3 and I_4 in the circuit shown in Fig. Q1.24.***

Solution

By KCL at node-A,

$$I_1 + 2 + 5 = 0 \quad \Rightarrow \quad I_1 = -7\,A$$

At node-B,

$$I_2 = 2 + 4 = 6\,A$$

At node-C,

$$4 + I_4 + 1 = 0 \quad \Rightarrow \quad I_4 = -5\,A$$

At node-D,

$$I_3 = 5 + 1 = 6\,A$$

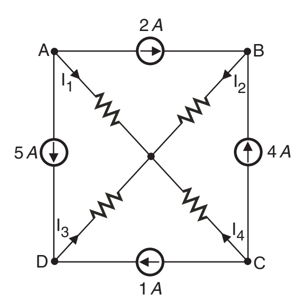


Fig. Q1.24.

Q1.25 ***Find the current I and voltage across 30 Ω resistor in the circuit shown in Fig. Q1.25.1.*** ***(AU June'14, 8 Marks)***

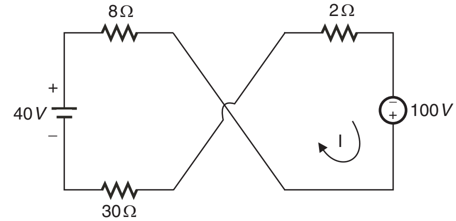


Fig. Q1.25.1.

Solution

The given circuit is redrawn as shown in Fig. Q1.25.2.

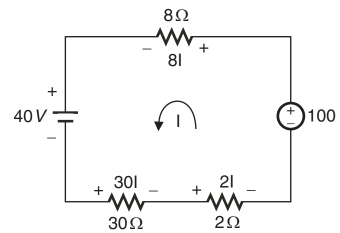


Fig. Q1.25.2.

Now, by KVL,

$$40 + 8\,I + 2\,I + 30\,I = 100$$

$$8\,I + 2\,I + 30\,I = 100 - 40$$

$$40\,I = 60 \Rightarrow I = \frac{60}{40} = 1.5\,A$$

Voltage across $30\,\Omega$ resistor $= 30I = 30 \times 1.5 = 45\,V$

Q1.26 ***Determine the current through each resistor in the circuit shown in Fig. Q1.26.*** ***(AU June'14, 6 Marks)***

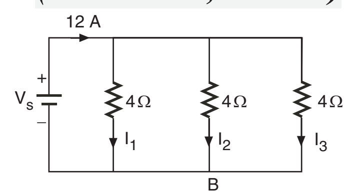


Fig. Q1.26.

Solution

Since three equal resistances are connected in parallel, the current will divide equally in three parallel paths.

$$\therefore\ I_1 = I_2 = I_3 = \frac{12}{3} = 4\,A$$

$$\therefore\ V_s = I_1 \times 4 = 4 \times 4 = 16\,V$$

Q1.27 ***What will be the length of a copper rod having a cross-section of 1 cm² and a resistance of 1Ω ? Take resistivity of copper as 2 × 10⁻⁸ Ω-m.***

Solution

Given that, $R = 1\Omega,\ a = 1\,cm^2 = 1 \times 10^{-4}\,m^2,\ \rho = 2 \times 10^{-8}\,\Omega\text{-}m$

We know that, Resistance, $R = \dfrac{\rho l}{a}$

$$\therefore \text{ Length, } l = \frac{Ra}{\rho}$$

$$= \frac{1 \times 1 \times 10^{-4}}{2 \times 10^{-8}} = 5000\,m = 5\,km$$

Q1.28 ***What is a single loop circuit?***

A single loop circuit is one which has only one closed path.

Q1.29 ***Define single node pair circuit.***

A single node pair circuit is one which has only one independent node and a reference node.

Q1.30 Distinguish between mesh and loop of a circuit. ***(AU June'16, 2 Marks)***

Loop: A loop is any closed path in a circuit in which no node is encountered more than once. A loop may contain other paths inside it.

Mesh : A mesh is a closed path which does not contain any other loops within it.

Consider the circuit shown in Fig. Q1.30.

The closed path ABCDA is a loop. It has two more closed paths ABDA and BCDB inside it.

The closed paths ABDA and BCDB are meshes which have no other closed paths inside them.

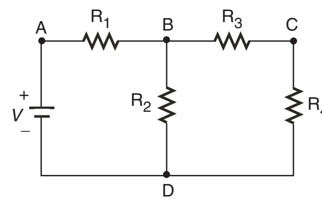


Fig. Q1.30.

Q1.31 What is mesh analysis?

Mesh analysis is a useful technique to solve independent current variables of a circuit.

Q1.32 When is mesh analysis preferred to solve the currents?

Mesh analysis is preferred to solve current variables when a circuit is excited by only voltage sources. Applying mesh analysis is straightforward and easier in circuits excited by only voltage sources. However, mesh analysis can also be extended to circuits excited by both voltage and current sources.

Q1.33 How is mesh analysis performed?

In a circuit with B branches and N nodes the number of independent currents is given by, $m = B - N + 1$. Hence, m number of meshes are selected in the given circuit and one mesh current is attached to each mesh.

For each mesh a KVL equation is formed and then the m number of mesh equations are solved by Cramer's rule to get a unique solution for mesh currents.

Q1.34 How are mesh currents solved using the mesh basis matrix equation?

Consider the mesh basis matrix equation,

$$\mathbf{R\,I = E}$$

On premultiplying the above equation both sides by $\mathbf{R}^{-1}$ we get,

$$\mathbf{R}^{-1}\mathbf{R\,I} = \mathbf{R}^{-1}\mathbf{E}$$

$$\mathbf{U\,I} = \mathbf{R}^{-1}\mathbf{E} \qquad \boxed{\mathbf{R}^{-1}\,\mathbf{R} = \mathbf{U} = \text{Unit matrix}}$$

$$\therefore\ \mathbf{I} = \mathbf{R}^{-1}\mathbf{E} \qquad \boxed{\mathbf{UI = I}}$$

The above equation will be the solution for mesh currents and the k^{th} mesh current is,

$$\boxed{I_k = \frac{\Delta_{1k}}{\Delta}E_{11} + \frac{\Delta_{2k}}{\Delta}E_{22} + \frac{\Delta_{3k}}{\Delta}E_{33} + \ldots\ldots + \frac{\Delta_{mk}}{\Delta}E_{mm} = \frac{1}{\Delta}\sum_{j=1}^{m}\Delta_{jk}E_{jj}}$$

The above equation for mesh currents is also called Cramer's rule.

Q1.35 *What is supermesh?*

When a current source lies common to two meshes then the common current source can be removed for analysis purpose and the resultant two meshes can be considered as one single mesh called a supermesh.

Q1.36 *What is the value of E in the circuit shown in Fig. Q1.36 if the value of I_2 is zero?*

Solution

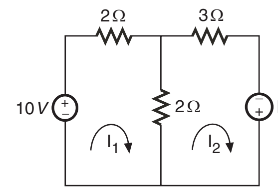


Fig. Q1.36.

The mesh basis matrix equation by inspection is,

$$\begin{bmatrix} 4 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 10 \\ E \end{bmatrix}$$

Here, $I_2 = \frac{\Delta_2}{\Delta}$. Therefore, for $I_2 = 0$, $\Delta_2 = 0$

Now, $\Delta_2 = \begin{vmatrix} 4 & 10 \\ -2 & E \end{vmatrix} = 4E + 20$

$$\therefore\ 4E + 20 = 0 \quad \Rightarrow \quad 4E = -20 \quad \Rightarrow \quad E = \frac{-20}{4} = -5\,V$$

Q1.37 *Find the value of E_1 and E_2 in the circuit shown in Fig. Q1.37.*

Solution

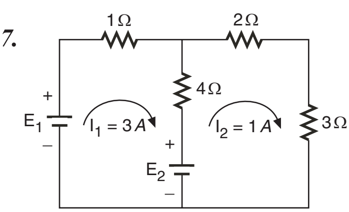


Fig. Q1.37.

The mesh basis matrix equation by inspection is,

$$\begin{bmatrix} 5 & -4 \\ -4 & 9 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} E_1 - E_2 \\ E_2 \end{bmatrix}$$

On substituting for $I_1 = 3A$ and $I_2 = 1A$ in the above equation we get,

$$\begin{bmatrix} 5 & -4 \\ -4 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} E_1 - E_2 \\ E_2 \end{bmatrix}$$

From row-2 we get,

$$-4 \times 3 + 9 \times 1 = E_2 \quad \Rightarrow \quad E_2 = -3\,V$$

From row-1 we get,

$$5 \times 3 - 4 \times 1 = E_1 - E_2 \quad \Rightarrow \quad 11 = E_1 - E_2$$

$$\therefore \quad E_1 = 11 + E_2 = 11 - 3 = 8\,V$$

Q1.38 *Find the value of I_2 and E_2 in the circuit shown in Fig. Q1.38.*

Solution

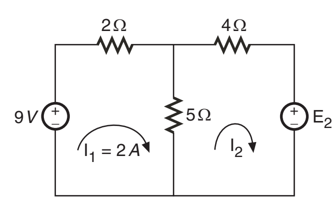


Fig. Q1.38.

The mesh basis matrix equation by inspection is,

$$\begin{bmatrix} 7 & -5 \\ -5 & 9 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 9 \\ -E_2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 7 & -5 \\ -5 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ I_2 \end{bmatrix} = \begin{bmatrix} 9 \\ -E_2 \end{bmatrix}$$

From row-1 we get,

$$7 \times 2 - 5I_2 = 9 \quad \Rightarrow \quad 5I_2 = 14 - 9 \quad \Rightarrow \quad I_2 = \frac{14 - 9}{5} = 1\,A$$

From row-2 we get,

$$-5 \times 2 + 9I_2 = -E_2 \quad \Rightarrow \quad -E_2 = -10 + 9 \times 1 = -1\,V \quad \Rightarrow \quad E_2 = 1V$$

Q1.39 In the circuit shown in Fig Q1.39, find the power delivered to 1Ω resistor.

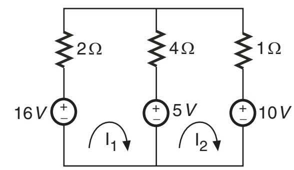


Fig. Q1.39.

Solution

The mesh basis matrix equation is,

$$\begin{bmatrix} 2+4 & -4 \\ -4 & 4+1 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 16-5 \\ 5-10 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 6 & -4 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$$

Now, $\Delta_2 = \begin{vmatrix} 6 & 11 \\ -4 & -5 \end{vmatrix} = 6\times(-5)-(-4)\times 11 = 14$

$$\Delta = \begin{vmatrix} 6 & -4 \\ -4 & 5 \end{vmatrix} = 6\times 5-(-4)^2 = 14$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{14}{14} = 1$$

Power delivered to 1Ω resistor $= I_2^2 \times 1 = 1 \times 1 = 1\,W$

Q1.40 Find the current I in the circuit shown in Fig. Q1.40.

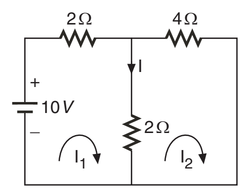


Fig. Q1.40.

Solution

The mesh basis matrix equation is,

$$\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

Now, $\Delta = \begin{vmatrix} 4 & -2 \\ -2 & 6 \end{vmatrix} = 4\times 6-(-2)^2 = 20$

$$\Delta_1 = \begin{vmatrix} 10 & -2 \\ 0 & 6 \end{vmatrix} = 10\times 6-0 = 60$$

$$\Delta_2 = \begin{vmatrix} 4 & 10 \\ -2 & 0 \end{vmatrix} = 0-(-2)\times 10 = 20$$

$$I = I_1 - I_2 = \frac{\Delta_1}{\Delta}-\frac{\Delta_2}{\Delta} = \frac{\Delta_1-\Delta_2}{\Delta} = \frac{60-20}{20} = 2\,A$$

Q1.41 What is node analysis?

Node analysis is a useful technique to solve independent voltage variables of a circuit.

Q1.42 When is node analysis preferred to solve voltages?

Node analysis is preferred to solve voltage variables when a circuit is excited by only current sources. Applying node analysis is straightforward and easier in circuits excited by only current sources. However, node analysis can also be extended to circuits excited by both voltage and current sources.

Q1.43 *How is node analysis performed?*

In a circuit with N nodes, one of the nodes is chosen as the reference node and its voltage is considered as zero. The voltages of remaining N – 1 nodes are independent voltages of the circuit with respect to the reference node. For each node (except the reference node) a KCL equation is formed and then the n (where, n = N – 1) number of node equations are solved by Cramer's rule to get a unique solution for node voltages.

Q1.44 *How are node voltages solved using the node basis matrix equation?*

Consider the node basis matrix equation,

$$\mathbf{G\,V = I}$$

On premultiplying the above equation both sides by $\mathbf{G}^{-1}$ we get,

$$\mathbf{G^{-1}\,G\,V = G^{-1}\,I}$$

$$\mathbf{U\,V} = \mathbf{G^{-1}\,I} \qquad \boxed{\mathbf{G^{-1}\,G = U} = \text{Unit matrix}}$$

$$\therefore\ \mathbf{V} = \mathbf{G^{-1}\,I} \qquad \boxed{\mathbf{UV = V}}$$

The above equation will be the solution for node voltages and the kth node voltage is,

$$\boxed{V_k = \frac{\Delta'_{1k}}{\Delta'} I_{11} + \frac{\Delta'_{2k}}{\Delta'} I_{22} + \frac{\Delta'_{3k}}{\Delta'} I_{33} + \ldots\ldots + \frac{\Delta'_{nk}}{\Delta'} I_{nn} = \frac{1}{\Delta'} \sum_{j=1}^{n} \Delta'_{jk} I_{jj}}$$

The above equation for node voltages is also called Cramer's rule.

Q1.45 *What is supernode?*

When a voltage source is connected between two nodes it can be short-circuited for analysis purpose and the short-circuited two nodes can be considered as one single node called a supernode.

Q1.46 *What is the value of I_{s2} in the circuit shown in Fig. Q1.46 if the value of V_2 is zero?*

Solution

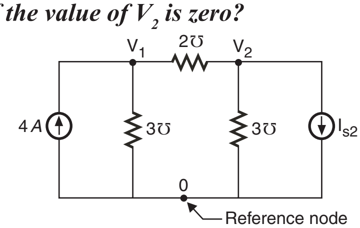


Fig. Q1.46.

The node basis matrix equation by inspection is,

$$\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -I_{s2} \end{bmatrix}$$

Now, $V_2 = \dfrac{\Delta'_2}{\Delta}$. Therefore, for $V_2 = 0$, $\Delta'_2 = 0$

Here, $\Delta'_2 = \begin{vmatrix} 5 & 4 \\ -2 & -I_{s2} \end{vmatrix} = -5I_{s2} + 8$

$$\therefore\ -5I_{s2} + 8 = 0 \Rightarrow 5I_{s2} = 8 \Rightarrow I_{s2} = \frac{8}{5} = 1.6\,A$$

Q1.47 *Find the value of I_{s1} and I_{s2} in the circuit shown in Fig. Q1.47.*

Solution

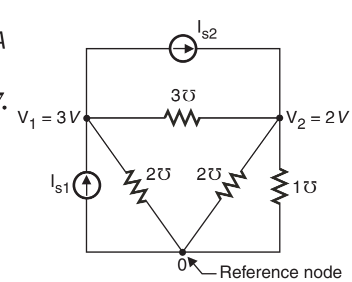


Fig. Q1.47.

The node basis matrix equation by inspection is,

$$\begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_{s1} - I_{s2} \\ I_{s2} \end{bmatrix}$$

On substituting for $V_1 = 3\,V$ and $V_2 = 2\,V$ in the above equation we get,

$$\begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} I_{s1} - I_{s2} \\ I_{s2} \end{bmatrix}$$

From row-2 we get,

$$-3 \times 3 + 6 \times 2 = I_{s2} \;\Rightarrow\; I_{s2} = 3\,A$$

From row-1 we get,

$$5 \times 3 - 3 \times 2 = I_{s1} - I_{s2} \;\Rightarrow\; I_{s1} = 9 + I_{s2} = 9 + 3 = 12\,A$$

Q1.48 ***Find the value of V_2 and I_{s2} in the circuit shown in Fig. Q1.48.***

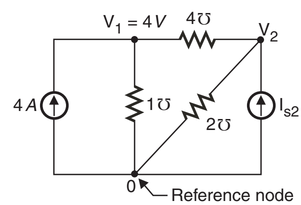


Fig. Q1.48.

Solution

The node basis matrix equation by inspection is,

$$\begin{bmatrix} 5 & -4 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 4 \\ I_{s2} \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} 5 & -4 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ V_2 \end{bmatrix} = \begin{bmatrix} 4 \\ I_{s2} \end{bmatrix}$$

From row-1 we get,

$$5 \times 4 - 4V_2 = 4 \;\Rightarrow\; 4V_2 = 16 \;\Rightarrow\; V_2 = \frac{16}{4} = 4\,V$$

From row-2 we get,

$$-4 \times 4 + 6V_2 = I_{s2} \;\Rightarrow\; I_{s2} = -16 + 6 \times 4 = 8\,A$$

Q1.49 ***In the circuit shown in Fig. Q1.49, find the power delivered to 2℧ conductance.***

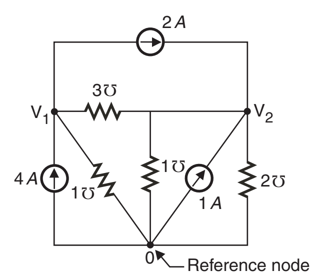


Fig. Q1.49.

Solution

The node basis matrix equation by inspection is,

$$\begin{bmatrix} 1+3 & -3 \\ -3 & 3+1+2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 4-2 \\ 2+1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Delta' = \begin{vmatrix} 4 & -3 \\ -3 & 6 \end{vmatrix} = 4 \times 6 - (-3)^2 = 15$$

$$\Delta'_2 = \begin{vmatrix} 4 & 2 \\ -3 & 3 \end{vmatrix} = 4 \times 3 - (-3) \times 2 = 18$$

$$\therefore\; V_2 = \frac{\Delta'_2}{\Delta'} = \frac{18}{15} = 1.2\,V$$

Power delivered to 2℧ conductance $= V_2^2 \times 2 = 1.2^2 \times 2 = 2.88\,W$

Q1.50 ***Find the voltage V_x in the circuit shown in Fig. Q1.50.***

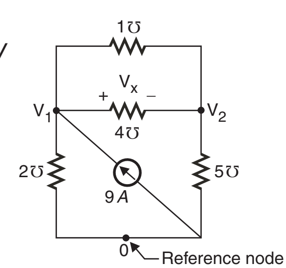


Fig. Q1.50.

Solution

The node basis matrix equation is,

$$\begin{bmatrix} 7 & -5 \\ -5 & 10 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

$$\Delta' = \begin{vmatrix} 7 & -5 \\ -5 & 10 \end{vmatrix} = 7 \times 10 - (-5)^2 = 45$$

$$\Delta'_1 = \begin{vmatrix} 9 & -5 \\ 0 & 10 \end{vmatrix} = 90 - 0 = 90$$

$$\Delta'_2 = \begin{vmatrix} 7 & 9 \\ -5 & 0 \end{vmatrix} = 0 - (-5) \times 9 = 45$$

$$V_x = V_1 - V_2 = \frac{\Delta'_1}{\Delta'} - \frac{\Delta'_2}{\Delta} = \frac{\Delta'_1 - \Delta'_2}{\Delta} = \frac{90 - 45}{45} = 1V$$

1.10 Exercises

I. Fill in the Blanks with Appropriate Words

1. The elements which consume energy either by absorbing/storing are called _________ elements.
2. The sources in which the current/voltage does not change with time are called _________.
3. The electrical energy supplied by _________source depends on another source of electrical energy.
4. The path between any two nodes is called _________.
5. In an electric circuit when elements are _________connected, the current will be the same.
6. In an electric circuit a path of infinite resistance is called _________.
7. In an electric circuit a path of zero resistance is called _________.
8. In an electric circuit the algebraic sum of _________ in a node is zero.
9. In an electric circuit the algebraic sum of _________ in a closed path is zero.
10. In an ideal _________ source the terminal voltage remains constant.
11. The _________ is given by the product of power and time.
12. Mesh analysis is used to solve ________ variables of a circuit.
13. A circuit with B branches and N nodes will have ________ independent currents.
14. In the mesh basis matrix equation, mesh currents are solved by ________.
15. The mesh equations are ________ equations of a circuit.
16. The solution of the mesh basis matrix equation **RI = E** will be in the form ________.
17. Node analysis is used to solve ________ variables of a circuit.
18. A circuit with N nodes will have ________ independent voltages.
19. In node basis matrix equation, node voltages are solved by ________.
20. The node equations are ________ equations of a circuit.
21. The solution of the node basis matrix equation **GV = I** will be in the form ________.

ANSWERS

1. passive	6. open circuit	11. energy	16. $\mathbf{I = R^{-1}E}$	**21**. $\mathbf{V = G^{-1}I}$
2. dc sources	7. short circuit	12. Current	17. Voltage	
3. dependent	8. currents	13. B – N + 1	18. N – 1	
4. branch	9. voltages	14. Cramer's rule	19. Cramer's rule	
5. series	10. voltage	15. KVL	20. KCL	

II. State Whether the Following Statements are True or False

1. The elements of electric circuits which can deliver energy are called active elements.
2. Inductance and capacitance absorb energy.
3. The electrical energy supplied by an independent source depends on another electrical source.
4. In an electric circuit the meeting point of two or more elements is called the principal node.
5. In parallel connected elements the voltage will be the same.
6. In an electric circuit, a voltage exists across open terminals.
7. In an electric circuit, a current flows through short-circuited terminals.
8. In an electric circuit, when an element delivers energy the current will leave from the negative terminal.
9. In an electric circuit, when an element absorbs energy the current will enter at the negative terminal.
10. An ideal voltage source can be converted into an ideal current source.
11. In an ideal current source the terminal voltage remains constant.
12. Power is rate of work done and energy is the total work done.
13. Mesh analysis can be used to solve voltage variables from the knowledge of current variables.
14. Mesh analysis can be applied only to circuits excited by voltage sources.
15. Mesh currents are independent current variables of a circuit.
16. Mesh analysis is applicable to non-planar circuits.
17. Mesh equations are formed using KCL.
18. Node analysis can be used to solve current variables from the knowledge of voltage variables.
19. Node analysis can be applied only to circuits excited by current sources.
20. Node voltages are independent voltage variables of a circuit.
21. For node analysis, any node can be chosen as the reference node.
22. Node equations are formed using KVL.

ANSWERS

1. True	6. True	11. False	16. False	21. True
2. False	7. True	12. True	17. False	22. False
3. False	8. False	13. True	18. True	
4. True	9. False	14. False	19. False	
5. True	10. False	15. True	20. True	

III. Choose the Right Answer for the Following Questions

1. ***The voltage V_{AB} across the open circuit in the circuit shown in Fig. 1 is,***

   a) $2\,V$

   b) $4\,V$

   c) $8\,V$

   d) $6\,V$

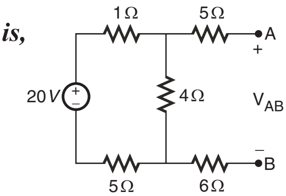


Fig. 1.

2. ***The current I_{sc} through the short circuit in the circuit shown in Fig. 2 is,***

   a) $4\,A$

   b) $3\,A$

   c) $2\,A$

   d) $1\,A$

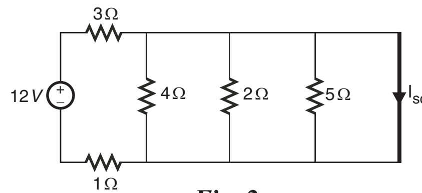


Fig. 2.

3. ***The voltage V_L in the circuit shown in Fig. 3 is,***

   a) $4\,V$

   b) $8\,V$

   c) $12\,V$

   d) $20\,V$

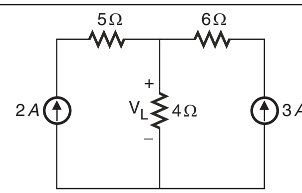


Fig. 3.

4. ***The current I_s delivered by the voltage source in the circuit shown in Fig. 4 is,***

   a) $6\,A$

   b) $5\,A$

   c) $4\,A$

   d) $2\,A$

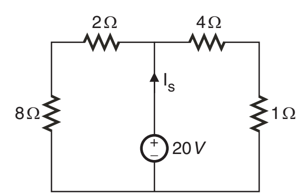


Fig. 4.

5. ***The node voltage V_A in the circuit shown in Fig. 5 is,***

   a) $8\,V$

   b) $6\,V$

   c) $10\,V$

   d) $2\,V$

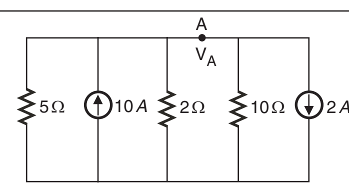


Fig. 5.

6. ***The values of voltage sources E_1 and E_2 in the circuit shown in Fig. 6 are,***

   a) $11\,V$, $16\,V$

   b) $16\,V$, $11\,V$

   c) $5\,V$, $7\,V$

   d) $7\,V$, $5\,V$

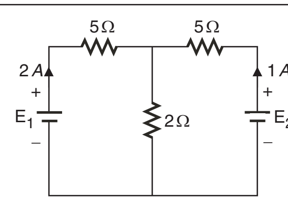


Fig. 6.

7. ***The currents I_1 and I_2 in the circuit shown in Fig. 7 are,***

   a) 7, 5

   b) 3, 7

   c) –5, –3

   d) –7, –5

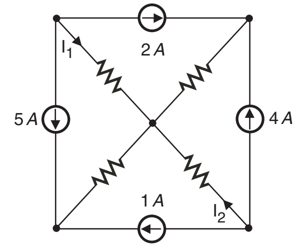


Fig. 7.

8. ***The equivalent current source for the voltage source in the circuit shown in Fig. 8 is,***

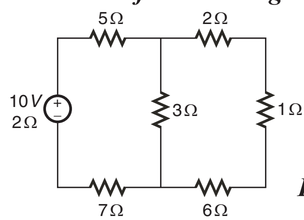
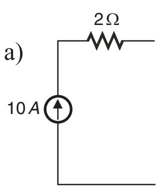
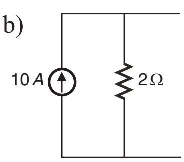
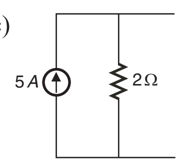
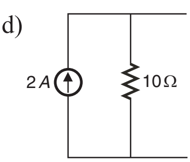


Fig. 8.

a)


b)


c)


d)


9. ***The equivalent voltage source for the current source in the circuit shown in Fig. 9 is,***

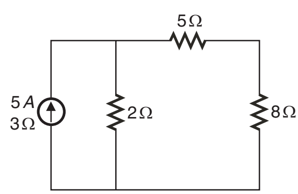
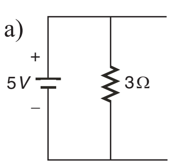
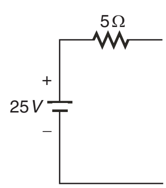
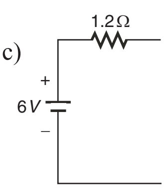
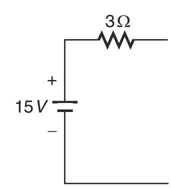


Fig. 9.

a)


b)


c)


d)


10. ***The equivalent current source for the dependent voltage source in the circuit shown in Fig. 10 with respect to terminals A -B is,***

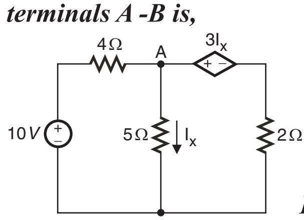
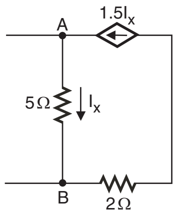
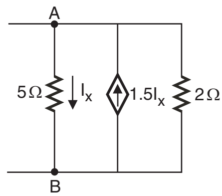
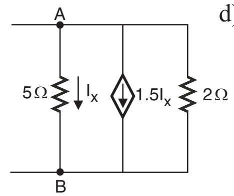


Fig. 10.

a) 

b) 

c) 

d) Not possible

11. The equivalent voltage source for the dependent current source in the circuit shown in Fig. 11 with respect to terminals A-B is,

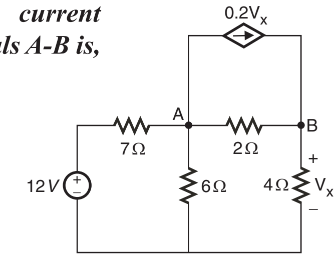
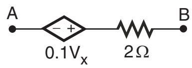
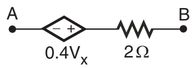


Fig. 11.

a) 

b) 

c) 

d) Not possible

12. In Fig. 12, the currents I_1, I_2 and I_3, respectively are,

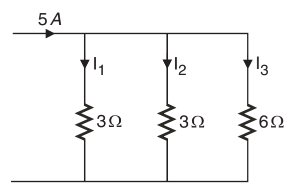


Fig. 12.

a) 1.5*A*, 1.5*A*, 2*A*

b) 1*A*, 1*A*, 3*A*

c) 1*A*, 2*A*, 3*A*

d) 2*A*, 2*A*, 1*A*

13. In Fig. 13, the voltage V_1, V_2 and V_3, respectively are,

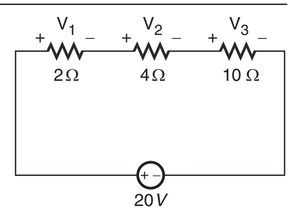


Fig. 13.

a) 4*V*, 6*V*, 10*V*

b) 3.5*V*, 6.5*V*, 10*V*

c) 2.5*V*, 5*V*, 12.5*V*

d) 2*V*, 6*V*, 12*V*

14. Mesh analysis is based on,

a) KCL b) KVL c) Ohm's law d) None of the above

15. If a planar circuit has six branches and four nodes then the number of meshes are,

a) 6 b) 5 c) 4 d) 3

16. In a circuit with m meshes, the k^th^ mesh current by Cramer's rule is given by,

a) $I_k = \frac{1}{\Delta}\sum_{j=1}^{k} \Delta_{jk} E_{jj}$

b) $I_k = \frac{1}{\Delta}\sum_{k=1}^{j} \Delta_{jk} E_{kk}$

c) $I_k = \frac{1}{\Delta}\sum_{j=1}^{m} \Delta_{jk} E_{jj}$

d) $I_k = \frac{1}{\Delta}\sum_{k=1}^{m} \Delta_{jm} E_{km}$

17. In mesh analysis, when all the mesh currents are chosen in the same orientation then the mutual-resistances are,

a) always negative b) always positive

c) positive or negative d) always zero

18. The mesh currents I_1, I_2 and I_3 in the circuit shown in Fig. 18 respectively are,

a) $2A, 4A, 6A$

b) $2A, 10A, 6A$

c) $2A, 6A, 0$

d) $2A, 6A, 4A$

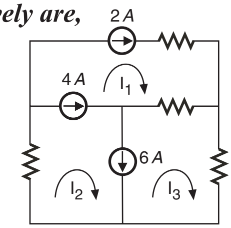


Fig. 18.

19. In the circuit shown in Fig. 19, the currents I_1 and I_2 respectively are,

a) $1A, -1A$

b) $2A, 2A$

c) $2A, 0$

d) $1A, 0$

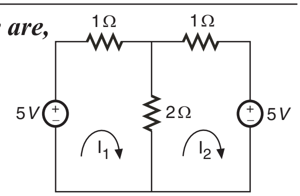


Fig. 19.

20. In the circuit shown in Fig. 20, the values of E_1 and I_1 respectively are,

a) $6V, 3A$

b) $4V, 2A$

c) $7V, 1A$

d) $5V, 3A$

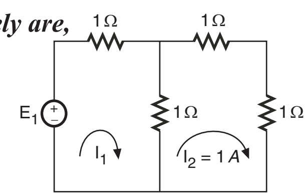


Fig. 20.

21. In the circuit shown in Fig. 21, what is the value of E_1 for the current I_1 to be zero?

a) $20V$

b) $10V$

c) $5V$

d) 0

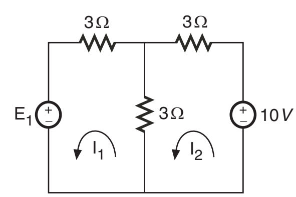


Fig. 21.

22. In the circuit shown in Fig. 22, what is the value of E_1 if the power delivered to 1Ω resistor is 1 W?

a) $10V$

b) $9V$

c) $11V$

d) $7V$

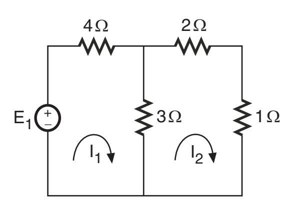


Fig. 22.

23. In the circuit shown in Fig. 23, the values of I_1 and I_2 respectively are,

a) $2A, 1A$

b) $3A, 2A$

c) $4A, 1A$

d) $2A, 5A$

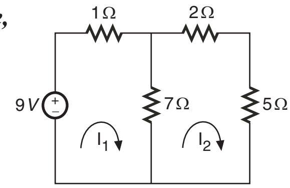


Fig. 23.

24. Node analysis is based on,

a) KCL b) KVL c) Ohm's law d) none of the above

25. If a circuit has eight branches and five nodes then the number of independent voltages are,

a) 7 b) 6 c) 5 d) 4

26. In a circuit with n independent voltages, the k^{th} node voltage by Cramer's rule is given by,

a) $V_k = \frac{1}{\Delta'} \sum_{j=1}^{k} \Delta'_{jk} I_{jj}$ b) $V_k = \frac{1}{\Delta'} \sum_{k=1}^{j} \Delta'_{jk} I_{kk}$

c) $V_k = \frac{1}{\Delta'} \sum_{j=1}^{n} \Delta'_{jk} I_{jj}$ d) $V_k = \frac{1}{\Delta'} \sum_{k=1}^{m} \Delta'_{jn} I_{kn}$

27. In node basis matrix equation, the mutual conductances are,

a) always positive b) always negative

c) positive or negative d) always zero

28. The node voltages V_1, V_2 and V_3 in the circuit shown in Fig. 28 respectively are,

a) $6V, 4V, -2V$

b) $10V, 4V, 2V$

c) $6V, 2V, -2V$

d) $10V, 2V, 2V$

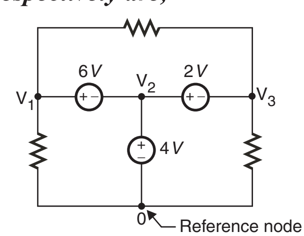


Fig. 28.

29. In the circuit shown in Fig. 29, the voltages V_1 and V_2 respectively are,

a) $4V, 2V$

b) $3V, 1V$

c) $3V, 3V$

d) $2V, 2V$

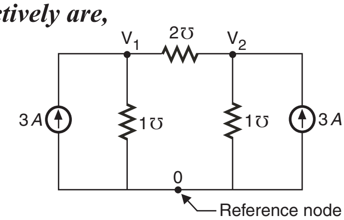


Fig. 29.

30. The node voltages V_1, V_2 and V_3 in the circuit shown in Fig. 30 are,

a) $1.5\,V,\ 2\,V,\ 1.67\,V$

b) $24\,V,\ 8\,V,\ 15\,V$

c) $2\,V,\ 2\,V,\ 2\,V$

d) $10\,V,\ 6\,V,\ 8\,V$

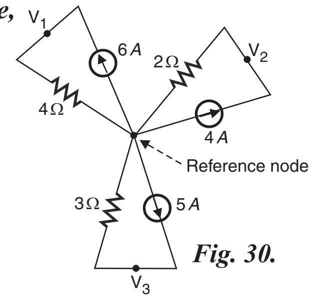


Fig. 30.

31. If the node voltage V_2 is zero then the value of I_{s2} in the cirucit shown in Fig. 31 is,

a) $-4\,A$

b) $4\,A$

c) $-2\,A$

d) $2\,A$

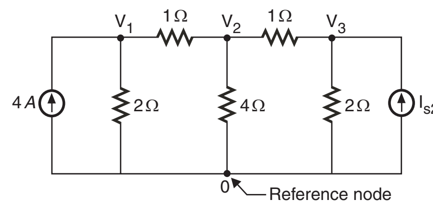


Fig. 31.

32. In the circuit shown in Fig. 32, the power P_1 and P_2 delivered by the current sources are,

a) $P_1 = 4\,W,\ P_2 = 4\,W$

b) $P_1 = 8\,W,\ P_2 = 4\,W$

c) $P_1 = 6\,W,\ P_2 = 2\,W$

d) $P_1 = 8\,W,\ P_2 = 0$

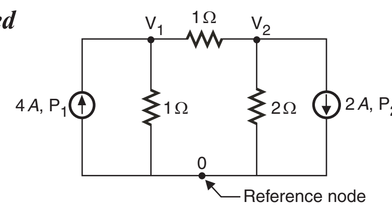


Fig. 32.

33. In the circuit shown in Fig. 33, what is the value of I_{s1} if the power delivered to 1 Ω resistor is 25 W?

a) $35\,A$

b) $17.5\,A$

c) $8.75\,A$

d) $4.375\,A$

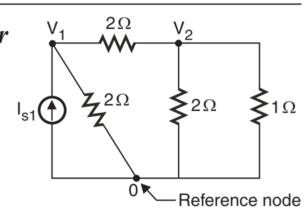


Fig. 33.

34. The number of links and twigs in the graph shown in Fig. 34 respectively are,

a) 5, 6

b) 5, 5

c) 6, 5

d) 11, 5

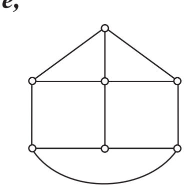


Fig. 34.

35. For the graph shown in Fig. 35, which of the following is not a proper tree?

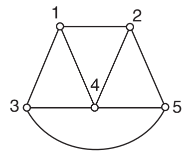
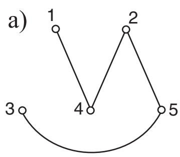
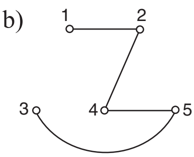
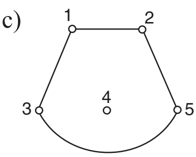
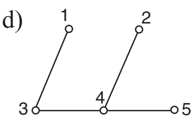


Fig. 35.

a) 

b) 

c) 

d) 

ANSWERS

1. c	8. c	15. d	22. c	29. c
2. b	9. d	16. c	23. a	30. b
3. d	10. b	17. a	24. a	31. a
4. a	11. c	18. c	25. d	32. d
5. c	12. d	19. a	26. c	33. b
6. b	13. c	20. d	27. b	34. a
7. d	14. b	21. c	28. b	35. c

IV. Unsolved Problems

E1.1 Find the node voltages in the circuit shown in Fig. E1.1.

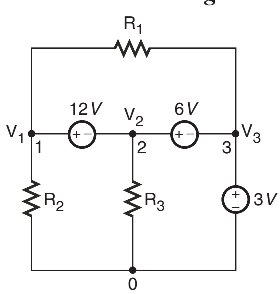


Fig. E1.1.

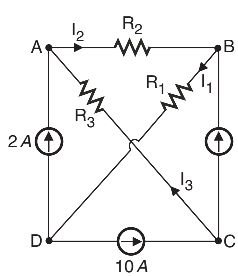


Fig. E1.2.

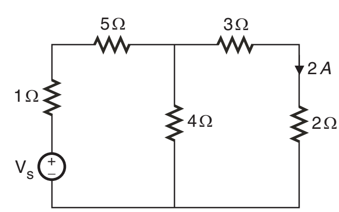


Fig. E1.3.

E1.2 Find the branch currents I_1, I_2 and I_3 in the circuit shown in Fig. E1.2.

E1.3 Find the value of the source voltage V_s in the circuit shown in Fig. E1.3.

E1.4 A 20 V source with internal resistance 0.2 Ω is connected in series with a 30 V source with internal resistance 0.3 Ω to deliver a load current of 10 A to resistive load. Calculate a) load power P_L and b) Power delivered by each source to load.

E1.5 ***Two current sources with internal resistance 50 Ω and 100 Ω, respectively, are connected in parallel to supply a 4.8 kW load at 200 V. If the generated source current of the source with 50 Ω internal resistance is 12 A, what is the generated source current of the other source?***

E1.6 ***What is the value of the emf E of the battery in the circuit shown in Fig. E1.6? Also say whether the battery is charging or discharging.***

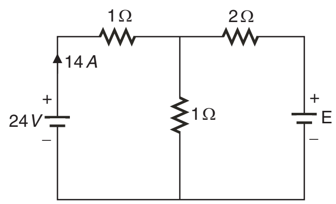


Fig. E1.6.

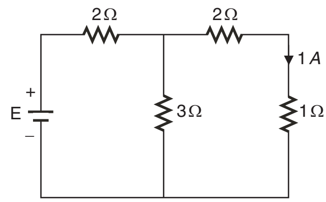


Fig. E1.7.

E1.7 ***What is the value of source voltage E in the circuit shown in Fig E1.7.***

E1.8 ***What is the value of load voltage V_L in the circuit shown in Fig. E1.8? Also calculate the power delivered by the current source.***

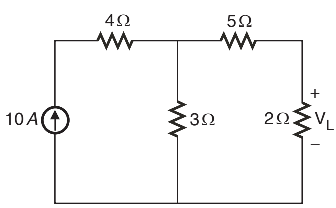


Fig. E1.8.

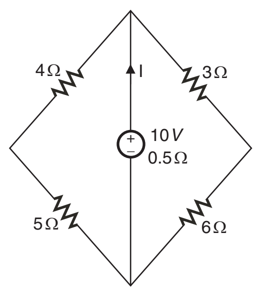


Fig. E1.9.

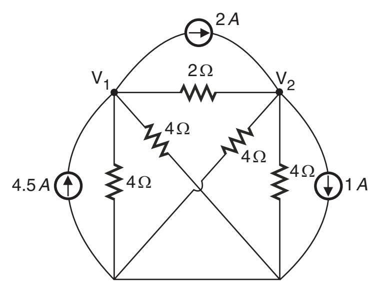


Fig. E1.10.

E1.9 ***Determine the current I delivered by the voltage source in the circuit shown in Fig. E1.9. Also calculate the power delivered by the voltage source.***

E1.10 ***Determine the voltages V_1 and V_2 in the circuit shown in Fig. E1.10.***

E1.11 ***Determine the mesh currents shown in the circuit of Fig. E1.11.***

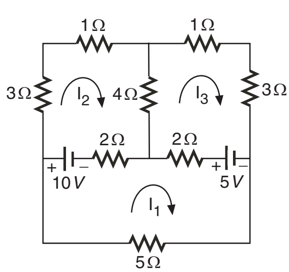


Fig. E1.11.

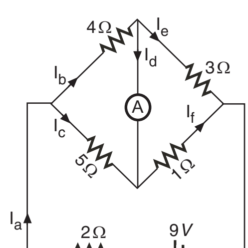


Fig. E1.12.

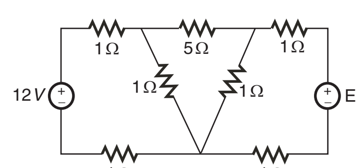


Fig. E1.13.

E1.12 ***Determine the current through various branches of the circuit in Fig. E1.12 by mesh analysis. Take resistance of ammeter as 0.2 Ω.***

E1.13 ***In the circuit shown in Fig. E1.13, find the value of E by mesh analysis such that the current through the 5 Ω resistor is zero.***

E1.14 By mesh analysis, determine the power delivered to 2 Ω and 1 Ω resistor in the circuit of Fig. E1.14.

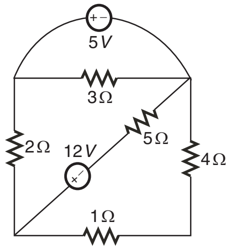


Fig. E1.14.

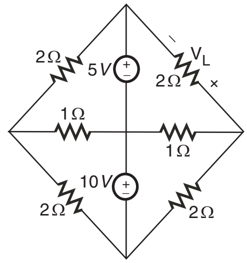


Fig. E1.15.

E1.15 In the circuit shown in Fig. E1.15, determine the voltage V_L by mesh analysis.

E1.16 Determine the mesh currents shown in the circuit of Fig. E1.16.

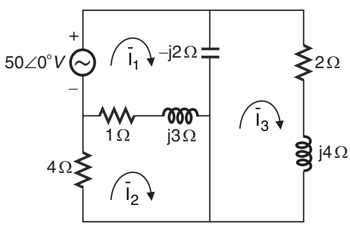


Fig. E1.16.

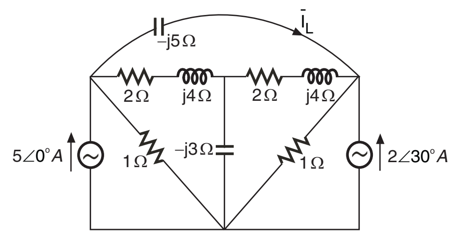


Fig. E1.17.

E1.17 Determine the current $\bar{I}_L$ in the circuit shown in Fig. E1.17 by mesh analysis.

E1.18 Determine the active and reactive power delivered to the 2 + j4 Ω impedance in the circuit of Fig. E1.18.

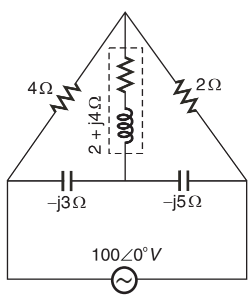


Fig. E1.18.

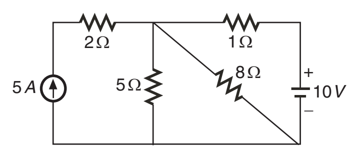


Fig. E1.19.

E1.19 Determine the power delivered to the 8 Ω resistor in the circuit of Fig. E1.19 using mesh analysis.

E1.20 In the circuit shown in Fig. E1.20, form two supermeshes and determine the current I_L.

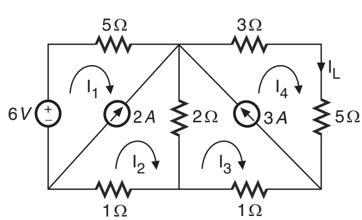


Fig. E1.20.

E1.21 Determine the power delivered by each source to the 2 Ω resistor in the circuit of Fig. E1.21 by mesh analysis.

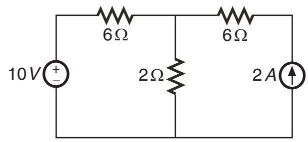


Fig. E1.21.

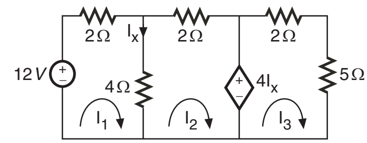


Fig. E1.22.

E1.22 Solve the mesh currents shown in the circuit of Fig. E1.22.

E1.23 Determine the voltage across the 6 Ω resistor in the circuit of Fig. E1.23 by mesh analysis.

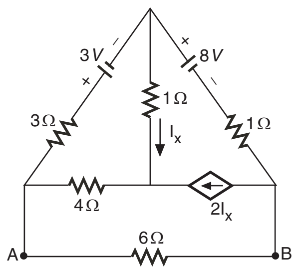


Fig. E1.23.

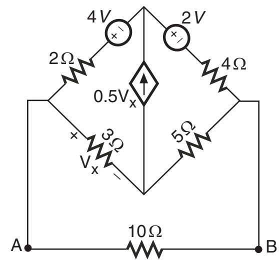


Fig. E1.24.

E1.24 In the circuit of Fig. E1.24 determine the current through the 10 Ω resistor by mesh analysis.

E1.25 Determine the power delivered by the dependent voltage source in the circuit of Fig. E1.25 by mesh analysis.

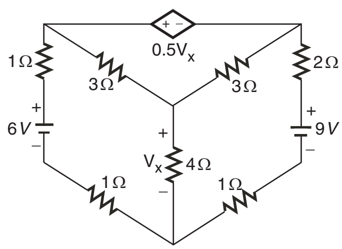


Fig. E1.25.

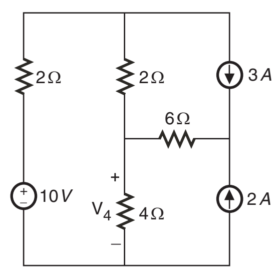


Fig. E1.26.

E1.26 Find the voltage across the 4 Ω resistor in the circuit shown in Fig. E1.26 by mesh analysis.

E1.27 Determine the branch voltages in the circuit shown in Fig. E1.27 by node method.

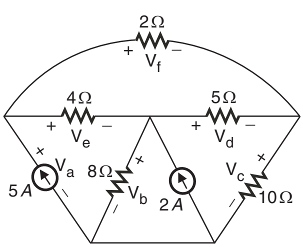


Fig. E1.27.

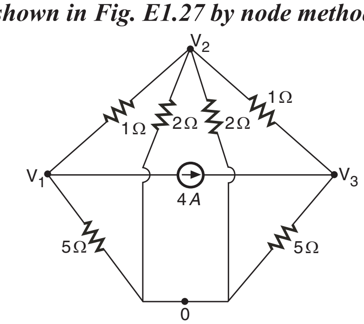


Fig. E1.28.

E1.28 Determine the node voltages in the circuit shown in Fig. E1.28.

E1.29 Determine the current I_L in the circuit of Fig. E1.29 by node method.

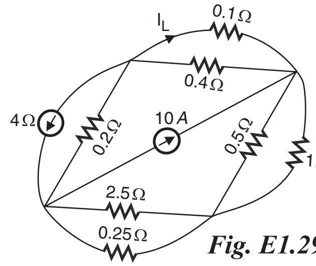


Fig. E1.29.

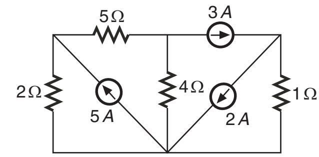


Fig. E1.30.

E1.30 Determine the power supplied/absorbed by the current sources in the circuit shown in Fig. E1.30.

E1.31 Determine the power absorbed by the 10 Ω resistor in the circuit of Fig. E1.31.

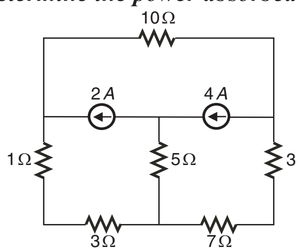


Fig. E1.31.

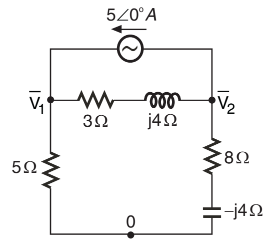


Fig. E1.32.

E1.32 Determine the node voltages $\overline{V}_1$ and $\overline{V}_2$ in the circuit shown in Fig. E1.32.

E1.33 Determine the node voltages in the circuit shown in Fig. E1.33.

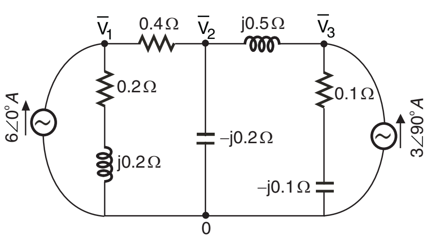


Fig. E1.33.

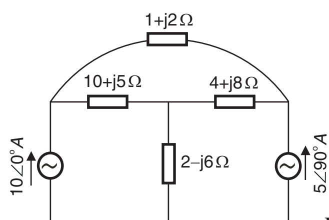


Fig. E1.34.

E1.34 In the circuit shown in Fig. E1.34, determine the active and reactive power in the impedance $1 + j2$ Ω by node method.

E1.35 In the circuit shown in Fig. E1.35, calculate the current $\overline{I}_L$ by node method.

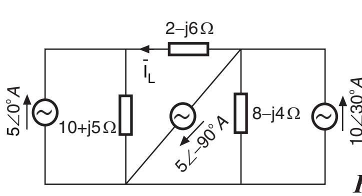


Fig. E1.35.

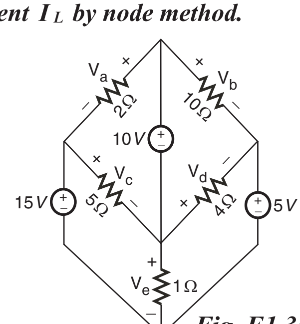


Fig. E1.36.

E1.36 Determine the voltages across the resistors in the circuit shown in Fig. E1.36, by node analysis.

E1.37 Determine the power delivered/absorbed by the sources in the circuit shown in Fig E1.37 by node analysis.

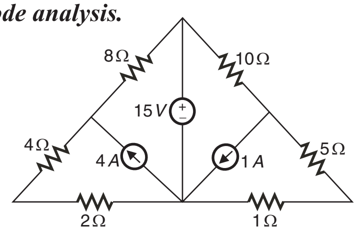


Fig. E1.37.

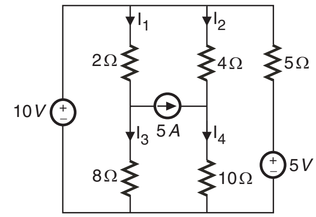


Fig. E1.38.

E1.38 Determine the currents I_1, I_2, I_3 and I_4 in the circuit shown in Fig. E1.38 by node analysis.

E1.39 Calculate the voltage V_x and, hence, estimate the power delivered or absorbed by the dependent source in the circuit shown in Fig. E1.39.

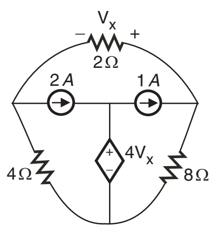


Fig. E1.39.

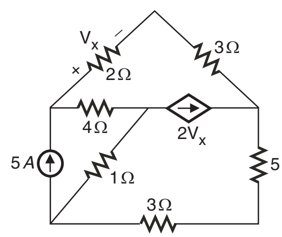


Fig. E1.40.

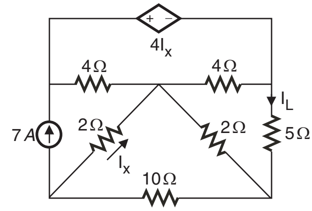


Fig. E1.41.

E1.40 Calculate the power delivered to the 5 Ω resistor in the circuit shown in Fig. E1.40 by node analysis.

E1.41 Calculate the current through the 5 Ω resistance in the circuit shown in Fig. E1.41 by node analysis.

ANSWERS

E1.1 $V_1 = 21V,\ V_2 = 9V,\ V_3 = 3V$

E1.2 $I_1 = 12A,\ I_2 = 7A,\ I_3 = 5A$

E1.3 $V_s = 37V$ **E1.4** $P_L = 450W,\ P_{D20} = 180W,\ P_{D30} = 270W$

E1.5 $I_{s2} = 18A$ **E1.6** $E = 2V$, Charging

E1.7 $E = 7V$ **E1.8** $V_L = 6V,\ P_s = 610W$

E1.9 $I = 2A,\ P_s = 18W$ **E1.10** $V_1 = 4V,\ V_2 = 3V$

E1.11 $I_1 = -1.3043A$; $I_2 = 0.9938A$; $I_3 = 0.6366A$

E1.12 $I_a = 1.8037A$; $I_b = 0.9907A$; $I_c = 0.813A$; $I_d = 0.5141A$;

$I_e = 0.4766A$; $I_f = 1.3271A$

E1.13 $E = 12V$ **E1.14** $P_{2\Omega} = 0.0987W$; $P_{1\Omega} = 1.1857W$

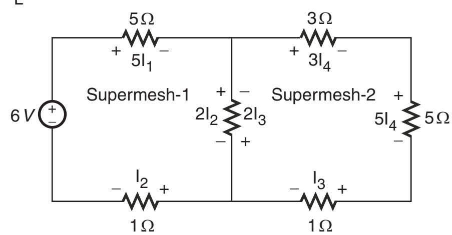
E1.15 $V_L = -6.25V$

E1.16 $\bar{I}_1 = 16.1971\angle 31^\circ A$; $\bar{I}_2 = 8.7841\angle 71.6^\circ A$; $\bar{I}_3 = 11.4531\angle -104^\circ A$

E1.17 $\bar{I}_L = 0.6114\angle 62.7^\circ A$ **E1.18** $P = 108.9\,W$; $Q = 217.8\,VAR$

E1.19 $P = 16.0201\,W$

E1.20 $I_L = 1.0952\,A$



E1.21 $P_{\frac{2\Omega}{10V}} = 4.125\,W$; $P_{\frac{2\Omega}{2A}} = 11\,W$

E1.22 $I_1 = 2\,A$; $I_2 = 0$; $I_3 = 1.1429\,A$ **E1.23** $V_{AB} = 6.7056\,V$

E1.24 $I_{AB} = 0.375\,A$ (current flowing from A to B) **E1.25** $P = 2.6315\,W$ (delivered)

E1.26 $V_4 = 12\,V$

E1.27 $V_a = 37.8947\,V$; $V_b = 30.8772\,V$; $V_c = 31.4035\,V$

$V_d = -0.5263\,V$; $V_e = 7.0175\,V$; $V_f = 6.4912\,V$

E1.28 $V_1 = -3.3333\,V$; $V_2 = 0$; $V_3 = 3.3333\,V$

E1.29 $I_L = -6.199\,A$

E1.30 $P_{5A} = 30\,W$ (Delivered) ; $P_{2A} = -2\,W$ (Absorbed) ; $P_{3A} = 15\,W$ (Delivered)

E1.31 $P_{10\Omega} = 40\,W$

E1.32 $\bar{V}_1 = 7.8125\angle 53.1^\circ V$; $\bar{V}_2 = 13.9754\angle -153.4^\circ V$

E1.33 $\bar{V}_1 = 1.5132\angle 15.8^\circ V$; $\bar{V}_2 = 1.1186\angle -34.2^\circ V$; $\bar{V}_3 = 0.2925\angle 106.6^\circ V$

E1.34 $P = 45.5555\,W$; $Q = 91.111\,VAR$

E1.35 $\bar{I}_L = 3.0403\angle 33^\circ A$

E1.36 $V_a = -1.9512\,V$; $V_b = 8.0488\,V$; $V_c = 11.9512\,V$; $V_d = -1.9512\,V$; $V_e = 3.0488\,V$

E1.37 $P_{4A} = 80.5624\,W$ (Delivered) ; $P_{1A} = -1.8748\,W$ (Absorbed) ; $P_{15V} = 10.0485\,W$ (Delivered)

E1.38 $I_1 = 5\,A$; $I_2 = -2.8572\,A$; $I_3 = 0$; $I_4 = 2.1429\,A$

E1.39 $P_{4V_x} = -9.1428\,W$ (Power absorbed)

E1.40 $P_{5\Omega} = 26.792\,W$

E1.41 $I_L = 3.1111\,A$

Chapter 2

NETWORK REDUCTION AND THEOREMS FOR AC AND DC CIRCUITS

2.1 Network Reduction *(AU May'15, 2 Marks)*

A typical network involves series, parallel, star and delta-connections of parameters like resistances, inductances and capacitances. Sometimes it may require to find the single equivalent value of the series/parallel/star-delta-connected parameters of the network. In such a case, the parameters of the network have to be reduced step by step, starting from a dead end. Basically, the network reduction will be attempted with respect to two terminals, and in any network reduction technique, the ratio of voltage and current should be the same even after reducing the network.

Reducing the series/parallel/star-delta-connected parameters to a single equivalent parameter, conversion of star-connected parameters to equivalent delta parameters and vice versa are explained in this chapter.

The network also involves series/parallel connection of sources for higher voltage/current requirement. The series and parallel connections of voltage sources and current sources and their reduction into a single equivalent are also discussed in this chapter.

2.1.1 Resistances in Series and Parallel

Equivalent of Series-connected Resistances

Consider a circuit with series combination of two resistances R_1 and R_2 connected to a dc source of voltage V as shown in Fig. 2.1(a). Let the current through the circuit be I. Now the voltage across R_1 and R_2 are IR_1 and IR_2, respectively.

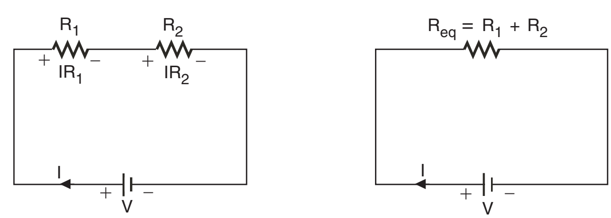


Fig. a : *Resistances in series.* ***Fig. b :*** *Equivalent circuit of Fig. a.*

Fig. 2.1 : *Resistances in series.*

By Kirchhoff's Voltage Law (KVL), we can write,

$$V = IR_1 + IR_2$$

$$= I(R_1 + R_2)$$

Let, $V = IR_{eq}$ ..... (2.1)

where, $R_{eq} = R_1 + R_2$

From equation (2.1), we can say that the series-connected resistances R_1 and R_2 can be replaced with an equivalent resistance R_{eq} given by the sum of individual resistances R_1 and R_2.

This concept can be extended to any number of resistances in series. Therefore, we can say that the resistances in series can be replaced with an equivalent resistance whose value is given by the sum of individual resistances.

"*When n number of identical resistances of value R are connected in series, the equivalent resistance* $R_{eq} = nR$".

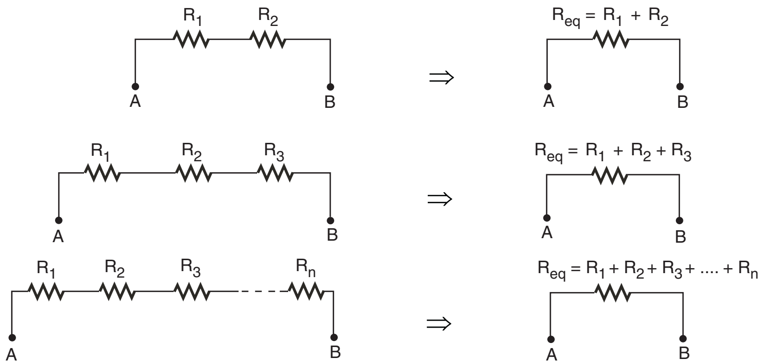


Fig. 2.2 : *Resistances in series and their equivalents.*

Equivalent of Parallel-connected Resistances

Consider a circuit in which two resistances in parallel are connected to a dc source of voltage V as shown in Fig. 2.3(a). Let I be the current supplied by the source and I_1 and I_2 be the currents through R_1 and R_2, respectively. Since the resistances are parallel to the source, the voltage across them will be the same.

By Ohm's law, we can write,

$$I_1 = \frac{V}{R_1} \qquad ..... (2.2)$$

$$I_2 = \frac{V}{R_2} \qquad ..... (2.3)$$

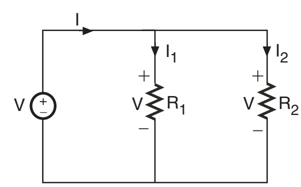


Fig. a : *Resistances in parallel.*

By Kirchhoff's Current Law (KCL), we can write,

$$I = I_1 + I_2$$

$$= \frac{V}{R_1} + \frac{V}{R_2} \qquad \boxed{\text{Using equations (2.2) and (2.3)}}$$

$$= V\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$\therefore \quad V = I\,\frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

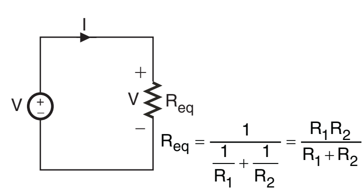


Fig. b : *Equivalent circuit of Fig. a.*

Fig. 2.3 : *Resistances in parallel.*

Let, $V = I R_{eq}$ ..... (2.4)

$$\text{where,} \quad R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 R_2}{R_1 + R_2} \qquad \text{..... (2.5)}$$

$$\text{Also,} \quad \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \qquad \text{..... (2.6)}$$

From equation (2.4), we can say that the parallel-connected resistances R_1 and R_2 can be replaced with an equivalent resistance given by equation (2.5). From equation (2.6), we can say that the inverse of the equivalent resistance of parallel-connected resistances is equal to the sum of the inverse of individual resistances.

This concept can be extended to any number of resistances in parallel. Therefore, we can say that the resistances in parallel can be replaced with an equivalent resistance whose value is given by the inverse of sum of the inverse of individual resistances.

"When n number of identical resistances of value R are connected in parallel, the equivalent resistance $R_{eq} = R/n$".

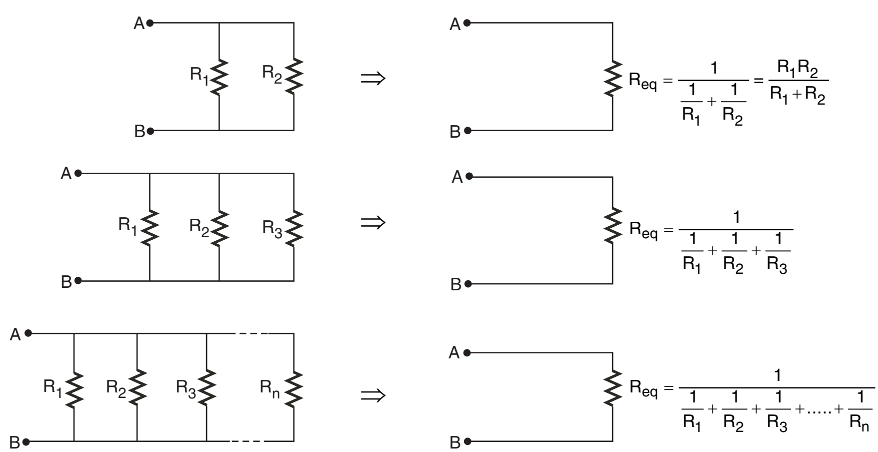


Fig. 2.4 : *Resistances in parallel and their equivalents.*

2.1.2 Voltage Sources in Series and Parallel

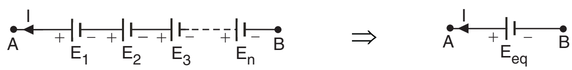
A voltage source is designed to deliver energy at a constant voltage called **rated voltage**. The current delivered by the voltage source depends on the load and the current is limited by the power rating of the source. When the voltage requirement of a load is higher, the voltage sources are connected in series. When the current requirement of a load is higher, the voltage sources are connected in parallel.

Series Connection of Voltage Sources

In series connection, voltage sources of different voltage ratings may be connected in series. However in series connection, the current delivered by the voltage sources is the same.

Case i : Series connection of ideal voltage sources

The series connection of ideal voltage sources is shown in Fig. 2.5(a). By applying KVL to series-connected sources of Fig. 2.5(a) it is possible to determine the single equivalent source as shown in Fig. 2.5(b).



$$\boxed{E_{eq} = E_1 + E_2 + E_3 + ... + E_n}$$

Fig. a : *Series connection.* ***Fig. b :*** *Equivalent voltage source.*

Fig. 2.5 : *Series connection of ideal voltage sources.*

Case ii : Series connection of voltage sources with internal resistance

The series-connected voltage sources with internal resistance shown in Fig. 2.6(a) can be represented as ideal sources with a series resistance as shown in Fig. 2.6(b). The series-connected resistance of Fig. 2.6(c) can be represented by an equivalent resistance as shown in Fig. 2.6(d). The series-connected voltage sources of Fig. 2.6(c) can be represented by an equivalent source as shown in Fig. 2.6(d).

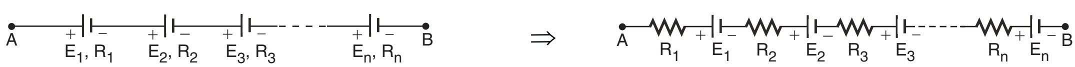


Fig. a : *Series connection.* ***Fig. b.***

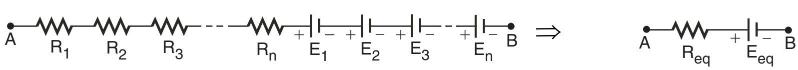


Fig. c. ***Fig. d :*** *Equivalent source.*

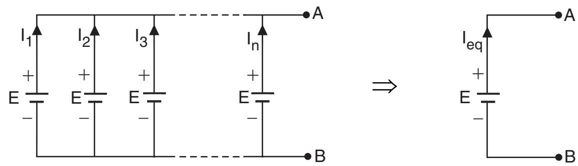
$$\boxed{\begin{aligned} E_{eq} &= E_1 + E_2 + E_3 + ... + E_n \\ R_{eq} &= R_1 + R_2 + R_3 + ... + R_n \end{aligned}}$$

Fig. 2.6 : *Series connection of voltage sources with internal resistance.*

Parallel Connection of Voltage Sources

Practically, voltage sources of an identical voltage rating should be connected in parallel, but the current delivered by the parallel-connected sources may be different. If voltage sources with different voltage ratings are connected in parallel then current will circulate within the sources which produce excess heat and this in-turn may damage the source.

Case i : Parallel connection of ideal voltage sources



$$\boxed{I_{eq} = I_1 + I_2 + I_3 + ... + I_n}$$

Fig. a : *Parallel connection.* ***Fig. b :*** *Equivalent source.*

Fig. 2.7 : *Parallel connection of ideal voltage sources.*

The parallel-connected ideal voltage sources with the same voltage rating, as shown in Fig. 2.7(a), can be converted into a single equivalent source using KCL, as shown in Fig. 2.7(b).

> ***Note :*** *The parallel connection of ideal voltage sources with different voltage rating is illegal.*

Case ii : Parallel connection of voltage sources with internal resistance

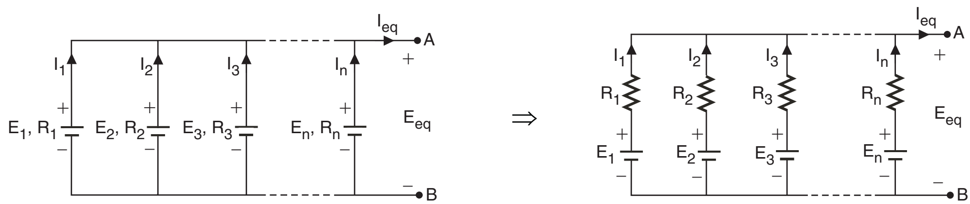


Fig. a : *Parallel connection.* ***Fig. b.***

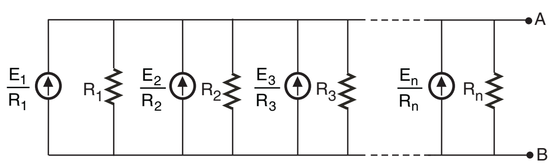


Fig. c.

$$I_{eq} = \frac{E_1}{R_1} + \frac{E_2}{R_2} + \frac{E_3}{R_3} + ... + \frac{E_n}{R_n}$$

$$R_{eq} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + ... + \frac{1}{R_n}}$$

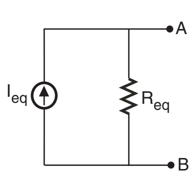


Fig. d.

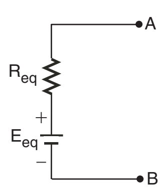


Fig. e.

$$E_{eq} = I_{eq} \times R_{eq}$$

$$= R_{eq}\left[\frac{E_1}{R_1} + \frac{E_2}{R_2} + \frac{E_3}{R_3} + ... + \frac{E_n}{R_n}\right]$$

Fig. 2.8 : *Parallel connection of voltage sources with internal resistance.*

The parallel-connected voltage sources with internal resistance shown in Fig. 2.8(a) can be represented as ideal sources with a series resistance as shown in Fig. 2.8(b).

Using source transformation technique, the voltage sources can be converted into current sources as shown in Fig. 2.8(c). Now, the parallel-connected current sources can be combined as an equivalent single current source as shown in Fig. 2.8(d). Finally, again using source transformation technique, the current source of Fig. 2.8(d) can be converted into an equivalent voltage source as shown in Fig. 2.8(e).

> ***Note :*** *The conversion of parallel-connected voltage sources into a single equivalent voltage source can also be obtained by **Millman's Theorem**.*

2.1.3 Current Sources in Series and Parallel

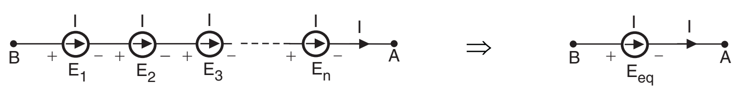
A current source is designed to deliver energy at a constant current called **rated current**. The voltage across the current source depends on the load and the voltage is limited by the power rating of the source. When the voltage requirement of a load is higher, the current sources are connected in series. When the current requirement of a load is higher, the current sources are connected in parallel.

Series Connection of Current Sources

Practically, current sources of an identical current rating should be connected in series, but the voltage across the series-connected sources may be different. If current sources with different current ratings are connected in series then sources with lesser current ratings are forced to carry higher currents which produce excess heat and this in-turn may damage the source.

Case i : Series connection of ideal current sources

The series-connected ideal current sources with an identical current rating, as shown in Fig. 2.9(a), can be converted into a single equivalent source using KVL, as shown in Fig. 2.9(b).



$E_{eq} = E_1 + E_2 + E_3 + ... + E_n$

Fig. a : *Series connection.* ***Fig. b :*** *Equivalent current source.*

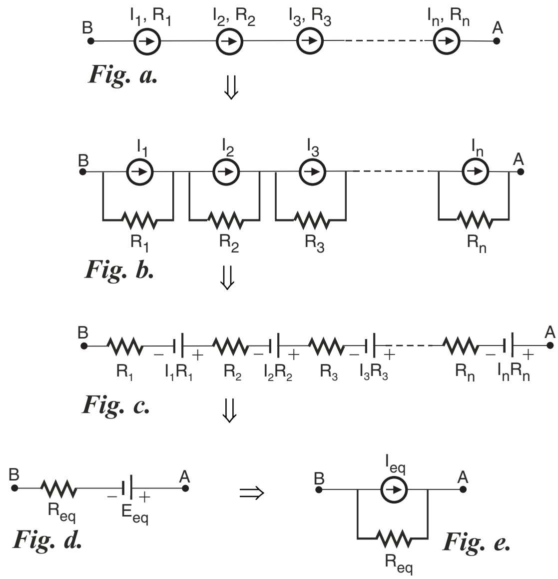
Fig. 2.9 : *Series connection of ideal current sources.*

> ***Note :*** *The series connection of ideal current sources with different current rating is illegal.*

Case ii : Series connection of current sources with internal resistance

The series-connected current sources with internal resistance shown in Fig. 2.10(a) can be represented as ideal sources with a parallel resistance as shown in Fig. 2.10(b).

Using source transformation technique, the current sources can be converted into voltage sources as shown in Fig. 2.10(c). Now, the series-connected voltage sources can be combined as an equivalent single voltage source as shown in Fig. 2.10(d). Finally, again using source transformation technique, the voltage source of Fig. 2.10(d) can be converted into an equivalent current source as shown in Fig. 2.10(e).



$R_{eq} = R_1 + R_2 + R_3 + ... + R_n$ $E_{eq} = I_1 R_1 + I_2 R_2 + I_3 R_3 + ... + I_n R_n$	$I_{eq} = \dfrac{E_{eq}}{R_{eq}}$ $= \dfrac{1}{R_{eq}} \left[I_1 R_1 + I_2 R_2 + I_3 R_3 + ... + I_n R_n \right]$

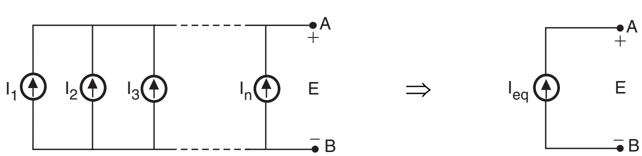
Fig. 2.10 : *Series connection of current sources with internal resistance.*

Parallel Connection of Current Sources

In parallel connection, current sources with different current ratings may be connected in parallel. However in parallel connection, the voltage across the sources is the same.

Case i : Parallel connection of ideal current source

The parallel-connected ideal current sources shown in Fig. 2.11(a) can be converted into a single equivalent source using KCL, as shown in Fig. 2.11(b).



$$I_{eq} = I_1 + I_2 + I_3 + ... + I_n$$

Fig. a : *Parallel connection.* ***Fig. b :*** *Equivalent of parallel connection.*

Fig. 2.11 : *Parallel connection of ideal current sources.*

Case ii : Parallel connection of current sources with internal resistance

The parallel-connected current sources with internal resistance shown in Fig. 2.12(a) can be represented as ideal sources with a parallel-resistance as shown in Fig. 2.12(b).

The parallel-connected resistances of Fig. 2.12(c) can be represented by an equivalent resistance as shown in Fig. 2.12(d). The parallel-connected current sources of Fig. 2.12(c) can be represented by an equivalent current source as shown in Fig. 2.12(d).

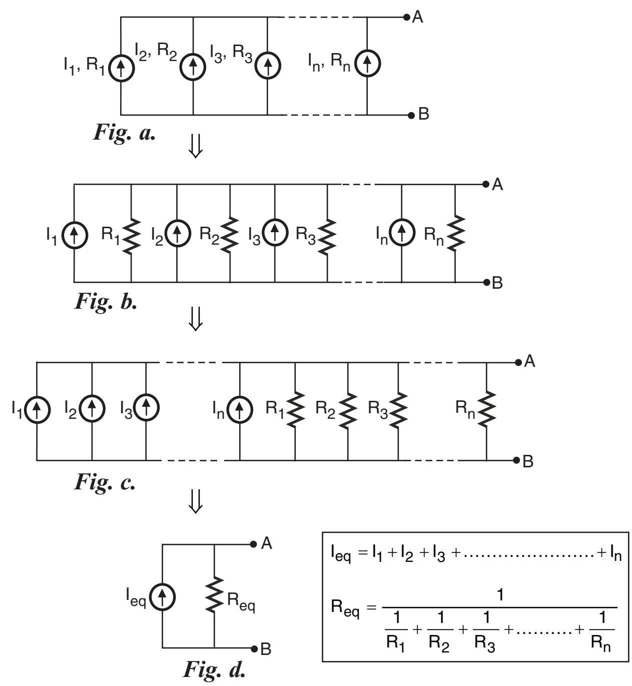


Fig. 2.12 : *Parallel connection of current sources with internal resistance.*

2.1.4 Inductances in Series and Parallel

Equivalent of Series-connected Inductances

Consider a circuit with series combination of two inductances L_1 and L_2 connected to an ac source of voltage v as shown in Fig. 2.13(a). Let the current through the circuit be i and voltages across L_1 and L_2 be v_1 and v_2, respectively.

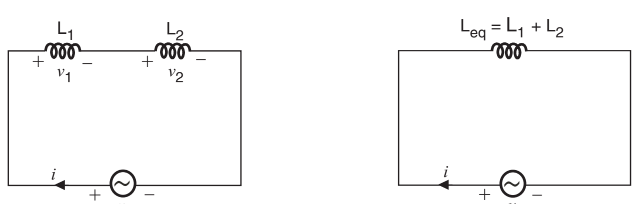


Fig. a : *Inductance in series.* ***Fig. b :*** *Equivalent circuit of Fig. a.*

Fig. 2.13 : *Inductance in series.*

By Faraday's Law, we can write,

$$v_1 = L_1 \frac{di}{dt} \quad \text{and} \quad v_2 = L_2 \frac{di}{dt} \qquad \text{..... (2.7)}$$

In Fig. 2.13(a), using Kirchhoff's Voltage Law, we can write,

$$v = v_1 + v_2$$

$$= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} \qquad \boxed{\text{Using equation (2.7)}}$$

$$= \left(L_1 + L_2\right) \frac{di}{dt}$$

Let, $v = L_{eq} \frac{di}{dt}$

$$\text{where, } L_{eq} = L_1 + L_2 \qquad \text{..... (2.8)}$$

From equation (2.8), we can say that the series-connected inductances L_1 and L_2 can be replaced with an equivalent inductance L_{eq} given by the sum of individual inductances L_1 and L_2.

This concept can be extended to any number of inductances in series. Therefore, we can say that inductances in series can be replaced with an equivalent inductance whose value is given by the sum of individual inductances.

"*When n number of identical inductances of value L are connected in series, the equivalent inductance $L_{eq} = nL$*".

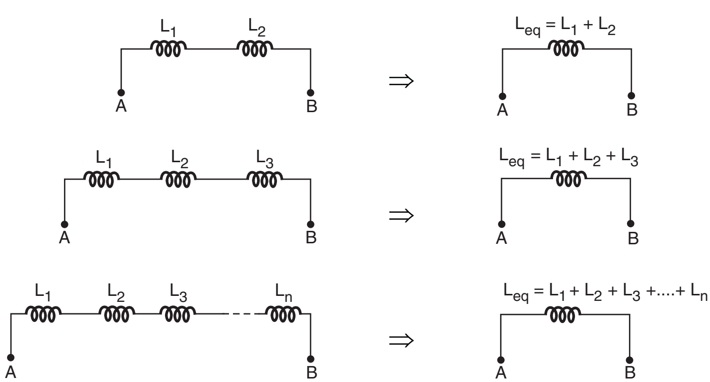


Fig. 2.14 : *Inductances in series and their equivalents.*

Equivalent of Parallel-connected Inductances

Consider a circuit with two inductances in parallel and connected to an ac source of voltage v as shown in Fig. 2.15(a). Let i be the current supplied by the source and i_1 and i_2 be the currents through L_1 and L_2, respectively. Since the inductances are parallel to the source, the voltage across them will be the same.

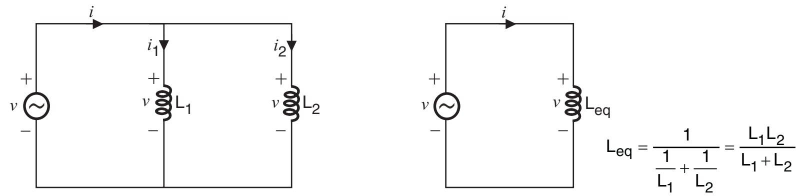


Fig. a : *Inductance in parallel.* ***Fig. b :*** *Equivalent circuit of Fig. a.*

Fig. 2.15 : *Inductance in parallel.*

We know that,

$$i_1 = \frac{1}{L_1}\int v\,dt \quad \text{and} \quad i_2 = \frac{1}{L_2}\int v\,dt \qquad .....(2.9)$$

By Kirchhoff's Current Law, we can write,

$$i = i_1 + i_2$$

$$= \frac{1}{L_1}\int v\,dt + \frac{1}{L_2}\int v\,dt \qquad \boxed{\text{Using equation (2.9)}}$$

$$= \left(\frac{1}{L_1} + \frac{1}{L_2}\right)\int v\,dt$$

On differentiating the above equation, we get,

$$\frac{di}{dt} = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)v \quad \Rightarrow \quad v = \frac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2}}\,\frac{di}{dt}$$

Let, $$v = L_{eq}\frac{di}{dt} \qquad .....(2.10)$$

$$\text{where, } L_{eq} = \frac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2}} = \frac{L_1 L_2}{L_1 + L_2} \qquad .....(2.11)$$

Also, $$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} \qquad .....(2.12)$$

From equation (2.10), we can say that the parallel-connected inductances L_1 and L_2 can be replaced with an equivalent inductance given by equation (2.11). From equation (2.12), we can say that the inverse of the equivalent inductance of parallel-connected inductances is equal to the sum of the inverse of individual inductances.

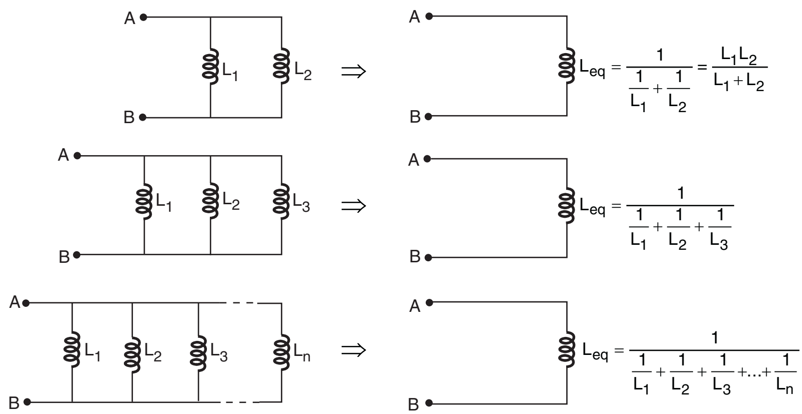


Fig. 2.16 : *Inductances in parallel and their equivalents.*

This concept can be extended to any number of inductances in parallel. Therefore, we can say that inductances in parallel can be replaced with an equivalent inductance whose value is given by the inverse of sum of the inverse of individual inductances.

"When n number of identical inductances of value L are connected in parallel, the equivalent inductance $L_{eq} = \dfrac{L}{n}$*".*

2.1.5 Capacitances in Series and Parallel

Equivalent of Series-connected Capacitances

Consider a circuit with series combination of two capacitances C_1 and C_2 connected to an ac source of voltage, v as shown in Fig. 2.17(a). Let the current through the circuit be i and voltages across C_1 and C_2 be v_1 and v_2, respectively.

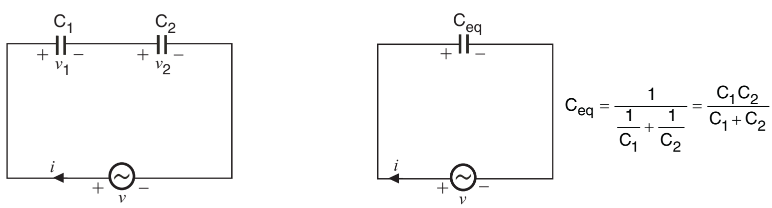


Fig. a : *Capacitances in series.* ***Fig. b :*** *Equivalent circuit of Fig. a.*

Fig. 2.17 : *Capacitances in series.*

We know that,

$$v_1 = \frac{1}{C_1}\int i\,dt \quad \text{and} \quad v_2 = \frac{1}{C_2}\int i\,dt \qquad .....(2.13)$$

With reference to Fig. 2.17(a) using Kirchhoff's Voltage Law, we can write,

$$v = v_1 + v_2$$

$$v = \frac{1}{C_1}\int i\,dt + \frac{1}{C_2}\int i\,dt \qquad \boxed{\text{Using equation (2.13)}}$$

$$= \left(\frac{1}{C_1}+\frac{1}{C_2}\right)\int i\,dt$$

Let, $$v = \frac{1}{C_{eq}}\int i\,dt \qquad .....(2.14)$$

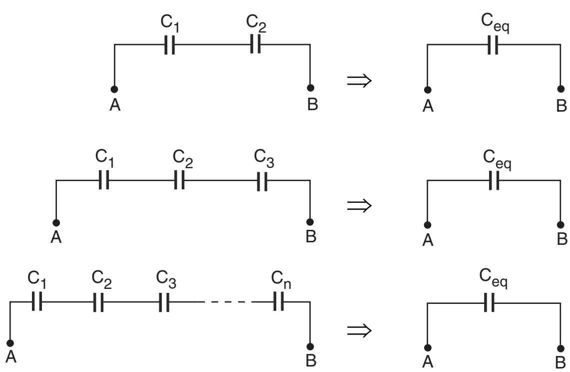
where, $$C_{eq} = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}} = \frac{C_1 C_2}{C_1 + C_2} \qquad .....(2.15)$$

Also, $$\frac{1}{C_{eq}} = \frac{1}{C_1}+\frac{1}{C_2} \qquad .....(2.16)$$

From equation (2.14), we can say that the series-connected capacitances C_1 and C_2 can be replaced with an equivalent capacitance given by equation (2.15). From equation (2.16), we can say that the inverse of the equivalent capacitance of series-connected capacitances is equal to the sum of the inverse of individual capacitances.

This concept can be extended to any number of capacitances in series. Therefore, we can say that capacitances in series can be replaced with an equivalent capacitance whose value is given by the inverse of sum of the inverse of individual capacitances.

"*When n number of identical capacitances of value C are connected in series, the equivalent capacitance, $C_{eq} = C/n$*".



$$C_{eq} = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}} = \frac{C_1 C_2}{C_1 + C_2}$$

$$C_{eq} = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}}$$

$$C_{eq} = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}+...+\frac{1}{C_n}}$$

Fig. 2.18 : *Capacitances in series and their equivalents.*

Equivalent of Parallel-connected Capacitances

Consider a circuit in which two capacitances in parallel are connected to an ac source of voltage, v as shown in Fig. 2.19(a). Let i be the current supplied by the source and i_1 and i_2 be the currents through C_1 and C_2, respectively. Since the capacitances are parallel to the source, the voltage across them will be the same.

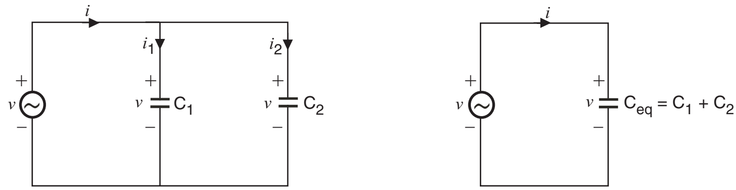


Fig. a : *Capacitances in parallel.* ***Fig. b :*** *Equivalent circuit of Fig. a.*

Fig. 2.19 : *Capacitances in parallel.*

We know that,

$$i_1 = C_1 \frac{dv}{dt} \quad \text{and} \quad i_2 = C_2 \frac{dv}{dt} \qquad .....(2.17)$$

By Kirchhoff's Current Law, we can write,

$$i = i_1 + i_2$$

$$= C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} \qquad \boxed{\text{Using equation (2.17)}}$$

$$= \left(C_1 + C_2\right)\frac{dv}{dt}$$

Let, $i = C_{eq} \dfrac{dv}{dt}$ .....(2.18)

where, $C_{eq} = C_1 + C_2$

From equation (2.18), we can say that the parallel-connected capacitances C_1 and C_2 can be replaced with an equivalent capacitance C_{eq} given by the sum of individual capacitances C_1 and C_2.

This concept can be extended to any number of capacitances in parallel. Therefore, we can say that capacitances in parallel can be replaced with an equivalent capacitance whose value is given by the sum of individual capacitances.

"*When n number of identical capacitances of value C are connected in parallel, the equivalent capacitance $C_{eq} = nC$*".

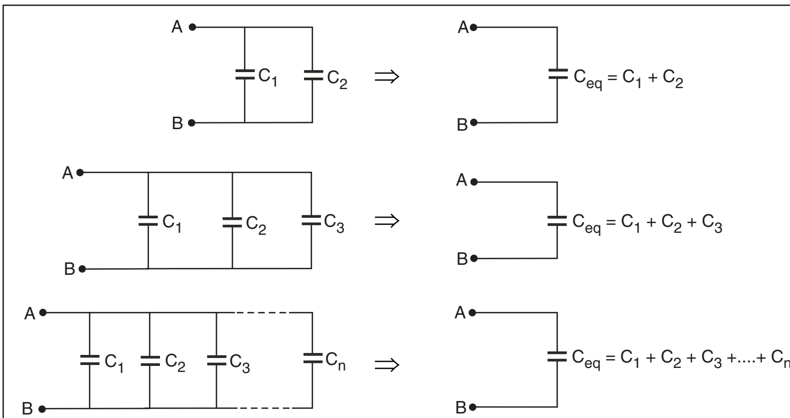


Fig. 2.20 : *Capacitances in parallel and their equivalents.*

2.1.6 Impedances in Series and Parallel

Equivalent of Series-connected Impedances

Consider a circuit with series combination of two impedances $\overline{Z}_1$ and $\overline{Z}_2$ connected to an ac source of voltage $\overline{V}$ volts rms value as shown in Fig. 2.21(a). Let the current through the elements be $\overline{I}$. Now, by Ohm's law the voltage across $\overline{Z}_1$ and $\overline{Z}_2$ are $\overline{I}\,\overline{Z}_1$ and $\overline{I}\,\overline{Z}_2$, respectively.

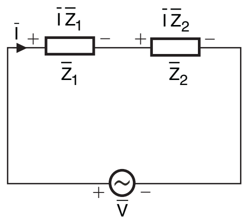


Fig. a : *Impedances in series.*

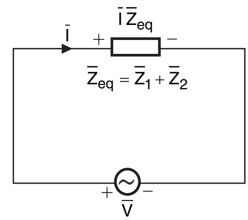


Fig. b : *Equivalent circuit of Fig. a.*

Fig. 2.21 : *Impedances in series and their equivalent.*

By Kirchhoff's Voltage Law, we can write,

$$\overline{V} = \overline{I}\,\overline{Z}_1 + \overline{I}\,\overline{Z}_2$$

$$= \overline{I}\left(\overline{Z}_1 + \overline{Z}_2\right)$$

Let, $\overline{V} = \overline{I}\,\overline{Z}_{eq}$ .....(2.19)

where, $\overline{Z}_{eq} = \overline{Z}_1 + \overline{Z}_2$

From equation (2.19), we can say that the series-connected impedances $\overline{Z}_1$ and $\overline{Z}_2$ can be replaced with an equivalent impedance $\overline{Z}_{eq}$ given by the sum of individual impedances $\overline{Z}_1$ and $\overline{Z}_2$.

This concept can be extended to any number of impedances in series. Therefore, we can say that impedances in series can be replaced with an equivalent impedance whose value is given by the sum of individual impedances.

"When n number of identical impedances of value $\overline{Z}$ are connected in series, the equivalent impedance, $\overline{Z}_{eq} = n\overline{Z}$".

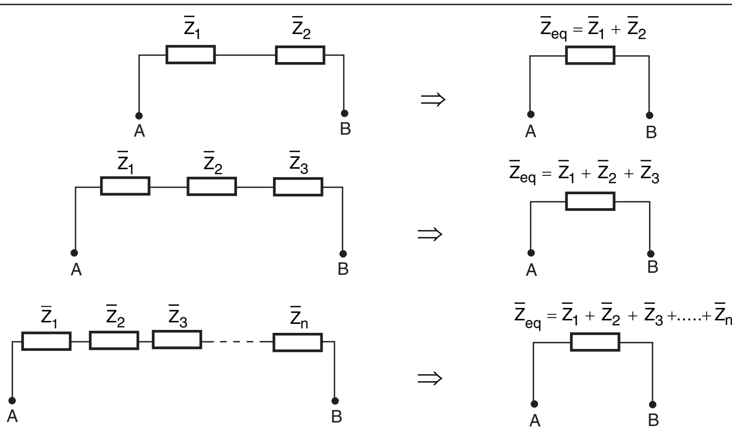


Fig. 2.22 : *Impedances in series and their equivalents.*

Equivalent of Parallel-connected Impedances

Consider a circuit in which two impedances in parallel are connected to an ac source of $\bar{V}$ volts rms value as shown in Fig. 2.23(a). Let $\bar{I}$ be the current supplied by the source and $\bar{I}_1$ and $\bar{I}_2$ be the currents through $\bar{Z}_1$ and $\bar{Z}_2$, respectively. Since the impedances are parallel to the source, the voltage across them will be the same as that of the source voltage.

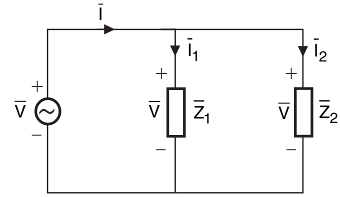
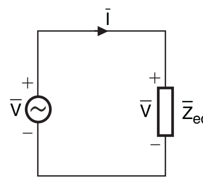


Fig. a : Impedances in parallel.



$$\boxed{\bar{Z}_{eq} = \frac{1}{\frac{1}{\bar{Z}_1} + \frac{1}{\bar{Z}_2}} = \frac{\bar{Z}_1 \bar{Z}_2}{\bar{Z}_1 + \bar{Z}_2}}$$

Fig. b : Equivalent circuit of Fig. a.
Fig. 2.23 : Impedances in parallel and their equivalent.

By Ohm's law, we can write,

$$\bar{I}_1 = \frac{\bar{V}}{\bar{Z}_1} \quad \text{and} \quad \bar{I}_2 = \frac{\bar{V}}{\bar{Z}_2} \qquad .....(2.20)$$

By Kirchhoff's Current Law, we can write,

$$\bar{I} = \bar{I}_1 + \bar{I}_2$$

$$= \frac{\bar{V}}{\bar{Z}_1} + \frac{\bar{V}}{\bar{Z}_2} \qquad \boxed{\text{Using equation (2.20)}}$$

$$= \bar{V}\left(\frac{1}{\bar{Z}_1} + \frac{1}{\bar{Z}_2}\right)$$

$$\therefore \quad \bar{V} = \bar{I}\left(\frac{1}{\frac{1}{\bar{Z}_1} + \frac{1}{\bar{Z}_2}}\right)$$

Let, $\bar{V} = \bar{I}\,\bar{Z}_{eq}$ .....(2.21)

$$\text{where,} \quad \bar{Z}_{eq} = \frac{1}{\frac{1}{\bar{Z}_1} + \frac{1}{\bar{Z}_2}} = \frac{\bar{Z}_1 \bar{Z}_2}{\bar{Z}_1 + \bar{Z}_2} \qquad ..... (2.22)$$

$$\text{Also,} \quad \frac{1}{\bar{Z}_{eq}} = \frac{1}{\bar{Z}_1} + \frac{1}{\bar{Z}_2} \qquad .....(2.23)$$

From equation (2.21), we can say that the parallel impedances $\bar{Z}_1$ and $\bar{Z}_2$ can be replaced with an equivalent impedance given by equation (2.22). From equation (2.23), we can say that the inverse of the equivalent impedance of parallel-connected impedances is equal to the sum of the inverse of individual impedances.

This concept can be extended to any number of impedances in parallel. Therefore, we can say that impedances in parallel can be replaced with an equivalent impedance whose value is given by the inverse of sum of the inverse of individual impedances.

"When n number of identical impedances of value $\overline{Z}$ are connected in parallel, the equivalent impedance $\overline{Z}_{eq} = \overline{Z}/n$".

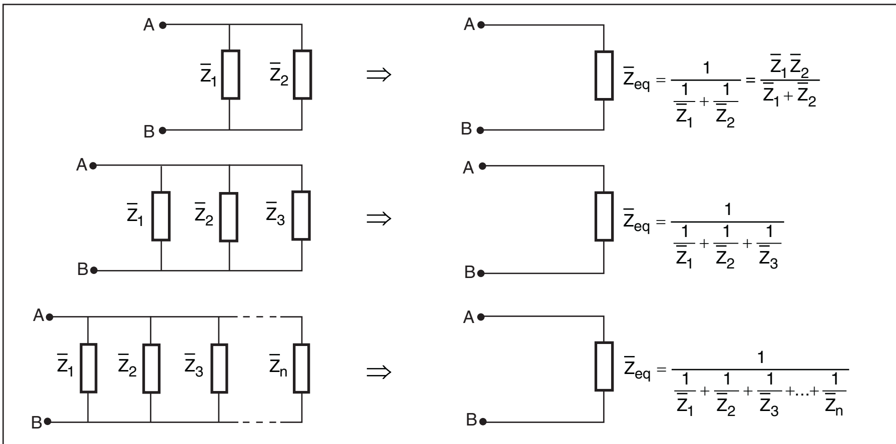


Fig. 2.24 : *Impedances in parallel and their equivalents.*

2.1.7 Reactances in Series and Parallel

The reactances in series and parallel combinations can be combined to give an equivalent reactance similar to that of impedance. In fact, the reactances are impedances with imaginary part alone. The equivalent reactances of series and parallel combinations of reactances are diagrammatically illustrated in Fig. 2.25.

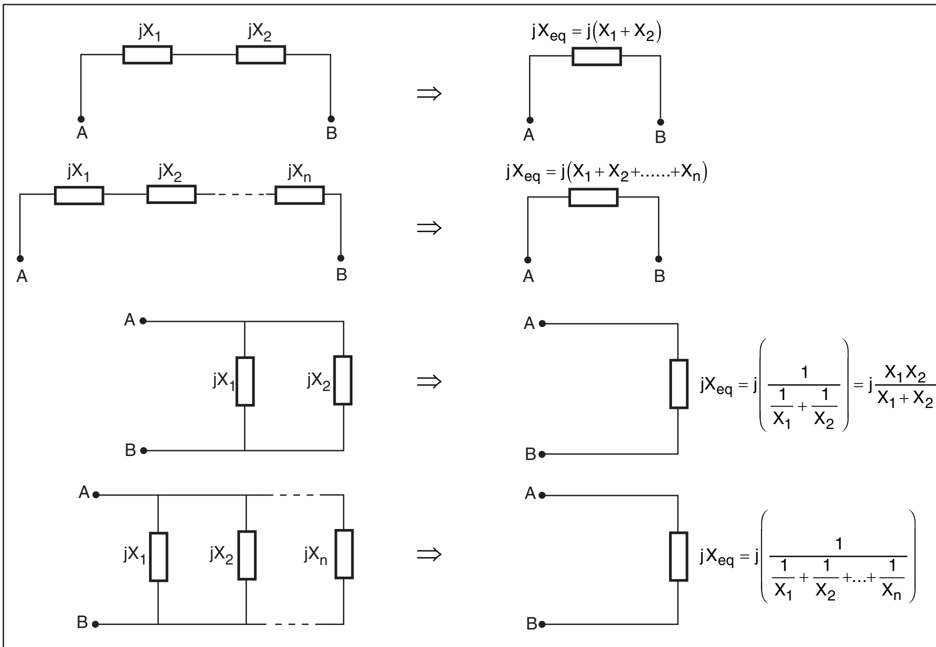


Fig. 2.25 : *Series and parallel combinations of reactances and their equivalents.*

2.1.8 Conductances in Series and Parallel

Equivalent of Series-connected Conductances

Consider a circuit with series combination of two conductances G_1 and G_2 connected to a dc source of voltage V volts as shown in Fig. 2.26(a). Let I be the current through the conductances and V_1 and V_2 be the voltages across G_1 and G_2, respectively.

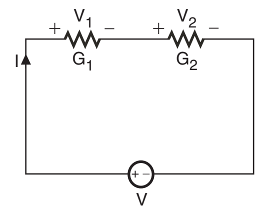
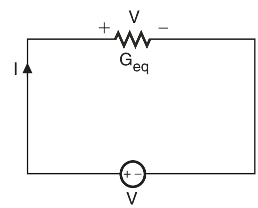


Fig. a : *Conductances in series.*



$$G_{eq} = \frac{1}{\frac{1}{G_1} + \frac{1}{G_2}} = \frac{G_1 G_2}{G_1 + G_2}$$

Fig. b : *Equivalent circuit of Fig. a.*

Fig. 2.26 : *Conductances in series and their equivalent.*

From Fig. 2.26, we can write,

$$V_1 = \frac{I}{G_1} \quad \text{and} \quad V_2 = \frac{I}{G_2} \qquad \text{..... (2.24)}$$

By Kirchhoff's Voltage Law, we can write,

$$V = V_1 + V_2$$

$$= \frac{I}{G_1} + \frac{I}{G_2} \qquad \boxed{\text{Using equation (2.24)}}$$

$$= I\left(\frac{1}{G_1} + \frac{1}{G_2}\right)$$

$$\therefore \quad I = V\left(\frac{1}{\frac{1}{G_1} + \frac{1}{G_2}}\right)$$

Let, $I = VG_{eq}$ ..... (2.25)

$$\text{where,} \quad G_{eq} = \frac{1}{\frac{1}{G_1} + \frac{1}{G_2}} = \frac{G_1 G_2}{G_1 + G_2} \qquad \text{..... (2.26)}$$

$$\text{Also,} \quad \frac{1}{G_{eq}} = \frac{1}{G_1} + \frac{1}{G_2} \qquad \text{.....(2.27)}$$

From equation (2.25), we can say that the series-connected conductances G_1 and G_2 can be replaced with an equivalent conductance given by equation (2.26). From equation (2.27), we can say that the inverse of the equivalent conductance of series-connected conductances is equal to sum of the inverse of individual conductances.

This concept can be extended to any number of conductances in series. Therefore, we can say that conductances in series can be replaced with an equivalent conductance whose value is given by the inverse of sum of the inverse of individual conductances.

"*When n number of identical conductances of value G are connected in series, the equivalent conductance $G_{eq} = G/n$*".

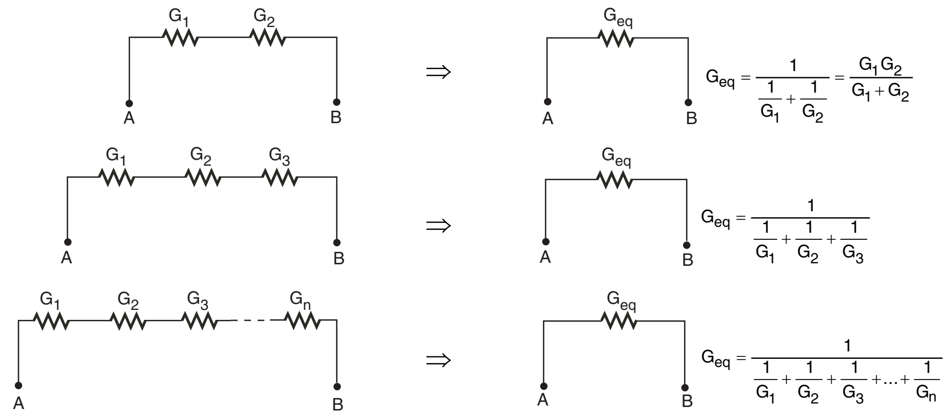


Fig. 2.27 : *Conductances in series and their equivalents.*

Equivalent of Parallel-connected Conductances

Consider a circuit with two conductances in parallel and connected to a source of voltage V volts as shown in Fig. 2.28(a). Let I be the current supplied by the source and I_1 and I_2 be the current through G_1 and G_2, respectively. Since the conductances are parallel to the source, the voltage across them will be the same as that of the source.

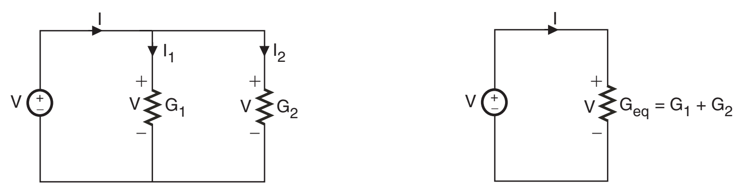


Fig. a : *Conductances in parallel.* ***Fig. b :*** *Equivalent circuit of Fig. a.*

Fig. 2.28 : *Conductances in parallel and their equivalent.*

From Fig. 2.28, we can write,

$$I_1 = VG_1 \quad \text{and} \quad I_2 = VG_2 \qquad \text{..... (2.28)}$$

By Kirchhoff's Current Law, we can write,

$$I = I_1 + I_2$$

$$= VG_1 + VG_2 \qquad \boxed{\text{Using equation (2.28)}}$$

$$= V(G_1 + G_2)$$

Let, $$I = VG_{eq} \qquad \text{..... (2.29)}$$

where, $G_{eq} = G_1 + G_2$

From equation (2.29), we can say that the parallel-connected conductances G_1 and G_2 can be replaced with an equivalent conductance G_{eq} given by the sum of individual conductances G_1 and G_2.

This concept can be extended to any number of conductances in parallel. Therefore, we can say that conductances in parallel can be replaced with an equivalent conductance whose value is given by the sum of individual conductances.

"*When n number of identical conductance of value G are connected in parallel, the equivalent conductance* $G_{eq} = nG$".

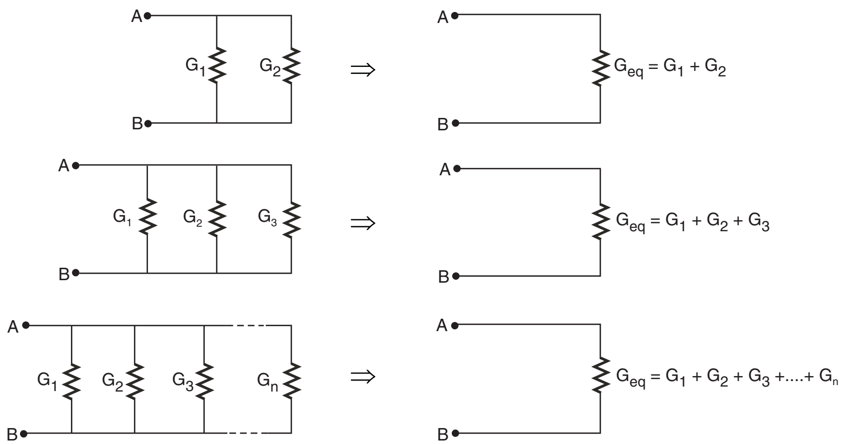


Fig. 2.29 : *Conductances in parallel and their equivalents.*

2.1.9 Admittances in Series and Parallel

Equivalent of Series-connected Admittances

Consider a circuit with series combination of two admittances $\overline{Y}_1$ and $\overline{Y}_2$ connected to an ac source of voltage $\overline{V}$ volts rms value as shown in Fig. 2.30(a). Let $\overline{I}$ be the current through the admittances and $\overline{V}_1$ and $\overline{V}_2$ be the voltages across $\overline{Y}_1$ and $\overline{Y}_2$, respectively.

From Fig. 2.30, we can write,

$$\overline{V}_1 = \frac{\overline{I}}{\overline{Y}_1} \quad \text{and} \quad \overline{V}_2 = \frac{\overline{I}}{\overline{Y}_2} \qquad .....(2.30)$$

By Kirchhoff's Voltage Law, we can write,

$$\overline{V} = \overline{V}_1 + \overline{V}_2$$

$$= \frac{\overline{I}}{\overline{Y}_1} + \frac{\overline{I}}{\overline{Y}_2} \qquad \boxed{\text{Using equation (2.30)}}$$

$$= \overline{I}\left(\frac{1}{\overline{Y}_1} + \frac{1}{\overline{Y}_2}\right)$$

$$\therefore \quad \overline{I} = \overline{V}\left(\frac{1}{\frac{1}{\overline{Y}_1} + \frac{1}{\overline{Y}_2}}\right)$$

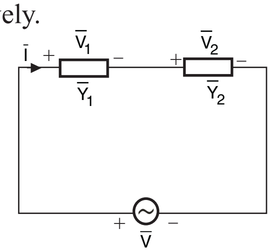


Fig. a : *Admittances in series.*

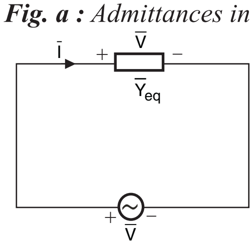


Fig. b : *Equivalent circuit of Fig. a.*

Fig. 2.30 : *Admittances in series and their equivalent.*

Let, $\bar{I} = \overline{V}\,\overline{Y}_{eq}$ .....(2.31)

$$\text{where,} \quad \overline{Y}_{eq} = \frac{1}{\frac{1}{\overline{Y}_1} + \frac{1}{\overline{Y}_2}} = \frac{\overline{Y}_1\,\overline{Y}_2}{\overline{Y}_1 + \overline{Y}_2} \qquad ..... (2.32)$$

$$\text{Also,} \quad \frac{1}{\overline{Y}_{eq}} = \frac{1}{\overline{Y}_1} + \frac{1}{\overline{Y}_2} \qquad ..... (2.33)$$

From equation (2.31), we can say that the series-connected admittances $\overline{Y}_1$ and $\overline{Y}_2$ can be replaced with an equivalent admittance given by equation (2.32). From equation (2.33), we can say that the inverse of the equivalent admittance of series-connected admittances is equal to the sum of the inverse of individual admittances.

This concept can be extended to any number of admittances in series. Therefore, we can say that admittances in series can be replaced with an equivalent admittance whose value is given by the inverse of sum of the inverse of individual admittances.

"*When n number of identical admittances of value $\overline{Y}$ are connected in series, the equivalent admittance $\overline{Y}_{eq} = \overline{Y}/n$*".

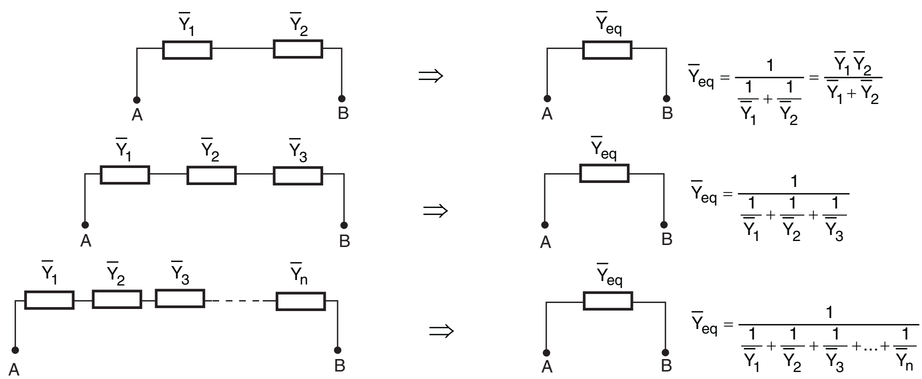


Fig. 2.31 : *Series combination of admittances and their equivalents.*

Equivalent of Parallel-connected Admittances

Consider a circuit with two admittances in parallel and connected to an ac source of voltage $\overline{V}$ volts rms value as shown in Fig. 2.32(a). Let $\bar{I}$ be the current supplied by the source and $\bar{I}_1$ and $\bar{I}_2$ be the currents through $\overline{Y}_1$ and $\overline{Y}_2$, respectively. Since the admittances are parallel to the source, the voltage across them will be the same as that of the source.

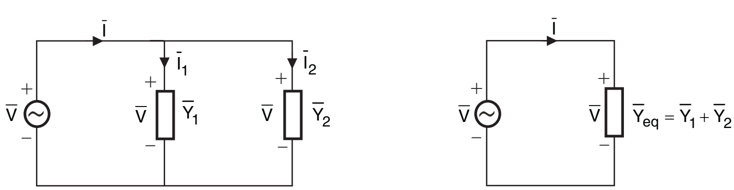


Fig. a : *Admittances in parallel.* ***Fig. b :*** *Equivalent circuit of Fig. a.*

Fig. 2.32 : *Admittances in parallel and their equivalent.*

From Fig. 2.32, we can write,

$$\bar{I}_1 = \bar{V}\,\bar{Y}_1 \text{ and } \bar{I}_2 = \bar{V}\,\bar{Y}_2 \qquad .....(2.34)$$

By Kirchhoff's Current Law, we can write,

$$\bar{I} = \bar{I}_1 + \bar{I}_2$$

$$= \bar{V}\,\bar{Y}_1 + \bar{V}\,\bar{Y}_2 \qquad \boxed{\text{Using equation (2.34)}}$$

$$= \bar{V}\left(\bar{Y}_1 + \bar{Y}_2\right)$$

Let, $\bar{I} = \bar{V}\,\bar{Y}_{eq}$ ..... (2.35)

where, $\bar{Y}_{eq} = \bar{Y}_1 + \bar{Y}_2$

From equation (2.35), we can say that the parallel-connected admittances $\bar{Y}_1$ and $\bar{Y}_2$ can be replaced with an equivalent admittance $\bar{Y}_{eq}$ given by the sum of individual admittances $\bar{Y}_1$ and $\bar{Y}_2$.

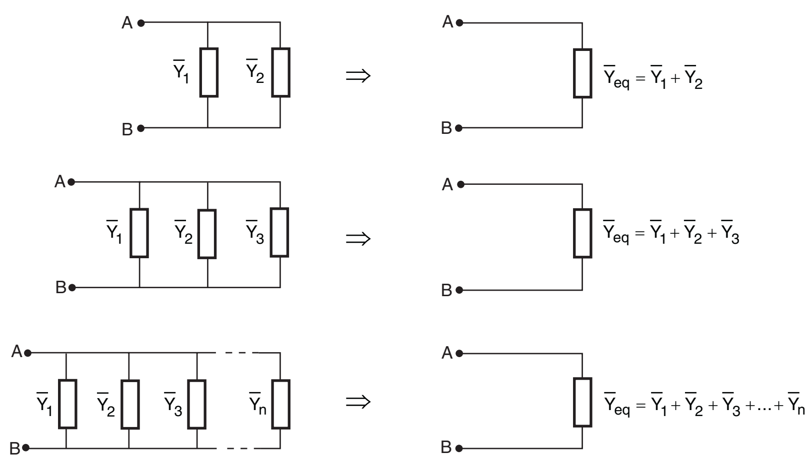


Fig. 2.33 : *Admittances in parallel and their equivalents.*

This concept can be extended to any number of admittances in parallel. Therefore, we can say that admittances in parallel can be replaced with an equivalent admittance whose value is given by the sum of individual admittances.

2.1.10 Susceptances in Series and Parallel

The susceptances in series and parallel combination can be combined to give an equivalent susceptance similar to that of admittance. In fact, the susceptances are admittances with imaginary part alone. The equivalent susceptances of series and parallel combinations of susceptances are diagrammatically illustrated in Fig. 2.34.

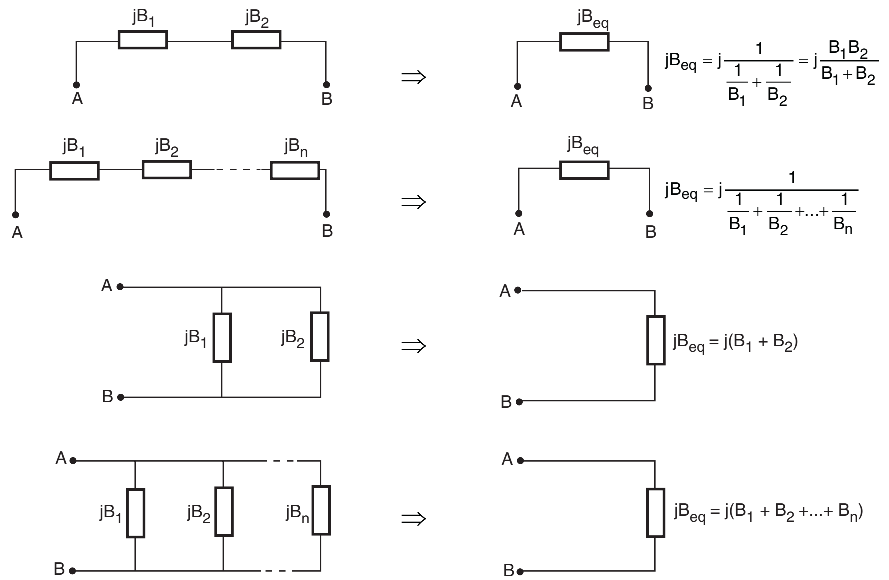


Fig. 2.34 : *Series and parallel combinations of susceptances and their equivalents.*

2.1.11 Generalised Concept of Reducing Series/Parallel-connected Parameters

In order to generalise the concept of reducing the series/parallel-connected parameters, they can be classified into two groups.

Let the parameters Resistance (R), Inductance (L), Reactance (X) and Impedance (Z) be group-1 parameters. Let the parameters Conductance (G), Capacitance (C), Susceptance (B) and Admittance (Y) be group-2 parameters.

The equivalent of series-connected group-1 parameters will be given by the sum of individual parameters. The equivalent of parallel-connected group-1 parameters will be given by the inverse of the sum of individual inverses. The equivalent of series and parallel-connected group-1 parameters are summarised in Table 2.1.

The equivalent of series-connected group-2 parameters will be given by the inverse of the sum of individual parameters. The equivalent of parallel-connected group-2 parameters will be given by the sum of individual parameters. The equivalent of series and parallel-connected group-2 parameters are summarised in Table 2.2.

Table 2.1 : Summary of Equivalent of Series/Parallel-Connected Group-1 Parameters

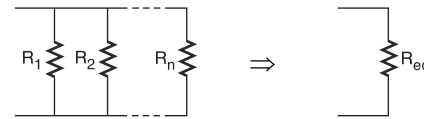
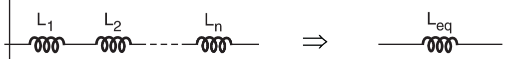
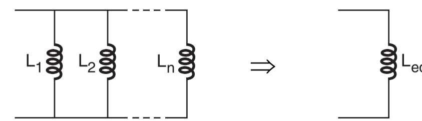
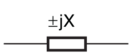
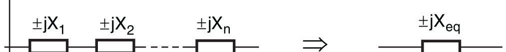
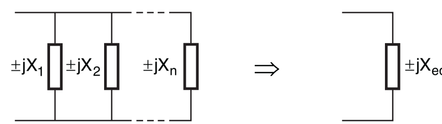
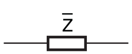
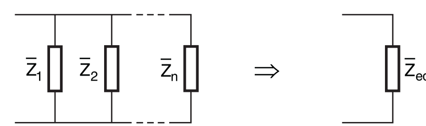
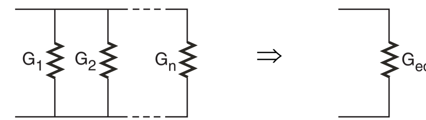
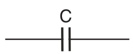
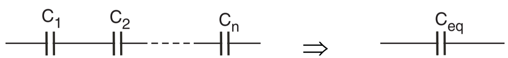
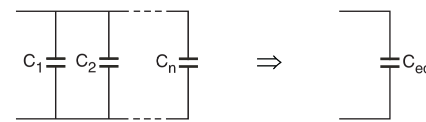
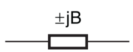
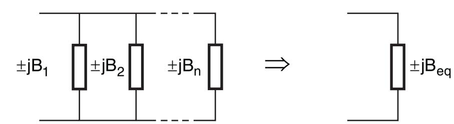
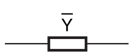
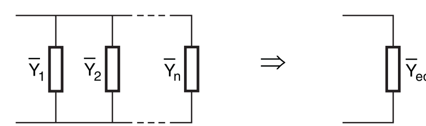
Group-1 Parameter	Series connection of parameters and their equivalent	Parallel connection of parameters and their equivalent
Resistance 	 $R_{eq} = R_1 + R_2 + ... + R_n$	 $R_{eq} = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + ... + \dfrac{1}{R_n}}$
Inductance 	 $L_{eq} = L_1 + L_2 + ... + L_n$	 $L_{eq} = \dfrac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2} + ... + \dfrac{1}{L_n}}$
Reactance 	 $\pm jX_{eq} = \pm jX_1 \pm jX_2 \pm ..... \pm jX_n$	 $\pm jX_{eq} = \dfrac{1}{\dfrac{1}{\pm jX_1} + \dfrac{1}{\pm jX_2} + ... + \dfrac{1}{\pm jX_n}}$
Impedance 	 $\overline{Z}_{eq} = \overline{Z}_1 + \overline{Z}_2 + ... + \overline{Z}_n$	 $\overline{Z}_{eq} = \dfrac{1}{\dfrac{1}{\overline{Z}_1} + \dfrac{1}{\overline{Z}_2} + ... + \dfrac{1}{\overline{Z}_n}}$

Table 2.2 : Summary of Equivalent of Series/Parallel-Connected Group-2 Parameters

Group-2 Parameter	Series connection of parameters and their equivalent	Parallel connection of parameters and their equivalent
Conductance 	 $G_{eq} = \dfrac{1}{\dfrac{1}{G_1}+\dfrac{1}{G_2}+.....+\dfrac{1}{G_n}}$	 $G_{eq} = G_1 + G_2 + ... + G_n$
Capacitance 	 $C_{eq} = \dfrac{1}{\dfrac{1}{C_1}+\dfrac{1}{C_2}+.....+\dfrac{1}{C_n}}$	 $C_{eq} = C_1 + C_2 + ... + C_n$
Susceptance 	 $\pm jB_{eq} = \dfrac{1}{\dfrac{1}{\pm jB_1}+\dfrac{1}{\pm jB_2}+.....+\dfrac{1}{\pm jB_n}}$	 $\pm jB_{eq} = \pm jB_1 \pm jB_2 \pm ..... \pm jB_n$
Admittance 	 $\bar{Y}_{eq} = \dfrac{1}{\dfrac{1}{\bar{Y}_1}+\dfrac{1}{\bar{Y}_2}+.....+\dfrac{1}{\bar{Y}_n}}$	 $\bar{Y}_{eq} = \bar{Y}_1 + \bar{Y}_2 + ... + \bar{Y}_n$

2.2 Voltage and Current Division

2.2.1 Voltage Division in Series-connected Resistances

Consider two resistances R_1 and R_2 in series which are connected to a dc source of V volts as shown in Fig. 2.35. Let I be the current supplied by the source and V_1 and V_2 be the voltages across R_1 and R_2, respectively. Since the resistances are in series, the current through them will be I amperes.

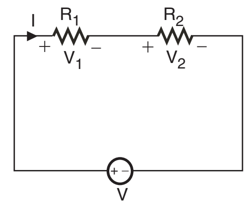


Fig. 2.35 : *Resistances in series.*

Equations (2.36) and (2.37) given below can be used to determine the voltages in series-connected resistances in terms of the total voltage across the series combination and the value of individual resistances. Hence, these equations are called the **voltage division rule**.

$$\boxed{V_1 = V \times \frac{R_1}{R_1 + R_2}} \quad \text{..... (2.36)}$$

$$\boxed{V_2 = V \times \frac{R_2}{R_1 + R_2}} \quad \text{..... (2.37)}$$

The following equation will be helpful to remember the voltage division rule.

In two series-connected resistances,

$$\boxed{\text{Voltage across one of the resistances} = \frac{\begin{matrix}\text{Total voltage across} \\ \text{series combination}\end{matrix} \times \text{Value of the resistance}}{\text{Sum of individual resistances}}}$$

Proof for Voltage Division Rule

With reference to Fig. 2.35, by Ohm's law, we can write,

$$V_1 = IR_1 \quad \text{..... (2.38)}$$

$$V_2 = IR_2 \quad \text{..... (2.39)}$$

By Kirchhoff's Voltage Law, we get,

$$V = V_1 + V_2 \quad \text{..... (2.40)}$$

On substituting for V_1 and V_2 from equations (2.38) and (2.39) in equation (2.40), we get,

$$V = IR_1 + IR_2 = I(R_1 + R_2)$$

$$\therefore \quad I = \frac{V}{R_1 + R_2} \quad \text{..... (2.41)}$$

On substituting for I from equation (2.41) in equation (2.38), we get,

$$V_1 = \frac{V}{R_1 + R_2} \times R_1 = V \times \frac{R_1}{R_1 + R_2}$$

On substituting for I from equation (2.41) in equation (2.39), we get,

$$V_2 = \frac{V}{R_1 + R_2} \times R_2 = V \times \frac{R_2}{R_1 + R_2}$$

2.2.2 Voltage Division in Series-connected Impedances

Consider two impedances $\overline{Z}_1$ and $\overline{Z}_2$ in series which are connected to an ac source of $\overline{V}$ volts as shown in Fig. 2.36. Let $\overline{I}$ be the current supplied by the source and $\overline{V}_1$ and $\overline{V}_2$ be the voltages across $\overline{Z}_1$ and $\overline{Z}_2$, respectively. Since the impedances are in series, the current through them will be $\overline{I}$ amperes.

Equations (2.42) and (2.43) can be used to solve the voltages in series-connected impedances in terms of the total voltage across the series combination and the value of individual impedances. Hence, these equations are called the **voltage division rule**.

$$\boxed{\overline{V}_1 = \overline{V} \times \frac{\overline{Z}_1}{\overline{Z}_1 + \overline{Z}_2}} \quad .....(2.42)$$

$$\boxed{\overline{V}_2 = \overline{V} \times \frac{\overline{Z}_2}{\overline{Z}_1 + \overline{Z}_2}} \quad .....(2.43)$$

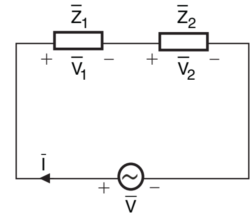


***Fig. 2.36 :** Impedances in series.*

The following equation will be helpful to remember the voltage division rule.

In two series-connected impedances,

$$\boxed{\text{Voltage across one of the impedances} = \frac{\text{Total voltage across series combination} \times \text{Value of the impedance}}{\text{Sum of individual impedances}}}$$

2.2.3 Current Division in Parallel-connected Resistances

Consider two resistances R_1 and R_2 in parallel are connected to a dc source of V volts as shown in Fig. 2.37. Let I be the current supplied by the source and I_1 and I_2 be the current through R_1 and R_2, respectively. Since the resistances are parallel to the source, the voltage across them will be V volts.

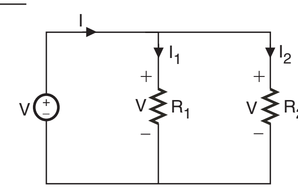


***Fig. 2.37 :** Resistances in parallel.*

Equations (2.44) and (2.45) given below can be used to determine the currents in parallel-connected resistances in terms of the total current drawn by the parallel combination and the values of individual resistances. Hence, these equations are called the **current division rule**.

$$\boxed{I_1 = I \times \frac{R_2}{R_1 + R_2}} \quad ..... (2.44)$$

$$\boxed{I_2 = I \times \frac{R_1}{R_1 + R_2}} \quad ..... (2.45)$$

The following equation will be helpful to remember the current division rule.

In two parallel-connected resistances,

$$\boxed{\text{Current through one of the resistances} = \frac{\text{Total current drawn by parallel combination} \times \text{Value of the other resistance}}{\text{Sum of individual resistances}}}$$

Proof for Current Division Rule

With reference to Fig. 2.37, by Ohm's law we can write,

$$I_1 = \frac{V}{R_1} \qquad \text{..... (2.46)}$$

$$I_2 = \frac{V}{R_2} \qquad \text{..... (2.47)}$$

By Kirchhoff's Current Law, we get,

$$I = I_1 + I_2$$

$$= \frac{V}{R_1} + \frac{V}{R_2} = V\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = V\left(\frac{R_2+R_1}{R_1 R_2}\right) \qquad \boxed{\text{Using equations (2.46) and (2.47)}}$$

$$\therefore \quad V = I \times \frac{R_1 R_2}{R_1+R_2} \qquad \text{..... (2.48)}$$

On substituting for V from equation (2.48) in equation (2.46), we get,

$$I_1 = I \times \frac{R_1 R_2}{R_1+R_2} \times \frac{1}{R_1} = I \times \frac{R_2}{R_1+R_2}$$

On substituting for V from equation (2.48) in equation (2.47), we get,

$$I_2 = I \times \frac{R_1 R_2}{R_1+R_2} \times \frac{1}{R_2} = I \times \frac{R_1}{R_1+R_2}$$

2.2.4 Current Division in Parallel-connected Impedances

Consider two impedances $\bar{Z}_1$ and $\bar{Z}_2$ in parallel are connected to an ac source of $\bar{V}$ volts as shown in Fig. 2.38. Let $\bar{I}$ be the current supplied by the source and $\bar{I}_1$ and $\bar{I}_2$ be the current through $\bar{Z}_1$ and $\bar{Z}_2$, respectively. Since the impedances are parallel to the source, the voltage across them will be $\bar{V}$ volts.

Equations (2.49) and (2.50) can be used to solve the currents in parallel-connected impedances in terms of the total current drawn by the parallel combination and the value of individual impedances. Hence, these equations are called the **current division rule**.

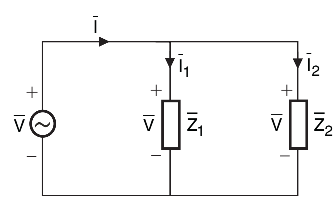


Fig. 2.38 : *Impedances in parallel.*

$$\boxed{\bar{I}_1 = \bar{I} \times \frac{\bar{Z}_2}{\bar{Z}_1 + \bar{Z}_2}} \qquad \text{.....(2.49)}$$

$$\boxed{\bar{I}_2 = \bar{I} \times \frac{\bar{Z}_1}{\bar{Z}_1 + \bar{Z}_2}} \qquad \text{.....(2.50)}$$

The following equation will be helpful to remember the current division rule.

In two parallel-connected impedances,

$$\boxed{\text{Current through one of the impedances} = \frac{\begin{matrix}\text{Total current drawn by} \\ \text{parallel combination}\end{matrix} \times \begin{matrix}\text{Value of the} \\ \text{other impedance}\end{matrix}}{\text{Sum of individual impedances}}}$$

2.3 Source Transformation

"The practical voltage source can be converted into an equivalent practical current source and vice versa, with the same terminal behaviour". In these conversions, the current and voltage at the terminal of the equivalent source will be the same as that of the original source, so that the power delivered to the load connected at the terminals of the original and equivalent source will be the same.

The voltage source with series resistance can be converted into an equivalent current source with parallel resistance as shown in Fig. 2.39. Similarly, the current source with parallel resistance can be converted to an equivalent voltage source with series resistance as shown in Fig. 2.40.

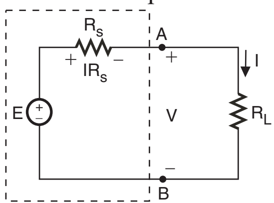


Fig. a : *Voltage source.*

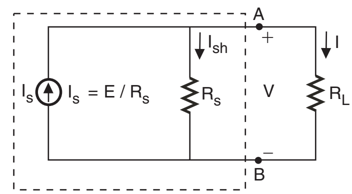


Fig. b : *Equivalent current source of the voltage source in Fig. a.*

Fig. 2.39 : *Conversion of voltage source to current source.*

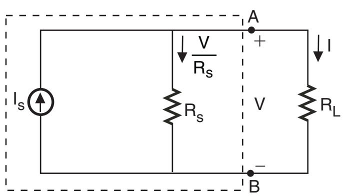


Fig. a : *Current source.*

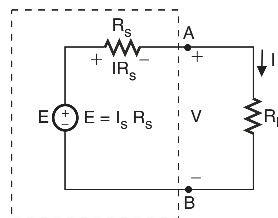


Fig. b : *Equivalent voltage source of the current source in Fig. a.*

Fig. 2.40 : *Conversion of current source to voltage source.*

Proof for Conversion of Voltage Source to Current Source

Consider a voltage source with source resistance R_s delivering a current I to a load resistance R_L as shown in Fig. 2.39(a).

In Fig. 2.39(a), using KVL, we can write,

$$E = IR_s + V \qquad \text{..... (2.51)}$$

On dividing equation (2.51) throughout by R_s, we get,

$$\frac{E}{R_s} = I + \frac{V}{R_s} \qquad \text{..... (2.52)}$$

Let, $\frac{E}{R_s} = I_s$ *and* $\frac{V}{R_s} = I_{sh}$ ..... (2.53)

From equations (2.52) and (2.53), we can write,

$$I_s = I + I_{sh} \qquad \text{..... (2.54)}$$

Equation (2.54) represents a current source with generated current I_s and delivering a load current I to the load resistance R_L. The I_{sh} is the current drawn by a parallel resistance of value R_s connected across the current source. Hence, equation (2.54) can be used to construct an equivalent current source as shown in Fig. 2.39(b). It can be observed that the current source of Fig. 2.7(b) is equivalent to the voltage source of Fig. 2.39(a) with respect to the terminals A-B.

Proof for Conversion of Current Source to Voltage Source

Consider a current source with source resistance R_s delivering a current I to a load resistance R_L as shown in Fig. 2.40(a).

In Fig. 2.40(a), using KCL, we can write,

$$I_s = \frac{V}{R_s} + I \qquad .....(2.55)$$

On multiplying equation (2.55) by R_s, we get,

$$I_s R_s = V + I R_s \qquad ..... (2.56)$$

Let, $I_s R_s = E$ ..... (2.57)

From equations (2.56) and (2.57), we can write,

$$E = V + I R_s \qquad ..... (2.58)$$

Equation (2.58) represent a voltage source with generated voltage E and delivering a load current I to the load resistance R_L. The IR_s is the voltage across a series resistance of value R_s. Hence, equation (2.58) can be used to construct an equivalent voltage source as shown in Fig. 2.40(b). It can be observed that the voltage source of Fig. 2.40(b) is equivalent to the current source of Fig. 2.40(a) with respect to the terminals A-B.

2.4 Star-Delta Conversion

2.4.1 Resistances in Star and Delta

The star-connected resistances can be converted into equivalent delta-connected resistances and vice versa. The conversion is valid if the ratio of voltage to current at any two terminals of the equivalent network is the same as that in the original network. This means that the **looking back resistance** at any two terminals of the original network is the same as that of the equivalent network.

Delta to Star Transformation

Consider three delta-connected resistances R_{12}, R_{23} and R_{31} as shown in Fig. 2.41(a). These resistances can be converted into equivalent star-connected resistances R_1, R_2 and R_3 of Fig. 2.41(b).

The equations used to determine the star equivalent of delta-connected resistances are given below:

$$\boxed{R_1 = \frac{R_{12} R_{31}}{R_{12} + R_{23} + R_{31}}}$$

$$\boxed{R_2 = \frac{R_{12} R_{23}}{R_{12} + R_{23} + R_{31}}}$$

$$\boxed{R_3 = \frac{R_{23} R_{31}}{R_{12} + R_{23} + R_{31}}}$$

From the above equations, we can say that when the three resistances in delta are equal to the value R, their equivalent star-connected resistances will consist of three equal resistances of value R/3.

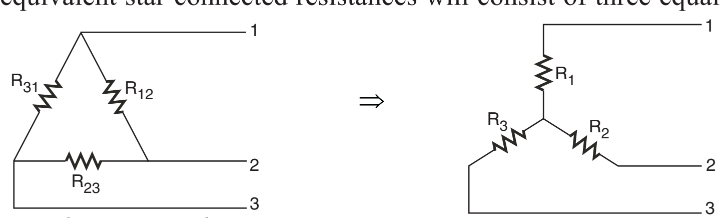


Fig. a : *Delta-connected resistances.* ***Fig. b :*** *Star-connected resistances.*

Fig. 2.41 : *Delta to star transformation.*

The following equation will be helpful to remember the delta to star conversion equations.

$$\text{Star equivalent resistance at one terminal} = \frac{\text{The product of resistances connected to the terminal in the delta network}}{\text{Sum of three resistances in the delta network}}$$

Star to Delta Transformation

Consider three star-connected resistances R_1, R_2 and R_3 as shown in Fig. 2.42(a). These resistances can be converted into equivalent delta-connected resistances R_{12}, R_{23} and R_{31} of Fig. 2.42(b).

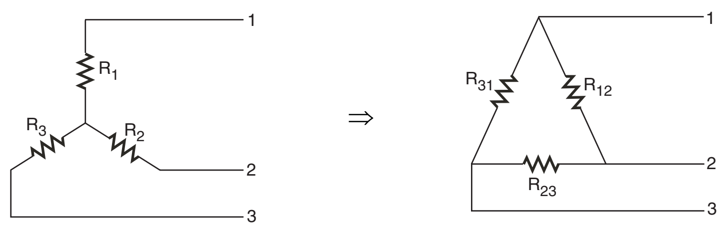


Fig. a : *Star-connected resistances.* ***Fig. b :*** *Delta-connected resistances.*

Fig. 2.42 : *Star to delta transformation.*

The equations used to determine the delta equivalent of star-connected resistances are given below:

$$\boxed{R_{12} = R_1 + R_2 + \frac{R_1 R_2}{R_3}}$$

$$\boxed{R_{23} = R_2 + R_3 + \frac{R_2 R_3}{R_1}}$$

$$\boxed{R_{31} = R_3 + R_1 + \frac{R_3 R_1}{R_2}}$$

From the above equations we can say that when the three resistances in star are equal to the value R, their equivalent delta-connected resistances will consist of three equal resistances of value 3R.

The following equation will be helpful to remember the star to delta conversion equations.

$$\text{Delta equivalent resistance between the two terminals} = \text{Sum of resistances connected to the two terminals in the star network} + \frac{\text{Product of the resistances connected to the two terminals in the star network}}{\text{The third resistance in the star network}}$$

2.4.2 Impedances in Star and Delta

The star-connected impedances can be converted into equivalent delta-connected impedances and vice versa. The conversion is valid if the ratio of voltage to current at any two terminals of the equivalent network is the same as that in the original network. This means that the **looking back impedance** at any two terminals of the original network is the same as that of the equivalent network.

Delta to Star Transformation

Consider three delta-connected impedances $\overline{Z}_{12}$, $\overline{Z}_{23}$ and $\overline{Z}_{31}$ as shown in Fig. 2.43(a). These impedances can be converted into equivalent star-connected impedances $\overline{Z}_1, \overline{Z}_2$ and $\overline{Z}_3$ of Fig. 2.43(b) using the equations given below:

$$\boxed{\overline{Z}_1 = \frac{\overline{Z}_{12}\,\overline{Z}_{31}}{\overline{Z}_{12}+\overline{Z}_{23}+\overline{Z}_{31}}} \quad ; \quad \boxed{\overline{Z}_2 = \frac{\overline{Z}_{12}\,\overline{Z}_{23}}{\overline{Z}_{12}+\overline{Z}_{23}+\overline{Z}_{31}}} \quad ; \quad \boxed{\overline{Z}_3 = \frac{\overline{Z}_{23}\,\overline{Z}_{31}}{\overline{Z}_{12}+\overline{Z}_{23}+\overline{Z}_{31}}}$$

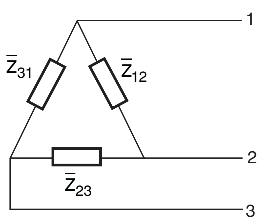


Fig. a : Delta-connected impedances.

$\Rightarrow$

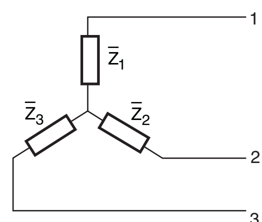


***Fig. b :** Star-connected impedances.*

***Fig. 2.43 :** Delta to star transformation.*

Star to Delta Transformation

Consider three star-connected impedances $\overline{Z}_1, \overline{Z}_2$ and $\overline{Z}_3$ as shown in Fig. 2.44(a). These impedances can be converted into equivalent delta-connected impedances $\overline{Z}_{12}, \overline{Z}_{23}$ and $\overline{Z}_{31}$ of Fig. 2.44(b) using the equations given below:

$$\boxed{\overline{Z}_{12} = \overline{Z}_1+\overline{Z}_2+\frac{\overline{Z}_1\,\overline{Z}_2}{\overline{Z}_3}} \quad ; \quad \boxed{\overline{Z}_{23} = \overline{Z}_2+\overline{Z}_3+\frac{\overline{Z}_2\,\overline{Z}_3}{\overline{Z}_1}} \quad ; \quad \boxed{\overline{Z}_{31} = \overline{Z}_3+\overline{Z}_1+\frac{\overline{Z}_3\,\overline{Z}_1}{\overline{Z}_2}}$$

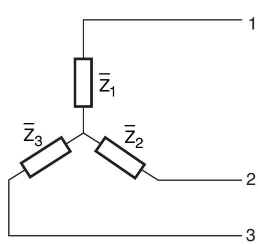


***Fig. a :** Star-connected impedances.*

$\Rightarrow$

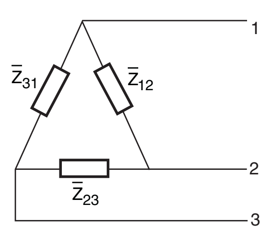


***Fig. b :** Delta-connected impedances.*

***Fig. 2.44 :** Star to delta transformation.*

2.5 Solved Problems in Network Reduction

EXAMPLE 2.1

In the circuit shown in Fig. 1, find the total resistance across the supply voltage.

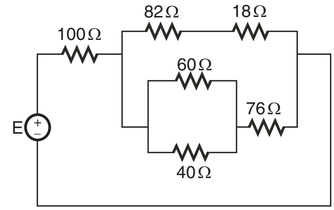


Fig. 1.

SOLUTION

The step-by-step reduction of the given network is shown in Figs 2 to 5.

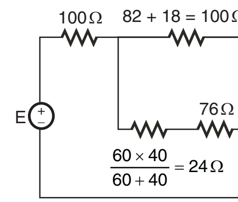


Fig. 2.

Step-1 :

The series-connected 82 Ω and 18 Ω resistances in Fig. 1 are combined to form a single equivalent resistance as shown in Fig. 2. Also the parallel-connected 60 Ω and 40 Ω in Fig. 1 are combined to form a single equivalent resistance as shown in Fig. 2.

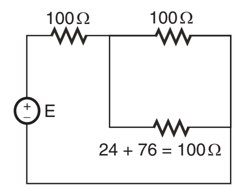


Fig. 3.

Step-2 :

The series-connected 24 Ω and 76 Ω resistances in Fig. 2 are combined to form a single equivalent resistance as shown in Fig. 3.

Step-3 :

The two parallel-connected 100 Ω resistances in Fig. 3 are combined to form a single equivalent resistance as shown in Fig. 4.

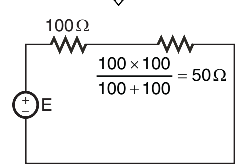


Fig. 4.

Step-4 :

The series-connected 100 Ω and 50 Ω resistances in Fig. 4 are combined to form a single equivalent resistance as shown in Fig. 5.

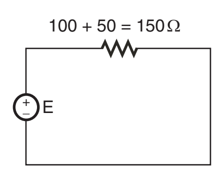


Fig. 5.

RESULT

With reference to Fig. 5, we can say that,

Total resistance across supply = 150 Ω

EXAMPLE 2.2

Find the total resistance as seen by the source in the circuit shown in Fig. 1.

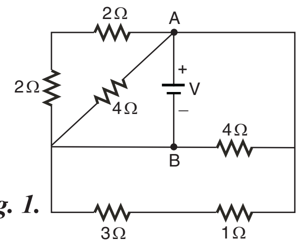


Fig. 1.

SOLUTION

The step-by-step reduction of the given circuit is shown in Figs 2 to 5.

Step-1 :

The given circuit is redrawn as shown in Fig. 2.

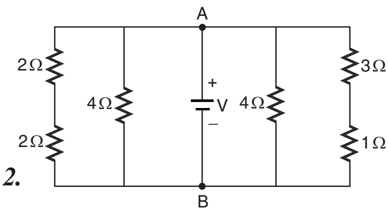


Fig. 2.

Step-2 :

The two series-connected 2 Ω resistances in Fig. 2 are combined to form a single equivalent resistance as shown in Fig. 3. Similarly, the 3 Ω and 1 Ω in series are converted into a single equivalent reisistance.

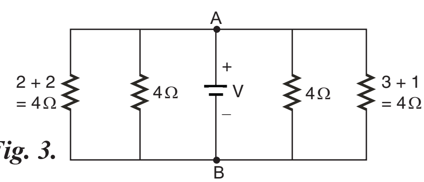


Fig. 3.

Step-3 :

The circuit of Fig. 3 is redrawn as shown in Fig. 4.

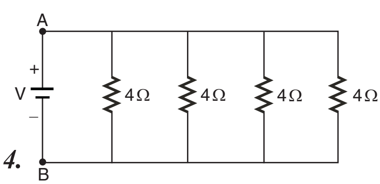


Fig. 4.

Step-4 :

The four parallel-connected 4 Ω resistances in Fig. 4 are combined to form a single equivalent resistance as shown in Fig. 5.

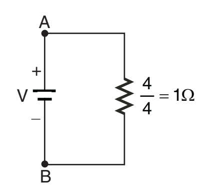


Fig. 5.

RESULT

With reference to Fig. 5, we can say that,

Total resistance across supply = 1 Ω

EXAMPLE 2.3 *(AU Dec'14, 4 Marks)*

Find the equivalent resistance of the network shown in Fig. 1.

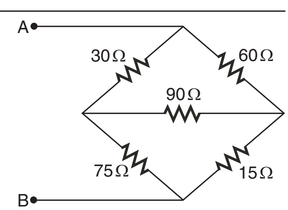


Fig. 1.

SOLUTION

The step-by-step reduction of the given network is shown in Figs 2 to 6.

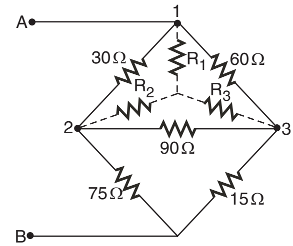


Fig. 2.

Step-1 :

The delta-connected 30 Ω, 90 Ω and 60 Ω resistances in Fig. 2 are converted into equivalent star-connected resistances as shown in Fig. 3. The resistances R_1, R_2 and R_3 connected by dotted lines in Fig. 2 are the star equivalent resistances.

$$R_1 = \frac{30 \times 60}{30 + 90 + 60} = 10\,\Omega$$

$$R_2 = \frac{30 \times 90}{30 + 90 + 60} = 15\,\Omega$$

$$R_3 = \frac{90 \times 60}{30 + 90 + 60} = 30\,\Omega$$

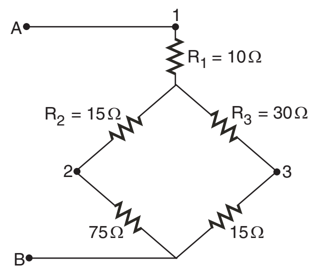


Fig. 3.

Step-2 :

The series-connected 15 Ω and 75 Ω resistances in Fig. 3 are combined to form a single equivalent resistance as shown in Fig. 4. Similarly, the 30 Ω and 15 Ω in series are converted into a single equivalent reisistance.

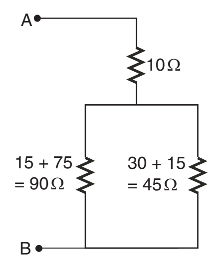


Fig. 4.

Step-3 :

The parallel-connected 90 Ω and 45 Ω resistances in Fig. 4 are combined to form a single equivalent resistance as shown in Fig. 5.

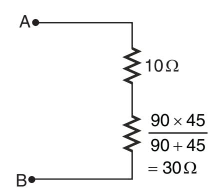


Fig. 5.

Step-4 :

The series-connected 10 Ω and 30 Ω resistances in Fig. 5 are combined to form a single equivalent resistance as shown in Fig. 6.

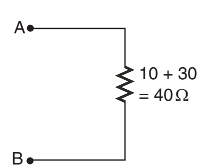


Fig. 6.

RESULT

With reference to Fig. 6, we can say that,

Equivalent resistance across A-B, $R_{AB} = 40\,\Omega$

EXAMPLE 2.4

Find the equivalent input resistance across terminals A and B of the bridged-T network shown in Fig. 1.

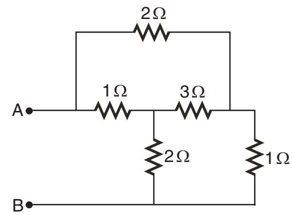


Fig. 1.

SOLUTION

The step-by-step reduction of the bridged-T network is shown in Figs 2 to 5.

Step-1 :

The delta-connected resistances 1 Ω, 3 Ω and 2 Ω in Fig. 2 are converted into equivalent star-connected resistances as shown in Fig. 3. The resistances R_1, R_2 and R_3 connected by dotted lines in Fig. 2 are star equivalent resistances.

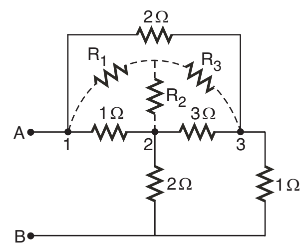


Fig. 2.

$$R_1 = \frac{2 \times 1}{2 + 1 + 3} = \frac{2}{6} = \frac{1}{3}\,\Omega$$

$$R_2 = \frac{1 \times 3}{2 + 1 + 3} = \frac{3}{6} = \frac{1}{2}\,\Omega$$

$$R_3 = \frac{2 \times 3}{2 + 1 + 3} = \frac{6}{6} = 1\Omega$$

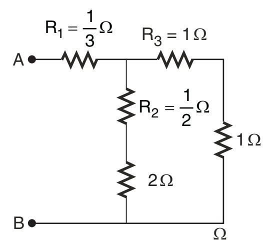


Fig. 3.

Step-2 :

The series-connected two 1 Ω resistances in Fig. 3 are combined to form a single equivalent resistance as shown in Fig. 4. Similarly, the 1/2 Ω and 2 Ω in series are converted into a single equivalent reisistance.

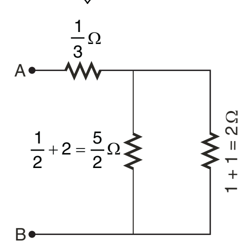


Fig. 4.

Step-3 :

The parallel-connected 5/2 Ω and 2 Ω resistances in Fig. 4 are converted into a single equivalent resistance as shown in Fig. 5.

Step-4 :

The series-connected 1/3 Ω and 10/9 Ω resistances in Fig. 5 are combined to form a single equivalent resistance as shown in Fig. 6.

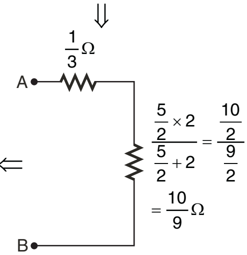


Fig. 5.

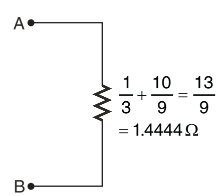


Fig. 6.

RESULT

With reference to Fig. 6, we can say that,

$$\text{Equivalent resistance across A-B} = \frac{13}{9}\Omega = 1.4444\Omega$$

EXAMPLE 2.5

Find the equivalent resistance across terminals A-B in the network shown in Fig. 1. All the resistances are 3 Ω.

SOLUTION

The step-by-step reduction of the network is shown in Figs 2 to 7.

Step-1 :

The innermost delta-connected resistances in Fig. 2 are converted into an equivalent star-connected resistance in Fig. 3. Since the delta-connected resistances are of equal value, the equivalent star-connected resistances will also have equal value, which is one-third of the delta-connected resistance.

Step-2 :

The series-connected 3 Ω and 1 Ω resistances in each branch of the star-connection in Fig. 3 are combined to form a single equivalent resistance of 3 Ω + 1 Ω = 4 Ω in each branch, as shown in Fig. 4.

Step-3 :

The star-connected resistances in Fig. 4 are converted into an equivalent delta-connected resistance as shown in Fig. 5. Since the star-connected resistances are of equal value, the delta-connected resistances will also have equal value, which is three times the star-connected resistance.

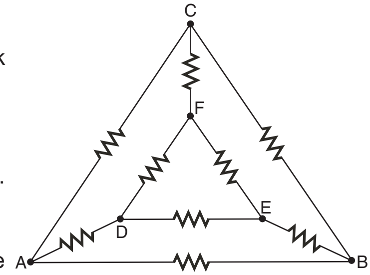


Fig. 1.

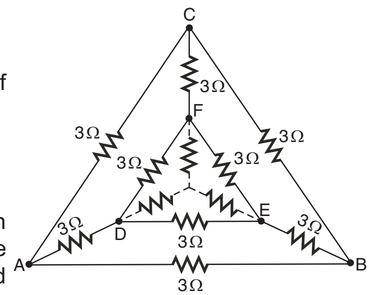


Fig. 2.

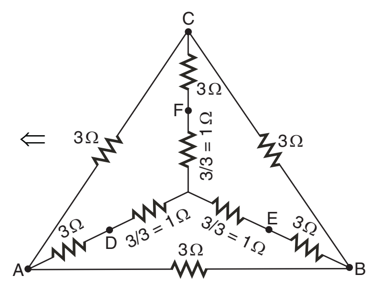


Fig. 3.

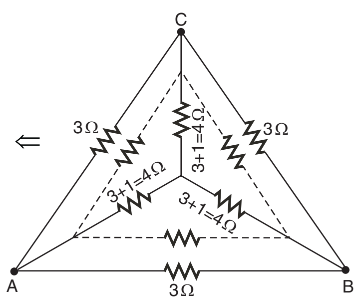


Fig. 4.

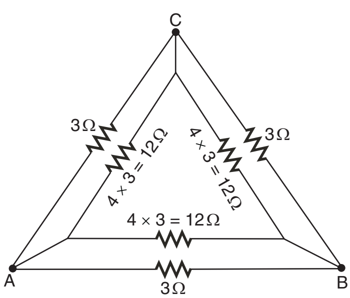


Fig. 5.

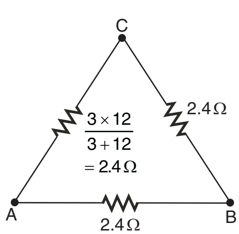


Fig. 6.

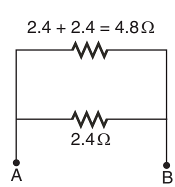


Fig. 7.

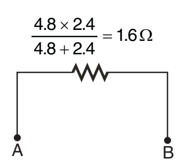


Fig. 8.

Step-4 :

The network of Fig. 5 has three similar parallel connections of 12 Ω and 3 Ω resistances. Each parallel connection is converted into a single equivalent resistance as shown in Fig. 6.

Step-5 :

Since we need resistance across nodes A and B, we can eliminate node C by combining the two series-connected resistances in the path ACB to a single equivalent resistance as shown in Fig. 7.

Step-6 :

The parallel-connected 4.8 Ω and 2.4 Ω resistances in Fig. 7 are converted into a single equivalent resistance in Fig. 8.

RESULT

With reference to Fig. 8, we can say that,

Resistance across A-B = 1.6 Ω

EXAMPLE 2.6

Determine the equivalent resistance at A-B in the network shown in Fig. 1.

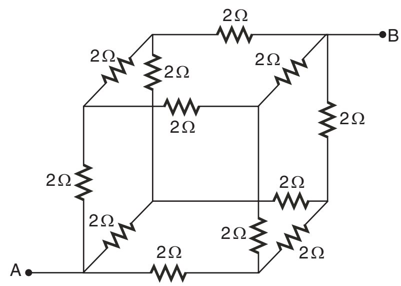


Fig. 1.

SOLUTION

Method-I

Let us connect a voltage source of value V volts across A-B as shown in Fig. 2. Let I be the current delivered by the source. Let R_{AB} be the resistance across A-B. Now, R_{AB} is given by,

$$R_{AB} = \frac{V}{I}$$

Due to the symmetry of the network, when a current enters a node it will divide equally in the outgoing path.

Similarly, the currents entering a node from incoming branches will also be equal. The currents that will flow in the various paths are shown in Fig. 2.

With reference to Fig. 2, by KVL in the path AMNOSBA, we get,

$$V = \left(\frac{I}{3}\times 2\right)+\left(\frac{I}{6}\times 2\right)+\left(\frac{I}{3}\times 2\right)=\left(\frac{2}{3}+\frac{2}{6}+\frac{2}{3}\right)I$$

$$= \left(\frac{4+2+4}{6}\right)I = \frac{10}{6}I = \frac{5}{3}I$$

$$\therefore \quad \frac{V}{I} = \frac{5}{3}$$

$$\therefore \quad R_{AB} = \frac{V}{I} = \frac{5}{3}\,\Omega$$

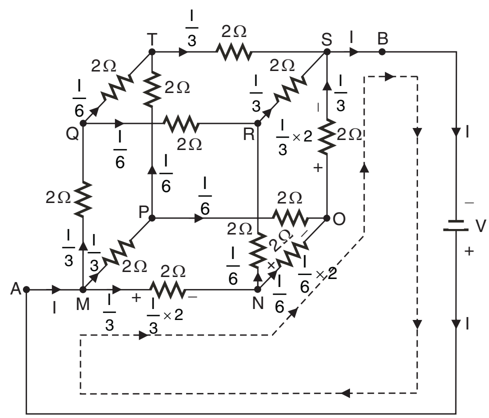


Fig. 2.

Method-II

The given network can be redrawn as shown in Fig. 3, in which two of the resistive branches are considered as the parallel combination of two 4 Ω resistances.

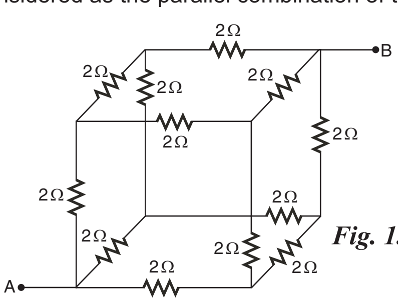


Fig. 1.

⇒

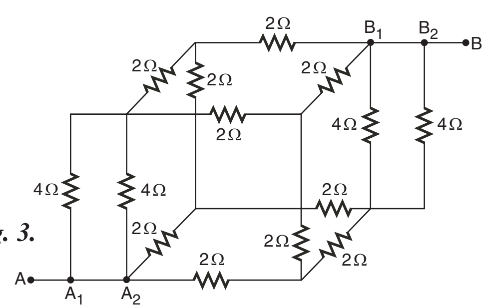


Fig. 3.

The network of Fig. 3 can be considered as the parallel combination of two identical networks as shown in Figs 4 and 5.

Let R_{A1B1} be the single equivalent resistance of the network shown in Fig. 4 and R_{A2B2} be the single equivalent resistance of the network shown in Fig. 5. Now, the equivalent resistance R_{AB} at A-B of the original network is given by the parallel combination of R_{A1B1} and R_{A2B2}. Since, the networks of Figs 4 and 5 are identical, R_{A1B1} will be equal to R_{A2B2} and so $R_{AB} = R_{A1B1}/2$ or $R_{AB} = R_{A2B2}/2$.

Therefore, it is sufficient if we reduce the network of Fig. 4 into a single equivalent resistance. The step-by-step reduction of the network of Fig. 4 is shown in Figs 6 to 10.

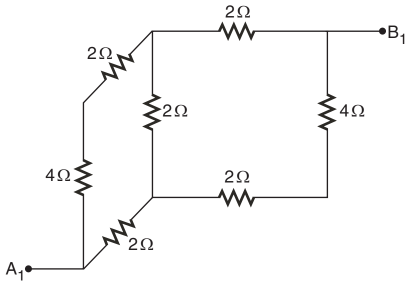


Fig. 4.

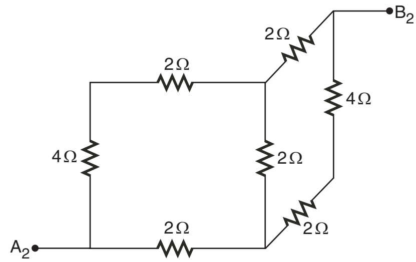


Fig. 5.

Step-1 :

The series-connected 4 Ω and 2 Ω resistances in Fig. 4 are combined to form a single equivalent resistance as shown in Fig. 6.

Step-2 :

The delta-connected resistances 2 Ω, 2 Ω and 6 Ω in Fig. 6 are converted into an equivalent star-connected resistance as shown in Fig. 7.

$$R_1 = \frac{2 \times 6}{2+2+6} = \frac{12}{10} = \frac{6}{5}\,\Omega$$

$$R_2 = \frac{2 \times 2}{2+2+6} = \frac{4}{10} = \frac{2}{5}\,\Omega$$

$$R_3 = \frac{2 \times 6}{2+2+6} = \frac{12}{10} = \frac{6}{5}\,\Omega$$

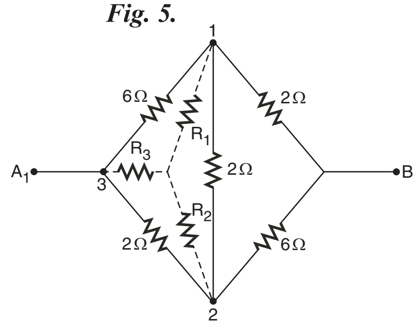


Fig. 6.

$\Downarrow$

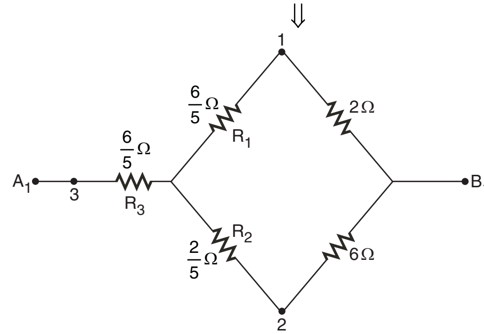


Fig. 7.

$\Leftarrow$

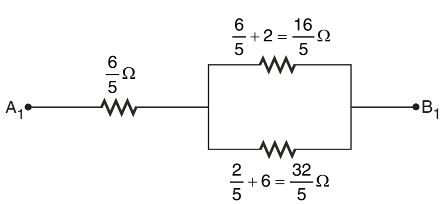


Fig. 8.

$\Downarrow$

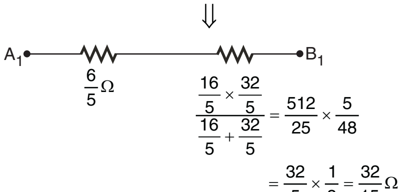


Fig. 9.

$\Rightarrow$

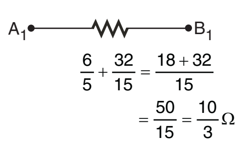


Fig. 10.

Step-3 :

The series-connected $\frac{6}{5}\,\Omega$ and $2\,\Omega$ resistances in Fig. 7 are combined to form a single equivalent resistance as shown in Fig. 8. Similarly, the series-connected $\frac{2}{5}\,\Omega$ and $6\,\Omega$ resistances are converted into a single equivalent.

Step-4 :

The parallel-connected $\frac{16}{5}\,\Omega$ and $\frac{32}{5}\,\Omega$ in Fig. 8 are combined to form a single equivalent resistance as shown in Fig. 9.

Step-5 :

The series-connected $\frac{6}{5}\,\Omega$ and $\frac{32}{15}\,\Omega$ resistances in Fig. 9 are combined to form a single equivalent as shown in Fig. 10.

RESULT

With reference to Fig. 10, we get,

$$R_{A1B1} = \frac{10}{3}\,\Omega$$

Let, R_{AB} be the equivalent resistance at A-B in the network of Fig. 1.

Now, $R_{AB} = \frac{R_{A1B1}}{2} = \frac{10}{3} \times \frac{1}{2} = \frac{5}{3}\Omega$

EXAMPLE 2.7 *(AU Dec'16, 8 Marks)*

Find the equivalent resistance and total current I_T in the network shown in Fig. 1.

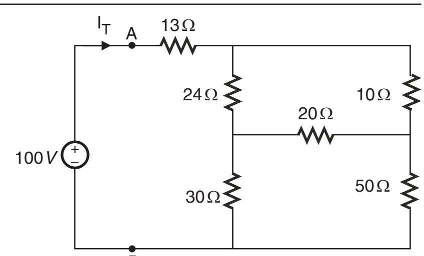


Fig. 1.

SOLUTION

The step-by-step reduction of the given network is shown in Figs 2 to 6.

Step-1 :

The delta-connected $30\,\Omega$, $20\,\Omega$ and $50\,\Omega$ resistances in Fig. 2 are converted into equivalent star-connected resistances as shown in Fig. 3. The resistances R_1, R_2 and R_3 connected by dotted lines in Fig. 2 are the star equivalent resistances.

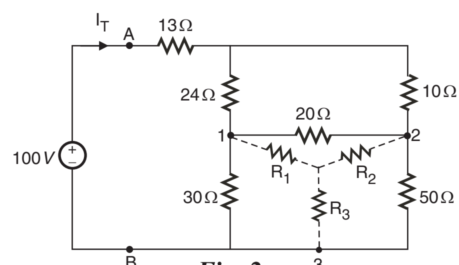


Fig. 2.

⇓

$$R_1 = \frac{20 \times 30}{20 + 30 + 50} = 6\,\Omega$$

$$R_2 = \frac{20 \times 50}{20 + 30 + 50} = 10\,\Omega$$

$$R_3 = \frac{30 \times 50}{20 + 30 + 50} = 15\,\Omega$$

Step-2 :

The series-connected $24\,\Omega$ and $6\,\Omega$ resistances in Fig. 3 are combined to form a single equivalent resistance as shown in Fig. 4. Similarly, the $10\,\Omega$ and $10\,\Omega$ in series are converted into a single equivalent reisistance.

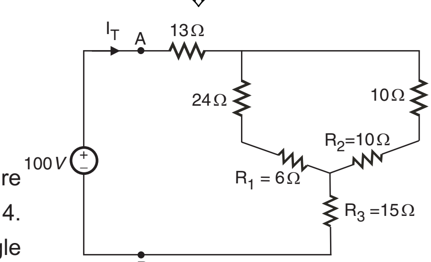


Fig. 3.

Step-3 :

The parallel-connected 30 Ω and 20 Ω resistances in Fig. 4 are combined to form a single equivalent resistance as shown in Fig. 5.

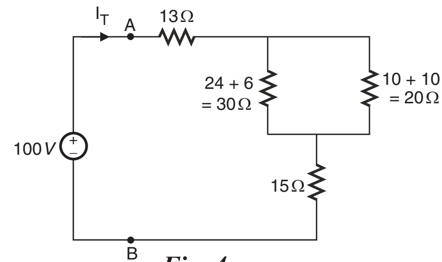


Fig. 4.

⇓

Step-4 :

The series-connected 13 Ω, 12 Ω and 15 Ω resistances in Fig. 5 are combined to form a single equivalent resistance as shown in Fig. 6.

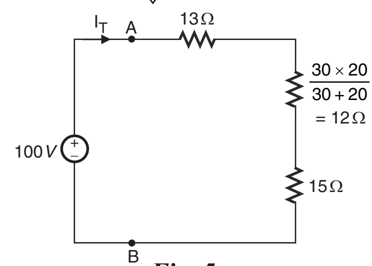


Fig. 5.

⇓

Let, R_{AB} be the resistance across A and B.

With reference to Fig. 6, we get,

$R_{AB} = 40\,\Omega$

To find the total current drawn from the source

Let, I_T be the total current drawn from the source.

Now, by Ohm's law, we get,

$$\text{Total current, } I_T = \frac{100}{R_{AB}} = \frac{100}{40} = 2.5\,A$$

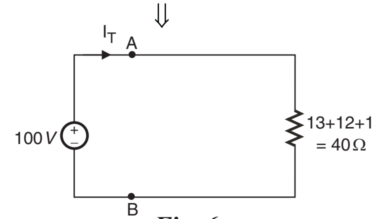


Fig. 6.

EXAMPLE 2.8

When a 6 V source is connected across A and B in the network shown in Fig. 1, find **a)** the total resistance between terminals A and B, **b)** the total current drawn from the source, **c)** the voltage across 3 $k\Omega$ resistance and **d)** the current through 4.7 $k\Omega$ resistance.

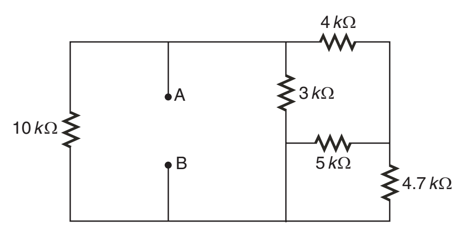


Fig. 1.

SOLUTION

a) To find the resistance between A and B

The step-by-step reduction of the given network to a single equivalent resistance is shown in Figs 2 to 4.

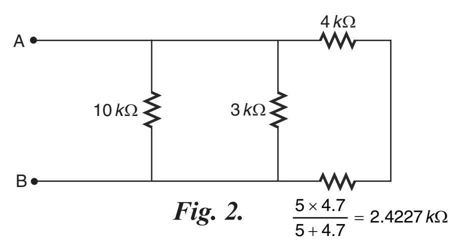


Fig. 2.

Step-1 :

The parallel-connected 5 $k\Omega$ and 4.7 $k\Omega$ resistances in Fig. 1 are converted into a single equivalent resistance as shown in Fig. 2

Step-2 :

The series-connected 4 $k\Omega$ and 2.4227 $k\Omega$ resistances in Fig. 2 are converted into a single equivalent resistance as shown in Fig. 3.

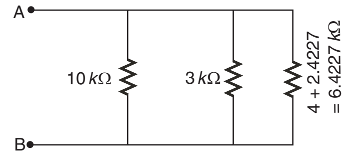


Fig. 3.

Step-3 :

The parallel-connected 10 $k\Omega$, 3 $k\Omega$ and 6.4227 $k\Omega$ resistances in Fig. 3 are converted into a single equivalent resistance as shown in Fig. 4.

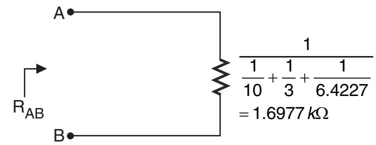


Fig. 4.

Let, R_{AB} be the resistance across A and B.

With reference to Fig. 4, we get,

$$R_{AB} = 1.6977\ k\Omega$$

b) To find the total current drawn from the source

The given network can be represented by a single equivalent resistance as shown in Fig. 4. Let us connect a 6 V source across A and B as shown in Fig. 5. Let, I_T be the total current drawn from the source.

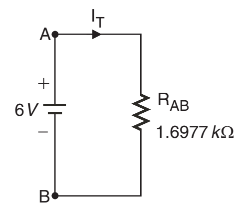


Fig. 5.

Now, by Ohm's law, we get,

$$\text{Total current, } I_T = \frac{6}{R_{AB}} = \frac{6}{1.6977 \times 10^3} = 3.5342 \times 10^{-3}\ A = 3.5342\ mA$$

c) To find the voltage across 3 kΩ resistance

In the network of Fig. 1, connect a 6 V source across A and B as shown in Fig. 6. Now, by inspection we can say that the 6 V source and 3 $k\Omega$ resistance are in parallel. Since the voltages are same in parallel connection, the voltage across 3 $k\Omega$ is 6 V.

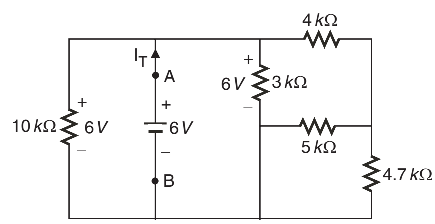


Fig. 6.

$\therefore$ Voltage across 3 $k\Omega$ resistance = 6 V

d) To find current through 4.7 kΩ resistance

The voltages and currents that will exist in various resistances when a 6 V source is connected across A and B of the network of Fig. 1 are shown in Fig. 7. For convenience, the circuit of Fig. 7 is redrawn as shown iin Fig. 8.

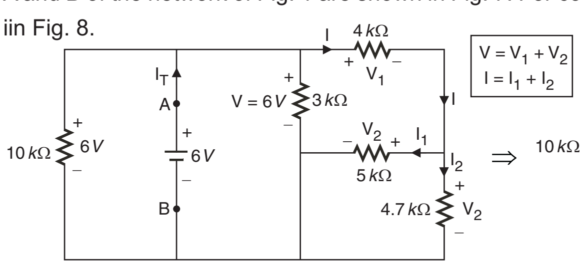


Fig. 7.

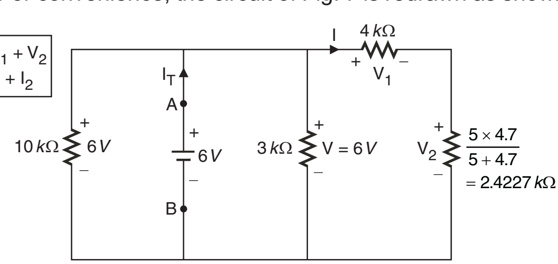


Fig. 8.

With reference to Fig. 8, by voltage division rule, we get,

$$V_2 = V \times \frac{2.4227}{4+2.4227} = 6 \times \frac{2.4227}{4+2.4227} = 2.2633\,V$$

With reference to Fig. 7, by Ohm's law, we get,

$$I_2 = \frac{V_2}{4.7\times10^3} = \frac{2.2633}{4.7\times10^3} = 0.4816\times10^{-3} = 0.4816\,mA$$

$\therefore$ Current through the 4.7 $k\Omega$ resistance = I_2 = 0.4816 mA

RESULT

Total resistance between A and B, R_{AB} = 1.6977 $k\Omega$

Total current, I_T = 3.5342 mA

Voltage across the 3 $k\Omega$ resistance = 6 V

Current through the 4.7 $k\Omega$ resistance = 0.4816 mA

EXAMPLE 2.9

Find the equivalent impedance of the network shown in Fig. 1.

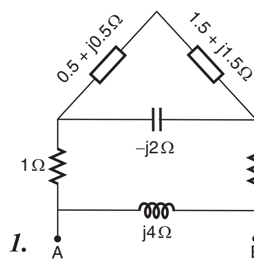


Fig. 1.

SOLUTION

The step-by-step reduction of the given network to a single equivalent impedance is shown in Figs 2 to 5.

Step-1 :

The series-connected 0.5 + j0.5 Ω and 1.5 + j1.5 Ω impedance in Fig.1 are converted into a single equivalent impedance as shown in Fig. 2.

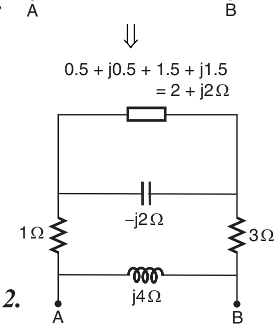


Fig. 2.

Step-2 :

The parallel-connected 2 + j2 Ω impedance and –j2 Ω capacitive reactance are converted into a single equivalent impedance as shown in Fig. 3.

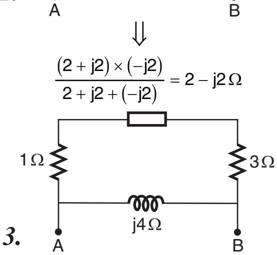


Fig. 3.

Step-3 :

The series-connected 1 Ω and 3 Ω resistances and 2 – j2 Ω impedance in Fig. 3 are converted into a single equivalent impedance as shown in Fig. 4.

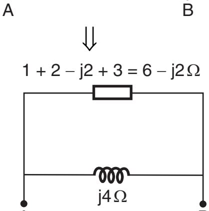


Fig. 4.

Step-4 :

The parallel-connected $6 - j2\,\Omega$ impedance and $j4\,\Omega$ inductive reactance are converted into a single equivalent impedance as shown in Fig. 5.

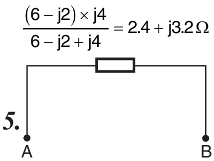


Fig.5.

RESULT

With reference to Fig. 5, the equivalent impedance $\bar{Z}_{AB}$ at terminals A-B is,

$$\bar{Z}_{AB} = 2.4 + j3.2\,\Omega$$

EXAMPLE 2.10

Obtain the single delta-connected equivalent of the network shown in Fig. 1.

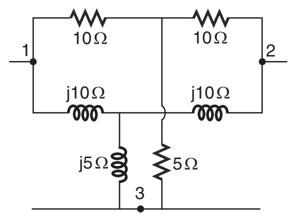


Fig. 1.

SOLUTION

The given network has two star networks between nodes 1, 2 and 3. We can convert the star networks one by one to delta.

Consider the star-connections of $10\,\Omega$, $10\,\Omega$ and $5\,\Omega$ shown in Fig. 2. The equivalent delta network is shown in Fig. 3.

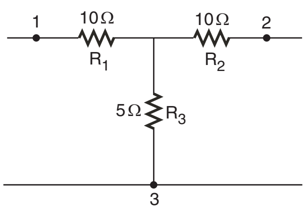


Fig. 2.

$\Rightarrow$

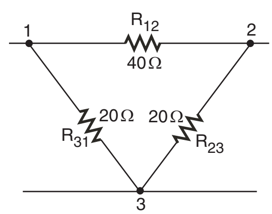


Fig. 3.

The star-connected resistances are denoted by R_1, R_2 and R_3. The equivalent delta-connected resistances are denoted by R_{12}, R_{23} and R_{31} and they are computed as shown below:

$$R_{12} = R_1 + R_2 + \frac{R_1 R_2}{R_3} = 10 + 10 + \frac{10 \times 10}{5} = 40\,\Omega$$

$$R_{23} = R_2 + R_3 + \frac{R_2 R_3}{R_1} = 10 + 5 + \frac{10 \times 5}{10} = 20\,\Omega$$

$$R_{31} = R_3 + R_1 + \frac{R_3 R_1}{R_2} = 5 + 10 + \frac{5 \times 10}{10} = 20\,\Omega$$

Consider the star connections of $j10\,\Omega$, $j10\,\Omega$ and $j5\,\Omega$ shown in Fig. 4. The equivalent delta network is shown in Fig. 5.

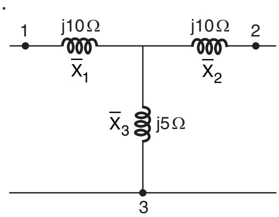


Fig. 4.

$\Rightarrow$

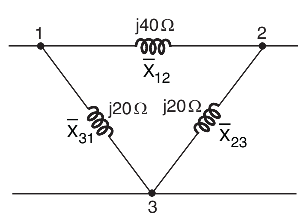


Fig. 5.

The star-connected reactances are denoted by $\overline{X}_1, \overline{X}_2$ and $\overline{X}_3$. The equivalent delta-connected reactances are denoted by $\overline{X}_{12}, \overline{X}_{23}$ and $\overline{X}_{31}$ and they are computed as shown below:

$$\overline{X}_{12} = \overline{X}_1 + \overline{X}_2 + \frac{\overline{X}_1 \overline{X}_2}{\overline{X}_3} = j10 + j10 + \frac{j10 \times j10}{j5} = j40\,\Omega$$

$$\overline{X}_{23} = \overline{X}_2 + \overline{X}_3 + \frac{\overline{X}_2 \overline{X}_3}{\overline{X}_1} = j10 + j5 + \frac{j10 \times j5}{j10} = j20\,\Omega$$

$$\overline{X}_{31} = \overline{X}_3 + \overline{X}_1 + \frac{\overline{X}_3 \overline{X}_1}{\overline{X}_2} = j5 + j10 + \frac{j5 \times j10}{j10} = j20\,\Omega$$

Using the delta equivalent shown in Figs 3 and 5, the network of Fig. 1 can be transformed into the type shown in Fig. 6 and we can observe that R_{12} and $\overline{X}_{12}$ are in parallel in Fig. 6. Similarly, R_{23} and $\overline{X}_{23}$ are in parallel and R_{31} and $\overline{X}_{31}$ are in parallel. The parallel combination of resistance and reactance can be combined to give a single equivalent impedance as shown below:

Let, $\overline{Z}_{12}$ = Parallel combination of R_{12} and $\overline{X}_{12}$

$\overline{Z}_{23}$ = Parallel combination of R_{23} and $\overline{X}_{23}$

$\overline{Z}_{31}$ = Parallel combination of R_{31} and $\overline{X}_{31}$

$$\overline{Z}_{12} = \frac{R_{12}\overline{X}_{12}}{R_{12} + \overline{X}_{12}} = \frac{40 \times j40}{40 + j40} = 20 + j20\,\Omega$$

$$\overline{Z}_{23} = \frac{R_{23}\overline{X}_{23}}{R_{23} + \overline{X}_{23}} = \frac{20 \times j20}{20 + j20} = 10 + j10\,\Omega$$

$$\overline{Z}_{31} = \frac{R_{31}\overline{X}_{31}}{R_{31} + \overline{X}_{31}} = \frac{20 \times j20}{20 + j20} = 10 + j10\,\Omega$$

The single delta equivalent of the network of Fig. 1 is shown in Fig. 7.

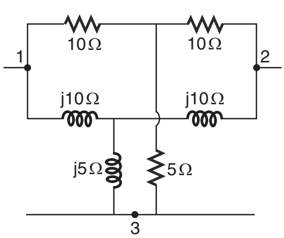


Fig. 1.

⇒

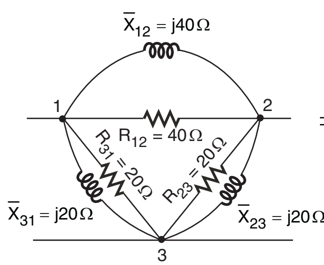


Fig. 6.

⇒

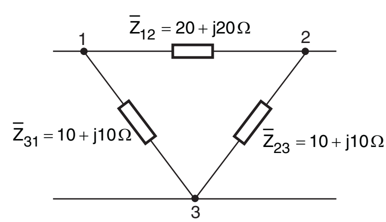


Fig. 7.

EXAMPLE 2.11 ***(AU Dec'16, 4 Marks)***

Find the equivalent capacitance of the network shown in Fig. 1.

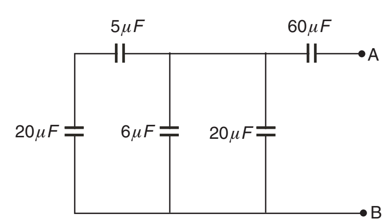


Fig. 1.

SOLUTION

The step-by-step reduction of the given circuit is shown in Figs 2 to 4.

Step-1 :

The series-connected 20 μf and 5 μf capacitances in Fig. 1 are combined to form a single equivalent capacitance as shown in Fig. 2.

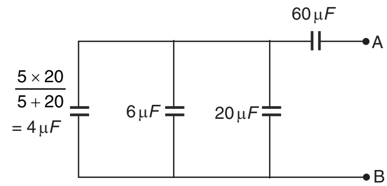


Fig. 2.

Step-2 :

The parallel-connected capacitances 4 μf, 6 μf and 20 μf in Fig. 2 are combined to form a single equivalent capacitance as shown in Fig. 3.

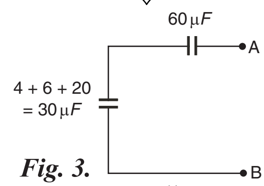


Fig. 3.

Step-3 :

The series-connected capacitances 30 μf and 60 μf in Fig. 3 are combined to form a single equivalent capacitance as shown in Fig. 4.

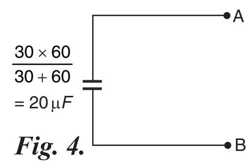


Fig. 4.

RESULT

With reference to Fig. 4, we can say that,

Equivalent capcitance A-B, C_{eq} = 20 μF

EXAMPLE 2.12

Find the equivalent capacitance across terminals A-B in the network shown in Fig. 1.

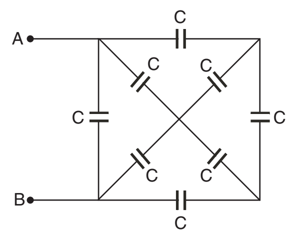


Fig. 1.

SOLUTION

Let us consider the capacitances as capacitive reactances as shown in Fig. 2 for convenience in applying reduction techniques.

The step-by-step reduction of capacitive reactances into a single equivalent is shown in Figs 2 to 7.

Step-1 :

The delta-conected capacitive reactances in Fig.2 are converted into star-connected capacitive reactances as shown in Fig. 3. Since the delta-connected reactances are of equal value, the equivalent star-connected reactances will also have equal value, which is one third of the delta-connected reactances.

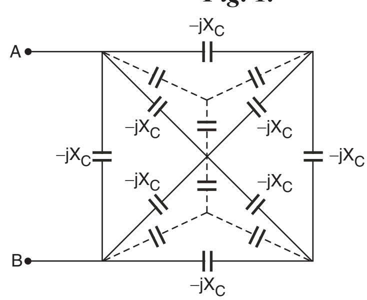


Fig. 2.

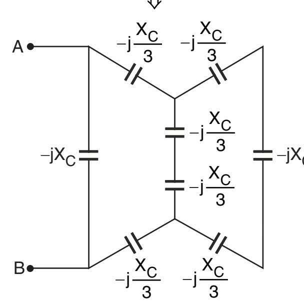


Fig. 3.

Step-2 :

The series-connected capacitive reactances $-jX_C/3$, $-jX_C$ and $-jX_C/3$ in Fig. 3 are converted into a single equivalent reactance as shown in Fig. 4. Similarly, the two $-jX_C/3$ reactances in series are converted into a single equivalent.

Step-3 :

The parallel-connected capacitive reactances $-j2X_C/3$ and $-j5X_C/3$ in Fig.4 are converted into a single equivalent capacitive reactance as shown in Fig.5.

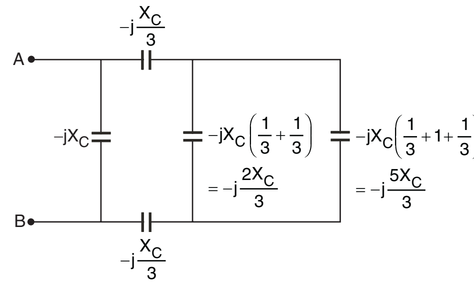


Fig. 4.

Step-4 :

The series-connected capacitive reactances $-jX_C/3$, $-j10X_C/21$, and $-jX_C/3$ in Fig.5 are converted into a single equivalent reactance as shown in Fig. 6.

Step-5 :

The parallel-connected capacitive reactances $-jX_C$ and $-j8X_C/7$ in Fig. 6 are converted into a single equivalent reactance as shown in Fig. 7.

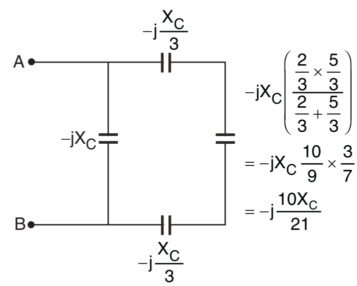


Fig. 5.

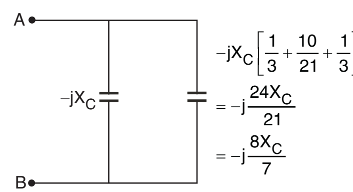


Fig. 6.

RESULT

Let, C_{AB} = Equivalent capacitance across terminals A-B

X_{CAB} = Equivalent capacitive reactance across terminals A-B

With reference to Fig. 7, we get,

$$X_{CAB} = \frac{8X_C}{15} \qquad .....(1)$$

We know that,

$$X_C = \frac{1}{2\pi fC} \qquad .....(2)$$

Using equation (2), equation (1) can be written as,

$$X_{CAB} = \frac{8}{15} \times \frac{1}{2\pi fC} \qquad .....(3)$$

$$\text{Let,} \quad X_{CAB} = \frac{1}{2\pi fC_{AB}} \qquad .....(4)$$

On equating equations (3) and (4), we get,

$$\frac{8}{15} \times \frac{1}{2\pi fC} = \frac{1}{2\pi fC_{AB}} \quad \Rightarrow \quad C_{AB} = \frac{15}{8}C = 1.875\,C$$

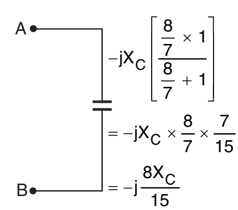


Fig. 7.

$\therefore$ Equivalent capacitance at A-B, $C_{AB} = 1.875\,C$ *farad.*

EXAMPLE 2.13

Find the equivalent admittance of the network shown in Fig. 1.

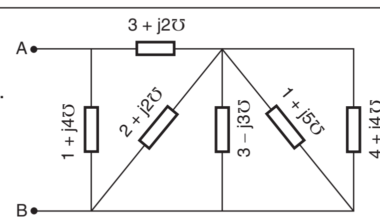


Fig. 1.

SOLUTION

The step-by-step reduction of the given network into a single equivalent admittance is shown in Figs 2 to 5.

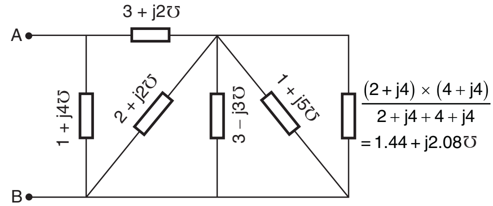


Fig. 2.

Step-1 :

The series-connected 2 + j4 ℧ and 4 + j4 ℧ admittances in Fig. 1 are converted into a single equivalent admittance as shown in Fig. 2.

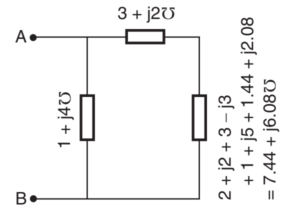


Fig. 3.

Step-2 :

The parallel-connected 2 + j2 ℧, 3 – j3 ℧, 1 + j5 ℧ and 1.44 + j2.08 ℧ admittances in Fig. 2 are converted into a single equivalent admittance as shown in Fig. 3.

Step-3 :

The series-connected 3 + j2 ℧ and 7.44 + j6.08 ℧ are converted into a single equivalent admittance as shown in Fig. 4.

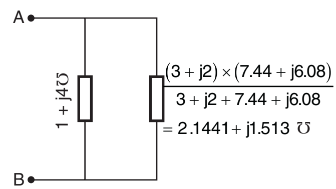


Fig. 4.

Step-4 :

The parallel-connected 1 + j4 ℧ and 2.1441 + j1.513 ℧ admittances are converted into a single equivalent admittance as shown in Fig. 5.

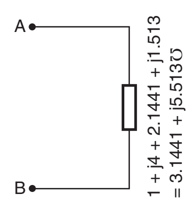


Fig. 5.

RESULT

With reference to Fig. 5, the equivalent admittance $\overline{Y}_{AB}$ at terminals A-B is,

$$\overline{Y}_{AB} = 3.1441 + j5.513 \,℧$$

EXAMPLE 2.14

Determine the equivalent susceptance for the circuit shown in Fig. 1.

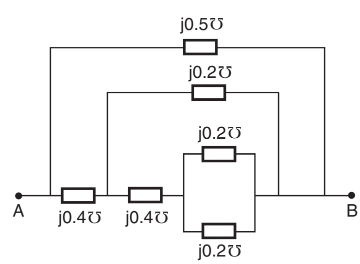


Fig. 1.

SOLUTION

The step-by-step reduction of the given network is shown in Figs 2 to 6. Let, jB_{AB} be the equivalent susceptance across A-B.

With reference to Fig. 6, we get,

$$jB_{AB} = j0.7 \,℧$$

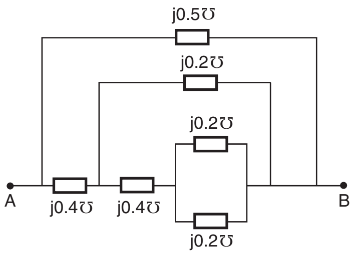


Fig. 1.

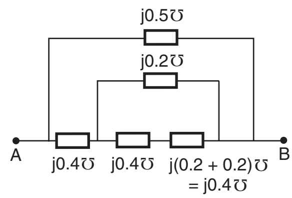


Fig. 2.

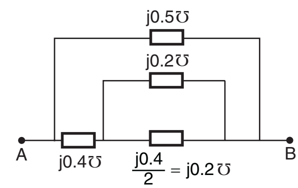


Fig. 3.

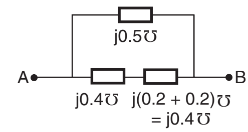


Fig. 4.

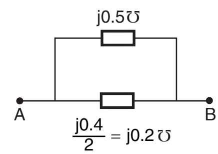


Fig. 5.

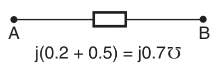


Fig. 6.

EXAMPLE 2.15

(AU May'17, 8 Marks)

Using source transformation technique, find the current I_o, through the 7 Ω resistor shown in Fig. 1.

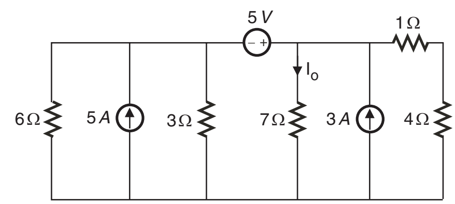


Fig. 1.

SOLUTION

The parallel-connected resistances 6 Ω and 3 Ω are converted into a single equivalent resistance as shown in Fig. 2. Similarly, the series-connected resistances 1 Ω and 4 Ω are converted into a single equivalent resistance as shown in Fig. 2.

Let us convert the 5 A source into a voltage source as shown in Fig. 3.

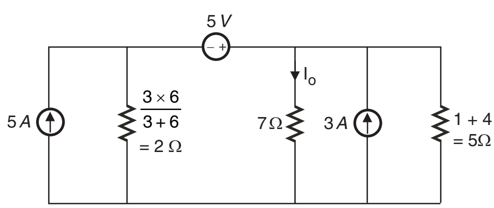


Fig. 2.

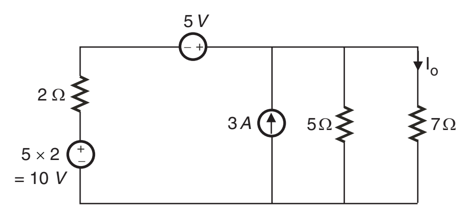


Fig. 3.

The voltage sources 10 V and 5 V in series are combined to form a single source as shown in Fig. 4.

Let us convert the 5 V source in series with the 2 Ω resistance into an equivalent current source in parallel with 2 Ω resistance as shown in Fig. 5.

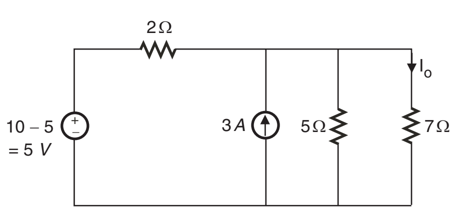


Fig. 4.

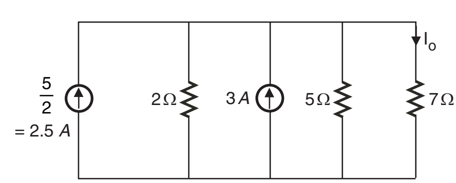


Fig. 5.

The current sources in parallel in Fig. 5 can be combined to form a single source as shown in Fig. 6. Also, the resistances $2\,\Omega$ and $5\,\Omega$ can be combined to form a single resistance.

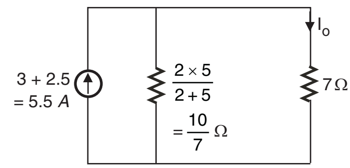


Fig. 6.

With reference to Fig. 6, by the current division rule,

$$I_o = 5.5 \times \frac{\frac{10}{7}}{\frac{10}{7}+7} = 5.5 \times \frac{\frac{10}{7}}{\frac{10+7\times7}{7}} = \frac{5.5\times10}{59}$$

$$= 0.9322\,A$$

2.6 Network Theorems

Theorems are useful tools for analysing circuits with lesser effort. They are derived from fundamental laws and concepts. A given circuit can be analysed by different methods, namely., KCL/KVL, mesh/node method, theorems, etc. In most of the cases, the analysis of circuits using theorems is much easier as compared to other methods. Remember that theorems do not always simplify the task of analysis, sometimes it may become more tough than other methods.

2.6.1 Thevenin's and Norton's Theorems

Thevenin's theorem will be useful to find the response of an element in a circuit by replacing the complicated part of the circuit by a simple equivalent voltage source. Similarly, Norton's theorem will be useful to find the response of an element in a circuit by replacing the complicated part of the circuit by a simple equivalent current source.

Consider a load impedance $\bar{Z}_L$ connected to two terminals A and B of a circuit represented as a box in Fig. 2.45(a). Using Thevenin's theorem, the circuit can be replaced with a voltage source in series with an impedance as shown in Fig. 2.45(b). Using Norton's theorem, the circuit can be replaced with a current source in parallel with an impedance as shown in Fig. 2.45(c).

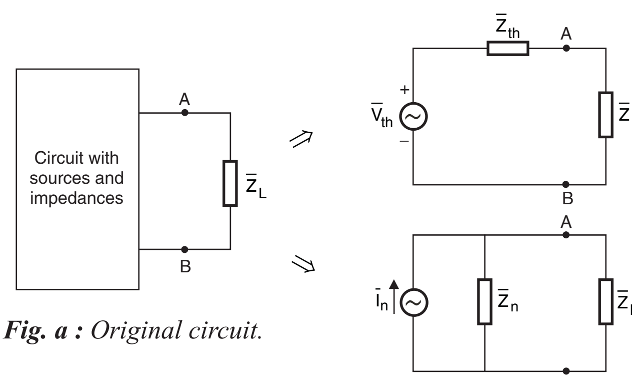


***Fig. a :** Original circuit.*

***Fig. b :** Thevenin's equivalent.*

***Fig. c :** Norton's equivalent.*

***Fig. 2.45 :** Thevenin's and Norton's equivalent of a circuit.*

These theorems can also be used to analyse a part of a circuit by replacing the complicated part of the circuit with a simple equivalent circuit.

Consider two parts of a circuit N_1 and N_2 connected through resistance-less wires as shown in Fig. 2.46(a). Now, one part of the circuit can be replaced with a simple equivalent circuit using Thevenin's/Norton's theorem for the analysis of another part of the circuit.

Using Thevenin's theorem, the circuit N_1 is replaced with a voltage source in series with an impedance as shown in Fig. 2.46(b). Using Norton's theorem, the circuit N_1 is replaced with a current source in parallel with an impedance as shown in Fig. 2.46(c).

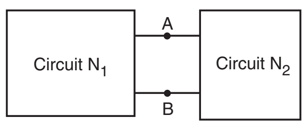


Fig. a : *Original circuit.*

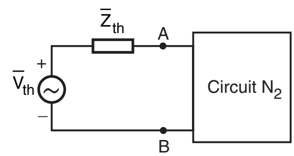


Fig. b : *Thevenin's equivalent.*

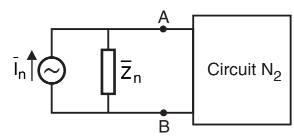


Fig. c : *Norton's equivalent.*

Fig. 2.46 : *Thevenin's and Norton's equivalent of a circuit.*

Thevenin's Theorem

Thevenin's theorem states that a circuit with two terminals can be replaced with an equivalent circuit, consisting of a voltage source in series with a resistance (or impedance).

The voltage source is called **Thevenin's voltage source** and its value is given by the voltage across the two open terminals of the circuit.

The series resistance (or impedance) is called **Thevenin's resistance** (or impedance) and it is given by looking back resistance (or impedance) at the two open terminals of the network. The **looking back resistance** (or impedance) is the resistance (or impedance) measured at the two open terminals of a circuit after replacing all the independent sources with zero value sources.

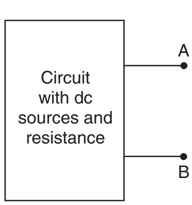


Fig. a : *Original circuit.*

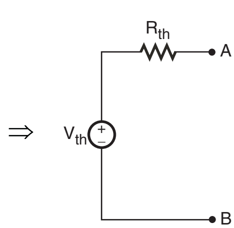


Fig. b : *Thevenin's equivalent.*

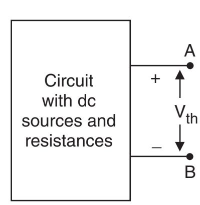


Fig. c : *To find Thevenin's voltage.*

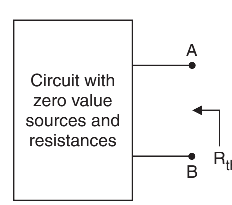


Fig. d : *To find Thevenin's resistance.*

Fig. 2.47 : *Thevenin's equivalent of a DC circuit.*

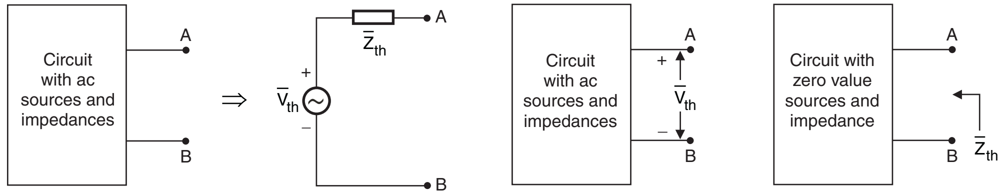


Fig. a : Original Circuit. *Fig. b : Thevenin's equivalent.* *Fig. c : To find Thevenin's voltage.* *Fig. d : To find Thevenin's impedance.*

Fig. 2.48 : Thevenin's equivalent of AC circuit.

In order to calculate Thevenin's resistance (or impedance), all the sources are replaced with zero value sources and the circuit is reduced to a single equivalent resistance (or impedance) with respect to two open terminals. The **zero value sources** are represented by their internal resistance (or impedance). For an ideal voltage source, the internal resistance (or impedance) is zero and so it is replaced with a short circuit. For an ideal current source, the internal resistance (or impedance) is infinite and so it is replaced with an open circuit.

Norton's Theorem

Norton's theorem states that a circuit with two terminals can be replaced with an equivalent circuit, consisting of a current source in parallel with a resistance (or impedance).

The current source is called **Norton's current source** and its value is given by the current flowing when the two terminals of the circuit are shorted. The parallel resistance (or impedance) is called **Norton's resistance** (or impedance) and it is given by looking back resistance (or impedance) at the two terminals of the circuit. The **looking back resistance** (or impedance) is the resistance (or impedance) measured at the two open terminals of a circuit after replacing all the independent sources by zero value sources.

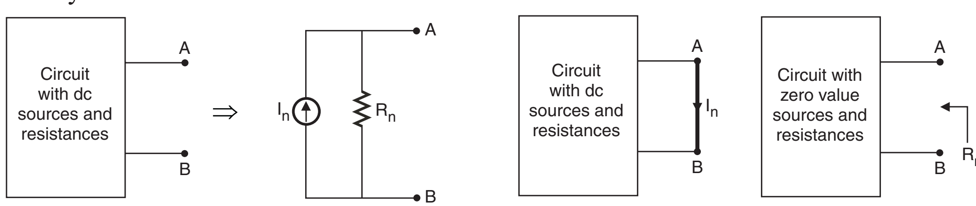


Fig. a : Original circuit. *Fig. b : Norton's equivalent.* *Fig. c : To find Norton's current.* *Fig. d : To find Norton's resistance.*

Fig. 2.49 : Norton's equivalent of a DC circuit.

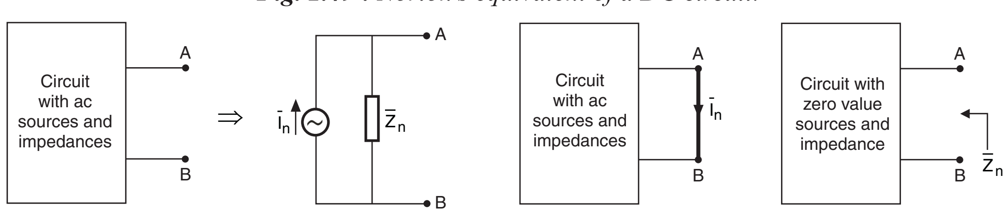


Fig. a : Original Circuit. *Fig. b : Norton's equivalent.* *Fig. c : To find Norton's current.* *Fig. d : To find Norton's impedance.*

Fig. 2.50 : Norton's equivalent of an AC circuit.

In order to calculate Norton's resistance (or impedance), all the sources are replaced by zero value sources and the circuit is reduced to a single equivalent resistance (or impedance) with respect to two open terminals. The **zero value sources** are represented by their internal resistance (or impedance). For an ideal voltage source, the internal resistance (or impedance) is zero and so it is replaced with a short circuit. For an ideal current source, the internal resistance (or impedance) is infinite and so it is replaced with an open circuit.

Relation Between Thevenin's and Norton's Equivalents

Consider Thevenin's equivalent of a given circuit as shown in Fig. 2.51.

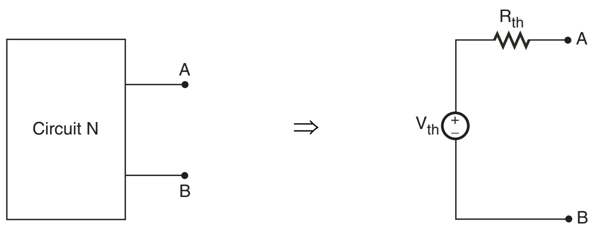


Fig. a : *Original circuit.* ***Fig. b :*** *Thevenin's equivalent.*

Fig. 2.51 : *A circuit and its Thevenin's equivalent.*

Let us find Norton's equivalent of the circuit N from its Thevenin's equivalent. To find Norton's current I_n, the terminals A and B are short-circuited as shown in Fig. 2.52(a). Now, I_n is the current flowing through the short circuit. By Ohm's law, we get, $I_n = V_{th} / R_{th}$.

To find Norton's resistance, the voltage source V_{th} is replaced with a short circuit as shown in Fig. 2.52(b). With reference to Fig. 2.52(b), we can say that Norton's resistance R_n is the same as that of Thevenin's resistance R_{th}. Norton's equivalent of the circuit N is shown in Fig. 2.52(c).

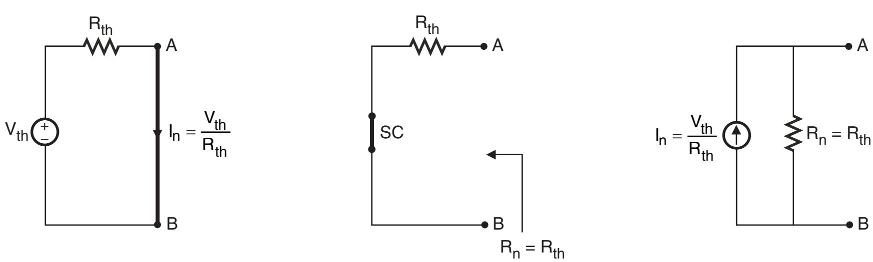


Fig. a : *To find Norton's current.* ***Fig. b :*** *To find Norton's resistance.* ***Fig. c :*** *Norton's equivalent.*

Fig. 2.52 : *Norton's equivalent of circuit N.*

Norton's equivalent of a circuit can also be directly obtained from its Thevenin's equivalent (or vice versa) using source transformation technique as shown in Fig. 2.53. In fact, *"Thevenin's equivalent is the voltage source model and Norton's equivalent is the current source model of a circuit".*

From the above discussion it is evident that $R_{th} = R_n$ (or $\overline{Z}_{th} = \overline{Z}_n$) and also that Thevenin's resistance (or impedance) is given by the ratio of Thevenin's voltage and Norton's current.

$$\boxed{\therefore\ R_{th} = R_n = \frac{V_{th}}{I_n}} \quad .....(2.59)$$

$$\boxed{\overline{Z}_{th} = \overline{Z}_n = \frac{\overline{V}_{th}}{\overline{I}_n}} \quad .....(2.60)$$

Equations (2.59) and (2.60) can be used to determine the looking back resistance (or impedance) from the knowledge of open circuit voltage (V_{th}) and short circuit current (I_n).

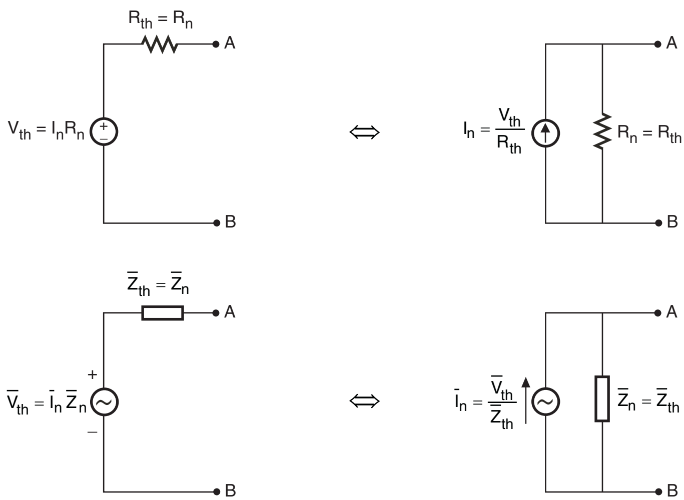


Fig. 2.53 : *Conversion of Thevenin's equivalent to Norton's equivalent (or vice versa) using source transformation technique.*

EXAMPLE 2.16

Determine Thevenin's and Norton's equivalents of the circuit shown in Fig. 1 with respect to terminals A and B.

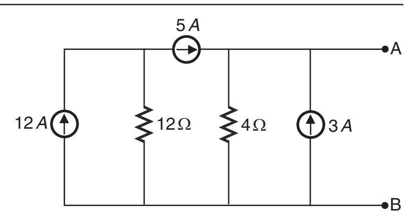


Fig. 1.

SOLUTION

To find Thevenin's voltage V_{th}

Thevenin's voltage V_{th} is the voltage across terminals A and B as shown in Fig. 2. The polarity of V_{th} is assumed such that terminal-A is positive and terminal-B is negative.

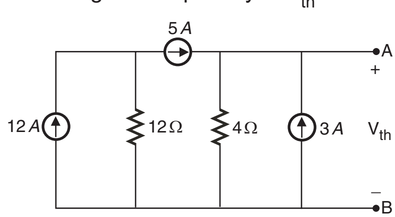


Fig. 2.

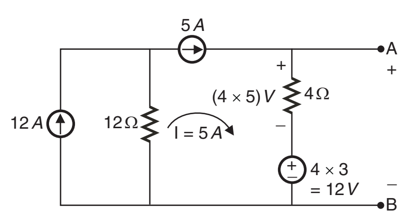


Fig. 3.

The 3 A current source in parallel with the 4 Ω resistance is converted into a voltage source in series with the $4\,\Omega$ resistance as shown in Fig. 3. Now, the $5\,A$ source is in series with the $4\,\Omega$ resistance and $12\,V$ source.

By KVL, we can write,

$$V_{th} = (4 \times 5) + 12 = 32\,V$$

To find Thevenin's resistance R_{th} (and Norton's resistance R_n)

The current sources are replaced with an open circuit as shown in Fig. 4. With reference to Fig. 5, Thevenin's resistance, $R_{th} = 4\,\Omega$

$$\therefore\ R_{th} = R_n = 4\,\Omega$$

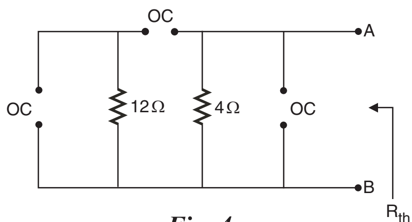


Fig. 4.

⇒

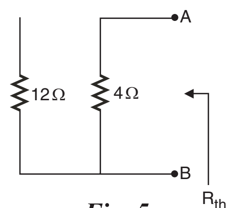


Fig. 5.

To find Norton's current I_n

The terminals A and B are shorted as shown in Fig. 6. Now, the $4\,\Omega$ resistance is short-circuited and so no current will flow through it. Hence, the $4\,\Omega$ resistance is removed and the circuit is redrawn as shown in Fig. 7. With reference to Fig. 7, by KCL at node-A, we can say that the current through the short circuit is $5 + 3 = 8\,A$.

$$\therefore\ I_n = 8\,A$$

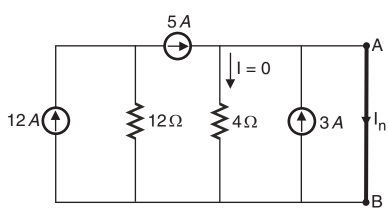


Fig. 6.

⇒

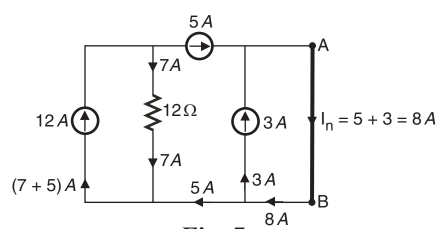


Fig. 7.

Thevenin's and Norton's equivalent

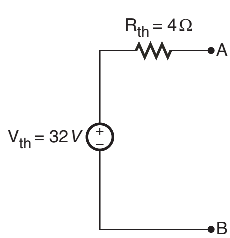


Fig. 8 : Thevenin's equivalent.

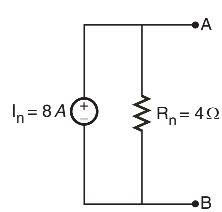


Fig. 9 : Norton's equivalent.

EXAMPLE 2.17

Find the current through the 10 Ω resistance of the circuit shown in Fig. 1 using Thevenin's theorem. Confirm the result by mesh analysis.

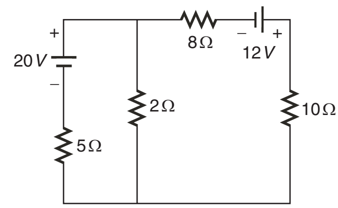


Fig. 1.

SOLUTION

Let us remove the 10 Ω resistance and mark the resulting open terminals as A and B as shown in Fig. 2.

Now, Thevenin's voltage is the voltage measured across A and B and Thevenin's resistance is the resistance measured between A and B. The polarity of V_{th} is assumed such that terminal-A is at a higher potential than terminal-B.

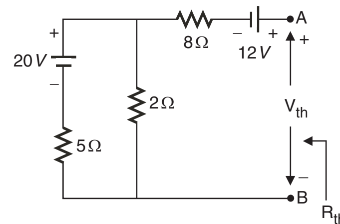


Fig. 2.

To find Thevenin's voltage V_{th}

With reference to Fig. 3, by Ohm's law, we get,

$$I = \frac{20}{5+2} = 2.8571\,A$$

With reference to Fig. 3, by KVL, we can write,

$$V_{th} = 2I + 12 \quad \Rightarrow \quad V_{th} = 2 \times 2.8571 + 12 = 17.7142\,V$$

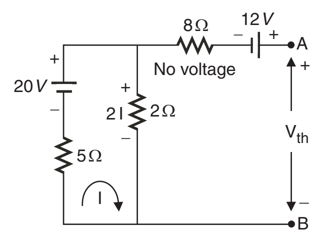


Fig. 3.

To find Thevenin's resistance R_{th}

The voltage sources are replaced with a short circuit as shown in Fig. 4.

In Fig. 4, the 5 Ω and 2 Ω resistances are in parallel and the parallel combination is in series with the 8 Ω resistance.

$$\therefore \; R_{th} = \left(5 \,||\, 2\right) + 8 = \frac{5 \times 2}{5+2} + 8 = 9.4286\,\Omega$$

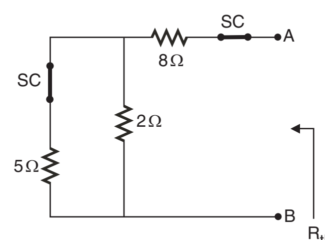


Fig. 4.

Thevenin's equivalent at A-B

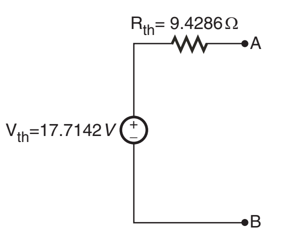


***Fig. 5 :** Thevenin's equivalent.*

To find current through 10 Ω resistance

The 10 Ω resistance is connected to terminals A and B as shown in Fig. 6. Let, I_L be the current through the 10 Ω resistance.

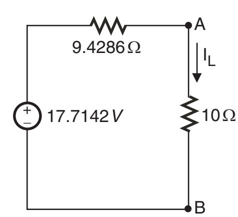


Fig. 6.

With reference to Fig. 6, by Ohm's law, we get,

$$I_L = \frac{17.7142}{9.4286 + 10} = 0.9118\,A$$

Cross-Check by Mesh Analysis

Let us assume mesh currents as shown in Fig. 7. The mesh basis matrix equation is,

$$\begin{bmatrix} 5+2 & -2 \\ -2 & 2+8+10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} 7 & -2 \\ -2 & 20 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 12 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 7 & -2 \\ -2 & 20 \end{vmatrix} = 7 \times 20 - (-2)^2 = 136$$

$$\Delta_2 = \begin{vmatrix} 7 & 20 \\ -2 & 12 \end{vmatrix} = 7 \times 12 - (-2 \times 20) = 124$$

$$\therefore \; I_2 = \frac{\Delta_2}{\Delta} = \frac{124}{136} = 0.9118\,A$$

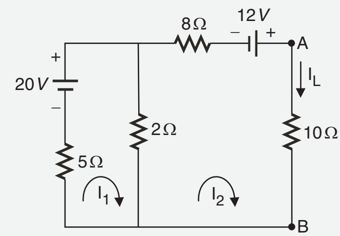


Fig. 7.

EXAMPLE 2.18

Find Thevenin's and Norton's equivalents of the circuit shown in Fig. 1 with respect to terminals A and B.

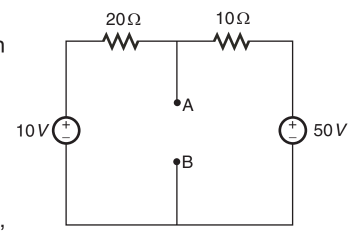


Fig. 1.

SOLUTION

To find Thevenin's voltage V_{th}

With reference to Fig. 2, in the closed path DACBD, by KVL we can write,

$$10I + 20I + 10 = 50 \;\Rightarrow\; 30I = 50 - 10$$

$$\therefore \; I = \frac{50-10}{30} = 1.3333\,A$$

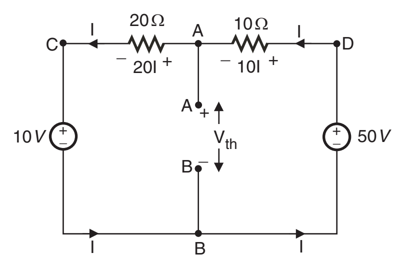


Fig. 2.

With reference to Fig. 2, in the path DABD, we get,

$$10I + V_{th} = 50 \;\Rightarrow\; V_{th} = 50 - 10I$$

$$\therefore \; V_{th} = 50 - 10 \times 1.3333 = 36.667\,V$$

To find Thevenin's resistance R_{th}

The voltage sources are replaced with a short circuit as shown in Fig. 3.

With reference to Fig. 4,

$$R_{th} = 20 \,||\, 10 = \frac{20 \times 10}{20 + 10} = 6.6667\,\Omega$$

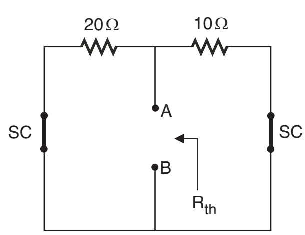


Fig. 3.

⇒

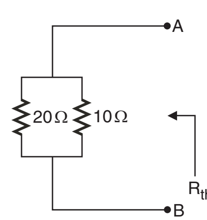


Fig. 4.

Thevenin's and Norton's equivalent

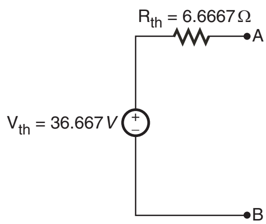


Fig. 5 : *Thevenin's equivalent.*

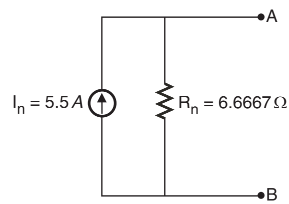


Fig. 6 : *Norton's equivalent.*

Using source transformation technique, Norton's equivalent is obtained from Thevenin's equivalent as shown in Fig. 5.

$$I_n = \frac{V_{th}}{R_{th}} = \frac{36.667}{6.6667} = 5.5\,A$$

$$R_n = R_{th} = 6.6667\,\Omega$$

Alternatively, Norton's current can be directly determined by shorting the terminals A and B and measuring the current through the short.

With reference to the circuit shown in Fig. 7, we can write,

$$I_n = I_1 + I_2 = \frac{10}{20} + \frac{50}{10} = 5.5\,A$$

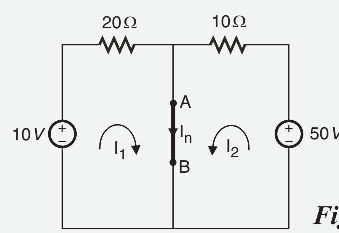


Fig. 7.

EXAMPLE 2.19

(AU May'15, 16 Marks)

Obtain the Thevenin and Norton equivalent circuits for the active network shown in Fig. 1.

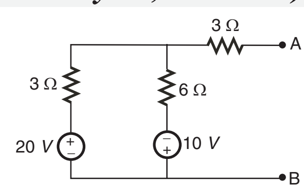


Fig. 1.

SOLUTION

To find Thevenin's voltage V_{th}

With reference to Fig. 2, by KVL we can write,

$$3I + 6I = 20 + 10 \Rightarrow 9I = 30 \Rightarrow I = \frac{30}{9} = \frac{10}{3}\,A$$

With reference to Fig. 2, we get,

$$V_{th} + 10 = 6I \Rightarrow V_{th} = 6I - 10 = 6 \times \frac{10}{3} - 10 = 10\,V$$

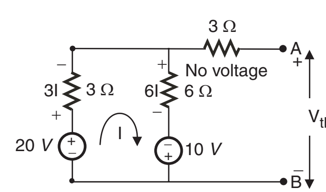


Fig. 2.

To find Thevenin's resistance R_{th}

The voltage sources are replaced with a short circuit as shown in Fig. 3.

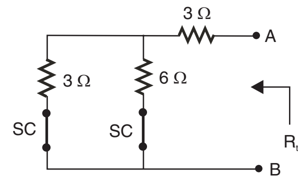


Fig. 3.

$$R_{th} = \frac{3 \times 6}{3+6} + 3 = 5\,\Omega$$

Thevenin's and Norton's equivalent

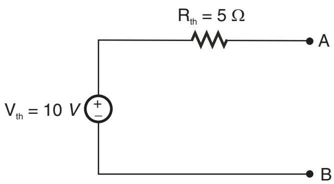


***Fig. 4 :** Thevenin's equivalent.*

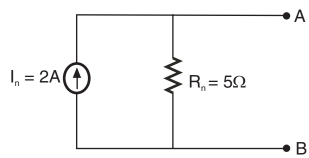


***Fig. 5 :** Norton's equivalent.*

Using source transformation technique, Norton's equivalent is obtained from Thevenin's equivalent as shown in Fig. 4.

$$I_n = \frac{V_{th}}{R_{th}} = \frac{10}{5} = 2\,A$$

$$R_n = R_{th} = 5\,\Omega$$

EXAMPLE 2.20

Using Thevenin's theorem, find the current I_L in the circuit shown in Fig. 1.

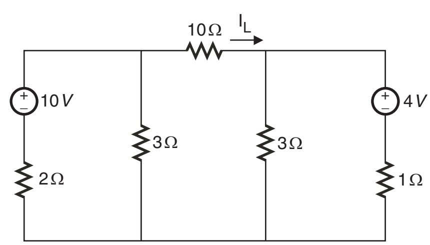


Fig. 1.

SOLUTION

Let us remove the 10 Ω resistance and mark the resulting open terminals as A and B as shown in Fig. 2.

Now, we have to determine Thevenin's equivalent of the circuit shown in Fig. 2, with respect to terminals A and B. Let us assume V_{th} as shown in Fig. 2 with terminal-A as positive and terminal-B as negative.

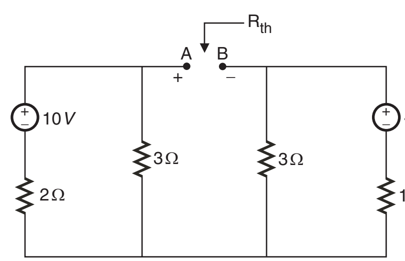


Fig. 2.

To find Thevenin's voltage V_{th}

In Fig. 3, by voltage division rule, we can write,

$$V_a = 10 \times \frac{3}{3+2} = 6\,V$$

$$V_c = 4 \times \frac{3}{3+1} = 3\,V$$

With reference to Fig. 3, using KVL in the path ABCA, we can write,

$$V_{th} + V_c = V_a$$

$$\therefore \; V_{th} = V_a - V_c = 6 - 3 = 3\,V$$

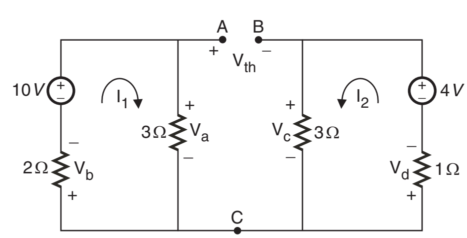


Fig. 3.

To find Thevenin's resistance R_{th}

The voltage sources are replaced with a short circuit as shown in Fig. 4. With reference to Fig. 5, we get,

$$R_{th} = \left(3\,||\,2\right) + \left(3\,||\,1\right)$$

$$= \frac{3\times 2}{3+2} + \frac{3\times 1}{3+1} = 1.95\,\Omega$$

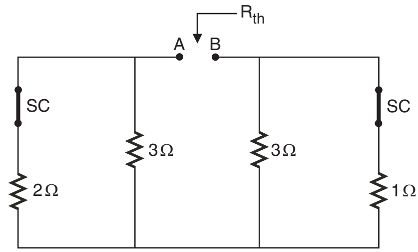


Fig. 4.

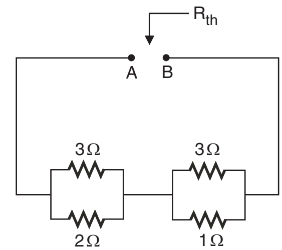


Fig. 5.

Thevenin's equivalent at A-B

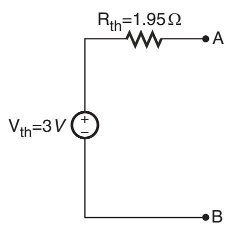


Fig. 6 : *Thevenin's equivalent.*

To find I_L

Connect the 10 Ω resistance between terminals A and B as shown in Fig. 7.

With reference to Fig. 7, by Ohm's law,

$$I_L = \frac{3}{1.95+10} = 0.251\,A$$

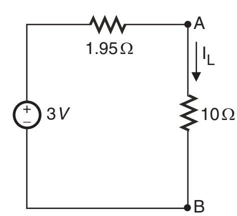


Fig. 7.

EXAMPLE 2.21 ***(AU Dec'14, 16 Marks)***

Using Thevenin's theorem, find the current I_L, through the 20 Ω resistor shown in Fig. 1.

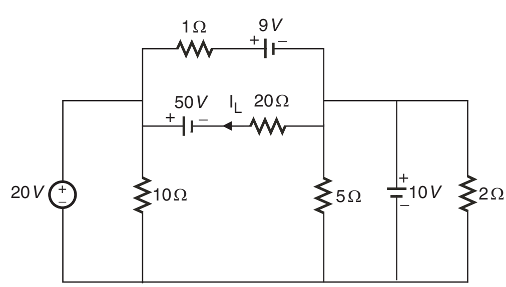


Fig. 1.

SOLUTION

Let us remove the 20 Ω resistance and mark the resulting open terminals as A and B as shown in Fig. 2.

Now, we have to determine Thevenin's equivalent of the circuit shown in Fig. 2, with respect to terminals A and B. Let us assume V_{th} as shown in Fig. 2 with terminal-A as positive and terminal-B as negative.

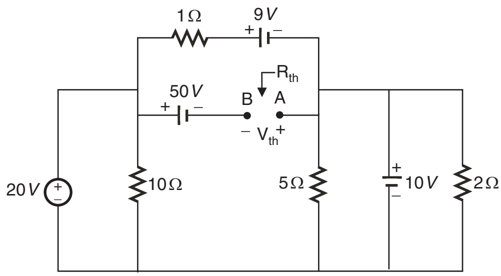


Fig. 2.

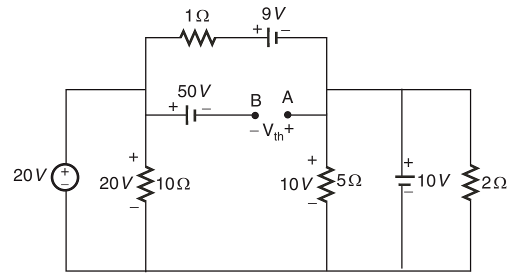


Fig. 3.

To find Thevenin's voltage V_{th}

With reference to Fig. 3, using KVL, we can write,

$$20 + V_{th} = 50 + 10$$

$$\therefore V_{th} = 50 + 10 - 20 = 40\ V$$

To find Thevenin's resistance R_{th}

The voltage sources are replaced with a short circuit as shown in Fig. 5. With reference to Fig. 6, we get,

$$R_{th} = 0\ \Omega$$

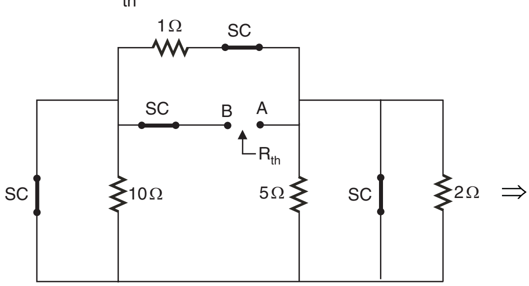


Fig. 4.

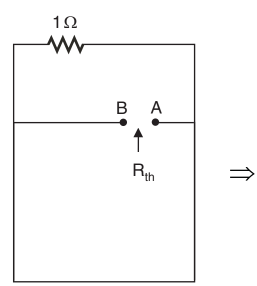


Fig. 5.

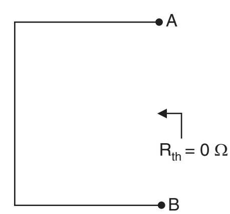


Fig. 6.

Thevenin's equivalent at A-B

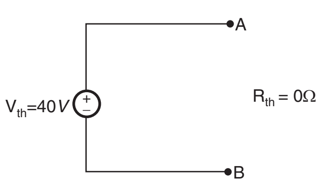


Fig. 7 : Thevenin's equivalent

To find I_L current through 20Ω

Connect the 20Ω resistance between terminals A and B as shown in Fig. 8.

With reference to Fig. 8, by Ohm's law,

$$I_L = \frac{40}{20} = 2\ A$$

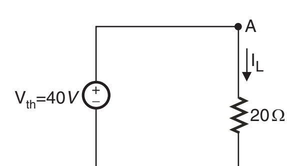


Fig. 8.

Cross-Check

With reference to Fig. 9, by KVL we get,

$$20 + V_{20} = 50 + 10$$

$$\therefore V_{20} = 60 - 20 = 40\,V$$

$$\therefore I_{20} = \frac{V_{20}}{20} = \frac{40}{20} = 2\,A$$

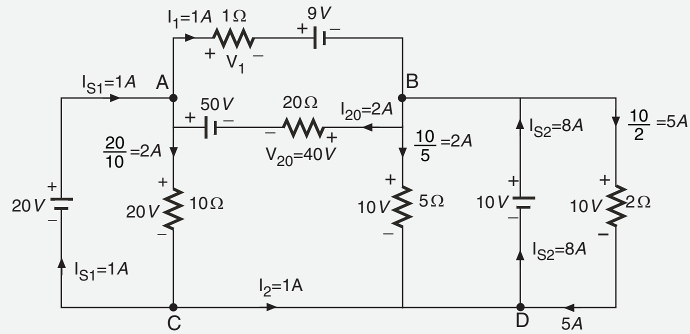


Fig. 9.

By KVL ,

$$V_1 + 9 + V_{20} = 50$$

$$\therefore V_1 = 50 - 9 - V_{20} = 50 - 9 - 40 = 1\,V$$

$$\therefore I_1 = \frac{V_1}{1} = \frac{1}{1} = 1\,A$$

By KCL, at node-A,

$$I_{S1} + I_{20} = I_1 + 2 \quad \Rightarrow \quad I_{S1} = I_1 + 2 - I_{20} = 1 + 2 - 2 = 1\,A$$

By KCL, at node-B,

$$I_{S2} + I_1 = I_{20} + 5 + 2 \quad \Rightarrow \quad I_{S2} = I_{20} + 5 + 2 - I_1 = 2 + 5 + 2 - 1 = 8\,A$$

By KCL, at node-C,

$$I_2 + I_{S1} = 2 \quad \Rightarrow \quad I_2 = 2 - I_{S1} = 2 - 1 = 1\,A$$

By KCL, at node-D,

$$I_2 + 2 + 5 = I_{S2} \quad \Rightarrow \quad I_2 = I_{S2} - 2 - 5 = 8 - 2 - 5 = 1\,A$$

EXAMPLE 2.22 ***(AU Dec'15, 16 Marks)***

Find Thevenin's equivalent of the circuit shown in Fig. 1 with respect to terminals A and B.

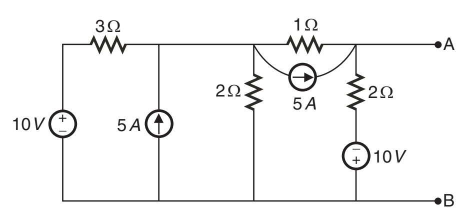


Fig. 1.

SOLUTION

To find Thevenin's voltage V_{th}

Let us convert the 10*V* source in series with the 3 Ω resistance to an equivalent current source in parallel with 3 Ω resistance as shown in Fig. 2.

Also, the 5 *A* current source in parallel with the 1 Ω resistance is converted into an equivalent voltage source in series with the 1 Ω resistance as shown in Fig. 3, and the modified circuit is shown in Fig. 4.

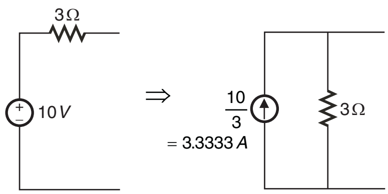


Fig. 2 : Conversion of voltage source to current source.

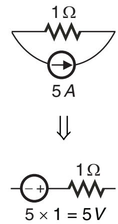


Fig. 3 : Conversion of current source to voltage source.

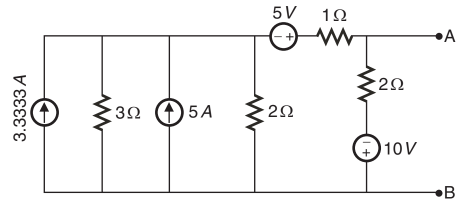


Fig. 4 : Modified circuit.

The current sources in parallel in Fig. 4 can be combined to form a single source as shown in Fig. 5. Also, the resistances 3 Ω and 2 Ω can be combined to form a single resistance.

The current source in Fig. 5 is converted into a voltage source in Fig. 6.

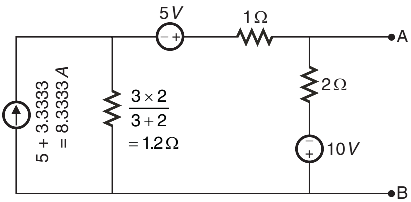


Fig. 5.

With reference to Fig. 6, using KVL, we can write,

$$1.2I + I + 2I = 10 + 5 + 10 \Rightarrow 4.2I = 25$$

$$\therefore \quad I = \frac{25}{4.2} = 5.9524\ A$$

Also, $V_{th} + 10 = 2I$

$$\therefore V_{th} = 2I - 10$$

$$= 2 \times 5.9524 - 10 = 1.9048\ V$$

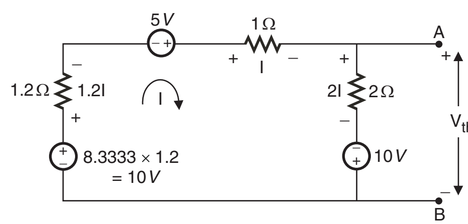


Fig. 6.

To find Thevenin's resistance R_{th}

In the given circuit, the voltage sources are replaced with a short circuit and the current sources are replaced with an open circuit as shown in Fig. 7. Thevenin's resistance R_{th} is obtained by using network reduction technique as shown below:

With reference to Fig. 11, we get,

$$R_{th} = 1.0476\ \Omega$$

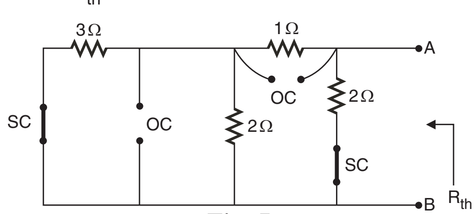


Fig. 7.

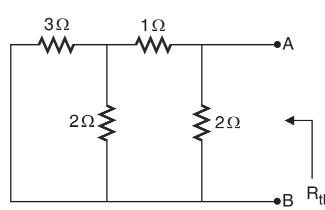


Fig. 8.

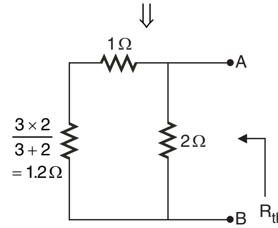


Fig. 9.

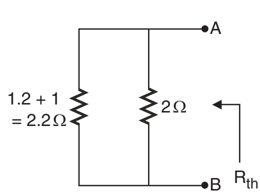


Fig. 10.

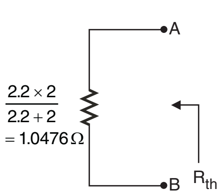


Fig. 11.

Thevenin's equivalent at A-B

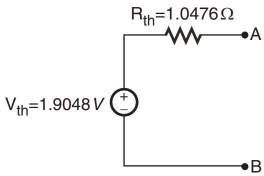


Fig. 12 : *Thevenin's equivalent.*

Alternate Method to Find V_{th}

The voltage sources in Fig. 1 are converted into current sources as shown in Fig. 13. The node basis matrix equation is formed using the circuit of Fig. 13, as shown below. Now, $V_{th} = V_2$

$$\begin{bmatrix} \frac{1}{3}+\frac{1}{2}+\frac{1}{1} & -\frac{1}{1} \\ -\frac{1}{1} & \frac{1}{2}+\frac{1}{1} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{10}{3}+5-5 \\ 5-5 \end{bmatrix}$$

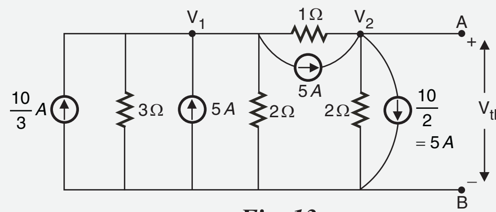


Fig. 13.

$$\begin{bmatrix} 1.8333 & -1 \\ -1 & 1.5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 3.3333 \\ 0 \end{bmatrix}$$

$$\Delta' = \begin{vmatrix} 1.8333 & -1 \\ -1 & 1.5 \end{vmatrix} = 1.8333 \times 1.5 - (-1)^2 = 1.74995$$

$$\Delta'_2 = \begin{vmatrix} 1.8333 & 3.3333 \\ -1 & 0 \end{vmatrix} = 0 - (-1) \times 3.3333 = 3.3333$$

$$\therefore \quad V_{th} = V_2 = \frac{\Delta'_2}{\Delta'} = \frac{3.3333}{1.74995} = 1.9048\,V$$

EXAMPLE 2.23

Determine Thevenin's and Norton's equivalents at P-Q for the circuit shown in Fig. 1.

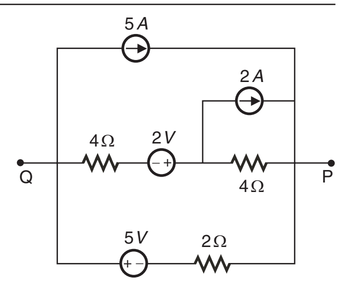


Fig. 1.

SOLUTION

Thevenin's and Norton's equivalents can be obtained by using source transformation techniques as shown below:

The 5 V source in series with the 2Ω resistance is converted into a current source as shown in Fig. 2. The 2 A source in parallel with the 4Ω resistance is converted into a voltage source as shown in Fig. 3. The modified circuit is shown in Fig 4.

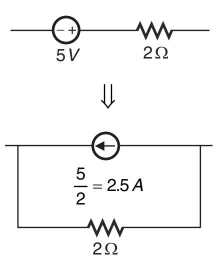


Fig. 2.

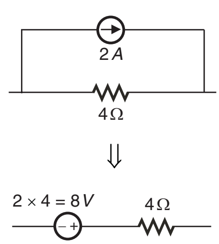


Fig. 3.

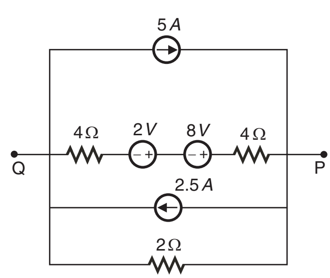


Fig. 4.

The series-connected voltage sources in Fig. 4 are combined to form a single source as shown in Fig. 5. Also, the $4\,\Omega$ resistances in series are represented by a single equivalent resistance.

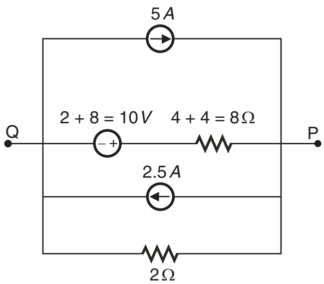


Fig. 5.

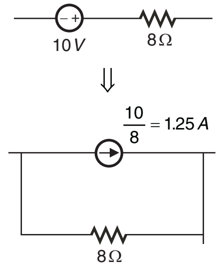


Fig. 6.

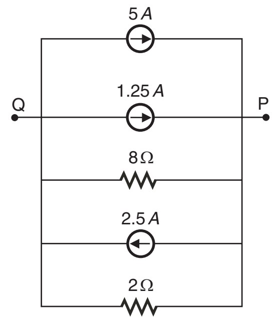


Fig. 7.

The $10\,V$ voltage source in series with the $8\,\Omega$ resistance can be converted into a current source as shown in Fig. 6. The modified circuit is shown in Fig. 7.

In Fig. 7, three current sources are in parallel and they can be combined to form a single current source as shown in Fig. 8. Similarly, the resistances $8\,\Omega$ and $2\,\Omega$ in parallel are also represented by a single equivalent resistance in Fig. 8.

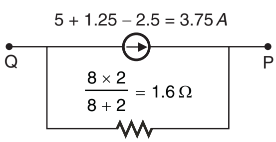


Fig. 8.

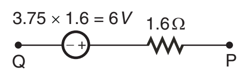


Fig. 9.

Here, Fig. 8 is Norton's equivalent which can be transformed into Thevenin's equivalent shown in Fig. 9, using source transformation technique.

Thevenin's and Norton's equivalent

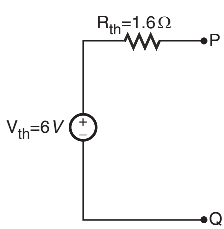


Fig. 10 : *Thevenin's equivalent at P-Q.*

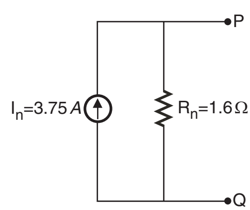


Fig. 11 : *Norton's equivalent at P-Q.*

EXAMPLE 2.24

Using Norton's theorem, determine the current through an ammeter connected across A and B of the circuit shown in Fig. 1. Take the resistance of the ammeter as $0.5\,\Omega$.

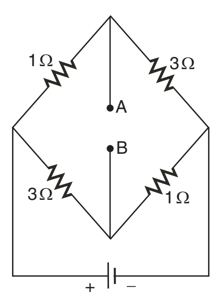


Fig. 1.

SOLUTION

To find Norton's current I_n

The terminals A and B are shorted as shown in Fig. 2. The direction of Norton's current is assumed such that it flows from terminal-A to terminal-B. The circuit of Fig. 2 is redrawn as shown in Fig. 3.

In the circuit of Fig. 3, the 3 Ω and 1 Ω resistances are in parallel and so they are represented as a single equivalent resistance as shown in Fig. 4.

In Fig. 4, the two equal resistances are in series with the 12 *V* source and so the source voltage 12 *V* divides equally between them. Since the voltage across the parallel resistances 1 Ω and 3 Ω is 6 *V*, the voltage across each resistance is also 6 *V*. By using Ohm's law, the current through each resistance is calculated and marked in Fig. 3.

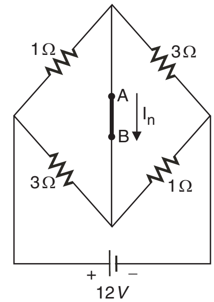


Fig. 2.

With reference to Fig. 3, at node A using KCL, we can write,

$$I_n + 2 = 6$$

$$\therefore \; I_n = 6 - 2 = 4\,A$$

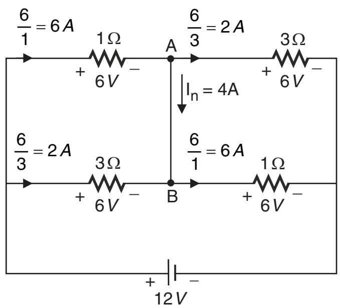


Fig. 3.

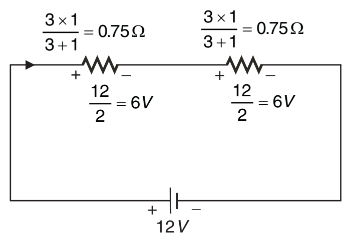


Fig. 4.

To find Norton's resistance R_n

The voltage source is replaced with a short circuit as shown in Fig. 5. Norton's resistance is determined using network reduction techniques as shown below:

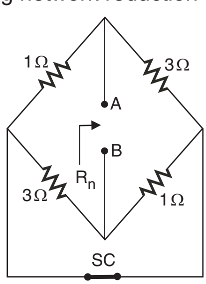


Fig. 5.

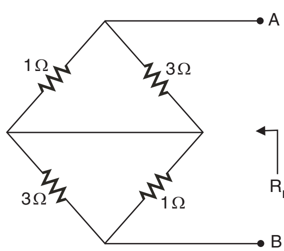


Fig. 6.

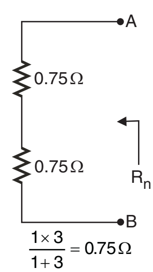


Fig. 7.

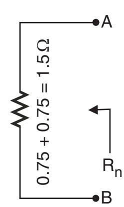


Fig. 8.

With reference to Fig. 8, we get,

Norton's resistance, $R_n = 1.5\,\Omega$

Norton's equivalent

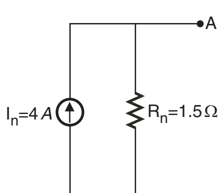


Fig. 9 : *Norton's equivalent at A-B.*

To find current through ammeter

Connect the ammeter across terminals A and B as shown in Fig. 10. The ammeter can be represented by its internal resistance as shown in Fig. 11.

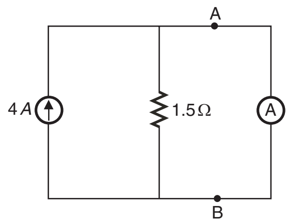


Fig. 10.

⇒

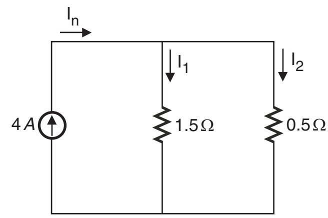


Fig. 11.

Let, I_2 be the current through the ammeter. Now, by current division rule,

$$\text{Current through ammeter}, I_2 = 4 \times \frac{1.5}{1.5 + 0.5} = 3\,A$$

EXAMPLE 2.25

In the circuit shown in Fig. 1, determine the power delivered to the 15 Ω resistance using Norton's theorem.

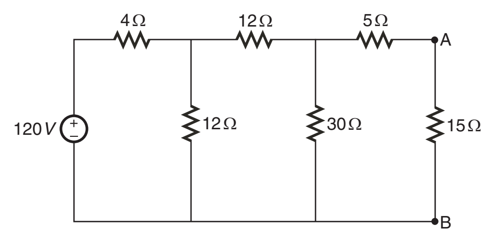


Fig. 1.

SOLUTION

Let us remove the 15 Ω resistance and determine Norton's equivalent with respect to terminals A and B. Norton's equivalent of the circuit of Fig. 2 is obtained by source transformation technique as shown below:

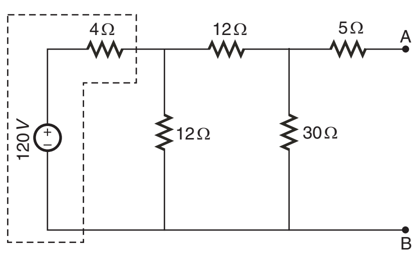


Fig. 2.

⇒

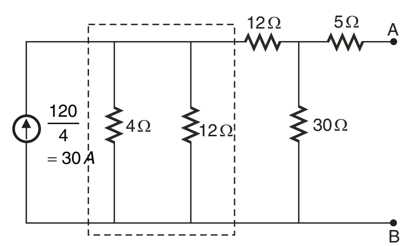


Fig. 3.

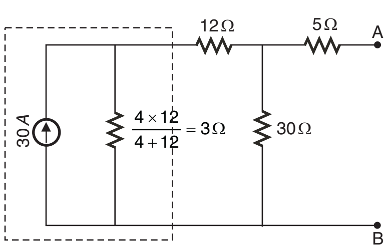


Fig. 4.

⇒

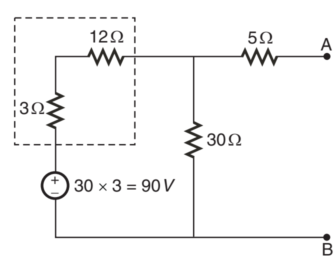


Fig. 5.

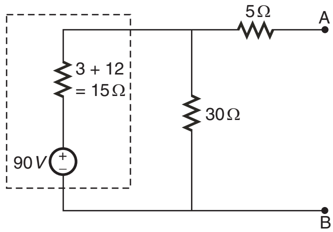


Fig. 6.

⇒

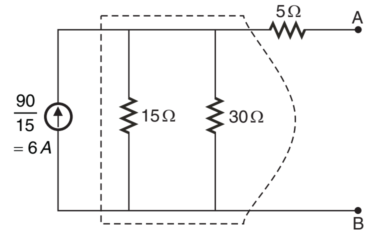


Fig. 7.

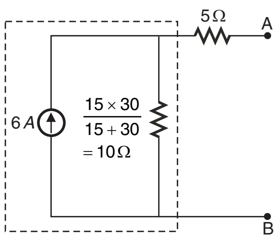


Fig. 8.

⇒

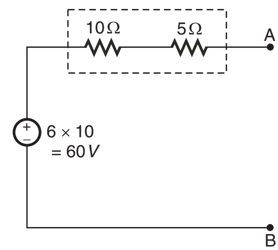


Fig. 9.

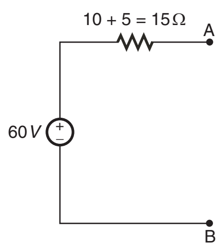


Fig. 10.

⇒

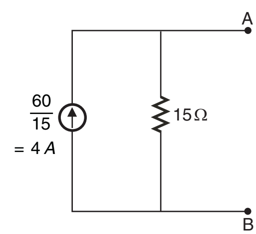


Fig. 11.

Here, Fig. 11 is Norton's equivalent with respect to terminals A-B of the circuit of Fig. 2.

Norton's equivalent

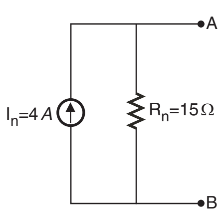


Fig. 12 : *Norton's equivalent.*

To find current through 15Ω resistance

Connect the 15Ω resistance to terminals A-B of Norton's equivalent as shown in Fig. 13. Now, the current 4 A divides equally between the parallel resistances.

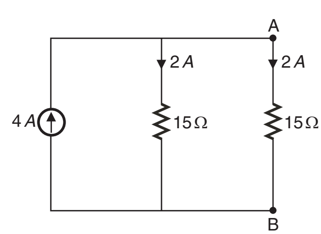


Fig. 13.

$\therefore$ Current through the 15 Ω resistance $= \dfrac{4}{2} = 2\ A$

Power through the 15Ω resistance = Current^2 × Resistance = $2^2 \times 15 = 60\ W$

EXAMPLE 2.26

In the network shown in Fig. 1, the resistance R is variable from zero to infinity. The current I through R can be expressed as I = a + bV where, V is the voltage across R with polarity as shown in Fig. 1 and a and b are constants. Determine a and b.

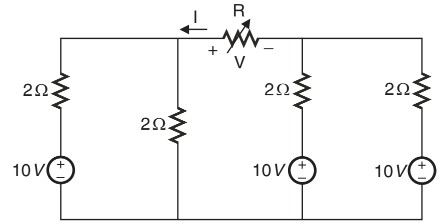


Fig. 1.

SOLUTION

Case i : Let R = 0 (zero)

When R = 0, the resistance can be represented as a short circuit as shown in Fig. 2.

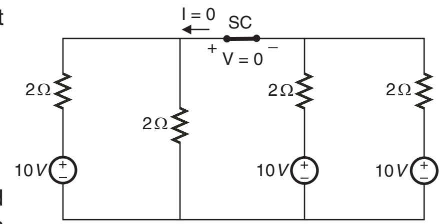


Fig. 2.

Given that, I = a + bV

Since, R = 0, V is also equal to zero.

$\therefore$ a = I

The voltage sources in the circuit of Fig. 2 are converted into current sources as shown in Fig. 3 and the modified circuit is shown in Fig. 4.

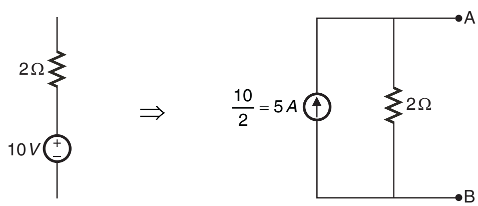


Fig. 3.

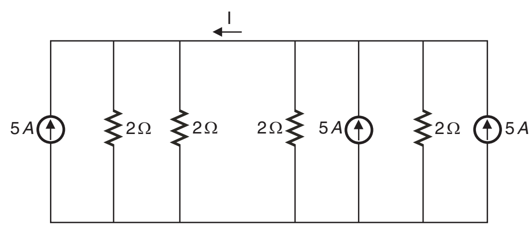


Fig. 4.

The current sources in parallel in Fig. 4 are combined to form a single equivalent source as shown in Fig. 5. Similarly, the resistances in parallel are combined to form a single equivalent resistance.

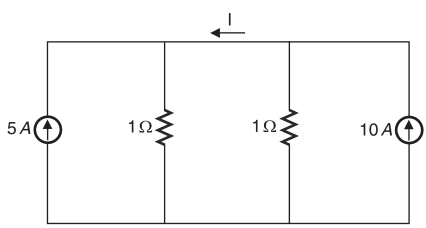


Fig. 5.

⇒

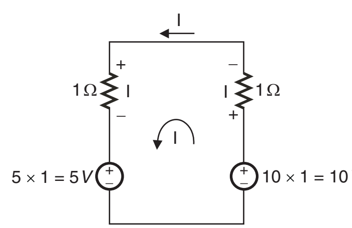


Fig. 6.

The current sources of Fig. 5, are converted into equivalent voltage sources as shown in Fig. 6. With reference to Fig. 6, by KVL, we can write,

$$5 + I + I = 10$$

$$\therefore\ 2I = 10 - 5 \Rightarrow 2I = 5 \Rightarrow I = \frac{5}{2} = 2.5\,A$$

Since, I = 2.5, a = 2.5

Case ii : Let R = ∞ (infinity)

When R = ∞, the resistance can be represented as an open circuit as shown in Fig. 7.

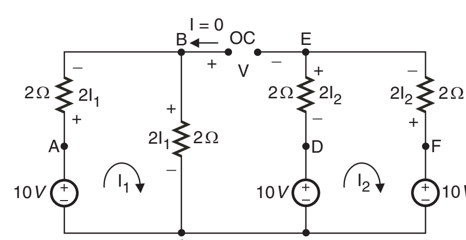


Fig. 7.

Given that, I = a + bV

Since, R = ∞, I is equal to zero.

$$\therefore \ a + bV = 0 \quad \Rightarrow \quad b = -\frac{a}{V}$$

With reference to Fig. 7, we can write the following KVL equations.

In the closed path ABCA, using KVL, we can write,

$$2I_1 + 2I_1 = 10$$

$$4I_1 = 10 \quad \Rightarrow \quad I_1 = \frac{10}{4} = 2.5\,A$$

In the closed path DEFCD, we can write,

$$10 = 10 + 2I_2 + 2I_2$$

$$\therefore 4I_2 = 10 - 10 \quad \Rightarrow \quad 4I_2 = 0 \quad \Rightarrow \quad I_2 = 0$$

In the path BEDCB,

$$V + 2I_2 + 10 = 2I_1$$

$$\therefore V = 2I_1 - 2I_2 - 10$$

$$= 2 \times 2.5 - 2 \times 0 - 10$$

$$= -5\,V$$

Since, $V = -5,\ b = \frac{-a}{V} = \frac{-2.5}{-5} = 0.5$

RESULT

$a = 2.5 \quad ; \quad b = 0.5$

$\therefore I = a + bV = 2.5 + 0.5V$

EXAMPLE 2.27

Find the current through the galvanometer shown in Fig. 1, using Thevenin's theorem.

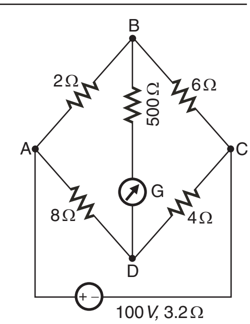


Fig. 1.

SOLUTION

Let us remove the galvanometer and denote the resultant open terminals as P and Q. The source is represented as an ideal source with its internal resistance (3.2 Ω) connected external to the source in series as shown in Fig. 1.

Let us represent the circuit of Fig. 2 by Thevenin's equivalent with respect to terminals P and Q. The polarity of Thevenin's voltage is assumed as shown in Fig. 2, with terminal P as positive.

To find Thevenin's voltage V_{th}

The circuit of Fig. 2 is redrawn as shown in Fig. 3.

Since the 500 Ω resistance is left open, the potential at node-P will be the same as that of node-B. Also the nodes Q and D are at the same potential. Hence, the circuit of Fig. 3 is redrawn as shown in Fig. 4. Now, V_{th} is the voltage across B and D.

Let I_s be the current supplied by the source and this current divides into I_1 and I_2 between two parallel paths as shown in Fig. 4.

The series resistances and their parallel combinations are represented as a single resistance in Fig. 5.

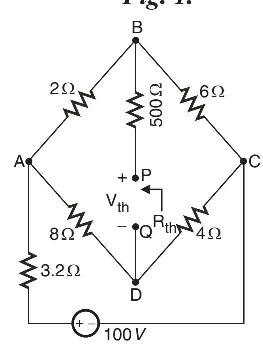


Fig. 2.

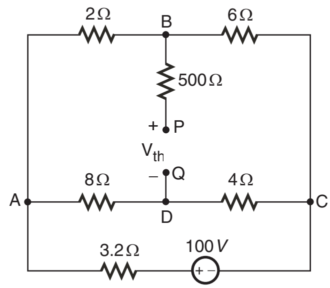


Fig. 3.

⇒

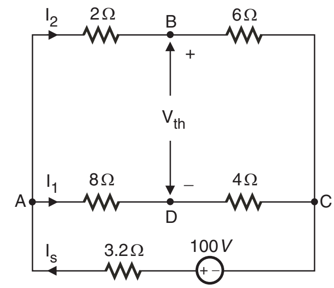


Fig. 4.

$$\frac{(2+6)\times(8+4)}{(2+6)+(8+4)} = \frac{8\times 12}{8+12} = 4.8\,\Omega$$

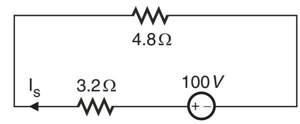


Fig. 5.

With reference to Fig. 5, by Ohm's law, we can write,

$$I_s = \frac{100}{3.2+4.8} = 12.5\,A$$

With reference to Fig. 4, by current division rule, we get,

$$I_1 = I_s \times \frac{(2+6)}{(8+4) + (2+6)} = 12.5 \times \frac{8}{12+8} = 5\,A$$

$$\therefore\ I_2 = I_s - I_1 = 12.5 - 5 = 7.5\,A$$

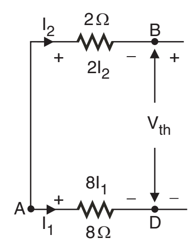


Fig. 6.

With reference to Fig. 6, by KVL, we can write,

$$2I_2 + V_{th} = 8I_1$$

$$\therefore\ V_{th} = 8I_1 - 2I_2$$

$$= 8 \times 5 - 2 \times 7.5 = 25\,V$$

<u>To find Thevenin's resistance R_{th}</u>

The ideal voltage source of Fig. 2 is replaced with a short circuit as shown in Fig. 7. The network of Fig. 7 is redrawn as shown in Fig. 8, and it is reduced to a single equivalent resistance with respect to P-Q. The step-by-step reduction of the network is shown in Figs 9 to 14.

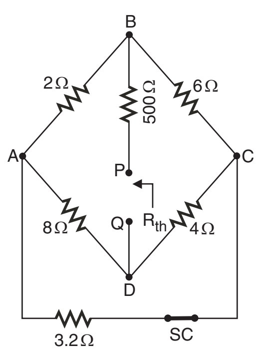


Fig. 7.

⇒

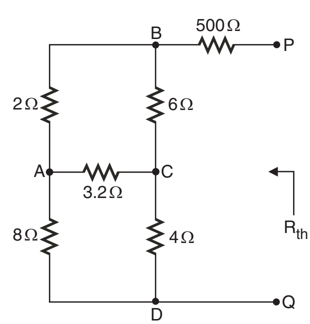


Fig. 8.

$$R_{12} = R_1 + R_2 + \frac{R_1 R_2}{R_3} = 2 + 3.2 + \frac{2 \times 3.2}{8} = 6\,\Omega$$

$$R_{23} = R_2 + R_3 + \frac{R_2 R_3}{R_1} = 3.2 + 8 + \frac{3.2 \times 8}{2} = 24\,\Omega$$

$$R_{31} = R_3 + R_1 + \frac{R_3 R_1}{R_2} = 8 + 2 + \frac{8 \times 2}{3.2} = 15\,\Omega$$

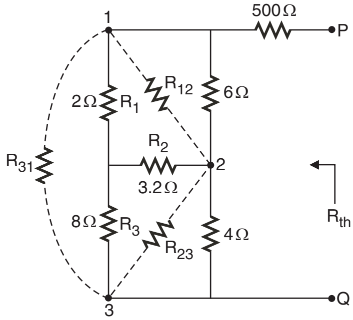


Fig. 9.

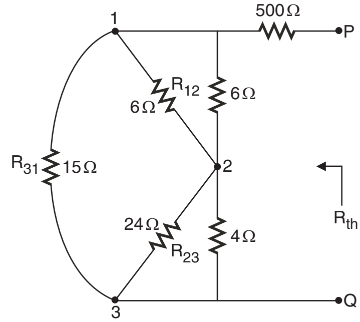


Fig. 10.

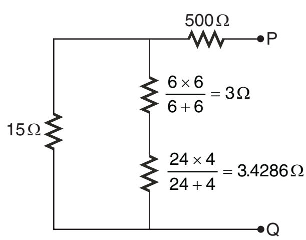


Fig. 11.

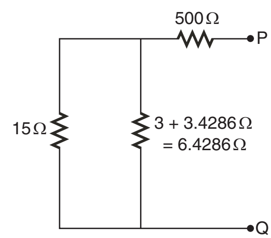


Fig. 12.

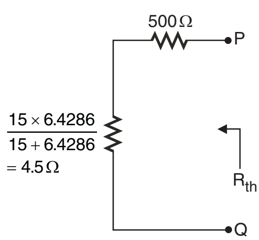


Fig. 13.

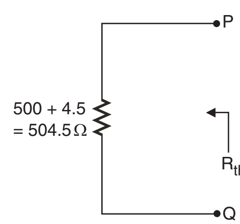


Fig. 14.

With reference to Fig. 14, Thevenin's resistance, R_{th} = 504.5 Ω

Thevenin's equivalent at P-Q

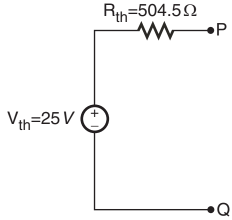


Fig. 15 : Thevenin's equivalent at P-Q.

To find current through galvanometer

Connect the galvanometer across P-Q as shown in Fig. 16. The galvanometer has negligible resistance and so it can be represented as a short circuit as shown in Fig. 17.

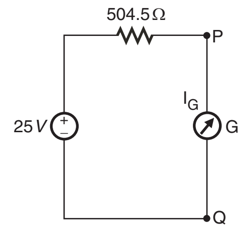


Fig. 16.

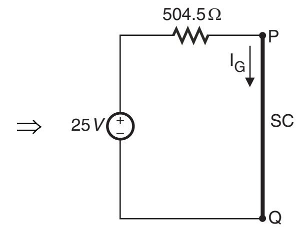


Fig. 17.

Let, I_G be the current through a Galvanometer. With reference to Fig. 17, by Ohm's law, we get,

$$I_G = \frac{25}{504.5} = 0.0496\,A = 49.6 \times 10^{-3} A = 49.6\,mA$$

EXAMPLE 2.28

Determine the current through $\bar{Z}_L$ in the circuit of Fig. 1, using Thevenin's theorem.

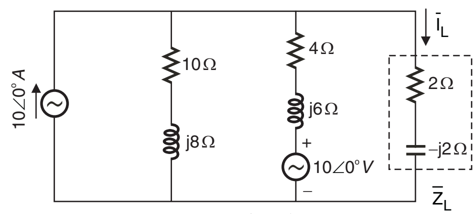


Fig. 1.

SOLUTION

Let us remove the load impedance and denote the resultant open terminals as A and B as shown in Fig. 2. Now, we have to determine Thevenin's equivalent of the circuit shown in Fig. 2.

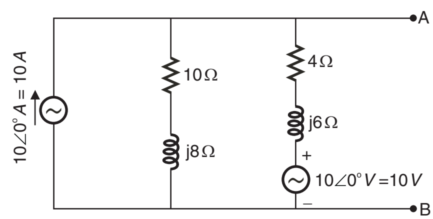


Fig. 2.

To find Thevenin's voltage $\bar{V}_{th}$

Let us convert the 10 *V* voltage source into a current source. The modified circuit is shown in Fig. 3.

With reference to Fig. 3, using KCL, we can write,

$$\frac{\bar{V}_{th}}{10+j8} + \frac{\bar{V}_{th}}{4+j6} = 10 + \frac{10}{4+j6}$$

$$\left(\frac{1}{10+j8} + \frac{1}{4+j6}\right)\bar{V}_{th} = 10 + \frac{10}{4+j6}$$

$$(0.1379 - j0.1642)\bar{V}_{th} = 10.7692 - j1.1538$$

$$\therefore \quad \bar{V}_{th} = \frac{10.7692 - j1.1538}{0.1379 - j0.1642}$$

$$= 36.4201 + j34.9992\,V = 50.5111\angle 43.9^\circ\,V$$

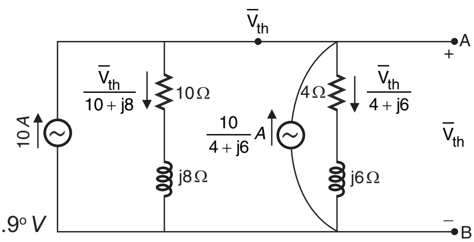


Fig. 3.

To find Thevenin's impedance $\bar{Z}_{th}$

The current source is represented by an open circuit and the voltage source is represented by a short circuit as shown in Fig. 4.

With reference to Fig. 4,

$$\bar{Z}_{th} = \frac{(10+j8)\times(4+j6)}{10+j8+4+j6} = 3 + j3.5714\,\Omega$$

$$= 4.6642\angle 50^\circ\,\Omega$$

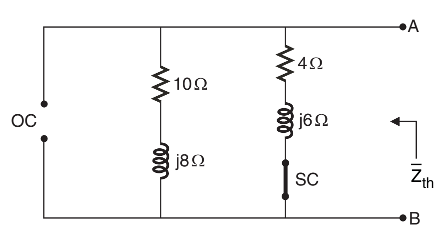
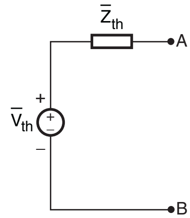


Fig. 4.

Thevenin's equivalent at A-B



$\bar{V}_{th} = 50.5111\angle 43.9^\circ V$

$= 36.4201 + j34.9992 V$

$\bar{Z}_{th} = 4.6642\angle 50^\circ \Omega$

$= 3 + j3.5714\,\Omega$

Fig. 5 : Thevenin's equivalent at A-B.

To find current through $\bar{Z}_L$

Connect the load impedance $\bar{Z}_L$ across terminals A and B of Thevenin's equivalent as shown in Fig. 6.

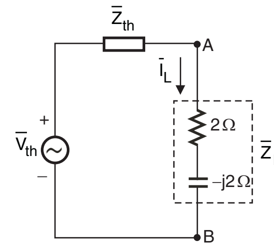


Fig. 6.

Now, by Ohm's law,

$$\bar{I}_L = \frac{\bar{V}_{th}}{\bar{Z}_{th} + \bar{Z}_L} = \frac{36.4201 + j34.9992}{3 + j3.5714 + 2 - j2}$$

$$= 8.6314 + j4.2872\ A$$

$$= 9.6375\angle 26.4^\circ\ A$$

EXAMPLE 2.29

Find the current flowing in the 5Ω resistance connected across terminals A and B of the circuit shown in Fig. 1, using Thevenin's theorem.

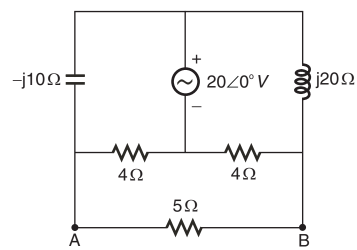


Fig. 1.

SOLUTION

Let us remove the 5Ω resistance and denote the resultant open terminals as A and B as shown in Fig. 2. Now, the circuit of Fig. 2 should be replaced with Thevenin's equivalent at terminals A and B. The polarity of Thevenin's voltage is assumed as shown in Fig. 2, with terminal-A as positive.

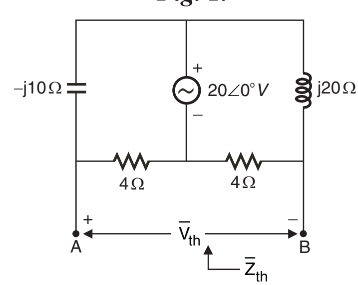


Fig. 2.

To find Thevenin's voltage $\bar{V}_{th}$

With reference to Fig. 3, the voltage across the series combination of $-j10\,\Omega$ and $4\,\Omega$ is $20\angle 0^\circ V$. Hence, by voltage division rule,

$$\bar{V}_b = 20\angle 0^\circ \times \frac{4}{4 - j10}$$

$$= \frac{20 \times 4}{4 - j10} = 2.7586 + j6.8966\,V$$

With reference to Fig. 3, the voltage across the series combination of $j20\,\Omega$ and $4\,\Omega$ is $20\angle 0^\circ V$. Hence, by voltage division rule,

$$\bar{V}_d = 20\angle 0^\circ \times \frac{4}{4 + j20} = \frac{20 \times 4}{4 + j20} = 0.7692 - j3.8462\,V$$

With reference to Fig. 3, by KVL, we can write,

$$\bar{V}_{th} + \bar{V}_d = \bar{V}_b$$

$$\therefore\ \bar{V}_{th} = \bar{V}_b - \bar{V}_d$$

$$= 2.7586 + j6.8966 - (0.7692 - j3.8462)$$

$$= 1.9894 + j10.7428\,V = 10.9255\angle 79.5^\circ\,V$$

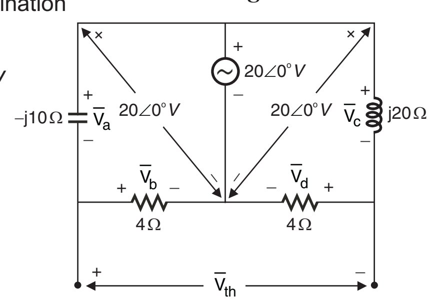


Fig. 3.

To find Thevenin's impedance $\overline{Z}_{th}$

The voltage source $20\angle 0^{\circ}\,V$ in the circuit of Fig. 2 is replaced with a short circuit as shown in Fig. 4, and is $\overline{Z}_{th}$ determined by reducing the network of Fig. 4 to a single equivalent impedance across A and B as shown below:

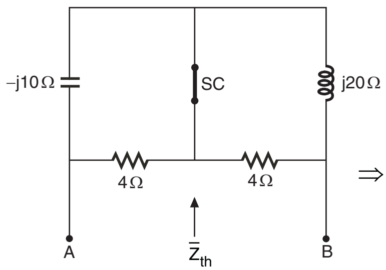


Fig. 4.

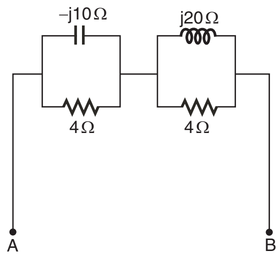


Fig. 5.

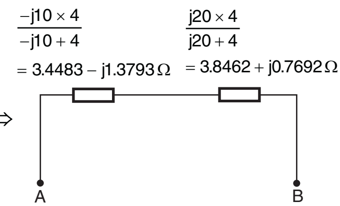
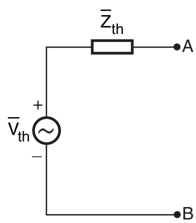


Fig. 6.

With reference to Fig. 6, we get,

$$\overline{Z}_{th} = 3.4483 - j1.3793 + 3.8462 + j0.7692$$
$$= 7.2945 - j0.6101\,\Omega = 7.32\angle -4.8^{\circ}\,\Omega$$

Thevenin's equivalent at A-B



$$\overline{V}_{th} = 1.9894 + j10.7428\,V = 10.9255\angle 79.5^{\circ}\,V$$
$$\overline{Z}_{th} = 7.2945 - j0.6101\,\Omega = 7.32\angle -4.8^{\circ}\,\Omega$$

Fig. 7 : Thevenin's equivalent at A-B.

To find current through 5Ω resistance

Connect the 5Ω resistance across A and B of Thevenin's equivalent as shown in Fig. 8. Let, $\overline{I}_L$ be the current through 5Ω resistance.

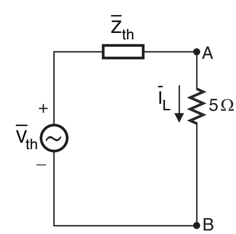


Fig. 8.

$$\text{Now,}\quad \overline{I}_L = \frac{\overline{V}_{th}}{\overline{Z}_{th} + 5}$$
$$= \frac{1.9894 + j10.7428}{7.2945 - j0.6101 + 5} = 0.1182 + j0.8797\,A$$
$$= 0.8876\angle 82.3^{\circ}\,A$$

RESULT

Current through the 5Ω resistance $= 0.8876\angle 82.3^{\circ}\,A$

EXAMPLE 2.30

Determine the voltage across terminals A and B in the circuit of Fig. 1, using Norton's theorem.

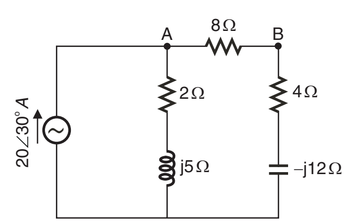


Fig. 1.

SOLUTION

Let us remove the 8Ω resistance and denote the resultant open terminals as A and B as shown in Fig. 2. Now, the circuit of Fig. 2 should be replaced by Norton's equivalent at terminals A and B.

To find Norton's current $\bar{I}_n$

Let us short circuit the terminals A and B in the circuit of Fig. 2 as shown in Fig. 3. The current flowing through the short circuit is Norton's current. Let us assume the direction of Norton's current as A to B.

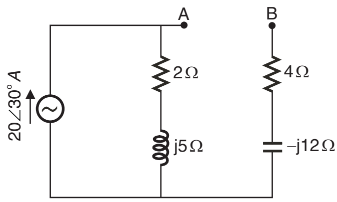


Fig. 2.

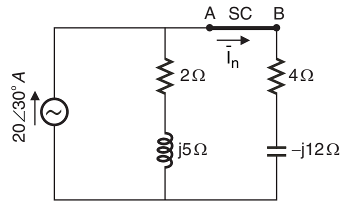


Fig. 3.

With reference to Fig. 3, by current division rule, we can write,

$$\text{Norton's current, } \bar{I}_n = 20\angle 30^\circ \times \frac{2+j5}{2+j5+4-j12} = -9.8632 + j6.26\ A$$
$$= 11.6821\angle 147.6^\circ A$$

To find Norton's impedance $\bar{Z}_n$

Let us replace the current source in Fig. 2, with an open circuit as shown in Fig. 4. $\bar{Z}_n$ is determined by reducing the network of Fig. 4 to a single equivalent impedance as shown below:

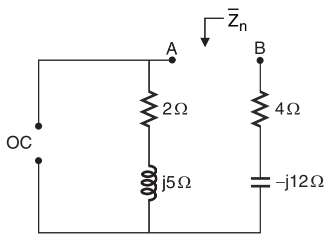


Fig. 4.

⇒

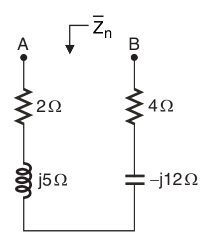


Fig. 5.

With reference to Fig. 5, we can write,

$$\text{Norton's impedance, } \bar{Z}_n = 2 + j5 - j12 + 4 = 6 - j7\ \Omega$$
$$= 9.2195\angle -49.4^\circ\ \Omega$$

Norton's equivalent at A-B

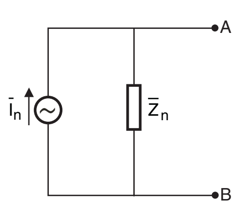


***Fig. 6 :** Norton's equivalent at A-B.*

$$\bar{I}_n = -9.8632 + j6.26\ A = 11.6821\angle 147.6^\circ\ A$$
$$\bar{Z}_n = 6 - j7\ \Omega = 9.2195\angle -49.4^\circ\ \Omega$$

To find voltage across 8 Ω resistance

Let us connect the 8 Ω resistance across A and B of Norton's equivalent as shown in Fig. 7. Let, $\bar{V}_L$ be the voltage across the 8 Ω resistance. With reference to Fig. 7, by KCL we can write,

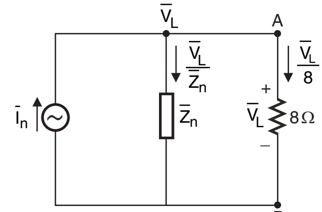


Fig. 7.

$$\frac{\bar{V}_L}{\bar{Z}_n} + \frac{\bar{V}_L}{8} = \bar{I}_n \quad \Rightarrow \quad \bar{V}_L\left(\frac{1}{\bar{Z}_n} + \frac{1}{8}\right) = \bar{I}_n$$

$$\therefore \quad \bar{V}_L = \frac{\bar{I}_n}{\frac{1}{\bar{Z}_n} + \frac{1}{8}} = \frac{\bar{I}_n}{\bar{Z}_n^{-1} + 8^{-1}} = \frac{-9.8632 + j6.26}{(6 - j7)^{-1} + 8^{-1}}$$

$$= -31.3876 + j45.2219\ V$$

$$= 55.0473\angle 124.8^\circ\ V$$

RESULT

Voltage across terminals A and B = $55.0473\angle 124.8^\circ$ V

2.6.2 Superposition Theorem

The superposition theorem states that the response in a circuit with multiple sources is given by the algebraic sum of responses due to individual sources acting alone. The superposition theorem is also referred to as the principle of superposition.

The superposition theorem is a useful tool for analysis of linear circuits with multiple sources. A linear circuit is a circuit composed entirely of independent sources, linear dependent sources and linear elements. A circuit element is said to be **linear**, if the voltage-current relationship is linear, i.e., $v \propto i$ or $v = ki$, where k is a constant.

The responses that can be determined by the superposition theorem are listed below:

i) Current in resistance, inductance and capacitance.

ii) Voltage across resistance, inductance and capacitance.

iii) Current delivered by independent voltage sources.

iv) Voltage across independent current sources.

v) Voltage and current of linear dependent sources.

While calculating the response due to an individual source, all other sources are made inactive or replaced by zero value sources (Sometimes the bloodthirsty term killed is used). A **zero value source** is represented by its internal resistance (or impedance). "*In an ideal voltage source, the internal resistance (or impedance) is zero, and in an ideal current source, the internal resistance (or impedance) is infinite*".

Therefore, while calculating the response due to one source, all other ideal voltage sources are replaced with a short circuit (or by their internal impedance) and all other ideal current sources are replaced with an open circuit (or by their internal impedance).

Procedure for Analysis Using Superposition Theorem

1. When the internal impedances of sources are specified, represent them as an external impedance and so the sources will become ideal sources. For a voltage source, the internal impedance is represented as an impedance in series with the ideal voltage source. For a current source, the internal impedance is represented as an impedance in parallel with the ideal current source.

2. The response is either voltage or current in the elements. The response when all the sources are acting is called the **total response**. If the polarity of the total voltage response or direction of the total current response are not specified in the problem, then assume a polarity for the total voltage response and direction for the total current response when all the sources are acting together.

3. Determine the response due to each independent source by allowing one source to act at a time. While determining the response due to one source, replace all other independent ideal voltage sources by a short circuit (SC) and all other independent ideal current sources by an open circuit (OC).

4. Denote the voltage response due to each source as V', V'', V'''... and the current response as I', I'', I'''.... While determining the response due to each source, maintain the polarity of voltage response the same as that of the total response. Similarly, maintain the direction of current response the same as that of the total response.

5. Determine the total response by taking the sum of individual responses.

> ***Note :*** *1. Power cannot be directly determined from the superposition theorem. Hence, determine the power only using the total current and voltage response.*
>
> *2. When all independent sources are deactivated, there will not be any current or voltage in any part of the circuit. Hence, dependent sources will not contribute to the response when all independent sources are deactivated (i.e., the response due to a dependent source acting alone will be zero).*

EXAMPLE 2.31

Find the current through the 5 Ω resistor in the circuit shown in Fig. 1 using the superposition theorem.

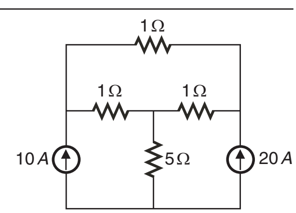


Fig. 1.

SOLUTION

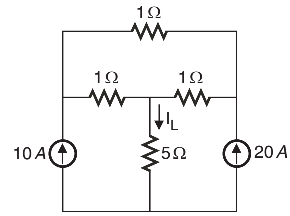


Fig. 2.

Let I_L be the current through the 5 Ω resistance when both the current sources are acting together as shown in Fig. 2.

Let, I'_L = Current through 5 Ω in the direction of I_L when the 10 *A* source alone is acting.

I''_L = Current through 5 Ω in the direction of I_L when the 20 *A* source alone is acting.

Now, by the superposition theorem,

$$I_L = I'_L + I''_L$$

To find the response I'_L when the 10 *A* source is acting alone

The 20 *A* current source is replaced by an open circuit as shown in Fig. 3. The circuit of Fig. 3 is redrawn as shown in Fig. 4.

In Fig. 4, the 5 Ω resistance is in series with the 10 *A* source and so the current through 5 Ω is also 10 *A*.

$$\therefore \quad I'_L = 10\,A$$

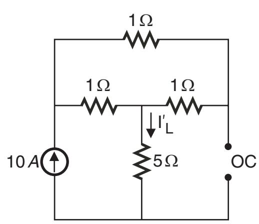


Fig. 3.

⇒

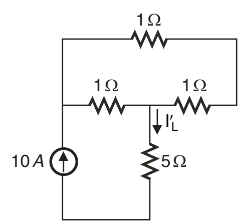


Fig. 4.

To find the response I''_L when the 20 *A* source is acting alone

The 10 *A* current source is replaced by an open circuit as shown in Fig. 5. The circuit of Fig. 5 is redrawn as shown in Fig. 6.

In Fig. 6, the 5 Ω resistance is in series with the 20 *A* source and so the current through 5 Ω is also 20 *A*.

$$\therefore \quad I''_L = 20\,A$$

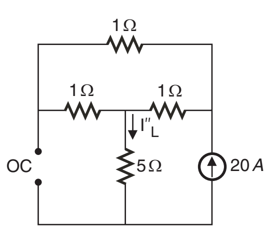


Fig. 5.

⇒

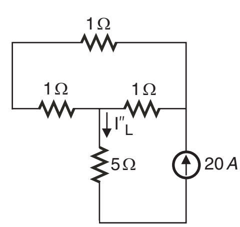


Fig. 6.

To find the total response I_L when both the sources are acting

By the superposition theorem,

$$I_L = I'_L + I''_L$$

$$\therefore \quad I_L = 10 + 20 = 30\,A$$

Cross-Check by Node Analysis

With reference to Fig. 7, the node basis matrix equation is,

$$\begin{bmatrix} \frac{1}{1}+\frac{1}{1} & -\frac{1}{1} & -\frac{1}{1} \\ -\frac{1}{1} & \frac{1}{1}+\frac{1}{1}+\frac{1}{5} & -\frac{1}{1} \\ -\frac{1}{1} & -\frac{1}{1} & \frac{1}{1}+\frac{1}{1} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2.2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 20 \end{bmatrix}$$

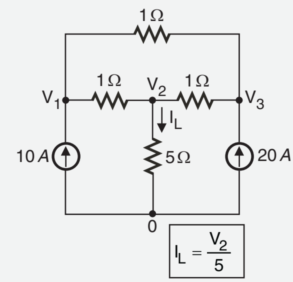


Fig. 7.

$$\Delta' = \begin{vmatrix} 2 & -1 & -1 \\ -1 & 2.2 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 2\times\left[2.2\times 2-(-1)^2\right]-(-1)\times\left[-1\times 2-(-1)^2\right]+(-1)\times\left[(-1)^2-(-1)\times 2.2\right]$$

$$= 6.8-3-3.2 = 0.6$$

$$\Delta'_2 = \begin{vmatrix} 2 & 10 & -1 \\ -1 & 0 & -1 \\ -1 & 20 & 2 \end{vmatrix} = 2\times\left[0-20\times(-1)\right]-10\times\left[-1\times 2-(-1)^2\right]+(-1)\times\left[-1\times 20-0\right]$$

$$= 40+30+20 = 90$$

$$V_2 = \frac{\Delta'_2}{\Delta'} = \frac{90}{0.6} = 150\,V$$

$$\therefore\ I_L = \frac{V_2}{5} = \frac{150}{5} = 30\,A$$

EXAMPLE 2.32

Using the superposition theorem, find the current through the $3\,\Omega$ resistance in the circuit shown in Fig. 1.

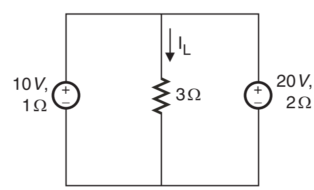


Fig. 1.

SOLUTION

The internal resistances of the voltage sources are represented as a series resistance external to the source and the voltage sources are represented as ideal sources as shown in Fig. 2.

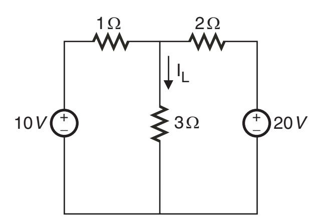


Fig. 2.

Let, I'_L = Current through $3\,\Omega$ in the direction of I_L when the $10\,V$ source alone is acting.

I''_L = Current through $3\,\Omega$ in the direction of I_L when the $20\,V$ source alone is acting.

Now, by the superposition theorem,

$$I_L = I'_L + I''_L$$

To find the response I'_L when the 10 *V* source is acting alone

The $20\,V$ source is replaced with a short circuit as shown in Fig. 3. Let I_{s1} be the current supplied by the $10\,V$ source. In Fig. 3, at node-A, the current I_{s1} divides between the parallel resistances $2\,\Omega$ and $3\,\Omega$.

The parallel-connected resistances $2\,\Omega$ and $3\,\Omega$ are combined to form a single equivalent as shown in Fig. 4.

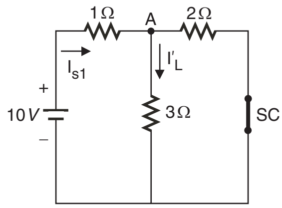


Fig. 3.

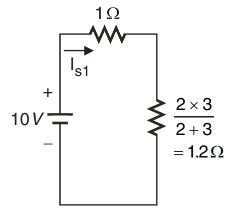


Fig. 4.

With reference to Fig. 4, by Ohm's law,

$$I_{s1} = \frac{10}{1+1.2} = 4.5455\,A$$

With reference to Fig. 3, by current division rule,

$$I'_L = I_{s1} \times \frac{2}{2+3} = 4.5455 \times \frac{2}{2+3} = 1.8182\,A$$

To find the response I''_L when the 20 V source is acting alone

The 10 *V* source is replaced with a short circuit as shown in Fig. 5. Let I_{s2} be the current supplied by the 20 *V* source. In Fig. 5, at node-A, the current I_{s2} divides between the parallel resistances 1 Ω and 3 Ω.

The parallel-connected resistances 1 Ω and 3 Ω are replaced with a single equivalent as shown in Fig. 6.

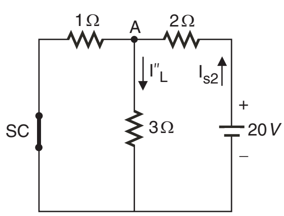


Fig. 5.

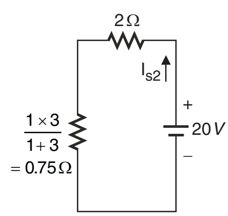


Fig. 6.

With reference to Fig. 6, by Ohm's law,

$$I_{s2} = \frac{20}{2+0.75} = 7.2727\,A$$

With reference to Fig. 5, by current division rule,

$$I''_L = I_{s2} \times \frac{1}{1+3} = 7.2727 \times \frac{1}{1+3} = 1.8182\,A$$

To find the response I_L when both the sources are acting

By the superposition theorem,

$$I_L = I'_L + I''_L$$

$$= 1.8182 + 1.8182 = 3.6364A$$

Cross-Check by Mesh Analysis

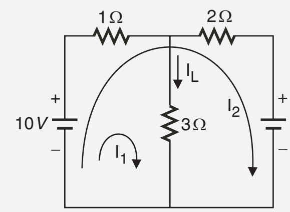


Fig. 7.

With reference to Fig. 7, the mesh basis matrix equation is,

$$\begin{bmatrix} 1+3 & 1 \\ 1 & 1+2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10-20 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} = 4 \times 3 - 1 \times 1 = 11$$

$$\Delta_1 = \begin{vmatrix} 10 & 1 \\ -10 & 3 \end{vmatrix} = 10 \times 3 - (-10) \times 1 = 40$$

$$\therefore \; I_L = I_1 = \frac{\Delta_1}{\Delta} = \frac{40}{11} = 3.6364\,A$$

EXAMPLE 2.33 ***(AU June'14, 8 Marks)***

Using the superposition theorem, find the current through the 5Ω resistor in the circuit shown in Fig.1.

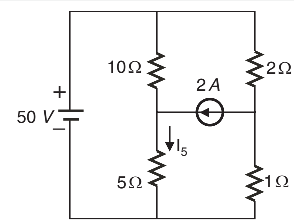


Fig. 1.

SOLUTION

Let , I_5 = Current through the 5 Ω resistance shown in Fig. 1.

Let, I'_5 = Current through the 5 Ω resistance in the direction of I_5 when the 50 *V* source alone is acting.

I''_5 = Current through the 5 Ω resistance in the direction of I_5 when the 2 *A* source alone is acting.

Now, by the superposition theorem,

$$I_5 = I'_5 + I''_5$$

To find the response I'_5 when the 50 *V* source is acting alone

The 2 *A* source is replaced with an open circuit as shown in Fig. 2.

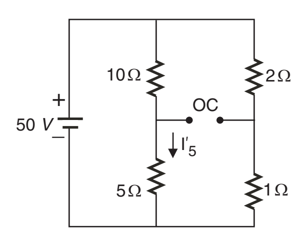


Fig. 2.

With reference to Fig. 2, by Ohm's law,

$$I'_5 = \frac{50}{10+5} = \frac{50}{15} = \frac{10}{3}\,A$$

To find the response I''_5 when the 2 *A* source is acting alone

The 50 *V* source is replaced with a short circuit as shown in Fig. 3. In the circuit of Fig. 4, the 2 *A* source current divides between parallel resistances 10 Ω and 5 Ω.

Therefore, by current division rule,

$$I_5'' = 2 \times \frac{10}{10+5} = \frac{20}{15} = \frac{4}{3}\,A$$

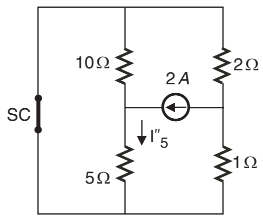


Fig. 3.

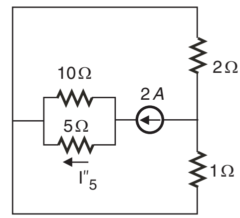


Fig. 4.

To find the total current I_5 when both the sources are acting

By superposition theorem,

$$I_5 = I_5' + I_5'' = \frac{10}{3} + \frac{4}{3} = \frac{14}{3} = 4.6667\,A$$

EXAMPLE 2.34 *(AU May'17, 8 Marks)*

Using the superposition theorem, find the power delivered by the 20 *V* source in the circuit shown in Fig. 1.

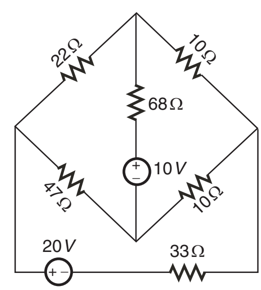


Fig. 1.

SOLUTION

Let, I_S = Current in the series branch with the 20 *V* source and 33 Ω in series.

I'_S = Current in the series branch when the 20 *V* source alone is acting.

I''_S = Current in the series branch when the 10 *V* source alone is acting.

Now, by the superposition theorem,

$$I_S = I'_S + I''_S$$

Note : *Since power is proportional to the square of voltage, it is not a linear quantity. So, power cannot be determined directly by the superposition theorem.*

To find the response I'_S when the 20 *V* source is acting alone

The 10 *V* source is replaced with a short circuit as shown in Fig. 2.

The delta-connected resistances 22 Ω, 47 Ω and 68 Ω in Fig. 2 are converted into a star-connected resistance as shown in Fig. 3.

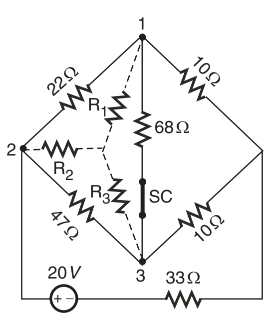


Fig. 2.

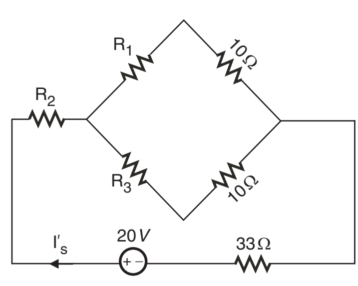


Fig. 3.

$$R_1 = \frac{22 \times 68}{22 + 47 + 68} = 10.9197\,\Omega$$

$$R_2 = \frac{22 \times 47}{22 + 47 + 68} = 7.5474\,\Omega$$

$$R_3 = \frac{47 \times 68}{22 + 47 + 68} = 23.3285\,\Omega$$

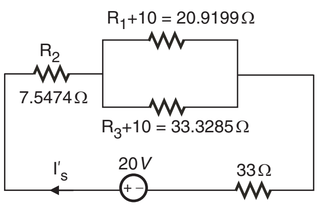


Fig. 4.

⇒

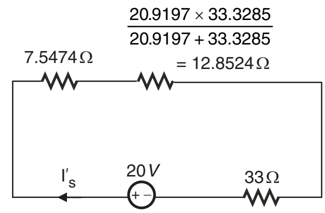


Fig. 5.

With reference to Fig. 5, by Ohm's law,

$$I'_S = \frac{20}{7.5474 + 12.8524 + 33} = 0.3745\,A$$

To find the response I''_S when the 10 *V* source is acting alone

The 20 *V* source is replaced with a short circuit as shown in Fig. 6. Now the current I''_S is solved by mesh analysis.

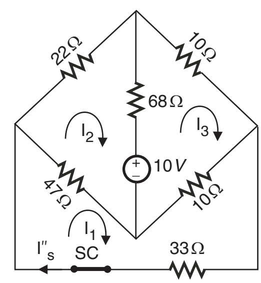


Fig. 6.

With reference to Fig. 6, the mesh basis matrix equation is,

$$\begin{bmatrix} 47+10+33 & -47 & -10 \\ -47 & 22+68+47 & -68 \\ -10 & -68 & 68+10+10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 90 & -47 & -10 \\ -47 & 137 & -68 \\ -10 & -68 & 88 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}$$

Let us define two determinants Δ and Δ_1 as shown below and mesh current I_1 is solved by Cramerr's rule.

$$\Delta = \begin{vmatrix} 90 & -47 & -10 \\ -47 & 137 & -68 \\ -10 & -68 & 88 \end{vmatrix} = 90 \times \left[137 \times 88 - (-68)^2\right] - (-47) \times \left[-47 \times 88 - (-10) \times (-68)\right]$$

$$+ (-10) \times \left[(-47) \times (-68) - (-10) \times 137\right]$$

$$= 668880 - 226352 - 45660 = 396868$$

$$\Delta_1 = \begin{vmatrix} 0 & -47 & -10 \\ -10 & 137 & -68 \\ -10 & -68 & 88 \end{vmatrix} = 0 - (-47) \times \left[-10 \times 88 - 10 \times (-68)\right]$$

$$+ (-10) \times \left[-10 \times (-68) - 10 \times 137)\right]$$

$$= -9400 - 6900 = -16300$$

$$\therefore I''_S = I_1 = \frac{\Delta_1}{\Delta} = \frac{-16300}{396868} = -0.0411\,A$$

To find the total response I_S and power

By the superposition theorem,

$$I_S = I'_S + I''_S$$

$$= 0.3745 + (-0.0411) = 0.3334\ A$$

$$\text{Power delivered by 20}\ V\ \text{source} = 20 \times I_1 = 20 \times 0.3334$$

$$= 6.668\ W$$

EXAMPLE 2.35

Using the superposition theorem, find the voltage V_L and the power consumed by the $6\,\Omega$ resistor in the circuit shown in Fig. 1.

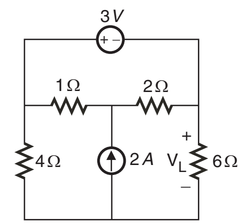


Fig. 1.

SOLUTION

Let, V'_L = Voltage across the $6\,\Omega$ resistance with polarity same as that of V_L when the $3\,V$ source alone is acting.

V''_L = Voltage across the $6\,\Omega$ resistance with polarity same as that of V_L when the $2\,A$ source alone is acting.

Now, by the superposition theorem,

$$V_L = V'_L + V''_L$$

$$\text{Power consumed by the } 6\,\Omega \text{ resistor, } P_L = \frac{V_L^2}{6}$$

> ***Note :*** *Since power is proportional to the square of voltage, it is not a linear quantity. So, power cannot be determined directly by the superposition theorem.*

To find the response V'_L when the $3\,V$ source is acting alone

The $2\,A$ current source is replaced with an open circuit as shown in Fig. 2. The circuit of Fig. 2 is redrawn as shown in Fig. 3.

In Fig. 3, the voltage across series combination of $4\,\Omega$ and $6\,\Omega$ is $3\,V$. This $3\,V$ divides into V_1 and V_2 and so by voltage division rule, we get,

$$V'_L = -V_2 = -3 \times \frac{6}{6+4} = -1.8\,V$$

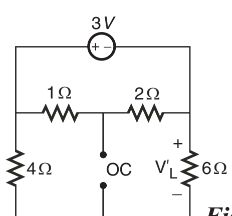


Fig. 2.

⇒

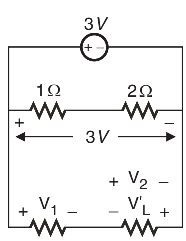


Fig. 3.

To find the response V''_L when 2 A source is acting alone

The 3 V source is replaced with a short circuit as shown in Fig. 4. The circuit of Fig. 4 is redrawn as shown in Figs. 5 and 6.

In the circuit of Fig. 6, the current 2 A divides between the parallel resistances 6 Ω and 4 Ω.

Hence, by current division rule,

$$I_1 = 2 \times \frac{4}{6+4} = 0.8\,A$$

By Ohm's law,

$$V''_L = I_1 \times 6 = 0.8 \times 6 = 4.8\,V$$

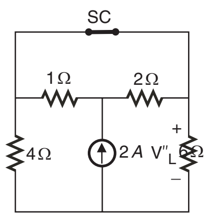


Fig. 4.

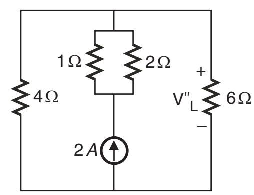


Fig. 5.

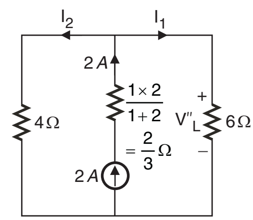


Fig. 6.

To find V_L and power in the 6 Ω resistor

By the superposition theorem,

$$V_L = V'_L + V''_L = -1.8 + 4.8 = 3\,V$$

Power consumed by the 6 Ω resistor, $P_L = \frac{V_L^2}{6} = \frac{3^2}{6} = 1.5\,W$

Cross-Check

Let, $P'_L = \frac{(V'_L)^2}{6} = \frac{(-1.8)^2}{6} = 0.54\,W$

$$P''_L = \frac{(V''_L)^2}{6} = \frac{4.8^2}{6} = 3.84\,W$$

Let, $P_{L,\,sup} = P'_L + P''_L = 0.54 + 3.84 = 4.38\,W$

Here, $P_{L,\,sup} \neq P_L$. So we can say that the power calculated directly by the superposition theorem is not equal to actual power.

EXAMPLE 2.36

Find the voltage across the 2 Ω resistance in the circuit of Fig. 1 using the principle of superposition.

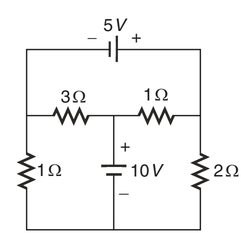


Fig. 1.

SOLUTION

Let V_L be the voltage across the 2 Ω resistance when both the voltage sources are acting together as shown in Fig. 2.

Let, V'_L = Voltage across the 2 Ω resistance with polarity same as that of V_L when the 5 V source alone is acting.

V''_L = Voltage across the 2 Ω resistance with polarity same as that of V_L when the 10 V source alone is acting.

Now, by the superposition theorem,

$$V_L = V'_L + V''_L$$

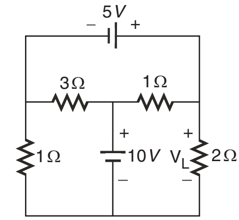


Fig. 2.

To find the response V'_L when the 5 V source is acting alone

The 10 V source is replaced with a short circuit as shown in Fig. 3. The circuit of Fig. 3 is redrawn as shown in Figs 4 and 5.

The parallel combinations of resistances are replaced with a single equivalent as shown in Fig. 5.

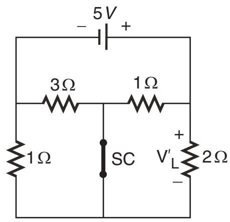


Fig. 3.

⇒

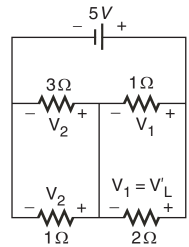


Fig. 4.

⇒

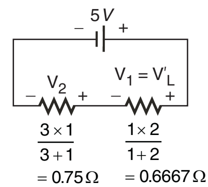


Fig. 5.

In Fig. 5, the source voltage 5 V divides between the series-connected resistances 0.6667 Ω and 0.75 Ω. Let, these voltages be V_1 and V_2. Since 1 Ω and 2 Ω are in parallel, the voltage across them will be the same and so $V_1 = V'_L$.

With reference to Fig. 5, by voltage division rule,

$$V'_L = V_1 = 5 \times \frac{0.6667}{0.6667 + 0.75} = 2.353\,V$$

To find the response V''_L when the 10 V source is acting alone

The 5 V source is replaced with a short circuit as shown in Fig. 6. The circuit of Fig. 6 is redrawn as shown in Figs. 7 and 8.

Figure 7 is same as that Fig. 4, except the 5 V source, which is the 10 V in this case.

Therefore, by voltage division rule,

$$V''_L = 10 \times \frac{0.6667}{0.6667 + 0.75} = 4.706\,V$$

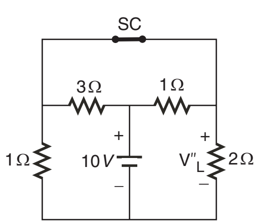


Fig. 6.

⇒

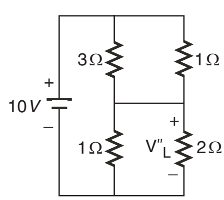


Fig. 7.

⇒

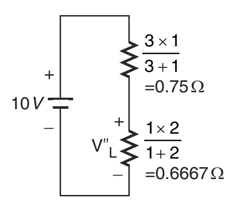


Fig. 8.

To find the response V_L when both the sources are acting

By the superposition theorem,

$$V_L = V'_L + V''_L$$

$$= 2.353 + 4.706 = 7.059\,V$$

EXAMPLE 2.37

Use the principle of superposition to find the current I_L through the $8\,\Omega$ resistance in the circuit shown in Fig. 1.

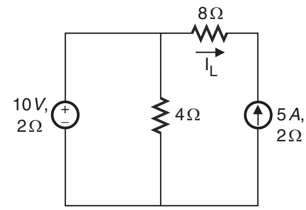


Fig. 1.

SOLUTION

The internal resistance of the voltage source is represented as a series resistance external to the source and the internal resistance of the current source is represented as a parallel resistance external to the source as shown in Fig. 2. Now, the sources can be treated as ideal sources.

Let, I'_L = Current through the $8\,\Omega$ resistance in the direction of I_L when the $10\,V$ source alone is acting.

I''_L = Current through the $8\,\Omega$ resistance in the direction of I_L when the $5\,A$ source alone is acting.

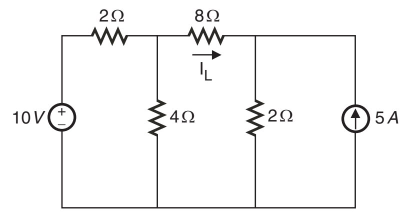


Fig. 2.

Now, by the superposition theorem,

$$I_L = I'_L + I''_L$$

To find the response I'_L when the 10 *V* source is acting alone

The $5\,A$ source is replaced with an open circuit as shown in Fig. 3. Let, I_{s1} be the total current supplied by the $10\,V$ source. This current divides into I_1 and I_2 and flows through the two parallel paths as shown in Fig. 3.

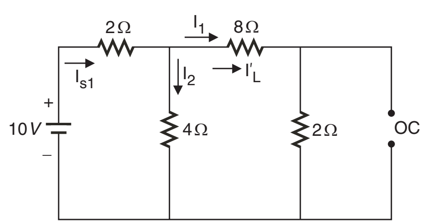


Fig. 3.

⇒

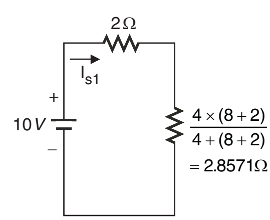


Fig. 4.

With reference to Fig. 4,

$$I_{s1} = \frac{10}{2 + 2.8571} = 2.0588\,A$$

With reference to Fig. 3, by current division rule,

$$I'_L = I_1 = I_{s1} \times \frac{4}{4 + (8 + 2)} = 2.0588 \times \frac{4}{4 + 10} = 0.5882\,A$$

To find the response I''_L when the 5 *A* source is acting alone

The $10\,V$ source is replaced with a short circuit as shown in Fig. 5. The parallel combination of $2\,\Omega$ and $4\,\Omega$ is replaced with a single equivalent as shown in Fig. 6. With reference to Fig. 6, we can say that the $5\,A$ current divides between the parallel resistances $2\,\Omega$ and $(8 + 1.3333)\,\Omega$.

By current division rule,

$$I''_L = -I_4 = -5 \times \frac{2}{2 + (8 + 1.3333)} = -0.8824\,A$$

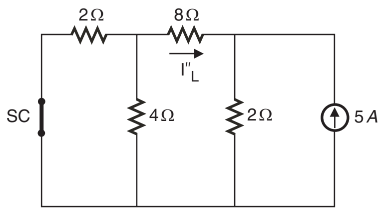


Fig. 5.

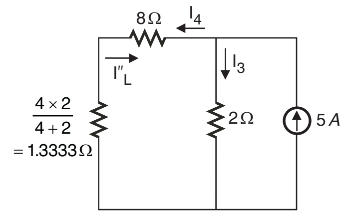


Fig. 6.

To find the response I_L when both the sources are acting

By the superposition theorem,

$$I_L = I'_L + I''_L$$

$$= 0.5882 + (-0.8824) = -0.2942\,A$$

EXAMPLE 2.38

(AU May'15, 8 Marks)

Compute the current I_L in the circuit of Fig. 1 using the superposition theorem.

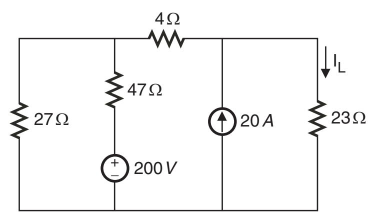


Fig. 1.

SOLUTION

Let, I'_L = Current through the $23\,\Omega$ resistance in the direction of I_L when the $200\,V$ source alone is acting.

I''_L = Current through the $23\,\Omega$ resistance in the direction of I_L when the $20\,A$ source alone is acting.

Now, by the superposition theorem,

$$I_L = I'_L + I''_L$$

To find the response I'_L when the 200 V source is acting alone

The $20\,A$ source is replaced by an open circuit as shown in Fig. 2. Let, I_{s1} be the total current supplied by the $200\,V$ source. This current divides equally between parallel-connected resistances $27\,\Omega$ and $(4 + 23)\,\Omega$.

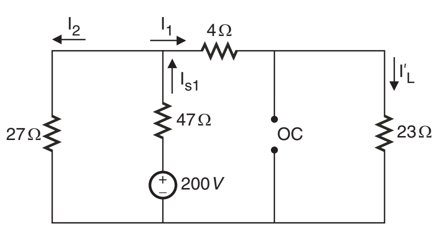


Fig. 2.

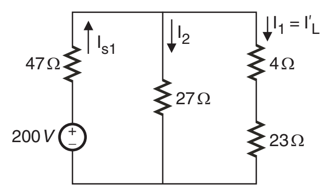


Fig. 3.

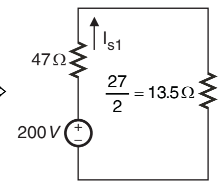


Fig. 4.

With reference to Fig. 3, by Ohm's law,

$$I_{s1} = \frac{200}{47 + 13.5} = 3.3058\,A$$

With reference to Fig. 2, by current division rule,

$$I'_L = I_1 = \frac{I_{s1}}{2} = \frac{3.3058}{2} = 1.6529\,A$$

To find the response I''_L when 20 *A* source is acting alone

The 200 *V* source is replaced with a short circuit as shown in Fig. 5. The parallel resistances 27 Ω and 47 Ω are replaced with a single equivalent as shown in Fig. 6. In the circuit of Fig. 6, the 20 *A* source current divides between parallel resistances 23 and (4 + 17.1486) Ω.

Therefore, by current division rule,

$$I''_L = I_3 = 20 \times \frac{(4+17.1486)}{23+(4+17.1486)} = 9.5806\ A$$

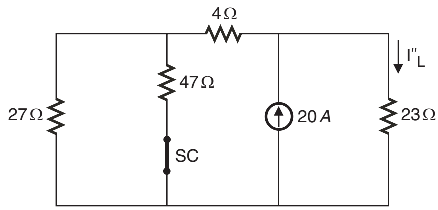


Fig. 5.

$\Rightarrow$

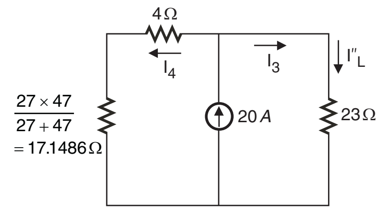


Fig. 6.

To find the total current I_L when both the sources are acting

By the superposition theorem,

$$I_L = I'_L + I''_L$$

$$= 1.6529 + 9.5806 = 11.2335\ A$$

EXAMPLE 2.39

Determine the current in the 5 Ω resistance in the circuit shown in Fig. 1 using the superposition theorem.

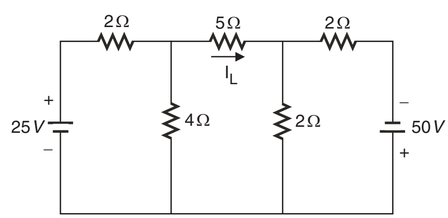


Fig. 1.

SOLUTION

Let, I'_L = Current through the 5 Ω resistance in the direction of I_L when the 25 *V* source alone is acting.

I''_L = Current through the 5 Ω resistance in the direction of I_L when the 50 *V* source alone is acting.

By the superposition theorem,

$$I_L = I'_L + I''_L$$

To find the response I'_L when the 25 *V* source is acting alone

The 50 *V* source is replaced with a short circuit as shown in Fig. 2. The 2 Ω resistances in parallel are replaced by a single equivalent and then I'_L is solved using mesh analysis.

With reference to Fig. 3, the mesh basis matrix equation is,

$$\begin{bmatrix} 2+4 & -4 \\ -4 & 4+5+1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 25 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & -4 \\ -4 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 25 \\ 0 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 6 & -4 \\ -4 & 10 \end{vmatrix} = 6 \times 10 - (-4)^2 = 44 \quad ; \quad \Delta_2 = \begin{vmatrix} 6 & 25 \\ -4 & 0 \end{vmatrix} = 0 - (-4) \times 25 = 100$$

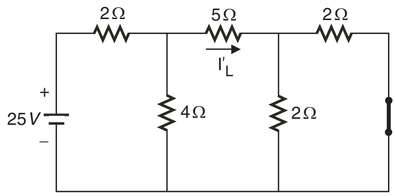


Fig. 2.

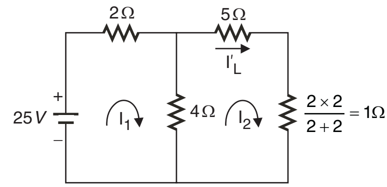


Fig. 3.

$$\therefore \quad I'_L = I_2 = \frac{\Delta_2}{\Delta} = \frac{100}{44} = 2.2727\,A$$

To find the response I''_L when 50 *V* source is acting alone

The 25 *V* source is replaced with a short circuit as shown in Fig. 4. The parallel-connected resistances 2Ω and 4Ω are replaced with a single equivalent as shown in Fig. 5 and then I''_L is solved using mesh analysis.

The mesh basis matrix equation is,

$$\begin{bmatrix} 5+2+1.3333 & -2 \\ -2 & 2+2 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 50 \end{bmatrix} \Rightarrow \begin{bmatrix} 8.3333 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 50 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 8.3333 & -2 \\ -2 & 4 \end{vmatrix} = 8.3333\times4-(-2)^2 = 29.3332 \quad ; \quad \Delta_1 = \begin{vmatrix} 0 & -2 \\ 50 & 4 \end{vmatrix} = 0-50\times(-2) = 100$$

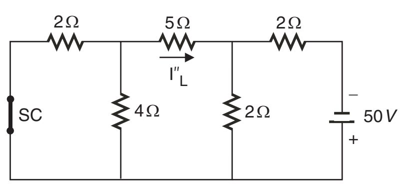


Fig. 4.

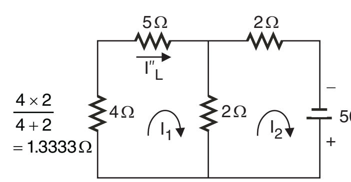


Fig. 5.

$$\therefore \quad I''_L = I_1 = \frac{\Delta_1}{\Delta} = \frac{100}{29.3332} = 3.4091\,A$$

To find the total response I_L when both the sources are acting

By the superposition theorem,

$$I_L = I'_L + I''_L$$

$$= 2.2727 + 3.4091 = 5.6818$$

EXAMPLE 2.40 ***(AU Dec'16, 16 Marks)***

Using the superposition theorem, find the current through the 3Ω resistor in the circuit shown in Fig.1.

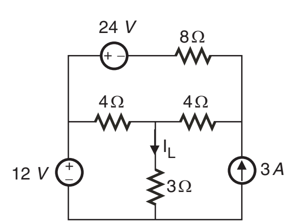


Fig. 1.

SOLUTION

Let, I'_L = Current through 3Ω in the direction of I_L when the 12 *V* source alone is acting.

I''_L = Current through $3\,\Omega$ in the direction of I_L when the 24 V source alone is acting.

I'''_L = Current through $3\,\Omega$ in the direction of I_L when the 3 A source alone is acting.

Now, by the superposition theorem,

$$I_L = I'_L + I''_L + I'''_L$$

To find the response I'_L when the 12 V source is acting alone

The 24 V source is replaced with a short circuit and the 3 A source is replaced with an open circuit as shown in Fig. 2.

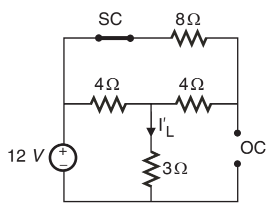


Fig. 2.

$\Rightarrow$

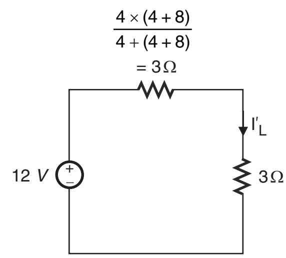


Fig. 3.

With reference to Fig. 3, by Ohm's law,

$$I'_L = \frac{12}{3+3} = 2\,A$$

To find the response I''_L when the 24 V source is acting alone

The 12 V source is replaced with a short circuit and the 3 A source is replaced with an open circuit as shown in Fig. 4.

The series-connected resistances $8\,\Omega$ and $4\,\Omega$ are replaced with a single equivalent and parallel-connected resistances $4\,\Omega$ and $3\,\Omega$ are replaced with a single equivalent in Fig. 5.

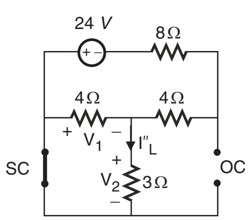


Fig. 4.

$\Rightarrow$

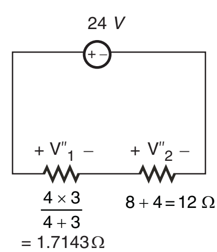


Fig. 5.

With reference to Fig. 5, by voltage division rule,

$$V_1'' = 24 \times \frac{1.7143}{1.7143 + 12} = 3\,V$$

With reference to Fig. 4, by Ohm's law,

$$I''_L = \frac{V''_1}{3} = \frac{3}{3} = 1\,A$$

To find the response I'''_L when the 3 *A* source is acting alone

The 12 *V* and 24 *V* are replaced with a short circuit as shown in Fig. 6.

The parallel-connected resistances 4 Ω and 3 Ω are replaced by a single equivalent in Fig. 7.

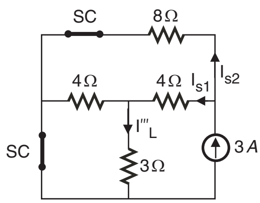


Fig. 6.

⇒

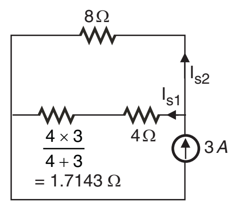


Fig. 7.

With reference to Fig. 7, by current division rule,

$$I_{s1} = 3 \times \frac{8}{(4 + 1.7143) + 8} = 1.75\,A$$

With reference to Fig. 6, by current division rule,

$$I'''_L = I_{s1} \times \frac{4}{4+3} = 1.75 \times \frac{4}{7} = 1\,A$$

To find the total current I_L when all the sources are acting

By superposition theorem,

$$I_L = I'_L + I''_L + I'''_L = 2 + 1 + 1 = 4\,A$$

EXAMPLE 2.41 *(AU Dec'14, 16 Marks)*

Use the principle of superposition to find the current I_L through the 5 Ω resistance in the circuit shown in Fig. 1.

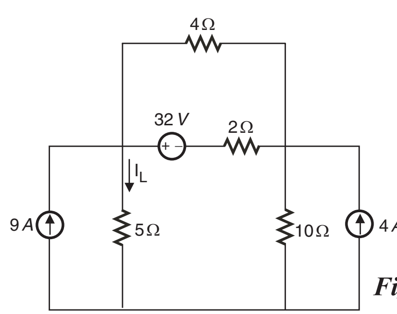


Fig. 1.

SOLUTION

Let, I'_L = Current through 5 Ω in the direction of I_L when the 9 *A* source alone is acting.

I''_L = Current through 5 Ω in the direction of I_L when the 4 *A* source alone is acting.

I'''_L = Current through 5 Ω in the direction of I_L when the 32 *V* source alone is acting.

Now, by the superposition theorem,

$$I_L = I'_L + I''_L + I'''_L$$

To find the response I'_L due to the 9 *A* source

The 32 *V* source is replaced with a short circuit and the 4 A source with an open circuit as shown in Fig. 2. The parallel combination of 2 Ω and 4 Ω is replaced with a single equivalent as shown in Fig. 3. With reference to Fig. 3, we can say that the 9 *A* current divides between the parallel resistances 5 Ω and 11.3333 Ω.

By current division rule,

$$I'_L = 9 \times \frac{11.3333}{5 + 11.3333} = 6.2449\ A$$

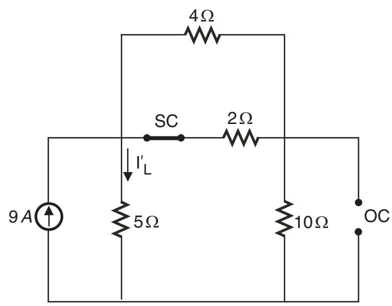


Fig. 2.

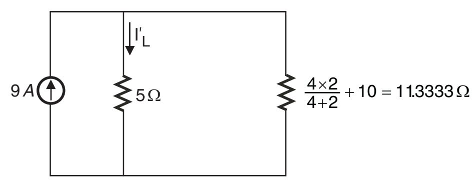


Fig. 3.

To find the response I''_L due to the 4 *A* source

The 32 *V* source is replaced with a short circuit and 9 A source with an open circuit as shown in Fig. 4. The parallel combination of 2 Ω and 4 Ω is replaced with a single equivalent in Fig. 5. With reference to Fig. 5, we can say that the 4 *A* current divides between the parallel resistances 10 Ω and (5+1.3333) Ω.

By current division rule,

$$I''_L = 4 \times \frac{10}{10 + (1.3333 + 5)} = 2.449\ A$$

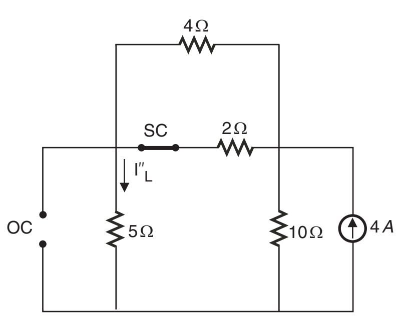


Fig. 4.

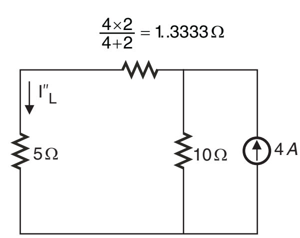


Fig. 5.

To find the response I'''_L due to the 32 *V* source

The current sources 9*A* and 4A are replaced with an open circuit as shown in Fig. 6. Let us assume two mesh currents I_1 and I_2 as shown in Fig. 6. Now, $I''_L = I_2$. The mesh basis matrix equation is,

$$\begin{bmatrix} 4+2 & +2 \\ +2 & 5+2+10 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 32 \\ 32 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 2 \\ 2 & 17 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 32 \\ 32 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 6 & 2 \\ 2 & 17 \end{vmatrix} = 6\times 17 - 2\times 2 = 98$$

$$\Delta_2 = \begin{vmatrix} 6 & 32 \\ 2 & 32 \end{vmatrix} = 6\times 32 - 2\times 32 = 128$$

$$I'''_L = I_2 = \frac{\Delta_2}{\Delta} = \frac{128}{98} = 1.3061\,A$$

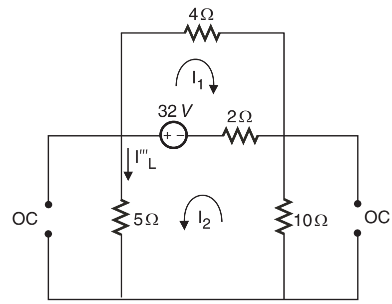


Fig. 6.

To find the total response I_L when all the sources are acting

By the superposition theorem,

$$I_L = I'_L + I''_L + I'''_L = 6.2449 + 2.449 + 1.3061 = 10\,A$$

EXAMPLE 2.42

In Fig. 1, find the component of V_x caused by each source acting alone. What is the value of V_x when all the sources are acting together?

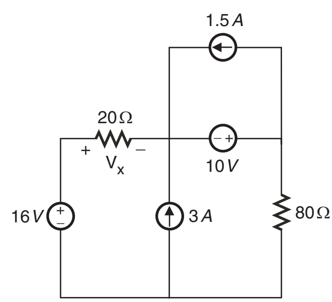


Fig. 1.

SOLUTION

Let, V'_x = Voltage across the 20Ω resistance when the16 *V* source is acting alone.

V''_x = Voltage across the 20Ω resistance when the 3*A* source is acting alone.

V'''_x = Voltage across the 20Ω resistance when the10 *V* source is acting alone.

V''''_x = Voltage across the 20Ω resistance when the 1.5*A* source is acting alone.

The polarity of voltages V'_x, V''_x, V'''_x and V''''_x are chosen the same as that of V_x. Now, by the superposition theorem,

$$V_x = V'_x + V''_x + V'''_x + V''''_x$$

To find the response V'_x due to the 16 *V* source

The 10 *V* source is replaced with a short circuit and the current sources are replaced with an open circuit as shown in Fig. 2.

With reference to Fig. 3, by voltage division rule,

$$V'_x = 16\times\frac{20}{20+80} = 3.2\,V$$

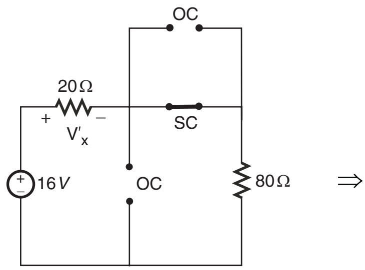


Fig. 2.

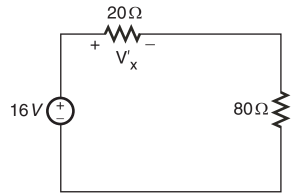


Fig. 3.

To find the response V''_x due to the 3 A source

The voltage sources are replaced with a short circuit and the 1.5 A source is replaced with an open circuit as shown in Fig. 4.

With reference to Fig. 5, by current division rule,

$$I_2 = 3 \times \frac{80}{20+80} = 2.4\,A$$

By Ohm's law,

$$V''_x = -20 \times I_2 = -20 \times 2.4 = -48\,V$$

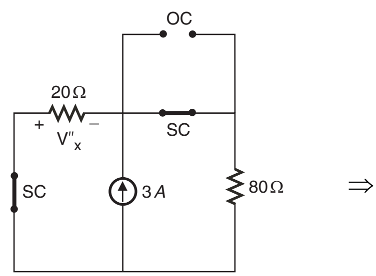


Fig. 4.

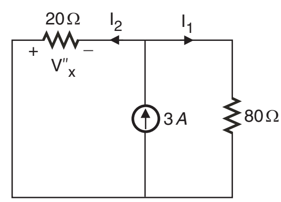


Fig. 5.

To find the response V'''_x due to the 10 V source

The 16 V source is replaced with a the short circuit and the current sources are replaced with an open circuit as shown in Fig. 6.

With reference to Fig. 7, by voltage division rule,

$$V'''_x = 10 \times \frac{20}{20+80} = 2\,V$$

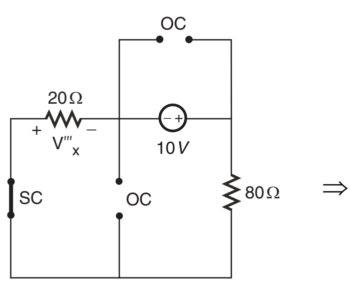


Fig. 6.

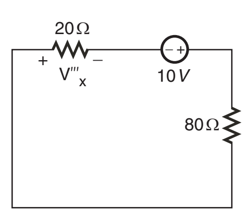


Fig. 7.

To find the response V''''_x due to the 1.5 *A* source

The voltage sources are replaced with a short circuit and the 3 *A* source is replaced with an open circuit as shown in Fig. 8.

With reference to Fig. 9, we can say that the current source is shorted and so no current will flow through the 20 Ω resistance.

$\therefore \quad V''''_x = 0$

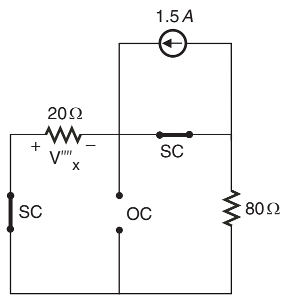


Fig. 8.

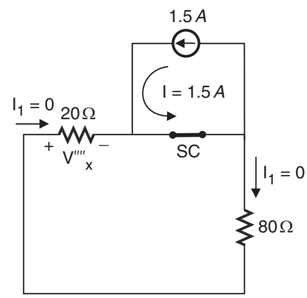


Fig. 9.

To find the response V_x due to all the sources

By the superposition theorem,

$$V_x = V'_x + V''_x + V'''_x + V''''_x = 3.2 + (-48) + 2 + 0 = -42.8\,V$$

RESULT

Component of V_x when the 16 *V* source alone is acting, $V'_x = 3.2\,V$

Component of V_x when the 3 *A* source alone is acting, $V''_x = -48\,V$

Component of V_x when the 10 *V* source alone is acting, $V'''_x = 2\,V$

Component of V_x when the 1.5 *A* source alone is acting, $V''''_x = 0\,V$

The value of V_x when all the sources are acting, $V_x = -42.8\,V$

EXAMPLE 2.43

Using the superposition theorem, find the current through 2 + j2 Ω impedance of the circuit shown in Fig. 1.

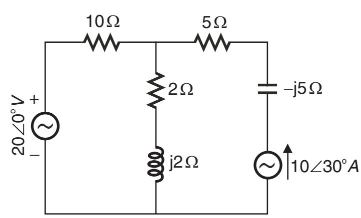


Fig. 1.

SOLUTION

Let $\bar{I}_L$ be the current through 2 + j2 Ω impedance branch as shown in Fig. 2.

Let, $\bar{I}'_L$ = Component of $\bar{I}_L$ due to $20\angle 0^\circ\,V$ source acting alone.

$\bar{I}''_L$ = Component of $\bar{I}_L$ due to $10\angle 30^\circ\,A$ source acting alone.

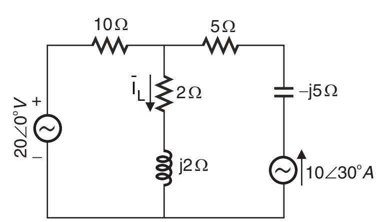


Fig. 2.

Now, by the superposition theorem,

$$\bar{I}_L = \bar{I}'_L + \bar{I}''_L$$

To find the response $\bar{I}'_L$ due to the $20\angle 0^\circ$ V source

The $10\angle 30^\circ\,A$ current source is replaced with an open circuit as shown in Fig. 3.

With reference to Fig. 4, by Ohm's law, we get,

$$\bar{I}'_L = \frac{20\angle 0^\circ}{10 + 2 + j2} = \frac{20}{12 + j2} = 1.6216 - j0.2703\,A$$

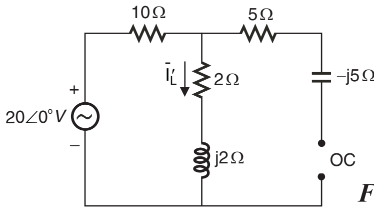


Fig. 3.

⇒

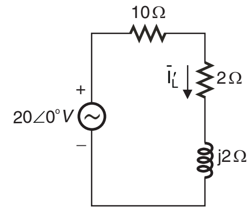


Fig. 4.

To find the response $\bar{I}''_L$ due to the $10\angle30^\circ A$ source

The $20\angle0^\circ V$ voltage source is replaced with a short circuit as shown in Fig. 5.

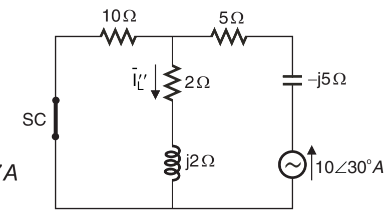


Fig. 5.

With reference to Fig. 5, by current division rule, we get,

$$\bar{I}''_L = 10\angle30^\circ \times \frac{10}{10+2+j2} = \frac{100\angle30^\circ}{12+j2} = 7.6975 + j2.8837 A$$

To find the response $\bar{I}_L$ due to both the sources

By the superposition theorem,

$$\bar{I}_L = \bar{I}'_L + \bar{I}''_L = 1.6216 - j0.2703 + 7.6975 + j2.8837$$
$$= 9.3191 + j2.6134\ A = 9.6786\angle15.7^\circ A$$

Cross-Check

The $20\angle0^\circ V$ voltage in series with the $10\,\Omega$ is converted into a current source as shown in Fig. 6.

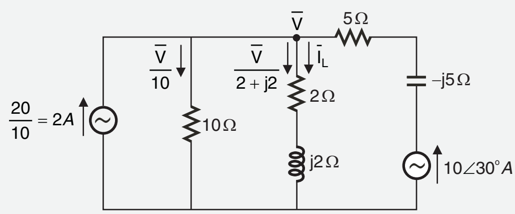


Fig. 6.

With reference to Fig. 6, by KCL at node A, we get,

$$\frac{\bar{V}}{10} + \frac{\bar{V}}{2+j2} = 2 + 10\angle30^\circ$$

$$\therefore\ \bar{V}\left(\frac{1}{10} + \frac{1}{2 + j2}\right) = 2 + 10\angle30^\circ \Rightarrow \left(0.35 - j0.25\right)\bar{V} = 2 + 10\angle30^\circ$$

$$\therefore\ \bar{V} = \frac{2+10\angle30^\circ}{0.35 - j0.25} = 13.4113 + j23.8652$$

Now, by Ohm's law,

$$\bar{I}_L = \frac{\bar{V}}{2+j2} = \frac{13.4113 + j23.8652}{2+j2} = 9.3191 + j2.6135\ A$$
$$= 9.6786\angle15.7^\circ A$$

EXAMPLE 2.44

Using the superposition theorem, find the voltage $\bar{V}_1$ across the capacitance in the circuit shown in Fig. 1.

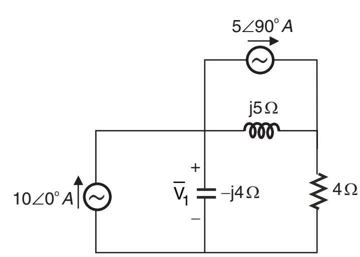


Fig. 1.

SOLUTION

Let, $\bar{V}'_1$ = Voltage across capacitance when the $10\angle0^\circ A$ source alone is acting.

$\bar{V}''_1$ = Voltage across capacitance when the $5\angle90^\circ A$ source alone is acting.

Now, by the superposition theorem,

$$\bar{V}_1 = \bar{V}'_1 + \bar{V}''_1$$

To find the response $\overline{V}'_1$ due to the $10\angle 0^{\circ} A$ source

The $5\angle 90^{\circ} A$ source is replaced with an open circuit as shown in Fig. 2. The circuit of Fig. 2 is redrawn as shown in Fig. 3.

With reference to Fig. 3, by current division rule,

$$\bar{I}_1 = 10 \times \frac{4+j5}{-j4+4+j5} = 12.3529 + j9.4118\ A$$

Now, by Ohm's law, we get,

$$\overline{V}'_1 = -j4 \times \bar{I}_1 = -j4 \times \left(12.3529 + j9.4118\right) = 37.6472 - j49.4116\ V$$

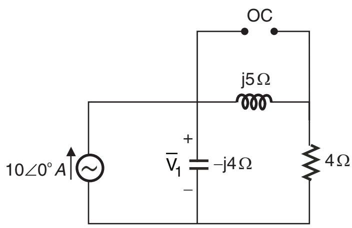


Fig. 2.

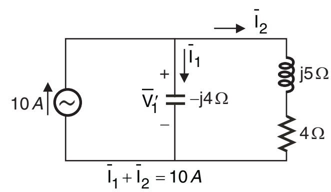


Fig. 3.

To find the response $\overline{V}''_1$ due to the $5\angle 90^{\circ} A$ source

The $10\angle 0^{\circ} A$ source is replaced with an open circuit as shown in Fig. 4. The circuit of Fig. 4 is redrawn as shown in Fig. 5.

With reference to Fig. 5, by current division rule,

$$\bar{I}_4 = j5 \times \frac{j5}{j5+(4-j4)} = -5.8824 + j1.4706\ A$$

Now, by Ohm's law, we get,

$$\begin{aligned} \overline{V}''_1 &= -(-j4 \times \bar{I}_4) \\ &= j4 \times \left(-5.8824 + j1.4706\right) = -5.8824 - j23.5296\ V \end{aligned}$$

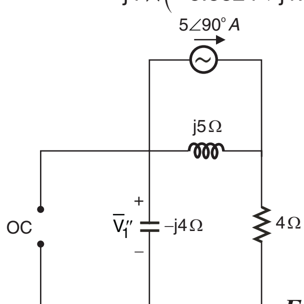


Fig. 4.

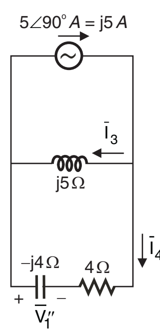


Fig. 5.

To find the total response $\overline{V}_1$ when both the sources are acting

By the superposition theorem,

$$\begin{aligned} \overline{V}_1 &= \overline{V}'_1 + \overline{V}''_1 \\ &= (37.6472 - j49.4116) + (-5.8824 - j23.5296) \\ &= 31.7648 - j72.9412\ V \\ &= 79.5577\angle -66.5^{\circ}\ V \end{aligned}$$

Cross-Check by Node Analysis

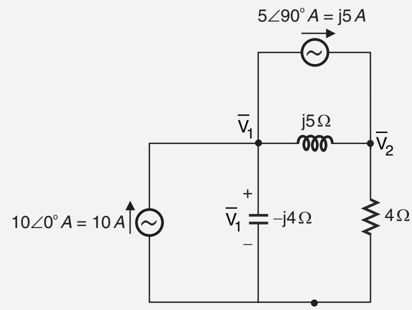


Fig. 6.

With reference to Fig. 6, the node basis matrix equation is,

$$\begin{bmatrix} \frac{1}{j5}+\frac{1}{-j4} & -\frac{1}{j5} \\ -\frac{1}{j5} & \frac{1}{4}+\frac{1}{j5} \end{bmatrix}\begin{bmatrix} \overline{V}_1 \\ \overline{V}_2 \end{bmatrix} = \begin{bmatrix} 10-j5 \\ j5 \end{bmatrix}$$

$$\begin{bmatrix} j0.05 & j0.2 \\ j0.2 & 0.25-j0.2 \end{bmatrix}\begin{bmatrix} \overline{V}_1 \\ \overline{V}_2 \end{bmatrix} = \begin{bmatrix} 10-j5 \\ j5 \end{bmatrix}$$

$$\Delta' = \begin{vmatrix} j0.05 & j0.2 \\ j0.2 & 0.25-j0.2 \end{vmatrix} = j0.05\times(0.25-j0.2)-(j0.2)^2 = 0.05+j0.0125$$

$$\Delta'_1 = \begin{vmatrix} 10-j5 & j0.2 \\ j5 & 0.25-j0.2 \end{vmatrix} = (10-j5)\times(0.25-j0.2)-(j5)\times(j0.2) = 2.5-j3.25$$

$$\overline{V}_1 = \frac{\Delta'_1}{\Delta'} = \frac{2.5-j3.25}{0.05+j0.0125} = 31.7647-j72.9412\,V = 79.5576\angle-66.5^\circ\,V$$

2.6.3 Maximum Power Transfer Theorem

All practical sources have internal resistance or impedance. When a source delivers current to a load, the current flows through the internal impedance also and so a part of power is consumed by the internal impedance of the source. Hence, when a load is connected to a source, the power generated by the source is shared between the internal impedance and the load.

In certain applications, it is desirable to have a maximum power transfer from the source to the load. The maximum power transfer to the load is possible only if the source and the load has matched impedance. This situation arises in electronics, communication and control circuits.

For example, an antenna used in a TV/radio receives a signal from the atmosphere and the power level of the signal is very low. This weak signal should be transferred to the input section of an amplifier to which it is connected. For good reception, the maximum power should be transferred from the antenna to the amplifier. This is possible only if the input impedance of the amplifier is matched with the antenna impedance.

The sources may be dc or ac and the loads may be resistive or reactive. Hence, the matched impedance for maximum power will be different for different combinations of source and load. The following important six combinations of source and load are discussed in this book.

Case i : *DC source with internal resistance connected to a resistive load.*

Case ii : *AC source with internal resistance connected to a resistive load.*

Case iii : *AC source with internal impedance connected to a resistive load.*

Case iv : *AC source with internal impedance connected to a load with variable resistance and variable reactance.*

Case v : *AC source with internal impedance connected to a load with variable resistance and fixed reactance.*

Case vi : *AC source with internal impedance connected to a load with fixed resistance and variable reactance.*

Case i : DC source with internal resistance connected to a resistive load

Theorem: ***(AU Dec'14, 2 Marks)***

"Maximum power is transferred from source to load, when load resistance is equal to source resistance".

Consider a dc source of emf E and internal resistance R_s connected to a variable load resistance R. Now, the condition for maximum power transfer from source to load is,

$$R = R_s$$

The maximum power P_{max} is,

$$P_{max} = \frac{E^2}{4R}$$

Proof:

Consider a dc source of emf E and internal resistance R_s connected to a load resistance R as shown in Fig. 2.54. Let, I be the current through the circuit. With reference to Fig. 2.54, by Ohm's law, we can write,

$$I = \frac{E}{R_s + R} \qquad \text{..... (2.61)}$$

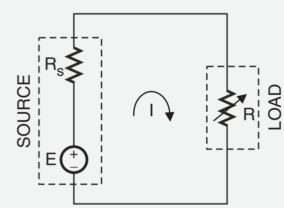


Fig. 2.54.

Let, P = Power delivered to load

Now, $P = I^2R$ ..... (2.62)

Using equation (2.61) in equation (2.62), we get,

$$P = \left(\frac{E}{R_s + R}\right)^2 R = \frac{E^2 R}{(R_s + R)^2} \qquad \text{..... (2.63)}$$

The condition for maximum power can be obtained by differentiating P with respect to R and equating (dP/dR) = 0

On differentiating equation (2.63) with respect to R, we get,

$$\boxed{\frac{du}{dv} = \frac{du \times v - u \times dv}{v^2}}$$

$$\frac{dP}{dR} = \frac{E^2 \times (R_s + R)^2 - E^2 R \times 2(R_s + R)}{(R_s + R)^4} \qquad \text{..... (2.64)}$$

For (dP/dR) = 0, the numerator of equation (2.64), should be zero.

$$\therefore \quad E^2(R_s + R)^2 - 2E^2R(R_s + R) = 0 \;\Rightarrow\; 2\,E^2\,R(R_s + R) = E^2\,(R_s + R)^2 \qquad \text{.... (2.65)}$$

On dividing equation (2.65) throughout by $E^2(R_s + R)$, we get,

$$2R = R_s + R \;\Rightarrow\; 2R - R = R_s \;\Rightarrow\; R = R_s \qquad \text{..... (2.66)}$$

Equation (2.66) is the condition for maximum power transfer to load, which states that the maximum power is transferred from source to load when load resistance is equal to source resistance.

On substituting R for R_s in equation (2.63), we can get an expression for maximum power.

$$\boxed{\therefore \quad \text{Maximum power, } P_{max} = P|_{R_s = R} = \frac{E^2 R}{(R + R)^2} = \frac{E^2 R}{(2R)^2} = \frac{E^2 R}{4R^2} = \frac{E^2}{4R}} \qquad \text{..... (2.67)}$$

Case ii : AC source with internal resistance connected to a resistive load

Theorem:

"Maximum power is transferred from source to load, when load resistance is equal to source resistance."

Consider an ac source of emf $\overline{E}$ and internal resistance R_s connected to a variable load resistance R.

Now, the condition for maximum power transfer from source to load is,

$$R = R_s$$

Proof:

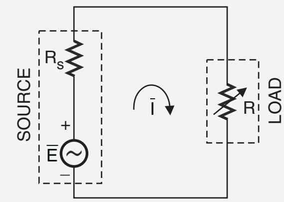


Fig. 2.55.

Consider an ac source of emf $\bar{E}$ and internal resistance R_s connected to a load resistance R as shown in Fig. 2.55. Let, $\bar{I}$ be the current through the circuit. With reference to Fig. 2.55, by Ohm's law, we can write,

$$\bar{I} = \frac{\bar{E}}{R_s + R}$$

$$\therefore \quad I = |\bar{I}| = \left|\frac{\bar{E}}{R_s + R}\right| = \frac{E}{R_s + R}$$

Let, P = Power delivered to load

$$\text{Now, } P = I^2 R = \frac{E^2 R}{(R_s + R)^2} \qquad \text{..... (2.68)}$$

Equation (2.68) is the same as equation (2.63) and so the condition for maximum power transfer will be the same as that of case (i). But here, I is rms value of current and E is rms value of source emf.

Case iii : AC source with internal impedance connected to a resistive load

Theorem:

"Maximum power is transferred from source to load, when load resistance is equal to magnitude of source impedance."

Consider an ac source of emf $\bar{E}$ and internal impedance $\bar{Z}_s$ ($\bar{Z}_s = R_s + jX_s$) connected to a variable load resistance R.

Now, the condition for maximum power transfer from source to load is,

$$R = |\bar{Z}_s| = \sqrt{R_s^2 + X_s^2}$$

Proof:

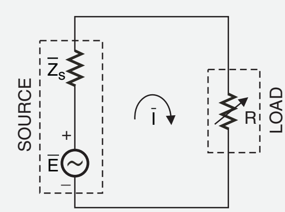


Fig. 2.56.

Consider an ac source of emf $\bar{E}$ and internal impedance $\bar{Z}_s$ connected to a load resistance R as shown in Fig. 2.56.

$$\text{Let, } \bar{Z}_s = R_s + jX_s$$

$$\therefore \quad \text{Magnitude of source impedance, } Z_s = |\bar{Z}_s| = \sqrt{R_s^2 + X_s^2} \qquad \text{.....(2.69)}$$

With reference to Fig. 2.56, by Ohm's law, we can write,

$$\bar{I} = \frac{\bar{E}}{\bar{Z}_s + R} = \frac{\bar{E}}{R_s + jX_s + R} = \frac{\bar{E}}{(R_s + R) + jX_s} \qquad \text{..... (2.70)}$$

$$\text{Magnitude of current, } I = |\bar{I}| = \frac{|\bar{E}|}{|(R_s + R) + jX_s|} = \frac{E}{\sqrt{(R_s + R)^2 + X_s^2}} \qquad \text{..... (2.71)}$$

Let, P = Power delivered to load

$$\text{Now, } P = |\bar{I}|^2 R = I^2 R \qquad \text{..... (2.72)}$$

Using equation (2.71) in equation (2.72), we can write,

$$P = \frac{E^2 R}{(R_s + R)^2 + X_s^2} \qquad \text{..... (2.73)}$$

The condition for maximum power can be obtained by differentiating P with respect to R and equating (dP/dR) = 0.

On differentiating equation (2.73) with respect to R, we get,

$$\boxed{\frac{du}{dv} = \frac{du \times v - u \times dv}{v^2}}$$

$$\frac{dP}{dR} = \frac{E^2 \times [(R_s+R)^2 + X_s^2] - E^2 R \times 2(R_s+R)}{[(R_s+R)^2+X_s^2]^2} \quad \text{..... (2.74)}$$

For $\frac{dP}{dR} = 0$, *the numerator of equation (2.74) should be zero.*

$$\therefore \; E^2[(R_s+R)^2+X_s^2] - E^2 R \times 2(R_s+R) \;\Rightarrow\; 2E^2R(R_s+R) = E^2[(R_s+R)^2+X_s^2]$$

On dividing the above equation throughout by E^2 *we get,*

$$2R(R_s+R) = (R_s+R)^2 + X_s^2$$

$$2RR_s + 2R^2 = R_s^2 + R^2 + 2RR_s + X_s^2$$

$$2RR_s + 2R^2 - R^2 - 2RR_s = R_s^2 + X_s^2$$

$$R^2 = R_s^2 + X_s^2$$

$$\boxed{\therefore \; R = \sqrt{R_s^2 + X_s^2}} \quad \text{.....(2.75)}$$

Here, $\sqrt{R_s^2+X_s^2} = Z_s =$ *Magnitude of source impedance*

Equation (2.75) is the condition for maximum power transfer. From equation (2.75) we can say that, maximum power is transferred to load when load resistance is equal to magnitude of source impedance.

Case iv : AC source with internal impedance connected to a load with variable resistance and variable reactance

Theorem:

"Maximum power is transferred from source to load, when load impedance is equal to complex conjugate of source impedance."

Consider an ac source of emf $\overline{E}$ and internal impedance $\overline{Z}_s$ ($\overline{Z}_s = R_s + jX_s$) connected to a load impedance $\overline{Z}$, where $\overline{Z} = R + jX$ with R and X are individually variable.

Now, the condition for maximum power transfer from source to load is,

$$\overline{Z} = \overline{Z}_s^* \;\Rightarrow\; R + jX = (R_s + jX_s)^* \;\Rightarrow\; R + jX = R_s - jX_s$$

Proof :

Consider an ac source of emf $\overline{E}$ *and internal impedance* $\overline{Z}_s$ *connected to a load impedance* $\overline{Z}$ *as shown in Fig. 2.57.*

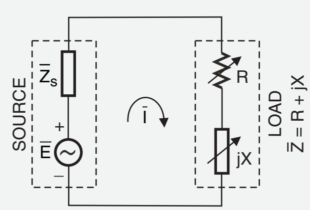


Fig. 2.57.

Let, $\overline{Z}_s = R_s + jX_s \;\Rightarrow\; \overline{Z}_s^* = R_s - jX_s$

$\overline{I}$ = *Current through the circuit*

$$\text{Now, } \overline{I} = \frac{\overline{E}}{\overline{Z}_s + \overline{Z}} = \frac{\overline{E}}{R_s + jX_s + R + jX} = \frac{\overline{E}}{(R_s+R) + j(X_s+X)}$$

$$\text{Magnitude of current, } I = |\overline{I}| = \frac{|\overline{E}|}{|(R_s+R)+j(X_s+X)|}$$

$$= \frac{E}{\sqrt{(R_s+R)^2 + (X_s+X)^2}} \quad \text{.....(2.76)}$$

Let, P = Power delivered to load

$$\text{Now, } P = |\bar{I}|^2 R = I^2 R \qquad .....(2.77)$$

Note : *In reactive loads, power is consumed only by resistance and active power in the reactance is zero.*

Using equation (2.76) in equation (2.77), we can write,

$$P = \frac{E^2 R}{(R_s + R)^2 + (X_s + X)^2} \qquad .....(2.78)$$

The condition for maximum power can be obtained by partially differentiating P with respect to X and then with respect to R and equating $(\partial P/\partial X) = 0$ and $(\partial P/\partial R) = 0$.

On partially differentiating equation (2.78) with respect to X, we get,

$$\boxed{\frac{du}{dv} = \frac{du \times v - u \times dv}{v^2}}$$

$$\frac{\partial P}{\partial X} = \frac{0 \times [(R_s + R)^2 + (X_s + X)^2] - E^2 R \times 2(X_s + X)}{[(R_s + R)^2 + (X_s + X)^2]^2}$$

$$= \frac{-2E^2 R (X_s + X)}{[(R_s + R)^2 + (X_s + X)^2]^2} \qquad .....(2.79)$$

For $\frac{\partial P}{\partial X} = 0$*, the numerator of equation (2.79) should be zero.*

$$\therefore \quad -2E^2 R (X_s + X) = 0 \qquad ..... (2.80)$$

In equation (2.80), $E \neq 0$ and $R \neq 0$, hence,

$$X_s + X = 0 \qquad ..... (2.81)$$

$$\boxed{\therefore \quad X = -X_s} \qquad ..... (2.82)$$

On partially differentiating equation (2.78) with respect to R, we get,

$$\frac{\partial P}{\partial R} = \frac{E^2 \times [(R_s + R)^2 + (X_s + X)^2] - E^2 R \times 2(R_s + R)}{[(R_s + R)^2 + (X_s + X)^2]^2} \qquad ..... (2.83)$$

For $\frac{\partial P}{\partial R} = 0$*, the numerator of equation (2.83) should be zero.*

$$\therefore E^2 [(R_s + R)^2 + (X_s + X)^2] - 2E^2 R (R_s + R) = 0$$

$$2E^2 R (R_s + R) = E^2 [(R_s + R)^2 + (X_s + X)^2]$$

On dividing throughout by E^2, we get,

$$2R (R_s + R) = (R_s + R)^2 + (X_s + X)^2 \qquad ..... (2.84)$$

From equation (2.81) we know that $X_s + X = 0$, hence equation (2.84) can be written as,

$$2R (R_s + R) = (R_s + R)^2 \Rightarrow 2R = R_s + R \Rightarrow 2R - R = R_s \Rightarrow \boxed{R = R_s} \qquad .....(2.85)$$

Equations (2.82) and (2.85) are the conditions for maximum power transfer. From these two equations, for maximum power transfer, we can say that, $R + jX = R_s - jX_s$.

Here, $R_s - jX_s = \bar{Z}_s^* =$ *Conjugate of source impedance*

Hence, for maximum power transfer, load impedance should be equal to conjugate of source impedance. An interesting observation is that when maximum power transfer condition is met, the circuit will behave as a purely resistive circuit and so the circuit will be in resonance.

Case v : AC source with internal impedance connected to a load with variable resistance and fixed reactance

Theorem:

> "Maximum power is transferred from source to load, when load resistance is equal to absolute value of the rest of the impedence of the circuit".

Consider an ac source of emf $\overline{E}$ and internal impedance $\overline{Z}_s$ ($\overline{Z}_s = R_s + jX_s$) connected to a load impedance $\overline{Z}$, where $\overline{Z} = R + jX$ with variable R and fixed X.

Now, the condition for maximum power transfer from source to load is,

$$R = \sqrt{R_s^2 + (X_s + X)^2}$$

Proof :

The statement of case (v) can be proved by proceeding similar to that of case (iv) and differentiating equation (2.79) with respect to R and equating (dP/dR) = 0.

From equation (2.84) we get,

$$2R(R_s + R) = (R_s + R)^2 + (X_s + X)^2$$

$$2RR_s + 2R^2 = R_s^2 + R^2 + 2RR_s + (X_s + X)^2$$

$$2RR_s + 2R^2 - R^2 - 2RR_s = R_s^2 + (X_s + X)^2 \Rightarrow R^2 = R_s^2 + (X_s + X)^2$$

$$\boxed{\therefore \; R = \sqrt{R_s^2 + (X_s + X)^2}} \qquad .....(2.86)$$

Equation (2.86) is the condition for maximum power transfer in case (v).

Case vi : AC source with internal impedance connected to a load with fixed resistance and variable reactance

Theorem:

> "Maximum power is transferred from source to load, when load reactance is equal to conjugate of source reactance".

Consider an ac source of emf $\overline{E}$ and internal impedance $\overline{Z}_s$ ($\overline{Z}_s = R_s + jX_s$) connected to a load impedance $\overline{Z}$, where $\overline{Z} = R + jX$ with fixed R and variable X.

Now, the condition for maximum power transfer is,

$$jX = -jX_s$$

Proof :

The statement of case (vi) can be proved by proceeding similar to that of case (iv) and differentiating equation (2.79) with respect to X and equating (dP/dX) = 0.

From equation (7.24) we get,

$$X = -X_s \qquad ..... (2.87)$$

The equation (2.87) is the condition for maximum power transfer in case (vi).

Table 2.3 : Summary of Conditions for Maximum Power Transfer *(AU June'16, 2 Marks)*

Case	Source emf	Source impedance	Load impedance	Variable element of load impedance	Condition for maximum power transfer
i	dc	R_s	R	R	$R = R_s$
ii	ac	R_s	R	R	$R = R_s$
iii	ac	$R_s + jX_s$	R	R	$R = \sqrt{R_s^2 + X_s^2}$
iv	ac	$R_s + jX_s$	$R + jX$	R, X	$R + jX = R_s - jX_s$
v	ac	$R_s + jX_s$	$R + jX$	R	$R = \sqrt{R_s^2 + (X_s + X)^2}$
vi	ac	$R_s + jX_s$	$R + jX$	X	$jX = -jX_s$

Applying maximum power transfer theorem to circuits

Generally it is desirable to have maximum power transfer to a particular element of a circuit. In this case remove the concerned element and create two open terminals. Then the circuit is represented by Thevenin's equivalent with respect to open terminals. Now, consider Thevenin's equivalent as the voltage source for load and apply the maximum power transfer theorem.

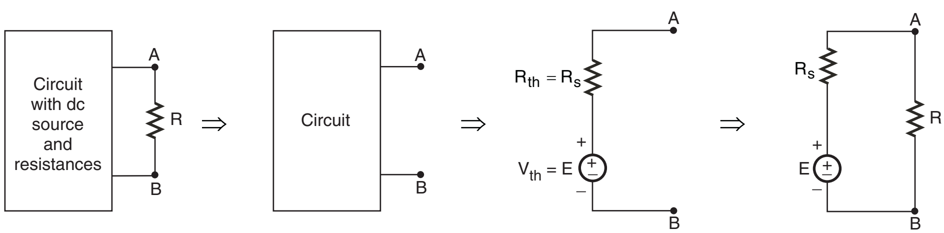


***Fig. a :** Circuit with dc source and resistances.*

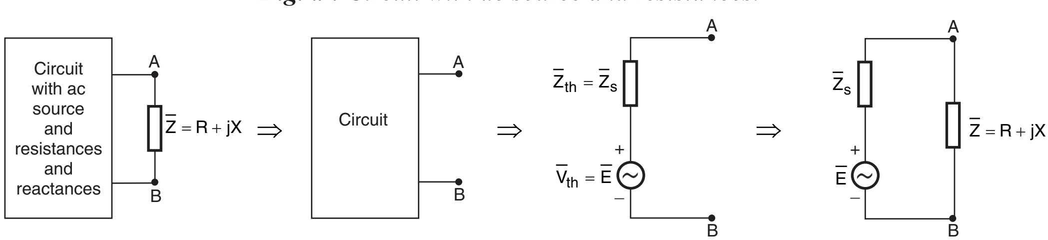


***Fig. b :** Circuit with ac source and resistances and reactances.*

***Fig. 2.58 :** Applying maximum power transfer theorem to an element of a circuit.*

Sometimes, it is desirable to have maximum power transfer to load by varying some parameter of a circuit. In this case, determine an expression for power, P delivered to load by relating the variable parameter to P. Let, Y be the variable parameter. Now differentiate P with respect to Y to get $\frac{dP}{dY}$. Then form an equation by equating $\frac{dP}{dY} = 0$ and solve the equation to get the condition for maximum power transfer. [Refer to Examples 2.58 and 2.59.]

EXAMPLE 2.45

In the circuit of Fig. 1, find the value of the adjustable resistor R for maximum power transfer to R. Also, calculate the maximum power.

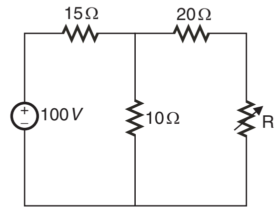


Fig. 1.

SOLUTION

Let us remove the adjustable resistance R and denote the two open terminals by A and B, as shown in Fig. 2. Now, the circuit of Fig. 2 should be replaced by Thevenin's equivalent. Let us assume the polarity of V_{th} as shown in Fig. 2.

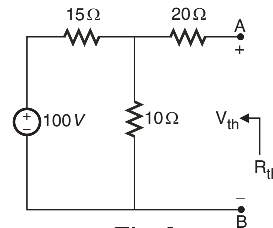


Fig. 2.

To find Thevenin's voltage V_{th}

In Fig. 3, the 20 Ω resistance is open and so no current will flow through it. Hence, the voltage across the 20 Ω resistance is zero.

With reference to Fig. 3, by voltage division rule, we can write,

$$V_{th} = 100 \times \frac{10}{15+10} = \frac{1000}{25} = 40\,V$$

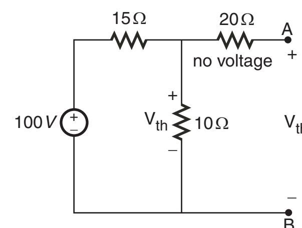


Fig. 3.

To find Thevenin's resistance R_{th}

The 100 V voltage source in the circuit of Fig. 2 is replaced with a short circuit and the resulting network is reduced to a single equivalent resistance as shown below:

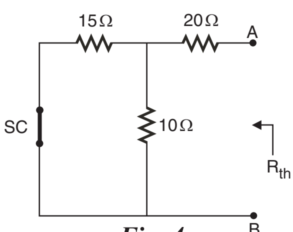


Fig. 4.

⇒

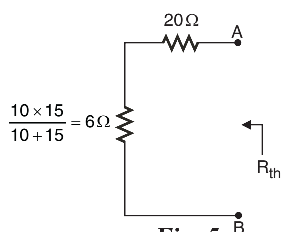


Fig. 5.

With reference to Fig. 5,

$$R_{th} = 20 + 6 = 26\ \Omega$$

To find R for maximum power and P_{max}

Thevenin's equivalent of Fig. 2 is shown in Fig. 6. Now, Thevenin's equivalent is the voltage generator for load resistance R.

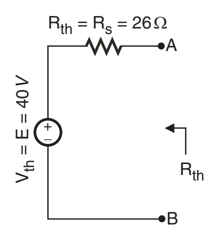


Fig. 6.

$$\therefore \quad V_{th} = E \quad ; \quad R_{th} = R_s$$

Let us connect the adjustable resistance R across A and B of Thevenin's equivalent as shown in Fig. 7.

With reference to Fig. 7, by maximum power transfer theorem, for maximum power in R the value of R should be equal to R_s.

$$\therefore \quad R = 26\,\Omega$$

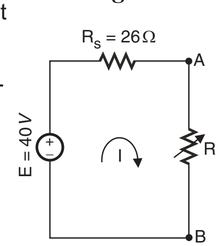


Fig. 7.

We know that, maximum power, $P_{max} = \frac{E^2}{4R}$

$$= \frac{40^2}{4 \times 26} = 15.3846\,W$$

$$\boxed{\text{Also, } P_{max} = I^2R = \left(\frac{40}{26+26}\right)^2 \times 26 = 15.3846\,W}$$

RESULT

The value of R for maximum power transfer = 26 Ω

The maximum power in R, P_{max} = 15.3846 *W*

EXAMPLE 2.46 *(AU Dec'16, 8 Marks)*

In the circuit of Fig. 1, find the value of R for maximum power transfer. Also, calculate the maximum power.

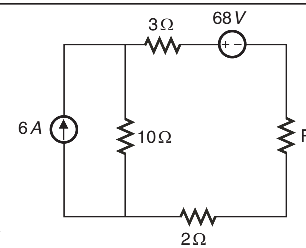


Fig. 1.

SOLUTION

Let us remove the resistance R and denote the two open terminals by A and B as shown in Fig. 2.

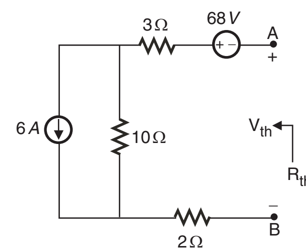


Fig. 2.

To find Thevenin's voltage V_{th}

The circuit of Fig. 2 should be replaced by Thevenin's equivalent. Let us assume the polarity of V_{th} as shown in Fig. 2. The current source in parallel with the 10 Ω is converted to a voltage source in series as shown in Fig. 3.

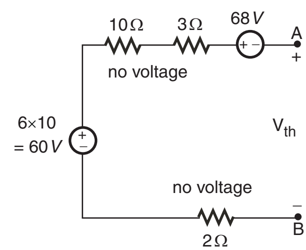


Fig. 3.

With reference to Fig. 3, by KVL, we can write,

$$V_{th} + 68 + 60I = 0$$

$$\therefore \; V_{th} = -128\ V$$

To find Thevenin's resistance R_{th}

Let us replace the voltage source with a short circuit and the current source with an open circuit as shown in Fig. 4.

With reference to Fig. 4,

$$R_{th} = 3 + 10 + 2 = 15\,\Omega$$

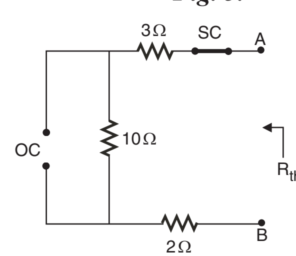


Fig. 4.

To find R for maximum power and P_{max}

Thevenin's equivalent of Fig. 2 is shown in Fig. 5. Now, the Thevenin's equivalent is the voltage generator for load resistance R.

$$\therefore \; V_{th} = E \qquad ; \qquad R_{th} = R_s$$

Let us connect the resistance R across A and B of Thevenin's equivalent as shown in Fig. 6.

With reference to Fig. 6, by maximum power transfer theorem, for maximum power transfer to R the value of R should be equal to R_s.

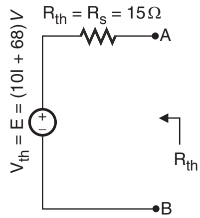


Fig. 5.

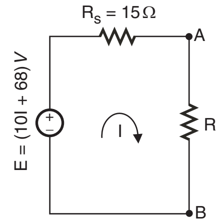


Fig. 6.

$$\therefore \quad R = 15\,\Omega$$

$$\text{Maximum power, } P_{max} = \frac{E^2}{4R} = \frac{(-128)^2}{4 \times 15} = 273.0667\,W$$

EXAMPLE 2.47

In the circuit of Fig. 1, find the value of R for maximum power transfer. Also, calculate the maximum power.

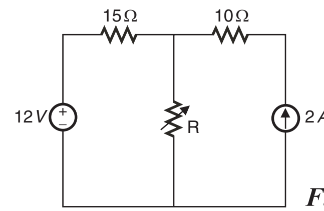


Fig. 1.

SOLUTION

Let us remove the resistance R and denote the two open terminals by A and B, as shown in Fig. 2. Now, the circuit of Fig. 2, should be replaced by Thevenin's equivalent. Let us assume the polarity of V_{th} as shown in Fig. 2.

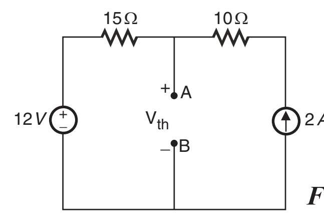


Fig. 2.

To find Thevenin's voltage V_{th}

With reference to Fig. 3, by KVL, we can write,

$$V_{th} = 2 \times 15 + 12$$

$$\therefore \quad V_{th} = 42\,V$$

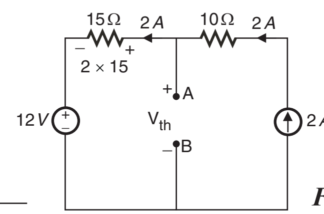


Fig. 3.

To find Thevenin's resistance R_{th}

Let us replace the voltage source with a short circuit and the current source with an open circuit as shown in Fig. 4.

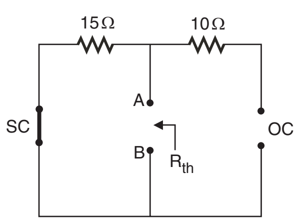


Fig. 4.

$\Rightarrow$

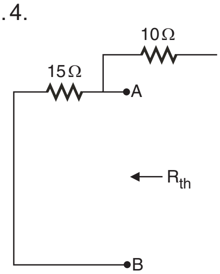


Fig. 5.

With reference to Fig. 5,

$$R_{th} = 15\,\Omega$$

To find R for maximum power and P_{max}

Thevenin's equivalent of Fig. 2 is shown in Fig. 6. Now, the Thevenin's equivalent is the voltage generator for load resistance R.

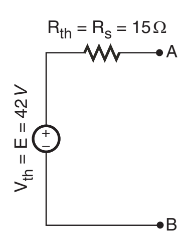


Fig. 6.

$$\therefore \quad V_{th} = E \quad ; \quad R_{th} = R_s$$

Let us connect the resistance R across A and B of Thevenin's equivalent as shown in Fig. 7.

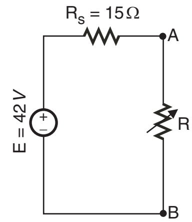


Fig. 7.

With reference to Fig. 7, by maximum power transfer theorem, for maximum power transfer to R the value of R should be equal to R_s.

$$\therefore \; R = 15\,\Omega$$

$$\text{Maximum power, } P_{max} = \frac{E^2}{4R} = \frac{42^2}{4 \times 15} = 29.4\,W$$

EXAMPLE 2.48 *(AU Dec'15, 16 Marks)*

Determine the value of resistance that may be connected across terminals A and B so that maximum power is transferred from the circuit to the resistance. Also, estimate the maximum power transferred to the resistance.

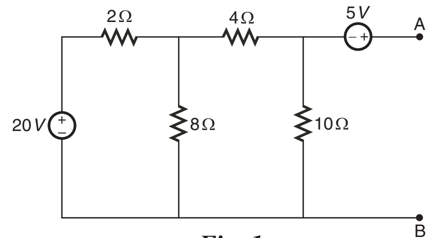


Fig. 1.

SOLUTION

The circuit of Fig. 1 should be replaced by Thevenin's equivalent as shown below:

To find Thevenin's voltage V_{th}

With reference to Fig. 2, the mesh basis matrix equation is,

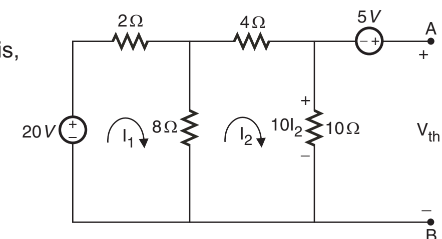


Fig. 2.

$$\begin{bmatrix} 2+8 & -8 \\ -8 & 8+4+10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -8 \\ -8 & 22 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 10 & -8 \\ -8 & 22 \end{vmatrix} = 10 \times 22 - (-8)^2 = 156$$

$$\Delta_2 = \begin{vmatrix} 10 & 20 \\ -8 & 0 \end{vmatrix} = 10 \times 0 - (-8) \times 20 = 160$$

$$\therefore \; I_2 = \frac{\Delta_2}{\Delta} = \frac{160}{156} = 1.0256\,A$$

With reference to Fig. 2 by KVL, we can write,

$$V_{th} = 5 + 10\,I_2 = 5 + 10 \times 1.0256$$

$$\therefore \; V_{th} = 15.256\,V$$

To find Thevenin's resistance R_{th}

Let us replace the voltage sources with a short circuit and reduce the resulting network to a single equivalent resistance as shown below:

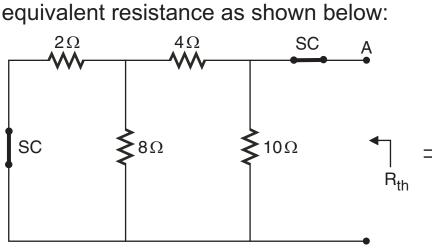


Fig. 3.

$\Rightarrow$

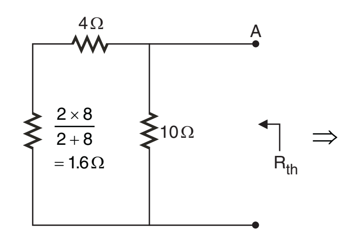


Fig. 4.

$\Rightarrow$

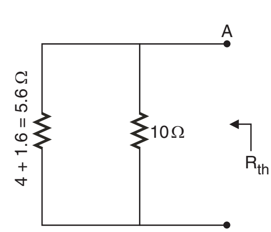


Fig. 5.

With reference to Fig. 5,

$$R_{th} = \frac{5.6 \times 10}{5.6 + 10} = 3.5897\,\Omega$$

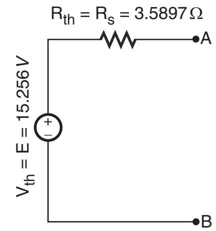


Fig. 6.

To find the value of R for maximum power transfer and P_{max}

Thevenin's equivalent of the given circuit is shown in Fig. 6. Now, Thevenin's equivalent is the voltage generator for the load that may be connected across A and B.

$$\therefore \quad E = V_{th} \quad ; \quad R_s = R_{th}$$

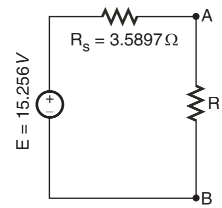


Fig. 7.

Let us connect a load resistance R across A and B of Thevenin's equivalent as shown in Fig. 7. Now, for maximum power transfer, the value of R should be equal to R_s.

$$\therefore \quad R = 3.5897\,\Omega$$

Maximum power, $P_{max} = \dfrac{E^2}{4R}$

$$= \frac{15.256^2}{4 \times 3.5897} = 16.2093\,W$$

EXAMPLE 2.49 ***(AU June'16, 16 Marks)***

In the circuit shown in Fig. 1, determine the maximum power delivered to R_L where $R_L = 100\,\Omega$ using Norton's theorem. Also, determine the value of R_L for maximum power transfer.

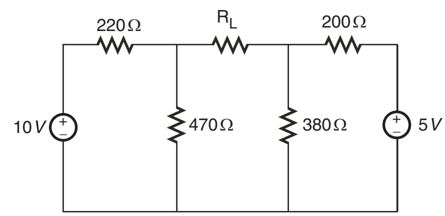


Fig. 1.

SOLUTION

Let us remove the resistance R_L and denote the two open terminals by A and B as shown in Fig. 2. The voltage sources in the circuit of Fig. 2 are converted into current sources as shown in Fig. 3. The parallel resistances in Fig. 3 are converted into a single equivalent resistance as shown in Fig. 4.

The current sources in Fig. 4 are converted into voltage sources in Fig. 5. Now short circuit the terminals A and B and the current through this short circuit is Norton's circuit I_N.

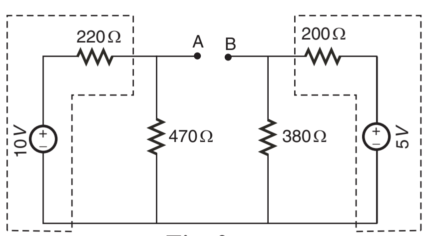


Fig. 2.

$\Rightarrow$

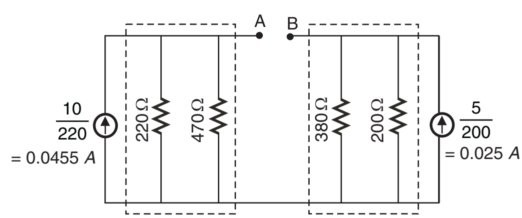


Fig. 3.

$\Downarrow$

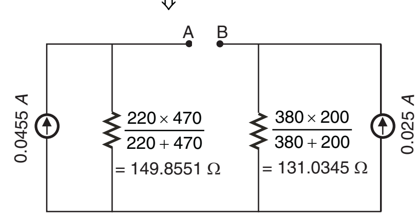


Fig. 4.

$\Leftarrow$

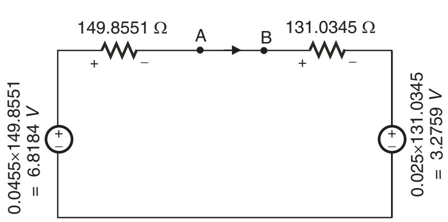


Fig. 5.

With reference to Fig. 5, by KVL, we can write,

$149.8551\ I_N + 131.0345\ I_N + 3.2759 = 6.8184$

$I_N(149.8551 + 131.0345) = 6.8184 - 3.2759$

$$\therefore\ I_N = \frac{6.8184 - 3.2759}{149.855 + 131.0345} = 0.0126\ A$$

To find Norton's resistance R_n

The voltage sources are replaced with a short circuit as shown in Fig. 6. Norton's resistance is determined by using network reduction techniques as shown below:

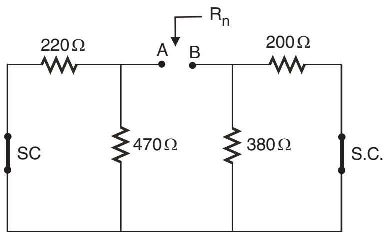


Fig. 6.

⇒

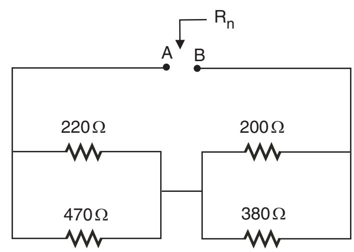


Fig. 7.

⇓

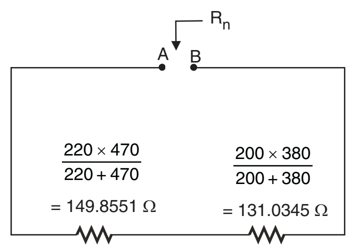


Fig. 8.

⇐

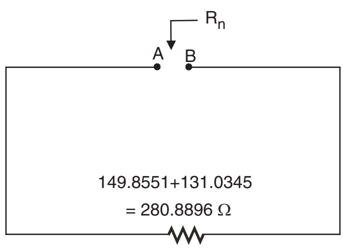


Fig. 9.

With reference to Fig. 9, we get,

Norton's resistance, $R_n = 280.8896\ \Omega$

Norton's equivalent

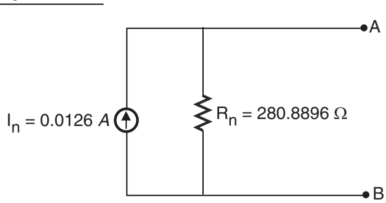


Fig. 10.

To find maximum power transfer

Norton's equivalent of the given circuit is shown in Fig. 11. Let us connect a load resistance R across A and B of Norton's equivalent as shown in Fig. 12.

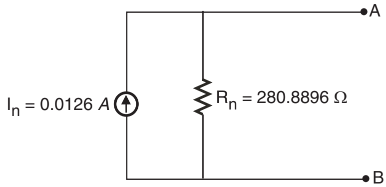


Fig. 11.

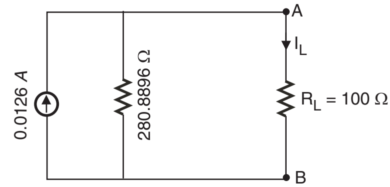


Fig. 12.

By current division rule,

$$I_L = I_N \times \frac{R_N}{R_N \times R_L} = 0.0126 \times \frac{280.8896}{280.8896 + 100} = 9.292 \times 10^{-3}\ A$$

Power, $P_L = I_L^2 R_L = (9.292 \times 10^{-3})^2 \times 100 = 8.6341 \times 10^{-3}\ W = 8.6341\ mW$

Now, for maximum power transfer the value of R_L should be equal to R_n.

$$\therefore \text{ Maximum power, } P_{max} = \left(\frac{I_N}{2}\right)^2 R_n = \left(\frac{0.0126}{2}\right)^2 \times 280.8896$$

$$= 11.1485 \times 10^{-3}\ \text{m} = 11.1485\ mW$$

> ***Note:*** When $R_L = R_n$, the Norton's current will divide equally between R_n and R_L and so current through R_L is $I_N/2$

EXAMPLE 2.50

Determine the value of R for maximum power transfer to it and the maximum power.

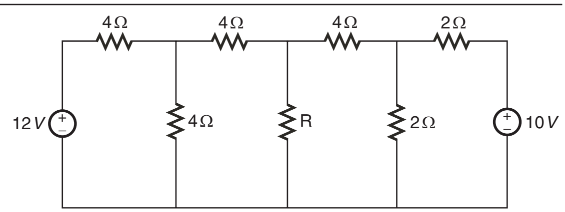


Fig. 1.

SOLUTION

The given circuit can be reduced to a single voltage source with a resistance in series with respect to terminals of load resistance R, by source transformation technique, as shown below:

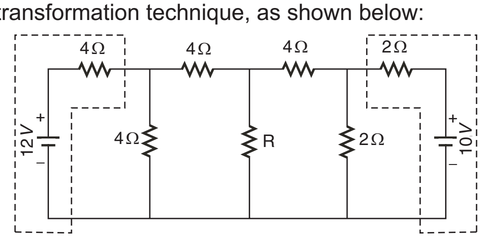


Fig. 2.

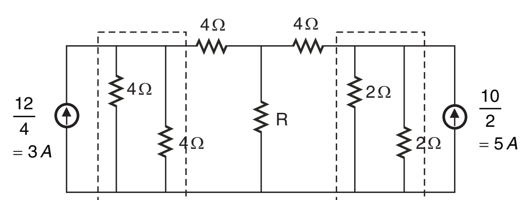


Fig. 3.

⇓

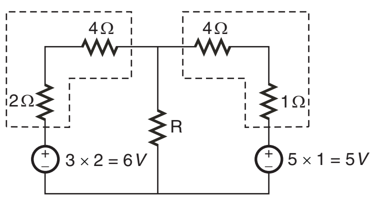


Fig. 5.

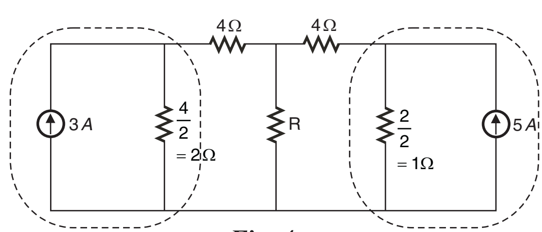


Fig. 4.

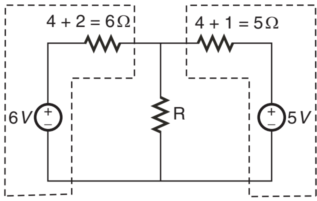


Fig. 6.

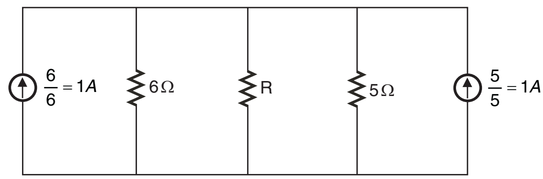


Fig. 7.

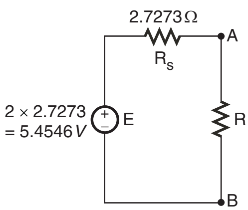


Fig. 10.

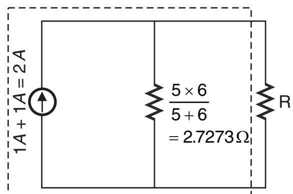


Fig. 9.

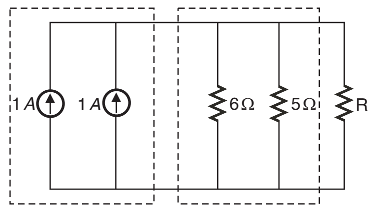


Fig. 8.

In Fig. 10, the given circuit has been reduced to the form of a voltage generator with respect to terminals of R. With reference to Fig. 10,

The value of R for maximum power transfer = R_s = 2.7273 Ω

$$\text{The maximum power, } P_{max} = \frac{E^2}{4 \times R} = \frac{5.4546^2}{4 \times 2.7273} = 2.7273\,W$$

EXAMPLE 2.51

Determine the value of R in the circuit of Fig. 1 for maximum power transfer to R from the rest of the circuit.

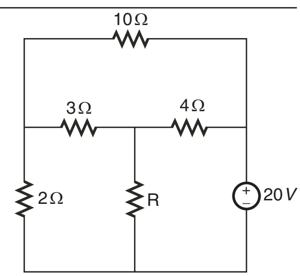


Fig. 1.

SOLUTION

The given circuit should be replaced by Thevenin's equivalent with respect to terminals of R as shown below:

Now, for maximum power transfer to R, the value of R should be equal to R_{th}. Hence, we have to find the value of Thevenin's resistance R_{th}.

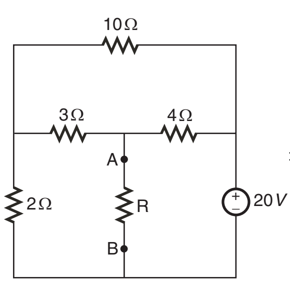


Fig. 2.

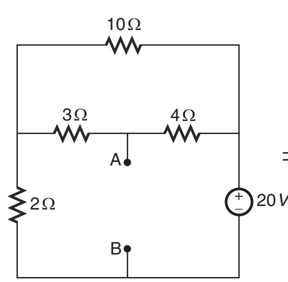


Fig. 3.

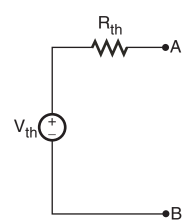


Fig. 4.

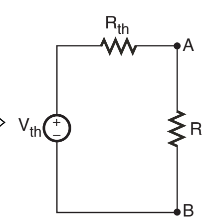


Fig. 5.

To find Thevenin's resistance R_{th}

Let us replace the voltage source with a short circuit and remove the resistance R and denote the resulting open terminals as A and B as shown in Fig. 6. The network of Fig. 6 is reduced to a single equivalent resistance as shown below:

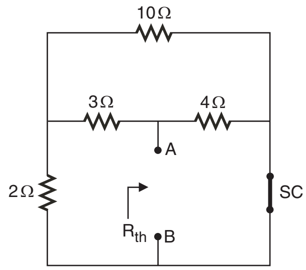


Fig. 6.

⇒

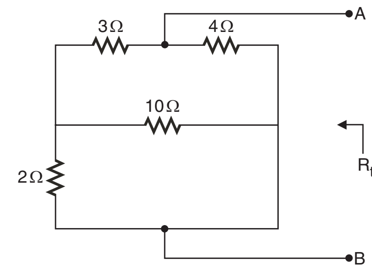


Fig. 7.

⇓

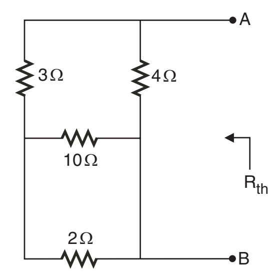


Fig. 8.

⇐

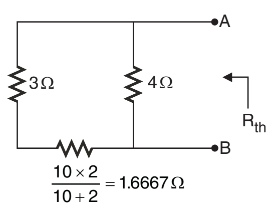


Fig. 9.

⇐

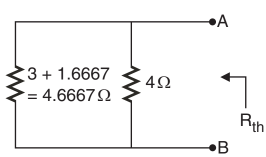


Fig. 10.

With reference to Fig. 10, we can write,

$$R_{th} = \frac{4.6667 \times 4}{4.6667 + 4} = 2.1539\,\Omega$$

RESULT

The value of R for maximum power transfer = 2.1539 Ω

EXAMPLE 2.52

Determine the value of R in the circuit of Fig. 1 for maximum power transfer to R from the rest of the circuit.

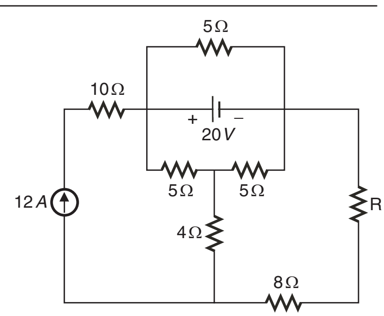


Fig. 1.

SOLUTION

The given circuit should be replaced by Thevenin's equivalent with respect to terminals of R as shown below:

Now, for maximum power transfer to R, the value of R should be equal to R_{th}. Hence, we have to find the value of Thevenin's resistance R_{th}.

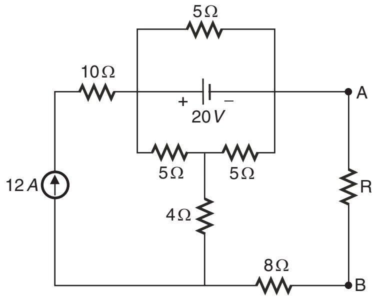


Fig. 2.

⇒

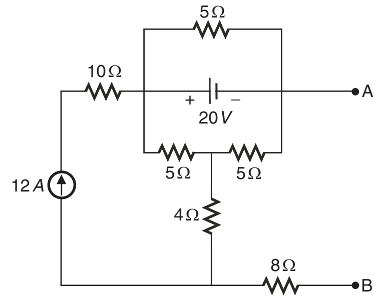


Fig. 3.

⇓

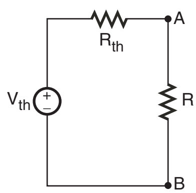


Fig. 5.

⇐

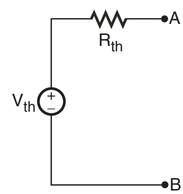


Fig. 4.

To find Thevenin's resistance R_{th}

Let us replace the voltage source with a short circuit and the current source with an open circuit as shown in Fig. 6. Remove the resistance R and denote the resulting open terminals as A and B as shown in Fig. 6. Now, the network of Fig. 6 is reduced to a single equivalent resistance as shown below:

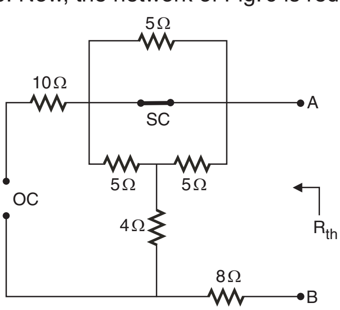


Fig. 6.

⇒

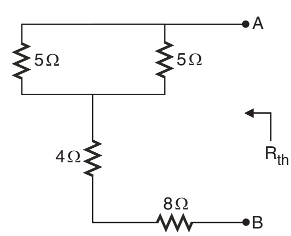


Fig. 7.

⇒

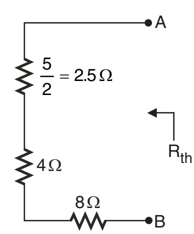


Fig. 8.

With reference to Fig. 8,

$$R_{th} = 2.5 + 4 + 8 = 14.5\,\Omega$$

RESULT

The value of R for maximum power transfer = 14.5 Ω

EXAMPLE 2.53

Determine the load impedance that can be connected across terminals A and B for maximum power transfer to load impedance. Also, calculate the maximum power transferred to load.

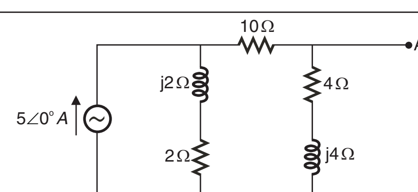


Fig. 1.

SOLUTION

Let us convert the given circuit into Thevenin's equivalent with respect to terminals A and B by using source transformation technique as shown below:

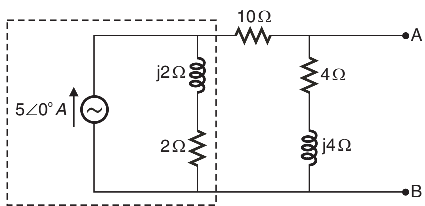


Fig. 2.

Convert current source to voltage source ⇒

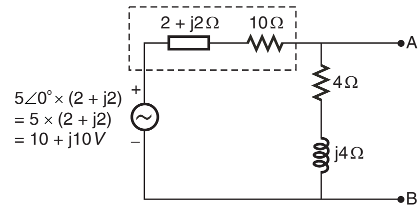


Fig. 3.

⇓ Combine series impedance

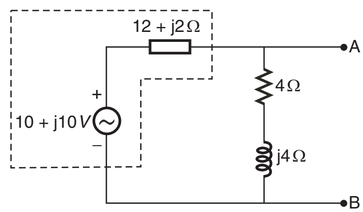


Fig. 4.

Convert voltage source to current source ⇐

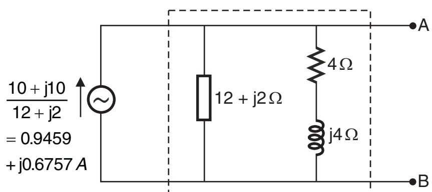


Fig. 5.

⇓ Combine parallel impedance

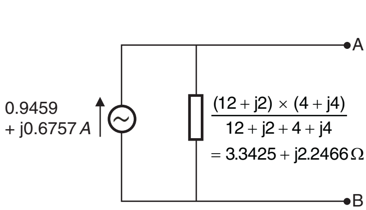


Fig. 6.

Convert current source to voltage source ⇒

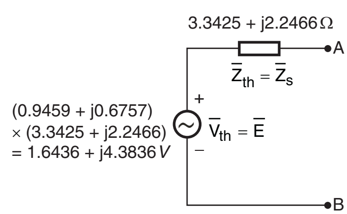


Fig. 7.

Now, the circuit of Fig. 7 is Thevenin's equivalent of the given circuit with respect to terminals A and B. Thevenin's source is the voltage source for load connected across terminals A and B and Thevenin's impedance is the source impedance.

$$\therefore \quad \bar{E} = 1.6436 + j4.3836\,V$$

$$\bar{Z}_s = 3.3425 + j2.2466\,\Omega$$

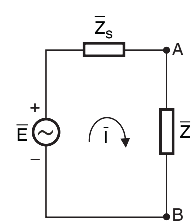


Fig. 8.

Let us connect a load impedance $\bar{Z}$ across terminals A and B of Thevenin's equivalent as shown in Fig. 8.

Now, for maximum power transfer,

$$\bar{Z} = (\bar{Z}_s)^* = (3.3425 + j2.2466)^*\,\Omega = 3.3425 - j2.2466\,\Omega = 4.0273\angle -33.9^\circ\Omega$$

With reference to Fig. 8, we can write,

$$\bar{I} = \frac{\bar{E}}{\bar{Z}_s + \bar{Z}} = \frac{1.6436 + j4.3836}{(3.3425 + j2.2466) + (3.3425 - j2.2466)}$$

$$= \frac{1.6436 + j4.3836}{2 \times 3.3425} = 0.2459 + j0.6557 = 0.7003\angle 69.4^\circ\ A$$

Maximum power delivered to load, $P_{max} = |\bar{I}|^2 \times \text{Real part of } \bar{Z}$

$$= 0.7003^2 \times 3.3425 = 1.6392\ W$$

RESULT

Load impedance across A and B for maximum power transfer = $3.3425 - j2.2466\ \Omega = 4.0273\angle -33.9^\circ\ \Omega$

Maximum power transferred to load = 1.6392 *W*

EXAMPLE 2.54

In the circuit of Fig. 1, determine the value of R so that maximum power is transferred to it.

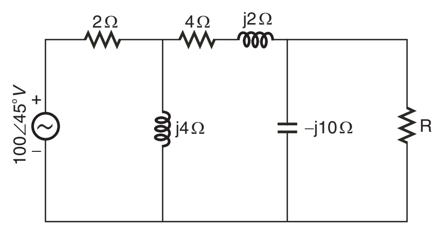


Fig. 1.

SOLUTION

Let us remove the resistance R and denote the resulting open terminals by A and B as shown in Fig. 2. The circuit of Fig. 2 should be converted into Thevenin's equivalent with respect to terminals A and B as shown in Fig. 3. Let us connect the load resistance R across A and B as shown in Fig. 4. Now, by maximum power transfer theorem, the value of R should be equal to magnitude of source impedance $\bar{Z}_s$, for maximum power transfer. Here, $\bar{Z}_s = \bar{Z}_{th}$.

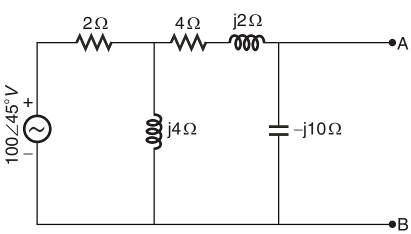


Fig. 2.

$\Rightarrow$

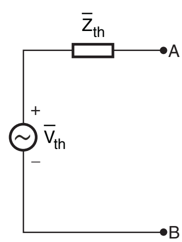


Fig. 3.

$\Rightarrow$

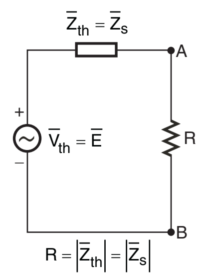


Fig. 4.

To find $\bar{Z}_{th}$ and the value of R for maximum power

Let us replace the voltage source in the circuit of Fig. 2 with a short circuit as shown in Fig. 5. The network of Fig. 5 is reduced to a single equivalent impedance as shown below:

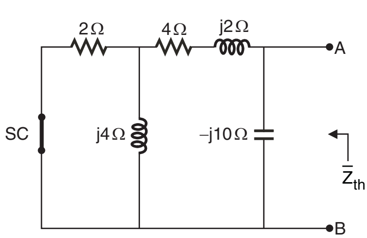


Fig. 5.

$\Rightarrow$

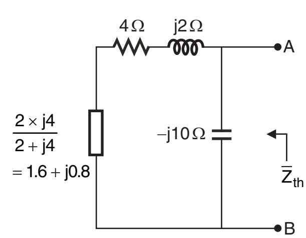


Fig. 6.

$\Rightarrow$

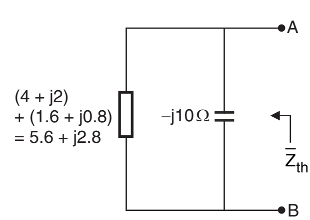


Fig. 7.

With reference to Fig. 7, we can write,

$$\bar{Z}_{th} = \frac{(5.6 + j2.8) \times (-j10)}{(5.6 + j2.8) + (-j10)}$$

$$= 6.7308 - j1.3462\,\Omega$$

$$= 6.8641\angle -11.3^\circ\,\Omega$$

Now, by maximum power transfer theorem,

$$R = |\bar{Z}_{th}| = 6.8641\Omega$$

RESULT

The value of R for maximum power transfer = 6.8641 Ω

EXAMPLE 2.55

In the circuit of Fig. 1, the load impedance $\bar{Z}$ has a fixed resistance of 2 Ω and a variable reactance jX. Determine the value of the reactance jX for maximum power transfer to load.

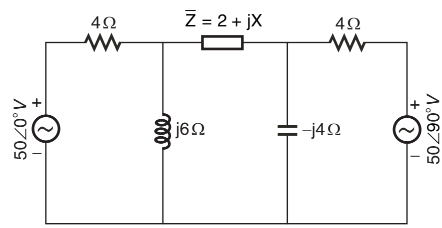


Fig. 1.

SOLUTION

Let us remove the load impedance $\bar{Z}$ and denote the resulting open terminals by A and B as shown in Fig. 2. The circuit of Fig. 2 should be reduced to Thevenin's equivalent with respect to terminals A and B as shown in Fig. 3. Let us connect the load impedance $\bar{Z}$ across terminals A and B as shown in Fig. 4.

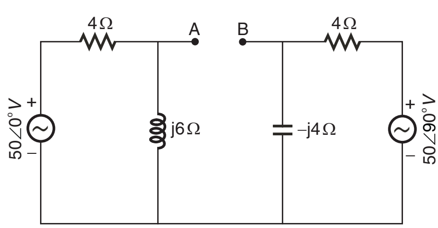


Fig. 2.

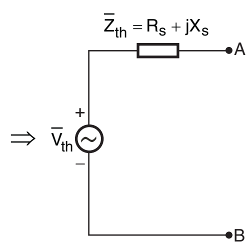


Fig. 3.

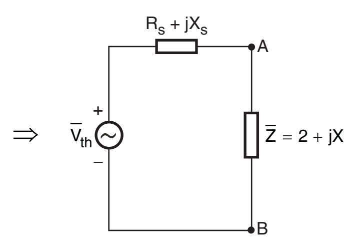


Fig. 4.

Now, by maximum power transfer theorem, for maximum power transfer to load, $jX = -jX_s$.

To find $\bar{Z}_{th}$ and the value of jX for maximum power

Let us replace the voltage sources in the circuit of Fig. 2 with a short circuit as shown in Fig. 5. The network of Fig. 5 is reduced to a single equivalent impedance as shown below:

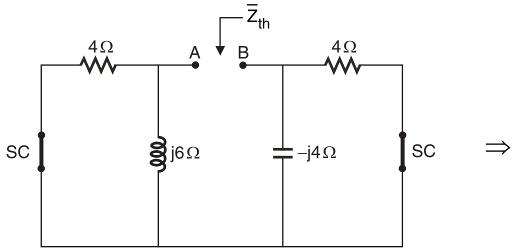


Fig. 5.

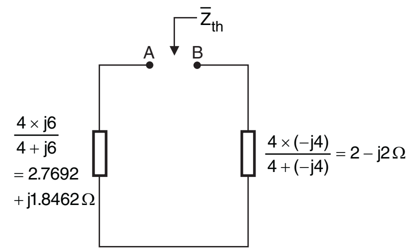


Fig. 6.

With reference to Fig. 6, we can write,

$$\bar{Z}_{th} = (2.7692 + j1.8462) + (2 - j2)$$
$$= 4.7692 - j0.1538\,\Omega$$

Here, $\bar{Z}_{th} = R_s + jX_s = 4.7692 - j0.1538$

$\therefore\ jX_s = -j0.1538$

For maximum power transfer to load,

$$jX = -jX_s = -(-j0.1538) = +j0.1538\,\Omega$$

$$\therefore\ jX = j0.1538\,\Omega$$

RESULT

For maximum power transfer to load, the value of $jX = j0.1583\,\Omega$

EXAMPLE 2.56

In the circuit of Fig. 1, determine the impedance that can be connected across terminals A and B for maximum power transfer. Also, estimate the maximum power.

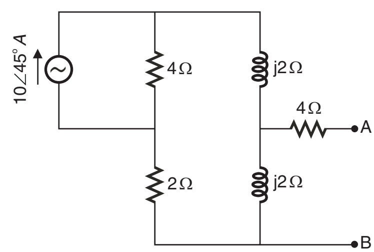


Fig. 1.

SOLUTION

Let us determine Thevenin's equivalent of the given circuit with respect to terminals A and B as shown below:

To find Thevenin's voltage $\bar{V}_{th}$

In the given circuit, 4 Ω resistance is open and so no current will flow through it. Hence, there is no voltage across the 4 Ω resistance. The given circuit is redrawn as shown in Fig. 3.

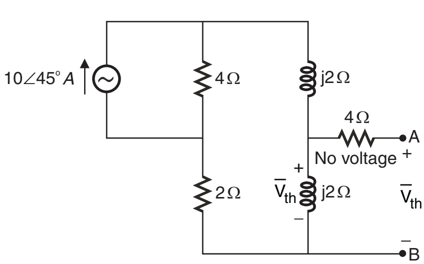


Fig. 2.

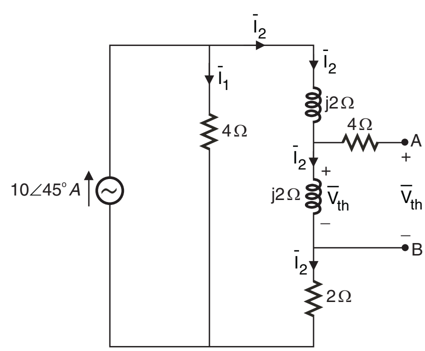


Fig. 3.

With reference to Fig. 3, by current division rule,

$$\bar{I}_2 = 10\angle 45^\circ \times \frac{4}{4+(2+j2+j2)} = \frac{40\angle 45^\circ}{6+j4} = \frac{40\angle 45^\circ}{7.2111\angle 33.7^\circ}$$
$$= 5.547\angle 11.3^\circ\ A$$

$$\bar{V}_{th} = \bar{I}_2 \times j2 = 5.547\angle 11.3^\circ \times j2 = 5.547\angle 11.3^\circ \times 2\angle 90^\circ$$
$$= 11.094\angle 101.3^\circ\ V$$

To find Thevenin's impedance $\bar{Z}_{th}$

Let us replace the current source in the given circuit by an open circuit as shown in Fig. 4. The network of Fig. 4 is reduced to a single equivalent impedance as shown below:

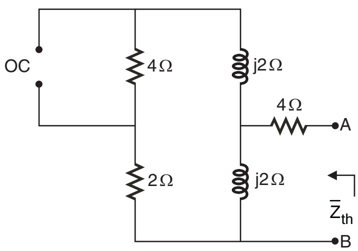


Fig. 4.

$\Rightarrow$

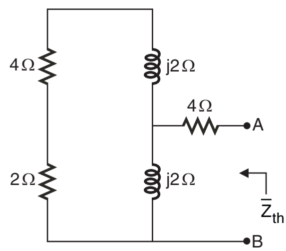


Fig. 5.

$\Rightarrow$

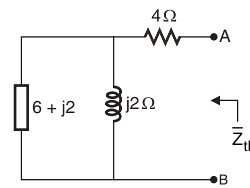


Fig. 6.

With reference to Fig. 6, we can write,

$$\bar{Z}_{th} = \left[\frac{(6+j2)\times j2}{(6+j2)+j2}\right]+4 = [0.4615+j1.6923]+4$$
$$= 4.4615+j1.6923\ \Omega$$

To find $\bar{Z}$ for maximum power and P_{max}

Thevenin's equivalent of the given circuit is shown in Fig. 7. Now, Thevenin's equivalent is the voltage generator for the load that may be connected across A and B.

$$\therefore\ \bar{E} = \bar{V}_{th} = 11.094\angle 101.3^\circ\ V$$

$$\bar{Z}_s = \bar{Z}_{th} = 4.4615+j1.6923\ \Omega$$

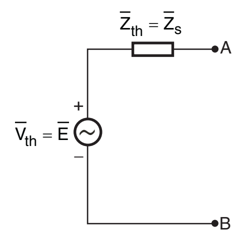


Fig. 7.

Let us connect a load impedance $\bar{Z}$ across A and B of Thevenin's equivalent as shown in Fig. 8. Now, for maximum power transfer to load, the load impedance $\bar{Z}$ should be equal to complex conjugate of source impedance $\bar{Z}_s$.

$$\therefore\ \bar{Z} = \bar{Z}_s^* = (4.4615+j1.6923)^* = 4.4615-j1.6923\ \Omega$$

With reference to Fig. 8, we can write,

$$\bar{I} = \frac{\bar{E}}{\bar{Z}_s+\bar{Z}} = \frac{11.094\angle 101.3^\circ}{4.4615+j1.6923\ +\ 4.4615-j1.6923}$$

$$= \frac{11.094\angle 101.3^\circ}{2\times 4.4615} = 1.2433\angle 101.3^\circ A$$

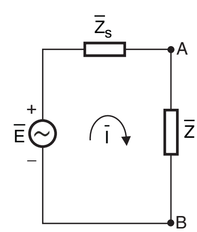


Fig. 8.

Maximum power in the load, $P_{max} = |\bar{I}|^2 \times \text{Real part of } \bar{Z}$

$$= 1.2433^2 \times 4.4615$$

$$= 6.8966\,W$$

RESULT

The load impedance for maximum power transfer, $\bar{Z} = 4.4615 - j1.6923\,\Omega$

Maximum power transferred to load, $P_{max} = 6.8966\,W$

EXAMPLE 2.57

(AU May'17, 16 Marks)

Find the current flowing in the $20\,\Omega$ resistance connected across terminals A and B of the circuit shown in Fig. 1 using Norton's theorem.

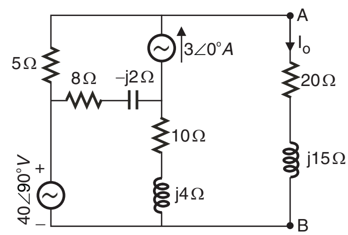


Fig. 1.

SOLUTION

Let us remove the $20 + j5\,\Omega$ and denote the resultant open terminals as A and B as shown in Fig. 2. Now, the circuit of Fig. 2 should be replaced by Norton's equivalent at terminals A and B.

To find Norton's current $\bar{I}_n$

Let us short circuit the terminals A and B in the circuit of Fig. 2 as shown in Fig. 3. The current flowing through the short circuit is Norton's current. Let us assume the direction of Norton's current as A to B.

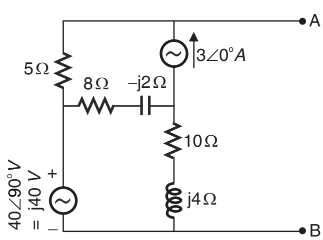


Fig. 2.

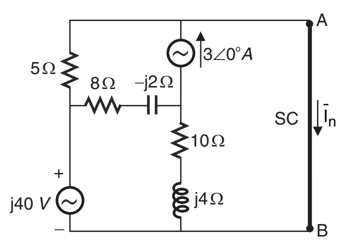


Fig. 3.

Let us calculate Norton's current by superposition theorem.

Let, $\bar{I}'_n$ = Current when the $j40\,V$ source alone is acting.

$\bar{I}''_n$ = Current when the $3\,A$ source alone is acting.

To find the response $\bar{I}'_n$ when j40 *V* source is acting alone

The $3\,A$ current source is replaced with a short circuit as shown in Fig. 4. Now the $10 + j4\,\Omega$ impedance is short-circuited and so it is removed and the circuit is redrawn as shown in Fig. 5.

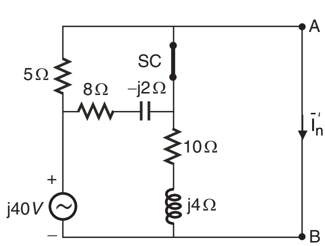


Fig. 4.

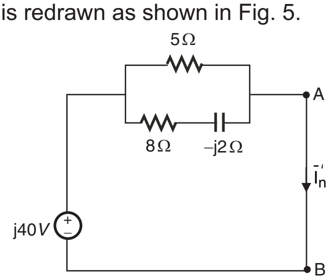


Fig. 5.

With reference to Fig. 6, by Ohm's law we get,

$$\bar{I}'_n = \frac{j40}{3.1214 - j0.289} = -1.1764 + j12.7058\ A$$

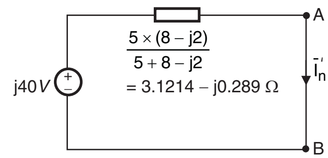


Fig. 6.

To find the response $\bar{I}''_n$ when the 3 A source is acting alone

The j40 V source is replaced with a short circuit as shown in Fig. 7. The circuit of Fig. 7 is redrawn as shown in Fig. 8. The 5 Ω resistance is short-circuited and so entire current 3 A flows through short circuit as shown in Fig. 8.

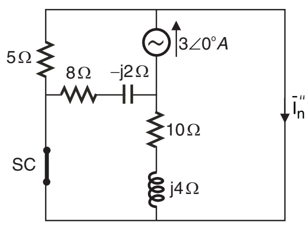


Fig. 7.

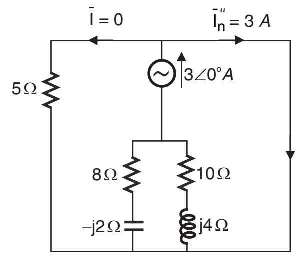


Fig. 8.

$\therefore\ \bar{I}''_n = 3\ A$

To find the response $\bar{I}_n$ when both the sources are acting

By the superposition theorem,

$$\bar{I}_n = \bar{I}'_n + \bar{I}''_n = -1.1764 + j12.7058 + 3$$

$$= 1.8236 + j12.7058\ A$$

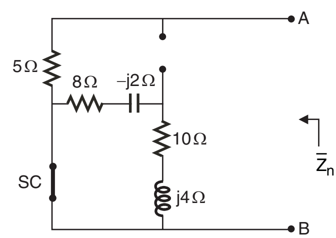


Fig. 9.

To find Norton's impedance $\bar{Z}_n$

Let us replace the voltage source with a short circuit and the current source with an open circuit as shown in Fig. 9.

With reference to Fig. 9, we can write,

$$\text{Norton's impedance, } \bar{Z}_n = 5 + \frac{(8 - j2) \times (10 + j4)}{8 - j2 + 10 + j4} = 5 + 4.9024 + j0.122$$

$$= 9.9024 + j0.122\ \Omega$$

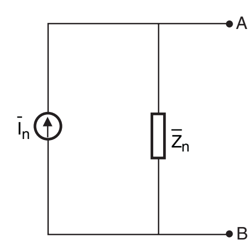


Fig. 10.

Norton's equivalent at A-B and current $\bar{I}_o$

The Norton's equivalent is shown in Fig. 10.

Let us connect the 20 + j15 Ω impedancce across A and B as shown in Fig. 11.

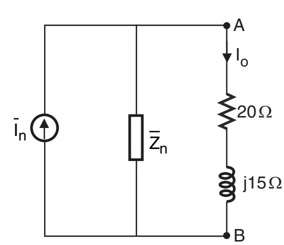


Fig. 11.

By current division rule, we get,

$$\bar{I}_o = \bar{I}_n \times \frac{\bar{Z}_n}{\bar{Z}_n + 20 + j5}$$

$$= (1.8236 + j12.7058) \times \frac{9.9024 + j0.122}{9.9024 + j0.122 + 20 + j15}$$

$$= (1.8236 + j12.7058) \times (0.2654 - j0.1301) = 2.137 + j3.1349\ A$$

$$= 3.794 \angle 55.7^\circ\ A$$

EXAMPLE 2.58

In the circuit of Fig. 1, the load impedance $\bar{Z}$ has a fixed reactance of $+j2\,\Omega$ and a variable resistance R. Determine the value of R for maximum power transfer.

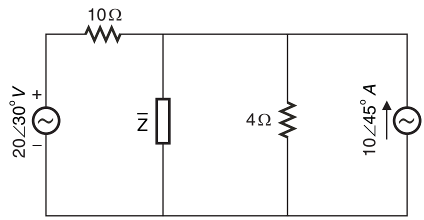


Fig. 1.

SOLUTION

The $20\angle 30^\circ V$ voltage source in series with $10\,\Omega$ is converted to an equivalent current source in Fig. 2.

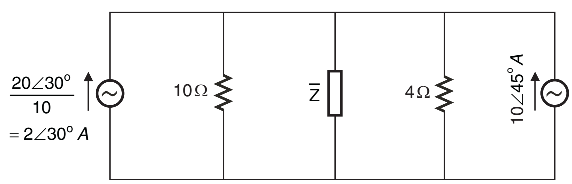


Fig. 2.

The circuit of Fig. 2 is redrawn as shown in Fig. 3. In this circuit, the current sources in parallel can be combined to give a single equivalent current source and the $10\,\Omega$ and $4\,\Omega$ resistances in parallel can be combined to give a single equivalent resistance as shown in Fig. 4.

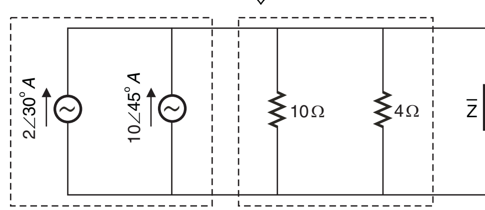


Fig. 3.

The current source in parallel with $2.8571\,\Omega$ resistance in the circuit of Fig. 4 is converted to a voltage source as shown in Fig. 5.

With reference to Fig. 5, we can write,

$$\bar{I} = \frac{34.1229\angle 42.5^\circ}{2.8571 + R + j2}$$

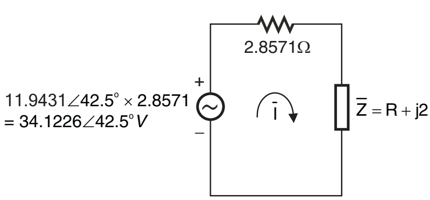


Fig. 5.

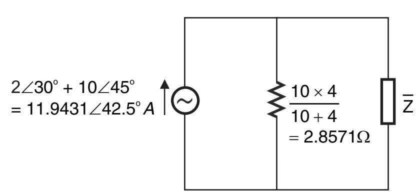


Fig. 4.

$$\therefore\ I = \left|\bar{I}\right| = \left|\frac{34.1226\angle 42.5^\circ}{2.8571 + R + j2}\right| = \frac{34.1226}{\sqrt{(2.8571 + R)^2 + 2^2}} = \frac{34.1226}{\sqrt{(2.8571 + R)^2 + 4}}$$

Let, P = Power delivered to load.

Now, $P = \left|\bar{I}\right|^2 \times \text{Real part of } \bar{Z}$

$$= \frac{34.1226^2}{(2.8571+R)^2+4} \times R = \frac{1164\,R}{(2.8571+R)^2+4}$$

The condition for maximum power can be obtained by differentiating P with respect to R and equating (dP/dR) = 0.

$$\therefore \quad \frac{dP}{dR} = \frac{\left[(2.8571+R)^2+4\right]\times 1164 - 1164\,R \times 2(2.8571+R)}{\left[(2.8571+R)^2+4\right]^2}$$

$$\boxed{d(uv) = \frac{v\,du - u\,dv}{v^2}}$$

For $\frac{dP}{dR} = 0$, the numerator of $\frac{dP}{dR}$ should be equal to zero.

$$\therefore \quad 1164[(2.8571 + R)^2 + 4] - 2 \times 1164\,R\,(2.8571 + R) = 0$$

$$2 \times 1164\,R\,(2.8571 + R) = 1164[(2.8571 + R)^2 + 4]$$

On dividing either side by 1164, we get,

$$2R\,(2.8571 + R) = (2.8571 + R)^2 + 4$$

$$5.7142\,R + 2R^2 = 8.163 + 5.7142\,R + R^2 + 4$$

$$\therefore \quad 5.7142\,R + 2R^2 - 5.7142\,R - R^2 = 8.163 + 4$$

$$\therefore \quad R^2 = 12.163 \quad \Rightarrow \quad R = \sqrt{12.163} = 3.4875\,\Omega$$

Note : *Here, the value of R for maximum power transfer is also given by,*

$$R = \sqrt{2.8571^2 + 2^2} = 3.4875\,\Omega$$

RESULT

The value of R for maximum power transfer = 3.4875 Ω

EXAMPLE 2.59

In the circuit of Fig. 1, the phase angle θ of the voltage source, $5\angle\theta^\circ\,V$ is continuously variable. Find the value of θ for maximum power transfer to 10 Ω resistance.

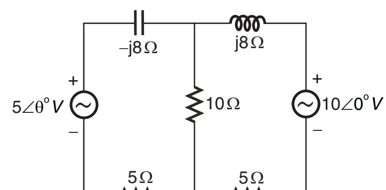


Fig. 1.

SOLUTION

Let us assume two mesh currents as shown in Fig. 2.

Let, $5\angle\theta^\circ\,V = 5\cos\theta + j5\sin\theta\,V$

$10\angle 0^\circ\,V = 10\cos 0^\circ + j10\sin 0^\circ = 10\,V$

With reference to Fig. 2, the mesh basis matrix equation is,

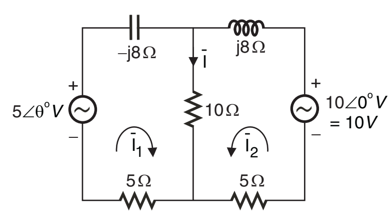


Fig. 2.

$$\begin{bmatrix} 15-j8 & 10 \\ 10 & 15+j8 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} 5\cos\theta + j5\sin\theta \\ 10 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 15-j8 & 10 \\ 10 & 15+j8 \end{vmatrix} = (15-j8)\times(15+j8) - 10\times 10 = 15^2 + 8^2 - 100 = 189$$

$$\Delta_1 = \begin{vmatrix} 5\cos\theta + j5\sin\theta & 10 \\ 10 & 15+j8 \end{vmatrix} = (5\cos\theta + j5\sin\theta)\times(15+j8) - 10\times 10$$
$$= 75\cos\theta + j40\cos\theta + j75\sin\theta - 40\sin\theta - 100$$
$$= (75\cos\theta - 40\sin\theta - 100) + j(40\cos\theta + 75\sin\theta)$$

$$\Delta_2 = \begin{vmatrix} 15-j8 & 5\cos\theta + j5\sin\theta \\ 10 & 10 \end{vmatrix} = (15-j8)\times 10 - (5\cos\theta + j5\sin\theta)\times 10$$
$$= 150 - j80 - 50\cos\theta - j50\sin\theta$$
$$= (150 - 50\cos\theta) + j(-50\sin\theta - 80)$$

Now, $\bar{I}_1 = \dfrac{\Delta_1}{\Delta}$; $\bar{I}_2 = \dfrac{\Delta_2}{\Delta}$

Let, $\bar{I}$ = current through $10\,\Omega$ resistance.

With reference to Fig. 2, by KCL, we can write,

$$\bar{I} = \bar{I}_1 + \bar{I}_2 = \frac{\Delta_1}{\Delta} + \frac{\Delta_2}{\Delta} = \frac{1}{\Delta}[\Delta_1 + \Delta_2]$$

$$= \frac{1}{189}\left[(75\cos\theta - 40\sin\theta - 100) + j(40\cos\theta + 75\sin\theta) + (150 - 50\cos\theta) + j(-50\sin\theta - 80)\right]$$

$$= \frac{1}{189}\left[(25\cos\theta - 40\sin\theta + 50) + j(40\cos\theta + 25\sin\theta - 80)\right]$$

$$= \frac{25}{189}\left[(\cos\theta - 1.6\sin\theta + 2) + j(1.6\cos\theta + \sin\theta - 3.2)\right]$$

$$\therefore\ I = \left|\bar{I}\right| = \frac{25}{189}\sqrt{(\cos\theta - 1.6\sin\theta + 2)^2 + (1.6\cos\theta + \sin\theta - 3.2)^2}$$

Let, P = Power delivered to $10\,\Omega$ resistance.

Now, $P = I^2 \times 10 = \dfrac{10\times 25^2}{189^2}\left[(\cos\theta - 1.6\sin\theta + 2)^2 + (1.6\cos\theta + \sin\theta - 3.2)^2\right]$

$$= 0.175\left[(\cos\theta - 1.6\sin\theta + 2)^2 + (1.6\cos\theta + \sin\theta - 3.2)^2\right]$$

The condition for maximum power transfer can be obtained by differentiating P with respect to θ and solving the equation obtained by equating $(dP/d\theta) = 0$.

$$\therefore\ \frac{dP}{d\theta} = 0.175\left[2(\cos\theta - 1.6\sin\theta + 2)\times(-\sin\theta - 1.6\cos\theta) + 2(1.6\cos\theta + \sin\theta - 3.2)\times(-1.6\sin\theta + \cos\theta)\right]$$

$$= 0.175\times 2[-\cos\theta\sin\theta - 1.6\cos^2\theta + 1.6\sin^2\theta + 2.56\cos\theta\sin\theta - 2\sin\theta - 3.2\cos\theta$$
$$-2.56\cos\theta\sin\theta + 1.6\cos^2\theta - 1.6\sin^2\theta + \cos\theta\sin\theta + 5.12\sin\theta - 3.2\cos\theta]$$

$$= 0.35\left[3.12\sin\theta - 6.4\cos\theta\right]$$

On equating $\dfrac{dP}{d\theta} = 0$, we get,

$$0.35\,[3.12\sin\theta - 6.4\cos\theta] = 0$$

$$\therefore\ 3.12\sin\theta - 6.4\cos\theta = 0$$

$$\therefore\ 3.12\sin\theta = 6.4\cos\theta \Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{6.4}{3.12} \Rightarrow \tan\theta = 2.0513$$

$$\therefore\ \theta = \tan^{-1}(2.0513) = 64^\circ$$

RESULT

The value of θ for maximum power transfer = 64°

2.6.4 Reciprocity Theorem *(AU June'14, '16 & Dec'16, 2 Marks)*

The reciprocity theorem states that in a linear, bilateral, single source circuit, the ratio of excitation to response is constant when the position of excitation and response are interchanged.

Here, the excitation is either a voltage source or current source and the response is either a current or voltage in an element (R, L or C).

"The reciprocity theorem will be satisfied only by circuits or networks which do not have dependent sources". In circuits without dependent sources, the Z and Y matrices formed for mesh and node basis analysis will be symmetric, i.e., the element of matrices $\overline{Z}_{jk} = \overline{Z}_{kj}$ and $\overline{Y}_{jk} = \overline{Y}_{kj}$. The networks which satisfy the reciprocity theorem are called **reciprocal networks**.

2.6.5 Reciprocity Theorem Applied to Mesh Basis Circuit

Consider a mesh basis circuit with a single voltage source E as shown in Fig. 2.59. Let I be the current in mesh-j when the source is in mesh-k as shown in Fig. 2.59(a). The reciprocity theorem implies that the same current I will be produced in mesh-k, if the source is shifted to mesh-j as shown in Fig. 2.59(b). It must be noted that currents in other parts of the circuit may not be the same.

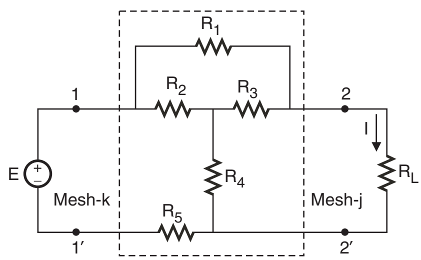


Fig. a : *Source in mesh-k and response in mesh-j.*

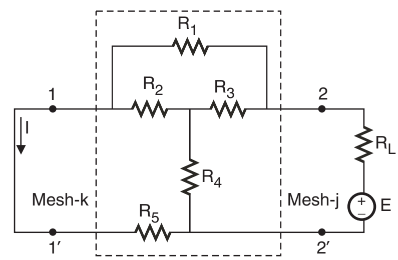


Fig. b : *Source in mesh-j and response in mesh-k.*

Fig. 2.59 : *Mesh basis circuit to demonstrate the reciprocity theorem.*

Proof by mesh analysis :

Consider a 2-port resistive network without sources shown in Fig. 2.60. In order to prove the reciprocity theorem we can connect a voltage source at port-1 and short circuit port-2 to observe the current response. Then the source can be shifted to port-2 and the same current response can be observed in short-circuited port-1.

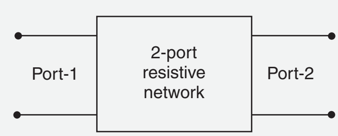


Fig. 2.60.

Case i : Excitation in port-1 and response at port-2

Let us connect a voltage source E to port-1 and short circuit the terminals of port-2 as shown in Fig. 2.61. Let I be the current through short circuit, which is the response due to excitation, E.

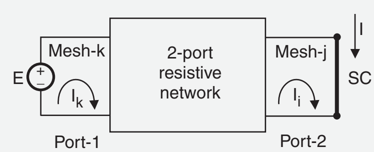


Fig. 2.61.

In the circuit of Fig. 2.61, the response I due to excitation E can be solved by mesh analysis. Let us consider two meshes mesh-k and mesh-j as shown in Fig. 2.61. Now the response, $I = I_j$.

In mesh analysis, the current in j^{th} mesh I_j is given by (Refer to equation 1.23 in Chapter 1),

$$I_j = \frac{\Delta_{1j}}{\Delta} E_{11} + \frac{\Delta_{2j}}{\Delta} E_{22} + \frac{\Delta_{3j}}{\Delta} E_{33} + \ldots.. + \frac{\Delta_{kj}}{\Delta} E_{kk} + \ldots.. \qquad \ldots..(2.88)$$

Since the circuit of Fig. 2.61, has only one source in mesh-k,

$$E_{11} = E_{22} = E_{33} = ... = 0 \text{ and } E_{kk} = E$$

Hence, equation (2.88) can be written as,

$$I_j = \frac{\Delta_{kj}}{\Delta} E_{kk} = \frac{\Delta_{kj}}{\Delta} E$$

$$\therefore \quad \text{The response, } I = I_j = \frac{\Delta_{kj}}{\Delta} E \qquad ..... (2.89)$$

From equation (2.89), the ratio of excitation to response is,

$$\frac{E}{I} = \frac{\Delta}{\Delta_{kj}} \qquad .....(2.90)$$

Case ii : Excitation in port-2 and response at port-1

Let us change the excitation source E to mesh-j as shown in Fig. 2.62, and estimate the current in mesh-k. [In mesh-k the voltage source is replaced with its internal impedance. Since the source is ideal the internal impedance is zero and so it is replaced with a short circuit.] Now the response, $I = I_k$.

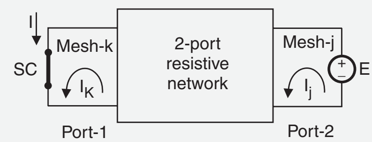


Fig. 2.62.

In mesh analysis the current in k^{th} mesh, I_k is given by (Refer to equation 1.23 in Chapter 1),

$$I_k = \frac{\Delta_{1k}}{\Delta} E_{11} + \frac{\Delta_{2k}}{\Delta} E_{22} + \frac{\Delta_{3k}}{\Delta} E_{33} + ...+ \frac{\Delta_{jk}}{\Delta} E_{jj} + ... \qquad .....(2.91)$$

Since the circuit of Fig. 2.62 has only one source in mesh-j,

$$E_{11} = E_{22} = E_{33} = ... = 0 \text{ and } E_{jj} = E$$

Hence, equation (2.91) can be written as,

$$I_k = \frac{\Delta_{jk}}{\Delta} E_{jj} = \frac{\Delta_{jk}}{\Delta} E$$

$$\therefore \quad \text{The response, } I = I_k = \frac{\Delta_{jk}}{\Delta} E \qquad ..... (2.92)$$

From equation (2.92), the ratio of excitation to response is,

$$\frac{E}{I} = \frac{\Delta}{\Delta_{jk}} \qquad ..... (2.93)$$

Conclusion

When the 2-port network does not have dependent sources $\Delta_{kj} = \Delta_{jk}$, equations (2.90) and (2.93) are same. Hence, the reciprocity theorem is proved.

2.6.6 Reciprocity Theorem Applied to Node Basis Circuit

Consider a node basis circuit with a single current source I_s as shown in Fig. 2.63. Let, V be the voltage in node-j when the current source is connected between node-k and reference as shown in Fig. 2.63a . The reciprocity theorem implies that the same voltage V will exist in node-k if the source is shifted to node-j as shown in Fig. 2.63b. It must be noted that the voltages in other parts of the circuit may not be the same.

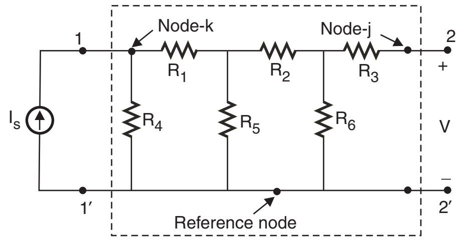


Fig. a : Source in node-k and response in node-j.

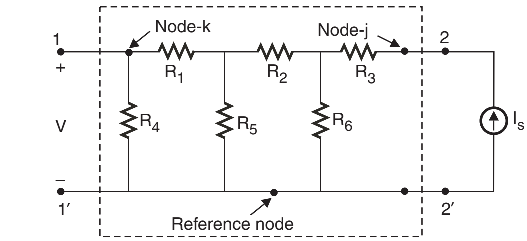


Fig. b : Source in node-j and response in node-k.

Fig. 2.63 : Node basis circuit to demonstrate the reciprocity theorem.

Proof by node analysis:

Consider a 2-port resistive network without sources shown in Fig. 2.64. In order to prove the reciprocity theorem we can connect a current source at port-1 and observe the open circuit voltage at port-2. Then the source can be shifted to port-2 and the same open circuit voltage can be observed in port-1.

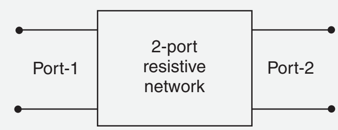


Fig. 2.64.

Case i : Excitation in port-1 and response at port-2

Let us connect a current source, I_s to port-1 as shown in Fig. 2.65. Let the voltage across the terminals of port-2 be V, which is the response due to excitation I_s.

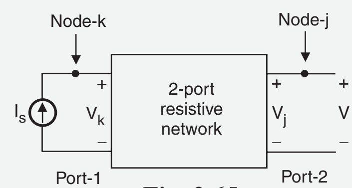


Fig. 2.65.

In the circuit of Fig. 2.65, the response V due to excitation I_s can be solved by node analysis. Let us consider two nodes, node-k and node-j as shown in Fig. 2.65. Now the response, $V = V_j$.

In node analysis, the voltage in j^{th} node V_j is given by (Refer to equation 1.37 in Chapter 1),

$$V_j = \frac{\Delta'_{1j}}{\Delta'} I_{11} + \frac{\Delta'_{2j}}{\Delta'} I_{22} + \frac{\Delta'_{3j}}{\Delta'} I_{33} + \ldots + \frac{\Delta'_{kj}}{\Delta'} I_{kk} + \ldots \qquad \text{..... (2.94)}$$

Since the circuit of Fig. 2.65, has only one source in node-k,

$$I_{11} = I_{22} = I_{33} = \ldots = 0 \;\; \text{and} \;\; I_{kk} = I_s$$

Hence, equation (2.94) can be written as,

$$V_j = \frac{\Delta'_{kj}}{\Delta'} I_{kk} = \frac{\Delta'_{kj}}{\Delta'} I_s$$

$$\therefore \;\; \text{The response,} \;\; V = V_j = \frac{\Delta'_{kj}}{\Delta'} I_s \qquad \text{..... (2.95)}$$

From equation (2.95), the ratio of excitation to response is,

$$\frac{I_s}{V} = \frac{\Delta'}{\Delta'_{kj}} \qquad \text{..... (2.96)}$$

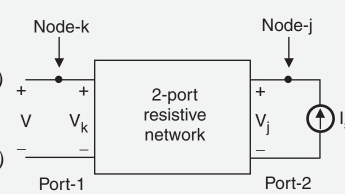


Fig. 2.66.

Case ii : Excitation in port-2 and response at port-1

Let us change the excitation source to node-j as shown in Fig. 2.66, and estimate the voltage at node-k. [In node-k the current source is replaced with its internal impedance. Since the source is ideal, the internal impedance is infinity and so it is replaced with an open circuit.] Now the response, $V = V_k$.

In node analysis the voltage in node-k, V_k is given by (Refer to equation 1.37 in Chapter 1),

$$V_k = \frac{\Delta'_{1k}}{\Delta'} I_{11} + \frac{\Delta'_{2k}}{\Delta'} I_{22} + \frac{\Delta'_{3k}}{\Delta'} I_{33} + \ldots + \frac{\Delta'_{jk}}{\Delta'} I_{jj} + \ldots \qquad \text{..... (2.95)}$$

Since the circuit of Fig. 2.66, has only one source in node-j,

$$I_{11} = I_{22} = I_{33} = \ldots = 0 \ \text{and} \ I_{jj} = I_s$$

Hence, equation (2.95) can be written as,

$$V_k = \frac{\Delta'_{jk}}{\Delta'} I_{jj} = \frac{\Delta'_{jk}}{\Delta'} I_s$$

$$\therefore \ \text{The response, } V = V_k = \frac{\Delta'_{jk}}{\Delta'} I_s \qquad \text{..... (2.96)}$$

From equation (2.96), the ratio of excitation to response is,

$$\frac{I_s}{V} = \frac{\Delta'}{\Delta'_{jk}} \qquad \text{..... (2.97)}$$

Conclusion

When the 2-port network does not have dependent sources $\Delta'_{kj} = \Delta'_{jk}$ *, equations (2.94) and (2.97) are the same. Hence, the reciprocity theorem is proved.*

EXAMPLE 2.60

Prove the reciprocity theorem for the two port network of Fig. 1.

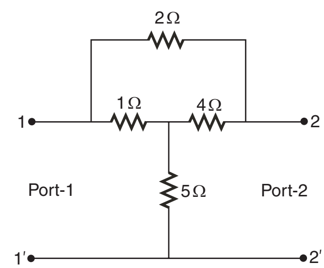


Fig. 1.

SOLUTION

Method-I : Proof by mesh analysis

Case i : Excitation in port-1 and response at port-2

Let us connect a voltage source of E volts to port-1 and short circuit the port-2 as shown in Fig. 2. Let, I be the current through short circuit which is the response due to excitation E.

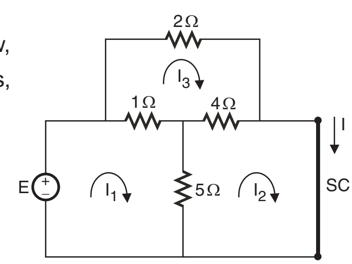


Fig. 2.

Let us assume three mesh currents I_1, I_2 and I_3 as shown in Fig. 2. Now, the response $I = I_2$. With reference to Fig. 2, the mesh basis matrix equation is,

$$\begin{bmatrix} 1+5 & -5 & -1 \\ -5 & 4+5 & -4 \\ -1 & -4 & 1+2+4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -5 & -1 \\ -5 & 9 & -4 \\ -1 & -4 & 7 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 6 & -5 & -1 \\ -5 & 9 & -4 \\ -1 & -4 & 7 \end{vmatrix} = 6\times\left[(9\times7)-(-4)^2\right]-(-5)\times\left[(-5\times7)-(-1\times(-4))\right] + (-1)\times\left[(-5\times(-4))-(-1\times9)\right]$$

$$= 282 - 195 - 29 = 58$$

$$\Delta_2 = \begin{vmatrix} 6 & E & -1 \\ -5 & 0 & -4 \\ -1 & 0 & 7 \end{vmatrix} = 0 - E\times\left[(-5\times7)-(-1\times(-4))\right]+0$$

$$= 39E$$

$$\therefore \ \text{The response, } I = I_2 = \frac{\Delta_2}{\Delta} = \frac{39E}{58}$$

$$\text{The ratio of excitation to response} = \frac{E}{I} = \frac{58}{39}$$

Case ii : Excitation in port-2 and response at port-1

Let us interchange the positions of source and response as shown in Fig. 3. Let us assume mesh currents I_a, I_b and I_c as shown in Fig. 3. Now the response, $I = I_b$. With reference to Fig. 3, the mesh basis matrix equation is,

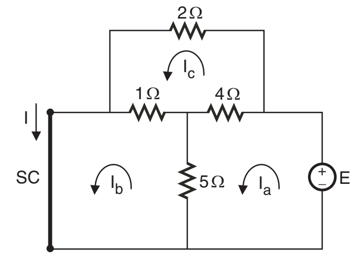


Fig. 3.

$$\begin{bmatrix} 5+4 & -5 & -4 \\ -5 & 5+1 & -1 \\ -4 & -1 & 1+4+2 \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -5 & -4 \\ -5 & 6 & -1 \\ -4 & -1 & 7 \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 9 & -5 & -4 \\ -5 & 6 & -1 \\ -4 & -1 & 7 \end{vmatrix} = 9\times\left[(6\times 7)-(-1)^2\right]-(-5)\times[(-5\times 7)-(-4\times(-1))] + (-4)\times[(-5\times(-1))-(-4\times 6)]$$

$$= 369 - 195 - 116 = 58$$

$$\Delta_b = \begin{vmatrix} 9 & E & -4 \\ -5 & 0 & -1 \\ -4 & 0 & 7 \end{vmatrix} = 0 - E\times[(-5\times 7)-(-4\times(-1))]+0 = 39E$$

$$\therefore \text{ The response, } I = I_b = \frac{\Delta_b}{\Delta} = \frac{39E}{58}$$

$$\text{The ratio of excitation to response} = \frac{E}{I} = \frac{58}{39}$$

Conclusion

It is observed that the ratio of excitation to response is the same when the positions of excitation and response are interchanged. Hence, the reciprocity theorem is proved.

Method II : Proof by node analysis

Case i : Excitation in port-1 and response at port-2

Let us connect a current source of I_s amperes to port-1 and open circuit the port-2 as shown in Fig. 4. Let, V be the voltage across the open terminals of port-2, which is the response due to excitation I_s.

Let us assume three node voltages V_1, V_2 and V_3 as shown in Fig. 4. Now the response, $V = V_2$. With reference to Fig. 4, the node basis matrix equation is,

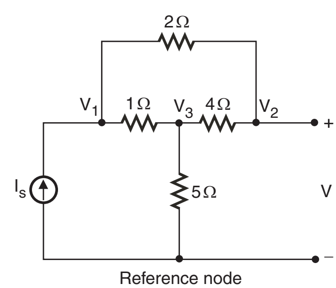


Fig. 4.

$$\begin{bmatrix} \frac{1}{1}+\frac{1}{2} & -\frac{1}{2} & -\frac{1}{1} \\ -\frac{1}{2} & \frac{1}{4}+\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{1} & -\frac{1}{4} & \frac{1}{1}+\frac{1}{4}+\frac{1}{5} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_s \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1.5 & -0.5 & -1 \\ -0.5 & 0.75 & -0.25 \\ -1 & -0.25 & 1.45 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_s \\ 0 \\ 0 \end{bmatrix}$$

$$\Delta' = \begin{vmatrix} 1.5 & -0.5 & -1 \\ -0.5 & 0.75 & -0.25 \\ -1 & -0.25 & 1.45 \end{vmatrix} = 1.5 \times \left[(0.75\times1.45)-(-0.25)^2\right]-(-0.5)\times\left[(-0.5\times1.45)-(-1\times(-0.25))\right]+(-1)\times\left[(-0.5\times(-0.25))-(-1\times0.75)\right]$$

$$= 1.5375 - 0.4875 - 0.875 = 0.175$$

$$\Delta'_2 = \begin{vmatrix} 1.5 & I_s & -1 \\ -0.5 & 0 & -0.25 \\ -1 & 0 & 1.45 \end{vmatrix} = 0 - I_s\times\left[(-0.5\times1.45)-(-1\times(-0.25))\right]+0 = 0.975\,I_s$$

$$\therefore \text{ The response, } V = V_2 = \frac{\Delta'_2}{\Delta} = \frac{0.975\,I_s}{0.175}$$

$$\text{The ratio of excitation to response} = \frac{I_s}{V} = \frac{0.175}{0.975} = 0.1795$$

Case ii : Excitation in port-2 and response at port-1

Let us interchange the positions of source and response as shown in Fig. 5. Let us assume node voltages V_a, V_b and V_c as shown in Fig. 5. Now the response, V = V_b. With reference to Fig. 5, the node basis matrix equation is,

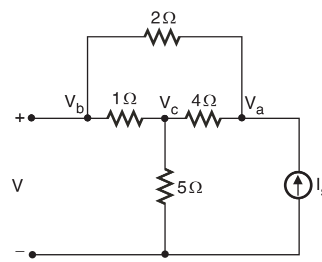


Fig. 5.

$$\begin{bmatrix} \frac{1}{4}+\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{1}+\frac{1}{2} & -\frac{1}{1} \\ -\frac{1}{4} & -\frac{1}{1} & \frac{1}{1}+\frac{1}{4}+\frac{1}{5} \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} I_s \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.75 & -0.5 & -0.25 \\ -0.5 & 1.5 & -1 \\ -0.25 & -1 & 1.45 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} I_s \\ 0 \\ 0 \end{bmatrix}$$

$$\Delta' = \begin{vmatrix} 0.75 & -0.5 & -0.25 \\ -0.5 & 1.5 & -1 \\ -0.25 & -1 & 1.45 \end{vmatrix} = 0.75\times\left[(1.5\times1.45)-(-1)^2\right]-(-0.5)\times\left[(-0.5\times1.45)-(-0.25\times(-1))\right]+(-0.25)\times\left[(-0.5\times(-1))-(-0.25\times1.5)\right]$$

$$= 0.88125 - 0.4875 - 0.21875$$

$$= 0.175$$

$$\Delta'_b = \begin{vmatrix} 0.75 & I_s & -0.25 \\ -0.5 & 0 & -1 \\ -0.25 & 0 & 1.45 \end{vmatrix} = 0 - I_s\times\left[(-0.5\times1.45)-(-0.25\times(-1))\right]+0 = 0.975\,I_s$$

$$\therefore \text{ The response, } V = V_b = \frac{\Delta'_b}{\Delta'} = \frac{0.975\,I_s}{0.175}$$

$$\text{The ratio of excitation to response} = \frac{I_s}{V} = \frac{0.175}{0.975} = 0.1795$$

Conclusion

It is observed that the ratio of excitation to response is the same when the positions of excitation and response are interchanged. Hence, the reciprocity theorem is proved.

EXAMPLE 2.61

In the circuit of Fig. 1, calculate I_x. Prove the reciprocity theorem by interchanging the position of the $10\,V$ source and I_x.

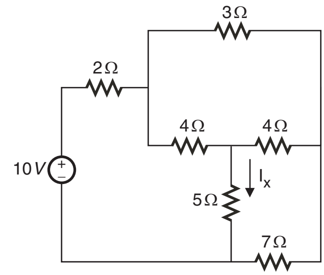


Fig. 1.

SOLUTION

Case i : To solve I_x in the given circuit

Let us assume three mesh currents as shown in Fig. 2. Now the response, $I_x = I_1 - I_2$

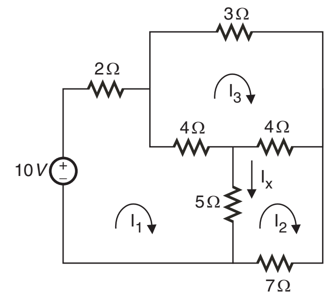


Fig. 2.

With reference to Fig. 2, the mesh basis matrix equation is,

$$\begin{bmatrix} 2+4+5 & -5 & -4 \\ -5 & 5+4+7 & -4 \\ -4 & -4 & 3+4+4 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 11 & -5 & -4 \\ -5 & 16 & -4 \\ -4 & -4 & 11 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \qquad \text{..... (1)}$$

$$\Delta = \begin{vmatrix} 11 & -5 & -4 \\ -5 & 16 & -4 \\ -4 & -4 & 11 \end{vmatrix} = 11 \times \left[(16 \times 11) - (-4)^2\right] - (-5) \times \left[(-5 \times 11) - (-4)^2\right] + (-4) \times \left[(-5 \times (-4)) - (-4 \times 16)\right]$$

$$= 1760 - 355 - 336 = 1069$$

$$\Delta_1 = \begin{vmatrix} 10 & -5 & -4 \\ 0 & 16 & -4 \\ 0 & -4 & 11 \end{vmatrix} = 10 \times \left[(16 \times 11) - (-4)^2\right] - 0 + 0 = 1600$$

$$\Delta_2 = \begin{vmatrix} 11 & 10 & -4 \\ -5 & 0 & -4 \\ -4 & 0 & 11 \end{vmatrix} = 0 - 10 \times \left[(-5 \times 11) - (-4)^2\right] + 0 = 710$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{1600}{1069} \quad ; \quad I_2 = \frac{\Delta_2}{\Delta} = \frac{710}{1069}$$

$$\therefore \ \text{The respone, } I_x = I_1 - I_2 = \frac{1600}{1069} - \frac{710}{1069} = \frac{1600-710}{1069} = \frac{890}{1069} = 0.8326\,A$$

Case ii : To prove the reciprocity theorem by interchanging the positions of source and response

Let us interchange the positions of source and response as shown in Fig. 3. Let us assume mesh currents I_a, I_b and I_c as shown in Fig. 3.

Now, the response, $I_x = I_a$.

With reference to Fig. 3, the mesh basis matrix equation is,

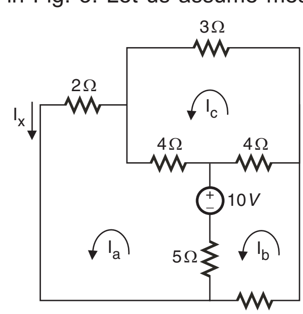


Fig. 3.

$$\begin{bmatrix} 2+4+5 & -5 & -4 \\ -5 & 5+4+7 & -4 \\ -4 & -4 & 3+4+4 \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 10 \\ -10 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 11 & -5 & -4 \\ -5 & 16 & -4 \\ -4 & -4 & 11 \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} = \begin{bmatrix} 10 \\ -10 \\ 0 \end{bmatrix} \qquad \text{..... (2)}$$

On comparing equations (2) and (1), we can say that the Δ is same in both the case.

$\therefore \quad \Delta = 1069$

$$\Delta_a = \begin{vmatrix} 10 & -5 & -4 \\ -10 & 16 & -4 \\ 0 & -4 & 11 \end{vmatrix} = 10\times[(16\times 11)-(-4)^2]-(-5)\times[(-10\times 11)-0] \\ -4\times[(-10\times(-4))-0]$$

$$= 1600 - 550 - 160 = 890$$

Now the response, $I_x = I_a = \dfrac{\Delta_a}{\Delta} = \dfrac{890}{1069} = 0.8326\,A$

Here, the response I_x remains the same after interchanging the positions of source and response. Hence, the reciprocity theorem is proved.

EXAMPLE 2.62

In the circuit of Fig. 1, calculate V_x. Prove the reciprocity theorem by interchanging the positions of the 12 *A* source and V_x.

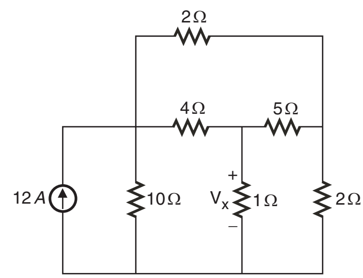


Fig. 1.

SOLUTION

Case i : To solve V_x in the given circuit

Let us assume three node voltages as shown in Fig. 2. Now the response, $V_x = V_2$. With reference to Fig. 2, the node basis matrix equation is,

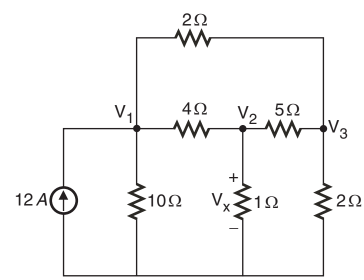


Fig. 2.

$$\begin{bmatrix} \frac{1}{4}+\frac{1}{10}+\frac{1}{2} & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4}+\frac{1}{1}+\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{2} & -\frac{1}{5} & \frac{1}{5}+\frac{1}{2}+\frac{1}{2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.85 & -0.25 & -0.5 \\ -0.25 & 1.45 & -0.2 \\ -0.5 & -0.2 & 1.2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} \quad \text{..... (1)}$$

$$\Delta' = \begin{vmatrix} 0.85 & -0.25 & -0.5 \\ -0.25 & 1.45 & -0.2 \\ -0.5 & -0.2 & 1.2 \end{vmatrix} = 0.85\times[(1.45\times 1.2)-(-0.2)^2]-(-0.25)\times[(-0.25\times 1.2)-(-0.5\times(-0.2))] \\ +(-0.5)\times[(-0.25\times(-0.2))-(-0.5\times 1.45)]$$

$$= 1.445 - 0.1 - 0.3875$$

$$= 0.9575$$

$$\Delta'_2 = \begin{vmatrix} 0.85 & 12 & -0.5 \\ -0.25 & 0 & -0.2 \\ -0.5 & 0 & 1.2 \end{vmatrix} = 0 - 12\times[(-0.25\times 1.2)-(-0.5\times(-0.2))]+0$$

$$= 4.8$$

$\therefore$ The response, $V_x = V_2 = \dfrac{\Delta'_2}{\Delta'} = \dfrac{4.8}{0.9575} = 5.0131\,V$

Case ii : To prove the reciprocity theorem by interchanging the positions of source and response

Let us interchange the positions of source and response as shown in Fig. 3. Let us assume node voltages as shown in Fig. 3. Now the response, $V_x = V_a$. With reference to Fig. 3, the node basis matrix equation is,

$$\begin{bmatrix} \frac{1}{4}+\frac{1}{10}+\frac{1}{2} & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4}+\frac{1}{1}+\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{2} & -\frac{1}{5} & \frac{1}{5}+\frac{1}{2}+\frac{1}{2} \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$$

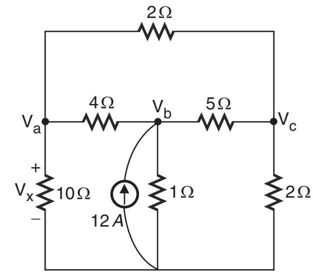


Fig. 3.

$$\begin{bmatrix} 0.85 & -0.25 & -0.5 \\ -0.25 & 1.45 & -0.2 \\ -0.5 & -0.2 & 1.2 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix} \qquad \text{..... (2)}$$

On comparing equations (1) and (2), we can say that the Δ' remains the same.

$$\therefore \quad \Delta' = 0.9575$$

$$\Delta'_a = \begin{vmatrix} 0 & -0.25 & -0.5 \\ 12 & 1.45 & -0.2 \\ 0 & -0.2 & 1.2 \end{vmatrix} \begin{aligned} &= 0-(-0.25)\times[12\times1.2-0]+(-0.5)\times[12\times(-0.2)-0] \\ &= 3.6+1.2 = 4.8 \end{aligned}$$

$$\therefore \quad \text{The response,} \quad V_x = V_a = \frac{\Delta'_a}{\Delta'} = \frac{4.8}{0.9575} = 5.0131\,V$$

Here, the response V_x remains the same after interchanging the positions of source and response. Hence, the reciprocity theorem is proved.

EXAMPLE 2.63

In the circuit of Fig. 1, compute $\bar{I}_x$. Demonstrate the reciprocity theorem by interchanging the positions of the source and $\bar{I}_x$.

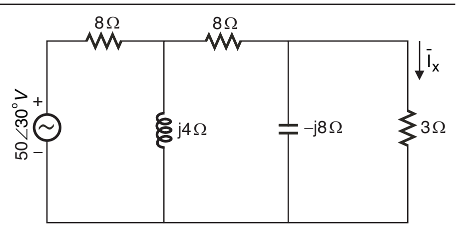


Fig. 1.

SOLUTION

Case i : To solve $\bar{I}_x$ in the given circuit

Let us assume three mesh currents $\bar{I}_1, \bar{I}_2$ and $\bar{I}_3$ as shown in Fig. 2. Now, the response, $\bar{I}_x = \bar{I}_3$.

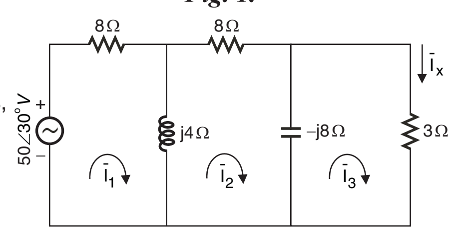


Fig. 2.

With reference to Fig. 2, the mesh basis matrix equation is,

$$\begin{bmatrix} 8+j4 & -j4 & 0 \\ -j4 & 8+j4-j8 & -(-j8) \\ 0 & -(-j8) & 3-j8 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} 50\angle 30^\circ \\ 0 \\ 0 \end{bmatrix}$$

Here, $50\angle 30^\circ = 50\cos 30^\circ + j50\sin 30^\circ = 43.3013 + j25\,V$

$$\begin{bmatrix} 8+j4 & -j4 & 0 \\ -j4 & 8-j4 & j8 \\ 0 & j8 & 3-j8 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} = \begin{bmatrix} 43.3013+j25 \\ 0 \\ 0 \end{bmatrix} \qquad \text{.....(1)}$$

$$\Delta = \begin{vmatrix} 8+j4 & -j4 & 0 \\ -j4 & 8-j4 & j8 \\ 0 & j8 & 3-j8 \end{vmatrix} \begin{aligned} &= (8+j4)\times[(8-j4)\times(3-j8)-(j8)^2]-(-j4)\times[-j4\times(3-j8)-0]+0 \\ &= 752-j384+48-j128 \\ &= 800-j512 \end{aligned}$$

$$\Delta_3 = \begin{vmatrix} 8+j4 & -j4 & 43.3013+j25 \\ -j4 & 8-j4 & 0 \\ 0 & j8 & 0 \end{vmatrix} \begin{aligned} &= 0-0+(43.3013+j25)\times[-j4\times j8-0] \\ &= 1385.6416+j800 \end{aligned}$$

$$\therefore \text{ The response, } \bar{I}_x = \bar{I}_3 = \frac{\Delta_3}{\Delta} = \frac{1385.6416+j800}{800-j512}$$
$$= 0.7747 + j1.4958\ A$$
$$= 1.6845\angle 62.6^\circ\ A$$

Case ii : To demonstrate the reciprocity theorem by interchanging the positions of source and response

Let us interchange the position of source and response as shown in Fig. 3. Let us assume mesh currents as shown in Fig. 3.

Now, the response, $\bar{I}_x = \bar{I}_a$.

With reference to Fig. 3, the mesh basis matrix equation is,

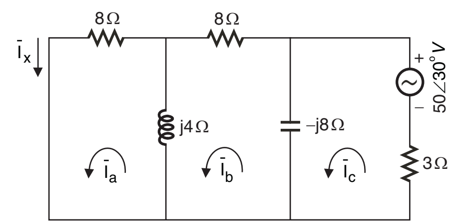


Fig. 3.

$$\begin{bmatrix} 8+j4 & -j4 & 0 \\ -j4 & 8+j4-j8 & -(-j8) \\ 0 & -(-j8) & 3-j8 \end{bmatrix} \begin{bmatrix} \bar{I}_a \\ \bar{I}_b \\ \bar{I}_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 50\angle 30^\circ \end{bmatrix}$$

$$\begin{bmatrix} 8+j4 & -j4 & 0 \\ -j4 & 8-j4 & j8 \\ 0 & j8 & 3-j8 \end{bmatrix} \begin{bmatrix} \bar{I}_a \\ \bar{I}_b \\ \bar{I}_c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 43.3013+j25 \end{bmatrix} \quad \text{..... (2)}$$

On comparing equations (1) and (2), we can say that the value of Δ remains the same in both the cases.

$\therefore\ \Delta = 800 - j512$

$$\Delta_a = \begin{vmatrix} 0 & -j4 & 0 \\ 0 & 8-j4 & j8 \\ 43.3013+j25 & j8 & 3-j8 \end{vmatrix} \begin{aligned} &= 0-(-j4)\times[0-(43.3013+j25)\times j8]+0 \\ &= 1385.6416+j800 \end{aligned}$$

$$\therefore \text{ The response, } \bar{I}_x = \bar{I}_a = \frac{\Delta_a}{\Delta} = \frac{1385.6416+j800}{800-j512}$$
$$= 0.7747 + j1.4958\ A$$
$$= 1.6845\angle 62.6^\circ\ A$$

It is observed that the response remains the same after interchanging the positions of source and response, which demonstrates the validity of the reciprocity theorem.

EXAMPLE 2.64

In the circuit of Fig. 1, compute $\bar{V}_x$. Demonstrate the reciprocity theorem by interchanging the positions of the source and response.

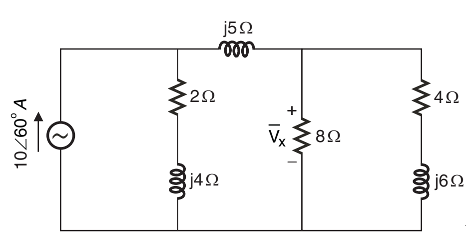


Fig. 1.

SOLUTION

Case i : To solve $\bar{V}_x$ in the given circuit

Let us assume two node voltages $\bar{V}_1$ and $\bar{V}_2$ as shown in Fig. 2. Now, the response, $\bar{V}_x = \bar{V}_2$.

With reference to Fig. 2, the node basis matrix equation is,

$$\begin{bmatrix} \frac{1}{2+j4}+\frac{1}{j5} & -\frac{1}{j5} \\ -\frac{1}{j5} & \frac{1}{j5}+\frac{1}{8}+\frac{1}{4+j6} \end{bmatrix} \begin{bmatrix} \bar{V}_1 \\ \bar{V}_2 \end{bmatrix} = \begin{bmatrix} 10\angle 60^\circ \\ 0 \end{bmatrix}$$

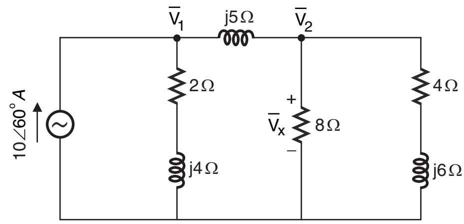


Fig. 2.

Here, $10\angle 60^\circ = 10\cos 60^\circ + j10\sin 60^\circ = 5 + j8.6603\ A$

$$\begin{bmatrix} 0.1-j0.4 & j0.2 \\ j0.2 & 0.2019-j0.3154 \end{bmatrix} \begin{bmatrix} \bar{V}_1 \\ \bar{V}_2 \end{bmatrix} = \begin{bmatrix} 5+j8.6603 \\ 0 \end{bmatrix} \quad \text{..... (1)}$$

$$\Delta' = \begin{vmatrix} 0.1-j0.4 & j0.2 \\ j0.2 & 0.2019-j0.3154 \end{vmatrix} = \left[(0.1-j0.4)\times(0.2019-j0.3154)\right]-(j0.2)^2 = -0.06597-j0.1123$$

$$\Delta'_2 = \begin{vmatrix} 0.1-j0.4 & 5+j8.6603 \\ j0.2 & 0 \end{vmatrix} = 0-\left[j0.2\times(5+j8.6603)\right] = 1.73206-j$$

$$\therefore \text{ The response, } \bar{V}_x = \bar{V}_2 = \frac{\Delta'_2}{\Delta'} = \frac{1.73206-j}{-0.06597-j0.1123} = -0.1158+j15.3555\ V = 15.3559\angle 90.4^\circ\ V$$

Case ii : To demonstrate the reciprocity theorem by interchanging the positions of source and response

Let us interchange the positions of source and response as shown in Fig. 3. Let us assume node voltages $\bar{V}_a$ and $\bar{V}_b$ as shown in Fig. 3.

Now, the response, $\bar{V}_x = \bar{V}_a$

With reference to Fig. 3, the node basis matrix equation is,

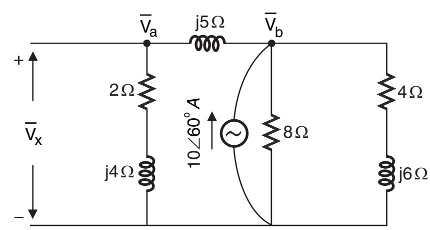


Fig. 3.

$$\begin{bmatrix} \frac{1}{2+j4}+\frac{1}{j5} & -\frac{1}{j5} \\ -\frac{1}{j5} & \frac{1}{j5}+\frac{1}{8}+\frac{1}{4+j6} \end{bmatrix} \begin{bmatrix} \bar{V}_a \\ \bar{V}_b \end{bmatrix} = \begin{bmatrix} 0 \\ 10\angle 60^\circ \end{bmatrix}$$

$$\begin{bmatrix} 0.1-j0.4 & j0.2 \\ j0.2 & 0.2019-j0.3154 \end{bmatrix} \begin{bmatrix} \bar{V}_a \\ \bar{V}_b \end{bmatrix} = \begin{bmatrix} 0 \\ 5+j8.6603 \end{bmatrix} \quad \text{..... (2)}$$

On comparing equations (1) and (2), we can say that the value of Δ' remains the same.

$$\therefore \Delta' = -0.06597 - j0.1123$$

$$\Delta'_a = \begin{vmatrix} 0 & j0.2 \\ 5+j8.6603 & 0.2019-j0.3154 \end{vmatrix} = 0-\left[(5+j8.6603)\times j0.2\right] = 1.73206-j$$

$$\therefore \text{ The response, } \bar{V}_x = \bar{V}_a = \frac{\Delta'_a}{\Delta'} = \frac{1.73206-j}{-0.06597-j0.1123} = -0.1158+j15.3555\ V = 15.3559\angle 90.4^\circ\ V$$

It is observed that the response remains the same after interchanging the positions of source and response, which demonstrates the validity of the reciprocity theorem.

2.6.7 Millman's Theorem

Millman's theorem will be useful to combine a number of voltage sources in parallel into a single equivalent source.

> **Millman's theorem states that if n number of voltage sources with internal impedance are in parallel then they can be combined to give a single voltage source with an equivalent emf and internal impedance.**

Consider n number of parallel connected voltage sources with internal impedance in series with an ideal source as shown in Fig. 2.67. Now by Millman's theorem, the voltage sources in parallel can be converted into a single source as shown in Fig. 2.68.

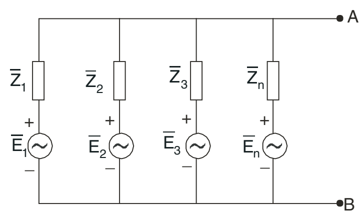


Fig 2.67 : *Voltage source in parallel.*

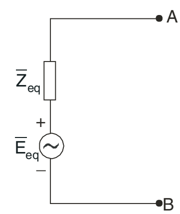


Fig 2.68 : *Millman's equivalent voltage source.*

Here, $\overline{E}_1, \overline{E}_2, \overline{E}_3 \ldots \overline{E}_n$ = Emf of voltage sources in parallel

$\overline{Z}_1, \overline{Z}_2, \overline{Z}_3 \ldots \overline{Z}_n$ = Internal impedance of voltage sources.

Let, $\overline{E}_{eq}$ = Emf of equivalent voltage source

$\overline{Z}_{eq}$ = Internal impedance of equivalent voltage source.

Now, by Millman's theorem,

$$\overline{E}_{eq} = \left(\frac{\overline{E}_1}{\overline{Z}_1} + \frac{\overline{E}_2}{\overline{Z}_2} + \frac{\overline{E}_3}{\overline{Z}_3} + \ldots + \frac{\overline{E}_n}{\overline{Z}_n}\right)\overline{Z}_{eq} \quad \text{..... (2.98)}$$

$$\overline{Z}_{eq} = \frac{1}{\dfrac{1}{\overline{Z}_1} + \dfrac{1}{\overline{Z}_2} + \dfrac{1}{\overline{Z}_3} + \ldots + \dfrac{1}{\overline{Z}_n}} \quad \text{..... (2.99)}$$

Since the admittance, $\overline{Y} = \frac{1}{\overline{Z}}$, equations (2.98) and (2.99) can be written in terms of admittance as shown below:

$$\overline{E}_{eq} = \frac{\overline{E}_1\overline{Y}_1 + \overline{E}_2\overline{Y}_2 + \overline{E}_3\overline{Y}_3 + \ldots + \overline{E}_n\overline{Y}_n}{\overline{Y}_{eq}} \quad \text{..... (2.100)}$$

$$\overline{Y}_{eq} = \overline{Y}_1 + \overline{Y}_2 + \overline{Y}_3 + \ldots + \overline{Y}_n \quad \text{..... (2.101)}$$

where, $\overline{Y}_{eq} = \frac{1}{\overline{Z}_{eq}}$; $\overline{Y}_1 = \frac{1}{\overline{Z}_1}$; $\overline{Y}_2 = \frac{1}{\overline{Z}_2}$; $\overline{Y}_3 = \frac{1}{\overline{Z}_3}$ and so on.

In case of dc sources with internal resistance, the impedance will become resistance and admittance will become conductance. Hence, equations (2.98) to (2.101) can be expressed as shown ahead for parallel connected dc sources.

$$E_{eq} = \left(\frac{E_1}{R_1} + \frac{E_2}{R_2} + \frac{E_3}{R_3} + \ldots + \frac{E_n}{R_n}\right) R_{eq} \qquad \text{..... (2.102)}$$

$$R_{eq} = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \ldots + \dfrac{1}{R_n}} \qquad \text{..... (2.103)}$$

$$E_{eq} = \frac{E_1 G_1 + E_2 G_2 + E_3 G_3 + \ldots + E_n G_n}{G_{eq}} \qquad \text{..... (2.104)}$$

$$G_{eq} = G_1 + G_2 + G_3 + \ldots + G_n \qquad \text{..... (2.105)}$$

Proof :

The voltage sources in Fig. 2.69 can be converted into current sources as shown in Fig. 2.70.

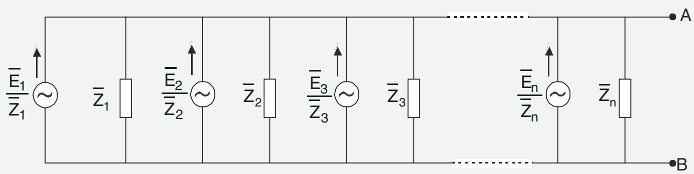


Fig 2.69.

$\Downarrow$

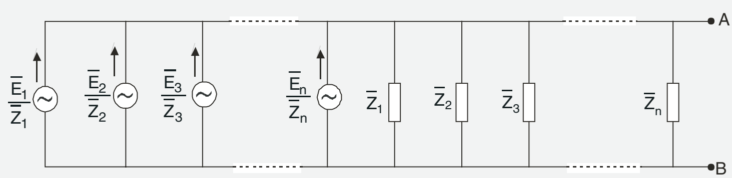


Fig 2.70.

The parallel current sources in Fig. 2.70 can be added to give a single equivalent current source $\overline{I}_{eq}$. The parallel impedances in Fig. 2.70 can be combined to give a single equivalent impedance $\overline{Z}_{eq}$.

Here,

$$\overline{I}_{eq} = \frac{\overline{E}_1}{\overline{Z}_1} + \frac{\overline{E}_2}{\overline{Z}_2} + \frac{\overline{E}_3}{\overline{Z}_3} + \ldots + \frac{\overline{E}_n}{\overline{Z}_n} \qquad \text{..... (2.106)}$$

$$\overline{Z}_{eq} = \frac{1}{\dfrac{1}{\overline{Z}_1} + \dfrac{1}{\overline{Z}_2} + \dfrac{1}{\overline{Z}_3} + \ldots + \dfrac{1}{\overline{Z}_n}} \qquad \text{..... (2.107)}$$

Therefore, the parallel connected current sources in Fig. 2.70 can be represented as shown in Fig. 2.71.

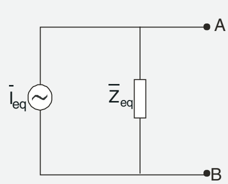


Fig 2.71.

$\Rightarrow$

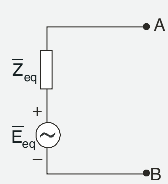


Fig 2.72.

Again, by source transformation technique, the current source $\overline{I}_{eq}$ in parallel with $\overline{Z}_{eq}$ can be converted into a voltage source in series with $\overline{Z}_{eq}$ as shown in Fig. 2.72. Here the voltage source in series with $\overline{Z}_{eq}$ is the Millman's equivalent source of the parallel connected voltage sources of Fig. 2.67.

Here,

$$\overline{Z}_{eq} = \frac{1}{\dfrac{1}{\overline{Z}_1} + \dfrac{1}{\overline{Z}_2} + \dfrac{1}{\overline{Z}_3} + ... + \dfrac{1}{\overline{Z}_n}} \quad ..... (2.108)$$

$$\overline{E}_{eq} = \overline{I}_{eq}\,\overline{Z}_{eq} \quad ..... (2.109)$$

On substituting for $\overline{I}_{eq}$ from equation (2.106), we get,

$$\overline{E}_{eq} = \left(\frac{\overline{E}_1}{\overline{Z}_1} + \frac{\overline{E}_2}{\overline{Z}_2} + \frac{\overline{E}_3}{\overline{Z}_3} + ... + \frac{\overline{E}_n}{\overline{Z}_n}\right)\overline{Z}_{eq} \quad ..... (2.110)$$

EXAMPLE 2.65

In the circuit of Fig. 1, use Millman's theorem to find current through the 4 Ω resistance.

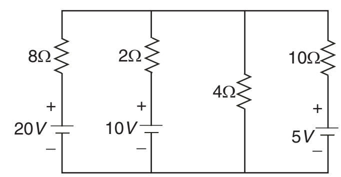


Fig 1.

SOLUTION

The given circuit can be redrawn as shown in Fig. 2. In the circuit of Fig. 2 each voltage source has a series resistance which can be considered as internal resistance of the source. Hence, the parallel connected voltage sources with internal resistance can be converted into a single equivalent source using Millman's theorem.

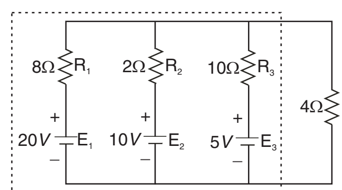


Fig 2.

Let, E_{eq} = Equivalent emf of parallel connected sources

R_{eq} = Equivalent internal resistance.

Now, by Millman's theorem,

$$R_{eq} = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}} = \frac{1}{\dfrac{1}{8} + \dfrac{1}{2} + \dfrac{1}{10}} = \frac{1}{0.725} = 1.3793\,\Omega$$

$$E_{eq} = \left(\frac{E_1}{R_1} + \frac{E_2}{R_2} + \frac{E_3}{R_3}\right)R_{eq}$$

$$= \left(\frac{20}{8} + \frac{10}{2} + \frac{5}{10}\right) \times 1.3793 = 11.0344\,V$$

The circuit of Fig. 2 can be redrawn as shown in Fig. 3. Let, I be the current through 4 Ω resistance. With reference to Fig. 3, by Ohm's law we can write,

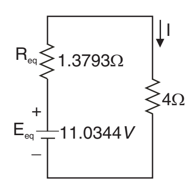


Fig 3.

$$I = \frac{E_{eq}}{R_{eq} + 4} = \frac{11.0344}{1.3793 + 4} = 2.0513\,A$$

RESULT

Current through 4 Ω resistance = 2.0513 *A*

EXAMPLE 2.66

In the circuit of Fig. 1, determine V_0 using Millman's theorem.

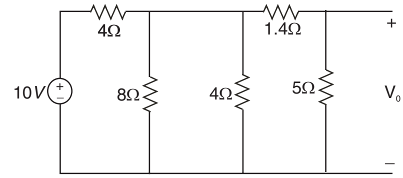


Fig 1.

SOLUTION

In the given circuit the parallel branches with 8 Ω and 4 Ω resistances can be assumed to have a zero value voltage source as shown in Fig. 2. In the circuit of Fig. 2, each source has a series resistance, which can be considered as internal resistance of the source. Therefore, the parallel connected voltage sources with internal resistance can be converted into a single equivalent source using Millman's theorem.

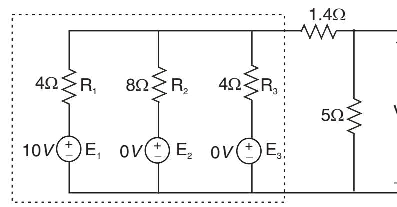


Fig 2.

Let, E_{eq} = Equivalent emf of parallel connected source

R_{eq} = Equivalent internal resistance.

Now, by Millman's theorem,

$$R_{eq} = \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}} = \frac{1}{\frac{1}{4}+\frac{1}{8}+\frac{1}{4}} = \frac{1}{0.625} = 1.6\,\Omega$$

$$E_{eq} = \left(\frac{E_1}{R_1}+\frac{E_2}{R_2}+\frac{E_3}{R_3}\right)R_{eq} = \left(\frac{10}{4}+\frac{0}{8}+\frac{0}{4}\right)\times 1.6 = 4\,V$$

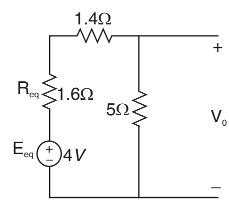


Fig 3.

The circuit of Fig. 2, can be redrawn as shown in Fig. 3. With reference to Fig. 3, by voltage division rule, we can write,

$$V_0 = E_{eq}\times\frac{5}{5+(1.6+1.4)} = 4\times\frac{5}{8} = 2.5\,V$$

EXAMPLE 2.67

In the circuit of Fig. 1, apply Millman's theorem to find Thevenin's equivalent at A-B. Hence, find $\bar{Z}_L$ for maximum power transfer.

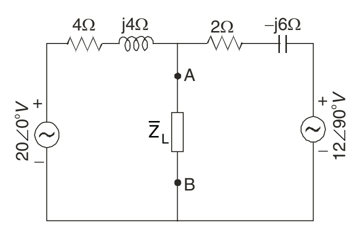


Fig 1.

SOLUTION

Let us remove $\bar{Z}_L$ and redraw the circuit of Fig. 1 as shown in Fig. 2. In the circuit of Fig. 2, each voltage source has a series impedance which can be considered as internal impedance of the source.

Therefore, the parallel connected voltage sources with internal impedance can be converted into a single equivalent source using Millman's theorem.

Let, $\bar{E}_{eq}$ = Equivalent emf of parallel connected sources

$\bar{Z}_{eq}$ = Equivalent internal impedance.

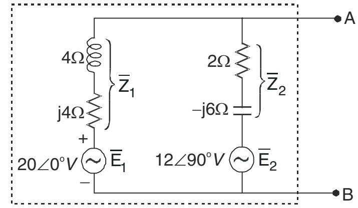


Fig 2.

Now by Millman's theorem,

$$\bar{Z}_{eq} = \frac{1}{\frac{1}{\bar{Z}_1}+\frac{1}{\bar{Z}_2}} = \frac{1}{\frac{1}{4+j4}+\frac{1}{2-j6}}$$

$$= \left[(4+j4)^{-1}+(2-j6)^{-1}\right]^{-1} = 5.6-j0.8\,\Omega$$

$$\bar{E}_{eq} = \left(\frac{\bar{E}_1}{\bar{Z}_1}+\frac{\bar{E}_2}{\bar{Z}_2}\right)\bar{Z}_{eq} = \left(\frac{20\angle 0^\circ}{4+j4}+\frac{12\angle 90^\circ}{2-j6}\right)\times(5.6-j0.8)$$

$$= \left(\frac{20}{4+j4}+\frac{j12}{2-j6}\right)\times(5.6-j0.8) = (0.7-j1.9)\times(5.6-j0.8)$$

$$= 2.4-j11.2 = 11.4543\angle -77.9^\circ\ V$$

Now, the parallel connected sources in Fig. 2 can be represented as shown in Fig. 3 by Millman's theorem.

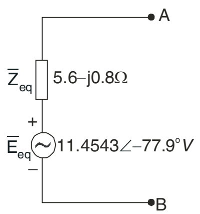


Fig 3.

⇒

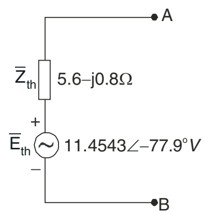


Fig 4 : Thevenins's equivalent at A-B.

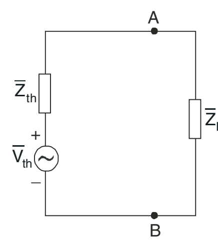


Fig 5.

Here, $\bar{E}_{eq} = \bar{V}_{th}$; $\bar{Z}_{eq} = \bar{Z}_{th}$

$$\therefore\ \bar{V}_{th} = 2.4-j11.2\,V = 11.4543\angle -77.9^\circ\ V$$

$$\bar{Z}_{th} = 5.6-j0.8\,\Omega$$

The Thevenin's equivalent of the given circuit at terminals A-B is shown in Fig. 4. Let us connect the load impedance $\bar{Z}_L$ at terminals A-B of Thevenin's equivalent as shown in Fig. 5. Now, by maximum power transfer theorem, for maximum power transfer to $\bar{Z}_L$, the value of $\bar{Z}_L$ should be conjugate of $\bar{Z}_{th}$.

$$\therefore\ \bar{Z}_L = \bar{Z}_{th}^* = (5.6-j0.8)^* = 5.6+j0.8\,\Omega$$

2.7 Summary of Important Concepts

1. Resistances in series can be replaced with an equivalent resistance whose value is given by the sum of individual resistances.

2. When n number of identical resistances of value R are connected in series, they can be replaced with a single equivalent resistance of value nR.

3. **Voltage division rule**: When a voltage V exists across a series combination of two resistances R_1 and R_2, the voltages V_1 across R_1 and V_2 across R_2 are given by,

$$V_1 = V\times\frac{R_1}{R_1+R_2} \quad ; \quad V_2 = V\times\frac{R_2}{R_1+R_2}$$

Similarly, voltages in two impedances $\bar{Z}_1$ and $\bar{Z}_2$ in series are,

$$\bar{V}_1 = \bar{V}\times\frac{\bar{Z}_1}{\bar{Z}_1+\bar{Z}_2} \quad ; \quad \bar{V}_2 = \bar{V}\times\frac{\bar{Z}_2}{\bar{Z}_1+\bar{Z}_2}$$

4. Resistances in parallel can be replaced with an equivalent resistance whose value is given by the inverse of sum of the inverse of individual resistances.

5. When n number of identical resistances of value R are connected in series, they can be replaced with a single equivalent resistance of value R/n.

6. **Current division rule** : When a total current I flows through a parallel combination of two resistances R_1 and R_2, the currents I_1 through R_1 and I_2 through R_2 are given by,

$$I_1 = I \times \frac{R_2}{R_1 + R_2} \quad ; \quad I_2 = I \times \frac{R_1}{R_1 + R_2}$$

Similarly, currents in two impedances $\bar{Z}_1$ and $\bar{Z}_2$ in parallel are,

$$\bar{I}_1 = \bar{I} \times \frac{\bar{Z}_2}{\bar{Z}_1 + \bar{Z}_2} \quad ; \quad \bar{I}_2 = \bar{I} \times \frac{\bar{Z}_1}{\bar{Z}_1 + \bar{Z}_2}$$

7. When three resistances R_{12}, R_{23} and R_{31} are in delta-connection with respect to terminals 1, 2 and 3, their equivalent star-connected resistances R_1, R_2 and R_3 with respect to the same terminals are given by,

$$R_1 = \frac{R_{12} R_{31}}{R_{12} + R_{23} + R_{31}} \quad ; \quad R_2 = \frac{R_{12} R_{23}}{R_{12} + R_{23} + R_{31}} \quad ; \quad R_3 = \frac{R_{23} R_{31}}{R_{12} + R_{23} + R_{31}}$$

Similarly, the star equivalent of delta-connected impedances are given by,

$$\bar{Z}_1 = \frac{\bar{Z}_{12}\,\bar{Z}_{31}}{\bar{Z}_{12} + \bar{Z}_{23} + \bar{Z}_{31}} \quad ; \quad \bar{Z}_2 = \frac{\bar{Z}_{12}\,\bar{Z}_{23}}{\bar{Z}_{12} + \bar{Z}_{23} + \bar{Z}_{31}} \quad ; \quad \bar{Z}_3 = \frac{\bar{Z}_{23}\,\bar{Z}_{31}}{\bar{Z}_{12} + \bar{Z}_{23} + \bar{Z}_{31}}$$

8. When three equal resistances of value R are in delta-connection, their equivalent star-connected resistances will consist of three equal resistances of value R/3.

9. When three resistances R_1, R_2, and R_3 are in star-connection with respect to terminals 1, 2 and 3, their equivalent delta-connected resistances R_{12}, R_{23} and R_{31} with respect to same terminals are given by,

$$R_{12} = R_1 + R_2 + \frac{R_1 R_2}{R_3} \quad ; \quad R_{23} = R_2 + R_3 + \frac{R_2 R_3}{R_1} \quad ; \quad R_{31} = R_3 + R_1 + \frac{R_3 R_1}{R_2}$$

Similarly, the delta equivalent of star-connected impedances are given by,

$$\bar{Z}_{12} = \bar{Z}_1 + \bar{Z}_2 + \frac{\bar{Z}_1 \bar{Z}_2}{\bar{Z}_3} \quad ; \quad \bar{Z}_{23} = \bar{Z}_2 + \bar{Z}_3 + \frac{\bar{Z}_2 \bar{Z}_3}{\bar{Z}_1} \quad ; \quad \bar{Z}_{31} = \bar{Z}_3 + \bar{Z}_1 + \frac{\bar{Z}_3 \bar{Z}_1}{\bar{Z}_2}$$

10. When three equal resistances of value R are in star-connection, their equivalent delta-connected resistances will consist of three equal reisistances of value 3R.

11. A voltage source E with a resistance R_S in series can be converted into current source I_S, (where $I_S = E/R_S$) with the resistance R_S in parallel.

12. A current source I_S with a resistance R_S in parallel can be converted into voltage source E, (where $E = I_S R_S$) with the resistance R_S in series.

13. Sources are connected in series for higher voltage ratings and connected in parallel for higher current ratings.

14. In group-1 parameters (resistance / inductance / impedance / reactance), the series combination of parameters can be replaced with an equivalent parameter whose value is given by the sum of individual parameters.

15. In group-1 parameters (resistance / inductance / impedance / reactance), the parallel combination of parameters can be replaced with an equivalent parameter whose value is given by the inverse of sum of the inverse of individual parameters.
16. In group-2 parameters (conductance / capacitance / admittance / susceptance), the series combination of parameters can be replaced by with equavalent parameter whose value is given by the inverse of sum of the inverse of individual parameters.
17. In group-2 parameters (conductance / capacitance / admittance / susceptance), the parallel combination of parameters can be replaced with an equivalent parameter whose value is given by the sum of individual parameters.
18. Thevenin's theorem states that a circuit with two terminals can be replaced with an equivalent circuit consisting of a voltage source in series with a resitance (or impedance).
19. Thevenin's voltage is given by the voltage across the two open terminals of a circuit.
20. Thevenin's impedance is given by looking back impedance at the two open terminals of a network.
21. The looking back impedance is the impedance measured at the two open terminals of a circuit after replacing all the sources by zero value sources.
22. Norton's theorem states that a circuit with two terminals can be replaced with an equivalent circuit consisting of a current source in parallel with a resistance (or impedance).
23. Norton's impedance is given by looking back impedance at the two open terminals of a network.
24. Thevenin's equivalent is the voltage source model and Norton's equivalent is the current source model of a circuit.
25. Thevenin's and Norton's impedances are the same and given by the ratio of Thevenin's voltage and Norton's current.
26. The superposition theorem states that the response in a circuit with multiple sources is given by the algebraic sum of responses due to individual sources acting alone.
27. The superposition theorem is also referred to as the principle of superposition.
28. A circuit element is said to be linear, if the voltage-current relationship is linear.
29. The principle of superposition is a combination of additivity property and homogeneity property.
30. The property of additivity says that the response in a circuit due to a number of sources is given by the sum of the responses due to individual sources acting alone.
31. The property of homogeneity says that if all the sources are mutiplied by a constant then the respones are also multiplied by the same constant.
32. While calculating the response due to one source, all other sources are made inactive or replaced by zero value sources.
33. A zero value source is represented by its internal impedance.
34. For an ideal voltage source, the internal impedance is zero and for an ideal current source, the internal impedance is infinite.

35. While calculating the response due to one source, all other ideal voltage sources are replaced with a short circuit and all other ideal current sources are replaced with an open circuit.
36. The maximum power transfer to load is possible only if the source and load has matched impedance.
37. In a dc source connected to resistive load, the maximum power transfer theorem states that maximum power is transferred from the source to load, when the load resistance is equal to the source resistance.
38. In an ac source connected to reactive load, where resistance and reactance are independently variable, the maximum power transfer theorem states that maximum power is transferred from the source to load, when the load impedance is equal to the complex conjugate of source impedance.
39. In general, the maximum power transfer theorem states that maximum power is transferred to a load impedance if the absolute value of the load impedance is equal to the absolute value of the looking back impedance of the circuit from the terminals of the load.
40. The reciprocity theorem states that in a linear, bilateral, single source circuit, the ratio of excitation to response is constant when the positions of excitation and response are interchanged.
41. The networks which satisfy the reciprocity theorem are called reciprocal networks.
42. The reciprocity theorem will be satisfied only by circuits or networks which do not have dependent sources.

2.8 Short-answer Questions

(AU Dec'15, '16, 2 Marks)

Q2.1 ***Determine the currents I_1 and I_2 in the circuit shown in Fig. Q2.1.***

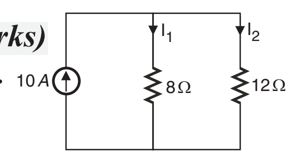


Fig. Q2.1.

Solution

By current division rule,

$$I_1 = 10 \times \frac{12}{8+12} = 6\ A$$

$$I_2 = 10 \times \frac{8}{8+12} = 4\ A$$

Q2.2 ***Determine the voltages V_1 and V_2 in the circuit shown in Fig. Q2.2.***

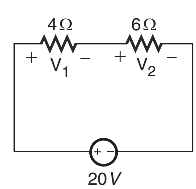


Fig. Q2.2.

Solution

By voltage division rule,

$$V_1 = 20 \times \frac{4}{4+6} = 8\,V$$

$$V_2 = 20 \times \frac{6}{4+6} = 12\,V$$

Q2.3 ***Determine the resistance across A-B in the circuit shown in Fig. Q2.3.1.***

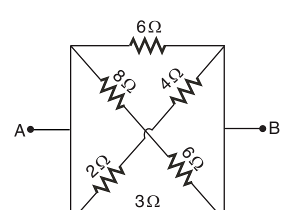


Fig. Q2.3.1.

Solution

The given network can be redrawn as shown in Fig. Q2.3.2.

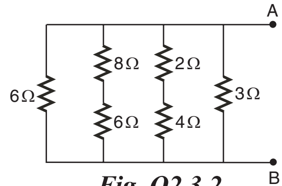


Fig. Q2.3.2.

With reference to Fig. Q2.3.2, we get,

$$R_{AB} = \frac{1}{\frac{1}{6}+\frac{1}{8+6}+\frac{1}{2+4}+\frac{1}{3}} = 1.3548\,\Omega$$

Q2.4 ***Determine the resistance across A-B in the circuit shown in Fig. Q2.4.1.*** ***(AU Dec'14, 2 Marks)***

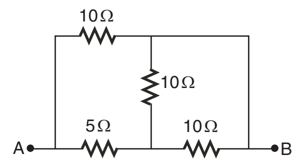


Fig. Q2.4.1.

⇓

Solution

The given network can be redrawn as shown in Fig. Q2.4.2

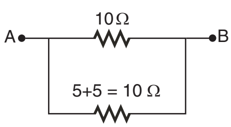


Fig. Q2.4.4.

⇐

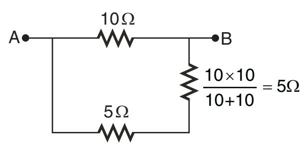


Fig. Q2.4.3.

⇐

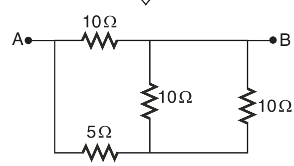


Fig. Q2.4.2.

With reference to Fig. Q2.4.4, we get,

$$R_{AB} = \frac{10}{2} = 5\,\Omega$$

Q2.5 ***The equivalent resistance of four resistors joined in parallel is 30 Ω. The current flowing through them are 0.5, 0.4, 0.6 and 0.1 A. Find the value of each resistor.***

(AU Dec'16, 2 Marks)

Solution

Let R_1, R_2, R_3 and R_4 be the resistances in parallel.

Let, R_T = Equivalent resistance of parallel combination.

Given that, R_T = 30 Ω

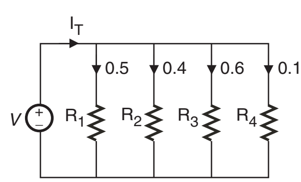


Fig. Q2.5.1.

⇒

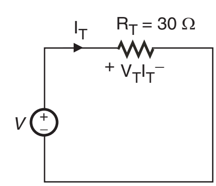


Fig. Q2.5.2.

With reference to Fig. Q2.5.2

$$I_T = 0.5 + 0.4 + 0.6 + 0.1 = 1.6\ A$$

$$V = I_T R_T = 1.6 \times 30 = 48\ V$$

Now, by Ohm's law,

$$R_1 = \frac{V}{0.5} = \frac{48}{0.5} = 96\,\Omega \ ; \ R_2 = \frac{V}{0.4} = \frac{48}{0.4} = 120\,\Omega$$

$$R_3 = \frac{V}{0.6} = \frac{48}{0.6} = 80\,\Omega \ ; \ R_4 = \frac{V}{0.1} = \frac{48}{0.1} = 480\,\Omega$$

Q2.6 ***Determine the resistance across A-B in the circuit shown in Fig. Q2.6.1.*** ***(AU June'14, 2 Marks)***

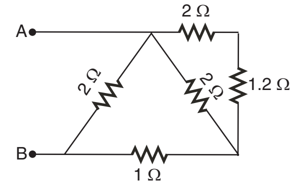


Fig. Q2.6.1.

Solution

The given network can be redrawn as shown in Fig. Q2.6.4.

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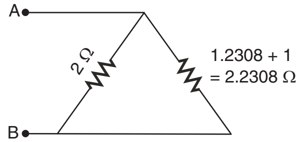


Fig. Q2.6.4.

⇐

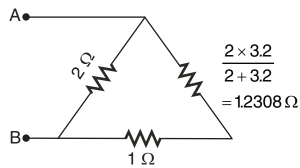


Fig. Q2.6.3.

⇐

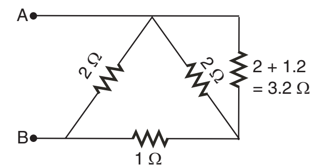


Fig. Q2.6.2.

With reference to Fig. Q2.6.4, we get,

$$R_{AB} = \frac{2 \times 2.2308}{2 + 2.2308} = 1.0546\ \Omega$$

Q2.7 ***Determine the resistance across A-B in the circuit shown in Fig. Q2.7.*** ***(AU Dec'15, 2 Marks)***

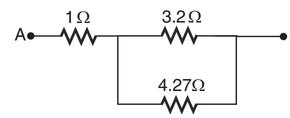


Fig. Q2.7.

Solution

In the given network, 3.2 Ω and 4.27 Ω resistances are in parallel and the parallel combination is in series with 1 Ω resistance.

$$\therefore R_{AB} = 1 + \frac{3.2 \times 4.27}{3.2 + 4.27} = 1 + 1.829 = 2.829\ \Omega$$

Q2.8 ***Determine the value of R in the circuit shown in Fig Q2.8.1.***

Solution ***(AU May'17, 2 Marks)***

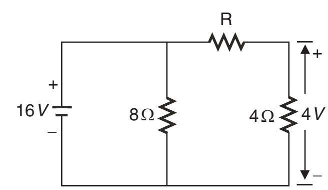


Fig. Q2.8.1.

The voltage and current in the various resistances are shown in Fig. 2.8.1

With reference to Fig. Q2.8.2, by KVL,

$$V_8 = 16\ V$$

$$V_8 = V_R + 4$$

On equating equations (1) and (2), we get,

$$V_R + 4 = 16 \qquad .....(1)$$

$$V_R = 16 - 4 = 12\ V \qquad .....(2)$$

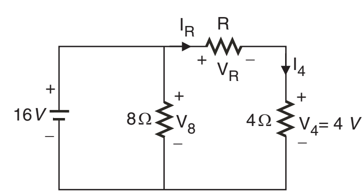


Fig. Q2.8.2.

With reference to Fig. Q2.8.2, by Ohm's law,

$$I_4 = \frac{V_4}{4} = \frac{4}{4} = 1\ A$$

$$\therefore R = \frac{V_R}{I_R} = \frac{V_R}{I_4} = \frac{12}{1} = 12\ \Omega$$

Q2.9 ***In the circuit shown in Fig. Q2.9, find the total resistance across A-B.***

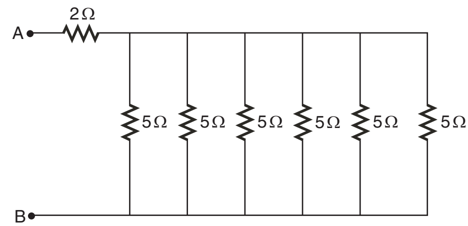


Fig. Q2.9.

Solution

In Fig. Q2.9, the parallel combination of six numbers of 5 Ω resistances is equivalent to a single resistance of $\frac{5}{6}\,\Omega$.

$$\therefore\ R_{AB} = 2 + \frac{5}{6} = 2.8333\,\Omega$$

Q2.10 ***Seven bulbs each rated at 75 W, 120 V are connected in parallel. Calculate the power and current consumed by them.***

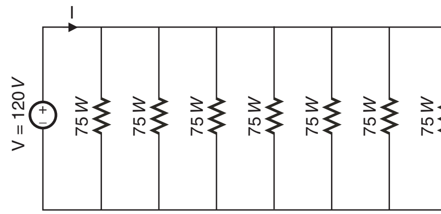


Fig. Q2.10.

Solution

The parallel connection of seven bulbs is equivalent to seven resistances in parallel as shown in Fig Q2.10.

Now, the total power is given by sum of power consumed by each bulb/resistance.

$$\therefore\ \text{Total power} = 7 \times 75 = 525\,\text{W}$$

We know that, in purely resistive loads, P = VI

$$\therefore\ \text{Total current,}\ \ I = \frac{P}{V} = \frac{525}{120} = 4.375\,A$$

(AU June'16, 8 Marks)

Q2.11 ***In the circuit shown in Fig. Q2.11, the power in resistance R_A is 9.6 kW, current through R_B is 60 A and the value of resistance R_C is 4.8 Ω. Determine the value of R_A, R_B, total current, total power and equivalent resistance.***

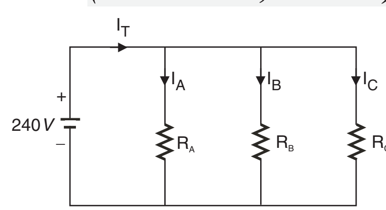


Fig. Q2.11.

Solution

Given that, $P_A = 9.6\text{ kW} = 9.6 \times 10^3\text{ W}$

Here, $P_A = V \times I_A$

$$\therefore\ I_A = \frac{P_A}{V} = \frac{9.6 \times 10^3}{240} = 40\,A$$

By Ohm's law,

$$I_C = \frac{V}{R_C} = \frac{240}{4.8} = 50\,A$$

By KCL,

Total Current, $I_T = I_A + I_B + I_C = 40 + 60 + 50 = 150\,A$

$\therefore$ Total Power, $P_T = V \times I_T = 240 \times 150 = 36000\,W = 36\,kW$

$$\therefore\ \text{Equivalent resistance,}\ R_{eq} = \frac{V}{I_T} = \frac{240}{150} = 1.6\,\Omega$$

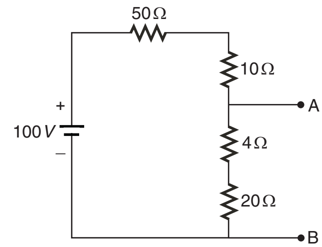


Fig. Q2.12.

Q2.12 ***Calculate the voltage across the terminals A-B for the circuit shown in Fig. Q2.12.***

Solution

By voltage division rule,

$$V_{AB} = 100 \times \frac{20+4}{50+10+4+20} = 28.5714\,V$$

Q2.13 ***Determine the resistance of each wire when the resistance of two wires is 25 Ω when connected in series and 6 Ω when connected in parallel.*** ***(AU June'16, 2 Marks)***

Solution

Let R_1 and R_2 be the resistance of two wires.

Equivalent resistance in series = $R_1 + R_2 = 25$ .....(1)

Equivalent resistance in parallel = $\frac{R_1 R_2}{R_1 + R_2} = 6$ .....(2)

From equation (1), we get,

$R_2 = 25 - R_1$ .....(3)

On substituting for R_2 from equation (3) in equation (1), we get,

$$\frac{R_1(25-R_1)}{R_1+25-R_1} = 6 \quad \Rightarrow \quad 25R_1 - R_1^2 = 150 \quad \Rightarrow \quad R_1^2 - 25R_1 + 150 = 0$$

$$\therefore\ R_1 = \frac{-(-25) \pm \sqrt{(-25)^2 - 4 \times 150}}{2} = \frac{25 \pm 5}{2} = 15, 10$$

If, $R_1 = 15\ \Omega$, then $R_2 = 25 - 15 = 10\ \Omega$

If, $R_1 = 10\ \Omega$, then $R_2 = 25 - 10 = 15\ \Omega$

$\therefore\ R_1 = 15\ \Omega$,

$R_2 = 10\ \Omega$

Q2.14 ***A star-connected network consists of three resistances 3 Ω, 6 Ω and 10 Ω. Convert the star-connected network into an equivalent delta-connected network.***

Solution

Let R_1, R_2 and R_3 be the resistances in star connection as shown in Fig. Q2.14.1 and R_{12} R_{23} and R_{31} be the resistances in equivalent delta connection as shown in Fig. Q2.14.2.

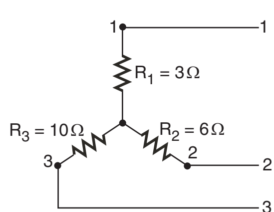


Fig. Q2.14.1.

⇒

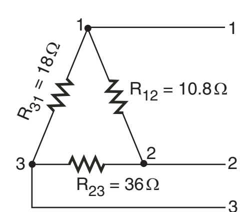


Fig. Q2.14.2.

$$R_{12} = R_1 + R_2 + \frac{R_1 R_2}{R_3} = 3 + 6 + \frac{3 \times 6}{10} = 10.8\,\Omega$$

$$R_{23} = R_2 + R_3 + \frac{R_2 R_3}{R_1} = 6 + 10 + \frac{6 \times 10}{3} = 36\,\Omega$$

$$R_{31} = R_3 + R_1 + \frac{R_3 R_1}{R_2} = 10 + 3 + \frac{10 \times 3}{6} = 18\,\Omega$$

Q2.15 ***A delta-connected network consists of three resistances 5 Ω, 6 Ω and 9 Ω. Convert the delta-connected network into an equivalent star-connected network.***

Solution

Let R_{12}, R_{23} and R_{31} be the resistances in delta connection as shown in Fig. Q2.15.1 and R_1, R_2 and R_3 be the resistances in star connection as shown in Fig. Q2.15.2.

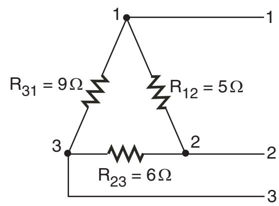


Fig. Q2.15.1.

⇒

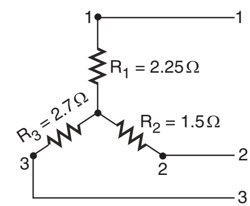


Fig. Q2.15.2.

$$R_1 = \frac{R_{12} R_{31}}{R_{12} + R_{23} + R_{31}} = \frac{5 \times 9}{5 + 6 + 9} = 2.25\,\Omega$$

$$R_2 = \frac{R_{12} R_{23}}{R_{12} + R_{23} + R_{31}} = \frac{5 \times 6}{5 + 6 + 9} = 1.5\,\Omega$$

$$R_3 = \frac{R_{23} R_{31}}{R_{12} + R_{23} + R_{31}} = \frac{6 \times 9}{5 + 6 + 9} = 2.7\,\Omega$$

Q2.16 ***What will be the equivalent inductance across A-B in the network shown in Fig. Q2.16.1.***

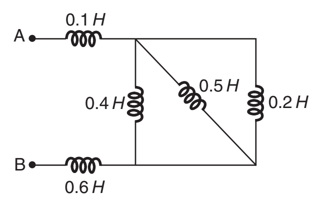


Fig. Q2.16.1.

⇓

Solution

First, the parallel combination of 0.4 *H*, 0.5 *H* and 0.2 *H* has been reduced to a single equivalent as shown in Fig. Q2.16.2.

In the network of Fig. Q2.16.2, all the inductances are in series.

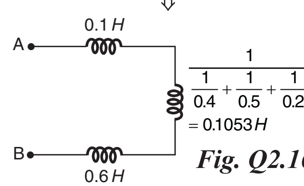


Fig. Q2.16.2.

Hence, the equivalent inductance L_{AB} across A-B is given by,

$L_{AB} = 0.1 + 0.1053 + 0.6 = 0.8053\,H$

Q2.17 ***What will be the equivalent capacitance across A-B in the network shown in Fig. Q2.17.***

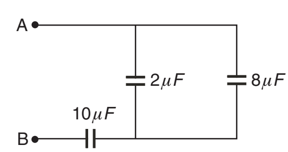


Fig. Q2.17.

Solution

In the given network 2 *μF* and 8 *μF* capacitances are in parallel and the parallel combination is in series with 10 *μF* capacitance. Hence, the equivalent capacitance C_{AB} across A-B is given by,

$$C_{AB} = \frac{(2+8) \times 10}{(2+8) + 10} = 5\,\mu F$$

Q2.18 ***Determine the currents $\overline{I}_1$ and $\overline{I}_2$ in the circuit shown in Fig. Q2.18.***

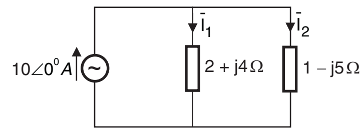


Fig. Q2.18.

Solution

By current division rule,

$$\bar{I}_1 = 10\angle 0^\circ \times \frac{1-j5}{2+j4+1-j5} = 8 - j14\ A = 16.1245\angle -60.3^\circ\ A$$

$$\bar{I}_2 = 10\angle 0^\circ \times \frac{2+j4}{2+j4+1-j5} = 2 + j14\ A = 14.1421\angle 81.9^\circ\ A$$

Q2.19 ***Determine the voltages $\overline{V}_1$ and $\overline{V}_2$ in the circuit shown in Fig. Q2.19.***

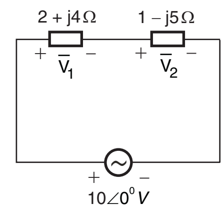


Fig. Q2.19.

Solution

By voltage division rule,

$$\overline{V}_1 = 10\angle 0^\circ \times \frac{2+j4}{2+j4+1-j5} = 2 + j14\ V = 14.1421\angle 81.9^\circ\ V$$

$$\overline{V}_2 = 10\angle 0^\circ \times \frac{1-j5}{2+j4+1-j5} = 8 - j14\ V = 16.1245\angle -60.3^\circ\ V$$

Q2.20 ***The resistance of each branch of a star-connected circuit is 5 Ω. What will be the branch resistance of equivalent delta-connected circuit?*** ***(AU May'17, 2 Marks)***

When three equal resistances are in star, equivalent delta resistance of each branch will be three times the star impedance.

Given that, $R_{star} = 5\ \Omega$

$\therefore\ R_{delta} = 3 \times R_{star} = 3 \times 5 = 15\ \Omega$

Q2.21 ***The impedance of each branch of a delta-connected circuit is $\sqrt{3}\,\overline{Z}$. What will be the branch impedance of equivalent star-connected circuit?***

Solution

When three equal impedances are in delta, equivalent star impedance of each branch will be 1/3 times the delta impedance.

Given that, $\overline{Z}_{delta} = \sqrt{3}\,\overline{Z}$

$$\therefore\ \overline{Z}_{star} = \frac{1}{3} \times \overline{Z}_{delta} = \frac{1}{3} \times \sqrt{3}\,\overline{Z} = \frac{\overline{Z}}{\sqrt{3}}$$

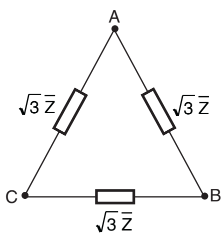


Fig. Q2.21.1.

$\Rightarrow$

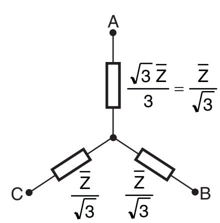


Fig. Q2.21.2.

Q2.22 ***Find the equivalent admittance at A-B in the network shown inFig Q2.22.***

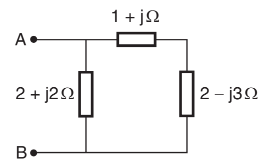


Fig. Q2.22.

Solution

Let, $\overline{Z}_{AB}$ = Equivalent impedance at A-B

$$\text{Now, } \overline{Z}_{AB} = \frac{(2+j2)\times(1+j+2-j3)}{(2+j2)+(1+j+2-j3)} = 2+j0.4\,\Omega$$

$$\therefore \text{ Equivalent admittance at A-B}, \overline{Y}_{AB} = \frac{1}{\overline{Z}_{AB}} = \frac{1}{2+j0.4}$$

$$= 0.4808 - j0.0962\,\mho$$

Q2.23 ***Find the equivalent admittance and impedance at A-B in the network shown inFig Q2.23.***

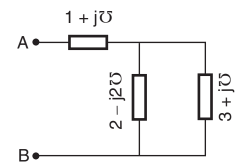


Fig. Q2.23.

Solution

Let, $\overline{Y}_{AB}$ = Equivalent admittance at A-B

$$\text{Now, } \overline{Y}_{AB} = \frac{(1+j)\times(2-j2+3+j)}{(1+j)+(2-j2+3+j)} = 1+j0.6667\,\mho$$

$$\therefore \text{ Equivalent impedance at A-B}, \overline{Z}_{AB} = \frac{1}{\overline{Y}_{AB}} = \frac{1}{1+j0.6667}$$

$$= 0.6923 - j0.4615\,\Omega$$

Q2.24 ***State the superposition theorem.***

The superposition theorem states that the response in a linear circuit with multiple sources is given by the algebraic sum of the responses due to individual sources acting alone.

Q2.25 ***What are the properties of additivity and homogeneity?***

The property of additivity says that the response in a circuit due to a number of sources is given by the sum of the response due to individual sources acting alone.

The property of homogeneity says that if all the sources are multiplied by a constant, the response is also multiplied by the same constant.

Q2.26 ***Find the current through the ammeter shown in Fig. Q2.26.1 by using the superposition theorem.***

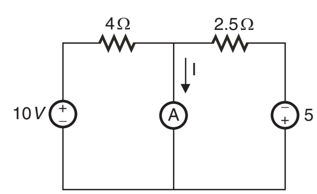


Fig. Q2.26.1.

Since the resistance of ammeter is not specified it can be represented by a short circuit. The condition of the given circuit when each source is acting separately is shown in Figs Q2.26.2 and Q2.26.3.

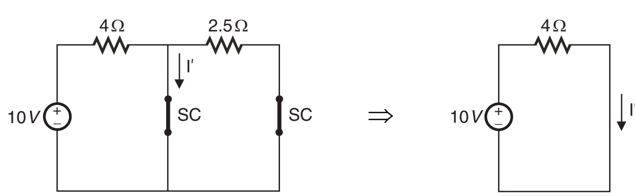


Fig. Q2.26.2.

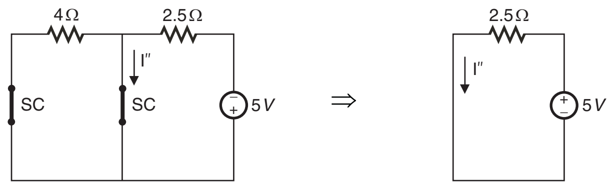


Fig. Q2.26.3.

With reference to Figs Q2.26.2 and Q2.26.3, we can write,

Response due to 10 V source, $I' = \dfrac{10}{4} = 2.5\,A$

Response due to 5 V source, $I'' = -\dfrac{5}{2.5} = -2\,A$

Total response, $I = I' + I'' = 2.5 + (-2) = 0.5\,A$

Q2.27 ***Find the voltage V_L in the circuit shown in Fig. Q2.27.1 using the principle of superposition.***

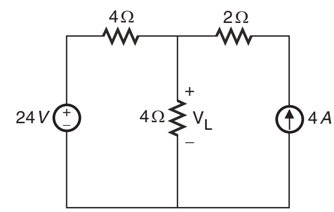


Fig. Q2.27.1.

The condition of the circuit when each source is acting separately is shown in Figs Q2.27.2 and Q2.27.3.

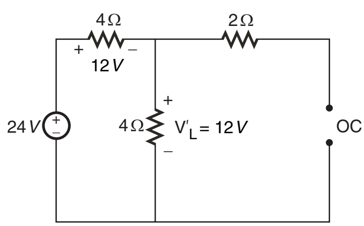


Fig. Q2.27.2.

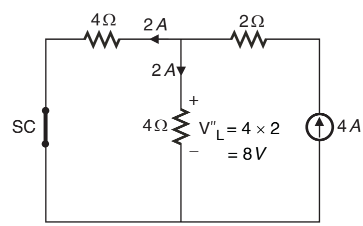


Fig. Q2.27.3.

With reference to Figs Q2.27.2 and Q2.27.3, we can write,

$$V'_L = 12\,V \quad ; \quad V''_L = 8\,V \quad ; \quad \therefore V_L = V'_L + V''_L = 12 + 8 = 20\,V$$

Q2.28 ***In the circuit shown in Fig. Q2.28, the power in resistance R is 9 W when V_1 is acting alone and 4W when V_2 is acting alone. What is the power in R when V_1 and V_2 are acting together ?***

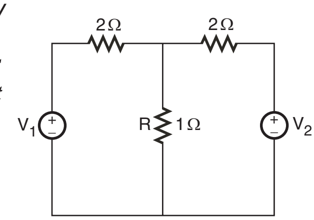


Fig. Q2.28.

Current through R when V_1 is acting, $I' = \sqrt{\dfrac{9}{1}} = 3\,A$

Current through R when V_2 is acting, $I'' = \sqrt{\dfrac{4}{1}} = 2\,A$

Total current when V_1 and V_2 are acting, $I = I' + I'' = 3 + 2 = 5\,A$

Power in R when V_1 and V_2 are acting = $I^2 R = 5^2 \times 1 = 25\,W$

$P = I^2R$	$\therefore \quad I = \sqrt{\dfrac{P}{R}}$

Q2.29 ***State Thevenin's theorem.***

Thevenin's theorem states that a circuit with two terminals can be replaced with an equivalent circuit consisting of a voltage source in series with a resistance (or impedance).

Q2.30 ***State the applications of thevenin's theorem.*** ***(AU Dec'15, 2 Marks)***

1. Thevenin's theorem can be used to represent a complicated part of a circuit by an equivalent voltage source by performing two simple measurements namely., open circuit voltage and short circuit current.
2. Thevenin's theorem can be used to estimate the matched resistance or impedance for implementing maximum power transfer condition between any two parts of a circuit.

Q2.31 ***State Norton's theorem.***

Norton's theorem states that a circuit with two terminals can be replaced with an equivalent circuit consisting of a current source in parallel with a resistance (or impedance).

Q2.32 ***Find Thevenin's voltage across terminals A and B in the circuit shown in Fig. Q2.32.***

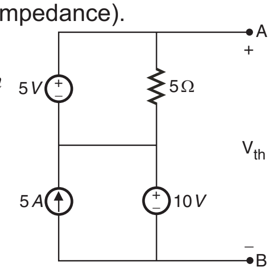


Fig. Q2.32.

Thevenin's voltage, V_{th} = 5 + 10 = 15 V

***Note :** Voltage across 5 Ω is 5 V.*

Q2.33. ***Find the value of I_n for the circuit shown in Fig. Q2.33.1*** ***(AU Dec'14, 2 Marks)***

Let us remove the resistance R_L and mark the resulting open terminals as A and B as shown in Fig. Q2.33.2.

The terminals A and B are shorted as shown in Fig. Q2.33.3. The 360 Ω resistance is short-circuited and so no current will flow through it. Hence, the circuit of Fig. Q2.33.3 is redrawn as shown in Fig. Q2.33.4.

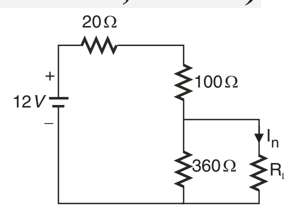


Fig. Q2.33.1.

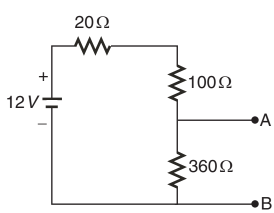


Fig. Q2.33.2.

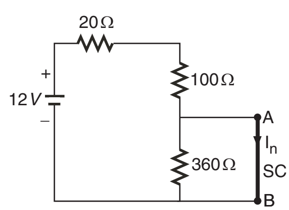


Fig. Q2.33.3.

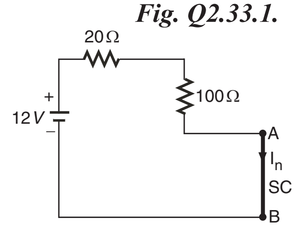


Fig. Q2.33.4.

With reference to Fig. Q2.33.4, by Ohm's law, we can write,

$$I_n = \frac{12}{20+100} = 0.1\,A$$

Q2.34 ***Find Norton's equivalent of the circuit shown in Fig. Q2.34.1.***

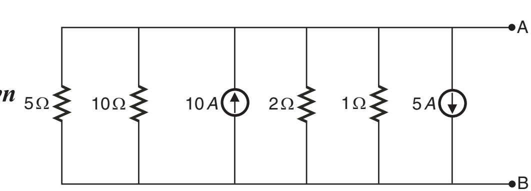


Fig. Q2.34.1.

Norton's current, I_n = 10 − 5 = 5 A

Norton's resistance,

$$R_n = \frac{1}{\frac{1}{5}+\frac{1}{10}+\frac{1}{2}+\frac{1}{1}} = 0.5556\,\Omega$$

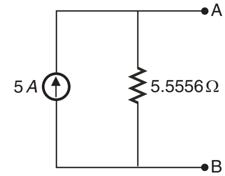


Fig. Q2.34.2 : *Norton's equivalent.*

Q2.35 Determine Thevenin's equivalent of the circuit shown in Fig. Q2.35.1.

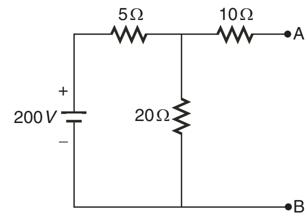


Fig. Q2.35.1.

Thevenin's voltage is the voltage across 20 Ω resistance.

By voltage division rule,

$$\text{Thevenin's voltage, } V_{th} = 200 \times \frac{20}{20+5} = 160\,V$$

To find Thevenin's resistance, the 200*V* source is replaced with a short circuit as shown in Fig. Q2.35.2.

With reference to Fig. Q2.35.2, we can write,

$$\text{Thevenin's resistance, } R_{th} = \frac{5 \times 20}{5+20} + 10 = 14\,\Omega$$

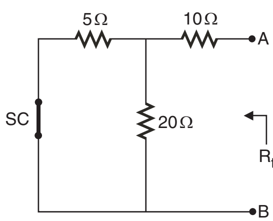


Fig. Q2.35.2.

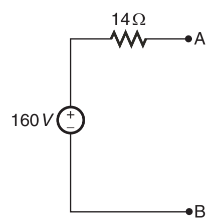


Fig. Q2.35.3 : *Thevenin's equivalent.*

Q2.36 In the circuit shown in Fig. Q2.36.1, using Thevenin's theorem, determine the voltage across 90 Ω resistance after the switch is closed.

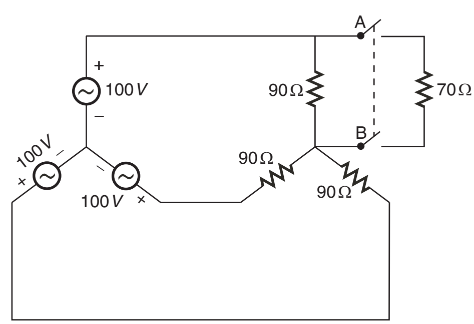


Fig. Q2.36.1.

Since the load is balanced when the switch is open, the voltage across 90 Ω is 100 *V*. This is also Thevenin's voltage at terminals A-B.

To find Thevenin's resistance, the voltage sources are replaced with a short circuit. When the voltage sources are shorted, the three 90 Ω resistances are in parallel.

$$\therefore \text{ Thevenin's resistance, } R_{th} = \frac{90}{3} = 30\,\Omega$$

The Thevenin's equivalent at A-B is shown in Fig. Q2.36.2. With reference to Fig. Q2.36.2 by voltage division rule,

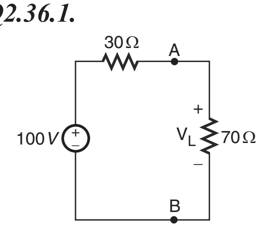


Fig. Q2.36.2.

$$\text{Voltage across 70}\,\Omega\text{ resistance, } V_L = 100 \times \frac{70}{70+30} = 70\,V$$

Q2.37 *Find Thevenin's equivalent of the circuit shown in Fig. Q.2.37.1.*

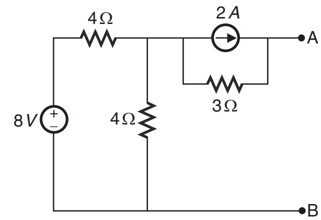


Fig. Q2.37.1.

To find Thevenin's voltage, the current source is converted into a voltage source as shown in Fig. Q2.37.2.

By voltage division rule, $V_1 = 8 \times \frac{4}{4+4} = 4\,V$

By KVL, $V_{th} = V_1 + 6 = 4 + 6 = 10\ V$

To find Thevenin's resistance, the voltage source is replaced with a short circuit and the current source is opened as shown in Fig. Q2.37.3.

$$R_{th} = \frac{4}{2} + 3 = 5\,\Omega$$

Thevenin's equivalent is shown in Fig. Q2.37.4.

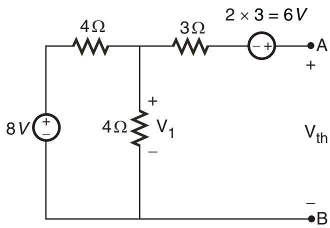


Fig. Q2.37.2.

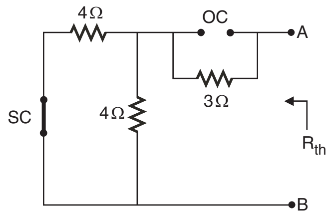


Fig. Q2.37.3.

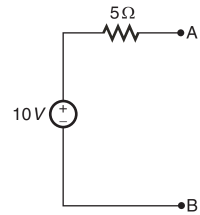


Fig. Q2.37.4 : Thevenin's equivalent.

Q2.38 *Find $\overline{V}_{th}$ at terminals A-B in the circuit shown in Fig. Q2.38.*

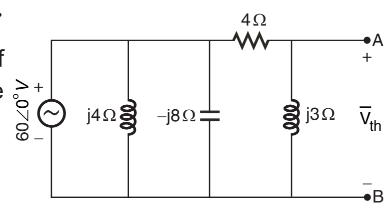


Fig. Q2.38.

In the given circuit the voltage across series combination of 4 Ω and j3 Ω elements is $60\angle 0°\,V$. Hence, by voltage division rule,

$$\overline{V}_{th} = 60\angle 0° \times \frac{j3}{4+j3} = 21.6 + j28.8\,V = 36\angle 53.1°\,V$$

Q2.39 *State maximum power transfer theorem.* ***(AU May'15, 2 Marks)***

In purely resistive circuits, maximum power transfer theorem states that maximum power is transferred from source to load when load resistance is equal to source resistance.

In general, the maximum power transfer theorem states that maximum power is transferred to a load impedance if the absolute value of the load impedance is equal to the absolute value of the looking back impedance of the circuit from the terminals of the load.

Q2.40 *Determine the maximum power transfer to the load where the load is connected to a network of the terminals for R_{th} = 10 Ω and V_{th} = 40 V* ***(AU May'17, 2 Marks)***

Given that R_{th} = 10 Ω and V_{th} = 40 V

For maximum power transform,

Load resistance, $R_L = R_{th} = 10\ \Omega$

$$\text{Maximum power},\ P_{max} = \frac{V_{th}^2}{4R_L} = \frac{40^2}{4 \times 10} = 40\,W$$

Q2.41 ***The VI characteristics of a network are shown in Fig. Q2.41. Determine the maximum power that can be supplied by the network to a resistance connected across A-B.***

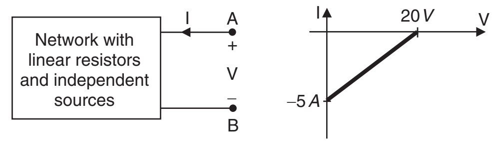


Fig. Q2.41.

When V = 0, I = −5 A

The condition V = 0 is equivalent to short circuiting terminals A−B and the current flowing through the short circuit is Norton's current.

$\therefore$ Norton's current, $I_n = -I = -(-5) = 5\,A$

When I = 0, V = 20 V

The condition I = 0 is equivalent to open terminals A-B and the voltage across the open terminals is Thevenin's voltage.

$\therefore$ Thevenin's voltage, $V_{th} = 20\,V$

$$\text{Thevenin's resistance, } R_{th} = \frac{V_{th}}{I_n} = \frac{20}{5} = 4\,\Omega$$

The resistance, R to be connected for maximum power transfer across terminals A-B is $R_{th.}$.

$$\text{Maximum power transferred to R, } P_{max} = \frac{V_{th}^2}{4R} = \frac{V_{th}^2}{4R_{th}} = \frac{20^2}{4 \times 4} = 25\,W$$

Q2.42 ***Determine the value of R in the circuit shown in Fig. Q2.42.1 for maximum power transfer.***

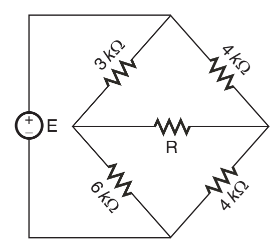


Fig. Q2.42.1.

The value of R for maximum power transfer is given by the looking back resistance (or Thevenin's resistance) from the terminals of R, which is determined as shown below:

$$R = R_{th} = \frac{3 \times 6}{3+6} + \frac{4}{2} = 4\,k\Omega$$

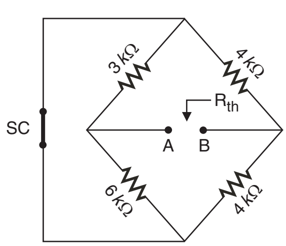


Fig. Q2.42.2.

⇒

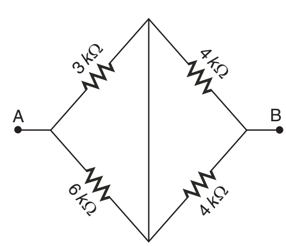


Fig. Q2.42.3.

⇒

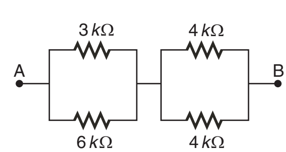


Fig. Q2.42.4.

Q2.43 ***Determine the value of R in the circuit shown in Fig. Q2.43.1 for maximum power transfer.***

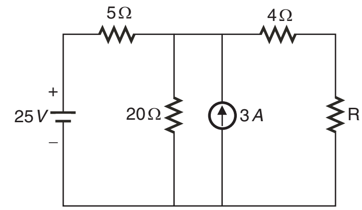


Fig. Q2.43.1.

The value of R for maximum power transfer is given by the looking back resistance (or Thevenins's resistance) from the terminals of R, which is determined as shown below:

$$R = R_{th} = \frac{5 \times 20}{5 + 20} + 4 = 8\,\Omega$$

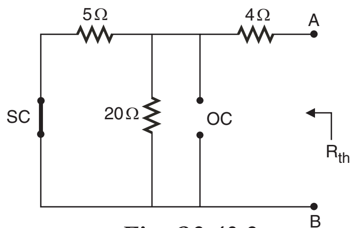


Fig. Q2.43.2.

⇒

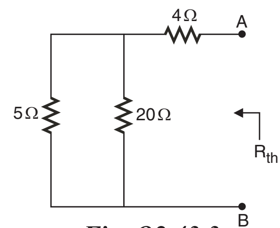


Fig. Q2.43.3.

Q2.44 ***Find the value of R for maximum power transfer in the circuit shown in Fig. Q2.44.***

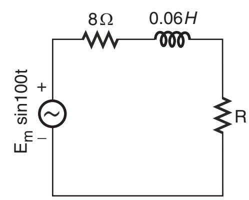


Fig. Q2.44.

For maximum power transfer, the value of R should be equal to the absolute value of the looking back impedance from the terminals of R.

Here, $E_m \sin \omega t = E_m \sin 100t$; $\therefore\ \omega = 100\ rad/s$

$$\therefore\ R = \sqrt{8^2 + (\omega \times 0.06)^2} = \sqrt{8^2 + (100 \times 0.06)^2} = 10\,\Omega$$

Q2.45 ***State the reciprocity theorem.***

The reciprocity theorem states that in a linear, bilateral, single source circuit, the ratio of excitation to the response is constant, when the positions of excitation and response are interchanged.

Q2.46 ***Two conditions of a passive, linear network are shown in Figs Q2.46.1 and Q2.46.2. Using the superposition and the reciprocity theorem, find I_x.***

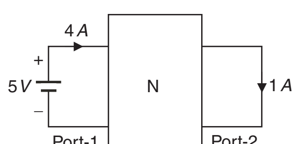


Fig. Q2.46.1.

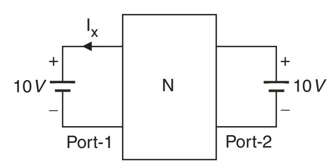


Fig. Q2.46.2.

Let us replace the 10*V* source in port-1 with a short circuit as shown in Fig. Q2.46.3.

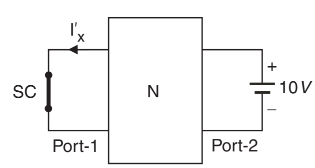


Fig. Q2.46.3.

On comparing Figs Q2.46.1 and Q2.46.3 using the reciprocity theorem we can write,

$$\frac{I'_x}{10\,V} = \frac{1\,A}{5\,V} \quad \Rightarrow \quad I'_x = \frac{1}{5} \times 10 = 2\,A$$

Let us replace the 10 V source in port-2 with a short circuit as shown in Fig. Q2.46.4. On comparing Figs Q2.46.1 and Q2.46.4, using homogeneity property we can write,

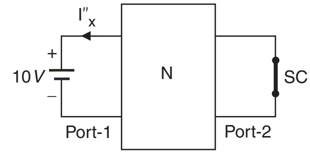


Fig. Q2.46.4.

$$I''_x = -4 \times 2 = -8\,A$$

By the principle of superposition,

$$I_x = I'_x + I''_x = 2 + (-8) = -6\,A$$

Q2.47 State Millman's theorem.

Millman's theorem states that if n number of voltage sources with internal impedance are in parallel then they can be combined to give a single voltage source with an equivalent emf and internal impedance.

Q2.48 Write the expression for Millman's equivalent source of n number of parallel connected voltage sources.

$$\bar{E}_{eq} = \left(\frac{\bar{E}_1}{\bar{Z}_1} + \frac{\bar{E}_2}{\bar{Z}_2} + \frac{\bar{E}_3}{\bar{Z}_3} + \ldots + \frac{\bar{E}_n}{\bar{Z}_n}\right)\bar{Z}_{eq}$$

$$\bar{Z}_{eq} = \frac{1}{\frac{1}{\bar{Z}_1} + \frac{1}{\bar{Z}_2} + \frac{1}{\bar{Z}_3} + \ldots. + \frac{1}{\bar{Z}_n}}$$

where, $\bar{E}_1, \bar{E}_2, \bar{E}_3 \ldots$ = Emf of voltage sources in parallel.

$\bar{Z}_1, \bar{Z}_2, \bar{Z}_3 \ldots$ = Internal impedance of voltage sources.

Q2.49 In the circuit shown in Fig. Q2.49.1 find the current I using Millman's theorem.

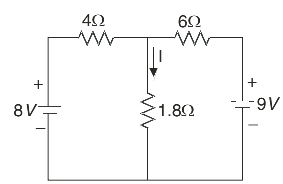


Fig Q2.49.1.

The circuit can be redrawn as shown in Fig. Q2.49.2.

Using Millman's theorem, the parallel connected voltage sources can be converted into a single source as shown in Fig. Q2.49.3.

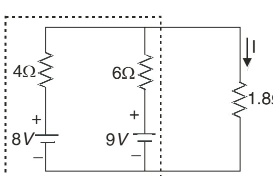


Fig Q2.49.2.

⇒

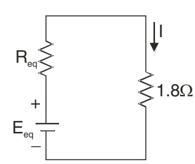


Fig Q2.49.3.

$$R_{eq} = \frac{1}{\frac{1}{4} + \frac{1}{6}} = 2.4\ \Omega \quad ; \quad E_{eq} = \left(\frac{8}{4} + \frac{9}{6}\right) \times 2.4 = 8.4\ \text{V}$$

With reference to Fig. Q2.49.3,

$$I = \frac{E_{eq}}{R_{eq} + 1.8} = \frac{8.4}{2.4 + 1.8} = 2\,A$$

Q2.50 ***Find the current I_L in the circuit shown in Fig. Q2.50.1 using Millman's theorem.***

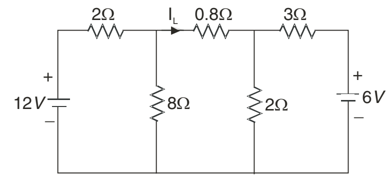


Fig Q2.50.1.

The circuit can be redrawn as shown in Fig. Q2.50.2 and then the parallel connected voltage sources can be converted into a single source as shown in Fig. Q2.50.3.

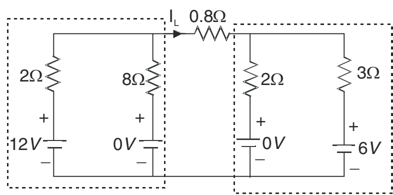


Fig Q2.50.2.

⇒

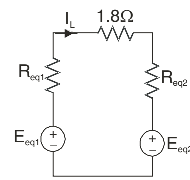


Fig Q2.50.3.

$$R_{eq1} = \frac{1}{\frac{1}{2}+\frac{1}{8}} = 1.6\ \Omega \quad ; \quad E_{eq1} = \left(\frac{12}{2}+\frac{0}{8}\right)\times 1.6 = 9.6\text{V}$$

$$R_{eq2} = \frac{1}{\frac{1}{2}+\frac{1}{3}} = 1.2\ \Omega \quad ; \quad E_{eq2} = \left(\frac{0}{2}+\frac{6}{3}\right)\times 1.2 = 2.4\text{V}$$

With reference to fig Q2.50.3,

$$I_L = \frac{E_{eq1}-E_{eq2}}{R_{eq1}+0.8+R_{eq2}} = \frac{9.6-2.4}{1.6+0.8+1.2} = 2\ A$$

2.9 Exercises

I. Fill in the Blanks with Appropriate Words

1. When n number of resistances of value R are connected in series, the equivalent resistance is given by ___________.
2. When n number of resistances of value R are connected in parallel, the equivalent resistance is given by ___________.
3. When three equal resistances of value R are in star-connection, its equivalent delta-connection will have three equal resistances of value ___________.
4. When n number of capacitances of value C are connected in series, the equivalent capacitance is equal to ___________.
5. The equivalent admittance of three identical parallel-connected admittances of value Y is equal to ________.
6. The ________ impedance is the looking back impedance from the open terminals of a network.
7. The ________ equivalent is the voltage generator model of a network.
8. Norton's equivalent is the ________ generator model of a network.

9. The principle of superposition is a combination of ________ and ________ property.
10. While finding response due to one source, all other sources are replaced with their ________.
11. While finding response due to one source, all other ideal ________ sources are replaced with a short circuit.
12. While finding response due to one source, all other ideal current sources are replaced with ________.
13. When load resistance and reactance are independently variable, maximum power transfer is achieved if load impedance is equal to ________ impedance.
14. In purely resistive circuits ________ is transferred to load when load resistance is equal to source resistance.
15. If a load impedance, R + jX with R alone variable is connected to a source with internal impedance $R_s + jX_s$ then the condition for maximum power transfer is ________ .
16. If a variable resistance R is connected to a source with impedance $R_s + jX_s$ then the condition for maximum power transfer is ________ .
17. The networks which satisfy the reciprocity theorem are called ________ networks.

ANSWERS

1. nR	6. Thevenin's/Norton's	11. voltage	16. $R = \sqrt{R_s^2 + X_s^2}$
2. $\frac{R}{n}$	7. Thevenin's	12. open circuit	17. reciprocal
3. 3R	8. current	13. conjugate of source	
4. $\frac{C}{n}$	9. additivity, homogeneity	14. maximum power	
5. 3 Y	10. internal impedances	15. $R = \sqrt{R_s^2 + (X_s + X)^2}$	

II. State Whether the Following Statements are True or False

1. When star-connected resistances are converted into a equivalent delta-connected resistances, the power consumed by the resistances remains the same.
2. Inductances connected in series can be replaced by an equivalent inductance whose value is given by the sum of individual inductances.
3. Capacitances connected in parallel can be replaced with an equivalent capacitance whose value is given by the inverse of sum of the inverse of individual capacitances.
4. Inductive susceptance is always negative and capacitive susceptance is always positive.
5. When the voltage requirement of the load is higher, the voltage sources should be connected in parallel.
6. Thevenin's equivalent can be determined only for the linear part of a circuit.
7. A circuit with a non-linear resistance can be analysed using Thevenin's theorem by replacing the rest of the circuit with Thevenin's equivalent.
8. Thevenin's and Norton's impedance are the same.
9. The superposition theorem can be extended to non-linear circuits by piecewise linear approximation.
10. The superposition theorem can be used to estimate power directly.
11. The superposition theorem is applicable to any network containing linear dependent sources.

12. The superposition theorem is applicable to any network containing a time varying resistor.
13. In reactive circuits with variable reactance, the maximum power transfer is achieved at resonance.
14. A reactive circuit with fixed reactance will behave as a purely resistive circuit when the maximum power transfer condition is satisfied.
15. The reciprocity theorem will be satisfied by a circuit with dependent sources.
16. The reciprocity theorem can be applied only to a circuit with a single source.
17. Ideal voltage sources in parallel cannot be converted into a single equivalent source using Millman's theorem.

ANSWERS

1.	True	6.	True	11.	True	16.	True
2.	True	7.	True	12.	False	17.	True
3.	False	8.	True	13.	True		
4.	True	9.	True	14.	False		
5.	False	10.	False	15.	False		

III. Choose the Right Answer for the Following Questions

1. When n number of identical resistances of value R are connected in parallel, the equivalent value of the parallel combination is,

a) nR b) $\frac{R}{n}$ c) $\frac{2R}{n}$ d) $\frac{R}{2n}$

2. The star equivalent of three identical resistances in delta of value R will be three identical resistances of value,

a) $\frac{R}{3}$ b) 3R c) $\frac{R^2}{3}$ d) $3R^2$

3. In the network shown in Fig. 3, if the value of all the resistances are 1 Ω then what will be the equivalent resistance at A-B?

a) 1 Ω

b) 0.5 Ω

c) 0.4 Ω

d) 0.2 Ω

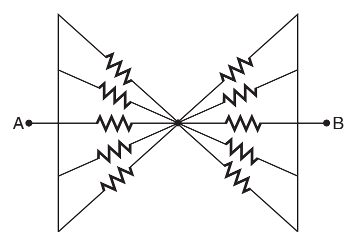


Fig. 3.

4. In the network shown in Fig. 4, if the value of all the resistances are 2 Ω then the equivalent resistance at AC and BD respectively are,

a) 0 Ω, 0.8 Ω

b) 0.8 Ω, 0 Ω

c) 0 Ω, 0.4 Ω

d) 0.8 Ω, 0.8 Ω

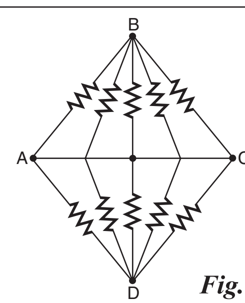


Fig. 4.

5. ***The equivalent value of two inductances 3 H and 6 H when connected in series and in parallel respectively are,***

a) $9H, 3H$ b) $9H, 2H$ c) $18H, 2H$ d) $2H, 3H$

6. ***In the network shown in Fig. 6, if the value of all the inductance is 2 H, then what is the equivalent inductance at A-B?***

a) $4H$

b) $3H$

c) $2H$

d) $1H$

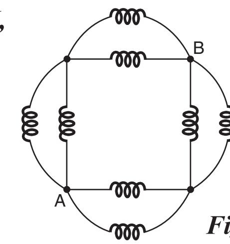


Fig. 6.

7. ***The equivalent value of two capacitances 6 μF and 12 μF when connected in series and in parallel respectively are,***

a) $6\mu F, 18\mu F$ b) $18\mu F, 4\mu F$ c) $4\mu F, 18\mu F$ d) $72\mu F, 18\mu F$

8. ***In the network shown in Fig. 8, if the values of all the capacitances is 2 μF then what is the equivalent capacitance at A-B?***

a) $4\mu F$

b) $3\mu F$

c) $2\mu F$

d) $1\mu F$

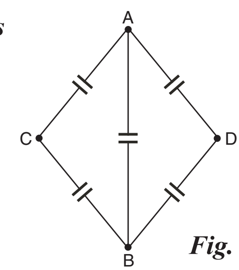


Fig. 8.

9. ***The equivalent reactance with respect to terminals A-B in the network shown in Fig. 9 is,***

a) j10Ω

b) j7.5Ω

c) j5Ω

d) j2.5Ω

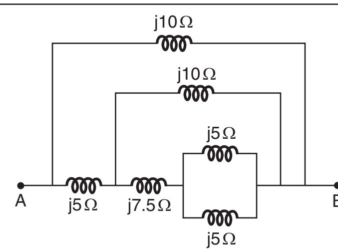


Fig. 9.

10. ***The equivalent value of two impedances 8 + j3 Ω and 2 – j3 Ω when connected in parallel is,***

a) 10 + j0Ω b) 6 + j6Ω c) 10 + j6Ω d) 2.5 – j1.8Ω

11. ***In the network shown in Fig. 11, what is the equivalent impedance at A-B?***

a) 3 + j3Ω

b) 1 + jΩ

c) 3 – j4Ω

d) 1 + j2Ω

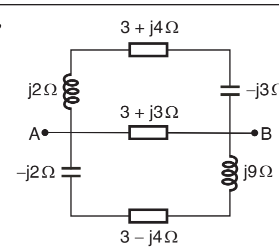


Fig. 11.

12. In the network shown in Fig. 12, if the values of all the conductances are 3 ℧ then what is the value of equivalent conductance at A-B?

a) 18 ℧

b) 2 ℧

c) 0.5 ℧

d) 0.25 ℧

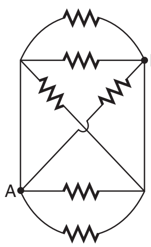


Fig. 12.

13. In the network shown in Fig. 13, what is the equivalent conductance at A-B?

a) 0.9083 ℧
b) 15.5 ℧

c) 29 ℧
d) 20 ℧

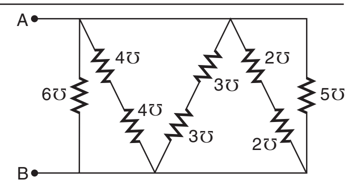


Fig. 13.

14. In the network shown in Fig. 14, what is the equivalent susceptance at A-B?

a) j0.5 ℧

b) j3 ℧

c) j6 ℧

d) j12 ℧

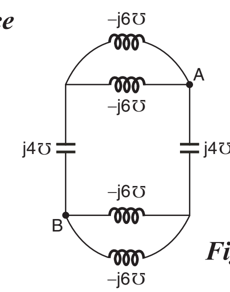


Fig. 14.

15. In the network shown in Fig. 15, what is the equivalent admittance at A-B?

a) 0 + j4 ℧

b) 4 + j4 ℧

c) 4 + j0 ℧

d) 4 – j4 ℧

d) 20 ℧

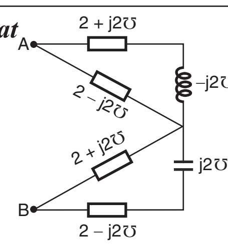


Fig. 15.

16. The current through the ideal ammeter in the circuit shown in Fig. 16 is,

a) 1 *A*

b) 1.5 *A*

c) 2 *A*

d) 4 *A*

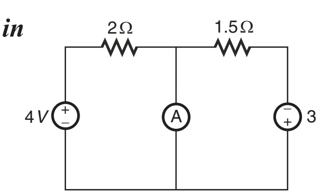


Fig. 16.

17. The voltage across the ideal voltmeter in the circuit shown in Fig. 17 is,

a) 10 V

b) 8 V

c) 6 V

d) 4 V

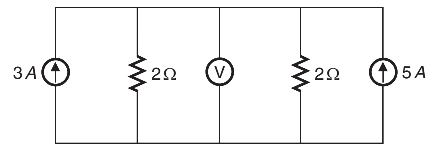


Fig. 17.

18. The voltage V_L in the circuit shown in Fig. 18 is,

a) 5 V

b) 6 V

c) 9 V

d) 12 V

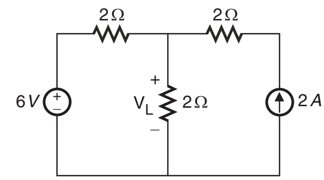


Fig. 18.

19. The current I_2 in the circuit shown in Fig. 19 is,

a) 8 A

b) 7 A

c) 4 A

d) 2 A

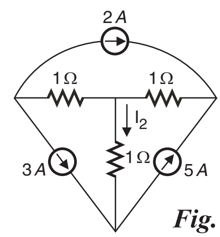


Fig. 19.

20. The value of V_{th} and R_{th} in the circuit shown in Fig. 20 is,

a) 20 V, 30 Ω

b) 5 V, 10 Ω

c) 25 V, 4 Ω

d) 10 V, 12 Ω

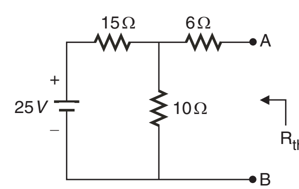


Fig. 20.

21. The value of V_{th} in the circuit shown in Fig. 21 is,

a) 7 V

b) 5 V

c) 3 V

d) 2 V

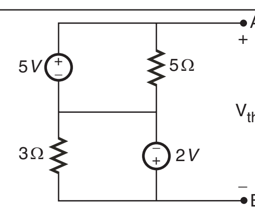


Fig. 21.

22. The value of R_{th} in the circuit shown in Fig. 22 is,

a) 6 Ω

b) 7 Ω

c) 13 Ω

d) 22 Ω

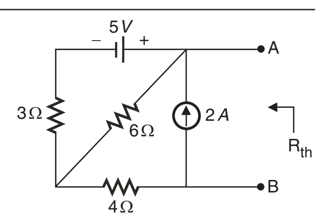


Fig. 22.

23. The value of R_N in the circuit shown in Fig. 23 is,

a) 4Ω

b) 1.6Ω

c) 1.33Ω

d) 1.2Ω

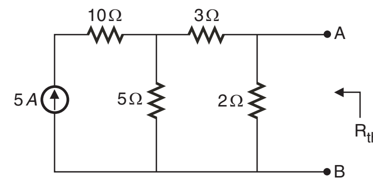


Fig. 23.

24. The value of I_N in the circuit shown in Fig. 24 is,

a) $2A$

b) $3.5A$

c) $1A$

d) $5A$

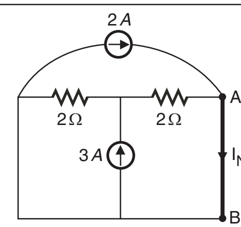


Fig. 24.

25. The value of I_N in the cirucit shown in Fig. 25 is,

a) $1.5A$

b) $3A$

c) $0.5A$

d) $1A$

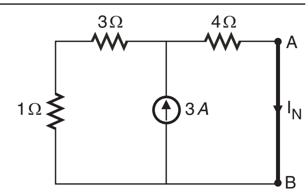


Fig. 25.

26. In the 2-terminal linear circuit shown in Fig. 26, the open circuit voltage measured across AB is 10 V and short circuit current through AB is 5 A. The value of resistance that can be connected across AB for maximum power transfer is,

a) 10Ω

b) 5Ω

c) 50Ω

d) 2Ω

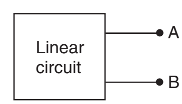


Fig. 26.

27. The value of R for maximum power transfer in the circuit shown in Fig. 27 is,

a) 8Ω

b) 10Ω

c) 8.25Ω

d) $-j0.06\Omega$

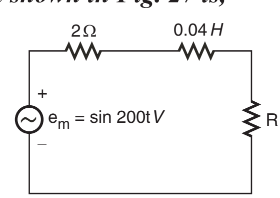


Fig. 27.

28. The value of impedance that can be connected across AB for maximum power transfer in the circuit shown in Fig. 28 is,

a) $2.4 + j1.2\Omega$ b) $2.4 - j1.2\Omega$

c) $1.2 + j2.4\Omega$ d) $1.2 - j2.4\Omega$

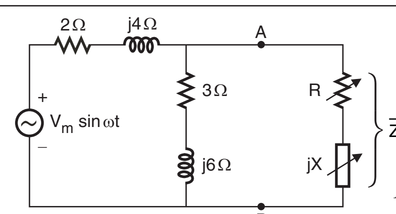


Fig. 28.

29. ***The positions of source and response in the circuit shown in Fig. 29a are changed as shown in Fig. 29b. What is the value of I_X?***

a) $2A$ b) $4A$ c) $1A$ d) $8A$

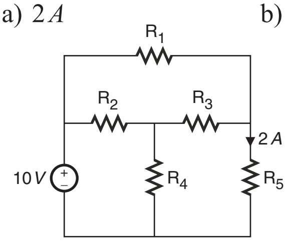


Fig. 29a.

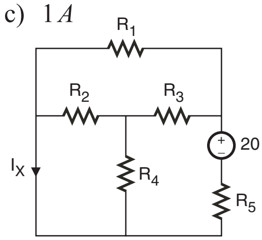


Fig. 29b.

30. ***The positions of source and response in the circuit shown in Fig. 30a are changed as shown in Fig. 30b. What is the value of V_X?***

a) $6V$ b) $3V$ c) $2.4V$ d) $2V$

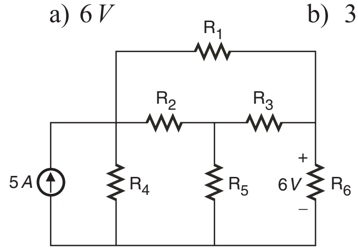


Fig. 30a.

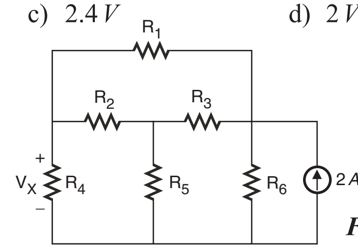


Fig. 30b.

ANSWERS

1.	b	7.	c	13.	b	19.	d	25.	a
2.	a	8.	a	14.	d	20.	d	26.	d
3.	c	9.	c	15	c	21.	c	27.	c
4.	a	10.	d	16.	d	22.	a	28.	d
5.	b	11.	b	17.	b	23.	b	29.	b
6.	d	12.	a	18.	a	24.	b	30.	c

IV. Unsolved Problems

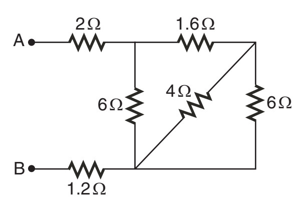


Fig. E2.1.

E2.1 ***Find the equivalent resistance with respect to terminals A-B in the network shown in Fig. E2.1.***

E2.2 ***Find the equivalent resistance across the source terminals A-B of the circuit shown in Fig. E2.2 and calculate the current delivered by the source.***

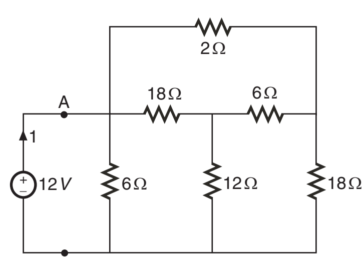


Fig. E2.2.

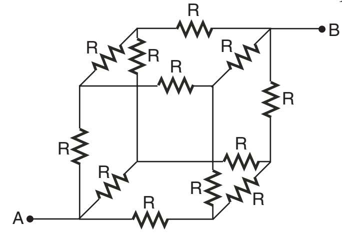


Fig. E2.3.

E2.3 ***Determine the equivalent resistance at A-B for the circuit shown in Fig. E2.3.***

E2.4 ***Determine the equivalent resistance for the circuit shown in Fig. E2.4.***

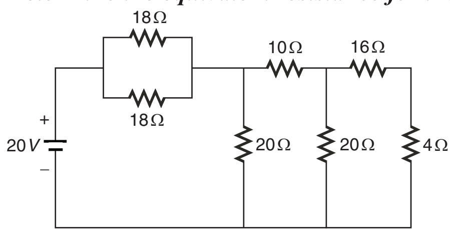


Fig. E2.4.

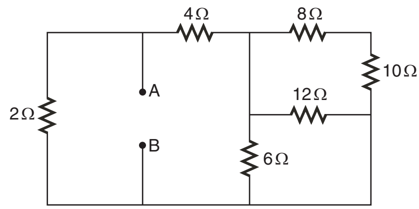


Fig. E2.5.

E2.5 ***In the network shown in Fig. E2.5, find the equivalent resistance between A and B.***

E2.6 ***For the network shown in Fig. E2.6, find the equivalent resistance across A-B.***

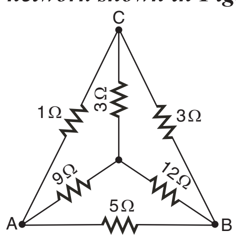


Fig. E2.6.

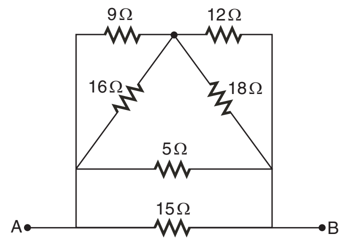


Fig. E2.7.

E2.7 ***In Fig. E2.7, find the equivalent resistance across A-B.***

E2.8 ***In circuit shown in Fig. E2.8, find the equivalent resistance across A-B.***

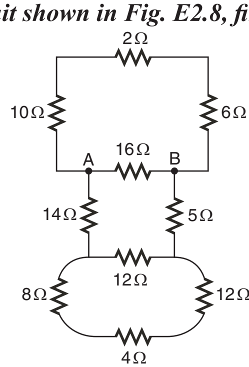


Fig. E2.8.

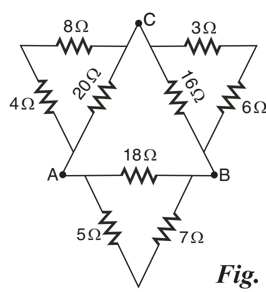


Fig. E2.9.

E2.9 ***In Fig. E2.9, find the equivalent resistance R_{AB}.***

E2.10 ***Convert the circuit shown in Fig.E2.10 with multiple sources into a single equivalent current source at terminal A-B, with a single equivalent resistance in parallel. Also, calculate the voltage across the equivalent resistance.***

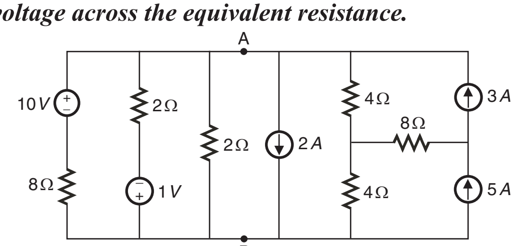


Fig. E2.10.

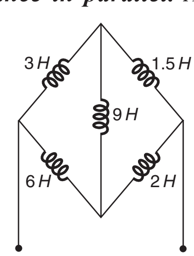


Fig. E2.11.

E2.11 ***Find the equivalent inductance across terminals A-B in the network shown in Fig. E2.11.***

E2.12 In Fig. E2.12, find the equivalent reactance across A-B.

E2.13 Find the equivalent capacitance across terminals A-B in the network shown in Fig. E2.13.

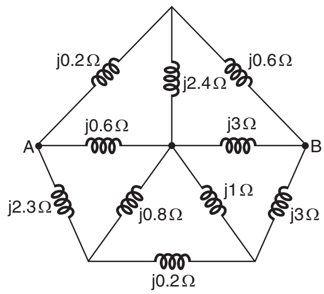


Fig. E2.12.

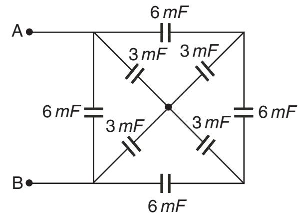


Fig. E2.13.

Note : *Consider 6 mF capacitor as reactance –jX and so reactance of 3mF capacitor is –j2X. Reduce the network by treating all the capacitors as reactances and finally convert the reactance to capacitor.*

E2.14 In Fig. E2.14, find the equivalent capacitive reactance across A-B.

E2.15 Find the equivalent impedance across A-B in the circuit shown in Fig. E2.15.

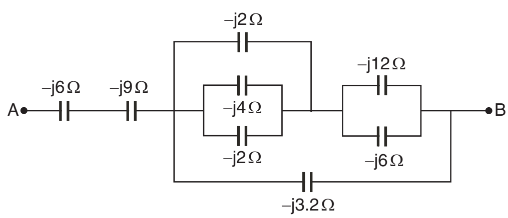


Fig. E2.14.

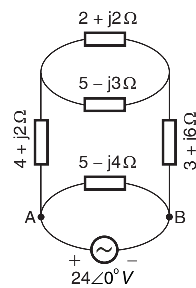


Fig. E2.15.

E2.16 In the circuit shown in Fig. E2.16, determine I_L using Thevenin's theorem.

E2.17 In the circuit shown in Fig. E2.17, determine V_L using Norton's theorem.

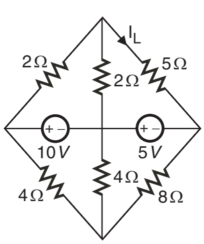


Fig. E2.16.

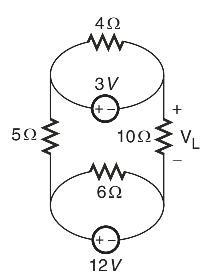


Fig. E2.17.

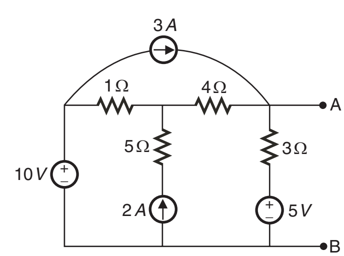


Fig. E2.18.

E2.18 Find Thevenin's and Norton's equivalents of the circuit shown in Fig. E2.18 with respect to terminals A and B.

E2.19 ***In the circuit shown in Fig. E2.19, determine the current $\overline{I}_L$ using Norton's theorem.***

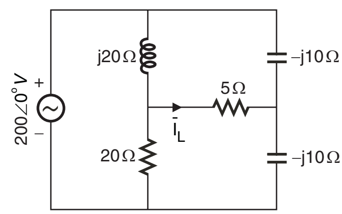


Fig. E2.19.

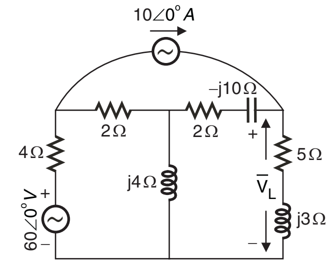


Fig. E2.20.

E2.20 ***In the circuit shown in Fig. E2.20, determine the voltage across 5 + j3 Ω impedance using Thevenin's theorem.***

E2.21 ***Using the superposition theorem, determine the current I_L in the circuit shown in Fig. E2.21.***

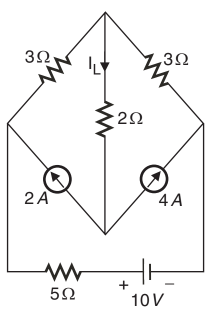


Fig. E2.21.

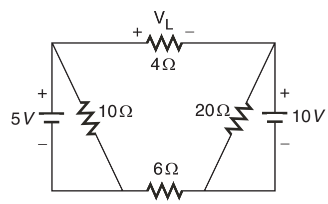


Fig. E2.22.

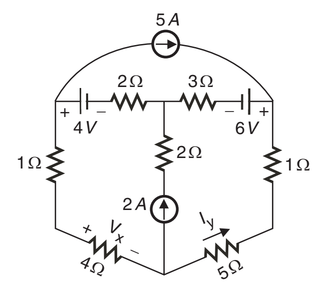


Fig. E2.23.

E2.22 ***Determine the voltage V_L in the circuit shown in Fig. E2.22 using the superposition theorem.***

E2.23 ***Calculate the current I_y and voltage V_x in the circuit shown in Fig. E2.23 using the superposition theorem.***

E2.24 ***Determine the current $\overline{I}_y$ and voltage $\overline{V}_x$ in the circuit shown in Fig. E2.24 using the superposition theorem.***

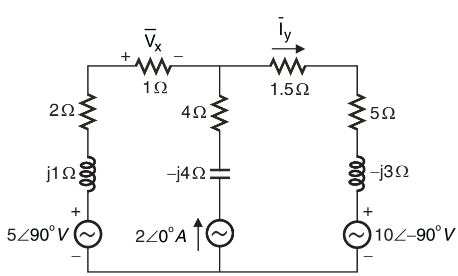


Fig. E2.24.

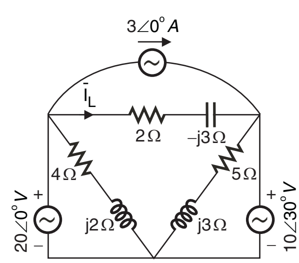


Fig. E2.25.

E2.25 ***In the circuit shown in Fig. E2.25, determine $\overline{I}_L$ using the superposition theorem and estimate the active and reactive power in 2 – j3 Ω impedance.***

E2.26 ***Determine the value of R in the circuit shown in Fig. E2.26 for maximum power transfer. Also find the value of the maximum power.***

E2.27 ***Determine the value of $\overline{Z}_L$ in the circuit shown in Fig. E2.27 for maximum power transfer.***

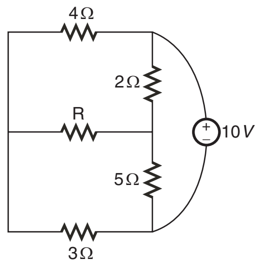


Fig. E2.26.

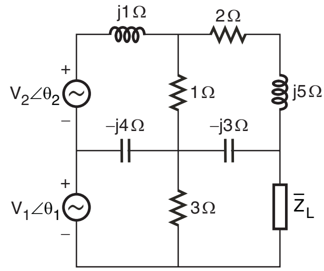


Fig. E2.27.

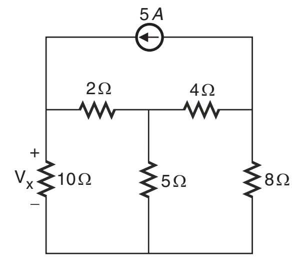


Fig. E2.28.

E2.28 ***In the circuit of Fig. E2.28, calculate V_x. Prove reciprocity theorem by interchanging the positions of the 5 A source and V_x.***

E2.29 ***In the circuit of Fig. E2.29, calculate I_x. Prove the reciprocity theorem by interchanging the positions of the 10 V source and I_x.***

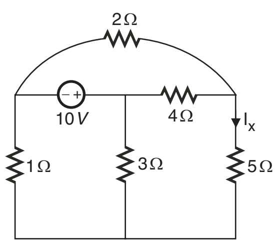


Fig. E2.29.

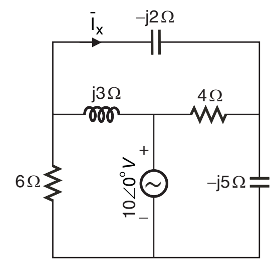


Fig. E2.30.

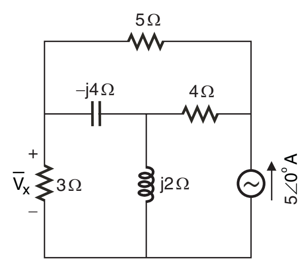


Fig. E2.31.

E2.30 ***In the circuit of Fig. E2.30, demonstrate the reciprocity theorem by interchanging the positions of the source and response, $\overline{I}_x$.***

E2.31 ***Demonstrate reciprocity theorem in the circuit shown in Fig. E2.31 by interchanging the positions of the source and response.***

E2.32 ***In the circuit shown in fig E2.32, apply Millman's theorem to find Thevenin's equivalent at A-B. Also, find the value of resistance R for maximum power transfer.***

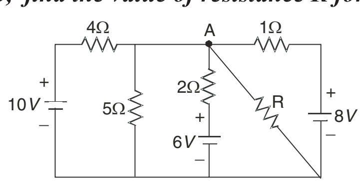


Fig E2.32.

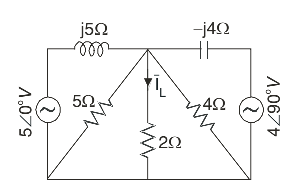


Fig E2.33.

E2.33 ***In the circuit shown in Fig. E2.33, find the current $\overline{I}_L$ using Millman's theorem.***

ANSWERS

E2.1 $R_{AB} = 5.6\,\Omega$

E2.2 $R_{AB} = 3.75\,\Omega$, $I = 3.2\,A$

E2.3 $R_{AB} = \frac{5}{6} R\,\Omega$

E2.4 $R_{eq} = 19\,\Omega$

E2.5 $R_{AB} = 1.5686\,\Omega$

E2.6 $R_{AB} = 1.9637\,\Omega$

E2.7 $R_{AB} = 2.9084\,\Omega$

E2.8 $R_{AB} = 6.4478\,\Omega$

E2.9 $R_{AB} = 4.6663\,\Omega$

E2.10 $I_{eq} = 2.75\,A$, $R_{eq} = 0.8\,\Omega$, $V = 2.2\,V$

E2.11 $L_{AB} = 2.875\,H$

E2.12 $jX_{AB} = j0.3522\,\Omega$

E2.13 $C_{AB} = 9.6432\,mF$

E2.14 $-jX_{AB} = -j16.92\,\Omega$

E2.15 $\bar{Z}_{AB} = 5.3021 - j1.2792\,\Omega = 5.4542\angle -13.6^\circ\,\Omega$

E2.16 $V_{th} = 10\,V$; $R_{th} = 1\,\Omega$; $I_L = 1.6667\,A$

E2.17 $I_n = 1.8\,A$; $R_n = 5\,\Omega$; $V_L = 6\,V$

E2.18 $V_{th} = 13.25\,V$; $R_{th} = R_n = 1.875\,\Omega$; $I_n = 7.0667\,A$

E2.19 $\bar{I}_n = -4 - j8\,A = 8.9443\angle -116.6^\circ\,A$; $\bar{Z}_n = 10 + j5\,\Omega = 11.1803\angle 26.6^\circ\,\Omega$

$\bar{I}_L = -2 - j6\,A = 6.3246\angle -108.4^\circ\,A$

E2.20 $\bar{V}_{th} = 44.6152 - j63.0768\,V = 77.2606\angle -54.7^\circ\,V$

$\bar{Z}_{th} = 3.8462 - j7.2308\,\Omega = 8.1901\angle -62^\circ\,\Omega$

$\bar{V}_L = 45.9196 + j1.44\,V = 45.9422\angle 1.8^\circ\,V$

E2.21 $I_L = I'_{L(2\,A)} + I''_{L(4\,A)} + I'''_{L(10\,V)} = 2 + 4 + 0 = 6\,A$

E2.22 $V_L = V'_{L(5\,V)} + V''_{L(10\,V)} = 2 + (-4) = -2\,V$

E2.23 $V_x = V'_{x(4\,V)} + V''_{x(6\,V)} + V'''_{x(2\,A)} + V''''_{x(5\,A)} = 1 + (-1.5) + 4.5 + (-6.25) = -2.25\,V$

$I_y = I'_{y(4\,V)} + I''_{y(6\,V)} + I'''_{y(2\,A)} + I''''_{y(5\,A)} = 0.25 + (-0.375) + (-0.875) + (-1.5625) = -2.5625\,A$

E2.24 $\bar{I}_y = \bar{I}'_{y(5\,V)} + \bar{I}''_{y(10\,V)} + \bar{I}'''_{y(2\,A)} = (-0.1061 + j0.504) + (-0.2122 + j1.008) + (0.5623 + j0.3289)$

$= 0.244 + j1.8409 = 1.857\angle 82.4^\circ\,A$

$\bar{V}_x = \bar{V}'_{x(5\,V)} + \bar{V}''_{x(10\,V)} + \bar{V}'''_{x(2\,A)} = (-0.1061 + j0.504) + (-0.2122 + j1.008) + (-1.4377 + j0.3289)$

$= -1.756 + j1.8409 = 2.5441\angle 133.6^\circ\,V$

E2.25 $\bar{I}_L = \bar{I}'_{L(20\,V)} + \bar{I}''_{L(10\,V)} + \bar{I}'''_{L(3\,A)} = (3.0769 + j4.6154) + (-0.1785 - j2.7678) + 0$

$= 2.8984 + j1.8476\,A = 3.4372\angle 32.5^\circ\,A$

$P = 23.6287\,W$; $Q = -35.443\,VAR$

E2.26 $R = 3.1429\,\Omega$, $P_{max} = 0.6492\,W$

E2.27 $\bar{Z}_L = 4.6432 + j3.443\,\Omega = 5.7804\angle 36.6^\circ\,\Omega$

E2.28 $V_x = 10.2273\,V$

E2.29 $I_x = 0.1176\,A$

E2.30 $\bar{I}_x = 0.3554 + j0.3928\,A = 0.5297\angle 47.9^\circ\,A$

E2.31 $\bar{V}_x = 1.5529 + j2.1887\,V = 2.6836\angle 54.6^\circ\,V$

E2.32 $V_{th} = 6.9228\,V$; $R_{th} = R = 0.5128\,\Omega$

E2.33 $\bar{E}_{eq} = -2.439 - j1.9512\,V = 3.1234\angle -141.3^\circ\,V$

$\bar{Z}_{eq} = 2.1951 - j0.2439\,\Omega\,A = 2.2086\angle -6.3^\circ\,\Omega$

$\bar{I}_L = -0.5525 - j0.4972\,A = 0.7433\angle -138^\circ\,A$

Chapter 3

TRANSIENT RESPONSE ANALYSIS

3.1 L and C Elements and Transient Response

Electrical devices are controlled by switches which are closed in order to connect supply to a device or opened in order to disconnect supply to a device. The switching operation changes the current and the voltage in a device. Purely resistive devices allow instantaneous change in current and voltage. Inductive devices do not allow a sudden change in current (or delay the change in current), and capacitive devices do not allow a sudden change in voltage (or delay the change in voltage). Hence, when a switching operation is performed in inductive or capacitive devices, the current and voltage in the device take some time to change from a pre-switching value to a steady value after switching. This phenomena can be observed in starting of a motor, which is an inductive device.

The study of switching condition in a circuit is called **transient analysis**. The state (or condition) of the circuit from the instant of switching to the attainment of a steady state is called **transient state** or simply, transient. The time duration from the instant of switching till the attainment of a steady state is called the **transient period**. The current and voltage of circuit elements during the transient period is called **transient response**.

Apart from switching, the transient will also occur due to a change in circuit elements (i.e., due to change in values of R, L and C). In electrical engineering, transient analysis is a useful tool for analysis of switching conditions in circuit breakers, relays, generators and various types of loads. It is also useful for the analysis of faulty conditions in electrical devices.

3.1.1 Natural and Forced Response

Transient response is the response (or output) of a circuit from the instant of switching to the attainment of a steady state. In order to study the response with respect to time, the switching instant is taken as time origin, i.e., t = 0. The time $t = 0^-$ is used to denote the time instant just prior to switching and the time $t = 0^+$ is used to denote the time instant immediately after switching.

The time difference between $t = 0^-$ and $t = 0^+$ is zero. It is necessary to define three time instants 0^-, 0 and 0^+ because current and voltage in certain elements may change suddenly.

A resistance will allow a sudden change in current and voltage.

$\therefore \quad i_R(0^+) \neq i_R(0^-) \quad$ and $\quad v_R(0^+) \neq v_R(0^-)$

An inductance will not allow a sudden change in current but allow a sudden change in voltage.

$\therefore \quad i_L(0^+) = i_L(0^-) \quad$ and $\quad v_L(0^+) \neq v_L(0^-)$

A capacitance will allow a sudden change in current but will not allow a sudden change in voltage.

$\therefore \quad i_C(0^+) \neq i_C(0^-) \quad$ and $\quad v_C(0^+) = v_C(0^-)$

Inductance and capacitance are energy storage devices. Hence, they may have stored energy prior to the switching instant. The response of a circuit due to stored energy alone (without an external source) is called **natural response or source-free response**. The response of a circuit due to an external source is called **forced response**.

In time domain analysis, the voltage-current relation of circuits is studied in the form of differential equations. Hence, the response of a circuit is the solution of its differential equations. In general, the solution of differential equation or the response has two parts, namely, **complementary function and particular solution**.

The **complementary function** becomes zero as t tends to infinity, and so it is the transient part of the solution. The complementary function is similar to natural response and so it is also called natural response (i.e., the complementary function and source-free response have similar form, which depends on the nature of the circuit).

The **particular solution** attains a steady value as t tends to infinity, and so it is the steady state part of the solution. The particular solution depends on the nature of the exciting source and so it is also called forced response.

3.1.2 First and Second Order Circuits

In time domain, the equations governing the circuits will be in the form of differential equations. The order of the differential equations governing the circuit will be the order of the circuit.

Consider the RL circuit shown in Fig. 3.1. By KVL, we can write,

$$Ri(t) + L\frac{di(t)}{dt} = e(t) \qquad .....(3.1)$$

Equation (3.1) is the differential equation governing the RL circuit. The order of equation (3.1) is one and so the RL circuit is a first order circuit.

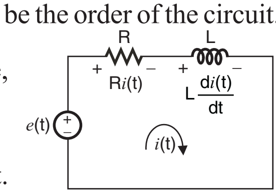


***Fig. 3.1 :** RL Circuit.*

Consider the RC circuit shown in Fig. 3.2. By KVL, we can write,

$$Ri(t) + \frac{1}{C}\int i(t)\,dt = e(t)$$

On differentiating the above equation with respect to t, we get,

$$R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{de(t)}{dt} \qquad .....(3.2)$$

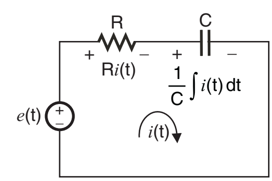


***Fig. 3.2 :** RC Circuit.*

Equation (3.2) is the differential equation governing the RC circuit. The order of equation (3.2) is one and so the RC circuit is a first order circuit.

Consider the RLC circuit shown in Fig. 3.3. By KVL, we can write,

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt = e(t)$$

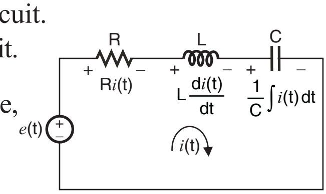


***Fig. 3.3 :** RLC Circuit.*

On differentiating the above equation with respect to t, we get,

$$R\frac{di(t)}{dt} + L\frac{d^2 i(t)}{dt^2} + \frac{1}{C}i(t) = \frac{de(t)}{dt} \qquad .....(3.3)$$

Equation (3.3) is the equation governing the RLC circuit. The order of equation (3.3) is two and so the RLC circuit is a second order circuit.

3.2 Transient Analysis Using Laplace Transform

In time domain, the voltage-current relation of circuits are governed by differential equations. The direct solution of differential equations gives the transient response.

Alternatively, the transient response can be obtained using the Laplace transform technique. In this method, the differential equations are converted into simple s-domain algebraic equations using Laplace transform. The solution of algebraic equations are simple as compared to the solution of differential equations. The s-domain algebraic equations are solved to obtain the s-domain response. But for practical purposes, we require the response in time domain, and so the time domain response is obtained by taking the inverse Laplace transform of the s-domain response. In this book, transient analysis is performed only via the Laplace transform technique.

In order to solve a circuit using the Laplace transform technique, the given circuit is first transformed into an s-domain circuit, then the sources and parameters are replaced by their s-domain equivalent.

3.2.1 Some Standard Voltage Functions

Impulse Voltage

Impulse voltage is a function with **infinite magnitude** and **zero duration**, but with an area of E. Mathematically, impulse voltage is defined as,

$$\text{Impulse voltage, } v(t) = \delta(t) = \infty\ ; \quad t = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t)\, dt = E$$
$$= 0\ ; \quad t \neq 0$$

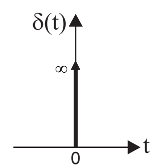


Fig. 3.4 : *Impulse voltage.*

Graphically, impulse voltage can be expressed as a pulse of width E/Δ and height Δ, as shown in Fig. 3.5.

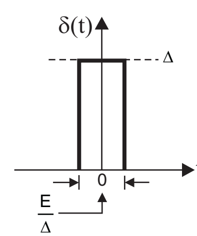


Fig. 3.5 : *Representation of impulse as a pulse.*

Unit impulse voltage is a function with infinite magnitude and zero duration, but with unit area. Mathematically, unit impulse voltage is defined as,

$$\text{Unit Impulse voltage, } v(t) = \delta(t) = \infty\ ; \quad t = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$
$$= 0\ ; \quad t \neq 0$$

Step Voltage

Step voltage is defined as,

$$\begin{aligned} v(t) &= E \quad &; t \geq 0 \\ &= 0 \quad &; t < 0 \end{aligned}$$

Unit step voltage is defined as,

$$\begin{aligned} v(t) = u(t) &= 1 \quad &; t \geq 0 \\ &= 0 \quad &; t < 0 \end{aligned}$$

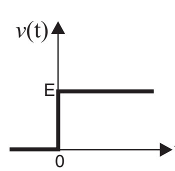


Fig. 3.6 : *Step voltage.*

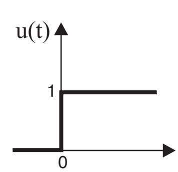


Fig. 3.7 : *Unit step voltage.*

Ramp Voltage

Ramp voltage is defined as,

$$\boxed{\begin{aligned} v(t) &= E\,t \quad ; t \geq 0 \\ &= 0 \quad ; t < 0 \end{aligned}}$$

Unit ramp voltage is defined as,

$$\boxed{\begin{aligned} v(t) &= t \quad ; t \geq 0 \\ &= 0 \quad ; t < 0 \end{aligned}}$$

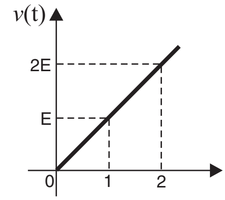


Fig. 3.8 : Ramp voltage.

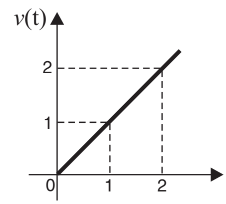


Fig. 3.9 : Unit ramp voltage.

Parabolic Voltage

Parabolic voltage is defined as,

$$\boxed{\begin{aligned} v(t) &= \frac{Et^2}{2} \quad ; \text{ for } t \geq 0 \\ &= 0 \quad ; \text{ for } t < 0 \end{aligned}}$$

Unit parabolic voltage is defined as,

$$\boxed{\begin{aligned} v(t) &= \frac{t^2}{2} \quad ; \text{ for } t \geq 0 \\ &= 0 \quad ; \text{ for } t < 0 \end{aligned}}$$

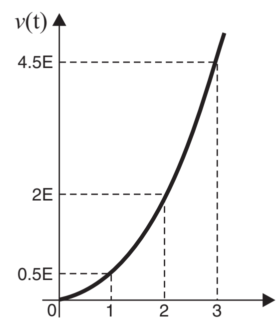


Fig. 3.10 : Parabolic voltage.

Exponential Voltage

Exponential voltage is defined as,

$$\boxed{v(t) = E\,e^{at}}$$

where, 'E' and 'a' are real.

Here, when 'a' is positive, the voltage $v(t)$ will be an exponentially rising voltage and when 'a' is negative, the voltage $v(t)$ will be an exponentially decaying voltage.

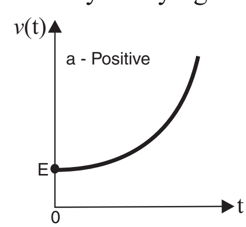


Fig. 3.11 : Rising exponential voltage.

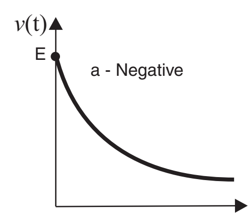


Fig. 3.12 : Decaying exponential voltage.

Sinusoidal and Cosinusoidal Voltage

Sinusoidal voltage is defined as,

$$\boxed{v(t) = E_m \sin(\omega t + \phi)}$$

Cosinusoidal voltage is defined as,

$$\boxed{v(t) = E_m \cos(\omega t + \phi)}$$

where, $\omega = 2\pi f = \frac{2\pi}{T}$ = Angular frequency in *rad/s*

f = Frequency in *cycles/second* or *Hz*

T = Time period in *seconds*

The waveforms of sinusoidal voltages for various choices of ϕ are shown in Chapter 4, Section 4.2, Fig. 4.10. The waveforms of cosinusoidal voltages are phase-shifted versions of sinusoidal voltages, shifted in phase by 90 degrees.

s-Domain Representation of Voltage Functions

The s-domain representation of voltage functions is obtained by taking the Laplace transform of their time domain representation.

Table 3.1 : Standard Voltage Functions ***(AU Dec'16, 2 Marks)***

Name of the voltage, $v(t)$	Time domain representation of the voltage, $v(t)$	Laplace transform of the voltage, $v(t)$ = V(s)
Impulse	$\delta(t)$	1
Step	E	$\frac{E}{s}$
Unit step	1	$\frac{1}{s}$
Ramp	E t	$\frac{E}{s^2}$
Unit ramp	t	$\frac{1}{s^2}$
Parabolic	$\frac{Et^2}{2}$	$\frac{E}{s^3}$
Unit parabolic	$\frac{t^2}{2}$	$\frac{1}{s^3}$
Rising exponential	$E\,e^{at}$	$E\frac{1}{s-a}$
Decaying exponential	$E\,e^{-at}$	$E\frac{1}{s+a}$
Sinusoidal	$E_m \sin \omega t$	$E_m\frac{\omega}{s^2+\omega^2}$
Cosinusoidal	$E_m \cos \omega t$	$E_m\frac{s}{s^2+\omega^2}$

3.2.2 s-Domain Representation of R, L, C Parameters

In an s-domain circuit, the R, L and C parameters of the circuit are also respresented by their s-domain equivalent. The s-domain equivalent of R, L and C parameters without initial energy is discussed in Chapter 4, Section 4.6. Since L and C parameters store energy, the s-domain equivalent of L and C parameters with initial energy is developed in this section.

Inductance with Initial Current

Inductance is an energy storage element. Hence, it can have an initial stored energy at the time of analysis which results in initial current (or flux). Consider an inductance connected to a circuit through position-1 of SPDT(**S**ingle **P**ole **D**ouble **T**hrow) switch as shown in Fig. 3.13. Let a steady current I_0 be established in the inductance. (Here, I_0 is the rms value in case of ac).

Let us change the switch position from 1 to 2 and the time instant be t = 0. At the time of closing the switch to position-2, a steady current I_0 is flowing through an inductance and this current is called initial current in the inductance for the analysis at (or after) t = 0, (i.e., for $t \geq 0$).

Note : $L = N\phi / I = \Psi / I, \quad \therefore I_0 = \Psi_0 / L$ *,where* Ψ_0 *is initial flux linkages in weber turns.*

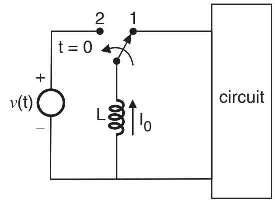


Fig. a : *Inductance with steady current I_0.*

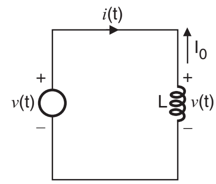


Fig. b : *Inductance with initial current in time domain.*

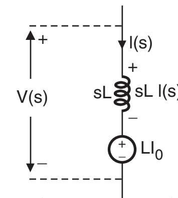


Fig. c : *Inductance with initial current in s-domain.*

Fig. 3.13 : *Voltage-current relation of inductance with initial current in time and s-domain.*

Let i(t) be the current through the inductance and v(t) be the voltage across the inductance for $t \geq 0$, as shown in Fig. 3.13(b).

$$\text{Now, } v(t) = L\frac{di(t)}{dt} \qquad .....(3.4)$$

On taking Laplace transform of the above equation, we get,

$$V(s) = L[sI(s) - i(0)]$$

$$\therefore V(s) = sLI(s) - Li(0)$$

$i(0) = -I_0$ = Initial condition. Here, I_0 is negative because it is opposite to i(t).

$$\boxed{V(s) = sLI(s) + LI_0} \qquad .....(3.5)$$

Using equation (3.5), the s-domain equivalent circuit of inductance with initial current opposite to i(t) is drawn as shown in Fig. 3.13(c).

When the direction of initial current is the same as that of the source current then $i(0) = +I_0$ and so equation (3.5) can be written as shown in equation (3.6).

$$\boxed{V(s) = sL\,I(s) - LI_0} \qquad .....(3.6)$$

Using equation (3.6), the s-domain equivalent circuit of inductance with initial current in direction i(t) is drawn as shown in Fig. 3.14.

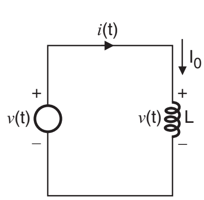


Fig. a : *Inductance with initial current in time domain.*

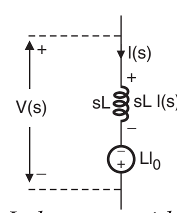


Fig. b : *Inductance with initial current in s-domain.*

Fig. 3.14 : *Voltage-current relation of inductance with initial current in the direction of source current.*

Capacitance with Initial Voltage

Capacitance is an energy storage element. Hence, it can have an initial stored energy at the time of analysis which results in initial voltage (or charge). Consider a capacitance connected to a circuit through position-1 of SPDT(**S**ingle **P**ole **D**ouble **T**hrow) switch as shown in Fig. 3.15. Let a steady voltage V_0 be established in the capacitance. (Here, V_0 is the rms value in case of ac).

Let us change the switch position from 1 to 2 and the time instant be t = 0. At the time of closing the switch to position-2, a steady voltage V_0 exists across capacitance and this voltage is called initial voltage in the capacitance for the analysis at (or after) t = 0 (i.e., for t ≥ 0).

Note : *C = Q/V ,* $\therefore$ $V_0 = Q_0/C$ *, where* Q_0 *is the initial charge.*

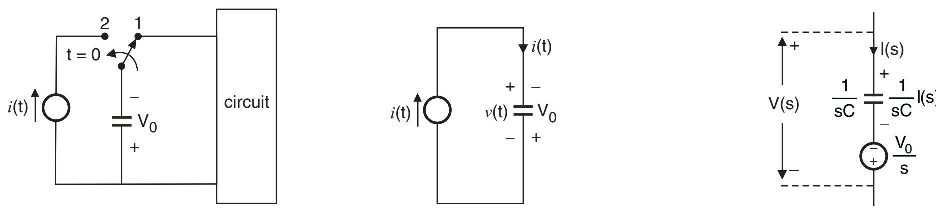


Fig. a : *Capacitance with steady voltage* V_0*.* ***Fig. b :*** *Capacitance with initial voltage in time domain.* ***Fig. c :*** *Capacitance with initial voltage in s-domain.*

Fig. 3.15 : *Voltage-current relation of capacitance with initial current in time and s-domain.*

Let $i(t)$ be the current through the capacitance and $v(t)$ be the voltage across the capacitance for t ≥ 0 as shown in Fig. 3.15(b).

$$\text{Now, } i(t) = C\frac{dv(t)}{dt} \qquad .....(3.7)$$

On taking Laplace transform of the above equation, we get,

$$I(s) = C[sV(s) - v(0)]$$

$v(0) = -V_0$ = Initial condition. Here, V_0 is negative because it is opposite to $v(t)$.

$$\therefore \; I(s) = sCV(s) + C\,V_0$$

$$sCV(s) = I(s) - C\,V_0$$

$$\boxed{V(s) = \frac{1}{sC}I(s) - \frac{V_0}{s}} \qquad .....(3.8)$$

Using equation (3.8), the s-domain equivalent circuit of capacitance with initial voltage opposite to $v(t)$ is drawn as shown in Fig. 3.15(c).

When the polarity of initial voltage is the same as that of voltage $v(t)$ then $v(0) = +V_0$ and so equation (3.8) can be written as shown in equation (3.9).

$$\boxed{V(s) = \frac{1}{sC}I(s) + \frac{V_0}{s}} \qquad .....(3.9)$$

Using equation (3.9), the s-domain equivalent circuit of capacitance with initial voltage of polarity same as v(t) is drawn as shown in Fig. 3.16.

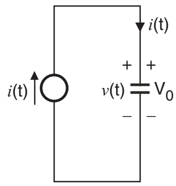


Fig. a : Capacitance with initial voltage in time domain.

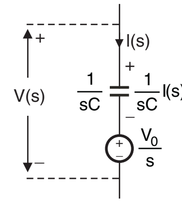
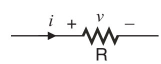
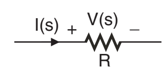
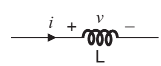
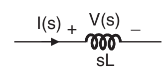
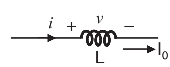
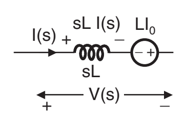
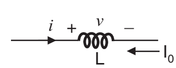
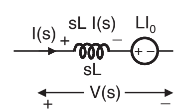
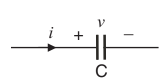
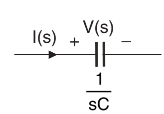
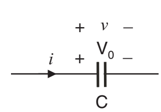
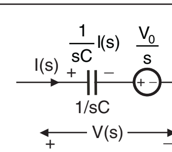
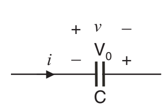
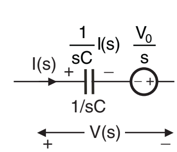


Fig. b : Capacitance with initial voltage in s-domain.

Fig. 3.16 : Voltage-current relation of capacitance with initial voltage of polarity same as v(t).

Table 3.2 : s-Domain Representation of R, L and C Parameters

S.No.	Parameter	Time domain	s-Domain
1.	Resistance, R		
2.	Inductance, L		
3.	Inductance, L with initial current		
			
4.	Capacitance, C		
5.	Capacitance, C with initial voltage		
			

3.2.3 Solving Initial and Final Conditions Using Laplace Transform

The initial condition in circuits can be solved using the initial value theorem of Laplace transform. The final condition in circuits can be solved using the final value theorem of Laplace transform.

Initial Condition

The initial value theorem of Laplace transform states that if F(s) is Laplace transform of f(t) then,

$$\underset{t \to 0}{\text{Lt}}\ f(t) = \underset{s \to \infty}{\text{Lt}}\ sF(s)$$

$$\boxed{\therefore\ \text{Initial value of } f(t) = f(0) = \underset{s \to \infty}{\text{Lt}}\ sF(s)}$$

Hence, by the initial value theorem, the initial value of a time domain function can be directly determined from its s-domain function. In transient analysis, the initial value theorem is useful to determine the following initial conditions in circuits.

1. Let, $\mathcal{L}\{i(t)\} = I(s)$

   Now,

   Initial current, $i(0^+) = \underset{t \to 0}{\text{Lt}}\ i(t) = \underset{s \to \infty}{\text{Lt}}\ sI(s)$

2. Let, $\mathcal{L}\{i(t)\} = I(s)$

   $$\therefore\ \mathcal{L}\left\{\frac{di(t)}{dt}\right\} = sI(s) - i(0^+)$$

   Initial rate of rise of current, $\left.\dfrac{di(t)}{dt}\right|_{t=0^+} = \underset{t \to 0}{\text{Lt}}\ \dfrac{di(t)}{dt} = \underset{s \to \infty}{\text{Lt}}\ s\left[sI(s) - i(0^+)\right]$

3. Let, $\mathcal{L}\{i(t)\} = I(s)$

   $$\therefore\ \mathcal{L}\left\{\frac{d^2 i(t)}{dt^2}\right\} = s^2 I(s) - si(0^+) - \left.\frac{di(t)}{dt}\right|_{t=0^+}$$

   Now,

   $$\left.\frac{d^2 i(t)}{dt^2}\right|_{t=0^+} = \underset{t \to 0}{\text{Lt}}\ \frac{d^2 i(t)}{dt^2} = \underset{s \to \infty}{\text{Lt}}\ s\left[s^2 I(s) - s\,i(0^+) - \left.\frac{di(t)}{dt}\right|_{t=0^+}\right]$$

Similarly, the initial voltage and initial rate of rise of voltage can be solved using the above equations after replacing i by v.

Final Condition

The final value theorem of Laplace transform states that if F(s) is Laplace transform of f(t) then,

$$\underset{t \to \infty}{\text{Lt}}\ f(t) = \underset{s \to 0}{\text{Lt}}\ sF(s)$$

$$\boxed{\therefore\ \text{Final value of } f(t) = f(\infty) = \underset{s \to 0}{\text{Lt}}\ sF(s)}$$

Hence, by the final value theorem, the final value of a time domain function can be directly determined from its s-domain function.

In transient analysis, the final value theorem is useful to determine the following final conditions in circuits.

1. Let, $\mathcal{L}\{i(\mathrm{t})\} = \mathrm{I(s)}$

   Now,

   Final current, $i(\infty) = \underset{\mathrm{t}\to\infty}{\mathrm{Lt}}\ i(\mathrm{t}) = \underset{\mathrm{s}\to 0}{\mathrm{Lt}}\ \mathrm{sI(s)}$

2. Let, $\mathcal{L}\{v(\mathrm{t})\} = \mathrm{V(s)}$

   Now,

   Final voltage, $v(\infty) = \underset{\mathrm{t}\to\infty}{\mathrm{Lt}}\ v(\mathrm{t}) = \underset{\mathrm{s}\to 0}{\mathrm{Lt}}\ \mathrm{sV(s)}$

3.3 Transient Response of RL Circuit

3.3.1 Natural or Source-Free Response of RL Circuit

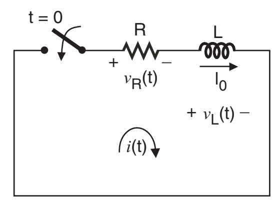


Fig. 3.17.

Consider the RL circuit with initial current $i(0^-) = \mathrm{I}_0$ through inductor as shown in Fig. 3.17. Let the switch be closed at t = 0.

Let, $i(\mathrm{t})$ = Current through the RL circuit

$v_R(\mathrm{t})$ = Voltage across resistance, R

$v_L(\mathrm{t})$ = Voltage across inductance, L

Now, the transient equations of the source-free RL circuit shown in Fig. 3.17 are,

$$i(\mathrm{t}) = \mathrm{I}_0 e^{-\frac{t}{\tau}} \qquad .....(3.10)$$

$$v_R(\mathrm{t}) = \mathrm{RI}_0 e^{-\frac{t}{\tau}}$$

$$v_L(\mathrm{t}) = -\mathrm{RI}_0 e^{-\frac{t}{\tau}}$$

$$\text{where, } \tau = \frac{L}{R} = \text{Time constant of the RL circuit} \qquad .....(3.11)$$

Equation (3.10) is called **source-free response** of RL circuit.

Note : *It can be proved that Henry/Ohm has the unit of time and so the unit of time constant is seconds.*

Proof:

The s-domain equivalent of the RL circuit is shown in Fig. 3.18.

With reference to Fig. 3.18, we can write,

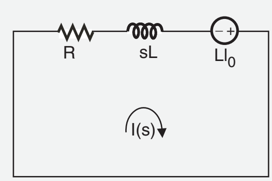


Fig. 3.18.

$$I(s) = \frac{L\,I_0}{R + sL} \Rightarrow I(s) = \frac{LI_0}{L\left(\frac{R}{L} + s\right)}$$

$$\therefore\ I(s) = \frac{I_0}{s + \frac{R}{L}}$$

Let us take the inverse Laplace transform of the above equation.

$$\therefore\ \mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{I_0}{s+\frac{R}{L}}\right\} \qquad \boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

$$\therefore\ i(t) = I_0 e^{-\frac{R}{L}t}$$

$$= I_0 e^{-\frac{t}{L/R}} = I_0 e^{-\frac{t}{\tau}}$$

$$\text{where, } \tau = \frac{L}{R} = \text{Time constant of the circuit.}$$

The equations for $v_R(t)$ and $v_L(t)$ can be obtained by using the equation for $i(t)$ as shown below:

$$v_R(t) = Ri(t) = R\,I_0 e^{-\frac{t}{\tau}}$$

$$v_L(t) = L\frac{di(t)}{dt} = L\frac{d}{dt}\left(I_0 e^{-\frac{t}{\tau}}\right) = LI_0 e^{-\frac{t}{\tau}} \times \left(-\frac{1}{\tau}\right) = -L\,I_0 e^{-\frac{t}{\tau}} \times \frac{R}{L} = -R\,I_0 e^{-\frac{t}{\tau}}$$

Let us calculate i(t) using equation (3.10) for various values of time in multiples of time constant as shown below:

$$\text{At t} = 1\tau,\ i(t) = I_0 e^{-\frac{1\tau}{\tau}} = I_0 e^{-1} = 0.368\,I_0$$

$$\text{At t} = 2\tau,\ i(t) = I_0 e^{-\frac{2\tau}{\tau}} = I_0 e^{-2} = 0.135\,I_0$$

$$\text{At t} = 3\tau,\ i(t) = I_0 e^{-\frac{3\tau}{\tau}} = I_0 e^{-3} = 0.05\,I_0$$

$$\text{At t} = 4\tau,\ i(t) = I_0 e^{-\frac{4\tau}{\tau}} = I_0 e^{-4} = 0.018\,I_0$$

$$\text{At t} = 5\tau,\ i(t) = I_0 e^{-\frac{5\tau}{\tau}} = I_0 e^{-5} = 0.007\,I_0$$

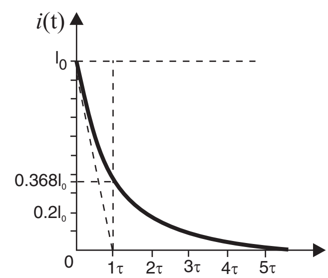


***Fig. 3.19 :** i(t) Vs t.*

From the above analysis we can say that the initial current exponentially decays to zero as time t tends to infinity.

3.3.2 Step Response of RL Circuit *(AU May'15, 8 Marks)*

(Response of RL Circuit Excited by DC Supply)

***Note :** A step voltage applied at t = 0 is equivalent to switching DC supply at t = 0.*

Consider the RL circuit with no initial inductor current and excited by a step voltage of E volts as shown in Fig. 3.20. Let the switch be closed at t = 0.

Let, i(t) = Current through RL circuit

v_R(t) = Voltage across resistance, R

v_L(t) = Voltage across inductance, L

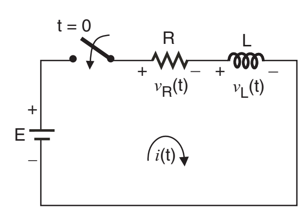


Fig. 3.20.

Now the transient equations of the RL circuit excited by a dc source shown in Fig. 3.20 are,

$$i(t) = \frac{E}{R}\left(1 - e^{-\frac{t}{\tau}}\right) \qquad .....(3.12)$$

$$v_L(t) = Ee^{-\frac{t}{\tau}}$$

$$v_R(t) = E\left(1 - e^{-\frac{t}{\tau}}\right)$$

where, $\tau = \frac{L}{R}$ = Time constant of the RL circuit

Equation (3.12) is called the **forced response** of the RL circuit when excited by a dc source.

Proof:

Let, $\mathcal{L}\{i(t)\} = I(s)$; $\mathcal{L}\{v_R(t)\} = V_R(s)$; $\mathcal{L}\{v_L(t)\} = V_L(s)$; $\mathcal{L}\{E\} = \frac{E}{s}$

The s-domain equivalent of the RL circuit is shown in Fig. 3.21.

By Ohm's law, we get,

$$V_R(s) = R\,I(s) \quad ; \quad V_L(s) = sL\,I(s) \qquad .....(3.13)$$

With reference to Fig. 3.21, by KVL, we can write,

$$V_R(s) + V_L(s) = \frac{E}{s}$$

$$RI(s) + sLI(s) = \frac{E}{s} \qquad \boxed{\text{Using equation (3.13)}}$$

$$(R + sL)\,I(s) = \frac{E}{s}$$

$$\therefore\; I(s) = \frac{E}{s(R + sL)}$$

$$= \frac{E}{s \times L\left(s + \frac{R}{L}\right)} = \frac{\frac{E}{L}}{s\left(s + \frac{R}{L}\right)}$$

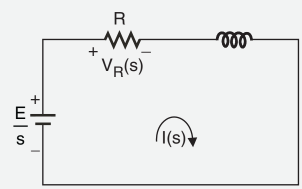


Fig. 3.21.

By partial fraction expansion technique, the above equation can be expressed as,

$$I(s) = \frac{\frac{E}{L}}{s\left(s + \frac{R}{L}\right)} = \frac{K_1}{s} + \frac{K_2}{s + \frac{R}{L}}$$

$$K_1 = \frac{\frac{E}{L}}{s\left(s + \frac{R}{L}\right)} \times s\,\Bigg|_{s=0} = \frac{\frac{E}{L}}{\left(s + \frac{R}{L}\right)}\Bigg|_{s=0} = \frac{\frac{E}{L}}{\frac{R}{L}} = \frac{E}{R}$$

$$K_2 = \frac{\frac{E}{L}}{s\left(s + \frac{R}{L}\right)} \times \left(s + \frac{R}{L}\right)\Bigg|_{s=-\frac{R}{L}} = \frac{\frac{E}{L}}{s}\Bigg|_{s=-\frac{R}{L}} = \frac{\frac{E}{L}}{-\frac{R}{L}} = -\frac{E}{R}$$

$$\therefore\; I(s) = \frac{\frac{E}{R}}{s} - \frac{\frac{E}{R}}{s + \frac{R}{L}} = \frac{E}{R}\frac{1}{s} - \frac{E}{R}\frac{1}{s + \frac{R}{L}}$$

Let us take the inverse Laplace transform of the above equation.

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{E}{R}\frac{1}{s} - \frac{E}{R}\frac{1}{s+\frac{R}{L}}\right\}$$

$$\boxed{\begin{array}{l}\mathcal{L}\{1\} = \dfrac{1}{s} \\ \mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}\end{array}}$$

$$\therefore\ i(t) = \frac{E}{R}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{E}{R}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{R}{L}}\right\}$$

$$= \frac{E}{R} - \frac{E}{R}e^{-\frac{R}{L}t}$$

$$= \frac{E}{R}\left(1 - e^{-\frac{t}{L/R}}\right)$$

$$\therefore\ i(t) = \frac{E}{R}\left(1 - e^{-\frac{t}{\tau}}\right) \qquad .....(3.14)$$

where, $\tau = \frac{L}{R}$ = *Time constant of the circuit.*

With reference to Fig. 3.20, by Ohm's law, we can write,

$$v_R(t) = R\,i(t)$$

$$= R \times \frac{E}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$\boxed{\text{Using equation (3.14)}}$$

$$\therefore\ v_R(t) = E\left(1 - e^{-\frac{t}{\tau}}\right) \qquad .....(3.15)$$

With reference to Fig. 3.20, by KVL, we can write,

$$v_R(t) + v_L(t) = E$$

$$\therefore\ v_L(t) = E - v_R(t)$$

$$= E - E\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$\boxed{\text{Using equation (3.15)}}$$

$$\therefore\ v_L(t) = Ee^{-\frac{t}{\tau}}$$

Initial and Final Conditions

Let us examine the values of current and voltages of an RL circuit excited by a dc source at $t = 0^+$ and at $t = \infty$.

$$\boxed{\begin{array}{l}
\text{At } t = 0^+,\ i(0^+) = \dfrac{E}{R}\left(1 - e^0\right) = \dfrac{E}{R}\left(1 - 1\right) = 0 \\
\text{At } t = 0^+,\ v_L(0^+) = E\,e^0 = E \times 1 = E \\
\text{At } t = 0^+,\ v_R(0^+) = E(1 - e^0) = E(1 - 1) = 0 \\
\text{At } t = \infty,\ i(\infty) = \dfrac{E}{R}\left(1 - e^{-\infty}\right) = \dfrac{E}{R}\left(1 - 0\right) = \dfrac{E}{R} \\
\text{At } t = \infty,\ v_L(\infty) = E\,e^{-\infty} = E \times 0 = 0 \\
\text{At } t = \infty,\ v_R(\infty) = E(1 - e^{-\infty}) = E(1 - 0) = E
\end{array}}$$

From the above analysis, we can infer that "*at t = 0⁺, the current through inductance is zero and so it behaves as an open circuit. At t = ∞, the voltage across inductance is zero and so it behaves as a short circuit*".

The condition of the circuit at $t = 0^+$ is called the **initial condition** and the condition of the circuit at $t = \infty$ is called the **final condition**.

In the transient equations of the RL circuit, the term containing $e^{-\frac{t}{\tau}}$ tends to zero as t tends to infinity and so the term containing $e^{-\frac{t}{\tau}}$ is the **natural response** (or **complementary function** or **transient part**).

The term E/R is the steady state value of $i(t)$ and the term E is the steady state value of $v_R(t)$. The steady state value of $v_L(t)$ is zero.

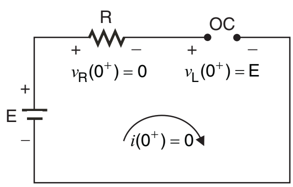


Fig. a : *Initial condition.*

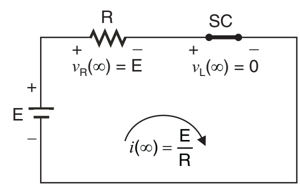


Fig. b : *Final condition.*

Fig. 3.22 : *Initial and final condition of RL circuit of Fig. 3.20.*

Let us examine the values of current and voltage when t is equal to one time constant.

$$\text{At } t = 1\tau, \quad i(t) = \frac{E}{R}\left(1 - e^{-\frac{1\tau}{\tau}}\right) = \frac{E}{R}\left(1 - e^{-1}\right) = 0.6321\frac{E}{R}$$

$$\text{At } t = 1\tau, \quad v_L(t) = e^{-\frac{1\tau}{\tau}} = E\, e^{-1} = 0.3679E$$

$$\text{At } t = 1\tau, \quad v_R(t) = E\left(1 - e^{-\frac{1\tau}{\tau}}\right) = E\left(1 - e^{-1}\right) = 0.6321E$$

From the above calculations, we can say that, "*the current through inductance rises from zero to 63.21% of steady state value in a time of one time constant*". Also we can say that, "*the voltage across the inductance falls from initial value to 36.79% of initial value in a time of one time constant*". These two measures are also used to define time constant.

Theoretically, the circuit attains steady state only at infinite time. But for practical purposes, we can show that the circuit attains steady state in a time of five time constants. Let us calculate $i(t)$ for various values of time in multiples of time constant as shown below:

$$\text{At } t = 1\tau, \quad i(t) = \frac{E}{R}\left(1 - e^{-\frac{1\tau}{\tau}}\right) = \frac{E}{R}\left(1 - e^{-1}\right) = 0.6321\frac{E}{R}$$

$$\text{At } t = 2\tau, \quad i(t) = \frac{E}{R}\left(1 - e^{-\frac{2\tau}{\tau}}\right) = \frac{E}{R}\left(1 - e^{-2}\right) = 0.8647\frac{E}{R}$$

$$\text{At t} = 3\tau, \quad i(\text{t}) = \frac{\text{E}}{\text{R}}\left(1 - \text{e}^{-\frac{3\tau}{\tau}}\right) = \frac{\text{E}}{\text{R}}\left(1 - \text{e}^{-3}\right) = 0.9502\frac{\text{E}}{\text{R}}$$

$$\text{At t} = 4\tau, \quad i(\text{t}) = \frac{\text{E}}{\text{R}}\left(1 - \text{e}^{-\frac{4\tau}{\tau}}\right) = \frac{\text{E}}{\text{R}}\left(1 - \text{e}^{-4}\right) = 0.9817\frac{\text{E}}{\text{R}}$$

$$\text{At t} = 5\tau, \quad i(\text{t}) = \frac{\text{E}}{\text{R}}\left(1 - \text{e}^{-\frac{5\tau}{\tau}}\right) = \frac{\text{E}}{\text{R}}\left(1 - \text{e}^{-5}\right) = 0.9933\frac{\text{E}}{\text{R}}$$

From the above calculations we can say that the current i(t) approximately reaches steady value in a time of five time constants. Hence, "*for all practical purposes the transient period is assumed to be for a duration of five time constants, and after five time constants, the circuit is said to be in a steady state*". **(AU May'17, 2 Marks)**

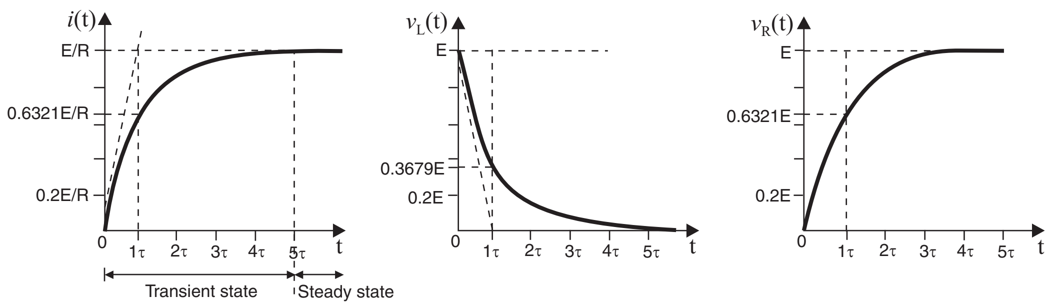


Fig. a : *i(t) vs t.* ***Fig. b :*** $v_L(t)$ *vs t.* ***Fig. c :*** $v_R(t)$ *vs t.*

Fig. 3.23 : *The sketch of transient current and voltages of the RL circuit of Fig. 3.20.*

Time Constant

The time constant of an RL circuit is defined as the time taken by the inductor current to reach a steady state if the initial rate of rise is maintained. It can be proved that the time constant, τ = L/R, for an RL circuit.

Proof:

In the RL circuit, the inductance delays the rate of rise of current. The rate of rise of current is obtained by differentiating i(t) with respect to t.

Here, $i(t) = \frac{E}{R}(1 - e^{-\frac{t}{\tau}})$

On differentiating i(t) with respect to t, we get,

$$\frac{di(t)}{dt} = \frac{E}{R}(-e^{-\frac{t}{\tau}})\left(-\frac{1}{\tau}\right) \quad \Rightarrow \quad \frac{di(t)}{dt} = \frac{E}{R\tau}e^{-\frac{t}{\tau}}$$

The value of $\frac{di(t)}{dt}$ *at* $t = 0^+$ *is the initial rate of rise of current.*

$$\therefore \left.\frac{di(t)}{dt}\right|_{t=0^+} = \frac{E}{R\tau}e^0 = \frac{E}{R\tau} = \frac{E}{R \times \frac{L}{R}} = \frac{E}{L}$$

For small time intervals, $\frac{di(t)}{dt} = \frac{\Delta i(t)}{\Delta t}$, *and so using the above equation we can write,*

$$\left.\frac{di(t)}{dt}\right|_{t=0^+} = \frac{\Delta i(t)}{\Delta t} = \frac{E}{L} \qquad \Rightarrow \qquad \Delta t = \frac{L}{E} \times \Delta i(t)$$

In the above equation if, $\Delta i(t) = \frac{E}{R}$

then, $\Delta t = \frac{L}{E} \times \Delta i(t) = \frac{L}{E} \times \frac{E}{R} = \frac{L}{R}$

From the above analysis we can say that if the initial rate of rise of current is maintained then the current would have reached the steady value of E/R in a time of $\Delta t = L/R$, *which is called time constant of RL circuit.*

$$\therefore \text{ Time constant, } \tau = \frac{L}{R} \qquad .....(3.16)$$

From the above discussions we can say that the time constant of the RL circuit may be defined in different ways. The various definitions of time constants are summarised below:

Definitions of Time Constant of RL Circuit

- The time constant of the RL circuit is defined as the time taken by the current through the inductance to reach a steady value if the initial rate of rise is maintained.
- The time constant of the RL circuit is defined as the ratio of inductance and resistance of the circuit.
- The time constant of the RL circuit is defined as the time taken by the current through the inductance to reach 63.21% of the final steady value.
- The time constant of the RL circuit is defined as the time taken by the voltage across the inductance to fall to 36.79% of the initial value.

3.3.3 RL Transient with Initial Current I_0

Case i : When I_0 is in a direction opposite to that of i(t)

Consider the RL circuit with initial current $i(0^-) = -I_0$ through inductor as shown in Fig. 3.24. Let the switch be closed at t = 0.

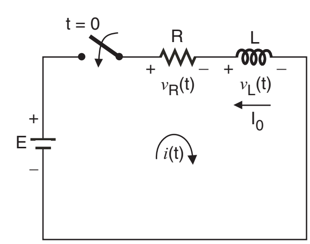


Fig. 3.24.

Let, $i(t)$ = Current through the RL circuit

$v_R(t)$ = Voltage across resistance, R

$v_L(t)$ = Voltage across inductance, L

Now, the response $i(t)$ of RL circuit shown in Fig. 3.24 is,

$$i(t) = \frac{E}{R}\left(1 - e^{-\frac{t}{\tau}}\right) - I_0 e^{-\frac{t}{\tau}} \qquad .....(3.17)$$

where, $\tau = \frac{L}{R}$ = Time constant

Proof:

Let, $\mathcal{L}\{i(t)\} = I(s)$; $\mathcal{L}\{v_R(t)\} = V_R(s)$;

$\mathcal{L}\{v_L(t)\} = V_L(s)$; $L\{E\} = \frac{E}{s}$

The s-domain equivalent of the RL circuit is shown in Fig. 3.25.

In Fig. 3.25, $V_R(s) = R\,I(s)$ .....(3.18)

$V_L(s) = sL\,I(s) + LI_0$ .....(3.19)

With reference to Fig. 3.25, by KVL, we can write,

$$V_R(s) + V_L(s) = \frac{E}{s}$$

$$R\,I(s) + sL\,I(s) + LI_0 = \frac{E}{s}$$

$$(R + sL)\,I(s) = \frac{E}{s} - LI_0$$

$$L\left(s + \frac{R}{L}\right) I(s) = \frac{E - sLI_0}{s}$$

$$\therefore\ I(s) = \frac{\frac{E}{L} - sI_0}{s\left(s + \frac{R}{L}\right)}$$

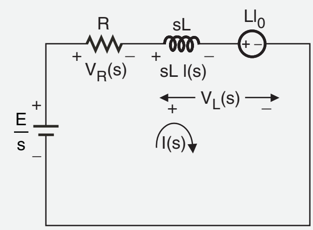


Fig. 3.25.

Using equations (3.18) and (3.19)

By partial fraction expansion technique, the above equation can be expressed as,

$$I(s) = \frac{\frac{E}{L} - sI_0}{s\left(s + \frac{R}{L}\right)} = \frac{K_1}{s} + \frac{K_2}{s + \frac{R}{L}}$$

$$K_1 = \left.\frac{\frac{E}{L} - sI_0}{s\left(s + \frac{R}{L}\right)} \times s\right|_{s=0} = \left.\frac{\frac{E}{L} - sI_0}{s + \frac{R}{L}}\right|_{s=0} = \frac{\frac{E}{L}}{\frac{R}{L}} = \frac{E}{R}$$

$$K_2 = \left.\frac{\frac{E}{L} - sI_0}{s\left(s + \frac{R}{L}\right)} \times \left(s + \frac{R}{L}\right)\right|_{s=-\frac{R}{L}} = \left.\frac{\frac{E}{L} - sI_0}{s}\right|_{s=-\frac{R}{L}} = \frac{\frac{E}{L} + \frac{R}{L}I_0}{\frac{-R}{L}} = -\left(\frac{E}{R} + I_0\right)$$

$$\therefore\ I(s) = \frac{\frac{E}{R}}{s} - \frac{\frac{E}{R} + I_0}{\left(s + \frac{R}{L}\right)} = \frac{E}{R}\frac{1}{s} - \left(\frac{E}{R} + I_0\right)\left(\frac{1}{s + \frac{R}{L}}\right)$$

Let us take the inverse Laplace transform of the above equation.

$$\therefore\ \mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{E}{R}\frac{1}{s} - \left(\frac{E}{R} + I_0\right)\left(\frac{1}{s + \frac{R}{L}}\right)\right\}$$

$$\therefore\ i(t) = \frac{E}{R}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \left(\frac{E}{R}+I_0\right)\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{R}{L}}\right\}$$

$$= \frac{E}{R} - \left(\frac{E}{R}+I_0\right)e^{-\frac{R}{L}t}$$

$$= \frac{E}{R} - \frac{E}{R}e^{-\frac{R}{L}t} - I_0 e^{-\frac{R}{L}t} \quad = \frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right) - I_0 e^{-\frac{R}{L}t}$$

$$\therefore\ i(t) = \frac{E}{R}\left(1-e^{-\frac{t}{\tau}}\right) - I_0 e^{-\frac{t}{\tau}}$$

$$\textit{where, } \tau = \frac{L}{R} = \textit{Time constant}$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

Let us examine the values of i(t) at $t = 0^+$ and $t = \infty$.

$$\text{At } t = 0^+,\quad i(0^+) = \frac{E}{R}\left(1-e^0\right) - I_0 e^0 = \frac{E}{R}\left(1-1\right) - I_0 \times 1 = -I_0$$

$$\text{At } t = \infty,\quad i(\infty) = \frac{E}{R}\left(1-e^{-\infty}\right) - I_0 e^{-\infty} = \frac{E}{R}\left(1-0\right) - I_0 \times 0 = \frac{E}{R}$$

From the above analysis we can say that the current i(t) has a component due to I_0, which decays exponentially to zero as time t tends to infinity.

Also, on comparing the above results with the RL transient without initial current, we can conclude that the steady state value is not affected by the initial current in the inductance.

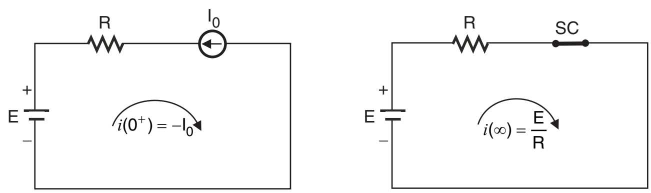


***Fig. a :** Initial condition.* ***Fig. b :** Final condition.*

***Fig. 3.26 :** Initial and final condition of the RL circuit of Fig. 3.24.*

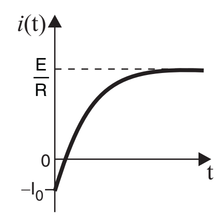


***Fig. 3.27 :** i(t) Vs t.*

Case ii : When I_0 is in the same direction as i(t)

Consider the RL circuit with initial current, $i(0^-) = +I_0$ through the inductor as shown in Fig. 3.28,

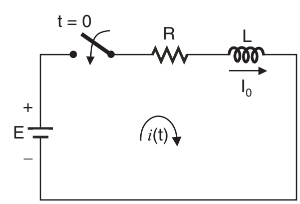


Fig. 3.28.

The response of this circuit can be obtained from case (i) by replacing I_0 by $-I_0$.

On replacing I_0 of equation (3.17) by $-I_0$, we get,

$$i(t) = \frac{E}{R}\left(1-e^{-\frac{t}{\tau}}\right) + I_0 e^{-\frac{t}{\tau}} \qquad .....(3.20)$$

Let us examine the values of i(t) at $t = 0^+$ and $t = \infty$.

$$\text{At } t = 0^+,\quad i(0^+) = \frac{E}{R}\left(1-e^0\right) + I_0 e^0 = \frac{E}{R}\left(1-1\right) + I_0 \times 1 = I_0$$

$$\text{At } t = \infty,\quad i(\infty) = \frac{E}{R}\left(1-e^{-\infty}\right) + I_0 e^{-\infty} = \frac{E}{R}\left(1-0\right) + I_0 \times 0 = \frac{E}{R}$$

From the above analysis we can say that the current $i(t)$ has a component due to I_0, which decays exponentially to zero as time t tends to infinity.

Also, on comparing the above results with the RL transient without initial current, we can conclude that the steady state value is not affected by the initial current in the inductance.

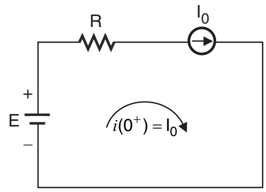


***Fig. a :** Initial condition.*

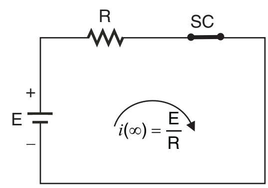


***Fig. b :** Final condition.*

***Fig. 3.29 :** Initial and final condition of the circuit of Fig. 3.28.*

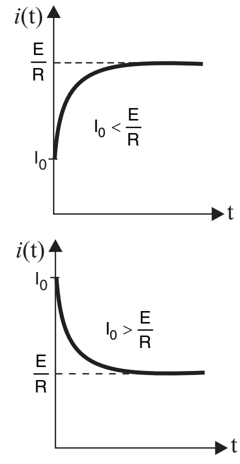


***Fig. 3.30 :** i(t) Vs t.*

> ***Note :*** *When initial flux is specified in terms of weber-turns Ψ then the initial current can be calculated using the relation,* $L = \frac{\Psi}{I}$.
>
> $\therefore$ *Initial current,* $I_0 = \frac{\Psi_0}{L}$, *where* Ψ_0 *is the initial flux linkage.*

3.4 Transient Response of RC Circuit

3.4.1 Natural or Source-Free Response of RC Circuit

Consider the RC circuit with initial voltage, $v_C(0^-) = -V_0$ across the capacitor as shown in Fig. 3.31. Let the switch be closed at t = 0.

Let, $i(t)$ = Current through the RC circuit

$v_R(t)$ = Voltage across resistance, R

$v_C(t)$ = Voltage across capacitance, C

Now the transient equations of the source-free RC circuit shown in Fig. 3.31 are,

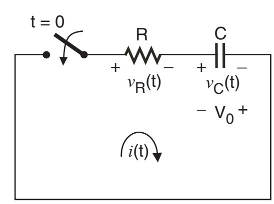


Fig. 3.31.

$$v_C(t) = -V_0 e^{-\frac{t}{\tau}} \quad .....(3.21)$$

$$i(t) = \frac{V_0}{R} e^{-\frac{t}{\tau}}$$

$$v_R(t) = V_0 e^{-\frac{t}{\tau}}$$

where, τ = RC = Time constant of the RC circuit ......(3.22)

Equation (3.21) is called **source-free response** of the RC circuit.

> ***Note :*** *It can be proved that Ohm × Farad has the unit of time and so the unit of time constant is seconds.*

Proof:

The s-domain equivalent of the RC circuit is shown in Fig. 3.32.

With reference to Fig. 3.32, we can write,

$$I(s) = \frac{\frac{V_0}{s}}{R + \frac{1}{sC}} \Rightarrow I(s) = \frac{V_0}{s\left(\frac{sRC+1}{sC}\right)}$$

$$\therefore\ I(s) = \frac{V_0}{\frac{1}{C}(sRC+1)} = \frac{V_0}{\frac{1}{C} \times RC\left(s + \frac{1}{RC}\right)}$$

$$= \frac{V_0}{R\left(s + \frac{1}{RC}\right)}$$

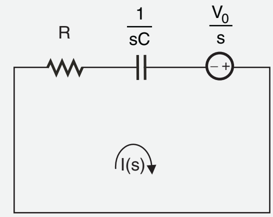


Fig. 3.32.

Let us take the inverse Laplace transform of the above equation.

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{V_0}{R\left(s+\frac{1}{RC}\right)}\right\} \Rightarrow i(t) = \frac{V_0}{R}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\right\}$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

$$\therefore\ i(t) = \frac{V_0}{R} e^{-\frac{t}{RC}} = \frac{V_0}{R} e^{-\frac{t}{\tau}}$$

where, $\tau = RC$ = *Time constant of the RC circuit.*

The equations for $v_C(t)$ *and* $v_R(t)$ *can be obtained using the equation of* $i(t)$*, as shown below:*

$$v_C(t) = \frac{1}{C}\int i(t)\,dt = \frac{1}{C}\int \frac{V_0}{R} e^{\frac{-t}{\tau}}\,dt = \frac{V_0}{RC}\frac{e^{\frac{-t}{\tau}}}{\frac{-1}{\tau}}$$

$$= -\frac{V_0}{RC}\tau\, e^{\frac{-t}{\tau}} = -\frac{V_0}{RC}RC\, e^{\frac{-t}{\tau}} = -V_0 e^{\frac{-t}{\tau}}$$

$$v_R(t) = Ri(t) = R \times \frac{V_0}{R} e^{\frac{-t}{\tau}} = V_0 e^{\frac{-t}{\tau}}$$

Let us calculate the voltage across the capacitance $v_C(t)$ for various values of time constant using equation (3.21).

$$\text{At } t = 1\tau,\quad v_C(t) = -V_0 e^{-\frac{1\tau}{\tau}} = -V_0 e^{-1} = -0.368V_0$$

$$\text{At } t = 2\tau,\quad v_C(t) = -V_0 e^{-\frac{2\tau}{\tau}} = -V_0 e^{-2} = -0.135V_0$$

$$\text{At } t = 3\tau,\quad v_C(t) = -V_0 e^{-\frac{3\tau}{\tau}} = -V_0 e^{-3} = -0.05V_0$$

$$\text{At } t = 4\tau,\quad v_C(t) = -V_0 e^{-\frac{4\tau}{\tau}} = -V_0 e^{-4} = -0.018V_0$$

$$\text{At } t = 5\tau,\quad v_C(t) = -V_0 e^{-\frac{5\tau}{\tau}} = -V_0 e^{-5} = -0.007V_0$$

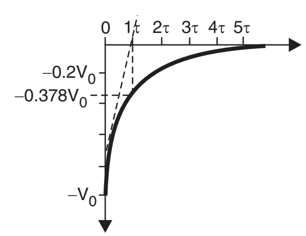


Fig. 3.33 : $v_C(t)$ *Vs t.*

From the above analysis, we can say that the initial voltage exponentially decays to zero as time t tends to infinity.

3.4.2 Step Response of RC Circuit *(AU May'15, 8 Marks)*

(Response of RC Circuit Excited by DC Supply)

Note : *A step voltage applied at t = 0 is equivalent to switching dc supply at t = 0.*

Consider the RC circuit with no initial capacitor voltage and excited by a step voltage of E volts as shown in Fig. 3.34. Let the switch be closed at t = 0.

Let, $i(t)$ = Current through the RC circuit

$v_R(t)$ = Voltage across resistance, R

$v_C(t)$ = Voltage across capacitance, C

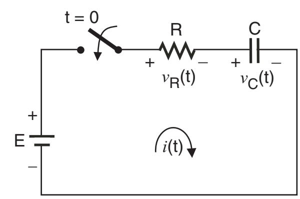


Fig. 3.34.

Now, the transient equations of the RC circuit excited by a dc source shown in Fig. 3.34 are,

$$v_C(t) = E\left(1 - e^{-\frac{t}{\tau}}\right) \qquad .....(3.23)$$

$$i(t) = \frac{E}{R}e^{-\frac{t}{\tau}}$$

$$v_R(t) = Ee^{-\frac{t}{\tau}}$$

where, $\tau = RC$ = Time constant of the RC circuit. .....(3.24)

Equation (3.23) is called **forced response** of the RC circuit when excited by a dc source.

Proof:

Let, $\mathcal{L}\{i(t)\} = I(s)$; $\mathcal{L}\{v_R(t)\} = V_R(s)$; $\mathcal{L}\{v_C(t)\} = V_C(s)$; $L\{E\} = \frac{E}{s}$

The s-domain equivalent of the RC circuit is shown in Fig. 3.35.

With reference to Fig. 3.35, by voltage division rule, we can write,

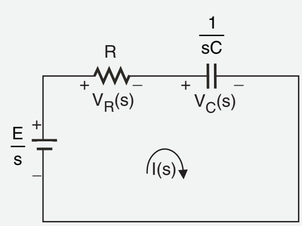


Fig. 3.35.

$$V_C(s) = \frac{E}{s} \times \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{E}{s} \times \frac{\frac{1}{sC}}{\frac{sRC+1}{sC}} = \frac{E}{s} \times \frac{1}{RC\left(s + \frac{1}{RC}\right)}$$

$$\therefore \; V_C(s) = \frac{\frac{E}{RC}}{s\left(s + \frac{1}{RC}\right)}$$

By partial fraction expansion technique, the above equation can be expressed as,

$$V_C(s) = \frac{\frac{E}{RC}}{s\left(s + \frac{1}{RC}\right)} = \frac{K_1}{s} + \frac{K_2}{s + \frac{1}{RC}}$$

$$K_1 = \left.\frac{\frac{E}{RC}}{s\left(s + \frac{1}{RC}\right)} \times s\right|_{s=0} = \left.\frac{\frac{E}{RC}}{s + \frac{1}{RC}}\right|_{s=0} = \frac{\frac{E}{RC}}{\frac{1}{RC}} = E$$

$$K_2 = \left.\frac{\frac{E}{RC}}{s\left(s+\frac{1}{RC}\right)}\times\left(s+\frac{1}{RC}\right)\right|_{s=-\frac{1}{RC}} = \left.\frac{\frac{E}{RC}}{s}\right|_{s=-\frac{1}{RC}} = \frac{\frac{E}{RC}}{\frac{-1}{RC}} = -E$$

$$\boxed{\begin{array}{l}\mathcal{L}\{1\} = \frac{1}{s}\\ \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}\end{array}}$$

$$\therefore\ V_C(s) = \frac{E}{s} - \frac{E}{s+\frac{1}{RC}}$$

Let us take the inverse Laplace transform of the above equation.

$$\mathcal{L}^{-1}\{V_C(s)\} = \mathcal{L}^{-1}\left\{\frac{E}{s} - \frac{E}{s+\frac{1}{RC}}\right\} \Rightarrow v_C(t) = E\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - E\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\right\}$$

$$v_C(t) = E - Ee^{-\frac{t}{RC}} \Rightarrow v_C(t) = E\left(1-e^{-\frac{t}{RC}}\right)$$

$$\therefore\ v_C(t) = E\left(1-e^{-\frac{t}{\tau}}\right) \qquad .....(3.25)$$

where, $\tau = RC =$ Time constant of the RC circuit.

If v is the voltage across capacitance then the current i through the capacitance is given by, $i = C\frac{dv}{dt}$.

$$\therefore\ i(t) = C\frac{d}{dt}v_C(t)$$

$$\boxed{\text{Using equation (3.25)}}$$

$$= C\frac{d}{dt}\left[E\left(1-e^{-\frac{t}{\tau}}\right)\right]$$

$$\therefore\ i(t) = C\times E\left(-e^{-\frac{t}{\tau}}\right)\left(-\frac{1}{\tau}\right)$$

$$= \frac{C}{\tau}\times Ee^{-\frac{t}{\tau}} = \frac{C}{RC}\times Ee^{-\frac{t}{\tau}}$$

$$\therefore\ i(t) = \frac{E}{R}e^{-\frac{t}{\tau}}$$

With reference to Fig. 3.34, by KVL, we can write,

$$v_R(t) + v_C(t) = E$$

$$\therefore\ v_R(t) = E - v_C(t)$$

$$= E - E\left(1-e^{-\frac{t}{\tau}}\right)$$

$$\therefore\ v_R(t) = Ee^{-\frac{t}{\tau}}$$

Initial and Final Conditions

Let us examine the values of current and voltages of an RC circuit excited by a dc source, at $t = 0^+$ and at $t = \infty$.

$$\boxed{\begin{array}{l}\text{At } t=0^+,\ v_C(0^+) = E(1-e^0) = E(1-1) = 0\\ \text{At } t=0^+,\ i(0^+) = \frac{E}{R}e^0 = \frac{E}{R}\times 1 = \frac{E}{R}\\ \text{At } t=0^+,\ v_R(0^+) = Ee^0 = E\times 1 = E\end{array}}$$

$$\text{At } t = \infty, \quad v_C(\infty) = E\left(1 - e^{-\infty}\right) = E\left(1 - 0\right) = E$$

$$\text{At } t = \infty, \quad i(\infty) = \frac{E}{R}e^{-\infty} = \frac{E}{R}\times 0 = 0$$

$$\text{At } t = \infty, \quad v_R(\infty) = Ee^{-\infty} = E \times 0 = 0$$

From the above analysis we can infer that, "*at t = 0⁺, the voltage across capacitance is zero and so it behaves as short circuit. At t = ∞, the current through the capacitance is zero and so it behaves as an open circuit*".

The condition of the circuit at $t = 0^+$ is called the **initial condition** and the condition of the circuit at $t = \infty$ is called the **final condition**.

Fig. a : *Initial condition.* ***Fig. b :*** *Final condition.*

Fig. 3.36 : *Initial and final condition of RC circuit of Fig. 3.34.*

In the transient equations of the RC circuit, the term containing $e^{-\frac{t}{\tau}}$ tends to zero as t tends to infinity and so the term containing $e^{-\frac{t}{\tau}}$ is the **natural response** (or **complementary function** or **transient part**).

The term E is the steady state value of $v_C(t)$ and steady state value of $i(t)$ and $v_R(t)$ are zero.

Let us examine the values of voltages and current when t is equal to one time constant.

$$\text{At } t = 1\tau, \quad v_C(t) = E\left(1 - e^{-\frac{1\tau}{\tau}}\right) = E\left(1 - e^{-1}\right) = 0.6321E$$

$$\text{At } t = 1\tau, \quad i(t) = \frac{E}{R}e^{-\frac{1\tau}{\tau}} = \frac{E}{R}e^{-1} = 0.3679\frac{E}{R}$$

$$\text{At } t = 1\tau, \quad v_R(t) = Ee^{-\frac{1\tau}{\tau}} = Ee^{-1} = 0.3679E$$

From the above calculations we can say that, "*the voltage across the capacitance rises from zero to 63.21% of steady value in a time of one time constant*". Also we can say that, "*the current through the capacitance falls from initial value to 36.79% of initial value in a time of one time constant*". These two measures are used to define the time constant.

Theoretically, the circuit attains steady state only at an infinite time. But for practical purposes, we can show that the circuit attains steady state in a time of five time constants. Let us calculate $v_C(t)$ for various values of time in multiples of the time constant as shown below:

$$\text{At } t = 1\tau, \quad v_C(t) = E\left(1 - e^{-\frac{1\tau}{\tau}}\right) = E\left(1 - e^{-1}\right) = 0.6321E$$

$$\text{At } t = 2\tau, \quad v_C(t) = E\left(1 - e^{-\frac{2\tau}{\tau}}\right) = E\left(1 - e^{-2}\right) = 0.8647E$$

$$\text{At } t = 3\tau, \quad v_C(t) = E\left(1 - e^{-\frac{3\tau}{\tau}}\right) = E\left(1 - e^{-3}\right) = 0.9502E$$

$$\text{At } t = 4\tau, \quad v_C(t) = E\left(1 - e^{-\frac{4\tau}{\tau}}\right) = E\left(1 - e^{-4}\right) = 0.9817E$$

$$\text{At } t = 5\tau, \quad v_C(t) = E\left(1 - e^{-\frac{5\tau}{\tau}}\right) = E\left(1 - e^{-5}\right) = 0.9933E$$

From the above calculations we can say that the voltage $v_C(t)$ approximately reaches steady value in a time of five time constants. Hence, for all practical purposes, the transient period is assumed to be for a duration of five time constants and after five time constants the circuit is said to be in a steady state.

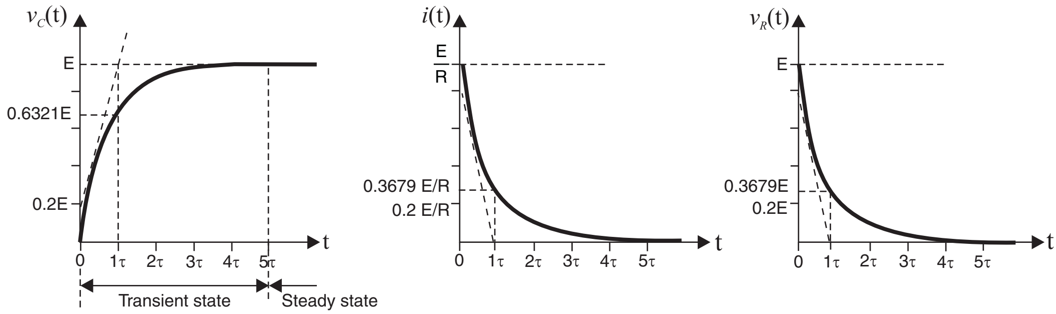


Fig. a : *$v_C(t)$ vs t.* **Fig. b :** *i(t) vs t.* **Fig. c :** *$v_R(t)$ vs t.*

Fig. 3.37 : *The sketch of transient current and voltages of the RC circuit of Fig. 3.34.*

Time Constant

The time constant of an RC circuit is defined as the time taken by the capacitor voltage to reach a steady state if the initial rate of rise is maintained. It can be proved that the time constant, $\tau = RC$, for an RC circuit.

Proof:

In the RC circuit, the capacitance delays the rate of rise of voltage. The rate of rise of voltage is obtained by differentiating $v_C(t)$ with respect to t.

Here, $v_c(t) = E\left(1 - e^{-\frac{t}{\tau}}\right)$

On differentiating $v_C(t)$ with respect to t, we get,

$$\frac{d}{dt}v_c(t) = \frac{d}{dt}E\left(1 - e^{-\frac{t}{\tau}}\right) \Rightarrow \frac{d}{dt}v_c(t) = E\left(-e^{-\frac{t}{\tau}}\right)\left(-\frac{1}{\tau}\right)$$

$$\therefore \frac{d}{dt}v_c(t) = \frac{E}{\tau}e^{-\frac{t}{\tau}}$$

The value of $\frac{dv_c(t)}{dt}$ *at t = 0⁺ is the initial rate of rise of voltage.*

$$\therefore \left.\frac{dv_c(t)}{dt}\right|_{t=0^+} = \frac{E}{\tau}e^0 = \frac{E}{\tau}\times 1 = \frac{E}{RC}$$

For small time intervals, $\frac{dv_c(t)}{dt} = \frac{\Delta v_c(t)}{\Delta t}$, *and so using the above equation we can write,*

$$\left.\frac{dv_c(t)}{dt}\right|_{t=0^+} = \frac{\Delta v_c(t)}{\Delta t} = \frac{E}{RC} \qquad \Rightarrow \qquad \Delta t = \frac{RC}{E} \times \Delta v_C(t)$$

In the above equation if, $\Delta v_C(t) = E$

$$\text{then,}\ \Delta t = \frac{RC}{E} \times \Delta v_C(t) = \frac{RC}{E} \times E = RC$$

From the above analysis we can say that if the initial rate of rise of voltage is maintained then the voltage would have reached the steady value of E in a time of $\Delta t = RC$*, which is called time constant of the RC circuit.*

$$\therefore\ \text{Time constant,}\ \tau = RC \qquad .....(3.26)$$

From the above discussions we can say that the time constant of the RC circuit may be defined in different ways. The various definitions of time constants are summarised here.

Definitions of Time Constant of RC Circuit

- The time constant of the RC circuit is defined as the time taken by the voltage across the capacitance to reach a steady value if the initial rate of rise is maintained.
- The time constant of the RC circuit is defined as the product of resistance and capacitance of the circuit.
- The time constant of the RC circuit is defined as the time taken by the voltage across the capacitance to reach 63.21% of the final steady value.
- The time constant of the RC circuit is defined as the time taken by the current through the capacitance to fall to 36.79% of the initial value.

3.4.3 RC Transient with Initial Voltage V_0

Case i : When polarity of V_0 is opposite to that of $v_C(t)$

Consider the RC circuit with initial voltage, $v_C(0^-) = -V_0$ across the capacitor as shown in Fig. 3.38. Let the switch be closed at $t = 0$.

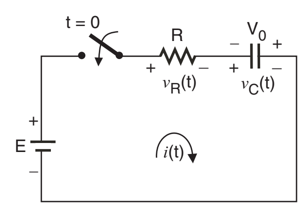


Fig. 3.38.

Let, $i(t)$ = Current through the RC circuit

$v_R(t)$ = Voltage across resistance, R

$v_C(t)$ = Voltage across capacitance, C

Now, the response $v_C(t)$ of the RC circuit shown in Fig. 3.38 is,

$$v_C(t) = E\left(1 - e^{-\frac{t}{\tau}}\right) - V_0\, e^{-\frac{t}{\tau}} \qquad .....(3.27)$$

where, $\tau = RC$ = Time constant.

Proof:

Let, $\mathcal{L}\{i(t)\} = I(s)$; $\mathcal{L}\{v_R(t)\} = V_R(s)$; $\mathcal{L}\{v_C(t)\} = V_C(s)$; $\mathcal{L}\{E\} = \frac{E}{s}$

The s-domain equivalent of the RC circuit is shown in Fig. 3.39.

With reference to Fig. 3.39, the expression for current I(s) can be written as,

$$I(s) = \frac{\frac{E}{s}+\frac{V_0}{s}}{R+\frac{1}{sC}} \Rightarrow I(s) = \left(\frac{E}{s}+\frac{V_0}{s}\right)\times\frac{1}{\frac{sRC+1}{sC}} = \left(\frac{E}{s}+\frac{V_0}{s}\right)\times\frac{sC}{sRC+1}$$

$$\therefore I(s) = \left(\frac{E}{s}+\frac{V_0}{s}\right)\frac{sC}{RC\left(s+\frac{1}{RC}\right)} = \left(\frac{E}{s}+\frac{V_0}{s}\right)\times\frac{s}{R\left(s+\frac{1}{RC}\right)}$$

$$= \frac{E}{R}\frac{1}{\left(s+\frac{1}{RC}\right)} + \frac{V_0}{R}\frac{1}{\left(s+\frac{1}{RC}\right)} \qquad .....(3.28)$$

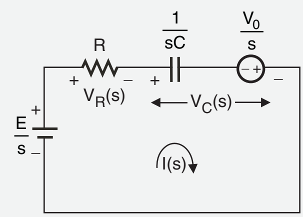


Fig. 3.39.

By Ohm's law, we can write, $V_R(s) = R\,I(s)$ .....(3.29)

With reference to Fig. 3.39, by KVL, we can write,

$$V_R(s)+V_C(s) = \frac{E}{s}$$

$$\therefore V_C(s) = \frac{E}{s}-V_R(s)$$

$$= \frac{E}{s}-RI(s) = \frac{E}{s}-R\left[\frac{E}{R}\frac{1}{\left(s+\frac{1}{RC}\right)}+\frac{V_0}{R}\frac{1}{\left(s+\frac{1}{RC}\right)}\right]$$

$$= \frac{E}{s}-E\frac{1}{\left(s+\frac{1}{RC}\right)}-V_0\frac{1}{\left(s+\frac{1}{RC}\right)}$$

Using equations (3.28) and (3.29)

Let us take the inverse Laplace transform of the above equation.

$$\therefore \mathcal{L}^{-1}\{V_C(s)\} = \mathcal{L}^{-1}\left\{\frac{E}{s}-E\frac{1}{\left(s+\frac{1}{RC}\right)}-V_0\frac{1}{\left(s+\frac{1}{RC}\right)}\right\}$$

$$\therefore v_C(t) = E\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-E\,\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\right\}-V_0\,\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\right\}$$

$$= E-Ee^{-\frac{t}{RC}}-V_0e^{-\frac{t}{RC}}$$

$$= E\left(1-e^{-\frac{t}{\tau}}\right)-V_0e^{-\frac{t}{\tau}}$$

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

where, $\tau = RC =$ *Time constant.*

Let us examine the values of $v_C(t)$ at $t = 0^+$ and $t = \infty$

$$\text{At } t = 0^+, \quad v_C(0^+) = E(1 - e^0) - V_0\, e^0 = E(1 - 1) - V_0 \times 1 = -V_0$$

$$\text{At } t = \infty, \quad v_C(\infty) = E(1 - e^{-\infty}) - V_0\, e^{-\infty} = E(1 - 0) - V_0 \times 0 = E$$

From the above analysis, we can say that the voltage $v_C(t)$ has a component due to V_0, which decays exponentially to zero as time t tends to infinity.

Also, on comparing the above results with the RC transient without initial voltage, we can conclude that the steady state value is not affected by the initial voltage in the capacitance.

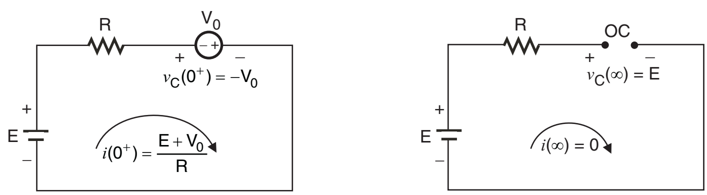


Fig. a : *Initial condition.* ***Fig. b :*** *Final condition.*

Fig. 3.40 : *Initial and final condition of RC circuit of Fig. 3.38.*

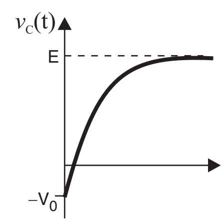


Fig. 3.41 : *$v_C(t)$ Vs t.*

Case ii : When the polarity of V_0 is same as that of $v_C(t)$

Consider the RC circuit with initial voltage, $v_C(0^-) = +V_0$ as shown in Fig. 3.42.

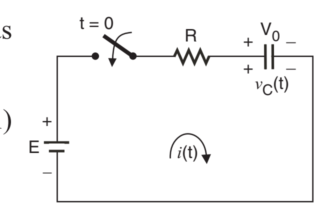


Fig. 3.42.

The response of this circuit can be obtained from case (i) by replacing V_0 by $-V_0$.

On replacing V_0 of equation (3.27) by $-V_0$, we get,

$$\boxed{v_C(t) = E\left(1 - e^{-\frac{t}{\tau}}\right) + V_0 e^{-\frac{t}{\tau}}} \qquad .....(3.30)$$

Let us examine the values of $v_C(t)$ at $t = 0^+$ and $t = \infty$.

$$\text{At } t = 0^+, \quad v_C(0^+) = E(1 - e^0) + V_0\, e^0 = E(1 - 1) + V_0 \times 1 = V_0$$

$$\text{At } t = \infty, \quad v_C(\infty) = E(1 - e^{-\infty}) + V_0\, e^{-\infty} = E(1 - 0) + V_0 \times 0 = E$$

From the above analysis we can say that the voltage $v_C(t)$ has a component due to V_0, which decays exponentially to zero as time t tends to infinity.

Also, on comparing the above results with the RC transient without initial voltage, we can conclude that the steady state value is not affected by the initial voltage in the capacitance.

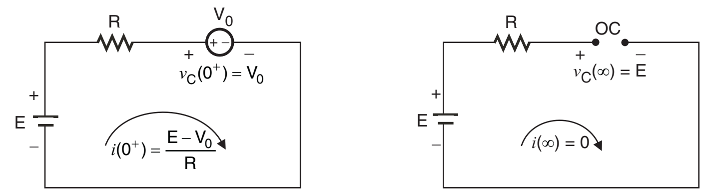


***Fig. a :** Initial condition.* ***Fig. b :** Final condition.*

***Fig. 3.43 :** Initial and final condition of the circuit of Fig. 3.42.*

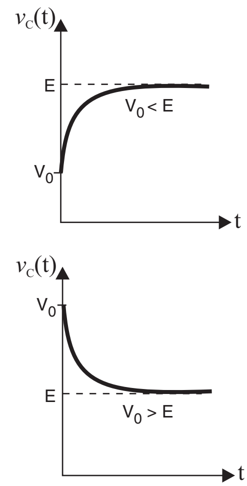


***Fig. 3.44 :** $v_C(t)$ Vs t.*

> ***Note :*** *When initial charge is specified, the initial voltage can be calculated using the relation,* $C = \frac{Q}{V}$.
>
> $\therefore$ *Initial voltage,* $V_0 = \frac{Q_0}{C}$, *where* Q_0 *is the initial charge.*

Table 3.3 : Initial and Final Condition of R, L and C when a Circuit is Excited by DC Supply

Element	**Initial condition** $t = 0^+$	**Final condition** $t = \infty$
R	R	R
L	OC	SC
L (I_0 →)	I_0	SC
L (I_0 ←)	I_0	SC
C	SC	OC
C (V_0, + –)	V_0 (+ –)	OC
C (V_0, – +)	V_0 (– +)	OC

3.5 Transient Response of RLC Circuit

3.5.1 Natural or Source-Free Response of RLC Circuit

Consider the RLC circuit with initial voltage across the capacitor as shown in Fig. 3.45. Let the switch be closed at t = 0. Let $i(t)$ be the current through the circuit. Let V_0 be the initial voltage across the capacitor before closing the switch.

The s-domain equivalent of the RLC circuit is shown in Fig. 3.46.

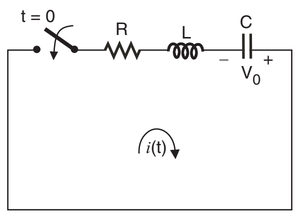


Fig. 3.45.

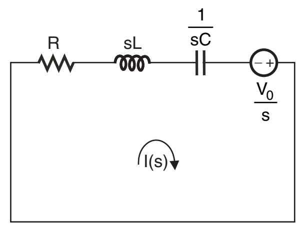


Fig. 3.46.

With reference to Fig. 3.46, we can write,

$$I(s) = \frac{\frac{V_0}{s}}{R + sL + \frac{1}{sC}}$$

$$I(s) = \frac{V_0}{sR + s^2L + \frac{1}{C}}$$

$$\boxed{\therefore\ I(s) = \frac{V_0}{L}\left(\frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right)} \qquad .....(3.31)$$

The denominator of equation (3.31) is a quadratic polynomial and the roots of the quadratic polynomial may be real or complex. Hence, the solution of equation (3.31) will depend on the roots of the denominator polynomial. The solution of equation (3.31) is the source-free response and this solution will be in the form similar to that of the step response discussed in Section 3.5.2.

3.5.2 Step Response of RLC Circuit (Response of RLC Circuit Excited by DC Supply)

Note : *A step voltage applied at t = 0 is equivalent to switching dc supply at t = 0.*

Consider the RLC circuit with no initial current or voltage and excited by a step voltage of E Volts as shown in Fig. 3.47. Let the switch be closed at t = 0.

Let, $i(t)$ = Current through the RLC circuit.

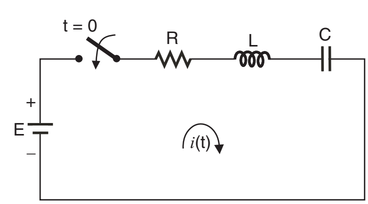


Fig. 3.47.

The response $i(t)$ of the RLC circuit excited by a dc source (as shown in Fig. 3.47) will take three different forms, as shown below:

Underdamped response: $$i(t) = \frac{E}{L\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \quad .....(3.32)$$

Critically damped response: $$i(t) = \frac{E}{L} t e^{-\omega_n t} \quad .....(3.33)$$

Overdamped response: $$i(t) = \frac{E}{2L\omega_n\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_n t}\left(e^{\omega_n\sqrt{\zeta^2-1}\,t} - e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right) \quad .....(3.34)$$

where, $$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}} = \text{Damping ratio} \quad .....(3.35)$$

$$\omega_n = \frac{1}{\sqrt{LC}} = \text{Natural frequency of oscillation} \quad .....(3.36)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \text{Damped frequency of oscillation} \quad .....(3.37)$$

Response of RLC Circuit Excited by DC Supply in s-domain

Let, $\mathcal{L}\{i(t)\} = I(s) \quad ; \quad \mathcal{L}\{E\} = \frac{E}{s}$

The s-domain equivalent of the RLC circuit is shown in Fig. 3.48.

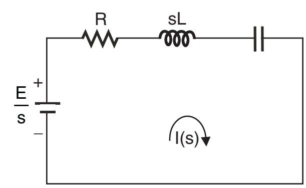


Fig. 3.48.

With reference to Fig. 3.49, we can write,

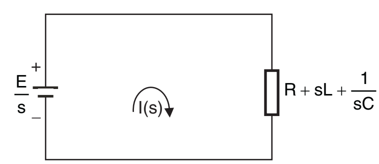


Fig. 3.49.

$$I(s) = \frac{\frac{E}{s}}{R + sL + \frac{1}{sC}} \Rightarrow I(s) = \frac{E}{sR + s^2L + \frac{1}{C}}$$

$$\therefore\ I(s) = \frac{E}{L}\left(\frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}}\right) \quad .....(3.38)$$

Equation (3.38) is the s-domain response of the RLC circuit excited by a dc supply. Equation (3.38) can be expressed in terms of damping ratio ζ and natural frequency of oscillation ω_n as shown below:

$$\frac{R}{L} = 2\times\frac{R}{2L} = 2\times\frac{R}{2\sqrt{L}\sqrt{L}} = 2\times\frac{R}{2\frac{\sqrt{L}}{\sqrt{C}}}\times\frac{1}{\sqrt{L}\sqrt{C}} = 2\times\frac{R}{2}\sqrt{\frac{C}{L}}\times\frac{1}{\sqrt{LC}} = 2\zeta\omega_n$$

$$\frac{1}{LC} = \frac{1}{(\sqrt{LC})^2} = \frac{1}{\omega_n^2}$$

$$\therefore\ I(s) = \frac{E}{L}\,\frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{E}{L}\,\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad .....(3.39)$$

Damping Ratio

Consider the s-domain response I(s) of the RLC circuit excited by a dc supply as shown below:

$$I(s) = \frac{E}{L}\,\frac{1}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

From equation (3.38)

Let us examine the roots of the second order polynomial, $s^2+\frac{R}{L}s+\frac{1}{LC}$ in the above equation.

Let, s_1 and s_2 be the roots of the second order polynomial, $s^2+\frac{R}{L}s+\frac{1}{LC} = 0$

$$\therefore\ s_1, s_2 = \frac{-\frac{R}{L}\pm\sqrt{\left(\frac{R}{L}\right)^2-\frac{4}{LC}}}{2} = -\frac{R}{2L}\pm\frac{1}{2}\sqrt{4\left[\frac{1}{4}\left(\frac{R}{L}\right)^2-\frac{1}{LC}\right]}$$

$$= -\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}} \qquad .....(3.40)$$

The above roots of the denominator polynomial may be complex or real depending on the value of R, L and C. Hence, we may come across the following three cases:

Case i *: When* $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$*, the term* $\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$ *will be imaginary and roots will be complex conjugate.*

Case ii *: When* $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$*, the term* $\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$ *will be zero and so the roots will be real and equal.*

Case iii: *When* $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$*, the term* $\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$ *will be positive and so the roots will be real and unequal.*

"*When the roots are complex, the current i (t) will be damped oscillatory (or damped sinusoid). When roots are real, the oscillations are completely damped (or eliminated).*"

When the term $\left(\frac{R}{2L}\right)^2$ is equal to $\frac{1}{LC}$, the oscillations are just eliminated and this condition is called **critical damping**. Critical damping can be achieved by choosing a value of R that makes the term $\left(\frac{R}{2L}\right)^2$ equal to $\frac{1}{LC}$ for a given value of L and C.

Let, R_C = Value of R for critical damping.

$\therefore$ At critical damping,

$$\left(\frac{R_C}{2L}\right)^2-\frac{1}{LC} = 0 \Rightarrow \left(\frac{R_C}{2L}\right)^2 = \frac{1}{LC} \Rightarrow \frac{R_C}{2L} = \sqrt{\frac{1}{LC}} \Rightarrow R_C = 2L\times\frac{1}{\sqrt{LC}}$$

$$\boxed{\therefore\ R_C = 2\sqrt{\frac{L}{C}}} \qquad .....(3.41)$$

The ratio of resistance of the circuit and resistance for critical damping (**critical resistance, R_c**) is called **damping ratio** and denoted by ζ.

$$\therefore \text{ Damping ratio, } \zeta = \frac{R}{R_C} = \frac{R}{2\sqrt{\frac{L}{C}}} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

$$\boxed{\therefore \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}} \qquad .....(3.42)$$

Natural Frequency of Oscillation

In an RLC circuit, if the resistance is zero then from equation (3.40), we can say that the roots are purely imaginary and the current will be completely oscillatory.

$$\text{When } R = 0, \quad s_1, s_2 = \pm\sqrt{-\frac{1}{LC}} = \pm j\omega_n^2$$

$$\text{where, } \boxed{\omega_n = \frac{1}{\sqrt{LC}} = \text{Natural frequency of oscillation.}} \qquad .....(3.43)$$

On substituting the condition R = 0 in equation (3.38), we get,

$$\boxed{\omega_n = \frac{1}{\sqrt{LC}}}$$

$$I(s) = \frac{E}{L}\frac{1}{s^2 + \frac{1}{LC}} = \frac{E}{L}\frac{1}{s^2 + \omega_n^2}$$

$$\boxed{\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}}$$

On taking the inverse Laplace transform of the above equation, we get,

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{E}{L}\frac{1}{s^2 + \omega_n^2}\right\} \Rightarrow i(t) = \frac{E}{L\omega_n}\mathcal{L}^{-1}\left\{\frac{\omega_n}{s^2 + \omega_n^2}\right\}$$

$$i(t) = \frac{E}{L\omega_n}\sin \omega_n t \qquad .....(3.44)$$

From equation (3.43) we can say that current $i(t)$ is completely oscillatory with a frequency ω_n in the absence of resistance and so this frequency of oscillation is called **natural frequency of oscillation**.

Also, when R = 0, the damping ratio, $\zeta = 0$ and so equation (3.44) is called **undamped response** of the RLC circuit excited by a dc supply.

Condition for Three Cases of Response in Terms of Damping Ratio

Consider the s-domain response I(s) of the RLC circuit excited by a dc supply as shown below:

$$I(s) = \frac{E}{L}\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \boxed{\text{From equation (3.39)}}$$

Let, s_1 and s_2 be the roots of the second order polynomial, $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$.

$$\therefore s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = -\frac{2\zeta\omega_n}{2} \pm \frac{1}{2}\sqrt{4\omega_n^2(\zeta^2 - 1)}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Depending on the value of ζ the roots may be real or complex.

Case i : *If $0 < \zeta < 1$, then $\zeta < 1$ and ζ^2 is also less than 1. Hence, $\sqrt{\zeta^2 - 1}$ is imaginary and so the roots are complex conjugate. In this case the current i (t) will be damped oscillatory.*

Case ii : *If $\zeta = 1$, then $\sqrt{\zeta^2 - 1} = 0$ and so the roots are real and equal. In this case, the oscillation of the current is just eliminated and so the response is called critically damped response.*

Case iii : *If $\zeta > 1$, then $\sqrt{\zeta^2 - 1}$ is real and so the roots are real and unequal. In this case, the response is called overdamped response.*

The equations for time domain response of the RLC circuit for the above three cases are presented here.

Note : *Here the current i(t) is referrd as time domain response.*

Case i : Underdamped response ($0 < \zeta < 1$)

The time domain response i(t) of the RLC circuit excited by a dc supply, as shown in Fig. 3.47, when damping ratio ζ lies between 0 to 1 is given by equation (3.45).

$$\boxed{i(t) = \frac{E}{L\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t} \qquad .....(3.45)$$

$$\text{where,} \quad \omega_d = \omega_n \sqrt{1 - \zeta^2} = \text{Damped frequency of oscillation.} \qquad .....(3.46)$$

Equation (3.45) has a sinusoidal component of frequency ω_d. The sinusoidal oscillations are damped by the exponential term $e^{-\zeta\omega_n t}$. Hence, "*the current i(t) will have damped oscillations, whose amplitude decays to zero as t tends to infinity*". Therefore, equation (3.45) is called **damped oscillatory response** or **underdamped response**.

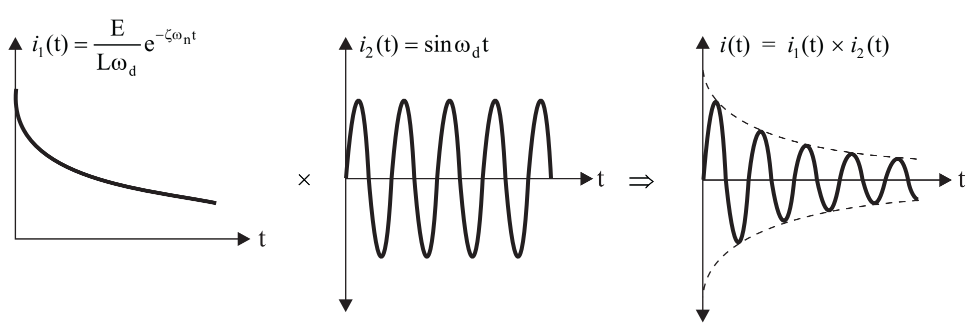


Fig. a : *Decaying exponential component.* ***Fig. b :*** *Sinusoidal component.* ***Fig. c :*** *Underdamped response of the RLC circuit.*

Fig. 3.50 : *Underdamped response of the RLC circuit of Fig. 3.47.*

Proof:

The s-domain response, I(s) of the RLC circuit excited by dc supply is,

$$I(s) = \frac{E}{L}\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$$ *From equation (3.39)*

$$= \frac{E}{L}\frac{1}{s^2+2\zeta\omega_n s+\zeta^2\omega_n^2-\zeta^2\omega_n^2+\omega_n^2}$$ *Add and subtract the term $\zeta^2\omega_n^2$*

$$\therefore\ I(s) = \frac{E}{L}\frac{1}{\left(s^2+2\zeta\omega_n s+\zeta^2\omega_n^2\right)+\omega_n^2(1-\zeta^2)}$$

$$= \frac{E}{L}\frac{1}{(s+\zeta\omega_n)^2+\omega_d^2} = \frac{E}{L\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$\omega_n\sqrt{1-\zeta^2} = \omega_d$$

$$(a+b)^2 = a^2+b^2+2ab$$

Let us take inverse Laplace transform of the above equation.

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{E}{L\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right\}$$

$$\therefore\ i(t) = \frac{E}{L\omega_d}\mathcal{L}^{-1}\left\{\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right\}$$ $$\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2+\omega^2}$$

$$= \frac{E}{L\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$$

Case ii : Critically damped response (ζ = 1)

The time domain response i(t) of the RLC circuit excited by a dc supply, as shown in Fig. 3.47, when damping ratio ζ is equal to 1 is given by equation (3.47).

$$\boxed{i(t) = \frac{E}{L}t\,e^{-\omega_n t}} \qquad .....(3.47)$$

Equation (3.47) has a ramp component and decaying exponential component. "*Initially the current rises due to ramp component and then it gradually decays to zero due to exponential component*". The sketch of i(t) for critically damped case is shown in Fig. 3.51.

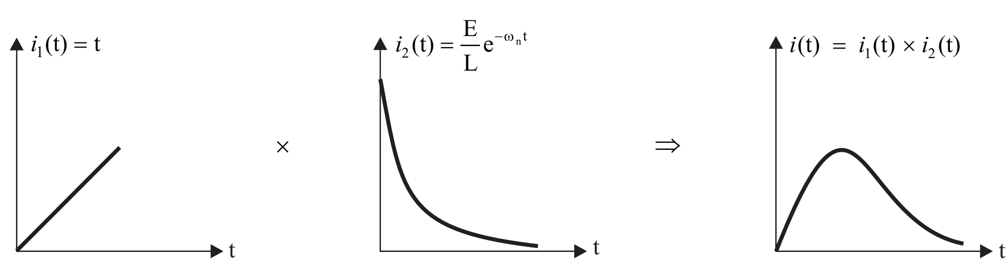


Fig. a : *Ramp component.* ***Fig. b :*** *Decaying exponential component.* ***Fig. c :*** *Critically damped response.*

Fig. 3.51 : *Critically damped response of the RLC circuit of Fig. 3.47.*

Proof:

The s-domain response, I(s) of the RLC circuit excited by dc supply is,

From equation (3.39)

For critical damped case $\zeta = 1$

$$I(s) = \frac{E}{L}\frac{1}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$= \frac{E}{L}\frac{1}{(s^2 + 2\omega_n s + \omega_n^2)}$$

$$(a+b)^2 = a^2 + b^2 + 2ab$$

$$= \frac{E}{L}\frac{1}{(s + \omega_n)^2}$$

Let us take inverse Laplace transform of I(s).

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{E}{L}\frac{1}{(s+\omega_n)^2}\right\}$$

$$\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$

$$\therefore\ i(t) = \frac{E}{L}\mathcal{L}^{-1}\left\{\frac{1}{(s+\omega_n)^2}\right\} = \frac{E}{L}\,te^{-\omega_n t}$$

Case iii : Overdamped response ($\zeta > 1$)

The time domain response $i(t)$ of the RLC circuit excited by a dc supply, as shown in Fig. 3.47, when damping ratio ζ is greater than 1 is given by equation (3.48).

$$\boxed{i(t) = \frac{E}{2L\omega_n\sqrt{\zeta^2 - 1}}\, e^{-\zeta\omega_n t}\left(e^{\omega_n\sqrt{\zeta^2-1}\,t} - e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right)} \quad .....(3.48)$$

Equation (3.48) is called **overdamped response** of the RLC circuit excited by a dc supply. The decaying exponential component $e^{-\zeta\omega_n t}$ in equation (3.48) will make the current $i(t)$ to zero as t tends to infinity. "*Since the circuit is overdamped, the current decays at a faster rate than the underdamped or critically damped RLC circuit*".

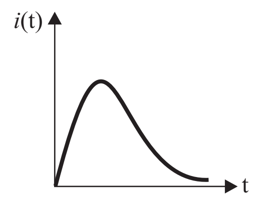


Fig. 3.52 : *Overdamped response of the RLC circuit.*

Proof:

The s-domain response I(s) of the RLC circuit excited by a dc supply is,

From equation (3.39)

$$I(s) = \frac{E}{L}\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Let, s_1 and s_2 be the roots of the second order polynomial, $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$.

$$\therefore\ s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = \frac{-2\zeta\omega_n}{2} \pm \frac{1}{2}\sqrt{4\omega_n^2(\zeta^2 - 1)} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Let, $$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \quad .....(3.49)$$

$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \quad .....(3.50)$$

Using the above roots of the denominator polynomial, the s-domain response I(s) can be written as,

$$I(s) = \frac{E}{L}\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{E}{L}\frac{1}{(s - s_1)(s - s_2)}$$

By partial fraction expansion technique, the above equation can be expressed as,

$$I(s) = \frac{E}{L}\frac{1}{(s - s_1)(s - s_2)} = \frac{K_1}{s - s_1} + \frac{K_2}{s - s_2} \quad .....(3.51)$$

$$K_1 = \frac{E}{L}\frac{1}{(s-s_1)(s-s_2)} \times (s-s_1)\bigg|_{s=s_1} = \frac{E}{L}\frac{1}{(s-s_2)}\bigg|_{s=s_1} = \frac{E}{L}\frac{1}{s_1-s_2}$$

$$= \frac{E}{L}\frac{1}{-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} - \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)} = \frac{E}{2L\omega_n\sqrt{\zeta^2-1}}$$

Using equations (3.49) and (3.50)

$$K_2 = \frac{E}{L}\frac{1}{(s-s_1)(s-s_2)} \times (s-s_2)\bigg|_{s=s_2} = \frac{E}{L}\frac{1}{(s-s_1)}\bigg|_{s=s_2} = \frac{E}{L}\frac{1}{s_2-s_1}$$

$$= \frac{E}{L}\frac{1}{-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} - \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)} = -\frac{E}{2L\omega_n\sqrt{\zeta^2-1}}$$

Using equations (3.49) and (3.50)

The time domain response, i(t) is obtained by taking inverse Laplace transform of equation (3.51) as shown below:

$$i(t) = \mathcal{L}^{-1}\{I(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{K_1}{s-s_1} + \frac{K_2}{s-s_2}\right\} = K_1\mathcal{L}^{-1}\left\{\frac{1}{s-s_1}\right\} + K_2\mathcal{L}^{-1}\left\{\frac{1}{s-s_2}\right\}$$

$$= K_1e^{s_1t} + K_2e^{s_2t}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{(s-a)}$$

On substituting the expressions for s_1, s_2, K_1 and K_2 in the above equation, we get

$$i(t) = \frac{E}{2L\omega_n\sqrt{\zeta^2-1}}\,e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} - \frac{E}{2L\omega_n\sqrt{\zeta^2-1}}\,e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t}$$

$$= \frac{E}{2L\omega_n\sqrt{\zeta^2-1}}\left(e^{-\zeta\omega_nt} \times e^{\omega_n\sqrt{\zeta^2-1}\,t} - e^{-\zeta\omega_nt} \times e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right)$$

$$= \frac{E}{2L\omega_n\sqrt{\zeta^2-1}}\,e^{-\zeta\omega_nt}\left(e^{\omega_n\sqrt{\zeta^2-1}\,t} - e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right)$$

3.5.3 s-Domain Current and Voltage Equation of RLC Circuit

Consider the RLC circuit excited by a dc supply and its s-domain equivalent as shown in Fig. 3.53.

Let, i(t) = Current through the RLC circuit in time domain
v_R(t) = Voltage across resistance in time domain
v_L(t) = Voltage across inductance in time domain
v_C(t) = Voltage across capacitance in time domain

In RLC series circuit we are interested in the initial and final value of i(t), v_R(t), v_L(t) and v_C(t). The initial and final condition can be evaluated using the initial and final value theorem of Laplace transform. In order to evaluate the initial and final conditions using Laplace transform technique, we require the current and voltage as s-domain functions.

Let, I(s) = Current through the RLC circuit in s-domain
V_R(t) = Voltage across resistance in s-domain
V_L(t) = Voltage across inductance in s-domain
V_C(t) = Voltage across capacitance in s-domain

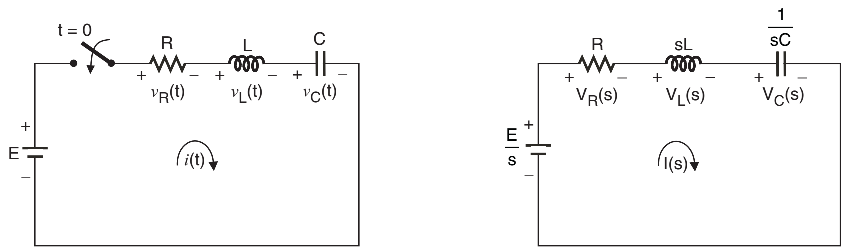


***Fig. a :** Time domain RLC circuit* ***Fig. b :** s-domain RLC circuit*

***Fig. 3.53 :** RLC circuit excited by dc supply.*

The s-domain response I(s) of this circuit is given by equation (3.39).

$$\therefore \; I(s) = \frac{E}{L(s^2 + 2\zeta\omega_n s + \omega_n^2)} \qquad .....(3.52)$$

With reference to Fig. 3.53, by using Ohm's law and equation (3.52), we can get the following equations for voltage across R, L and C.

$$V_R(s) = R \times I(s) = R \times \frac{E}{L(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{ER}{L(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$V_L(s) = sL \times I(s) = sL \times \frac{E}{L\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \frac{sE}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$V_C(s) = \frac{1}{sC} \times I(s) = \frac{1}{sC} \times \frac{E}{L\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \frac{E}{sLC\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

In Summary,

$$\boxed{V_R(s) = \frac{ER}{L(s^2 + 2\zeta\omega_n s + \omega_n^2)}} \qquad .....(3.53)$$

$$\boxed{V_L(s) = \frac{sE}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \qquad .....(3.54)$$

$$\boxed{V_C(s) = \frac{E}{sLC(s^2 + 2\zeta\omega_n s + \omega_n^2)}} \qquad .....(3.55)$$

3.5.4 Initial Conditions in RLC Circuit

The initial value theorem of Laplace transform states that if F(s) is the Laplace transform of f(t) then the initial value of f(t) is given by,

$$\underset{t \to 0}{\text{Lt}} \; f(t) = \underset{s \to \infty}{\text{Lt}} \; sF(s)$$

$$\therefore \; \text{Initial value, } f(0) = \underset{s \to \infty}{\text{Lt}} \; sF(s)$$

The initial value of current and voltage of the RLC circuit shown in Fig. 3.53 can be solved using the initial value theorem of Laplace transform as shown ahead.

Initial Currrent in RLC Circuit

Let, $i(0^+) = i(t)\Big|_{t=0^+}$ = Initial value of current $i(t)$.

Now, by initial value theorem of Laplace transform,

$$i(0^+) = \underset{s\to\infty}{\mathrm{Lt}}\ sI(s)$$

$$i(0^+) = \underset{s\to\infty}{\mathrm{Lt}}\ s\times\frac{E}{L(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$\boxed{\text{Using equation (3.52)}}$

$$= \underset{s\to\infty}{\mathrm{Lt}}\ s\times\frac{E}{Ls^2\left(1+\dfrac{2\zeta\omega_n}{s}+\dfrac{\omega_n^2}{s^2}\right)} = \underset{s\to\infty}{\mathrm{Lt}}\ \frac{E}{L}\times\frac{1}{s}\times\frac{1}{1+\dfrac{2\zeta\omega_n}{s}+\dfrac{\omega_n^2}{s^2}}$$

$$= \frac{E}{L}\times\frac{1}{\infty}\times\frac{1}{1+\dfrac{2\zeta\omega_n}{\infty}+\dfrac{\omega_n^2}{\infty}} = \frac{E}{L}\times 0\times\frac{1}{1+0+0} = 0$$

Initial Voltage Across Resistor

Let, $v_R(0^+) = v_R(t)\Big|_{t=0^+}$ = Initial value of resistor voltage $v_R(t)$.

Now, by initial value theorem of Laplace transform,

$$v_R(0^+) = \underset{s\to\infty}{\mathrm{Lt}}\ sV_R(s)$$

$\boxed{\text{Using equation (3.53)}}$

$$= \underset{s\to\infty}{\mathrm{Lt}}\ s\times\frac{ER}{L(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$= \underset{s\to\infty}{\mathrm{Lt}}\ s\times\frac{ER}{Ls^2\left(1+\dfrac{2\zeta\omega_n}{s}+\dfrac{\omega_n^2}{s^2}\right)} = \underset{s\to\infty}{\mathrm{Lt}}\ \frac{ER}{L}\times\frac{1}{s}\times\frac{1}{1+\dfrac{2\zeta\omega_n}{s}+\dfrac{\omega_n^2}{s^2}}$$

$$= \frac{ER}{L}\times\frac{1}{\infty}\times\frac{1}{1+\dfrac{2\zeta\omega_n}{\infty}+\dfrac{\omega_n^2}{\infty}} = \frac{ER}{L}\times 0\times\frac{1}{1+0+0} = 0$$

Initial Voltage Across Inductor

Let, $v_L(0^+) = v_L(t)\Big|_{t=0^+}$ = Initial value of inductor voltage $v_L(t)$.

Now, by initial value theorem of Laplace transform,

$$v_L(0^+) = \underset{s\to\infty}{\mathrm{Lt}}\ sV_L(s)$$

$\boxed{\text{Using equation (3.54)}}$

$$= \underset{s\to\infty}{\mathrm{Lt}}\ s\times\frac{sE}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$= \underset{s\to\infty}{\mathrm{Lt}}\ \frac{s^2E}{s^2\left(1+\dfrac{2\zeta\omega_n}{s}+\dfrac{\omega_n^2}{s^2}\right)} = \frac{E}{1+\dfrac{2\zeta\omega_n}{\infty}+\dfrac{\omega_n^2}{\infty}} = \frac{E}{1+0+0} = E$$

Initial Voltage Across Capacitor

Let, $v_C(0^+) = v_C(t)\Big|_{t=0^+}$ = Initial value of capacitor voltage $v_C(t)$.

Now, by initial value theorem of Laplace transform,

$$v_C(0^+) = \underset{s\to\infty}{\mathrm{Lt}}\ sV_C(s)$$

$$= \underset{s\to\infty}{\mathrm{Lt}}\ s\times\frac{E}{sLC(s^2+2\zeta\omega_n s+\omega_n^2)} \qquad \boxed{\text{Using equation (3.55)}}$$

$$= \underset{s\to\infty}{\mathrm{Lt}}\ \frac{E}{LCs^2\left(1+\dfrac{2\zeta\omega_n}{s}+\dfrac{\omega_n^2}{s^2}\right)} = \underset{s\to\infty}{\mathrm{Lt}}\ \frac{E}{LC}\times\frac{1}{s^2}\times\frac{1}{1+\dfrac{2\zeta\omega_n}{s}+\dfrac{\omega_n^2}{s^2}}$$

$$= \frac{E}{LC}\times\frac{1}{\infty}\times\frac{1}{1+\dfrac{2\zeta\omega_n}{\infty}+\dfrac{\omega_n^2}{\infty}} = \frac{E}{LC}\times 0\times\frac{1}{1+0+0} = 0$$

Initial Condition Circuit

From the above analysis we can make the following conclusions:

- At t = 0^+, the current through the inductance is zero and so it behaves as an open circuit.
- At t = 0^+, the voltage across the capacitance is zero and so it behaves as a short circuit.

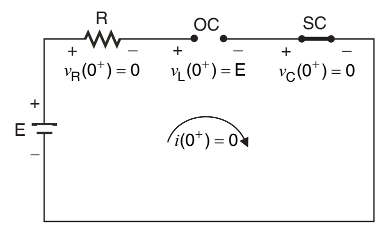


***Fig. 3.54 :** Initial condition circuit of the RLC circuit shown in Fig. 3.53.*

3.5.5 Final Conditions in RLC Circuit

The final value theorem of Laplace transform states that if F(s) is Laplace transform of f(t) then the final value of f(t) is given by,

$$\underset{t\to\infty}{\mathrm{Lt}}\ f(t) = \underset{s\to 0}{\mathrm{Lt}}\ sF(s)$$

$$\therefore\ \text{Final value,}\ f(\infty) = \underset{s\to 0}{\mathrm{Lt}}\ sF(s)$$

The final value of current and voltage of the RLC circuit shown in Fig. 3.53 can be solved using the final value theorem of Laplace transform as shown below:

Final Current in RLC Circuit

Let, $i(\infty) = i(t)\Big|_{t=\infty}$ = Final value of current $i(t)$.

Now, by final value theorem of Laplace transform,

$$i(\infty) = \underset{s\to 0}{\mathrm{Lt}}\ sI(s)$$

$$= \underset{s\to 0}{\mathrm{Lt}}\ s\times\frac{E}{L(s^2+2\zeta\omega_n s+\omega_n^2)} \qquad \boxed{\text{Using equation (3.52)}}$$

$$= 0\times\frac{E}{L(0+0+\omega_n^2)} = 0$$

Final Voltage Across Resistor

Let, $v_R(\infty) = v_R(t)\Big|_{t=\infty}$ = Final value of resistor voltage $v_R(t)$.

Now, by final value theorem of Laplace transform,

$$v_R(\infty) = \underset{s \to 0}{\text{Lt}}\ sV_R(s)$$

$$= \underset{s \to 0}{\text{Lt}}\ s \times \frac{ER}{L(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

Using equation (3.53)

$$= 0 \times \frac{ER}{L(0 + 0 + \omega_n^2)} = 0$$

Final Voltage Across Inductor

Let, $v_L(\infty) = v_L(t)\Big|_{t=\infty}$ = Final value of inductor voltage $v_L(t)$.

Now, by final value theorem of Laplace transform,

$$v_L(\infty) = \underset{s \to 0}{\text{Lt}}\ sV_L(s)$$

$$= \underset{s \to 0}{\text{Lt}}\ s \times \frac{sE}{s^2 + 2\zeta\,\omega_n s + \omega_n^2}$$

Using equation (3.54)

$$= \underset{s \to 0}{\text{Lt}}\ \frac{s^2E}{s^2 + 2\zeta\,\omega_n s + \omega_n^2} = \frac{0}{0 + 0 + \omega_n^2} = 0$$

Final Voltage Across Capacitor

Let, $v_C(\infty) = v_C(t)\Big|_{t=\infty}$ = Final value of capacitor voltage $v_C(t)$.

Now, by final value theorem of Laplace transform,

$$v_C(\infty) = \underset{s \to 0}{\text{Lt}}\ sV_C(s)$$

Using equation (3.55)

$$= \underset{s \to 0}{\text{Lt}}\ s \times \frac{E}{sLC(s^2 + 2\zeta\,\omega_n s + \omega_n^2)}$$

$$= \underset{s \to 0}{\text{Lt}}\ \frac{E}{LC(s^2 + 2\zeta\,\omega_n s + \omega_n^2)} = \frac{E}{LC(0 + 0 + \omega_n^2)} = \frac{E}{LC(\omega_n^2)} = E$$

$$\boxed{\omega_n^2 = \frac{1}{LC}}$$

Final Condition Circuit

From the above analysis we can make the following conclusions:

- At t = ∞, the voltage across the inductance is zero and so it behaves as a short circuit.
- At t = ∞, the current through the capacitance is zero and so it behaves as an open circuit.

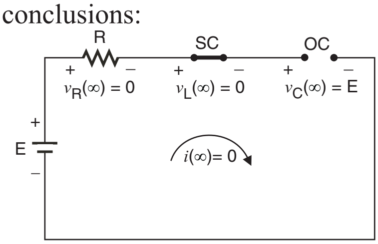


Fig. 3.55 : Final condition circuit of the RLC circuit shown in Fig. 3.53.

3.6 Transient Response of Circuits Excited by Sinusoidal Source

3.6.1 RL Circuit Excited by Sinusoidal Source

Consider the RL circuit with no initial current and excited by a sinusoidal source $e(t) = E_m \sin \omega t$, as shown in Fig. 3.56. Let the switch be closed at $t = 0$.

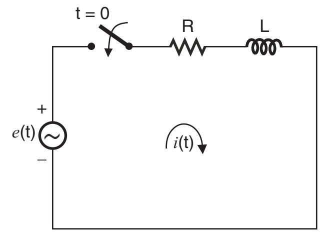


Fig. 3.56.

Let, $i(t)$ = Current through the RL circuit

Now, the response, $i(t)$ of the RL circuit excited by sinusoidal source as shown in Fig. 3.56 is,

$$\boxed{i(t) = \underbrace{\frac{\omega L E_m}{Z^2} e^{-\frac{R}{L}t}}_{\text{Transient part}} + \underbrace{\frac{E_m}{Z} \sin(\omega t - \phi)}_{\text{Steady state part}}} \quad .....(3.56)$$

$$\text{where, } Z = \sqrt{R^2 + (\omega L)^2} \quad ; \quad \phi = \tan^{-1}\frac{\omega L}{R}$$

Proof:

Let, $\mathcal{L}\{i(t)\} = I(s) \quad ; \quad \mathcal{L}\{e(t)\} = E(s)$

$$\therefore E(s) = \mathcal{L}\{e(t)\} = \mathcal{L}\{E_m \sin \omega t\} = \frac{\omega E_m}{s^2 + \omega^2}$$

The s-domain equivalent of the RL circuit is shown in Fig. 3.57. With reference to Fig. 3.57, we can write,

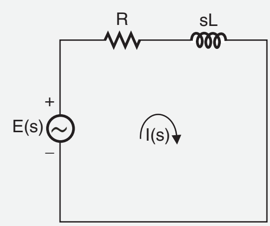


Fig. 3.57.

$$I(s) = \frac{E(s)}{R + sL} = E(s) \times \frac{1}{L\left(s + \frac{R}{L}\right)} = \frac{\omega E_m}{s^2 + \omega^2} \times \frac{1}{L\left(s + \frac{R}{L}\right)}$$

$$\therefore I(s) = \frac{\frac{\omega E_m}{L}}{\left(s + \frac{R}{L}\right)(s^2 + \omega^2)}$$

By partial fraction expansion technique, the above equation of I(s) can be expressed as,

$$I(s) = \frac{\frac{\omega E_m}{L}}{\left(s + \frac{R}{L}\right)(s^2 + \omega^2)} = \frac{K_1}{s + \frac{R}{L}} + \frac{K_2 s + K_3}{s^2 + \omega^2} \quad .....(3.57)$$

$$K_1 = \left.\frac{\frac{\omega E_m}{L}}{\left(s + \frac{R}{L}\right)(s^2 + \omega^2)} \times \left(s + \frac{R}{L}\right)\right|_{s = -\frac{R}{L}} = \frac{\frac{\omega E_m}{L}}{\left(-\frac{R}{L}\right)^2 + \omega^2} = \frac{\frac{\omega E_m}{L}}{\frac{R^2 + \omega^2 L^2}{L^2}}$$

$$= \frac{\omega L E_m}{R^2 + (\omega L)^2} = \frac{\omega L E_m}{Z^2}$$

In frequency domain, magnitude of the impedance of RL circuit is given by,

$$Z = \sqrt{R^2 + (\omega L)^2}$$
$$\therefore Z^2 = R^2 + (\omega L)^2$$

On cross-multiplying equation (3.57), we get,

$$\frac{\omega E_m}{L} = K_1 (s^2 + \omega^2) + (K_2 s + K_3)\left(s + \frac{R}{L}\right)$$

$$\frac{\omega E_m}{L} = K_1 s^2 + K_1 \omega^2 + K_2 s^2 + K_2 \frac{R}{L} s + K_3 s + K_3 \frac{R}{L}$$

$$\frac{\omega E_m}{L} = (K_1 + K_2) s^2 + \left(K_2 \frac{R}{L} + K_3\right) s + K_1 \omega^2 + K_3 \frac{R}{L} \quad .....(3.58)$$

On equating coefficients of s^2 term of equation (3.58), we get,

$$K_1 + K_2 = 0$$

$$\therefore\ K_2 = -K_1 = -\frac{\omega L E_m}{Z^2}$$

On equating coefficients of s term of equation (3.58), we get,

$$K_2 \frac{R}{L} + K_3 = 0$$

$$\therefore\ K_3 = -K_2 \frac{R}{L} = -\left(-\frac{\omega L E_m}{Z^2}\right)\frac{R}{L} = \frac{\omega R E_m}{Z^2}$$

By using the expressions for K_1, K_2 and K_3, equation (3.57) of I(s) can be expressed as,

$$I(s) = \frac{\frac{\omega L E_m}{Z^2}}{s+\frac{R}{L}} + \frac{-\frac{\omega L E_m}{Z^2}s + \frac{\omega R E_m}{Z^2}}{s^2+\omega^2} = \frac{\omega L E_m}{Z^2} \times \frac{1}{s+\frac{R}{L}} - \frac{\omega L E_m}{Z^2} \times \frac{s}{s^2+\omega^2} + \frac{R E_m}{Z^2} \times \frac{\omega}{s^2+\omega^2}$$

Let us take the inverse Laplace transform of I(s).

$$\mathcal{L}^{-1}\{I(s)\} = \frac{\omega L E_m}{Z^2} \times \mathcal{L}^{-1}\left\{\frac{1}{s+\frac{R}{L}}\right\} - \frac{\omega L E_m}{Z^2} \times \mathcal{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} + \frac{R E_m}{Z^2} \times \mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\}$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}} \quad \boxed{\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}} \quad \boxed{\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}}$$

$$\therefore\ i(t) = \frac{\omega L E_m}{Z^2} e^{-\frac{R}{L}t} - \frac{\omega L E_m}{Z^2}\cos\omega t + \frac{R E_m}{Z^2}\sin\omega t$$

$$= \frac{\omega L E_m}{Z^2} e^{-\frac{R}{L}t} + \frac{E_m}{Z^2}\left[\sin\omega t \times R - \cos\omega t \times \omega L\right] \quad .....(3.59)$$

Let us construct a right-angled triangle with R and ωL as two sides as shown in Fig. 3.58. With reference to Fig. 3.58, we can write,

$$\tan\phi = \frac{\omega L}{R} \Rightarrow \phi = \tan^{-1}\frac{\omega L}{R} \quad .....(3.60)$$

Also,

$$\cos\phi = \frac{R}{Z} \Rightarrow R = Z\cos\phi \quad .....(3.61)$$

$$\sin\phi = \frac{\omega L}{Z} \Rightarrow \omega L = Z\sin\phi \quad .....(3.62)$$

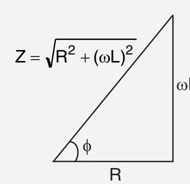


Fig. 3.58.

Using equations (3.60) to (3.62), equation (3.59) can be written as,

$$i(t) = \frac{\omega L E_m}{Z^2} e^{-\frac{R}{L}t} + \frac{E_m}{Z^2}\left[\sin\omega t\ Z\cos\phi - \cos\omega t\ Z\sin\phi\right]$$

$$= \frac{\omega L E_m}{Z^2} e^{-\frac{R}{L}t} + \frac{E_m}{Z^2} \times Z\left[\sin\omega t\cos\phi - \cos\omega t\sin\phi\right]$$

$$\boxed{\sin(A-B) = \sin A\cos B - \cos A\sin B}$$

$$= \underbrace{\frac{\omega L E_m}{Z^2} e^{-\frac{R}{L}t}}_{\text{Transient part}} + \underbrace{\frac{E_m}{Z}\sin(\omega t - \phi)}_{\text{Steady state part}}$$

$$\text{where, } Z = \sqrt{R^2 + (\omega L)^2} \quad ; \quad \phi = \tan^{-1}\frac{\omega L}{R}$$

3.6.2 RC Circuit Excited by Sinusoidal Source

Consider the RC circuit with no initial voltage and excited by a sinusoidal source $e(t) = E_m \sin \omega t$, as shown in Fig. 3.59. Let the switch be closed at t = 0.

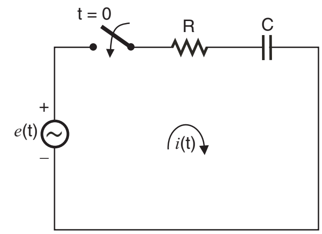


Fig. 3.59.

Let, $i(t)$ = Current through the RC circuit

Now, the response $i(t)$ of the RC circuit excited by a sinusoidal source as shown in Fig. 3.59 is,

$$\boxed{i(t) = \underbrace{-\frac{E_m}{\omega C Z^2}\, e^{-\frac{t}{RC}}}_{\text{Transient part}} + \underbrace{\frac{E_m}{Z}\sin(\omega t + \phi)}_{\text{Steady state part}}} \quad .....(3.63)$$

$$\text{where, } Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} \quad ; \quad \phi = \tan^{-1}\frac{1/\omega C}{R}$$

Proof:

Let, $\mathcal{L}\{i(t)\} = I(s) \quad ; \quad \mathcal{L}\{e(t)\} = E(s)$

$$\therefore\ E(s) = \mathcal{L}\{e(t)\} = \mathcal{L}\{E_m \sin \omega t\} = \frac{\omega E_m}{s^2 + \omega^2}$$

The s-domain equivalent of the RC circuit is shown in Fig. 3.60. With reference to Fig. 3.60, we can write,

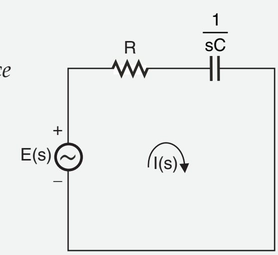


Fig. 3.60.

$$I(s) = \frac{E(s)}{R + \frac{1}{sC}} = E(s) \times \frac{1}{\frac{R}{s}\left(s + \frac{1}{RC}\right)} = \frac{\omega E_m}{s^2 + \omega^2} \times \frac{\frac{s}{R}}{s + \frac{1}{RC}}$$

$$\therefore\ I(s) = \frac{\frac{\omega E_m}{R}s}{\left(s + \frac{1}{RC}\right)\left(s^2 + \omega^2\right)}$$

By partial fraction expansion technique, the above equation of I(s) can be expressed as,

$$I(s) = \frac{\frac{\omega E_m}{R}s}{\left(s + \frac{1}{RC}\right)\left(s^2 + \omega^2\right)} = \frac{K_1}{s + \frac{1}{RC}} + \frac{K_2 s + K_3}{s^2 + \omega^2} \quad .....(3.64)$$

$$K_1 = \left.\frac{\frac{\omega E_m}{R}s}{\left(s + \frac{1}{RC}\right)\left(s^2 + \omega^2\right)} \times \left(s + \frac{1}{RC}\right)\right|_{s = -\frac{1}{RC}} = \frac{\frac{\omega E_m}{R} \times \left(\frac{-1}{RC}\right)}{\left(\frac{-1}{RC}\right)^2 + \omega^2} = \frac{-\frac{\omega E_m}{R^2 C}}{\frac{\omega^2}{R^2}\left[\frac{1}{\omega^2 C^2} + R^2\right]}$$

$$= \frac{-\frac{E_m}{\omega C}}{R^2 + \left(\frac{1}{\omega C}\right)^2} = -\frac{E_m}{\omega C Z^2}$$

In frequency domain, magnitude of the impedance of RC circuit is given by,

$$Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

$$\therefore\ Z^2 = R^2 + \left(\frac{1}{\omega C}\right)^2$$

On cross-multiplying equation (3.64), we get,

$$\frac{\omega E_m}{R}s = K_1\left(s^2 + \omega^2\right) + \left(K_2 s + K_3\right)\left(s + \frac{1}{RC}\right)$$

$$\frac{\omega E_m}{R}s = K_1 s^2 + K_1 \omega^2 + K_2 s^2 + \frac{K_2}{RC}s + K_3 s + \frac{K_3}{RC}$$

$$\frac{\omega E_m}{R}s = (K_1 + K_2)s^2 + \left(K_3 + \frac{K_2}{RC}\right)s + K_1\omega^2 + \frac{K_3}{RC} \quad .....(3.65)$$

On equating coefficients of s^2 term of equation (3.65), we get,

$$K_1 + K_2 = 0$$

$$\therefore\ K_2 = -K_1 = \frac{E_m}{\omega C Z^2}$$

On equating coefficients of s term of equation (3.65), we get,

$$K_3 + \frac{K_2}{RC} = \frac{\omega E_m}{R} \quad \Rightarrow \quad K_3 = \frac{\omega E_m}{R} - \frac{K_2}{RC}$$

$$\therefore\ K_3 = \frac{\omega E_m}{R} - \frac{E_m}{\omega C Z^2} \times \frac{1}{RC} = \frac{\omega E_m (\omega C^2 Z^2) - E_m}{\omega R C^2 Z^2} = \frac{E_m (\omega^2 C^2 Z^2 - 1)}{\omega R C^2 Z^2}$$

$$= \frac{E_m\left[\omega^2 C^2 \left(R^2 + \dfrac{1}{\omega^2 C^2}\right) - 1\right]}{\omega R C^2 Z^2} = \frac{E_m\left[\omega^2 C^2 R^2 + 1 - 1\right]}{\omega R C^2 Z^2}$$

$$= \frac{E_m \omega^2 C^2 R^2}{\omega R C^2 Z^2} = \frac{\omega R E_m}{Z^2}$$

$$\boxed{Z^2 = R^2 + \frac{1}{\omega^2 C^2}}$$

By using the expressions for K_1, K_2 and K_3, equation (3.64) of I(s) can be expressed as,

$$I(s) = \frac{-\dfrac{E_m}{\omega C Z^2}}{s + \dfrac{1}{RC}} + \frac{\dfrac{E_m}{\omega C Z^2} s + \dfrac{\omega R E_m}{Z^2}}{s^2 + \omega^2} = \frac{-E_m}{\omega C Z^2} \times \frac{1}{s + \dfrac{1}{RC}} + \frac{E_m}{\omega C Z^2} \times \frac{s}{s^2 + \omega^2} + \frac{R E_m}{Z^2} \times \frac{\omega}{s^2 + \omega^2}$$

Let us take the inverse Laplace transform of I(s).

$$\mathcal{L}^{-1}\{I(s)\} = \frac{-E_m}{\omega C Z^2} \times \mathcal{L}^{-1}\left\{\frac{1}{s + \dfrac{1}{RC}}\right\} + \frac{E_m}{\omega C Z^2} \times \mathcal{L}^{-1}\left\{\frac{s}{s^2 + \omega^2}\right\} + \frac{R E_m}{Z^2} \times \mathcal{L}^{-1}\left\{\frac{\omega}{s^2 + \omega^2}\right\}$$

$$\therefore\ i(t) = \frac{-E_m}{\omega C Z^2} e^{-\frac{t}{RC}} + \frac{E_m}{\omega C Z^2} \cos \omega t + \frac{R E_m}{Z^2} \sin \omega t$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}} \quad \boxed{\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}} \quad \boxed{\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}}$$

$$= \frac{-E_m}{\omega C Z^2} e^{-\frac{t}{RC}} + \frac{E_m}{Z^2}\left[\sin \omega t \times R + \cos \omega t \times \frac{1}{\omega C}\right] \qquad .....(3.66)$$

Let us construct a right-angled triangle with R and $\frac{1}{\omega C}$ as two sides as shown in Fig. 3.61. With reference to Fig. 3.61, we can write,

$$\tan \phi = \frac{\dfrac{1}{\omega C}}{R} \quad \Rightarrow \quad \phi = \tan^{-1} \frac{\dfrac{1}{\omega C}}{R} \qquad .....(3.67)$$

Also, $$\cos \phi = \frac{R}{Z} \quad \Rightarrow \quad R = Z \cos \phi \qquad .....(3.68)$$

$$\sin \phi = \frac{\dfrac{1}{\omega C}}{Z} \quad \Rightarrow \quad \frac{1}{\omega C} = Z \sin \phi \qquad .....(3.69)$$

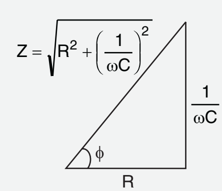


Fig. 3.61.

Using equations (3.67) to (3.69), equation (3.66) can be written as,

$$i(t) = \frac{-E_m}{\omega C Z^2} e^{-\frac{t}{RC}} + \frac{E_m}{Z^2} \times \left[\sin \omega t \times Z \cos \phi + \cos \omega t \times Z \sin \phi\right]$$

$$= \frac{-E_m}{\omega C Z^2} e^{-\frac{t}{RC}} + \frac{E_m}{Z^2} \times Z\left[\sin \omega t \cos \phi + \cos \omega t \sin \phi\right]$$

$$= \underbrace{\frac{-E_m}{\omega C Z^2} e^{-\frac{t}{RC}}}_{\text{Transient part}} + \underbrace{\frac{E_m}{Z} \sin(\omega t + \phi)}_{\text{Steady state part}}$$

$$\boxed{\sin(A + B) = \sin A \cos B + \cos A \sin B}$$

$$\text{where, } Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} \quad ; \quad \phi = \tan^{-1} \frac{1/\omega C}{R}$$

3.6.3 RLC Circuit Excited by Sinusoidal Source

Consider the RLC circuit with no initial current and voltage and excited by a sinusoidal source $e(t) = E_m \sin \omega t$, as shown in Fig. 3.62. Let the switch be closed at t = 0.

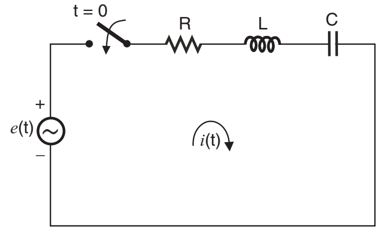


Fig. 3.62.

Let, $i(t)$ = Current through the RLC circuit

$$\mathcal{L}\{i(t)\} = I(s) \quad ; \quad \mathcal{L}\{e(t)\} = E(s)$$

$$\therefore \; E(s) = \mathcal{L}\{e(t)\} = \mathcal{L}\{E_m \sin \omega t\} = \frac{\omega E_m}{s^2 + \omega^2}$$

The s-domain equivalent of the RLC circuit is shown in Fig. 3.63. With reference to Fig. 3.63, we can write,

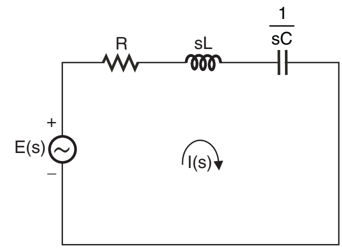


Fig. 3.63.

$$I(s) = \frac{E(s)}{R + sL + \frac{1}{sC}} = E(s) \times \frac{1}{R + sL + \frac{1}{sC}}$$

$$I(s) = \frac{\omega E_m}{s^2 + \omega^2} \times \frac{1}{\frac{L}{s}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

$$= \frac{\frac{\omega E_m}{L} s}{(s^2 + \omega^2)\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)} \qquad .....(3.70)$$

The roots of the quadratic factor in the denominator of equation (3.70) may be real or complex. Hence, we may come across the following three cases of response.

Case i : The roots of quadratic are real and equal

Let, $\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = (s+a)^2$, where $s = -a$, is the real root of the quadratic factor.

Now by partial expansion technique, equation (3.70) can be expressed as shown below :

$$I(s) = \frac{\frac{\omega E_m}{L} s}{(s^2 + \omega^2)\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)} = \frac{\frac{\omega E_m}{L} s}{(s^2 + \omega^2)(s+a)^2}$$

$$= \frac{K_1}{s+a} + \frac{K_2}{(s+a)^2} + \frac{K_3 s + K_4}{s^2 + \omega^2} \qquad .....(3.71)$$

On taking inverse Laplace transform of equation (3.71), we get the time domain response shown in equation (3.72).

$$i(t) = \underbrace{K_1 e^{-at} + K_2 t e^{-at}}_{\text{Transient part}} + \underbrace{I_{m1} \sin(\omega t \pm \phi_1)}_{\text{Steady state part}} \qquad .....(3.72)$$

where, I_{m1} = Maximum value of steady state current.

Note : *The evaluation of a, K_1, K_2, K_3, K_4, I_{m1} and ϕ_1 are left as exercise to the readers.*

Case ii : The roots of quadratic are real and unequal

Let, $\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right) = (s+b)(s+c)$, where $s=-b$ and $s=-c$ are the real roots of the quadratic factor.

Now, by partial fraction expansion technique, equation (3.70) can be expressed as shown below:

$$I(s) = \frac{\frac{\omega E_m}{L}s}{(s^2+\omega^2)\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)} = \frac{\frac{\omega E_m}{L}s}{(s^2+\omega^2)(s+b)(s+c)}$$

$$= \frac{K_5}{s+b}+\frac{K_6}{s+c}+\frac{K_7 s+K_8}{s^2+\omega^2} \quad .....(3.73)$$

On taking inverse Laplace transform of equation (3.73), we get the time domain response shown in equation (3.74).

$$\boxed{i(t) = \underbrace{K_5 e^{-bt}+K_6 e^{-ct}}_{\text{Transient part}} + \underbrace{I_{m2}\sin(\omega t \pm \phi_2)}_{\text{Steady state part}}} \quad .....(3.74)$$

where, I_{m2} = Maximum value of steady state current.

***Note :** The evaluation of b, c, K_5, K_6, K_7, K_8, I_{m2} and ϕ_2 are left as exercise to the readers.*

Case iii : The roots of quadratic are complex conjugate

When the roots are complex conjugate, the quadratic factor can be rearranged as shown in equation (3.75), and then by partial fraction expansion technique, equation (3.70) can be expressed as shown in equation (3.76).

Let, $\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right) = \left[s^2+\frac{R}{L}s+\left(\frac{R}{2L}\right)^2\right]+\left[\frac{1}{LC}-\left(\frac{R}{2L}\right)^2\right]$ $\boxed{\text{Add and subtract the term } \left(\frac{R}{2L}\right)^2}$

$$= (s+d)^2+\omega_d^2 \quad .....(3.75)$$

where, $d = \frac{R}{2L}$ and $\omega_d = \sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}$

$$\text{Now, } I(s) = \frac{\frac{\omega E_m}{L}s}{(s^2+\omega^2)\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)} = \frac{\frac{\omega E_m}{L}s}{(s^2+\omega^2)((s+d)^2+\omega_d^2)} \quad .....(3.76)$$

$$= \frac{K_9 s+K_{10}}{(s+d)^2+\omega_d^2}+\frac{K_{11}s+K_{12}}{s^2+\omega^2} \quad .....(3.77)$$

On taking inverse Laplace transform of equation (3.77), we get the time domain response shown in equation (3.78).

$$\boxed{i(t) = \underbrace{I_0 e^{-dt}\sin(\omega_d t+\theta)}_{\text{Transient part}} + \underbrace{I_{m3}\sin(\omega t \pm \phi_3)}_{\text{Steady state part}}} \quad .....(3.78)$$

where, I_{m3} = Maximum value of steady state current.

***Note :** The evaluation of K_9, K_{10}, K_{11}, K_{12}, I_{m3}, θ and ϕ_3 are left as exercise to the readers.*

3.7 Solved Problems in RL Transient

EXAMPLE 3.1

In the circuit of Fig. 1, a steady current of 5 *A* is established through the inductance by connecting it to a current source. At time t = 0, the current source is disconnected and switch S_1 is closed to connect a 20 Ω resistance across the inductance. Find the expression for *i*(t) and sketch the response. Also draw the initial and final condition of the circuit.

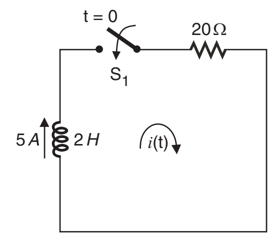


Fig. 1.

SOLUTION

Let, $\mathcal{L}\{i(t)\} = I(s)$

The s-domain equivalent circuit is shown in Fig. 2. With reference to Fig. 2, we can write,

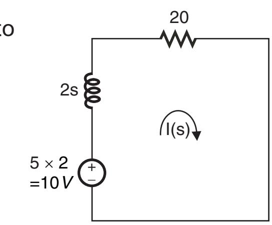


Fig. 2.

$$I(s) = \frac{10}{2s+20} = \frac{10}{2\left(s+\frac{20}{2}\right)}$$

$$\therefore\ I(s) = \frac{5}{s+10} \qquad .....(1)$$

Equation (1) is the s-domain response of the circuit. Let us take the inverse Laplace transform of I(s) to get the time domain response.

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{5}{s+10}\right\}$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

$$\therefore\ i(t) = 5\,e^{-10t}\,A$$

Time constant, $\tau = \frac{L}{R} = \frac{2}{20} = 0.1\,second$

$$\therefore\ i(t) = 5\,e^{-\frac{t}{1/10}} = 5\,e^{-\frac{t}{0.1}}\,A$$

At $t = 0^+$, $i(0^+) = 5 \times e^0 = 5A$

At $t = \tau = 0.1\,sec$, $i(t) = 5 \times e^{-\frac{0.1}{0.1}} = 5 \times e^{-1} = 5 \times 0.3679 = 1.8395\,A$

At $t = \infty$, $i(\infty) = 5 \times e^{-\infty} = 5 \times 0 = 0$

$$v_L(t) = L\frac{di(t)}{dt}$$
$$= 2 \times \frac{d}{dt}(5\,e^{-10t})$$
$$= -100\,e^{-10t}\ V$$
$$v_L(\infty) = -100\,e^{-\infty} = 0$$

since, $v_L(\infty) = 0$, the inductance is represented by short circuit in final condition circuit.

From the above analysis, we can say that at $t = 0^+$, the initial current is 5 *A* and this current of 5 *A* exponentially decays to zero as t tends to infinity.

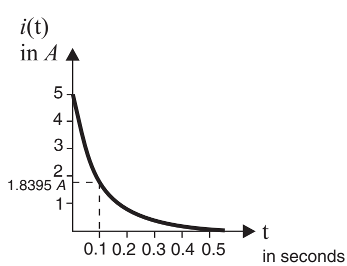


Fig. 3 : *i(t) Vs t.*

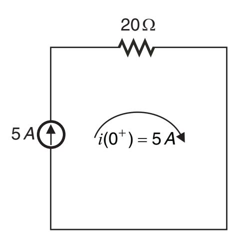


Fig. 4 : *Initial condition.*

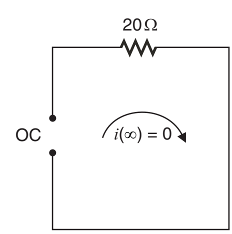


Fig. 5 : *Final condition.*

EXAMPLE 3.2 *(AU June'14 & Dec'16, 16 Marks)*

In the RL circuit of Fig. 1, the switch is closed at $t = 0$. Find the current $i(t)$ and the voltage across resistance and inductance.

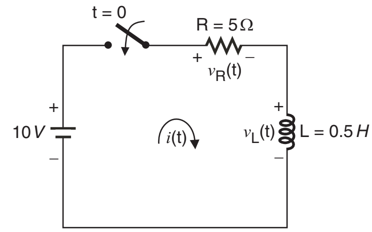


Fig. 1.

SOLUTION

Let, $\mathcal{L}\{i(t)\} = I(s)$, $\mathcal{L}\{v_R(t)\} = V_R(s)$ and $\mathcal{L}\{v_L(t)\} = V_L(s)$

Also, $\mathcal{L}\{10\} = \dfrac{10}{s}$

The s-domain equivalent circuit is shown in Fig. 2. With reference to Fig. 2, we can write,

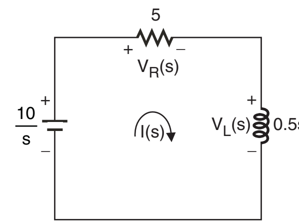


Fig. 2.

$$I(s) = \frac{\frac{10}{s}}{5 + 0.5s}$$

$$= \frac{10}{s} \times \frac{1}{0.5\left(s + \frac{5}{0.5}\right)} = \frac{20}{s(s+10)}$$

By partial fraction expansion, I(s) can be expressed as,

$$I(s) = \frac{20}{s(s+10)} = \frac{K_1}{s} + \frac{K_2}{s+10}$$

$$K_1 = \frac{20}{s(s+10)} \times s\Bigg|_{s=0} = \frac{20}{s+10}\Bigg|_{s=0} = \frac{20}{10} = 2$$

$$K_2 = \frac{20}{s(s+10)} \times (s+10)\Bigg|_{s=-10} = \frac{20}{s}\Bigg|_{s=-10} = \frac{20}{-10} = -2$$

$$\therefore\ I(s) = \frac{2}{s} - \frac{2}{s+10}$$

On taking the inverse Laplace transform of I(s), we get,

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{2}{s+10}\right\}$$

$$\boxed{\begin{aligned}\mathcal{L}\{A\} &= \frac{A}{s}\\ \mathcal{L}\{e^{-at}\} &= \frac{1}{s+a}\end{aligned}}$$

$$\therefore\ i(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\}$$

$$= 2 - 2\,e^{-10t}\,A$$

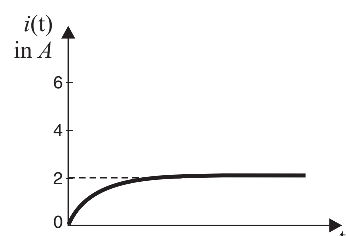


Fig. 3 : i(t) Vs t.

Time constant, $\tau = \dfrac{L}{R} = \dfrac{0.5}{5} = 0.1\,second$

$$\therefore\ i(t) = 2\left(1 - e^{-\frac{t}{1/10}}\right) = 2\left(1 - e^{-\frac{t}{0.1}}\right)A$$

With reference to Fig. 1, by Ohm's law, we can write,

$$v_R(t) = R\,i(t)$$

$$\therefore\ v_R(t) = 5 \times 2\left(1 - e^{-\frac{t}{0.1}}\right) = 10\left(1 - e^{-\frac{t}{0.1}}\right)V$$

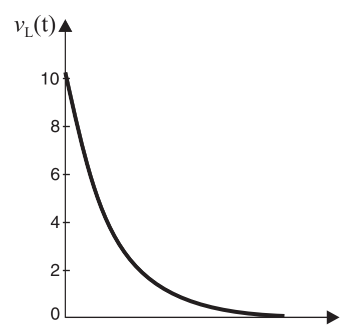


Fig. 4 : i(t) Vs $v_L(t)$.

With reference to Fig. 1, by KVL, we can write,

$$v_R(t) + v_L(t) = 10$$

$$\therefore \ v_L(t) = 10 - v_R(t)$$

$$= 10 - 10\left(1 - e^{-\frac{t}{0.1}}\right) = 10\, e^{-\frac{t}{0.1}}\ V$$

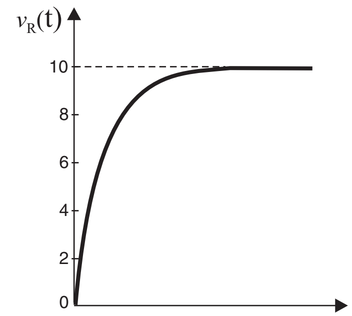


Fig. 5 : i(t) Vs $v_R(t)$.

RESULT

Current through RL series circuit, $i(t) = 2\left(1 - e^{-\frac{t}{0.1}}\right) A$

Voltage across resistance, $v_R(t) = 10\left(1 - e^{-\frac{t}{0.1}}\right) V$

Voltage across inductance, $v_L(t) = 10\, e^{-\frac{t}{0.1}}\ V$

EXAMPLE 3.3

In the RL circuit of Fig. 1, the switch is closed at position-1 for a long time and then thrown to position-2 at time t = 0. Determine the response $i(t)$.

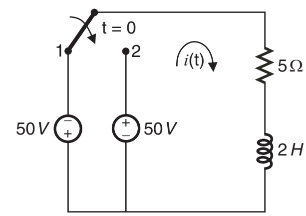


Fig. 1.

SOLUTION

Case i : Switch in position-1

Since the switch remains closed at position-1 for a long time, the circuit might have attained a steady state. The steady state (final condition) of the RL circuit with switch in position-1 is shown in Fig. 2. Let the current in the circuit be I_0.

Now, $I_0 = \frac{50}{5} = 10\, A$

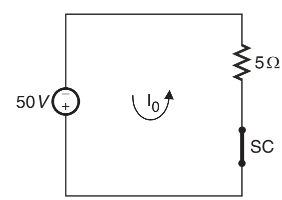


Fig. 2 : Final condition with switch in position-1.

Case ii : Switch in position-2

When the switch is changed from position-1 to position-2, a steady current I_0 is flowing in the inductance. Since the inductance does not allow a sudden change in current, this steady current I_0 will be the initial current when the switch is closed to position-2.

$$\therefore \ i(0^-) = i(0^+) = I_0 = 10\, A$$

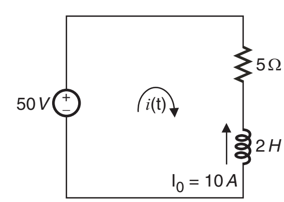


Fig. 3.

The time domain RL circuit with switch in position-2 is shown in Fig. 3.

Let, $\mathcal{L}\{i(t)\} = I(s)$

Also, $\mathcal{L}\{50\} = \frac{50}{s}$

The s-domain equivalent of the circuit of Fig. 3 is shown in Fig. 4.

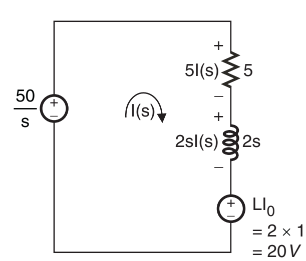


Fig. 4.

With reference to Fig. 4, by KVL, we can write,

$$5I(s) + 2sI(s) + 20 = \frac{50}{s}$$

$$I(s)[5+2s] = \frac{50}{s} - 20$$

$$I(s) \times 2\left(s+\frac{5}{2}\right) = \frac{50-20s}{s}$$

$$\therefore \; I(s) = \frac{50-20s}{s \times 2\ (s+2.5)}$$

$$= \frac{25-10s}{s\,(s+2.5)}$$

By partial fraction technique, I(s) can be expressed as,

$$I(s) = \frac{25-10s}{s\,(s+2.5)} = \frac{K_1}{s} + \frac{K_2}{s+2.5}$$

$$K_1 = \frac{25-10s}{s\,(s+2.5)} \times s\Bigg|_{s\,=\,0} = \frac{25-10s}{s+2.5}\Bigg|_{s\,=\,0} = \frac{25}{2.5} = 10$$

$$K_2 = \frac{25-10s}{s\,(s+2.5)} \times (s+2.5)\Bigg|_{s\,=\,-2.5} = \frac{25-10s}{s}\Bigg|_{s\,=\,-2.5} = \frac{25-10\times(-2.5)}{-2.5} = -20$$

$$\therefore \; I(s) = \frac{10}{s} - \frac{20}{s+2.5}$$

$$\boxed{\mathcal{L}\{A\} = \frac{A}{s} \qquad \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

On taking the inverse Laplace transform of I(s), we get,

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{10}{s} - \frac{20}{s+2.5}\right\}$$

$$\therefore \; i(t) = 10 - 20e^{-2.5t}$$

Time constant, $\tau = \dfrac{L}{R} = \dfrac{2}{5} = 0.4$ *second*

$$\therefore \; i(t) = 10\left(1-2\,e^{-\frac{t}{1/2.5}}\right) = 10\left(1-2\,e^{-\frac{t}{0.4}}\right) A$$

At $t = 0^+$, $i(0^+) = 10(1-2\times e^0) = -10\,A$

At $t = \infty$, $i(\infty) = 10(1-2\times e^{-\infty}) = 10(1-2\times 0) = 10\,A$

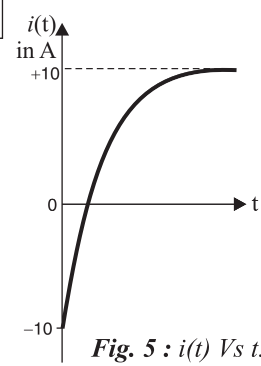


Fig. 5 : i(t) Vs t.

EXAMPLE 3.4

In the RL circuit of Fig. 1, the switch is closed at t = 0. Find the current $i(t)$ for $t \geq 0$. Also determine $\dfrac{di(t)}{dt}$, $\dfrac{d^2 i(t)}{dt^2}$ at $t = 0^+$.

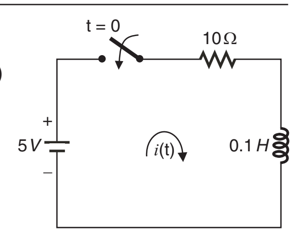


Fig. 1.

SOLUTION

Let, $\mathcal{L}\{i(t)\} = I(s)$. Also, $\mathcal{L}\{5\} = \dfrac{5}{s}$

The s-domain equivalent circuit is shown in Fig. 2. With reference to Fig. 2, we can write,

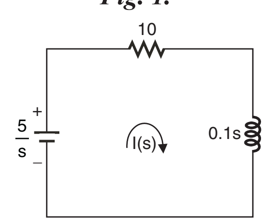


Fig. 2.

$$I(s) = \frac{\frac{5}{s}}{10+0.1s}$$

$$= \frac{5}{s} \times \frac{1}{0.1\left(s+\frac{10}{0.1}\right)} = \frac{50}{s(s+100)}$$

By partial fraction expansion, I(s) can be expressed as,

$$I(s) = \frac{50}{s(s+100)} = \frac{K_1}{s} + \frac{K_2}{s+100}$$

$$K_1 = \frac{50}{s(s+100)} \times s\bigg|_{s=0} = \frac{50}{s+100}\bigg|_{s=0} = \frac{50}{100} = 0.5$$

$$K_2 = \frac{50}{s(s+100)} \times (s+100)\bigg|_{s=-100} = \frac{50}{s}\bigg|_{s=-100} = \frac{50}{-100} = -0.5$$

$$\therefore\ I(s) = \frac{0.5}{s} - \frac{0.5}{s+100}$$

On taking the inverse Laplace transform of I(s), we get,

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{0.5}{s} - \frac{0.5}{s+100}\right\}$$

$$\boxed{\begin{array}{c}\mathcal{L}\{A\} = \dfrac{A}{s} \\ \mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}\end{array}}$$

$$\therefore\ i(t) = 0.5\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 0.5\mathcal{L}^{-1}\left\{\frac{1}{s+100}\right\}$$

$$= 0.5 - 0.5\,e^{-100t}\ A$$

$$\text{Time constant, } \tau = \frac{L}{R} = \frac{0.1}{10} = 0.01\,sec$$

$$\therefore\ i(t) = 0.5\left(1 - e^{-\frac{t}{1/100}}\right) = 0.5\left(1 - e^{-\frac{t}{0.01}}\right) A \ \text{ for } t \geq 0.$$

On differentiating $i(t)$ with respect to t, we get,

$$\frac{di(t)}{dt} = 0.5\left(0 - e^{-\frac{t}{0.01}} \times \left(-\frac{1}{0.01}\right)\right) \Rightarrow \frac{di(t)}{dt} = 50\,e^{-\frac{t}{0.01}}\ A/s$$

On differentiating $\frac{di(t)}{dt}$ with respect to t, we get,

$$\frac{d^2i(t)}{dt^2} = 50\,e^{-\frac{t}{0.01}}\left(-\frac{1}{0.01}\right) \Rightarrow \frac{d^2i(t)}{dt^2} = -5000\,e^{-\frac{t}{0.01}}\ A/s^2$$

$$\text{At } t = 0^+;\quad \frac{di(t)}{dt} = \frac{di(0^+)}{dt} = 50\,e^{-\frac{t}{0.01}}\bigg|_{t=0} = 50 \times e^0 = 50\ A/s$$

$$\text{At } t = 0^+;\quad \frac{d^2i(t)}{dt^2} = \frac{d^2i(0^+)}{dt^2} = -5000\,e^{-\frac{t}{0.01}}\bigg|_{t=0} = -5000 \times e^0 = -5000\ A/s^2$$

RESULT

$$i(t) = 0.5\left(1 - e^{-\frac{t}{0.01}}\right) A \quad \text{for } t \geq 0$$

$$\frac{di(0^+)}{dt} = 50\ A/s$$

$$\frac{d^2i(0^+)}{dt^2} = -5000\ A/s^2$$

EXAMPLE 3.5 ***(AU Dec'15, 16 Marks)***

In the RL circuit shown in Fig. 1, the switch is closed to position-1 at t = 0. After t = 100 *ms*, the switch is changed to position-2. Find $i(t)$ and sketch the transient.

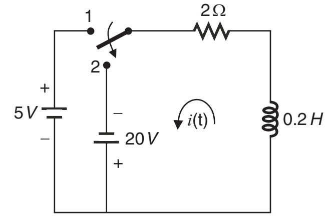


Fig. 1.

SOLUTION

Case i : Switch in position-1

Let, $\mathcal{L}\{i(t)\} = I(s)$

The s-domain equivalent of the RL circuit with switch in position-1 is shown in Fig. 2.

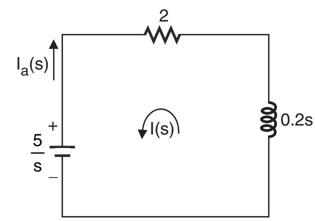


Fig. 2.

Let, $I_a(s)$ be the current delivered by 5 *V* source.

With reference to Fig. 2, we can write,

$$I_a(s) = \frac{\frac{5}{s}}{2+0.2s} = \frac{5}{s} \times \frac{1}{0.2\left(s+\frac{2}{0.2}\right)} = \frac{25}{s(s+10)}$$

By partial fraction expansion, $I_a(s)$ can be expressed as,

$$I_a(s) = \frac{25}{s(s+10)} = \frac{K_1}{s} + \frac{K_2}{s+10}$$

$$\therefore \; K_1 = \frac{25}{s(s+10)} \times s\bigg|_{s=0} = \frac{25}{s+10}\bigg|_{s=0} = \frac{25}{10} = 2.5$$

$$K_2 = \frac{25}{s(s+10)} \times (s+10)\bigg|_{s=-10} = \frac{25}{s}\bigg|_{s=-10} = \frac{25}{-10} = -2.5$$

$$\therefore \; I_a(s) = \frac{2.5}{s} - \frac{2.5}{s+10}$$

Here, $I(s) = -I_a(s) = -\frac{2.5}{s} + \frac{2.5}{s+10}$

On taking the inverse Laplace transform of $I(s)$, we get,

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{-\frac{2.5}{s} + \frac{2.5}{s+10}\right\}$$

$$\boxed{\begin{array}{l}\mathcal{L}\{A\} = \frac{A}{s} \\ \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}\end{array}}$$

$$\therefore \; i(t) = -2.5 + 2.5\,e^{-10t}\ A$$

$$= -2.5\left(1 - e^{-\frac{t}{1/10}}\right) A$$

$$= -2.5\left(1 - e^{-\frac{t}{0.1}}\right) A$$

At $t = 100\,ms$, $i(t) = -2.5\left(1 - e^{-\frac{100\times10^{-3}}{0.1}}\right) = -2.5\left(1-e^{-1}\right) = -1.5803\ A$

Let, $i(t)\Big|_{t=100ms} = I_0 = -1.5803\ A$

This current I_0 will be the initial current for circuit transient when the switch is moved from position-1 to position-2.

Case ii : Switch in position-2

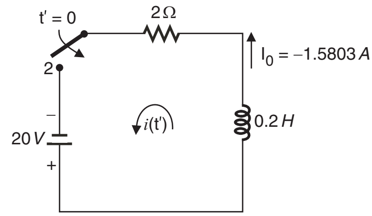


Fig. 3.

Let us assume another time frame, where time is denoted by t′ and time t′ is related to the original time frame by the relation t′ = t − 0.1 (Because 100 *ms* = 0.1 *second*).

Now the switch is moved from position-1 to position-2 at the time instant t′= 0, and at this instant there is a current of $I_0 = -1.5803\,A$ flowing in the direction of *i*(t′). The time domain circuit is shown in Fig. 3.

The s-domain equivalent of the RL circuit with switch in position-2 is shown in Fig. 4.

In the s-domain equivalent, the initial current I_0 can be represented by a voltage source of value LI_0.

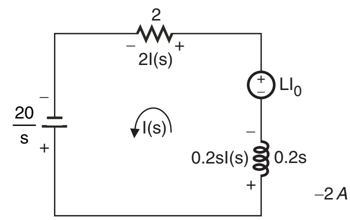


Fig. 4.

With reference to Fig. 4, by using KVL, we can write,

$$0.2s\,I(s) + 2I(s) = \frac{20}{s} + LI_0$$

$$I(s)[0.2s + 2] = \frac{20 + sLI_0}{s}$$

$$\therefore\ I(s) = \frac{20 + sLI_0}{s(0.2s+2)} = \frac{20 + sLI_0}{s \times 0.2\left(s + \frac{2}{0.2}\right)} = \frac{\frac{20}{0.2} + \frac{sLI_0}{0.2}}{s(s+10)}$$

$$= \frac{100 + \frac{s \times 0.2 \times (-1.5803)}{0.2}}{s(s+10)} = \frac{100 - 1.5803s}{s(s+10)}$$

By partial fraction expansion, I(s) can be expressed as,

$$I(s) = \frac{100 - 1.5803s}{s\,(s+10)} = \frac{K_1}{s} + \frac{K_2}{s+10}$$

$$K_1 = \frac{100 - 1.5803s}{s\,(s+10)} \times s\Bigg|_{s\,=\,0} = \frac{100 - 1.5803s}{s+10}\Bigg|_{s\,=\,0} = \frac{100}{10} = 10$$

$$K_2 = \frac{100 - 1.5803s}{s\,(s+10)} \times (s+10)\Bigg|_{s\,=\,-10} = \frac{100 - 1.5803s}{s}\Bigg|_{s\,=\,-10} = \frac{100 - 1.5803(-10)}{-10} = -11.5803$$

$$\therefore\ I(s) = \frac{10}{s} - \frac{11.5803}{s+10}$$

On taking the inverse Laplace transform of I(s), we get,

$$\boxed{\begin{aligned} \mathcal{L}\{A\} &= \frac{A}{s} \\ \mathcal{L}\{e^{-at}\} &= \frac{1}{s+a} \end{aligned}}$$

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{10}{s} - \frac{11.5803}{s+10}\right\}$$

$$\therefore\ i(t') = 10 - 11.5803\,e^{-10t'}$$

$$= 10\left(1 - \frac{11.5803}{10}\,e^{-\frac{t'}{1/10}}\right)$$

$$\therefore\ i(t') = 10\left(1 - 1.15803\,e^{-\frac{t'}{0.1}}\right) A \quad \text{for } t' \geq 0$$

Put $t' = t - 0.1$,

$$\therefore \; i(t-0.1) = 10\left(1 - 1.15803\, e^{-\frac{(t-0.1)}{0.1}}\right)A \quad ; \quad \text{for } t \geq 100\,ms$$

For convenience, $i(t-0.1)$ can be written as $i(t)$ for $t \geq 100\,ms$.

$$\therefore \; i(t) = 10\left(1 - 1.15803\, e^{-\frac{(t-0.1)}{0.1}}\right)A \quad ; \quad \text{for } t \geq 100\,ms$$

To sketch *i*(t)

$$\text{Here,} \quad i(t) = -2.5\left(1 - e^{\frac{-t}{0.1}}\right)A \quad ; \quad \text{for } 0 \leq t \leq 100\,ms$$

$$\text{and} \quad i(t) = 10\left(1 - 1.15803\, e^{\frac{-(t-0.1)}{0.1}}\right)A \quad ; \quad \text{for } 100\,ms \leq t \leq \infty$$

$\therefore$ At $t = 0^+$, $\quad i(t) = i(0^+) = -2.5(1-e^0) = 0$

At $t = 100\,ms$, $\quad i(t) = -1.5803\,A$

At $t = \infty$, $\quad i(t) = i(\infty) = 10(1 - 1.15803e^{-\infty}) = 10\,A$

Initially, the current is zero and it rises to $-1.5803\,A$ in a time of $100\,ms$. After the switch is moved to position-2, the negative current becomes zero and rises to $+10\,A$. Let us find the time instant $t = t_0$ at which the current is zero, when it changes from negative to positive.

$$i(t)\Big|_{t=t_0} = i(t_0) = 10\left(1 - 1.15803\, e^{-\frac{(t_0-0.1)}{0.1}}\right)$$

Here, at $t = t_0$, $i(t) = i(t_0) = 0$

$$\therefore \; 10\left(1 - 1.15803\, e^{\frac{-(t_0-0.1)}{0.1}}\right) = 0$$

$$\therefore \; 1.15803\, e^{-\frac{(t_0-0.1)}{0.1}} = 1$$

$$e^{-\frac{(t_0-0.1)}{0.1}} = \frac{1}{1.15803}$$

On taking the natural logarithm, we get,

$$-\left(\frac{t_0 - 0.1}{0.1}\right) = \ln\left(\frac{1}{1.15803}\right)$$

$$t_0 - 0.1 = -0.1\left[\ln\frac{1}{1.15803}\right]$$

$$\therefore \; t_0 = -0.1\left[\ln\frac{1}{1.15803}\right] + 0.1$$

$$= 0.1147\ \textit{second}$$

$$= 0.1147 \times 1000\,ms$$

$$= 114.7\,ms$$

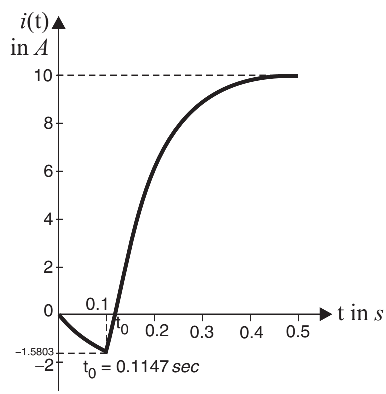


Fig. 5 : *i(t) Vs t.*

RESULT

$$i(t) = -2.5\left(1 - e^{-\frac{t}{0.1}}\right)A \; ; \quad \text{for } 0 \leq t \leq 100\,ms$$

$$i(t) = 10\left(1 - 1.15803\, e^{-\frac{(t-0.1)}{0.1}}\right)A \; ; \quad \text{for } t \geq 100\,ms$$

Time at which *i(t)* is zero, $t_0 = 0.1147$ *second*

EXAMPLE 3.6

In the circuit shown in Fig. 1, the switch is kept open for a long time. The switch is closed at t = 0. Find $i(t)$ and sketch.

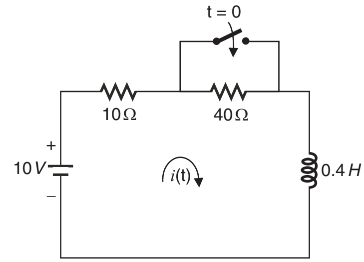


Fig. 1.

SOLUTION

Case i : Switch remains open

Since the switch remains open for a long time, the circuit might have attained steady state. The steady state (final condition) of the RL circuit with switch remains open as shown in Fig. 2. Let the steady current flowing in the circuit be I_0.

$$\text{Now, } I_0 = \frac{10}{10+40} = 0.2\,A$$

Case ii : Switch remains closed

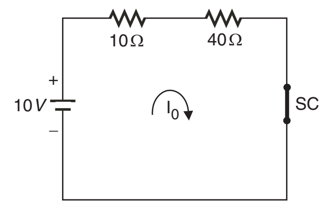


Fig. 2.

When the switch is closed the total circuit resistance reduced to 10 Ω because the 40 Ω resistance is short-circuited.

The time instant at which the switch is closed is considered as the origin of time and at this instant, and there is an initial current of $I_0 = 0.2\,A$ flowing in the direction of $i(t)$.

The s-domain equivalent of the circuit with the switch closed is shown in Fig. 3.

The initial current I_0 is represented by a voltage source of value LI_0 in the s-domain equivalent circuit.

With reference to Fig. 3, by KVL, we can write,

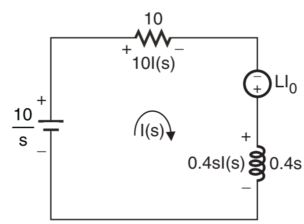


Fig. 3.

$$10\,I(s) + 0.4sI(s) = \frac{10}{s} + LI_0$$

$$I(s)\left[10 + 0.4s\right] = \frac{10 + sLI_0}{s}$$

$$\therefore \quad I(s) = \frac{10 + sLI_0}{s(10 + 0.4s)} = \frac{10 + sLI_0}{s \times 0.4 \times \left(\frac{10}{0.4} + s\right)} = \frac{\frac{10}{0.4} + \frac{sLI_0}{0.4}}{s(s+25)}$$

$$= \frac{25 + \frac{s \times 0.4 \times 0.2}{0.4}}{s(s+25)} = \frac{25 + 0.2s}{s(s+25)}$$

By partial fraction expansion, $I(s)$ can be expressed as,

$$I(s) = \frac{25 + 0.2s}{s(s+25)} = \frac{K_1}{s} + \frac{K_2}{s+25}$$

$$K_1 = \left.\frac{25 + 0.2s}{s(s+25)} \times s\right|_{s\,=\,0} = \left.\frac{25 + 0.2s}{(s+25)}\right|_{s\,=\,0} = \frac{25}{25} = 1$$

$$K_2 = \left.\frac{25 + 0.2s}{s(s+25)} \times (s+25)\right|_{s\,=\,-25} = \left.\frac{25 + 0.2s}{s}\right|_{s\,=\,-25} = \frac{25 + 0.2(-25)}{-25} = -0.8$$

$$\therefore \quad I(s) = \frac{1}{s} - \frac{0.8}{s+25}$$

On taking the inverse Laplace transform of I(s), we get,

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{0.8}{s+25}\right\}$$

$$i(t) = 1 - 0.8\, e^{-25t}\, A$$

$$= 1 - 0.8\, e^{-\frac{t}{1/25}}\, A$$

$$= 1 - 0.8\, e^{-\frac{t}{0.04}}\, A$$

At $t = 0$, $i(t) = i(0) = 1 - 0.8\, e^{0} = 0.2\,A$

At $t = \infty$, $i(t) = i(\infty) = 1 - 0.8\, e^{-\infty} = 1\,A$

$$\mathcal{L}\{A\} = \frac{A}{s}$$
$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

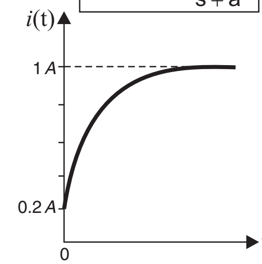


Fig. 4 : i(t) Vs t.

EXAMPLE 3.7

In the circuit shown in Fig. 1, the switch is closed to position-1 at t = 0 and at t = t′ the switch is moved to position-2. Find the time instant t = t′ such that the current i(t) remains constant at 1 A for t ≥ t′.

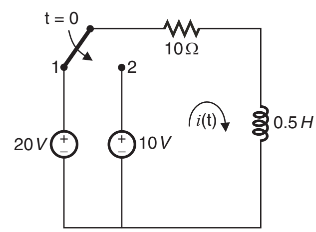


Fig. 1.

SOLUTION

Case i : Switch in position-1

The s-domain equivalent of the RL circuit with switch in position-1 is shown in Fig. 2. With reference to Fig. 2, we can write,

$$I(s) = \frac{\frac{20}{s}}{10 + 0.5s} = \frac{20}{s \times 0.5\left(\frac{10}{0.5} + s\right)} = \frac{40}{s(s+20)}$$

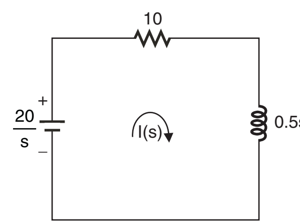


Fig. 2.

By partial fraction expansion, I(s) can be expressed as,

$$I(s) = \frac{40}{s\,(s+20)} = \frac{K_1}{s} + \frac{K_2}{s+20}$$

$$K_1 = \frac{40}{s\,(s+20)} \times s\Bigg|_{s\,=\,0} = \frac{40}{s+20}\Bigg|_{s\,=\,0} = \frac{40}{20} = 2$$

$$K_2 = \frac{40}{s\,(s+20)} \times (s+20)\Bigg|_{s\,=\,-20} = \frac{40}{s}\Bigg|_{s\,=\,-20} = \frac{40}{-20} = -2$$

$$\therefore\ I(s) = \frac{2}{s} - \frac{2}{s+20}$$

$$\mathcal{L}\{A\} = \frac{A}{s}$$
$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

On taking the inverse Laplace transform of I(s), we get,

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{2}{s+20}\right\}$$

$$i(t) = 2 - 2e^{-20t}$$

$$= 2\left(1 - e^{-\frac{t}{1/20}}\right)$$

$$= 2\left(1 - e^{-\frac{t}{0.05}}\right) A$$

Case ii : Switch in position-2

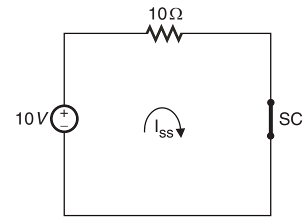


Fig. 3.

When the switch is moved to position-2 there should not be any transient. This is possible only if the initial current is equal to the steady state current in position-2. The steady state condition of the RL circuit with switch in postion-2 is shown in Fig. 3. With reference to Fig. 3,

$$\text{Steady state current, } I_{SS} = \frac{10}{10} = 1\,A$$

Now we have to find a time instant t = t′ at which the current *i*(t) with switch in position-1 is 1 *A*.

$$i(t)\Big|_{t\,=\,t'} = i(t') = 2\left(1 - e^{-\frac{t'}{0.05}}\right)$$

Here, at t = t′, *i*(t′) = 1 *A*

$$\therefore\; 2\left(1 - e^{\frac{-t'}{0.05}}\right) = 1$$

$$\therefore\; 1 - e^{\frac{-t'}{0.05}} = \frac{1}{2} \;\Rightarrow\; -e^{\frac{-t'}{0.05}} = \frac{1}{2} - 1 \;\Rightarrow\; e^{\frac{-t'}{0.05}} = \frac{1}{2}$$

On taking natural logarithm, we get,

$$-\frac{t'}{0.05} = \ln\left(\frac{1}{2}\right)$$

$$\therefore\; t' = -\ln\left(\frac{1}{2}\right) \times 0.05 = 0.0347 \textit{ second}$$

RESULT

At t = t′ = 0.0347 *second*, the switch can be moved from position-1 to position-2 to maintain the current as 1 *A* with switch in position-2 .

EXAMPLE 3.8 ***(AU June'16 & May'17, 16 Marks)***

An RL series circuit excited by a sinusoidal source *e*(t) = 10 sin 100t *V* by closing the switch at t = 0. Take R = 10 Ω and L = 0.1 *H*. Determine the current i(t) flowing through the RL circuit.

SOLUTION

Given that, *e*(t) = 10 sin 100t *V*

Let, E(s) = $\mathcal{L}$ {*e*(t)}

$$\therefore\; E(s) = \mathcal{L}\{e(t)\} = \mathcal{L}\{10\sin 100t\} = 10 \times \frac{100}{s^2 + 100^2} = \frac{10^3}{s^2 + 10^4}$$

The time domain and s-domain RL circuits excited by a sinusoidal source are shown in Figs 1 and 2.

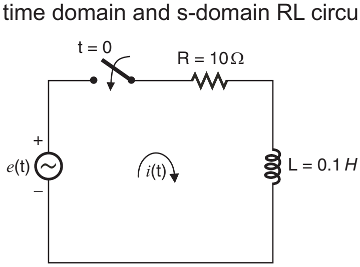


Fig. 1.

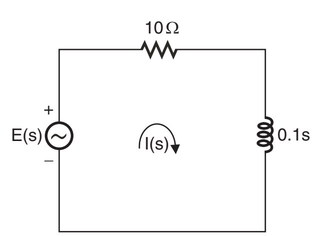


Fig. 2.

With reference to Fig. 2, we can write,

$$I(s) = \frac{E(s)}{10 + 0.1s}$$

$$= \frac{\dfrac{10^3}{s^2 + 10^4}}{0.1\left(\dfrac{10}{0.1} + s\right)} = \frac{10^3/0.1}{(s+100)(s^2+10^4)} = \frac{10^4}{(s+100)(s^2+10^4)}$$

By partial fraction expansion technique, I(s) can be expressed as,

$$I(s) = \frac{10^4}{(s+100)(s^2+10^4)} = \frac{K_1}{s+100} + \frac{K_2 s + K_3}{s^2+10^4} \quad .....(1)$$

$$K_1 = \frac{10^4}{\cancel{(s+100)}(s^2+10^4)} \times \cancel{(s+100)}\Bigg|_{s=-100} = \frac{10^4}{s^2+10^4}\Bigg|_{s=-100} = \frac{10^4}{(-100)^2+10^4} = 0.5$$

On cross-multiplying equation (1), we get,

$$10^4 = K_1 (s^2 + 10^4) + (K_2 s + K_3)(s+100)$$

$$10^4 = K_1 s^2 + 10^4 K_1 + K_2 s^2 + 100 K_2 s + K_3 s + 100 K_3$$

$$10^4 = (K_1 + K_2)s^2 + (100 K_2 + K_3)s + (10^4 K_1 + 100 K_3) \quad .....(2)$$

On equating coefficients of s^2 of equation (2), we get,

$$K_1 + K_2 = 0$$

$$\therefore \quad K_2 = -K_1 = -0.5$$

On equating coefficients of s of equation (2), we get,

$$100 K_2 + K_3 = 0$$

$$\therefore \quad K_3 = -100 K_2 = -100 \times (-0.5) = 50$$

$$\therefore \quad I(s) = \frac{0.5}{s+100} + \frac{-0.5s+50}{s^2+10^4}$$

$$= 0.5\frac{1}{s+100} - 0.5\frac{s}{s^2+100^2} + \frac{50}{100}\frac{100}{s^2+100^2}$$

$$= 0.5\frac{1}{s+100} - 0.5\frac{s}{s^2+100^2} + 0.5\frac{100}{s^2+100^2}$$

$$\boxed{\begin{aligned} \mathcal{L}\{A\} &= \frac{A}{s} \\ \mathcal{L}\{\cos\omega t\} &= \frac{s}{s^2+\omega^2} \\ \mathcal{L}\{\sin\omega t\} &= \frac{\omega}{s^2+\omega^2} \end{aligned}}$$

Let us take the inverse Laplace transform of I(s).

$$\therefore \mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{0.5\frac{1}{s+100} - 0.5\frac{s}{s^2+100^2} + 0.5\frac{100}{s^2+100^2}\right\}$$

$$\therefore i(t) = 0.5\,\mathcal{L}^{-1}\left\{\frac{1}{s+100}\right\} - 0.5\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+100^2}\right\} + 0.5\,\mathcal{L}^{-1}\left\{\frac{100}{s^2+100^2}\right\}$$

$$= 0.5e^{-100t} - 0.5\cos 100t + 0.5\sin 100t$$

$$= 0.5e^{-100t} + [\sin 100t \times 0.5 - \cos 100t \times 0.5]$$

Let us construct a right-angled triangle with 0.5 as two sides as shown in Fig. 3.

With reference to Fig. 3,

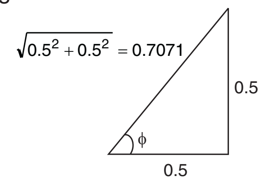


Fig. 3.

$$\tan\phi = \frac{0.5}{0.5} = 1$$

$$\therefore \phi = \tan^{-1} 1 = 45^\circ$$

$$\text{Also, } \cos\phi = \frac{0.5}{0.7071} \Rightarrow 0.5 = 0.7071\cos\phi = 0.7071\cos 45^\circ$$

$$\sin\phi = \frac{0.5}{0.7071} \Rightarrow 0.5 = 0.7071\sin\phi = 0.7071\sin 45^\circ$$

$$\therefore i(t) = 0.5e^{-100t} + [\sin 100t \times 0.7071 \cos 45^\circ - \cos 100t \times 0.7071 \sin 45^\circ]$$

$$= 0.5e^{-100t} + 0.7071 [\sin 100t \cos 45^\circ - \cos 100t \sin 45^\circ]$$

$$= \underbrace{0.5e^{-100t}}_{\text{Transient part}} + \underbrace{0.7071\sin(100t - 45^\circ)}_{\text{Steady state part}} A$$

$\boxed{\sin(A - B) = \sin A \cos B - \cos A \sin B}$

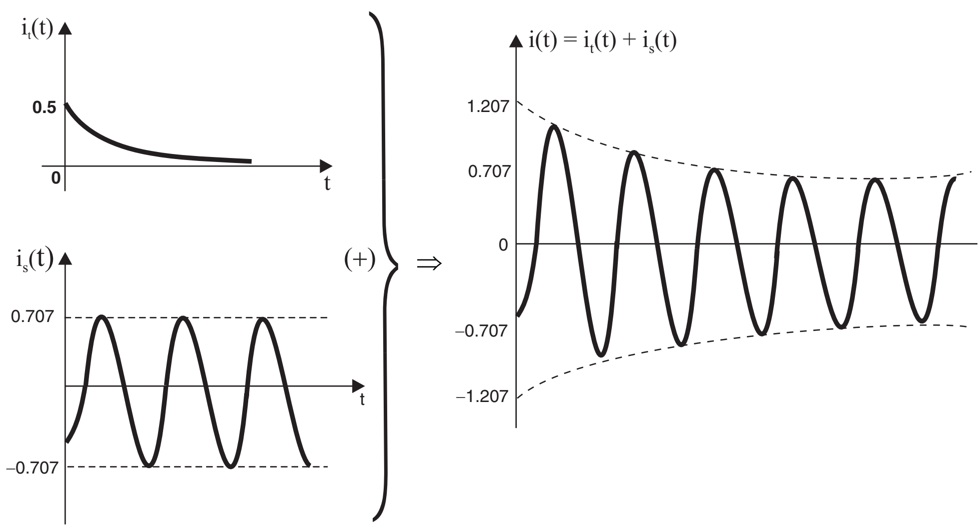


Fig 4: *Waveform of current through inductance, $i_t(t)$.*

$$v_L(t) = L\frac{di(t)}{dt} = L\frac{d}{dt}\left[0.5e^{-100t} + 0.707\sin(100t - 45^\circ)\right]$$

$$= 0.1\left[0.5e^{-100t} \times (-100) + 0.707\cos(100t - 45^\circ) \times 100\right]$$

$$= -5e^{-100t} + 7.07\cos(100t - 45^\circ)$$

$$= -5e^{-100t} + 7.07\sin(100t - 45^\circ + 90^\circ)$$

$\boxed{\because \cos\theta = \sin(\theta + 90^\circ)}$

$$= \underbrace{-5e^{-100t}}_{\text{Transient part}} + \underbrace{7.07\sin(100t + 45^\circ)}_{\text{Steady state part}} V$$

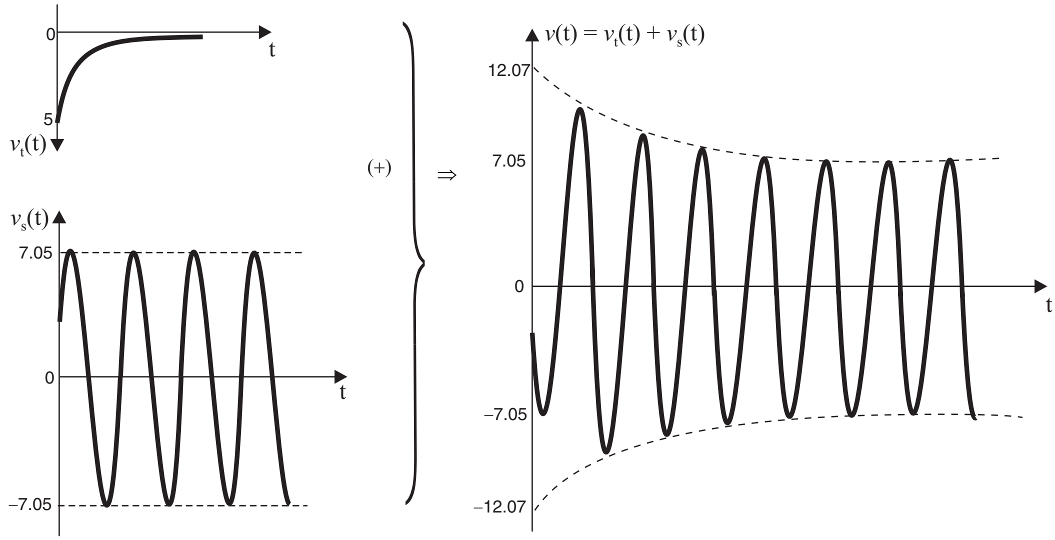


***Fig 5:** Waveform of voltage across inductance, v(t).*

EXAMPLE 3.9

In the circuit of Fig. 1, the switch remains in position-1 until steady state is reached. At time t = 0, the switch is changed to position-2. Find *i*(t).

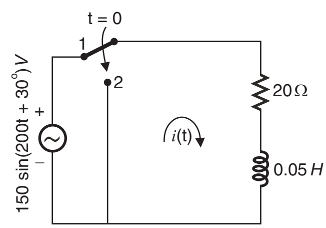


Fig. 1.

SOLUTION

Case i : Switch in position-1

In position-1, the circuit has attained a steady state. Hence, we can perform steady state analysis. The steady state of the RL circuit with switch in position-1 is shown in Fig. 2. The standard form of sinusoidal source is $E_m \sin(\omega t \pm \phi)$.

Here, $E_m \sin(\omega t \pm \phi) = 150 \sin(200t + 30^\circ)\ V$

$$\therefore\ E_m = 150\ V, \quad \omega = 200 \text{ rad/s}, \quad \phi = 30^\circ$$

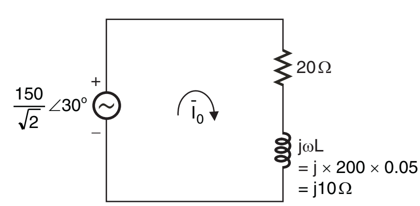


Fig. 2.

Let, I_0 be the magnitude of rms value of current flowing in the circuit.

$$\text{Now, } I_0 = \frac{\frac{150}{\sqrt{2}}}{\sqrt{20^2 + 10^2}} = 4.7434\ A$$

This steady rms current I_0 will be the initial current when the switch is moved from position-1 to position-2.

Case ii : Switch in position-2

When the switch is changed from position-1 to position-2, a steady current of I_0 is flowing in the inductance. Since the inductance does not allow sudden change in current, this steady current I_0 will be the initial current when the switch is closed to position-2.

$$\therefore\ i(0^-) = i(0^+) = I_0 = 4.7434\ A$$

The time domain and s-domain RL circuits with switch in position-2 are shown in Figs 3 and 4, respectively.

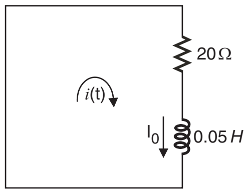


Fig. 3.

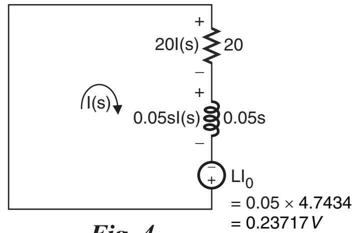


Fig. 4.

With reference to Fig. 4, by KVL, we can write,

$$20\,I(s) + 0.05s\,I(s) = 0.23717$$

$$I(s) \times 0.05\left(\frac{20}{0.05} + s\right) = 0.23717$$

$$I(s) \times 0.05\,(400 + s) = 0.23717$$

$$\therefore\ I(s) = \frac{0.23717}{0.05(400 + s)}$$

$$= \frac{4.7434}{s + 400}$$

Let us take inverse Laplace transform of I(s).

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{4.7434}{s + 400}\right\}$$

$$i(t) = 4.7434e^{-400t}\ A$$

EXAMPLE 3.10

In the circuit of Fig. 1, the switch is closed at a particular value of α so that there is no transient in the RL circuit. Find the value of α.

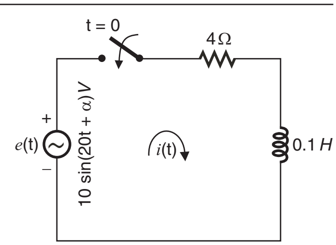


Fig. 1.

SOLUTION

Given that, $e(t) = 10\sin(20t + \alpha)\ V$

Let, $\mathcal{L}\{e(t)\} = E(s)$

$\boxed{\sin(A + B) = \sin A \cos B + \cos A \sin B}$

$$\therefore\ E(s) = \mathcal{L}\{e(t)\} = \mathcal{L}\{10\sin(20t + \alpha)\}$$

$$= 10 \times \mathcal{L}\{\sin 20t \cos\alpha + \cos 20t \sin\alpha\}$$

$$= 10 \times \cos\alpha \times \mathcal{L}\{\sin 20t\} + 10 \times \sin\alpha \times \mathcal{L}\{\cos 20t\}$$

$$= 10 \times \cos\alpha \times \frac{20}{s^2 + 20^2} + 10 \times \sin\alpha \times \frac{s}{s^2 + 20^2}$$

$$= \frac{200\cos\alpha + 10\sin\alpha\, s}{s^2 + 20^2} \quad .....(1)$$

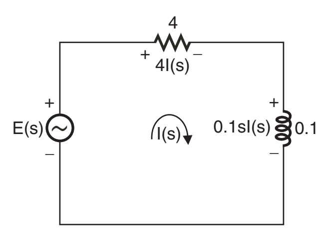


Fig. 2.

The s-domain equivalent of the given circuit is shown in Fig. 2.

With reference to Fig. 2, by KVL, we can write,

$$4\,I(s) + 0.1s\,I(s) = E(s)$$

$$\therefore\ I(s)\times 0.1\left(\frac{4}{0.1}+s\right) = \frac{200\cos\alpha + 10\sin\alpha\ s}{s^2+20^2}$$ Using equation (1)

$$I(s)\times 0.1(s+40) = \frac{200\cos\alpha + 10\sin\alpha\ s}{s^2+20^2}$$

$$\therefore\ I(s) = \frac{200\cos\alpha + 10\sin\alpha\ s}{0.1(s+40)(s^2+20^2)}$$

$$= \frac{2000\cos\alpha + 100\sin\alpha\ s}{(s+40)(s^2+20^2)}$$

By partial fraction expansion, I(s) can be expressed as,

$$I(s) = \frac{2000\cos\alpha + 100\sin\alpha\ s}{(s+40)(s^2+20^2)} = \frac{K_1}{s+40} + \frac{K_2 s + K_3}{s^2+20^2}$$

$$\therefore\ I(s) = \frac{K_1}{s+40} + \frac{K_2 s + K_3}{s^2+20^2}$$

$$= \frac{K_1}{s+40} + K_2\frac{s}{s^2+20^2} + \frac{K_3}{20}\frac{20}{s^2+20^2}$$

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$$

Let us take the inverse Laplace transform of I(s).

$$\therefore\ \mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{K_1}{s+40} + K_2\frac{s}{s^2+20^2} + \frac{K_3}{20}\frac{20}{s^2+20^2}\right\}$$

$$i(t) = \underbrace{K_1 e^{-40t}}_{\text{Transient part}} + \underbrace{K_2\cos 20t + \frac{K_3}{20}\sin 20t}_{\text{Steady state part}}$$

For no transient in i(t), the transient part should be zero.

$$\therefore\ K_1 e^{-40t} = 0$$

At t = 0, $e^{-40t} = e^0 = 1$, and so e^{-40t} cannot be zero. Therefore, $K_1 = 0$.

Let us determine an expression for K_1 and equate to zero as shown below:

$$K_1 = \left.\frac{2000\cos\alpha + 100\sin\alpha\ s}{(s+40)(s^2+20^2)}\times(s+40)\right|_{s=-40}$$

$$= \left.\frac{2000\cos\alpha + 100\sin\alpha\ s}{(s^2+20^2)}\right|_{s=-40} = \frac{2000\cos\alpha + 100\sin\alpha\,(-40)}{(-40)^2+20^2}$$

$$= \frac{2000\cos\alpha - 4000\sin\alpha}{2000} = \cos\alpha - 2\sin\alpha$$

Here, $K_1 = 0$; $\therefore$ $\cos\alpha - 2\sin\alpha = 0$

$$\therefore\ 2\sin\alpha = \cos\alpha$$

$$\frac{\sin\alpha}{\cos\alpha} = \frac{1}{2}$$

$$\tan\alpha = 0.5$$

$$\alpha = \tan^{-1}(0.5) = 26.6^\circ$$

RESULT

For no transient in $i(t)$, $\alpha = 26.6^\circ$

3.8 Solved Problems in RC Transient

EXAMPLE 3.11

In the RC circuit of Fig. 1, a charge of 100 μC is stored in the capacitor before closing the switch. At t = 0, the switch is closed to discharge the charge through 10 $k\Omega$ resistance. Determine and sketch $v_C(t)$. Also draw the initial and final condition of the circuit. Take C = 25 μF.

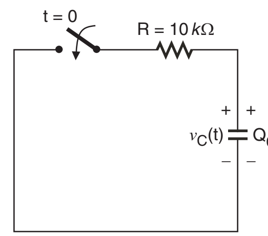


Fig. 1.

SOLUTION

Given that, $Q_0 = 100\,\mu C$ and $C = 25\,\mu F$

$$\therefore \text{ Initial voltage, } V_0 = \frac{Q_0}{C} = \frac{100\times10^{-6}}{25\times10^{-6}} = 4\,V$$

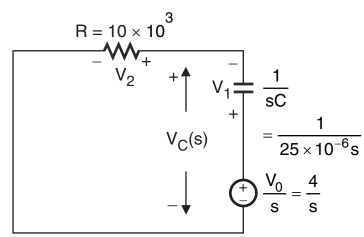


Fig. 2.

The s-domain equivalent of the given circuit is shown in Fig. 2. With reference to Fig. 2, by using voltage division rule, we get,

$$V_1 = \frac{V_0}{s}\times\frac{\frac{1}{sC}}{R+\frac{1}{sC}}$$

$$\therefore\ V_1 = \frac{V_0}{s}\times\frac{1}{sRC+1} \qquad .....(1)$$

With reference to Fig. 2, using KVL, we can write,

$$V_C(s)+V_1 = \frac{V_0}{s} \quad\Rightarrow\quad V_C(s) = \frac{V_0}{s}-V_1$$

$$\therefore\ V_C(s) = \frac{V_0}{s}-\frac{V_0}{s}\times\frac{1}{sRC+1}$$

$$= \frac{V_0}{s}\left(1-\frac{1}{sRC+1}\right) = \frac{V_0}{s}\left(\frac{sRC+1-1}{sRC+1}\right) \qquad \boxed{\text{Using equation (1)}}$$

$$= \frac{V_0}{s}\,\frac{sRC}{RC\left(s+\frac{1}{RC}\right)} = \frac{V_0}{s+\frac{1}{RC}} = \frac{4}{s+\frac{1}{10\times10^3\times25\times10^{-6}}} = \frac{4}{s+4}$$

$$\therefore\ V_C(s) = \frac{4}{s+4}$$

On taking the inverse Laplace transform of $V_C(s)$, we get,

$$\mathcal{L}^{-1}\{V_C(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{s+4}\right\}$$

$$\therefore \quad v_C(t) = 4e^{-4t} \quad \Rightarrow \quad v_C(t) = 4e^{-\frac{t}{1/4}}$$

$$\therefore v_C(t) = 4e^{-\frac{t}{0.25}} V$$

At $t = 0^+$, $v_C(0^+) = 4e^0 = 4\,V$

At $t = \infty$, $v_C(\infty) = 4e^{-\infty} = 4 \times 0 = 0$

From the above analysis, we can say that at $t = 0^+$, the initial voltage is 4 *V* and this voltage of 4 *V*, exponentially decays to zero as t tends to infinity.

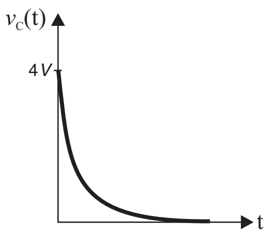


Fig. 3 : *$v_C(t)$ Vs t.*

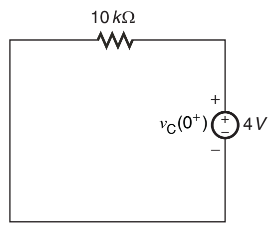


Fig. 4 : *Initial condition.*

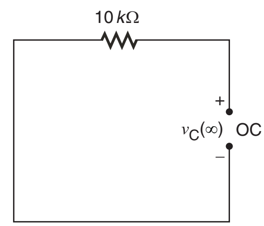


Fig. 5 : *Final condition.*

EXAMPLE 3.12

(AU June'16, 16 Marks)

In the RC circuit of Fig. 1, the switch is closed at t = 0. Find the current i(t). The initial charge in capacitor, $Q_o = 100\,\mu C$. Take R = 15Ω and C = 200 *μF*. Sketch i(t).

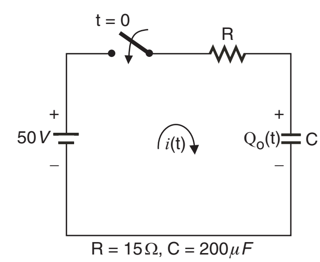


Fig. 1.

SOLUTION

Given that, $Q_0 = 100\,\mu C$ and $C = 200\,\mu F$

$$\therefore \text{ Initial voltage}, V_0 = \frac{Q_0}{C} = \frac{100 \times 10^{-6}}{200 \times 10^{-6}} = 0.5\,V$$

The s-domain equivalent of the RC circuit is shown in Fig. 2.

With reference to Fig. 2, by KVL we can write,

$$R\,I(s) + \frac{1}{sC} I(s) + \frac{0.5}{s} = \frac{50}{s}$$

$$\therefore I(s)\left[R + \frac{1}{sC}\right] = \frac{50}{s} - \frac{0.5}{s}$$

$$\therefore I(s)\left[\frac{sRC + 1}{sC}\right] = \frac{50 - 0.5}{s}$$

$$I(s)\left[\frac{RC\left(s + \frac{1}{RC}\right)}{sC}\right] = \frac{49.5}{s}$$

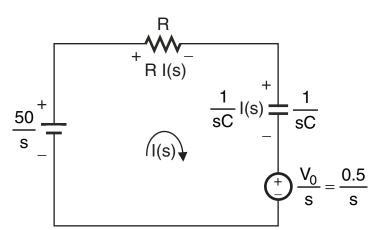


Fig. 2.

$$\therefore \; I(s) = \frac{49.5}{s} \times \left[\frac{sC}{RC\left(s + \frac{1}{RC}\right)} \right]$$

$$\therefore \; I(s) = \frac{49.5}{R} \frac{1}{s + \frac{1}{RC}}$$

On taking the inverse Laplace transform of above equation, we get,

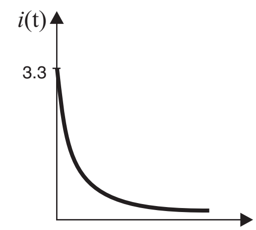


Fig. 3 : i(t) Vs t.

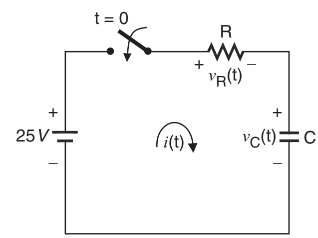
$$i(t) = \frac{49.5}{R} e^{-\frac{1}{RC}t}$$

$$= \frac{49.5}{15} e^{-\frac{1}{15 \times 200 \times 10^{-6}}t}$$

$$= 3.3 \, e^{-333.33t} \; A$$

EXAMPLE 3.13

In the RC circuit of Fig. 1, the switch is closed at t = 0. Find the current i(t), and the voltage across resistance and capacitance.



R = 100 Ω, C = 5000 μF

Fig. 1.

SOLUTION

Let, $\mathcal{L}\{i(t)\} = I(s)$, $\mathcal{L}\{v_R(t)\} = V_R(s)$ and $\mathcal{L}\{v_C(t)\} = V_C(s)$

Also, $\mathcal{L}\{25\} = \frac{25}{s}$

The s-domain equivalent circuit is shown in Fig. 2 (Please refer to Table 3.2 for the s-domain equivalent of R and C parameters.) With reference to Fig. 2, we can write,

$$I(s) = \frac{\frac{25}{s}}{R + \frac{1}{sC}}$$

$$= \frac{25}{sR + \frac{1}{C}} = \frac{25}{R\left(s + \frac{1}{RC}\right)} = \frac{25}{100\left(s + \frac{1}{RC}\right)}$$

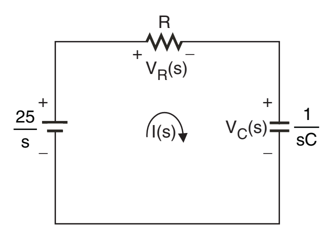


Fig. 2.

Here, Time constant, $\tau = RC = 100 \times 5000 \times 10^{-6} = 0.5$ *second*

$$\therefore \; I(s) = \frac{0.25}{\left(s + \frac{1}{0.5}\right)}$$

Let us take the inverse Laplace transform of I(s).

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{0.25}{s + \frac{1}{0.5}}\right\}$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

$$\therefore \; i(t) = 0.25 \, e^{-\frac{t}{0.5}} \; A$$

With reference to Fig. 1, by Ohm's law, we can write,

$$v_R(t) = i(t) \times R$$

$$= 0.25\, e^{-\frac{t}{0.5}} \times 100 = 25\, e^{-\frac{t}{0.5}}\ V$$

With reference to Fig. 1, by KVL, we can write,

$$v_R(t) + v_C(t) = 25$$

$$\therefore\ v_C(t) = 25 - v_R(t)$$

$$= 25 - 25\, e^{-\frac{t}{0.5}} = 25\left(1 - e^{-\frac{t}{0.5}}\right) V$$

RESULT

$$i(t) = 0.25\, e^{-\frac{t}{0.5}}\ A$$

$$v_R(t) = 25\, e^{-\frac{t}{0.5}}\ V$$

$$v_C(t) = 25\left(1 - e^{-\frac{t}{0.5}}\right) V$$

EXAMPLE 3.14 ***(AU Dec'14, 16 Marks)***

In the RC circuit of Fig. 1,the switch in the circuit is moved from position 1 to 2 at t = 0. Find the voltage across resistance and capacitance, and energy in the capacitor for t > 0.

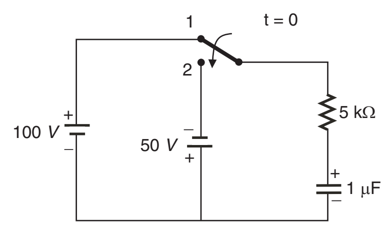


Fig. 1.

SOLUTION

Case i : Switch in position-1

Given , $C = 1 \times 10^{-6}\ F$

Let , V_0 = Voltage across capacitor with switch in position-1

W_0 = Energy stored in capacitor with switch in position-1

With switch in position-1, the circuit attains steady state. The steady state circuit is shown in Fig. 2. Therefore, the capacitor gets charged to 100 *V*.

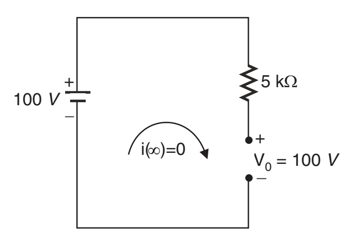


Fig. 2.

$$\therefore\ V_0 = 100\ V$$

$$W_0 = \frac{1}{2} CV_0^2 = \frac{1}{2} \times 1 \times 10^{-6} \times 100^2 = 5 \times 10^{-3}\ Joules$$

Case ii : Switch in position-2

When the switch is changed from position-1 to position-2, a steady voltage of V_0 exists across the capacitance. Since the capacitance does not allow a sudden change in voltage, this steady voltage V_0 will be the initial voltage and W_0 will be the initial energy when the switch is closed to position-2.

Let, $v_C(t)$ = Voltage across capacitor

$v_R(t)$ = Voltage across resistance

$i(t)$ = Current through capacitor.

The time domain and s-domain RC circuits are shown in Figs 3 and 4, respectively.

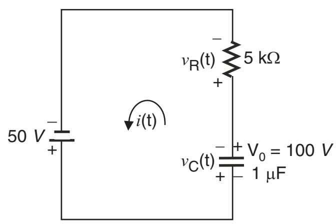


Fig. 3.

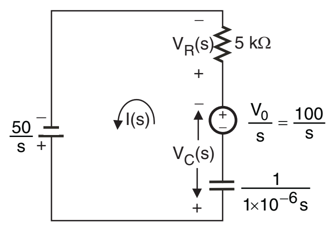


Fig. 4.

Let, $I(s) = \mathcal{L}\{i(t)\}$

With reference to Fig. 4, we can write,

$$I(s) = \frac{\frac{50}{s} + \frac{100}{s}}{5 \times 10^3 + \frac{1}{1 \times 10^{-6}s}} = \frac{\frac{150}{s}}{5000 + \frac{1}{1 \times 10^{-6}s}} = \frac{150}{5000s + \frac{1}{1 \times 10^{-6}}}$$

$$= \frac{150}{5000\left(s + \frac{1}{1 \times 10^{-6} \times 5000}\right)} = \frac{0.03}{s + 200}$$

On taking the inverse Laplace transform of the above equation, we get,

$$\therefore\ i(t) = \mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{0.03}{s+200}\right\} = 0.03e^{-200t}\ A$$

With reference to Fig. 3, by Ohm's law, we can write,

$$v_R(t) = i(t) \times R = 0.03e^{-200t} \times 5000 = 150\,e^{-200t}\ V$$

With reference to Fig. 3, by KVL,we can write,

$$v_C(t) = 50 - v_R(t) = 50 - 150e^{-200t}\ V$$

Let, $p_C(t)$ = Power in capacitor for t > 0

$w_C(t)$ = Energy in capacitor for t > 0

$$\therefore\ p_C(t) = v_C(t) \times i(t) = (50 - 150e^{-200t}) \times 0.03e^{-200t}$$

$$= 1.5e^{-200t} - 4.5e^{-400t}\ W$$

$$w_C(t) = \int_0^t p_C(t)\ dt = \int_0^t (1.5e^{-200t} - 4.5\,e^{-400t})\ dt$$

W_o = Integral constant
= Initial energy

$$= \left[\frac{1.5e^{-200t}}{-200} - \frac{4.5e^{-400t}}{-400}\right]_0^t + W_0$$

$$= \frac{1.5e^{-200t}}{-200} - \frac{4.5e^{-400t}}{-400} - \frac{1.5e^0}{-200} + \frac{4.5e^0}{-400} + 5\times10^{-3}$$

$$= -7.5\times10^{-3}\,e^{-200t} + 11.25\times10^{-3}\,e^{-400t} + 7.5\times10^{-3} - 11.25\times10^{-3} + 5\times10^{-3}$$

$$= (1.25 - 7.5\,e^{-200t} + 11.25\,e^{-400t})\times10^{-3}\ \textit{Joules}$$

EXAMPLE 3.15

In the circuit shown in Fig. 1, the capacitor C_1 has an initial charge of $12 \times 10^{-4}\,C$. Find the current through the RC circuit if the switch is closed at t = 0.

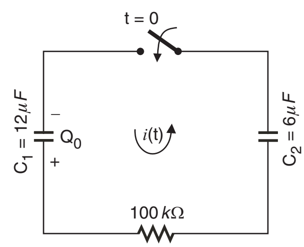


Fig. 1.

SOLUTION

Given that, $Q_0 = 12 \times 10^{-4}\,C$ and $C_1 = 12 \times 10^{-6}\,F$.

Initial voltage in capacitor C_1 is, $V_0 = \dfrac{Q_0}{C_1} = \dfrac{12\times10^{-4}}{12\times10^{-6}} = 100\,V$

Let, $i(t)$ = Current through the circuit when the switch is closed at t = 0.

Let, $\mathcal{L}\{i(t)\} = I(s)$

The s-domain equivalent of the RC circuit is shown in Fig. 2. With reference to Fig. 2, we can write,

$$I(s) = \frac{\frac{V_0}{s}}{R + \frac{1}{sC_1} + \frac{1}{sC_2}} = \frac{V_0}{sR + \frac{1}{C_1} + \frac{1}{C_2}} = \frac{V_0}{R\left(s + \frac{1}{R}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)\right)}$$

$$= \frac{V_0}{R\left(s + \frac{1}{R}\left(\frac{C_1 + C_2}{C_1\,C_2}\right)\right)} = \frac{V_0}{R\left(s + \cfrac{1}{R\left(\frac{C_1\,C_2}{C_1 + C_2}\right)}\right)}$$

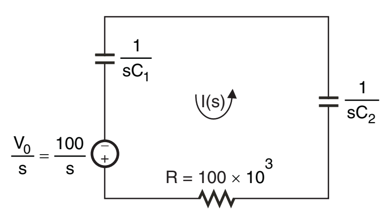


Fig. 2.

Here, $R\left(\dfrac{C_1\,C_2}{C_1 + C_2}\right) = 100\times10^3 \times \dfrac{12\times10^{-6}\times6\times10^{-6}}{12\times10^{-6} + 6\times10^{-6}} = 0.4$

$$\therefore\ I(s) = \frac{100}{100\times10^3\left(s + \frac{1}{0.4}\right)} = \frac{10^{-3}}{s + \frac{1}{0.4}}$$

Let us take the inverse Laplace transform of I(s).

$$\therefore\ \mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{10^{-3}}{s+\frac{1}{0.4}}\right\}$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

$$\therefore\ i(t) = 10^{-3}\,e^{-\frac{t}{0.4}}\,A = e^{-\frac{t}{0.4}}\,mA$$

EXAMPLE 3.16

In the circuit of Fig. 1, the switch is open for a long time. On closing the switch at t = 0, the capacitor voltage rises to 70 *V* in 10 *ms*. After the steady state is reached, the switch is opened again and found that capacitor voltage is 90 *V* in 0.5 *second*. Find the value of R and C.

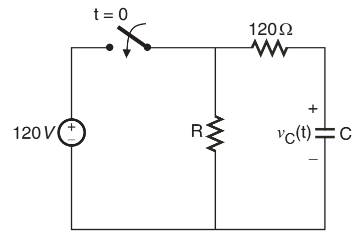


Fig. 1.

SOLUTION

Case i : Switch is closed at t = 0

When the switch is closed at t = 0, the source voltage 120 *V* is applied across the series combination of 120 Ω and C. The s-domain equivalent is shown in Fig. 2.

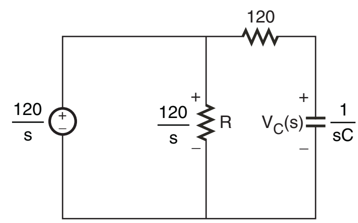


Fig. 2.

Here, $\mathcal{L}\{120\} = \frac{120}{s}$ and $\mathcal{L}\{v_C(t)\} = V_C(s)$

With reference to Fig. 2, by voltage division rule, we can write,

$$V_C(s) = \frac{120}{s} \times \frac{\frac{1}{sC}}{120 + \frac{1}{sC}}$$

$$= \frac{120}{s} \times \frac{1}{120 \times Cs + 1}$$

$$= \frac{120}{s} \times \frac{1}{120C\left(s + \frac{1}{120C}\right)} = \frac{\frac{1}{C}}{s\left(s+\frac{1}{120C}\right)}$$

$$\text{Let, } V_C(s) = \frac{\frac{1}{C}}{s\left(s+\frac{1}{120C}\right)} = \frac{K_1}{s} + \frac{K_2}{s+\frac{1}{120C}}$$

$$\boxed{\text{Using partial fraction expansion technique.}}$$

$$K_1 = \left.\frac{\frac{1}{C}}{s\left(s+\frac{1}{120C}\right)} \times s\right|_{s=0} = \left.\frac{\frac{1}{C}}{s+\frac{1}{120C}}\right|_{s=0} = \frac{\frac{1}{C}}{\frac{1}{120C}} = 120$$

$$K_2 = \left.\frac{\frac{1}{C}}{s\cancel{\left(s+\frac{1}{120C}\right)}} \times \cancel{\left(s+\frac{1}{120C}\right)}\right|_{s=\frac{-1}{120C}} = \left.\frac{\frac{1}{C}}{s}\right|_{s=\frac{-1}{120C}} = \frac{\frac{1}{C}}{-\frac{1}{120C}} = -120$$

$$\therefore\ V_C(s) = \frac{120}{s} - \frac{120}{s+\frac{1}{120C}}$$

Let us take the inverse Laplace transform of $V_C(s)$.

$$\mathcal{L}\left\{\frac{A}{s}\right\} = A \qquad \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

$$\mathcal{L}^{-1}\{V_C(s)\} = \mathcal{L}^{-1}\left\{\frac{120}{s} - \frac{120}{s+\frac{1}{120C}}\right\}$$

$$\therefore\ v_C(t) = 120 - 120\,e^{-\frac{t}{120C}}$$

$$= 120\left(1 - e^{-\frac{t}{120C}}\right)$$

Given that, at t = 10 *ms*, $v_C(t) = 70\,V$

$$\therefore\ 70 = 120\left(1 - e^{-\frac{10\times10^{-3}}{120C}}\right) \quad\Rightarrow\quad \frac{70}{120} = 1 - e^{-\frac{10\times10^{-3}}{120C}}$$

$$\therefore\ e^{-\frac{10\times10^{-3}}{120C}} = 1 - \frac{70}{120}$$

On taking the natural logarithm of the above equation, we get,

$$-\frac{10\times10^{-3}}{120C} = \ln\left(1 - \frac{70}{120}\right) \quad\Rightarrow\quad -\frac{10\times10^{-3}}{120C} = -0.8755 \quad\Rightarrow\quad \frac{120C}{10\times10^{-3}} = \frac{1}{0.8755}$$

$$\therefore\ C = \frac{10\times10^{-3}}{0.8755\times120} = 9.5184\times10^{-5}\,F = 95.184\times10^{-6}\,F = 95.184\,\mu F$$

Steady state in Case i

The steady state condition of the circuit with the switch closed is shown in Fig. 3.

With reference to Fig. 3, we can say that,

At t = ∞, $v_C(t) = 120\,V$

This steady state voltage 120 *V* will be the initial voltage across the capacitor for the analysis in Case ii.

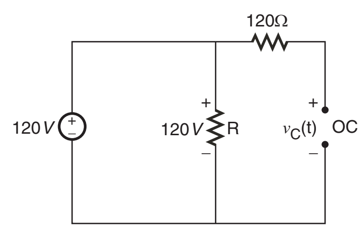


Fig. 3.

Case ii : Switch is opened after steady state in Case i

Let us denote the time by t′ in Case ii. Here the switch is opened at t′ = 0 and at this instant the capacitor has an initial voltage of 120 *V*. The time domain and s-domain circuits are shown in Figs 4 and 5, respectively.

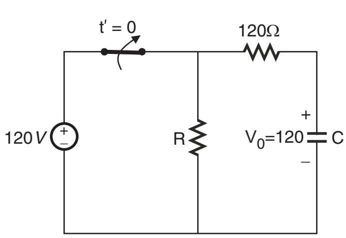


Fig. 4.

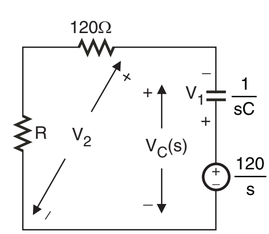


Fig. 5.

Here, $V_0 = 120\,V$

$$\therefore\ \mathcal{L}\{V_0\} = \frac{V_0}{s} = \frac{120}{s}$$

With reference to Fig. 4, by voltage division rule, we can write,

$$V_1 = \frac{120}{s} \times \frac{\frac{1}{sC}}{(120+R)+\frac{1}{sC}}$$

Let, $120 + R = R_1$

$$\therefore\ V_1 = \frac{120}{s} \times \frac{1}{sR_1C+1} \qquad .....(1)$$

By KVL, we can write,

$$V_C(s) + V_1 = \frac{120}{s}$$

$$\therefore\ V_C(s) = \frac{120}{s} - V_1 = \frac{120}{s} - \frac{120}{s} \times \frac{1}{sR_1C+1}$$

$\boxed{\text{Using equation (1)}}$

$$= \frac{120}{s}\left[1 - \frac{1}{sR_1C+1}\right] = \frac{120}{s}\left[\frac{sR_1C+1-1}{sR_1C+1}\right]$$

$$= \frac{120}{s}\left[\frac{sR_1C}{R_1C\left(s+\frac{1}{R_1C}\right)}\right]$$

$$\therefore\ V_C(s) = \frac{120}{s+\frac{1}{R_1C}}$$

On taking the inverse Laplace transform of $V_C(s)$, we get,

$$v_C(t) = 120\,e^{-\frac{t'}{R_1C}}$$

Given that, at $t' = 0.5\,second$, $v_C(t) = 90\,V$

$$\therefore\ 90 = 120\,e^{-\frac{0.5}{R_1C}} \Rightarrow e^{-\frac{0.5}{R_1C}} = \frac{90}{120}$$

On taking the natural logarithm of the above equation, we get,

$$-\frac{0.5}{R_1C} = \ln\left(\frac{90}{120}\right) \Rightarrow -\frac{0.5}{R_1C} = -0.2877 \Rightarrow \frac{R_1C}{0.5} = \frac{1}{0.2877}$$

$$\therefore\ R_1 = \frac{0.5}{0.2877C} = \frac{0.5}{0.2877 \times 95.184 \times 10^{-6}} = 18259\,\Omega$$

Here, $R_1 = 120 + R$

$$\therefore\ R = R_1 - 120 = 18259 - 120 = 18139\,\Omega = 18.139\,k\Omega$$

RESULT

$R = 18.139\,k\Omega$

$C = 95.184\,\mu F$

EXAMPLE 3.17

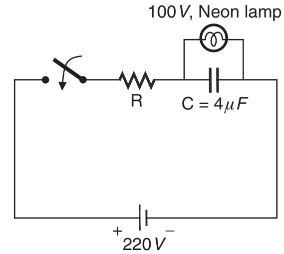


Fig. 1.

In Fig. 1, the neon lamp connected across the capacitor strikes at 100 V. Find the value of R for the lamp to strike at 5 seconds after the switch is closed. If R = 5 $M\,\Omega$, how long will it take for the lamp to strike?

SOLUTION

Let, $v_C(t)$ = Voltage across the capacitor.

$$\therefore\ \mathcal{L}\{v_C(t)\} = V_C(s). \quad \text{Also,}\ \mathcal{L}\{220\} = \frac{220}{s}$$

The s-domain equivalent of the given circuit is shown in Fig. 2. (For simplicity in analysis, the R, L and C parameters of the lamp are neglected.)

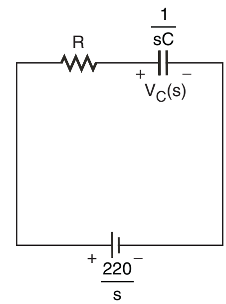


Fig. 2.

With reference to Fig. 2, by voltage division rule,

$$V_C(s) = \frac{220}{s} \times \frac{\frac{1}{sC}}{R + \frac{1}{sC}}$$

$$V_C(s) = \frac{220}{s} \times \frac{1}{sRC+1} = \frac{220}{s} \times \frac{1}{RC\left(s + \frac{1}{RC}\right)} = \frac{\frac{220}{RC}}{s\left(s + \frac{1}{RC}\right)}$$

By partial fraction expansion techniques, $V_C(s)$ can be expressed as,

$$V_C(s) = \frac{\frac{220}{RC}}{s\left(s + \frac{1}{RC}\right)} = \frac{K_1}{s} + \frac{K_2}{s + \frac{1}{RC}}$$

$$K_1 = \left.\frac{\frac{220}{RC}}{\cancel{s}\left(s + \frac{1}{RC}\right)} \times \cancel{s}\right|_{s=0} = \left.\frac{\frac{220}{RC}}{s + \frac{1}{RC}}\right|_{s=0} = \frac{\frac{220}{RC}}{\frac{1}{RC}} = 220$$

$$K_2 = \left.\frac{\frac{220}{RC}}{s\cancel{\left(s + \frac{1}{RC}\right)}} \times \cancel{\left(s + \frac{1}{RC}\right)}\right|_{s=-\frac{1}{RC}} = \left.\frac{\frac{220}{RC}}{s}\right|_{s=-\frac{1}{RC}} = \frac{\frac{220}{RC}}{-\frac{1}{RC}} = -220$$

$$\therefore\ V_C(s) = \frac{220}{s} - \frac{220}{s + \frac{1}{RC}}$$

On taking the inverse Laplace transform, we get,

$$\mathcal{L}^{-1}\{V_C(s)\} = \mathcal{L}^{-1}\left\{\frac{220}{s} - \frac{220}{s + \frac{1}{RC}}\right\}$$

$$\therefore\ v_C(t) = 220 - 220\,e^{-\frac{t}{RC}}$$

$$= 220\left(1 - e^{-\frac{t}{RC}}\right) V \qquad \text{.....(1)}$$

Case i : To find R for lamp to strike after 5 seconds

From equation (1), we get,

$$v_C(t) = 220\left(1 - e^{-\frac{t}{RC}}\right) \Rightarrow \frac{v_C(t)}{220} = 1 - e^{-\frac{t}{RC}} \Rightarrow e^{-\frac{t}{RC}} = 1 - \frac{v_C(t)}{220}$$

On taking the natural logarithm of the above equation, we get,

$$-\frac{t}{RC} = \ln\left(1 - \frac{v_C(t)}{220}\right) \qquad .....(2)$$

$$\therefore \frac{RC}{t} = -\frac{1}{\ln\left(1 - \frac{v_C(t)}{220}\right)} \Rightarrow R = \frac{t}{C}\left[-\frac{1}{\ln\left(1 - \frac{v_C(t)}{220}\right)}\right]$$

Given that, t = 5 *seconds* ; C = 4 μF ; The lamp will strike when $v_C(t) = 100\,V$

$$\therefore R = \frac{5}{4 \times 10^{-6}}\left[-\frac{1}{\ln\left(1 - \frac{100}{220}\right)}\right]$$

$$= 2062244.125\,\Omega = \frac{2062244.125}{10^6}\,M\Omega = 2.0622\,M\Omega$$

Case ii : To find time for lamp to strike if R = 5 MΩ

From equation (2), we get,

$$\text{Time, } t = -RC\left[\ln\left(1 - \frac{v_C(t)}{220}\right)\right]$$

Here, $R = 5\ M\Omega = 5 \times 10^6\,\Omega$, $C = 4\,\mu F = 4 \times 10^{-6}\,F$, $v_C(t) = 100\,V$

$$\therefore t = -5 \times 10^6 \times 4 \times 10^{-6}\left[\ln\left(1 - \frac{100}{220}\right)\right]$$

$$= 12.1227\ \textit{seconds}$$

RESULT

1. For lamp to strike at 5 *seconds* after switching, R = 2.0622 $M\Omega$
2. When R = 5 $M\Omega$, the time for lamp to strike after switching, t = 12.1227 *seconds*

EXAMPLE 3.18

In the RC circuit of Fig. 1, when the switch is closed at t = 0, the current through the circuit is $i(t) = 0.075\,e^{-50t}\,A$. Find the value of Q_0 and its polarity.

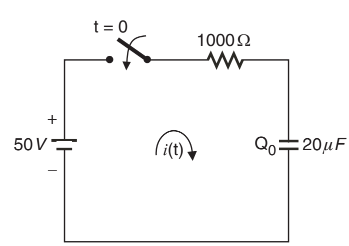


Fig. 1.

SOLUTION

Given that,

Initial charge = Q_0

$$\therefore \text{Initial voltage, } V_0 = \frac{Q_0}{C}$$

Let, $\mathcal{L}\{i(t)\} = I(s)$, $\mathcal{L}\{50\} = \frac{50}{s}$

The s-domain equivalent of the given circuit is shown in Fig. 2.

In Fig. 2, the polarity of Q_0 is assumed to be opposing the direction of current.

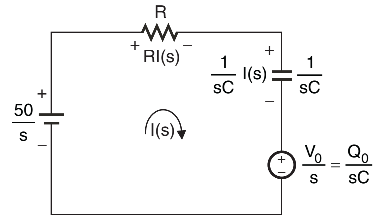


Fig. 2.

With reference to Fig. 2, by KVL, we can write,

$$RI(s) + \frac{1}{sC}I(s) + \frac{Q_0}{sC} = \frac{50}{s}$$

$$I(s)\left[R + \frac{1}{sC}\right] = \frac{50}{s} - \frac{Q_0}{sC}$$

$$I(s)\left[\frac{sRC+1}{sC}\right] = \frac{50C - Q_0}{sC} \quad \Rightarrow \quad I(s) = \frac{50C - Q_0}{sRC + 1}$$

$$\therefore\ I(s) = \left(\frac{50C - Q_0}{RC}\right)\frac{1}{\left(s + \frac{1}{RC}\right)}$$

On taking the inverse Laplace transform of I(s), we get,

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\left(\frac{50C - Q_0}{RC}\right)\frac{1}{\left(s + \frac{1}{RC}\right)}\right\}$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

$$\therefore\quad i(t) = \frac{50C - Q_0}{RC}\, e^{-\frac{t}{RC}}\ A \qquad .....(1)$$

Given that, $i(t) = 0.075\ e^{-50t}\ A$ .....(2)

On comparing equations (1) and (2), we get,

$$\frac{50C - Q_0}{RC} = 0.075$$

$$50C - Q_0 = 0.075 \times RC$$

$$-Q_0 = 0.075RC - 50C$$

$$\therefore\ Q_0 = -0.075RC + 50C$$

$$= (-0.075 \times 1000 \times 20 \times 10^{-6}) + (50 \times 20 \times 10^{-6})$$

$$= -5 \times 10^{-4}\,C = -500 \times 10^{-6}\,C$$

$$= -500\,\mu C$$

Since the initial charge is negative, the actual polarity of Q_0 is opposite to that of the assumed polarity. The actual polarity of Q_0 is shown in Fig. 3.

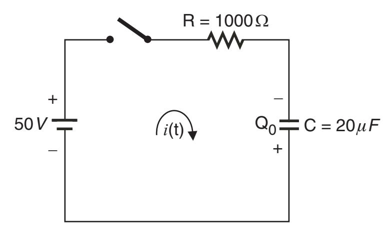


Fig. 3. $Q_0 = 500\,\mu C$

RESULT

Initial charge, $Q_0 = 500\ \mu C$ (with polarity as shown in Fig. 3)

EXAMPLE 3.19

In the RC circuit of Fig. 1, the capacitor has an initial charge of $Q_0 = 120\,\mu C$. When the switch is closed at t = 0, find the time taken for the capacitor voltage to drop from 50 *V* to 10 *V*.

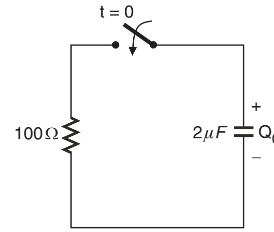


Fig. 1.

SOLUTION

Given that,

Initial charge, $Q_0 = 120\,\mu C$

$$\therefore \text{ Initial voltage, } V_0 = \frac{Q_0}{C} = \frac{120 \times 10^{-6}}{2 \times 10^{-6}} = 60\,V$$

Let, $v_C(t)$ be the voltage across the capacitor with a polarity same as that of Q_0, as shown in Fig. 2. The s-domain equivalent circuit is shown in Fig. 3.

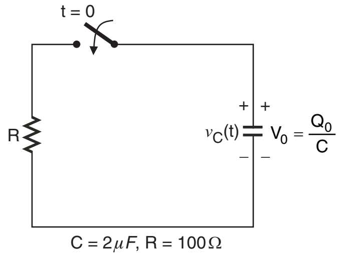


Fig. 2.

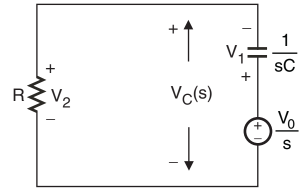


Fig. 3.

With reference to Fig. 1, by voltage division rule,

$$V_1 = \frac{V_0}{s} \times \frac{\frac{1}{sC}}{R + \frac{1}{sC}}$$

$$= \frac{V_0}{s} \times \frac{1}{sRC + 1} \qquad .....(1)$$

With reference to Fig. 3, by KVL, we can write,

$$V_C(s) + V_1 = \frac{V_0}{s} \quad \Rightarrow \quad V_C(s) = \frac{V_0}{s} - V_1$$

$$\therefore V_C(s) = \frac{V_0}{s} - \frac{V_0}{s} \times \frac{1}{sRC + 1} \qquad \boxed{\text{Using equation (1)}}$$

$$= \frac{V_0}{s}\left(1 - \frac{1}{sRC + 1}\right) = \frac{V_0}{s}\left(\frac{sRC + 1 - 1}{sRC + 1}\right) = \frac{V_0}{s}\,\frac{sRC}{RC\left(s + \frac{1}{RC}\right)}$$

$$\therefore V_C(s) = \frac{V_0}{s + \frac{1}{RC}} \qquad \boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

On taking the inverse Laplace transform of the above equation, we get,

$$v_C(t) = V_0\, e^{-\frac{t}{RC}} \quad \Rightarrow \quad \frac{v_C(t)}{V_0} = e^{-\frac{t}{RC}} \quad \Rightarrow \quad \ln\left(\frac{v_C(t)}{V_0}\right) = -\frac{t}{RC}$$

$$\therefore t = -RC\left[\ln\left(\frac{v_C(t)}{V_0}\right)\right]$$

Here, RC = $100 \times 2 \times 10^{-6} = 200 \times 10^{-6}$ *seconds*

$V_0 = 60\,V$

$$\therefore \quad t = -200 \times 10^{-6} \left[\ln \left(\frac{v_C(t)}{60} \right) \right] \qquad .....(2)$$

Let, the time instant at which $v_C(t) = 50\,V$ be t_1.

From equation (2),

$$t_1 = -200 \times 10^{-6} \left[\ln \left(\frac{50}{60} \right) \right] = 3.6464 \times 10^{-5} = 36.464 \times 10^{-6} \textit{ seconds}$$

Let the time instant at which $v_C(t) = 10\,V$ be t_2.

From equation (2),

$$t_2 = -200 \times 10^{-6} \left[\ln \left(\frac{10}{60} \right) \right] = 3.5835 \times 10^{-4} = 358.35 \times 10^{-6} \textit{ seconds}$$

$$\therefore \left. \begin{array}{l} \text{Time for capacitor voltage} \\ \text{to drop from } 50\,V \text{ to } 10\,V \end{array} \right\} = t_2 - t_1$$

$$= 358.35 \times 10^{-6} - 36.464 \times 10^{-6}$$

$$= 321.886 \times 10^{-6} \textit{ seconds}$$

$$= 321.886\,\mu s$$

EXAMPLE 3.20

A capacitor has an initial charge of Q_0. A resistance R is connected across the capacitor at t = 0, to discharge the charge. The power transient of the capacitor $p_C(t) = 800e^{-4000t}\,W$. Find the value of R and Q_0. Take C = $10\,\mu F$.

SOLUTION

Let, $v_C(t)$ = Voltage across capacitor

$i(t)$ = Current through capacitor

$\therefore$ Power, $p(t) = v_C(t) \times i(t)$

The time domain and s-domain RC circuits are shown in Figs 1 and 2, respectively.

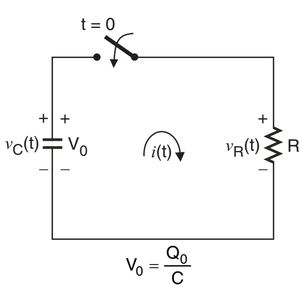


Fig. 1.

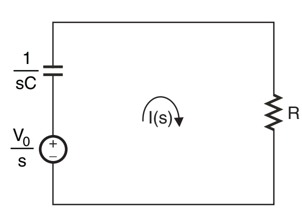


Fig. 2.

With reference to Fig. 2, we can write,

$$I(s) = \frac{\frac{V_0}{s}}{\frac{1}{sC}+R} = \frac{V_0}{s} \times \frac{1}{\frac{1}{sC}+R} = \frac{V_0}{\frac{1}{C}+sR} = \frac{V_0}{R\left(s+\frac{1}{RC}\right)}$$

$$\therefore\ I(s) = \frac{V_0}{R}\frac{1}{s+\frac{1}{RC}}$$

On taking the inverse Laplace transform of the above equation, we get,

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{V_0}{R}\frac{1}{s+\frac{1}{RC}}\right\}$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

$$\therefore\ i(t) = \frac{V_0}{R}e^{-\frac{t}{RC}}\ A$$

With reference to Fig. 1, by Ohm's law, we can write,

$$v_R(t) = i(t) \times R$$

$$\therefore\ v_R(t) = \frac{V_0}{R}e^{-\frac{t}{RC}} \times R$$

$$= V_0\,e^{-\frac{t}{RC}}\ V$$

Also, $v_R(t) = v_C(t)$

$$\therefore\ v_C(t) = V_0\,e^{-\frac{t}{RC}}\ V$$

Power in capacitor, $p_C(t) = v_C(t) \times i(t)$

$$= V_0\,e^{-\frac{t}{RC}} \times \frac{V_0}{R}e^{-\frac{t}{RC}}$$

$$= \frac{V_0^2}{R}e^{-\frac{2t}{RC}}\ W \qquad .....(1)$$

Given that, $p_C(t) = 800e^{-4000t}\ W$ .....(2)

On comparing equations (1) and (2), we get,

$$\frac{2}{RC} = 4000 \quad \Rightarrow \quad \frac{RC}{2} = \frac{1}{4000}$$

$$\therefore\ R = \frac{2}{C} \times \frac{1}{4000} = \frac{2}{10 \times 10^{-6}} \times \frac{1}{4000} = 50\,\Omega$$

$$\boxed{\text{Given that, } C = 10\ \mu F}$$

Again on comparing equations (1) and (2), we get,

$$\frac{V_0^2}{R} = 800$$

$$\therefore\ V_0 = \sqrt{800 \times R} = \sqrt{800 \times 50} = 200\ V$$

We know that, $Q_0 = V_0C$

$$\therefore\ Q_0 = 200 \times 10 \times 10^{-6} = 2000 \times 10^{-6}\ C$$

$$= 2000\ \mu C$$

RESULT

$R = 50\,\Omega, \quad Q_0 = 2000\ \mu C$

EXAMPLE 3.21

In the circuit of Fig. 1, the switch is closed for a long time. At t = 0, the switch is opened. Find $v_C(t)$.

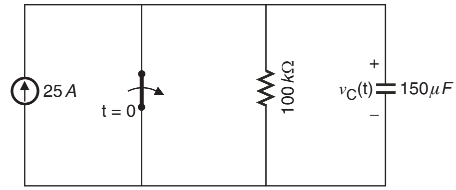


Fig. 1.

SOLUTION

When the switch is closed the current source is shorted and so no current will flow to R and C in parallel. Since the switch is closed for a long time, the charges in the capacitor might have discharged through 100 $k\Omega$ in parallel. Hence, there is no initial charge in the capacitor.

Let, $\mathcal{L}\{v_C(t)\} = V_C(s)$, Also, $\mathcal{L}\{25\} = \dfrac{25}{s}$

The s-domain equivalent of the RC circuit is shown in Fig. 2.

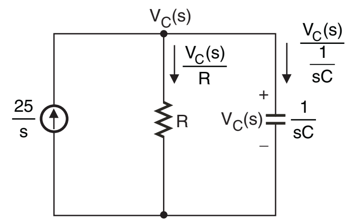


Fig. 2.

With reference to Fig. 2, by KCL we can write,

$$\frac{V_C(s)}{R} + \frac{V_C(s)}{\frac{1}{sC}} = \frac{25}{s}$$

$$V_C(s)\left(\frac{1}{R} + sC\right) = \frac{25}{s}$$

$$V_C(s)\,C\left(\frac{1}{RC} + s\right) = \frac{25}{s}$$

$$V_C(s) = \frac{25}{s} \times \frac{1}{C\left(s + \frac{1}{RC}\right)}$$

$$\therefore\ V_C(s) = \frac{\frac{25}{C}}{s\left(s + \frac{1}{RC}\right)}$$

By partial fraction expansion technique, $V_C(s)$ can be expressed as,

$$V_C(s) = \frac{\frac{25}{C}}{s\left(s + \frac{1}{RC}\right)} = \frac{K_1}{s} + \frac{K_2}{s + \frac{1}{RC}}$$

$$K_1 = \left.\frac{\frac{25}{C}}{s\left(s + \frac{1}{RC}\right)} \times s\right|_{s=0} = \left.\frac{\frac{25}{C}}{s + \frac{1}{RC}}\right|_{s=0} = \frac{\frac{25}{C}}{\frac{1}{RC}} = 25R$$

$$K_2 = \left.\frac{\frac{25}{C}}{s\left(s + \frac{1}{RC}\right)} \times \left(s + \frac{1}{RC}\right)\right|_{s=-\frac{1}{RC}} = \left.\frac{\frac{25}{C}}{s}\right|_{s=-\frac{1}{RC}} = \frac{\frac{25}{C}}{-\frac{1}{RC}} = -25R$$

$$\therefore\ V_C(s) = \frac{25R}{s} - \frac{25R}{s + \frac{1}{RC}}$$

Let us take the inverse Laplace transform of $V_C(s)$.

$$\mathcal{L}\left\{\frac{A}{s}\right\} = A$$
$$\mathcal{L}\left\{\frac{A}{e^{-at}}\right\} = \frac{A}{s+a}$$

$$\mathcal{L}^{-1}\{V_C(s)\} = \mathcal{L}^{-1}\left\{\frac{25R}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{25R}{s+\frac{1}{RC}}\right\}$$

$$\therefore\ v_C(t) = 25R - 25R\,e^{-\frac{t}{RC}}$$

$$= 25R\left(1 - e^{-\frac{t}{RC}}\right) V$$

Here, $RC = 100 \times 10^3 \times 150 \times 10^{-6} = 15\,seconds$

$$\therefore\ v_C(t) = 25 \times 100 \times 10^3 \left(1 - e^{-\frac{t}{15}}\right) V$$

$$= 2.5 \times 10^6 \left(1 - e^{-\frac{t}{15}}\right) V = 2.5\left(1 - e^{-\frac{t}{15}}\right) MV$$

EXAMPLE 3.22 *(AU June'14, 10 Marks)*

When a dc voltage is applied to the capacitor in the circuit of Fig. 1 the voltage across its terminals is found to build up in accordance with $v_c(t) = 50(1 - e^{-100t})\ V$. After 0.01 second the current flow is equal to 2 *mA*.

(1) Find the value of capacitance in Farad.

(2) How much energy is stored in the electric field?

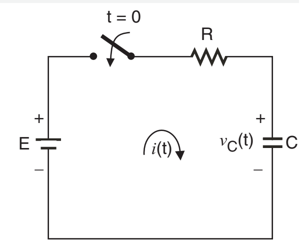


Fig. 1.

SOLUTION

The capacitor voltage and current in an RC series circuit is,

$$v_C(t) = E\left(1 - e^{-\frac{t}{RC}}\right) \quad .....(1)$$

$$i(t) = \frac{E}{R}e^{-\frac{t}{RC}} \quad .....(2)$$

Given that,

$$v_C(t) = 50\left(1 - e^{-100t}\right) \quad .....(3)$$

On comparing equations (1) and (3), we get,

$$E = 50\,V \quad ; \quad \frac{1}{RC} = 100$$

On substituting the above values in equation (2), we get,

$$i(t) = \frac{50}{R}e^{-100t} \quad .....(4)$$

Given that, at t = 0.01 *second*, $i(t) = 2\,mA = 2 \times 10^{-3}\,A$

On substituting the above values in equation (4), we get,

$$2 \times 10^{-3} = \frac{50}{R} \times e^{-100 \times 0.01} \quad \Rightarrow \quad R = \frac{50}{2 \times 10^{-3}} \times e^{-1} = 9196.986\,\Omega$$

$$\therefore\ R = 9196.986\,\Omega \approx 9197\,\Omega$$

Here, $\frac{1}{RC} = 100$

$$\therefore\ C = \frac{1}{R \times 100} = \frac{1}{9197 \times 100} = 1.0873 \times 10^{-6}\,F = 1.0873\,\mu F$$

The energy stored in the capacitor is,

$$W_C = \frac{1}{2} C\, v_C^2(\infty)$$

Here, $v_C(\infty) = v_C(t)\Big|_{t=\infty} = 50\left(1 - e^{-\infty}\right) = 50\,V$

$$\text{Energy, } W_C = \frac{1}{2} C\, v_C^2(\infty) = \frac{1}{2} \times 1.0873 \times 10^{-6} \times 50^2 = 1.3591 \times 10^{-3}\ \textit{Joules}$$

EXAMPLE 3.23

An RC circuit is excited by a sinusoidal source of voltage $50 \sin 314t$ *V*. Find the voltage and current in the capacitor. Take $R = 100\,\Omega$ and $C = 20\,\mu F$.

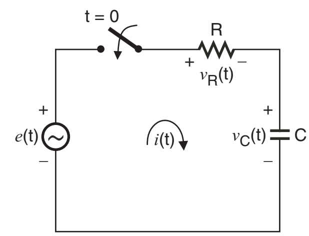


Fig. 1.

SOLUTION

Given that, $e(t) = 50 \sin 314t$ *V*

$$R = 100\,\Omega \text{ and } C = 20\,\mu F = 20 \times 10^{-6}\,F$$

The time domain RC circuit is shown in Fig. 1.

Let, $\mathcal{L}\{e(t)\} = E(s)$ and $\mathcal{L}\{i(t)\} = I(s)$

$$\therefore\ E(s) = \mathcal{L}\{e(t)\} = \mathcal{L}\{50 \sin 314t\}$$

$$= 50 \times \frac{314}{s^2 + 314^2} = \frac{15700}{s^2 + 314^2}$$

The s-domain equivalent of the RC circuit is shown in Fig. 2. With reference to Fig. 2, we can write,

$$I(s) = \frac{E(s)}{R + \frac{1}{sC}}$$

$$= \frac{E(s)}{\frac{sRC + 1}{sC}} = E(s) \times \frac{sC}{RC\left(s + \frac{1}{RC}\right)}$$

$$= \frac{15700}{s^2 + 314^2} \times \frac{s}{R\left(s + \frac{1}{RC}\right)}$$

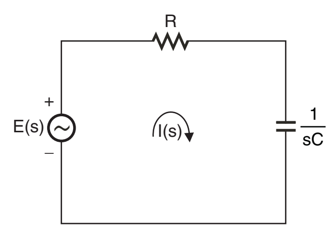


Fig. 2.

Here, $RC = 100 \times 20 \times 10^{-6} = 0.002$ *second*.

$$\therefore\ I(s) = \frac{15700}{s^2 + 314^2} \times \frac{s}{100\left(s + \frac{1}{0.002}\right)}$$

$$= \frac{157s}{(s + 500)(s^2 + 314^2)}$$

By partial fraction expansion technique, we can write,

$$I(s) = \frac{157s}{(s+500)(s^2+314^2)} = \frac{K_1}{s+500} + \frac{K_2 s + K_3}{s^2+314^2} \qquad ..... (1)$$

$$K_1 = \frac{157s}{(s+500)(s^2+314^2)} \times (s+500)\Bigg|_{s=-500} = \frac{157\times(-500)}{(-500)^2+314^2} = -0.2252$$

On cross-multiplying equation (1), we get,

$$157s = K_1(s^2+314^2) + (K_2 s + K_3)(s+500)$$

$$157s = K_1 s^2 + 314^2 K_1 + K_2 s^2 + 500\,K_2 s + K_3 s + 500K_3$$

$$157s = (K_1+K_2)s^2 + (500\,K_2+K_3)s + 314^2 K_1 + 500K_3 \qquad ..... (2)$$

On equating coefficients of s^2 of equation (2), we get,

$$K_1 + K_2 = 0$$

$$\therefore\ K_2 = -K_1 = 0.2252$$

On equating coefficients of s of equation (2), we get,

$$500K_2 + K_3 = 157$$

$$\therefore\ K_3 = 157 - 500K_2$$

$$= 157 - 500 \times 0.2252 = 44.4$$

$$\therefore\ I(s) = \frac{-0.2252}{s+500} + \frac{0.2252s+44.4}{s^2+314^2}$$

$$= \frac{-0.2252}{s+500} + 0.2252\frac{s}{s^2+314^2} + \frac{44.4}{314}\times\frac{314}{s^2+314^2}$$

$$= -0.2252\times\frac{1}{s+500} + 0.2252\times\frac{s}{s^2+314^2} + 0.1414\times\frac{314}{s^2+314^2}$$

Let us take the inverse Laplace transform of I(s).

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\begin{aligned}&-0.2252\times\frac{1}{s+500} + 0.2252\times\frac{s}{s^2+314^2}\\ &\qquad\qquad + 0.1414\times\frac{314}{s^2+314^2}\end{aligned}\right\}$$

$$\boxed{\begin{aligned}\mathcal{L}\{e^{-at}\} &= \frac{1}{s+a}\\ \mathcal{L}\{\cos\omega t\} &= \frac{s}{s^2+\omega^2}\\ \mathcal{L}\{\sin\omega t\} &= \frac{\omega}{s^2+\omega^2}\end{aligned}}$$

$$\therefore\ i(t) = -0.2252\,e^{-500t} + 0.2252\cos 314t + 0.1414\sin 314t$$

$$i(t) = -0.2252\,e^{-500t} + [\sin 314t \times 0.1414 + \cos 314t \times 0.2252]$$

Let us construct a right-angled triangle with 0.1414 and 0.2252 as two sides as shown in Fig. 3.

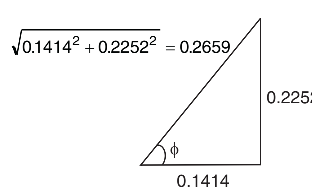


Fig. 3.

With reference to Fig. 3, we can write,

$$\tan\phi = \frac{0.2252}{0.1414} = 1.5926$$

$$\therefore\ \phi = \tan^{-1}(1.5926) = 57.9° \approx 58°$$

Also, $\cos\phi = \dfrac{0.1414}{0.2659} \Rightarrow 0.1414 = 0.2659\cos\phi = 0.2659\cos 58^\circ$

$\sin\phi = \dfrac{0.2252}{0.2659} \Rightarrow 0.2252 = 0.2659\sin\phi = 0.2659\sin 58^\circ$

$$\therefore\ i(t) = -0.2252\,e^{-500t} + [\sin 314t \times 0.2659\cos 58^\circ + \cos 314t \times 0.2659\sin 58^\circ]$$

$$= -0.2252\,e^{-500t} + 0.2659\,[\sin 314t\cos 58^\circ + \cos 314t\sin 58^\circ]$$

$$= -0.2252\,e^{-500t} + 0.2659\sin(314t + 58^\circ)\,A \qquad \boxed{\sin(A+B) = \sin A\cos B + \cos A\sin B}$$

If $i(t)$ is the current through the capacitor then the voltage across the capacitor $v_C(t)$ is given by,

$$v_C(t) = \frac{1}{C}\int i(t)\,dt$$

$$\therefore\ v_C(t) = \frac{1}{C}\int[-0.2252\,e^{-500t} + 0.2659\sin(314t + 58^\circ)]\,dt$$

$$= \frac{1}{20\times10^{-6}}\left[\frac{-0.2252\,e^{-500t}}{-500} - \frac{0.2659\cos(314t+58^\circ)}{314}\right]$$

$$\boxed{\sin(\theta - 90^\circ) = -\cos\theta}$$

$$= \frac{-0.2252}{20\times10^{-6}\times(-500)}\,e^{-500t} + \frac{0.2659}{20\times10^{-6}\times314}\sin(314t + 58^\circ - 90^\circ)$$

$$= 22.52\,e^{-500t} + 42.34\sin(314t - 32^\circ)\,V$$

RESULT

$$i(t) = -0.2252\,e^{-500t} + 0.2659\sin(314t + 58^\circ)\,A$$

$$v_C(t) = 22.52\,e^{-500t} + 42.34\sin(314t - 32^\circ)\,V$$

EXAMPLE 3.24

In the circuit of Fig. 1, the switch remains in position-1 for a long time. At t = 0, the switch is moved from position-1 to position-2. Find an expression for the current through the RC circuit.

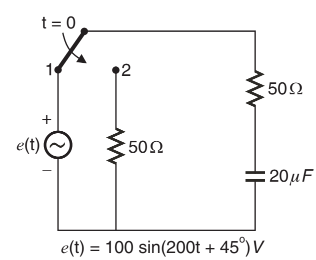


Fig. 1.

SOLUTION

Case i : Switch in position-1

In position-1, the circuit has attained a steady state. Hence, we can perform steady state analysis. The steady state of the RC circuit with switch in position-1 is shown in Fig. 2.

The standard form of sinusoidal source is, $E_m \sin(\omega t \pm \phi)$.

Here, $E_m \sin(\omega t \pm \phi) = 100\sin(200t + 45^\circ)\,V$

$\therefore\ E_m = 100\,V,\ \ \omega = 200$ rad/s, $\ \phi = 45^\circ$

$\bar{E}_m = 100\angle 45^\circ\,V$

$\therefore$ Rms value of voltage, $\bar{E} = \dfrac{\bar{E}_m}{\sqrt{2}} = \dfrac{100}{\sqrt{2}}\angle 45^\circ\,V$

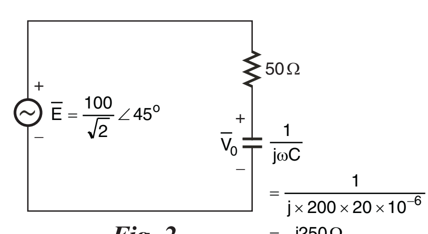


Fig. 2.

Let $\overline{V}_0$ be the voltage across the capacitor in steady state. By voltage division rule,

$$\overline{V}_0 = \frac{100}{\sqrt{2}}\angle 45^\circ \times \frac{-j250}{50-j250}$$

$$= \frac{100}{\sqrt{2}}\angle 45^\circ \times \frac{250\angle -90^\circ}{254.951\angle -78.7^\circ}$$

$$= 69.3375\angle 33.7^\circ\, V$$

Let, $V_0 = \left|\overline{V}_0\right| = 69.3375\, V$

This steady state voltage $V_0 = 69.3375\, V$ will be the initial voltage when the switch is moved from position-1 to position-2.

Case ii : Switch in position-2

When the switch is changed from position-1 to position-2, a steady voltage of V_0 (and hence a charge of Q_0) exists across the capacitance. Since the capacitance does not allow a sudden change in voltage, this steady voltage V_0 will be the initial voltage when the switch is closed to position-2.

$$\therefore\ v_C(0^-) = v_C(0^+) = V_0 = 69.3375\, V$$

The time domain and s-domain RC circuits with the switch in position-2 are shown in Figs 3 and 4, respectively.

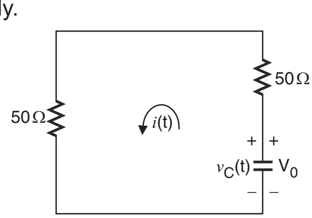


Fig. 3.

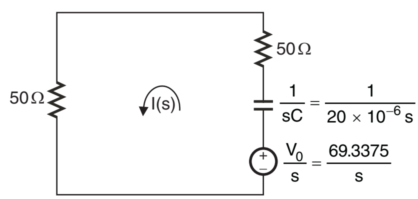


Fig. 4.

Let, $i(t)$ be the current through the RC circuit, when the switch is closed to position-2 at t = 0.

Let, $I(s) = \mathcal{L}\{i(t)\}$

With reference to Fig. 4, we can write,

$$I(s) = \frac{\frac{69.3375}{s}}{50+50+\frac{1}{20\times 10^{-6}s}}$$

$$= \frac{69.3375}{s}\times\frac{1}{100+\frac{1}{20\times 10^{-6}s}}$$

$$= \frac{69.3375}{100s+\frac{1}{20\times 10^{-6}}} = \frac{69.3375}{100\left(s+\frac{1}{100\times 20\times 10^{-6}}\right)} = \frac{0.693375}{(s+500)}$$

$$\therefore\ I(s) = \frac{0.693375}{s+500}$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

On taking the inverse Laplace transform of I(s), we get,

$$i(t) = 0.693375\, e^{-500t}\, A$$

3.9 Solved Problems in RLC Transient

EXAMPLE 3.25 *(AU June'14, 16 Marks)*

An RLC series circuit is excited by a dc source of 100 V. Find an expression for current and voltage in the elements of the circuit. Take R = 60 Ω, L = 0.2 H and C = 50 μF. Also draw the initial and final state of the circuits.

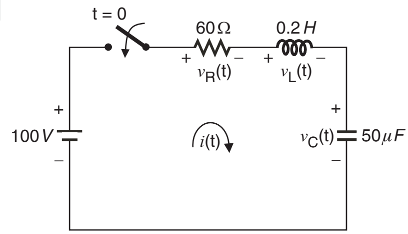


Fig. 1.

SOLUTION

The time domain RLC circuit is shown in Fig. 1. Let the switch be closed at t = 0 and i(t) be the current through the circuit.

Here, $\mathcal{L}\{i(t)\} = I(s)$ and $\mathcal{L}\{100\} = \dfrac{100}{s}$

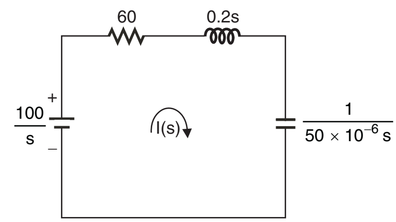


Fig. 2.

The s-domain RLC circuit is shown in Fig. 2. With reference to Fig. 2, we can write,

$$I(s) = \frac{\frac{100}{s}}{60 + 0.2s + \frac{1}{50 \times 10^{-6} s}}$$

$$= \frac{100}{60s + 0.2s^2 + \frac{1}{50 \times 10^{-6}}} = \frac{100}{0.2\left(s^2 + \frac{60}{0.2}s + \frac{1}{0.2 \times 50 \times 10^{-6}}\right)}$$

$$= \frac{500}{s^2 + 300s + 10^5}$$

Let us examine the roots of the denominator polynomial of I(s).

The roots of quadratic $s^2 + 300s + 10^5 = 0$ are,

$$s = \frac{-300 \pm \sqrt{300^2 - 4 \times 10^5}}{2} = \frac{-300 \pm \sqrt{-310000}}{2}$$

$$= \frac{-300 \pm \sqrt{-1}\sqrt{310000}}{2} = -150 \pm j278.3882$$

Since the roots are complex conjugate the current i(t) will be damped sinusoid. Let us rearrange the terms of denominator polynomial of I(s) as shown below:

$$I(s) = \frac{500}{(s^2 + 2 \times 150s + 150^2) + 10^5 - 150^2} = \frac{500}{(s+150)^2 + 77500}$$

$\boxed{\text{Add and subtract } 150^2}$

$\boxed{(a + b)^2 = a^2 + 2ab + b^2}$

$$= \frac{500}{(s+150)^2 + (\sqrt{77500})^2} = \frac{500}{(s+150)^2 + 278.4^2}$$

$$= \frac{500}{278.4} \times \frac{278.4}{(s+150)^2 + 278.4^2} = 1.796 \times \frac{278.4}{(s+150)^2 + 278.4^2}$$

Let us take the inverse Laplace transform of I(s).

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{1.796 \times \frac{278.4}{(s+150)^2 + 278.4^2}\right\}$$

$$\boxed{\mathcal{L}\{e^{-at}\sin \omega t\} = \frac{\omega}{(s+a)^2 + \omega^2}}$$

$\therefore$ $i(t) = 1.796\, e^{-150t} \sin 278.4t\ A$ ..... (1)

With reference to Fig. 1, by Ohm's law, we can write,

$$v_R(t) = R \times i(t)$$

$$= 60 \times 1.796\, e^{-150t} \sin 278.4t \qquad \boxed{\text{Using equation (1)}}$$

$$= 107.76\, e^{-150t} \sin 278.4t\ V \qquad \text{..... (2)}$$

We know that if $i(t)$ is the current through inductance then voltage across the inductance is given by,

$$v_L(t) = L \frac{d}{dt} i(t) \qquad \boxed{\text{Using equation (1)}}$$

$$= 0.2 \frac{d}{dt} [1.796\, e^{-150t} \sin 278.4t] \qquad \boxed{d(uv) = u\,dv - v\,du}$$

$$= 0.2 \times 1.796\, [e^{-150t} \cos 278.4t \times 278.4 + e^{-150t} \times (-150) \sin 278.4t]$$

$$= 100\, e^{-150t} \cos 278.4t - 53.88\, e^{-150t} \sin 278.4t \qquad \text{..... (3)}$$

$$= e^{-150t} [\cos 278.4t \times 100 - \sin 278.4t \times 53.88] \qquad \text{..... (4)}$$

Let us construct a right-angled triangle with 100 and 53.88 as two sides as shown in Fig. 3.

With reference to Fig. 3, we can write,

$$\tan\phi = \frac{53.88}{100} = 0.5388$$

$$\therefore\ \phi = \tan^{-1} 0.5388 = 28.3^\circ$$

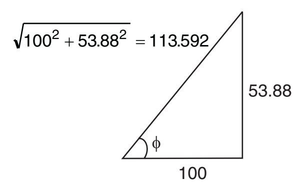


Fig. 3.

Also, $$\sin\phi = \frac{53.88}{113.592} \Rightarrow 53.88 = 113.592 \sin\phi$$

$$\therefore\ 53.88 = 113.592 \sin 28.3^\circ \qquad \text{.....(5)}$$

$$\cos\phi = \frac{100}{113.592} \Rightarrow 100 = 113.592 \cos\phi$$

$$\therefore\ 100 = 113.592 \cos 28.3^\circ \qquad \text{.....(6)}$$

Using equations (5) and (6), equation (4) can be written as,

$$v_L(t) = e^{-150t} [\cos 278.4t \times 113.592 \cos 28.3^\circ - \sin 278.4t \times 113.592 \sin 28.3^\circ]$$

$$= e^{-150t}\, 113.592\, [\cos 278.4t \cos 28.3^\circ - \sin 278.4t \sin 28.3^\circ]$$

$$= 113.592\, e^{-150t} \cos (278.4t + 28.3^\circ) \qquad \boxed{\cos(A + B) = \cos A \cos B - \sin A \sin B}$$

$$= 113.592\, e^{-150t} \sin (278.4t + 28.3^\circ + 90^\circ) \qquad \boxed{\cos\theta = \sin(\theta + 90^\circ)}$$

$$= 113.592\, e^{-150t} \sin (278.4t + 118.3^\circ)\ V$$

With reference to Fig. 1, by KVL, we get,

$$v_R(t) + v_L(t) + v_C(t) = 100$$

$$\therefore\ v_C(t) = 100 - v_R(t) - v_L(t)$$ $\boxed{\text{Using equations (2) and (3)}}$

$$= 100 - 107.76e^{-150t}\sin 278.4t - 100\,e^{-150t}\cos 278.4t + 53.88\,e^{-150t}\sin 278.4t$$

$$= 100 - 53.88\,e^{-150t}\sin 278.4t - 100\,e^{-150t}\cos 278.4t$$

$$= 100 - e^{-150t}[\sin 278.4t \times 53.88 + \cos 278.4t \times 100] \qquad .....(7)$$

Let us construct a right-angled triangle with 53.88 and 100 as two sides as shown in Fig. 4.

With reference to Fig. 4, we get,

$$\tan\phi = \frac{100}{53.88} = 1.856$$

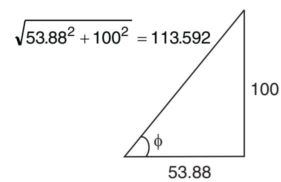


Fig. 4.

$$\therefore\ \phi = \tan^{-1} 1.856 = 61.7^\circ$$

Also, $\cos\phi = \dfrac{53.88}{113.592} \Rightarrow 53.88 = 113.592 \times \cos\phi$

$$\therefore\ 53.88 = 113.592\cos 61.7^\circ \qquad .....(8)$$

$\sin\phi = \dfrac{100}{113.592} \Rightarrow 100 = 113.592\sin\phi$

$$\therefore\ 100 = 113.592\sin 61.7^\circ \qquad .....(9)$$

Using equations (8) and (9), equation (7) can be written as,

$$v_C(t) = 100 - e^{-150t}[\sin 278.4t \times 113.592\cos 61.7^\circ + \cos 278.4t \times 113.592\sin 61.7^\circ]$$

$$= 100 - 113.592\,e^{-150t}[\sin 278.4t\cos 61.7^\circ + \cos 278.4t\sin 61.7^\circ]$$

$$= 100 - 113.592\,e^{-150t}\sin(278.4t + 61.7^\circ)$$ $\boxed{\sin(A + B) = \sin A\ \cos B + \cos A\ \sin B}$

In summary

$$\boxed{\begin{aligned} i(t) &= 1.796\,e^{-150t}\sin 278.4t\ A \\ v_R(t) &= 107.76\,e^{-150t}\sin 278.4t\ V \\ v_L(t) &= 113.592\,e^{-150t}\sin(278.4t + 118.3^\circ)\ V \\ v_C(t) &= 100 - 113.592\,e^{-150t}\sin(278.4t + 61.7^\circ)\ V \end{aligned}}$$

Initial state circuit

At $t = 0^+$, $i(t) = i(0^+) = 1.796 \times e^0 \times \sin 0 = 0$

At $t = 0^+$, $v_R(t) = v_R(0^+) = 107.76 \times e^0 \times \sin 0 = 0$

At $t = 0^+$, $v_L(t) = v_L(0^+) = 113.592 \times e^0 \times \sin 118.3^\circ = 100.0152\,V \approx 100\,V$

At $t = 0^+$, $v_C(t) = v_C(0^+) = 100 - 113.592 \times e^0 \times \sin 61.7^\circ$

$$= 100 - 100.0152 \approx 0$$

From the above analysis we can say that at $t = 0^+$, the inductance behaves as an open circuit and the capacitance behaves as a short circuit.

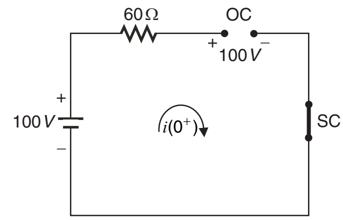


Fig. 5 : Initial state circuit.

Final state circuit

$$\text{At } t = \infty,\ i(t) = i(\infty) = 1.796 \times e^{-\infty} \times \sin(\infty) = 0$$

$$\text{At } t = \infty,\ v_R(t) = v_R(\infty) = 107.76 \times e^{-\infty} \times \sin(\infty) = 0$$

$$\text{At } t = \infty,\ v_L(t) = v_L(\infty) = 113.592 \times e^{-\infty} \times \sin(\infty) = 0$$

$$\text{At } t = \infty,\ v_C(t) = v_C(\infty) = 100 - 113.592 \times e^{-\infty} \times \sin(\infty) = 100 - 0 = 100\,V$$

From the above analysis we can say that at $t = \infty$, i.e., at steady state, the inductance behaves as a short circuit and the capacitance behaves as an open circuit.

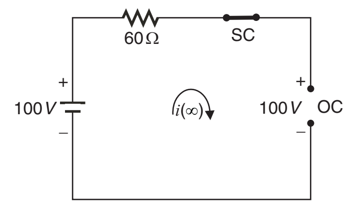


Fig. 6 : Final state circuit.

EXAMPLE 3.26

In the RLC circuit of Fig. 1, the capacitor has an initial voltage of 40 V, when the switch is closed at t = 0. Find an expression for the current $i(t)$.

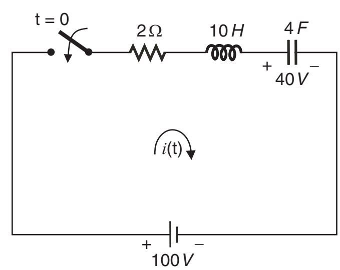


Fig. 1.

SOLUTION

Let, $\mathcal{L}\{i(t)\} = I(s)$

Also, $\mathcal{L}\{100\} = \dfrac{100}{s}$

The s-domain equivalent of the given RLC circuit is shown in Fig. 2. With reference to Fig. 2, we can write,

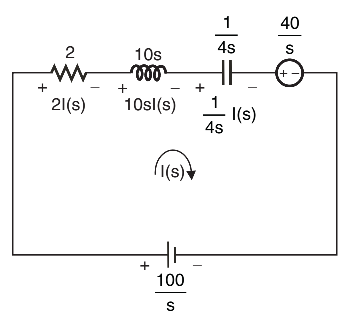


Fig. 2.

$$2I(s) + 10sI(s) + \frac{1}{4s}I(s) + \frac{40}{s} = \frac{100}{s}$$

$$\therefore\ I(s)\left[2 + 10s + \frac{1}{4s}\right] = \frac{100}{s} - \frac{40}{s}$$

$$I(s)\left[2 + 10s + \frac{1}{4s}\right] = \frac{60}{s}$$

$$\therefore\ I(s) = \frac{60}{s} \times \frac{1}{2 + 10s + \frac{1}{4s}}$$

$$= \frac{60}{2s + 10s^2 + \frac{1}{4}} = \frac{60}{10\left(s^2 + \frac{2}{10}s + \frac{1}{4 \times 10}\right)}$$

$$= \frac{6}{s^2 + 0.2s + 0.025}$$

Let us examine the roots of the denominator polynomial of $I(s)$.

The roots of quadratic $s^2 + 0.2s + 0.025 = 0$ are,

$$s = \frac{-0.2 \pm \sqrt{0.2^2 - 4 \times 0.025}}{2}$$

$$= \frac{-0.2 \pm \sqrt{-0.06}}{2} = \frac{-0.2 \pm \sqrt{-1}\sqrt{0.06}}{2} = -0.1 \pm j0.1225$$

Since the roots are complex conjugate, the current $i(t)$ will be damped sinusoid. Let us rearrange the terms of denominator polynomial of I(s) as shown below:

$$I(s) = \frac{6}{(s^2 + 2 \times 0.1s + 0.1^2) + 0.025 - 0.1^2}$$

$\boxed{\text{Add and subtract } 0.1^2}$

$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$

$$= \frac{6}{(s+0.1)^2 + 0.015} = \frac{6}{(s+0.1)^2 + (\sqrt{0.015})^2}$$

$$= \frac{6}{(s+0.1)^2 + 0.1225^2} = \frac{6}{0.1225} \times \frac{0.1225}{(s+0.1)^2 + 0.1225^2}$$

$$= 48.9796 \times \frac{0.1225}{(s+0.1)^2 + 0.1225^2}$$

Let us take the inverse Laplace transform of I(s).

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{48.9796 \times \frac{0.1225}{(s+0.1)^2 + 0.1225^2}\right\}$$

$\boxed{\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2 + \omega^2}}$

$$\therefore \; i(t) = 48.9796\, e^{-0.1t} \sin 0.1225t \; A$$

EXAMPLE 3.27

In the RLC circuit of Fig. 1, the inductor has an initial current of 10 *A*, when the switch is closed at t = 0. Find an expression for the current $i(t)$.

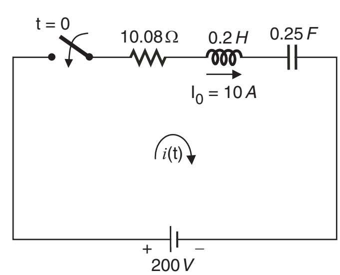


Fig. 1.

SOLUTION

Let, $\mathcal{L}\{i(t)\} = I(s)$

Also, $\mathcal{L}\{200\} = \frac{200}{s}$

The s-domain equivalent of the given RLC circuit is shown in Fig. 2. With reference to Fig. 2, we can write,

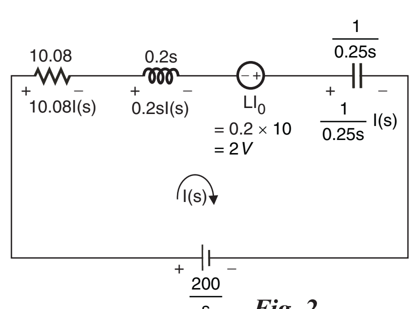


Fig. 2.

$$10.08I(s) + 0.2s\,I(s) + \frac{1}{0.25s}I(s) = \frac{200}{s} + 2$$

$$I(s)\left[10.08 + 0.2s + \frac{1}{0.25s}\right] = \frac{200 + 2s}{s}$$

$$\therefore \; I(s) = \frac{200+2s}{s} \times \frac{1}{10.08 + 0.2s + \frac{1}{0.25s}}$$

$$= \frac{200+2s}{10.08s + 0.2s^2 + \frac{1}{0.25}} = \frac{200+2s}{0.2\left(s^2 + \frac{10.08}{0.2}s + \frac{1}{0.2 \times 0.25}\right)} = \frac{1000+10s}{s^2 + 50.4s + 20}$$

The roots of quadratic, $s^2 + 50.4s + 20 = 0$ are,

$$s = \frac{-50.4 \pm \sqrt{50.4^2 - 4 \times 20}}{2}$$

$$= \frac{-50.4 \pm 49.6}{2} = -0.4, \ -50$$

$$\therefore \ I(s) = \frac{1000 + 10s}{(s+0.4)(s+50)}$$

By partial fraction expansion, I(s) can be expressed as,

$$I(s) = \frac{1000 + 10s}{(s+0.4)(s+50)} = \frac{K_1}{s+0.4} + \frac{K_2}{s+50}$$

$$K_1 = \left.\frac{1000 + 10s}{(s+0.4)(s+50)} \times (s+0.4)\right|_{s=-0.4}$$

$$= \left.\frac{1000 + 10s}{s+50}\right|_{s=-0.4} = \frac{1000 + 10 \times (-0.4)}{-0.4 + 50} = 20.0806 \approx 20$$

$$K_2 = \left.\frac{1000 + 10s}{(s+0.4)(s+50)} \times (s+50)\right|_{s=-50}$$

$$= \left.\frac{1000 + 10s}{s+0.4}\right|_{s=-50} = \frac{1000 + 10 \times (-50)}{-50 + 0.4} = -10.0806 \approx -10$$

$$\therefore \ I(s) = \frac{20}{s+0.4} - \frac{10}{s+50}$$

Let us take the inverse Laplace transform of I(s).

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{20}{s+0.4} - \frac{10}{s+50}\right\}$$

$$\boxed{\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}}$$

$$\therefore \ i(t) = 20\, e^{-0.4t} - 10\, e^{-50t} A$$

$$= 10\left(\frac{20}{10} e^{-0.4t} - e^{-50t}\right) A$$

$$= 10\,(2\, e^{-0.4t} - e^{-50t})\, A$$

EXAMPLE 3.28

Find i(t) in the circuit of Fig. 1, if the switch is closed at t = 0.

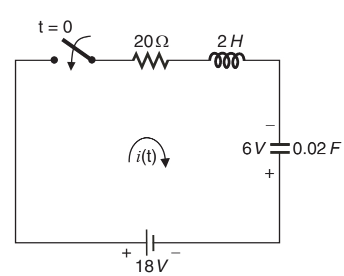


Fig. 1.

SOLUTION

Let, $\mathcal{L}\{i(t)\} = I(s)$

Also, $\mathcal{L}\{18\} = \frac{18}{s}$

The s-domain equivalent of the given circuit is shown in Fig. 2. With reference to Fig. 2, by KVL, we can write,

$$20\,I(s) + 2s\,I(s) + \frac{1}{0.02s}\,I(s) = \frac{18}{s} + \frac{6}{s}$$

$$I(s)\left[20 + 2s + \frac{1}{0.02s}\right] = \frac{24}{s}$$

$$\therefore\ I(s) = \frac{24}{s} \times \frac{1}{20 + 2s + \frac{1}{0.02s}}$$

$$= \frac{24}{20s + 2s^2 + \frac{1}{0.02}} = \frac{24}{2\left(s^2 + \frac{20}{2}s + \frac{1}{2 \times 0.02}\right)}$$

$$= \frac{12}{s^2 + 10s + 25}$$

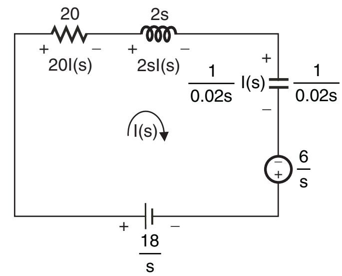


Fig. 2.

The roots of quadratic, $s^2 + 10s + 25 = 0$ are,

$$s = \frac{-10 \pm \sqrt{10^2 - 4 \times 25}}{2} = \frac{-10}{2} = -5$$

$$\therefore\ I(s) = \frac{12}{(s+5)^2}$$

Let us take the inverse Laplace transform of I(s).

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{12}{(s+5)^2}\right\}$$

$$\boxed{\mathcal{L}\{t\,e^{-at}\} = \frac{1}{(s+a)^2}}$$

$$i(t) = 12t\,e^{-5t}\,A$$

EXAMPLE 3.29

In the circuit of Fig. 1, the switch is closed to position-1 for a long time. At t = 0, the switch position is changed from position-1 to position-2. Find an expression for *i*(t).

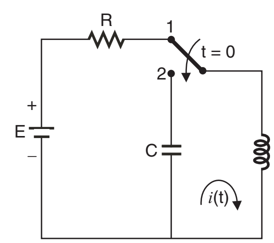


Fig. 1.

SOLUTION

Case i : Switch in position-1

Since the switch is closed for a long time, the circuit might have attained steady state. The steady state circuit with switch in position-1 is shown in Fig. 2. Here a steady current of I_0 flows through the inductance.

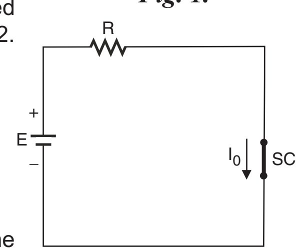


Fig. 2.

With reference to Fig. 2, we get,

$$I_0 = \frac{E}{R}$$

This current I_0 will be the initial current through the inductance when the switch is moved to position-2.

Case ii : Switch in position-2

Let, $\mathcal{L}\{i(t)\} = I(s)$

The s-domain equivalent of the given circuit with switch in position-2 is shown in Fig. 3.

With reference to Fig. 3, we get,

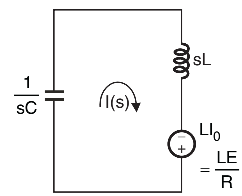


Fig. 3.

$$I(s) = \frac{\frac{LE}{R}}{\frac{1}{sC} + sL}$$

$$= \frac{LE}{R} \times \frac{1}{\frac{1 + s^2LC}{sC}} = \frac{LE}{R} \times \frac{sC}{LC\left(s^2 + \frac{1}{LC}\right)}$$

$$= \frac{E}{R}\,\frac{s}{s^2 + \frac{1}{LC}} = \frac{E}{R}\,\frac{s}{s^2 + \left(\frac{1}{\sqrt{LC}}\right)^2}$$

Let us take the inverse Laplace transform of I(s).

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{E}{R}\,\frac{s}{s^2 + \left(\frac{1}{\sqrt{LC}}\right)^2}\right\}$$

$$\boxed{\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}}$$

$$\therefore\ i(t) = \frac{E}{R} \cos \frac{1}{\sqrt{LC}} t\ A$$

CONCLUSION

The current is sinusoidal in nature and does not decay because the ideal inductance and capacitance do not consume energy.

EXAMPLE 3.30

In the circuit of Fig. 1, the switch remains in position-1 for a long time. At t = 0, the switch is closed to position-2. Determine the current response.

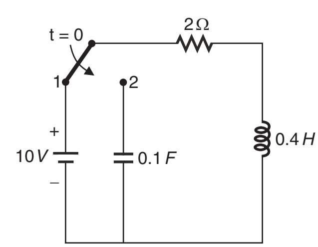


Fig. 1.

SOLUTION

Case i : Switch in position-1

In position-1, the circuit should have attained steady state because the switch is closed for a long time. The steady state condition of the given circuit with switch in position-1 is shown in Fig. 2. Here a steady current of I_0 flows through the inductance.

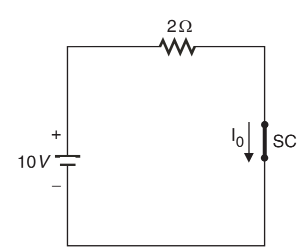


Fig. 2.

With reference to Fig. 2, we can write,

$$I_0 = \frac{10}{2} = 5\ A$$

This current I_0 will be the initial current when the switch is changed from position-1 to position-2.

Case ii : Switch in position-2

Let, $i(t)$ be the current through the circuit when the switch is closed to position-2.

Let, $I(s) = \mathcal{L}\{i(t)\}$

The time domain and s-domain circuits with switch in position-2 are shown in Figs 3 and 4, respectively.

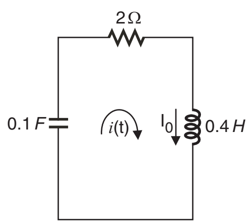


Fig. 3.

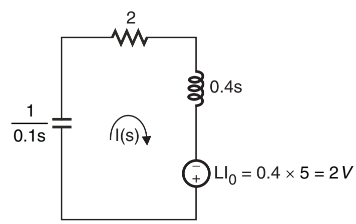


Fig. 4.

With reference to Fig. 4, we can write,

$$I(s) = \frac{2}{2 + 0.4s + \dfrac{1}{0.1s}}$$

$$= \frac{2}{\dfrac{2 \times 0.1s + 0.4s \times 0.1s + 1}{0.1s}} = \frac{2 \times 0.1s}{0.2s + 0.04s^2 + 1}$$

$$= \frac{0.2s}{0.04\left(s^2 + \dfrac{0.2}{0.04}s + \dfrac{1}{0.04}\right)} = \frac{5s}{s^2 + 5s + 25}$$

The roots of the quadratic, $s^2 + 5s + 25 = 0$ are,

$$s = \frac{-5 \pm \sqrt{5^2 - 4 \times 25}}{2} = \frac{-5 \pm \sqrt{-75}}{2}$$

$$= \frac{-5 \pm \sqrt{-1}\sqrt{75}}{2} = -2.5 \pm j4.3301$$

Since the roots are complex conjugate, the response will be damped sinusoid. Let us rearrange the terms of denominator polynomial of I(s).

$$I(s) = \frac{5s}{(s^2 + 2 \times 2.5s + 2.5^2) + 25 - 2.5^2} = \frac{5s}{(s + 2.5)^2 + 18.75}$$

Add and subtract 2.5^2

$(a + b)^2 = a^2 + 2ab + b^2$

$$= \frac{5s}{(s + 2.5)^2 + (\sqrt{18.75})^2} = \frac{5(s + 2.5 - 2.5)}{(s + 2.5)^2 + 4.33^2}$$

$$= \frac{5(s + 2.5)}{(s + 2.5)^2 + 4.33^2} - \frac{2.5 \times 5}{4.33} \times \frac{4.33}{(s + 2.5)^2 + 4.33^2}$$

$$= 5 \times \frac{(s + 2.5)}{(s + 2.5)^2 + 4.33^2} - 2.8868 \times \frac{4.33}{(s + 2.5)^2 + 4.33^2}$$

Let us take the inverse Laplace transform of I(s).

$$\mathcal{L}^{-1}\{I(s)\} = 5\times\mathcal{L}^{-1}\left\{\frac{(s+2.5)}{(s+2.5)^2+4.33^2}\right\} - 2.8868\times\mathcal{L}^{-1}\left\{\frac{4.33}{(s+2.5)^2+4.33^2}\right\}$$

$$\therefore\ i(t) = 5e^{-2.5t}\cos 4.33t - 2.8868\,e^{-2.5t}\sin 4.33t$$

$$= e^{-2.5t}\,[\cos 4.33t\times 5 - \sin 4.33t\times 2.8868] \qquad .....(1)$$

Let us construct a right-angled triangle with 5 and 2.8868 as two sides as shown in Fig. 5.

With reference to Fig. 5, we can write,

$$\tan\phi = \frac{2.8868}{5} = 0.5774$$

$$\therefore\ \phi = \tan^{-1} 0.5774 = 30^\circ$$

Also, $\cos\phi = \dfrac{5}{5.7735} \Rightarrow 5 = 5.7735\cos\phi$

$$\therefore\ 5 = 5.7735\ \cos 30^\circ \qquad .....(2)$$

$$\sin\phi = \frac{2.8868}{5.7735} \Rightarrow 2.8868 = 5.7735\sin\phi$$

$$\therefore\ 2.8868 = 5.7735\ \sin 30^\circ \qquad .....(3)$$

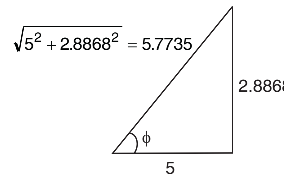


Fig. 5.

Using equations (2) and (3), equation (1) can be written as,

$$i(t) = e^{-2.5t}\,[\cos 4.33t\times 5.7735\cos 30^\circ - \sin 4.33t\times 5.7735\sin 30^\circ]$$

$$= 5.7735\,e^{-2.5t}\,[\cos 4.33t\cos 30^\circ - \sin 4.33t\sin 30^\circ]$$

$$= 5.7735\,e^{-2.5t}\cos(4.33t + 30^\circ)$$

$$= 5.7735\,e^{-2.5t}\sin(4.33t + 30^\circ + 90^\circ)$$

$$= 5.7735\,e^{-2.5t}\sin(4.33t + 120^\circ)\,A$$

$\boxed{\cos(A+B) = \cos A\ \cos B - \sin A\sin B}$

$\boxed{\cos\theta = \sin(\theta + 90^\circ)}$

EXAMPLE 3.31

In the circuit of Fig. 1, the switch is closed to position-1 for a long time. At t = 0, the switch position is changed from 1 to 2. Determine the current response.

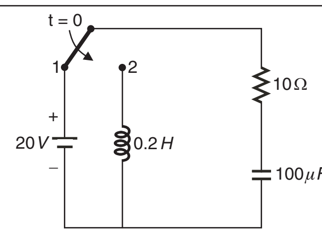


Fig. 1.

SOLUTION

Case i : Switch in position-1

In position-1, the circuit should have attained steady state because the switch is closed for a long time. The steady state condition of the given circuit with switch in position-1 is shown in Fig. 2. Here the capacitor is fully charged and behaves as an open circuit. Hence, the voltage across the capacitor V_0 will be equal to the supply voltage 20 *V*.

$$\therefore\ V_0 = 20\ V$$

This voltage V_0 will be the initial voltage when the switch position is moved from 1 to 2.

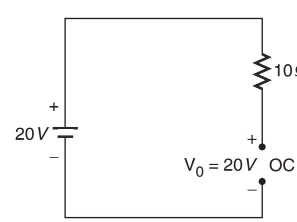


Fig. 2.

Case ii : Switch in position-2

Let, $i(t)$ be the current through the circuit when the switch is closed to position-2.

Let, $I(s) = \mathcal{L}\{i(t)\}$

The time domain and s-domain circuits with the switch in position-2 are shown in Figs 3 and 4, respectively.

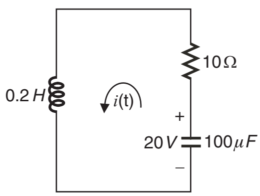


Fig. 3.

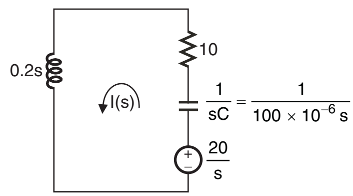


Fig. 4.

With reference to Fig. 4, we can write,

$$I(s) = \frac{\frac{20}{s}}{0.2s + 10 + \frac{1}{100 \times 10^{-6}s}}$$

$$= \frac{20}{0.2s^2 + 10s + \frac{1}{100 \times 10^{-6}}} = \frac{20}{0.2\left(s^2 + \frac{10}{0.2}s + \frac{1}{0.2 \times 100 \times 10^{-6}}\right)}$$

$$= \frac{100}{s^2 + 50s + 50000} = \frac{100}{(s^2 + 2 \times 25s + 25^2) + 50000 - 25^2}$$

$\boxed{\text{Add and subtract } 25^2}$

$\boxed{(a + b)^2 = a^2 + 2ab + b^2}$

$$= \frac{100}{(s+25)^2 + 49375} = \frac{100}{(s+25)^2 + (\sqrt{49375})^2} = \frac{100}{(s+25)^2 + (222.2)^2}$$

$$= \frac{100}{222.2} \times \frac{222.2}{(s+25)^2 + (222.2)^2}$$

$$= 0.45 \times \frac{222.2}{(s+25)^2 + 222.2^2}$$

Let us take the inverse Laplace transform of I(s).

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{0.45 \times \frac{222.2}{(s+25)^2 + 222.2^2}\right\}$$

$\boxed{\mathcal{L}\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2 + \omega^2}}$

$$i(t) = 0.45\, e^{-25t} \sin(222.2t)\ A$$

3.10 Summary of Important Concepts

1. The study of switching condition in a circuit is called transient analysis.
2. The state of a circuit from the instant of switching to attainment of steady state is called transient.
3. The duration from the instant of switching till the attainment of steady state is called transient period.

4. The current and voltage of circuit elements during the transient period is called transient response.
5. The transients are due to inductance and capacitance. There is no transient in purely resistive circuits.
6. The response of a circuit due to stored energy alone is called natural response or source-free response.
7. The response of a circuit due to an external source is called forced response.
8. In s-domain or Laplace domain, the inductive reactance is represented as sL and capacitive reactance as 1/sC.
9. In transient analysis using Laplace transform, the intial current I_0 in inductance can be represented by a voltage source of value LI_0 delivering current in the direction of I_0.
10. In transient analysis using Laplace transform, the initial voltage V_0 in capacitance can be represented by a voltage source of value V_0/s with the same polarity/sign as that of V_0.
11. The inital value theorem of Laplace transform can be used to determine the initial value of voltages and currents form their s-domain equations.
12. The initial value theorem of Laplace transform states that if F(s) is Laplace transform of f(t) then

$$\underset{t \to 0}{\mathrm{Lt}}\ f(t) = \underset{s \to \infty}{\mathrm{Lt}}\ s\,F(s)$$

$$\therefore\ \text{Initial value of } f(t) = f(0) = \underset{s \to \infty}{\mathrm{Lt}}\ s\,F(s)$$

13. The final value theorem of Laplace transform can be used to determine the final values of voltages and currents from their s-domain equations.
14. The final value theorem of Laplace transform states that if F(s) is Laplace transform of f(t) then

$$\underset{t \to \infty}{\mathrm{Lt}}\ f(t) = \underset{s \to 0}{\mathrm{Lt}}\ s\,F(s)$$

$$\therefore\ \text{Final value of } f(t) = f(\infty) = \underset{s \to 0}{\mathrm{Lt}}\ s\,F(s)$$

15. When switched to dc supply, an inductance initially behaves as a current source if there is an initial current or as an open circuit if there is no initial current and finally behaves as a short circuit irrespective of inital current.
16. When switched to dc supply, a capacitance initially behaves as a voltage source if there is an initial voltage or as a short circuit if there is no initial voltage and finally behaves as an open circuit irrespective of inital voltage.
17. The source-free response $i(t)$ in an RL circuit due to initial current I_0 is given by,

$$i(t) = I_0\, e^{-\frac{t}{\tau}}\ ; \qquad \text{where, } \tau = \frac{L}{R} = \text{Time constant.}$$

18. The transient equations of an RL circuit excited by a dc supply of E volts are,

$$i(t) = \frac{E}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$v_L(t) = E\, e^{-\frac{t}{\tau}}$$

$$v_R(t) = E\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$\text{where, } \tau = \frac{L}{R} = \text{Time constant.}$$

19. The time constant of an RL circuit can be defined in the following four different ways.
    - The time constant of an RL circuit is defined as the time taken by the current through the inductance to reach steady value if the initial rate of rise is maintained.
    - The time constant of an RL circuit is defined as the ratio of inductance and resistance of the circuit.
    - The time constant of an RL circuit is defined as the time taken by the current through the inductance to reach 63.21% of the final steady value.
    - The time constant of an RL circuit is defined as the time taken by the voltage across the inductance to fall to 36.79% of the initial value.

20. The source-free response $v_C(t)$ in an RC circuit due to inital voltage V_0 is given by,

$$v_C(t) = -V_0\, e^{-\frac{t}{\tau}} \; ; \qquad \text{where, } \tau = RC = \text{Time constant.}$$

21. The transient equations of an RC circuit excited by a dc supply of E volts are,

$$v_C(t) = E\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$i(t) = \frac{E}{R}\, e^{-\frac{t}{\tau}}$$

$$v_R(t) = E\, e^{-\frac{t}{\tau}}$$

where, $\tau = RC$ = Time constant.

22. The time constant of an RC circuit can be defined in the follwing four different ways:
    - The time constant of an RC circuit is defined as the time taken by the voltage across the capacitance to reach steady value if the initial rate of rise is maintained.
    - The time constant of an RC circuit is defined as the product of resistance and capacitance of the circuit.
    - The time constant of an RC circuit is defined as the time taken by the voltage across the capacitance to reach 63.21% of the final steady value.
    - The time constant of an RC circuit is defined as the time taken by the current through the capacitance to fall to 36.79% of the initial value.

23. The response $i(t)$ of an RLC circuit excited by a dc supply of E volts will take any one of the following three forms depending on the value of R, L and C:

$$i(t) = \frac{E}{L\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \; ; \; 0 < \zeta < 1 \; ; \quad \text{Underdamped response.}$$

$$i(t) = \frac{E}{L} t\, e^{-\omega_n t} \; ; \; \zeta = 1 \; ; \quad \text{Critically damped response.}$$

$$i(t) = \frac{E}{2L\omega_n\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_n t}\left(e^{\omega_n\sqrt{\zeta^2-1}\,t} - e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right); \zeta > 1; \text{Overdamped response.}$$

where, $\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$ = Damping ratio.

$\omega_n = \frac{1}{\sqrt{LC}}$ = Natural frequency of oscillation.

$\omega_d = \omega_n\sqrt{1 - \zeta^2}$ = Damped frequency of oscillation.

24. The ratio of resistance of a circuit and resistance for critical damping (or critical resistance) is called damping ratio, ζ.

$$\therefore \text{ Damping ratio, } \zeta = \frac{R}{R_C} = \frac{R}{2\sqrt{\frac{L}{C}}} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

25. Critical resistance R_C is the value of the resistance of a circuit to achieve critical damping and it is given by, $R_C = 2\sqrt{\frac{L}{C}}$.

3.11 Short-answer Questions

Q3.1 What is transient?

The state (or condition) of a circuit from the instant of switching to attainment of steady state is called transient state or simply transient.

Q3.2 Why does transient occur in electric circuits?

The inductance does not allow a sudden change in current and the capacitance does not allow a sudden change in voltage. Hence, in inductive and capacitive circuits (or in general in reactive circuits), transient occurs during a switching operation.

Q3.3 What are free and forced response? ***(AU June'14, 2 Marks)***

The response of a circuit due to stored energy alone is called free response and the response of a circuit due to an external source is called forced response.

Q3.4 What is a complementary function?

The part of the response or solution which becomes zero as t tends to infinity is called complementary function. It is the transient part of the response or solution.

Q3.5 What is a particular solution?

The part of the solution or response which attains a steady value as t tends to infinity is called particular solution. It is the steady state part of the solution.

Q3.6 Define time constant of an RL circuit. ***(AU June'14,'16, Dec'15, & May'17, 2 Marks)***

The time constant of an RL circuit is defined as the time taken by the current through the inductance to reach a steady value if the initial rate of rise is maintained.

Q3.7 Distinguish between steady state and transient state. ***(AU Dec'15, 2 Marks)***

In transient state, the values of voltage and current will not be constant, whereas in steady state, the values of voltage and current will attain a steady value.

Q3.8 What is the time constant of an RL circuit with $R = 10\,\Omega$ and $L = 20\,mH$?

$$\text{Time constant, } \tau = \frac{L}{R} = \frac{20 \times 10^{-3}}{10} = 2 \times 10^{-3} \textit{ second} = 2\,ms$$

Q3.9 ***What is the time constant of an RL circuit shown in Fig. Q3.9.1?***

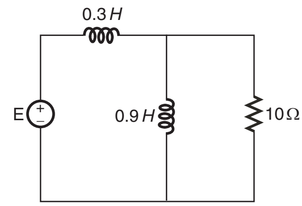


Fig. Q3.9.1.

Let us find an equivalent inductance at terminals A-B using Thevenin's theorem as shown below:

Using Thevenin's equivalent, the given network can be drawn as shown in Fig. Q3.9.5.

Now, Time constant, $\tau = \frac{L_{eq}}{R} = \frac{0.225}{10} = 0.0225\ second = 22.5\ ms$

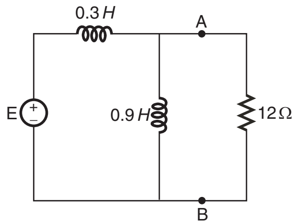


Fig. Q3.9.2.

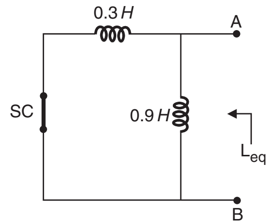


Fig. Q3.9.3.

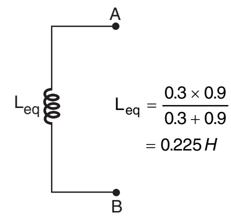


Fig. Q3.9.4.

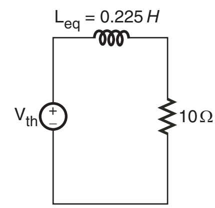


Fig. Q3.9.5.

Q3.10 ***Define time constant of an RC circuit.*** ***(AU June'14 & Dec'15, 2 Marks)***

The time constant of an RC circuit is defined as the time taken by the voltage across the capacitance to reach a steady value if the initial rate of rise is maintained.

Q3.11 ***What is the time constant of an RC circuit with R = 10kΩ and C = 40μF ?***

Time constant, $\tau = RC = 10 \times 10^{3} \times 40 \times 10^{-6} = 0.4\ second.$

Q3.12 ***What is the time constant of an RC circuit shown in Fig. Q3.12.1?***

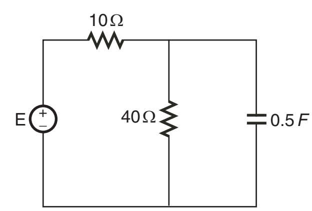


Fig. Q3.12.1.

Let us find an equivalent resistance at terminals A-B using Thevenin's theorem as shown below:

Using Thevenin's equivalent, the given network can be drawn as shown in Fig. Q3.12.5.

Now, Time constant, $\tau = R_{eq}C = 8 \times 0.5 = 4\ seconds.$

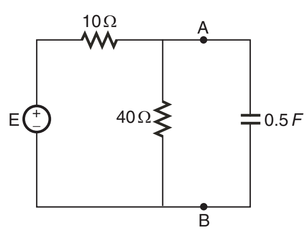


Fig. Q3.12.2.

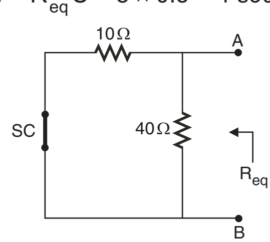


Fig. Q3.12.3.

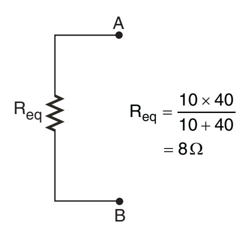


Fig. Q3.12.4.

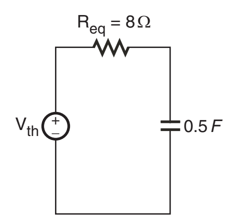


Fig. Q3.12.5.

Q3.13 ***What is damping ratio?***

The ratio of resistance of a circuit and resistance for critical damping is called damping ratio.

Q3.14 ***What is critical damping?***

Critical damping is the condition of a circuit at which the oscillations in the response are just eliminated. This is possible by increasing the value of resistance in the circuit.

Q3.15 What is critical resistance?

Critical resistance is the value of the resistance of a circuit to achieve critical damping.

Q3.16 Write the expression for critical resistance and damping ratio of an RLC series circuit.

$$\text{Critical resistance, } R_C = 2\sqrt{\frac{L}{C}}$$

$$\text{Damping ratio, } \zeta = \frac{R}{R_C} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

Q3.17 What is natural and damped frequency?

The response of a circuit is completely oscillatory with a frequency ω_n in the absence of resistance and this frequency ω_n is called natural frequency.

The response of an underdamped circuit is oscillatory with a frequency of ω_d and these oscillations are damped as t tends to infinity. The frequency of damped oscillatory response is called damped frequency.

Q3.18 Write the condition for underdamping and critical damping in an RLC series circuit.

$$\text{The condition for underdamping is, } \left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$$

$$\text{The condition for critical damping is, } \left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$$

Q3.19 An RLC series circuit with L = 2 H and C = 5 μF. Determine the value of R to give critical damping.

(AU June'16, 2 Marks)

Let, R_C = Value of R for critical damping

$$R_C = 2\sqrt{\frac{L}{C}} = 2\times\sqrt{\frac{2}{5\times10^{-6}}} = 1264.911\,\Omega$$

Q3.20 An RLC series circuit with R = 10Ω and L = 2 H. Determine the value of C to give critical damping.

(AU Dec'16, 2 Marks)

For critical damping, the value of R is given by,

$$R = 2\sqrt{\frac{L}{C}} \Rightarrow R^2 = 4\frac{L}{C} \Rightarrow C = \frac{4L}{R^2}$$

Let, C_C = Value of capacitor for critical damping.

$$\therefore C_C = \frac{4L}{R^2} = \frac{4\times2}{10^2} = 0.08\,F = 80\times10^{-3}F = 80\,mF$$

Q3.21 An RL series circuit with R = 10Ω is excited by a dc voltage source of 30 V by closing the switch at t = 0. Determine the current in the circuit at t = 2 τ.

$$\text{Current, } i(t) = \frac{E}{R}(1-e^{-\frac{t}{\tau}})$$

$$= \frac{30}{10}(1-e^{-2}) = 2.594\,A$$

Q3.22 ***An RC series circuit is excited by a dc voltage source of 80 V by closing the switch at t = 0. Determine the voltage across the capacitor in a time of one time constant.***

$$\text{Voltage across capacitor, } v_C(t) = E\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$= 80\left(1 - e^{-1}\right) = 50.5696\,V$$

Q3.23 ***A 50 μF capacitor is discharged through a 100 kΩ resistor. If the capacitor is initially charged to 400 V, determine the initial energy.*** ***(AU Dec'14, 2 Marks)***

$$\text{Initial energy} = \frac{1}{2}CV_0^2 = \frac{1}{2} \times 50 \times 10^{-6} \times 400^2$$

$$= 4\ \textit{Joules}$$

Q3.24 ***An RLC series circuit with R = 5 Ω is excited by a dc source of 10 V by closing the switch at t = 0. Draw the initial and final conditions of the circuit.***

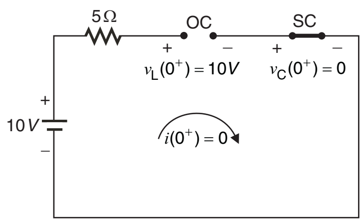


Fig. Q3.24.1 : *Initial condition.*

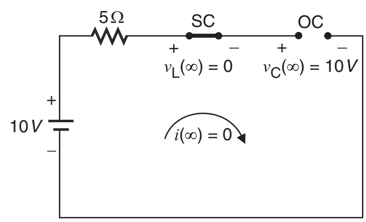


Fig. Q3.24.2 : *Final condition.*

3.12 EXERCISES

I. Fill in the Blanks With Appropriate Words

1. The time duration from the instant of switching till the attainment of steady state is called ________ .
2. The current and voltage of circuit elements during transient period is called ________ .
3. The response of a circuit due to ________ alone is called natural response.
4. The complementary function is also called ________ .
5. In _______, the current at $t = 0^-$ is equal to the current at $t = 0^+$.
6. In circuits excited by a dc source at steady state, the _______ behaves as a short circuit and ________ behaves as an open circuit.
7. In circuits excited by a dc source when there is no stored energy at initial state, ________ behaves as a short circuit and ________ behaves as an open circuit.
8. The time constant of an RL circuit with R = 5 Ω and L = 0.2 *H* is ________ .
9. The time constant of an RC circuit with R = 200 Ω and C = 10 *μF* is ________ .
10. The steady state current in an RC series circuit with R = 100 Ω excited by a dc source of 10 *V* is ________ .
11. The steady state voltage across the inductance in an RL circuit with R = 5 Ω, excited by a dc source of 20 *V* is ________ .
12. The ratio of resistance of a circuit and resistance for critical damping is called ________ .
13. The ________ is the condition for critical damping in an RLC series circuit.

14. The frequency of oscillatory response of a circuit with zero resistance is called ________ .
15. The resistance of a circuit at critical damping is called ________ .

ANSWERS

1. transient period	6. inductance, capacitance	11. zero
2. transient response	7. capacitance, inductance	12. damping ratio
3. stored energy	8. 0.04 second	13. $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$
4. natural response	9. 2 *ms*	14. natural frequency
5. inductance	10. zero	15. critical resistance

II. State Whether the Following Statements are True or False

1. Transients are due to energy storage elements.
2. There is no transient in resistive circuits.
3. The complementary function depends on the nature of the exciting source.
4. The particular solution depends on the nature of the circuit.
5. In a capacitance, the voltage at $t = 0^-$ is equal to the voltage at $t = 0^+$.
6. The steady state value does not depend on initial conditions.
7. The transient in a circuit exists for a period of five time constant.
8. The time constant of a circuit does not depend on R, L and C parameters.
9. In circuits with energy storage elements, the response can attain steady state in a time of one time constant if the initial rate of change is maintained.
10. An inductance with stored energy behaves as a current source at $t = 0$.
11. A capacitance with stored energy behaves as a voltage source at $t = 0$.
12. The oscillations in response can be reduced by reducing the resistance of a circuit.
13. Damping ratio can be adjusted by varying the capacitance of a circuit.
14. Critical resistance does not depend on inductance and capacitance.
15. Natural frequency depends on damping ratio.

ANSWERS

1. True	4. False	7. True	10. True	13. True
2. True	5. True	8. False	11. True	14. False
3. False	6. True	9. True	12. False	15. False

III. Choose the Right Answer for the Following Questions

1. ***The time constant of an RL series circuit with $R = 5\,\Omega$ and $L = 0.1\,H$ is,***

   a) $0.05\,ms$ b) $20\,ms$ c) $0.5\,ms$ d) $2\,ms$

2. ***The current through an RL circuit excited by a 10 V dc source is given by $i(t) = 2\,(1 - e^{-10t})$ A. What is the value of R and L ?***

   a) $5\,\Omega, 0.5\,H$ b) $2\,\Omega, 0.1\,H$ c) $5\,\Omega, 0.2\,H$ d) $20\,\Omega, 0.5\,H$

3. ***An RL series circuit with $R = 10\,\Omega$ and $L = 0.2\,H$ is excited by a dc supply of 15 V by closing the switch at $t = 0$. The voltage across the inductance is,***

   a) $1.5\ e^{-50\,t}\,V$ b) $1.5\ e^{-0.02\,t}\,V$ c) $15\ e^{-50\,t}\,V$ d) $15\ e^{-0.02\,t}\,V$

4. ***The steady state voltage across the inductance in an RL circuit with $R = 5\,\Omega$ and excited by a dc source of 20 V is,***

   a) $20\,V$ b) $-20\,V$ c) $4\,V$ d) $0\,V$

5. ***An RL circuit with $R = 12\,\Omega$ and $L = 0.2\,H$ is excited by a dc source of 24 V by closing the switch at $t = 0$. The initial and final currents through the circuit respectively are,***

   a) $2A,\ 0A$ b) $0A,\ 2A$ c) $0.50A,\ 0A$ d) $0A,\ 0.5A$

6. ***The time constant of an RC series circuit with $R = 200\,\Omega$ and $C = 100\,\mu F$ is,***

   a) $0.05\,ms$ b) $5.0\,ms$ c) $2\,ms$ d) $20\,ms$

7. ***The current through an RC circuit excited by a 5 V dc source is given by, $i(t) = 2.5\ e^{-20\,t}$ A. What is the value of R and C ?***

   a) $2\,\Omega, 0.025\,F$ b) $0.5\,\Omega, 20\,F$ c) $5\,\Omega, 0.2\,F$ d) $12.5\,\Omega, 0.1\,F$

8. ***An RC series circuit with $R = 10k\Omega$ and $C = 1\mu F$ is excited by a dc supply of 20 V by closing the switch at $t = 0$. The voltage across the resistance is,***

   a) $2\ e^{-100\,t}\,mV$ b) $20\ e^{-0.01\,t}\,mV$ c) $20\ e^{-100\,t}\,V$ d) $2\ e^{-0.01\,t}\,V$

9. ***The steady state current through the capacitance in an RC circuit with $R = 100\,\Omega$ and excited by a DC source of 10 V is,***

   a) $10A$ b) $-10A$ c) $0.1A$ d) $0A$

10. ***An RC circuit with $R = 150\,\Omega$ and $C = 2\ \mu F$ is excited by a dc source of 15 V by closing the switch at $t = 0$. The initial and final voltages across the capacitor respectively are,***

    a) $10V,\ 15V$ b) $15V,\ 0V$ c) $0V,\ 15V$ d) $0V,\ 10V$

11. What is the time constant of the RC circuit shown in Fig. 11?

a) $4 ms$

b) $40 ms$

c) $0.25 ms$

d) $25 ms$

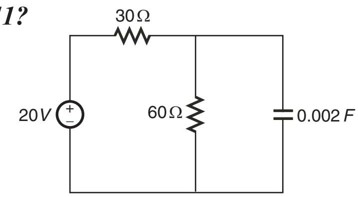


Fig. 11.

12. What is the time constant of the RC circuit shown in Fig. 12?

a) 3 RC

b) $\frac{2}{9}$RC

c) $\frac{9}{2}$RC

d) 2 RC

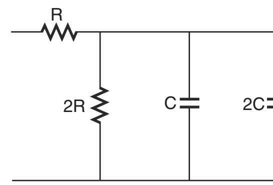


Fig. 12.

13. What is the steady state current through the inductance in the circuit shown in Fig. 13?

a) $0.5 A$

b) $2 A$

c) $3 A$

d) $4 A$

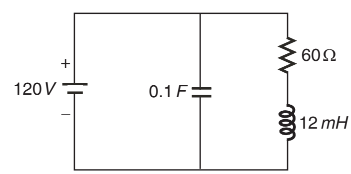


Fig. 13.

14. In a series RLC circuit, what is the condition for critically damped response?

a) $\frac{R}{2L} = \frac{1}{LC}$ b) $\frac{R}{2L} = \frac{L}{C}$ c) $\frac{R}{2L} = \left(\frac{1}{LC}\right)^2$ d) $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$

15. An RLC series circuit with R = 10Ω, L = 0.01 H and C = 1 μF is excited by a dc source of 16V by closing the switch at t = 0. The initial and final voltages across the capacitor respectively are,

a) $0 V,\ 1.6 V$ b) $0 V,\ 16 V$ c) $1.6 V,\ 16 V$ d) $1.6 V,\ 0 V$

ANSWERS

1. b	5. b	9. d	13. b
2. a	6. d	10. c	14. d
3. c	7. a	11. b	15. b
4. d	8. c	12. d	

IV. Unsolved Problems

E3.1 ***A steady current of 12.5 A is established through an inductance of 0.4 H by connecting it to a current source. At time t = 0, the inductance is disconnected from the source and connected to a resistance of 32 Ω. Find an expression for the current through the resistance. Also draw the initial and final conditions of the circuit.***

E3.2 ***In the RL circuit shown in Fig. E3.2, the switch is closed at t = 0. Find the current through the circuit and voltage across inductance and resistance. Also determine*** $\dfrac{di(t)}{dt}$ ***and*** $\dfrac{d^2 i(t)}{dt^2}$ ***at*** $t = 0^+$.

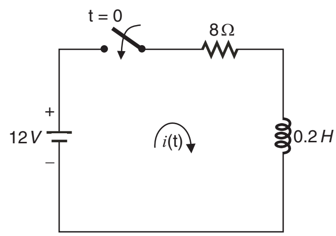


Fig. E3.2.

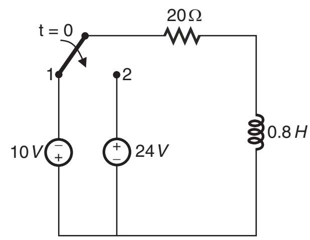


Fig. E3.3.

E3.3 ***In the RL circuit shown in Fig. E3.3, the switch is closed at position-1 for a long time and then switched to position-2 at t = 0. Determine the response i(t). Also draw the initial and final condition of the circuit.***

E3.4 ***In the RL circuit of Fig. E3.4, the switch is closed to position-1 at t = 0. Then at t = 0.24 second, the switch is moved to position-2. Determine the response i(t) and sketch the response. Also determine the time at which i(t) is zero.***

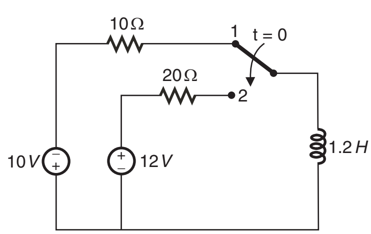


Fig. E3.4.

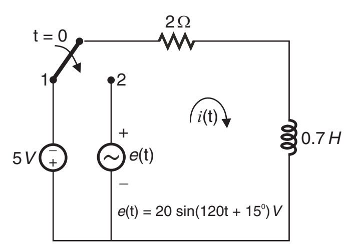


Fig. E3.5.

E3.5 ***The switch in the circuit of Fig. E3.5 is closed at position-1 for a long time. At time t = 0, the switch is moved to position-2. Find i(t) for t ≥ 0.***

E3.6 ***A voltage of 12 V is established across a capacitor of 100μF by connecting it to a voltage source. At time t = 0, the capacitor is disconnected from the source and connected to a resistance of 10 kΩ. Find an expression for current through the resistance. Also draw the initial and final conditions of the circuit.***

E3.7 ***In the RC circuit shown in Fig. E3.7, the switch is closed at t = 0. Determine the current and voltage in the resistance and capacitance. Also determine*** $\frac{dv_C(t)}{dt}$ ***and*** $\frac{d^2 v_C(t)}{dt^2}$ ***at*** $t = 0^+$.

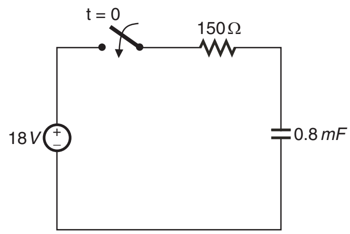


Fig. E3.7.

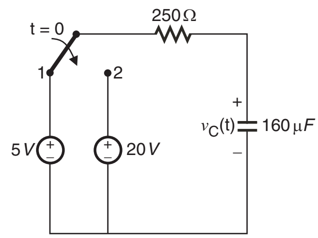


Fig. E3.8.

E3.8 ***In the RC circuit shown in Fig. E3.8, the switch is closed to position-1 for a long time. At time t = 0, the switch is moved to position-2. Determine and sketch the voltage across the capacitance for t ≥ 0. Also draw the initial and final conditions of the circuit.***

E3.9 ***A capacitor of 100 μF has to be charged to a voltage of 160 V by connecting it to a dc source of 200 V. Determine the value of series resistance required to charge the capacitor in 0.1 second.***

E3.10 ***In the RC circuit shown in Fig. E3.10, the capacitor has an initial charge of 300μC. If the switch is closed at t = 0, determine the time at which the capacitor voltage is zero. Also estimate and sketch the current and voltage in the capacitor for t ≥ 0.***

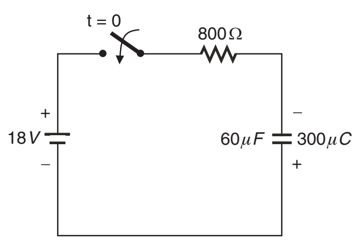


Fig. E3.10.

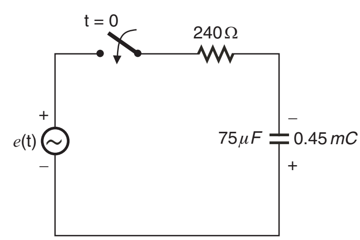


Fig. E3.11.

E3.11 ***The RC circuit shown in Fig. E3.11 is excited by a sinusoidal voltage source*** $e(t) = 60 \sin(144t + \phi)$ ***V by closing the switch at*** $\phi = 20^o$. ***Determine the current and voltage in the capacitor.***

E3.12 ***An RLC series circuit is excited by a dc source of 80 V. If the initial current through the inductance is 2 A opposing the circuit current, determine the current through the circuit and voltage across the capacitor. Take R = 10Ω, L = 0.1 H and C = 4 mF. Also draw the initial and final state of the circuits.***

E3.13 ***An RLC series circuit with R = 40Ω, L = 0.8 H and C = 200μF is connected to a dc source of 100 V by closing the switch at t = 0. Determine the current and voltage in the inductance. Take initial charge on the capacitor as 4 mC opposing the capacitor voltage.***

E3.14 ***From the RLC circuit shown in Fig. E3.14, find an expression for i(t) for $t \geq 0$. Also calculate the value of $\frac{di(t)}{dt}$ and $\frac{d^2 i(t)}{dt^2}$ at $t = 0^+$.***

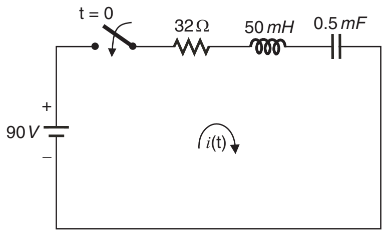


Fig. E3.14.

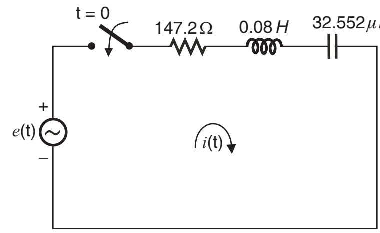


Fig. E3.15.

E3.15 ***The RLC series circuit shown in Fig. E3.15 is excited by a sinusoidal source of value $e(t) = 40 \sin (415t + \phi)$ V by closing the switch at $\phi = 0$. Find an expression for the response i(t).***

ANSWERS

E3.1 $i(t) = 12.5\, e^{-80t}\, A = 12.5\, e^{-\frac{t}{0.0125}}\, A$

$i(0^+) = 12.5\, A \quad ; \quad i(\infty) = 0$

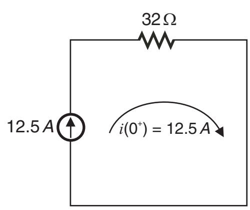


Fig : *Initial condition.*

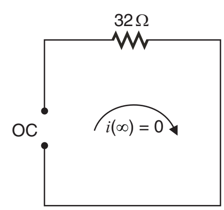


Fig : *Final condition.*

E3.2 $i(t) = 1.5\left(1 - e^{-\frac{t}{1/40}}\right) A = 1.5\left(1 - e^{-\frac{t}{0.025}}\right) A$

$v_R(t) = 12\left(1 - e^{-\frac{t}{0.025}}\right) V \quad ; \quad v_L(t) = 12\, e^{-\frac{t}{0.025}}\, V$

$\frac{di(0^+)}{dt} = 60\, A/s \quad ; \quad \frac{d^2 i(0^+)}{dt^2} = -2400\, A/s^2$

E3.3 $i(t) = 1.2 - 1.7\, e^{-\frac{t}{0.04}}\, A \quad ; \quad i(0^+) = -0.5\, A \quad ; \quad i(\infty) = 1.2\, A$

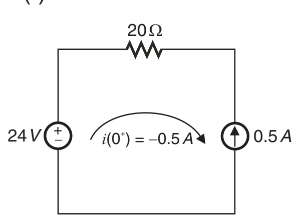


Fig : *Initial condition.*

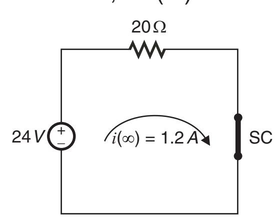
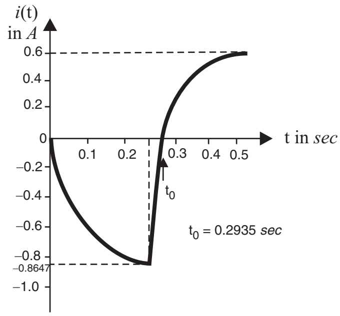


Fig : *Final condition.*

E3.4 $i(t) = -1 + e^{-\frac{t}{0.12}}\,A$; for $0 \le t \le 0.24$ *sec*

$i(t) = 0.6 - 1.4647\,e^{-\frac{(t-0.24)}{0.06}}\,A$; for $t \ge 0.24$ *sec*

$t_0 = 0.2935$ *sec*, (Time at which $i(t)$ is zero)



E3.5 $i(t) = -2.272e^{-2.857t} + 0.238\sin(120t - 73.6^\circ)\,A$

E3.6 $i(t) = 1.2\,e^{-t}\,mA$; $i(0^+) = 1.2\,mA$; $i(\infty) = 0$

$v_C(t) = 12\,e^{-t}\,V$; $v_C(0^+) = 12\,V$; $v_C(\infty) = 0$

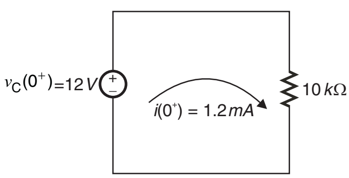


Fig : *Initial condition.*

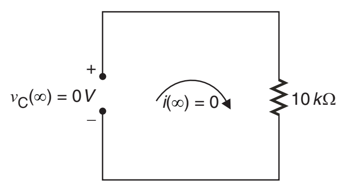


Fig : *Final condition.*

E3.7 $i(t) = 0.12\,e^{-\frac{t}{0.12}}\,A$; $v_R(t) = 18\,e^{-\frac{t}{0.12}}\,V$

$v_C(t) = 18\left(1 - e^{-\frac{t}{0.12}}\right)V$; $\dfrac{dv_C(0^+)}{dt} = 150\,V/s$

$\dfrac{d^2 v_C(0^+)}{dt^2} = -1250\,V/s^2$

E3.8 $v_C(t) = 20 - 15\,e^{-\frac{t}{0.04}}\,V$

$v_C(0^+) = 5\,V$; $v_C(\infty) = 20\,V$

$i(0^+) = 0.06\,A$; $i(\infty) = 0$

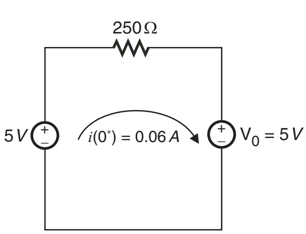


Fig : *Initial condition.*

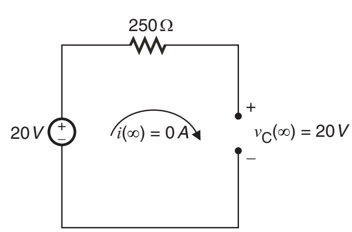


Fig : *Final condition.*

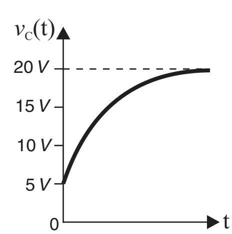


Fig : *Capacitor voltage.*

E3.9 $R = 1609\,\Omega$

E3.10 $v_C(t) = 18 - 23\,e^{-\frac{t}{0.048}}\,V$

$i(t) = 28.75\,e^{-\frac{t}{0.048}}\,mA$

$t_0 = 0.0118$ *second*, (Time at which $v_C(t) = 0$)

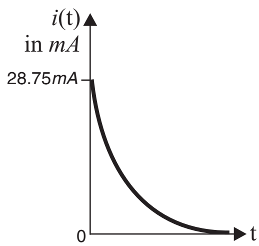


***Fig :** Capacitor current.*

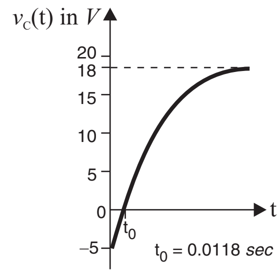


***Fig :** Capacitor voltage.*

E3.11 $v_C(t) = 10.278e^{-55.556t} + 21.593\sin(144t - 48.9°)\,V$

$i(t) = -\,0.043e^{-55.556t} + 0.233\sin(144t + 41.1°)\,A$

E3.12 $i(t) = 900t\,e^{-50t} - 2\,e^{-50t}\,A$; $v_R(t) = 9000t\,e^{-50t} - 20\,e^{-50t}\,V$; $v_L(t) = 100\,e^{-50t} - 4500t\,e^{-50t}\,V$

$v_C(t) = 80 - 4500t\,e^{-50t} - 80e^{-50t}\,V$

$i(0^+) = -2\,A$; $v_R(0^+) = -20\,V$; $v_L(0^+) = 100\,V$; $v_C(0^+) = 0$

$i(\infty) = 0$; $v_R(\infty) = 0$; $v_L(\infty) = 0$; $v_C(\infty) = 80\,V$

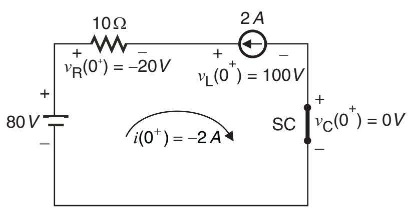


***Fig :** Initial condition circuit.*

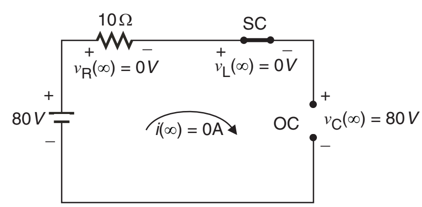


***Fig :** Final condition circuit.*

E3.13 $i(t) = 2\,e^{-25t}\sin 75t\,A$

$v_L(t) = 126.491\,e^{-25t}\sin(75t + 108.4°)\,V$

E3.14 $i(t) = 3.603\,(e^{-70.2t} - e^{-569.8t})\,A$

$$\frac{di(0^+)}{dt} = 1800\,A/s \quad ; \quad \frac{d^2 i(0^+)}{dt^2} = -\,1152038\,A/s^2$$

E3.15 $i(t) = -\,0.159\,e^{-240t} + 0.0896\,e^{-1600t} + 0.261\sin(415t + 15.6°)\,A$

Chapter 4

AC SINGLE AND THREE-PHASE CIRCUITS

4.1 AC Circuits

The sources in which the current/voltage sinusoidally varies with time are called **sinusoidal sources**. In sinusoidal sources, the voltage/current undergoes cyclic changes and the number of cycles per second is called **frequency**. The time for one cycle is called **time period**. Since the nature of variation is identical in every period, the sinusoidal voltages and currents are also called periodic voltages and currents.

In one period/cycle of sinusoidal quantity, the value of the quantity (i.e., voltage or current) is positive for one half period/cycle and the value of the quantity is negative for another half period/cycle. The nature of variations in the positive half cycle is identical to that of the negative half cycle but with opposite polarity. Since the sinusoidal voltage/current has alternate identical positive and negative half cycles, it is called **alternating voltage/current**. Therefore, the sinusoidal sources are called alternating current sources or in short **ac sources**.

The circuits excited by sinusoidal sources are called **ac circuits**. In ac circuits, the current and voltage varies with time and so all the three basic parameters, i.e., resistance, inductance and capacitance exist in ac circuits. The basic concepts of resistance parameter are discussed in Chapter 1 and the basic concepts of inductance and capacitance are discussed in this chapter.

4.1.1 AC Voltage and Current Source

Voltage and current are the two quantities which control the energy supplied by the sources of electrical energy. Usually, the sources are operated by maintaining one of the two quantities as constant and by allowing the other quantity to vary depending on the load.

In ac sources, when the rms value of voltage is maintained constant and the rms value of current is allowed to vary, the source is called a **voltage source**. When the rms value of current is maintained constant and the rms value of voltage is allowed to vary, the source is called a **current source**.

The voltage across an **ideal voltage source** should be constant for whatever current is delivered by the source. Similarly, the **ideal current source** should deliver a constant current for whatever voltage exists across its terminals.

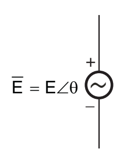


Fig. a : *Ac voltage source.*

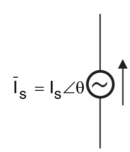


Fig. b : *Ac current source.*

Fig. 4.1 : *Symbols for ac source.*

In reality, such ideal conditions never exist (but for analysis purposes the sources can be considered ideal). In **practical ac voltage source,** the voltage across the source decreases with increasing load current and the reduction in voltage is due to its **internal impedance**. In **practical ac current source,** the current delivered by the source decreases with increasing load voltage and the reduction in current is due to its **internal impedance**.

Here, E, V = Magnitude of rms value of voltage.

I, I_S = Magnitude of rms value of current.

***Fig. a :** Characteristics of an ideal voltage source* ***Fig. b :** Characteristics of an ideal current source.*

***Fig. 4.2 :** Characteristics of ideal sources.*

***Fig. a :** Characteristics of a practical voltage source.* ***Fig. b :** Characteristics of a practical current source.*

***Fig. 4.3 :** Characteristics of practical sources.*

Let, $\bar{E}$ = Voltage across ideal source (or internal voltage of the source)

$\bar{I}_s$ = Current delivered by ideal source (or current generated by the source)

$\bar{V}$ = Voltage across the terminals of the source

$\bar{I}$ = Current delivered through the terminals of the source

$\bar{Z}_s$ = Source impedance (or internal impedance).

The **practical voltage source** can be considered as a series combination of an ideal voltage source and a **source impedance** $\bar{Z}_s$. The reduction in voltage across the terminals with increasing load current is due to the voltage drop in the source impedance. When the value of source impedance is zero, the ideal condition is achieved in voltage sources. Hence, the source impedance for an ideal voltage source is zero.

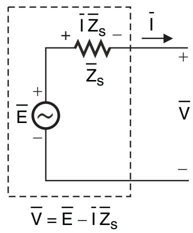


Fig. 4.4 : *Model of a practical ac voltage source.*

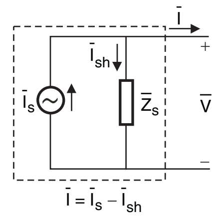


Fig. 4.5 : *Model of a practical ac current source.*

The **practical current source** can be considered as a parallel combination of an ideal current source and a **source impedance** $\overline{Z}_s$. The reduction in current delivered by the source is due to the current drawn by the parallel source impedance. When the value of source impedance is infinite, the ideal condition is achieved in current sources. Hence, the source impedance for an ideal current source is infinite.

4.1.2 AC Source Transformation

The practical voltage source can be converted into an equivalent practical current source and vice-versa, with the same terminal behaviour. In these conversions, the current and voltage at the terminal of the equivalent source will be the same as that of the original source, so that the power delivered to a load connected at the terminals of the original and equivalent source is same.

The voltage source with series impedance can be converted into an equivalent current source with parallel impedance. Similarly, the current source with parallel impedance can be converted into an equivalent voltage source with series impedance.

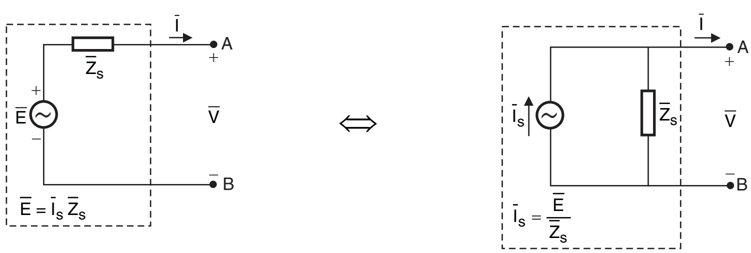


Fig. 4.6 : *Conversion of an ac voltage source to a current source and vice versa.*

4.2 Sinusoidal Voltage

A **sinusoidal voltage** can be considered as a vector of length V_m rotating in space with a uniform angular speed ω *rad/s* as shown in Fig. 4.7. At any time instant, the vector can be resolved into x_{comp} and y_{comp}. Now, y_{comp} gives the value of sinusoidal voltage at any time instant. Therefore, the instantaneous value (i.e., the value at any particular time instant) of a sinusoidal voltage is given by,

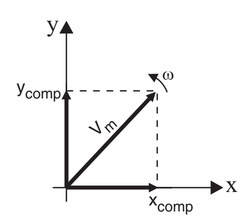


Fig. 4.7.

$$v = V_m \sin \omega t$$

Since sinusoidal voltage is a rotating vector, the value of the voltage repeats after an angular rotation of 2π radians or 360°. The number of revolutions (or rotations) per second is called **frequency** and it is denoted by f. The unit of frequency is **Hertz** and denoted by *Hz* (or cycles per second). One rotation of the voltage vector is also called **cycle** because the value of voltage repeats in every revolution.

One revolution is equal to an angular motion of 2π radians. Hence, frequency can also be expressed in radians per second (*rad/s*) which is denoted by ω and popularly called **angular frequency**.

The relation between angular frequency (ω) and frequency (f) is,

$$\boxed{\omega = 2\pi f}$$

The time taken for one revolution or cycle is called **time period** (or simply period), and it is denoted by T. The unit of time period (T) is seconds.

We know that,

Frequency, f = Number of cycles per second.

Hence, Time for one cycle $= \frac{1}{f}$

$$\boxed{\therefore \quad \text{Time period, } T = \frac{1}{f} \text{ seconds}}$$

The plot of the instantaneous value of sinusoidal voltage with respect to ωt is called **waveform**. The instantaneous value of sinusoidal voltage is computed for one cycle and the waveform is plotted in Fig. 4.8.

when, $\omega t = 0$; $\quad v = V_m \sin 0 = 0$

when, $\omega t = \frac{\pi}{4}$; $\quad v = V_m \sin \frac{\pi}{4} = \frac{V_m}{\sqrt{2}}$

when, $\omega t = \frac{2\pi}{4} = \frac{\pi}{2}$; $\quad v = V_m \sin \frac{\pi}{2} = V_m$

when, $\omega t = \frac{3\pi}{4}$; $\quad v = V_m \sin \frac{3\pi}{4} = \frac{V_m}{\sqrt{2}}$

when, $\omega t = \frac{4\pi}{4} = \pi$; $\quad v = V_m \sin \pi = 0$

when, $\omega t = \frac{5\pi}{4}$; $\quad v = V_m \sin \frac{5\pi}{4} = -\frac{V_m}{\sqrt{2}}$

when, $\omega t = \frac{6\pi}{4} = \frac{3\pi}{2}$; $\quad v = V_m \sin \frac{3\pi}{2} = -V_m$

when, $\omega t = \frac{7\pi}{4}$; $\quad v = V_m \sin \frac{7\pi}{4} = -\frac{V_m}{\sqrt{2}}$

when, $\omega t = \frac{8\pi}{4} = 2\pi$; $\quad v = V_m \sin 2\pi = 0$

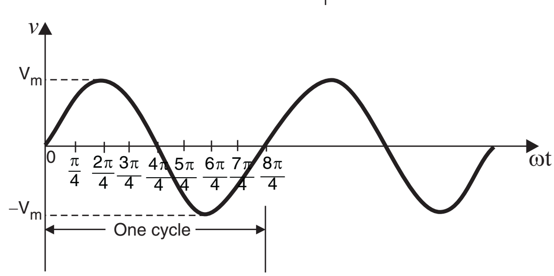


***Fig. 4.8 :** Sinusoidal voltage waveform on an angular scale.*

Since $\omega = 2\pi f = 2\pi/T$, the instantaneous value of sinusoidal voltage can also be expressed as shown in equation (4.1).

$$v = V_m \sin \frac{2\pi}{T} t \qquad \text{..... (4.1)}$$

Using equation (4.1), the instantaneous value of voltage can be calculated for various time instants and the waveform is plotted as a function of t in Fig. 4.9. It can be observed that the waveform of Fig. 4.9 is the same as that of Fig. 4.8 except the angular scale, which is replaced by the time scale.

when, $t = 0$; $\quad v = V_m \sin 0 = 0$

when, $t = \frac{T}{8}$; $\quad v = V_m \sin \frac{2\pi}{T} \times \frac{T}{8} = \frac{V_m}{\sqrt{2}}$

when, $t = \frac{2T}{8} = \frac{T}{4}$; $\quad v = V_m \sin \frac{2\pi}{T} \times \frac{T}{4} = V_m$

when, $t = \frac{3T}{8}$; $\quad v = V_m \sin \frac{2\pi}{T} \times \frac{3T}{8} = \frac{V_m}{\sqrt{2}}$

when, $t = \frac{4T}{8} = \frac{T}{2}$; $\quad v = V_m \sin \frac{2\pi}{T} \times \frac{T}{2} = 0$

when, $t = \frac{5T}{8}$; $\quad v = V_m \sin \frac{2\pi}{T} \times \frac{5T}{8} = -\frac{V_m}{\sqrt{2}}$

when, $t = \frac{6T}{8} = \frac{3T}{4}$; $\quad v = V_m \sin \frac{2\pi}{T} \times \frac{3T}{4} = -V_m$

when, $t = \frac{7T}{8}$; $\quad v = V_m \sin \frac{2\pi}{7} \times \frac{7T}{8} = -\frac{V_m}{\sqrt{2}}$

when, $t = \frac{8T}{8} = T$; $\quad v = V_m \sin \frac{2\pi}{T} \times T = 0$

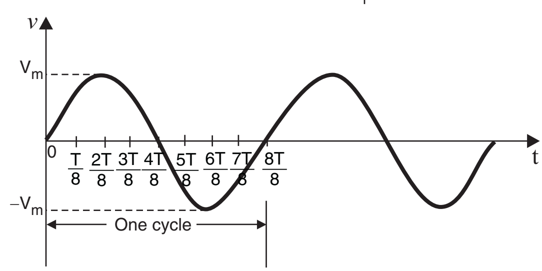


Fig. 4.9 : *Sinusoidal voltage waveform on a time scale.*

The sinusoidal voltage described by equation (4.1) starts at $\omega t = 0$ (i.e., the origin is at $\omega t = 0$). At the origin, the value of the voltage is zero. Sometimes, the voltage may be non-zero at the origin (i.e., at $\omega t = 0$). Such a sinusoidal voltage can be described by equation (4.2).

$$v = V_m \sin(\omega t + \phi) \qquad \text{..... (4.2)}$$

Equation (4.2) for sinusoidal voltage is the more generalised equation.

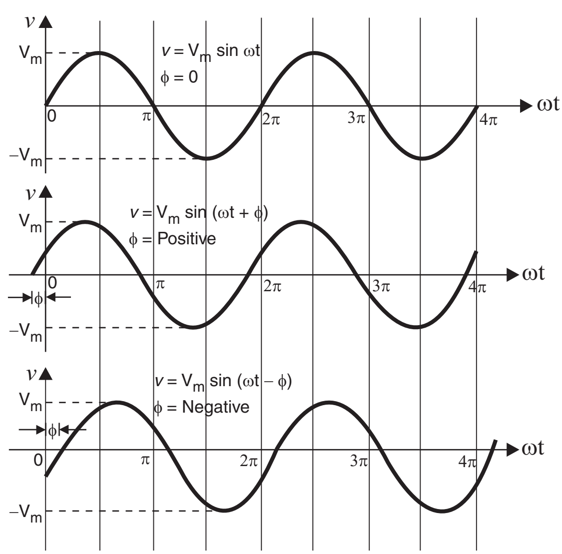


Fig. 4.10 : *Sinusoidal voltage waveforms.*

4.2.1 Average Value

The **average value** of a time varying quantity is the average of the instantaneous value for a particular time period. Usually, for periodic waveforms, the average is taken for one time period. In alternating quantities, the average value for one time period is zero because in one period it has equal positive and negative values. Therefore, for alternating quantities, the average is taken over half a period.

The instantaneous value of sinusoidal voltage is expressed by, $v = V_m \sin\omega t = V_m \sin\theta$, where, $\theta = \omega t$. The total value over half a period (π) is obtained by integrating the instantaneous value between limits 0 to π. Then the average is obtained by dividing this total value by half a period (π).

Let, V_{ave} = Average value of sinusoidal voltage or alternating voltage.

Now, by definition of average value, we can write,

$$V_{ave} = \frac{1}{\pi}\int_0^{\pi} v\, d\theta = \frac{1}{\pi}\int_0^{\pi} V_m \sin\theta\, d\theta = \frac{V_m}{\pi}\int_0^{\pi} \sin\theta\, d\theta$$

$$= \frac{V_m}{\pi}\left[-\cos\theta\right]_0^{\pi} = \frac{V_m}{\pi}\left[-\cos\pi + \cos 0\right]$$

$$= \frac{V_m}{\pi}\left[1 + 1\right] = \frac{2V_m}{\pi}$$

$$\boxed{\therefore \; V_{ave} = \frac{2V_m}{\pi}} \qquad .....(4.3)$$

4.2.2 RMS Value

(AU June'14, 2 Marks)

The **rms value** of a time varying quantity is the equivalent dc value of that quantity. (The rms value is also known as **effective value**.) For example, a 5 *V* dc is equivalent to 5 *V* rms value of ac. The rms stands for **r**oot-**m**ean-**s**quare, which means that the value is obtained by taking the root of the mean of the squared function. Hence, to obtain the rms value, a function is squared and the mean (average) of the squared function is determined. And the root of this mean value is taken. For periodic waveforms, rms value is computed for one period. For alternating quantities, rms value will be the same if it is computed for half a period or one period.

The instantaneous value of sinusoidal voltage is expressed by, $v = V_m \sin\omega t = V_m \sin\theta$, where, $\theta = \omega t$. The total value of the squared function over half a period (π) is obtained by integrating v^2 between limits 0 to π . The mean (average) is obtained by dividing the total value of the squared function by half a period (π). The rms value is obtained by taking the square root of this mean value.

Let, V = Rms value of sinusoidal voltage or alternating voltage.

Now, by definition of rms value, we can write,

$$V = \sqrt{\frac{1}{\pi}\int_0^{\pi} v^2 d\theta} = \sqrt{\frac{1}{\pi}\int_0^{\pi} (V_m \sin\theta)^2 d\theta}$$

$$= \sqrt{\frac{V_m^2}{\pi}\int_0^{\pi} \sin^2\theta\, d\theta} = \sqrt{\frac{V_m^2}{\pi}\int_0^{\pi} \frac{(1 - \cos 2\theta)}{2}\, d\theta} \qquad \boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$$

$$\therefore\ V = \sqrt{\frac{V_m^2}{2\pi}\int_0^{\pi}(1 - \cos 2\theta)d\theta} = \sqrt{\frac{V_m^2}{2\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}}$$

$$= \sqrt{\frac{V_m^2}{2\pi}\left[\pi - \frac{\sin 2\pi}{2} - 0 + \frac{\sin 0}{2}\right]} = \sqrt{\frac{V_m^2}{2\pi}[\pi]} = \frac{V_m}{\sqrt{2}}$$

$$\boxed{\therefore\ V = \frac{V_m}{\sqrt{2}}} \quad .....(4.4)$$

4.2.3 Form Factor and Peak Factor

Form factor is defined as the ratio of rms value and average value of a periodic waveform. **Peak factor** is defined as the ratio of peak value (or maximum value) and the rms value of a periodic waveform.

$$\boxed{\therefore\ \text{Form factor, } k_f = \frac{\text{rms value}}{\text{Average value}}} \quad \boxed{\therefore\ \text{Peak factor, } k_p = \frac{\text{Maximum value}}{\text{rms value}}}$$

Form and peak factors are constant for a particular waveform. If the average value of a waveform is known then its rms value can be estimated using the form factor (or vice-versa). If the rms value of a particular waveform is known then its peak value can be estimated using the peak factor (or vice-versa).The form and peak factors for sinusoidal voltage can be estimated using equations (4.5) and (4.6) as shown below:

$$\boxed{\text{Form factor, } k_f = \frac{V_m/\sqrt{2}}{2V_m/\pi} = \frac{\pi}{2\sqrt{2}} = 1.111 \quad (\text{For full sine wave})} \quad ..... (4.5)$$

$$\boxed{\text{Peak factor, } k_p = \frac{V_m}{V_m/\sqrt{2}} = \sqrt{2} = 1.414 \quad (\text{For full sine wave})} \quad ..... (4.6)$$

The form factor and peak factor of equations (4.5) and (4.6) are applicable for full sinusoidal waveform of voltage or current or any other quantity.

4.3 Sinusoidal Current

The discussions and analysis presented in Section 4.2 for sinusoidal voltage are applicable to sinusoidal currents. The equations derived for sinusoidal voltage in Section 4.2 are applicable to sinusoidal currents if we change the variable *v* by *i*.

The instantaneous value of sinusoidal current is given by,

$$i = I_m \sin \omega t = I_m \sin \theta, \quad \text{where, } \theta = \omega t$$

The general equation for instantaneous value of sinusoidal current is given by,

$$i = I_m \sin(\omega t + \phi)$$

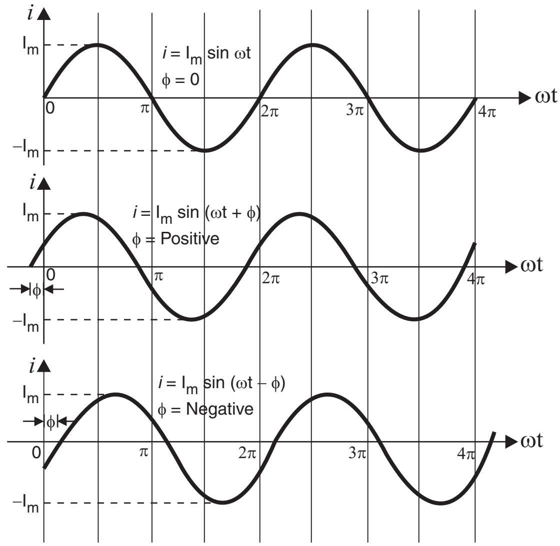


Fig. 4.11 : Sinusoidal current waveforms.

Average value of sinusoidal current, $I_{ave} = \frac{1}{\pi}\int_0^{\pi} i\, d\theta = \frac{1}{\pi}\int_0^{\pi} I_m \sin\theta\, d\theta = \frac{2I_m}{\pi}$

rms value of sinusoidal current, $I = \sqrt{\frac{1}{\pi}\int_0^{\pi} i^2\, d\theta} = \sqrt{\frac{1}{\pi}\int_0^{\pi} (I_m \sin\theta)^2\, d\theta} = \frac{I_m}{\sqrt{2}}$

$$\boxed{\text{Form factor, } k_f = \frac{I_m/\sqrt{2}}{2I_m/\pi} = \frac{\pi}{2\sqrt{2}} = 1.111}$$

$$\boxed{\text{Peak factor, } k_p = \frac{I_m}{1_m/\sqrt{2}} = \sqrt{2} = 1.414}$$

4.4 Inductance

Inductance is the property of an element (or matter) by which it opposes any change in flux or current. The opposition is offered by the way of an induced emf opposing the current flow. Hence, in an element with inductance property, current cannot change instantaneously, i.e., change in current will be delayed. (However, in purely resistive elements, current can change instantaneously).

> ***Note :*** *Flux and current are inseparable in nature. Whenever flux exists in an element, it is due to motion of electrons (i.e., current). Whenever current flows in an element, flux is created in the element.*

Flux in straight current-carrying conductors is negligible, but flux in conductors in the form of a coil is appreciable. Hence, the property of inductance is predominant in coils and so commercial inductors are also made in the form of coils. Typically, a coil consists of a number of turns of copper or aluminium conductor wound on an iron core (Sometimes the coils may not have a core).

The unit of inductance is **Henry** and denoted by *H*. The **inductance** of a coil is defined as the ratio of flux linkages (weber-turns) and current through the coil. The **weber-turns** refers to the product of flux ϕ and number of turns N of a coil. Hence, the inductance of a coil with N turns and carrying a current of I amperes is given by,

$$\boxed{\text{Inductance, } L = \frac{N\phi}{I}} \qquad ..... (4.7)$$

In equation (4.7), if N = 1 turn, ϕ = 1 *Weber*, I = 1 *Ampere*, then L = 1 *Henry*.

Therefore, "*a coil is said to have an inductance of one* ***Henry*** *if a current of one ampere flowing through it produces a flux linkage of one weber-turn in it*". Practically, one Henry is a large value and so the smaller values, *mH* (milli-Henry) and μ*H* (micro-Henry) are used.

Note : *In this section, inductance refers to self-inductance.*

4.4.1 Voltage-Current Relation in an Inductance

We know that a current-carrying conductor will always have a flux associated with current. **Faraday** has observed that whenever the flux linkage of a conductor changes, an emf is induced in the conductor. He proposed this phenomena as **Laws of magnetic induction**.

Law I : Whenever the magnetic flux linked with a conductor changes, an emf is always induced in it.

Law II : The magnitude of the induced emf is equal to the rate of change of flux linkage.

Consider an inductor with N turns carrying a current i as shown in Fig. 4.12. Let, ϕ be the instantaneous value of flux in the inductor. When the current i is varied the flux ϕ will also vary and so an emf is induced in the coil in a direction opposing the current flow. (The direction of the induced emf can also be found using Len's law or Fleming's right-hand rule).

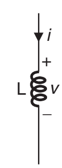


Fig. 4.12.

Now, by Faraday's Law we can write,

$$v = N\frac{d\phi}{dt} \quad \Rightarrow \quad v = N\frac{d}{dt}\left(\frac{Li}{N}\right) \qquad \boxed{\text{Using equation (4.7)}}$$

$$v = N \times \frac{L}{N}\frac{di}{dt} \qquad \boxed{\text{N and L are constants}}$$

$$\therefore\ v = L\frac{di}{dt} \qquad .....(4.8)$$

Equation (4.8) gives the voltage-current relation in an inductance. From equation (4.8), we can say that when i = constant, $\frac{di}{dt} = 0$ and so $v = 0$. Hence, for constant or direct current, the inductance will behave as a short circuit (in steady state). Therefore, in steady state analysis of circuits excited by dc sources, inductances are considered as short circuits (or simply the inductances are neglected).

On rearranging equation (4.8), we get,

$$v\,dt = L\,di$$

Integrating on both sides, we get,

$$\int v\,dt = \int L\,di \quad \Rightarrow \quad \int v\,dt = L\int di \quad \Rightarrow \quad \frac{1}{L}\int v\,dt = i$$

$$\boxed{\therefore\ i = \frac{1}{L}\int v\,dt} \qquad .....(4.9)$$

4.4.2 Energy Stored in an Inductance

Energy is stored as magnetic field in an inductance. Let, dw be the energy stored in the inductance in the time interval dt. Now the energy dw is given by the product of power p and time dt.

$$\therefore\ dw = p\,dt$$

$$= vi\,dt \qquad \boxed{\text{Put, } p = vi}$$

$$= L\frac{di}{dt}i\,dt \qquad \boxed{\text{Put, } v = L\frac{di}{dt}}$$

$$= Li\,di \qquad ..... (4.10)$$

Let the current i rise from zero to a steady value of I (where I is the rms value in case of ac) to establish an energy of W. Now, the stored energy (or total work) is obtained by integrating equation (4.10) between limits 0 to I.

$$\therefore\ W = \int_0^I Li\,di = L\int_0^I i\,di = L\left[\frac{i^2}{2}\right]_0^I = L\frac{I^2}{2}$$

$$\boxed{\therefore\ W = \frac{1}{2}LI^2} \qquad ..... (4.11)$$

Equation (4.11) can be used to compute the energy stored in an inductance when a steady current I flows through it.

4.5 Capacitance

Capacitance is the property of an element (or matter) by which it opposes any change in charge or voltage. Since charge is a physical quantity, it cannot change from one value to another instantaneously. Whenever charge exists at a point, there should be some voltage at that point. Hence, in an element with capacitance property, voltage cannot change instantaneously, i.e., change in voltage will be delayed. (However, in purely resistive elements, voltage can change instantaneously).

> ***Note:*** *Charge and voltage are inseparable in nature. Whenever charge exists in an element, there should be some voltage in it, or whenever voltage exists in an element, there should be some charge stored in that element.*

Capacitance will exist between any two conductors separated by a dielectric. Commercial capacitors consist of two conducting plates separated by a dielectric.

The unit of capacitance is **farad** and it is denoted by *F*. The **capacitance** of a capacitor is defined as the ratio of stored charge and the potential difference across its plates. The capacitance of a capacitor with a charge of Q coulombs and a potential difference of V volts across its plate is given by,

$$\boxed{\text{Capacitance, } C = \frac{Q}{V}} \qquad ..... (4.12)$$

In equation (4.12), if Q = 1 *coulomb* and V = 1 *volt,* then C = 1 *farad*.

Therefore, "*a capacitor is said to have a capacitance of one* ***farad*** *if a charge of one Coulomb establishes a potential difference of one volt between its plates*". Practically, one Farad is a large value and so the smaller values, μ*F* (micro-Farad), *nF* (nano-Farad) and *pF* (pico-Farad) are used.

4.5.1 Voltage-Current Relation in a Capacitance

We know that a potential difference exists between two charged conductors. Charge is directly proportional to voltage and the proportionality constant is capacitance C.

Consider a capacitor with capacitance C and carrying a current of i as shown in Fig. 4.13. Let, q and v be the instantaneous value of charge and voltage in the capacitor, respectively. Since charge is directly proportional to voltage, we can write,

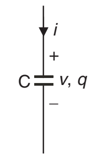


Fig. 4.13.

$$q \propto v$$

$$\therefore \quad q = Cv$$

On differentiating the above equation, we get,

$$\frac{dq}{dt} = C\frac{dv}{dt} \qquad \boxed{\frac{dq}{dt} = i \;;\; \text{Refer equation (1.5)}}$$

$$\therefore \quad i = C\,\frac{dv}{dt} \qquad \text{..... (4.13)}$$

Equation (4.13) gives the voltage-current relation in a capacitance. From equation (4.13), we can say that when v = constant, $\frac{dv}{dt} = 0$ and so $i = 0$. Hence, for constant or dc voltage, the capacitance will behave as an open circuit (in steady state). Therefore, in steady state analysis of circuits excited by dc sources, capacitances are considered as open circuits (or simply the capacitances are neglected).

On rearranging equation (4.13), we get,

$$i\,dt = C\,dv$$

On integrating both sides, we get,

$$\int i\,dt = \int C\,dv \quad \Rightarrow \quad \int i\,dt = C\int dv \quad \Rightarrow \quad \int i\,dt = Cv$$

$$\boxed{\therefore\; v = \frac{1}{C}\int i\,dt} \qquad \text{..... (4.14)}$$

4.5.2 Energy Stored in a Capacitance

Energy is stored as electric field in a capacitance. Let, dw be the energy stored in the capacitance in the time interval dt. Now the energy dw is given by the product of power p and time dt.

$$\therefore\; dw = p\,dt$$

$$= vi\,dt \qquad \boxed{\text{Put, } p = vi}$$

$$= vC\frac{dv}{dt}\,dt \qquad \boxed{\text{Put, } i = C\frac{dv}{dt}}$$

$$= C\,v\,dv \qquad \text{..... (4.15)}$$

Let the voltage v rise from zero to a steady value of V (where V is the rms value in case of ac) to establish an energy of W. Now, the stored energy (or total work) is obtained by integrating equation (4.15) between limits 0 to V.

$$\therefore\ W = \int_0^V Cv\,dv = C\int_0^V v\,dv = C\left[\frac{v^2}{2}\right]_0^V = \frac{CV^2}{2}$$

$$\boxed{\therefore\ W = \frac{1}{2}CV^2} \qquad ..... (4.16)$$

Equation (4.16), can be used to compute the energy stored in a capacitance when a steady voltage V exists across it.

4.6 Voltage-Current Relation of R, L and C in Various Domains

The circuit variables like voltage, current, power and energy are functions of time t. Time domain is a practical domain where we can physically realise any system or phenomena or activity. The voltage-current relations of fundamental parameters in time domain are differential equations. The solutions of differential equations are tedious when compared to algebraic equations. Hence, it will be convenient if we transform the differential equation into algebraic equations. One such transform is Laplace transform. A brief discussion about Laplace transform is presented in Appendix 3.

Let, $i = i(t)$ = Current in time domain

$v = v(t)$ = Voltage in time domain

$\mathcal{L}\{v(t)\} = V(s)$ = Voltage in s-domain or Laplace domain

$\mathcal{L}\{i(t)\} = I(s)$ = Current in s-domain or Laplace domain.

4.6.1 Voltage-Current Relation of Resistance

Consider a resistance R connected to a source of voltage $v(t)$ as shown in Fig. 4.14.

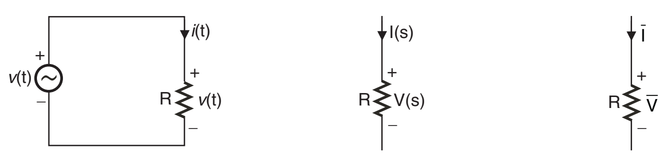


***Fig. a :** Resistance in time domain.*

***Fig. b :** Resistance in s-domain.*

***Fig. c :** Resistance in frequency domain.*

***Fig. 4.14 :** Voltage-current relation of resistance in various domains.*

Let, $i(t)$ = Current through the resistance

$v(t)$ = Voltage across the resistance.

By Ohm's law, we can write

$$v(t) = R\,i(t) \xrightarrow{\text{For simplicity}} v = R\,i \qquad ..... (4.17)$$

On taking Laplace transform of equation (4.17), we get,

$$V(s) = R\,I(s) \qquad ..... (4.18)$$

On substituting $s = j\omega$ in equation (4.18), we get,

$$\bar{V}(j\omega) = R\,\bar{I}(j\omega) \xrightarrow{\text{For simplicity}} \bar{V} = R\bar{I}$$

In Summary,

$v = R\,i$	; Voltage-current relation of resistance in time domain.
$V(s) = R\,I(s)$	; Voltage-current relation of resistance in s-domain.
$\bar{V} = R\,\bar{I}$	; Voltage-current relation of resistance in frequency domain.

4.6.2 Voltage-Current Relation of Inductance

Consider an inductance L connected to a source of voltage $v(t)$ as shown in Fig. 4.15.

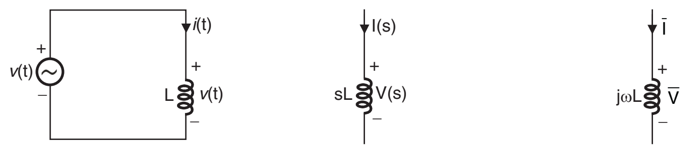


***Fig. a :** Inductance in time domain.* ***Fig. b :** Inductance in s-domain.* ***Fig. c :** Inductance in frequency domain.*

***Fig. 4.15 :** Voltage-current relation of inductance in various domains.*

Let, $i(t)$ = Current through the inductance

$v(t)$ = Voltage across the inductance

By Faraday's Law, we can write,

$$v(t) = L\frac{d}{dt}i(t) \xrightarrow{\text{For simplicity}} v = L\frac{di}{dt} \qquad .....(4.19)$$

On taking Laplace transform of equation (4.19) with zero initial conditions, we get,

$$V(s) = L\,sI(s)$$

$$\therefore V(s) = sL\,I(s) \qquad .....(4.20)$$

where, **sL = Inductive reactance in s-domain**

On substituting $s = j\omega$ in equation (4.20), we get,

$$\bar{V}(j\omega) = j\omega L\,\bar{I}(j\omega) \xrightarrow{\text{For simplicity}} \bar{V} = j\omega L\,\bar{I}$$

where, ω**L = X$_L$ = Inductive reactance**

In Summary,

$v = L\dfrac{di}{dt}$	; Voltage-current relation of inductance in time domain.
$V(s) = sL\,I(s)$	; Voltage-current relation of inductance in s-domain.
$\bar{V} = j\omega L\bar{I}$	; Voltage-current relation of inductance in frequency domain.

4.6.3 Voltage-Current Relation of Capacitance

Consider a capacitance C connected to a source of current *i*(t) as shown in Fig. 4.16.

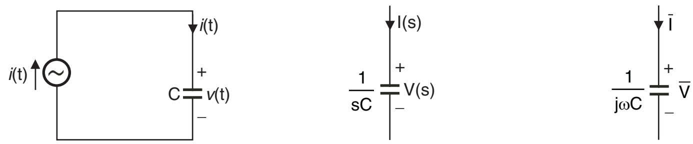


***Fig. a :** Capacitance in time domain.* ***Fig. b :** Capacitance in s-domain.* ***Fig. c :** Capacitance in frequency domain.*

***Fig. 4.16 :** Voltage-current relation of capacitance in various domains.*

Let, $i(t)$ = Current through the capacitance

$v(t)$ = Voltage across the capacitance.

$$\text{Now, } i(t) = C\frac{dv(t)}{dt} \xrightarrow{\text{For simplicity}} i = C\frac{dv}{dt} \quad \text{..... (4.21)}$$

On integrating and rearranging equation (4.21), we get,

$$v = \frac{1}{C}\int i\,dt$$

On taking Laplace transform of equation (4.21) with zero initial conditions, we get,

$$I(s) = CsV(s)$$

$$\therefore\ V(s) = \frac{1}{sC}I(s) \quad \text{..... (4.22)}$$

where, $\mathbf{\frac{1}{sC}}$ = **Capacitive reactance in s-domain**

On substituting s = jω in equation (4.22), we get,

$$\bar{V}(j\omega) = \frac{1}{j\omega C}\bar{I}(j\omega) \xrightarrow{\text{For simplicity}} \bar{V} = \frac{1}{j\omega C}\bar{I} = -j\frac{1}{\omega C}\bar{I}$$

where, $\mathbf{\frac{1}{\omega C}} = \mathbf{X_C}$ = **Capacitive reactance**

In Summary,

$v = \frac{1}{C}\int i\,dt$	; Voltage-current relation of capacitance in time domain.
$V(s) = \frac{1}{sC}I(s)$	; Voltage-current relation of capacitance in s-domain.
$\bar{V} = -j\frac{1}{\omega C}\bar{I}$	; Voltage-current relation of capacitance in frequency domain.

Table 4.1 : R, L, C Representation in Various Domains

S.No.	Parameter	Time domain	s-domain	Frequency domain
1.	Resistance, R	i + v − R	I(s) + V(s) − R	$\bar{I}$ + $\bar{V}$ − R
2.	Inductance, L	i + v − L	I(s) + V(s) − sL	$\bar{I}$ + $\bar{V}$ − $j\omega L$
3.	Capacitance, C	i + v − C	I(s) + V(s) − $\frac{1}{sC}$	$\bar{I}$ + $\bar{V}$ − $\frac{1}{j\omega C}$

4.7 Sinusoidal Voltage and Current in Frequency Domain

The instantaneous value of sinusoidal voltage in time domain is represented as,

$$v(t) = V_m \sin(\omega t \pm \phi)$$

In frequency domain, the rms value of sinusoidal voltage can be represented as,

$$\bar{V}(j\omega) = V\angle \pm \phi \xrightarrow{\text{For simplicity}} \bar{V} = V\angle \pm \phi$$

where, $V = V_m/\sqrt{2}$

$$\therefore\ V_m \sin(\omega t \pm \phi) \Rightarrow \boxed{\bar{V} = \frac{V_m}{\sqrt{2}}\angle \pm \phi = V\angle \pm \phi} \quad .....(4.23)$$

The instantaneous value of sinusoidal current in time domain is represented as,

$$i(t) = I_m \sin(\omega t \pm \phi)$$

In frequency domain, the rms value of sinusoidal current can be represented as,

$$\bar{I}(j\omega) = I\angle \pm \phi \xrightarrow{\text{For simplicity}} \bar{I} = I\angle \pm \phi$$

where, $I = I_m/\sqrt{2}$.

$$\therefore\ I_m \sin(\omega t \pm \phi) \Rightarrow \boxed{\bar{I} = \frac{I_m}{\sqrt{2}}\angle \pm \phi = I\angle \pm \phi} \quad .....(4.24)$$

4.8 Phasor Diagram

4.8.1 Phase and Phase Difference

Sinusoidal quantities are rotating vectors and the y-component of the rotating vector gives the instantaneous value. The plot of the instantaneous value on an angular scale or time scale gives the waveform of a sinusoidal quantity. These waveforms are continuous (i.e., with no beginning and end). For analysis purposes, we assume a starting point (or origin) for the waveform. Normally, the origin of the waveform is considered as time t = 0. The position of the rotating vector at time t = 0 decides the phase (or phase angle) of the vector. Three different positions of a vector at time t = 0 and their waveforms are shown in Fig. 4.17.

"*The **phase** (or **phase angle**) of a vector is the angular position of the vector with respect to reference at time t = 0*". In Fig. 4.17(a), the vector lies on the reference line at time t = 0. Hence, the phase angle of the vector is zero.

In Fig. 4.17(b), the vector is lying at an angle ϕ ahead of reference at t = 0. Hence, the phase angle of the vector is ϕ degrees (or radians) **leading**.

In Fig. 4.17(c), the vector is lying at an angle ϕ behind the reference at t = 0. Hence, the phase angle of the vector is ϕ degrees (or radians) **lagging**.

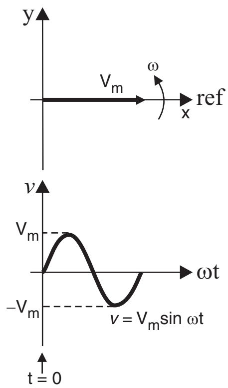


Fig. a : *The vector lies on the reference line at time t = 0.*

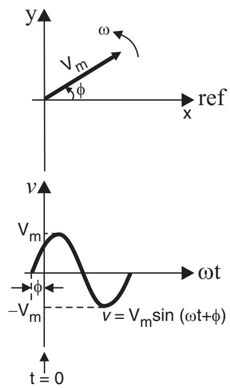


Fig. b : *The vector is lying ahead of the reference (in the direction of rotation) at an angle ϕ at time t = 0.*

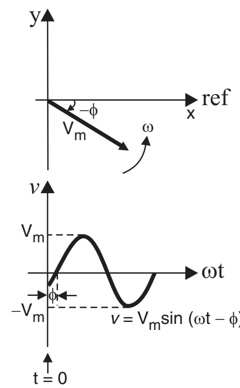


Fig. c : *The vector is lying behind the reference (in the direction of rotation) at an angle ϕ at time t = 0.*

Fig. 4.17 : *Different positions of a vector at t = 0.*

In electrical engineering (or circuit theory) we are interested in phase difference between two sine waves more than the phase of a single sine wave. "*The **phase difference** between two (or more) sine waves can be estimated only if the frequency of sine waves is the same (or speed of rotation is the same), but the magnitudes need not be the same*".

Consider two voltage vectors with maximum values V_{m1} and V_{m2} rotating at the same speed but having an angular spacing of ϕ radians between each other, as shown in Fig. 4.18(a). The two vectors represent two sinusoidal voltages of same frequency ω, as shown in Fig. 4.18(b).

With reference to Fig. 4.18, we can say that phase of voltage V_{m1} is ϕ_1 and that of voltage V_{m2} is ϕ_2. The phase difference between the two voltages is ϕ, where $\phi = \phi_2 - \phi_1$. "*The phase difference ϕ can be expressed either in degrees or in radians*". Also we can say that wave-2 is leading wave-1 by an angle ϕ or wave-1 is lagging wave-2 by an angle ϕ.

If "*the phase difference between two sinusoids is zero then they are said to be **in-phase***", because both the sinusoids have the same phase.

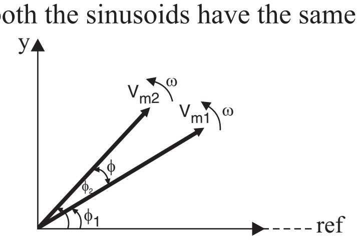


Fig. a : *Position of two vectors rotating at the same speed at t = 0.*

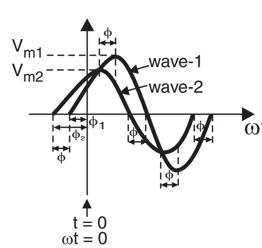


Fig. b : *Waveform of the vectors shown in Fig. a.*

Fig. 4.18 : *Two sinusoidal voltages with a phase difference of ϕ.*

4.8.2 Phasor Representation of Sinusoidal Quantities

Sinusoidal quantities are **rotating vectors**. When represented in a complex plane, the position of the vector at any time instant can be specified by two quantities namely., magnitude and phase angle. The magnitude of a rotating vector is constant but the phase angle varies with rotation. Hence, "*rotating vectors are functions of phase angle and so they are called **phasors***".

The sinusoidal quantity is given by the y-component (vertical component) of a rotating vector and so the sinusoidal quantity is also called a phasor (or sinor or vector). The x-component (horizontal component) of a rotating vector gives the cosinusoidal wave. In general, both the sine wave and cosine wave are called **phasors** (or **sinors** or **vectors**). Throughout this book, the terms, phasor and vector are used synonymously to denote sinusoidal quantity.

Since the sinusoidal quantity or phasor is a rotating vector, we can take the position of the phasor at t = 0 for analysis purposes, as shown in Fig. 4.19.

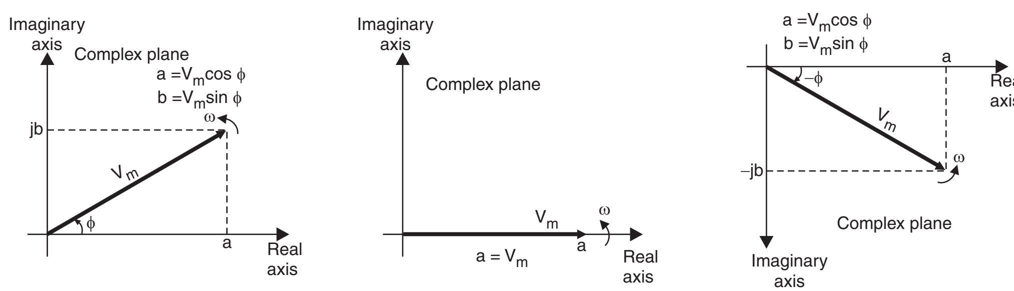


Fig. a : *A leading phasor.* ***Fig. b :*** *A phasor with zero phase.* ***Fig. c :*** *A lagging phasor.*

Fig. 4.19 : *Various positions of a sinusoidal voltage phasor at t = 0.*

From the theory of complex numbers, the phasors (or vectors) shown in Fig. 4.19 can be represented as shown below:

$V_m \angle\phi$	; Polar form
$V_m \cos\phi + j\, V_m \sin\phi$	; Rectangular form
$V_m\, e^{j\phi}$	; Eular form

In the above representation, the phase angle ϕ can be positive, zero or negative.

- **When ϕ is positive, the voltage vector will be in the position shown in Fig. 4.19(a) at t = 0.**
- **When ϕ is zero, the voltage vector will be in the position shown in Fig. 4.19(b) at t = 0.**
- **When ϕ is negative, the voltage vector will be in the position shown in Fig. 4.19(c) at t = 0.**

We know that rms values (or effective values) are used for practical measurements/ applications. For example, the rated voltage 230 *V* of a ceiling fan at home refers to the rms value of sinusoidal voltage. The relation between rms value and maximum value for a sinusoidal voltage is $V = V_m/\sqrt{2}$, where V is rms value and V_m is maximum value.

From the relation, $V = V_m/\sqrt{2}$, we can say that the rms value is the scaled down value of maximum value. Hence, the rms value can also be represented as a phasor of the same phase as that of maximum value but with reduced magnitude, as shown in Fig. 4.20. However, remember that the rotation of rms value of phasor will not produce the instantaneous value.

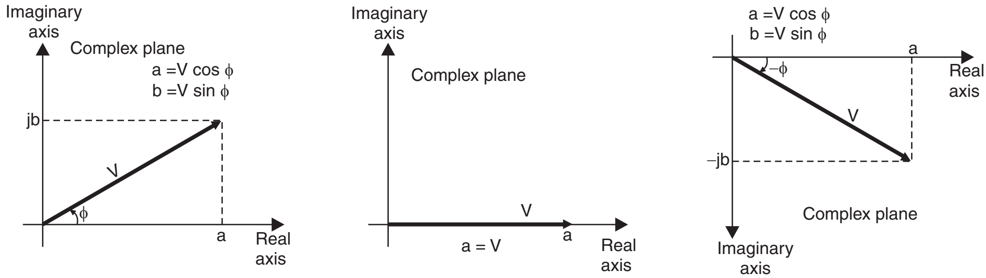


Fig. a : *A leading phasor.* ***Fig. b :*** *A phasor with zero phase.* ***Fig. c :*** *A lagging phasor.*

Fig. 4.20 : *Phasor representation of rms values of sinusoidal voltage.*

"In order to draw the ***rms vector*** *take a snapshot of the rotating phasor at t = 0 and divide its magnitude by* $\sqrt{2}$*".*

4.8.3 Phasor Diagram of a Circuit

When the excitation source in a circuit is sinusoidal, the voltage, current and power in various elements of the circuit will also be sinusoidal. Hence, the various voltages and currents in a circuit can be represented by phasors. While drawing these phasors, one quantity (either voltage or current) is chosen as reference and the phasors of the other quantity are drawn in relation to the chosen reference phasor. Such a diagram is called a **phasor diagram**.

Note : *Phasor diagrams throughout this book are drawn only using rms phasors.*

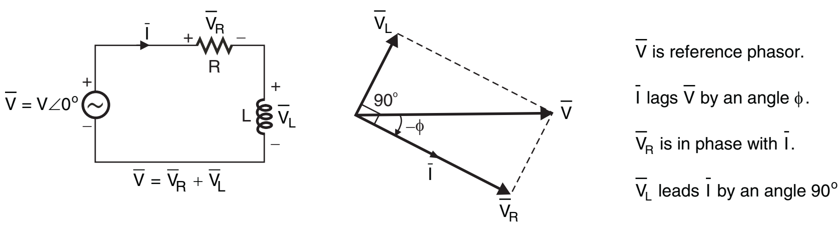


Fig. a : *RL series circuit and its phasor diagram.*

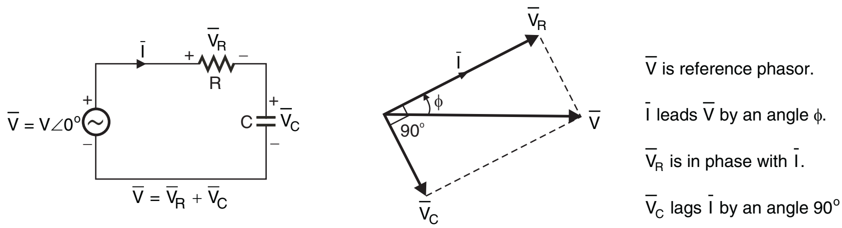


Fig. b : *RC series circuit and its phasor diagram.*

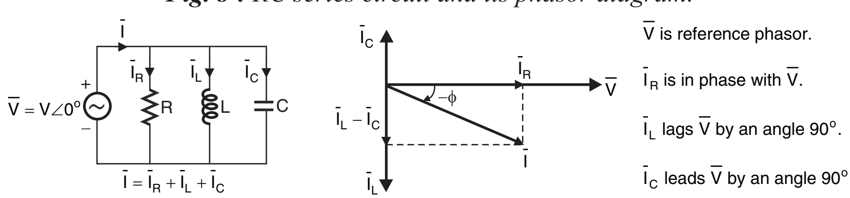


Fig. c : *RLC parallel circuit and its phasor diagram.*

Fig. 4.21 : *Phasor diagrams of simple circuits.*

4.9 Power, Energy and Power Factor

Power is the rate at which work is done or **power** is the rate of energy transfer.

Let, w = Instantaneous value of energy

q = Instantaneous value of charge.

Now, Instantaneous power, $p = \frac{dw}{dt} = \frac{dw}{dq} \times \frac{dq}{dt}$

We know that, $\frac{dw}{dq} = v$ and $\frac{dq}{dt} = i$

Refer Chapter - 1
Equations (1.5) and (1.7)

$\boxed{\therefore \text{ Instantaneous power, } p = v\,i}$

i.e., power is the product of voltage and current. In circuits excited by dc sources, voltage and current are constant and so power is also constant. This constant power is called **average power** or power and it is denoted by P.

$\therefore$ In dc circuits,

$\boxed{\text{Power, } P = VI}$

In circuits excited by ac sources, voltage and current are sinusoidal quantities which vary with time. When voltage and current are time varying quantities, power is also a time varying quantity.

For time varying quantities, power is defined as the average over a period of time. Since the average values of sinusoidal voltage and current are zero, we can take the rms values of voltage and current. We know that the rms values of voltage and current are complex and so power is also complex. "*C**omplex power** is denoted by* $\bar{S}$ *and it is defined as the product of rms voltage and the conjugate of rms current*".

$\boxed{\therefore \text{ Complex power, } \bar{S} = \bar{V}\,\bar{I}^*}$

where, $\bar{I}^*$ = Conjugate of $\bar{I}$

Let, $\bar{V} = V\angle\delta$

$\bar{I} = I\angle\gamma$

then, $\bar{I}^* = I\angle -\gamma$

where, δ is phase of voltage and γ is phase of current.

$$\therefore \bar{S} = \bar{V}\,\bar{I}^* = V\angle\delta \times I\angle -\gamma = VI\angle(\delta - \gamma)$$

Let, $\delta - \gamma = \phi$

where, ϕ = Phase difference between $\bar{V}$ and $\bar{I}$

$$\therefore \bar{S} = VI\angle\phi$$

Let, $S = |\bar{S}| = VI$

where, S = Apparent power and expressed in *Volt-Ampere,* i.e., *VA*.

(The larger units of S are *kVA* and *MVA*).

"***Apparent power*** *S is defined as the product of the magnitude of rms voltage and rms current*".

Since $\bar{S}$ is complex, it can be expressed as a vector in a complex plane as shown in Fig. 4.22.

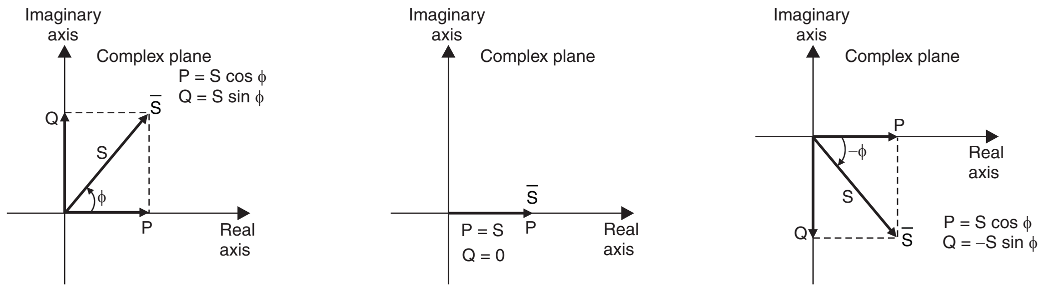


Fig. a : *Vector of $\bar{S}$ when ϕ is positive.*

Fig. b : *Vector of $\bar{S}$ when ϕ is zero.*

Fig. c : *Vector of $\bar{S}$ when ϕ is negative.*

Fig. 4.22 : *Vector of complex power $\bar{S}$ for various values of ϕ.*

The real part of $\bar{S}$ is called **active power** or simply power. The imaginary part of $\bar{S}$ is called **reactive power**. Power is denoted by P and expressed in watts, *W*. Reactive power is denoted by Q and expressed in volt-ampere-reactive, *VAR*.

With reference to Fig. 4.22, we can write,

$$\bar{S} = |\bar{S}|\cos\phi + j|\bar{S}|\sin\phi$$

Let, $$\bar{S} = P + jQ$$

$$\therefore P = |\bar{S}|\cos\phi$$

$$Q = |\bar{S}|\sin\phi$$

We know that, $|\bar{S}| = S = VI$

$$\boxed{\begin{aligned} \therefore P &= VI\cos\phi \text{ in } W \\ Q &= VI\sin\phi \text{ in } VAR \end{aligned}}$$

In Fig. 4.22, the triangle formed by P, Q and S is also called a **power triangle**.

The larger units of power P is *kW* or *MW* and larger units of reactive power Q is *kVAR* or *MVAR*.

$$\boxed{\begin{aligned} \therefore P &= VI\cos\phi \text{ in } W = \frac{VI\cos\phi}{10^3} \text{ in } kW = \frac{VI\cos\phi}{10^6} \text{ in } MW \\ Q &= VI\sin\phi \text{ in } VAR = \frac{VI\sin\phi}{10^3} \text{ in } kVAR = \frac{VI\sin\phi}{10^6} \text{ in } MVAR \end{aligned}}$$

In dc circuits, V and I are constants and there is no phase difference between V and I. Hence, $\phi = 0$ and so, $\cos\phi = 1$ and $\sin\phi = 0$. Therefore, in dc circuits, complex power or apparent power is equal to active power, and reactive power is zero.

In ac circuits, the phase angle ϕ may be positive, zero or negative. (Remember that ϕ is phase difference between $\overline{V}$ and $\overline{I}$.)

When ϕ is positive,

- the current lags voltage.
- the circuit is inductive.
- the active power is positive.
- the reactive power is positive.

When ϕ is zero,

- the current is in-phase with voltage.
- the circuit is resistive.
- the active power is positive.
- the reactive power is zero.

When ϕ is negative,

- the current leads the voltage.
- the circuit is capacitive.
- the active power is positive.
- the reactive power is negative.

In summary, we can say that active power P is always positive. Reactive power Q is positive in inductive circuits and negative in capacitive circuits. In resistive circuits, active power is equal to apparent power.

Power is rate of work done and energy is the total work done. Hence, "***energy** is given by the product of power and time*". When time is expressed in seconds, the unit of energy is *watt-second* and when time is expressed in hours, the unit of energy is *watt-hour*.

$$\boxed{\therefore \text{ Energy, E} = \text{Pt in } W\text{-}s \text{ or } W\text{-}h}$$

The larger unit of energy is *kWh* and commercially one *kWh* of electrical energy is called one **unit**.

"*The ratio of active power and apparent power is defined as **power factor***". Power factor is a measure of active power in the apparent power.

$$\boxed{\therefore \text{ Power factor} = \frac{\text{Active power}}{\text{Apparent power}} = \frac{P}{S}}$$

From the power triangle of Fig. 3.22, we get,

$$P = S \cos \phi$$

$$\boxed{\therefore \text{ Power factor} = \frac{P}{S} = \frac{S \cos \phi}{S} = \cos \phi}$$

Here, ϕ is the phase difference between voltage and current. Hence, from the above equation we can say that, " ***power factor** is also defined as cosine of the phase difference between voltage and current*".

4.10 Resistance Connected to Sinusoidal Source

Consider a resistance R connected to an ac source of voltage, $v = V_m \sin \omega t$ as shown in Fig. 4.23. Since the resistance is connected across (or parallel to) the source, the instantaneous value of voltage across resistance is also v volts.

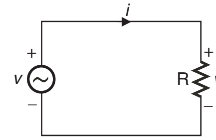


***Fig 4.23 :** Resistance connected to an ac source.*

$$\therefore \text{ Voltage, } v = V_m \sin \omega t \qquad .....(4.25)$$

By Ohm's law, the instantaneous current through the resistance is given by,

$$\text{Current, } i = \frac{v}{R}$$

$$= \frac{V_m \sin \omega t}{R} = \frac{V_m}{R} \sin \omega t \qquad \boxed{v = V_m \sin \omega t}$$

$$= I_m \sin \omega t \qquad ..... (4.26)$$

$$\text{where, } \frac{V_m}{R} = I_m = \text{Maximum value of current}$$

From equations (4.25) and (4.26), we can say that the voltage and current in a resistance are sinusoidal quantities of the same frequency and are in-phase (i.e., the phase difference between voltage and current in a resistance is zero).

Sinusoidal voltage and current can be expressed in polar form as shown below:

$$\bar{V}_m = V_m \angle 0^\circ$$

$$\bar{I}_m = I_m \angle 0^\circ$$

Since the rms values are practically used (than the maximum values), the rms values of voltage and current are shown in the circuit of Fig. 4.24.

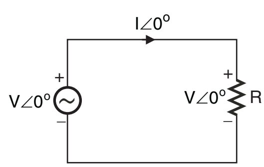


***Fig. a :** Circuit showing rms value of voltage and current in resistance.*

***Fig. b :** Phasor diagram of voltage and current in circuit of Fig. a.*

***Fig. 4.24 :** Resistance connected to an ac source.*

Rms value of voltage across resistance, $\bar{V} = V \angle 0^\circ$

Rms value of current through resistance, $\bar{I} = I \angle 0^\circ$

The instantaneous value of power in a resistance is given by the product of the instantaneous value of voltage and the current in the resistance.

$$\therefore \text{ Instantaneous power, } p = v \times i = V_m \sin \omega t \times I_m \sin \omega t$$

$$= V_m I_m \sin^2 \omega t = V_m I_m \sin^2\theta = P_m \sin^2\theta \qquad ..... (4.27)$$

$$\text{where, } \quad \theta = \omega t \;\; ; \;\; P_m = V_m I_m$$

From equation (4.27), we can say that instantaneous power is squared sine wave and so it is always positive (because on squaring the negative cycle it becomes positive).

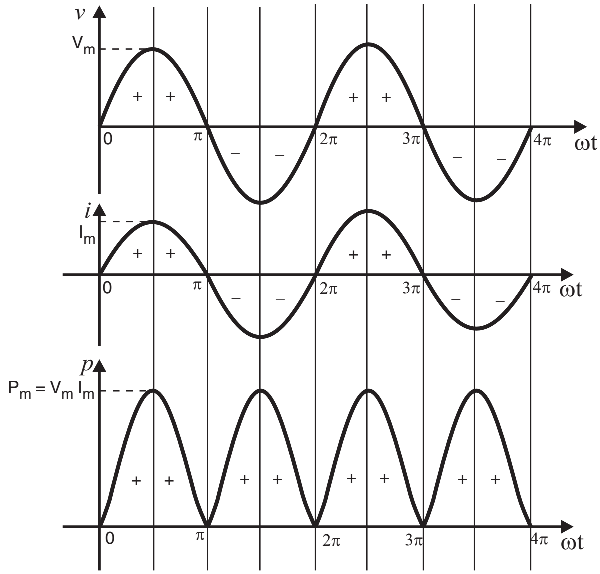


Fig. 4.25 : *Waveform of voltage, current and power in a resistance.*

Practically, power is measured as an average value obtained by taking the average value of equation (4.27) over one period of voltage or current (or the average can be taken for half a period of voltage or current).

$$\therefore \text{ Power, } P = \frac{1}{\pi}\int_0^{\pi} p\, d\theta = \frac{1}{\pi}\int_0^{\pi} V_m I_m \sin^2\theta\, d\theta$$

$$= \frac{V_m I_m}{\pi}\int_0^{\pi} \frac{1 - \cos 2\theta}{2}\, d\theta$$

$$= \frac{V_m I_m}{2\pi}\int_0^{\pi} (1 - \cos 2\theta)\, d\theta$$

$$= \frac{V_m I_m}{2\pi}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{V_m I_m}{2\pi}\left[\pi - \frac{\sin 2\pi}{2} - 0 + \frac{\sin 0}{2}\right] \qquad \boxed{\sin 2\pi = 0 \text{ and } \sin 0 = 0}$$

$$= \frac{V_m I_m}{2} = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}} = VI$$

$$\therefore P = VI$$

Alternatively, the expression for power can be obtained from complex power.

Complex power, $\overline{S} = \overline{V}\,\overline{I}^* = V\angle 0^\circ \times (I\angle 0^\circ)^* = V\angle 0^\circ \times I\angle 0^\circ = VI\angle 0^\circ$

We know that, $|\bar{S}| = S = VI$ and $\angle\bar{S} = \phi = 0^o$

$\therefore$ Power, $P = S\cos\phi = VI\cos 0^o = VI$

Reactive power, $Q = S\sin\phi = VI\sin 0^o = 0$

"*Resistance consumes only active power and the reactive power in a resistance is zero*".

4.11 Inductance Connected to Sinusoidal Source

Consider an inductance L connected to an ac source of voltage, $v = V_m \sin\omega t$ as shown in Fig. 4.26. Since the inductance is connected across (or parallel to) the source, the instantaneous value of voltage across inductance is also v volts.

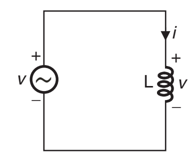


Fig. 4.26 : *Inductance connected to an ac source.*

$\therefore$ Voltage, $v = V_m \sin\omega t$ ..... (4.28)

The current i through the inductance is given by,

$$i = \frac{1}{L}\int v\,dt$$

$$= \frac{1}{L}\int V_m \sin\omega t\,dt \qquad \boxed{v = V_m \sin\omega t}$$

$$= \frac{V_m}{L}\int \sin\omega t\,dt = \frac{V_m}{L}\left[\frac{-\cos\omega t}{\omega}\right]$$

$$= \frac{V_m}{\omega L}(-\cos\omega t)$$

$$= \frac{V_m}{X_L}(-\cos\omega t)$$

$$= \frac{V_m}{X_L}\sin(\omega t - 90^o) \qquad \boxed{\sin(A - 90^o) = -\cos A}$$

$$= I_m \sin(\omega t - 90^o) \qquad .....(4.29)$$

where, $\omega L = X_L$ = Inductive reactance

$$\frac{V_m}{X_L} = I_m = \text{Maximum value of current} \qquad ..... (4.30)$$

From equations (4.28) and (4.29), we can say that the voltage and current in an inductance are sinusoidal quantities of the same frequency, but have a phase difference of 90°. "*The current in an inductance lags behind the voltage by 90°*" (or the voltage across the inductance leads the current in an inductance by 90°).

The sinusoidal voltage and current of equations (4.28) and (4.29) can be expressed in polar form as shown below:

$$\bar{V}_m = V_m\angle 0^o$$

$$\bar{I}_m = I_m\angle -90^o$$

Since rms values are practically used (than the maximum values), the rms values of voltage and current are shown in circuit of Fig. 4.27.

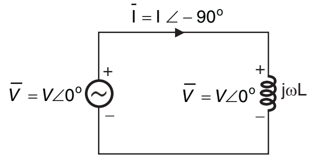


Fig. a : *Circuit showing rms values of voltage and current in inductance.*

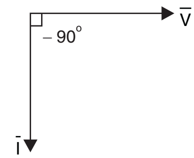


Fig. b : *Phasor diagram of voltage and current in circuit of Fig. a.*

Fig. 4.27 : *Inductance connected to ac source.*

Rms value of voltage across inductance, $\bar{V} = V\angle 0^\circ$

Rms value of current through inductance, $\bar{I} = I\angle -90^\circ$

From equation (4.30), we get,

$$X_L = \frac{V_m}{I_m}$$

Since, $V = V_m/\sqrt{2}$ and $I = I_m/\sqrt{2}$ we can write,

$$X_L = \frac{V}{I}$$

Now, $$\frac{\bar{V}}{\bar{I}} = \frac{V\angle 0^\circ}{I\angle -90^\circ} = \frac{V}{I}\angle 90^\circ = X_L\angle 90^\circ = jX_L = j\omega L \quad ..... (4.31)$$

From equation (4.31), we can say that, "***inductive reactance*** *is given by the ratio of sinusoidal voltage to current in an inductance*".

The instantaneous value of power in an inductance is given by the product of the instantaneous value of voltage and the current in the inductance.

$\therefore$ Instantaneous power, $p = v \times i$

$$= V_m \sin \omega t \times I_m \sin(\omega t - 90^\circ) \qquad \boxed{\sin(A - 90^\circ) = -\cos A}$$

$$= V_m I_m \sin \omega t\,(-\cos \omega t)$$

$$= -V_m I_m \sin \omega t \cos \omega t$$

$$\therefore \quad p = -V_m I_m \frac{\sin 2\omega t}{2}$$

$$= -\frac{V_m I_m}{2}\sin 2\omega t = -\frac{V_m I_m}{2}\sin 2\theta \quad ..... (4.32)$$

where, $\theta = \omega t$

From equation (4.32), we can say that instantaneous power is also a sinusoidal quantity whose frequency is double that of voltage or current.

Practically, power is measured as an average value. With reference to Fig. 4.28, we can say that for half a period of voltage or current wave, the power wave undergoes one full cycle and it is sinusoidal. Hence, the average value of power over half a period or a full period of voltage will be zero.

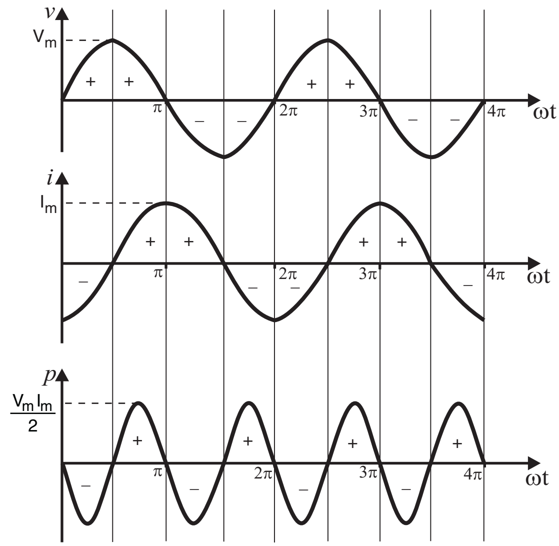


Fig. 4.28 : *Waveform of voltage, current and power in an inductance.*

$$\text{Power, } P = \frac{1}{\pi}\int_0^{\pi} p\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{-V_m I_m}{2} \sin 2\theta\, d\theta$$

$$= \frac{-V_m I_m}{2\pi}\int_0^{\pi} \sin 2\theta\, d\theta$$

$$= \frac{-V_m I_m}{2\pi}\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{-V_m I_m}{2\pi}\left[-\frac{\cos 2\pi}{2} + \frac{\cos 0}{2}\right]$$

$$= \frac{-V_m I_m}{2\pi}\left[-\frac{1}{2} + \frac{1}{2}\right] = 0$$

Alternatively, the expression for power can be obtained from complex power.

$$\text{Complex Power, } \bar{S} = \bar{V}\,\bar{I}^* = V\angle 0^\circ \times (I\angle -90^\circ)^*$$

$$= V\angle 0^\circ \times I\angle +90^\circ$$

$$= VI\angle 90^\circ$$

We know that, $|\bar{S}| = S = VI$ and $\angle\bar{S} = \phi = 90^\circ$

$$\therefore \text{ Power, } P = S\cos\phi = VI\cos 90^\circ = 0$$

$$\text{Reactive Power, } Q = S\sin\phi = VI\sin 90^\circ = VI$$

Inductance consumes only reactive power and "*the active power in pure inductance is zero*." The reactive power of inductance is positive, which means that it absorbs reactive power.

4.12 Capacitance Connected to Sinusoidal Source

Consider a capacitance C connected to an ac source of voltage, $v = V_m \sin \omega t$ as shown in Fig. 4.29. Since the capacitance is connected across (or parallel to) the source, the instantaneous value of voltage across the capacitance is also v volts.

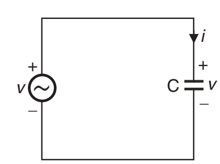


Fig. 4.29 : *Capacitance connected to an ac source.*

$$\therefore \text{ Voltage, } v = V_m \sin \omega t \quad .....(4.33)$$

The current i through the capacitance is given by,

$$i = C\frac{dv}{dt}$$

$$= C\frac{d}{dt}V_m \sin\omega t \qquad \boxed{v = V_m \sin\omega t}$$

$$= C\,V_m \frac{d}{dt}\sin\omega t$$

$$= C\,V_m \cos\omega t \times \omega$$

$$= \frac{V_m}{1/\omega C}\cos\omega t$$

$$= \frac{V_m}{X_C}\cos\omega t$$

$$= \frac{V_m}{X_C}\sin(\omega t + 90^o)$$

$$= I_m \sin(\omega t + 90^o) \qquad .....(4.34)$$

where, $\frac{1}{\omega C} = X_C =$ Capacitive reactance.

$$\frac{V_m}{X_C} = I_m = \text{Maximum value of current.} \qquad .....(4.35)$$

From equations (4.33) and (4.34), we can say that voltage and current in a capacitance are sinusoidal quantities of the same frequency, but have a phase difference of 90°. "*The current in a capacitance leads the voltage across it by 90°*" (or the voltage across the capacitance lags the current in a capacitance by 90°).

The sinusoidal voltage and current of equations (4.33) and (4.34) can be expressed in polar form as shown below:

$$\bar{V}_m = V_m \angle 0^o$$

$$\bar{I}_m = I_m \angle +90^o$$

Since rms values are practically used (than the maximum values), the rms values of voltage and current are shown in circuit of Fig. 4.30.

Rms value of voltage across capacitance, $\bar{V} = V\angle 0^o$

Rms value of current through capacitance, $\bar{I} = I\angle +90^o$

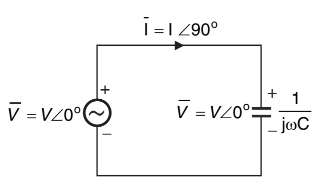


Fig. a : *Circuit showing rms values of voltage and current in capacitance.*

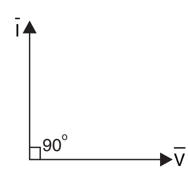


Fig. b : *Phasor diagram of voltage and current in circuit of Fig. a.*

Fig. 4.30 : *Capacitance connected to an ac source.*

From equation (4.35), we get,

$$X_C = \frac{V_m}{I_m}$$

Since, $V = V_m/\sqrt{2}$ and $I = I_m/\sqrt{2}$ we can write,

$$X_C = \frac{V}{I}$$

Now, $$\frac{\overline{V}}{\overline{I}} = \frac{V\angle 0^\circ}{I\angle 90^\circ} = \frac{V}{I}\angle -90^\circ = X_C\angle -90^\circ = -jX_C = -j\frac{1}{\omega C} = \frac{1}{j\omega C} \quad ..... (4.36)$$

From equation (4.36), we can say that, "***capacitive reactance** is given by the ratio of sinusoidal voltage to current in a capacitance*".

The instantaneous value of power in a capacitance is given by the product of the instantaneous value of voltage and the current in the capacitance.

Let, p = Instantaneous power in the capacitance

$$\therefore \; p = v \times i$$

$$= V_m \sin\omega t \times I_m \sin(\omega t + 90^\circ) \qquad \boxed{\sin(A+90^\circ) = \cos A}$$

$$= V_m I_m \sin\omega t \cos\omega t \qquad \boxed{\sin 2A = 2\sin A\cos A}$$

$$= V_m I_m \frac{2\sin\omega t\cos\omega t}{2}$$

$$= \frac{V_m I_m}{2}\sin 2\omega t \qquad \boxed{\text{where}, \theta = \omega t}$$

$$= \frac{V_m I_m}{2}\sin 2\theta \quad ..... (4.37)$$

From equation (4.37), we can say that instantaneous power is also a sinusoidal quantity whose frequency is double that of voltage or current.

Practically, power is measured as an average value. With reference to Fig. 4.31, we can say that for half a period of voltage or current wave, the power wave undergoes one full cycle and it is sinusoidal. Hence, the average value of power over half a period or a full period of voltage will be zero.

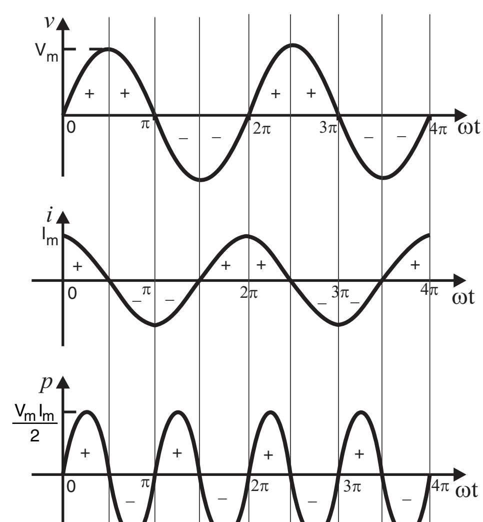


***Fig. 4.31 :** Waveform of voltage, current and power in a capacitance.*

$$\text{Power, } P = \frac{1}{\pi}\int_0^{\pi} p\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{V_m I_m}{2}\sin 2\theta\, d\theta = \frac{V_m I_m}{2\pi}\int_0^{\pi} \sin 2\theta\, d\theta$$

$$\therefore \; P = \frac{V_m I_m}{2\pi}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{V_m I_m}{2\pi}\left[-\frac{\cos 2\pi}{2} + \frac{\cos 0}{2}\right]$$

$$= \frac{V_m I_m}{2\pi}\left[-\frac{1}{2} + \frac{1}{2}\right] = 0$$

Alternatively, the expression for power can be obtained from complex power.

$$\text{Complex Power, } \bar{S} = \bar{V}\,\bar{I}^* = V\angle 0^\circ \times (I\angle 90^\circ)^*$$

$$= V\angle 0^\circ \times I\angle -90^\circ$$

$$= VI\angle -90^\circ$$

We know that, $|\bar{S}| = S = VI$ and $\angle\bar{S} = \phi = -90^\circ$

$$\therefore \text{ Power, } P = S\cos\phi = VI\cos(-90^\circ) = 0$$

$$\text{Reactive power, } Q = S\sin\phi = VI\sin(-90^\circ) = -VI$$

Capacitance has only reactive power and "*active power in pure capacitance is zero*". The reactive power of capacitance is negative, which means that it delivers reactive power.

4.13 Impedance

"***Impedance** is defined as the ratio of (sinusoidal) voltage and current*". It is a frequency domain parameter but not a sinusoidal quantity. "***Impedance** is also defined as the total opposition offered to flow of (sinusoidal) current*". Hence, impedance is measured in Ohms (same as the unit of resistance).

Impedance is a complex quantity and denoted by $\bar{Z}$. The real part of impedance is resistance and the imaginary part of impedance is reactance. The unit of resistance, reactance and impedance are Ohm. There are two types of reactances, namely., the inductive reactance and capacitive reactance. Inductive reactance is denoted by X_L and equal to ωL. Capacitive reactance is denoted by X_C and equal to $1/\omega C$. Inductive and capacitive reactances have the exact opposite behaviours. Therefore, when expressed as a complex quantity, inductive reactance takes a positive value and capacitive reactance takes a negative value [Refer to equations (4.31) and (4.36)].

$$\therefore \text{ Impedance, } \bar{Z} = R + jX \qquad ..... (4.38)$$

where, R = Resistance

X = Reactance

Also, $\bar{Z} = R + jX = R + jX_L$; when reactance is inductive.

$\bar{Z} = R + jX = R - jX_C$; when reactance is capacitive.

$\bar{Z} = R + jX = R + j(X_L - X_C)$; when reactance is the sum of inductance and capacitance.

The symbol used to represent impedance is a rectangle as shown in Fig. 4.32. Impedance is connected to other parts of circuits using resistance-less wires.

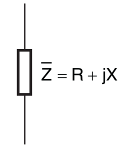


***Fig. 4.32 :** Symbol for impedance.*

The magnitude of impedance is denoted by *Z* (i.e., without an overbar). The argument of impedance is called **impedance angle** and it is denoted by θ.

In equation (4.38), complex impedance is expressed in rectangular form. It can be expressed in polar form as shown below:

$$\bar{Z} = R + jX = \sqrt{R^2 + X^2} \angle \tan^{-1}\frac{X}{R} = Z\angle\theta \qquad ..... (4.39)$$

$$\text{where, } Z = |\bar{Z}| = \sqrt{R^2 + X^2} \quad \text{and} \quad \theta = \angle\bar{Z} = \tan^{-1}\frac{X}{R}$$

$$\therefore \text{ Magnitude of impedance, } Z = |\bar{Z}| = \sqrt{R^2 + X^2}$$

$$\text{Impedance angle, } \theta = \angle\bar{Z} = \tan^{-1}\frac{X}{R}$$

Since impedance is a complex quantity it can be represented as a point in a complex plane with polar coordinates Z and θ, as shown in Fig. 4.33. The line joining the origin and $\bar{Z}$ will be a vector of length Z and making an angle θ with the reference. Now, the vector $\bar{Z}$ can be resolved into horizontal and vertical components. The horizontal component is resistance R and vertical component is reactance X. The right-angled triangle formed by R, X and Z is called the **impedance triangle**.

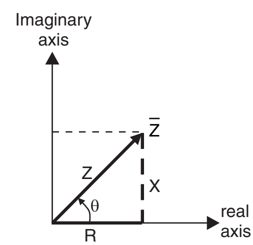


***Fig. 4.33 :** Impedance triangle.*

4.13.1 Impedance Connected to Sinusoidal Source

Consider an impedance $\bar{Z}$ connected to an ac source of voltage $\bar{V}$ volts rms value as shown in Fig. 4.34. Since the impedance is connected across (or parallel to) the source, the voltage across the impedance is also $\bar{V}$ volts.

By Ohm's law, the current through the impedance is given by,

$$\bar{I} = \frac{\bar{V}}{\bar{Z}} \qquad ..... (4.40)$$

Let, $\bar{V}$ be the reference phasor and θ be the impedance angle.

$$\therefore \quad \bar{V} = V\angle 0^\circ \quad \text{and} \quad \bar{Z} = Z\angle \pm\theta \qquad ..... (4.41)$$

(+for inductive reactance and – for capacitive reactance)

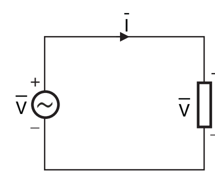


***Fig. 4.34 :** Impedance connected to an ac source.*

From equations (4.40) and (4.41), we can write,

$$\bar{I} = \frac{\bar{V}}{\bar{Z}} = \frac{V\angle 0^\circ}{Z\angle \pm\theta} = \frac{V}{Z}\angle \mp\theta = I\angle \mp\theta$$

$$\text{where, } I = \frac{V}{Z}$$

The **power factor angle** is the phase difference between $\overline{V}$ and $\overline{I}$. Let, ϕ be the phase difference between $\overline{V}$ and $\overline{I}$.

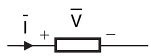
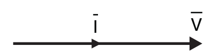
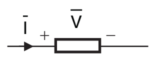
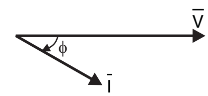
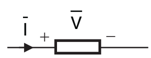
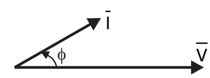
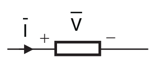
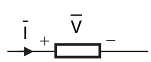
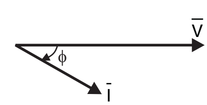
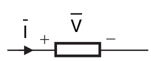
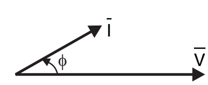
Now, $\phi = \angle\overline{V} - \angle\overline{I} = 0 - (\mp\theta) = \pm\theta$

"It is interesting to observe that the power factor angle is the same as that for the impedance angle".

From the above analysis, we can say that the current through the impedance leads or lags the voltage by an angle ϕ.

Impedance is a combination of resistances and reactances. Depending on the combination, the phase of current may be in-phase or leading or lagging the voltage.

Table 4.2 : Various Combinations of Resistance and Reactance

Impedance, $\overline{Z}$	**Phase of current with respect to voltage**	**Phasor diagram of $\overline{V}$ and $\overline{I}$**
 $\overline{Z} = R$	$\overline{I}$ in-phase with $\overline{V}$	
 $\overline{Z} = R + j X_L$	$\overline{I}$ lags $\overline{V}$	
 $\overline{Z} = R - j X_C$	$\overline{I}$ leads $\overline{V}$	
 $\overline{Z} = R + j X_L - jX_C$ and $X_L = X_C$ (This condition is called resonance)	$\overline{I}$ in-phase with $\overline{V}$	
 $\overline{Z} = R + j X_L - jX_C$ and $X_L > X_C$	$\overline{I}$ lags $\overline{V}$	
 $\overline{Z} = R + j X_L - jX_C$ and $X_L < X_C$	$\overline{I}$ leads $\overline{V}$	

4.14 Conductance, Susceptance and Admittance

In certain circuits, inverse parameters are useful for analysis. The inverse of resistance is **conductance**, the inverse of reactance is **susceptance** and the inverse of impedance is **admittance**. The MKS unit of conductance, susceptance and admittance is **mho**, (Ohm spelt in the reverse order). The SI unit of conductance, susceptance and admittance is **Siemens**, S. The circuit symbol used for an inverse parameter is the same as that for the original (or parent) parameter. The letters used to denote conductance, susceptance and admittance are G, B and Y, respectively.

$$\text{Conductance, } G = \frac{1}{R}\ ; \qquad \text{Susceptance, } B = \frac{1}{X}\ ; \qquad \text{Admittance, } \overline{Y} = \frac{1}{Z}$$

4.14.1 Conductance

The relationship between voltage and current in a conductance can be obtained from Ohm's law. Consider a conductance carrying a current of I amperes as shown in Fig. 4.35. The conductance of the element is G and resistance is R = 1/G. By Ohm's law, the voltage across the element is given by the product of current and resistance.

$$\therefore \ \text{Voltage across conductance, } V = IR = \frac{I}{G}$$

$$\therefore \ I = VG \quad \text{or} \quad V = \frac{I}{G} \qquad ..... (4.42)$$

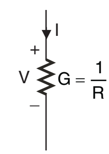


Fig. 4.35 : *Conductance.*

Equation (4.42), gives the relation between voltage and current in a conductance.

4.14.2 Admittance

Admittance is the inverse of impedance and so "***admittance*** *is defined as the ratio of (sinusoidal) current and voltage*". The unit of admittance is mho or Siemens. Admittance is a complex quantity and denoted by $\overline{Y}$. The real part of admittance is **conductance** and the imaginary part of admittance is **susceptance**. The unit of conductance, susceptance and admittance is mho or Siemens.

There are two types of susceptance, namely., inductive susceptance and capacitive susceptance. Inductive susceptance is denoted by B_L and equal to $1/\omega L$. Capacitive susceptance is denoted by B_C and equal to ωC. Inductive and capacitive susceptances have the exact opposite behaviours. Therefore, when expressed as a complex quantity, the inductive susceptance takes a negative value and the capacitive susceptance takes a positive value [Refer equations (4.31) and (4.36)].

$$\text{Admittance, } \overline{Y} = G + jB \qquad ..... (4.43)$$

where, G = Conductance

B = Susceptance

Also, $\overline{Y} = G + jB = G - jB_L$; when susceptance is inductive.

$\overline{Y} = G + jB = G + jB_C$; when susceptance is capacitive.

$\overline{Y} = G + jB = G + j(-B_L + B_C)$; when susceptance is the sum of inductance and capacitance.

The symbol used to represent admittance is a rectangle as shown in Fig. 4.36. Admittance is connected to other parts of circuits using resistance-less wires.The magnitude of admittance is denoted by Y (i.e., without overbar). The argument of admittance is called **admittance angle** and it is denoted by θ.

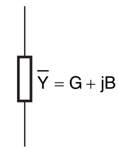


***Fig. 4.36 :** Symbol for admittance.*

In equation (4.43), complex admittance is expressed in rectangular form. It can be expressed in polar form as shown below:

$$\bar{Y} = G + jB = \sqrt{G^2 + B^2} \angle \tan^{-1}\frac{B}{G} = Y\angle\theta$$

$$\text{where, } Y = |\bar{Y}| = \sqrt{G^2 + B^2} \quad \text{and} \quad \theta = \angle\bar{Y} = \tan^{-1}\frac{B}{G}$$

$$\therefore \text{ Magnitude of admittance, } Y = |\bar{Y}| = \sqrt{G^2 + B^2}$$

$$\text{Admittance angle, } \theta = \angle\bar{Y} = \tan^{-1}\frac{B}{G}$$

Since the admittance is a complex quantity it can be represented as a point in a complex plane with polar coordinates Y and θ, as shown in Fig. 4.37. The line joining the origin and $\bar{Y}$ will be a vector of length Y and making an angle θ with the reference. Now, the vector $\bar{Y}$ can be resolved into horizontal and vertical components. The horizontal component is conductance G and the vertical component is susceptance B. The right-angled triangle formed by G, B and Y is called the **admittance triangle**.

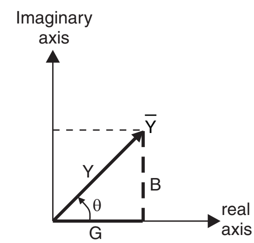


***Fig. 4.37 :** Admittance triangle.*

4.14.3 Admittance Connected to Sinusoidal Source

Consider an admittance $\bar{Y}$ connected to an ac source of voltage $\bar{V}$ volts rms value as shown in Fig. 4.38. Let, $\bar{Z} = 1/\bar{Y}$ or $\bar{Y} = 1/\bar{Z}$. Since the admittance is connected across (or parallel to) the source, the voltage across the admittance is also $\bar{V}$ volts.

By Ohm's law, the current through the admittance is given by,

$$\bar{I} = \frac{\bar{V}}{\bar{Z}} = \frac{\bar{V}}{1/\bar{Y}} = \bar{V}\,\bar{Y} \qquad \text{..... (4.44)}$$

Let, $\bar{V}$ be the reference vector and θ be the admittance angle.

$$\therefore \quad \bar{V} = V\angle 0^\circ \quad \text{and} \quad \bar{Y} = Y\angle\mp\theta \qquad \text{..... (4.45)}$$

(− for inductive susceptance and + for capacitive susceptance).

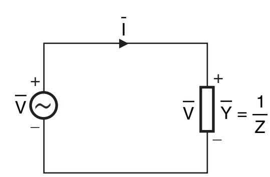


***Fig. 4.38 :** Admittance connected to an ac source.*

From equations (4.44) and (4.45), we can write,

$$\bar{I} = \bar{V}\,\bar{Y} = V\angle 0^\circ \; Y\angle\mp\theta = VY\angle\mp\theta = I\angle\mp\theta$$

$$\text{where, } I = VY$$

The **power factor angle** is the phase difference between $\overline{V}$ and $\overline{I}$. Let, ϕ be the phase difference between $\overline{V}$ and $\overline{I}$.

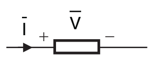
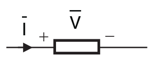
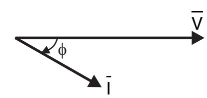
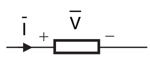
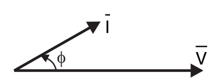
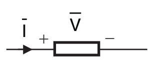
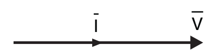
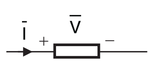
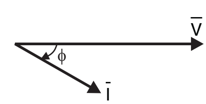
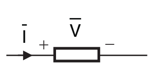
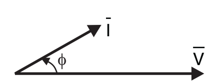
Now, $\phi = \angle\overline{V} - \angle\overline{I} = 0^\circ - (\mp\theta) = \pm\theta$

"*It is interesting to observe that the power factor angle is the same as that for the admittance angle*".

From the above analysis, we can say that the current through the admittance leads or lags the voltage by an angle ϕ.

Admittance is a combination of conductances and susceptances. Depending on the combination, the phase of current may be in-phase or leading or lagging the voltage.

Table 4.3 : Various Combinations of Conductance and Susceptance

Admittance, $\overline{Y}$	**Phase of current with respect to voltage**	**Phasor diagram of $\overline{V}$ and $\overline{I}$**
 $\overline{Y} = G$	$\overline{I}$ in-phase with $\overline{V}$	
 $\overline{Y} = G - j\,B_L$	$\overline{I}$ lags $\overline{V}$	
 $\overline{Y} = G + j\,B_C$	$\overline{I}$ leads $\overline{V}$	
 $\overline{Y} = G - jB_L + j\,B_C$ and $B_L = B_C$ (This condition is also called resonance)	$\overline{I}$ in-phase with $\overline{V}$	
 $\overline{Y} = G - jB_L + j\,B_C$ and $B_L > B_C$	$\overline{I}$ lags $\overline{V}$	
 $\overline{Y} = G - jB_L + j\,B_C$ and $B_L < B_C$	$\overline{I}$ leads $\overline{V}$	

4.15 KVL, KCL and Ohm's Law Applied to AC Circuits

In ac circuits, the total opposition to the flow of sinusoidal current is called **impedance** $\bar{Z}$. **Impedance** is given by the ratio of sinusoidal voltage and current.

$$\boxed{\therefore \bar{Z} = \frac{\bar{V}}{\bar{I}}} \quad \boxed{\text{or } \bar{V} = \bar{I}\,\bar{Z}} \quad \boxed{\text{or } \bar{I} = \frac{\bar{V}}{\bar{Z}}}$$

The above equation is called **Ohm's law of ac circuits**. From above equation we can say that when a current $\bar{I}$ flows through an impedance $\bar{Z}$, the voltage $\bar{V}$ across the impedance is given by the product of current and impedance, i.e., $\bar{V} = \bar{I}\,\bar{Z}$.

In ac circuits, voltage and current will vary with time. Hence, while applying KCL and KVL to ac circuits we have to consider the signs of voltage and current at a particular time instant. The sign conventions for ac circuits are applicable to a particular time instant.

4.16 Current and Voltage Division Rules for Impedances

4.16.1 Current Division in Parallel Connected Impedances

Consider two impedances $\bar{Z}_1$ and $\bar{Z}_2$ in parallel and connected to an ac source of $\bar{V}$ volts as shown in Fig. 4.39. Let, $\bar{I}$ be the current supplied by the source and $\bar{I}_1$ and $\bar{I}_2$ be the current through $\bar{Z}_1$ and $\bar{Z}_2$, respectively. Since the impedances are parallel to the source, the voltage across them will be $\bar{V}$ volts.

Equations (4.46) and (4.47) can be used to solve currents in parallel connected impedances in terms of the total current drawn by the parallel combination and the values of individual impedances. Hence, these equations are called the **current division rule**.

$$\boxed{\bar{I}_1 = \bar{I} \times \frac{\bar{Z}_2}{\bar{Z}_1 + \bar{Z}_2}} \quad ..... (4.46)$$

$$\boxed{\bar{I}_2 = \bar{I} \times \frac{\bar{Z}_1}{\bar{Z}_1 + \bar{Z}_2}} \quad ..... (4.47)$$

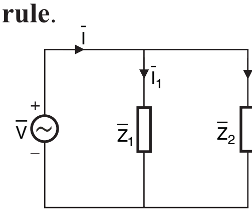


Fig. 4.39 : *Impedances in parallel.*

The following equation will be helpful to remember the current division rule.

In two parallel connected impedances,

$$\boxed{\text{Current through one of the impedances} = \frac{\text{Total current drawn by parallel combination} \times \text{Value of the other impedance}}{\text{Sum of the individual impedances}}}$$

4.16.2 Voltage Division in Series Connected Impedances

Consider two impedances $\bar{Z}_1$ and $\bar{Z}_2$ in series and connected to an ac source of $\bar{V}$ volts as shown in Fig. 4.40. Let, $\bar{I}$ be the current supplied by the source and $\bar{V}_1$ and $\bar{V}_2$ be the voltages across $\bar{Z}_1$ and $\bar{Z}_2$, respectively. Since the impedances are in series, the current through them will be $\bar{I}$ amperes.

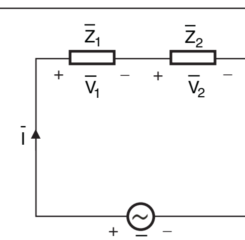


Fig. 4.40 : *Impedances in series.*

Equations (4.48) and (4.49) can be used to solve the voltages in series connected impedances in terms of the total voltage across the series combination and the values of individual impedances. Hence, these equations are called **voltage division rule**.

$$\boxed{\bar{V}_1 = \bar{V} \times \frac{\bar{Z}_1}{\bar{Z}_1 + \bar{Z}_2}} \qquad ..... (4.48)$$

$$\boxed{\bar{V}_2 = \bar{V} \times \frac{\bar{Z}_2}{\bar{Z}_1 + \bar{Z}_2}} \qquad ..... (4.49)$$

The following equation will be helpful to remember the voltage division rule.

In two series connected impedances,

$$\boxed{\text{Voltage across one of the impedances} = \frac{\begin{matrix}\text{Total voltage across}\\ \text{series combination}\end{matrix} \times \begin{matrix}\text{Value of the}\\ \text{impedance}\end{matrix}}{\text{Sum of the individual impedances}}}$$

4.17 Solved Problems in Single Phase Circuits

EXAMPLE 4.1

Find the node voltages $\bar{V}_1$ and $\bar{V}_2$ in the circuit shown in Fig. 1.

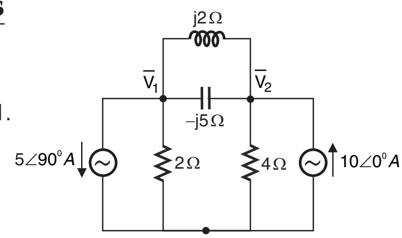


Fig. 1.

SOLUTION

In the circuit of Fig. 1, the reference node is 0. The voltage of the reference node is zero. To find the voltages $\bar{V}_1$ and $\bar{V}_2$ with respect to the reference node, write two KCL equations at these nodes and solve the two equations for a unique solution.

With reference to Fig. 2, at node-1, using KCL we get,

Currents leaving node-1 : $5\angle 90^\circ, \dfrac{\bar{V}_1}{2}, \dfrac{\bar{V}_1 - \bar{V}_2}{j2}, \dfrac{\bar{V}_1 - \bar{V}_2}{-j5}$

Current entering node-1 : Nil

$$\therefore\ 5\angle 90^\circ + \frac{\bar{V}_1}{2} + \frac{\bar{V}_1 - \bar{V}_2}{j2} + \frac{\bar{V}_1 - \bar{V}_2}{-j5} = 0$$

$$j5 + \frac{\bar{V}_1}{2} + \frac{\bar{V}_1}{j2} - \frac{\bar{V}_2}{j2} - \frac{\bar{V}_1}{j5} + \frac{\bar{V}_2}{j5} = 0$$

$$\therefore\ \left(\frac{1}{2} + \frac{1}{j2} - \frac{1}{j5}\right)\bar{V}_1 + \left(-\frac{1}{j2} + \frac{1}{j5}\right)\bar{V}_2 = -j5$$

$$(0.5 - j0.3)\,\bar{V}_1 + j0.3\,\bar{V}_2 = -j5 \qquad ..... (1)$$

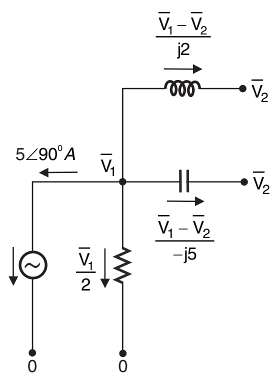


Fig. 2.

With reference to Fig. 3, at node-2, using KCL we get,

Currents leaving node-2 : $\dfrac{\overline{V}_2 - \overline{V}_1}{j2}, \ \dfrac{\overline{V}_2}{4}, \ \dfrac{\overline{V}_2 - \overline{V}_1}{-j5}$

Current entering node-2 : $10\angle 0^\circ A$

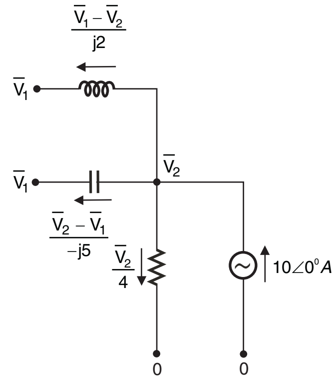


Fig. 3.

$$\therefore \ \frac{\overline{V}_2 - \overline{V}_1}{j2} + \frac{\overline{V}_2}{4} + \frac{\overline{V}_2 - \overline{V}_1}{-j5} = 10\angle 0^\circ$$

$$\frac{\overline{V}_2}{j2} - \frac{\overline{V}_1}{j2} + \frac{\overline{V}_2}{4} - \frac{\overline{V}_2}{j5} + \frac{\overline{V}_1}{j5} = 10$$

$$\left(-\frac{1}{j2} + \frac{1}{j5}\right)\overline{V}_1 + \left(\frac{1}{j2} + \frac{1}{4} - \frac{1}{j5}\right)\overline{V}_2 = 10$$

$$j0.3\,\overline{V}_1 + (0.25 - j0.3)\,\overline{V}_2 = 10$$

$$\therefore \ (0.25 - j0.3)\,\overline{V}_2 = 10 - j0.3\,\overline{V}_1$$

$$\therefore \ \overline{V}_2 = \frac{10}{0.25 - j0.3} - \frac{j0.3}{0.25 - j0.3}\,\overline{V}_1$$

$$= 16.3934 + j19.6721 + (0.5902 - j0.4918)\,\overline{V}_1 \qquad \text{..... (2)}$$

In order to solve $\overline{V}_1$, let us substitute for $\overline{V}_2$ from equation (2) in equation (1).

$$\therefore \ (0.5 - j0.3)\,\overline{V}_1 + j0.3\left[16.3934 + j19.6721 + (0.5902 - j0.4918)\,\overline{V}_1\right] = -j5$$

$$(0.5 - j0.3)\,\overline{V}_1 - 5.9016 + j4.918 + (0.1475 + j0.1771)\,\overline{V}_1 = -j5$$

$$(0.5 - j0.3 + 0.1475 + j0.1771)\,\overline{V}_1 = -j5 + 5.9016 - j4.918$$

$$(0.6475 - j0.1229)\,\overline{V}_1 = 5.9016 - j9.918$$

$$\therefore \ \overline{V}_1 = \frac{5.9016 - j9.918}{0.6475 - j0.1229} = 11.6037 - j13.1149\,V = 17.5113\angle -48.5^\circ\,V$$

From equation (2), we get,

$$\overline{V}_2 = 16.3934 + j19.6721 + (0.5902 - j0.4918)\,\overline{V}_1$$

$$= 16.3934 + j19.6721 + (0.5902 - j0.4918) \times (11.6037 - j13.1149)$$

$$= 16.792 + j6.225\,V$$

$$= 17.9087\angle 20.3^\circ\,V$$

RESULT

The node voltages are,

$$\overline{V}_1 = 17.5113\angle -48.5^\circ\,V$$

$$\overline{V}_2 = 17.9087\angle 20.3^\circ\,V$$

EXAMPLE 4.2

In the circuit shown in Fig. 1, find the node voltages $\bar{V}_1$ and $\bar{V}_2$ and the current $\bar{I}_s$ supplied by the voltage source.

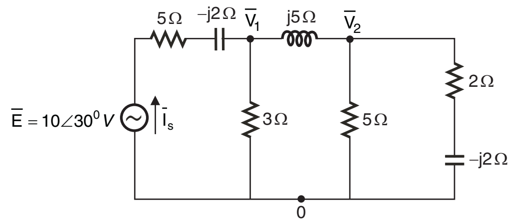


Fig. 1.

SOLUTION

In the circuit of Fig. 1, the reference node is 0. The voltage of the reference node is zero. To find the voltages $\bar{V}_1$ and $\bar{V}_2$ with respect to the reference node, write two KCL equations at these nodes and solve the two equations for a unique solution.

Given that, $\bar{E} = 10\angle 30^\circ V = 10\,(\cos 30^\circ + j\sin 30^\circ) = 8.6602 + j5\,V$ .....(1)

With reference to Fig. 2, using KCL we get,

Currents leaving node-1 : $\dfrac{\bar{V}_1 - \bar{E}}{5 - j2}, \dfrac{\bar{V}_1 - \bar{V}_2}{j5}, \dfrac{\bar{V}_1}{3}$

Current entering node-1 : Nil

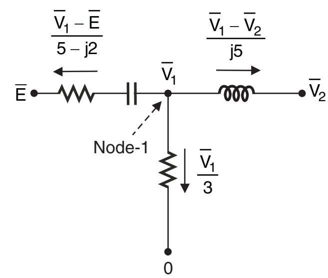


Fig. 2.

$$\therefore \quad \frac{\bar{V}_1 - \bar{E}}{5 - j2} + \frac{\bar{V}_1 - \bar{V}_2}{j5} + \frac{\bar{V}_1}{3} = 0$$

$$\frac{\bar{V}_1}{5 - j2} - \frac{\bar{E}}{5 - j2} + \frac{\bar{V}_1}{j5} - \frac{\bar{V}_2}{j5} + \frac{\bar{V}_1}{3} = 0$$

$$\left(\frac{1}{5 - j2} + \frac{1}{j5} + \frac{1}{3}\right)\bar{V}_1 - \frac{\bar{V}_2}{j5} = \frac{\bar{E}}{5 - j2}$$

$$\left(\frac{1}{5 - j2} + \frac{1}{j5} + \frac{1}{3}\right)\bar{V}_1 - \frac{\bar{V}_2}{j5} = \frac{8.6602 + j5}{5 - j2}$$

Using equation (1)

$$(0.5057 - j0.131)\,\bar{V}_1 + j0.2\,\bar{V}_2 = 1.1483 + j1.4593 \qquad \text{..... (2)}$$

With reference to Fig. 3, at node-2, using KCL we get,

Currents leaving node-2 : $\dfrac{\bar{V}_2 - \bar{V}_1}{j5}, \dfrac{\bar{V}_2}{5}, \dfrac{\bar{V}_2}{2 - j2}$

Current entering node-2 : Nil

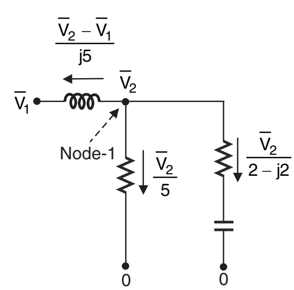


Fig. 3.

$$\therefore \quad \frac{\bar{V}_2 - \bar{V}_1}{j5} + \frac{\bar{V}_2}{5} + \frac{\bar{V}_2}{2 - j2} = 0$$

$$\frac{\bar{V}_2}{j5} - \frac{\bar{V}_1}{j5} + \frac{\bar{V}_2}{5} + \frac{\bar{V}_2}{2 - j2} = 0$$

$$\left(\frac{1}{j5} + \frac{1}{5} + \frac{1}{2 - j2}\right)\bar{V}_2 - \frac{\bar{V}_1}{j5} = 0$$

$$(0.45 + j0.05)\bar{V}_2 + j0.2\,\bar{V}_1 = 0$$

$$\therefore \quad \bar{V}_2 = \frac{-j0.2}{0.45 + j0.05}\bar{V}_1 \;\Rightarrow\; \bar{V}_2 = (-0.0488 - j0.439)\bar{V}_1 \qquad \text{..... (3)}$$

In order to solve $\bar{V}_1$, let us substitute for $\bar{V}_2$ from equation (3) in equation (2).

$$\therefore\ (0.5057 - j0.131)\,\bar{V}_1 + j0.2\,(-0.0488 - j0.439)\bar{V}_1 = 1.1483 + j1.4593$$

$$[\,0.5057 - j0.131 + j0.2\,(-0.0488 - j0.439\,)\,]\bar{V}_1 = 1.1483 + j1.4593$$

$$[\,0.5935 - j0.1408\,]\,\bar{V}_1 = 1.1483 + j1.4593$$

$$\therefore\ \bar{V}_1 = \frac{1.1483 + j1.4593}{0.5935 - j0.1408}$$

$$= 1.2795 + j2.7623\,V \qquad \text{..... (4)}$$

$$= 3.0442\angle 65.1^\circ\,V$$

In order to solve $\bar{V}_2$, let us substitute for $\bar{V}_1$ from equation (4) in equation (3).

$$\therefore\ \bar{V}_2 = (\,-0.0488 - j0.439\,) \times (\,1.2795 + j2.7623\,)$$

$$= 1.1502 - j0.6965\,V$$

$$= 1.3446\angle -31.2^\circ\,V \qquad \text{..... (5)}$$

With reference to Fig. 1, we can write,

$$\text{Current supplied by the voltage source, } \bar{I}_s = \frac{\bar{E} - \bar{V}_1}{5 - j2}$$

$$= \frac{8.6602 + j5 - (1.2795 + j2.7623)}{5 - j2}$$

$$= 1.1182 + j0.8948\,A = 1.4321\angle 38.7^\circ\,A$$

RESULT

The node voltages are,

$$\bar{V}_1 = 3.0442\angle 65.1^\circ\,V$$

$$\bar{V}_2 = 1.3446\angle -31.2^\circ\,V$$

Current delivered by the source, $\bar{I}_s = 1.4321\angle 38.7^\circ\,A$

EXAMPLE 4.3

Determine the current $\bar{I}_2$ in the circuit shown in Fig. 1.

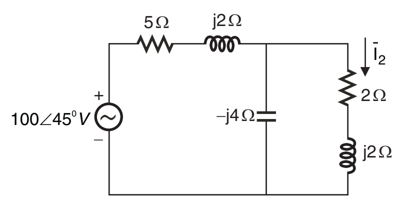


Fig. 1.

SOLUTION

Let, $\bar{I}_T$ be the total current supplied by the source. This current $\bar{I}_T$ divides into $\bar{I}_1$ and $\bar{I}_2$ and flows through parallel impedances $-j4\,\Omega$ and $2 + j2\,\Omega$ as shown in Fig. 2. The current $\bar{I}_T$ is given by the ratio of source voltage and total impedance at the source terminals.

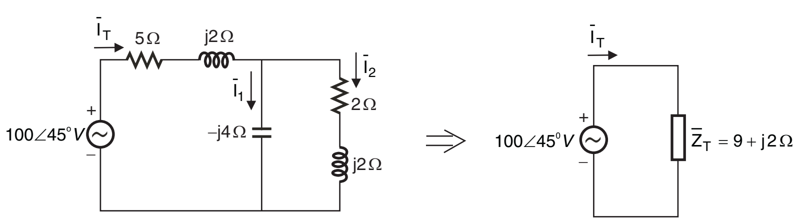


Fig. 2.

The total impedance $\bar{Z}_T$ at the source terminal is given by the parallel combination of $-j4\,\Omega$ and $2 + j2\,\Omega$ in series with $5 + j2\,\Omega$.

$$\therefore \quad \bar{Z}_T = (5 + j2) + [-j4 \,||\, (2 + j2)] = (5 + j2) + \left[\frac{(-j4) \times (2 + j2)}{-j4 + 2 + j2}\right]$$

$$= 5 + j2 + 4 + j0 = 9 + j2\,\Omega$$

Now, $\bar{I}_T = \dfrac{100\angle 45^\circ}{9 + j2} = \dfrac{100\,(\cos 45^\circ + j\sin 45^\circ)}{9 + j2} = 9.1508 + j5.8232\ A$

By current division rule,

$$\bar{I}_2 = \bar{I}_T \times \frac{-j4}{-j4 + 2 + j2} = \frac{(9.1508 + j5.8232) \times (-j4)}{2 - j2}$$

$$= 14.974 - j3.3276 = 15.3393\angle -12.5^\circ A$$

EXAMPLE 4.4

Determine the current $\bar{I}_L$ in the circuit shown in Fig. 1.

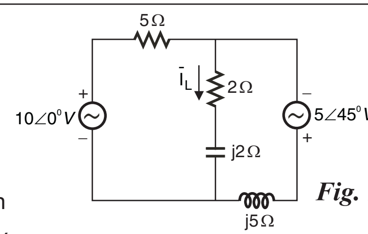


Fig. 1.

SOLUTION

Let us mark the nodes of the circuit as A, B, C and D as shown in Fig. 2. Let, $\bar{I}_1$ and $\bar{I}_2$ be the current delivered by $10\angle 0^\circ\,V$ and $5\angle 45^\circ\ V$ sources, respectively, from their positive end as shown in Fig. 2.

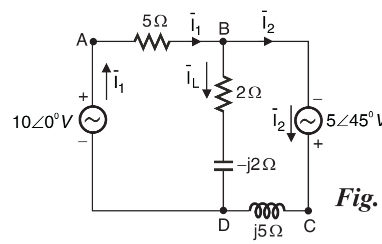


Fig. 2.

By KCL at node-B, we can write,

$$\bar{I}_L + \bar{I}_2 = \bar{I}_1 \qquad \text{..... (1)}$$

The currents $\bar{I}_1$ and $\bar{I}_2$ can be solved by writing KVL equations in the closed paths ABDA and BCDB.

Consider the closed path ABDA shown in Fig. 3. By KVL, we can write,

$$5\bar{I}_1 + (2 - j2)\bar{I}_L = 10\angle 0^\circ$$

$$5(\bar{I}_L + \bar{I}_2) + (2 - j2)\bar{I}_L = 10$$

$$5\bar{I}_2 + (7 - j2)\,\bar{I}_L = 10$$

$$\therefore\ 5\bar{I}_2 = 10 - (7 - j2)\,\bar{I}_L$$

$$\therefore\ \bar{I}_2 = \frac{10}{5} - \frac{7 - j2}{5}\,\bar{I}_L$$

$$= 2 - (1.4 - j0.4)\,\bar{I}_L \qquad \text{.....(2)}$$

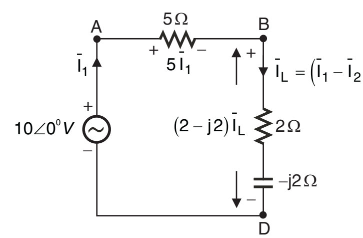


Fig. 3.

Using equation (1)

Consider the closed path BCDB shown in Fig. 4. By KVL, we can write,

$$j5\bar{I}_2 = 5\angle 45^\circ + (2 - j2)\bar{I}_L$$

$$j5\bar{I}_2 - (2 - j2)\bar{I}_L = 5(\cos 45^\circ + j\sin 45^\circ)$$

$$j5[2 - (1.4 - j0.4)]\bar{I}_L - (2 - j2)\bar{I}_L = 5(\cos 45^\circ + j\sin 45^\circ)$$

$$j10 - (2 + j7)\bar{I}_L - (2 - j2)\bar{I}_L = 3.5355 + j3.5355$$

$$(-4 - j5)\bar{I}_L = 3.5355 + j3.5355 - j10$$

Using equation (1)

$$\therefore\ \bar{I}_L = \frac{3.5355 + j3.5355 - j10}{-4 - j5}$$

$$= 0.4434 + j1.0618\ A$$

$$= 1.1507\angle 67.3^\circ\ A$$

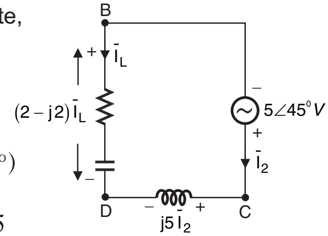


Fig. 4.

EXAMPLE 4.5

(AU June'16, 8 Marks)

In the circuit shown in Fig. 1, Find total current $\bar{I}_T$ and power factor. Take frequency of supply as 100 *Hz*.

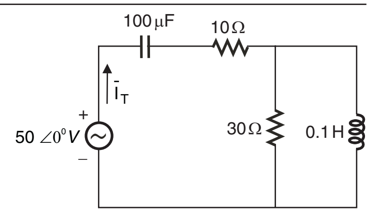


Fig. 1.

SOLUTION

Given that, V = 50 *V*, f = 100 *Hz*, L = 0.1 *H* and C = 100 μ*F*

Capacitive reactance, $X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 100 \times 100 \times 10^{-6}} = 15.9155\ \Omega$

Inductive reactance, $X_L = 2\pi fL = 2\pi \times 100 \times 0.1 = 62.8319\ \Omega$

The frequency domain representation of the circuit is shown in Fig. 2.

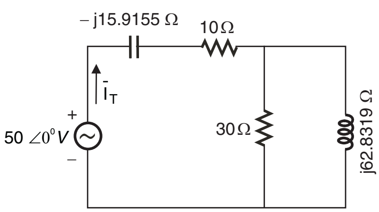


Fig. 2.

⇒

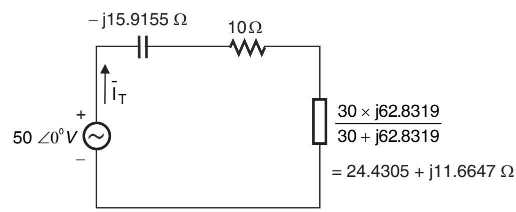


Fig. 3.

With reference to Fig.4, by Ohm's law,

$$\bar{I}_T = \frac{50}{34.4305 - j4.2508}$$

$$= 1.4304 + j0.1766\ A$$

$$= 1.4413\angle 7^\circ\ A$$

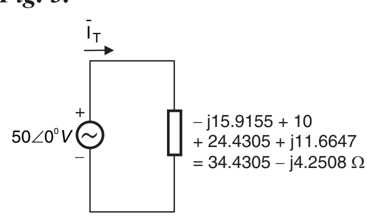


Fig. 4.

Power factor angle, $\phi = \angle\bar{V} - \angle\bar{I}_T = 0^\circ - 7^\circ = -7^\circ$

$\therefore$ Power factor = $\cos\phi = \cos(-7^\circ) = 0.9925$ *lead*

Since the current leads the voltage, the power factor is lead.

EXAMPLE 4.6

A series combination of 10 Ω resistance and 50 *mH* inductance is connected to a 220 *V*, 50 *Hz* supply. Estimate the current, active power, reactive power and apparent power. Also estimate the voltage across R and L and draw the phasor diagram.

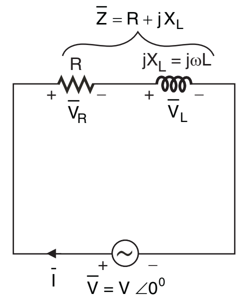


Fig. 1.

SOLUTION

Given that, $V = 220\,V, \quad f = 50\,Hz$

$R = 10\,\Omega, \quad L = 50\,mH$

The RL series circuit excited by a sinusoidal source is shown in Fig. 1.

Inductive reactance $= jX_L = j\omega L = j2\pi fL = j2\pi \times 50 \times 50 \times 10^{-3} = j15.708\,\Omega$

Impedance, $\bar{Z} = R + jX_L = 10 + j15.708\,\Omega = 18.621\angle 57.5^\circ\,\Omega$

Let the supply voltage be the reference phasor.

$$\therefore \ \bar{V} = V\angle 0^\circ = 220\angle 0^\circ\,V$$

Let, $\bar{I}$ be the current through the RL circuit. Now by Ohm's law,

$$\text{Current, } \bar{I} = \frac{\bar{V}}{\bar{Z}} = \frac{220\angle 0^\circ}{18.621\angle 57.5^\circ} = 11.8146\angle -57.5^\circ A$$

$$\therefore \ I = |\bar{I}| = 11.8146\,A$$

Power factor angle, $\phi = \angle\bar{V} - \angle\bar{I} = 0^\circ - (-57.5^\circ) = 57.5^\circ$

$\therefore$ Power factor $= \cos\phi = \cos 57.5^\circ = 0.5373$ lag

Since the current lags the voltage, the power factor is lag.

Apparent power, $S = VI = 220 \times 11.8146 = 2599.2\,VA$

$$= \frac{2599.2}{1000}\,kVA = 2.5992\,kVA$$

Active power, $P = VI\cos\phi = 220 \times 11.8146 \times \cos 57.5^\circ$

(or power)

$$= 1396.6\,W = \frac{1396.6}{1000}\,kW = 1.3966\,kW$$

Reactive power, $Q = VI\sin\phi = 220 \times 11.8146 \times \sin 57.5^\circ$

$$= 2192.2\,VAR = \frac{2192.2}{1000}\,kVAR = 2.1922\,kVAR$$

Voltage across resistance, $\bar{V}_R = \bar{I} \times R = 11.8146\angle{-57.5^\circ} \times 10$

$= 118.146\angle{-57.5^\circ}\ V$

Voltage across inductance, $\bar{V}_L = \bar{I} \times jX_L = 11.8146\angle{-57.5^\circ} \times j15.708$

$= 11.8146\angle{-57.5^\circ} \times 15.708\angle 90^\circ$

$= 185.5837\angle 32.5^\circ\ V$

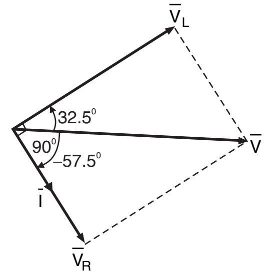


Fig. 2 : Phasor diagram.

The phasor diagram of the RL circuit with $\bar{V}$ as the reference phasor is shown in Fig. 2.

EXAMPLE 4.7

A current of $50\angle{-30^\circ} A$ is flowing through a circuit which consists of series connected elements, when excited by a source of $230\angle 45^\circ\ V$, $50\ Hz$. Determine the elements of the circuit and power. Also draw the phasor diagram.

SOLUTION

Given that, $\bar{V} = 230\angle 45^\circ V$ and $\bar{I} = 50\angle{-30^\circ} A$

$$\therefore \text{ Impedance, } \bar{Z} = \frac{\bar{V}}{\bar{I}} = \frac{230\angle 45^\circ}{50\angle -30^\circ} = 4.6\angle 75^\circ\Omega = 1.1906 + j4.4433\,\Omega$$

Since the reactance is positive, the circuit is RL series circuit. (Also the current is lagging and so the circuit is inductive.)

We know that, $\bar{Z} = R + jX_L$

$\therefore\ R = 1.1906\,\Omega$

$X_L = 4.4433\,\Omega$

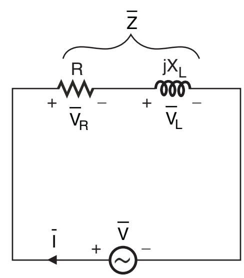


Fig. 1.

We know that, $X_L = 2\pi f L$

$$\therefore\ L = \frac{X_L}{2\pi f} = \frac{4.4433}{2\pi \times 50} = 0.0141 H = 14.1 mH$$

The complex power, $\bar{S} = \bar{V}\,\bar{I}^* = 230\angle 45^\circ \times (50\angle -30^\circ)^*$

$= 230\angle 45^\circ \times 50\angle 30^\circ = 11500\angle 75^\circ = 2976.4 + j11108.1\ VA$

Also, $\bar{S} = P + jQ \Rightarrow P + jQ = 2976.4 + j11108.1$

$$\therefore \text{ Active power, } P = 2976.4\,W = \frac{2976.4}{1000} kW = 2.9764\,kW$$

$$\text{Reactive power, } Q = 11108.1\,VAR = \frac{11108.1}{1000} kVAR = 11.1081\,kVAR$$

$$\text{Apparent power, } S = |\bar{S}| = 11500\,VA = \frac{11500}{1000} kVA = 11.5\,kVA$$

The RL series circuit is shown in Fig. 1. Let, $\bar{V}_R$ and $\bar{V}_L$ be the voltage across R and L.

Now, by Ohm's law,

$\bar{V}_R = \bar{I} \times R = 50\angle{-30^\circ} \times 1.1906 = 59.53\angle{-30^\circ} V$

$\bar{V}_L = \bar{I} \times jX_L = 50\angle{-30^\circ} \times j4.4433 = 50\angle{-30^\circ} \times 4.4433\angle 90^\circ$

$= 222.165\angle 60^\circ V$

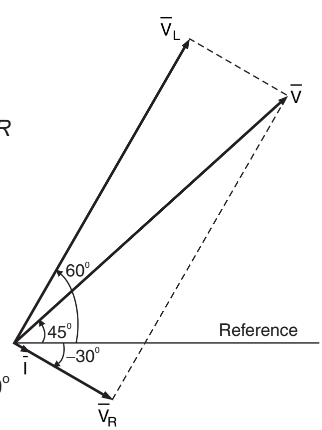


Fig. 2 : Phasor diagram.

The phasor diagram of the RL series circuit is shown in Fig. 2.

EXAMPLE 4.8

Consider an RL series circuit with an impedance angle of 50° at a frequency of 60 *Hz*. At what frequency will the magnitude of the impedance be twice the magnitude of the impedance at 60 *Hz*?

SOLUTION

Let, R = Resistance of RL series circuit

L = Inductance of RL series circuit

Case i **:** $f_1 = 60\,Hz,\ \ \omega_1 = 2\pi f_1$

Let, $\overline{Z}_1$ = Impedance of RL circuit at f_1

Now, $$\overline{Z}_1 = R + j\omega_1 L = \sqrt{R^2 + \omega_1^2 L^2}\angle \tan^{-1}\frac{\omega_1 L}{R} = Z_1\angle\theta_1$$

$$\text{where, } Z_1 = |\overline{Z}_1| = \sqrt{R^2 + \omega_1^2 L^2} \qquad .....(1)$$

$$\theta_1 = \angle\overline{Z}_1 = \tan^{-1}\frac{\omega_1 L}{R}$$

Given that, $\phi_1 = 50°$

$$\therefore\ \tan^{-1}\frac{\omega_1 L}{R} = 50° \ \Rightarrow\ \frac{\omega_1 L}{R} = \tan 50° \ \Rightarrow\ \frac{\omega_1 L}{R} = 1.1918 \qquad .....(2)$$

Case ii **:** Frequency $= f_2,\ \ \omega_2 = 2\pi f_2$

Let, f_2 = Frequency at which magnitude of impedance doubles

$\overline{Z}_2$ = Impedance of RL circuit at f_2

Now, $$\overline{Z}_2 = R + j\omega_2 L = \sqrt{R^2 + \omega_2^2 L^2}\angle \tan^{-1}\frac{\omega_2 L}{R} = Z_2\angle\theta_2$$

$$\text{where, } Z_2 = |\overline{Z}_2| = \sqrt{R^2 + \omega_2^2 L^2} \qquad .....(3)$$

$$\theta_2 = \angle\overline{Z}_2 = \tan^{-1}\frac{\omega_2 L}{R}$$

To solve for f_2 :

Given that, $Z_2 = 2Z_1$

$$\therefore\ \sqrt{R^2 + \omega_2^2 L^2} = 2 \times \sqrt{R^2 + \omega_1^2 L^2} \qquad \boxed{\text{Using equations (1) and (3)}}$$

On squaring the above equation, we get,

$$R^2 + \omega_2^2 L^2 = 4(R^2 + \omega_1^2 L^2)$$

On dividing by R^2, we get,

$$1 + \frac{\omega_2^2 L^2}{R^2} = 4\left(1 + \frac{\omega_1^2 L^2}{R^2}\right) \ \Rightarrow\ \left(\frac{\omega_2 L}{R}\right)^2 = 4\left[1 + \left(\frac{\omega_1 L}{R}\right)^2\right] - 1$$

$$\therefore\ \frac{\omega_2 L}{R} = \sqrt{4\left[1 + \left(\frac{\omega_1 L}{R}\right)^2\right] - 1} = \sqrt{4(1 + 1.1918^2) - 1} = 2.9464 \qquad \boxed{\text{Using equation (2)}}$$

Now, $$\frac{\frac{\omega_2 L}{R}}{\frac{\omega_1 L}{R}} = \frac{2.9464}{1.1918} \ \Rightarrow\ \frac{\omega_2}{\omega_1} = 2.4722 \ \Rightarrow\ \frac{2\pi f_2}{2\pi f_1} = 2.4722$$

$$\therefore\ f_2 = 2.4722 \times f_1 = 2.4722 \times 60 = 148.332\,Hz$$

RESULT

The frequency at which the magnitude of the impedance doubles = 148.332 *Hz*

Cross-Check

Let, $R = 2\,\Omega$

Here, $\dfrac{\omega_1 L}{R} = 1.1918$

$$\therefore\quad L = \frac{1.1918\,R}{\omega_1} = \frac{1.1918\,R}{2\pi f_1} = \frac{1.1918 \times 2}{2\pi \times 60} = 6.3227 \times 10^{-3}H = 6.3227 \times 10^{-3}H$$

$$Z_1 = \sqrt{R^2 + (\omega_1 L)^2} = \sqrt{R^2 + (2\pi f_1 L)^2} = \sqrt{2^2 + (2\pi \times 60 \times 6.3227 \times 10^{-3})^2} = 3.1115\,\Omega$$

$$Z_2 = \sqrt{R^2 + (\omega_2 L)^2} = \sqrt{R^2 + (2\pi f_2 L)^2} = \sqrt{2^2 + (2\pi \times 148.332 \times 6.3227 \times 10^{-3})^2} = 6.2229\,\Omega$$

$$\frac{Z_2}{Z_1} = \frac{6.2229}{3.1115} = 1.999968 = 2$$

EXAMPLE 4.9

A resistance of 16 Ω is connected in parallel to an inductance of 20 *mH* and the parallel combination is connected to an ac supply of 230 *V*, 50 *Hz*. Determine the current through the elements and power delivered by the source. Draw the phasor diagram.

SOLUTION

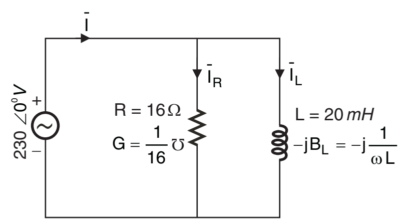


Fig. 1.

Given that, $R = 16\,\Omega$, $\quad L = 20\,mH$

$V = 230\,V$, $\quad f = 50\,Hz$

The parallel RL circuit is shown in Fig. 1.

$$\text{Conductance, } G = \frac{1}{R} = \frac{1}{16} = 0.0625\,\mho$$

$$\text{Inductive susceptance} = -jB_L = -j\frac{1}{\omega L} = -j\frac{1}{2\pi f L} = -j\frac{1}{2\pi \times 50 \times 20 \times 10^{-3}} = -j0.1592\,\mho$$

$$\text{Admittance, } \bar{Y} = G - jB_L = 0.0625 - j0.1592\,\mho = 0.171\angle -68.6^\circ\,\mho$$

Let the supply voltage be the reference phasor.

$$\therefore\quad \bar{V} = V\angle 0^\circ = 230\angle 0^\circ\,V$$

Let, $\bar{I}$ be the current through the RL circuit. Now, by Ohm's law,

$$\text{Current, } \bar{I} = \frac{\bar{V}}{\bar{Z}} = \bar{V}\,\bar{Y} = 230\angle 0^\circ \times 0.171\angle -68.6^\circ$$

$$= 39.33\angle -68.6^\circ\,A$$

$$\therefore\quad I = |\bar{I}| = 39.33\,A$$

$$\text{Power factor angle, } \phi = \angle\bar{V} - \angle\bar{I} = 0^\circ - (-68.6^\circ) = 68.6^\circ$$

$$\therefore\quad \text{Power factor} = \cos\phi = \cos 68.6^\circ = 0.3649 \text{ lag}$$

Since the current lags the voltage, the power factor is lag.

Power, $P = VI\cos\phi = 230 \times 39.33 \times 0.3649 = 3300.8\ W = \frac{3300.8}{1000}\ kW = 3.3008\ kW$

Current through resistance, $\bar{I}_R = \frac{\bar{V}}{R} = \bar{V} \times G$

$= 230\angle 0^\circ \times 0.0625 = 14.375\angle 0^\circ\ A$

Current through inductance, $\bar{I}_L = \frac{\bar{V}}{j\omega L} = \bar{V} \times (-jB_L) = 230\angle 0^\circ \times (-j0.1592)$

$= 230\angle 0^\circ \times 0.1592\angle -90^\circ = 36.616\angle -90^\circ\ A$

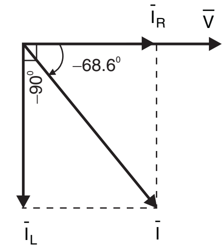


Fig.2: Phasor diagram.

The phasor diagram of the RL parallel circuit with $\bar{V}$ as the reference phasor is shown in Fig. 2.

Alternate method

Inductive reactance $= jX_L = j\omega L = j2\pi fL = j2\pi \times 50 \times 20 \times 10^{-3} = j6.2832\,\Omega$

Impedance, $\bar{Z} = \frac{R \times jX_L}{R + jX_L} = \frac{16 \times j6.2832}{16 + j6.2832} = 2.1377 + j5.4437 = 5.8484\angle 68.6^\circ\ \Omega$

Let, $\bar{V}$ be reference phasor, $\therefore\ \bar{V} = V\angle 0^\circ = 230\angle 0^\circ\ V$

Current, $\bar{I} = \frac{\bar{V}}{\bar{Z}} = \frac{230\angle 0^\circ}{5.8484\angle 68.6^\circ} = 39.327\angle -68.6^\circ\ A$; $\therefore\ I = |\bar{I}| = 39.327\ A$

Power factor angle, $\phi = \angle\bar{V} - \angle\bar{I} = 0^\circ - (-68.6^\circ) = 68.6^\circ$

$\therefore$ Power factor $= \cos\phi = \cos 68.6^\circ = 0.3649$ lag

Power, $P = VI\cos\phi = 230 \times 39.327 \times 0.3649 = 3300.6\ W = 3.3006\ kW$

Current through resistance, $\bar{I}_R = \frac{\bar{V}}{R} = \frac{230\angle 0^\circ}{16} = 14.375\angle 0^\circ\ A$

Current through inductance, $\bar{I}_L = \frac{\bar{V}}{j\omega L} = \frac{230\angle 0^\circ}{j6.2832} = \frac{230\angle 0^\circ}{6.2832\angle 90^\circ} = 36.606\angle -90^\circ\ A$

EXAMPLE 4.10

A series combination of 12 Ω resistance and 600 *μF* capacitance is connected to a 220 *V*, 50 *Hz* supply. Estimate the current, active power, reactive power and apparent power. Also estimate the voltage across R and C and draw the phasor diagram.

SOLUTION

Given that, $V = 220\ V$, $f = 50\ Hz$

$R = 12\,\Omega$, $C = 600\ \mu F$

The RC series circuit excited by a sinusoidal source is shown in Fig. 1.

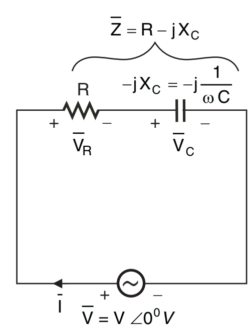


Fig. 1.

Capacitive reactance $= -jX_C = -j\frac{1}{\omega C} = -j\frac{1}{2\pi fC}$

$= -j\frac{1}{2\pi \times 50 \times 600 \times 10^{-6}} = -j5.3052\,\Omega$

Impedance, $\bar{Z} = R - jX_C = 12 - j5.3052\,\Omega = 13.1204\angle -23.9^\circ\ \Omega$

Let the supply voltage be the reference phasor.

$$\therefore \; \bar{V} = V\angle 0^\circ = 220\angle 0^\circ \, V$$

Let, $\bar{I}$ be the current through the RC circuit. Now, by Ohm's law,

$$\text{Current, } \bar{I} = \frac{\bar{V}}{\bar{Z}} \; \frac{220\angle 0^\circ}{13.1204\angle -23.9^\circ} = 16.7678\angle 23.9^\circ A$$

$$\therefore I = |\bar{I}| = 16.7678 \, A$$

$$\text{Power factor angle, } \phi = \angle\bar{V} - \angle\bar{I} = 0^\circ - 23.9^\circ = -23.9^\circ$$

$$\therefore \text{Power factor} = \cos\phi = \cos(-23.9^\circ) = 0.9143 \text{ lead}$$

Since the current leads the voltage, the power factor is lead.

$$\text{Apparent power, } S = VI = 220 \times 16.7678 = 3688.9 \, VA$$

$$= \frac{3688.9}{1000} \, kVA = 3.6889 \, kVA$$

Active power, (or power)

$$P = VI\cos\phi = 220 \times 16.7678 \times \cos(-23.9^\circ)$$

$$= 3372.6 \, W = \frac{3372.6}{1000} \, kW = 3.3726 \, kW$$

Alternatively,
$Q = 1.4945$ *kVAR*-capacitive

$$\text{Reactive power, } Q = VI\sin\phi = 220 \times 16.7678 \times \sin(-23.9^\circ)$$

$$= -1494.5 \, VAR = \frac{-1494.5}{1000} \, kVAR = -1.4945 \, kVAR$$

$$\text{Voltage across resistance, } \bar{V}_R = \bar{I} \times R = 16.7678\angle 23.9^\circ \times 12$$

$$= 201.2136\angle 23.9^\circ \, V$$

$$\text{Voltage across capacitance, } \bar{V}_C = \bar{I} \times (-jX_C) = 16.7678\angle 23.9^\circ \times (-j5.3052)$$

$$= 16.7678\angle 23.9^\circ \times 5.3052\angle -90^\circ$$

$$= 88.9565\angle -66.1^\circ \, V$$

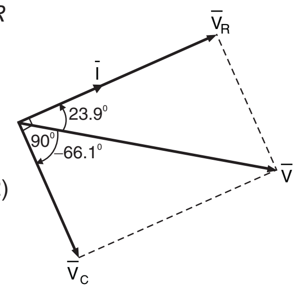


***Fig. 2 :** Phasor diagram.*

The phasor diagram of the RC circuit with $\bar{V}$ as reference phasor is shown in Fig. 2

EXAMPLE 4.11

A current of $60\angle 25^\circ A$ is flowing through a circuit which consists of parallel connected elements when excited by a source of $230\angle -20^\circ V$, 50 *Hz*. Determine the elements of the circuit, active power and reactive power. Also calculate the current through the elements and draw the phasor diagram.

SOLUTION

$$\text{Given that, } \bar{V} = 230\angle -20^\circ V \quad \text{and} \quad \bar{I} = 60\angle 25^\circ A$$

$$\therefore \; \text{Admittance, } \bar{Y} = \frac{\bar{I}}{\bar{V}} = \frac{60\angle 25^\circ}{230\angle -20^\circ} = 0.2609\angle 45^\circ \, \mho$$

$$= 0.1845 + j0.1845 \, \mho$$

Since the susceptance is positive, the circuit is an RC parallel circuit. (Also the current is leading and so the circuit is capacitive.)

We know that, $\overline{Y} = G + jB_C$

$$\therefore \quad G = 0.1845 \, \mho \quad \text{and} \quad R = \frac{1}{G} = \frac{1}{0.1845} = 5.4201 \, \Omega$$

$$B_C = 0.1845 \, \mho$$

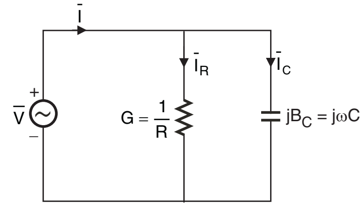


Fig. 1.

We know that, $B_C = \omega C = 2\pi f C$

$$\therefore \quad C = \frac{B_C}{2\pi f} = \frac{0.1845}{2\pi \times 50} = 5.8728 \times 10^{-4} \, F$$

$$= 0.5873 \times 10^{-3} \, F$$

$$= 0.5873 \, mF$$

The complex power, $\overline{S} = \overline{V} \, \overline{I}^*$

$$= 230\angle{-20^\circ} \times (60\angle 25^\circ)^* = 230\angle{-20^\circ} \times 60\angle{-25^\circ}$$

$$= 13800\angle{-45^\circ} \, VA$$

$$= 9758.1 - j9758.1 \, VA$$

Alternatively,
Q = 9.7581 *kVAR*-capacitive

Also, $\overline{S} = P + jQ$

$$\therefore \quad \text{Active power,} \quad P = 9758.1 W = \frac{9758.1}{1000} \, kW = 9.7581 \, kW$$

$$\text{Reactive power,} \quad Q = -\, 9758.1 VAR = -\frac{9758.1}{1000} \, kVAR = -9.7581 \, kVAR$$

Apparent power, $S = |\overline{S}| = 13800 \, VA$

The RC parallel circuit is shown in Fig. 1. Let, $\overline{I}_R$ and $\overline{I}_C$ be the current through R and C.

Now, by Ohm's law,

$$\overline{I}_R = \frac{\overline{V}}{R} = \overline{V} \times G = 230\angle{-20^\circ} \times 0.1845 = 42.435\angle{-20^\circ} \, A$$

$$\overline{I}_C = \frac{\overline{V}}{-jX_C} = \overline{V} \times jB_C = 230\angle{-20^\circ} \times j0.1845$$

$$= 230\angle{-20^\circ} \times 0.1845\angle 90^\circ = 42.435\angle 70^\circ \, A$$

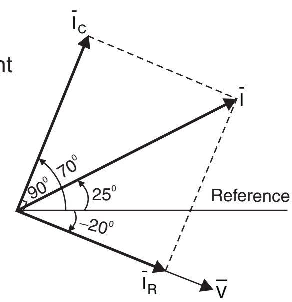


Fig 2 : Phasor diagram.

The phasor diagram of the RC parallel circuit is shown in Fig. 2.

EXAMPLE 4.12

An RLC series circuit consists of R = 75 Ω, L = 125 *mH* and C = 200 *μF*. The circuit is excited by a sinusoidal source of value 115 *V*, 60 *Hz*. Determine the voltage across the various elements. Calculate the current and power. Draw the phasor diagram.

SOLUTION

Given that, V = 115 *V* , f = 60 *Hz* , R = 75 Ω , L = 125 *mH* and C = 200 *μF*

$$\text{Inductive reactance} = jX_L = j\omega L = j2\pi f L = j2\pi \times 60 \times 125 \times 10^{-3} = j47.1239 \, \Omega$$

$$\text{Capacitive reactance} = -jX_C = -j\frac{1}{\omega C} = -j\frac{1}{2\pi f C} = -j\frac{1}{2\pi \times 60 \times 200 \times 10^{-6}} = -j13.2629 \, \Omega$$

$$\text{Total reactance} = jX = jX_L - jX_C = j47.1239 - j13.2629 = j33.861 \, \Omega$$

$$\therefore \quad \text{Impedance,} \quad \overline{Z} = R + jX = 75 + j33.861 \, \Omega = 82.2895 \angle 24.3^\circ \, \Omega$$

The RLC series circuit is shown in Fig. 1. Let, $\bar{I}$ be the current through the circuit and $\bar{V}_R$, $\bar{V}_L$ and $\bar{V}_C$ be the voltage across R, L and C respectively. Let, $\bar{V}$ be the reference phasor.

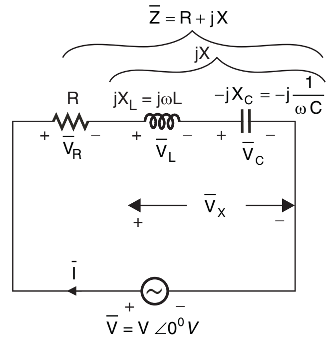


Fig. 1.

$$\therefore\ \bar{V} = V\angle 0^o\,V$$

Now, by Ohm's law,

$$\bar{I} = \frac{\bar{V}}{\bar{Z}} = \frac{115\angle 0^o}{82.2895\angle 24.3^o} = 1.3975\angle -24.3^o\,A$$

$$\therefore\ I = |\bar{I}| = 1.3975\,A$$

$$\bar{V}_R = \bar{I} \times R = 1.3975\angle -24.3^o \times 75 = 104.8125\angle -24.3^o\,V$$

$$\bar{V}_L = \bar{I} \times jX_L = 1.3975\angle -24.3^o \times j47.1239$$

$$= 1.3975\angle -24.3^o \times 47.1239\angle 90^o = 65.8557\angle 65.7^o\,V$$

$$\bar{V}_C = \bar{I} \times (-jX_C) = 1.3975\angle -24.3^o \times (-j13.2629)$$

$$= 1.3975\angle -24.3^o \times 13.2629\angle -90^o = 18.5349\angle -114.3^o\,V$$

Apparent power, $S = VI = 115 \times 1.3975 = 160.7125\,VA$

Power factor angle, $\phi = \angle V - \angle I = 0^o - (-24.3^o) = 24.3^o$

Active power, $P = VI\cos\phi = 115 \times 1.3975 \times \cos 24.3^o = 146.4739\,W$

Reactive power, $Q = VI\sin\phi = 115 \times 1.3975 \times \sin 24.3^o$

$= 66.1355\,VAR$

Alternatively,
Q = 66.1355 *VAR*-inductive

The phasor diagram of the RLC series circuit is shown in Fig. 2.

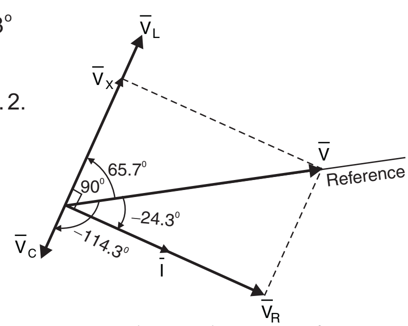


Fig. 2 : *Phasor diagram of RLC series circuit.*

Here, $\bar{V}_X = \bar{V}_L + \bar{V}_C$

$$= \bar{I}jX_L + \bar{I}(-jX_C)$$

$$= \bar{I} \times j(X_L - X_C)$$

$$= \bar{I} \times jX = 1.3975\angle -24.3^o \times j33.861$$

$$= 1.3975\angle -24.3^o \times 33.861\angle 90^o$$

$$= 47.3207\angle 65.7^o\,V$$

EXAMPLE 4.13

An RLC parallel circuit consists of $R = 50\,\Omega$, $L = 150\,mH$ and $C = 100\,\mu F$. The circuit is excited by a current source of $5\angle 0^o A$, $100\,Hz$. Calculate the voltage and current in the various elements. Determine the apparent, active and reactive power delivered by the source. Draw the phasor diagram.

SOLUTION

Given that, $\bar{I} = 5\angle 0^o\,A$, $f = 100\,Hz$

$R = 50\,\Omega$, $L = 150\,mH$ and $C = 100\,\mu F$.

Let us analyse the parallel circuit in terms of admittance.

Conductance, $G = \frac{1}{R} = \frac{1}{50} = 0.02\,\mho$

Inductive susceptance $= -jB_L = -j\frac{1}{2\pi fL} = -j \times \frac{1}{2\pi \times 100 \times 150 \times 10^{-3}} = -j0.0106\,\mho$

Capacitive susceptance $= jB_C = j2\pi fC = j2\pi \times 100 \times 100 \times 10^{-6} = j0.0628\,\mho$

Total susceptance $= jB = jB_C - jB_L = j0.0628 - j0.0106 = j0.0522\,\mho$

Admittance, $\bar{Y} = G + jB = 0.02 + j0.0522 = 0.0559\angle 69^\circ\,\mho$

The RLC parallel circuit excited by a current source is shown in Fig. 1. Let, $\bar{V}$ be the voltage across the source and parallel connected elements. Let, $\bar{I}_R$, $\bar{I}_L$ and $\bar{I}_C$ be the current through R, L and C, respectively.

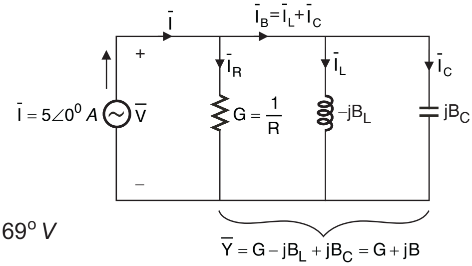


Fig. 1.

Now, by Ohm's law,

$$\bar{V} = \bar{I}\,\bar{Z} = \frac{\bar{I}}{\bar{Y}} = \frac{5\angle 0^\circ}{0.0559\angle 69^\circ} = 89.4454\angle -69^\circ\ V$$

$$\bar{I}_R = \bar{V} \times G = 89.4454\angle -69^\circ \times 0.02 = 1.7889\angle -69^\circ\ A$$

$$\bar{I}_L = \bar{V} \times (-jB_L) = 89.4454\angle -69^\circ \times (-j0.0106)$$
$$= 89.4454\angle -69^\circ \times 0.0106\angle -90^\circ = 0.9481\angle -159^\circ\ A$$

$$\bar{I}_C = \bar{V} \times jB_C = 89.4454\angle -69^\circ \times j0.0628$$
$$= 89.4454\angle -69^\circ \times 0.0628\angle 90^\circ = 5.6172\angle 21^\circ\ A$$

Let, $\bar{I}_B = \bar{I}_L + \bar{I}_C = \bar{V} \times (-jB_L) + \bar{V} \times jB_C = \bar{V} \times j(B_C - B_L) = \bar{V} \times jB$

$$= 89.4454\angle -69^\circ \times j0.0522 = 89.4454\angle -69^\circ \times 0.0522\angle 90^\circ$$

$$= 4.669\angle 21^\circ A$$

Complex power, $\bar{S} = \bar{V}\,\bar{I}^* = 89.4454\angle -69^\circ \times (5\angle 0^\circ)^*$

$$= 89.4454\angle -69^\circ \times 5\angle 0^\circ$$
$$= 447.227\angle -69^\circ\ VA$$
$$= 160.2718 - j417.5224\ VA$$

Apparent power, $S = |\bar{S}| = 447.227\ VA$

Also, $\bar{S} = P + jQ$

$\therefore$ Active power, $P = 160.2718\ W$

Reactive power, $Q = -417.5224\ VAR$

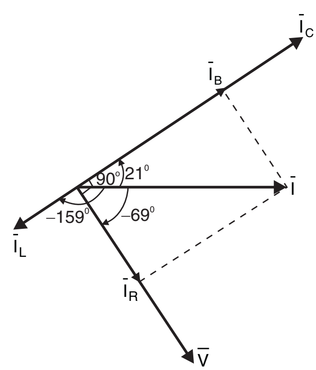


***Fig. 2 :** Phasor diagram.*

The phasor diagram of the RLC parallel circuit with $\bar{I}$ as the reference phasor is shown in Fig. 2.

EXAMPLE 4.14

A load absorbs 2.5 *kW* at a power factor of 0.707 lagging from a 230 *V*, 50 *Hz* source. A capacitor is connected in parallel to the load in order to improve the power factor to 0.9lag. Determine the value of the capacitor.

SOLUTION

Method - I

Case i

Given that, $V = 230\,V$, $P_1 = 2.5\,kW$, $\cos\phi_1 = 0.707$ lag , $f = 50\,Hz$

Apparent power, $S_1 = \dfrac{P}{\cos\phi_1} = \dfrac{2.5}{0.707} = 3.5361\,kVA$

Alternatively,
$Q_1 = 2.5\ kVAR$-inductive

Power factor angle, $\phi_1 = \cos^{-1} 0.707 = 45^\circ$

Reactive power, $Q_1 = S_1 \sin\phi_1 = 3.5361 \times \sin 45^\circ = 2.5\,kVAR$

Case ii

Given that, $V = 230\,V$, $f = 50\,Hz$, $\cos\phi_2 = 0.9$ lag

The addition of capacitor to the load does not alter the active power but decreases the reactive power supplied by the source. Hence, the active power remains the same as that of 2.5 kW.

$\therefore\ P_2 = 2.5\,kW$

Alternatively,
$Q_2 = 1.209\ kVA$-inductive

Apparent power, $S_2 = \dfrac{P_2}{\cos\phi_2} = \dfrac{2.5}{0.9} = 2.7778\ kVA$

Here, Q_2 is inductive because the power factor is still lagging.

Power factor angle, $\phi_2 = \cos^{-1} 0.9 = 25.8^\circ$

Reactive power, $Q_2 = S_2 \sin\phi_2 = 2.7778 \times \sin 25.8^\circ = 1.209\,kVAR$

Now, the reactive power supplied by the capacitor Q_C is given by,

$$Q_C = Q_2 - Q_1 = 1.209 - 2.5 = -1.291\,kVAR = -1291\,VAR$$

We know that, $|Q_C| = |\bar{V}\,\bar{I}_C| \;\Rightarrow\; |Q_C| = VI_C \;\Rightarrow\; I_C = \dfrac{|Q_C|}{V}$

where, I_C is the magnitude of current through capacitor.

Capacitive reactance, $X_C = \dfrac{V}{I_C} = \dfrac{V}{|Q_C|/V} = \dfrac{V^2}{|Q_C|} = \dfrac{230^2}{1291} = 40.976$

Also, $X_C = \dfrac{1}{2\pi fC}$

$$\therefore \text{ Capacitance, } C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi \times 50 \times 40.976} = 77.682 \times 10^{-6}\,F = 77.682\,\mu F$$

Method - II

Case i

Given that, $V = 230\,V$, $f = 50\,Hz$, $P_1 = 2.5\,kW$, $\cos\phi_1 = 0.707$ lag

Let, $\bar{I}_L$ be the current through the inductive load as shown in Fig. 1.

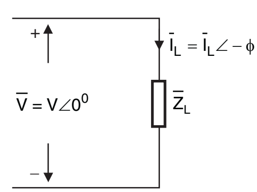


Fig. 1.

We know that, $P_1 = VI_L \cos\phi_1$

$$\therefore\ I_L = |\bar{I}_L| = \frac{P_1}{V\cos\phi_1} = \frac{2.5 \times 10^3}{230 \times 0.707} = 15.3742\,A \quad .....(1)$$

Let the supply voltage $\bar{V}$ be the reference phasor.

$$\therefore \bar{V} = V\angle 0^o = 230\angle 0^o\ V$$

Since the power factor is lagging, the current $\bar{I}_L$ will lag the supply voltage $\bar{V}$ by an angle ϕ_1, where $\phi_1 = \cos^{-1}0.707$

$$\therefore \bar{I}_L = I_L\angle -\cos^{-1}\phi_1$$ Using equation (1)

$$= 15.3742\angle -\cos^{-1}0.707 = 10.8696 - j10.8728\ A$$

Case ii : When capacitor is added to inductive load

An inductive load with capacitance in parallel is shown in Fig. 2. Here, the current through the load remains the same as that of $\bar{I}_L$. Let, $\bar{I}$ be the current supplied by the source and $\bar{I}_C$ be the current through the capacitor.

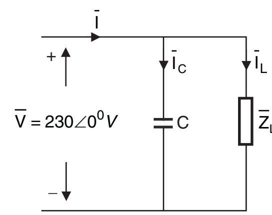


Fig. 2.

Active power remains the same even after addition of the capacitor.

$$\therefore P_2 = 2.5\,kW$$

We know that, $P_2 = VI\cos\phi_2$

$$\therefore I = |\bar{I}| = \frac{P_2}{V\cos\phi_2} = \frac{2.5\times 10^3}{230\times 0.9} = 12.0773\ A \qquad .....(2)$$

$$\therefore \bar{I} = |\bar{I}|\angle -\cos^{-1}\phi_2$$

$$= 12.0773\angle -\cos^{-1}0.9 = 10.8696 - j5.2644\ A$$ Using equation (2)

By KCL, we can write,

$$\bar{I} = \bar{I}_C + \bar{I}_L$$

$$\therefore \bar{I}_C = \bar{I} - \bar{I}_L = 10.8696 - j10.8728 - (10.8696 - j5.2644) = -j5.6084\ A$$

Magnitude of capacitor current, $I_C = 5.6084\ A$

Capacitive reactance, $X_C = \dfrac{V}{I_C} = \dfrac{230}{5.6084} = 41.0092\,\Omega$

Also, $X_C = \dfrac{1}{2\pi fC}$

$$\therefore \text{Capacitance, } C = \frac{1}{2\pi fX_C} = \frac{1}{2\pi\times 50\times 41.0092} = 77.619\times 10^{-6}\,F = 77.619\,\mu F$$

Note : *The slight difference in the capacitance value is due to approximation in calculations.*

EXAMPLE 4.15

An inductive coil of power factor 0.8 lagging is connected in series with a 120 μF capacitor. When the series circuit is connected to a source of frequency 50 Hz, it is observed that the magnitude of voltage across the coil and capacitor are equal. Determine the parameters of the coil.

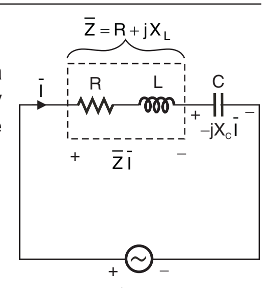


Fig. 1.

SOLUTION

The given circuit is an RLC series circuit as shown in Fig. 1.

Let, $\bar{Z}$ = Impedance of the coil

$\bar{I}$ = Current through the RLC series circuit

Given that,

$$|\bar{Z}\,\bar{I}| = |-jX_C\bar{I}| \Rightarrow \quad Z = X_C$$

$$\text{where, } Z = \sqrt{R^2 + X_L^2} = \text{Magnitude of impedance of the coil}$$

$$\therefore \text{ Magnitude of impedance, } \quad Z = X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 120 \times 10^{-6}} = 26.5258\,\Omega$$

Given that, power factor of the coil = 0.8 lag

$$\therefore \text{ Power factor angle of the coil, } \phi = \cos^{-1} 0.8 = 36.9^\circ$$

Let us construct an impedance triangle for Z using R and X_L as the two sides, as shown in Fig. 2. Here, the impedance angle is the same as the power factor angle.

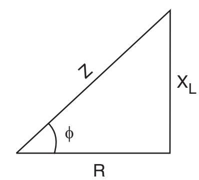


***Fig. 2 :** Impedance triangle.*

With reference to Fig. 2, we can write,

$$\cos\phi = \frac{R}{Z} \quad \text{and} \quad \sin\phi = \frac{X_L}{Z}$$

$\therefore$ Resistance, $R = Z\cos\phi = 26.5258 \times \cos 36.9^\circ = 21.2123\,\Omega$

Inductive reactance, $X_L = Z\sin\phi = 26.5258 \times \sin 36.9^\circ = 15.9266\,\Omega$

We know that, $\quad X_L = \omega L$

$$\therefore \text{ Inductance, } L = \frac{X_L}{\omega} = \frac{X_L}{2\pi f} = \frac{15.9266}{2\pi \times 50} = 0.0507\,H = 50.7\,mH$$

RESULT

The parameters of the coil are R and L.

Resistance of the coil, $R = 21.2123\,\Omega$

Inductance of the coil, $L = 50.7\,mH$

EXAMPLE 4.16

Three impedances 12 Ω, 5 + j8 Ω and −j7 Ω are connected in parallel. This parallel combination is connected in series with an impedance of 4 + j6 Ω across a 230 *V* source. Determine the current through each impedance and the power.

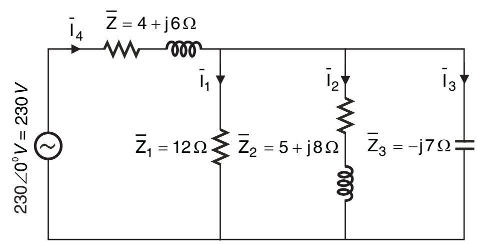


Fig. 1.

SOLUTION

The series-parallel connections of the impedances are shown in Fig. 1. Let us name the impedances as $\bar{Z}_1, \bar{Z}_2, \bar{Z}_3$ and $\bar{Z}_4$, as shown in Fig. 1. Let, the current through the impedances be $\bar{I}_1, \bar{I}_2, \bar{I}_3$ and $\bar{I}_4$ as shown in Fig. 1. Let, supply voltage be the reference phasor.

$$\therefore \text{ Supply voltage } = 230\angle 0^\circ V$$

Let, $\bar{Z}_{eq}$ be the equivalent impedance of the parallel combination of $\bar{Z}_1$, $\bar{Z}_2$ and $\bar{Z}_3$ and the circuit can be modified as shown in Fig. 2.

$$\text{Now, } \bar{Z}_{eq} = \frac{1}{\frac{1}{\bar{Z}_1} + \frac{1}{\bar{Z}_2} + \frac{1}{\bar{Z}_3}} = \frac{1}{\frac{1}{12} + \frac{1}{5 + j8} + \frac{1}{-j7}}$$

$$= [12^{-1} + (5 + j8)^{-1} + (-j7)^{-1}]^{-1} = 6.2647 - j2.3785\,\Omega$$

Let, $\bar{V}_a$ and $\bar{V}_b$ be the voltage across $\bar{Z}_4$ and $\bar{Z}_{eq}$ as shown in Fig. 2.

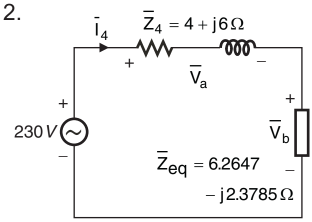


Fig. 2.

Now, by voltage division rule, we can write,

$$\bar{V}_a = 230 \times \frac{4 + j6}{4 + j6 + 6.2647 - j2.3785}$$

$$= 121.8879 + j91.4379\,V$$

$$= 152.3731\angle 36.9^\circ\,V$$

By KVL, we can write,

$$\bar{V}_b = 230 - \bar{V}_a = 230 - (121.8879 + j91.4379) = 108.1121 - j91.4379\,V$$

$$= 141.5949\angle{-40.2^\circ}\,V$$

Here, the voltage across $\bar{Z}_4$ is $\bar{V}_a$ and the voltage across $\bar{Z}_1$, $\bar{Z}_2$ and $\bar{Z}_3$ are $\bar{V}_b$ (because $\bar{Z}_1$, $\bar{Z}_2$ and $\bar{Z}_3$ are in parallel). Now, the current through the impedances can be evaluated using Ohm's law, as shown below:

$$\bar{I}_4 = \frac{\bar{V}_a}{\bar{Z}_4} = \frac{152.3731\angle 36.9^\circ}{4 + j6} = \frac{152.3731\angle 36.9^\circ}{7.2111\angle 56.3^\circ} = 21.1304\angle -19.4^\circ A$$

$$\bar{I}_1 = \frac{\bar{V}_b}{\bar{Z}_1} = \frac{141.5949\angle -40.2^\circ}{12} = 11.7996\angle -40.2^\circ A$$

$$\bar{I}_2 = \frac{\bar{V}_b}{\bar{Z}_2} = \frac{141.5949\angle -40.2^\circ}{5 + j8} = \frac{141.5949\angle -40.2^\circ}{9.434\angle 58^\circ} = 15.009\angle -98.2^\circ A$$

$$\bar{I}_3 = \frac{\bar{V}_b}{\bar{Z}_3} = \frac{141.5949\angle -40.2^\circ}{-j7} = \frac{141.5949\angle -40.2^\circ}{7\angle -90^\circ} = 20.2278\angle 49.8^\circ A$$

We know that complex power is given by the product of voltage and conjugate of current. Hence, the complex power in each impedance can be obtained from the product of voltage and conjugate of current in the impedance.

Let, $\bar{S}_1, \bar{S}_2, \bar{S}_3$ and $\bar{S}_4$ be the complex power of the impedances $\bar{Z}_1, \bar{Z}_2, \bar{Z}_3$ and $\bar{Z}_4$, respectively. Let, P_1, P_2, P_3 and P_4 be the active power and Q_1, Q_2, Q_3 and Q_4 be the reactive power of the impedances.

For impedance $\bar{Z}_4$,

$$\bar{S}_4 = \bar{V}_a \times \bar{I}_4^* = 152.3731\angle 36.9^\circ \times (21.1304\angle -19.4^\circ)^*$$

$$= 152.3731\angle 36.9^\circ \times 21.1304\angle 19.4^\circ = 3219.7\angle 56.3^\circ\,VA$$

$$= 1786.4 + j2678.6 = P_4 + jQ_4$$

$$\therefore S_4 = 3219.7\,VA = 3.2197\,kVA$$

$$P_4 = 1786.4\,W = 1.7864\,kW$$

$$Q_4 = 2678.6\,VAR = 2.6786\,kVAR$$

For impedance $\bar{Z}_1$,

$$\bar{S}_1 = \bar{V}_b \times \bar{I}_1^* = 141.5949\angle -40.2^\circ \times (11.7996\angle -40.2^\circ)^*$$

$$= 141.5949\angle -40.2^\circ \times 11.7996\angle 40.2^\circ = 1670.8\angle 0^\circ\,VA$$

$$= 1670.8 + j0 = P_1 + jQ_1$$

$$\therefore S_1 = 1670.8\,VA = 1.6708\,kVA$$

$$P_1 = 1670.8\,W = 1.6708\,kW$$

$$Q_1 = 0$$

For impedance $\bar{Z}_2$,

$$\bar{S}_2 = \bar{V}_b \times \bar{I}_2^* = 141.5949\angle -40.2^\circ \times (15.009\angle -98.2^\circ)^*$$

$$= 141.5949\angle -40.2^\circ \times 15.009\angle 98.2^\circ$$

$$= 2125.2\angle 58^\circ\, VA$$

$$= 1126.2 + j1802.3\, VA = P_2 + jQ_2$$

$$\therefore\ S_2 = 2125.2\, VA = 2.1252\, kVA$$

$$P_2 = 1126.2\, W = 1.1262\, kW$$

$$Q_2 = 1802.3\, VAR = 1.8023\, kVAR$$

For impedance $\bar{Z}_3$,

$$\bar{S}_3 = \bar{V}_b \times \bar{I}_3^* = 141.5949\angle -40.2^\circ \times (20.2278\angle 49.8^\circ)^*$$

$$= 141.5949\angle -40.2^\circ \times 20.2278\angle -49.8^\circ$$

$$= 2864\angle -90^\circ = 0 - j2864 = P_3 + jQ_3$$

$$\therefore\ S_3 = 2864\, VA = 2.864\, kVA$$

$$P_3 = 0$$

$$Q_3 = -2864\, VAR = -2.864\, kVAR$$

EXAMPLE 4.17

Two reactive circuits have an impedance of 20 Ω each. One of them has a power factor of 0.75 lagging and the other 0.65 leading. Find the voltage necessary to send a current of 12 *A* through the two circuits in series. Also determine the current drawn from the 200 *V* supply if they are connected in parallel to the supply.

SOLUTION

Case i : Impedances in series

Let, $\bar{Z}_1$ and $\bar{Z}_2$ be the impedances of the two circuits. The series combination of two circuits excited by a voltage source can be represented by the circuit shown in Fig. 1.

Now, the impedances $\bar{Z}_1$ and $\bar{Z}_2$ can be expressed as shown below:

$$\bar{Z}_1 = 20\angle \cos^{-1} 0.75$$

$$= 20\angle 41.4^\circ\, \Omega$$

$$= 15.0022 + j13.2262\, \Omega$$

When the current is lagging, the impedance is inductive and the impedance angle is the same as the power factor angle.

$$\bar{Z}_2 = 20\angle -\cos^{-1} 0.65$$

$$= 20\angle -49.5^\circ\, \Omega$$

$$= 12.989 - j15.2081\, \Omega$$

When the current is leading, the impedance is capacitive and the impedance angle is the negative of the power factor angle.

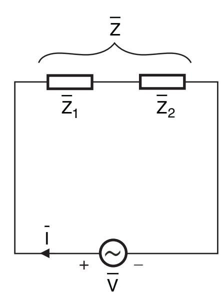


Fig. 1.

Let, $\bar{Z}$ be the total impedance of series combination.

Now, $\bar{Z} = \bar{Z}_1 + \bar{Z}_2 = 15.0022 + j13.2262 + 12.989 - j15.2081$

$$= 27.9912 - j1.9819 = 28.0613\angle -4.1^\circ\, \Omega$$

Magnitude of impedance, $Z = |\bar{Z}| = 28.0613\,\Omega$

Given that, the magnitude of current, $I = |\bar{I}| = 12\,A$

$\therefore$ Magnitude of supply voltage, $V = IZ = 12 \times 28.0613 = 336.7356\,V$

If $\bar{V}$ is the reference phasor then,

$$\bar{V} = 336.7356\angle 0^\circ\,V \quad \text{and} \quad \bar{I} = 12\angle 4.1^\circ\,A$$

If $\bar{I}$ is the reference phasor then,

$$\bar{V} = 336.7356\angle -4.1^\circ\,V \quad \text{and} \quad \bar{I} = 12\angle 0^\circ\,A$$

Case ii : When the impedances are in parallel

Two circuits in parallel and excited by a 200 V source can be represented by the circuit shown in Fig. 2. Let, $\bar{I}_1$ and $\bar{I}_2$ be the current through the impedances $\bar{Z}_1$ and $\bar{Z}_2$, respectively. Let, $\bar{I}$ be the total current supplied by the source.

Let, the supply voltage $\bar{V}$ be the reference phasor.

$$\therefore \bar{V} = V\angle 0^\circ = 200\angle 0^\circ\,V$$

Now, $\bar{I}_1 = \dfrac{\bar{V}}{\bar{Z}_1} = \dfrac{200\angle 0^\circ}{20\angle 41.4^\circ} = 10\angle -41.4^\circ\,A$

$$\bar{I}_2 = \frac{\bar{V}}{\bar{Z}_2} = \frac{200\angle 0^\circ}{20\angle -49.5^\circ} = 10\angle 49.5^\circ\,A$$

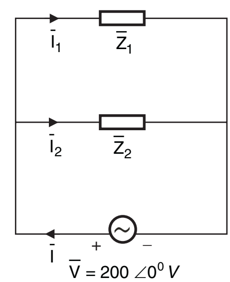


Fig. 2.

By KCL, we can write,

$$\bar{I} = \bar{I}_1 + \bar{I}_2 = 10\angle -41.4^\circ + 10\angle 49.5^\circ$$

$$= 7.5011 - j6.6131 + 6.4945 + j7.6041$$

$$= 13.9956 + j0.991\,A = 14.0306\angle 4.1^\circ\,A$$

EXAMPLE 4.18

In the circuit shown in Fig. 1, **(a)** determine the currents in all the branches, **(b)** calculate the power and power factor of the source, **(c)** show that power delivered by the source is equal to power consumed by the 2 Ω resistor.

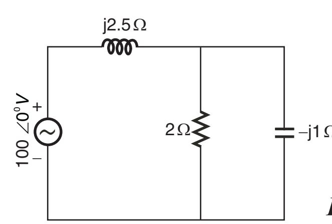


Fig. 1.

SOLUTION

a) To find branch currents

The circuit has three branches. Let us assume three branch currents $\bar{I}_a$, $\bar{I}_b$ and $\bar{I}_c$ as shown in Fig. 2. Let us denote the nodes as A, B and C.

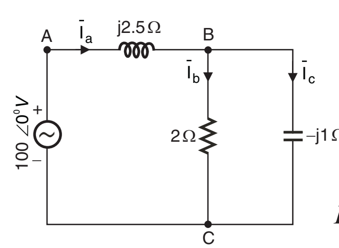


Fig. 2.

By KCL at node-B, we can write,

$$\bar{I}_b + \bar{I}_c = \bar{I}_a$$

$$\therefore \bar{I}_c = \bar{I}_a - \bar{I}_b \qquad \text{..... (1)}$$

With reference to Fig. 3, using KVL in the closed path ABCA, we can write,

$$j2.5\bar{I}_a + 2\bar{I}_b = 100\angle 0^\circ$$

$$\therefore \; j2.5\bar{I}_a + 2\bar{I}_b = 100 \qquad ..... (2)$$

With reference to Fig. 4, using KVL in the closed path BCB, we can write,

$$-j\bar{I}_c = 2\bar{I}_b$$

$$\therefore \; 2\bar{I}_b + j\bar{I}_c = 0 \qquad ..... (3)$$

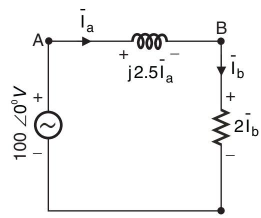


Fig. 3.

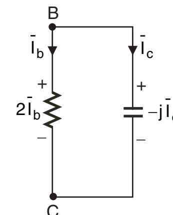


Fig. 4.

On substituting for $\bar{I}_c$ from equation (1) in equation (3), we get,

$$2\bar{I}_b + j(\bar{I}_a - \bar{I}_b) = 0$$

$$\therefore \; j\bar{I}_a + (2 - j)\bar{I}_b = 0 \qquad ..... (4)$$

Equation (2) × 1 $\Rightarrow$ $j2.5\bar{I}_a + 2\bar{I}_b = 100$

Equation (4) × (−2.5) $\Rightarrow$ $-j2.5\bar{I}_a - 2.5\,(2 - j)\bar{I}_b = 0$

On adding $[2 - 2.5\,(2 - j)]\bar{I}_b = 100$

$$\therefore \; \bar{I}_b = \frac{100}{2 - 2.5(2 - j)} = \frac{100}{-3 + j2.5} = -19.6721 - j16.3934 = 25.6073\angle -140.19^\circ$$

From equation (2), we can write,

$$\bar{I}_a = \frac{100 - 2\bar{I}_b}{j2.5} = \frac{100 - 2 \times (-19.6721 - j16.3934)}{j2.5} = \frac{139.3442 + j32.7868}{j2.5}$$

$$= 13.1147 - j55.7377 = 57.2598\angle -76.76^\circ \, A$$

From equation (1), we get,

$$\bar{I}_c = \bar{I}_a - \bar{I}_b$$

$$= 13.1147 - j55.7377 - (-19.6721 - j16.3934)$$

$$= 32.7868 - j39.3443 = 51.2147\angle -50.19^\circ \, A$$

b) To find power and power factor of the source

Let, $\bar{E}$ = Source voltage.

$\bar{I}_s$ = Current delivered by the source.

ϕ = Phase difference between $\bar{E}$ and $\bar{I}_s$.

Given that, $\bar{E} = 100\angle 0^\circ \, V$

With reference to Fig. 2, we can write,

$$\bar{I}_s = \bar{I}_a = 57.2598\angle -76.76^\circ \, A$$

Now, $\phi = \angle\bar{E} - \angle\bar{I}_s$

$= 0^\circ - (-76.76^\circ) = 76.76^\circ$

$\therefore$ Power factor of the source $= \cos\phi = \cos(76.76^\circ) = 0.229$ lag

Power delivered of the source $= |\bar{E}| \times |\bar{I}_s| \times \cos\phi$

$$= 100 \times 57.2598 \times 0.229 = 1311\,W \quad \text{..... (5)}$$

c) To find power consumed by 2 Ω resistor

Power consumed by 2 Ω resistor $= |\bar{I}_b|^2 \times 2$

$$= 25.6073^2 \times 2 = 1311\,W \quad \text{..... (6)}$$

From equations (5) and (6) we can say that the power delivered by the source is equal to the power consumed by the 2 Ω resistor. (Remember that reactive elements do not consume power).

RESULT

a) The branch currents are,

$\bar{I}_a = 57.2598\angle-76.76^\circ\,A$; $\bar{I}_b = 25.6073\angle-140.19^\circ\,A$; $\bar{I}_c = 51.2147\angle-50.19^\circ\,A$

b) The power delivered by the source = 1311 *W*

The power factor of the source = 0.229 lag

c) The power consumed by 2 Ω resistor = 1311 *W*

EXAMPLE 4.19

(AU Dec'14, 16 Marks)

Impedances $\bar{Z}_1$ and $\bar{Z}_2$ are parallel and this combination is in series with an impedance $\bar{Z}_3$, connected to a 100 *V*, 50 *Hz* ac supply. $\bar{Z}_1 = 5 - jX_c\,\Omega$, $\bar{Z}_2 = 5 + j0\,\Omega$, $\bar{Z}_3 = 6.25 + j1.25\,\Omega$. Determine the value of capacitance such that the total current of the circuit will be in phase with the total voltage. Find the circuit current and power.

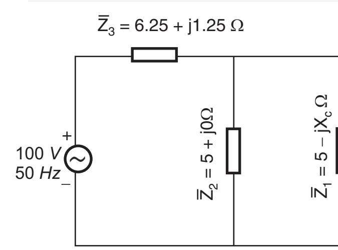


Fig. 1.

SOLUTION

Given that, $\bar{Z}_1 = 5 - jX_c\,\Omega$; $\bar{Z}_2 = 5 + j0\,\Omega$; $\bar{Z}_3 = 6.25 + j1.25\,\Omega$

$V = 100\,V$; $f = 50\,Hz$

With reference to Fig. 1, equivalent impedance is obtained as shown below:

$$\bar{Z}_{eq} = \bar{Z}_3 + \frac{\bar{Z}_1 \times \bar{Z}_2}{\bar{Z}_1 + \bar{Z}_2}$$

$$= 6.25 + j1.25 + \frac{(5 - jX_C) \times 5}{5 - jX_C + 5}$$

$$\therefore\ \bar{Z}_{eq} = 6.25 + j1.25 + \frac{25 - j5X_C}{10 - jX_C} \times \frac{10 + jX_C}{10 + jX_C}$$

$$= 6.25 + j1.25 + \frac{250 + j25X_C - j50X_C + 5X_C^2}{10^2 + X_C^2}$$

$$= 6.25 + j1.25 + \frac{250 + 5X_C^2}{10^2 + X_C^2} - j\frac{25X_C}{10^2 + X_C^2}$$

For voltage and current to be in phase, $\bar{Z}_{eq}$ should be real and so the imaginary part of $\bar{Z}_{eq}$ should be zero.

$$\therefore\ j1.25 - j\frac{25X_C}{10^2 + X_C^2} = 0 \quad \Rightarrow \quad j1.25 = \frac{j25X_C}{10^2 + X_C^2} \quad \Rightarrow \quad 10^2 + X_C^2 = \frac{j25X_C}{j1.25}$$

$$10^2 + X_C^2 = 20\,X_C \quad \Rightarrow \quad X_C^2 - 20\,X_C + 100 = 0$$

$$\therefore\ X_C = \frac{20 \pm \sqrt{20^2 - 4 \times 100}}{2} = \frac{20}{2} = 10\,\Omega$$

We know that,

$$X_C = \frac{1}{2\pi fC} \quad \Rightarrow \quad C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi \times 50 \times 10} = 3.1831 \times 10^{-4}\,F = 318.31 \times 10^{-6}\,F$$

$$= 318.31\ \mu F$$

When X_c = 10 Ω,

$$\bar{Z}_{eq} = 6.25 + \frac{250 + 5X_C^2}{10^2 + X_C^2} = 6.25 + \frac{250 + 5 \times 10^2}{10^2 + 10^2}$$

$$= 6.25 + 3.75 = 10\,\Omega$$

$$\therefore\ \text{Current, I} = \frac{V}{Z_{eq}} = \frac{100}{10} = 10\,A$$

$$\text{Power, P} = V \times I = 100 \times 10 = 1000\,W = 1\,kW$$

(or Power, P = $I^2 \times$ real part of $\bar{Z}_{eq} = 10^2 \times 10 = 1000$ *W*)

4.18 Three-Phase Circuits

(AU Dec'15 & '16, 2 Marks)

In many countries electrical energy is generated and distributed as a three-phase ac supply. For many reasons three-phase system has been adopted for generation and distribution of electrical energy. The three-phase system is more economical than the single-phase system. Also, three-phase motors are self-starting and produce more uniform torque as compared to single-phase motors.

Three-phase sources are **three-phase alternators** generating three emfs having equal magnitude but with a phase difference of 120° with respect to each other. Each generated emf is called **a phase** and so the three generated emfs are called **three phase**.

The three phases are named as R phase, Y phase and B phase in British convention and A phase, B phase and C phase in American convention.

The letters R,Y,B, have been chosen from the first letters, of three colours **R**ed, **Y**ellow, and **B**lue. In practical wiring, these coloured wires are used for wiring of three phase circuits in order to provide clear visual distinction between different phase connections to loads.

In a polyphase system, when the magnitude of emfs are equal and the phase difference between consecutive emfs are equal, the system is called a **balanced system** and the emfs are called **balanced emfs**. Three-phase sources are designed to generate balanced emfs. Hence, in the analysis of three-phase systems, the source emfs are always considered as a balanced set of emfs.

4.19 Three-Phase Sources

The three-phase sources can be represented by three independent sources. For operational convenience the three sources can be connected either in star or delta as shown in Fig. 4.41. The emfs generated by the sources vary sinusoidally and so instantaneous values of the emfs are represented by the following equations.

$$e_R = E_{R,m} \sin \omega t = E_m \sin \omega t$$

$$e_Y = E_{Y,m} \sin (\omega t - 120^\circ) = E_m \sin (\omega t - 120^\circ)$$

$$e_B = E_{B,m} \sin (\omega t - 240^\circ) = E_m \sin (\omega t - 240^\circ)$$

$$\boxed{E_{R,m} = E_{Y,m} = E_{B,m} = E_m}$$

where, $E_{R,m}$, $E_{y,m}$ and $E_{B,m}$ are maximum values of generated emfs.

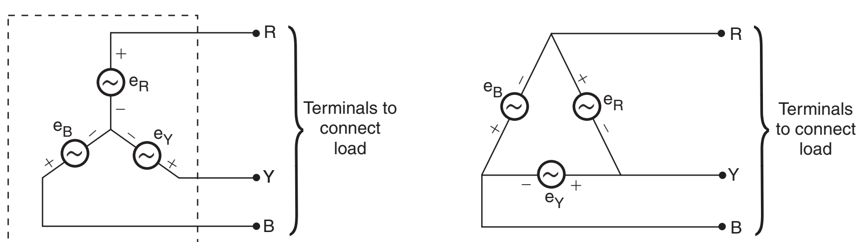


***Fig. a :** Star-connected source.* ***Fig. b :** Delta-connected source.*

***Fig. 4.41 :** Three phase ac sources.*

In star connection, the meeting point of the three sources is called **neutral** and there is no such neutral point in delta connection. The neutral point is denoted by N. The two types of three phase sources offer three terminals for connecting the load. The transmission lines or connecting wires from the source terminals to the load terminals are called **lines**, as shown in Fig. 4.42.

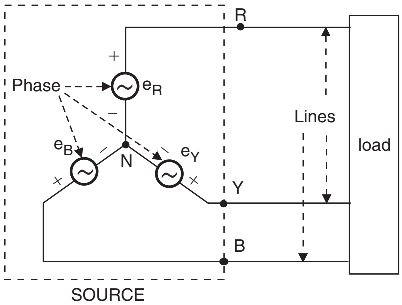


Fig. a : Star-connected source with load.

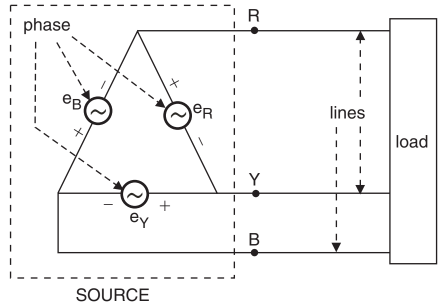


Fig. b : Delta-connected source with load.

***Fig. 4.42 :** Phase and lines in a three-phase system.*

The voltage generated by each phase of a three-phase source is called **phase voltage** and the voltage between the lines connecting the load is called **line voltage**.

The current delivered by each phase of the three-phase source is called **phase current** and the current flowing through the line is called **line current**.

4.20 Representation of Three-Phase EMFs

The three-phase emf can be represented by three rotating phasors (or vectors) having equal magnitude but maintaining a phase difference of 120° with respect to each other. The phasor will rotate at a constant angular speed of ω *rad/s*. The rotation of the phasors can be clockwise or anticlockwise. It is conventional practice to choose anticlockwise rotation for phasors (throughout this book anticlockwise rotation is followed for phasors). The reference point is chosen as $\omega t = 0$. For the three-phase rotating phasors, the order of reaching the reference point is called **phase sequence**.

In the three-phase rotating phasors, when the order of reaching the reference is R phase, Y phase and B phase, the phasors are said to have **normal phase sequence** or **RYB sequence**. When the order of reaching the reference is R phase, B phase and Y phase, the phasors are said to have **reversed phase sequence** or **RBY sequence**. The normal and reversed sequence three-phase emfs are shown in Fig. 4.43.

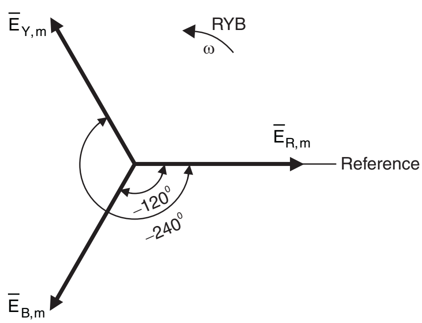


***Fig. a :** Phasors rotating in RYB sequence.*

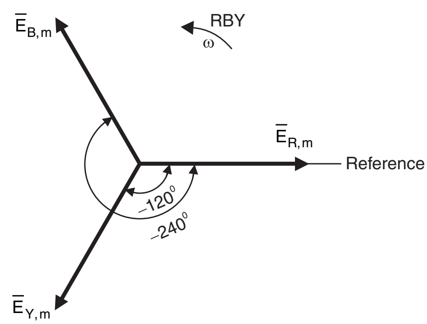


***Fig. b :** Phasors rotating in RBY sequence.*

***Fig. 4.43 :** Phasors representation of three phase emf.*

The instantaneous values of three emfs for RYB sequence with R-phase emf as reference are,

$$e_R = E_{R,m} \sin \omega t = E_m \sin \omega t$$

$$e_Y = E_{Y,m} \sin(\omega t - 120^\circ) = E_m \sin(\omega t - 120^\circ)$$

$$e_B = E_{B,m} \sin(\omega t - 240^\circ) = E_m \sin(\omega t - 240^\circ)$$

$$\boxed{E_{R,m} = E_{Y,m} = E_{B,m} = E_m}$$

The instantaneous values of three emfs for RBY sequence with R-phase emf as reference are,

$$e_R = E_{R,m} \sin \omega t = E_m \sin \omega t$$

$$e_B = E_{B,m} \sin(\omega t - 120^\circ) = E_m \sin(\omega t - 120^\circ)$$

$$e_Y = E_{Y,m} \sin(\omega t - 240^\circ) = E_m \sin(\omega t - 240^\circ)$$

$$\boxed{E_{R,m} = E_{B,m} = E_{Y,m} = E_m}$$

The rms phasors can be drawn by taking a "snapshot" of rotating phasors at $\omega t = 0$ and reducing the length by $\sqrt{2}$. However remember that the rms phasors are not rotating phasors. The three-phase rms voltage phasors for RYB and RBY sequence are shown in Fig. 4.44. (Note that the phasor diagram in this book are drawn using rms phasors.)

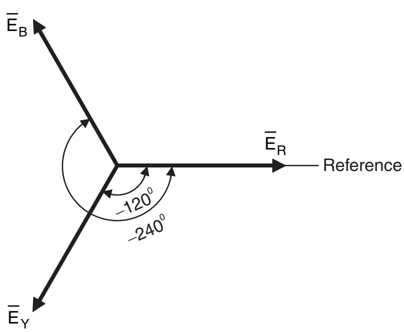


Fig. a : *Rms phasors in RYB sequence.*

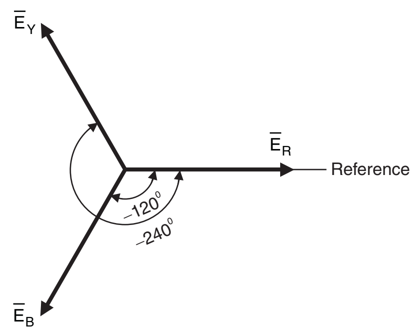


Fig. b : *Rms phasors in RBY sequence.*

Fig. 4.44 : *Phasors representation of three phase rms value of emf.*

The rms values of three emfs for RYB sequence in polar form with R-phase emf as reference are,

$$\bar{E}_R = E_R \angle 0^\circ = E \angle 0^\circ$$

$$\bar{E}_Y = E_Y \angle -120^\circ = E \angle -120^\circ$$

$$\bar{E}_B = E_B \angle -240^\circ = E \angle -240^\circ$$

$$\boxed{E_R = E_Y = E_B = E}$$

$$\boxed{E = E_m/\sqrt{2}}$$

The rms values of three emfs for RBY sequence in polar form with R-phase emf as reference are,

$$\boxed{E_R = E_B = E_Y = E}$$

$$\bar{E}_R = E_R \angle 0^\circ = E \angle 0^\circ$$

$$\bar{E}_B = E_B \angle -120^\circ = E \angle -120^\circ$$

$$\bar{E}_Y = E_Y \angle -240^\circ = E \angle -240^\circ$$

$$\boxed{E = E_m/\sqrt{2}}$$

4.21 Analysis of Three-Phase Star and Delta-connected Source

4.21.1 Star-connected Source Three-Wire System *(AU May'15, 16 Marks)*

Let the phase sequence of the source emfs be RYB.

e_R, e_Y, e_B	=	Instantaneous values of source emfs
$\bar{E}_R, \bar{E}_Y, \bar{E}_B$	=	rms values of source phase emfs
E_R, E_Y, E_B	=	Magnitude of rms values of source phase emfs
$\bar{V}_{RY}, \bar{V}_{YB}, \bar{V}_{BR}$	=	Rms values of line voltages
V_{RY}, V_{YB}, V_{BR}	=	Magnitude of rms values of line voltages
$\bar{I}_R, \bar{I}_Y, \bar{I}_B$	=	Rms values of phase and line currents
I_R, I_Y, I_B	=	Magnitude of rms values of phase and line currents.

The three-phase star-connected source with conventional polarity (or sign) of voltages and direction of currents for RYB sequence are shown in Fig. 4.45.

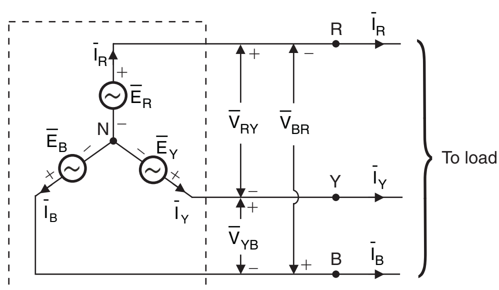


***Fig. 4.45 :** Three phase star-connected source with conventional polarity of voltages and direction of currents for RYB sequence.*

The instantaneous values of three emfs for RYB sequence with R-phase emf as reference are,

$$\boxed{\begin{aligned} e_R &= E_m \sin \omega t \\ e_Y &= E_m \sin (\omega t - 120^\circ) \\ e_B &= E_m \sin (\omega t - 240^\circ) \end{aligned}}$$

Here, E_m is the maximum value of rotating phasors. Hence, the rms value E of the rotating phasor is $\frac{E_m}{\sqrt{2}}$.

The phase emfs in the polar form are,

$$\boxed{\begin{aligned} \bar{E}_R &= E_R\angle 0^\circ &&= E\angle 0^\circ \\ \bar{E}_Y &= E_Y\angle -120^\circ &&= E\angle -120^\circ \\ \bar{E}_B &= E_B\angle -240^\circ &&= E\angle -240^\circ \end{aligned}}$$

$$E_R = E_Y = E_B = E = \text{Magnitude of phase emf.}$$

Also, $E = E_m/\sqrt{2}$.

In a balanced star-connected supply, the magnitude of line voltage is $\sqrt{3}$ times the magnitude of phase voltage and the line voltage leads the phase voltage by 30°. Therefore, the line voltages of a star-connected source can be written as shown below:

$$\boxed{\begin{aligned} \overline{V}_{RY} &= \sqrt{3}\,E\angle 30^{\circ} = V_{RY}\angle 30^{\circ} = V_L\angle 30^{\circ} \\ \overline{V}_{YB} &= \sqrt{3}\,E\angle -90^{\circ} = V_{YB}\angle -90^{\circ} = V_L\angle -90^{\circ} \\ \overline{V}_{BR} &= \sqrt{3}\,E\angle -210^{\circ} = V_{BR}\angle -210^{\circ} = V_L\angle -210^{\circ} \end{aligned}}$$

where, $V_{RY} = V_{YB} = V_{BR} = \sqrt{3}\,E = V_L$ = Magnitude of line voltage.

Proof for line voltages in star-connected source:

Consider the star-connected source shown in Fig. 4.45.

Now, the phase emf of the source are,

$$\overline{E}_R = E\angle 0^o$$

$$\overline{E}_Y = E\angle -120^o$$

$$\overline{E}_B = E\angle -240^o$$

With reference to Fig. 4.46 in the path RNYR using KVL, we can write,

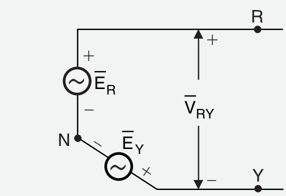


Fig. 4.46.

$$\overline{E}_Y + \overline{V}_{RY} = \overline{E}_R$$

$$\therefore\ \overline{V}_{RY} = \overline{E}_R - \overline{E}_Y$$

$$= E\angle 0^o - E\angle -120^o$$

$$= E\,(1\angle 0^o - 1\angle -120^o) = E(1.5 + j0.866)$$

$$= E(1.732\angle 30^o) = E(\sqrt{3}\angle 30^o)$$

$$\therefore\ \overline{V}_{RY} = \sqrt{3}\,E\angle 30^o$$

With reference to Fig. 4.47 in the path YNBY using KVL, we can write,

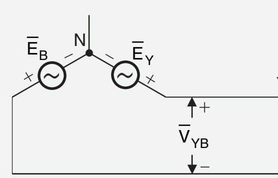


Fig. 4.47.

$$\overline{E}_B + \overline{V}_{YB} = \overline{E}_Y$$

$$\therefore\ \overline{V}_{YB} = \overline{E}_Y - \overline{E}_B$$

$$= E\angle -120^o - E\angle -240^o$$

$$= E\,(1\angle -120^o - 1\angle -240^o) = E(0 - j1.732)$$

$$= E(1.732\angle -90^o) = E(\sqrt{3}\angle -90^o)$$

$$\therefore\ \overline{V}_{YB} = \sqrt{3}\,E\angle -90^o$$

With reference to Fig. 4.48 in the path BNRB using KVL, we can write,

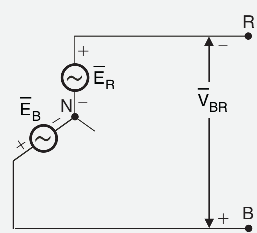


Fig. 4.48.

$$\overline{E}_R + \overline{V}_{BR} = \overline{E}_B$$

$$\therefore\ \overline{V}_{BR} = \overline{E}_B - \overline{E}_R$$

$$= E\angle -240^o - E\angle 0^o$$

$$= E\,(1\angle -240^o - 1\angle 0^o) = E(-1.5 + j0.866)$$

$$= E(1.732\angle 150^o) = E(\sqrt{3}\angle -210^o)$$

$$\therefore\ \overline{V}_{BR} = \sqrt{3}\,E\angle -210^o$$

From the above analysis, we can make the following conclusions,

1. The magnitude of line voltage is $\sqrt{3}$ times the phase voltage.
2. The line voltage leads the phase voltage by 30°.

It is conventional practice to assume one of the line voltages as reference. "*When we take line voltage as reference we can say that the phase voltage lags the line voltage by 30°*". By taking the line voltage $\bar{V}_{RY}$ as reference, we can rewrite the line and phase voltages as shown below:

Line voltages	**Phase voltages**
$\bar{V}_{RY} = V_{RY}\angle 0^\circ = V_L\angle 0^\circ$	$\bar{E}_R = E_R\angle -30^\circ = E\angle -30^\circ$
$\bar{V}_{YB} = V_{YB}\angle -120^\circ = V_L\angle -120^\circ$	$\bar{E}_Y = E_Y\angle -150^\circ = E\angle -150^\circ$
$\bar{V}_{BR} = V_{BR}\angle -240^\circ = V_L\angle -240^\circ = V_L\angle 120^\circ$	$\bar{E}_B = E_B\angle -270^\circ = E\angle -270^\circ = E\angle 90^\circ$

Here, $E_R = E_Y = E_B = E$

$$V_{RY} = V_{YB} = V_{BR} = V_L = \sqrt{3}\,E$$

The rms phasors of the line and phase voltages of a star-connected source for RYB sequence are shown in Fig. 4.49.

> ***Note :*** *To remember suffix for line voltages of RYB sequence write as shown below:*
>
> *R Y B R Y B*
>
> *Now the underlined letters will be the suffix for three consecutive line voltages. The suffix are RY, YB, BR.*

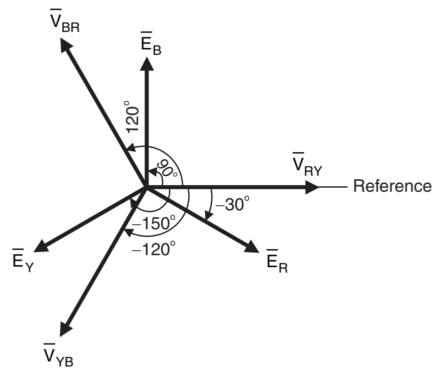


Fig. 4.49 : *Phasor diagram of rms value of line and phase voltages of a star-connected source for RYB sequence.*

From Fig. 4.50, we can say that the current delivered by the source will also flow through the lines. Hence, we can say that the phase current and line current are the same in a star-connected system. The phase difference between the voltage and current depends on the load impedance. Also, we can say that the source current is balanced if the load impedance is balanced. The source current will be unbalanced if the load impedance is unbalanced.

Let ϕ_R, ϕ_Y and ϕ_B be the phase difference between phase voltage and current. Now, the phase currents can be written as,

$$\bar{I}_R = I_R\angle(-30^\circ \pm \phi_R)$$
$$\bar{I}_Y = I_Y\angle(-150^\circ \pm \phi_Y)$$
$$\bar{I}_B = I_B\angle(-270^\circ \pm \phi_B)$$

"–" for lagging power factor
"+" for leading power factor.

For a balanced system,

$$I_R = I_Y = I_B = I$$
$$\phi_R = \phi_Y = \phi_B = \phi$$

Line and Phase Voltages of RBY Sequence

The line voltages of RBY sequence are $\overline{V}_{RB}$, $\overline{V}_{BY}$ and $\overline{V}_{YR}$. By taking $\overline{V}_{RB}$ as the reference phasor, the line and phase voltages of RBY sequence can be expressed as shown below:

Line voltages	Phase voltages
$\overline{V}_{RB} = V_{RB}\angle 0^\circ = V_L\angle 0^\circ$	$\overline{E}_R = E_R\angle -30^\circ = E\angle -30^\circ$
$\overline{V}_{BY} = V_{BY}\angle -120^\circ = V_L\angle -120^\circ$	$\overline{E}_B = E_B\angle -150^\circ = E\angle -150^\circ$
$\overline{V}_{YR} = V_{YR}\angle -240^\circ = V_L\angle -240^\circ = V_L\angle 120^\circ$	$\overline{E}_Y = E_Y\angle -270^\circ = E\angle -270^\circ = E\angle 90^\circ$

Here, $E_R = E_B = E_Y = E$

$$V_{RB} = V_{BY} = V_{YR} = V_L = \sqrt{3}\,E$$

The rms phasors of the line and phase voltages of a star-connected source for RBY sequence are shown in Fig. 4.50.

> ***Note :** To remember suffix for line voltages of RBY sequence write as shown below:*
>
> *R B Y R B Y*
>
> *Now the underlined letters will be the suffix for the three consecutive line voltages. The suffix are RB, BY and YR.*

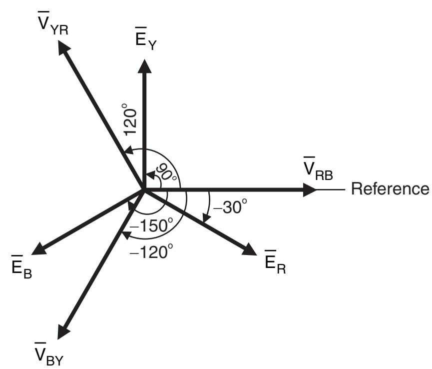


***Fig. 4.50 :** Phasor diagram of rms value of line and phase voltages of a star-connected source for RBY sequence.*

4.21.2 Star-connected Source Four-Wire System

Since neutral point is available in a star-connected system, it is possible to have a four-wire supply system. In a four-wire star-connected source, in addition to three terminals for phase, there will be a fourth terminal called neutral, as shown in Fig. 4.51.

With reference to Fig. 4.51, using KCL we can write,

$$\overline{I}_R + \overline{I}_Y + \overline{I}_B = \overline{I}_N$$

"In a balanced system, the sum of three-phase currents is zero and so the neutral current is zero".

Therefore, in a balanced system,

$$\boxed{\overline{I}_R + \overline{I}_Y + \overline{I}_B = 0 \text{ and so } \overline{I}_N = 0}$$

"In an unbalanced system, the sum of three-phase currents is not zero and so there is a definite neutral current flowing through the neutral wire and the neutral current is equal to the sum of all the phase currents".

Therefore, in an unbalanced system,

$$\boxed{\text{Neutral current, } \bar{I}_N = \bar{I}_R + \bar{I}_Y + \bar{I}_B}$$

In a four-wire system, the relation between phase and line voltage and the relation between phase and line current remains the same as that of a three-wire system.

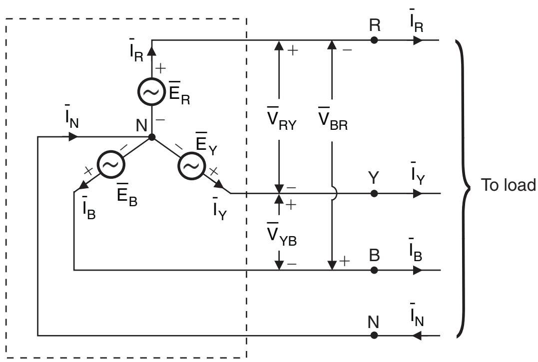


***Fig. 4.51 :** Three-phase four-wire star-connected source with conventional polarity of voltages and direction of currents for RYB sequence.*

Points to remember

1. The voltages are always balanced in star-connected sources.
2. The currents may be balanced or unbalanced depending on load.
3. The phase and line currents are the same in a star system.
4. The magnitude of line voltage is $\sqrt{3}$ times the magnitude of phase voltage.
5. The phase difference between the line voltage and phase voltage of source is 30°. The phase voltage of source lags line voltage by 30°.
6. In a balanced four-wire system, the neutral current is zero.

4.21.3 Delta-connected Source

Let the phase sequence of source emfs be RYB.

Let, e_R, e_Y, e_B = Instantaneous values of source emfs

$\bar{E}_R, \bar{E}_Y, \bar{E}_B$ = Rms values of source phase emfs

E_R, E_Y, E_B = Magnitude of rms values of source phase emfs

$\bar{V}_{RY}, \bar{V}_{YB}, \bar{V}_{BR}$ = Rms values of line voltages

V_{RY}, V_{YB}, V_{BR} = Magnitude of rms values of line voltages

$\bar{I}_{RY}, \bar{I}_{YB}, \bar{I}_{BR}$ = Rms values of phase currents

I_{RY}, I_{YB}, I_{BR} = Magnitude of rms values of phase currents

$\bar{I}_R, \bar{I}_Y, \bar{I}_B$ = Rms values of line currents

I_R, I_Y, I_B = Magnitude of rms values of line currents.

A three-phase delta-connected source with polarity (or sign) of voltages and direction of currents are shown in Fig. 4.52.

The instantaneous values of three emfs for RYB sequence with R-phase emf as reference are,

$$e_R = E_m \sin \omega t$$
$$e_Y = E_m \sin (\omega t - 120^\circ)$$
$$e_B = E_m \sin (\omega t - 240^\circ)$$

Here, E_m is the maximum value of rotating phasors. Hence, the rms value E of the rotating phasor is $E_m/\sqrt{2}$.

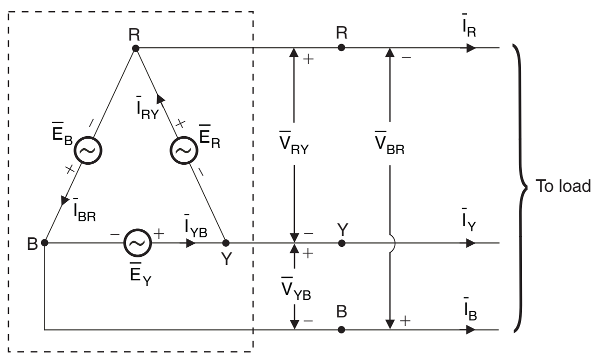


Fig. 4.52 : *Three-phase delta-connected source with conventional polarity of voltages and direction of currents for RYB sequence.*

The phase emfs in polar form are,

$$\boxed{\begin{aligned} \bar{E}_R &= E_R\angle 0^\circ &&= E\angle 0^\circ \\ \bar{E}_Y &= E_Y\angle -120^\circ &&= E\angle -120^\circ \\ \bar{E}_B &= E_B\angle -240^\circ &&= E\angle -240^\circ \end{aligned}}$$

Here, $E_R = E_Y = E_B = E$ = Magnitude of phase emf.

Also, $E = E_m/\sqrt{2}$.

With reference to Fig. 4.52, we can say that the phase voltage $\bar{E}_R$ is the same as line voltage $\bar{V}_{RY}$. Similarly, the phase voltage $\bar{E}_Y$ is the same as line voltage $\bar{V}_{YB}$ and the phase voltage $\bar{E}_B$ is the same as line voltage $\bar{V}_{BR}$. Hence, we can conclude that the phase and line voltages are the same in delta-connected sources. Therefore, the line voltages are,

$$\boxed{\begin{aligned} \bar{V}_{RY} &= V_{RY}\angle 0^{\circ} &&= V_L \angle 0^{\circ} \\ \bar{V}_{YB} &= V_{YB}\angle -120^{\circ} &&= V_L\angle -120^{\circ} \\ \bar{V}_{YB} &= V_{YB}\angle -240^{\circ} &&= V_L\angle -240^{\circ} \end{aligned}}$$

Here, $V_{RY} = V_{YB} = V_{BR} = V_L$ = Magnitude of line voltage.

Also, $V_L = E$.

The rms phasors of the line voltage of a delta-connected source are shown in Fig. 4.53.

(AU May'17, 2 Marks)

With reference to Fig. 4.52, we can say that the phase and line currents are not the same. The line currents can be computed with the knowledge of phase currents by writing KCL equations at nodes R, Y and B.

The phase and line currents can be balanced or unbalanced depending on the load impedance. If the load impedance is balanced then the currents are also balanced. If the load impedance is unbalanced then the currents will be unbalanced. Also, the phase difference between voltage and current depends on the nature of load impedance.

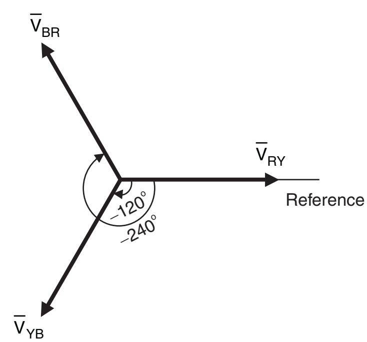


Fig. 4.53 : *Phasor diagram of rms value of line voltages of delta-connected source.*

Let, ϕ_{RY}, ϕ_{YB}, and ϕ_{BR} be the phase differences between phase voltage and current. Now, the phase currents can be written as,

$$\boxed{\begin{aligned} \bar{I}_{RY} &= I_{RY}\angle(0^{\circ} \pm \phi_{RY}) \\ \bar{I}_{YB} &= I_{YB}\angle(-120^{\circ} \pm \phi_{YB}) \\ \bar{I}_{BR} &= I_{BR}\angle(-240^{\circ} \pm \phi_{BR}) \end{aligned}}$$

"–" for lagging power factor "+" for leading power factor.

For a balanced system,

$$I_{RY} = I_{YB} = I_{BR} = I$$

$$\phi_{RY} = \phi_{YB} = \phi_{BR} = \phi$$

The relation between phase and line currents in a balanced delta system can be studied by considering the load as purely resistive, so that the phase voltage and current are in-phase.

Therefore, $\phi_{RY} = \phi_{YB} = \phi_{BR} = \phi = 0^{\circ}$.

Now, the balanced phase currents for RYB sequence can be written as,

$$\bar{I}_{RY} = I_{RY}\angle 0^{\circ} = I\angle 0^{\circ}$$

$$\bar{I}_{YB} = I_{YB}\angle -120^{\circ} = I\angle -120^{\circ}$$

$$\bar{I}_{BR} = I_{BR}\angle -240^{\circ} = I\angle -240^{\circ}$$

where, $I_{RY} = I_{YB} = I_{BR} = I$ = Magnitude of phase current.

In a balanced delta-connected source, the magnitude of line current is $\sqrt{3}$ times the magnitude of source current and the line current lags the phase current by 30°. Therefore, for a purely resistive load, the line currents can be expressed as shown below:

$$\boxed{\begin{aligned} \bar{I}_R &= \sqrt{3}\, I\angle -30^\circ &= I_R\angle -30^\circ &= I_L\angle -30^\circ \\ \bar{I}_Y &= \sqrt{3}\, I\angle -150^\circ &= I_Y\angle -150^\circ &= I_L\angle -150^\circ \\ \bar{I}_B &= \sqrt{3}\, I\angle -270^\circ &= I_B\angle -150^\circ &= I_L\angle -270^\circ \end{aligned}}$$

where, $I_R = I_Y = I_B = I_L = \sqrt{3}\, I$

Proof for line currents in delta-connected source:

Consider the delta-connected source shown in Fig. 4.52.

Now, the phase currents when the load is purely resistive are,

$$\bar{I}_{RY} = I\angle 0^o$$

$$\bar{I}_{YB} = I\angle -120^o$$

$$\bar{I}_{BR} = I\angle -240^o$$

With reference to Fig. 4.54 in the node-R using KCL, we can write,

$$\bar{I}_R + \bar{I}_{BR} = \bar{I}_{RY}$$

$$\therefore\ \bar{I}_R = \bar{I}_{RY} - \bar{I}_{BR}$$

$$= I\angle 0^o - I\angle -240^o$$

$$= I\,(1\angle 0^o - 1\angle -240^o) = I(1.5 - j0.866)$$

$$= I(1.732\angle -30^o) = I(\sqrt{3}\angle -30^o)$$

$$\boxed{\therefore\ \bar{I}_R = \sqrt{3}\, I\angle -30^o}$$

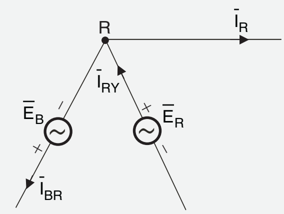


Fig. 4.54.

With reference to Fig. 4.55 in the node-Y using KCL, we can write,

$$\bar{I}_Y + \bar{I}_{RY} = \bar{I}_{YB}$$

$$\therefore\ \bar{I}_Y = \bar{I}_{YB} - \bar{I}_{RY}$$

$$= I\angle -120^o - I\angle 0^o$$

$$= I\,(1\angle -120^o - 1\angle 0^o) = I(-1.5 - j0.866)$$

$$= I(1.732\angle -150^o) = I(\sqrt{3}\angle -150^o)$$

$$\boxed{\therefore\ \bar{I}_Y = \sqrt{3}\, I\angle -150^o}$$

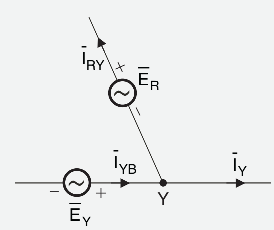


Fig. 4.55.

With reference to Fig. 4.56 in the node-B using KCL, we can write,

$$\bar{I}_B + \bar{I}_{YB} = \bar{I}_{BR}$$

$$\therefore \bar{I}_B = \bar{I}_{BR} - \bar{I}_{YB}$$

$$= I\angle -240^o - I\angle -120^o$$

$$= I(1\angle -240^o - 1\angle -120^o) = I(0 + j1.732)$$

$$= I(1.732\angle 90^o) = I(\sqrt{3}\angle -270^o)$$

$$\boxed{\therefore \bar{I}_B = \sqrt{3}\, I\angle -270^o}$$

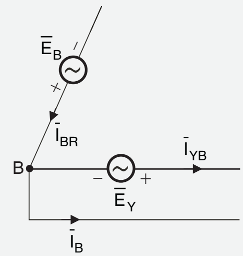


Fig. 4.56.

From the above analysis we can make following conclusion for balanced loads:

1. The magnitude of line current is $\sqrt{3}$ times the phase current.
2. The line current lags the phase current by 30°.

Line and Phase Voltages of RBY Sequence

The line voltages of RBY sequence are $\bar{V}_{RB}$, $\bar{V}_{BY}$, and $\bar{V}_{YR}$. By taking $\bar{V}_{RB}$ as the reference phasor, the line and phase voltages of RBY sequence can be expressed as shown below:

Line voltages	**Phase voltages**
$\bar{V}_{RB} = V_{RB}\angle 0^o = V_L\angle 0^o$	$\bar{E}_R = E_R\angle 0^o = E\angle 0^o$
$\bar{V}_{BY} = V_{BY}\angle -120^o = V_L\angle -120^o$	$\bar{E}_B = E_B\angle -120^o = E\angle -120^o$
$\bar{V}_{YR} = V_{YR}\angle -240^o = V_L\angle -240^o$	$\bar{E}_Y = E_Y\angle -240^o = E\angle -240^o$

(AU May'17, 2 Marks)

Here, $V_{RB} = V_{BY} = V_{YR} = V_L$

$E_R = E_B = E_Y = E = V_L$

The rms phasors of the line voltages of a delta-connected source for RBY sequence are shown in Fig. 4.57.

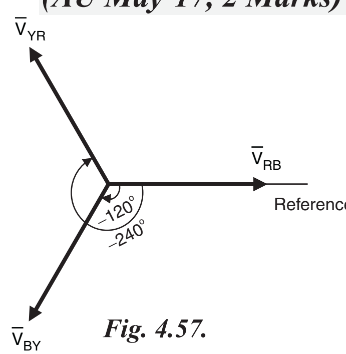


Fig. 4.57.

Points to remember

1. The voltages are always balanced in delta-connected sources.
2. The currents may be balanced or unbalanced depending on load.
3. The phase and line voltages are the same in a delta system.
4. The magnitude of line current is $\sqrt{3}$ times the magnitude of phase current.
5. The line current lags the phase current by 30° in a balanced delta system.

4.22 Three-Phase Loads

The three-phase loads can be connected in star or delta and the load impedance may be balanced or unbalanced. In a **balanced load**, the magnitude of load impedance of each phase will be equal and also the load impedance angle of each phase will be the same. In an **unbalanced load**, the load impedance of each phase may have different magnitude and/or different impedance angle. Three-phase loads can be classified as shown below. The balanced and unbalanced loads of star and delta systems are shown in Fig. 4.58.

Three phase load

- Balanced load
  - Four-wire star-connected load
  - Three-wire star-connected load
  - Delta-connected load
- Unbalanced load
  - Four-wire star-connected load
  - Three-wire star-connected load
  - Delta-connected load

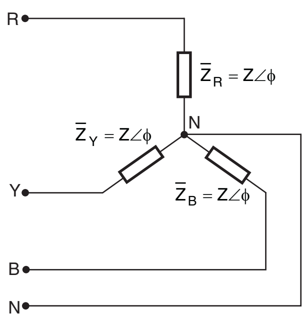


Fig. a : *Four-wire star-connected balanced load.*

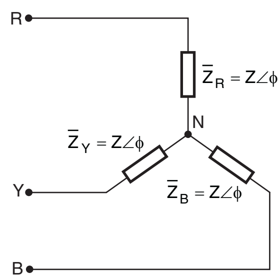


Fig. b : *Three-wire star-connected balanced load.*

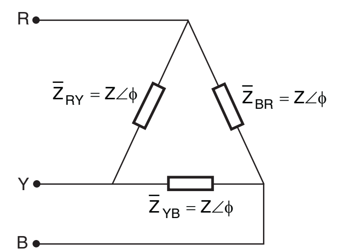


Fig. c : *Delta-connected balanced load.*

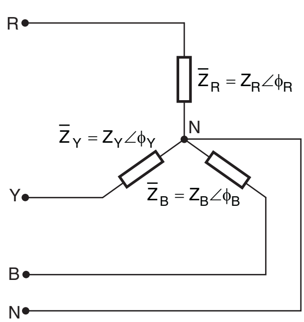


Fig. d : *Four-wire star-connected unbalanced load.*

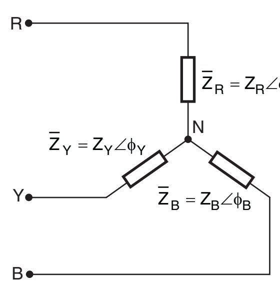


Fig. e : *Three-wire star -connected unbalanced load.*

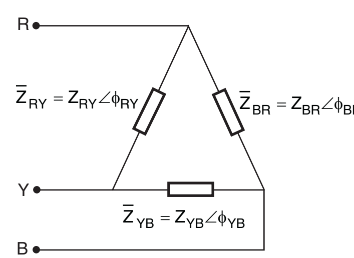


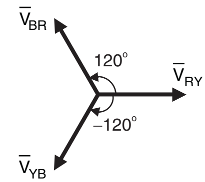
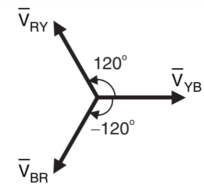
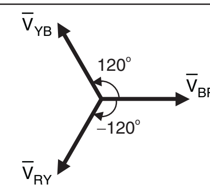
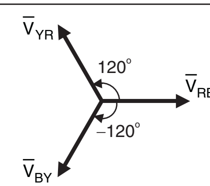
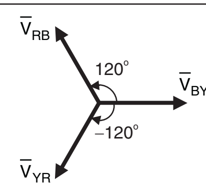
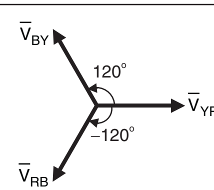
Fig. f : *Delta-connected unbalanced load.*

Fig. 4.58 : *Three-phase loads.*

4.22.1 Choice of Reference Phasor in Analysis of Three-Phase Circuits

In the analysis of three-phase circuits, it is conventional practice to choose one of the line voltages of the source as the **reference phasor**. In an RYB sequence, there are three line voltages and so we have three choices for reference phasor. In an RBY sequence, there are three line voltages and so we have another three choices for reference phasor. The line voltages for six possible choices for the reference phasor are listed in Table 4.4.

Table 4.4 : Choice of Reference Phasor

Phase sequence	Reference phasor	Line voltages	Phasor diagram
RYB	$\overline{V}_{RY}$	$\overline{V}_{RY} = V_L\angle 0^o$ $\overline{V}_{YB} = V_L\angle -120^o$ $\overline{V}_{BR} = V_L\angle -240^o = V_L\angle 120^o$	
RYB	$\overline{V}_{YB}$	$\overline{V}_{YB} = V_L\angle 0^o$ $\overline{V}_{BR} = V_L\angle -120^o$ $\overline{V}_{RY} = V_L\angle -240^o = V_L\angle 120^o$	
RYB	$\overline{V}_{BR}$	$\overline{V}_{BR} = V_L\angle 0^o$ $\overline{V}_{RY} = V_L\angle -120^o$ $\overline{V}_{YB} = V_L\angle -240^o = V_L\angle 120^o$	
RBY	$\overline{V}_{RB}$	$\overline{V}_{RB} = V_L\angle 0^o$ $\overline{V}_{BY} = V_L\angle -120^o$ $\overline{V}_{YR} = V_L\angle -240^o = V_L\angle 120^o$	
RBY	$\overline{V}_{BY}$	$\overline{V}_{BY} = V_L\angle 0^o$ $\overline{V}_{YR} = V_L\angle -120^o$ $\overline{V}_{RB} = V_L\angle -240^o = V_L\angle 120^o$	
RBY	$\overline{V}_{YR}$	$\overline{V}_{YR} = V_L\angle 0^o$ $\overline{V}_{RB} = V_L\angle -120^o$ $\overline{V}_{BY} = V_L\angle -240^o = V_L\angle 120^o$	

4.23 Analysis of Balanced Loads

4.23.1 Four-Wire Star-connected Balanced Load

Let us assume a phase sequence of RYB. Let the reference phasor be $\bar{V}_{RY}$. The three-phase four-wire star-connected load with conventional polarity of voltages and direction of currents is shown in Fig. 4.59.

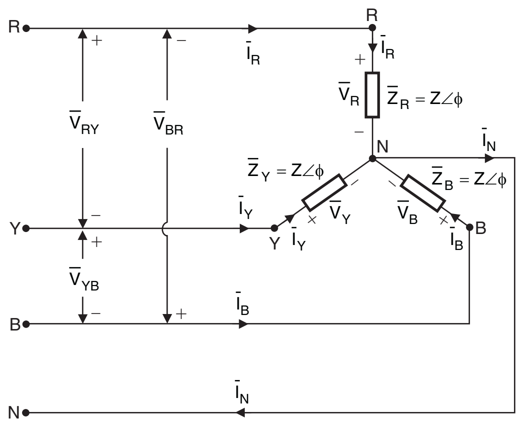


Fig. 4.59 : Four-wire star-connected balanced load with conventional polarity of voltages and direction of currents for RYB sequencce.

The line voltages of the supply/source for the RYB sequence are,

$$\bar{V}_{RY} = V_L\angle 0^o$$

$$\bar{V}_{YB} = V_L\angle -120^o$$

$$\bar{V}_{BR} = V_L\angle -240^o$$

where, V_L= Magnitude of line voltage.

Since the load impedance is balanced, the phase voltages of the load will be balanced. Since the load neutral is tied to source neutral, the phase voltages of the load will be the same as that of the phase voltages of the source. Hence, we can say that the magnitude of the phase voltage is $1/\sqrt{3}$ times the magnitude of the line voltage and the phase voltage lag behind the line voltage by 30^o. Therefore, the phase voltages of the load are,

$$\boxed{\begin{aligned} \bar{V}_R &= \frac{V_L}{\sqrt{3}}\angle(0^o - 30^o) &&= V\angle -30^o \\ \bar{V}_Y &= \frac{V_L}{\sqrt{3}}\angle(-120^o - 30^o) &&= V\angle -150^o \\ \bar{V}_B &= \frac{V_L}{\sqrt{3}}\angle(-240^o - 30^o) &&= V\angle -270^o \end{aligned}}$$

where, $V = \frac{V_L}{\sqrt{3}}$ = Magnitude of phase voltgage.

The phase currents are given by the ratio of phase voltage and phase impedance (Ohm's law applied to ac circuit). Therefore, the phase currents are,

$$\boxed{\begin{aligned}
\bar{I}_R &= \frac{\bar{V}_R}{\bar{Z}_R} = \frac{V\angle -30^\circ}{Z\angle\phi} = \frac{V}{Z}\angle(-30^\circ-\phi) = I\angle(-30^\circ-\phi)\\
\bar{I}_Y &= \frac{\bar{V}_Y}{\bar{Z}_Y} = \frac{V\angle -150^\circ}{Z\angle\phi} = \frac{V}{Z}\angle(-150^\circ-\phi) = I\angle(-150^\circ-\phi)\\
\bar{I}_B &= \frac{\bar{V}_B}{\bar{Z}_B} = \frac{V\angle -270^\circ}{Z\angle\phi} = \frac{V}{Z}\angle(-270^\circ-\phi) = I\angle(-270^\circ-\phi)
\end{aligned}}$$

where, $I = \frac{V}{Z}$ Magnitude of phase currets.

Since the load is balanced, the neutral current will be zero.

$$\boxed{\therefore \quad \text{Neutral current, } \bar{I}_N = 0.}$$

In a star system, the line currents are the same as phase currents. Therefore, the line currents are,

$$\boxed{\begin{aligned}
\bar{I}_R &= I_L\angle(-30^\circ-\phi)\\
\bar{I}_Y &= I_L\angle(-150^\circ-\phi)\\
\bar{I}_B &= I_L\angle(-270^\circ-\phi)
\end{aligned}}$$

where, $I_L = I$ = Magnitude of line current.

The power P consumed by a balanced three-phase star-connected load is given by,

$$\boxed{P = 3\,V\,I\cos\phi} \quad \text{or} \quad \boxed{P = \sqrt{3}\,V_L\,I_L\cos\phi}$$

Proof for power consumed by a balanced star-connected load:

Let, P = *Power consumed by a balanced three-phase load*

ϕ_1 = *Phase difference between* $\bar{V}_R$ *and* $\bar{I}_R$

ϕ_2 = *Phase difference between* $\bar{V}_Y$ *and* $\bar{I}_Y$

ϕ_3 = *Phase difference between* $\bar{V}_B$ *and* $\bar{I}_B$

Now,

$$P = \begin{matrix}\text{Power consumed}\\ \text{by R-phase load}\end{matrix} + \begin{matrix}\text{Power consumed}\\ \text{by Y-phase load}\end{matrix} + \begin{matrix}\text{Power consumed}\\ \text{by B-phase load}\end{matrix}$$

$$\boxed{\begin{aligned}
|\bar{V}_R| = |\bar{V}_Y| = |\bar{V}_B| = V\\
|\bar{I}_R| = |\bar{I}_Y| = |\bar{I}_B| = I\\
\phi_1 = \phi_2 = \phi_3 = \phi
\end{aligned}}$$

$$= |\bar{V}_R||\bar{I}_R|\cos\phi_1 + |\bar{V}_Y||\bar{I}_Y|\cos\phi_2 + |\bar{V}_B||\bar{I}_B|\cos\phi_3$$

$$= VI\cos\phi + VI\cos\phi + VI\cos\phi$$

$$= 3\,VI\cos\phi \qquad \text{..... (4.50)}$$

$$\boxed{\text{In balanced star system, } V = \frac{V_L}{\sqrt{3}} \text{ and } I = I_L}$$

$$= 3\frac{V_L}{\sqrt{3}}\,I_L\cos\phi$$

$$= \sqrt{3}\,V_L\,I_L\cos\phi \qquad \text{..... (4.51)}$$

4.23.2 Three-Wire Star-connected Balanced Load

Let us assume a phase sequence of RYB. Let the reference phasor be $\overline{V}_{RY}$. The three-phase three-wire star-connected load with conventional polarity of voltages and direction of currents is shown in Fig. 4.60.

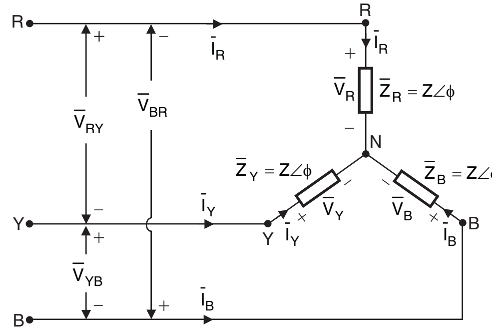


Fig. 4.60 : *Three-wire star-connected balanced load with conventional polarity of voltages and direction of currents for RYB sequence.*

The analysis of three-wire star-connected balanced load and four-wire star-connected balanced load are one and the same because both the source and load neutrals will be at zero potential when the source and load are balanced. In balanced loads, the physical connection of source and load neutral by the neutral line has no significance. Hence, for the analysis of three-wire balanced star-connected loads, follow the steps given in Section 4.23.1.

4.23.3 Delta-connected Balanced Load

Let us assume a phase sequence of RYB. Let the reference phasor be $\overline{V}_{RY}$. The three-phase delta-connected load with conventional polarity of voltages and direction of currents is shown in Fig. 4.61.

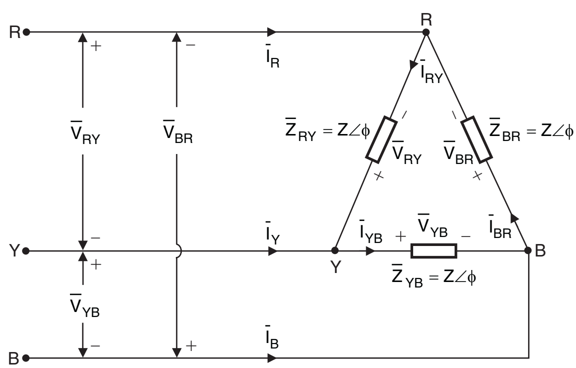


Fig. 4.61 : *Delta-connected balanced load with conventional polarity of voltages and direction of currents for RYB sequence.*

The line voltages of the supply/source for RYB sequence are,

$$\boxed{\begin{aligned}\overline{V}_{RY} &= V_L\angle 0^\circ\\ \overline{V}_{YB} &= V_L\angle -120^\circ\\ \overline{V}_{BR} &= V_L\angle -240^\circ\end{aligned}}$$

where, V_L = Magnitude of line voltage.

In delta-connected loads, the impedances are connected between two lines. Hence, the voltage across the impedance connected between two lines will be same as that of the line voltage between those two lines. Therefore, the phase voltages will be the same as that of the line voltages of the source. The phase voltages are,

$$\boxed{\begin{aligned}\overline{V}_{RY} &= V\angle 0^o \\ \overline{V}_{YB} &= V\angle -120^o \\ \overline{V}_{BR} &= V\angle -240^o\end{aligned}}$$

where, $V = V_L$ = Magnitude of phase voltage.

The phase currents are given by the ratio of phase voltage and phase impedance (Ohm's law applied to an ac circuit). Therefore the phase currents are,

$$\boxed{\begin{aligned}\bar{I}_{RY} &= \frac{\overline{V}_{RY}}{\overline{Z}_{RY}} = \frac{V\angle 0^o}{Z\angle\phi} = \frac{V}{Z}\angle -\phi = I\angle -\phi \\ \bar{I}_{YB} &= \frac{\overline{V}_{YB}}{\overline{Z}_{YB}} = \frac{V\angle -120^o}{Z\angle\phi} = \frac{V}{Z}\angle(-120^o-\phi) = I\angle(-120^o-\phi) \\ \bar{I}_{BR} &= \frac{\overline{V}_{BR}}{\overline{Z}_{BR}} = \frac{V\angle -240^o}{Z\angle\phi} = \frac{V}{Z}\angle(-240^o-\phi) = I\angle(-240^o-\phi)\end{aligned}}$$

where, $I = \dfrac{V}{Z}$ = Magnitude of phase current.

In balanced delta-connected loads, the relation between line and phase currents will be the same as that in balanced delta-connected source. Hence, we can say that the magnitude of line current will be $\sqrt{3}$ times the phase current and the line current will lag the phase current by 30^o. Therefore, the line currents are,

$$\boxed{\begin{aligned}\bar{I}_R &= \sqrt{3}\, I\angle(-\phi-30^o) = I_L\angle(-30^o-\phi) \\ \bar{I}_Y &= \sqrt{3}\, I\angle(-120^o-\phi-30^o) = I_L\angle(-150^o-\phi) \\ \bar{I}_B &= \sqrt{3}\, I\angle(-240^o-\phi-30^o) = I_L\angle(-270^o-\phi)\end{aligned}}$$

where, $I_L = \sqrt{3}\, I$ = Magnitude of line current.

Alternatively, line currents can be computed from the following relation:

$$\boxed{\begin{aligned}\bar{I}_R &= \bar{I}_{RY} - \bar{I}_{BR} \\ \bar{I}_Y &= \bar{I}_{YB} - \bar{I}_{RY} \\ \bar{I}_B &= \bar{I}_{BR} - \bar{I}_{YB}\end{aligned}}$$

The power P consumed by a balanced three-phase delta-connected load is given by,

$$\boxed{P = 3\,V\,I\cos\phi} \quad \text{or} \quad \boxed{P = \sqrt{3}\,V_L\,I_L\cos\phi}$$

Proof for power consumed by a balanced delta-connected load:

Let, P = Power consumed by a balanced three-phase load

ϕ_1 = Phase difference between $\overline{V}_{RY}$ and $\overline{I}_{RY}$

ϕ_2 = Phase difference between $\overline{V}_{YB}$ and $\overline{I}_{YB}$

ϕ_3 = Phase difference between $\overline{V}_{BR}$ and $\overline{I}_{BR}$

Now,

$$P = \begin{matrix}\text{Power consumed}\\ \text{by R-phase load}\end{matrix} + \begin{matrix}\text{Power consumed}\\ \text{by Y-phase load}\end{matrix} + \begin{matrix}\text{Power consumed}\\ \text{by B-phase load}\end{matrix}$$

$$= |\overline{V}_{RY}||\overline{I}_{RY}|\cos\phi_1 + |\overline{V}_{YB}||\overline{I}_{YB}|\cos\phi_2 + |\overline{V}_{BR}||\overline{I}_{BR}|\cos\phi_3$$

$$\boxed{\begin{matrix}|\overline{V}_{RY}| = |\overline{V}_{YB}| = |\overline{V}_{BR}| = V\\ |\overline{I}_{RY}| = |\overline{I}_{YB}| = |\overline{I}_{BR}| = I\\ \phi_1 = \phi_2 = \phi_3 = \phi\end{matrix}}$$

$$= VI\cos\phi + VI\cos\phi + VI\cos\phi$$

$$= 3VI\cos\phi \qquad \text{..... (4.52)}$$

$$= 3V_L\frac{I_L}{\sqrt{3}}\cos\phi$$

$$\boxed{\begin{matrix}\text{In balanced delta system,}\\ I = \dfrac{I_L}{\sqrt{3}} \text{ and } V = V_L\end{matrix}}$$

$$= \sqrt{3}\,V_L I_L\cos\phi \qquad \text{..... (4.53)}$$

Note : *From equations (4.50) to (4.53), we can say that the expression (or equation) for calculating power in a balanced star and delta load is the same.*

4.23.4 Power Consumed by Three Equal Impedances in Star and Delta

For same load impedance and supply voltage the power consumed by a delta-connected load will be three times the power consumed by a star-connected load. Alternatively, the power consumed by a star-connected load will be one-third the power consumed by a delta-connected load.

Let, P_D = Power consumed in delta connection

P_S = Power consumed in star connection

$$\text{Now,}\quad \boxed{P_D = 3\,P_S} \quad \text{or} \quad \boxed{P_S = \frac{1}{3}P_D} \qquad \text{..... (4.54)}$$

Proof:

Let us consider three equal impedances $Z\angle\phi$ connected in star. Let the supply voltage be V_L volts.

$$\text{Now, phase current in star, } I_S = \frac{\frac{V_L}{\sqrt{3}}}{Z} = \frac{V_L}{\sqrt{3}\,Z}$$

$$\text{Line current in star, } I_{L,S} = I_S = \frac{V_L}{\sqrt{3}\,Z}$$

$$\therefore \quad P_S = \sqrt{3}\,V_L I_{L,S}\cos\phi = \sqrt{3}\,V_L\frac{V_L}{\sqrt{3}\,Z}\cos\phi = \frac{V_L^2}{Z}\cos\phi \qquad \text{..... (4.55)}$$

Let us reconnect the three equal impedances $Z\angle\phi$ in delta. Let the supply voltage be V_L volts.

Now, phase voltage in delta, $V_D = V_L$

$$\text{Phase current in delta, } I_D = \frac{V_D}{Z} = \frac{V_L}{Z}$$

$$\text{Line current in delta, } I_{L,D} = \sqrt{3}\, I_D = \sqrt{3}\,\frac{V_L}{Z}$$

$$\therefore \quad P_D = \sqrt{3}\, V_L I_{L,D} \cos\phi = \sqrt{3}\, V_L \sqrt{3}\,\frac{V_L}{Z}\cos\phi = 3\frac{V_L^2}{Z}\cos\phi \qquad \text{..... (4.56)}$$

From equations (4.55) and (4.56), we can say that,

$$P_D = 3\, P_S$$

$$\text{or} \quad P_S = \frac{1}{3} P_D$$

4.24 Analysis of Unbalanced Loads

In the analysis of unbalanced loads, the supply/source is always assumed to be balanced. Moreover, the line voltages of the source and load are the same. Therefore, we can say that the line voltages are always balanced, for any type of load.

4.24.1 Four-Wire Star-connected Unbalanced Load

Let us assume a phase sequence of RYB. Let the reference phasor be $\overline{V}_{RY}$. The three-phase, four-wire unbalanced star-connected load with conventional polarity of voltages and direction of currents is shown in Fig. 4.62.

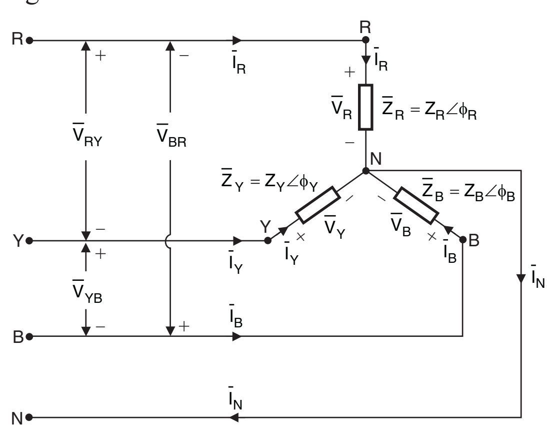


Fig. 4.62 : *Four-wire star-connected unbalanced load with conventional polarity of voltages and direction of currents.*

The line voltages of the supply/source for the RYB sequence are,

$$\boxed{\begin{aligned} \overline{V}_{RY} &= V_L\angle 0^\circ \\ \overline{V}_{YB} &= V_L\angle -120^\circ \\ \overline{V}_{BR} &= V_L\angle -240^\circ \end{aligned}}$$

where, V_L = Magnitude of line voltage.

In a four-wire system, the load neutral is tied to the source neutral and so the phase voltages of source and load are the same. Since the source voltages are balanced, the phase voltages of the load are also balanced. Here, *"the significance of connecting the load and source neutral is that, it will not allow voltage unbalance even though the load is unbalanced"*.

Therefore, in a four-wire three-phase star-connected unbalanced load system, the relation between line and phase voltages will be the same as that in balanced loads. The magnitude of phase voltage will be $1/\sqrt{3}$ times the line voltage and the phase voltage will lag behind the line voltage by 30^o. The phase voltages are,

$$\boxed{\begin{aligned} \bar{V}_R &= \frac{V_L}{\sqrt{3}} \angle(0^o - 30^o) = V\angle -30^o \\ \bar{V}_Y &= \frac{V_L}{\sqrt{3}} \angle(-120^o - 30^o) = V\angle -150^o \\ \bar{V}_B &= \frac{V_L}{\sqrt{3}} \angle(-240^o - 30^o) = V\angle -270^o \end{aligned}}$$

$$\text{where, } V = \frac{V_L}{\sqrt{3}} = \text{Magnitude of phase voltage.}$$

The phase currents are given by the ratio of phase voltage and phase impedance (Ohm's law applied to an ac circuit). Therefore, the phase currents are,

$$\boxed{\begin{aligned} \bar{I}_R &= \frac{\bar{V}_R}{\bar{Z}_R} = \frac{V\angle -30^o}{Z_R\angle\phi_R} = \frac{V}{Z_R}\angle(-30^o - \phi_R) = I_R\angle(-30^o - \phi_R) \\ \bar{I}_Y &= \frac{\bar{V}_Y}{\bar{Z}_Y} = \frac{V\angle -150^o}{Z_Y\angle\phi_Y} = \frac{V}{Z_Y}\angle(-150^o - \phi_Y) = I_Y\angle(-150^o - \phi_Y) \\ \bar{I}_B &= \frac{\bar{V}_B}{\bar{Z}_B} = \frac{V\angle -270^o}{Z_B\angle\phi_B} = \frac{V}{Z_B}\angle(-270^o - \phi_B) = I_B\angle(-270^o - \phi_B) \end{aligned}}$$

where, I_R, I_Y and I_B are magnitude of R-phase, Y-phase and B-phase currents respectively.

$$\boxed{\text{Neutral current, } \bar{I}_N = \bar{I}_R + \bar{I}_Y + \bar{I}_B} \quad \text{..... (4.57)}$$

In star system, the line currents are the same as phase currents. Therefore, the line currents are,

$$\bar{I}_R = I_R\angle(-30^o - \phi_R)$$

$$\bar{I}_Y = I_Y\angle(-150^o - \phi_Y)$$

$$\bar{I}_B = I_B\angle(-270^o - \phi_B)$$

Let, P = Power consumed by three-phase load.

$$\text{Here, } P = \begin{matrix}\text{Power consumed}\\ \text{by R-phase load}\end{matrix} + \begin{matrix}\text{Power consumed}\\ \text{by Y-phase load}\end{matrix} + \begin{matrix}\text{Power consumed}\\ \text{by B-phase load}\end{matrix}$$

$$= |\bar{V}_R||\bar{I}_R|\cos\phi_1 + |\bar{V}_Y||\bar{I}_Y|\cos\phi_2 + |\bar{V}_B||\bar{I}_B|\cos\phi_3$$

$$= V_L I_R \cos\phi_1 + V_L I_Y \cos\phi_2 + V_L I_B \cos\phi_3 \quad \text{..... (4.58)}$$

where, $|\bar{V}_R| = |\bar{V}_Y| = |\bar{V}_B| = V_L$

ϕ_1 = Phase difference between $\bar{V}_R$ and $\bar{I}_R$

ϕ_2 = Phase difference between $\bar{V}_Y$ and $\bar{I}_Y$

ϕ_3 = Phase difference between $\bar{V}_B$ and $\bar{I}_B$

Also, $\phi_1 = \phi_R$; $\phi_2 = \phi_Y$; $\phi_3 = \phi_B$

> ***Note :*** *The equation,* $P = \sqrt{3}\, V_L I_L \cos\phi$ *cannot be used to calculate power in unbalanced loads.*

4.24.2 Three-Wire Star-connected Unbalanced Load *(AU May'15, 16 Marks)*

Let us assume a phase sequence of RYB. Let the reference phasor be $\bar{V}_{RY}$.

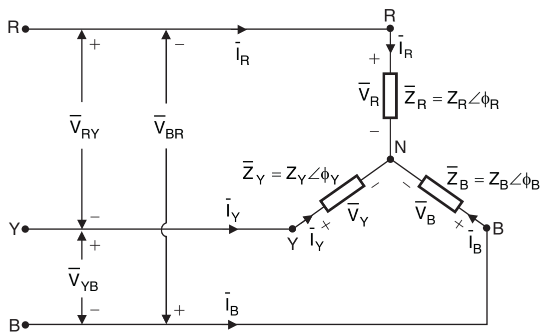


Fig. 4.63 : *Three-wire star-connected unbalanced load with conventional polarity of voltages and direction of currents for an RYB sequence.*

The line voltages of the supply/source for an RYB sequence are,

$$\bar{V}_{RY} = V_L\angle 0^\circ$$

$$\bar{V}_{YB} = V_L\angle -120^\circ$$

$$\bar{V}_{BR} = V_L\angle -240^\circ$$

where, V_L = Magnitude of line voltage.

In an unbalanced star-connected load, it will be easier to solve the line currents by assuming two sources across the lines whose values are equal to the corresponding line voltages. Consider the circuit shown in Fig. 4.64 in which a voltage source of value $\bar{V}_{RY}$ is connected across lines R and Y and a voltage source of value $\bar{V}_{YB}$ is connected across lines Y and B.

The circuit of Fig. 4.64 has two meshes. Hence, we can assume two mesh currents $\bar{I}_1$ and $\bar{I}_2$ as shown in Fig. 4.64. The mesh basis matrix equation is,

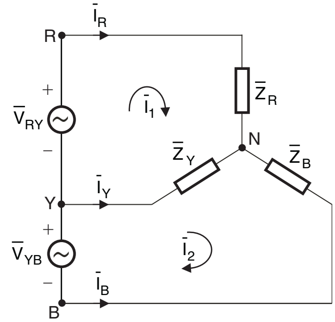


Fig. 4.64.

$$\begin{bmatrix} \bar{Z}_R + \bar{Z}_Y & -\bar{Z}_Y \\ -\bar{Z}_Y & \bar{Z}_B + \bar{Z}_Y \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{V}_{RY} \\ \bar{V}_{YB} \end{bmatrix}$$

Let, $\Delta = \begin{vmatrix} \bar{Z}_R + \bar{Z}_Y & -\bar{Z}_Y \\ -\bar{Z}_Y & \bar{Z}_B + \bar{Z}_Y \end{vmatrix}$; $\Delta_1 = \begin{vmatrix} \bar{V}_{RY} & -\bar{Z}_Y \\ \bar{V}_{YB} & \bar{Z}_B + \bar{Z}_Y \end{vmatrix}$; $\Delta_2 = \begin{vmatrix} \bar{Z}_R + \bar{Z}_Y & \bar{V}_{RY} \\ -\bar{Z}_Y & \bar{V}_{YB} \end{vmatrix}$

Now, $\bar{I}_1 = \dfrac{\Delta_1}{\Delta}$ and $\bar{I}_2 = \dfrac{\Delta_2}{\Delta}$

From the mesh currents, the line currents can be obtained as shown below:

$$\boxed{\begin{aligned} \bar{I}_R &= \bar{I}_1 \\ \bar{I}_Y &= \bar{I}_2 - \bar{I}_1 \\ \bar{I}_B &= -\bar{I}_2 \end{aligned}}$$

In a star-connected load, the phase currents are the same as line currents. Therefore, the phase and line currents in polar form can be written as,

$$\boxed{\begin{aligned} \bar{I}_R &= I_R \angle \gamma_R \\ \bar{I}_Y &= I_Y \angle \gamma_Y \\ \bar{I}_B &= I_B \angle \gamma_B \end{aligned}}$$

where, I_R, I_Y, and I_B are the magnitudes of line and phase currents and γ_R, γ_Y and γ_B are phase angles of line and phase currents with respect to the reference phasor.

Now, the phase voltages are given by the product of phase current and phase impedance (Ohm's law applied to an ac circuit.) Therefore, the phase voltages are,

$$\boxed{\begin{aligned} \bar{V}_R &= \bar{I}_R \bar{Z}_R = V_R \angle \delta_R \\ \bar{V}_Y &= \bar{I}_Y \bar{Z}_Y = V_Y \angle \delta_Y \\ \bar{V}_B &= \bar{I}_B \bar{Z}_B = V_B \angle \delta_B \end{aligned}}$$

where, V_R, V_Y and V_B are magnitudes of phase voltages and δ_R, δ_Y and δ_B are phase angles of phase voltages with respect to the reference phasor.

Let, P = Power consumed by the three-phase load.

$$P = \begin{array}{c}\text{Power consumed} \\ \text{by R-phase load}\end{array} + \begin{array}{c}\text{Power consumed} \\ \text{by Y-phase load}\end{array} + \begin{array}{c}\text{Power consumed} \\ \text{by B-phase load}\end{array}$$

$$= |\bar{V}_R||\bar{I}_R| \cos\phi_1 + |\bar{V}_Y||\bar{I}_Y| \cos\phi_2 + |\bar{V}_B||\bar{I}_B| \cos\phi_3$$

$$= V_R I_R \cos\phi_1 + V_Y I_Y \cos\phi_2 + V_B I_B \cos\phi_3 \quad \text{..... (4.59)}$$

where, $\phi_1 =$ Phase difference between $\bar{V}_R$ and $\bar{I}_R$

$\phi_2 =$ Phase difference between $\bar{V}_Y$ and $\bar{I}_Y$

$\phi_3 =$ Phase difference between $\bar{V}_B$ and $\bar{I}_B$

Here, $\phi_1 = \delta_R - \gamma_R$; $\phi_2 = \delta_Y - \gamma_Y$; $\phi_3 = \delta_B - \gamma_B$.

Also, $\phi_1 = \phi_R$; $\phi_2 = \phi_Y$; $\phi_3 = \phi_B$

4.24.3 Neutral Shift in Star-connected Load

In a three-wire star-connected load, the load neutral is not connected to the source neutral. Hence, when the load is unbalanced, the load neutral is not at zero potential. The voltage of load neutral with respect to source neutral is called **neutral shift voltage** (or **neutral displacement voltage**).

The neutral shift voltage can be obtained by subtracting a phase voltage of load from the corresponding phase emf of the source. Consider the R-phase source and load as shown in Fig. 4.65. Let, N′ be the source neutral and N be the load neutral.

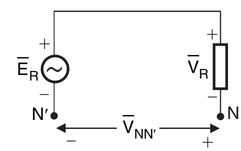


Fig. 4.65.

Let, $\bar{V}_{NN'} =$ Load neutral shift voltage with respect to source neutral.

In a star-connected source, the phase emf will lag behind the line voltage by 30° and the magnitude of phase emf will be $1/\sqrt{3}$ times the magnitude of line voltage. Hence, $\bar{E}_R$ will have a magnitude of $V_L/\sqrt{3}$ and lag behind $\bar{V}_{RY}$ by 30°

$$\therefore \quad \bar{E}_R = \frac{V_L}{\sqrt{3}} \angle -30^\circ$$

With reference to Fig. 4.65, using KVL, we can write,

$$\bar{V}_{NN'} + \bar{V}_R = \bar{E}_R$$

$$\therefore \quad \text{Neutral shift voltage, } \bar{V}_{NN'} = \bar{E}_R - \bar{V}_R \qquad \text{..... (4.60)}$$

4.24.4 Delta-connected Unbalanced Load

Let us assume a phase sequence of RYB. Let the reference phasor be $\bar{V}_{RY}$.

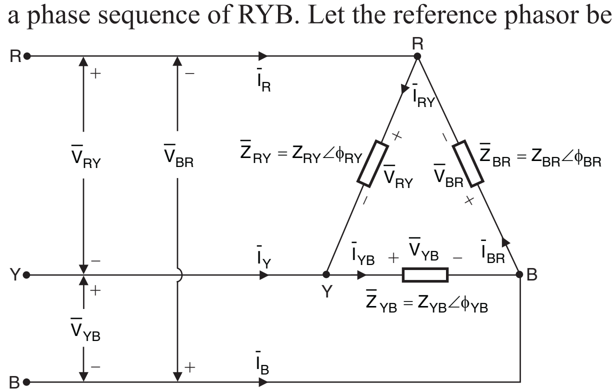


Fig. 4.66 : *Three-phase delta-connected unbalanced load with conventional polarity of voltages and direction of currents for an RYB sequence.*

The line voltages of the supply/source for the RYB sequence are,

$$\boxed{\begin{aligned}\bar{V}_{RY} &= V_L\angle 0^o\\ \bar{V}_{YB} &= V_L\angle -120^o\\ \bar{V}_{BR} &= V_L\angle -240^o\end{aligned}}$$

where, V_L = Magnitude of line voltage.

In delta-connected loads, the impedances are connected between two lines. Hence, the voltage across the impedance connected between two lines will be the same as that of line voltage between those two lines. Therefore, the phase voltages will be the same as that of line voltages of the source. Since the line voltages are balanced, the phase voltages of the load are also balanced even though load impedances are unbalanced. Therefore, the phase voltages are,

$$\boxed{\begin{aligned}\bar{V}_{RY} &= V\angle 0^o\\ \bar{V}_{YB} &= V\angle -120^o\\ \bar{V}_{BR} &= V\angle -240^o\end{aligned}}$$

where, $V = V_L$ = Magnitude of phase voltage.

The phase currents are given by the ratio of phase voltage and phase impedance (Ohm's law applied to an ac circuit). Therefore, the phase currents are,

$$\boxed{\begin{aligned}\bar{I}_{RY} &= \frac{\bar{V}_{RY}}{\bar{Z}_{RY}} = I_{RY}\angle\gamma_{RY}\\ \bar{I}_{YB} &= \frac{\bar{V}_{YB}}{\bar{Z}_{YB}} = I_{YB}\angle\gamma_{YB}\\ \bar{I}_{BR} &= \frac{\bar{V}_{BR}}{\bar{Z}_{BR}} = I_{BR}\angle\gamma_{BR}\end{aligned}}$$

where, I_{RY}, I_{YB} and I_{BR} are magnitude of phase currents and

γ_{RY}, γ_{YB} and γ_{BR} are phase angle of phase currents.

Using KCL at nodes R, Y, and B, the line currents can be calculated as shown below. The line currents are,

$$\boxed{\begin{aligned}\bar{I}_R &= \bar{I}_{RY} - \bar{I}_{BR}\\ \bar{I}_Y &= \bar{I}_{YB} - \bar{I}_{RY}\\ \bar{I}_B &= \bar{I}_{BR} - \bar{I}_{YB}\end{aligned}}$$

Let, P = Power consumed by the three-phase load.

$$P = \begin{matrix}\text{Power consumed}\\ \text{by R-phase load}\end{matrix} + \begin{matrix}\text{Power consumed}\\ \text{by Y-phase load}\end{matrix} + \begin{matrix}\text{Power consumed}\\ \text{by B-phase load}\end{matrix}$$

$$= |\bar{V}_{RY}||\bar{I}_{RY}|\cos\phi_1 + |\bar{V}_{YB}||\bar{I}_{YB}|\cos\phi_2 + |\bar{V}_{BR}||\bar{I}_{BR}|\cos\phi_3$$

$$= V_L I_{RY}\cos\phi_1 + V_L I_{YB}\cos\phi_2 + V_L I_{BR}\cos\phi_3 \qquad \text{..... (4.61)}$$

where, $|\overline{V}_{RY}| = |\overline{V}_{YB}| = |\overline{V}_{BR}| = V_L$

ϕ_1 = Phase difference between $\overline{V}_{RY}$ and $\bar{I}_{RY}$

ϕ_2 = Phase difference between $\overline{V}_{YB}$ and $\bar{I}_{YB}$

ϕ_3 = Phase difference between $\overline{V}_{BR}$ and $\bar{I}_{BR}$

Here, $\phi_1 = 0^\circ - \gamma_{RY}$; $\phi_2 = -120^\circ - \gamma_{YB}$; $\phi_3 = -240^\circ - \gamma_{BR}$

Also, $\phi_1 = \phi_{RY}$; $\phi_2 = \phi_{YB}$; $\phi_3 = \phi_{BR}$

4.25 Power Measurement in Three-Phase Circuits *(AU Dec'15 & '14, 16 Marks)*

Power is generally measured using wattmeters. Logically we may require one wattmeter for measuring power in one-phase and so we may require three wattmeters to measure power in three-phase. But it can be proved that, "*the power in any three-phase load (balanced/unbalanced and star/delta) can be measured using only two wattmeters*".

A wattmeter will have a **c**urrent **c**oil (CC) and a **p**ressure **c**oil (PC). The pressure coil is also called voltage coil. The current coils of the two wattmeters employed for measurement are connected such that they carry any two line currents. The third line in which there is no current coil is called the **common line**.

The voltage coil of a wattmeter is connected between the line to which its current coil is connected and the common line. Similarly, the voltage coil of the other wattmeter is connected between the line to which its current coil is connected and the common line. The possible connections of two wattmeters to measure the three-phase power are shown in Fig. 4.67.

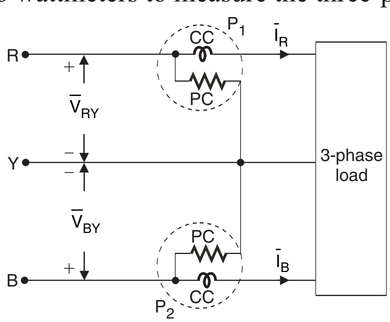


Fig. a : *Wattmeters in lines R and B.*

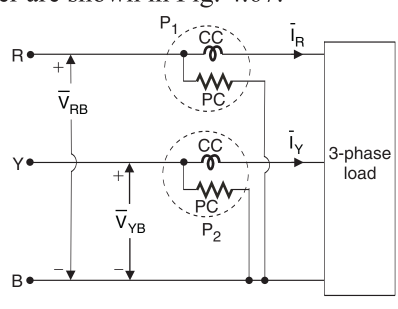


Fig. b : *Wattmeters in lines R and Y.*

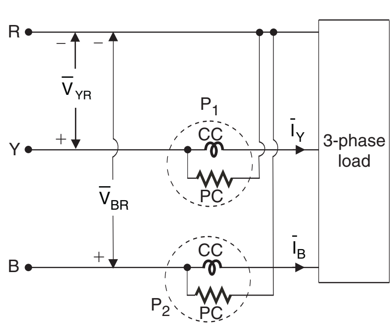


Fig. c : *Wattmeters in lines Y and B.*

Fig. 4.67 : *Possible connections of two wattmeters for measurement of three-phase power.*

4.25.1 Power Measurement in Balanced Load

Consider a balanced three-phase load (star or delta-connected). Let us connect wattmeters in the lines R and B and line Y be the common line for connecting the voltage coil as shown in Fig. 4.68.

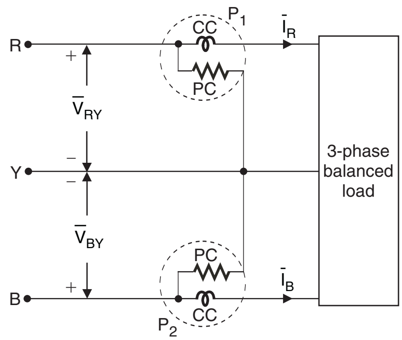


Fig. 4.68 : *Power measurement in a balanced load.*

Let, P_1 = Power measured by wattmeter-1

P_2 = Power measured by wattmeter-2

P = Power consumed by load

ϕ = Load power factor angle

Now, the load power and power factor angle in terms of wattmeter readings P_1 and P_2 are,

$$\boxed{\text{Power, } P = P_1 + P_2} \quad \text{..... (4.62)}$$

$$\boxed{\text{Power factor angle, } \phi = \tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)} \quad \text{..... (4.63)}$$

$$\boxed{\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right]} \quad \text{..... (4.64)}$$

Proof for power measurement in a three-phase load using two wattmeters :

Consider the circuit shown in Fig. 4.68 for power measurement in a three-phase load using two wattmeters.

Now, the current through wattmeter-1 is $\bar{I}_R$ and voltage across its pressure coil is $\bar{V}_{RY}$. Hence, power P_1 measured by wattmeter-1 is,

$$P_1 = |\bar{V}_{RY}| \times |\bar{I}_R| \times \cos\theta_1 \quad \text{..... (4.65)}$$

where, θ_1 = Phase difference between $\bar{V}_{RY}$ and $\bar{I}_R$.

The current through wattmeter-2 is $\bar{I}_B$ and voltage across its pressure coil is $\bar{V}_{BY}$. Hence, power P_2 measured by wattmeter-2 is,

$$P_2 = |\bar{V}_{BY}| \times |\bar{I}_B| \times \cos\theta_2 \quad \text{..... (4.66)}$$

where, θ_2 = Phase difference between $\bar{V}_{BY}$ and $\bar{I}_B$.

For balanced star and delta-connected loads, the line voltages and currents are,

Line voltages	Line currents
$\overline{V}_{RY} = V_L\angle 0^o$	$\overline{I}_R = I_L\angle(-30^o - \phi)$
$\overline{V}_{YB} = V_L\angle -120^o$	$\overline{I}_Y = I_L\angle(-150^o - \phi)$
$\overline{V}_{BR} = V_L\angle -240^o$	$\overline{I}_B = I_L\angle(-270^o - \phi)$

Refer section (4.23.1), (4.23.2) and (4.23.3)

$$\therefore \quad |\overline{V}_{RY}| = V_L \;; \quad |\overline{I}_R| = I_L \text{ and } \quad |\overline{I}_B| = I_L$$

Here, $\overline{V}_{BY} = -\overline{V}_{YB} = 1\angle 180^o \times \overline{V}_{YB} = 1\angle 180^o \times V_L\angle -120^o = V_L\angle(180^o - 120^o)$

$$\therefore \quad \overline{V}_{BY} = V_L\angle 60^o$$

$$\therefore \quad |\overline{V}_{BY}| = V_L$$

Now, $\theta_1 = 0^o - (-30^o - \phi) = 30^o + \phi = \phi + 30^o$

$$\theta_2 = 60^o - (-270^o - \phi) = 330^o + \phi = -30^o + \phi = \phi - 30^o$$

From the above discussions, equations (4.65) and (4.66) for balanced loads (star/delta-connected) can be written as,

$$P_1 = V_L I_L \cos(\phi + 30^o) \qquad \text{..... (4.67)}$$

$$P_2 = V_L I_L \cos(\phi - 30^o) \qquad \text{..... (4.68)}$$

Let us add the power measured by two wattmeters.

$\cos(C+D) + \cos(C-D) = 2\cos C \cos D$
$\cos 30^o = \sqrt{3}/2$

$$\therefore \quad P_1 + P_2 = V_L I_L \cos(\phi + 30^o) + V_L I_L \cos(\phi - 30^o)$$

$$= V_L I_L [\cos(\phi + 30^o) + \cos(\phi - 30^o)]$$

$$= V_L I_L [2\cos\phi \cos 30^o]$$

$$= V_L I_L \left[2\cos\phi \times \frac{\sqrt{3}}{2}\right]$$

$$= \sqrt{3}\, V_L I_L \cos\phi \qquad \text{..... (4.69)}$$

Equation (4.69) is the same as the equation for power in a balanced load. Hence, we can say that the sum of power measured by two wattmeters is equal to the power in a three-phase load.

Power factor

From equations, (4.67) and (4.68) we can write,

$\cos(C+D) - \cos(C-D) = -2\sin C \sin D$

$$P_1 - P_2 = V_L I_L \cos(\phi + 30^o) - V_L I_L \cos(\phi - 30^o)$$

$$= V_L I_L [\cos(\phi + 30^o) - \cos(\phi - 30^o)]$$

$$= V_L I_L [-2\sin\phi \sin 30^o]$$

$$\therefore \quad P_1 - P_2 = V_L I_L \left[-2\sin\phi \times \frac{1}{2}\right]$$

$$= -V_L I_L \sin\phi$$

$$\therefore \quad P_2 - P_1 = V_L I_L \sin\phi \qquad \text{..... (4.70)}$$

On dividing equation (4.70) by equation (4.69) we get,

$$\frac{V_L I_L \sin\phi}{\sqrt{3}\, V_L I_L \cos\phi} = \frac{P_2 - P_1}{P_1 + P_2}$$

$$\boxed{\sin 30^o = \frac{1}{2}}$$

$$\therefore \quad \tan\phi = \sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}$$

$$\therefore \quad \text{Power factor angle,} \quad \phi = \tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right) \qquad \text{..... (4.71)}$$

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right] \qquad \text{..... (4.72)}$$

Equation (4.72) gives the power factor of a balanced load in terms of wattmeter reading. One drawback in this method of power factor estimation is that we cannot determine whether the power factor is lagging or leading (But practically most of the loads are inductive in nature and so we can safely assume that the power factor is lag).

4.25.2 Relation Between Power Factor and Wattmeter Readings

Case i : Wattmeter readings are equal

Let, $P_1 = P_2 = P_x$

$$\therefore \text{ Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right] = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_x - P_x}{P_x + P_x}\right)\right]$$

$$= \cos\left[\tan^{-1} 0\right] = \cos 0^\circ = 1$$

Conclusion: When the wattmeter readings are equal, the power factor is unity.

Case ii : One of the wattmeter readings is zero

Let, $P_1 = 0$ and $P_2 \neq 0$

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right] = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - 0}{0 + P_2}\right)\right]$$

$$= \cos\left[\tan^{-1}\left(\sqrt{3}\right)\right] = \cos 60^\circ = 0.5$$

Let, $P_2 = 0$ and $P_1 \neq 0$

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right] = \cos\left[\tan^{-1}\sqrt{3}\,\frac{0 - P_1}{P_1 + 0}\right]$$

$$= \cos\left[\tan^{-1}\left(-\sqrt{3}\right)\right] = \cos\left[-60^\circ\right] = 0.5$$

Conclusion: When one of the wattmeter readings is zero, the power factor is 0.5.

Case iii : One of the wattmeter readings is negative

Let, $P_1 = -P_x$ and $P_2 = +P_y$

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right] = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_y + P_x}{-P_x + P_y}\right)\right]$$

$$= \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_y + P_x}{P_y - P_x}\right)\right]$$

Let, $P_1 = +P_x$ and $P_2 = -P_y$

$$\therefore \text{ Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right] = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{-P_y - P_x}{P_x - P_y}\right)\right]$$

$$= \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_y + P_x}{P_y - P_x}\right)\right]$$

Let, $\left(\sqrt{3}\,\frac{P_y + P_x}{P_y - P_x}\right) = A$

$\therefore \quad \cos\phi = \cos(\tan^{-1}A)$

Here, $(P_y + P_x) > (P_y - P_x)$, $\quad \therefore\ A > \sqrt{3}$ and $\tan^{-1}A > 60^\circ$

$\boxed{\tan 60^\circ = \sqrt{3}}$

Since, $\tan^{-1}A$ is greater than 60°, $\cos(\tan^{-1}A)$ will be less than 0.5.

$$\boxed{\therefore \text{ Power factor, } \cos\phi < 0.5}$$

Note: The value of cos 60° to cos 90° will lie in the range of 0.5 to 0.

Conclusion: When one of the wattmeter readings is negative, the power factor will be less than 0.5.

Case iv : Both the wattmeter readings are positive

Let, $P_1 = +P_x$ and $P_2 = +P_y$

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right] = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_y - P_x}{P_x + P_y}\right)\right]$$

$$= \cos[\tan^{-1}B]$$

$$\text{where, } B = \sqrt{3}\,\frac{P_y - P_x}{P_x + P_y}$$

$\boxed{\tan 60^\circ = \sqrt{3}}$

Here, $(P_y - P_x) < (P_x + P_y)$, $\quad \therefore\ B < \sqrt{3}$ and $\tan^{-1}B < 60^\circ$

Since, $\tan^{-1}B$ is less than 60°, $\cos(\tan^{-1}B)$ will be greater than 0.5.

$$\boxed{\therefore \text{ Power factor, } \cos\phi > 0.5}$$

Note: The value of cos 60° to cos 0° will lie in the range of 0.5 to 1.

Conclusion: When both the wattmeter readings are positive, the power factor will be greater than 0.5.

A Note on Power and Power Factor Estimation using Two Wattmeters

It is possible to prove that the sum of power measured by two wattmeters is equal to the power in three-phase, even if the load is unbalanced star/delta.

Power factor is defined as ratio of active power and apparent power. In balanced loads, the ratio of real and apparent power of each phase is the same, and so the power factor is the same for each phase. Also the three-phase load power factor is the same as power factor of each phase because the ratio of one phase active power and apparent power is the same as the ratio of three-phase active power and apparent power.

In unbalanced loads, the ratio of active and apparent power of each phase is different and so the power factor of each phase is different. Using two wattmeter readings we cannot determine the power factor of an individual phase load. Therefore, estimation of apparent power and hence, the estimation of three-phase load power factor in unbalanced loads is not possible by two wattmeter method of power measurement.

Note : *VAR meters are available for measurement of reactive power in three-phase loads.*

4.26 Solved Problems in Three-Phase Circuits

EXAMPLE 4.20

An unbalanced four-wire star-connected load has a balanced supply voltage of 400 *V*. The load impedances are $\bar{Z}_R = 4 + j8\,\Omega$, $\bar{Z}_Y = 3 + j4\,\Omega$ and $\bar{Z}_B = 15 + j10\,\Omega$. Calculate the line currents, neutral current and the total power. Also draw the phasor diagram.

SOLUTION

Let the phase sequence by RYB. Let the reference phasor be $\bar{V}_{RY}$. The star-connected load with polarity of voltages and direction of currents is shown in Fig. 1.

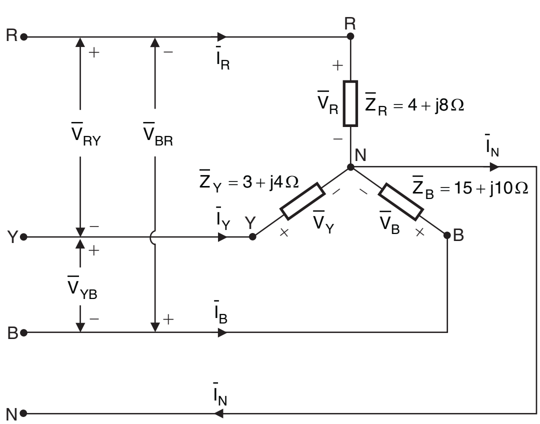


Fig. 1.

The line voltages for the RYB sequence, with $\bar{V}_{RY}$ as the reference phasor are,

$$\bar{V}_{RY} = 400\angle 0^\circ V$$

$$\bar{V}_{YB} = 400\angle -120^\circ V$$

$$\bar{V}_{BR} = 400\angle -240^\circ V = 400\angle 120^\circ V$$

In a four-wire star-connected unbalanced load, the phase voltages are balanced because of neutral connection. Hence, the phase voltages will have a magnitude of $1/\sqrt{3}$ times the line voltage and phase voltages lag 30° with respect to line voltages.

Therefore, the phase voltages are,

$$\bar{V}_R = \frac{400}{\sqrt{3}}\angle(0^\circ - 30^\circ) = 230.9401\angle -30^\circ V$$

$$\bar{V}_Y = \frac{400}{\sqrt{3}}\angle(-120^\circ - 30^\circ) = 230.9401\angle -150^\circ V$$

$$\bar{V}_B = \frac{400}{\sqrt{3}}\angle(-240^\circ - 30^\circ) = 230.9401\angle -270^\circ V = 230.9401\angle 90^\circ V$$

In a star system, the line and phase currents are the same. The phase currents are given by the ratio of phase voltage and phase impedance. Therefore, the phase and line currents are,

$$\bar{I}_R = \frac{\bar{V}_R}{\bar{Z}_R} = \frac{230.9401\angle -30^\circ}{4 + j8} = \frac{230.9401\angle -30^\circ}{8.9443\angle 63.4^\circ} = 25.8198\angle -93.4^\circ A$$

$$\bar{I}_Y = \frac{\bar{V}_Y}{\bar{Z}_Y} = \frac{230.9401\angle -150^\circ}{3 + j4} = \frac{230.9401\angle -150^\circ}{5\angle 53.1^\circ} = 46.188\angle -203.1^\circ A$$

$$\bar{I}_B = \frac{\bar{V}_B}{\bar{Z}_B} = \frac{230.9401\angle -270^\circ}{15 + j10} = \frac{230.9401\angle -270^\circ}{18.0278\angle 33.7^\circ} = 12.8102\angle -303.7^\circ A$$

$$= 12.8102\angle 56.3^\circ A$$

Neutral current, $\bar{I}_N = \bar{I}_R + \bar{I}_Y + \bar{I}_B$

$$= 25.8198\angle -93.4^\circ + 46.188\angle -203.1^\circ + 12.8102\angle -303.7^\circ$$

$$= -36.9083 + j3.0044 = 37.0304\angle 175.3^\circ A$$

$\boxed{-303.7^\circ + 360^\circ = 56.3^\circ}$

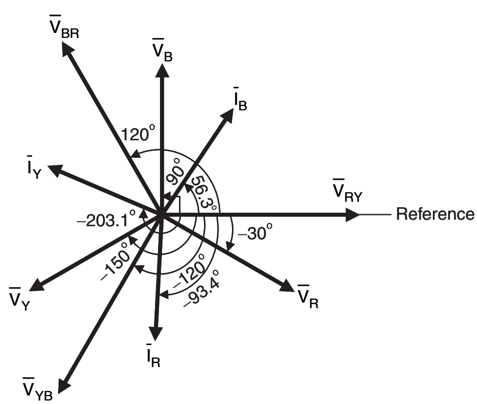


Fig. 2 : *Rms phasors of voltages and currents.*

$$\text{Power, } P = \begin{array}{c}\text{Power consumed}\\ \text{by R-phase load}\end{array} + \begin{array}{c}\text{Power consumed}\\ \text{by Y-phase load}\end{array} + \begin{array}{c}\text{Power consumed}\\ \text{by B-phase load}\end{array}$$

$$= |\bar{V}_R||\bar{I}_R|\cos\phi_1 + |\bar{V}_Y||\bar{I}_Y|\cos\phi_2 + |\bar{V}_B||\bar{I}_B|\cos\phi_3$$

ϕ_1 = Phase difference between $\bar{V}_R$ and $\bar{I}_R$ = $-30^\circ - (-93.4^\circ) = 63.4^\circ$

ϕ_2 = Phase difference between $\bar{V}_Y$ and $\bar{I}_Y$ = $-150^\circ - (-203.1^\circ) = 53.1^\circ$

ϕ_3 = Phase difference between $\bar{V}_B$ and $\bar{I}_B$ = $-270^\circ - (-303.7^\circ) = 33.7^\circ$

Here, $|\bar{V}_R| = |\bar{V}_Y| = |\bar{V}_B| = V = 230.9401\,V$

$$\therefore\ P = VI_R\cos\phi_1 + VI_Y\cos\phi_2 + VI_B\cos\phi_3$$

$$= V[I_R\cos\phi_1 + I_Y\cos\phi_2 + I_B\cos\phi_3]$$

$$= 230.9401 \times [25.8198 \times \cos 63.4^\circ + 46.188 \times \cos 53.1^\circ + 12.8102 \times \cos 33.7^\circ]$$

$$= 11535.6\,W = \frac{11535.6}{1000}\,kW = 11.5356\,kW$$

EXAMPLE 4.21

A three-phase four-wire symmetrical 440 V RBY system supplies power to a star-connected load in which $\bar{Z}_R = 10\angle 0^\circ\ \Omega$, $\bar{Z}_Y = 10\angle 26.8^\circ\,\Omega$ and $\bar{Z}_B = 10\angle -26.8^\circ\,\Omega$. Find the line currents, neutral current and the total power. Draw the phasor diagram.

SOLUTION

The phase sequence is RBY. The line voltages for the RBY sequence are $\bar{V}_{RB}$, $\bar{V}_{BY}$ and $\bar{V}_{YR}$. Let, $\bar{V}_{RB}$ be the reference phasor. The star-connected load with polarity of voltages and direction of currents is shown in Fig. 1.

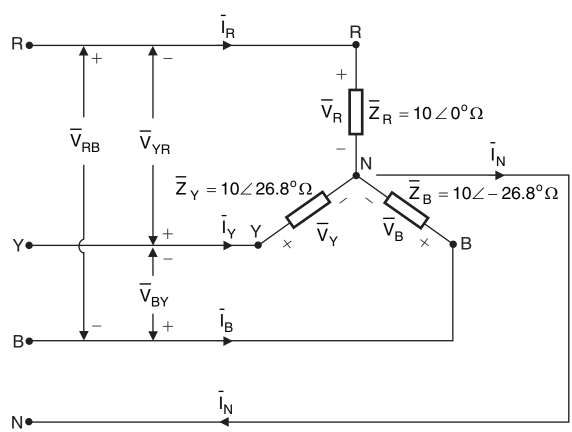


Fig. 1.

The line voltages for RBY sequence, with $\bar{V}_{RB}$ as reference phasor are,

$$\bar{V}_{RB} = 440\angle 0^\circ\,V$$

$$\bar{V}_{BY} = 440\angle -120^\circ\,V$$

$$\bar{V}_{YR} = 440\angle -240^\circ\,V = 440\angle 120^\circ\,V$$

In a four-wire star-connected unbalanced load, the phase voltages are balanced because of neutral connection. Hence, the phase voltages will have a magnitude of $1/\sqrt{3}$ times the line value and phase voltages lag 30^o with respect to line voltages. Therefore, the phase voltages are,

$$\bar{V}_R = \frac{440}{\sqrt{3}}\angle(0^o - 30^o) = 254.0341\angle -30^o \; V$$

$$\bar{V}_B = \frac{440}{\sqrt{3}}\angle(-120^o - 30^o) = 254.0341\angle -150^o \; V$$

$$\bar{V}_Y = \frac{440}{\sqrt{3}}\angle(-240^o - 30^o) = 254.0341\angle -270^o \; V = 254.0341\angle 90^o \; V$$

In a star system, the line and phase currents are the same. The phase currents are given by the ratio of phase voltage and phase impedance. Therefore, the phase and line currents are,

$$\bar{I}_R = \frac{\bar{V}_R}{\bar{Z}_R} = \frac{254.0341\angle -30^o}{10\angle 0^o} = 25.4034\angle -30^o \; A$$

$$\bar{I}_B = \frac{\bar{V}_B}{\bar{Z}_B} = \frac{254.0341\angle -150^o}{10\angle -26.8^o} = 25.4034\angle -123.2^o \; A$$

$$\bar{I}_Y = \frac{\bar{V}_Y}{\bar{Z}_Y} = \frac{254.0341\angle -270^o}{10\angle 26.8^o} = 25.4034\angle -296.8^o \; A = 25.4034\angle 63.2^o \; A$$

$\boxed{-296.8^o + 360^o = 63.2^o}$

$$\text{Neutral current, } \bar{I}_N = \bar{I}_R + \bar{I}_B + \bar{I}_Y$$

$$= 25.4034\angle -30^o + 25.4034\angle -123.2^o + 25.4034\angle -296.8^o$$

$$= 19.5438 - j11.2836$$

$$= 22.5672\angle -30^o \; A$$

$$\text{Power, } P = \begin{matrix}\text{Power consumed}\\ \text{by R-phase load}\end{matrix} + \begin{matrix}\text{Power consumed}\\ \text{by B-phase load}\end{matrix} + \begin{matrix}\text{Power consumed}\\ \text{by Y-phase load}\end{matrix}$$

$$= \left|\bar{V}_R\right|\left|\bar{I}_R\right|\cos\phi_1 + \left|\bar{V}_B\right|\left|\bar{I}_B\right|\cos\phi_2 + \left|\bar{V}_Y\right|\left|\bar{I}_Y\right|\cos\phi_3$$

$$\phi_1 = \text{Phase difference between } \bar{V}_R \text{ and } \bar{I}_R = -30^o - (-30^o) = 0^o$$

$$\phi_2 = \text{Phase difference between } \bar{V}_B \text{ and } \bar{I}_B = -150^o - (-123.2^o) = -26.8^o$$

$$\phi_3 = \text{Phase difference between } \bar{V}_Y \text{ and } \bar{I}_Y = -270^o - (-296.8^o) = 26.8^o$$

$$\text{Here, } \left|\bar{V}_R\right| = \left|\bar{V}_B\right| = \left|\bar{V}_Y\right| = V = 254.0341 V$$

$$\left|\bar{I}_R\right| = \left|\bar{I}_B\right| = \left|\bar{I}_Y\right| = I = 25.4034 \; A$$

$$\therefore \; P = VI\cos\phi_1 + VI\cos\phi_2 + VI\cos\phi_3$$

$$= VI[\cos\phi_1 + \cos\phi_2 + \cos\phi_3]$$

$$= 254.0341 \times 25.4034 \times [\cos 0^o + \cos(-26.8^o) + \cos(26.8^o)]$$

$$= 17973.6 \; W = \frac{17973.6}{1000} \; kW = 17.9736 \; kW$$

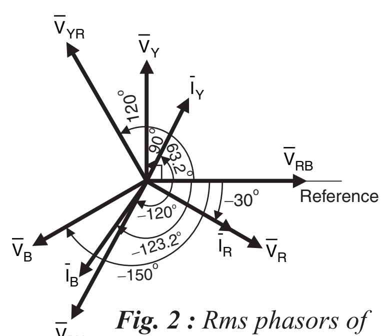


Fig. 2 : Rms phasors of voltages and currents.

EXAMPLE 4.22

In a four-wire three-phase system, two phases have currents of 10 *A* and 6 *A* of lagging power factor of 0.8 and 0.6, respectively, while the third phase is open-circuited. Calculate the current in neutral and sketch the phasor diagram.

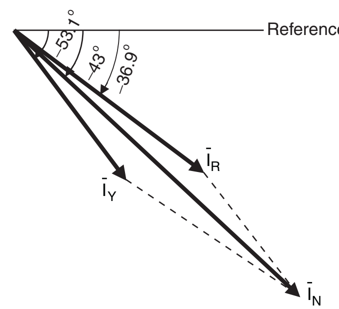


Fig. 1 : Phasor diagram.

SOLUTION

Let the given two phase currents be $\bar{I}_R$ and $\bar{I}_Y$. Let the phase-B be open-circuited and so the phase-B current $\bar{I}_B$ is zero.

$$\therefore \quad \bar{I}_R = 10\,A \text{ at lagging pf of } 0.8 = 10\angle{-\cos^{-1}0.8} = 10\angle{-36.9^\circ}\,A$$

$$\bar{I}_Y = 6\,A \text{ at lagging pf of } 0.6 = 6\angle{-\cos^{-1}0.6} = 6\angle{-53.1^\circ}\,A$$

$$\bar{I}_B = 0$$

In 4-wire 3-phase system, the neutral current is given by sum of three-phase currents.

$$\therefore \text{ Neutral current, } \bar{I}_N = \bar{I}_R + \bar{I}_Y + \bar{I}_B$$

$$= 10\angle{-36.9^\circ} + 6\angle{-53.1^\circ} + 0$$

$$= 11.5994 - j10.8023$$

$$= 15.8504\ \angle{-43^\circ}A$$

EXAMPLE 4.23

(AU June'16 &'14, 8 Marks) (AU Dec'14, '15 & '16, 8 Marks)

A balanced star-connected load of impedance 15 + j20 Ω per phase is connected to a three-phase, 440 *V*, 50*Hz* supply. Find line currents and power absorbed by the load. Assume the RYB sequence. Draw the phasor diagram.

SOLUTION

The phase sequence is RYB. The line voltages for the RYB sequence are $\bar{V}_{RY}, \bar{V}_{YB}$, and $\bar{V}_{BR}$. Let us choose $\bar{V}_{RY}$ as the reference phasor. The star-connected load with polarity of voltages and direction of currents is shown in Fig. 1.

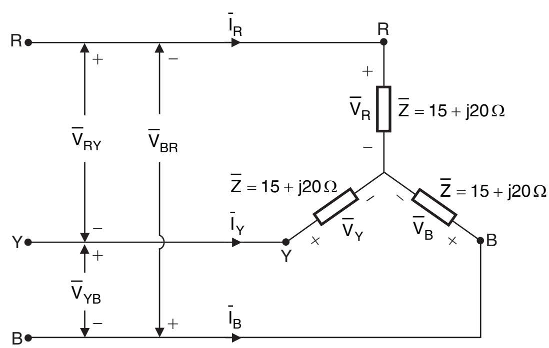


Fig. 1.

The line voltages for the RYB sequence with $\bar{V}_{RY}$ as the reference phasor are,

$$\bar{V}_{RY} = 440\angle 0^\circ\,V$$

$$\bar{V}_{YB} = 440\angle{-120^\circ}\,V$$

$$\bar{V}_{BR} = 440\angle{-240^\circ}\,V = 440\angle 120^\circ\,V$$

Here, $|\bar{V}_{RY}| = |\bar{V}_{YB}| = |\bar{V}_{BR}| = V_L = 440\,V$

In a three-phase balanced star-connected load, the magnitude of phase voltage will be $1/\sqrt{3}$ times the magnitude of line voltage. Also the phase voltage lags behind the line voltage by 30°. Therefore, the phase voltages are,

$$\bar{V}_R = \frac{440}{\sqrt{3}}\angle(0^\circ - 30^\circ) = 254.0341\angle{-30^\circ}\,V$$

$$\bar{V}_Y = \frac{440}{\sqrt{3}}\angle(-120^\circ - 30^\circ) = 254.0341\angle{-150^\circ}\,V$$

$$\bar{V}_B = \frac{440}{\sqrt{3}}\angle(-240^\circ - 30^\circ) = 254.0341\angle{-270^\circ}\,V = 254.0341\angle 90^\circ\,V$$

Here, $|\bar{V}_R| = |\bar{V}_Y| = |\bar{V}_B| = V = 254.0341 V$

Given that, load impedance per phase, $\bar{Z} = 15 + j20\ \Omega = 25\angle 53.1^\circ\ \Omega$

In a star system, the line and phase currents are the same. Therefore, the line and phase currents are,

$$\bar{I}_R = \frac{\bar{V}_R}{\bar{Z}} = \frac{254.0341\angle -30^\circ}{25\angle 53.1^\circ} = 10.1614\angle -83.1^\circ\ A$$

$$\bar{I}_Y = \frac{\bar{V}_Y}{\bar{Z}} = \frac{254.0341\angle -150^\circ}{25\angle 53.1^\circ} = 10.1614\angle -203.1^\circ\ A$$

$$\bar{I}_B = \frac{\bar{V}_B}{\bar{Z}} = \frac{254.0341\angle -270^\circ}{25\angle 53.1^\circ} = 10.1614\angle -323.1^\circ\ A$$

$$= 10.1614\angle 36.9^\circ\ A$$

Here, $|\bar{I}_R| = |\bar{I}_Y| = |\bar{I}_B| = I = I_L = 10.1614\ A$

Power, $P = \sqrt{3}\ V_L I_L \cos\phi$

Here, ϕ = Impedance angle = 53.1°

$\therefore$ $P = \sqrt{3} \times 440 \times 10.1614 \times \cos 53.1^\circ$

$$= 4649.7\ W = \frac{4649.7}{1000}\ kW = 4.6497\ kW$$

Reactive power, $Q = \sqrt{3}\ V_L I_L \sin\phi$

$$= \sqrt{3} \times 440 \times 10.1614 \times \sin 53.1^\circ$$

$$= 6192.8\ VAR = \frac{6192.8}{1000}\ kVAR = 6.1928\ kVAR$$

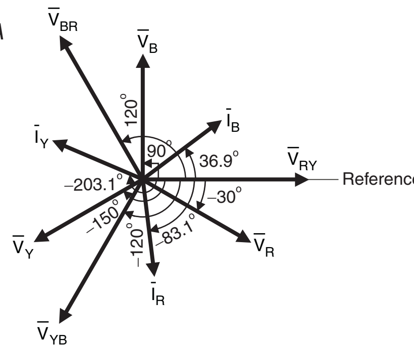


***Fig. 2 :** Rms phasors of voltages and currents.*

EXAMPLE 4.24

The power consumed in a three-phase balanced star-connected load is 2 *kW* at a power factor of 0.8 lagging. The supply voltage is 400 *V*, 50 *Hz*. Calculate the resistance and reactance of each phase.

SOLUTION

Given that,

Power factor, $\cos\phi = 0.8$ *lag*

Power, $P = 2\ kW = 2 \times 1000\ W = 2000\ W$

Line voltage, $V_L = 400\ V$

We know that,

$$P = \sqrt{3} V_L I_L \cos\phi$$

$$\therefore \text{ Line current, } I_L = \frac{P}{\sqrt{3} V_L \cos\phi} = \frac{2000}{\sqrt{3} \times 400 \times 0.8} = 3.6084\ A$$

In a star-connected load, the line and phase current are the same.

$\therefore$ Phase current, $I = I_L = 3.6084\ A$

The magnitude of phase voltage in a star-connected balanced load will be $1/\sqrt{3}$ times the magnitude of line voltage.

$$\therefore \quad \text{Phase voltage, } V = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.9401\,V$$

Now,

$$\text{Magnitude of impedance per phase, } Z = \frac{V}{I} = \frac{230.9401}{3.6084} = 64\,\Omega$$

We know that the impedance angle θ is the same as the power factor angle ϕ.

$$\therefore \quad \text{Impedance angle, } \theta = \cos^{-1}0.8 = 36.9^\circ$$

$$\therefore \quad \text{Impedance per phase, } \bar{Z} = Z\angle\theta = 64\angle 36.9^\circ\,\Omega$$

Let us express the impedance in rectangular form.

Here, the power factor is lagging and so the impedance angle is positive.

$$\therefore \quad \bar{Z} = 64\angle 36.9^\circ\,\Omega = 51.1798 + j38.4269\,\Omega$$

We know that, $\bar{Z} = R + jX$

$$\therefore \quad \text{Resistance of load, } R = \text{Real part of } \bar{Z} = 51.1798\,\Omega/phase$$

$$\text{Resistance of load, } X = \text{Imaginary part of } \bar{Z} = 38.4269\,\Omega/phase$$

EXAMPLE 4.25

For the circuit shown in Fig. 1, calculate the line current, the power and power factor when the supply voltage is 300 *V*, 50 *Hz*. The values of R, L and C in each phase are 10 Ω, 1 *H* and 100 *μF*, respectively.

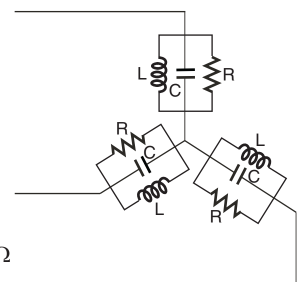


Fig. 1.

SOLUTION

$$\text{Inductive reactance, } X_L = 2\pi fL = 2\pi \times 50 \times 1 = 314.1593\,\Omega$$

$$\text{Capacitive reactance, } X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 100 \times 10^{-6}} = 31.831\,\Omega$$

Let, $\bar{Z}$ = Impedance per phase.

$$\text{Here, } \frac{1}{\bar{Z}} = \frac{1}{R} + \frac{1}{jX_L} + \frac{1}{-jX_C} = \frac{1}{R} - j\frac{1}{X_L} + j\frac{1}{X_C}$$

$$\therefore \quad \bar{Z} = \frac{1}{\frac{1}{R} - j\frac{1}{X_L} + j\frac{1}{X_C}} = \frac{1}{\frac{1}{10} - j\frac{1}{314.1593} + j\frac{1}{31.831}}$$

$$= \frac{1}{0.1 - j0.0032 + j0.0314} = \frac{1}{0.1 + j0.0282}$$

$$= \frac{1}{0.1039\angle 15.7^\circ} = 9.6246\angle -15.7^\circ\ \Omega/phase$$

$$\therefore \quad \text{Magnitude of impedance, } Z = 9.6246\,\Omega$$

$$\text{Impedance angle,} \qquad \theta = -15.7^\circ$$

Given that, Line voltage, $V_L = 300\,V$

$$\therefore \quad \text{Phase voltage, } V = \frac{V_L}{\sqrt{3}} = \frac{300}{\sqrt{3}} = 173.2051\,V$$

$$\text{Current per phase, } I = \frac{V}{Z} = \frac{173.2051}{9.6246} = 17.9961\,A$$

In star system, the line and phase currents are the same.

$\therefore$ Line current, $I_L = I = 17.9961\ A$

Here, the impedance angle is negative and so the load is capacitive. Therefore, the power factor is lead.

Power factor, $\cos\phi = \cos\theta = \cos(-15.7°) = 0.9627$ *lead*

$$\text{Power, } P = \sqrt{3}\, V_L I_L \cos\phi$$

$$= \sqrt{3} \times 300 \times 17.9961 \times 0.9627 = 9002.3\,W$$

$$= \frac{9002.3}{1000}\,kW = 9.0023\,kW$$

EXAMPLE 4.26

Three equal inductors connected in star takes 5 *kW* at 0.7 power factor, when connected to a 400 *V*, 50 *Hz*, three-phase, three-wire supply. Calculate the line currents, (i) if one of the inductor is disconnected and (ii) if one of inductor is short-circuited.

SOLUTION

Given that, Power, $P = 5\,kW = 5 \times 1000\,W = 5000\,W$

Power factor, $\cos\phi = 0.7$

Line voltage, $V_L = 400\,V$

We know that, $P = \sqrt{3}\, V_L I_L \cos\phi$

$$\therefore \quad \text{Line current, } I_L = \frac{P}{\sqrt{3}\, V_L \cos\phi} = \frac{5000}{\sqrt{3} \times 400 \times 0.7} = 10.3098\,A$$

In a star system the line and phase currents are the same.

$\therefore$ Phase current, $I = I_L = 10.3098\,A$

In a balanced star-connected load, the magnitude of phase voltage is $1/\sqrt{3}$ times the line voltage.

$$\therefore \quad \text{Phase voltage, } V = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.9401\,V$$

$$\text{Magnitude of impedance per phase, } Z = \frac{V}{I} = \frac{230.9401}{10.3098} = 22.4001\,\Omega/phase$$

We know that the impedance angle θ is the same as the power factor angle, ϕ.

$\therefore$ Impedance angle, $\theta = \cos^{-1} 0.7 = 45.6°$

Here the impedance angle is positive because the load is inductive.

Impedance per phase, $\bar{Z} = 22.4001\angle 45.6°\ \Omega/phase$

Case i : When one of the inductors is disconnected

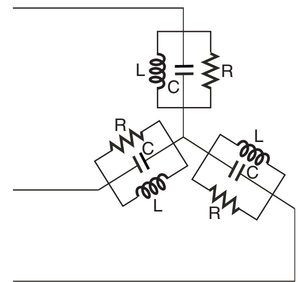


Fig. 1.

Let B-phase inductor be disconnected. With reference to Fig. 1, we can say that when one of the inductors is disconnected, the load becomes a two-phase load. Let us assume the RYB sequence. The line voltages of the RYB sequence are $\bar{V}_{RY}$, $\bar{V}_{YB}$ and $\bar{V}_{BR}$.

Let $\bar{V}_{RY}$ be the reference phasor.

$$\therefore \quad \bar{V}_{RY} = 400\angle 0^{\circ}\, V$$

With reference to Fig. 1, we can say that the voltage across the load is $\bar{V}_{RY}$ and the total load impedance is $2\,\bar{Z}$. Also $\bar{I}_R = -\bar{I}_Y$.

By Ohm's law, we can write,

$$\bar{I}_R = \frac{\bar{V}_{RY}}{2\,\bar{Z}} = \frac{400\angle 0^{\circ}}{2 \times 22.4001\angle 45.6^{\circ}} = 8.9285\angle -45.6^{\circ}\, A$$

$$\therefore \quad \bar{I}_Y = -\bar{I}_R = -1 \times 8.9285\angle -45.6^{\circ} = 1\angle 180^{\circ} \times 8.9285\angle -45.6^{\circ}$$

$$= 8.9285\angle 134.4^{\circ}\, A$$

Since, phase-B is open, $\bar{I}_B = 0$

In summary, the line currents when one of the inductors is disconnected are,

$$\bar{I}_R = 8.9285\angle -45.6^{\circ}\, A$$

$$\bar{I}_Y = 8.9285\angle 134.4^{\circ}\, A$$

$$\bar{I}_B = 0$$

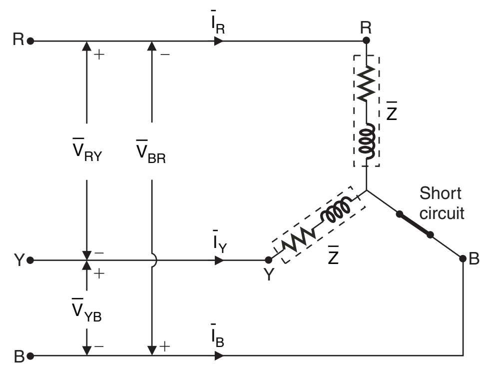


Fig. 2.

Case ii : When one inductor is short-circuited

Let B-phase inductor be short-circuited as shown in Fig. 2. Now the load can be treated as an unbalanced load and analysed using mesh method. Let the phase sequence be RYB and $\bar{V}_{RY}$ be the reference phasor.

The line voltages are,

$$\bar{V}_{RY} = 400\angle 0^{\circ}\, V$$

$$\bar{V}_{YB} = 400\angle -120^{\circ}\, V$$

$$\bar{V}_{BR} = 400\angle -240^{\circ}\, V = 400\angle 120^{\circ}\, V$$

Let us connect the two voltage sources $\bar{V}_{RY}$ and $\bar{V}_{YB}$ to an unbalanced load as shown in Fig. 3. Let us assume the two mesh currents $\bar{I}_1$ and $\bar{I}_2$ as shown in Fig. 3. Now, the line currents in terms of mesh currents are,

$$\bar{I}_R = \bar{I}_1$$

$$\bar{I}_Y = \bar{I}_2 - \bar{I}_1$$

$$\bar{I}_B = -\bar{I}_2$$

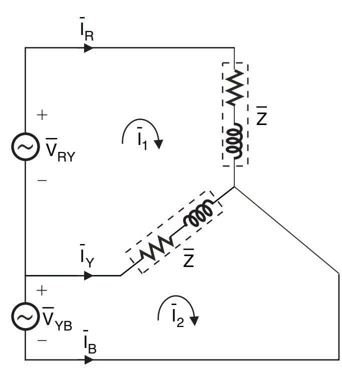


Fig. 3.

The mesh basis matrix equation for the circuit of Fig. 3 is,

$$\begin{bmatrix} \bar{Z}+\bar{Z} & -\bar{Z} \\ -\bar{Z} & \bar{Z} \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{V}_{RY} \\ \bar{V}_{YB} \end{bmatrix}$$

$$\begin{bmatrix} 2\bar{Z} & -\bar{Z} \\ -\bar{Z} & \bar{Z} \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{V}_{RY} \\ \bar{V}_{YB} \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 2\bar{Z} & -\bar{Z} \\ -\bar{Z} & \bar{Z} \end{vmatrix} = 2\bar{Z} \times \bar{Z} - (-\bar{Z}) \times (-\bar{Z}) = 2\bar{Z}^2 - \bar{Z}^2 = \bar{Z}^2$$

$$\Delta_1 = \begin{vmatrix} \bar{V}_{RY} & -\bar{Z} \\ \bar{V}_{YB} & \bar{Z} \end{vmatrix} = \bar{V}_{RY} \times \bar{Z} - \bar{V}_{YB} \times (-\bar{Z}) = \bar{Z}\,(\bar{V}_{RY} + \bar{V}_{YB})$$

$$\Delta_2 = \begin{vmatrix} 2\bar{Z} & \bar{V}_{RY} \\ -\bar{Z} & \bar{V}_{YB} \end{vmatrix} = 2\bar{Z} \times \bar{V}_{YB} + \bar{Z} \times \bar{V}_{RY} = \bar{Z}\,(2\bar{V}_{YB} + \bar{V}_{RY})$$

$$\bar{I}_1 = \frac{\Delta_1}{\Delta} = \frac{\bar{Z}\,(\bar{V}_{RY} + \bar{V}_{YB})}{\bar{Z}^2} = \frac{\bar{V}_{RY} + \bar{V}_{YB}}{\bar{Z}} = \frac{400 + 400\angle-120^\circ}{22.4001\angle 45.6^\circ}$$

$$= -4.8021 - j17.1993\ A$$

$$\bar{I}_2 = \frac{\Delta_2}{\Delta} = \frac{\bar{Z}\,(2\,\bar{V}_{YB} + \bar{V}_{RY})}{\bar{Z}^2} = \frac{2\,\bar{V}_{YB} + \bar{V}_{RY}}{\bar{Z}} = \frac{2 \times 400\angle-120^\circ + 400}{22.4001\angle 45.6^\circ}$$

$$= -22.0982 - j21.6401\ A$$

Now, the line currents are,

$$\bar{I}_R = \bar{I}_1 = -4.8021 - j17.1993 = 17.8571\angle-105.6^\circ\ A$$

$$\bar{I}_Y = \bar{I}_2 - \bar{I}_1 = -22.0982 - j21.6401 - (-4.8021 - j17.1993)$$

$$= -17.2961 - j4.4408 = 17.8571\angle-165.6^\circ A$$

$$\bar{I}_B = -\bar{I}_2 = -(-22.0982 - j21.6401) = 30.9293\angle 44.4^\circ\ A$$

EXAMPLE 4.27

Three similar resistors are connected in star across 400 *V*, three-phase lines. The line current is 5 *A*. Calculate the value of each resistor. To what value should the line voltage be changed to obtain the same line current with the resistances delta-connected?

SOLUTION

Case i : Star connection

Line voltage in star, $V_{L,\,S}$ = 400 *V*

$$\therefore \text{ Phase voltage, } V = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.9401V$$

Given that, line current, I_L = 5 *A*

In a star system, the line and phase currents are the same.

$\therefore$ Phase current, $I = I_L = 5\,A$

$$\text{Resistance per phase, } R = \frac{V}{I} = \frac{230.9401}{5} = 46.188\,\Omega$$

Case ii : Delta connection

The line current in a delta connection should be maintained the same as that in a star connection.

$\therefore$ Line current, $I_L = 5\,A$

In a delta connection, the phase current is $1/\sqrt{3}$ times the line current.

$$\therefore \text{ Phase current, } I = \frac{I_L}{\sqrt{3}} = \frac{5}{\sqrt{3}} = 2.8868\,A$$

The resistance per phase in a delta connection is the same as that in a star connection.

Resistance per phase, $R = 46.188\ \Omega$

By Ohm's law,

Phase voltage, $V = IR = 2.8868 \times 46.188 = 133.3355\,V$

In a delta connection, the line voltage is the same as phase voltage.

$\therefore$ Line voltage in delta, $V_{L,\,D} = V = 133.3355\,V$

Conclusion

The line voltage in star, $V_{L,\,S} = 400\,V$

The line voltage in delta, $V_{L,\,D} = 133.3355\,V$

Here, $V_{L,\,S} = 3\,V_{L,\,D}$, (i.e., $3 \times 133.3355 = 400\,V$)

Hence, we can say that when three equal impedances in star are reconnected to delta, then in order to maintain the same line current, the line voltage should be reduced to one-third. Alternatively, when three equal impedances in delta are reconnected to star, then in order to maintain the same line current, the line voltage should be increased by three times.

EXAMPLE 4.28

A total three-phase power of $100\,kW$ is transmitted over transmission line of impedance $1 + j2\ \Omega/phase$. The line voltage of the balanced three-phase load is $11\,kV$. The load pf is 0.8 lagging. Find the line voltage and power factor at sending end.

SOLUTION

Given that,

Active power of load, $P = 100\,kW = 100 \times 10^3\,W$

Power factor of load, $\cos\phi = 0.8\ lag$

$\therefore\ \phi = \cos^{-1}0.8$

$\therefore\ \tan\phi = \tan(\cos^{-1}0.8) = 0.75$

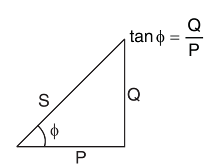


***Fig. 1 :** Power triangle.*

With reference to the power triangle shown in Fig. 1, Reactive power of load, $Q = P \tan\phi$

$$= 100 \times 10^3 \times 0.75 = 75 \times 10^3\ VAR = 75\ kVAR$$

We know that, $P = \sqrt{3}\, V_L I_L \cos\phi$

$$\therefore \text{ Line current, } I_L = \frac{P}{\sqrt{3} V_L \cos\phi} = \frac{100 \times 10^3}{\sqrt{3} \times 11 \times 10^3 \times 0.8} = 6.5608\ A$$

Let, $\overline{V}_d$ = Voltage drop in the line.

Given that, line impedance, $\overline{Z}_{line} = 1 + j2\,\Omega$

Since the current through the transmission line is $\bar{I}_L$, the line drop is given by,

$$\overline{V}_d = \bar{I}_L \overline{Z}_{line} \qquad \text{..... (1)}$$

Let, $\overline{S}_{line}$ = Complex power consumed by line impedance.

Now, $\overline{S}_{line} = \overline{V}_d \bar{I}_L^*$

$$\boxed{\text{Using equation (1)}}$$

$$\boxed{\bar{I}_L \bar{I}_L^* = \left|\bar{I}_L\right|^2 = I_L^2}$$

$$\therefore \overline{S}_{line} = \bar{I}_L \overline{Z}_{line} \times \bar{I}_L^*$$

$$= (\bar{I}_L \bar{I}_L^*)\, \overline{Z}_{line} = I_L^2\, \overline{Z}_{line}$$

$$= 6.5608^2 \times (1 + j2) = 43 + j86\ VA$$

Let, P_s = Real power at sending end.

Q_s = Reactive power at sending end.

Now, $P_s = P + \text{Real part of } [\overline{S}_{line}]$

$$= 100 \times 10^3 + 43 = 100043\ W$$

$$Q_s = Q + \text{Imaginary part of } [\overline{S}_{line}]$$

$$= 75 \times 10^3 + 86 = 75086\ VAR$$

With reference to the power triangle shown in Fig. 2,

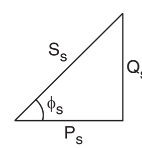


Fig. 2.

$$\tan\phi_s = \frac{Q_s}{P_s}$$

$$\therefore \phi_s = \tan^{-1}\frac{Q_s}{P_s}$$

Power factor at sending end, $\cos\phi_s = \cos\left(\tan^{-1}\frac{Q_s}{P_s}\right)$

$$= \cos\left(\tan^{-1}\frac{75086}{100043}\right) = 0.7998\ lag$$

We know that, $P_s = \sqrt{3}\, V_{L,s} I_L \cos\phi_s$

$$\therefore \text{ Line voltage at sending end, } V_{L,s} = \frac{P_s}{\sqrt{3} I_L \cos\phi_s}$$

$$= \frac{100043}{\sqrt{3} \times 6.5608 \times 0.7998}$$

$$= 11007.5\ V = \frac{11007.5}{1000}\ kV$$

$$= 11.0075\ V$$

EXAMPLE 4.29 ***(AU May'17, 16 Marks)***

A symmetrical three-phase, 100 *V*, three-wire supply feeds an unbalanced star-connected load with impedances of the load as $\bar{Z}_R = 5\angle 0^\circ\ \Omega$, $\bar{Z}_Y = 2\angle 90^\circ\ \Omega$ and $\bar{Z}_B = 4\angle -90^\circ\ \Omega$. Find the line currents, voltage across the impedances and the displacement neutral voltage. Also calculate the power consumed by the load. Sketch the phasor diagram.

SOLUTION

Let the phase sequence be RYB. The line voltages for the RYB sequence are $\bar{V}_{RY}, \bar{V}_{YB}$ and $\bar{V}_{BR}$. The star-connected unbalanced load, connected to a balanced source is shown in Fig. 1.

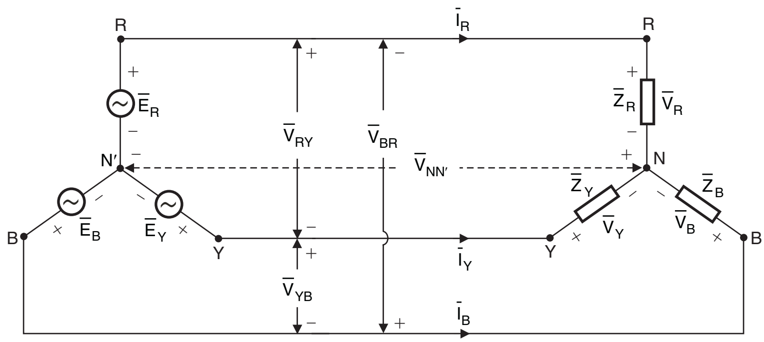


Fig. 1.

The line voltages for the RYB sequence with $\bar{V}_{RY}$ as the reference phasor are,

$$\bar{V}_{RY} = 100\angle 0^\circ\ V = 100\ V$$

$$\bar{V}_{YB} = 100\angle -120^\circ\ V$$

$$\bar{V}_{BR} = 100\angle -240^\circ\ V = 100\angle 120^\circ\ V$$

Let us solve the line currents using the mesh method. Consider two voltage sources, $\bar{V}_{RY}$ and $\bar{V}_{YB}$ connected across the load as shown in Fig. 2. Let $\bar{I}_1$ and $\bar{I}_2$ be the mesh currents. Now, the line currents in terms of mesh currents are,

$$\bar{I}_R = \bar{I}_1$$

$$\bar{I}_Y = \bar{I}_2 - \bar{I}_1$$

$$\bar{I}_B = -\bar{I}_2$$

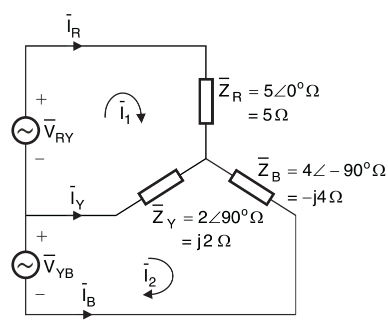


Fig. 2.

The mesh basis matrix equation for the circuit of Fig. 2 is,

$$\begin{bmatrix} \bar{Z}_R + \bar{Z}_Y & -\bar{Z}_Y \\ -\bar{Z}_Y & \bar{Z}_B + \bar{Z}_Y \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{V}_{RY} \\ \bar{V}_{YB} \end{bmatrix}$$

$$\therefore\ \Delta = \begin{vmatrix} \bar{Z}_R + \bar{Z}_Y & -\bar{Z}_Y \\ -\bar{Z}_Y & \bar{Z}_B + \bar{Z}_Y \end{vmatrix} = (\bar{Z}_R + \bar{Z}_Y)(\bar{Z}_B + \bar{Z}_Y) - \bar{Z}_Y^2$$

$$= \bar{Z}_R\bar{Z}_B + \bar{Z}_R\bar{Z}_Y + \bar{Z}_Y\bar{Z}_B + \bar{Z}_Y^2 - \bar{Z}_Y^2$$

$$= \bar{Z}_R\bar{Z}_B + \bar{Z}_R\bar{Z}_Y + \bar{Z}_Y\bar{Z}_B$$

$$= 5 \times (-j4) + 5 \times j2 + j2 \times (-j4) = 8 - j10$$

$$\Delta_1 = \begin{vmatrix} \bar{V}_{RY} & -\bar{Z}_Y \\ \bar{V}_{YB} & \bar{Z}_B + \bar{Z}_Y \end{vmatrix} = \bar{V}_{RY}(\bar{Z}_B + \bar{Z}_Y) + \bar{V}_{YB}\bar{Z}_Y$$

$$= \bar{V}_{RY}\bar{Z}_B + \bar{V}_{RY}\bar{Z}_Y + \bar{V}_{YB}\bar{Z}_Y = 100 \times (-j4) + 100 \times j2 + 100\angle -120^\circ \times j2$$

$$= 173.2051 - j300$$

$$\Delta_2 = \begin{vmatrix} \bar{Z}_R + \bar{Z}_Y & \bar{V}_{RY} \\ -\bar{Z}_Y & \bar{V}_{YB} \end{vmatrix} = (\bar{Z}_R + \bar{Z}_Y)\bar{V}_{YB} + \bar{Z}_Y\bar{V}_{RY}$$

$$= \bar{Z}_R\bar{V}_{YB} + \bar{Z}_Y\bar{V}_{YB} + \bar{Z}_Y\bar{V}_{RY}$$

$$= 5 \times 100\angle{-120^\circ} + j2 \times 100\angle{-120^\circ} + j2 \times 100$$

$$= -76.7949 - j333.0127$$

$$\therefore \quad \bar{I}_1 = \frac{\Delta_1}{\Delta} = \frac{173.2051 - j300}{8 - j10} = 26.7417 - j4.0729\,A$$

$$\bar{I}_2 = \frac{\Delta_2}{\Delta} = \frac{-76.7949 - j333.0127}{8 - j10} = 16.5596 - j20.9271\,A$$

Now, the line currents are,

$$\bar{I}_R = \bar{I}_1 = 26.7417 - j4.0729\,A = 27.0501\angle -8.7^\circ\,A$$

$$\bar{I}_Y = \bar{I}_2 - \bar{I}_1 = 16.5596 - j20.9271 - (26.7417 - j4.0729) = -10.1821 - j16.8542\,A$$

$$= 19.6911\angle -121.1^\circ A$$

$$\bar{I}_B = -\bar{I}_2 = -(16.5596 - j20.9271) = -16.5596 + j20.9271\,A$$

$$= 26.6864\angle 128.4^\circ\,A$$

The voltages across the impedance (i.e., phase voltages) are,

$$\bar{V}_R = \bar{I}_R\bar{Z}_R = 27.0501\angle -8.7^\circ \times 5\angle 0^\circ = 135.2505\angle -8.7^\circ\,V$$

$$\bar{V}_Y = \bar{I}_Y\bar{Z}_Y = 19.6911\angle -121.1^\circ \times 2\angle 90^\circ = 39.3822\angle -31.1^\circ\,V$$

$$\bar{V}_B = \bar{I}_B\bar{Z}_B = 26.6864\angle 128.4^\circ \times 4\angle -90^\circ = 106.7456\angle 38.4^\circ\,V$$

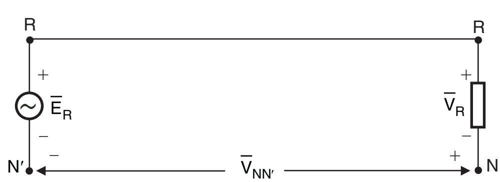


Fig. 3.

Let, $V_{NN'}$ = Neutral shift voltage.

With reference to Fig. 3, using KVL we can write,

$$V_{NN'} + \bar{V}_R = \bar{E}_R$$

$\therefore \; V_{NN'} = \bar{E}_R - \bar{V}_R$

Since the source is balanced, $\bar{E}_R$ will have a magnitude of $1/\sqrt{3}$ times the line voltage and lag behind $\bar{V}_{RY}$ by 30°.

$$\therefore \; \bar{E}_R = \frac{100}{\sqrt{3}} \angle(0^\circ - 30^\circ) = 57.735\angle - 30^\circ \, V$$

$$\therefore \text{ Neutral shift voltage, } V_{NN'} = \bar{E}_R - \bar{V}_R = 57.735\angle - 30^\circ - 135.2505\angle - 8.7^\circ$$

$$= -83.6943 - j8.4094 \, V$$

$$= 84.1157\angle{-174.3^\circ} \, V$$

$$\text{Power, } P = |\bar{V}_R||\bar{I}_R|\cos\phi_1 + |\bar{V}_Y||\bar{I}_Y|\cos\phi_2 + |\bar{V}_B||\bar{I}_B|\cos\phi_3$$

Here, ϕ_1 = Phase difference between $\bar{V}_R$ and $\bar{I}_R = -8.7 - (-8.7) = 0^\circ$

ϕ_2 = Phase difference between $\bar{V}_Y$ and $\bar{I}_Y = -31.1^\circ - (-121.1^\circ) = 90^\circ$

ϕ_3 = Phase difference between $\bar{V}_B$ and $\bar{I}_B = 38.4^\circ - 128.4^\circ = -90^\circ$

$$\therefore \; P = V_R I_R \cos\phi_1 + V_Y I_Y \cos\phi_2 + V_B I_B \cos\phi_3$$

Here, $\phi_2 = +90^\circ$, $\therefore \; \cos\phi_2 = \cos(+90^\circ) = 0$

and $\phi_3 = -90^\circ$, $\therefore \; \cos\phi_3 = \cos(-90^\circ) = 0$

$$\therefore \; P = V_R I_R \cos\phi_1 = 135.2505 \times 27.0501 \times \cos 0^\circ$$

$$= 3658.5 \text{ W} = \frac{3658.5}{1000} \text{ kW} = 3.6585 \, kW$$

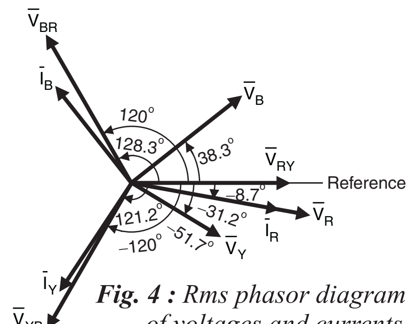


Fig. 4 : Rms phasor diagram of voltages and currents.

> ***Note :*** *Since Y-phase and B-phase loads are purely reactive loads, the power consumed by them is zero.*

EXAMPLE 4.30

A three-phase, three-wire unbalanced load is star-connected. The phase voltages of two of the arms are, $\bar{V}_R = 100\angle - 10^\circ \, V$ and $\bar{V}_Y = 150\angle 100^\circ \, V$. Calculate the voltage between star point of the load and the supply neutral.

SOLUTION

Let us assume the RYB sequence. The line voltages for the RYB sequence are $\bar{V}_{RY}$, $\bar{V}_{YB}$ and $\bar{V}_{BR}$. The line voltage $\bar{V}_{RY}$ can be calculated as shown below:

With reference to Fig. 1, using KVL we can write,

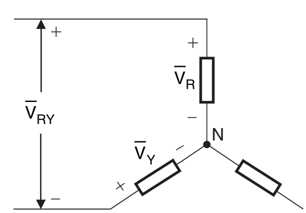


Fig. 1.

$$\bar{V}_{RY} + \bar{V}_Y = \bar{V}_R$$

$$\therefore \; \bar{V}_{RY} = \bar{V}_R - \bar{V}_Y$$

$$= 100\angle{-10^\circ} - 150\angle 100^\circ$$

$$= 124.528 - j165.086$$

$$= 206.7864\angle{-53^\circ} \, V$$

Usually the source is balanced. Let us assume a star-connected balanced source. We know that the line voltage of source and load are the same. Hence, the R-phase emf $\bar{E}_R$ of the source will have a magnitude of $1/\sqrt{3}$ times the magnitude of $\bar{V}_{RY}$ and phase lag by 30° with respect to $\bar{V}_{RY}$.

$$\therefore\ \bar{E}_R = \frac{206.7864}{\sqrt{3}}\angle(-53^\circ - 30^\circ) = 119.3882\angle -83^\circ\ V$$

The neutral shift voltage can be estimated from the knowledge of $\bar{E}_R$ and $\bar{V}_R$. Consider the circuit shown in Fig. 2. By KVL we can write,

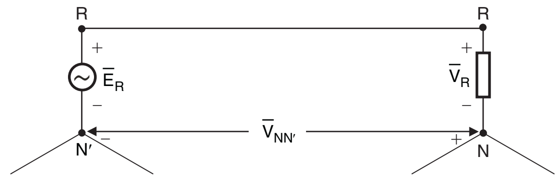


Fig. 2.

$$V_{NN'} + \bar{V}_R = \bar{E}_R$$

$$\therefore \text{ Neutral shift voltage, } V_{NN'} = \bar{E}_R - \bar{V}_R$$

$$= 119.3882\angle{-83^\circ} - 100\angle{-10^\circ}$$

$$= -83.931 - j101.1335$$

$$= 131.4245\angle{-129.7^\circ}\ V$$

EXAMPLE 4.31 ***(AU May'17, 8 Marks)***

A three-phase balanced delta-connected load of 4 + j8 Ω is connected across a 400 *V*, three-phase balanced supply. Determine the phase currents and line currents. Assume the phase sequence to be RYB. Also calculate the power drawn by the load. Sketch the phasor diagram.

SOLUTION

The phase sequence is RYB. The line voltages for the RYB sequence are $\bar{V}_{RY}, \bar{V}_{YB}$ and $\bar{V}_{BR}$. Let us choose $\bar{V}_{RY}$ as the reference phasor. The delta-connected load with polarity of voltages and direction of currents is shown in Fig. 1.

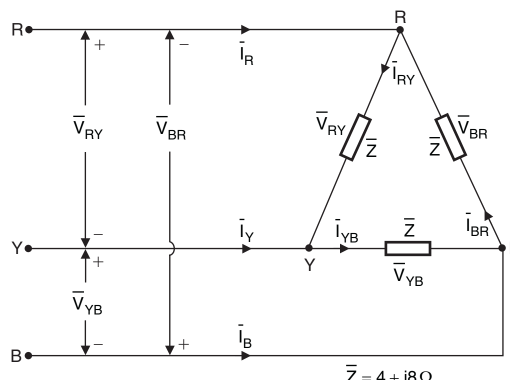


Fig. 1.

The line voltages of the RYB sequence with $\bar{V}_{RY}$ as reference phasor are,

$$\bar{V}_{RY} = 400\angle 0^\circ\ V$$

$$\bar{V}_{YB} = 400\angle -120^\circ\ V$$

$$\bar{V}_{BR} = 400\angle -240^\circ\ \text{V} = 400\angle 120^\circ\ V$$

In a delta-connected load, the phase and line voltages are the same.

Here, $|\bar{V}_{RY}| = |\bar{V}_{YB}| = |\bar{V}_{BR}| = V_L = V = 400\ V$

Now, the phase currents $\bar{I}_{RY}, \bar{I}_{YB}$ and $\bar{I}_{BR}$ are given by the ratio of phase voltage and impedance.

Given that, impedance per phase, $\bar{Z} = 4 + j8\,\Omega = 8.9443\angle 63.4^\circ\,\Omega$

Now, the phase currents are,

$$\bar{I}_{RY} = \frac{\bar{V}_{RY}}{\bar{Z}} = \frac{400\angle 0^\circ}{8.9443\angle 63.4^\circ} = 44.7212\angle -63.4^\circ$$

$$\bar{I}_{YB} = \frac{\bar{V}_{YB}}{\bar{Z}} = \frac{400\angle -120^\circ}{8.9443\angle 63.4^\circ} = 44.7212\angle -183.4^\circ$$

$$\bar{I}_{BR} = \frac{\bar{V}_{BR}}{\bar{Z}} = \frac{400\angle -240^\circ}{8.9443\angle 63.4^\circ} = 44.7212\angle -303.4^\circ = 44.7212\angle 56.6^\circ\ A$$

Here, $|\bar{I}_{RY}| = |\bar{I}_{YB}| = |\bar{I}_{BR}| = I = 44.7212\ A$

In a delta-connected balanced load, the line currents will have a magnitude of $\sqrt{3}$ times the magnitude of phase current and lag the phase current by 30°.

Therefore, the line currents are,

$$\bar{I}_R = 44.7212 \times \sqrt{3}\angle(-63.4^\circ - 30^\circ) = 77.4594\angle -93.4^\circ\ A$$

$$\bar{I}_Y = 44.7212 \times \sqrt{3}\angle(-183.4^\circ - 30^\circ) = 77.4594\angle -213.4^\circ\ A$$
$$= 77.4594\angle 146.6^\circ\ A$$

$$\bar{I}_B = 44.7212 \times \sqrt{3}\angle(-303.4^\circ - 30^\circ) = 77.4594\angle -333.4^\circ\ A$$
$$= 77.4594\angle 26.6^\circ\ A$$

Here, $|\bar{I}_R| = |\bar{I}_Y| = |\bar{I}_B| = I_L = 77.4594\ A$

Power, $P = \sqrt{3}\,V_L\,I_L\cos\phi$

Here, ϕ = Impedance angle = 63.4°

$$\therefore\ P = \sqrt{3} \times 400 \times 77.4594 \times \cos 63.4^\circ$$

$$= 24029.2\,W = \frac{24029.2}{1000} = 24.0292\,kW$$

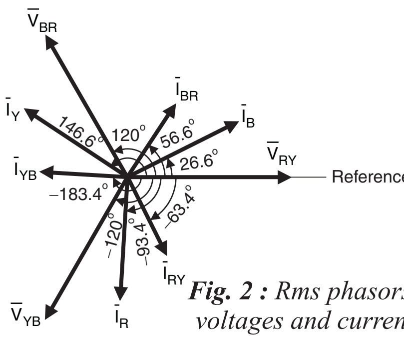


***Fig. 2 :** Rms phasors of voltages and currents.*

EXAMPLE 4.32

A delta-connected balanced three-phase load is supplied from a three-phase, 400 *V* supply. The line current is 20 *A* and the power taken by the load is 10,000 *W*. Find a) impedance in each branch, b) phase current, c) power factor and d) the power consumed if the same load is connected in star.

SOLUTION

Case i : Delta connection

Given that, Line voltage, $V_L = 400\ V$

Line current, $I_L = 20\ A$

Power, $P = 10{,}000\ W$

We know that, $P = \sqrt{3}\,V_L\,I_L\cos\phi$

$$\therefore\ \text{Power factor, } \cos\phi = \frac{P}{\sqrt{3}\,V_L\,I_L} = \frac{10{,}000}{\sqrt{3} \times 400 \times 20} = 0.7217$$

In a delta-connected balanced load, the phase current is $1/\sqrt{3}$ times the line current and phase voltage is the same as line voltage.

$\therefore$ Phase current, $I = \frac{I_L}{\sqrt{3}} = \frac{20}{\sqrt{3}} = 11.547\ A$

Phase voltage, $V = V_L = 400\ V$

Now, impedance per phase, $Z = \frac{V}{I} = \frac{400}{11.547} = 34.641\,\Omega$

Case ii : Star connection

In a star connection, the phase voltage is $1/\sqrt{3}$ times the line voltage.

$\therefore$ Phase voltage, $V = \frac{V_L}{\sqrt{3}} = \frac{400}{\sqrt{3}} = 230.9401\,V$

The impedance and power factor in a star connection is the same as that in a delta connection.

$\therefore$ Phase current, $I = \frac{V}{Z} = \frac{230.9401}{34.641} = 6.6667\ A$

In a star connection, the line current is the same as phase current.

$\therefore$ Line current, $I_L = I = 6.6667\,A$

Now, Power, $P = \sqrt{3}\,V_L\,I_L\cos\phi$

$= \sqrt{3} \times 400 \times 6.6667 \times 0.7217 = 3333.4\,W$

Conclusion

The power consumption in a star connection is one-third of the power in a delta connection. Also, the line current in star is one-third of the line current in delta. This concept is utilised in star-delta starters of induction motors in order to reduce the starting current.

EXAMPLE 4.33

Three capacitors of 100 μF each are connected in delta to a 440 V, three-phase, 50 Hz supply. What will be the capacitance of each of the three capacitors if the same three capacitors are connected in star across the same supply to draw the same line current?

SOLUTION

Case i : When capacitors are connected in delta

Let, C_D = Capacitance per phase in delta.

Given that, $C_D = 100\,\mu F = 100 \times 10^{-6}\ F$

$\therefore$ Capacitive reactance per phase in delta connection $\Big\}$ $X_{C,D} = \frac{1}{2\pi f C_D} = \frac{1}{2\pi \times 50 \times 100 \times 10^{-6}}$

$= 31.831\,\Omega/phase$

Given that, line voltage, $V_L = 440\,V$

In a delta connection, the phase voltage is the same as line voltage.

$\therefore$ Phase voltage, $V = V_L = 440\,V$

Now, phase current, $I = \frac{V}{X_{C,D}} = \frac{440}{31.831} = 13.823\ A$

In a balanced delta-connected load, the line current will be $\sqrt{3}$ times the phase current.

$\therefore$ Line current, $I_L = \sqrt{3}\,I = \sqrt{3} \times 13.823 = 23.9421\,A$

Case ii : When capacitors are connected in star

The line voltage and current in a star connection should be maintained the same as that in a delta connection.

$\therefore$ Line voltage, $V_L = 440\,V$

Line current, $I_L = 23.9421\,A$

In a star connection, the phase current is the same as line current and phase voltage is $1/\sqrt{3}$ times the line voltage.

$\therefore$ Phase current, $I = I_L = 23.9421\,A$

$$\text{Phase voltage, } V = \frac{V_L}{\sqrt{3}} = \frac{440}{\sqrt{3}} = 254.0341\,V$$

$$\left.\begin{array}{l}\text{Now, capacitive reactance}\\ \text{per phase in star connection}\end{array}\right\} \; X_{C,S} = \frac{V}{I} = \frac{254.0341}{23.9421} = 10.6104\;\Omega/phase$$

$$\text{We know that, } X_{C,S} = \frac{1}{2\pi f C_S}$$

where, C_S = Capacitance per phase in star connection

$$\therefore \text{ Capacitance, } C_S = \frac{1}{2\pi f X_{C,S}} = \frac{1}{2\pi \times 50 \times 10.6104} = 3 \times 10^{-4}\,F$$

$$= 300 \times 10^{-6}\,F = 300\,\mu F$$

Conclusion

$$\text{It is observed that, } X_{C,S} = \frac{X_{C,D}}{3} \quad \text{or} \quad X_{C,D} = 3X_{C,S}$$

When three equal impedances in delta are converted into star, in order to maintain the same line current for the same line voltage, the star-connected impedance should have a value one-third the delta-connected impedance.

Conversely, when three equal impedances in star are converted into delta, in order to maintain the same line current for the same line voltage, the delta-connected impedance should have a value three times that of the star-connected impedance.

$$\text{Also it is observed that, } C_D = \frac{C_S}{3} \quad \text{or} \quad C_S = 3C_D$$

Hence, for a given line voltage and same line current when three equal elements are converted from star to delta or vice versa, the following relations will hold good:

$$R_S = \frac{1}{3}R_D \qquad R_D = 3R_S$$

$$L_S = \frac{1}{3}L_D \qquad L_D = 3L_S$$

$$C_S = 3\,C_D \qquad C_D = \frac{1}{3}C_S$$

$$X_S = \frac{1}{3}X_D \qquad X_D = 3X_S$$

$$Z_S = \frac{1}{3}Z_D \qquad Z_D = 3Z_S$$

The suffix S stands for star-connected element per phase. The suffix D stands for delta-connected element per phase.

EXAMPLE 4.34

A three-phase delta-connected load has $\bar{Z}_{RY} = 100 + j50\,\Omega$, $\bar{Z}_{YB} = 20 - j75\,\Omega$ and $\bar{Z}_{BR} = 70.7 + j70.7\,\Omega$ and it is connected to balanced three-phase, 400 V supply. Determine the line currents, power consumed by the load. Sketch the phasor diagram. Assume the RYB sequence and take $\bar{V}_{YB}$ as the reference phasor.

SOLUTION

The phase sequence is RYB. The line voltages for the RYB sequence are $\bar{V}_{RY}, \bar{V}_{YB}$, and $\bar{V}_{BR}$. The three-phase delta-connected load with conventional polarity of voltages and direction of currents is shown in Fig. 1.

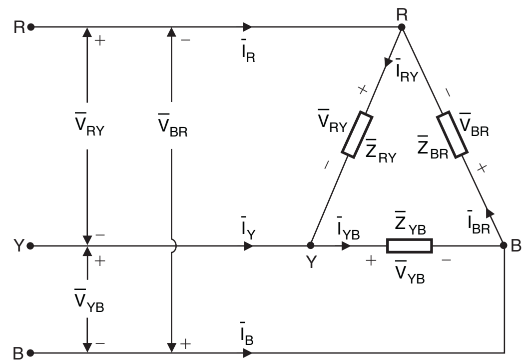


Fig. 1.

The line voltages of the RYB sequence with $\bar{V}_{YB}$ as the reference phasor are,

$$\bar{V}_{YB} = 400\angle 0^\circ$$

$$\bar{V}_{BR} = 400\angle -120^\circ$$

$$\bar{V}_{RY} = 400\angle -240^\circ$$

Here, $|\bar{V}_{YB}| = |\bar{V}_{BR}| = |\bar{V}_{RY}| = V_L = 400\,V$

In a delta-connected load, the phase voltage is the same as line voltage. Now, the phase currents $\bar{I}_{RY}, \bar{I}_{YB}$, and $\bar{I}_{BR}$ are given by the ratio of respective phase voltages and impedance.

Therefore, the phase currents are,

$$\bar{I}_{RY} = \frac{\bar{V}_{RY}}{\bar{Z}_{RY}} = \frac{400\angle -240^\circ}{100 + j50} = -0.2144 + j3.5713\,A$$

$$= 3.5777\angle 93.4^\circ\,A$$

$$\bar{I}_{YB} = \frac{\bar{V}_{YB}}{\bar{Z}_{YB}} = \frac{400\angle 0^\circ}{20 - j75} = 1.3278 + j4.9793\,A$$

$$= 5.1533\angle 75.1^\circ\,A$$

$$\bar{I}_{BR} = \frac{\bar{V}_{BR}}{\bar{Z}_{BR}} = \frac{400\angle -120^\circ}{70.7 + j70.7} = -3.8643 - j1.0354\,A$$

$$= 4.0006\angle -165^\circ A$$

The line currents can be computed by writing KCL equations at nodes R, Y and B.

With reference to Fig. 2, at node-R by KCL we can write,

$$\bar{I}_R + \bar{I}_{BR} = \bar{I}_{RY}$$

$$\therefore\ \bar{I}_R = \bar{I}_{RY} - \bar{I}_{BR}$$

$$= -0.2144 + j3.5713 - (-3.8643 - j1.0354)$$

$$= 3.6499 + j4.6067 = 5.8774\angle 51.6^\circ A$$

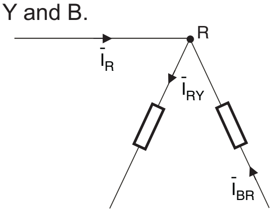


Fig. 2.

With reference to Fig. 3, at node-Y by KCL we can write,

$$\bar{I}_Y + \bar{I}_{RY} = \bar{I}_{YB}$$

$$\therefore\ \bar{I}_Y = \bar{I}_{YB} - \bar{I}_{RY}$$

$$= 1.3278 + j4.9793 - (-0.2144 + j3.5713)$$

$$= 1.5422 + j1.408 = 2.0883\angle 42.4^\circ A$$

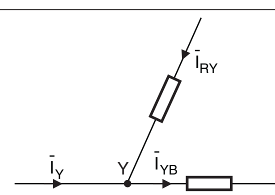


Fig. 3.

With reference to Fig. 4, at node-B by KCL we can write,

$$\bar{I}_B + \bar{I}_{YB} = \bar{I}_{BR}$$

$$\therefore \quad \bar{I}_B = \bar{I}_{BR} - \bar{I}_{YB}$$

$$= -3.8643 - j1.0354 - (1.3278 + j4.9793)$$

$$= -5.1921 - j6.0147 = 7.9457\angle{-130.8^\circ}\,A$$

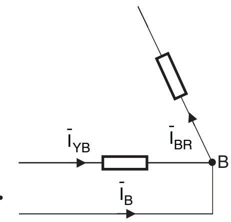


Fig. 4.

In summary, the line currents are,

$$\bar{I}_R = 5.8774\angle 51.6^\circ\,A \quad ; \quad \bar{I}_Y = 2.0883\angle 42.4^\circ\,A \quad ; \quad \bar{I}_B = 7.9457\angle -130.8^\circ\,A$$

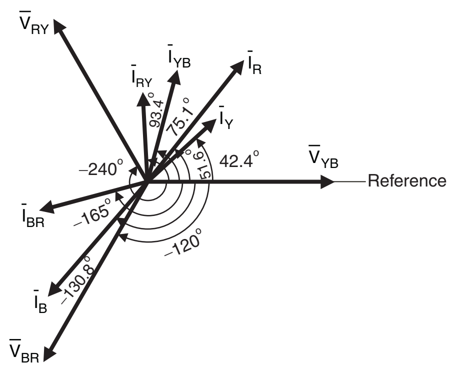


***Fig. 5 :** Rms phasors of voltages and currents.*

$$\text{Power, } P = \left|\bar{V}_{RY}\right|\left|\bar{I}_{RY}\right|\cos\phi_1 + \left|\bar{V}_{YB}\right|\left|\bar{I}_{YB}\right|\cos\phi_2 + \left|\bar{V}_{BR}\right|\left|\bar{I}_{BR}\right|\cos\phi_3$$

$$\text{Here, } \left|\bar{V}_{RY}\right| = \left|\bar{V}_{YB}\right| = \left|\bar{V}_{BR}\right| = V_L = 400\,V$$

$$\phi_1 = \text{Phase difference between } \bar{V}_{RY} \text{ and } \bar{I}_{RY} = -240^\circ - 93.4^\circ = -333.4^\circ$$

$$\phi_2 = \text{Phase difference between } \bar{V}_{YB} \text{ and } \bar{I}_{YB} = 0^\circ - 75.1^\circ = -75.1^\circ$$

$$\phi_3 = \text{Phase difference between } \bar{V}_{BR} \text{ and } \bar{I}_{BR} = -120^\circ - (-165^\circ) = 45^\circ$$

$$\therefore \quad P = V_L I_{RY}\cos\phi_1 + V_L I_{YB}\cos\phi_2 + V_L I_{BR}\cos\phi_3$$

$$= V_L[I_{RY}\cos\phi_1 + I_{YB}\cos\phi_2 + I_{BR}\cos\phi_3]$$

$$= 400 \times [3.5777 \times \cos(-333.4^\circ) + 5.1533 \times \cos(-75.1^\circ) + 4.0006 \times \cos 45^\circ]$$

$$= 2941.2\text{ W} = \frac{2941.2}{1000}\text{ kW} = 2.9412\,kW$$

EXAMPLE 4.35

A 300 V, three-phase balanced source is connected to a delta-connected load of impedance $\bar{Z}_{YR} = 10\angle 45^\circ\Omega$, $\bar{Z}_{BY} = 8\angle 0^\circ\Omega$ and $\bar{Z}_{RB} = 5\angle -45^\circ\Omega$. Determine the line currents and power consumed by the load. Assume the RBY sequence. Sketch the phasor diagram.

SOLUTION

The phase sequence is RBY. The line voltages for the RBY sequence are $\overline{V}_{RB}, \overline{V}_{BY}$ and $\overline{V}_{YR}$. Let us take $\overline{V}_{BY}$ as the reference phasor. The three-phase delta-connected load with conventional polarity of voltages and direction of currents is shown in Fig. 1.

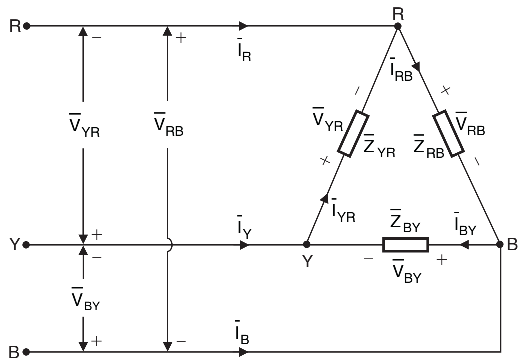


Fig. 1.

The line voltages for the RBY sequence, with $\overline{V}_{BY}$ as the reference phasor are,

$$\overline{V}_{BY} = 300\angle 0^{\circ}$$

$$\overline{V}_{YR} = 300\angle -120^{\circ}$$

$$\overline{V}_{RB} = 300\angle -240^{\circ}$$

Here, $\left|\overline{V}_{BY}\right| = \left|\overline{V}_{YR}\right| = \left|\overline{V}_{RB}\right| = V_L = 300\,V$

In a delta connection, the phase voltage is the same as line voltage. Now, the phase currents $\bar{I}_{RB}, \bar{I}_{BY}$ and $\bar{I}_{YR}$ are given by the ratio of respective phase voltages and impedance.

Therefore, the phase currents are,

$$\bar{I}_{RB} = \frac{\overline{V}_{RB}}{\overline{Z}_{RB}} = \frac{300\angle -240^{\circ}}{5\angle -45^{\circ}} = 60\angle -195^{\circ}\,A = -57.9555 + j15.5291\,A$$

$$\bar{I}_{BY} = \frac{\overline{V}_{BY}}{\overline{Z}_{BY}} = \frac{300\angle 0^{\circ}}{8\angle 0^{\circ}} = 37.5\angle 0^{\circ}\,A = 37.5 + j0\,A$$

$$\bar{I}_{YR} = \frac{\overline{V}_{YR}}{\overline{Z}_{YR}} = \frac{300\angle -120^{\circ}}{10\angle 45^{\circ}} = 30\angle -165^{\circ}\,A = -28.9778 - j7.7646\,A$$

The line currents can be computed by writing KCL equations at nodes R, Y and B.

With reference to Fig. 2, at node-R by writing KCL equation we get,

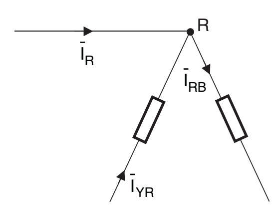


Fig. 2.

$$\bar{I}_R + \bar{I}_{YR} = \bar{I}_{RB}$$

$$\therefore\ \bar{I}_R = \bar{I}_{RB} - \bar{I}_{YR}$$

$$= -57.9555 + j15.5291 - (-28.9778 - j7.7646)$$

$$= -28.9777 + j23.2937\,A = 37.1793\angle 141.2^{\circ}\,A$$

With reference to Fig. 3, at node-Y by writing KCL equation we get,

$$\bar{I}_Y + \bar{I}_{BY} = \bar{I}_{YR}$$

$$\therefore \quad \bar{I}_Y = \bar{I}_{YR} - \bar{I}_{BY}$$

$$= -28.9778 - j7.7646 - (37.5 + j0)$$

$$= -66.4778 - j7.7646\,A = 66.9297\angle -173.3^\circ\,A$$

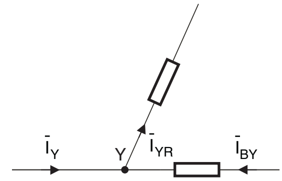


Fig. 3.

With reference to Fig. 4, at node-B by writing KCL equation we get,

$$\bar{I}_B + \bar{I}_{RB} = \bar{I}_{BY}$$

$$\therefore \quad \bar{I}_B = \bar{I}_{BY} - \bar{I}_{RB}$$

$$= 37.5 + j0 - (-57.9555 + j15.5291)$$

$$= 95.4555 - j15.5291\,A = 96.7104\angle -9.2^\circ\,A$$

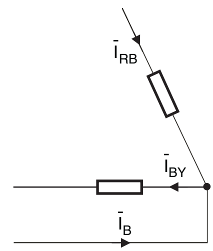


Fig. 4.

In summary, the line currents are,

$$\bar{I}_R = 37.1793\angle 141.2^\circ\,A$$

$$\bar{I}_Y = 66.9297\angle -173.3^\circ\,A$$

$$\bar{I}_B = 96.7104\angle -9.2^\circ\,A$$

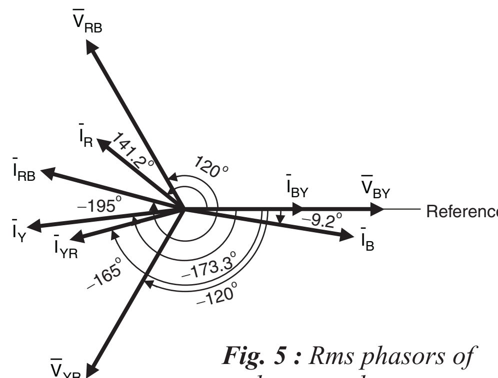


Fig. 5 : Rms phasors of voltages and currents.

Power, $\quad P = |\bar{V}_{RB}||\bar{I}_{RB}|\cos\phi_1 + |\bar{V}_{BY}||\bar{I}_{BY}|\cos\phi_2 + |\bar{V}_{YR}||\bar{I}_{YR}|\cos\phi_3$

Here, $\quad \left|\bar{V}_{RB}\right| = \left|\bar{V}_{BY}\right| = \left|\bar{V}_{YR}\right| = V_L = 300\,V$

ϕ_1 = Phase difference between $\bar{V}_{RB}$ and $\bar{I}_{RB}$ $= -240^\circ - (-195^\circ) = -45^\circ$

ϕ_2 = Phase difference between $\bar{V}_{BY}$ and $\bar{I}_{BY}$ $= 0^\circ - 0^\circ = 0^\circ$

ϕ_3 = Phase difference between $\bar{V}_{YR}$ and $\bar{I}_{YR}$ $= -120^\circ - (-165^\circ) = 45^\circ$

$$\therefore \quad P = V_L I_{RB}\cos\phi_1 + V_L I_{BY}\cos\phi_2 + V_L I_{YR}\cos\phi_3$$

$$= V_L[I_{RB}\cos\phi_1 + I_{BY}\cos\phi_2 + I_{YR}\cos\phi_3]$$

$$= 300 \times [60 \times \cos(-45^\circ) + 37.5 \times \cos 0^\circ + 30 \times \cos 45^\circ]$$

$$= 30341.9\,W = \frac{30341.9}{1000}\,kW = 30.3419\,kW$$

EXAMPLE 4.36 ***(AU June'14, 8 Marks)***

A symmetrical three-phase, three wire, 400*V* supply is connected to a delta-connected load. Impedance in each branch are $\overline{Z}_{RY} = 10\angle 30^0\ \Omega$, $\overline{Z}_{YB} = 10\angle 45^0\ \Omega$ and $\overline{Z}_{BR} = 2.5\angle 60^0\ \Omega$. Find its equivalent star-connected load.

SOLUTION

The equivalent star load can be obtained by using delta to star transformation as shown below :

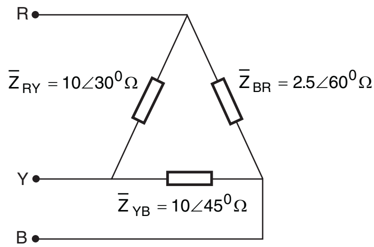


Fig. 1

$\Rightarrow$

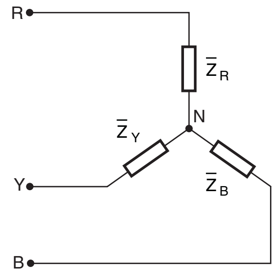


Fig. 2

$$\overline{Z}_R = \frac{\overline{Z}_{RY} \times \overline{Z}_{BR}}{\overline{Z}_{RY} + \overline{Z}_{YB} + \overline{Z}_{BR}} = \frac{10\angle 30^o \times 2.5\angle 60^o}{10\angle 30^o + 10\angle 45^o + 2.5\angle 60^o} = 0.7248 + j0.8646\ \Omega$$

$$= 1.1282\angle 50^o\ \Omega$$

$$\overline{Z}_Y = \frac{\overline{Z}_{YB} \times \overline{Z}_{RY}}{\overline{Z}_{RY} + \overline{Z}_{YB} + \overline{Z}_{BR}} = \frac{10\angle 45^o \times 10\angle 30^o}{10\angle 30^o + 10\angle 45^o + 2.5\angle 60^o} = 3.6955 + j2.5901\Omega$$

$$= 4.5128\angle 35^o\ \Omega$$

$$\overline{Z}_B = \frac{\overline{Z}_{BR} \times \overline{Z}_{YB}}{\overline{Z}_{RY} + \overline{Z}_{YB} + \overline{Z}_{BR}} = \frac{2.5\angle 60^o \times 10\angle 45^o}{10\angle 30^o + 10\angle 45^o + 2.5\angle 60^o} = 0.4763 + j1.0227\ \Omega$$

$$= 1.1282\angle 65^o\ \Omega$$

EXAMPLE 4.37 ***(AU June'14, 8 Marks)***

A symmetrical three-phase three wire 440V supply is connected to a star-connected load. The impedance in each branch is $\overline{Z}_R = 2 + j3\ \Omega$, $\overline{Z}_Y = 1 - j2\ \Omega$ and $\overline{Z}_B = 3 + j4\ \Omega$. Find its equivalent delta connected load.

SOLUTION

The standard equations for converting star connected impedances into delta or vice-versa are based on the concept that the power drawn by the impedances in the star or delta connection is the same. Hence, the equivalent delta load can be obtained by using star to delta transformation.

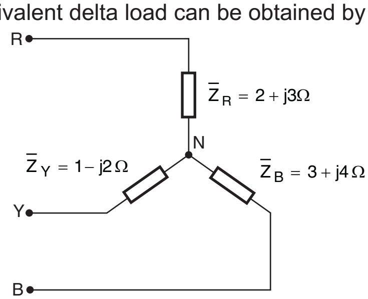


Fig. 1

$\Rightarrow$

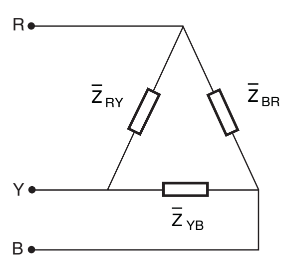


Fig. 2

$$\bar{Z}_{RY} = \bar{Z}_R + \bar{Z}_Y + \frac{\bar{Z}_R \bar{Z}_Y}{\bar{Z}_B} = 2 + j3 + 1 - j2 + \frac{(2+j3)\times(1-j2)}{3+j4}$$

$$= 2 + j3 + 1 - j2 + 0.8 - j1.4$$

$$= 3.8 - j0.4\,\Omega = 3.821\angle -6^0\,\Omega$$

$$\bar{Z}_{YB} = \bar{Z}_Y + \bar{Z}_B + \frac{\bar{Z}_Y \bar{Z}_B}{\bar{Z}_R} = 1 - j2 + 3 + j4 + \frac{(1-j2)\times(3+j4)}{2+j3}$$

$$= 1 - j2 + 3 + j4 + 1.2308 - j\,2.8462$$

$$= 5.2308 - j\,0.8462\,\Omega$$

$$= 5.2988\angle -9.2^0\,\Omega$$

$$\bar{Z}_{BR} = \bar{Z}_B + \bar{Z}_R + \frac{\bar{Z}_B \bar{Z}_R}{\bar{Z}_Y} = 3 + j4 + 2 + j3 + \frac{(3+j4)\times(2+j3)}{1-j2}$$

$$= 3 + j4 + 2 + j3 - 8 + j$$

$$= -3 + j\,8\,\Omega = 8.544\angle 110.6^0\,\Omega$$

The equivalent delta connected load impedances are,

$$\bar{Z}_{RY} = 3.8 - j\,0.4\,\Omega; \quad \bar{Z}_{YB} = 5.2308 - j\,0.8462\,\Omega; \quad \bar{Z}_{BR} = -3 + j\,8\,\Omega$$

EXAMPLE 4.38 *(AU June'14, 8 Marks)*

A three phase, balanced delta–connected load of $4 + j8\ \Omega$, is connected across a 400 *V*, three-phase balanced supply. Determine the phase currents and line currents (Phase sequence is RYB).

SOLUTION

The phase sequence is RYB. The line voltages for the RYB sequence are $\bar{V}_{RY}, \bar{V}_{YB}$ and $\bar{V}_{BR}$. Let us take $\bar{V}_{RY}$ as the reference phasor. The three phase delta-connected load with conventional polarity of voltages and direction of currents is shown in Fig. 1.

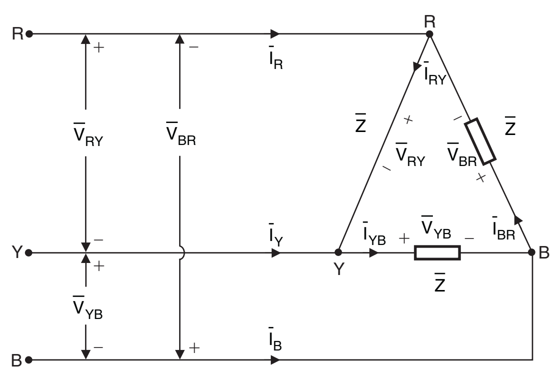


Fig. 1

The line voltages for the RYB sequence, with $\bar{V}_{RY}$ as the reference phasor are,

$$\bar{V}_{RY} = V_L\angle 0^0 = 400\angle 0^0\ V$$

$$\bar{V}_{YB} = V_L\angle -120^0 = 400\angle -120^0\ V$$

$$\bar{V}_{BR} = V_L\angle -240^0 = 400\angle -240^0\ V$$

Here, $\left|\bar{V}_{RY}\right| = \left|\bar{V}_{YB}\right| = \left|\bar{V}_{BR}\right| = V_L = 400\,V$

In a delta connection, the phase voltage is the same as line voltage. Now, the phase currents $\bar{I}_{RY}, \bar{I}_{YB}$ and $\bar{I}_{BR}$ are given by the ratio of respective phase voltages and impedance.

The load impedance per phase, $\bar{Z} = 4 + j8\,\Omega = 8.9443\angle 63.4^0\,\Omega$.

Therefore, the phase currents are,

$$\bar{I}_{RY} = \frac{\bar{V}_{RY}}{\bar{Z}} = \frac{400\angle 0^0}{8.9443\angle 63.4^0} = 44.7212\angle -63.4^0\ A$$

$$\bar{I}_{YB} = \frac{\bar{V}_{YB}}{\bar{Z}} = \frac{400\angle -120^0}{8.9443\angle 63.4^0} = 44.7212\angle -183.4^0\ A$$

$$\bar{I}_{BR} = \frac{\bar{V}_{BR}}{\bar{Z}} = \frac{400\angle -240^0}{8.9443\angle 63.4^0} = 44.7212\angle -303.4^0\ A$$

The line currents can be computed by writing KCL equations at nodes R, Y and B.

With reference to Fig. 2, at node-R by writing KCL equation we get,

$$\bar{I}_R = \bar{I}_{RY} - \bar{I}_{BR}$$
$$= 44.7212\angle -63.4^o - 44.7212\angle -303.4^o$$
$$= 77.4594\angle -93.4^o\ A$$

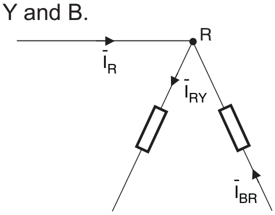


Fig. 2.

With reference to Fig. 3, at node-Y by writing KCL equation we get,

$$\bar{I}_Y = \bar{I}_{YB} - \bar{I}_{RY}$$
$$= 44.7212\angle -183.4^o - 44.7212\angle -63.4^o$$
$$= 77.4594\angle 146.6^o = 77.4594\angle -213.4^o\ A$$

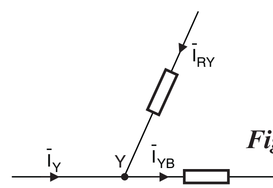


Fig. 3.

With reference to Fig. 4, at node-B by writing KCL equation we get,

$$\bar{I}_B = \bar{I}_{BR} - \bar{I}_{YB}$$
$$= 44.7212\angle -303.4^o - 44.7212\angle -183.4^o$$
$$= 77.4594\angle 26.6^o = 77.4594\angle -333.4^o\ A$$

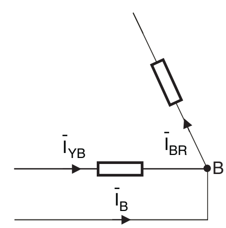


Fig. 4.

EXAMPLE 4.39

(AU Dec'14, 8 Marks)

Three star-connected impedances $\bar{Z}_1 = 20 + j37.7\ \Omega$/phase are in parallel with three delta-connected impedances $\bar{Z}_2 = 20 + j37.7\ \Omega$/phase. The line voltage is 398 *V*. Find the line current, power factor, power and reactive volt-ampere taken by the combination.

SOLUTION

Given that, $\bar{Z}_1 = 20 + j37.7\ \Omega$/phase

$\bar{Z}_2 = 20 + j37.7\ \Omega$/phase

Line voltage, V_L = 398 *V*

The star and delta-connected parallel loads are shown in Fig.1

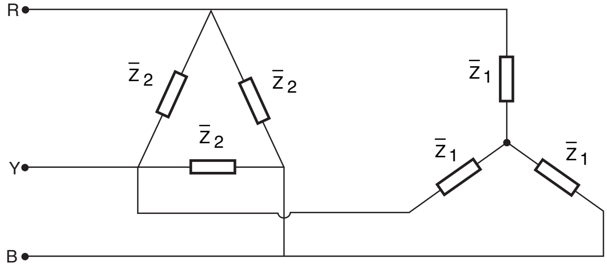


Fig. 1.

The delta-connected load can be converted into star, so that the total load is a parallel combination of two star-connected loads as shown in Fig. 2.

The two parallel star-connected loads shown in Fig. 2 are combined to give a single equivalent star-connected load as shown in Fig. 3

Let, $\bar{Z}_{eq}$ = Equivalent impedance per phase in star connection.

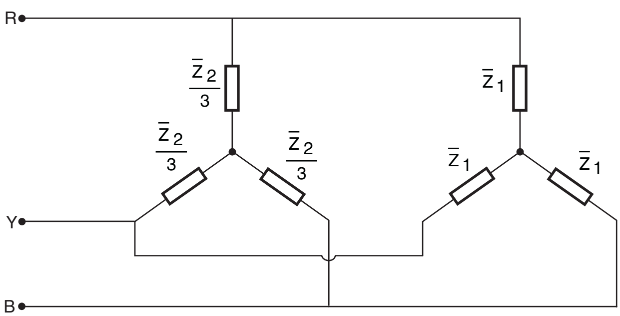


Fig. 2.

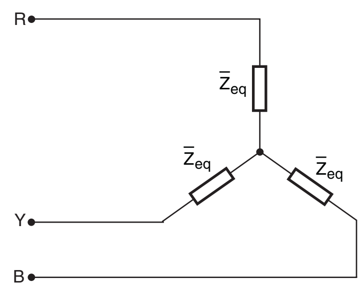


Fig. 3.

$$\therefore\ \bar{Z}_{eq} = \frac{\bar{Z}_1 \times \frac{\bar{Z}_2}{3}}{\bar{Z}_1 + \frac{\bar{Z}_2}{3}} = \frac{(20 + j37.7) \times \frac{(20 + j37.7)}{3}}{(20 + j37.7) + \frac{(20 + j37.7)}{3}}$$

$$= \frac{(20 + j37.7) \times (6.6667 + j12.5667)}{20 + j37.7 + 6.6667 + j12.5667} = 5 + j9.425 = 10.6691\angle 62.1^{\circ}\ \Omega$$

> ***Note :*** *Since the loads are balanced there is no shift in neutral and so both the star-connected phase impedances can be considered as parallel.*

We know that,

Line current, $I_L = \frac{V_L/\sqrt{3}}{|\bar{Z}_{eq}|} = \frac{398/\sqrt{3}}{10.6691} = 21.5375\ A$

Power factor, $\cos\phi = \cos\angle\bar{Z}_{eq} = \cos 62.1^{\circ} = 0.4679$ lag

Power, $P = \sqrt{3}\,V_L I_L \cos\phi = \sqrt{3} \times 398 \times 21.5375 \times \cos 62.1^{\circ}$

$= 6947.4\ W = 6.9474\ kW$

Reactive power, $Q = \sqrt{3}\,V_L I_L \sin\phi = \sqrt{3} \times 398 \times 21.5357 \times \sin 62.1^{\circ}$

$= 13121.2\ VAR = 13.1212\ kVAR$

EXAMPLE 4.40

Determine the power and power factor of a three-phase load if the two wattmeters used for power measurement read i) 1000 W each, both positive and ii) 1000 W each of opposite sign.

SOLUTION

Case i : Wattmeter readings are equal and positive

Let, $P_1 = P_2 = 1000\,W$

Power, $P = P_1 + P_2 = 1000 + 1000 = 2000\text{ W} = 2\,kW$

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right]$$

$$= \cos\left[\tan^{-1}\left(\sqrt{3}\times\frac{1000-1000}{1000+1000}\right)\right] = \cos\left[\tan^{-1}0\right] = \cos 0^\circ = 1$$

Case ii : Wattmeter readings are equal but opposite in sign

Let, $P_1 = 1000\,W$, $P_2 = -1000\,W$

Power, $P = P_1 + P_2 = 1000 - 1000 = 0\,W$

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right] = \cos\left[\tan^{-1}\left(\sqrt{3}\times\frac{-1000-1000}{1000-1000}\right)\right]$$

$$= \cos\left[\tan^{-1}\left(\sqrt{3}\times\frac{-2000}{0}\right)\right] = \cos\left[\tan^{-1}(-\infty)\right] = \cos(-90^\circ) = 0$$

EXAMPLE 4.41 *(AU Dec'15, 8 Marks)*

In a balanced load, the readings of two wattmeters are 2000 W and 500 W. Calculate the power factor of the load i) when both 2000 W and 500 W are positive and ii) when the later is obtained after reversing the connections to the current coil of one instrument.

SOLUTION

Case i : When both wattmeter readings are positive

Let, $P_1 = 2000\,W$

$P_2 = 500\,W$

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right]$$

$$= \cos\left[\tan^{-1}\left(\sqrt{3}\times\frac{500-2000}{2000+500}\right)\right] = \cos\left[\tan^{-1}(-1.0392)\right] = 0.6934$$

Case ii : When current coil of one wattmeter is reversed

When the current coil of one wattmeter is reversed, its reading is considered as negative. Let us take 500 W as a negative reading.

Let, $P_1 = 2000\,W$

$P_2 = -500\,W$

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right] = \cos\left[\tan^{-1}\left(\sqrt{3} \times \frac{-500 - 2000}{2000 - 500}\right)\right]$$

$$= \cos\left[\tan^{-1}\left(\sqrt{3} \times \frac{-2500}{1500}\right)\right] = \cos\left[\tan^{-1}(-2.8868)\right] = 0.3273$$

Note : *When a wattmeter reading is negative, power factor is less than 0.5*

EXAMPLE 4.42 *(AU May'17, 8 Marks)*

Each of two wattmeters connected to measure the power input to a three-phase circuit read 10 *kW* on a balanced load, when the power factor is unity. What does the instrument read when the power factor falls to i) 0.866 lagging, and ii) 0.5 lagging, the total three-phase power remaining unaltered?

SOLUTION

When power factor is unity, the power input to the load is 10 + 10 = 20 *kW*. This power remains the same for any power factor.

$$\therefore \; P_1 + P_2 = 10 + 10 = 20\,kW \qquad \text{..... (1)}$$

We know that,

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right]$$

$$\therefore \; \cos^{-1}(\cos\phi) = \tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)$$

$$\tan\left[\cos^{-1}(\cos\phi)\right] = \sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}$$

$$\therefore \; P_2 - P_1 = \frac{(P_1 + P_2)}{\sqrt{3}} \times \tan\left[\cos^{-1}(\cos\phi)\right] = \frac{20}{\sqrt{3}}\tan\left[\cos^{-1}(\cos\phi)\right]$$

$$= 11.547\;\tan\left[\cos^{-1}(\cos\phi)\right] \qquad \text{..... (2)}$$

Case i : Power factor, $\cos\phi$ = 0.866 lag

From equation (1), we get,

$$P_1 + P_2 = 20\,kW \qquad \text{..... (3)}$$

From equation (2), we get,

$$P_2 - P_1 = 11.547\tan\left[\cos^{-1}(\cos\phi)\right]$$
$$= 11.547\tan\left[\cos^{-1}(0.866)\right]$$
$$= 11.547\tan 30^\circ$$
$$= 6.6667\,kW \qquad \text{..... (4)}$$

On adding equations (3) and (4), we get,

$$P_1 + P_2 = 20$$
$$P_2 - P_1 = 6.6667$$
$$2P_2 = 26.6667$$

$$\therefore \; P_2 = \frac{26.6667}{2} = 13.3334\,kW$$

From equation (3), we can write,

$$P_1 = 20 - P_2$$

$$\therefore \quad P_1 = 20 - 13.3334 = 6.6666\, kW$$

The readings of wattmeter are,

$$P_1 = 6.6666\, kW \quad \text{and} \quad P_2 = 13.3334\, kW$$

Case ii : Power factor, cosϕ = 0.5 lag

From equation (1), we get,

$$P_1 + P_2 = 20\, kW \qquad \text{..... (5)}$$

From equation (2) we get,

$$P_2 - P_1 = 11.547 \tan[\cos^{-1}(\cos\phi)]$$
$$= 11.547 \tan[\cos^{-1}(0.5)]$$
$$= 11.547 \tan 60^\circ$$
$$= 20\, kW \qquad \text{..... (6)}$$

On adding equations (5) and (6), we get,

$$P_1 + P_2 = 20$$
$$P_2 - P_1 = 20$$
$$2P_2 = 40$$

$$\therefore \quad P_2 = \frac{40}{2} = 20\, kW$$

From equation (5), we get,

$$P_1 = 20 - P_2$$

$$\therefore \quad P_1 = 20 - 20 = 0$$

The readings of wattmeter are,

$$P_1 = 0 \quad \text{and} \quad P_2 = 20 \text{ kW}$$

EXAMPLE 4.43

A 500 *V*, three-phase motor has an output of 3.73 *kW* and operates at a power factor of 0.85, with an efficiency of 90%. Calculate the reading on each of the two wattmeters connected to measure the input.

SOLUTION

Given that, Power output = 3.73 *kW*

Efficiency = 90%

$$\text{Power input} = \frac{\text{Power output}}{\text{Efficiency}} = \frac{3.73}{\frac{90}{100}} = \frac{3.73}{0.9} = 4.1444\, kW$$

We know that in two wattmeter method of power measurement, the power input is equal to the sum of two wattmeter readings.

$$\therefore \quad P_1 + P_2 = 4.1444\, kW \qquad \text{..... (1)}$$

We know that,

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right]$$

$$\therefore \quad \cos^{-1}(\cos\phi) = \tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)$$

$$\tan\left[\cos^{-1}(\cos\phi)\right] = \sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}$$

$$\therefore \quad P_2 - P_1 = \frac{P_1 + P_2}{\sqrt{3}}\tan\left[\cos^{-1}(\cos\phi)\right]$$

$$= \frac{P_1 + P_2}{\sqrt{3}}\tan\left[\cos^{-1}0.85\right]$$

$\boxed{\cos\phi = 0.85}$

$$= \frac{4.1444}{\sqrt{3}} \times \tan\left[\cos^{-1}0.85\right]$$

$\boxed{\text{Using equation (1)}}$

$$= 1.4829\,kW \qquad \text{..... (2)}$$

On adding equations (1) and (2), we get,

$$P_1 + P_2 = 4.1444$$

$$P_2 - P_1 = 1.4829$$

$$2P_2 = 5.6273$$

$$\therefore \quad P_2 = \frac{5.6273}{2} = 2.8137\,kW$$

From equation (1), we get,

$$P_1 = 4.1444 - P_2$$

$$\therefore \quad P_1 = 4.1444 - 2.8137$$

$$= 1.3307\text{ kW}$$

Therefore, the wattmeter readings are,

$$P_1 = 1.3307\,kW \quad \text{and} \quad P_2 = 2.8137\,kW$$

EXAMPLE 4.44

A delta-connected load consists of $12\angle 30^\circ\,\Omega$/phase is connected to a 120 V, three-phase supply of phase sequence RYB. Calculate phase power and wattmeter readings.

SOLUTION

Given that, $V_L = 120\,V$ and $\bar{Z} = 12\angle 30^\circ\ \Omega$/phase

Let, $\bar{Z} = Z\angle\theta = 12\angle 30^\circ$

where, $Z = |\bar{Z}| = 12\,\Omega$

$\theta = \angle\bar{Z} = 30^\circ$

$\boxed{\text{In delta, phase voltage is the same as line voltage.}}$

$$\therefore \quad \text{Phase current, } I = \frac{V}{Z} = \frac{V_L}{Z} \qquad \text{.....(1)}$$

$$\therefore \quad \text{Line current } I_L = \sqrt{3}\,I = \frac{\sqrt{3}\,V_L}{Z} = \frac{\sqrt{3}\times 120}{12} \qquad \text{.....(2)}$$

$\boxed{\text{Impedance angle } \theta \text{ is the same as the power factor angle } \phi.}$

$$\text{Power factor, } \cos\phi = \cos\theta = \cos 30^\circ = 0.866 \qquad \text{.....(3)}$$

The total load power P can be calculated using equations (1), (2) and (3) as shown below:

Total load power, $P = \sqrt{3}\, V_L I_L \cos\phi$

$$= \sqrt{3} \times 120 \times \frac{\sqrt{3} \times 120}{12} \times 0.866$$

$$= \frac{3 \times 120^2}{12} \times \cos 30^o = 3117.7\,W \; = \; 3.1177\,kW \qquad .....(4)$$

Let, P_1 and P_2 be two wattmeter readings.

Now, $P_1 + P_2 = P$

$$\therefore \; P_1 + P_2 = 3.1177 \qquad .....(5)$$

We know that,

Power factor, $\cos\phi \; = \; \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right]$

$$\therefore \; \phi \; = \; \tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)$$

$$\therefore \tan\phi \; = \; \sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}$$

$$\therefore \quad P_2 - P_1 \; = \; \frac{P_1 + P_2}{\sqrt{3}} \tan\phi$$

$$= \frac{3117.7}{\sqrt{3}} \times \tan 30^o$$

$\boxed{\phi \; = \; 30^o}$

$\boxed{\text{Using equation (5)}}$

$$= 1039.2\,W \; = \; 1.0392\,kW \qquad ..... (6)$$

On adding equations (5) and (6), we get,

$$P_1 + P_2 = 3.1177$$

$$P_2 - P_1 = 1.0392$$

$$2P_2 = 4.1569$$

$$\therefore \quad P_2 \; = \; \frac{4.1569}{2} \; = \; 2.07845\,kW$$

From equation (5), we get,

$$P_1 = 3.1177 - P_2$$

$$\therefore \; P_1 = 3.1177 - 2.07845$$

$$= 1.03925\,kW$$

Therefore, the wattmeter readings are,

$$P_1 = 1.03925\,kW \quad \text{and} \quad P_2 = 2.07845\,kW$$

EXAMPLE 4.45

The input power to a three-phase motor was measured by two wattmeter method. The readings are 5.2 *kW* and 1.7 *kW*. The later reading have been obtained after reversing the current coil connections. The line voltage was 400 *V*. Calculate **a)** the total power, **b)** power factor and **c)** line current.

SOLUTION

Given that, $P_1 = 5.2\,kW$

$P_2 = -1.7\,kW$

Power, $P = P_1 + P_2$

$= 5.2 + (-1.7) = 3.5\,kW$

Here, P_2 is negative because the reading has been obtained after reversing the current coil connections.

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right]$$

$$= \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{-1.7 - 5.2}{5.2 + (-1.7)}\right)\right]$$

$$= \cos[\tan^{-1}(-3.4146)]$$

$$= \cos(-73.7^\circ) = 0.2807$$

We know that, $P = \sqrt{3}\, V_L I_L \cos\phi$

$$\therefore \text{ Line current, } I_L = \frac{P}{\sqrt{3}\, V_L \cos\phi}$$

$$= \frac{3.5 \times 1000}{\sqrt{3} \times 400 \times 0.2807} = 17.9972\,A$$

EXAMPLE 4.46

(AU Dec'16, 16 Marks)

Two wattmeters are used to measure power in a three-phase load. The wattmeter readings are 1560 *W* and 2100 *W* and line voltage is 220 *V*. Calculate **a)** average power per phase, **b)** total reactive power, **c)** power factor and **d)** phase impedance. Determine whether the impedance is inductive or capacitive.

SOLUTION

Given that, $P_1 = 1560\,W$

$P_2 = 2100\,W$

Power, $P = P_1 + P_2 = 1560 + 2100 = 3660\,W$

Power per phase (Average power per phase) $= \dfrac{P}{3} = \dfrac{3660}{3} = 1220\,W$

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right]$$

$$= \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{2100 - 1560}{1560 + 2100}\right)\right]$$

$$= \cos[\tan^{-1}(0.2555)] = 0.9689$$

The power factor $\cos\phi$ is defined as the ratio of active power P and apparent power S.

$$\therefore \cos\phi = \frac{P}{S}$$

$$\therefore\ S = \frac{P}{\cos\phi} = \frac{3660}{0.9689} = 3777.5\ VA$$

We know that, $\overline{S} = P + jQ$, where Q is reactive power,

$$\therefore\ S = |\overline{S}| = \sqrt{P^2 + Q^2} \qquad \Rightarrow \qquad S^2 = P^2 + Q^2$$

$$\therefore\ \text{Reactive power, } Q = \sqrt{S^2 - P^2}$$

$$= \sqrt{3777.5^2 - 3660^2} = 934.8\ VAR$$

Since the given load is delta-connected, the line voltage V_L and phase voltage V are the same.

Given that, $V_L = 220$ V

$V = V_L = 220$ V

Now, power per phase = VI cos ϕ .....(1)

where I = current per phase

We know that,

$$\text{Impedance per phase, } Z = \frac{V}{I}$$

$$\therefore\ I = \frac{V}{Z} \qquad .....(2)$$

From equations (1) and (2), we get,

$$\text{Power per phase} = V \times \frac{V}{Z} \times \cos\phi$$

$$\therefore\ \text{Impedance per phase, } Z = \frac{V^2 \cos\phi}{\text{power per phase}}$$

$$= \frac{220^2 \times 0.9689}{1220} = 38.4383\ \Omega/phase$$

In two wattmeter method of power measurement, the lag and lead nature of power factor cannot be determined. And so the nature of impedance, whether inductive or capacitive, cannot be determined.

EXAMPLE 4.47

Two wattmeters are used to measure power in a three-phase load. The wattmeter readings are 400 *W* and −35 *W*. Calculate **a)** total active power, **b)** power factor, **c)** reactive power and **d)** volt-amperes.

SOLUTION

Given that, $P_1 = 400\ W$

$P_2 = -35\ W$

Total active power, $P = P_1 + P_2$

$= 400 + (-35) = 365\ W$

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right]$$

$$= \cos\left[\tan^{-1}\left(\sqrt{3} \times \frac{-35 - 400}{400 - 35}\right)\right] = 0.436$$

Here, $\cos\phi = 0.436$

$\therefore\ \phi = \cos^{-1} 0.436 = 64.2°$

Consider the power triangle shown in Fig. 1.

With reference to Fig. 1, we can write,

$$\tan\phi = \frac{Q}{P}$$

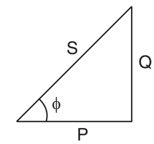


Fig. 1 : Power triangle.

$\therefore$ Reactive power, $Q = P \tan\phi$

$$= 365 \times \tan(64.2^\circ) = 755\, VAR$$

$$\text{Volt-amperes, } S = \sqrt{P^2 + Q^2} = \sqrt{365^2 + 755^2} = 838.6\, VA$$

EXAMPLE 4.48

Calculate the total power input and reading of the two wattmeters connected to measure power in a three-phase balanced load, if the reactive power input is 15 *kVAR* and the load power factor is 0.8. Also compute load *kVA*.

SOLUTION

Given that, Reactive power, $Q = 15\, kVAR$

Power factor, $\cos\phi = 0.8$

$\therefore$ Power factor angle, $\phi = \cos^{-1} 0.8 = 36.9^\circ$

Consider the power triangle shown in Fig. 1. With reference to Fig. 1, we can write,

$$\tan\phi = \frac{Q}{P}$$

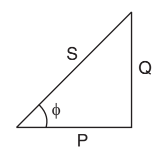


Fig. 1 : Power triangle.

$$\therefore \text{ Active power, } P = \frac{Q}{\tan\phi}$$

$$= \frac{15}{\tan 36.9^\circ} = 19.9781\, kW$$

$$\text{Load kVA, } S = \sqrt{P^2 + Q^2} = \sqrt{19.9781^2 + 15^2} = 24.9825\, kVA$$

In two-wattmeter method of power measurement, we know that the sum of two wattmeter readings is equal to active power, P.

$$\therefore P_1 + P_2 = P = 19.9781\, kW \qquad \text{..... (1)}$$

We know that,

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right]$$

$$\cos^{-1}(\cos\phi) = \tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)$$

$$\tan\left[\cos^{-1}(\cos\phi)\right] = \sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}$$

$$\therefore P_2 - P_1 = \frac{P_1 + P_2}{\sqrt{3}} \tan\left[\cos^{-1}(\cos\phi)\right]$$

$$= \frac{19.9781}{\sqrt{3}} \times \tan\left[\cos^{-1}(0.8)\right] = 8.6508\, kW \qquad \text{..... (2)}$$

On adding equations (1) and (2), we get,

$$2P_2 = 19.9781 + 8.6508$$

$$\therefore P_2 = \frac{19.9781 + 8.6508}{2} = 14.3145\, kW$$

From equation (1), we get,

$$P_1 = 19.9781 - P_2$$

$$= 19.9781 - 14.3145 = 5.6636\, kW$$

EXAMPLE 4.49

Find the reading of the wattmeter in the circuit shown in Fig. 1. Assume a symmetrical 360 *V* supply with RYB sequence. Draw the phasor diagram.

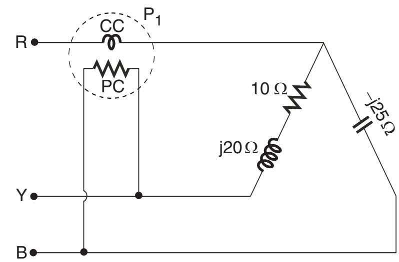


Fig. 1.

SOLUTION

The phase sequence is RYB. The line voltages for the RYB sequence are $\bar{V}_{RY}, \bar{V}_{YB}$ and $\bar{V}_{BR}$. Let us take $\bar{V}_{RY}$ as the reference phasor.

Now, $\bar{V}_{RY} = 360\angle 0^\circ\,V$

$$\bar{V}_{YB} = 360\angle -120^\circ\,V$$

$$\bar{V}_{BR} = 360\angle -240^\circ\,V$$

The polarity of voltages and direction of currents are shown in Fig. 2.

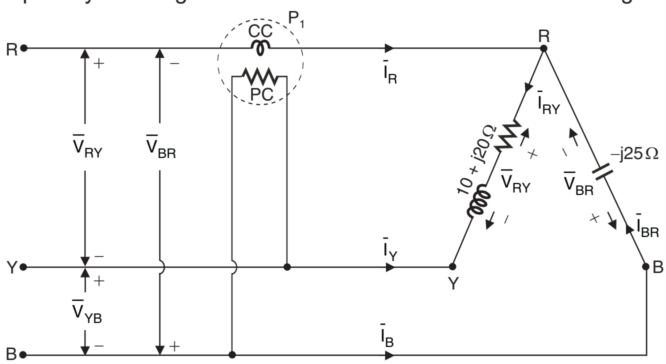


Fig. 2.

With reference to Fig. 2,

$$\bar{I}_{RY} = \frac{\bar{V}_{RY}}{10+j20} = \frac{360\angle 0^\circ}{10+j20} = 7.2 - j14.4\,A = 16.0997\angle -63.4^\circ\,A$$

$$\bar{I}_{BR} = \frac{\bar{V}_{BR}}{-j25} = \frac{360\angle -240^\circ}{-j25} = -12.4708 - j7.2\,A = 14.4\angle -150^\circ\,A$$

By KCL at node-R, we get,

$$\bar{I}_R + \bar{I}_{BR} = \bar{I}_{RY}$$

$$\therefore\ \bar{I}_R = \bar{I}_{RY} - \bar{I}_{BR}$$

$$= 7.2 - j14.4\,A - (-12.4708 - j7.2)$$

$$= 19.6708 - j7.2$$

$$= 20.9471\angle -20.1^\circ\,A$$

The wattmeter reading, $P_1 = |\bar{V}_{YB}||\bar{I}_R| \times \cos\theta_1$

Here, $|\bar{V}_{YB}| = 360\,V$

$$|\bar{I}_R| = 20.9471\,A$$

$$\theta_1 = -120^\circ - (-20.1^\circ) = -99.9^\circ$$

$$\therefore\ P_1 = 360 \times 20.9471 \times \cos(-99.9^\circ) = -1296.5\,W$$

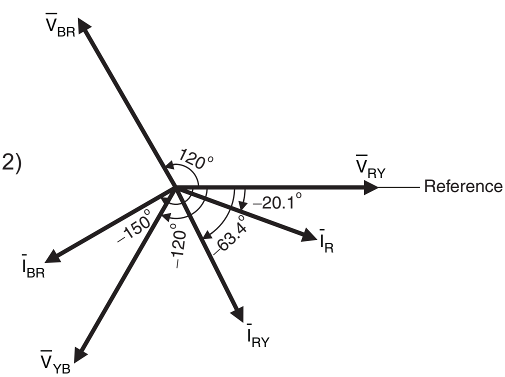


Fig. 3 : Rms phasors of voltages and currents.

EXAMPLE 4.50

A 440V, three-phase, three-wire system has a current of $10\angle -30^\circ A$ in R-phase, $14\angle -60^\circ A$ in the Y-phase. Find, **a)** the current in B-phase and **b)** the reading of two wattmeters with current coils in R and Y phase and voltage coils connected to B-phase. Phase sequence is RYB with $\bar{V}_{RY}$ as the reference phasor.

SOLUTION

The line currents are $\bar{I}_R$, $\bar{I}_Y$ and $\bar{I}_B$.

Given that, $\bar{I}_R = 10\angle -30^\circ A$

$\bar{I}_Y = 14\angle -60^\circ A$

We know that in a three-phase three-wire system, the sum of the line currents is zero.

$$\therefore \quad \bar{I}_R + \bar{I}_Y + \bar{I}_B = 0$$

$$\therefore \quad \bar{I}_B = -(\bar{I}_R + \bar{I}_Y)$$

$$= -(10\angle -30^\circ + 14\angle -60^\circ)$$

$$= -15.6603 + j17.1244$$

$$= 23.2054\angle 132.4^\circ A$$

The line voltages for the RYB sequence with $\bar{V}_{RY}$ as the reference phasor are,

$$\bar{V}_{RY} = 440\angle 0^\circ V$$

$$\bar{V}_{YB} = 440\angle -120^\circ V$$

$$\bar{V}_{BR} = 440\angle -240^\circ V$$

The connections of two wattmeters with polarity of voltages and direction of currents are shown in Fig. 1.

With reference to Fig. 1, the reading of wattmeter, P_1 is given by,

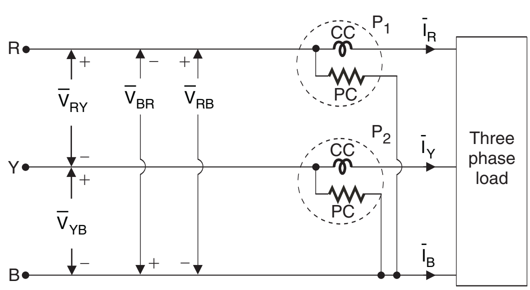


Fig. 1.

$$P_1 = |\bar{V}_{RB}||\bar{I}_R| \times \cos\theta_1$$

where, θ_1 = phase difference between $\bar{V}_{RB}$ and $\bar{I}_R$.

$$\bar{V}_{RB} = -\bar{V}_{BR} = 1\angle 180^\circ \times \bar{V}_{BR} = 1\angle 180^\circ \times 440\angle -240^\circ = 440\angle -60^\circ V$$

$$|\bar{V}_{RB}| = 440 V$$

$$|\bar{I}_R| = 10 A$$

$$\theta_1 = -60^o - (-30^o) = -30^o$$

$$\therefore \; P_1 = 440 \times 10 \times \cos(-30^o) = 3810.5\,W = \frac{3810.5}{1000}kW = 3.8105\,kW$$

With reference to Fig. 1, the reading of wattmeter, P_2 is given by,

$$P_2 = |\bar{V}_{YB}| \times |\bar{I}_Y| \times \cos\theta_2$$

where, θ_2 = phase difference between $\bar{V}_{YB}$ and $\bar{I}_Y$.

Here, $\bar{V}_{YB} = 440\angle-120^o\,V$

$$\bar{I}_Y = 14\angle-60^o\,A$$

$$\therefore \; |\bar{V}_{YB}| = 440\,V, \; |\bar{I}_Y| = 14\,A, \quad \theta_2 = -120^o - (-60^o) = -60^o$$

$$\therefore \; P_2 = 440 \times 14 \times \cos(-60^o) = 3080\,W = \frac{3080}{1000}kW = 3.08\,kW$$

EXAMPLE 4.51

A delta-connected load consists of $\bar{Z}_{RY} = 10 + j10\,\Omega$, $\bar{Z}_{YB} = 15 - j15\,\Omega$, and $\bar{Z}_{BR} = 20 + j10\,\Omega$ and it is connected to a 400 V, three-phase supply of phase sequence RYB. Calculate the readings of the wattmeter with current coil in line R and B.

SOLUTION

The phase sequence is RYB.The line voltages for the RYB sequence are $\bar{V}_{RY}$, $\bar{V}_{YB}$ and $\bar{V}_{BR}$. Let us take $\bar{V}_{RY}$ as the reference phasor. In delta connection, the line and phase voltages are the same. Hence, the line and phase voltages are,

$$\bar{V}_{RY} = 400\angle 0^o\,V$$

$$\bar{V}_{YB} = 400\angle-120^o\,V$$

$$\bar{V}_{BR} = 400\angle-240^o\,V$$

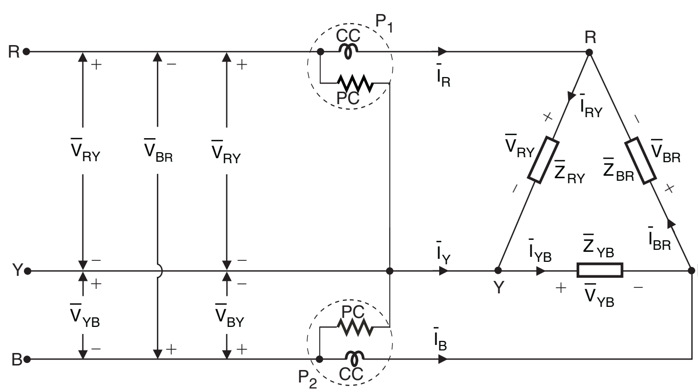


Fig. 1.

The connections of wattmeters with polarity of voltages and direction of currents are shown in Fig. 1.

With reference to Fig. 1, we can say that the current through the wattmeters are $\bar{I}_R$ and $\bar{I}_B$. The voltage across the wattmeters are $\bar{V}_{RY}$ and $\bar{V}_{BY}$. Hence, the wattmeter readings P_1 and P_2 are,

$$P_1 = |\bar{V}_{RY}| \times |\bar{I}_R| \times \cos\theta_1$$

$$P_2 = |\bar{V}_{BY}| \times |\bar{I}_B| \times \cos\theta_2$$

where, θ_1 = Phase difference between $\bar{V}_{RY}$ and $\bar{I}_R$.

θ_2 = Phase difference between $\bar{V}_{BY}$ and $\bar{I}_B$.

Using Ohm's law, the phase currents can be obtained as a ratio of phase voltage and impedance.

Therefore, phase currents are,

$$\bar{I}_{RY} = \frac{\bar{V}_{RY}}{\bar{Z}_{RY}} = \frac{400\angle 0^\circ}{10 + j10} = 20 - j20\ A$$

$$\bar{I}_{YB} = \frac{\bar{V}_{YB}}{\bar{Z}_{YB}} = \frac{400\angle -120^\circ}{15 - j15} = 4.8803 - j18.2137\ A$$

$$\bar{I}_{BR} = \frac{\bar{V}_{BR}}{\bar{Z}_{BR}} = \frac{400\angle -240^\circ}{20 + j10} = -1.0718 + j17.8564\ A$$

With reference to Fig. 2, by KCL, the line current $\bar{I}_R$ is given by,

$$\boxed{\bar{I}_R + \bar{I}_{BR} = \bar{I}_{RY}}$$

$$\bar{I}_R = \bar{I}_{RY} - \bar{I}_{BR}$$

$$= 20 - j20 - (-1.0718 + j17.8564)$$

$$= 21.0718 - j37.8564\ A$$

$$= 43.3258\angle -60.9^\circ\ A$$

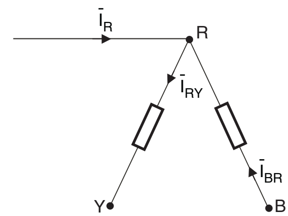


Fig. 2.

With reference to Fig. 3, by KCL, the line current $\bar{I}_B$ is given by,

$$\boxed{\bar{I}_B + \bar{I}_{YB} = \bar{I}_{BR}}$$

$$\bar{I}_B = \bar{I}_{BR} - \bar{I}_{YB}$$

$$= -1.0718 + j17.8564 - (4.8803 - j18.2137)$$

$$= -5.9521 + j36.0701\ A$$

$$= 36.5579\angle 99.4^\circ\ A$$

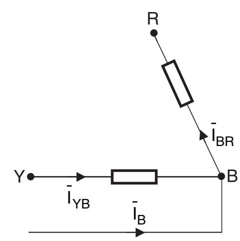


Fig. 3.

The wattmeter reading P_1 is given by,

$$P_1 = |\bar{V}_{RY}| \times |\bar{I}_R| \times \cos\theta_1$$

Here, $\bar{V}_{RY} = 400\angle 0^\circ\ V$ and $\bar{I}_R = 43.3258\angle -60.9^\circ\ A$

$$\therefore\ |\bar{V}_{RY}| = 400\ V$$

$$|\bar{I}_R| = 43.3258\ A$$

$$\theta_1 = 0^\circ - (-60.9^\circ) = 60.9^\circ$$

$$\therefore\ P_1 = 400 \times 43.3258 \times \cos 60.9^\circ$$

$$= 8428.3\ W = \frac{8428.3}{1000}\ kW = 8.4283\ kW$$

The wattmeter reading P_2 is given by,

$$P_2 = |\bar{V}_{BY}| \times |\bar{I}_B| \times \cos\theta_2$$

Here, $\bar{V}_{BY} = -\bar{V}_{YB} = -1 \times \bar{V}_{YB} = 1\angle 180^\circ \times 400\angle -120^\circ = 400\angle 60^\circ\ V$ and

$$\bar{I}_B = 36.5579\angle 99.4^\circ\ A$$

$$\therefore\ |\bar{V}_{BY}| = 400\ V$$

$$|\bar{I}_B| = 36.5579$$

$$\theta_2 = 60^\circ - 99.4^\circ = -39.4^\circ$$

$$\therefore\ P_2 = 400 \times 36.5579 \times \cos(-39.4^\circ)$$

$$= 11299.8\ W = \frac{11299.8}{1000}\ kW = 11.2998\ kW$$

The readings of wattmeter are,

$$P_1 = 8.4283\ kW$$

$$P_2 = 11.2998\ kW$$

EXAMPLE 4.52 ***(AU Dec'16, 8 Marks)***

A delta-connected generator with phase sequence of RBY is connected to a delta-connected load with phase sequence RYB as shown in Fig. 1. Determine the voltages of generator and load by taking $\bar{V}_{RY} = 120\angle 0^\circ\ V$. Also calculate the phase and line currents of the load.

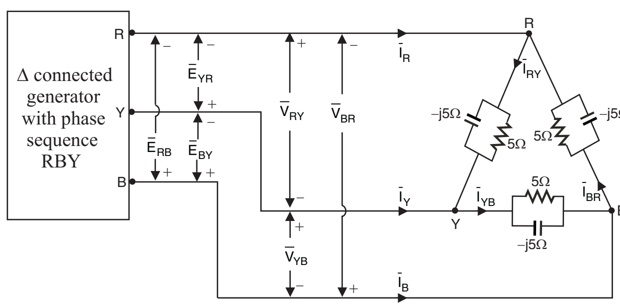


Fig. 1.

SOLUTION

The phase sequence of generator is RBY (or reversed sequence). Therefore, the phase and line volttages of the generator are,

$$\bar{E}_{RB} = E\angle\delta_1$$

$$\bar{E}_{BY} = E\angle\delta_2$$

$$\bar{E}_{YR} = E\angle\delta_3$$

In a delta connection, the phase and line voltages are the same.

The phase sequence of load is RYB (or normal sequence). Also it is given that $\bar{V}_{RY} = 120\angle 0^\circ$ and so it is the reference phasor. By taking $\bar{V}_{RY}$ as the reference phasor, the phase and line voltages of load are,

$$\bar{V}_{RY} = 120\angle\theta_1 = 120\angle 0^\circ\ V$$

$$\bar{V}_{YB} = 120\angle\theta_2 = 120\angle -120^\circ\ V$$

$$\bar{V}_{BR} = 120\angle\theta_3 = 120\angle -240^\circ\ V$$

The phasor diagram of generator and load voltages are shown in Fig. 2.

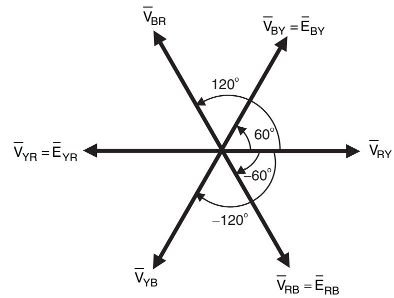


Fig. 2: *Rms phasors of voltages.*

With reference to Fig. 2, we get,

$$\bar{E}_{RB} = E\angle\delta_1 = 120\angle -60^o\ V$$

$$\bar{E}_{BY} = E\angle\delta_2 = 120\angle +60^o\ V$$

$$\bar{E}_{YR} = E\angle\delta_3 = 120\angle \pm 180^o\ V$$

Here the load impedances of all the phase are equal.

$$\therefore\ \bar{Z}_{RY} = \bar{Z}_{YB} = \bar{Z}_{BR} = \frac{5\times(-j5)}{5+(-j5)} = 2.5 - j2.5\ \Omega$$

Now, the phase currents $\bar{I}_{RY}, \bar{I}_{YB}$, and $\bar{I}_{BR}$. are given by the ratio of respective phase voltage and impedance.

In delta connection the phase and line voltages are same.

Therefore the phase currents are,

$$\bar{I}_{RY} = \frac{\bar{V}_{RY}}{\bar{Z}_{RY}} = \frac{120\angle 0^o}{2.5 - j2.5} = 24 + j24\ A = 33.9411\angle 45^o\ A$$

$$\bar{I}_{YB} = \frac{\bar{V}_{YB}}{\bar{Z}_{YB}} = \frac{120\angle -120^o}{2.5 - j2.5} = 8.7846 - j32.7846\ A = 33.9411\angle -75^o\ A$$

$$\bar{I}_{BR} = \frac{\bar{V}_{BR}}{\bar{Z}_{BR}} = \frac{120\angle -240^o}{2.5 - j2.5} = -32.7846 + j8.7846\ A = 33.9411\angle 165^o\ A$$

The line currents $\bar{I}_R, \bar{I}_Y, \bar{I}_B$ of load can be computed by writing KCL equations at nodes R, Y and B.

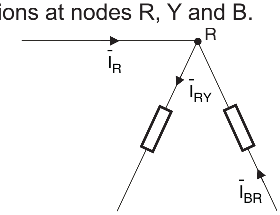


Fig. 3.

With reference to Fig. 3, the line current $\bar{I}_R$ is given by,

$$\bar{I}_R = \bar{I}_{RY} - \bar{I}_{BR} = (24 + j24) - (-32.7846 + j8.7846)$$

$$= 56.7846 + j15.2154\ A$$

$$= 58.7877\angle 15^o\ A$$

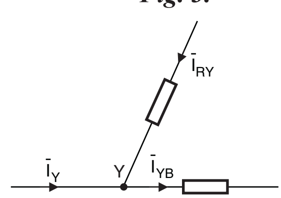


Fig. 4.

With reference to Fig. 4, the line current $\bar{I}_Y$ is given by,

$$\bar{I}_Y = \bar{I}_{YB} - \bar{I}_{RY} = (8.7846 - j32.7846) - (24 + j24)$$

$$= -15.2154 - j56.7846\ A$$

$$= 58.7877\angle -105^o\ A$$

With reference to Fig. 5, the line current $\bar{I}_B$ is given by,

$$\bar{I}_B = \bar{I}_{BR} - \bar{I}_{YB} = (-32.7846 + j8.7846) - (8.7846 - j32.7846)$$

$$= -41.5692 + j41.5692\ A$$

$$= 58.7877\angle 135^o\ A$$

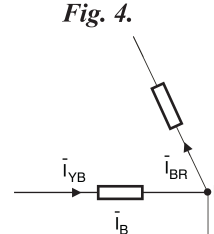


Fig. 5.

EXAMPLE 4.53

A balanced delta-connected load takes a line current of 15 *A* when connected to a balanced three-phase, 400 *V* system. A wattmeter with its current coil in one line and its pressure coil between the two remaining lines reads 2000 *W*. Describe the load impedance.

SOLUTION

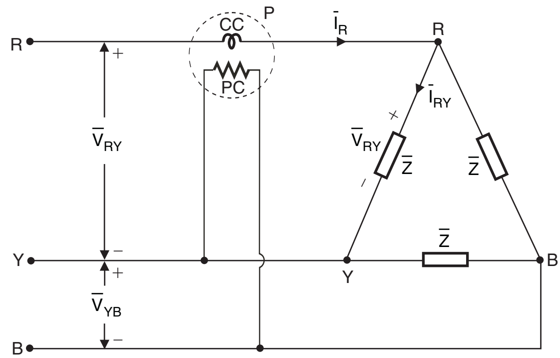


Fig. 1.

Let the load impedance, $\bar{Z} = Z\angle\phi\ \Omega/phase$

Given that, Line voltage, $V_L = 400\,V$

Line current, $I_L = 15\,A$

In a delta connection, the phase and line voltages are the same, and the phase current is $1/\sqrt{3}$ times the line current.

$\therefore$ Phase voltage, $V = V_L = 400$ V

Phase current, $I = \dfrac{I_L}{\sqrt{3}} = \dfrac{15}{\sqrt{3}} = 8.6603\,A$

Now, the magnitude of load impedance, Z is given by,

$$Z = \frac{V}{I} = \frac{400}{8.6603} = 46.1878\ \Omega/phase$$

Let the current coil of wattmeter be connected in line-R and its voltage coil be connected across line-Y and B as shown in Fig. 1.

Now, the reading of wattmeter, P is,

$$P = |\bar{V}_{YB}| \times |\bar{I}_R| \times \cos\theta$$

where, θ = phase difference between $\bar{V}_{YB}$ and $\bar{I}_R$.

Let the phase sequence of supply be RYB and $\bar{V}_{RY}$ be the reference phasor. Now, the line and phase voltages can be expressed as,

$$\bar{V}_{RY} = V_L\angle 0^\circ = V\angle 0^\circ$$

$$\bar{V}_{YB} = V_L\angle -120^\circ = V\angle -120^\circ$$

$$\bar{V}_{BR} = V_L\angle -240^\circ = V\angle -240^\circ$$

Now, the phase current $\bar{I}_{RY}$ for an impedance of $Z\angle\phi\ \Omega/phase$ can be expressed as,

$$\bar{I}_{RY} = \frac{\bar{V}_{RY}}{Z\angle\phi} = \frac{V\angle 0^\circ}{Z\angle\phi} = \frac{V}{Z}\angle(0^\circ - \phi) = I\angle -\phi$$

where, $\dfrac{V}{Z} = I =$ phase current

In a balanced delta-connected load, the line current $\bar{I}_R$ will lag behind the phase current $\bar{I}_{RY}$ by 30°. Also the magnitude of line current is $\sqrt{3}$ times the phase current. Hence, the line current $\bar{I}_R$ can be expressed as,

$$\bar{I}_R = \sqrt{3}\,I\angle(-\phi - 30^\circ) = I_L\angle(-\phi - 30^\circ) = 15\angle(-\phi - 30^\circ)\ A$$

Now, $|\bar{V}_{YB}| = V_L = 400\,V$

$$|\bar{I}_R| = I_L = 15\,A$$

$$\boxed{\bar{I}_L = 15\,A}$$

$$\theta_1 = -120^\circ - (-\phi - 30^\circ) = -120^\circ + \phi + 30^\circ = \phi - 90^\circ$$

$$\therefore \text{ Power, } P = 400 \times 15 \times \cos(\phi - 90^\circ) = 400 \times 15 \times \sin\phi$$

$$\boxed{\cos(\phi - 90^\circ) = \sin\phi}$$

Given that, $P = 2000\,W$

$$\therefore 2000 = 400 \times 15 \times \sin\phi$$

$$\therefore \sin\phi = \frac{2000}{400 \times 15}$$

Now impedance angle, $\phi = \sin^{-1}\left(\dfrac{2000}{400 \times 15}\right) = 19.5^\circ$

$\therefore$ Load impedance, $\bar{Z} = Z\angle\phi = 46.1878\angle 19.5^\circ\ \Omega/phase$

$$= 230.9401\angle -30^\circ - 200\angle 16^\circ$$

EXAMPLE 4.54 *(AU June'16, 8 Marks)*

If P_1 and P_2 are the readings of two wattmeters which measured power in a three-phase balanced system and if $P_1/P_2 = a$, show that the power factor of the circuit is given by,

$$\cos\phi = \frac{a+1}{2\sqrt{a^2 - a + 1}}$$

SOLUTION

In two wattmeter method of power measurement, the power factor cos ϕ is given by,

$$\cos = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right]$$

$$\therefore \phi = \tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)$$

$$\therefore \tan\phi = \sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}$$

$$\therefore \frac{\sin\phi}{\cos\phi} = \sqrt{3}\,\frac{P_2\left(1 - \dfrac{P_1}{P_2}\right)}{P_2\left(\dfrac{P_1}{P_2} + 1\right)}$$

$$\boxed{\frac{P_1}{P_2} = a}$$

$$\frac{\sin\phi}{\cos\phi} = \frac{\sqrt{3}\,(1 - a)}{a + 1}$$

$$\therefore (a+1)\sin\phi = \sqrt{3}\,(1-a)\cos\phi$$

On squaring both sides we get,

$$(a+1)^2\sin^2\phi = 3(1-a)^2\cos^2\phi$$

$$(a + 1)^2 (1 - \cos^2\phi) = 3(1 - a)^2 \cos^2\phi$$

$$\boxed{\begin{array}{l} \sin^2\phi + \cos^2\phi = 1 \\ \therefore \sin^2\phi = 1 - \cos^2\phi \end{array}}$$

$$(a + 1)^2 - (a + 1)^2 \cos^2\phi = 3(1 - a)^2 \cos^2\phi$$

$$(a + 1)^2 = 3(1 - a)^2 \cos^2\phi + (a + 1)^2 \cos^2\phi$$

$$(a + 1)^2 = [3(1 - a)^2 + (a + 1)^2] \cos^2\phi$$

$$\therefore \cos^2\phi = \frac{(a+1)^2}{3(1-a)^2 + (a+1)^2}$$

$$= \frac{(a+1)^2}{3(1+a^2-2a) + a^2 + 2a + 1}$$

$$= \frac{(a+1)^2}{3 + 3a^2 - 6a + a^2 + 2a + 1}$$

$$= \frac{(a+1)^2}{4a^2 - 4a + 4}$$

$$= \frac{(a+1)^2}{4(a^2 - a + 1)}$$

On taking square root on both sides we get,

$$\cos\phi = \frac{a+1}{2\sqrt{a^2 - a + 1}}$$

4.27 Summary of Important Concepts

1. The sources in which current/voltage sinusoidally varies with time are called sinusoidal sources.
2. Frequency is the number of cyclic changes that a sinewave will undergo in one second.
3. The time for one cycle is called time period and it is also given by the inverse of frequency.
4. The angular rotation of a sinusoidal vector in one second is called angular frequency.
5. The circuits excited by sinusoidal sources are called ac circuits.
6. The relation between frequency f and angular frequency ω is given by, $\omega = 2\pi f$.
7. The plot of the instantaneous value of the sinusoidal voltage/current with respect to ωt or t is called waveform.
8. In an ac source, when the rms value of voltage is constant and the rms value of current varies, it is called an ac voltage source.
9. In an ac source, when the rms value of current is constant and the rms value of voltage varies, it is called an ac current source.
10. In an ideal ac voltage source, the source impedance is zero.
11. In an ideal ac current source, the source impedance is infinite.
12. An ac voltage source with internal impedance $\overline{Z}_s$ can be represented by an ideal ac voltage source in series with an external impedance of value $\overline{Z}_s$.

13. An ac current source with internal impedance $\overline{Z}_s$ can be represented by an ideal ac current source in parallel with an external impedance of value $\overline{Z}_s$.

14. An ac voltage source $\overline{E}$ in series with impedance $\overline{Z}_s$ can be converted into an equivalent current source $\bar{I}_s\left(\bar{I}_s = \overline{E}/\overline{Z}_s\right)$ in parallel with impedance $\overline{Z}_s$.

15. An ac current source $\bar{I}_s$ in parallel with impedance $\overline{Z}_s$ can be converted into an equivalent voltage source $\overline{E}\left(\overline{E} = \bar{I}\,\overline{Z}_s\right)$ in series with impedance $\overline{Z}_s$.

16. The instantaneous value of a sinusoidal voltage(v) is expressed as, $v = V_m \sin(\omega t \pm \phi)$.

17. The instantaneous value of a sinusoidal current(i) is expressed as, $i = I_m \sin(\omega t \pm \phi)$.

18. The average value of a time varying quantity is the average of the instantaneous value for a particular time period.

19. The rms value of a time varying quantity is the equivalent dc value of that quantity.

20. Form factor is defined as the ratio of rms value and average value of a periodic waveform.

21. Peak factor is defined as the ratio of peak value and rms value of a periodic waveform.

22. Inductance is the property of an element (or matter) by which it opposes any change in flux or current.

23. The inductance of a coil is defined as the ratio of flux linkages and current through the coil.

24. A coil is said to have an inductance of one Henry if a current of one ampere flowing through it produces a flux linkage of one weber-turn in it.

25. Faraday's law says that an emf is induced in a conductor when there is a change in flux linkage and the emf is equal to the rate of change of flux linkage.

26. The v-i relation in an inductor is governed by Faraday's law. Therefore, the voltage acorss an inductor is directly proportional to the rate of change of current through it.

27. Energy W stored in an inductance L carrying a steady current I is given by, $W = \dfrac{LI^2}{2}$.

28. Capacitance is the property of an element (or matter) by which it opposes any change in charge or voltage.

29. The capacitance of a capacitor is defined as the ratio of stored charge and potential difference across it.

30. A capacitor is said to have a capacitance of one Farad if a charge of one Coulomb establishes a potential difference of one volt across it.

31. The current through a capacitor is directly proportional to the rate of change of voltage across it.

32. The energy W stored in a capacitance C having a steady voltage V across it is given by, $W = \frac{CV^2}{2}$.

33. The phase (or phase angle) of a vector is the angular position of the vector with respect to reference at time t = 0.

34. Rotating vectors are functions of phase angle and so they are called phasors.

35. The diagram in which all the voltage and current phasors of a circuit are drawn with respect to the reference phasor is called a phasor diagram.

36. The complex power $\bar{S}$ is defined as the product of rms voltage $\bar{V}$ and conjugate of rms current $\bar{I}^*$ $\left(\text{i.e., } \bar{S} = \bar{V}\,\bar{I}^*\right)$.

37. The magnitude of complex power $\bar{S}$ is called apparent power S. It is also given by the product of voltage and current $\left(\text{i.e., } S = \left|\bar{S}\right| = \left|\bar{V}\,\bar{I}^*\right| = VI\right)$.

38. The unit of apparent power is volt-ampere (*VA*). The higher units are *kVA* and *MVA*.

39. The complex power $\bar{S}$ can be expressed as $\bar{S} = VI\angle\pm\phi = VI\cos\phi \pm jVI\sin\phi = P \pm jQ$, where, ϕ is the phase difference between $\bar{V}$ and $\bar{I}$, $P = VI\cos\phi$ and $Q = VI\sin\phi$.

40. The real part of complex power $\bar{S}$ is called active power P (or simply power) and the unit of power is Watts (*W*). The higher units are *kW* and *MW*.

41. The imaginary part of complex power $\bar{S}$ is called reactive power Q and the unit of reactive power is Volt-Ampere-Reactive (*VAR*). The higher units are *kVAR* and *MVAR*.

42. Power factor is defined as the ratio of (active) power and apparent power. It is also given by cosine of the phase difference between voltage and current.

43. When a resistance is excited by an ac source, the current and voltage will be in-phase.

44. Resistance consumes only active power and the reactive power in a resistance is zero.

45. When an inductance is excited by an ac source, the current lags the voltage by 90°.

46. Inductance consumes only reactive power and the active power in an inductance is zero.

47. When a capacitance is excited by an ac source, the current leads the voltage by 90°.

48. Capacitance delivers only reactive power and the active power in a capacitance is zero.

49. The impedance of an element is defined as the ratio of sinusoidal voltage and current in that element.

50. Impedance is a complex quantity. The real part of impedance is resistance and the imaginary part is reactance.

51. Admittance is the inverse of impedance and it is defined as the ratio of sinusoidal current and voltage.

52. Admittance is a complex quantity. The real part of admittance is conductance and the imaginary part is susceptance.

53 The three-phase sources are three-phase alternators generating three emfs having equal magnitude but with a phase difference of 120^o with respect to each other.

54. The three-phases are named R phase, Y phase and B phase in British convention and A phase, B phase and C phase in American convention.

55. In a polyphase system, when the magnitude of emfs is equal and the phase difference between consecutive emfs is equal then the system is called a balanced system and the emfs are called balanced emfs.

56. Three-phase sources are always designed to generate balanced emfs.

57. For operational convenience and cost effective system, the three-phase sources are operated in star/delta connection.

58. In a star connection, the meeting point of the three sources is called neutral and there is no such neutral point in a delta connection.

59. The transmission lines or connecting wires from the source terminals to the load terminals are called lines.

60. The voltage generated by each phase of a three-phase source is called phase voltage and the voltage between the lines connecting the load is called line voltage.

61. The current delivered by each phase of a three-phase source is called phase current and the current flowing through the line is called line current.

62. In three-phase rotating phasors, the order of reaching the reference point is called phase sequence.

63. In three-phase rotating phasors, when the order of reaching the reference is R phase, Y phase and B phase, the phasors are said to have normal phase sequence or RYB sequence. When the order of reaching the reference is R phase, B phase and Y phase, the phasors are said to have reversed phase sequence or RBY sequence.

64. The maximum value phasors are rotating phasors and the rms phasors are non-rotating phasors.

65. The rms phasors can be drawn by taking a snapshot of rotating phasors at $\omega t = 0$ and reducing the length by $\sqrt{2}$.

66. The salient features of a star-connected three-/four-wire source are,

- ✦ The voltages are always balanced in star-connected sources.
- ✦ The currents may be balanced or unbalanced depending on load.
- ✦ The phase and line currents are the same in a star system.
- ✦ The magnitude of line voltage is $\sqrt{3}$ times the magnitude of phase voltage.
- ✦ The phase voltage of source lags the line voltage by 30^o.
- ✦ In a balanced four-wire system, the neutral current is zero.

67. The line and phase voltages of a star-connected three-/four-wire source for the RYB sequence with $\overline{V}_{RY}$ as the reference phasor are,

Line voltages	Phase voltages
$\overline{V}_{RY} = V_{RY}\angle 0^o = V_L\angle 0^o$	$\overline{E}_R = E_R\angle -30^o = E\angle -30^o$
$\overline{V}_{YB} = V_{YB}\angle -120^o = V_L\angle -120^o$	$\overline{E}_Y = E_Y\angle -150^o = E\angle -150^o$
$\overline{V}_{BR} = V_{BR}\angle -240^o = V_L\angle -240^o = V_L\angle 120^o$	$\overline{E}_B = E_B\angle -270^o = E\angle -270^o = E\angle 90^o$

where, $E_R = E_Y = E_B = E$

$V_{RY} = V_{YB} = V_{BR} = V_L = \sqrt{3}\,E$

68. The line and phase voltages of a star-connected three-/four-wire source for the RBY sequence with $\overline{V}_{RB}$ as the reference phasor are,

Line voltages	Phase voltages
$\overline{V}_{RB} = V_{RB}\angle 0^o = V_L\angle 0^o$	$\overline{E}_R = E_R\angle -30^o = E\angle -30^o$
$\overline{V}_{BY} = V_{BY}\angle -120^o = V_L\angle -120^o$	$\overline{E}_B = E_B\angle -150^o = E\angle -150^o$
$\overline{V}_{YR} = V_{YR}\angle -240^o = V_L\angle -240^o = V_L\angle 120^o$	$\overline{E}_Y = E_Y\angle -270^o = E\angle -270^o = E\angle 90^o$

where, $E_R = E_B = E_Y = E$

$V_{RB} = V_{BY} = V_{YR} = V_L = \sqrt{3}\,E$

69. In a star-connected four-wire system,

Neutral current, $\bar{I}_N = 0$ - for balanced system

Neutral current, $\bar{I}_N = \bar{I}_R + \bar{I}_Y + \bar{I}_B$ - for unbalanced system

70. The salient features of a delta-connected source are,

- ✦ The voltages are always balanced in delta-connected sources.
- ✦ The currents may be balanced or unbalanced depending on load.
- ✦ The phase and line voltages are the same in a delta system.
- ✦ The magnitude of line current is $\sqrt{3}$ times the magnitude of phase current.
- ✦ The line current lags the phase current by 30^o in a balanced delta system.

71. The line and phase voltages of a delta-connected source for the RYB sequence with $\overline{V}_{RY}$ as the reference phasor are,

Line voltages	Phase voltages
$\overline{V}_{RY} = V_{RY}\angle 0^o = V_L\angle 0^o$	$\overline{E}_R = E_R\angle 0^o = E\angle 0^o$
$\overline{V}_{YB} = V_{YB}\angle -120^o = V_L\angle -120^o$	$\overline{E}_Y = E_Y\angle -120^o = E\angle -120^o$
$\overline{V}_{BR} = V_{BR}\angle -240^o = V_L\angle -240^o$	$\overline{E}_B = E_B\angle -240^o = E\angle -240^o$

where, $E_R = E_Y = E_B = E = V_L$

$V_{RY} = V_{YB} = V_{BR} = V_L = E$

72. The line and phase voltages of a delta-connected source for the RBY sequence with $\overline{V}_{RB}$ as the reference phasor are,

Line voltages	Phase voltages
$\overline{V}_{RB} = V_{RB}\angle 0^o = V_L\angle 0^o$	$\overline{E}_R = E_R\angle 0^o = E\angle 0^o$
$\overline{V}_{BY} = V_{BY}\angle -120^o = V_L\angle -120^o$	$\overline{E}_B = E_B\angle -120^o = E\angle -120^o$
$\overline{V}_{YR} = V_{YR}\angle -240^o = V_L\angle -240^o$	$\overline{E}_Y = E_Y\angle -240^o = E\angle -240^o$

where, $E_R = E_B = E_Y = E = V_L$

$V_{RB} = V_{BY} = V_{YR} = V_L = E$

73. In a balanced load, the magnitude of load impedance of each phase will be equal and also the load impedance angle of each phase will be the same.

74. In an unbalanced load, the load impedance of each phase may have different magnitudes and/or different impedance angle.

75. Using analysis of three-phase circuits, it is conventional practice to choose one of the line voltages of the source as the reference phasor. There are six choices for the reference phasor. (Refer Table 12.1 for various choices for the reference phasor).

76. The apparent, active and reactive power in three-phase balanced star/delta-connected load are given by,

Apparent power, $S = \sqrt{3}\,V_L I_L$

(Active) Power, $P = \sqrt{3}\,V_L I_L \cos\phi$

Reactive power, $Q = \sqrt{3}\,V_L I_L \sin\phi$

77. For the same load impedance and supply voltage, the power consumed by a delta-connected load will be three times the power consumed by a star-connected load. Alternatively, power consumed by a star- connected load will be one-third the power consumed by a delta-connected load.

Let, P_D = Power consumed by a delta-connected load.

P_Y = Power consumed by a star-connected load.

Now,

$$P_D = 3P_Y \quad \text{(or)} \quad P_Y = \frac{1}{3}P_D$$

78. The power in an unbalanced star-connected load is,

$$\text{Power, } P = |\overline{V}_R||\overline{I}_R|\cos\phi_1 + |\overline{V}_Y||\overline{I}_Y|\cos\phi_2 + |\overline{V}_B||\overline{I}_B|\cos\phi_3$$

where, $|\overline{V}_R| = |\overline{V}_Y| = |\overline{V}_B| = V_L$

ϕ_1 = Phase difference between $\overline{V}_R$ and $\overline{I}_R$.

ϕ_2 = Phase difference between $\overline{V}_Y$ and $\overline{I}_Y$.

ϕ_3 = Phase difference between $\overline{V}_B$ and $\overline{I}_B$.

79. In a three-wire star-connected unbalanced load, the voltage of load neutral with respect to source neutral is called neutral shift voltage or neutral displacement voltage.

80. The neutral shift voltage can be obtained by subtracting a phase voltage of load from the corresponding phase emf of the source.

Let, $\bar{E}_R$ = R-phase source emf

$\bar{V}_R$ = R-phase load emf.

Now,

$$\text{Neutral shift voltage} = \bar{E}_R - \bar{V}_R$$

81. The power in an unbalanced delta-connected load is,

$$\text{Power, } P = |\bar{V}_{RY}||\bar{I}_{RY}|\cos\phi_1 + |\bar{V}_{YB}||\bar{I}_{YB}|\cos\phi_2 + |\bar{V}_{BR}||\bar{I}_{BR}|\cos\phi_3$$

where, $|\bar{V}_{RY}| = |\bar{V}_{YB}| = |\bar{V}_{BR}| = V_L$

ϕ_1 = Phase difference between $\bar{V}_{RY}$ and $\bar{I}_{RY}$.

ϕ_2 = Phase difference between $\bar{V}_{YB}$ and $\bar{I}_{YB}$.

ϕ_3 = Phase difference between $\bar{V}_{BR}$ and $\bar{I}_{BR}$.

82. The power in any three-phase load (balanced/unbalanced and star/delta) can be measured using only two wattmeters. The power is given by the sum of the two wattmeter readings.

83. In two wattmeter method of power measurement, the power factor of a balanced three-phase load in terms of two wattmeter readings P_1 and P_2 is,

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right]$$

$$\text{Also, Power factor angle, } \phi = \tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)$$

84. In two wattmeter method of power measurement, the following observations can be made regarding the power factor of a balanced load.

- When wattmeter readings are equal, the power factor is unity.
- When one of the wattmeter readings is zero, the power factor is 0.5.
- When one of the wattmeter readings is negative, the power factor is less than 0.5.
- When both the wattmeter readings are positive, the power factor is greater than 0.5.

4.28 Short-answer Questions

Q4.1 A sinusoidal voltage is represented by the equation 100 sin(503t + 30°) volts. What is the frequency and time period?

Solution

The general form of sinusoidal voltage is,

$$v(t) = V_m \sin(\omega t + \phi) \quad ......(1)$$

Given that,

$$v(t) = 100 \sin(503t + 30^\circ) \qquad ......(2)$$

On comparing equations (1) and (2), we get,

$$\omega = 503 \ rad/s$$

Since, $\omega = 2\pi f$

Frequency, $f = \frac{\omega}{2\pi} = \frac{503}{2\pi} = 80 \ Hz$

Time period, $T = \frac{1}{f} = \frac{1}{80} = 0.0125 \ second = 12.5 \times 10^{-3} \ second = 12.5 \ ms$

Q4.2 ***A sinusoidal current is given by the equation i(t) = 7.072 sin 314t A. What is the rms and average value of the current?***

Solution

The general form of sinusoidal current is, $i(t) = I_m \sin(\omega t + \phi)$ .....(1)

Given that, $i(t) = 7.072 \sin 314t$ .....(2)

On comparing equations (1) and (2), we get, $I_m = 7.072 \ A$

$\therefore$ Rms value of current, $I = \frac{I_m}{\sqrt{2}} = \frac{7.072}{\sqrt{2}} = 5 \ A$

Average value of current, $I_{ave} = \frac{2I_m}{\pi} = \frac{2 \times 7.072}{\pi} = 4.502 \ A$

Q4.3 ***Determine the current i(t) for the circuit shown in Fig. Q4.3.1.*** ***(AU May'17, 2 Marks)***

Solution

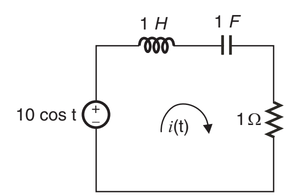


Fig. Q4.3.1

The general form of cosinusoidal voltage is,

$$v(t) = V_m \cos(\omega t \pm \phi) \qquad .....(1)$$

Given that, $v(t) = 10 \cos t$ .....(2)

On comparing equations (1) and (2), we get, $\omega = 1 \ rad/sec$

The circuit with L and C represented by their reactance is shown in Fig. Q4.3.2. Now it can be observed that the inductive and capacitive reactance cancel each other and the circuit is purely resistive as shown in Fig. Q4.3.3.

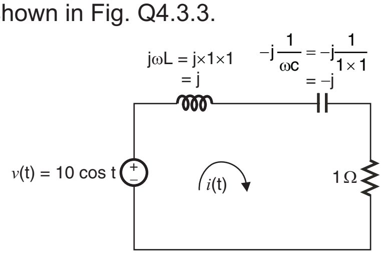


Fig. Q4.3.2.

$\Rightarrow$

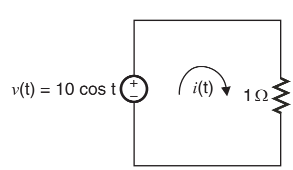


Fig. Q4.3.3.

With reference to Fig. 3, by Ohm's law,

$$i(t) = \frac{v(t)}{R} = \frac{10 \cos t}{1} = 10 \cos t \ A$$

Q4.4 Define form factor.

Form factor is defined as the ratio of rms value and average value of a periodic waveform.

$$\therefore \text{ Form factor, } k_f = \frac{\text{Rms value}}{\text{Average value}}$$

Q4.5 Define peak factor.

Peak factor is defined as the ratio of peak value (or maximum value) and the rms value of a periodic waveform.

$$\therefore \text{ Peak factor, } k_p = \frac{\text{Maximum value}}{\text{Rms value}}$$

Q4.6 What will be the inductance of a coil with 1000 turns while carrying a current of 2A and producing a flux of 0.5 mWb?

Solution

Given that, $N = 1000, \; \phi = 0.5\,mWb, \; I = 2A$

$$\text{Inductance, } L = \frac{N\phi}{I} = \frac{1000 \times 0.5 \times 10^{-3}}{2} = 0.25\,H$$

Q4.7 A steady current of 3 A flows through an inductance of 0.2 H. What will be the energy stored in the inductance?

Solution

Given that, $I = 3A, \; L = 0.2H$

$$\text{Energy stored in inductance, } W = \frac{1}{2} LI^2 = \frac{1}{2} \times 0.2 \times 3^2 = 0.9\,\textit{Joules}$$

Q4.8 A 100 μF capacitance is charged to a steady voltage of 500 V. What will be the energy stored in the capacitance?

Solution

Given that, $C = 100\,\mu F, \; V = 500\,V$

$$\text{Energy stored in capacitance, } W = \frac{1}{2} CV^2 = \frac{1}{2} \times 100 \times 10^{-6} \times 500^2 = 12.5\,\textit{Joules}$$

Q4.9 When a sinusoidal voltage v = 200 sin (377t + 30°) V is applied to a load, it draws a current of 10 (sin 377t + 60°) A. Determine the active and reactive power of the load.

The rms current and voltage phasors in polar form are,

$$\bar{V} = \frac{200}{\sqrt{2}} \angle 30^\circ\, V \qquad ; \qquad \bar{I} = \frac{10}{\sqrt{2}} \angle 60^\circ\, A$$

$$\text{Complex power, } \bar{S} = \bar{V}\,\bar{I}^* = \frac{200}{\sqrt{2}} \angle 30^\circ \times \left(\frac{10}{\sqrt{2}} \angle 60^\circ\right)^* = \frac{200}{\sqrt{2}} \angle 30^\circ \times \frac{10}{\sqrt{2}} \angle -60^\circ$$

$$= \frac{2000}{2} \angle (30^\circ - 60^\circ) = 1000 \angle -30^\circ = 866 - j500\,VA$$

Since, $\bar{S} = P + jQ$; Active power, $P = 866\,W$

Reactive power, $Q = -500\,VAR$ (or $500\,VAR$-Capacitive)

Q4.10 A load consisting of 3 Ω resistance and 4 Ω inductive reactance draws a current of 10 A when connected to a sinusoidal source. Determine the voltage and power in the load.

Magnitude of impedance, $Z = \sqrt{R^2 + X_L^2} = \sqrt{3^2 + 4^2} = 5\,\Omega$

$$\text{Voltage, } V = IZ = 10 \times 5 = 50\,V$$

$$\text{Power, } P = I^2R = 10^2 \times 3 = 300\,W$$

Q4.11 When a sinusoidal voltage of 120 V is applied across a load, it draws a current of 8 A with a phase lead of 30°. Determine the resistance, reactance and impedance of the load.

Let, $\bar{V}$ be the reference phasor.

$$\therefore\ \bar{V} = 120\angle 0^\circ\,V \quad \text{and} \quad \bar{I} = 8\angle +30^\circ\,A$$

$$\text{Impedance, } \bar{Z} = \frac{\bar{V}}{\bar{I}} = \frac{120\angle 0^\circ}{8\angle 30^\circ} = 15\angle -30^\circ = 12.99 - j7.5\,\Omega$$

Since, $\bar{Z} = R + jX$; Resistance, $R = 12.99\,\Omega$

Reactance, $jX = -j7.5\,\Omega$ (or $7.5\,\Omega$-Capacitive)

Q4.12 When a sinusoidal voltage of 100V is applied across a load, it draws a current of 10 A with 30° phase lag. Determine the conductance, susceptance and admittance of the load.

Let, $\bar{V}$ be the reference phasor.

$$\therefore\ \bar{V} = 100\angle 0^\circ\,V \quad \text{and} \quad \bar{I} = 10\angle -30^\circ\,A$$

$$\text{Impedance, } \bar{Y} = \frac{\bar{I}}{\bar{V}} = \frac{10\angle -30^\circ}{100\angle 0^\circ} = 0.1\angle -30^\circ = 0.0866 - j0.05\,℧$$

Since, $\bar{Y} = G + jB$; Conductance, $G = 0.0866\,℧$

Susceptance, $jB = j0.05\,℧$ (or $0.05\,℧$ - inductive)

Q4.13 An inductive load consumes 1000 W power and draws 10 A current when connected to a 250 V, 25 Hz supply. Determine the resistance and inductance of the load.

We know that, $P = I^2R$

$$\therefore\ \text{Resistance, } R = \frac{P}{I^2} = \frac{1000}{10^2} = 10\,\Omega$$

$$\therefore\ \text{Impedance, } Z = \frac{V}{I} = \frac{250}{10} = 25\,\Omega$$

We know that, $Z = \sqrt{R^2 + X_L^2}$

$$\therefore\ \text{Inductive reactance, } X_L = \sqrt{Z^2 - R^2} = \sqrt{25^2 - 10^2} = 22.9129\,\Omega$$

$$\boxed{X_L = \omega L = 2\pi f L}$$

$$\text{Inductance, } L = \frac{X_L}{2\pi f} = \frac{22.9129}{2\pi \times 25} = 0.1459\,H$$

Q4.14 In an RC series circuit excited by a sinusoidal source, the voltages across resistance and capacitance are 60 V and 80 V, respectively. What will be the supply voltage?

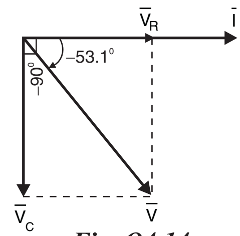


Fig. Q4.14.

Let current, $\bar{I}$ through the RC series circuit be the reference phasor. With reference to the phasor diagram shown in Fig. Q4.14, we can write,

$$\text{Supply voltage, } \bar{V} = \bar{V}_R + \bar{V}_C = 60\angle 0^\circ + 80\angle -90^\circ$$

$$= 60 - j80 = 100\angle{-53.1^\circ}\,V$$

Q4.15 In an RL series circuit with R = 20 Ω and X_L = 30 Ω, for what value of R will the impedance of RL series combination be doubled ?

$$\text{When } R = 20\,\Omega \text{ and } X_L = 30\,\Omega,\ Z = \sqrt{R^2 + X_L^2} = \sqrt{20^2 + 30^2} = 36.0555\,\Omega$$

Let, R_2 be the value of resistance when impedance is doubled.

$$\text{Now, } R_2 = \sqrt{(2Z)^2 - X_L^2} = \sqrt{(2 \times 36.0555)^2 - 30^2} = 65.5744\,\Omega$$

Q4.16 An RC series circuit with R = 1.2 kΩ and C = 0.1 μF is excited by a sinusoidal source of 45 V and frequency 1kHz. Find the apparent power.

$$\boxed{X_C = \frac{1}{2\pi fC}}$$

Magnitude of impedance,

$$Z = \sqrt{R^2 + X_C^2} = \sqrt{(1.2 \times 10^3)^2 + \left(\frac{1}{2\pi \times 10^3 \times 0.1 \times 10^{-6}}\right)^2} = 1993.246\,\Omega$$

$$\text{Magnitude of current, } I = \frac{V}{Z}$$

$$\text{Apparent power, } S = VI = V \times \frac{V}{Z} = \frac{V^2}{Z} = \frac{45^2}{1993.246} = 1.0159\,VA$$

Q4.17 Determine the impedance of the RLC parallel circuit shown in Fig. Q4.17.

$$R = \frac{100}{4} = 25\,\Omega \quad ; \quad X_L = \frac{100}{1} = 100\,\Omega \quad ; \quad X_C = \frac{100}{4} = 25\,\Omega$$

$$\bar{Z}\ \frac{1}{\frac{1}{R} + \frac{1}{jX_L} + \frac{1}{-jX_C}} = \frac{1}{\frac{1}{25} + \frac{1}{j100} + \frac{1}{-j25}} = 16 - j12\,\Omega$$

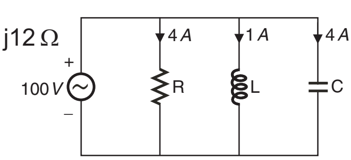


Fig. Q4.17.

$$Z = |\bar{Z}| = \sqrt{16^2 + 12^2} = 20\,\Omega$$

<u>Alternate method</u> :

Let total current be $\bar{I}$.

$$\bar{I} = 4 + j1 - j4 = 4 - j3\,A \quad ; \quad I = |\bar{I}| = \sqrt{4^2 + 3^2} = 5\,A$$

$$\text{Now, } Z = \frac{V}{I} = \frac{100}{5} = 20\,\Omega$$

Q4.18 Determine the power factor of an RLC series circuit with $R = 5\Omega$, $X_L = 8\Omega$ and $X_C = 12\Omega$.

Impedance, $\overline{Z} = R + jX_L - jX_C = 5 + j8 - j12 = 5 - j4\,\Omega$

Power factor angle, $\phi = \tan^{-1}\frac{-4}{5} = -38.7^\circ$

If, $Z = R + jX$ then, $\phi = \tan^{-1}\frac{X}{R}$

Power factor = $\cos\phi = \cos(-38.7^\circ) = 0.7804$ lead

Since impedance angle is negative, pf is lead.

Q4.19 What is balanced voltage?

In a polyphase system when the magnitude of phase voltages is equal and the phase difference between consecutive phasors is equal, the voltages are called balanced voltages.

Q4.20 What is balanced impedance?

When the impedances of all the phases of a three-phase load are equal, the impedances are called balanced impedances.

Q4.21 What is phase sequence? ***(AU Dec'16, 2 Marks)***

In a set of rotating phasors, the order of reaching the reference point is called phase sequence.

Q4.22 When is a three-phase system is called balanced supply system? ***(AU Dec'15, 2 Marks)***

A three-phase supply system is called a balanced supply system if the magnitude of voltages is the same and phase difference between any two phase is 120° with respect to each other.

Q4.23 Write the relation between the line and phase values of voltage and current in a balanced star-connected source/load.

In a star-connected system, the line current and phase current are the same. In a balanced star-connected system, the magnitude of line voltage is $\sqrt{3}$ times the phase voltage and it leads the phase voltage by 30°.

Q4.24 Write the relation between the line and phase values of voltage and current in a balanced delta-connected source/load.

In a delta-connected system, the line voltage and phase voltage are the same. In a balanced delta-connected system, the magnitude of line current is $\sqrt{3}$ times the phase current and it lags the phase current by 30°.

Q4.25 Distinguish between unbalanced source and unbalanced load. ***(AU May'15, June'14 & May'17, 2 Marks)***

In a three-phase source, when the phase difference between any two phases is not equal to 120^0, the sources are called unbalanced sources. Whereas in a three-phase load if the impedance angles of all the phases are not equal then the load is called an unbalanced load.

Q4.26 Write the equations for the phasor difference between the potentials of the delta connected networks. ***(AU Dec'14, 2 Marks)***

The phasor voltages in delta-connected network are,

$$\overline{V}_{RY} = V_L \angle 0^\circ\ V$$

$$\overline{V}_{YB} = V_L \angle -120^\circ\ V$$

$$\overline{V}_{BR} = V_L \angle -240^\circ\ V$$

Q4.27 ***Write the distortion power factor equation of three-phase circuits.***

(AU May'15, 2 Marks)

$$\text{Distortion power factor} = \frac{1}{\sqrt{1+\text{THD}^2}}$$

where, THD = Total Harmonic Distortion

When excited by a voltage source, the current gets distorded and so THD is calculated for current as shown below :

$$\text{THD} = \frac{\sqrt{I_2^2+I_3^2+I_4^2+.....+I_n^2}}{I_1}$$

Where, I_1 = RMS value of fundamental

I_n = RMS value of n^{th} harmonic.

True power factor = Power factor of fundamental × Distortion power factor

Q4.28 ***A star-connected load has 6 + j8 Ω impedance per phase. Determine the line current if it is connected to 400 V, 3φ, 50 Hz supply.***

(AU June'16, 2 Marks)

$$\text{Impedance, } Z = \sqrt{6^2 + 8^2} = 10\ \Omega/\textit{phase}$$

$$\text{Phase current, } I = \frac{V}{Z} = \frac{V_L/\sqrt{3}}{Z} = \frac{400}{\sqrt{3}\times 10} = 23.094\ A$$

$$\therefore \text{ Line current, } I_L = I = 23.094\ A$$

Q4.29 ***A star-connected load having a resistance of 20 Ω and an inductive reactance of 15 Ω is connected to a 400 V, 3-phase, and 50 Hz supply. What is the line current, power factor and power supplied?***

(AU Dec'14, 2 Marks)

$$\text{Phase current, } I = \frac{V_{phase}}{Z} = \frac{V_L/\sqrt{3}}{\sqrt{R^2+X_L^2}} = \frac{400/\sqrt{3}}{\sqrt{20^2+15^2}} = 9.2376\ A$$

In a star connection, the line current and phase current are the same.

$$\therefore \text{ Line current, } I_L = I = 9.2376\ A$$

$$\text{Power factor, } \cos\phi = \frac{R}{Z} = \frac{20}{\sqrt{20^2+15^2}} = 0.8 \text{ lag}$$

$$\text{Power, } P = \sqrt{3}V_L I_L \cos\phi = \sqrt{3}\times 400\times 9.2376\times 0.8$$

$$= 5120\ W = 5.12\ kW$$

Q4.30 ***A delta-connected load has 30 – j40 Ω impedance per phase. Determine the line current if it is connected to 415 V, 3 φ, 50 Hz supply.***

$$\text{Impedance, } Z = \sqrt{30^2 + 40^2} = 50\ \Omega/\textit{phase}$$

$$\text{Phase current, } I = \frac{V}{Z} = \frac{V_L}{Z} = \frac{400}{50} = 8\ A$$

$$\therefore \text{ Line current, } I_L = \sqrt{3}\,I = \sqrt{3}\times 8 = 13.8564\ A$$

Q4.31 ***A star-connected balanced load draws a current of 35 A per phase when connected to a 440 V supply. Determine the apparent power.***

$$\text{Apparent power, } S = \sqrt{3}V_L I_L = \sqrt{3}\times 440\times 35$$

$$= 26673.6\ VA = 26.6736\ kVA$$

Q4.32 ***A balanced delta-connected load of 4 – j6 Ω impedance is connected to 400 V 3-phase supply. What is the power and power factor of the load?***

$$Z = \sqrt{4^2 + 6^2} = 7.2111\,\Omega/phase$$

$$I = \frac{V}{Z} = \frac{400}{7.2111} = 55.47\,A$$

$$\text{Power factor} = \cos\left(\tan^{-1}\frac{-6}{4}\right) = 0.5547\ lead$$

$$\text{Power, } P = \sqrt{3}V_L I_L \cos\phi = 3VI\cos\phi = 3\times 400\times 55.47\times 0.5547$$

$$= 36923.1W = 36.9231\,kW$$

Q4.33 ***In a three-wire system, the two line currents of an unbalanced load are $\bar{I}_R = 10\angle -67^o$ A and $\bar{I}_B = 5\angle 136^o$ A. Determine the line current $\bar{I}_Y$.***

We know that, $\bar{I}_R + \bar{I}_Y + \bar{I}_B = 0$

$$\therefore\ \bar{I}_Y = -(\bar{I}_R + \bar{I}_B) = -(10\angle -67^\circ + 5\angle 136^\circ)$$

$$= -0.3106 + j5.7318\,A = 5.7402\angle 93.1^\circ\,A$$

Q4.34 ***What is neutral shift voltage?***

In a three-wire star-connected load, the load neutral is not connected to the source neutral. Therefore, when the load is unbalanced, the load neutral will not be at zero potential. The voltage of the load neutral with respect to source the neutral is called neutral shift voltage or neutral displacement voltage.

Q4.35 ***When a three-phase star-connected unbalanced load is connected to a 400 V supply, the R-phase voltage is 200 ∠16° V. Determine the neutral shift voltage.***

Let $\bar{V}_{RY}$ be the reference phasor.

The R-phase source voltage, $\bar{E}_R = \dfrac{400}{\sqrt{3}}\angle -30^\circ\,V$

With reference to Fig. Q4.35,

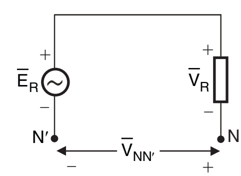


Fig. Q4.35.

Neutral shift voltage, $\bar{V}_{NN'} = \bar{E}_R - \bar{V}_R$

$$= \frac{400}{\sqrt{3}}\angle -30^\circ - 200\angle 16^\circ$$

$$= 230.9401\angle -30^\circ - 200\angle 16^\circ$$

$$= 7.7477 - j170.5975 = 170.7733\angle -87.4^\circ\,V$$

Q4.36 ***In a four-wire star-connected system, $\bar{I}_R = 5\angle 10^o$ A, $\bar{I}_Y = 7\angle 85^o$ A, $\bar{I}_B = 3\angle 200^o$ A. What is the neutral current?***

Neutral current, $\bar{I}_N = \bar{I}_R + \bar{I}_Y + \bar{I}_B$

$$= 5\angle 10^\circ + 7\angle 85^\circ + 3\angle 200^\circ$$

$$= 2.7151 + j6.8155\,A$$

$$= 7.3364\angle 68.3^\circ\,A$$

Q4.37 ***Write the expression for the power measured by two wattmeters used in 3-phase balanced load, in terms of voltage, current and power factor.***

$P_1 = V_L I_L \cos(\phi + 30^o)$

$P_2 = V_L I_L \cos(\phi - 30^o)$

where, ϕ is the power factor angle.

Q4.38 ***Write the expression for power factor in two-wattmeter method of power measurement.***

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right]$$

Q4.39 ***Write the relation between the power factor and wattmeter readings in two wattmeter method of power measurement.***

1. When the wattmeter readings are equal, the power factor is unity.
2. When one of the wattmeter reading is zero, the power factor is 0.5.
3. When one of the wattmeter reading is negative, the power factor will be less than 0.5.
4. When both the wattmeter readings are positive, the power factor will be greater than 0.5.

Q4.40 ***If P_1 and P_2 are two wattmeter readings and ϕ is the power factor angle of a three-phase load, write the relation between wattmeter readings and power factor angle.***

1. When $P_1 = P_2$, then $\phi = 0^o$.
2. When $P_1 = 0$ and $P_2 \neq 0$ (or $P_1 \neq 0$ and $P_2 = 0$), then $\phi = 60^o$.
3. When P_1 is negative and P_2 is positive (or P_2 is negative and P_1 is positive), then $\phi > 60^o$.
4. When P_1 and P_2 are positive, then $\phi < 60^o$.

Q4.41 ***The readings of two wattmeter used for 3-phase power measurement are 5.2 kW and –1.6 kW. Determine the power and power factor of the load.***

Power, $P = P_1 + P_2 = 5.2 + (-1.6) = 3.6\,kW$

$$\text{Power factor, } \cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\frac{P_2 - P_1}{P_1 + P_2}\right)\right] = \cos\left[\tan^{-1}\left(\sqrt{3} \times \frac{-1.6 - 5.2}{5.2 - 1.6}\right)\right] = 0.2923$$

Q4.42 ***In a star-connected 3-phase balanced load the total power measured by two wattmeter is 2400 W. What will be the power measured by two wattmeters if the load impedance is reconnected in delta.***

For same load impedance, the power consumed by delta-connected load will be three times the power consumed by star-connected load. Therefore the total power measured by wattmeters for delta-connected load will be 7200 *W*.

Q4.43 ***The line current of a delta-connected balanced load is 18A when connected to a 400 V supply. What will be the line current when the impedances are reconnected in star?***

When the balanced delta-connected impedances are reconnected in star, for the same supply voltage, the current drawn by the star-connected load will be one-third the current drawn by the delta-connected load.

$$\therefore \text{ Line current in a star connection} = \frac{18}{3} = 6\,A$$

4.29 Exercises

I. Fill in the Blanks with Appropriate Words

1. The sources in which the current/voltage do not change with time are called ________.
2. In sinusoidal voltage/current, the number of cycles per second is called ________ .
3. The ratio of rms value and average value of a periodic waveform is called ________.
4. The ratio of peak value and rms value of a periodic waveform is called ________.
5. In a current-carrying conductor, the induced emf is equal to the rate of change of ________.
6. In a capacitor, energy is stored as ________.
7. In an inductive circuit, current ________ and in a capacitive circuit, current ________ the supply voltage.
8. The ________ is defined as the ratio of active power and apparent power.
9. In an RLC circuit, when total reactance is negative, the current ________ the voltage.
10. In an RLC parallel circuit, the inductance and capacitance ________ are always in phase opposition.
11. In a balanced star-connected load, the line voltage is ________ times the phase voltage.
12. The line voltage ________ phase voltage by ________ in a balanced star-connected system.
13. The ________ current is zero in a balanced four-wire system.
14. In a balanced delta-connected load, the phase current is ________ times the line current.
15. The line current ________ phase current by ________ in a balanced delta system.
16. In a balanced three-phase load, the power consumed in a delta connection is ________ times the power consumed in a star connection.
17. The voltage of load neutral with respect to source neutral is called ________ voltage.
18. In a balanced three-phase load, for the same line voltage, the line current in a star connection is ________ times the line current in a delta connection.
19. The ________ current is equal to the sum of three line currents in a four-wire system.
20. In a three-wire unbalanced star-connected load, the sum of a ________ voltage of load and ________ voltage is equal to the corresponding phase voltage of the source.
21. In a balanced three-phase load, the total power consumed is equal to ________ times the power in one phase.
22. In three-phase load ________ can be measured by using only ________ wattmeters.
23. In two wattmeter method of power measurement, when pf is ________ , one wattmeter reading will be zero.
24. In two wattmeter method of power measurement, the readings of two wattmeters are ________ when the pf is unity.
25. In two wattmeter method of power measurement, one wattmeter reading is negative when pf angle is ________ than ________ .

ANSWERS

1.	dc sources	8.	power factor	15.	lags, 30°	22.	power, two
2.	frequency	9.	leads	16.	three	23.	0.5
3.	form factor	10.	current	17.	neutral shift	24.	equal
4.	peak factor	11.	$\sqrt{3}$	18.	1/3	25.	greater, 60°
5.	flux linkages	12.	leads, 30°	19.	neutral		
6.	electric field	13.	neutral	20.	phase, neutral shift		
7.	lags, leads	14.	$1/\sqrt{3}$	21.	three		

II. State Whether the Following Statements are True or False

1. The average value of sinusoidal voltage over one period is zero.
2. Flux and current are inseparable in nature.
3. Charge and voltage are inseparable in nature.
4. In an inductance, energy is stored as electric field.
5. In a capacitance, energy is stored as magnetic field.
6. In a resistance, voltage and current are always in phase.
7. In an inductance, voltage and current are always in phase quadrature.
8. The reactive power in a capacitive circuit is positive.
9. In an RLC circuit, when the total susceptance is positive, the current lags the voltage.
10. In an RLC series circuit, the inductance and capacitance voltage are always in phase opposition.
11. Rms phasors are rotating phasors.
12. In a balanced load, the magnitude and argument of the impedances of all the phases are equal.
13. In a three-phase system, the line voltages of the source and load are the same.
14. In a star-connected load, the line current and phase current are not the same.
15. In a delta-connected load, the line voltage and phase voltage are the same.
16. In a three-wire system, the sum of three line currents is zero.
17. In a four-wire system, the sum of three line currents and neutral current is zero.
18. In an unbalanced star-connected load, the sum of the three-phase voltages is zero.
19. In an unbalanced delta-connected load, the sum of the three-phase voltages is zero.
20. In a four-wire system, the voltages are always balanced irrespective of balanced or unbalanced load.
21. Two wattmeters are sufficient for measuring power in balanced and unbalanced loads.
22. In a balanced load, the power factor is the same in all the phases.
23. In an unbalanced load, the power factor is the same in all the phases.
24. The power factor of unbalanced load can be estimated from two wattmeter readings.
25. In two wattmeter method of power measurement we cannot determine whether the power factor is lagging or leading.

ANSWERS

1. True	6. True	11. False	16. True	21. True
2. True	7. True	12. True	17. True	22. True
3. True	8. False	13. True	18. False	23. False
4. False	9. False	14. False	19. True	24. False
5. False	10. True	15. True	20. True	25. True

III. Choose the Right Answer for the Following Questions

1. ***The frequency of the sinusoidal voltage, $v = 163 \sin 377t\,V$ is,***

a) $40\,Hz$ b) $50\,Hz$ c) $60\,Hz$ d) $70\,Hz$

2. ***The average and rms value of the sinusoidal current, $i = 10 \sin 314t\,A$ respectively are,***

a) $3.183\,A$, $6.366\,A$ b) $6.366\,A$, $7.071\,A$

c) $15.708\,A$, $14.142\,A$ d) $7.854\,A$, $9.003\,A$

3. ***The form factor of the sinusoidal voltage, $v = 230 \sin 314t\,V$ is,***

a) 1.111 b) 2.222 c) 0.901 d) 0.451

4. ***The peak factor of the sinusoidal voltage, $v = 400 \sin 377t\,V$ is,***

a) 1.732 b) 0.866 c) 0.707 d) 1.414

5. ***The inductance of a coil with a flux linkage of 0.202 Wb-turn and carrying a current of 0.2 A is,***

a) $0.404\,H$ b) $0.101\,H$ c) $0.002\,H$ d) $0.402\,H$

6. ***The energy stored in a coil carrying a current of 20 A and having an inductance of 5 mH is,***

a) 2 *Joule* b) 1 *Joule* c) 0.5 *Joule* d) 0.1 *Joule*

7. ***What is the charge in a 0.01 F capacitor when a voltage of 100 V exists in it?***

a) 10 *Coulomb* b) 1 *Coulomb* c) 0.1 *Coulomb* d) 0.01 *Coulomb*

8. ***What is the energy in a 0.01 F capacitor when a voltage of 10 V exists in it?***

a) 2 *Joule* b) 1 *Joule* c) 0.5 *Joule* d) 0.1 *Joule*

9. ***The phase difference between the voltages $v_1 = 230 \sin(377t + 30^o)\,V$ and $v_2 = 230 \sin(377t - 30^o)\,V$ is,***

a) 60^o with v_1 leading v_2 b) 30^o with v_2 leading v_1

c) 60^o with v_1 lagging v_2 d) 30^o with v_2 lagging v_1

10. ***When a load is connected to $230\angle 10^o\,V$, it draws a current of $10\angle -50^o\,A$. What is the real and reactive power of the load?***

a) $1992\,W$, $1150\,VAR$ b) $1150\,W$, $1150\,VAR$

c) 1626 *W*, 2086 *VAR* d) 1150 *W*, 1992 *VAR*

11. The power factor of a load with active power 120 W and reactive power 100 VAR is,

a) 0.64 lag b) 0.64 lead c) 0.768 lag d) 0.768 lead

12. What is the value of the impedance drawing a current of $8\angle -37^o$ A, when connected to $220\angle 10^o$ V supply?

a) $27.5\angle 47^o\,\Omega$ b) $27.5\angle -47^o\,\Omega$ c) $27.5\angle 27^o\,\Omega$ d) $27.5\angle -27^o\,\Omega$

13. The resistance and reactance of the impedance $4\angle 60^o\,\Omega$ respectively are,

a) $2\,\Omega$, $3.46\,\Omega$-capacitive b) $2\,\Omega$, $3.46\,\Omega$-inductive

c) $3.46\,\Omega$, $2\,\Omega$-capacitive d) $3.46\,\Omega$, $2\,\Omega$-inductive

14. What is the value of the admittance drawing a current of $20\angle -26^o$ A when connected to $40\angle 10^o$ V supply?

a) $0.5\angle -36^o\,\mho$ b) $0.5\angle 36^o\,\mho$ c) $2\angle -16^o\,\mho$ d) $2\angle 16^o\,\mho$

15. The conductance and susceptance of the admittance $8\angle 30^o\,\mho$ respectively are,

a) $4\,\mho$, $6.93\,\mho$ - inductive b) $4\,\mho$, $6.93\,\mho$ - capacitive

c) $6.93\,\mho$, $4\,\mho$ - inductive d) $6.93\,\mho$, $4\,\mho$ - capacitive

16. In a star-connected three-phase system, the relation between the line and phase values is,

a) $V_L = V,\ I_L = \sqrt{3}\,I$ b) $V_L = \sqrt{3}\,V,\ I_L = I$

c) $V_L = \frac{V}{\sqrt{3}},\ I_L = I$ d) $V_L = V,\ I_L = \frac{I}{\sqrt{3}}$

17. In a delta-connected three-phase system, the relation between the line and phase values is,

a) $V_L = V,\ I_L = \sqrt{3}\,I$ b) $V_L = \sqrt{3}\,V,\ I_L = I$

c) $V_L = \frac{V}{\sqrt{3}},\ I_L = I$ d) $V_L = V,\ I_L = \frac{I}{\sqrt{3}}$

18. Three identical impedances $3 + j6\,\Omega$ are connected in delta. What will be the equivalent star-connected impedance that draws the same current when connected to the same supply voltage?

a) $3 + j6\,\Omega$ b) $6 + j12\,\Omega$ c) $1 + j2\,\Omega$ d) $1.5 + j3\,\Omega$

19. Three identical impedances $2 + j5\,\Omega$ are connected in star. What will be the equivalent delta-connected impedance that draw same current when connected to the same supply voltage?

a) $2 + j5\,\Omega$ b) $1 + j2.5\,\Omega$ c) $\frac{2 + j5}{3}\,\Omega$ d) $6 + j15\,\Omega$

20. *A star-connected balanced load with impedance $4+j5\Omega$/phase is connected to a three-phase 200 V supply. What is the apparent power?*

a) 975.6 *VA* b) 1281 *VA* c) 6247 *VA* d) 8200 *VA*

21. *A balanced delta-connected load with impedance $5+j2\Omega$/phase is connected to a three-phase 100 V supply. What is the power?*

a) 2586.3 *W* b) 2986.3 *W* c) 5172.5 *W* d) 4479.6 *W*

22. *In a three-phase unbalanced load, the line currents $\bar{I}_R = 5\angle 30^o$ A and $\bar{I}_Y = 7\angle 180^o$ A. What is the value of the line current $\bar{I}_B$?*

a) $11.6\angle{-167.6^o}$ *A* b) $3.66\angle{-43.1^o}$ *A* c) $3.66\angle 136.9^o$ *A* d) $11.6\angle 12.4^o$ *A*

23. *The line currents of a four-wire star-connected load are $\bar{I}_R = j20$ A, $\bar{I}_Y = 4$ A and $\bar{I}_B = -2-j3$ A. What is the neutral current?*

a) $2-j23$ *A* b) $-6+j17$ *A* c) $2+j17$ *A* d) $-2-j17$ *A*

24. *A three-phase balanced inductive load draws a current of 10 A and consumes 6 kW when connected to a 400 V supply. What is the power factor of the load?*

a) 0.866 *lag* b) 0.5 *lag* c) 0.866 *lead* d) 0.5 *lead*

25. *A symmetrical delta-connected load draws a power of 8 kW when connected to a 380 V supply. If the power factor of the load is 0.866 lead then the line current is,*

a) 10.5 *A* b) 24.3 *A* c) 8.1 *A* d) 14 *A*

26. *In two-wattmeter method of power measurement the expression for power factor angle is,*

a) $\tan\sqrt{3}\ \dfrac{P_2-P_1}{P_1+P_2}$ b) $\tan^{-1}\sqrt{3}\ \dfrac{P_2-P_1}{P_1+P_2}$

c) $\tan\sqrt{3}\ \dfrac{P_1+P_2}{P_1-P_2}$ d) $\tan^{-1}\sqrt{3}\ \dfrac{P_1+P_2}{P_1-P_2}$

27. *In two-wattmeter method of power measurement what is the value of wattmeter readings when the load power is 8 kW with unity power factor?*

a) 8 *kW*, 0 *kW* b) 4 *kW*, 4 *kW* c) 12 *kW*, –4 *kW* d) 16*kW*,–8*kW*

28. *In two-wattmeter method of power measurement what is the value of wattmeter readings when the load power is 7 kW with 0.5 power factor?*

a) 7 *kW*, 0 *kW* b) 3.5 *kW*, 3.5 *kW* c) 9 *kW*, –2 *kW* d) 14*kW*,–7*kW*

29. ***In two-wattmeter method, the readings of wattmeter for a balanced load are, $P_1 = 500\,W$, $P_2 = 400\,W$. What is the power factor of the load?***

a) 0.347 b) 0.949 c) 0.999 d) 0.982

30. ***In two-wattmeter method, the readings of wattmeter for a balanced load are, $P_1 = 800\,W$, $P_2 = 600\,W$. What is the reactive power of the load?***

a) 1200 *VAR* b) 600 *VAR* c) 346 *VAR* d) 692 *VAR*

ANSWERS

1. c	6. b	11. c	16. b	21. b	26. b
2. b	7. b	12. a	17. a	22. b	27. b
3. a	8. c	13. b	18. c	23. c	28. a
4. d	9. a	14. a	19. d	24. a	29. d
5. b	10. d	15. d	20. c	25. d	30. c

IV. Unsolved Problems

E4.1 In the circuit shown in Fig. E4.1, determine the currents in all the branches.

E4.2 Determine the current $\bar{I}_2$ in the circuit shown in Fig. E4.2.

E4.3 Determine the voltage $\bar{V}_2$ in the circuit shown in Fig. E4.3.

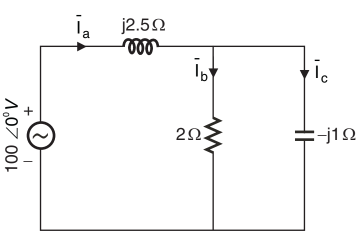


Fig. E4.1.

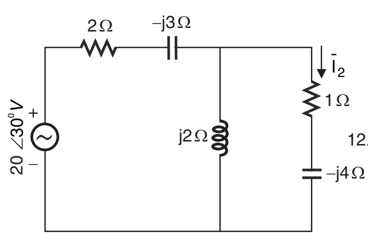


Fig. E4.2.

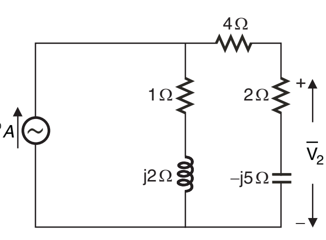


Fig. E4.3.

E4.4 ***A series combination of 5 Ω and 10 mH inductance is connected to a 115 V, 60 Hz supply. Estimate the voltage and current in the elements. Also calculate the active, reactive and apparent power.***

E4.5 ***A parallel RL circuit connected to a 230 V, 50 Hz supply has active power of 2.5 kW and reactive power of 3.12 kVAR. Calculate the current through the elements, total current supplied by the source and the value of R and L.***

E4.6 ***A current of $10\angle 30^o$ A flowing through a circuit consists of series connected elements when excited by a source of $200\angle{-30^o}$ V, 50 Hz. Determine the elements of the circuit, voltage across the elements and active, reactive and apparent power.***

E4.7 ***A parallel combination of 10 Ω resistance and 400 μF capacitance is connected to a 160 V, 50 Hz supply. Estimate the current through the elements and the total current drawn from the supply. Also calculate the apparent, active and reactive power.***

E4.8 *An RLC series circuit consists of $R = 40\,\Omega$, $L = 70\,mH$ and $C = 450\,\mu F$. The circuit is excited by a sinusoidal source of value 100 V, 50 Hz. Determine the current and voltage in the elements. Also estimate the apparent, active and reactive power.*

E4.9 *A load absorbs 4 kW at a pf of 0.65 lagging from a 160 V, 50 Hz source. A capacitor is connected in parallel to the load to improve the pf to 0.8 lag. Determine the value of the capacitor.*

E4.10 *An unbalanced four-wire star-connected load is connected to a balanced supply of 380 V. Estimate the line currents, neutral current and the power consumed by the load. Also draw the phasor diagram. Take $\overline{Z}_R = 2 + j4\,\Omega$; $\overline{Z}_Y = 3 + j2\,\Omega$; $\overline{Z}_B = 6 + j10\,\Omega$*

E4.11 *A balanced star-connected load of $4 + j8\,\Omega$ per phase is connected to a three-phase, 400 V, 60 Hz supply. Find the line currents, active and reactive power of the load. Draw the phasor diagram*

E4.12 *The power consumed by a three-phase balanced star-connected load is 2.25 kW at a power factor of 0.72 leading. The supply voltage is 360 V, 50 Hz. Describe the impedance per phase.*

E4.13 *An unbalanced three-wire star-connected load is connected to a symmetrical supply of 415 V. If the load impedances are $\overline{Z}_R = 7\angle 30^o\,\Omega$, $\overline{Z}_Y = 9\angle 60^o\,\Omega$ and $\overline{Z}_B = 8\angle 45^o\,\Omega$, calculate the line currents, phase voltages and displacement neutral voltage.*

E4.14 *Determine the values of the three impedances $\overline{Z}_R, \overline{Z}_Y$ and $\overline{Z}_B$ connected in star to a 415 V, three-phase supply, if the neutral shift voltage is $180\angle 75^o V$ and the line currents $\overline{I}_Y = 12\angle -60^o\,A$ and $\overline{I}_B = 16\angle 90^o\,A$.*

E4.15 *A delta-connected balanced three-phase load is supplied from a three-phase, 415 V supply. The line current is 19 A and the power consumed by the load is 12 kW. Find a) impedance per phase, b) current per phase, c) power factor and d) the power consumed if the same load is connected in star.*

E4.16 *A balanced delta load of $14 - j10\,\Omega$ per phase is connected to a 415 V, three-phase supply. Determine the line and phase currents for the RBY sequence, by taking $\overline{V}_{BY}$ as the reference phasor. Sketch the phasor diagram. Also calculate the power factor, active and reactive power of the load.*

E4.17 *An unbalanced delta load is connected to a three-phase 415 V supply. Determine the phase and line currents for the RYB sequence by taking as the reference phasor. The impedances of the three arms are $70 + j45\,\Omega$, $10 - j20\,\Omega$ and $30 + j10\,\Omega$.*

E4.18 *The readings of two wattmeters connected to measure a three-phase power is 8 kW each, when the power factor is unity. What will be the reading of wattmeters if the power factor falls to i) 0.7 lagging, ii) 0.5 lagging and iii) 0.4 lagging, the total three-phase power remaining unaltered.*

E4.19 *A 415 V, three-phase motor has an output of 5.595 kW and operates at a power factor of 0.82 with an efficiency of 85%. Calculate the readings of the two wattmeters connected to measure the input power.*

E4.20 *A three-phase load connected to a 415 V, three-phase, three-wire system draw a current of $12\angle -40^o A$ in R phase and $16\angle -200^o A$ in the B phase. Determine the readings of the two wattmeters connected to measure the power if current coil of one wattmeter is connected to R phase and that of other to B phase.*

E4.21 *The three impedances of star-connected load are $4 + j6\,\Omega$, $8 - j10\,\Omega$ and $12 + j4\,\Omega$. Calculate the readings of two wattmeters connected to measure the power consumed by the load if the current coils are connected to R and Y phase. Take supply voltage as 600 V, 50 Hz, RYB sequence.*

ANSWERS

E4.1 $\bar{I}_a = 57.2598\angle{-76.8^\circ}\,A$; $\bar{I}_b = 25.5711\angle{-140.2^\circ}\,A$; $\bar{I}_c = 51.2147\angle{-50.2^\circ}\,A$

E4.2 $\bar{I}_2 = 6.2469\angle 171.3^\circ\,A$

E4.3 $\overline{V}_2 = 18.9734\angle 18.4^\circ\,V$

E4.4 $\bar{I} = 18.3648\angle{-37^\circ}\,A$; $\overline{V}_R = 91.8242\angle{-37^\circ}\,V$; $\overline{V}_L = 69.2336\angle 53^\circ\,V$

S = 2.112 *kVA* ; Q = 1.271 *kVAR* ; P = 1.6867 *kW*

E4.5 $\bar{I} = 17.3828\angle{-51.3^\circ}\,A$; $\bar{I}_R = 10.8696\angle 0^\circ\,A$; $\bar{I}_L = 13.5652\angle{-90^\circ}$ A

R = 21.1599 Ω ; L = 54 *mH*

E4.6 R = 10 Ω ; C = 183.78 *μF* ; $\overline{V}_R = 100\angle 30^\circ\,V$; $\overline{V}_C = 173.205\angle{-60^\circ}$V

S = 2 *kVA* ; P = 1 *kW* ; Q = −1.7321 *kVAR*

E4.7 $\bar{I} = 25.6955\angle 51.5^\circ\,A$; $\bar{I}_R = 16\angle 0^\circ\,A$; $\bar{I}_c = 20.1062\angle 90^\circ\,A$

S = 4.1113 *kVA* ; P = 2.5593 *kW* ; Q = –3.2175 *kVAR*

E4.8 $\bar{I} = 2.3424\angle{-20.5^\circ}\,A$; $\overline{V}_R = 93.696\angle{-20.5^\circ}\,V$; $\overline{V}_L = 51.512\angle 69.5^\circ\,V$; $\overline{V}_C = 16.5692\angle{-110.5^\circ}\,V$

S = 234.24 *VA* ; P = 219.4061 *W* ; Q = 82.0326 *VAR*

E4.9 C = 207.31 *μF*

E4.10 RYB sequence with $\overline{V}_{RY}$ as reference phasor

$\bar{I}_R = 49.0582\angle -93.4^\circ\,A$; $\bar{I}_Y = 60.8479\angle -183.7^\circ\,A$; $\bar{I}_B = 18.8128\angle -329^\circ\,A = 18.8128\angle 31^\circ\,A$

$\bar{I}_N = 59.2178\angle -143.3^\circ\,A$ P = 18.0513 *kW*

E4.11 RYB sequence with $\overline{V}_{RY}$ as reference phasor

$\bar{I}_R = 25.8198\angle -93.4^\circ\,A$; $\bar{I}_Y = 25.8198\angle -213.4^\circ\,A$; $\bar{I}_B = 25.8198\angle 26.6^\circ\,A$

P = 8.0097 *kW* ; Q = 15.9951 *kVAR*

E4.12 $\overline{Z} = 41.4722\angle -43.9^\circ\,\Omega/phase$

E4.13 RYB sequence with $\bar{V}_{RY}$ as reference phasor

$\bar{I}_R = 34.9248\angle -70.2^o\ A$; $\bar{I}_Y = 30.8186\angle 154.6^o\ A$; $\bar{I}_B = 25.3461\angle 50.9^o\ A$

$\bar{V}_{NN'} = 43.3044\angle 61.3^o\ V$; $\bar{V}_R = 244.4736\angle -40.2^o\ V$; $\bar{V}_Y = 277.3674\angle -145.4^o\ V$

$\bar{V}_B = 202.7688\angle 95.9^o\ V$

E4.14 $\bar{Z}_R = 40.7744\angle 75.6^o\ \Omega$; $\bar{Z}_Y = 32.3609\angle -70.9^o\ \Omega$; $\bar{Z}_B = 5.0356\angle 35.3^o\ \Omega$

E4.15 a) $\bar{Z} = 37.8315\angle 28.5^o\ \Omega/phase$; b) $I = 10.9697\ A$

c) $\cos\phi = 0.8787$; d) $P_{star} = 4\ kW$

E4.16 $\bar{I}_{BY} = 24.1213\angle 35.5^o\ A$; $\bar{I}_{YR} = 24.1213\angle -84.5^o\ A$; $\bar{I}_{RB} = 24.1213\angle -204.5^o\ A$

$\bar{I}_B = 41.7793\angle 5.5^o\ A$; $\bar{I}_Y = 41.7793\angle -114.5^o\ A$; $\bar{I}_R = 41.7793\angle -234.5^o\ A$

$\cos\phi = 0.8141\,lag$; $P = 24.4482\ kW$; $Q = -17.4391\ kVAR$

E4.17 $\bar{I}_{BR} = 13.1235\angle -18.4^o\ A$; $\bar{I}_{RY} = 4.987\angle -152.7^o\ A$; $\bar{I}_{YB} = 18.5593\angle -176.6^o\ A$

$\bar{I}_B = 31.1246\angle -5.6^o\ A$; $\bar{I}_R = 16.9856\angle 173.7^o\ A$; $\bar{I}_Y = 14.1418\angle 175.2^o\ A$

E4.18 i) $P_1 = 3.2879\ kW$; $P_2 = 12.7121\ kW$; ii) $P_1 = 0$; $P_2 = 16\ kW$

iii) $P_1 = -2.583\ kW$; $P_2 = 18.583\ kW$

E4.19 $P_1 = 1.9648\ kW$; $P_2 = 4.6176\ kW$

E4.20 $P_1 = 3.8149\ kW$; $P_2 = -5.0865\ kW$

E4.21 $P_1 = 15.1432\ kW$; $P_2 = 19.1062\ kW$

Chapter 5

RESONANCE AND COUPLED CIRCUITS

5.1 Resonance

In RLC circuits excited by sinusoidal sources, the inductive and capacitive reactances have opposite signs. Hence, when the reactances are varied, there is a possibility that the inductive reactance may cancel the capacitive reactance and the circuit may behave as a purely resistive circuit. This condition of an RLC circuit is called resonance. **Resonance** may be defined as a circuit condition at which the circuit behaves as a purely resistive circuit.

The inductive reactance, $X_L = \omega L = 2\pi fL$, and so the inductive reactance can be varied by varying either frequency (f) or inductance (L).

The capacitive reactance, $X_C = \dfrac{1}{\omega C} = \dfrac{1}{2\pi fC}$, and so the capacitive reactance can be varied by varying either frequency (f) or capacitance (C).

When the frequency of the sinusoidal source exciting the RLC circuit is varied, there is a possibility that "*the inductive reactance is equal and opposite to the capacitive reactance at a particular frequency*". Therefore, the total reactance is zero and the circuit will behave as a purely resistive circuit. Now, the circuit will be in resonance and the frequency at which resonance occurs is called **resonant frequency**.

5.2 Series Resonance

In a series RLC circuit, the resonance condition can be achieved by varying the frequency of exciting sinusoidal source. When the frequency is varied, at a particular frequency the inductive reactance will cancel the capacitive reactance and the circuit will behave as a resistive circuit. This condition of an RLC series circuit is called **series resonance**.

5.2.1 Resonance Frequency of Series RLC Circuit

(AU May'17, 8 Marks)

Consider the RLC series circuit shown in Fig. 5.1 excited by a sinusoidal source of variable frequency.

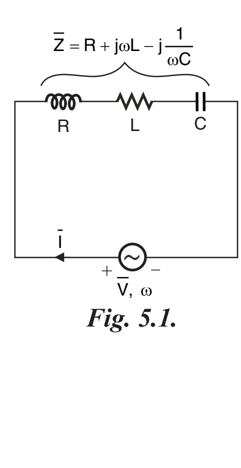


Fig. 5.1.

When the frequency of the source is varied by maintaining the voltage of the source as constant, the resonance occurs at a particular frequency. The expressions for resonance frequency in the RLC series circuit of Fig. 5.1 are given below:

$$\boxed{\text{Resonant angular frequency, } \omega_r = \frac{1}{\sqrt{LC}} \text{ in } rad/s} \quad .....(5.1)$$

$$\boxed{\text{Resonant frequency, } f_r = \frac{\omega_r}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \text{ in } Hz} \quad .....(5.2)$$

Proof for resonance frequency in series RLC

Consider the RLC series circuit shown in Fig 5.1.

Let, $\overline{Z}$ = *Impedance of RLC series circuit.*

Here, $\overline{Z} = R + j\omega L - j\dfrac{1}{\omega C} = R + j\left(\omega L - \dfrac{1}{\omega C}\right)$ .....(5.3)

At resonance frequency ω_r, *the total reactance is zero.*

Equating imaginary part of equation (5.3) to zero

$$\therefore \quad At\ \omega = \omega_r, \qquad \omega_r L - \frac{1}{\omega_r C} = 0 \qquad .....(5.4)$$

$$\therefore \quad \omega_r L = \frac{1}{\omega_r C} \quad \Rightarrow \quad \omega_r^2 = \frac{1}{LC}$$

$$\therefore \quad \omega_r = \frac{1}{\sqrt{LC}} \qquad .....(5.5)$$

$$\therefore \quad f_r = \frac{\omega_r}{2\pi} \quad \Rightarrow \quad f_r = \frac{1}{2\pi\sqrt{LC}}$$

$\omega = 2\pi f$

5.2.2 Frequency Response of Series RLC Circuit

Consider the RLC circuit shown in Fig. 5.1.

Let, $\overline{V} = V\angle 0^\circ\, V$ = Supply voltage.

$\overline{I}$ = Current through the RLC series circuit.

$\overline{I}_r$ = Current at resonance.

$\overline{Z}_r$ = Impedance at resonance.

The impedance at resonance is obtained by substituting $\omega = \omega_r$ in equation (5.3).

$$\therefore \ \overline{Z}_r = R + j\omega_r L - j\frac{1}{\omega_r C} = R$$

Using equation (5.4)

$$\therefore \ \text{Impedance at resonance, } \overline{Z}_r = R \qquad ..... (5.6)$$

$$\text{Current at resonance, } \overline{I}_r = \frac{\overline{V}}{\overline{Z}_r} = \frac{V\angle 0^\circ}{R} = \frac{V}{R}\angle 0^\circ = I_r\angle 0^\circ\ A \qquad .....(5.7)$$

$$\text{where, } I_r = \frac{V}{R} = \text{Magnitude of current at resonance.} \qquad .....(5.8)$$

Let us examine the variation of impedance $\overline{Z}$ of the RLC series circuit with frequency. At frequencies lower than resonant frequency, the capacitive reactance will be more than the inductive reactance and so the total reactance will be capacitive. Since the capacitive reactance is inversely proportional to frequency, the capacitive reactance and hence, the total reactance will increase when the frequency is decreased from the resonant frequency. Therefore, the impedance of the series RLC circuit will increase when the frequency is decreased from the resonant value.

At frequencies higher than resonant frequency, the inductive reactance will be more than the capacitive reactance and so the total reactance will be inductive. Since the inductive reactance is directly proportional to frequency, the inductive reactance and hence, the total reactance will increase when the frequency is increased from the resonant frequency. Therefore, the impedance of the series RLC circuit will increase when the frequency is increased from the resonant value.

At resonant frequency, the impedance of the RLC series circuit is equal to the resistance and this value of the impedance is minimum. Since the impedance is minimum, the current is maximum at resonance. Also the current at resonance will be in-phase with the supply voltage $\overline{V}$.

Since the impedance increases for frequencies lesser or higher than resonant value, the current decreases when frequency is increased or decreased from the resonant value.

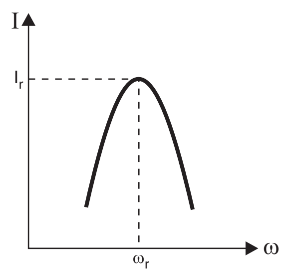


***Fig. a :** Current vs frequency.*

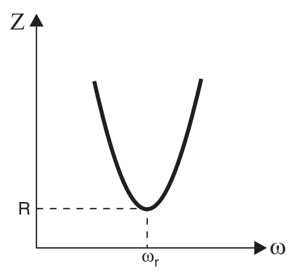


***Fig. b :** Impedance vs frequency.*

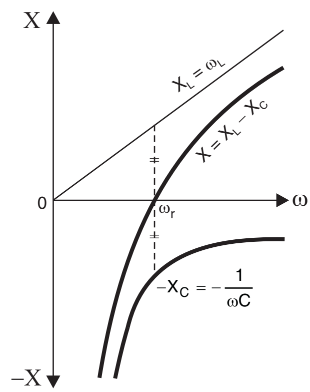


***Fig. c :** Reactance vs frequency.*

***Fig. 5.2 :** Characteristics of series resonance.*

5.2.3 Q-Factor (Quality Factor) of RLC Series Circuit

When a circuit consisting of a resistor, inductor and capacitor is excited by a sinusoidal source, the resistor dissipates energy in the form of heat, the inductor stores energy in the magnetic field associated with it and the capacitor stores energy in the electric field associated with it. In the steady state (after the transient period), there is a possibility that the sum of the energy stored in the inductor and capacitor is greater than the energy dissipated in the resistor.

In a series RLC circuit, due to larger stored energy in the inductor and capacitor, the voltage across these devices will be greater than the supply voltage. In other words we can say that there is a voltage magnification or amplification. The **voltage magnification** can be expressed by a factor called **Quality factor (Q)**, which is defined as the ratio of maximum energy stored to the energy dissipated in one period.

$$\therefore \text{ Quality factor, } Q = 2\pi \times \frac{\text{Maximum energy stored}}{\text{Energy dissipated in one period}} \qquad .....(5.9)$$

The term 2π is introduced to simplify the expression for quality factor.

Quality factor at resonance, $Q_r = 2\pi \times \dfrac{\text{Maximum energy stored at resonance}}{\text{Energy dissipated in one period at resonance}}$ .....(5.10)

Here, $\boxed{Q_r = \dfrac{\omega_r L}{R}}$ .....(5.11)

At resonance, $\omega_r = \dfrac{1}{\sqrt{LC}}$; $\omega_r^2 = \dfrac{1}{LC}$ and $L = \dfrac{1}{\omega_r^2 C}$

Therefore, the quality factor at resonance, Q_r can also be expressed as shown below:

$$Q_r = \frac{\omega_r L}{R} = \frac{\omega_r}{R} \times \frac{1}{\omega_r^2 C} = \frac{1}{\omega_r CR} \qquad \boxed{\therefore\ Q_r = \frac{1}{\omega_r CR}} \qquad .....(5.12)$$

$$Q_r = \frac{\omega_r L}{R} = \frac{1}{\sqrt{LC}}\frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}} \qquad \boxed{\therefore\ Q_r = \frac{1}{R}\sqrt{\frac{L}{C}}} \qquad .....(5.13)$$

For frequencies less than resonant frequency, the RLC series circuit behaves as a capacitive circuit.

$\boxed{\therefore \text{ When } \omega \le \omega_r ,\ Q = \dfrac{1}{\omega CR}}$ .....(5.14)

For frequencies higher than resonant frequency, the RLC series circuit behaves as an inductive circuit.

$\boxed{\therefore \text{ When } \omega \le \omega_r ,\ Q = \dfrac{\omega L}{R}}$ .....(5.15)

> ***Note :*** *In an RLC series circuit, when the inductor stores energy the capacitor discharges and vice-versa. At resonance, the sum of energy stored in the inductor and capacitor is maximum. For $\omega < \omega_r$, the energy stored in the capacitor is maximum and for $\omega > \omega_r$, the energy stored in the inductor is maximum.*
>
> *Equation (5.15) can be used to calculate the Q-factor of a coil or an RL series circuit and the equation (5.14) can be used to calculate the Q-factor of an RC series circuit.*

Proof for quality factor at resonance, Q_r in RLC series circuit

Consider an RLC series circuit shown in Fig 5.3, excited by a sinusoidal voltage source of frequency, ω.

Let, $\bar{I}$ be the reference phasor.

$\therefore\ \bar{I} = I\angle 0^o A$

Let, i = Instantaneous value of current.

$\therefore\ i = I_m \sin wt$ .....(5.16)

Let, w_L = *Instantaneous value of energy stored in inductor*

w_C = *Instantaneous value of enrgy stored in capacitor*

w = *Total instantaneous energy stored in the RLC circuit*

w_r = *Total intantaneous energy stored in the RLC circuit at resonance.*

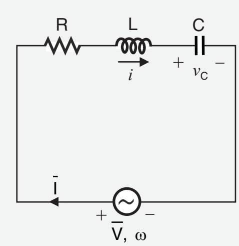


Fig. 5.3.

We know that,

$$w_L = \frac{1}{2} L\, i^2 \qquad \boxed{\text{Using equation (4.11) of Chapter - 4}}$$

$$= \frac{1}{2} L \left(I_m \sin \omega t\right)^2 \qquad \boxed{\text{Using equation (5.16)}}$$

$$= \frac{1}{2} L\, I_m^2 \sin^2 \omega t \qquad \text{.....(5.17)}$$

We know that,

$$w_C = \frac{1}{2} C \nu_c^2 \qquad \boxed{\text{Using equation (4.16) of Chapter - 4}}$$

$$= \frac{1}{2} C \left[\frac{1}{C} \int i\, dt\right]^2 \qquad \boxed{\nu_C = \frac{1}{C} \int i\, dt}$$

$$= \frac{1}{2} C \left[\frac{1}{C} \int I_m \sin \omega t\, dt\right]^2 \qquad \boxed{\text{Using equation (5.16)}}$$

$$\therefore \quad \omega_C = \frac{1}{2} C \left[\frac{1}{C}\left(-I_m \frac{\cos \omega t}{\omega}\right)\right]^2 = \frac{I_m^2}{2\omega^2 C} \cos^2 \omega t \qquad \text{.....(5.18)}$$

$$\text{Now, } w = w_L + w_C = \frac{1}{2} L\, I_m^2 \sin^2 \omega t + \frac{I_m^2}{2\omega^2 C} \cos^2 \omega t \qquad \boxed{\text{Using equations (5.17) and (5.18)}}$$

$$= \frac{I_m^2}{2} \left[L \sin^2 \omega t + \frac{1}{\omega^2 C} \cos^2 \omega t\right]$$

$$\therefore \quad w_r = w\Big|_{\omega = \omega_r} = \frac{I_m^2}{2} \left[L \sin^2 \omega t + \frac{1}{\omega^2 C} \cos^2 \omega t\right]\Bigg|_{\omega = \omega_r}$$

$$= \frac{I_m^2}{2} \left[L \sin^2 \omega_r t + \frac{1}{\omega_r^2 C} \cos^2 \omega_r t\right]$$

$$= \frac{I_m^2}{2} \left[L \sin^2 \omega_r t + \frac{LC}{C} \cos^2 \omega_r t\right] \qquad \boxed{\omega_r = \frac{1}{\sqrt{LC}} \Rightarrow \frac{1}{\omega_r^2} = LC}$$

$$= \frac{I_m^2 L}{2} \left[\sin^2 \omega_r t + \cos^2 \omega_r t\right] \qquad \boxed{\sin^2 + \cos^2 \theta = 1}$$

$$\therefore \quad \omega_r = \frac{I_m^2 L}{2} \qquad \text{.....(5.19)}$$

From equation (5.19) we can say that, the energy stored in the RLC circuit at resonance is independent of time and it is constant. Therefore, the instantaneous energy is the maximum energy stored at resonance.

Let, W_{mr} = *Maximum energy stored at resonance*

$$\therefore \quad W_{mr} = w_r = \frac{I_m^2 L}{2} \qquad \text{.....(5.20)}$$

In the RLC series circuit, the enrgy is dissipated by the resistor.

Let, W_R = *Energy dissipated in resistor in one period.*

W_{Rr} = *Energy dissipated in resistor in one period at resonance.*

$$\therefore \quad W_R = \text{Power} \times \text{Time period}$$

$$= I^2 R \times T \qquad \boxed{I = \text{RMS value of current}}$$

$$\therefore W_R = \left(\frac{I_m}{\sqrt{2}}\right)^2 R \times \frac{1}{f}$$

$$\boxed{I = \frac{I_m}{\sqrt{2}}} \quad \boxed{T = \frac{1}{f}}$$

$$= \frac{I_m^2}{2} R \times \frac{2\pi}{\omega}$$

$$\boxed{\omega = 2\pi f \Rightarrow \frac{1}{f} = \frac{2\pi}{\omega}}$$

$$= \frac{\pi I_m^2 R}{\omega} \qquad .....(5.21)$$

$$\therefore W_{Rr} = W_R\Big|_{\omega=\omega_r} = \frac{\pi I_m^2 R}{\omega_r} \qquad .....(5.22)$$

$$\therefore Q_r = 2\pi \times \frac{W_{mr}}{W_{Rr}} = 2\pi \times W_{mr} \times \frac{1}{W_{Rr}}$$

$$= 2\pi \times \frac{I_m^2 L}{2} \times \frac{\omega_r}{\pi I_m^2 R}$$

Using equations (5.20) and (5.22)

$$= \frac{\omega_r L}{R} \qquad .....(5.23)$$

5.2.4 Bandwidth of Series RLC Circuit *(AU June'16, 16 Marks)* *(AU May'17, 8 Marks)*

From the current response (Refer Fig. 5.2(a)) of an RLC series circuit we can say that the current is maximum at resonance and it decreases when frequency is decreased or increased from the resonant value. For practical applications, we have to define a range of frequencies over which the current response is appreciable and this range of frequency is called **bandwidth**.

Since the resistance is the load resistance in the practical circuits, the range of frequencies over which the response is appreciable can be decided based on the power in the resistance. At resonance, the current is maximum and so, power is maximum. For practical applications, the frequency range in which the power is greater than or equal to 50% of the maximum power is chosen as a useful range. It can be proved that when power is 50% of maximum value (or 1/2 times of maximum value), the current is $1/\sqrt{2}$ times of maximum value.

"*The current response is maximum at resonance*" and it decreases for increasing or decreasing frequency from the resonance value. Therefore, when frequency is decreased from the resonant value, we come across a frequency at which power is 1/2 times that of maximum value (or current is $1/\sqrt{2}$ times that of maximum value), and this frequency is called **lower cut-off frequency**, ω_l.

When frequency is increased from the resonant value, we come across a frequency at which power is 1/2 times that of maximum value (or current is $1/\sqrt{2}$ times that of maximum value), and this frequency is called **higher cut-off frequency**, ω_h.

The two cut-off frequencies are also called **half-power frequencies**, and they lie on either side of the resonant frequency as shown in Fig. 5.4. It can be proved that resonant frequency is given by the geometric mean of the two half-power frequencies, i.e., $\omega_r = \sqrt{\omega_l \, \omega_h}$

Now, "***bandwidth** can be defined as the range of frequencies over which power is greater than or equal to 1/2 times the maximum power*".

Alternatively, "***bandwidth** can be defined as the range of frequencies over which the current is greater than or equal to 1/$\sqrt{2}$ times the maximum current*".

Bandwidth is given by the difference between the cut-off frequencies and it can be denoted by β. The unit of bandwidth is *rad/s* or *Hz*.

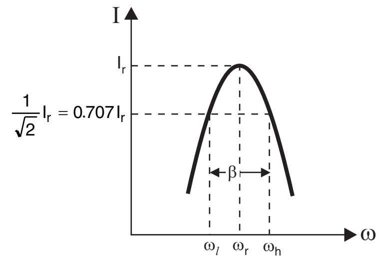


***Fig. 5.4 :** Current response of RLC series circuit*

The equations for cut-off frequencies and bandwidth are given below:

$$\boxed{\text{Higher cut-off angular frequency, } \omega_h = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \text{ in } rad/s} \qquad .....(5.24)$$

$$\boxed{\text{Lower cut-off angular frequency, } \omega_l = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \text{ in } rad/s} \qquad .....(5.25)$$

Alternatively,

$$\boxed{\omega_h = \omega_r\left[\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \text{ in } rad/s} \qquad .....(5.26)$$

$$\boxed{\omega_l = \omega_r\left[-\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \text{ in } rad/s} \qquad .....(5.27)$$

$$\boxed{\text{Higher cut-off frequency, } f_h = \frac{1}{2\pi}\left[\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right] \text{ in } Hz} \qquad .....(5.28)$$

$$\boxed{\text{Lower cut-off frequency, } f_l = \frac{1}{2\pi}\left[-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right] \text{ in } Hz} \qquad .....(5.29)$$

$$\boxed{\text{Bandwidth, } \beta = \frac{R}{L} \text{ in } rad/s} \qquad .....(5.30)$$

Alternatively,

$$\boxed{\text{Bandwidth, } \beta = \frac{\omega_r}{Q_r} \text{ in } rad/s} \qquad .....(5.31)$$

$$\boxed{\text{Bandwidth in } Hz = \frac{\beta}{2\pi} = \frac{R}{2\pi L} \text{ in } Hz} \qquad .....(5.32)$$

Proof for cut-off frequencies and bandwidth

Consider an RLC series circuit shown in Fig. 5.5, excited by a sinusoidal voltage source of frequency, ω.

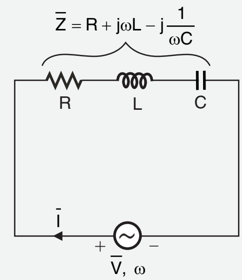


Fig. 5.5.

Let, $\bar{V} = V\angle 0^o$ = *Supply voltage.*

$\bar{I}$ = *Current through the RLC series circuit.*

$\bar{Z}$ = *Impedance of RLC series circuit.*

Here, $$\bar{Z} = R + j\omega L - j\frac{1}{\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$\therefore\ Z = \left|\bar{Z}\right| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \qquad .....(5.33)$$

Here, $$\bar{I} = \frac{\bar{V}}{\bar{Z}}$$

$$\therefore\ I = \left|\bar{I}\right| = \left|\frac{\bar{V}}{\bar{Z}}\right| = \frac{\left|\bar{V}\right|}{\left|\bar{Z}\right|} = \frac{V}{Z}$$

$$\therefore\ I = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \qquad \boxed{\text{Using equation (5.33)}} \qquad .....(5.34)$$

Let, P = *Power in RLC circuit.*

P_r = *Power in RLC circuit at resonance.*

Here, $$P = I^2 R = \frac{V^2 R}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \qquad \boxed{\text{Using equation (5.34)}} \qquad .....(5.35)$$

At resonance, $\omega L - \frac{1}{\omega C} = 0.$ *On substituting this condition in equation (5.35), we get power at resonance,* P_r.

$$\therefore\ P_r = \frac{V^2 R}{R^2} = \frac{V^2}{R} \qquad .....(5.36)$$

> **Note :** *Alternatively,* $P_r = I_r^2 R = \left(\frac{V}{R}\right)^2 R = \frac{V^2}{R}$

At half-power frequencies or cut-off frequencies, the power will be equal to half the power at resonance.

$$\therefore\ \frac{1}{2} \times P_r = P$$

$$\frac{1}{2} \times \frac{V^2}{R} = \frac{V^2 R}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \qquad \boxed{\text{Using equations (5.35) and (5.36)}}$$

On cross multiplying the above equation, we get,

$$R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 2R^2 \quad \Rightarrow \quad \left(\omega L - \frac{1}{\omega C}\right)^2 = 2R^2 - R^2$$

$$\therefore\ \left(\omega L - \frac{1}{\omega C}\right)^2 = R^2$$

On taking square root of above equation, we get,

$$\omega L - \frac{1}{\omega C} = \pm R \qquad .....(5.37)$$

> ***Note :*** *Equation (5.37) implies that the absolute value of total reactance at half-power frequencies is equal to the resistance of the circuit.*
>
> $$\therefore\ \omega_h L - \frac{1}{\omega_h C} = R\ ;\quad \omega_l L - \frac{1}{\omega_l C} = -R$$

On multiplying equation (5.37) by $\frac{\omega}{L}$*, we get,*

$$\omega^2 - \frac{1}{LC} = \pm\frac{R}{L}\omega \quad \Rightarrow \quad \omega^2 \mp \frac{R}{L}\omega - \frac{1}{LC} = 0$$

$$\therefore\ \omega^2 - \frac{R}{L}\omega - \frac{1}{LC} = 0 \quad \textit{and} \quad \omega^2 + \frac{R}{L}\omega - \frac{1}{LC} = 0$$

The roots of quadratic $\omega^2 - \frac{R}{L}\omega - \frac{1}{LC} = 0$ *are,*

$$\omega = \frac{\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}}{2}$$

$$= \frac{R}{2L} \pm \frac{1}{2}\sqrt{4\left[\frac{1}{4}\left(\frac{R}{L}\right)^2 + \frac{1}{LC}\right]}$$

$$= \frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

The roots of quadratic $\omega^2 + \frac{R}{L}\omega - \frac{1}{LC} = 0$ *are,*

$$\omega = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}}{2}$$

$$= -\frac{R}{2L} \pm \frac{1}{2}\sqrt{4\left[\frac{1}{4}\left(\frac{R}{L}\right)^2 + \frac{1}{LC}\right]}$$

$$= -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

The cut-off frequencies are given by the positive roots of the two quadratic.

$$\therefore\ \textit{Higher cut-off angular frequency, } \omega_h = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\ \textit{in rad/s} \qquad .....(5.38)$$

$$\textit{Lower cut-off angular frequency, } \omega_l = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\ \textit{in rad/s} \qquad .....(5.39)$$

Since, $\omega = 2\pi f$ *and* $f = \frac{\omega}{2\pi}$*, the cut-off frequency in Hz can be expressed as shown below:*

$$\therefore\ \textit{Higher cut-off frequency, } f_h = \frac{1}{2\pi}\left[\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right]\textit{in Hz} \qquad .....(5.40)$$

$$\textit{Lower cut-off frequency, } f_l = \frac{1}{2\pi}\left[-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right]\textit{in Hz} \qquad .....(5.41)$$

The bandwidth, β *is given by the difference between cut-off frequencies.*

$$\therefore\ \textit{Bandwidth, } \beta = \omega_h - \omega_l$$

$$= \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} - \left[-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right]$$

$$= \frac{R}{2L} + \frac{R}{2L} = 2\frac{R}{2L} = \frac{R}{L}\ \textit{in rad/s} \qquad .....(5.42)$$

$$\therefore\ \textit{Bandwidth in Hz} = \frac{\beta}{2\pi} = \frac{R}{2\pi L}\ \textit{in Hz} \qquad .....(5.43)$$

Alternatively, the bandwidth and cut-off frequencies can be expressed in terms of angular resonant frequency, ω_r *and quality factor,* Q_r *as shown below:*

From equation (5.42), we get, $\beta = \frac{R}{L}$

From equation (5.23), we get, $Q_r = \dfrac{\omega_r L}{R} \Rightarrow \dfrac{R}{L} = \dfrac{\omega_r}{Q_r}$

On comparing the above two equations, we get,

$$\beta = \frac{\omega_r}{Q_r} \quad .....(5.44)$$

From equation (5.38), we get,

$$\boxed{\omega_r = \frac{1}{\sqrt{LC}}} \quad \boxed{\frac{R}{L} = \beta} \quad \boxed{\beta = \frac{\omega_r}{Q_r}}$$

$$\omega_h = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = \frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + \omega_r^2}$$

$$= \frac{\omega_r}{2Q_r} + \sqrt{\frac{\omega_r^2}{4Q_r^2} + \omega_r^2} = \frac{\omega_r}{2Q_r} + \sqrt{\omega_r^2\left(1 + \frac{1}{4Q_r^2}\right)}$$

$$= \frac{\omega_r}{2Q_r} + \omega_r\sqrt{1 + \frac{1}{4Q_r^2}} = \omega_r\left[\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \quad .....(5.45)$$

Similarly,

$$\omega_l = \omega_r\left[-\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \quad .....(5.46)$$

On multiplying equations (5.45) and (5.46), we get,

$$\omega_h\omega_l = \omega_r\left[\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \times \omega_r\left[-\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right]$$

$$= \omega_r^2\left[\sqrt{1 + \frac{1}{4Q_r^2}} + \frac{1}{2Q_r}\right]\left[\sqrt{1 + \frac{1}{4Q_r^2}} - \frac{1}{2Q_r}\right]$$

$$= \omega_r^2\left[1 + \frac{1}{4Q_r^2} - \frac{1}{4Q_r^2}\right] \qquad \boxed{(a + b)\,(a - b) = a^2 - b^2}$$

$$= \omega_r^2$$

$$\therefore\ \omega_r = \sqrt{\omega_h\omega_l} \quad .....(5.47)$$

From equation (5.47) we can say that resonant frequency is given by the geometric mean of the cut-off frequencies.

5.2.5 Selectivity of Series RLC Circuit

RLC circuits are operated at resonance condition in order to select a particular frequency signal from a group of signals, or they are employed to pass a particular frequency of signal from one part of the circuit to the other. In general, resonance circuits are frequency selective circuits.

The best example of resonant circuits are tuned circuits used in TV/Radio. Each TV/Radio station broadcasts its signal at a particular frequency allotted to it. The **tuning circuit** is basically an RLC circuit with variable capacitance. In order to select a channel/station, the capacitance is varied and the circuit is made to resonate at the frequency of desired channel/station. Under resonance condition, the response is maximum at the desired frequency. Hence, the tuning circuit selects a particular frequency and rejects all the other frequencies.

A resonant circuit should be capable of selecting the desired frequency and rejecting all the other frequencies. This is possible only by achieving a very narrow response curve with smaller bandwidth as shown in Fig. 5.6. When the response curve is very narrow the circuit is highly selective. A term called **selectivity** can be defined in order to measure the ability of the resonant circuit to discriminate between the desired and unwanted frequencies.

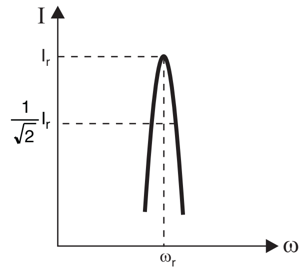


***Fig. 5.6 :** Current response curve of highly selective RLC series resonant circuit.*

Selectivity is defined as the ratio of bandwidth and resonant frequency.

$$\boxed{\therefore \text{ Selectivity } = \frac{\beta}{\omega_r} = \frac{\omega_h - \omega_l}{\omega_r}} \qquad .....(5.48)$$

From the definition of selectivity we can say that a circuit is highly selective when the value of selectivity is low, which is possible with a smaller value of bandwidth, β.

From equation (5.44), we get,

$$\beta = \frac{\omega_r}{Q_r}$$

$$\boxed{\therefore \text{ Selectivity } = \frac{\beta}{\omega_r} = \frac{\omega_r}{Q_r} \times \frac{1}{\omega_r} = \frac{1}{Q_r}} \qquad .....(5.49)$$

From equation (5.49), we can say that selectivity is the inverse of the quality factor. Therefore, when the quality factor is high, the selectivity will be small and so the circuit will be highly selective.

Also, from the expression of bandwidth we can say that when the quality factor is high, the bandwidth will be small and the circuit will be highly selective. The current response of a series RLC resonant circuit for various values of Q is shown in Fig. 5.7.

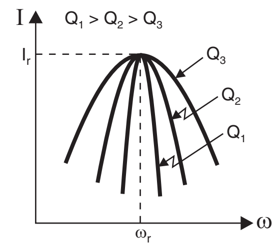


***Fig. 5.7 :** Current response of RLC series resonant circuit for various values of Q.*

5.2.6 Solved Problems in Series Resonance

EXAMPLE 5.1 ***(AU June'14, 16 Marks)***

For the RLC circuit shown in Fig. 1, determine the impedance at **a)** resonant frequency, **b)** 10 *Hz* below resonant frequency and **c)** 10 *Hz* above resonant frequency.

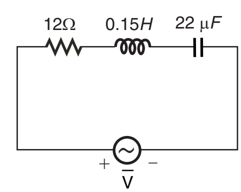


Fig. 1.

SOLUTION

Given that, $R = 12\,\Omega$; $L = 0.15\,H$ and $C = 22\,\mu F = 22 \times 10^{-6}\,F$

$$\text{Angular resonant frequency, } \omega_r = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.15 \times 22 \times 10^{-6}}} = 550.4819 \textit{ rad/s}$$

$$\text{Resonant frequency, } f_r = \frac{\omega_r}{2\pi} = \frac{550.4819}{2\pi} = 87.6119\,Hz$$

Let, $\bar{Z}$ be the impedance of an RLC series circuit. With reference to Fig. 1, we get,

$$\bar{Z} = R + j\omega L - j\frac{1}{\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

a) Impedance at resonance

Let, $\overline{Z}_r$ be the impedance at the resonance frequency. At resonant frequency ω_r, the total reactance is zero, i.e., $\omega_r L - \frac{1}{\omega_r C} = 0$ and the impedance is equal to resistance.

$$\therefore \text{ Impedance of resonance, } \overline{Z}_r = R = 12\,\Omega$$

b) Impedance at 10 *Hz* below the resonant frequency

Let, $\overline{Z}_1$ be the impedance at 10 *Hz* below the resonant frequency and ω_1 be the corresponding frequency.

Now, $\omega_1 = 2\pi\,(f_r - 10) = 2\pi \times (87.6119 - 10) = 487.6499$ *rad/s*

$$\therefore \overline{Z}_1 = R + j\left(\omega_1 L - \frac{1}{\omega_1 C}\right)$$

$$= 12 + j\left(487.6499 \times 0.15 - \frac{1}{487.6499 \times 22 \times 10^{-6}}\right)$$

$$= 12 - j20.0639\,\Omega = 23.3786\angle{-59.1^\circ}\,\Omega$$

c) Impedance at 10 *Hz* above the resonant frequency

Let, $\overline{Z}_2$ be the impedance at 10 *Hz* above the resonant frequency and ω_2 be the corresponding frequency.

Now, $\omega_2 = 2\pi\,(f_r + 10) = 2\pi \times (87.6119 + 10) = 613.3137$ *rad/s*

$$\therefore \overline{Z}_2 = R + j\left(\omega_2 L - \frac{1}{\omega_2 C}\right)$$

$$= 12 + j\left(613.3137 \times 0.15 - \frac{1}{613.3137 \times 22 \times 10^{-6}}\right)$$

$$= 12 + j17.884\,\Omega = 21.5369\angle 56.1^\circ\,\Omega$$

EXAMPLE 5.2

(AU Dec'16, 16 Marks)

For the circuit shown in Fig. 1,find I, V_R, V_L and V_C at resonance. Also calculate bandwidth, Q factor, half-power frequencies and power dissipated at resonance and at the half-power frequencies. Take resonant frequency as 5000 Hz.

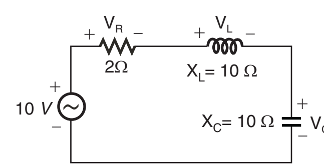


Fig. 1.

SOLUTION

At resonance the voltage across the resistor is equal to the supply voltage.

$$\therefore \overline{V}_R = \overline{E} = 10\angle 0^\circ\,V$$

Current at resonance, $\bar{I}_r = \frac{\overline{V}_R}{R} = \frac{\overline{E}}{R} = \frac{10\angle 0^\circ}{2} = 5\angle 0^\circ\,A$

Now, by Ohm's law,

$$\overline{V}_L = \bar{I}_r \times jX_L = 5\angle 0^\circ \times j10 = 5\angle 0^\circ \times 10\angle 90^\circ = 50\angle 90^\circ\,V$$

$$\overline{V}_C = \bar{I}_r \times (-jX_C) = 5\angle 0^\circ \times (-j10) = 5\angle 0^\circ \times 10\angle{-90^\circ} = 50\angle{-90^\circ}\,V$$

Quality factor at resonance, $Q_r = \frac{\omega_r L}{R} = \frac{X_L}{R} = \frac{10}{2} = 5$

Bandwidth, $\beta = \frac{f_r}{Q_r} = \frac{5000}{5} = 1000\ Hz$

Higher cut-off frequency , $f_h = f_r\left[\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right]$

$$= 5000 \times \left[\frac{1}{2 \times 5} + \sqrt{1 + \frac{1}{4 \times 5^2}}\right] = 5524.9378\ Hz$$

Lower cut-off frequency , $f_l = f_r\left[-\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right]$

$$= 5000 \times \left[-\frac{1}{2 \times 5} + \sqrt{1 + \frac{1}{4 \times 5^2}}\right] = 4524.9378\ Hz$$

Power dissipated at resonance, $P_r = I_r^2 R = 5^2 \times 2 = 50\ W$

Power dissipated at half-power frequency = $\frac{P_r}{2} = \frac{50}{2} = 25\ W$

EXAMPLE 5.3 *(AU Dec'15, 16 Marks)*

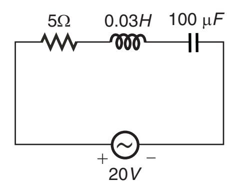


Fig. 1.

For the circuit shown in Fig. 1, determine the frequency at which the circuit resonates. Also find the quality factor, voltage across inductance and voltage across capacitance at resonance.

SOLUTION

Given that, $R = 5\Omega$; $L = 0.03\,H$; $C = 100\,\mu F$ and Supply voltage, $V = 20\,V$

Angular frequency of resonance, $\omega_r = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.03 \times 100 \times 10^{-6}}} = 577.3503\ rad/s$

Resonant frequency, $f_r = \frac{\omega_r}{2\pi} = \frac{577.3503}{2\pi} = 91.8882\ Hz$

Quality factor at resonance, $Q_r = \frac{\omega_r L}{R} = \frac{577.3503 \times 0.03}{5} = 3.4641$

Voltage across inductance at resonance, $\overline{V}_{Lr} = jQ_r V = j3.4641 \times 20$

$$= j69.282\ V = 69.282\angle 90^\circ\ V$$

Voltage across capacitance at resonance, $\overline{V}_{Cr} = -jQ_r V = -j3.4641 \times 20$

$$= -j69.282\ V = 69.282\angle -90^\circ\ V$$

Alternatively, $\overline{V}_{Lr}$ and $\overline{V}_{Cr}$ can be computed as shown below:

Current at resonance, $I_r = \frac{V}{R} = \frac{20}{5} = 4\,A$

$$\overline{V}_{Lr} = I_r \times j\omega_r L = 4 \times j577.3503 \times 0.03 = j69.282\,V = 69.282\angle 90^\circ\,V$$

$$\overline{V}_{Cr} = I_r \times \left(-j\frac{1}{\omega_r C}\right) = 4 \times \left(-j\frac{1}{577.3503 \times 100 \times 10^{-6}}\right) = -j69.282\,V = 69.282\angle -90^\circ\,V$$

EXAMPLE 5.4

A series RLC circuit has an impedance of 40 Ω at a frequency of 200 *rad/s*. When the circuit is made to resonate by connecting a 10 *V* source of variable frequency the current at resonance is 0.5 *A* and the quality factor at resonance is 10. Determine the circuit parameters.

SOLUTION

Given that, supply voltage, $V = 10\,V \quad ; \quad I_r = 0.5\,A \quad ; \quad Q_r = 10 \quad ; \quad Z = 40\,\Omega\Big|_{\omega = 200\,rad/s}$

We know that, $I_r = \frac{V}{R} \qquad \therefore R = \frac{V}{I_r} = \frac{10}{0.5} = 20\,\Omega$

We know that, $Q_r = \frac{1}{R}\sqrt{\frac{L}{C}} \quad ; \quad \therefore Q_r^2 = \frac{1}{R^2}\frac{L}{C} \quad \Rightarrow \quad C = \frac{1}{Q_r^2}\frac{1}{R^2}L$

$$\therefore C = \frac{1}{10^2} \times \frac{1}{20^2} \times L = 2.5 \times 10^{-5} L$$

The magnitude of impedance, Z of the RLC series circuit is given by,

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \Rightarrow Z^2 = R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 \Rightarrow Z^2 - R^2 = \left(\omega L - \frac{1}{\omega C}\right)^2$$

$$\therefore \omega L - \frac{1}{\omega C} = \sqrt{Z^2 - R^2}$$

On substituting $\omega = 200$ *rad/s*, $Z = 40\,\Omega$, $R = 20\,\Omega$ and $C = 2.5 \times 10^{-5}L$ in the above equation we get,

$$200\,L - \frac{1}{200 \times 2.5 \times 10^{-5} L} = \sqrt{40^2 - 20^2} \quad \Rightarrow \quad 200\,L - \frac{200}{L} = 34.641$$

On multiplying throughout by L/200, we get,

$$\frac{L}{200} \times 200L - \frac{L}{200} \times \frac{200}{L} = \frac{L}{200} \times 34.641 \quad \Rightarrow \quad L^2 - 1 = 0.1732\,L$$

$$\therefore L^2 - 0.1732\,L - 1 = 0$$

$$\therefore L = \frac{0.1732 \pm \sqrt{0.1732^2 + 4}}{2} = 1.0903\,H$$

Taking positive value

$$\therefore C = 2.5 \times 10^{-5} L = 2.5 \times 10^{-5} \times 1.0903 = 2.72575 \times 10^{-5} F$$

$$= 27.2575 \times 10^{-6} F = 27.2575\,\mu F$$

RESULT

The circuit parameters R, L and C are,

$R = 20\,\Omega \quad ; \quad L = 1.0903\,H \quad$ and $\quad C = 27.2575\,\mu F$

EXAMPLE 5.5 *(AU June'16, 16 Marks)*

An RLC series circuit consists of R = 16 Ω, L = 5 *mH* and C = 2 *μF*. Calculate the quality factor, bandwidth and half-power frequencies.

SOLUTION

Angular resonant frequency, $\omega_r = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{5 \times 10^{-3} \times 2 \times 10^{-6}}} = 10,000 \; rad/s$

Quality factor at resonance, $Q_r = \dfrac{\omega_r L}{R} = \dfrac{10,000 \times 5 \times 10^{-3}}{16} = 3.125$

Bandwidth, $\beta = \dfrac{\omega_r}{Q_r} = \dfrac{10,000}{3.125} = 3200 \; rad/s$

Bandwidth in *Hz* $= \dfrac{\beta}{2\pi} = \dfrac{3200}{2\pi} = 509.2958 \; Hz$

Lower angular cut-off frequency, $\omega_r = \dfrac{-R}{2L} + \sqrt{\left(\dfrac{R}{2L}\right)^2 + \dfrac{1}{LC}}$

Higher angular cut-off frequency, $\omega_h = \dfrac{R}{2L} + \sqrt{\left(\dfrac{R}{2L}\right)^2 + \dfrac{1}{LC}}$

Here, $\dfrac{R}{2L} = \dfrac{16}{2 \times 5 \times 10^{-3}} = 1600$

$$\frac{1}{LC} = \frac{1}{5 \times 10^{-3} \times 2 \times 10^{-6}} = 10^8$$

$$\therefore \; \omega_l = -1600 + \sqrt{1600^2 + 10^8} = 8527.1911 \; rad/s$$

$$\omega_h = 1600 + \sqrt{1600^2 + 10^8} = 11727.1911 \; rad/s$$

Lower cut-off frequency, $f_l = \dfrac{\omega_l}{2\pi} = \dfrac{8527.1911}{2\pi} = 1357.1446 \; Hz$

Higher cut-off frequency, $f_h = \dfrac{\omega_h}{2\pi} = \dfrac{11727.1911}{2\pi} = 1866.4404 \; Hz$

EXAMPLE 5.6

An RLC series circuit is to be designed to produce a magnification of 10 at 100 rad/s. The 100 *V* source connected to an RLC series circuit can supply a maximum current of 10 *A*. The half-power frequency impedance of the circuit should not be more than 14.14 Ω. Find the values of R, L and C.

SOLUTION

Given that ω_r = 100 *rad/s*, Q_r = 10, V = 100 *V*, I_r = 10 *A*.

The current will be maximum only at resonance. Hence, 10 *A* current can be considered as current at resonant condition.

Current at resonance, $I_r = \dfrac{V}{R}$

$$\therefore \; R = \frac{V}{I_r} = \frac{100}{10} = 10 \, \Omega$$

Quality factor at resonance, $Q_r = \frac{\omega_r L}{R}$

$$\therefore \; L = \frac{Q_r R}{\omega_r} = \frac{10 \times 10}{100} = 1H$$

Angular higher cut - off frequency, $\omega_h = \omega_r \left[\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}} \right]$

$$= 100 \left[\frac{1}{2 \times 10} + \sqrt{1 + \frac{1}{4 \times 10^2}} \right] = 105.1249 \; rad/s$$

Let, Z = Impedance of RLC series circuit.

Here, $Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$

$$\therefore \; Z\Big|_{\omega = \omega_h} = \sqrt{R^2 + \left(\omega_h L - \frac{1}{\omega_h C}\right)^2} = \sqrt{\left(\omega_h L - \frac{1}{\omega_h C}\right)^2 + \left(\omega_h L - \frac{1}{\omega_h C}\right)^2}$$

$$= \sqrt{2\left(\omega_h L - \frac{1}{\omega_h C}\right)^2} = \sqrt{2}\left(\omega_h L - \frac{1}{\omega_h C}\right)$$

At higher cut-off frequency, $R = \omega_h L - \frac{1}{\omega_h C}$

Here, $\sqrt{2}\left(\omega_h L - \frac{1}{\omega_h C}\right) \le 14.14 \;\Rightarrow\; \sqrt{2}\left(\omega_h L - \frac{1}{\omega_h C}\right) = 14.14 \;\Rightarrow\; \frac{1}{\omega_h C} = \omega_h L - \frac{14.14}{\sqrt{2}}$

$$\therefore \; C = \frac{1}{\omega_h \left(\omega_h L - \frac{14.14}{\sqrt{2}}\right)} = \frac{1}{105.1249 \times \left(105.1249 - \frac{14.14}{\sqrt{2}}\right)}$$

$$= 99.998 \times 10^{-6} \, F = 99.998 \, \mu F \approx 100 \, \mu F$$

RESULT

The values of the parameters of an RLC series circuit are,

$R = 10\,\Omega$, $L = 1\,H$ and $C = 100\,\mu F$

5.3 Parallel Resonance *(AU Dec'14, 16 Marks)*

Like series RLC circuits, the resonance condition can be achieved in parallel RLC circuits by varying the frequency of the exciting source. In a parallel RLC circuit, the circuit will behave as a purely resistive circuit at resonance and this circuit condition is called **parallel resonance.** However, the current supplied by the source is minimum in parallel resonance, which is why parallel resonance is also called **anti-resonance**.

For simplicity in analysis, parallel circuits can be analysed in terms of admittance. In admittance, the inductive and capacitive susceptances have opposite signs and they are functions of frequency. Hence, when the susceptances are varied by varying the frequency of the exciting source, there is a possibility that the inductive susceptance cancels the capacitive susceptance at a particular frequency. Therefore, the total susceptance is zero and the circuit will behave as a purely resistive circuit. Now, the circuit will be in resonance and the frequency at which resonance occurs is called **resonant frequency**.

The effective resistance of the RLC parallel circuit at resonance is called **dynamic resistance**. At resonance, the admittance of the RLC parallel circuit is purely real and so the dynamic resistance is given by the inverse of admittance at resonance.

In a parallel RLC circuit, we may come across the following four combinations of R, L and C circuits, as also shown in Fig. 5.8.

Case i : *R, L and C are in parallel.*

Case ii : *A branch with R_1 and L in series is parallel with another branch with R_2 and C in series.*

Case iii : *A branch with R_1 and L in series is parallel with C.*

Case iv : *A branch with R_2 and C in series is parallel with L.*

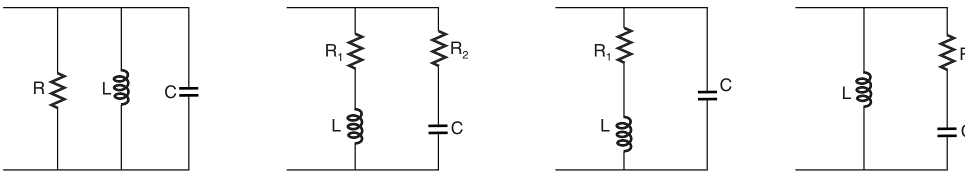


Fig. a : *R, L and C are in parallel.* ***Fig. b :*** *RL parallel with RC.* ***Fig. c :*** *RL parallel with C.* ***Fig. d :*** *L parallel with RC.*

Fig. 5.8 : *RLC parallel resonant networks.*

5.3.1 Resonant Frequency of Parallel RLC Circuits

Case i : Parallel combination of R, L and C

Consider the RLC parallel circuit shown in Fig. 5.9, excited by a sinusoidal source of variable frequency. When the frequency of the source is varied, the resonance occurs at a particular frequency. The expressions for resonance frequency and dynamic resistance for the RLC parallel circuit of Fig. 5.9 are given below:

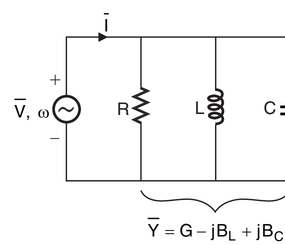


Fig. 5.9.

$$\boxed{\text{Resonant angular frequency, } \omega_r = \frac{1}{\sqrt{LC}} \text{ in } rad/s} \qquad .....(5.50)$$

$$\boxed{\text{Resonant frequency, } f_r = \frac{\omega_r}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \text{ in } Hz} \qquad .....(5.51)$$

$$\boxed{\text{Dynamic resistance, } R_{dynamic} = R} \qquad .....(5.52)$$

Proof for resonance frequency in RLC parallel circuit

Consider the parallel combination of R, L and C shown in Fig. 5.9.

Let, $\overline{Y}$ = *Admittance of the parallel combination of R, L and C.*

$$Now,\ \overline{Y} = G - jB_L + jB_C = \frac{1}{R} - j\frac{1}{\omega L} + j\omega C$$

$$\boxed{G = \frac{1}{R} \ ; \ B_L = \frac{1}{\omega L} \ ; \ B_C = \omega C}$$

$$= \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) \qquad .....(5.53)$$

At resonant frequency ω_r, the circuit behaves as a purely resistive circuit and so the imaginary part of admittance is zero.

$$\therefore \quad At \ \omega = \omega_r \ , \quad \omega_r C - \frac{1}{\omega_r L} = 0 \qquad .....(5.54)$$

$$\therefore \quad \frac{1}{\omega_r L} = \omega_r C$$

$$\therefore \quad \omega_r^2 = \frac{1}{LC}$$

$$\therefore \quad \omega_r = \frac{1}{\sqrt{LC}}$$

$$\therefore \quad f_r = \frac{\omega_r}{2\pi} \quad \Rightarrow \quad f_r = \frac{1}{2\pi\sqrt{LC}} \qquad \boxed{\omega = 2\pi f}$$

Let, $\overline{Y}_r$ be the admittance at resonance and it is obtained by substituting $\omega = \omega_r$ in equation (5.53).

$$\therefore \quad \overline{Y}_r = \frac{1}{R} + j\left(\omega_r C - \frac{1}{\omega_r L}\right) = \frac{1}{R} \qquad \boxed{\text{Using equation (5.54)}}$$

Dynamic resistance is the inverse of the admittance at resonance.

$$\therefore \quad \text{Dynamic resistance, } R_{dynamic} = \frac{1}{\overline{Y}_r} = R$$

Case ii : RL parallel with RC

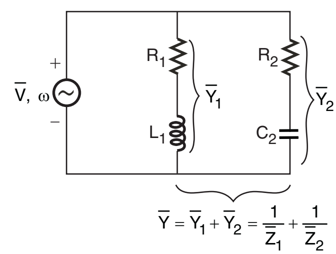


Fig. 5.10.

Consider the RLC parallel circuit shown in Fig. 5.10, excited by a sinusoidal source of variable frequency. When the frequency of the source is varied, the resonance occurs at a particular frequency. The expression for resonance frequency and dynamic resistance for the RLC parallel circuit of Fig. 5.10 are given below:

$$\boxed{\text{Resonant angular frequency, } \omega_r = \frac{1}{\sqrt{LC}}\sqrt{\frac{L - CR_1^2}{L - CR_2^2}} \text{ in } rad/s} \qquad .....(5.55)$$

$$\boxed{\text{Resonant frequency, } f_r = \frac{\omega_r}{2\pi} = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{L - CR_1^2}{L - CR_2^2}} \text{ in } Hz} \qquad .....(5.56)$$

$$\text{Dynamic resistance, } R_{dynamic} = \frac{1}{\dfrac{R_1}{R_1^2 + X_{Lr}^2} + \dfrac{R_2}{R_2^2 + X_{Cr}^2}} \text{ in } \Omega \qquad .....(5.57)$$

$$\text{where, } X_{Lr} = \omega_r L \quad ; \quad X_{Cr} = \frac{1}{\omega_r C}$$

Proof for resonance frequency in parallel RLC circuit in which RL parallel with RC

Consider the parallel resonant circuit shown in Fig.5.10.

Let, $\overline{Y}$ = Total admittance of the RLC parallel network.

$$\text{Now, } \overline{Y} = \overline{Y}_1 + \overline{Y}_2 = \frac{1}{\overline{Z}_1} + \frac{1}{\overline{Z}_2} = \frac{1}{R_1 + j\omega L} + \frac{1}{R_2 - j\dfrac{1}{\omega C}}$$

Let us separate the real and imaginary parts by multiplying the numerator and denominator of each term by the conjugate of the denominator.

$$\therefore \quad \overline{Y} = \frac{1}{R_1 + j\omega L} \times \frac{R_1 - j\omega L}{R_1 - j\omega L} + \frac{1}{R_2 - j\dfrac{1}{\omega C}} \times \frac{R_2 + j\dfrac{1}{\omega C}}{R_2 + j\dfrac{1}{\omega C}}$$

$$= \frac{R_1 - j\omega L}{R_1^2 + \omega^2 L^2} + \frac{R_2 + j\dfrac{1}{\omega C}}{R_2^2 + \dfrac{1}{\omega^2 C^2}}$$

$$= \frac{R_1}{R_1^2 + \omega^2 L^2} - j\frac{\omega L}{R_1^2 + \omega^2 L^2} + \frac{R_2}{R_2^2 + \dfrac{1}{\omega^2 C^2}} + j\frac{\dfrac{1}{\omega C}}{R_2^2 + \dfrac{1}{\omega^2 C^2}}$$

$$= \frac{R_1}{R_1^2 + \omega^2 L^2} + \frac{R_2}{R_2^2 + \dfrac{1}{\omega^2 C^2}} + j\left[\frac{\dfrac{1}{\omega C}}{R_2^2 + \dfrac{1}{\omega^2 C^2}} - \frac{\omega L}{R_1^2 + \omega^2 L^2}\right] \qquad .....(5.58)$$

At resonant frequency ω_r, the circuit behaves as a purely resistive circuit and so the imaginary part of the admittance is zero.

$$\therefore \text{ At } \omega = \omega_r \; ; \quad \frac{\dfrac{1}{\omega_r C}}{R_2^2 + \dfrac{1}{\omega_r^2 C^2}} - \frac{\omega_r L}{R_1^2 + \omega_r^2 L^2} = 0$$

$$\therefore \quad \frac{1}{\omega_r C\left(R_2^2 + \dfrac{1}{\omega_r^2 C^2}\right)} = \frac{\omega_r L}{R_1^2 + \omega_r^2 L^2}$$

On cross-multiplying the above equation, we get,

$$R_1^2 + \omega_r^2 L^2 = \omega_r^2 LC\left(R_2^2 + \frac{1}{\omega_r^2 C^2}\right) \Rightarrow R_1^2 + \omega_r^2 L^2 = \omega_r^2 R_2^2 LC + \frac{L}{C} \qquad .....(5.59)$$

$$\therefore \quad \omega_r^2 L^2 - \omega_r^2 R_2^2 LC = \frac{L}{C} - R_1^2 \Rightarrow \omega_r^2 LC\left(\frac{L}{C} - R_2^2\right) = \frac{L}{C} - R_1^2$$

$$\therefore \quad \omega_r^2 = \frac{\dfrac{L}{C} - R_1^2}{LC\left(\dfrac{L}{C} - R_2^2\right)} = \frac{L - CR_1^2}{LC\left(L - CR_2^2\right)}$$

$$\therefore \quad \omega_r = \frac{1}{\sqrt{LC}}\sqrt{\frac{L - CR_1^2}{L - CR_2^2}}$$

$$\therefore \quad f_r = \frac{\omega_r}{2\pi} = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{L - CR_1^2}{L - CR_2^2}}$$

The value of admittance at resonance is obtained from the real part of equation (5.58) at $\omega = \omega_r$.

$$\therefore \text{ Admittance at resonance, } \overline{Y}_r = Y_r = \frac{R_1}{R_1^2 + \omega_r^2 L^2} + \frac{R_2}{R_2^2 + \dfrac{1}{\omega_r^2 C^2}} = \frac{R_1}{R_1^2 + X_{Lr}^2} + \frac{R_2}{R_2^2 + X_{Cr}^2}$$

where, $X_{Lr} = \omega_r L \; ; \; X_{Cr} = \dfrac{1}{\omega_r C}$

Dynamic resistance is the inverse of the admittance at resonance.

$$\therefore \quad R_{dynamic} = \frac{1}{Y_r} = \frac{1}{\dfrac{R_1}{R_1^2 + X_{Lr}^2} + \dfrac{R_2}{R_2^2 + X_{Cr}^2}}$$

Resonance at All Frequency

The parameters R_1, R_2, L and C of the parallel resonant circuit shown in Fig. 5.11 can be chosen such that the circuit behaves as a purely resistive circuit at all frequencies. Hence, the circuit will be in resonance at all frequencies.

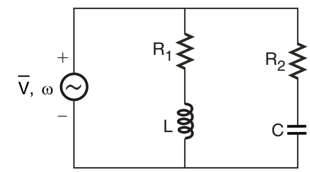


Fig. 5.11.

From equation (5.59), we can write,

$$R_1^2 - \frac{L}{C} + \omega_r^2 L^2 - \omega_r^2 R_2^2 LC = 0$$

$$\therefore \left(R_1^2 - \frac{L}{C}\right) - \omega_r^2 LC\left(R_2^2 - \frac{L}{C}\right) = 0$$

Let, $R_1 = R_2 = R$

$$\therefore \left(R^2 - \frac{L}{C}\right) - \omega_r^2 LC\left(R^2 - \frac{L}{C}\right) = 0$$

$$\left(R^2 - \frac{L}{C}\right)(1 - \omega_r^2 LC) = 0$$

The above equation is zero, if $\left(R^2 - \frac{L}{C}\right) = 0$

Let, $R^2 - \frac{L}{C} = 0$

$$\therefore R^2 = \frac{L}{C}$$

$$\therefore R = \sqrt{\frac{L}{C}} \qquad .....(5.60)$$

Equation (5.60) is the condition for resonance at all frequencies. Therefore, we can say that when $R_1 = R_2 = \sqrt{\frac{L}{C}}$, the imaginary part of the admittance will be zero for all frequencies and the circuit of Fig. 5.11 will behave as a resistive circuit at all frequencies, i.e., resonate at all frequencies.

Case iii : RL parallel with C

Consider the RLC parallel circuit shown in Fig. 5.12, excited by a sinusoidal source of variable frequency. When the frequency of the source is varied, the resonance occurs at a particular frequency. The expressions for resonance frequency and dynamic resistance for the RLC parallel circuit of Fig. 5.12 are given below:

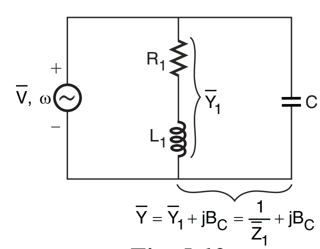


Fig. 5.12.

$$\boxed{\text{Resonant angular frequency, } \omega_r = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{CR_1^2}{L}}} \qquad .....(5.61)$$

$$\boxed{\text{Resonant frequency, } f_r = \frac{\omega_r}{2\pi} = \frac{1}{2\pi\sqrt{LC}}\sqrt{1 - \frac{CR_1^2}{L}}} \qquad .....(5.62)$$

$$\boxed{\text{Dynamic resistance, } R_{dynamic} = \frac{L}{R_1 C} \text{ in } \Omega} \qquad .....(5.63)$$

> ***Note :*** *Equation (5.61) for resonant frequency can be obtained from equation (5.55), if we put $R_2 = 0$.*

Proof for resonance frequency in parallel RLC circuit in which RL is parallel with C

Consider the parallel resonant circuit shown in Fig. 5.12.

Let, $\overline{Y}$ = *Total admittance of the RLC parallel network.*

$$\text{Now, } \overline{Y} = \overline{Y}_1 + jB_C = \frac{1}{\overline{Z}_1} + jB_C$$

$$= \frac{1}{R_1 + j\omega L} + j\omega C = \frac{1}{R_1 + j\omega L} \times \frac{R_1 - j\omega L}{R_1 - j\omega L} + j\omega C$$

$$= \frac{R_1 - j\omega L}{R_1^2 + \omega^2 L^2} + j\omega C = \frac{R_1}{R_1^2 + \omega^2 L^2} - j\frac{\omega L}{R_1^2 + \omega^2 L^2} + j\omega C$$

$$= \frac{R_1}{R_1^2 + \omega^2 L^2} + j\left(\omega C - \frac{\omega L}{R_1^2 + \omega^2 L^2}\right) \qquad .....(5.64)$$

At resonant frequency ω_r, the circuit behave as a purely resistive circuit and so the imaginary part of the admittance is zero.

$$\therefore \text{ At } \omega = \omega_r \text{ ; } \omega_r C - \frac{\omega_r L}{R_1^2 + \omega_r^2 L^2} = 0$$

$$\therefore \omega_r C = \frac{\omega_r L}{R_1^2 + \omega_r^2 L^2} \Rightarrow R_1^2 + \omega_r^2 L^2 = \frac{\omega_r L}{\omega_r C}$$

$$\therefore \omega_r^2 L^2 = \frac{L}{C} - R_1^2 \qquad .....(5.65)$$

On dividing the above equation by L^2, we get,

$$\omega_r^2 = \frac{1}{LC} - \frac{R_1^2}{L^2} \Rightarrow \omega_r^2 = \frac{1}{LC}\left(1 - \frac{CR_1^2}{L}\right)$$

$$\therefore \omega_r = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{CR_1^2}{L}}$$

$$\therefore f_r = \frac{\omega_r}{2\pi} = \frac{1}{2\pi\sqrt{LC}}\sqrt{1 - \frac{CR_1^2}{L}}$$

The value of admittance at resonance is obtained from the real part of equation (5.64) at $\omega = \omega_r$.

$$\therefore \text{ Admittance at resonance, } \overline{Y}_r = Y_r = \frac{R_1}{R_1^2 + \omega_r^2 L^2}$$

Dynamic resistance is given by the inverse of the admittance at resonance.

$$R_{dynamic} = \frac{1}{Y_r} = \frac{R_1^2 + \omega_r^2 L^2}{R_1} \qquad \boxed{\text{Using equation (5.65)}}$$

$$= \frac{R_1^2 + \frac{L}{C} - R_1^2}{R_1} = \frac{L}{R_1 C}$$

Case iv : L parallel with RC

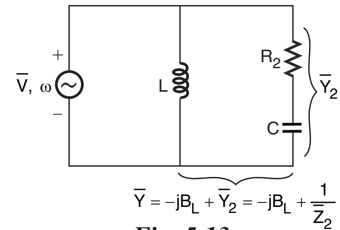


Fig. 5.13.

Consider the RLC parallel circuit shown in Fig. 5.13, excited by a sinusoidal source of variable frequency. When the frequency of the source is varied, resonance occurs at a paricular frequency. The expressions for resonance frequency and dynamic resistance for the RLC parallel circuit of Fig. 5.13 are given below:

$$\boxed{\text{Resonant angular frequency, } \omega_r = \frac{1}{\sqrt{LC}}\sqrt{\frac{L}{L - CR_2^2}} \text{ in } rad/s} \qquad .....(5.66)$$

$$\boxed{\text{Resonant frequency, } f_r = \frac{\omega_r}{2\pi} = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{L}{L - CR_2^2}} \text{ in } Hz} \qquad .....(5.67)$$

$$\boxed{\text{Dynamic resistance, } R_{dynamic} = \frac{L}{R_2 C} \text{ in } \Omega} \qquad .....(5.68)$$

> ***Note :*** *Equation (5.66) for resonant frequency can be obtained from equation (5.55), if we put $R_1 = 0$.*

Proof for resonance frequency in RLC parallel circuit in which L parallel with RC

Consider the parallel resonant circuit shown in Fig.5.13.

Let, $\overline{Y}$ = *Total admittance of the RLC parallel network.*

$$\text{Now, } \overline{Y} = -jB_L + \overline{Y}_2 = -jB_L + \frac{1}{\overline{Z}_2}$$

$$= -j\frac{1}{\omega L} + \frac{1}{R_2 - j\frac{1}{\omega C}}$$

$$\therefore \ \overline{Y} = -j\frac{1}{\omega L} + \frac{1}{R_2 - j\frac{1}{\omega C}} \times \frac{R_2 + j\frac{1}{\omega C}}{R_2 + j\frac{1}{\omega C}}$$

$$= -j\frac{1}{\omega L} + \frac{R_2 + j\frac{1}{\omega C}}{R_2^2 + \frac{1}{\omega^2 C^2}}$$

$$= -j\frac{1}{\omega L} + \frac{R_2}{R_2^2 + \frac{1}{\omega^2 C^2}} + j\frac{\frac{1}{\omega C}}{R_2^2 + \frac{1}{\omega^2 C^2}}$$

$$= \frac{R_2}{R_2^2 + \frac{1}{\omega^2 C^2}} + j\left(\frac{1}{\omega C\left(R_2^2 + \frac{1}{\omega^2 C^2}\right)} - \frac{1}{\omega L}\right) \qquad .....(5.69)$$

At resonant frequency ω_r, the circuit behaves as resistive circuit and so the imaginary part of the admittance is zero.

$$\therefore \text{ At } \omega = \omega_r \;; \quad \frac{1}{\omega_r C\left(R_2^2 + \dfrac{1}{\omega_r^2 C^2}\right)} - \frac{1}{\omega_r L} = 0$$

$$\therefore \frac{1}{\omega_r C\left(R_2^2 + \dfrac{1}{\omega_r^2 C^2}\right)} = \frac{1}{\omega_r L} \quad \Rightarrow \quad \frac{\omega_r L}{\omega_r C} = R_2^2 + \frac{1}{\omega_r^2 C^2}$$

$$\therefore \frac{L}{C} - R_2^2 = \frac{1}{\omega_r^2 C^2} \qquad .....(5.70)$$

$$\therefore \frac{L - CR_2^2}{C} = \frac{1}{\omega_r^2 C^2} \quad \Rightarrow \quad \omega_r^2 = \frac{1}{C^2} \times \frac{C}{\left(L - CR_2^2\right)}$$

On multiplying and dividing the above equation by L, we get,

$$\omega_r^2 = \frac{1}{LC} \times \frac{L}{L - CR_2^2} \quad \Rightarrow \quad \omega_r = \frac{1}{\sqrt{LC}} \sqrt{\frac{L}{L - CR_2^2}}$$

$$\therefore \omega_r = \frac{1}{\sqrt{LC}} \sqrt{\frac{L}{L - CR_2^2}}$$

$$\therefore f_r = \frac{\omega_r}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \sqrt{\frac{L}{L - CR_2^2}}$$

The value of admittance at resonance is obtained from the real part of equation (5.69) at $\omega = \omega_r$.

$$\therefore \text{ Admittance at resonance, } \overline{Y}_r = Y_r = \frac{R_2}{R_2^2 + \dfrac{1}{\omega_r^2 C^2}}$$

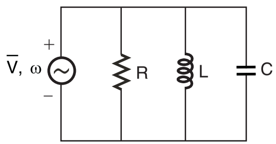
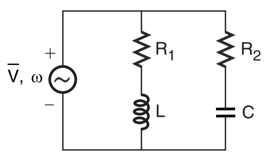
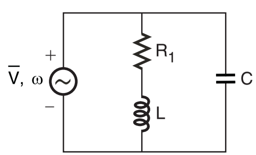
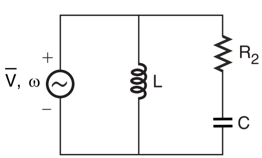
Dynamic resistance is the inverse of the admittance at resonance.

$$\therefore \text{ Dynamic resistance, } R_{dynamic} = \frac{1}{Y_r} = \frac{R_2^2 + \dfrac{1}{\omega_r^2 C^2}}{R_2}$$

Using equation (5.70)

$$= \frac{R_2^2 + \dfrac{L}{C} - R_2^2}{R_2} = \frac{L}{R_2 C} \qquad .....(5.71)$$

Table 5.1 : Summary of Equations for Resonance and Dynamic Resistance in Parallel RLC Circuits

RLC Parallel Circuit	Resonant frequency		Dynamic resistance
	ω_r in *rad/s*	f_r in *Hz*	$R_{dynamic}$ in Ω
	$\omega_r = \dfrac{1}{\sqrt{LC}}$	$f_r = \dfrac{1}{2\pi\sqrt{LC}}$	$R_{dynamic} = R$
	$\omega_r = \dfrac{1}{\sqrt{LC}}\sqrt{\dfrac{L - CR_1^2}{L - CR_2^2}}$	$f_r = \dfrac{1}{2\pi\sqrt{LC}}\sqrt{\dfrac{L - CR_1^2}{L - CR_2^2}}$	$R_{dynamic} = \dfrac{1}{\dfrac{R_1}{R_1^2 + X_{Lr}^2} + \dfrac{R_2}{R_2^2 + X_{Cr}^2}}$ $X_{Lr} = \omega_r L \;\; ; \;\; X_{Cr} = \dfrac{1}{\omega_r C}$
	$\omega_r = \dfrac{1}{\sqrt{LC}}\sqrt{1 - \dfrac{CR_1^2}{L}}$	$f_r = \dfrac{1}{2\pi\sqrt{LC}}\sqrt{1 - \dfrac{CR_1^2}{L}}$	$R_{dynamic} = \dfrac{L}{R_1 C}$
	$\omega_r = \dfrac{1}{\sqrt{LC}}\sqrt{\dfrac{L}{L - CR_2^2}}$	$f_r = \dfrac{1}{2\pi\sqrt{LC}}\sqrt{\dfrac{L}{L - CR_2^2}}$	$R_{dynamic} = \dfrac{L}{R_2 C}$

5.3.2 Frequency Response of Parallel RLC Circuit

(AU Dec'16, 2 Marks)

Consider the parallel RLC circuit excited by a sinusoidal source of variable frequency as shown in Fig. 5.14. The admittance $\overline{Y}$ of the circuit is given by,

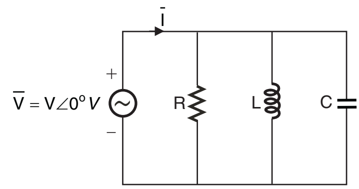


Fig. 5.14.

$$\overline{Y} = G - jB_L + jB_C = G + j(B_C - B_L) \quad .....(5.72)$$

Let, $\bar{I}_r$ = Current at resonance.

$\overline{Y}_r$ = Admittace at resonance.

At resonance, $B_L = B_C$

Therefore, from equation (5.72), we get,

$$\text{Admittance at resonance, } \overline{Y}_r = \frac{1}{R} \quad .....(5.73)$$

$$\therefore \quad \text{Current at resonance, } \bar{I}_r = \overline{V}\,\overline{Y}_r = V\angle 0^\circ \times \frac{1}{R} = \frac{V}{R}\angle 0^\circ = I_r \angle 0^\circ \text{A} \quad .....(5.74)$$

$$\text{where, } I_r = \frac{V}{R} = \text{Magnitude of current at resonance} \quad .....(5.75)$$

Let us examine the variation of admittance of the RLC parallel circuit with frequency. At frequencies lesser than resonant frequency, the inductive susceptance will be more than the capacitive susceptance and so the total susceptance will be inductive. Since the inductive susceptance is inversely proportional to frequency, the inductive susceptance and hence the total susceptance increases when the frequency is decreased from the resonant frequency. Therefore, the admittance of the RLC parallel circuit increases when the frequency is decreased from the resonant value.

At frequencies higher than resonant frequency, the capacitive susceptance will be more than the inductive susceptance and so the total susceptance will be capacitive. Since the capacitive susceptance is directly proportional to frequency, the capacitive susceptance and hence the total susceptance increases when the frequency is increased from the resonant frequency. Therefore, the admittance of the RLC parallel circuit increases when the frequency is increased from the resonant value.

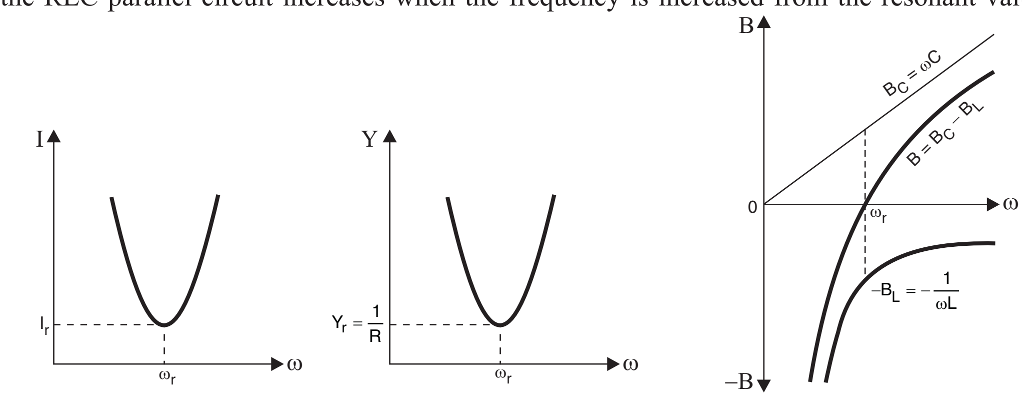


***Fig. a :** Current vs frequency.*

***Fig. b :** Admittance vs frequency.*

***Fig. c :** Susceptance vs frequency.*

***Fig. 5.15 :** Characteristics of parallel resonance.*

At resonant frequency, the admittance of the RLC parallel circuit is equal to the conductance and this value of admittance is minimum. Since the admittance is minimum, the current is minimum at resonance. Also the current at resonance will be in-phase with the supply voltage.

Since the admittance increases for frequencies lesser or higher than resonant value, the current increases when the frequency is increased or decreased from the resonant value. The variation of current, susceptances and admittance of an RLC parallel circuit with frequency are shown in Fig. 5.15.

When the RLC parallel circuit is excited by a constant current source as shown in Fig. 5.16(a), it can be justified from the above discussion that the voltage across the parallel elements is maximum at resonance and decreases when the frequency is increased or decreased from the resonance value. The variation of voltage with frequency when excited by a constant current source is shown in Fig. 5.16 (b).

Let, $\overline{V}_{ri}$ = Voltage across parallel elements at resonance when excited by a current source.

$\overline{Y}_{ri}$ = Admittance of RLC circuit at resonance.

Here, $$\overline{Y}_{ri} = G = \frac{1}{R} \qquad .....(5.76)$$

$$\overline{V}_{ri} = \frac{\bar{I}}{\overline{Y}_{ri}} = \frac{I\angle 0^\circ}{G} = IR\angle 0^\circ = V_{ri}\angle 0^\circ \qquad .....(5.77)$$

where, V_{ri} = IR = Magnitude of voltage at resonance .....(5.78)

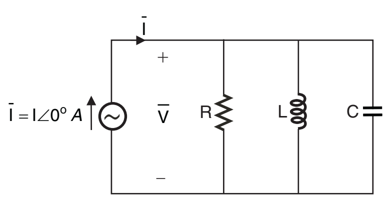


Fig. a : *RLC parallel circuit.*

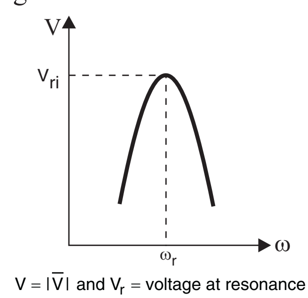


Fig. b : *Voltage vs frequency.*

Fig. 5.16 : *Characteristics of an RLC parallel circuit excited by a constant current source.*

5.3.3 Q-Factor(Quality Factor) of RLC Parallel Circuit *(AU June'14, 16 Marks)*

In an RLC parallel circuit, due to stored energy in the inductor and capacitor, the current through these devices will be greater than the current supplied by the source. In other words, we can say that there is a current magnification or amplification. **Current magnification** can be expressed by a factor called **Quality factor (Q)**, which is defined as the ratio of maximum energy stored to the energy dissipated in one period.

$$\therefore \quad \text{Quality factor, } Q = 2\pi \times \frac{\text{Maximum energy stored}}{\text{Energy dissipated in one period}} \qquad .....(5.79)$$

The term 2π is introduced to simplify the expression for quality factor.

$$\text{Quality factor at resonance, } Q_r = 2\pi \times \frac{\text{Maximum energy stored at resonance}}{\text{Energy dissipated in one period at resonance}} \qquad .....(5.80)$$

Here, $\boxed{Q_r = \omega_r CR}$ .....(5.81)

At resonance, $\omega_r = \dfrac{1}{\sqrt{LC}}, \quad \omega_r^2 = \dfrac{1}{LC}$ and $C = \dfrac{1}{\omega_r^2 L}$

Therefore, the quality factor at resonance, Q_r can also be expressed as shown below:

$$Q_r = \omega_r CR = \omega_r \times \frac{1}{\omega_r^2 L} \times R = \frac{R}{\omega_r L} \qquad \boxed{\therefore \quad Q_r = \frac{R}{\omega_r L}} \qquad .....(5.82)$$

$$= \omega_r CR = \frac{1}{\sqrt{LC}} \times CR = R\sqrt{\frac{C}{L}} \qquad \boxed{\therefore \quad Q_r = R\sqrt{\frac{C}{L}}} \qquad .....(5.83)$$

For frequencies less than the resonant frequency, the RLC parallel circuit behaves as an inductive circuit.

$$\boxed{\therefore \quad \text{When} \quad \omega_r \le \omega_r, \quad Q = \frac{R}{\omega L}} \qquad .....(5.84)$$

For frequencies higher than the resonant frequency, the RLC parallel circuit behaves as a capacitive circuit.

$$\boxed{\therefore \quad \text{When} \quad \omega \ge \omega_r \;, \quad Q = \omega CR} \qquad .....(5.85)$$

Note : *Equation (5.84) can be used to calculate the Q-factor of an RL parallel circuit and equation (5.85), can be used to calculate the Q-factor of a capacitor or RC parallel circuit.*

Proof for quality factor at resonance, Q_r in RLC parallel circuit

Consider a RLC parallel circuit shown in Fig. 5.17, excited by a sinusoidal voltage source of frequency, ω.

Let, $\bar{V}$ be the reference phasor.

$$\therefore \bar{V} = V\angle 0^o\, V$$

Let, v = Instantaneous value of voltage

$$\therefore v = V_m \sin \omega t \qquad .....(5.86)$$

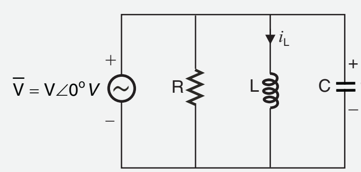


Fig. 5.17.

Let, w_L = *Instantaneous value of energy stored in inductor*

w_C = *Instantaneous value of energy stored in capacitor*

w = *Total instantaneous energy stored in the RLC circuit*

w_r = *Total instantaneous energy stored in the RLC circuit at resonance.*

We know that,

$$w_L = \frac{1}{2} L i_L^2 \qquad \boxed{\text{Using equation (4.11) of Chapter 4}}$$

$$= \frac{1}{2} L \left[\frac{1}{L}\int v\, dt\right]^2 \qquad \boxed{i_L = \frac{1}{L}\int v\, dt}$$

$$= \frac{1}{2} L \left[\frac{1}{L}\int V_m \sin \omega t\, dt\right]^2 \qquad \boxed{\text{Using equation (5.86)}}$$

$$= \frac{1}{2} L \left[\frac{1}{L}\left(-V_m \frac{\cos \omega t}{\omega}\right)\right]^2$$

$$= \frac{V_m^2}{2\omega^2 L} \cos^2 \omega t \qquad .....(5.87)$$

We know that,

$$w_C = \frac{1}{2}CV^2 \qquad \boxed{\text{Using equation (4.16) of Chapter - 4}}$$

$$= \frac{1}{2}C\left(V_m \sin \omega t\right)^2 \qquad \boxed{\text{Using equation (5.86)}}$$

$$= \frac{1}{2}C\, V_m^2 \sin^2 \omega t \qquad .....(5.88)$$

Now, $w = w_L + w_C = \frac{V_m^2}{2\omega^2 L}\cos^2 \omega t + \frac{1}{2}C\, V_m^2 \sin^2 \omega t$ $\boxed{\text{Using equations (5.87) and (5.88)}}$

$$= \frac{V_m^2}{2}\left[C \sin^2 \omega t + \frac{1}{\omega^2 L}\cos^2 \omega t\right]$$

$$\therefore\ w_r = w\Big|_{\omega=\omega_r} = \frac{V_m^2}{2}\left[C\sin^2 \omega t + \frac{1}{\omega^2 L}\cos^2 \omega t\right]\Bigg|_{\omega=\omega_r} = \frac{V_m^2}{2}\left[C\sin^2 \omega_r t + \frac{1}{\omega_r^2 L}\cos^2 \omega_r t\right]$$

$$= \frac{V_m^2}{2}\left[C\sin^2 \omega_r t + \frac{LC}{L}\cos^2 \omega_r t\right] \qquad \boxed{\omega_r = \frac{1}{\sqrt{LC}} \Rightarrow \frac{1}{\omega_r^2} = LC}$$

$$= \frac{V_m^2 C}{2}\left[\sin^2 \omega_r t + \cos^2 \omega_r t\right] \qquad \boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$= \frac{V_m^2 C}{2} \qquad .....(5.89)$$

From equation (5.89) we can say that the energy stored in the RLC circuit at resonance is independent of time and it is constant. Therefore, the instantaneous energy is the maximum energy stored at resonance.

Let, W_{mr} = *Maximum energy stored at resonance*

$$\therefore\ W_{mr} = w_r = \frac{V_m^2 C}{2} \qquad .....(5.90)$$

In the RLC parallel circuit, the energy is dissipated by the resistor

Let, W_R = *Energy dissipated in resistor in one period*

W_{Rr} = *Energy dissipated in resistor in one period at resonance*

$\therefore$ W_R = *Power × Time Period.*

$$= \frac{V^2}{R} \times T \qquad \boxed{V = \text{RMS value of voltage}}$$

$$= \left(\frac{V_m}{\sqrt{2}}\right)^2 \times \frac{1}{R} \times \frac{1}{f} \qquad \boxed{V = \frac{V_m}{\sqrt{2}}} \quad \boxed{T = \frac{1}{f}}$$

$$= \frac{V_m^2}{2} \times \frac{1}{R} \times \frac{2\pi}{\omega} \qquad \boxed{\omega = 2\pi f \Rightarrow \frac{1}{f} = \frac{2\pi}{\omega}}$$

$$= \frac{\pi V_m^2}{\omega R} \qquad .....(5.91)$$

$$\therefore\ W_{Rr} = W_R\Big|_{\omega=\omega_r} = \frac{\pi V_m^2}{\omega_r R} \qquad .....(5.92)$$

$$\therefore\ Q_r = 2\pi \times \frac{W_{mr}}{W_{Rr}} = 2\pi \times W_{mr} \times \frac{1}{W_{Rr}}$$

$$= 2\pi \times \frac{V_m^2 C}{2} \times \frac{\omega_r R}{\pi V_m^2} \qquad \boxed{\text{Using equations (5.90) and (5.92)}}$$

$$\therefore\ Q_r = w_r CR \qquad .....(5.93)$$

5.3.4 Bandwidth of RLC Parallel Circuit

When an RLC parallel circuit is excited by a constant voltage source as shown in Fig. 5.18(a), the current is the response as shown in Fig. 5.18(b). From this current response we can say that the RLC parallel circuit acts as a **rejector circuit** where a band of frequencies are rejected. This is because for the band of frequencies around resonant frequencies, the current response is minimum.

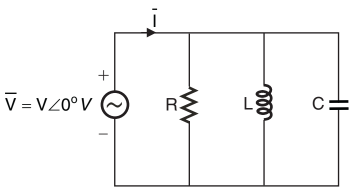


Fig. a.

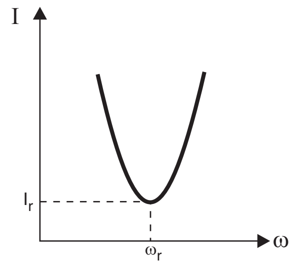


Fig. b : *Current vs frequency.*

Fig. 5.18 : *RLC parallel circuit excited by voltage source.*

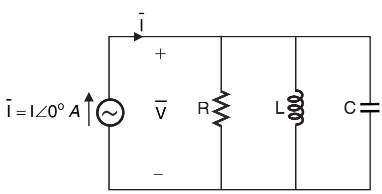


Fig. a : *RLC parallel circuit.*

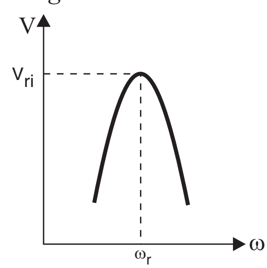


Fig. b : *Voltage vs frequency.*

Fig. 5.19 : *Parallel circuit excited by constant current source.*

On the other hand, when an RLC parallel circuit is excited by a constant current source as shown in Fig. 5.19(a), the voltage is the response as shown in Fig. 5.19(b). From this voltage response, we can say that the RLC parallel circuit acts as a **selector circuit** where a band of frequencies are selected. This is because for the band of frequencies around resonant frequencies, the voltage response is maximum. Bandwidth based on current response and voltage response is discussed here.

Bandwidth Based on Current Response

The current response of an RLC parallel circuit is shown in Fig. 5.20. The current response is minimum at resonance and it increases for increasing or decreasing frequency from the resonance value. Since current is minimum, power is also minimum at resonance. Therefore, when the frequency is decreased from the resonant value, we come across a frequency at which power is twice that of the minimum value (or the current is $\sqrt{2}$ times the minimum value), and this frequency is called **lower cut-off frequency**, ω_l. When frequency is increased from the resonant value, we come across a frequency at which power is twice that of the minimum value (or the current is $\sqrt{2}$ times the minimum value), and this frequency is called **higher cut-off frequency**, ω_h.

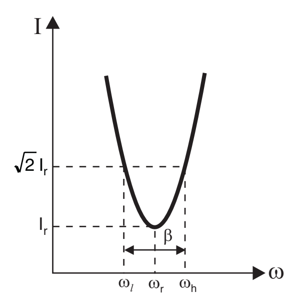


Fig. 5.20.

The two cut-off frequencies lie on either side of the resonant frequency as shown in Fig. 5.20. It can be proved that "*the resonant frequency is given by the geometric mean of the two cut-off frequencies*", i.e., $\omega_r = \sqrt{\omega_l \, \omega_h}$.

The bandwidth based on current response can be defined as the range of frequencies over which power is less than or equal to twice the minimum power.

Alternatively, the bandwidth based on current response can be defined as the range of frequencies over which current is less than or equal to $\sqrt{2}$ times the minimum current.

The bandwidth is given by the difference between the cut-off frequencies and it can be denoted by β. The unit of bandwidth is *rad/s* or *Hz*.

The equations for cut-off frequency and bandwidth are given below:

$$\boxed{\text{Higher cut - off angular frequency, } \omega_h = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \text{ in } rad/s} \quad \text{..... (5.94)}$$

$$\boxed{\text{Lower cut - off angular frequency, } \omega_l = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \text{ in } rad/s} \quad \text{..... (5.95)}$$

Alternatively,

$$\boxed{\omega_h = \omega_r\left[\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \text{in } rad/s} \quad \text{.....(5.96)}$$

$$\boxed{\omega_l = \omega_r\left[-\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \text{in } rad/s} \quad \text{.....(5.97)}$$

$$\boxed{\text{Higher cut - off frequency, } f_h = \frac{\omega_h}{2\pi} = \frac{1}{2\pi}\left[\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}\right] \text{in } Hz} \quad \text{.....(5.98)}$$

$$\boxed{\text{Lower cut - off frequency, } f_l = \frac{\omega_h}{2\pi} = \frac{1}{2\pi}\left[-\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}\right] \text{in } Hz} \quad \text{.....(5.99)}$$

$$\boxed{\text{Bandwidth, } \beta = \frac{1}{RC} \text{ in } rad/s} \quad \text{.....(5.100)}$$

Alternatively,

$$\boxed{\text{Bandwidth, } \beta = \frac{\omega_r}{Q_r} \text{ in } rad/s} \quad \text{.....(5.101)}$$

$$\boxed{\text{Bandwidth in } Hz = \frac{\beta}{2\pi} = \frac{1}{2\pi RC} \text{ in } Hz} \quad \text{.....(5.102)}$$

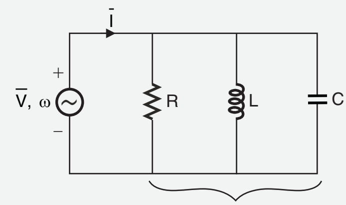
Proof for cut-off frequency and bandwidth based on current response

Consider an RLC parallel circuit shown in Fig. 5.21, excited by a sinusoidal voltage source of frequency ω.

Let, $\bar{V} = V\angle 0^o$ = Supply voltage

$\bar{I}$ = Current delivered by the source

$\bar{Y}$ = Admittance of RLC parallel circuit.



$$\bar{Y} = \frac{1}{R} - j\frac{1}{\omega L} + j\omega C$$

Fig. 5.21.

Here, $\bar{Y} = \frac{1}{R} - j\frac{1}{\omega L} + j\omega C = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$

$$\therefore \quad Y = \left|\bar{Y}\right| = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} \qquad .....(5.103)$$

Here, $\bar{I} = \bar{V}\,\bar{Y}$

$$\therefore \quad I = \left|\bar{I}\right| = \left|\bar{V}\,\bar{Y}\right| = \left|\bar{V}\right|\left|\bar{Y}\right| = VY$$

$$\therefore \quad I = V \times \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} \qquad .....(5.104)$$

Using equation (5.103)

Let, P = Power in RLC circuit

P_r = Power in RLC circuit at resonance.

Here, $P = I^2 R = V^2\left(\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2\right) R$

Using equation (5.104)

$$\therefore \quad P = V^2\left(\frac{1}{R} + R\left(\omega C - \frac{1}{\omega L}\right)^2\right) \qquad .....(5.105)$$

At resonance, $\omega C = 1/\omega L$. On substituting this condition in equation (5.105), we get power at resonance, P_r.

$$\therefore \quad P_r = \frac{V^2}{R} \qquad .....(5.106)$$

Note : Alternatively, $P_r = I_r^2 R = \left(\frac{V}{R}\right)^2 R = \frac{V^2}{R}$

At half-power frequencies or cut-off frequencies, the power will be equal to twice the power at resonance.

$$\therefore \quad 2 \times P_r = P$$

$$2 \times \frac{V^2}{R} = V^2\left(\frac{1}{R} + R\left(\omega C - \frac{1}{\omega L}\right)^2\right)$$

Using equations (5.105) and (5.106)

$$\therefore \quad \frac{2}{R} = \frac{1}{R} + R\left(\omega C - \frac{1}{\omega L}\right)^2 \quad \Rightarrow \quad \frac{2}{R} - \frac{1}{R} = R\left(\omega C - \frac{1}{\omega L}\right)^2 \quad \Rightarrow \quad \frac{1}{R} = R\left(\omega C - \frac{1}{\omega L}\right)^2$$

$$\therefore \quad \left(\omega C - \frac{1}{\omega L}\right)^2 = \frac{1}{R^2}$$

On taking the square root of the above equation, we get,

$$\omega C - \frac{1}{\omega L} = \pm\frac{1}{R} \qquad .....(5.107)$$

Note : Equation (5.107) implies that the absolute value of the total susceptance at cut-off frequencies is equal to the conductance of the circuit.

$$\therefore \; \omega_h C - \frac{1}{\omega_h L} = \frac{1}{R} \quad ; \quad \omega_l C - \frac{1}{\omega_l L} = -\frac{1}{R}$$

On multiplying the equation (5.107) by $\frac{\omega}{C}$, we get,

$$\omega^2 - \frac{1}{LC} = \pm\frac{1}{RC}\omega \quad \Rightarrow \quad \omega^2 \mp \frac{1}{RC}\omega - \frac{1}{LC} = 0$$

$$\therefore \quad \omega^2 - \frac{1}{RC}\omega - \frac{1}{LC} = 0 \quad \text{and} \quad \omega^2 + \frac{1}{RC}\omega - \frac{1}{LC} = 0$$

The roots of quadratic

$$\omega^2 - \frac{1}{RC}\omega - \frac{1}{LC} = 0 \text{ are,}$$

$$\omega = \frac{\frac{1}{RC} \pm \sqrt{\left(\frac{1}{RC}\right)^2 + \frac{4}{LC}}}{2}$$

$$= \frac{1}{2RC} \pm \frac{1}{2}\sqrt{4\left[\frac{1}{4}\left(\frac{1}{RC}\right)^2 + \frac{1}{LC}\right]}$$

$$= \frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

The roots of quadratic

$$\omega^2 + \frac{1}{RC}\omega - \frac{1}{LC} = 0 \text{ are,}$$

$$\omega = \frac{-\frac{1}{RC} \pm \sqrt{\left(\frac{1}{RC}\right)^2 + \frac{4}{LC}}}{2}$$

$$= -\frac{1}{2RC} \pm \frac{1}{2}\sqrt{4\left[\frac{1}{4}\left(\frac{1}{RC}\right)^2 + \frac{1}{LC}\right]}$$

$$= -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

The cut-off frequencies are given by the positive roots of the two quadratic.

$$\boxed{\therefore \text{ Higher cut-off angular frequency}, \omega_h = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \text{ in rad/s}} \quad ..... (5.108)$$

$$\boxed{\text{Lower cut-off angular frequency}, \omega_l = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \text{ in rad/s}} \quad ..... (5.109)$$

Since, $\omega = 2\pi f$ and $f = \frac{\omega}{2\pi}$, the cut-off frequency in Hz can be expressed as shown below:

$$\boxed{\therefore \text{ Higher cut-off frequency}, f_h = \frac{1}{2\pi}\left[\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}\right] \text{in Hz}} \quad ..... (5.110)$$

$$\boxed{\text{Lower cut-off frequency}, f_l = \frac{1}{2\pi}\left[-\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}\right] \text{in Hz}} \quad ..... (5.111)$$

The bandwidth, β is given by difference between cut-off frequencies.

$$\therefore \text{ Bandwidth}, \beta = \omega_h - \omega_l$$

$$= \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} - \left[-\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}\right]$$

$$= \frac{1}{2RC} + \frac{1}{2RC} = \frac{1}{RC} \text{ in rad/s} \quad ..... (5.112)$$

$$\therefore \text{ Bandwidth in Hz} = \frac{\beta}{2\pi} = \frac{1}{2\pi RC} \text{ in Hz} \quad ..... (5.113)$$

Alternatively, bandwidth and cut-off frequencies can be expressed in terms of angular resonant frequency, ω_r and quality factor, Q_r as shown below:

From equation (5.112), we get, $\beta = \frac{1}{RC}$

From equation (5.81), we get, $Q_r = \omega_r CR \;\Rightarrow\; \frac{1}{RC} = \frac{\omega_r}{Q_r}$

On comparing the above two equations, we get,

$$\beta = \frac{\omega_r}{Q_r} \quad ..... (5.114)$$

From equation (5.108), we get,

$$\boxed{\omega_r = \frac{1}{\sqrt{LC}}} \quad \boxed{\beta = \frac{1}{RC}}$$

$$\boxed{\beta = \frac{\omega_r}{Q_r}}$$

$$\omega_h = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} = \frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} + \omega_r^2}$$

$$= \frac{\omega_r}{2Q_r} + \sqrt{\frac{\omega_r^2}{4Q_r^2} + \omega_r^2} = \frac{\omega_r}{2Q_r} + \sqrt{\omega_r^2\left(1 + \frac{1}{4Q_r^2}\right)}$$

$$= \frac{\omega_r}{2Q_r} + \omega_r\sqrt{1 + \frac{1}{4Q_r^2}} = \omega_r\left[\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \quad ..... (5.115)$$

Similarly,

$$\omega_l = \omega_r\left[-\frac{1}{2Q_r}+\sqrt{1+\frac{1}{4Q_r^2}}\right] \qquad \text{..... (5.116)}$$

On multiplying equations (5.115) and (5.116), we get,

$$\omega_h\omega_l = \omega_r\left[\frac{1}{2Q_r}+\sqrt{1+\frac{1}{4Q_r^2}}\right] \times \omega_r\left[-\frac{1}{2Q_r}+\sqrt{1+\frac{1}{4Q_r^2}}\right]$$

$$= \omega_r^2\left[\sqrt{1+\frac{1}{4Q_r^2}}+\frac{1}{2Q_r}\right]\left[\sqrt{1+\frac{1}{4Q_r^2}}-\frac{1}{2Q_r}\right]$$

$$= \omega_r^2\left[1+\frac{1}{4Q_r^2}-\frac{1}{4Q_r^2}\right] \qquad \boxed{(a+b)\,(a-b) = a^2-b^2}$$

$$= \omega_r^2$$

$$\therefore\ \omega_r = \sqrt{\omega_h\omega_l} \qquad \text{..... (5.117)}$$

From equation (5.117) we can say that, resonant frequency is given by the geometric mean of the cut-off frequencies.

Bandwidth Based on Voltage Response

Consider a parallel RLC circuit excited by a constant current source as shown in Fig. 5.22(a). Now, the voltage response will be as shown in Fig. 5.22(b). The voltage response is maximum at resonance and it decreases for increasing or decreasing frequency from the resonance value. Since current is maximum, power is also maximum at resonance.

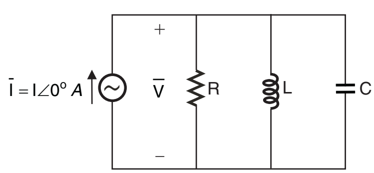


Fig. a : *RLC parallel circuit.*

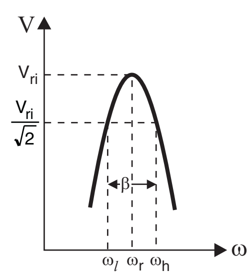


Fig. b : *Voltage response.*

Fig. 5.22 : *Voltage response of an RLC parallel circuit excited by a current source.*

Therefore, when frequency is decreased from the resonant value, we come across a frequency at which power is 1/2 times the maximum power (or voltage is $1/\sqrt{2}$ times the maximum value) and this frequency is called **lower cut-off frequency**, ω_l. When frequency is increased from the resonant value, we come across a frequency at which power is 1/2 times the maximum power (or voltage is $1/\sqrt{2}$ times the maximum value) and this frequency is called **higher cut-off frequency**, ω_h.

Now, bandwidth based on voltage response can be defined as the range of frequencies over which power is greater than or equal to 1/2 times the maximum power.

Alternatively, bandwidth based on voltage response can be defined as the range of frequencies over which voltage is greater than or equal to $1/\sqrt{2}$ times the maximum voltage.

Bandwidth is given by the difference between cut-off frequencies and is denoted by β. The unit of bandwidth is *rad/s* or *Hz*.

The expression for cut-off frequencies and bandwidth in an RLC parallel circuit is independent of the excitation source and depends only on parameters R, L and C. Therefore, equations (5.94) to (5.102) for cut-off frequencies and bandwidth are applicable for an RLC parallel circuit excited by a current source.

Proof for cut-off frequency and bandwidth based on voltage response

Consider an RLC parallel circuit shown in Fig. 5.23, excited by a sinusoidal current source of frequency, ω.

Let, $\bar{I} = I\angle 0^o$ = *Supply current.*

$\bar{V}$ = *Voltage across parallel combination of R, L and C.*

$\bar{Y}$ = *Admittance of RLC parallel circuit.*

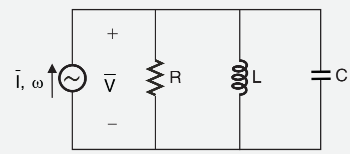


Fig. 5.23.

Here, $\bar{Y} = \frac{1}{R} - j\frac{1}{\omega L} + j\omega C = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$

$$\therefore\ Y = \left|\bar{Y}\right| = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} \qquad \text{..... (5.118)}$$

Here, $\bar{V} = \frac{\bar{I}}{\bar{Y}}$

$$\therefore\ V = \left|\bar{V}\right| = \left|\frac{\bar{I}}{\bar{Y}}\right| = \frac{\left|\bar{I}\right|}{\left|\bar{Y}\right|} = \frac{I}{Y}$$

$$\therefore\ V = \frac{I}{\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}} \qquad \text{..... (5.119)}$$

Using equation (5.118)

Let, P = *Power in RLC circuit.*

P_r = *Power in RLC circuit at resonance.*

$$\text{Here, } P = \frac{V^2}{R} = \frac{1}{R}\frac{I^2}{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} \qquad \text{..... (5.120)}$$

Using equation (5.119)

At resonance, $\omega C = \frac{1}{\omega L}$. *On substituting this condition in equation (5.120), we get power at resonance,* P_r.

$$\therefore\ P_r = \frac{1}{R}\frac{I^2}{\frac{1}{R^2}} = I^2 R \qquad \text{..... (5.121)}$$

At half-power frequencies or cut-off frequencies, power will be equal to 1/2 times the power at resonance.

$$\therefore\ P = \frac{1}{2}P_r$$

$$\frac{1}{R}\,\frac{I^2}{\dfrac{1}{R^2}+\left(\omega C-\dfrac{1}{\omega L}\right)^2} = \frac{1}{2}I^2R$$

Using equations (5.120) and (5.121)

$$\therefore\ \frac{1}{\dfrac{1}{R^2}+\left(\omega C-\dfrac{1}{\omega L}\right)^2} = \frac{R^2}{2}$$

On inverting the above equation, we get,

$$\frac{1}{R^2}+\left(\omega C-\frac{1}{\omega L}\right)^2 = \frac{2}{R^2} \quad\Rightarrow\quad \left(\omega C-\frac{1}{\omega L}\right)^2 = \frac{2}{R^2}-\frac{1}{R^2}$$

$$\therefore\ \left(\omega C-\frac{1}{\omega L}\right)^2 = \frac{1}{R^2}$$

On taking square root of the above equation, we get,

$$\omega C-\frac{1}{\omega L} = \pm\frac{1}{R} \qquad \text{..... (5.122)}$$

Note : *Equation (5.122) implies that the absolute value of the total susceptance at cut-off frequencies is equal to the conductance of the circuit.*

Equation (5.122) is same as equation (5.107), and so the rest of the proof will be similar to the proof of cut-off frequency and bandwidth based on current response.

5.3.5 Solved Problems in Parallel Resonance

EXAMPLE 5.7

The parameters of an RLC parallel circuit excited by a current source are R = 40 Ω, L = 2 *mH* and C = 3 *μF*. Determine the resonant frequency, quality factor, bandwidth and cut-off frequencies.

SOLUTION

Given that, R = 40 Ω, L = 2 *mH* and C = 3 *μF*

$$\text{Angular resonant frequency, } \omega_r = \frac{1}{\sqrt{LC}}$$

$$= \frac{1}{\sqrt{2\times10^{-3}\times3\times10^{-6}}} = 12909.9445 \approx 12910\ rad/s$$

$$\text{Resonant frequency, } f_r = \frac{\omega_r}{2\pi} = \frac{12910}{2\pi} = 2054.7\ Hz = \frac{2054.7}{1000}\ kHz = 2.0547\ kHz$$

$$\text{Quality factor at resonance, } Q_r = \frac{R}{\omega_r L} = \frac{40}{12910\times2\times10^{-3}} = 1.5492$$

$$\text{Bandwidth, } \beta = \frac{1}{RC} = \frac{1}{40\times3\times10^{-6}} = 8333.3333\ rad/s \approx 8333\ rad/s$$

$$\text{Bandwidth in } Hz = \frac{\beta}{2\pi} = \frac{8333.3333}{2\pi} = 1326.3\,Hz = \frac{1326.3}{1000}\,kHz = 1.3263\,kHz$$

$$\text{Higher cut-off frequecy,} \quad f_h = f_r\left[\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right]$$

$$= 2.0547 \times \left[\frac{1}{2 \times 1.5492} + \sqrt{1 + \frac{1}{4 \times 1.5492^2}}\right]$$

$$= 2.8222\,kHz$$

$$\text{Lower cut-off frequecy,} \quad f_l = f_r\left[-\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right]$$

$$= 2.0547 \times \left[-\frac{1}{2 \times 1.5492} + \sqrt{1 + \frac{1}{4 \times 1.5492^2}}\right]$$

$$= 1.4959\,kHz$$

EXAMPLE 5.8

(AU May'17, 8 Marks)

The RLC parallel circuit shown in Fig. 1, consists of R = 8 kΩ, L = 0.2 *mH* and C = 8 *μF*. Determine the resonant frequency, quality factor, bandwidth and cut-off frequencies.

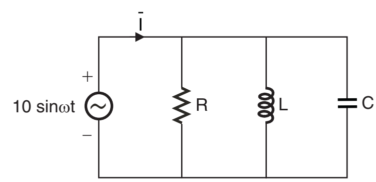


Fig. 1.

SOLUTION

Given that, R = 8 kΩ, L = 0.2 *mH* and C = 8 *μF*

$$\text{Resonant frequency in } rad/sec, \quad \omega_r = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2 \times 10^{-3} \times 8 \times 10^{-6}}} = 25000\,rad/s$$

$$\text{Quality factor at resonance, } Q_r = \omega_r CR = 25000 \times 8 \times 10^{-6} \times 8 \times 10^{3} = 1600$$

$$\text{Bandwidth, } \beta = \frac{\omega_r}{Q_r} = \frac{25000}{1600} = 15.625\,rad/s$$

$$\text{Higher cut-off frequecy,} \quad \omega_h = \omega_r\left[\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right]$$

$$= 25000 \times \left[\frac{1}{2 \times 1600} + \sqrt{1 + \frac{1}{4 \times 1600^2}}\right]$$

$$= 25007.8\,rad/s$$

$$\text{Lower cut-off frequecy,} \quad \omega_l = \omega_r\left[-\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right]$$

$$= 25000 \times \left[-\frac{1}{2 \times 1600} + \sqrt{1 + \frac{1}{4 \times 1600^2}}\right]$$

$$= 24992.2\,rad/s$$

EXAMPLE 5.9

A coil of inductance 31.8 *mH* and resistance 10 Ω is connected in parallel with a capacitor across a 250 *V*, 50 *Hz* supply. Determine the value of the capacitor if no reactive current is taken from the supply.

SOLUTION

The parallel combination of the coil and capacitor excited by the voltage source is shown in Fig. 1.

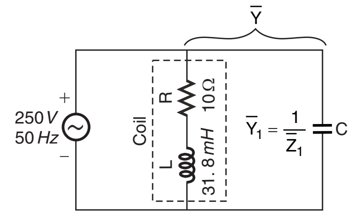


Fig. 1.

Let, $\overline{Y}$ = Admittance of parallel combination of coil and capacitor.

$\overline{Z}_1$ = Impedance of coil.

$\overline{Y}_1 = \dfrac{1}{\overline{Z}_1}$ = Admittance of coil.

B_C = Susceptance of capacitor.

The current supplied by the source does not have any reactive component. This happens only at resonance. At resonance, the circuit behaves as a purely resistive, which means that the imaginary part of admittance $\overline{Y}$ is zero. Therefore, the value of the capacitor can be determined by equating the imaginary part of admittance to zero.

$$\therefore\ \overline{Y} = \overline{Y}_1 + jB_C = \frac{1}{\overline{Z}_1} + jB_C = \frac{1}{R + j\omega L} + jB_C = \frac{1}{R + j2\pi fL} + jB_C$$

$$= \frac{1}{10 + j(2\pi \times 50 \times 31.8 \times 10^{-3})} + jB_C$$

$$= 0.05 - j0.05 + jB_C = 0.05 + j(B_C - 0.05)$$

On letting the imaginary part of admittance to zero, we get,

$$B_C - 0.05 = 0 \quad \Rightarrow \quad B_C = 0.05$$

Since, $B_C = \omega C$ we get,

$$C = \frac{B_C}{\omega} = \frac{B_C}{2\pi f} = \frac{0.05}{2\pi \times 50} = 1.5915 \times 10^{-4}\,F$$

$$= 159.15 \times 10^{-6}\,F = 159.15\,\mu F$$

EXAMPLE 5.10

In the RLC network shown in Fig. 1, determine the value of R_C for resonance. Also calculate the dynamic resistance.

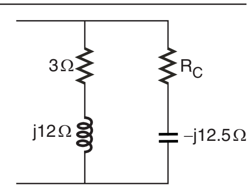


Fig. 1.

SOLUTION

The given network has two parallel branches and the admittances of the parallel branches are named as shown in Fig. 2

Let, $\overline{Z}_1, \overline{Z}_2$ = Impedance of parallel branches.

$\overline{Y}_1, \overline{Y}_2$ = Admittance of parallel branches.

$\overline{Y}$ = Total admittance of the RLC parallel network.

$$\boxed{\overline{Y}_1 = \frac{1}{\overline{Z}_1}} \quad \boxed{\overline{Y}_2 = \frac{1}{\overline{Z}_2}}$$

Now, $\overline{Y} = \overline{Y}_1 + \overline{Y}_2 = \dfrac{1}{\overline{Z}_1} + \dfrac{1}{\overline{Z}_2} = \dfrac{1}{3 + j12} + \dfrac{1}{R_C - j12.5}$

Let us separate the real and imaginary parts of the admittance by multiplying the numerator and denominator of each term of the admittance by the complex conjugate of the denominator.

$$\therefore\ \bar{Y} = \frac{1}{3+j12} \times \frac{3-j12}{3-j12} + \frac{1}{R_C - j12.5} \times \frac{R_C + j12.5}{R_C + j12.5}$$

$$= \frac{3-j12}{3^2+12^2} + \frac{R_C + j12.5}{R_C^2 + 12.5^2} = \frac{3-j12}{153} + \frac{R_C + j12.5}{R_C^2 + 156.25}$$

$$= \frac{3}{153} - j\frac{12}{153} + \frac{R_C}{R_C^2+156.25} + j\frac{12.5}{R_C^2+156.25}$$

$$= 0.0196 - j0.0784 + \frac{R_C}{R_C^2+156.25} + j\frac{12.5}{R_C^2+156.25}$$

$$= \left(0.0196 + \frac{R_C}{R_C^2+156.25}\right) + j\left(\frac{12.5}{R_C^2+156.25} - 0.0784\right)$$

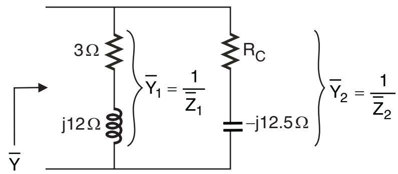


Fig. 2.

At resonance, the imaginary part of the admittance $\bar{Y}$ will be zero.

$$\therefore\ \frac{12.5}{R_C^2+156.25} - 0.0784 = 0 \quad \Rightarrow \quad \frac{12.5}{R_C^2+156.25} = 0.0784$$

$$\therefore\ \frac{12.5}{0.0784} = R_C^2 + 156.25 \quad \Rightarrow \quad R_C^2 = \frac{12.5}{0.0784} - 156.25$$

$$\therefore\ R_C = \sqrt{\frac{12.5}{0.0784} - 156.25} = 1.7857\,\Omega$$

Dynamic resistance is given by the inverse of the real part of the admittance at resonance.

$$\therefore\ R_{dynamic} = \frac{1}{0.0196 + \dfrac{R_C}{R_C^2+156.25}} = \frac{1}{0.0196 + \dfrac{1.7857}{1.7857^2 + 156.25}} = 32.4676\,\Omega$$

RESULT

For resonance, $R_C = 1.7857\,\Omega$

Dynamic resistance, $R_{dynamic} = 32.4676\,\Omega$

EXAMPLE 5.11

Determine the value of R_L for resonance in the network shown in Fig. 1. Also calculate the dynamic resistance.

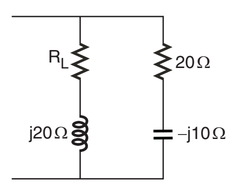


Fig. 1.

SOLUTION

The given network has two parallel branches and the admittances of the parallel branches are named as shown in Fig. 2.

Let, $\bar{Z}_1, \bar{Z}_2$ = Impedance of parallel branches

$\bar{Y}_1, \bar{Y}_2$ = Admittance of parallel branches

$\bar{Y}$ = Total admittance of the RLC parallel network.

$$\text{Now,}\ \bar{Y} = \bar{Y}_1 + \bar{Y}_2 = \frac{1}{\bar{Z}_1} + \frac{1}{\bar{Z}_2} = \frac{1}{R_L + j20} + \frac{1}{20 - j10}$$

$$\boxed{\bar{Y}_1 = \frac{1}{\bar{Z}_1}} \quad \boxed{\bar{Y}_2 = \frac{1}{\bar{Z}_2}}$$

Let us separate the real and imaginary parts of the admittance by multiplying the numerator and denominator of each term of the admittance by the complex conjugate of the denominator.

$$\therefore\ \bar{Y} = \frac{1}{R_L + j20} \times \frac{R_L - j20}{R_L - j20} + \frac{1}{20 - j10} \times \frac{20 + j10}{20 + j10}$$

$$= \frac{R_L - j20}{R_L^2 + 20^2} + \frac{20 + j10}{20^2 + 10^2} = \frac{R_L - j20}{R_L^2 + 400} + \frac{20 + j10}{500}$$

$$= \frac{R_L}{R_L^2 + 400} - j\frac{20}{R_L^2 + 400} + \frac{20}{500} + j\frac{10}{500}$$

$$= \frac{R_L}{R_L^2 + 400} + 0.04 + j\left(0.02 - \frac{20}{R_L^2 + 400}\right)$$

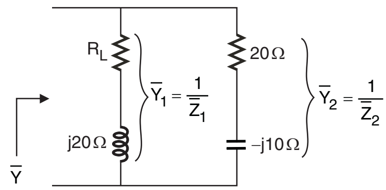


Fig. 2.

At resonance, the imaginary part of the admittance $\bar{Y}$ will be zero.

$$\therefore\ 0.02 - \frac{20}{R_L^2 + 400} = 0 \quad \Rightarrow \quad 0.02 = \frac{20}{R_L^2 + 400}$$

$$\therefore\ R_L^2 + 400 = \frac{20}{0.02} \quad \Rightarrow \quad R_L^2 = \frac{20}{0.02} - 400$$

$$\therefore\ R_L = \sqrt{\frac{20}{0.02} - 400} = 24.4949\,\Omega$$

Dynamic resistance is given by the inverse of the real part of the admittance at resonance.

$$\therefore\ R_{dynamic} = \frac{1}{\dfrac{R_L}{R_L^2 + 400} + 0.04} = \frac{1}{\dfrac{24.4949}{24.4949^2 + 400} + 0.04} = 15.5051\,\Omega$$

RESULT

For resonance, R_L = 24.4949 Ω

Dynamic resistance, $R_{dynamic}$ = 15.5051 Ω

EXAMPLE 5.12 *(AU Dec'14, 16 Marks)*

Determine the equivalent parallel network for a series RL combination.

SOLUTION

The series RL network excited by source $\bar{E}$ and its parallel equivalent are shown in Figs 1 and 2. The networks are equivalent if the impedances (or the admittances) with respect to source terminals are equal.

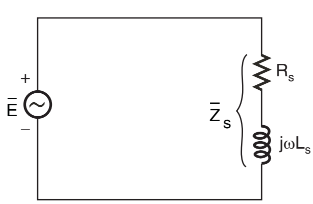


Fig. 1.

≡

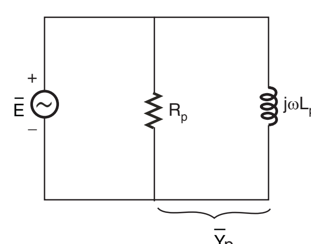


Fig. 2.

With reference to Fig. 1.

$$\bar{Y}_s = \frac{1}{\bar{Z}_s} = \frac{1}{R_s + j\omega L_s}$$

$$\therefore \bar{Y}_s = \frac{1}{R_s + j\omega L_s} \times \frac{R_s - j\omega L_s}{R_s - j\omega L_s}$$

Multiplying numerator and denominator by conjugate of denominator.

$$= \frac{R_s - j\omega L_s}{R_s^2 + \omega^2 L_s^2}$$

$$= \frac{R_s}{R_s^2 + \omega^2 L_s^2} - j\frac{\omega L_s}{R_s^2 + \omega^2 L_s^2} \qquad .....(1)$$

With reference to Fig. 2.

$$\bar{Y}_p = \frac{1}{R_p} + \frac{1}{j\omega L_p} \qquad .....(2)$$

On equating the real part of equations (1) and (2), we get,

$$\frac{1}{R_p} = \frac{R_s}{R_s^2 + \omega^2 L_s^2}$$

$$\therefore R_p = \frac{R_s^2 + \omega^2 L_s^2}{R_s} = R_s + \frac{\omega^2 L_s^2}{R_s}$$

On equating the imaginary part of equations (1) and (2), we get,

$$\frac{1}{j\omega L_p} = -j\frac{\omega L_s}{R_s^2 + \omega^2 L_s^2}$$

$$\therefore L_p = \frac{R_s^2 + \omega^2 L_s^2}{\omega^2 L_s} = L_s + \frac{R_s^2}{\omega^2 L_s}$$

EXAMPLE 5.13

In the RLC network shown in Fig. 1, determine the two possible values of C for the network to resonate at 2000 *rad/s*. Also, determine the value of C for resonance at all frequencies.

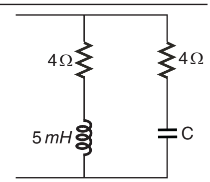


Fig. 1.

SOLUTION

The given network has two parallel branches and the admittance of the parallel branches are named as shown in Fig. 2.

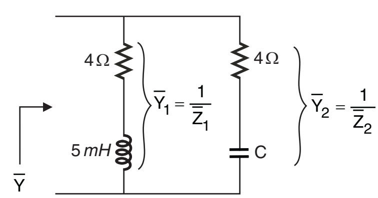


Fig. 2.

Let, $\bar{Z}_1, \bar{Z}_2$ = Impedance of parallel branches

$\bar{Y}_1, \bar{Y}_2$ = Admittance of parallel branches

$\bar{Y}$ = Total admittance of the RLC parallel network.

$$\boxed{\bar{Y}_1 = \frac{1}{\bar{Z}_1}} \quad \boxed{\bar{Y}_2 = \frac{1}{\bar{Z}_2}}$$

Now, $\bar{Y} = \bar{Y}_1 + \bar{Y}_2 = \dfrac{1}{\bar{Z}_1} + \dfrac{1}{\bar{Z}_2} = \dfrac{1}{R + jX_L} + \dfrac{1}{R - jX_C}$

$$= \frac{1}{R + j\omega L} + \frac{1}{R - jX_C} = \frac{1}{4 + j2000 \times 5 \times 10^{-3}} + \frac{1}{4 - jX_C}$$

$$= \frac{1}{4 + j10} + \frac{1}{4 - jX_C}$$

Let us separate the real and imaginary parts of the admittance by multiplying the numerator and denominator of each term of the admittance by the complex conjugate of the denominator.

$$\therefore\ \overline{Y} = \frac{1}{4+j10} \times \frac{4-j10}{4-j10} + \frac{1}{4-jX_C} \times \frac{4+jX_C}{4+jX_C}$$

$$= \frac{4-j10}{4^2+10^2} + \frac{4+jX_C}{4^2-X_C^2} = \frac{4-j10}{116} + \frac{4+jX_C}{16+X_C^2}$$

$$= \frac{4}{116} - j\frac{10}{116} + \frac{4}{16+X_C^2} + j\frac{X_C}{16+X_C^2}$$

$$= \left(0.0345 + \frac{4}{16+X_C^2}\right) + j\left(\frac{X_C}{16+X_C^2} - \frac{10}{116}\right)$$

At resonance, the imaginary part of the admittance $\overline{Y}$ will be zero.

$$\therefore\ \frac{X_C}{16+X_C^2} - \frac{10}{116} = 0 \quad \Rightarrow \quad \frac{X_C}{16+X_C^2} = \frac{10}{116}$$

$$\therefore\ \frac{116}{10}X_C = 16 + X_C^2 \quad \Rightarrow \quad 11.6X_C = 16 + X_C^2$$

$$\therefore\ X_C^2 - 11.6X_C + 16 = 0$$

The roots of the quadratic equation are,

$$X_C = \frac{-(-11.6) \pm \sqrt{(-11.6)^2 - 4\times 16}}{2}$$

$$= \frac{11.6 \pm 8.4}{2} = 10 \text{ or } 1.6\,\Omega$$

We know that, $X_C = \frac{1}{\omega C}$, $\therefore\ C = \frac{1}{\omega X_C}$

When, $X_C = 10\,\Omega$, $\quad C = \frac{1}{\omega X_C} = \frac{1}{2000\times 10} = 5\times 10^{-5}\,F$

$$= 50\times 10^{-6}\,F = 50\,\mu F$$

When, $X_C = 1.6\,\Omega$, $\quad C = \frac{1}{\omega X_C} = \frac{1}{2000\times 1.6} = 3.125\times 10^{-4}\,F$

$$= 312.5\times 10^{-6}\,F = 312.5\,\mu F$$

The condition for resonance at all frequencies is, $R = \sqrt{\frac{L}{C}}$.

$$\therefore\ R^2 = \frac{L}{C} \quad \Rightarrow \quad C = \frac{L}{R^2}$$

The value of C for resonance at all frequencies is, $C = \frac{L}{R^2} = \frac{5\times 10^{-3}}{4^2} = 3.125\times 10^{-4}\,F$

$$= 312.5\times 10^{-6}\,F = 312.5\,\mu F$$

RESULT

Two possible values of C for resonance at 2000 *rad/s* $= 50\,\mu F$ and $312.5\,\mu F$

The value of C for resonance at all frequencies $= 312.5\,\mu F$

EXAMPLE 5.14

In the RLC network shown in Fig. 1, determine the two possible values of L for the network to resonate at 4000 *rad/s.*

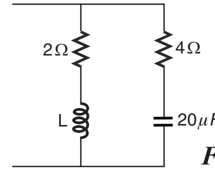


Fig. 1.

SOLUTION

The given network has two parallel branches and the admittances of the parallel branches are named as shown in Fig. 2.

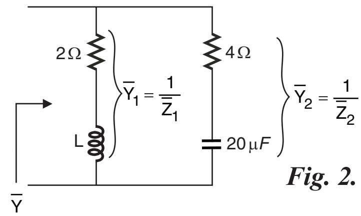


Fig. 2.

Let, $\overline{Z}_1, \overline{Z}_2$ = Impedance of the parallel branches

$\overline{Y}_1, \overline{Y}_2$ = Admittance of parallel branches

$\overline{Y}$ = Total admittance of the RLC parallel network.

Now, $$\overline{Y} = \overline{Y}_1 + \overline{Y}_2 = \frac{1}{\overline{Z}_1} + \frac{1}{\overline{Z}_2} = \frac{1}{R_1 + jX_L} + \frac{1}{R_2 - jX_C}$$ $$\boxed{\overline{Y}_1 = \frac{1}{\overline{Z}_1}} \quad \boxed{\overline{Y}_2 = \frac{1}{\overline{Z}_2}}$$

$$= \frac{1}{R_1 + jX_L} + \frac{1}{R_2 - j\frac{1}{\omega C}} = \frac{1}{2 + jX_L} + \frac{1}{4 - j\frac{1}{4000 \times 20 \times 10^{-6}}}$$

$$= \frac{1}{2 + jX_L} + \frac{1}{4 - j12.5}$$

Let us separate the real and imaginary parts of the admittance by multiplying the numerator and denominator of each term of the admittance by the complex conjugate of the denominator.

$$\overline{Y} = \frac{1}{2 + jX_L} \times \frac{2 - jX_L}{2 - jX_L} + \frac{1}{4 - j12.5} \times \frac{4 + j12.5}{4 + j12.5}$$

$$= \frac{2 - jX_L}{2^2 + X_L^2} + \frac{4 + j12.5}{4^2 + 12.5^2} = \frac{2 - jX_L}{4 + X_L^2} + \frac{4 + j12.5}{172.25}$$

$$= \frac{2}{4 + X_L^2} - j\frac{X_L}{4 + X_L^2} + \frac{4}{172.25} + j\frac{12.5}{172.25}$$

$$= \frac{2}{4 + X_L^2} - j\frac{X_L}{4 + X_L^2} + 0.0232 + j0.0726$$

$$= \left(0.0232 + \frac{2}{4 + X_L^2}\right) + j\left(j0.0726 - \frac{X_L}{4 + X_L^2}\right)$$

At resonance, the imaginary part of the admittance $\overline{Y}$ will be zero.

$$\therefore \quad 0.0726 - \frac{X_L}{4 + X_L^2} = 0 \quad \Rightarrow \quad 0.0726 = \frac{X_L}{4 + X_L^2}$$

$$\therefore \quad 4 + X_L^2 = \frac{X_L}{0.0726} \quad \Rightarrow \quad 4 + X_L^2 = 13.7741\,X_L$$

$$\therefore \quad X_L^2 - 13.7741\,X_L + 4 = 0$$

The roots of the quadratic equation are,

$$X_L = \frac{-(-13.7741) \pm \sqrt{(-13.7741)^2 - 4 \times 4}}{2}$$

$$= \frac{13.7741 \pm 13.1805}{2} = 13.4773\,\Omega \quad \text{or} \quad 0.2968\,\Omega$$

We know that, $X_L = \omega L$, $\therefore\ L = \dfrac{X_L}{\omega}$

When, $X_L = 13.4773\,\Omega,\quad L = \dfrac{X_L}{\omega} = \dfrac{13.4773}{4000} = 3.3693 \times 10^{-3}\,H = 3.3693\,mH$

When, $X_L = 0.2968\,\Omega,\quad L = \dfrac{X_L}{\omega} = \dfrac{0.2968}{4000} = 7.42 \times 10^{-5}\,H = 74.2 \times 10^{-6}\,H = 74.2\,\mu H$

RESULT

Two possible values of L for resonance at 4000 *rad/s* = 3.3693 *mH* and 74.2 μ*H*

5.4 Coupled Circuits

Coupled circuits refer to circuits involving elements with magnetic coupling. If the flux produced by an element of a circuit links (or pass through) other elements of the same circuit or a nearby circuit then the elements are said to have **magnetic coupling** (or connection through magnetic flux). Likewise, circuits with a number of branches and loops are said to have **conductive coupling**.

In magnetically coupled elements, the power (or energy) transfer occurs through the **magnetic flux**. In conductively coupled elements, the power (or energy) transfer takes place through the current. In general, **coupled circuits** refer to circuits with magnetically coupled elements.

Whenever current passes through a coil flux is set up in the coil. If the current is varying then the flux will also follow the variations in the current. Due to change in flux, an emf will be induced in the coil. The direction of the induced emf will be such as to oppose the current through the coil. This emf is called **self-induced emf**.

When a circuit has two or more coils then there is a possibility that the flux produced by one coil links the other coils. If the flux produced by coil-1 of a circuit links (or pass through) coil-2 of the circuit then an emf is induced in coil-2 due to change in flux (or current) in coil-1. This emf is called **mutual induced emf** . The term **mutual** is used here for induced emf because the action is reversible, i.e., if the flux produced by coil-2 links coil-1 then a change in flux in coil-2 will induce an emf in coil-1. The two coils linked by magnetic flux are called **coupled coils**.

A transformer is the best example of a coupled circuit. A transformer consists of two coils wound on a common core. The two coils are electrically isolated but linked magnetically. The two coils are called **primary winding** and **secondary winding**. When the primary is connected to an ac source, the current flows through the primary which creates flux. Since the coils are wound on a common core, the flux produced by primary current links both the coils. Here, the current is sinusoidal in nature and so the flux will also be sinusoidal. The **sinusoidal flux** varies with respect to time and so emfs are induced in primary and secondary.

The induced emf in primary is called self-induced emf and the induced emf in secondary is called mutual induced emf. The mutual induced emf acts as a source for the load connected to secondary. Hence, the power flows from primary to secondary through magnetic flux linking primary and secondary.

Note : *A transformer is a linear device as long as the flux in the core is not saturated. Normally, the flux density of a practical transformer is maintained to avoid saturation and so the transformer is also called linear transformer.*

5.5 Self-Inductance and Mutual Inductance

5.5.1 Self-Inductance

Consider a coil with N turns carrying a current i as shown in Fig. 5.24. Let ϕ be the flux in the coil. When the current in the coil is variable in nature, the flux ϕ also varies and so an emf is induced in the coil in a direction opposing the current flow. This emf is called self-induced emf.

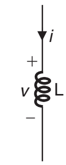


Fig. 5.24.

Self-induced emf is directly proportional to the rate of change of current and the constant of proportionality is **self-inductance**, L. (Refer to Chapter 4, Section 4.4, equation (4.8)).

$$\therefore \quad v \propto \frac{di}{dt}$$

$$\text{and} \quad v = L\frac{di}{dt} \qquad \text{..... (5.123)}$$

When the permeability is constant, self-inductance is given by the ratio of weber-turn and current. (Refer to Chapter 4, Section 4.4, equation (4.7)).

$$\boxed{\therefore \quad \text{Self - inductance, } L = \frac{N\phi}{i}} \qquad \text{..... (5.124)}$$

5.5.2 Mutual Inductance *(AU Dec'14, 6 Marks)* *(AU May'15, 16 Marks)*

Consider two coils with turns N_1 and N_2 placed very close to each other as shown in Fig. 5.25. Let L_1 and L_2 be the self-inductances of the coils 1 and 2, respectively.

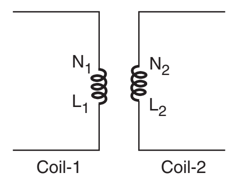


Fig. 5.25.

Let us excite the coil-1 by connecting a voltage source across its terminals as shown in Fig. 5.26. Now, current passes through coil-1 and so flux is set up in coil-1. Since coil-2 is placed very close to coil-1, a part of this flux will also pass through (or links) coil-2.

Let, i_1 = Current through coil-1

ϕ_1 = Flux set-up in coil-1

ϕ_{12} = Flux produced by coil-1 linking coil-2.

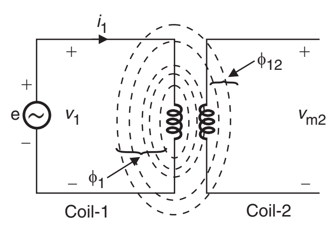


Fig. 5.26.

When the current i_1 is variable, the flux will also be variable. Now an emf is induced in coil-1 due to ϕ_1, which is called self-induced emf and an emf is induced in coil-2 due to ϕ_{12},which is called mutual induced emf.

Self-induced emf is proportional to the rate of change of current in coil-1 and the constant of proportionality is self-inductance of coil-1(i.e., L_1).

$$\boxed{\therefore \quad \text{Self - induced emf in coil-1,} \quad v_1 = L_1\frac{di_1}{dt}} \qquad \text{..... (5.125)}$$

$$\text{where,} \quad \boxed{L_1 = \frac{N_1\phi_1}{i_1}} \qquad \text{..... (5.126)}$$

Mutual induced emf in coil-2 is proportional to the rate of change of current in coil-1 and the constant of proportionality is the mutual inductance between coils 1 and 2.

Let, M_{12} = Mutual inductance between coils 1 and 2.

$$\boxed{\therefore \text{ Mutual induced emf in coil-2, } \quad v_{m2} = M_{12}\frac{di_1}{dt}} \qquad \text{..... (5.127)}$$

Similar to self-inductance, mutual inductance is also given by the ratio of weber-turn (flux linkage) and current.

$$\boxed{\therefore \; M_{12} = \frac{N_2 \phi_{12}}{i_1}} \qquad \text{..... (5.128)}$$

In another case, let us excite coil-2 by connecting a voltage source across its terminals as shown in Fig. 5.27. Now current passes through coil-2 and so flux is set up in coil-2. Since coil-1 is placed very close to coil-2, a part of this flux will also pass through (or links) coil-1.

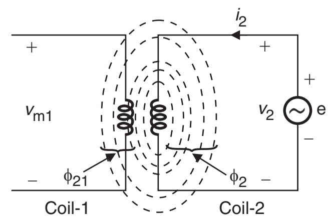


Fig. 5.27.

Let, i_2 = Current through coil-2

ϕ_2 = Flux set-up in coil-2

ϕ_{21} = Flux produced by coil-2 linking coil-1.

When the current i_2 is variable, the flux will also be variable. Now an emf is induced in coil-2 due to ϕ_2, which is called self-induced emf and an emf is induced in coil-1 due to ϕ_{21}, which is called mutual induced emf.

Self-induced emf is proportional to the rate of change of current in coil-2 and the constant of proportionality is self-inductance of coil-2 (i.e., L_2).

$$\boxed{\therefore \text{ Self - induced emf in coil-2, } \quad v_2 = L_2\frac{di_2}{dt}} \qquad \text{..... (5.129)}$$

$$\text{where, } \boxed{L_2 = \frac{N_2 \phi_2}{i_2}} \qquad \text{..... (5.130)}$$

Mutual induced emf in coil-1 is proportional to the rate of change of current in coil-2 and the constant of proportionality is the mutual inductance between coils 2 and 1.

Let, M_{21} = Mutual inductance between coils 2 and 1.

$$\boxed{\therefore \text{ Mutual induced emf in coil-1, } \quad v_{m1} = M_{21}\frac{di_2}{dt}} \qquad \text{..... (5.131)}$$

Similar to self-inductance, mutual inductance is also given by the ratio of weber-turn (flux linkage) and current.

$$\boxed{\therefore \; M_{21} = \frac{N_1 \phi_{21}}{i_2}} \qquad \text{..... (5.132)}$$

When variable current flows in both the coils, we have mutual induced emf in coil-2 due to current in coil-1 and mutual induced emf in coil-1 due to current in coil-2. Now, the turns ratio, the ratio of induced emfs and the ratio of currents are related as shown below:

$$\boxed{\frac{N_1}{N_2} = \frac{v_{m1}}{v_{m2}} = \frac{i_2}{i_1}} \qquad ..... (5.133)$$

From equation (5.133), we can write,

$$\frac{N_1}{i_2} = \frac{N_2}{i_1} \qquad ..... (5.134)$$

In most of the practical coupled coils, the flux linking coils 1 and 2 is the same as that of flux linking coils 2 and 1.

$$\therefore\ \phi_{12} = \phi_{21} = \phi \qquad ..... (5.135)$$

where, ϕ = Flux linking coils 1 and 2.

From equations (5.134) and (5.135), we can say that,

$$\boxed{M_{12} = M_{21} = M} \qquad ..... (5.136)$$

where, M = Mutual inductance.

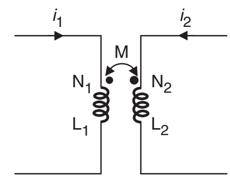


Fig. 5.28.

"*The existence of magnetic coupling and hence, mutual inductance between two coils is represented by a double-headed arrow as shown in Fig. 5.28*".

$$\therefore\ \text{Mutual inductance, } M = \frac{N_1\phi}{i_2} = \frac{N_2\phi}{i_1} \qquad ..... (5.137)$$

where, ϕ = Flux linking coils 1 and 2.

5.5.3 Coefficient of Coupling *(AU May'15, 16 Marks)*

The coefficient of coupling can be defined for two coils linked by magnetic flux. It is a measure of **flux linkages** between the two coils. The **coefficient of coupling** is defined as the fraction of the total flux produced by one coil linking another and it is denoted by k.

Let, ϕ_1 = Flux produced by coil-1

ϕ_2 = Flux produced by coil-2

ϕ_{12} = Flux produced by coil-1 linking coil-2

ϕ_{21} = Flux produced by coil-2 linking coil-1.

$$\text{Now, Coefficient of coupling, } k = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{21}}{\phi_2} \qquad ..... (5.138)$$

When k = 1, all the flux produced by one coil links the other coil and the coils are said to be tightly coupled or closely coupled coils. On the other hand, when k = 0, the flux produced by one coil does not link the other coil and the coils are said to be magnetically isolated. When the value of k is very low, the coils are said to be loosely coupled.

> ***Note :*** *When* $\dfrac{\phi_{12}}{\phi_1} \neq \dfrac{\phi_{21}}{\phi_2}$ *then* $k_1 = \dfrac{\phi_{12}}{\phi_1}$ *and* $k_2 = \dfrac{\phi_{21}}{\phi_2}$.
>
> *Now,* $k = \sqrt{k_1 k_2}$

The expression relating the coefficient of coupling with self- and mutual inductance is given by,

$$\boxed{\text{Coefficient of coupling, } k = \frac{M}{\sqrt{L_1 L_2}}} \quad ..... (5.139)$$

Also, from the above equation we can write, ***(AU Dec'14, 6 Marks)***

$$\boxed{\text{Mutual inductance, } M = k\sqrt{L_1 L_2}} \quad ..... (5.140)$$

Proof:

On multiplying equations (5.128) and (5.132), we get,

$$M_{12} \times M_{21} = \frac{N_2 \phi_{12}}{i_1} \times \frac{N_1 \phi_{21}}{i_2} \Rightarrow M^2 = \frac{N_2 \phi_{12}}{i_1} \times \frac{N_1 \phi_{21}}{i_2}$$ Using equation (5.136)

$$\therefore M^2 = \frac{N_2 k\phi_1}{i_1} \times \frac{N_1 k\phi_2}{i_2} = k^2 \frac{N_1 \phi_1}{i_1} \times \frac{N_2 \phi_2}{i_2}$$ Using equation (5.138) $\phi_{12} = k\phi_1$; $\phi_{21} = k\phi_2$

$$\therefore M^2 = k^2 L_1 L_2 \quad ..... (5.141)$$ Using equations (5.126) and (5.130)

On taking the square root of equation (5.141), we get,

$$M = k\sqrt{L_1 L_2}$$

$$\therefore k = \frac{M}{\sqrt{L_1 L_2}}$$

5.6 Analysis of Coupled Coils

In magnetically coupled coils, there are two induced emfs in each coil. One is self-induced emf and the other is mutual induced emf. "*The self-induced emf is due to current (or flux) in the same coil and the polarity (or sign) of self-induced emf will be always opposing the current through the coil*".

The mutual induced emf is due to current or flux in a nearby coil. The polarity (or sign) of mutual induced emf may be either the same as that of the self-induced emf or opposite to that of the self-induced emf.

In a pair of coupled coils, if the flux produced by one coil aids the flux produced by another coil then the polarity of self- and mutual induced emf will be the same in both the coils. On the other hand, if the flux produced by one coil opposes the flux produced by the other coil then the polarity of self- and mutual induced emf will be opposite in both the coils.

Consider a coil wound on a core carrying current i. Now, the direction of flux lines can be found using the right-hand rule. Hold the coil in the palm of the right-hand and fold the fingers except the thumb in the direction of current through the turns. Now, the thumb indicates the direction of flux.

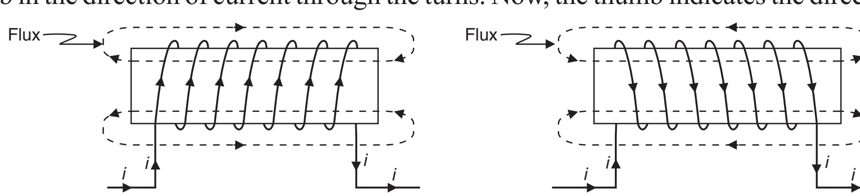


Fig. 5.29 : *Direction of flux in a coil with different winding sense.*

Two possible coil orientation (or winding sense) and resulting direction of flux for the same direction of (external) current are shown in Fig. 5.29. Coil orientation is also known as **winding sense**.

Let us apply the above concept to two coupled coils. The polarity of self- and mutual induced emf for some possible winding sense and direction of current are shown in Fig. 5.30.

In the coupled coils shown in Fig. 5.30, the polarity of self-induced emf is fixed first by assigning positive to the terminal where the current enters the coil (because the self-induced emf will always oppose the current).

Then the polarity of mutual induced emf is fixed depending on whether the fluxes are aiding or opposing. "*If the fluxes are aiding then the polarity of mutual induced emf is the same as that of the self-induced emf. If the fluxes are opposing then the polarity of the mutual induced emf is opposite to that of the self-induced emf*".

> ***Note :*** *In Fig. 5.30, separate cores are shown for coupled coils in order to explain the concepts. However, most coupled coils are wound on a common core (or on a single core).*

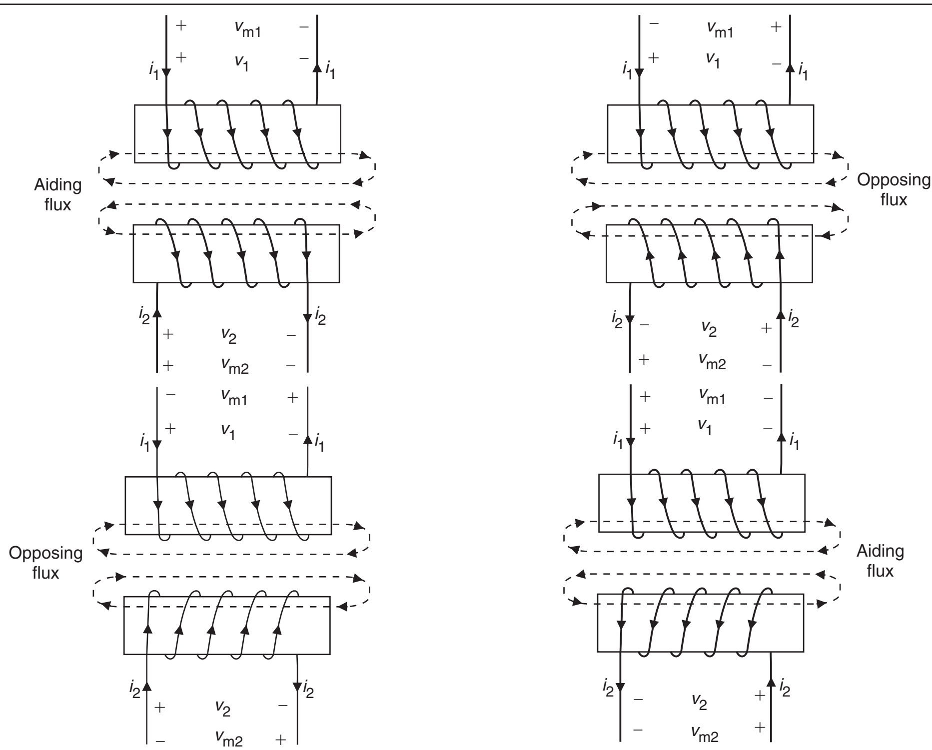


Fig. 5.30 : *Polarity of self- and mutual induced emf for different winding sense and direction of current.*

5.6.1 Dot Convention for Coupled Coils

The polarity (or sign) of mutual induced emf depends on coil orientation (or winding sense) and the direction of current through the coil. Most manufacturers of coupled coils/transformers mark one end of each coupled coil by a dot, which represents the polarity of mutual induced emf. The **dot rule** can be stated for the polarity of mutual induced emf as shown ahead.

Dot Rule

(AU May'15, 2 Marks)

"In coupled coils, current entering at the dotted terminal of one coil induces an emf in the second coil, which is positive at the dotted terminal of the second coil. Conversely, current entering at the undotted terminal of one coil induces an emf in the second coil, which is positive at the undotted terminal of the second coil".

Note : *When dot markings are not provided in coupled coils, they can be determined experimentally by measuring the additive and subtractive voltages of coupled coils using a voltmeter.*

The method of assigning dots to terminals of coupled coils is discussed in detail here.

Consider two coils wound on a common core as shown in Fig. 5.31(a). The terminals of coil-1 are marked as 1 and 1′ and the terminals of coil-2 are marked as 2 and 2′. Let us excite the coil-1 by connecting a time varying voltage source e_1 through current limiting resistance R_1. Now a current i_1 flows through coil-1 and a flux ϕ_1 is set up in the core. The direction of flux can be determined using the right-hand rule.

The flux ϕ_1 links both coils 1 and 2. Since the exciting source is a time varying source, the current and hence, flux are variable. Therefore, emfs are induced in coils 1 and 2. The emf in coil-1 is called self-induced emf and the emf in coil-2 is called mutual induced emf.

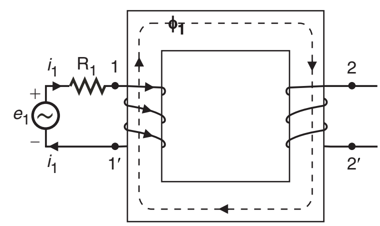


Fig. a.

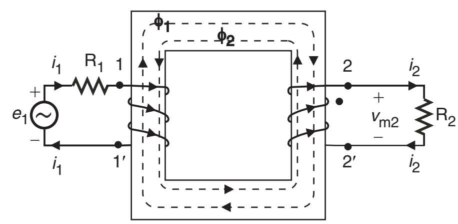


Fig. b.

Fig. 5.31 : *Assigning dot to coupled coils.*

(AU May'15, 16 Marks)

Let us connect a resistance, R_2 across the terminals of coil-2. Now a closed path is provided in coil-2 and a current flows in coil-2. The coupled coil shown in Fig. 5.31(b) is basically a transformer (or **linear transformer** or **natural transformer**) and so the current is also known as natural current.

By **Lenz's law**, the direction of current is to oppose the cause producing it. The cause is the flux ϕ_1 which induces mutual emf. Hence, the current i_2 will establish a flux ϕ_2 in order to oppose ϕ_1. (In a practical sense, in this case, ϕ_2 can never be greater than ϕ_1).

By taking the direction of flux ϕ_2 as opposite to that of ϕ_1 and applying the right-hand rule, the direction of current through coil-2 can be determined. It is found that i_2 leaves the coil at terminal-2 and enters the coil at terminal-2′. The mutual induced emf is the source for current i_2 and the terminal at which current leaves is positive for mutual induced emf. A dot is placed at terminal-2 of coil-2 to mark the positive polarity of the mutual induced emf.

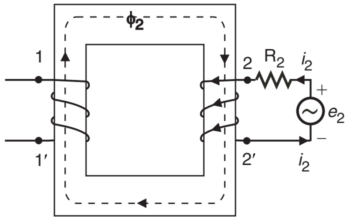


Fig. a.

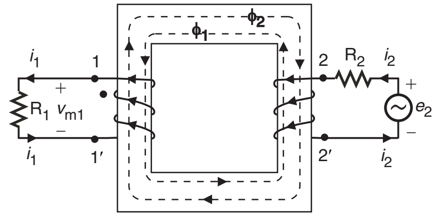


Fig. b.

Fig. 5.32 : *Assigning dot to coupled coils.*

By similar analysis on exciting the coil-2 by a source of e_2 and connecting a resistance across coil-1 as shown in Fig. 5.32, it is found that the polarity of mutual induced emf is positive at terminal-1 of coil-1. Hence, a dot is placed at terminal-1 of coil-1 to mark the positive polarity of mutual induced emf.

Let us assign the polarity of self- and mutual induced emf for the coupled coils as shown in Fig. 5.31(b) and 5.32(b). Now, the self-induced emf will oppose the current in the same coil. Since the fluxes produced by coils 1 and 2 are opposing, the polarity of mutual induced emf will be opposite to that of the self-induced emf. The coupled coils of Figs 5.31(b) and 5.32(b) with polarity of self- and mutual induced emf are shown in Figs 5.33 and 5.34, respectively.

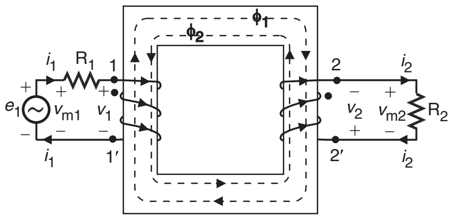


Fig. 5.33 : *Polarities of self- and mutual induced emf when coil-1 is excited.*

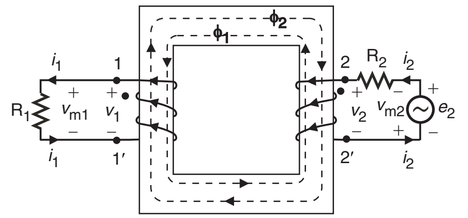


Fig. 5.34 : *Polarities of self- and mutual induced emf when coil-2 is excited.*

Here it is observed that the current enters at the dotted terminal in one coil and leaves at the dotted terminal in the other coil. In this situation, the polarity (or signs) of self- and mutual induced emfs are opposite.

Let us consider two other cases of exciting the coupled coils of Figs 5.33 and 5.34 with the same dot marking by two voltage sources e_1 and e_2 as shown in Figs 5.35 and 5.36, respectively. The self-induced emf will always oppose the current in the same coil. Since, the fluxes produced by the two coils are aiding each other, the mutual induced emf will have the same sign as that of the self-induced emf.

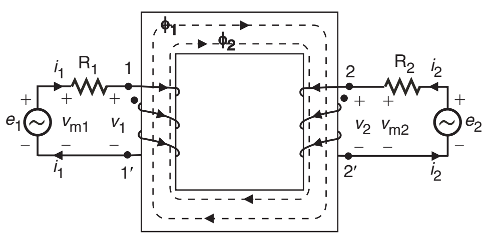


Fig. 5.35 : *Polarities of self- and mutual induced emf when current enters at the dotted terminal in both the coils.*

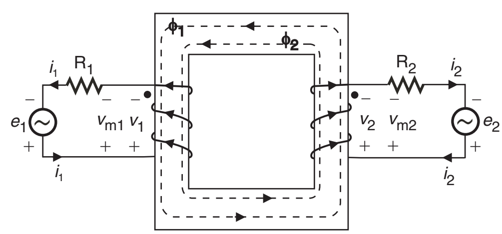


Fig. 5.36 : *Polarities of self- and mutual induced emf when current leaves at the dotted terminal in both the coils.*

From the above discussions, the dot rule can be interpreted as follows to fix the polarity of self- and mutual induced emf in coupled coils.

For analysis of coupled coils with dot marking, let us assume an arbitrary direction for current in coils 1 and 2. Now the current may either enter at the dotted end or leave at the dotted end and so we may come across the following four cases of current direction in relation to the dot marking.

Case i : *The current in coil-1 enters at the dotted end and current in coil-2 leaves at the dotted end.*
Case ii : *The current in coil-1 leaves at the dotted end and current in coil-2 enters at the dotted end.*
Case iii : *The current enters at the dotted end in both coils 1 and 2.*
Case iv : *The current leaves at the dotted end in both coils 1 and 2.*

In case (i) and (ii), the polarity (or sign) of self- and mutual induced emfs are opposite. (Refer to Figs 5.33 and 5.34). In case (iii) and (iv), the polarity (or signs) of self- and mutual induced emf are the same (Refer Figs. 5.35 and 5.36). These four cases of coupled coils are shown in Fig. 5.37, without details of winding orientation (or winding sense).

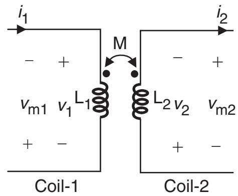


Fig. a : i_1 entering and i_2 leaving at dotted end.

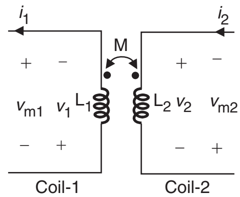


Fig. b : i_2 entering and i_1 leaving at dotted end.

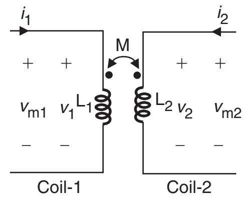


Fig. c : Both the current entering at dotted end.

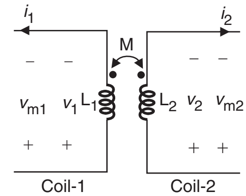


Fig. d : Both the current leaving at dotted end.

Fig. 5.37 : Polarity (or signs) of self- and mutual induced emf for various choices of current direction.

> *Note : In circuits with more than two coils, the coupling between various coils are denoted by different symbols like ●, ■, ▲, ◆, etc.*

5.6.2 Expression for Self- and Mutual Induced EMF in Various Domains

Time Domain

Consider the coupled coils as shown in Fig. 5.38, with i_1 entering at the dotted end and i_2 leaving at the dotted end. Now, the polarity of self- and mutual induced emfs are opposite. The polarity of self-induced emf are fixed such that they oppose the current through the same coil and the polarity of mutual induced emf are fixed such that they are opposite to that of the self-induced emf. From the discussions made in Section 5.5, the equation for self- and mutual induced emf are,

$$v_1 = L_1\frac{di_1}{dt} \; ; \quad v_2 = L_2\frac{di_2}{dt} \; ; \quad v_{m1} = M\frac{di_2}{dt} \; ; \quad v_{m2} = M\frac{di_1}{dt} \qquad ..... (5.142)$$

[Refer equations (5.125), (5.127), (5.129), (5.131) and (5.136)].

The coupled coils with time domain expression for self- and mutual emfs are shown in Fig. 5.39.

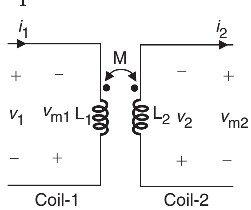


Fig. 5.38.

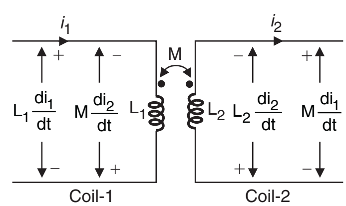


Fig. 5.39 : Coupled coils in time domain.

Laplace Domain

Let, $\mathcal{L}\{v_1\} = V_1(s)$ $\qquad \mathcal{L}\{v_2\} = V_2(s)$

$\mathcal{L}\{v_{m1}\} = V_{m1}(s)$ $\qquad \mathcal{L}\{v_{m2}\} = V_{m2}(s)$

$\mathcal{L}\{i_1\} = I_1(s)$ $\qquad \mathcal{L}\{i_2\} = I_2(s)$

On taking Laplace transform of time domain expression for self- and mutual induced emfs with zero initial conditions, we get,

$$\left.\begin{aligned} V_1(s) &= sL_1 I_1(s) \qquad & V_{m1}(s) &= sM\, I_2(s) \\ V_2(s) &= sL_2 I_2(s) \qquad & V_{m2}(s) &= sM\, I_1(s) \end{aligned}\right\} \qquad ..... (5.143)$$

The coupled coils with Laplace domain expression for self- and mutual induced emfs are shown in Fig. 5.40.

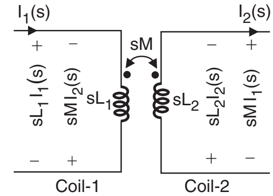


Fig. 5.40 : *Coupled coils in Laplace domain.*

Frequency Domain

On substituting $s = j\omega$, in the s-domain expression for emfs (i.e., in equation (5.143)), we get the expression for frequency domain emfs.

$$\left.\begin{aligned} V_1(j\omega) &= j\omega L_1 I_1(j\omega) \qquad & V_{m1}(j\omega) &= j\omega M\, I_2(j\omega) \\ V_2(j\omega) &= j\omega L_2 I_2(j\omega) \qquad & V_{m2}(j\omega) &= j\omega M\, I_1(j\omega) \end{aligned}\right\} \qquad ... (5.144)$$

For simplicity, $F(j\omega)$ is denoted as $\overline{F}$. Therefore, equation (5.144) can be written as shown below:

$$\begin{aligned} \overline{V}_1 &= j\omega L_1 \overline{I}_1 \qquad & \overline{V}_{m1} &= j\omega M\, \overline{I}_2 \\ \overline{V}_2 &= j\omega L_2 \overline{I}_2 \qquad & \overline{V}_{m2} &= j\omega M\, \overline{I}_1 \end{aligned}$$

The coupled coils with frequency domain expression for self- and mutual induced emfs are shown in Fig. 5.41.

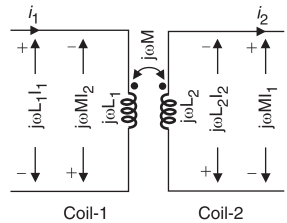


Fig. 5.41 : *Coupled coils in frequency domain.*

5.6.3 Writing Mesh Equations for Coupled Coils

(AU May'16, 16 Marks)

Consider the coupled coils shown in Fig. 5.42. Let us choose mesh currents $\overline{I}_1$ and $\overline{I}_2$ as shown in Fig. 5.42.

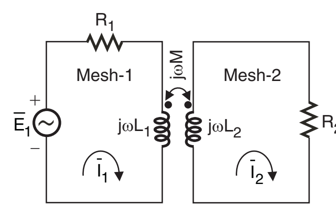


Fig. 5.42.

The assumed mesh currents enter at the dotted end in coil-1 and leave at the dotted end in coil-2. Hence, the polarity (or signs) of the mutual induced emf will be opposite to that of the self-induced emf. The voltages across the various elements of mesh-1 and mesh-2 are shown in Fig. 5.43.

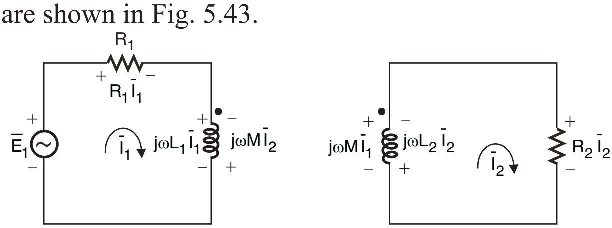


Fig. a : *Mesh-1.* ***Fig. b :*** *Mesh-2.*

Fig. 5.43 : *Voltage across various elements in mesh-1 and mesh-2.*

By using KVL in mesh-1, we can write,

$$R_1 \overline{I}_1 + j\omega L_1 \overline{I}_1 = \overline{E}_1 + j\omega M \overline{I}_2$$

$$\therefore \quad (R_1 + j\omega L_1)\overline{I}_1 - j\omega M \overline{I}_2 = \overline{E}_1 \qquad ..... (5.145)$$

By using KVL in mesh-2, we can write,

$$j\omega L_2 \overline{I}_2 + R_2 \overline{I}_2 = j\omega M \overline{I}_1$$

$$\therefore \quad -j\omega M\, \overline{I}_1 + (R_2 + j\omega L_2)\, \overline{I}_2 = 0 \qquad ..... (5.146)$$

On arranging equations (5.145) and (5.146) in the matrix form, we get,

$$\begin{bmatrix} R_1 + j\omega L_1 & -j\omega M \\ -j\omega M & R_2 + j\omega L_2 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{E}_1 \\ 0 \end{bmatrix} \qquad ..... (5.147)$$

From equation (5.147), it can be observed that the mutual reactance, ($j\omega M$) is introduced as an element common to meshes 1 and 2 and it is negative. However, it is not included in the self-impedance (i.e., not included in $\overline{Z}_{11}$ and $\overline{Z}_{22}$) of meshes 1 and 2.

In general, we can say that if there is a coupling between meshes j and k, first form the mesh basis matrix equation without considering the mutual reactance, and then add the mutual reactance to the impedances $\overline{Z}_{jk}$ and $\overline{Z}_{kj}$. The sign of mutual reactance is negative if current enters at the dotted end in one coil and leaves at the dotted end in another coil. The sign of mutual reactance is positive if current enters (or leaves) at the dotted end in both the coils.

5.6.4 Electrical Equivalent of Magnetic Coupling (Electrical Equivalent of a Transformer or Linear Transformer)

Magnetic coupling between two coils can be replaced by an electrical equivalent. Consider mesh-1 of the coupled circuit shown in Fig. 5.42. The mutual reactance introduce a voltage in mesh-1 due to current in mesh-2, but it does not introduce a voltage in mesh-1 due to mesh-1 current. The (self-) inductive reactance $j\omega L_1$ introduce a voltage in mesh-1 due to mesh-1 current.

Similarly in mesh-2, the mutual reactance introduce a voltage in mesh-2 due to current in mesh-1 and (self-) inductive reactance $j\omega L_2$ introduce a voltage in mesh-2 due to mesh-2 current.

From the above discussion, we can conclude that self-reactances of coupled coils can be introduced as elements associated with respective meshes alone and mutual reactance as an element common to two meshes. However, when we introduce mutual reactance as an element common to meshes 1 and 2, it will introduce voltage in mesh-1 due to both mesh-1 and mesh-2 currents. The voltage introduced by mesh-1 current can be eliminated by introducing a negative mutual reactance in mesh-1 as an element associated with mesh-1 alone. By similar argument, a negative mutual reactance has to be introduced in mesh-2 as an element associated with mesh-2 alone.

Therefore, in general, when a magnetic coupling exists between meshes j and k, the coupled coils can be replaced by introducing the following elements in mesh-j and mesh-k, provided the direction of current in coils j and k are not specified.

i) The mutual reactance $j\omega M$ is introduced as an element common to mesh-j and mesh-k.

ii) The reactance $j\omega(L_j - M)$ is introduced as an element associated with mesh-j alone.

iii) The reactance $j\omega(L_k - M)$ is introduced as an element associated with mesh-k alone.

where, L_j and L_k are self-inductances of the coils in mesh-j and mesh-k, respectively, and M is the mutual inductance between these two coils.

The electrical equivalent of coupled coils is shown in Fig. 5.44. The electrical equivalent of the coupled circuit of Fig. 5.42 is shown in Fig. 5.44. "*This type of magnetic coupling exists in a transformer and so the electrical equivalent of coupled coils shown in Figs 5.44 and 5.45 are applicable for a transformer or linear transformer or natural transformer*".

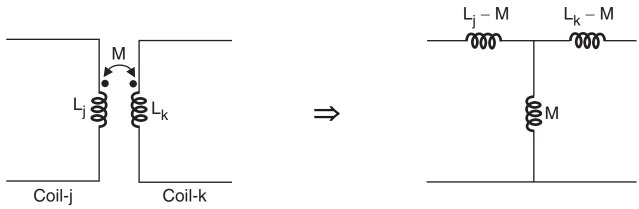


Fig. 5.44 : *Electrical equivalent of coupled coils.*

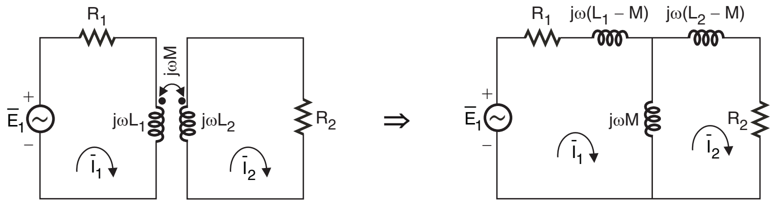


Fig. 5.45 : *Electrical equivalent of a coupled circuit.*

Note : *Physically realisable electrical equivalent is possible as long as* $M < L_1$ *and* $M < L_2$.

Equivalent of Coupled Coils with Resistances

So far we have considered only ideal coils, in which the resistance of the coil is zero. Sometimes, the resistance of the coil is specified along with the inductance (practically, the resistance of the coil is the resistance of the wire that is used to construct the coil). In such cases, the resistance of the coil can be represented by a resistance external to the coil in series as shown in Fig. 5.46.

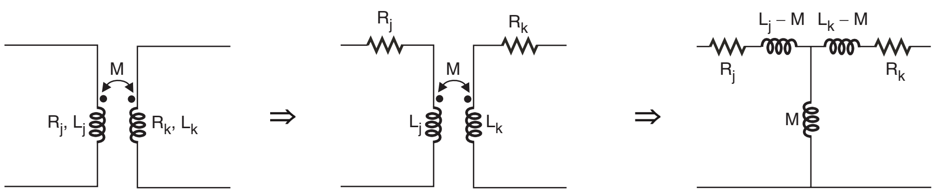


Fig. 5.46 : *Coupled coils with resistance of the coils.*

Equivalent of Coupled Coils When Currents and Dots are Specified

When the direction of current through the coils is specified along with dot convention, the electrical equivalent of coupled coils (shown in Fig. 5.44) will not be applicable for certain combination of dot marking and mesh currents. When mesh currents are specified, we come across the following two groups of coupled coils:

Group I : Mesh currents in opposite orientation and fluxes are aiding.
Mesh currents in same orientation and fluxes are opposing.

Group II : Mesh currents in same orientation and fluxes are aiding.
Mesh currents in opposite orientation and fluxes are opposing.

For group-I coupled coils, the electrical equivalent will be as shown in Fig. 5.47. For group-II coupled coils, the electrical equivalent is obtained by replacing M by –M in the electrical equivalent shown in Fig. 5.48.

Note : *1. When fluxes are aiding, emf due to self- and mutual inductances have the same sign.*
2. When fluxes are opposing, emf due to self- and mutual inductances have the opposite sign.

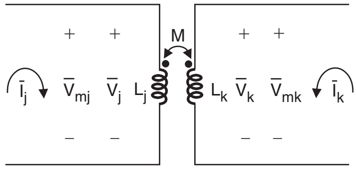


Fig. a : *Currents' opposite orientation and fluxes aiding.*

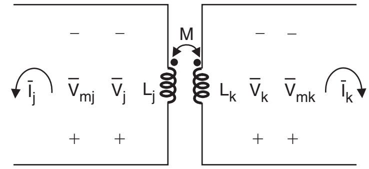
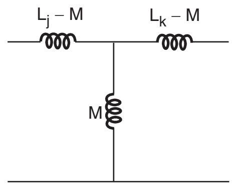


Fig. b : *Currents' opposite orientation and fluxes aiding.*



Note : *In the electrical equivalent, the mesh currents' orientation should be the same as that of the original circuit.*

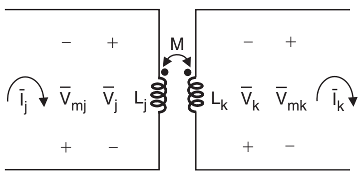


Fig. c : *Currents' same orientation and fluxes opposing.*

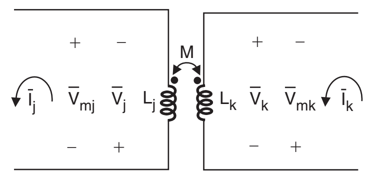


Fig. d : *Currents' same orientation and fluxes opposing.*

Fig. 5.47 : *Electrical equivalent of group-I coupled coils.*

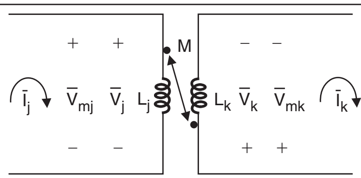


Fig. a : *Currents' same orientation and fluxes aiding.*

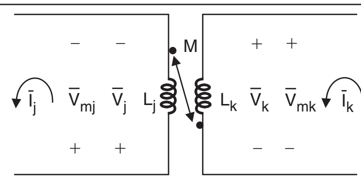
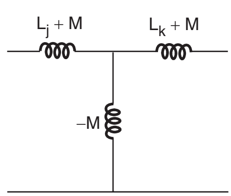


Fig. b : *Currents' same orientation and fluxes aiding.*



Note : *In the electrical equivalent, the mesh currents' orientation should be the same as that of the original circuit.*

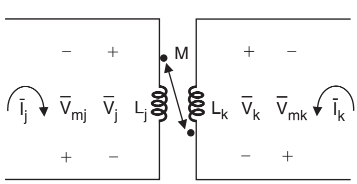


Fig. c : *Currents' opposite orientation and fluxes opposing.*

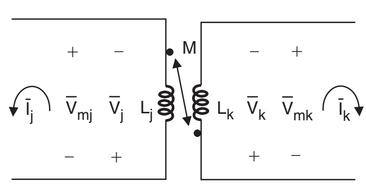


Fig. d : *Currents' opposite orientation and fluxes opposing.*

Fig. 5.48 : *Electrical equivalent of group-II coupled coils.*

5.6.5 Writing Mesh Equations in Circuits with Electrical Connection and Magnetic Coupling

Consider the circuit shown in Fig. 5.49(a) in which two electrically connected coils have magnetic coupling between them. Let us choose two mesh currents $\bar{I}_1$ and $\bar{I}_2$ as shown in Fig. 5.49(b). The voltages across various elements excluding the voltages due to mutual inductance are shown in Fig. 5.49(b).

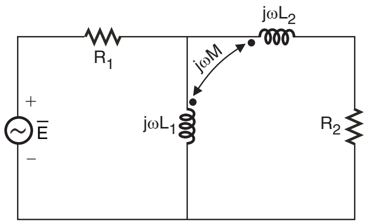


Fig. a.

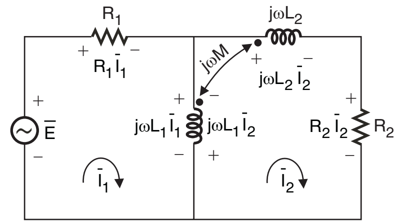


Fig. b.

Fig. 5.49 : *Magnetic coupling in electrically connected coils.*

Here $\bar{I}_2$ is the current through coil-2. Let us take the current through coil-1 as $\bar{I}_1 - \bar{I}_2$. Now due to mutual inductance, the current $\bar{I}_1 - \bar{I}_2$ entering at the dotted end in coil-1 will induce an emf $j\omega M(\bar{I}_1 - \bar{I}_2)$ in coil-2 such that the induced emf is positive at the dotted end in coil-2. Similarly, the current $\bar{I}_2$ entering at the dotted end in coil-2 will induce an emf $j\omega M\bar{I}_2$ in coil-1 such that the induced emf is positive at the dotted end in coil-1. The mutual induced emf along with self-induced emf are shown in Fig. 5.50.

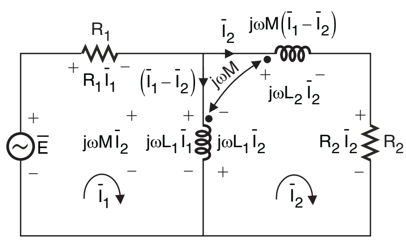


Fig. 5.50.

With reference to Fig. 5.50, in mesh-1 by KVL, we can write,

$$R_1\bar{I}_1 + j\omega M\bar{I}_2 \ + \ j\omega L_1\bar{I}_1 \ = \ \bar{E} + j\omega L_1\bar{I}_2$$

$$\therefore \ \ (R_1 + j\omega L_1)\bar{I}_1 \ + \ (-j\omega L_1 + j\omega M)\bar{I}_2 \ = \ \bar{E} \qquad ..... (5.148)$$

With reference to Fig. 5.50, in mesh-2 by KVL, we can write,

$$R_2\bar{I}_2 + j\omega L_1\bar{I}_2 + j\omega L_2\bar{I}_2 + j\omega M(\bar{I}_1 - \bar{I}_2) = \ j\omega M\bar{I}_2 + j\omega L_1\bar{I}_1$$

$$\therefore \ (-j\omega L_1 + j\omega M)\,\bar{I}_1 + (R_2 + j\omega L_2 + j\omega L_1 - j2\omega M)\,\bar{I}_2 \ = \ 0 \qquad ..... (5.149)$$

On arranging equations (5.148) and (5.149) in the matrix form, we get,

$$\begin{bmatrix} R_1 + j\omega L_1 & -j\omega L_1 + j\omega M \\ -j\omega L_1 + j\omega M & R_2 + j\omega L_2 + j\omega L_1 - j2\omega M \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{E} \\ 0 \end{bmatrix} \qquad ..... (5.150)$$

5.7 Series and Parallel Connections of Coupled Coils

In certain circuits there may be a magnetic coupling between series-connected or parallel-connected coils. In such case it is possible to replace the series or parallel combination by a single equivalent inductive reactance. In series (or parallel) connection, the mutual induced emf will have either the same polarity or opposite polarity as that of the self-induced emf.

In series connection, when the mutual and self-induced emf have the same polarity (or sign), the connection is called **series aiding**. When the mutual and self-induced emf have an opposite polarity, (or sign) the connection is called **series opposing**.

Similarly, in parallel connection, when the mutual and self-induced emf have the same polarity (or sign) the connection is called **parallel aiding**. When the mutual and self-induced emf have an opposite polarity (or sign) the connection is called **parallel opposing**.

Physically, in series or parallel aiding the flux produced by one coil aids the other (i.e., both the coil will set up flux in the same direction). Alternatively, in series or parallel opposing, the flux produced by one coil will oppose the other.

5.7.1 Series Aiding Connection of Coupled Coils

Consider the coupled coils connected in series as shown in Fig. 5.51(a). Here, the current enters at the dotted end in both the coils and so the self- and mutual induced emf will have the same polarity (or sign). Therefore, the connection shown in Fig. 5.51(a) is series aiding. Now, the coupled coils can be represented by a single equivalent inductance as shown in Fig 5.51(b), where equivalent inductance, L_{eq} is given by, ***(AU May'17, 2 Marks)***

$$L_{eq} = L_1 + L_2 + 2M \qquad .....(5.151)$$

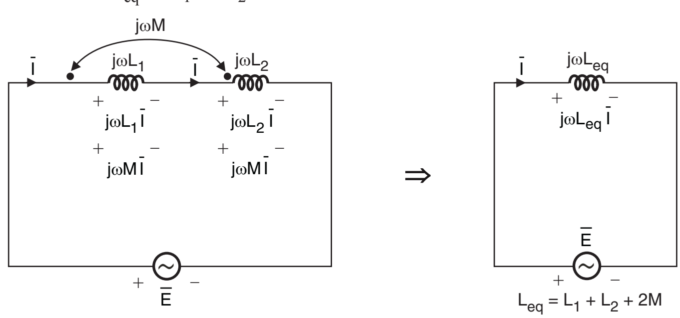


Fig. a : *Series aiding connection.* ***Fig. b :*** *Equivalent of coupled coils in series aiding.*

Fig. 5.51 : *Series aiding connection of coupled coils and its equivalent.*

Proof: Equivalent inductance of series aiding connection.

With reference to Fig. 5.51(a), by KVL, we can write,

$$j\omega L_1\bar{I} + j\omega M\bar{I} + j\omega L_2\bar{I} + j\omega M\bar{I} + j\omega M\bar{I} = \bar{E}$$

$$j\omega(L_1 + L_2 + 2M)\bar{I} = \bar{E}$$

$$\therefore \quad j\omega L_{eq}\bar{I} = \bar{E} \qquad ..... (5.152)$$

$$\text{where,} \quad L_{eq} = L_1 + L_2 + 2M$$

From equation (5.152), we can say that the coupled coils of Fig. 5.51(a) can be replaced by an equivalent reactance as shown in Fig. 5.51(b).

Equation (5.151), can be written as,

$$L_{eq} = (L_1 + M) + (L_2 + M) \qquad ..... (5.153)$$

From equation (5.153), we can say that, "*the series aiding connection of coupled coils can be viewed as a series connection of two inductances (L_1 + M) and (L_2 + M)*", as shown in Fig. 5.52.

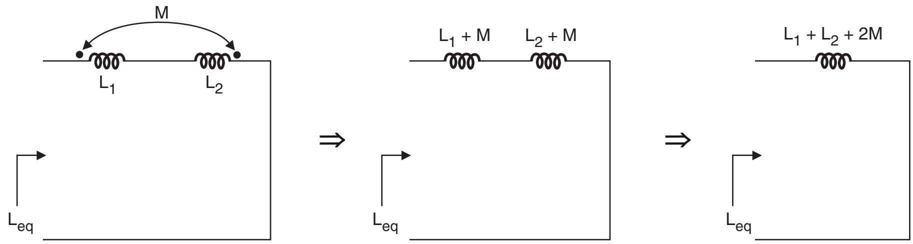


***Fig. 5.52 :** Alternate representation for series aiding connection.*

5.7.2 Series Opposing Connection of Coupled Coils **(AU Dec'16, 2 Marks)**

Consider the coupled coils connected in series as shown in Fig. 5.53(a). Here the current enters at the dotted end in one coil and leaves at the dotted end in the other coil. Hence, the polarity (or sign) of self- and mutual induced emf will be opposite. Therefore, the connection shown in Fig. 5.53(a) is series opposing. Now, the coupled coils can be represented by a single equivalent inductance as shown in Fig. 5.53(b), where equivalent inductance, L_{eq} is given by,

$$L_{eq} = L_1 + L_2 - 2M \qquad .....(5.154)$$

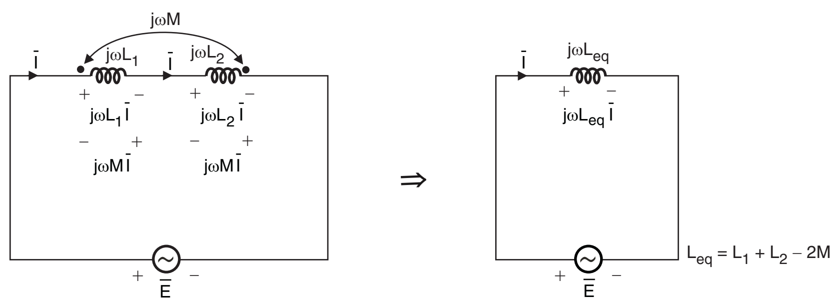


***Fig. a :** Series opposing connection.* ***Fig. b :** Equivalent of coupled coils in series opposing.*

***Fig. 5.53 :** Series opposing connection of coupled coils and its equivalent.*

Proof : Equivalent inductance of series opposing connection.

With reference to Fig. 5.53(a), by KVL, we can write,

$$j\omega L_1\bar{I} + j\omega L_2\bar{I} = \bar{E} + j\omega M\bar{I} + j\omega M\bar{I} \quad \Rightarrow \quad j\omega L_1\bar{I} + j\omega L_2\bar{I} - j\omega 2M\bar{I} = \bar{E}$$

$$\therefore \quad j\omega(L_1 + L_2 - 2M)\bar{I} = \bar{E}$$

$$j\omega L_{eq}\bar{I} = \bar{E} \qquad ..... (5.155)$$

$$\text{where, } L_{eq} = L_1 + L_2 - 2M$$

From equation (5.155), we can say that the coupled coils of Fig. 5.53(a) can be replaced by an equivalent reactance as shown in Fig. 5.53(b).

Equation (5.154) can be written as,

$$L_{eq} = (L_1 - M) + (L_2 - M) \qquad \text{..... (5.156)}$$

From equation (5.156), we can say that, "*the series opposing connection of coupled coils can be viewed as a series connection of two inductances (L_1– M) and (L_2 – M)*", as shown in Fig. 5.54.

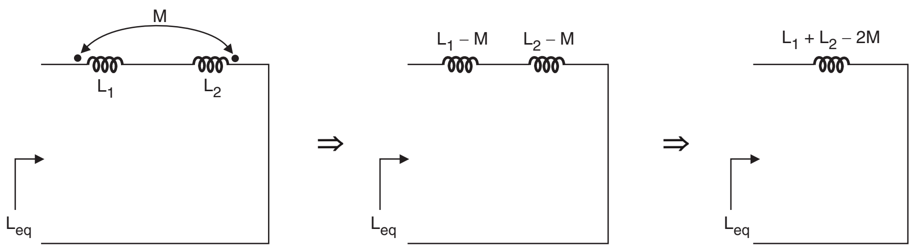


Fig. 5.54 : *Alternate representation for series opposing connection.*

5.7.3 Parallel Aiding Connection of Coupled Coils

Consider the coupled coils connected in parallel as shown in Fig. 5.55(a). Here, the current enters at the dotted end in both the coils and so the self- and mutual induced emfs will have the same polarity (or sign). Therefore, the connection shown in Fig. 5.55(a) is parallel aiding. Now, the coupled coils can be represented by a single equivalent inductance as shown in Fig. 5.55(b), where equivalent inductance, L_{eq} is given by,

$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M} \qquad \text{..... (5.157)}$$

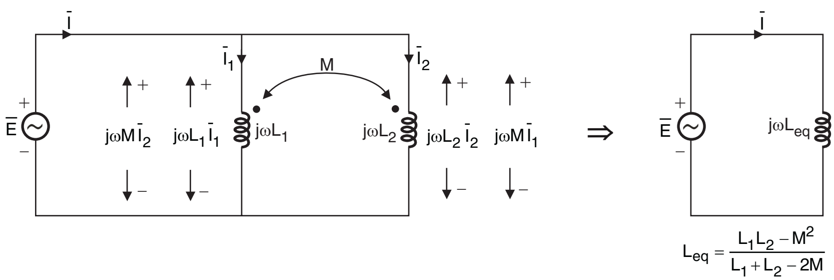


Fig. a : *Parallel aiding connection.* ***Fig. b :*** *Equivalent of coupled coils in parallel aiding.*

Fig. 5.55 : *Parallel aiding connection of coupled coils and its equivalent.*

Proof: Equivalent inductance of parallel aiding connection. ***(AU June'16, 8 Marks)***

With reference to Fig. 5.55(a), by using KVL in the parallel arms, we get the following two equations.

$$j\omega L_1 \bar{I}_1 + j\omega M \bar{I}_2 = \bar{E} \qquad \text{..... (5.158)}$$

$$j\omega M \bar{I}_1 + j\omega L_2 \bar{I}_2 = \bar{E} \qquad \text{..... (5.159)}$$

Let us arrange equations (5.158) and (5.159) in the matrix form, as shown below:

$$\begin{bmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{E} \\ \bar{E} \end{bmatrix}$$

Now, $\Delta = \begin{vmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{vmatrix} = j\omega L_1 \times j\omega L_2 - j\omega M \times j\omega M$

$$= -\omega^2 L_1 L_2 + \omega^2 M^2 = -\omega^2 (L_1 L_2 - M^2)$$

$$\Delta_1 = \begin{vmatrix} \overline{E} & j\omega M \\ \overline{E} & j\omega L_2 \end{vmatrix} = \overline{E} \times j\omega L_2 - \overline{E} \times j\omega M$$

$$= \overline{E}\, j\omega\, (L_2 - M)$$

$$\Delta_2 = \begin{vmatrix} j\omega L_1 & \overline{E} \\ j\omega M & \overline{E} \end{vmatrix} = j\omega L_1 \times \overline{E} - j\omega M \times \overline{E}$$

$$= \overline{E}\, j\omega\, (L_1 - M)$$

$$\therefore\ \overline{I}_1 = \frac{\Delta_1}{\Delta} = \frac{\overline{E}\, j\omega\, (L_2 - M)}{-\omega^2 (L_1 L_2 - M^2)}$$

$$\overline{I}_2 = \frac{\Delta_2}{\Delta} = \frac{\overline{E}\, j\omega\, (L_1 - M)}{-\omega^2 (L_1 L_2 - M^2)}$$

Now by KCL, $\overline{I} = \overline{I}_1 + \overline{I}_2$

$$\therefore\ \overline{I} = \frac{\overline{E}\, j\omega\, (L_2 - M)}{-\omega^2 (L_1 L_2 - M^2)} + \frac{\overline{E}\, j\omega\, (L_1 - M)}{-\omega^2 (L_1 L_2 - M^2)}$$

$$= \overline{E}\, \frac{j\omega}{-\omega^2} \left[\frac{L_2 - M + L_1 - M}{L_1 L_2 - M^2} \right]$$

$$= \overline{E}\, \frac{1}{j\omega} \left[\frac{L_1 + L_2 - 2M}{L_1 L_2 - M^2} \right]$$

$$\therefore\ \overline{E} = \overline{I}\, j\omega \left[\frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M} \right]$$

or $$\overline{E} = \overline{I}\, j\omega L_{eq} \qquad \text{..... (5.160)}$$

$$\text{where, } L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

From equation (5.160), we can say that the coupled coils of Fig. 5.55(a) can be replaced by an equivalent reactance as shown in Fig. 5.55(b).

Consider the network shown in Fig. 5.56(a). It can be reduced to a single equivalent inductance as shown in Fig. 5.56(d). Here this value of equivalent inductance is the same as that of equation (5.157). Therefore, the parallel aiding connection of coupled coils can be represented by an equivalent network as shown in Fig. 5.56(b).

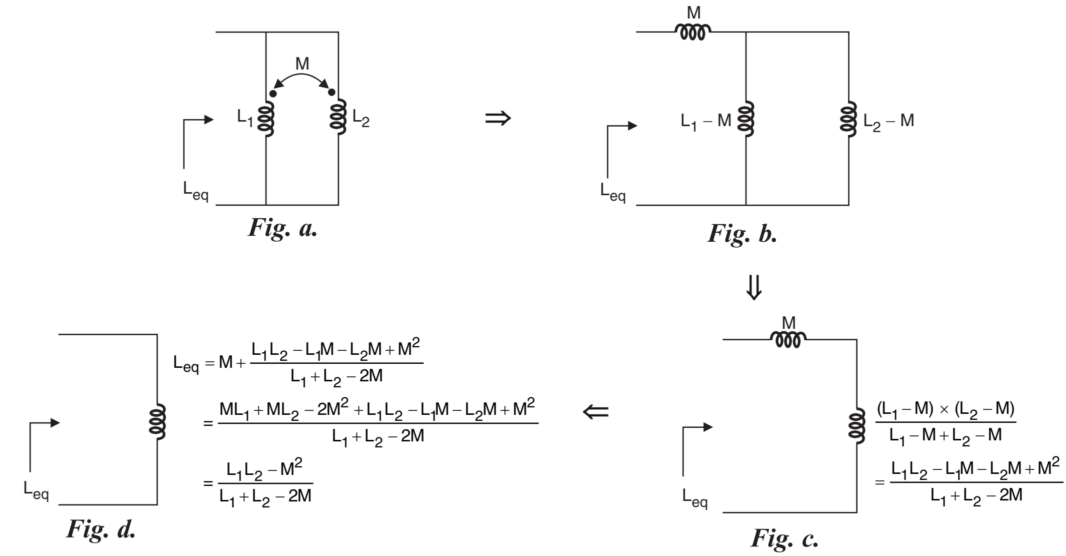


Fig. 5.56 : *Alternate representation for parallel aiding connection.*

5.7.4 Parallel Opposing Connection of Coupled Coils

Consider the coupled coils connected in parallel as shown in Fig. 5.57(a). Here, the current enters at the dotted end in one coil and leaves at the dotted end in the other coil. So the self- and mutual induced emf will have an opposite polarity (or sign). Therefore, the connection shown in Fig. 5.57(a) is parallel opposing. Now, the coupled coils can be represented by a single equivalent inductance as shown in Fig. 5.57(b), where equivalent inductance, L_{eq} is given by,

$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M} \qquad .....(5.161)$$

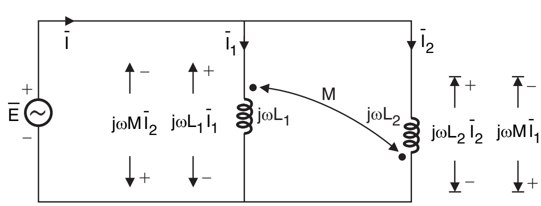


Fig. a : *Parallel opposing connection.*

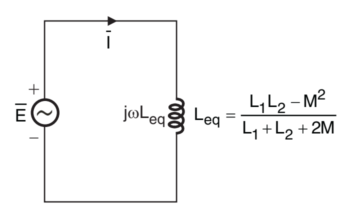


Fig. b : *Equivalent of coupled coils in parallel opposing.*

Fig. 5.57 : *Parallel opposing connection of coupled coils and its equivalent.*

Proof: Equivalent inductance of paralled opposing connection.

With reference to Fig. 5.57(a), by using KVL in the parallel arms, we get the following two equations.

$$j\omega L_1 \bar{I}_1 - j\omega M \bar{I}_2 = \bar{E} \qquad ..... (5.162)$$

$$-j\omega M \bar{I}_1 + j\omega L_2 \bar{I}_2 = \bar{E} \qquad ..... (5.163)$$

Let us arrange equations (5.162) and (5.163) in the matrix form, as shown below:

$$\begin{bmatrix} j\omega L_1 & -j\omega M \\ -j\omega M & j\omega L_2 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{E} \\ \bar{E} \end{bmatrix}$$

Now, $\Delta = \begin{vmatrix} j\omega L_1 & -j\omega M \\ -j\omega M & j\omega L_2 \end{vmatrix} = j\omega L_1 \times j\omega L_2 - (-j\omega M)^2$

$$= -\omega^2 L_1 L_2 + \omega^2 M^2 = -\omega^2 (L_1 L_2 - M^2)$$

$$\Delta_1 = \begin{vmatrix} \bar{E} & -j\omega M \\ \bar{E} & j\omega L_2 \end{vmatrix} = \bar{E} \times j\omega L_2 - \bar{E} \times (-j\omega M)$$

$$= \bar{E}\, j\omega (L_2 + M)$$

$$\Delta_2 = \begin{vmatrix} j\omega L_1 & \bar{E} \\ -j\omega M & \bar{E} \end{vmatrix} = j\omega L_1 \times \bar{E} - (-j\omega M) \times \bar{E}$$

$$= \bar{E}\, j\omega (L_1 + M)$$

$$\therefore\ \bar{I}_1 = \frac{\Delta_1}{\Delta} = \frac{\bar{E}\, j\omega (L_2 + M)}{-\omega^2 (L_1 L_2 - M^2)}$$

$$\bar{I}_2 = \frac{\Delta_2}{\Delta} = \frac{\bar{E}\, j\omega (L_1 + M)}{-\omega^2 (L_1 L_2 - M^2)}$$

Now by KCL, $\bar{I} = \bar{I}_1 + \bar{I}_2$

$$\therefore\ \bar{I} = \frac{\bar{E}\, j\omega (L_2 + M)}{-\omega^2 (L_1 L_2 - M^2)} + \frac{\bar{E}\, j\omega (L_1 + M)}{-\omega^2 (L_1 L_2 - M^2)}$$

$$= \bar{E}\, \frac{j\omega}{-\omega^2} \left[\frac{L_2 + M + L_1 + M}{L_1 L_2 - M^2} \right]$$

$$= \bar{E}\, \frac{1}{j\omega} \left[\frac{L_1 + L_2 + 2M}{L_1 L_2 - M^2} \right]$$

$$\therefore\ \bar{E} = \bar{I}\, j\omega \left[\frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M} \right]$$

or $\bar{E} = \bar{I}\, j\omega L_{eq}$ ..... (5.164)

where, $L_{eq} = \dfrac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$

From equation (5.164), we can say that the coupled coils of Fig. 5.57(a) can be replaced by an equivalent reactance as shown in Fig. 5.57(b).

Consider the network shown in Fig. 5.58(a). It can be reduced to a single equivalent inductance as shown in Fig. 5.58(d). Here this value of equivalent inductance is the same as that of equation (5.161). Therefore, the parallel opposing connection of coupled coils can be represented by an equivalent network as shown in Fig. 5.58(b).

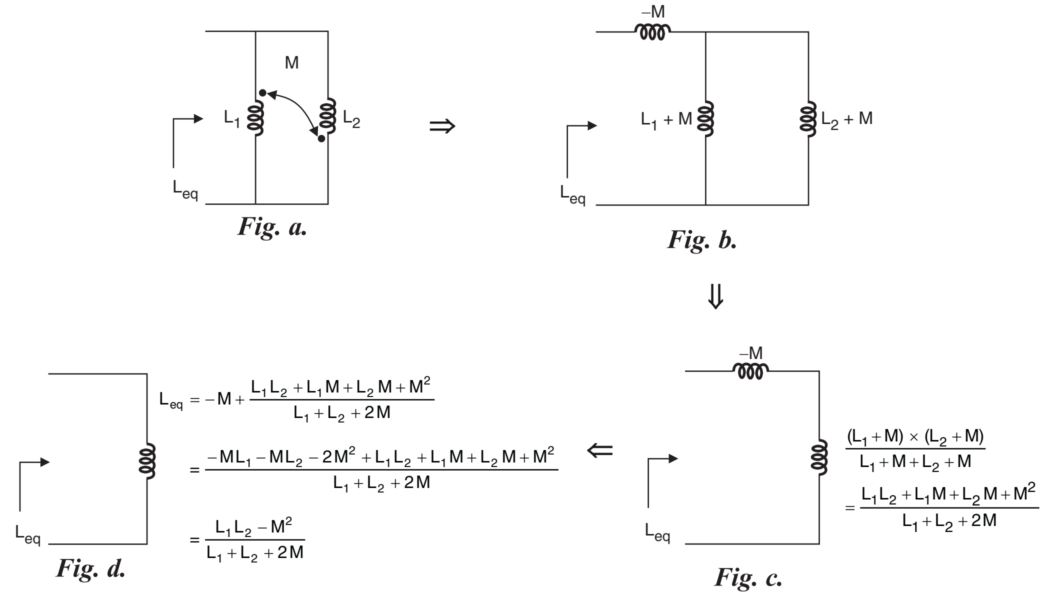


Fig. 5.58 : *Alternate representation for parallel opposing connection.*

5.8 Tuned Coupled Circuits

The coupled circuits are mainly used to transfer energy from a weak source to a load or employed for maximum power transfer from one circuit to another. This is possible only when both the coils work at resonance condition.

In coupled coils, the coil to which the source is connected is called **primary** and the coil to which the load is connected is called **secondary**. The coupled coils can be brought to resonance by adding capacitors to the primary and secondary coils. When the primary inductive reactance is very low or negligible, it is sufficient if we resonate the secondary coil alone by adding a capacitor to the secondary coil, to achieve maximum power transfer condition.

When a capacitor is added only to the secondary coil, the coupled coils are called **single tuned coupled coils** and the resultant circuit is called a **single tuned coupled circuit**.

When capacitors are added to both secondary and primary coils, the coupled coils are called **double tuned coupled coils** and the resultant circuit is called a **double tuned coupled circuit**. Normally, in a double tuned circuit, the primary and secondary are tuned to the same frequency. However, sometimes, intentionally the primary and secondary are tuned to slightly different frequencies and such double tuned circuits are called **stagger tuned circuits**.

5.8.1 Single Tuned Coupled Circuits

Let us connect a voltage source $\overline{E}$ with internal resistance R_g to single tuned coupled coils as shown in Fig. 5.59(a). Here R_p and R_s are resistances of primary and secondary coil and L_p and L_s are inductances of primary and secondary coil.

Let us represent the total resistance of primary as R_1 and that of secondary as R_2 as shown in Fig. 5.59(b). Also, let us denote the inductances of primary and secondary as L_1 and L_2.

$\therefore \quad R_1 = R_g + R_p \quad ; \quad R_2 = R_s \quad ; \quad L_1 = L_p \quad \text{and} \quad L_2 = L_s$

In a single tuned coupled circuit, the secondary is tuned by varying the capacitance in secondary. The secondary is equivalent to an RLC series circuit. Therefore, the condition for resonance and the resonace frequency of secondary will be the same as an RLC series resonance.

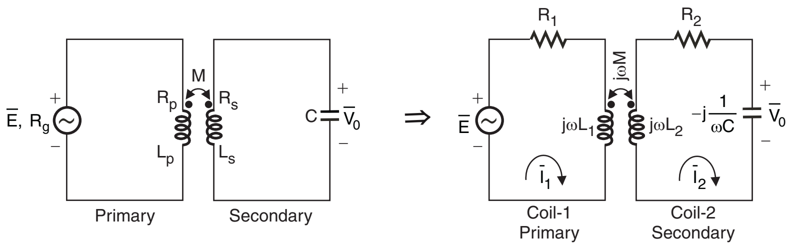


Fig. a. ***Fig. b.***

***Fig. 5.59 :** Single tuned coupled circuit.*

Now, the condition for resonance is,

$$\omega_r L_2 = \frac{1}{\omega_r C} \quad \Rightarrow \quad \omega_r L_2 - \frac{1}{\omega_r C} = 0 \qquad .....(5.165)$$

$$\therefore \text{ Resonant frequency, } \omega_r = \frac{1}{\sqrt{L_2 C}} \qquad .....(5.166)$$

Now, the secondary current and output voltage at resonance are given by the following equations:

$$\text{Secondary current at resonance, } \bar{I}_{2,r} = \frac{j\omega_r M\bar{E}}{R_1 R_2 + \omega_r^2 M^2} \qquad .....(5.167)$$

$$\text{Output voltage at resonance, } \bar{V}_{o,r} = \frac{\frac{M\bar{E}}{C}}{R_1 R_2 + \omega_r^2 M^2} \qquad .....(5.168)$$

Proof:

Let us assume two mesh currents $\bar{I}_1$ and $\bar{I}_2$ as shown in Fig. 5.59(b). With reference to Fig. 5.59(b), the mesh basis matrix equation is given below:

$$\begin{bmatrix} R_1 + j\omega L_1 & -j\omega M \\ -j\omega M & R_2 + j\omega L_2 - j\frac{1}{\omega C} \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{E} \\ 0 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} R_1 + j\omega L_1 & -j\omega M \\ -j\omega M & R_2 + j\omega L_2 - j\frac{1}{\omega C} \end{vmatrix} = (R_1 + j\omega L_1)\left(R_2 + j\omega L_2 - j\frac{1}{\omega C}\right) - (-j\omega M)^2$$

$$= (R_1 + j\omega L_1)\left(R_2 + j\omega L_2 - j\frac{1}{\omega C}\right) + \omega^2 M^2$$

$$\Delta_2 = \begin{vmatrix} R_1 + j\omega L_1 & \bar{E} \\ -j\omega M & 0 \end{vmatrix} = j\omega M\bar{E}$$

$$\therefore \quad \bar{I}_2 = \frac{\Delta_2}{\Delta} = \frac{j\omega M\bar{E}}{(R_1 + j\omega L_1)\left(R_2 + j\omega L_2 - j\frac{1}{\omega C}\right) + \omega^2 M^2} \qquad ..... (5.169)$$

Single tuned coupled circuits are employed only when the primary inductive reactance is negligible.

Hence, in the primary circuit, $R_1 >> \omega L_1$

$$\therefore \ R_1 + j\omega L_1 \approx R_1$$

On neglecting the primary inductive reactance, equation (5.169) can be written as shown below:

$$\bar{I}_2 = \frac{j\omega M\bar{E}}{R_1 R_2 + \omega^2 M^2 + jR_1\left(\omega L_2 - \dfrac{1}{\omega C}\right)}$$

$$\therefore \ \bar{I}_{2,r} = \bar{I}_2\Big|_{\omega=\omega_r} = \frac{j\omega_r M\bar{E}}{R_1 R_2 + \omega_r^2 M^2 + jR_1\left(\omega_r L_2 - \dfrac{1}{\omega_r C}\right)}$$

$$= \frac{j\omega_r M\bar{E}}{R_1 R_2 + \omega_r^2 M^2} \qquad \boxed{\text{Using equation(5.165)}}$$

Let, $\bar{V}_0$ *be the output voltage in secondary as shown in Fig. 5.59.*

$$\text{Now,} \ \bar{V}_0 = \bar{I}_2\left(-j\frac{1}{\omega C}\right) = \frac{j\omega M\bar{E}\left(-j\dfrac{1}{\omega C}\right)}{R_1 R_2 + \omega^2 M^2 + jR_1\left(\omega L_2 - \dfrac{1}{\omega C}\right)}$$

$$= \frac{\dfrac{M\bar{E}}{C}}{R_1 R_2 + \omega^2 M^2 + jR_1\left(\omega L_2 - \dfrac{1}{\omega C}\right)}$$

$$\therefore \ \bar{V}_{o,r} = \bar{V}_o\Big|_{\omega=\omega_r} = \frac{\dfrac{M\bar{E}}{C}}{R_1 R_2 + \omega_r^2 M^2 + jR_1\left(\omega_r L_2 - \dfrac{1}{\omega_r C}\right)}$$

$$= \frac{\dfrac{M\bar{E}}{C}}{R_1 R_2 + \omega_r^2 M^2} \qquad \boxed{\text{Using equation(5.165)}}$$

Critical Coupling

From equations (5.167) and (5.168), we can say that the secondary current and the output voltage at resonance are functions of mutual inductance. Hence, it is possible to vary the secondary current and output voltage by varying mutual inductance, which in turn can be varied by varying coefficient of coupling k.

It can be proved that for a particular value of k called **critical coupling** k_C, the output voltage at resonance will be maximum. The mutual inductance at critical coupling is denoted by M_C. It is also called **optimum mutual inductance**, M_{opt}.

Now, critical coupling and mutual inductance at critical coupling are given by the following equations:

$$\text{Critical coupling, } k_c = \frac{M_c}{\sqrt{L_1 L_2}} \qquad .....(5.170)$$

$$\text{Mutual inductance at critical coupling, } M_c = \frac{\sqrt{R_1 R_2}}{\omega_r} \qquad .....(5.171)$$

Proof:

Cosider the output voltage of a single tuned coupled circuit at resonance.

$$\overline{V}_{o,r} = \frac{\frac{M\overline{E}}{C}}{R_1 R_2 + \omega_r^2 M^2} \qquad .....(5.172)$$

The condition for maximum $\overline{V}_{0,r}$ can be obtained by differentiating $\overline{V}_{0,r}$ with respect to M and equating $\frac{d\overline{V}_{0,r}}{dM}$ equal to zero when M = M_C.

On differentiating equation (5.172) with respect to M, we get,

$$\frac{d\overline{V}_{0,r}}{dM} = \frac{(R_1 R_2 + \omega_r^2 M^2)\frac{\overline{E}}{C} - \frac{M\overline{E}}{C}(2\omega_r^2 M)}{(R_1 R_2 + \omega_r^2 M^2)^2}$$

For $\frac{d\overline{V}_{0,r}}{dM} = 0$ the numerator should be zero. Let us equate the numerator of $\frac{d\overline{V}_{0,r}}{dM}$ as zero after replacing M by M_C.

$$\therefore \quad \frac{\overline{E}}{C}(R_1 R_2 + \omega_r^2 M_C^2) - \frac{M_C\overline{E}}{C}(2\omega_r^2 M_C) = 0$$

$$-\frac{M_C\overline{E}}{C}(2\omega_r^2 M_C) = \frac{-\overline{E}}{C}(R_1 R_2 + \omega_r^2 M_C^2)$$

$$2\omega_r^2 M_C^2 = R_1 R_2 + \omega_r^2 M_C^2$$

$$2\omega_r^2 M_C^2 - \omega_r^2 M_C^2 = R_1 R_2$$

$$\therefore \omega_r^2 M_C^2 = R_1 R_2$$

$$\therefore \quad M_C = \frac{\sqrt{R_1 R_2}}{\omega_r}$$

We know that,

$$\text{Coefficient of coupling,} \quad k = \frac{M}{\sqrt{L_1 L_2}}$$

$$\therefore \text{ Critical coefficient of coupling,} \quad k_C = \frac{M_C}{\sqrt{L_1 L_2}}$$

From equations (5.170) and (5.171), we can write,

$$k_C = \frac{\frac{\sqrt{R_1 R_2}}{\omega_r}}{\sqrt{L_1 L_2}}. \qquad .... (5.173)$$

We know that,

$$\text{Quality factor of primary coil, } Q_1 = \frac{\omega_r L_1}{R_1}; \quad \therefore \frac{R_1}{L_1} = \frac{\omega_r}{Q_1}$$

$$\text{Quality factor of secondary coil, } Q_2 = \frac{\omega_r L_2}{R_2}; \quad \therefore \frac{R_2}{L_2} = \frac{\omega_r}{Q_2}$$

Using the above equations we can write equation (5.173) as shown below:

$$k_C = \frac{1}{\omega_r}\sqrt{\frac{R_1 R_2}{L_1 L_2}} = \frac{1}{\omega_r}\sqrt{\frac{\omega_r}{Q_1}\frac{\omega_r}{Q_2}} \Rightarrow k_C = \frac{1}{\sqrt{Q_1 Q_2}} \qquad .... (5.174)$$

Let, $\bar{I}_{2,C}$ = Secondary current at critical coupling.

The expression for secondary current at critical coupling can be obtained from equation (5.168), by replacing M by M_C.

$$\therefore \bar{I}_{2,C} = \frac{j\omega_r M_C \bar{E}}{R_1 R_2 + \omega_r^2 M_C^2} = \frac{j\omega_r \frac{\sqrt{R_1 R_2}}{\omega_r}\bar{E}}{R_1 R_2 + \omega_r^2 \frac{R_1 R_2}{\omega_r^2}} = \frac{j\sqrt{R_1 R_2}\,\bar{E}}{2R_1 R_2}$$

$$= \frac{j\,\bar{E}}{2\sqrt{R_1 R_2}} \qquad ..... (5.175)$$

Let, $\bar{V}_{0,C}$ = Output voltage at critical coupling.

The expression for output voltage at critical coupling can be obtained from equation (5.168), by replacing M by M_C.

$$\therefore \bar{V}_{0,C} = \frac{\frac{M_C \bar{E}}{C}}{R_1 R_2 + \omega_r^2 M_C^2} = \frac{\frac{\sqrt{R_1 R_2}}{\omega_r}\frac{\bar{E}}{C}}{R_1 R_2 + \omega_r^2 \frac{R_1 R_2}{\omega_r^2}} = \frac{\sqrt{R_1 R_2}\,\frac{\bar{E}}{\omega_r C}}{2R_1 R_2}$$

$$= \frac{\bar{E}}{2\omega_r C\sqrt{R_1 R_2}} \qquad .....(5.176)$$

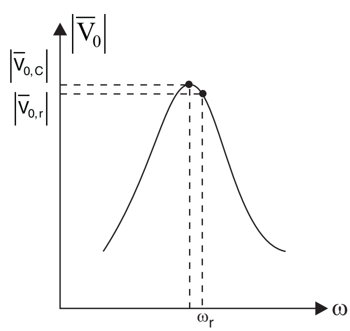


Fig. a : $|\bar{V}_0|$ *Vs* ω.

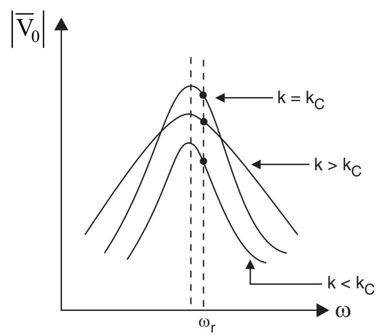


Fig. b : $|\bar{V}_0|$ *Vs* ω *for different values of k.*

Fig. 5.60 : *Frequency response of single tuned coupled circuit..*

The variation of output voltage $\bar{V}_0$ with angular frequency ω is shown in Fig. 5.60(a). From Fig. 5.60(a), it can be observed that the maximum value of $\bar{V}_0$ occurs at a frequency slightly less than the resonant frequency.

The variation of output voltage $\bar{V}_0$ with angular frequency ω for different values of k are shown in Fig. 5.60(b). From the curves of Fig. 5.60(b), it can be observed that when the value of k is above k_c, the curve becomes broader and when the value of k is less than k_c, the curve becomes narrower. In practical circuits k_c will be less than 0.5. For better selectivity, the curve should be narrow and so the value of k is less than 0.5 (or less than k_c).

5.8.2 Double Tuned Coupled Circuits

In a double tuned circuit, the capacitance may be connected either in series or parallel to the primary coil. When the capacitance is connected in series with the primary coil, the tuned circuit is called a **series fed double tuned circuit** and when the capacitance is connected in parallel with the primary coil, the tuned circuit is called a **parallel fed double tuned circuit**.

Series Fed Double Tuned Circuit

Let us connect a voltage source $\bar{E}$ with internal resistance R_g to a series fed double tuned coupled coils as shown in Fig. 5.61(a). Here, R_p and R_s are resistances of primary and secondary coil and L_p and L_s are inductances of primary and secondary coil.

Let us represent the total resistance of primary as R_1 and that of secondary as R_2 as shown in Fig. 5.61(b). Also, let us denote the inductances of primary and secondary as L_1 and L_2.

$$\therefore \quad R_1 = R_g + R_p \quad ; \quad R_2 = R_s \quad ; \quad L_1 = L_p \quad \text{and} \quad L_2 = L_s$$

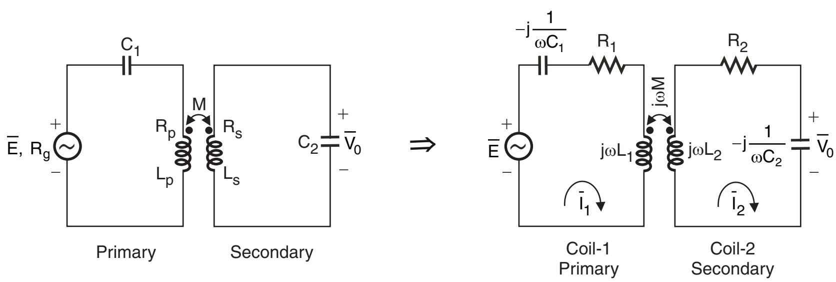


Fig. a. ***Fig. b.***

Fig. 5.61 : *Series fed double tuned coupled circuit.*

In a double tuned coupled circuit both the primary and secondary are tuned to resonate at the same frequency. Let ω_r be the frequency of resonance. At resonance, both the primary and secondary coils will behave as a purely resistive circuit.

Hence at resonance, $\omega_r L_1 - \dfrac{1}{\omega_r C_1} = 0$ and $\omega_r L_2 - \dfrac{1}{\omega_r C_2} = 0$ ..... (5.177)

$$\therefore \text{ Resonant frequency, } \omega_r = \frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{L_2 C_2}} \qquad .....(5.178)$$

Now, the secondary current and output voltage at resonance are given by the following equations:

$$\text{Secondary current at resonance, } \bar{I}_{2,r} = \frac{j\omega_r M\bar{E}}{R_1 R_2 + \omega_r^2 M^2} \qquad ..... (5.179)$$

$$\text{Output voltage at resonance, } \bar{V}_{0,r} = \frac{\dfrac{M\bar{E}}{C_2}}{R_1 R_2 + \omega_r^2 M^2} \qquad ..... (5.180)$$

Proof:

Let us assume two mesh currents $\bar{I}_1$ and $\bar{I}_2$ as shown in Fig. 5.61(b). With reference to Fig. 5.61(b), the mesh basis matrix equation is given below:

$$\begin{bmatrix} R_1 + j\omega L_1 - j\dfrac{1}{\omega C_1} & -j\omega M \\ -j\omega M & R_2 + j\omega L_2 - j\dfrac{1}{\omega C_2} \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{E} \\ 0 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} R_1 + j\omega L_1 - j\dfrac{1}{\omega C_1} & -j\omega M \\ -j\omega M & R_2 + j\omega L_2 - j\dfrac{1}{\omega C_2} \end{vmatrix}$$

$$\Delta = \left(R_1 + j\omega L_1 - j\frac{1}{\omega C_1}\right) \times \left(R_2 + j\omega L_2 - j\frac{1}{\omega C_2}\right) - (-j\omega M)^2$$

$$= \left(R_1 + j\left(\omega L_1 - \frac{1}{\omega C_1}\right)\right) \times \left(R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)\right) + \omega^2 M^2$$

$$\Delta_2 = \begin{vmatrix} R_1 + j\omega L_1 - j\dfrac{1}{\omega C_1} & \bar{E} \\ -j\omega M & 0 \end{vmatrix} = j\omega M\bar{E}$$

$$\therefore \bar{I}_2 = \frac{\Delta_2}{\Delta} = \frac{j\omega M\bar{E}}{\left(R_1 + j\left(\omega L_1 - \dfrac{1}{\omega C_1}\right)\right)\left(R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)\right) + \omega^2 M^2}$$

$$\therefore \bar{I}_{2,r} = \bar{I}_2\Big|_{\omega = \omega_r} = \frac{j\omega M\bar{E}}{\left(R_1 + j\left(\omega_r L_1 - \dfrac{1}{\omega_r C_1}\right)\right)\left(R_2 + j\left(\omega_r L_2 - \dfrac{1}{\omega_r C_2}\right)\right) + \omega_r^2 M^2}$$

$$= \frac{j\omega_r M\bar{E}}{R_1 R_2 + \omega_r^2 M^2} \qquad \boxed{\text{Using equation (5.177)}}$$

Let, $\bar{V}_0$ be the output voltage in secondary as shown in Fig. 5.61.

$$\text{Now, } \bar{V}_0 = \bar{I}_2\left(-j\frac{1}{\omega C_2}\right) = \frac{j\omega M\bar{E}\left(-j\dfrac{1}{\omega C_2}\right)}{\left(R_1 + j\left(\omega L_1 - \dfrac{1}{\omega C_1}\right)\right)\left(R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)\right) + \omega^2 M^2}$$

$$= \frac{\dfrac{M\bar{E}}{C_2}}{\left(R_1 + j\left(\omega L_1 - \dfrac{1}{\omega C_1}\right)\right)\left(R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)\right) + \omega^2 M^2}$$

$$\therefore \bar{V}_{o,r} = \bar{V}_o\Big|_{\omega = \omega_r} = \frac{\dfrac{M\bar{E}}{C_2}}{\left(R_1 + j\left(\omega_r L_1 - \dfrac{1}{\omega_r C_1}\right)\right)\left(R_2 + j\left(\omega_r L_2 - \dfrac{1}{\omega_r C_2}\right)\right) + \omega_r^2 M^2}$$

$$= \frac{\dfrac{M\bar{E}}{C_2}}{R_1 R_2 + \omega_r^2 M^2} \qquad \boxed{\text{Using equation (5.177)}}$$

Equations (5.179) and (5.180) are similar to equations (5.167) and (5.168) of single tuned circuit, respectively. Hence, the condition for maximum output voltage and the expression for critical coupling, k_C will be the same as that of a single tuned circuit. Therefore, by an analysis similar to that of a single tuned circuit, we can obtain the following equations:

Mutual inductance at critical coupling, $M_C = \dfrac{\sqrt{R_1 R_2}}{\omega_r}$ ..... (5.181)

Critical coefficient of coupling, $k_C = \dfrac{M_C}{\sqrt{L_1 L_2}}$ ..... (5.182)

or $k_C = \dfrac{1}{\omega_r}\sqrt{\dfrac{R_1 R_2}{L_1 L_2}}$

or $k_C = \dfrac{1}{\sqrt{Q_1 Q_2}}$

Secondary current at critical coupling $\left.\right\}$ $\bar{I}_{2,C} = \dfrac{j\bar{E}}{2\sqrt{R_1 R_2}}$ ..... (5.183)

Output voltage at critical coupling $\left.\right\}$ $\bar{V}_{0,C} = \dfrac{\bar{E}}{2\omega_r C_2 \sqrt{R_1 R_2}}$ ..... (5.184)

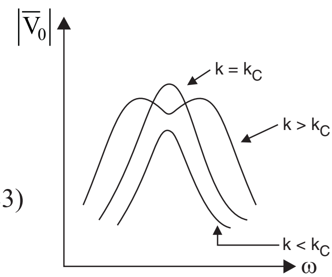


***Fig. 5.62 :** Frequency response of a double tuned circuit.*

In double tuned circuits, it can be proved that the frequency response will exhibit a double hump when the coefficient of coupling is greater than critical coupling. The variation of output voltage $\bar{V}_0$ with angular frequency ω for different values of k are shown in Fig. 5.62.

Parallel Fed Double Tuned Circuit

Let us connect a voltage source $\bar{E}$ with internal resistance R_g to a parallel fed double tuned coupled coils as shown in Fig. 5.63. Here R_p and R_s are resistances of primary and secondary coil and L_p and L_s are inductances of primary and secondary coil.

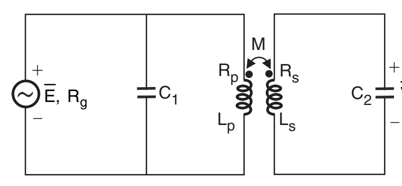


***Fig. 5.63 :** Parallel fed double tuned coupled circuit.*

The parallel fed double tuned circuit can be converted into an equivalent series fed double tuned circuit as shown below. Then the circuit can be analysed in a manner similar to that of a series fed double tuned circuit.

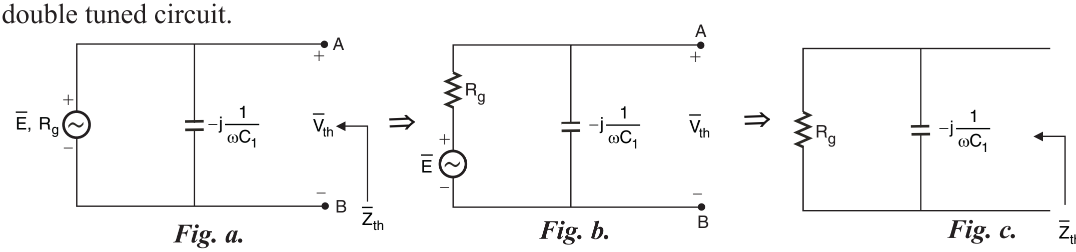


Fig. a. *Fig. b.* *Fig. c.*

***Fig. 5.64 :** To find Thevenin's equivalent of the exciting source in parallel with capacitance.*

With reference to Fig. 5.64(b) by, voltage division rule, we can write,

$$\bar{V}_{th} = \bar{E} \times \frac{-j\frac{1}{\omega C_1}}{R_g - j\frac{1}{\omega C_1}}$$

Let, $R_g >> \frac{1}{\omega C_1}$

$$\therefore \ \bar{V}_{th} = \bar{E} \times \frac{-j\frac{1}{\omega C_1}}{R_g} = -j\frac{\bar{E}}{\omega R_g C_1}$$

With reference to Fig. 5.64(c), we can write,

$$\bar{Z}_{th} = \frac{R_g\left(-j\frac{1}{\omega C_1}\right)}{R_g - j\frac{1}{\omega C_1}} = \frac{R_g\left(-j\frac{1}{\omega C_1}\right)}{R_g - j\frac{1}{\omega C_1}} \times \frac{R_g + j\frac{1}{\omega C_1}}{R_g + j\frac{1}{\omega C_1}} = \frac{-j\frac{R_g^2}{\omega C_1} + \frac{R_g}{(\omega C_1)^2}}{R_g^2 + \left(\frac{1}{\omega C_1}\right)^2}$$

Let, $R_g >> \frac{1}{\omega C_1}$

$$\therefore \ \bar{Z}_{th} = \frac{-j\frac{R_g^2}{\omega C_1} + \frac{R_g}{(\omega C_1)^2}}{R_g^2} = \frac{1}{R_g(\omega C_1)^2} - j\frac{1}{\omega C_1}$$

Using Thevenin's equivalent of the voltage source in parallel with capacitance, the parallel fed double tuned circuit of Fig. 5.63 can be drawn as shown in Fig. 5.65(a).

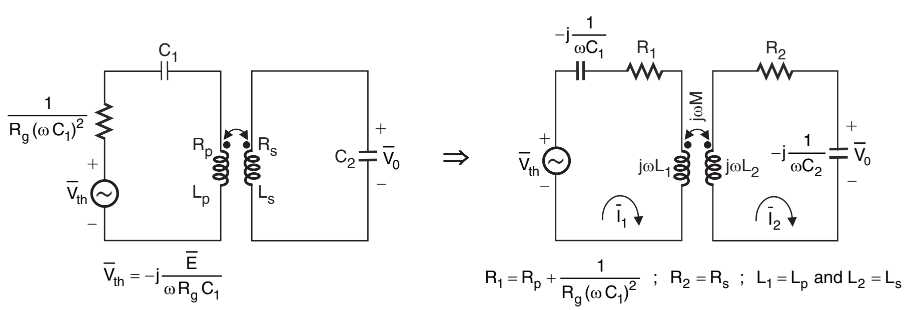


Fig. a. ***Fig. b.***

Fig. 5.65 : *Series fed equivalent of parallel fed double tuned circuit.*

The circuit of Fig. 5.65(b) is similar to that of a series fed double tuned circuit. Therefore, the analysis will be similar to that of a series fed double tuned circuit.

5.9 Solved Problems in Coupled Circuits

EXAMPLE 5.15

A coil having an inductance of 100 *mH* is magnetically coupled to another coil having an inductance of 900 *mH*. The coefficient of coupling between the coils is 0.45. Calculate the equivalent inductance if the two coils are connected in **a)** series aiding, **b)** series opposing, **c)** parallel aiding and **d)** parallel opposing.

SOLUTION

Given that, $L_1 = 100\,mH$, $L_2 = 900\,mH$ and $k = 0.45$

$\therefore$ Mutual-inductance, $M = k\sqrt{L_1 L_2} = 0.45\sqrt{100 \times 900} = 135\,mH$

a) Equivalent inductance in series aiding,
$$L_{eq} = L_1 + L_2 + 2M$$
$$= 100 + 900 + (2 \times 135) = 1270\,mH$$

b) Equivalent inductance in series opposing,
$$L_{eq} = L_1 + L_2 - 2M$$
$$= 100 + 900 - (2 \times 135) = 730\,mH$$

c) Equivalent inductance in parallel aiding,
$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$
$$= \frac{(100 \times 900) - 135^2}{100 + 900 - (2 \times 135)} = 98.3219\,mH$$

d) Equivalent inductance in parallel opposing,
$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$
$$= \frac{(100 \times 900) - 135^2}{100 + 900 + (2 \times 135)} = 56.5157\,mH$$

EXAMPLE 5.16

A primary coil having an inductance of 100 μH is connected in series with a secondary coil having an inductance of240 μH and the total inductance of the combination is measured as 146 μH. Determine the coefficient of coupling.

SOLUTION

Given that, $L_1 = 100\,\mu H$, $L_2 = 240\,\mu H$

Equivalent inductance in series = 146 μH.

Since the equivalent inductance in series is less than the sum of individual inductances, the series connection should be series opposing.

In series opposing connection,

$$L_{eq} = L_1 + L_2 - 2M$$
$$\therefore\ 2M = L_1 + L_2 - L_{eq}$$
$$\therefore\ M = \frac{L_1 + L_2 - L_{eq}}{2} = \frac{100 + 240 - 146}{2} = 97\,\mu H$$

Coefficient of coupling, $k = \dfrac{M}{\sqrt{L_1 L_2}} = \dfrac{97}{\sqrt{100 \times 240}} = 0.6261$

EXAMPLE 5.17

Two coils connected in series have an equivalent inductance of $0.8\,H$ when connected in aiding and an equivalent inductance of $0.4\,H$ when connected in opposing. Determine the mutual inductance. Calculate the self-inductance of the coils by taking $k = 0.55$.

SOLUTION

We know that,

$$\text{Equivalent inductance in series aiding, } L_{eq} = L_1 + L_2 + 2M$$

$$\text{Equivalent inductance in series opposing, } L_{eq} = L_1 + L_2 - 2M$$

Given that,

$$L_1 + L_2 + 2M = 0.8 \qquad \text{..... (1)}$$

$$L_1 + L_2 - 2M = 0.4 \qquad \text{..... (2)}$$

Let us subtract equation (2) from (1).

$$\begin{array}{rcl} L_1 + L_2 + 2M & = & 0.8 \\ L_1 + L_2 - 2M & = & 0.4 \\ (-)\quad (-)\quad (+) & & (-) \\ \hline 4M & = & 0.4 \\ \hline \end{array}$$

$$\therefore M = \frac{0.4}{4} = 0.1\,H \qquad \text{..... (3)}$$

We know that,

$$M = k\sqrt{L_1 L_2} \quad \Rightarrow \quad M^2 = k^2 L_1 L_2$$

$$\therefore L_2 = \frac{M^2}{k^2 L_1} = \frac{0.1^2}{0.55^2 \times L_1} = \frac{0.0331}{L_1} \qquad \text{...... (4)}$$

On substituting for M and L_2 from equations (3) and (4) in equation (1), we get,

$$L_1 + \frac{0.0331}{L_1} + (2 \times 0.1) = 0.8$$

On multiplying the above equation by L_1, we get,

$$L_1^2 + 0.0331 + 0.2L_1 = 0.8L_1$$

$$L_1^2 + 0.2L_1 - 0.8L_1 + 0.0331 = 0$$

$$\therefore \quad L_1^2 - 0.6L_1 + 0.0331 = 0$$

The above equation is a quadratic function of L_1. The roots of the quadratic equation will give the values of L_1.

The roots of the quadratic equation are,

$$L_1 = \frac{-(-0.6) \pm \sqrt{(-0.6)^2 - 4 \times 0.0331}}{2} = \frac{0.6 \pm 0.4771}{2}$$

$$= 0.53855 \quad \text{or} \quad 0.06145$$

$$\therefore L_1 = 0.53855\,H \quad \text{or} \quad L_1 = 0.06145\,H$$

From equation (1), we get,

$$L_2 = 0.8 - L_1 - 2M$$

When $L_1 = 0.53855H$, $L_2 = 0.8 - 0.53855 - 2 \times 0.1 = 0.06145H$

When $L_1 = 0.06145H$, $L_2 = 0.8 - 0.06145 - 2 \times 0.1 = 0.53855H$

RESULT

Mutual inductance, $M = 0.1H$

Self-inductance of coil-1, $L_1 = 0.53855H$

Self-inductance of coil-2, $L_2 = 0.06145H$

EXAMPLE 5.18

Two coupled coils with self-inductances $0.9H$ and $0.4H$ have a coupling coefficient of 0.3. Find the mutual inductance and turns ratio. What will be the maximum possible value of mutual inductance ?

SOLUTION

Given that, $L_1 = 0.9H$, $L_2 = 0.4H$ and $k = 0.3$

Mutualinductance, $M = k\sqrt{L_1 L_2} = 0.3\sqrt{0.9 \times 0.4} = 0.18\,H$

We know that, $L_1 = \dfrac{N_1 \phi_1}{I_1}$ and $L_2 = \dfrac{N_2 \phi_2}{I_2}$

$$\therefore \frac{L_1}{L_2} = \frac{\dfrac{N_1\phi_1}{I_1}}{\dfrac{N_2\phi_2}{I_2}} = \frac{N_1\phi_1}{N_2\phi_2} \times \frac{I_2}{I_1}$$

$$= \frac{N_1}{N_2} \times \frac{I_2}{I_1}$$

$$= \frac{N_1}{N_2} \times \frac{N_1}{N_2}$$

$$= \left(\frac{N_1}{N_2}\right)^2$$

In coupled coils, (or in transformer) the flux will be same.
$\therefore \phi_1 = \phi_2$

In coupled coils, $\dfrac{I_2}{I_1} = \dfrac{N_1}{N_2}$

$$\therefore \text{Turns ratio, } \frac{N_1}{N_2} = \sqrt{\frac{L_1}{L_2}} = \sqrt{\frac{0.9}{0.4}} = 1.5$$

The maximum possible value of M is achieved when $k = 1$.

$$\therefore \text{Maximum possible value of } M = k\sqrt{L_1 L_2} = 1 \times \sqrt{0.9 \times 0.4} = 0.6\,H$$

RESULT

When $k = 0.3$, Mutual inductance, $M = 0.18H$

Maximum possible value of $M = 0.6H$

Turns ratio, $\dfrac{N_1}{N_2} = 1.5$

EXAMPLE 5.19

Two coils A and B of 1200 turns and 1500 turns, respectively, lie in a parallel plane, so that 40% of flux produced by coil-A links with coil-B. A current of 4 *A* in coil-A produces a flux of 0.5×10^{-4} *Wb* while the same current in coil-B produces a flux of 0.8×10^{-4} *Wb*. Determine the coefficient of coupling between the coils.

SOLUTION

Self-inductance of coil-A, $L_A = \dfrac{N_A \phi_A}{I_A} = \dfrac{1200 \times 0.5 \times 10^{-4}}{4} = 0.015\,H$

Self-inductance of coil-B, $L_B = \dfrac{N_B \phi_B}{I_B} = \dfrac{1500 \times 0.8 \times 10^{-4}}{4} = 0.03\,H$

Mutual inductance between coil A and B, $M = \dfrac{N_B \phi_{AB}}{I_B}$

Here 40% of flux produced by coil-A links coil-B.

$$\therefore \frac{\phi_{AB}}{\phi_A} = 40\% = \frac{40}{100} = 0.4 \quad \Rightarrow \quad \phi_{AB} = 0.4\phi_A$$

$$\therefore M = \frac{N_B\,\phi_{AB}}{I_A} = \frac{1500 \times 0.4 \times 0.5 \times 10^{-4}}{4} = 0.0075\,H$$

Coefficient of coupling, $k = \dfrac{M}{\sqrt{L_1 L_2}} = \dfrac{0.0075}{\sqrt{0.015 \times 0.03}} = 0.3536$

EXAMPLE 5.20

(AU May'17 & Dec'16, 8 Marks)

Consider two coils A and B consisting of 500 turns and 1500 turns, respectively. A current of 5 *A* in coil-A produces a flux of 0.6×10^{-3} *Wb* and the flux linking coil-B is 0.3×10^{-3} *Wb*. Determine the inductance, coefficient of coupling and mutual inductance of the coils.

SOLUTION

Self-inductance of coil-A, $L_A = \dfrac{N_A \phi_A}{I_A} = \dfrac{500 \times 0.6 \times 10^{-3}}{5} = 0.06\,H = 60 \times 10^{-3}\,H = 60\,mH$

Self-inductance of coil-B, $L_B = \dfrac{N_B \phi_B}{I_B} = \dfrac{1500 \times 0.3 \times 10^{-3}}{5} = 0.09\,H = 90 \times 10^{-3} H = 90\,mH$

Coefficient of coupling, $k = \dfrac{\text{Flux linking coil–A and coil–B}}{\text{Total flux}} = \dfrac{0.3 \times 10^{-3}}{0.6 \times 10^{-3}} = 0.5$

$\therefore$ Mutual-inductance, $M = k\sqrt{L_A L_B} = 0.5\sqrt{60 \times 90} = 36.7423\,mH$

EXAMPLE 5.21

Two magnetically coupled coils are connected in series and their total effective inductance is found to be 4.4 *mH*. When one coil is reversed in connection, the combined inductance drops to 1.6 *mH*. Here all the flux due to the first coil links the second coil but only 40% of the flux due to the second coil links with the first coil. Find the self-inductance of each coil and the mutual inductance between the coils.

SOLUTION

We know that,

Equivalent inductance in series aiding, $L_{eq} = L_1 + L_2 + 2M$

Equivalent inductance in series opposing, $L_{eq} = L_1 + L_2 - 2M$

In series aiding connection, the equivalent inductance will be more than that in series opposing connection. Therefore, 4.4 *mH* is the equivalent inductance in series aiding and 1.6 *mH* is the equivalent inductance in series opposing.

$$\therefore\ L_1 + L_2 + 2M = 4.4 \qquad .....(1)$$

$$L_1 + L_2 - 2M = 1.6 \qquad .....(2)$$

Let us subtract equation (2) from (1).

$$\begin{array}{rcl} L_1 + L_2 + 2M &=& 4.4 \\ L_1 + L_2 - 2M &=& 1.6 \\ (-)\quad (-)\quad (+) & & (-) \\ \hline 4M &=& 2.8 \\ \hline \end{array}$$

$$\therefore\ M = \frac{2.8}{4} = 0.7\ mH \qquad .....(3)$$

Here all the flux produced by coil-1 links coil-2.

$$\therefore\ \text{Coefficient of coupling between coils 1 and 2, } k_1 = \frac{\phi_{12}}{\phi_1} = 1$$

Here only 40% of the flux produced by coil-2 links coil-1.

$$\therefore\ \text{Coefficient of coupling between coils 2 and 1, } k_2 = \frac{\phi_{21}}{\phi_2} = 0.4$$

Now coefficient of coupling, $k = \sqrt{k_1 k_2} = \sqrt{1 \times 0.4} = \sqrt{0.4}$

We know that,

$$M = k\sqrt{L_1 L_2} \quad \Rightarrow \quad M^2 = k^2 L_1 L_2$$

$$\therefore\ L_2 = \frac{M^2}{k^2 L_1} = \frac{0.7^2}{\left(\sqrt{0.4}\right)^2 L_1} = \frac{1.225}{L_1} \qquad .....(4)$$

On substituting for M and L_2 from equations (3) and (4) in equation (1), we get,

$$L_1 + \frac{1.225}{L_1} + \left(2 \times 0.7\right) = 4.4$$

On multiplying the above equation by L_1, we get,

$$L_1^2 + 1.225 + 1.4L_1 = 4.4L_1 \quad \Rightarrow \quad L_1^2 + 1.4L_1 - 4.4L_1 + 1.225 = 0$$

$$\therefore\ L_1^2 - 3L_1 + 1.225 = 0$$

The above equation is a quadratic function of L_1. The roots of the quadratic equation will give the values of L_1.

The roots of the quadratic equation are,

$$L_1 = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 1.225}}{2} = \frac{3 \pm 2.0248}{2}$$

$$= 2.5124 \quad \text{or} \quad 0.4876$$

$$\therefore\ L_1 = 2.5124\ mH \quad \text{or} \quad L_1 = 0.4876\ mH$$

From equation (1), we get,

$$L_2 = 4.4 - L_1 - 2M$$

When $L_1 = 2.5124\ mH$, $L_2 = 4.4 - 2.5124 - 2 \times 0.7 = 0.4876\ mH$

When $L_1 = 0.4876\ mH$, $L_2 = 4.4 - 0.4876 - 2 \times 0.7 = 2.5124\ mH$

RESULT

Mutual inductance, $M = 0.7\,mH$

Self-inductance of coil-1, $L_1 = 2.5124\,mH$

Self-inductance of coil-2, $L_2 = 0.4876\,mH$

EXAMPLE 5.22

Determine the effective inductance of the series-connected coupled coils shown in Figs 1, 2 and 3.

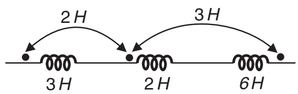


Fig. 1.

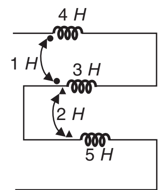


Fig. 2.

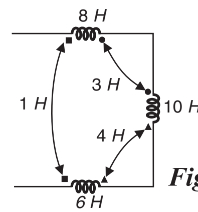


Fig. 3.

SOLUTION

a) To find the equivalent inductance of series-connected coils in Fig. 1

Consider the series-connected coils shown in Fig. 1. There are two mutual couplings. Let us remove them one by one.

Let $\bar{I}$ be the current through the series-connected coils as shown in Fig. 4.

The coupling between $3\,H$ and $2\,H$ coils is additive because the current $\bar{I}$ enters at the dotted end in both the coils. Hence, the magnetic coupling is eliminated by adding the mutual inductance $2\,H$ to the self-inductances as shown in Fig. 5.

The coupling between $4\,H$ and $6\,H$ coil of Fig. 5 is opposive, because the current $\bar{I}$ enters at the dotted end in one coil and leaves at the dotted end in the other coil. Hence, the magnetic coupling is eliminated by subtracting the mutual inductance $3\,H$ from the self-inductances as shown in Fig. 6.

Now, the series-connected inductances of Fig. 6 can be added to give an equivalent inductance as shown in Fig. 7.

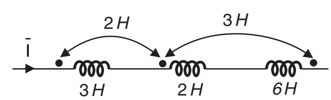


Fig. 4.

⇓

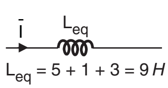


Fig. 7.

⇐

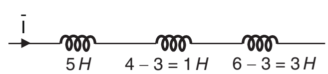


Fig. 6.

⇐

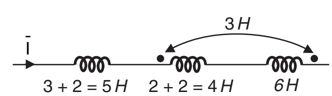


Fig. 5.

b) To find the equivalent inductance of series-connected coils in Fig. 2

In Fig. 2, there are two mutual couplings. Let us remove them one by one as shown below:

Let $\bar{I}$ be the current through the series combination as shown in Fig. 8. Here both the couplings are opposive because the current enters at the dotted end in one coil and leaves at the dotted end in the other coil.

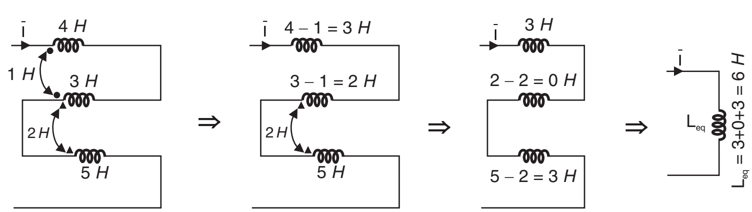


***Fig. 8 :** Equivalent inductance of the series connected coils shown in Fig. 2.*

c) To find the equivalent inductance of series-connected coils in Fig. 3

In Fig. 3, there are three mutual couplings. Let us remove them one by one as shown below:

Let $\bar{I}$ be the current through the series combination as shown in Fig. 9. Here all the couplings are opposive because the current enters at the dotted end in one coil and leaves at the dotted end in the other coil.

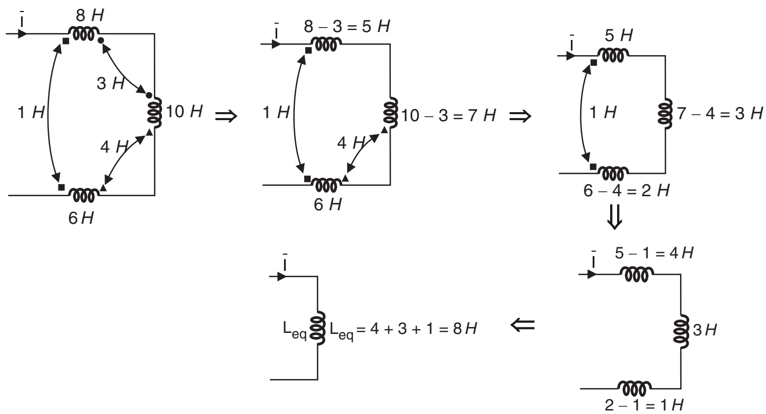


Fig. 9 : Equivalent inductance of the series connected coils shown in Fig. 3.

Alternate Method

Alternatively, the magnetic coupling in two series-connected coils can be represented by an additional inductance of value +2M or −2M in series with the coils.

When current enters (or leaves) at the dotted ends in both the coils, the flux is aiding and so it is represented by an additional inductance of +2M in series with the coils.

When current enters at the dotted end in one coil and leaves at the dotted end in other coil, the flux is opposing and so it is represented by an additional inductance of −2M in series with the coils.

The estimation of equivalent inductance of the series-connected coils by this method is illustrated diagrammatically here.

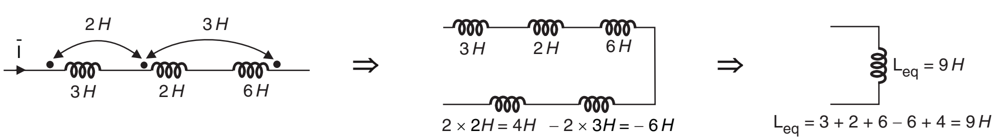


Fig. 10 : Equivalent inductance of the series connected coils shown in Fig. 1.

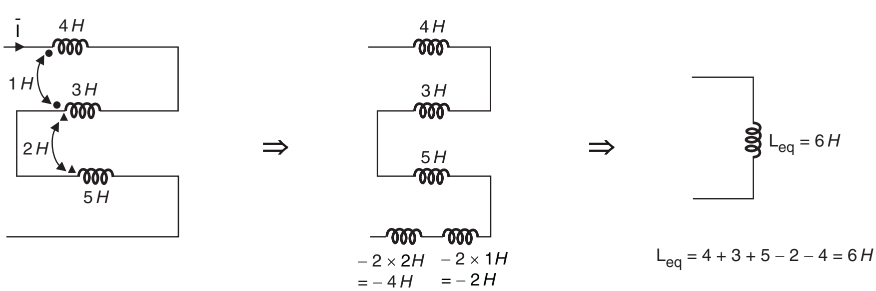


Fig. 11 : Equivalent inductance of the series connected coils shown in Fig. 2.

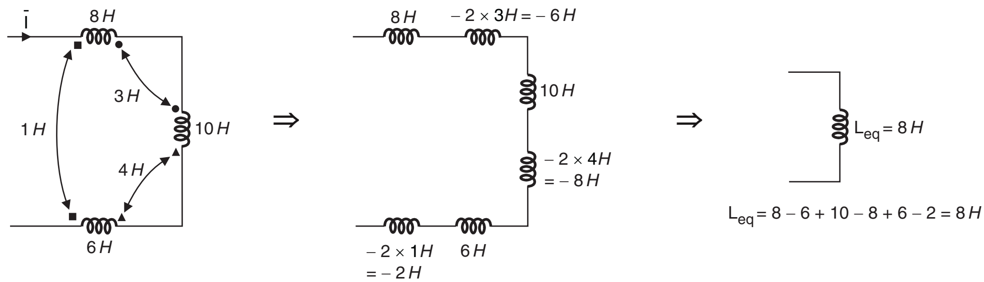


***Fig. 12 :** Equivalent inductance of the series connected coils shown in Fig. 3.*

EXAMPLE 5.23

Determine the equivalent inductance of the series-parallel-connected coupled coils shown in Fig. 1.

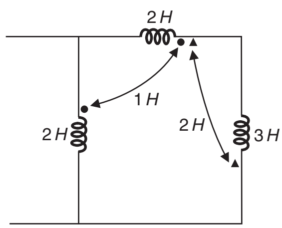


Fig. 1.

SOLUTION

Let current through the parallel branches be $\bar{I}_a$ and $\bar{I}_b$ as shown in Fig. 2.

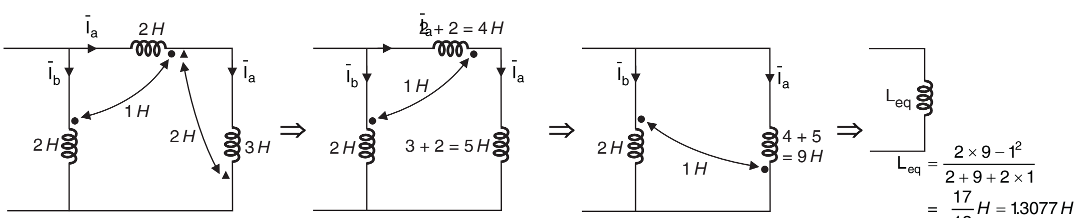


***Fig. 2 :** Equivalent inductance of the inductive network of Fig. 1.*

The coupling between the series-connected coils $2H$ and $3H$ is additive because the current $\bar{I}_a$ leaves at the dotted end in both the coils. Hence, the magnetic coupling can be eliminated by adding the mutual inductance to the self-inductances as shown in Fig. b. Then the series-connected inductances $4H$ and $5H$ are combined to form a single equivalent as shown in Fig. c.

The coupling between parallel-connected coils $2H$ and $9H$ is opposive because the current enters at the dotted end in one coil and leaves at the dotted end in the other coil. This parallel-connected coupled coil is combined to a single equivalent as shown in Fig. d.

The equivalent inductance in parallel opposing, $L_{eq} = \dfrac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$

Alternate Method

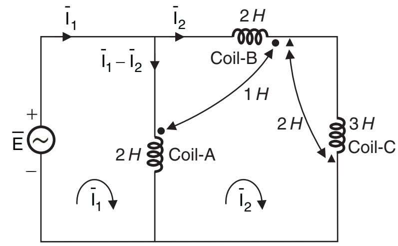


Fig. 3.

Alternatively, we can estimate the looking back impedance (inductive reactance) from the two terminals of the inductive network by connecting a source of value $\bar{E}$ at the two terminals of the network. The looking back inductive reactance is the equivalent inductive reactance at the two terminals, which is given by the ratio of voltage to current at the two terminals.

Let us connect a sinusoidal voltage source of value $\bar{E}$ as shown in Fig. 3. Let $\bar{I}_1$ and $\bar{I}_2$ be the mesh currents.

Now the equivalent inductive reactance, $j\omega L_{eq} = \dfrac{\bar{E}}{\bar{I}_1}$

Let us name the coils as coil-A, coil-B and coil-C as shown in Fig. 3. Now, $\bar{I}_1 - \bar{I}_2$ be the current through coil-A and $\bar{I}_2$ be the current through coil-B and coil-C. The current flowing in each coupled coil will induce an emf in the other coil. Therefore, in the circuit of Fig. 3, there will be four mutual induced emfs as explained below :

Emf-1 : The current $\bar{I}_1 - \bar{I}_2$ entering at the dotted end in coil-A will induce an emf $j\omega\left(\bar{I}_1 - \bar{I}_2\right)$ in coil-B such that the dotted end is positive.

Emf-2 : The current $\bar{I}_2$ entering at the undotted end in coil-B will induce an emf $j\omega\bar{I}_2$ in coil-A such that the undotted end is positive.

Emf-3 : The current $\bar{I}_2$ entering at the undotted end in coil-B will induce an emf $j2\omega\bar{I}_2$ in coil-C such that the undotted end is positive.

Emf-4 : The current $\bar{I}_2$ entering at the undotted end in coil-C will induce an emf $j2\omega\bar{I}_2$ in coil-B such that the undotted end is positive.

The self- and mutual induced emfs in the coils are shown in Fig. 4.

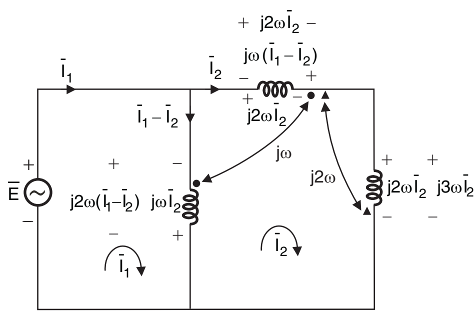


Fig. 4.

By KVL in mesh-1,

$$j2\omega(\bar{I}_1 - \bar{I}_2) = j\omega\bar{I}_2 + \bar{E}$$

$$\therefore\ j2\omega\bar{I}_1 - j3\omega\bar{I}_2 = \bar{E}$$

.....(1)

By KVL in mesh-2,

$$j\omega\bar{I}_2 + j2\omega\bar{I}_2 + j2\omega\bar{I}_2 + j2\omega\bar{I}_2 + j3\omega\bar{I}_2 = j2\omega(\bar{I}_1 - \bar{I}_2) + j\omega(\bar{I}_1 - \bar{I}_2)$$

$$\therefore\ j3\omega\bar{I}_1 + j13\omega\bar{I}_2 = 0 \qquad .....(2)$$

On arranging equations (1) and (2) in matrix form, we get,

$$\begin{bmatrix} j2\omega & -j3\omega \\ -j3\omega & j13\omega \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{E} \\ 0 \end{bmatrix}$$

$$\text{Now, } \Delta_1 = \begin{vmatrix} \bar{E} & -j3\omega \\ 0 & j13\omega \end{vmatrix} = \bar{E} \times j13\omega - 0 = j13\omega\bar{E}$$

$$\Delta = \begin{vmatrix} j2\omega & -j3\omega \\ -j3\omega & j13\omega \end{vmatrix} = j2\omega \times j13\omega - \left(-j3\omega\right)^2$$

$$= -26\omega^2 + 9\omega^2 = -17\omega^2$$

$$\bar{I}_1 = \frac{\Delta_1}{\Delta} = \frac{j13\omega\bar{E}}{-17\omega^2} = \frac{13\bar{E}}{j17\omega}$$

$$\therefore \frac{\bar{E}}{\bar{I}_1} = \frac{j17\omega}{13}$$

$$\text{Here, } j\omega L_{eq} = \frac{\bar{E}}{\bar{I}_1} \quad , \quad \therefore j\omega L_{eq} = \frac{j17\omega}{13} \quad \Rightarrow \quad L_{eq} = \frac{17}{13}\,H = 1.3077\,H$$

RESULT

$$\text{Equivalent inductance, } L_{eq} = \frac{17}{13}\,H = 1.3077\,H$$

EXAMPLE 5.24

Determine the equivalent inductance of the inductive network with coupled coils shown in Fig. 1.

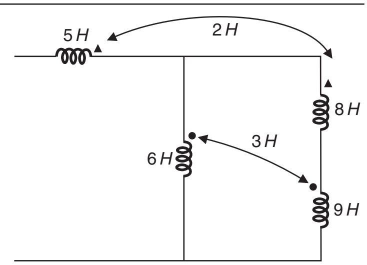


Fig. 1.

SOLUTION

Let us connect a sinusoidal voltage source of value, $\bar{E}$ as shown in Fig. 2. Let, $\bar{I}_1$ and $\bar{I}_2$ be the mesh currents.

$$\text{Now, Equivalent inductive reactance, } j\omega L_{eq} = \frac{\bar{E}}{\bar{I}_1}$$

Let us name the coils as shown in Fig. 2. Now, $\bar{I}_1$ is the current through coil-A, $\bar{I}_1 - \bar{I}_2$ is the current through coil-B and $\bar{I}_2$ is the current through coils C and D.

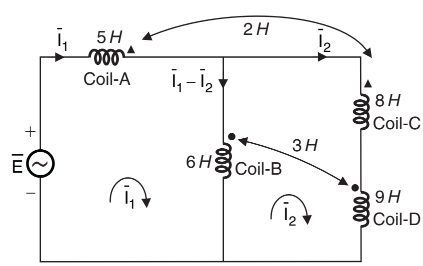


Fig. 2.

The current flowing in each coupled coil will induce an emf in the other coil. Therefore, in the circuit of Fig. 2, there will be four mutual induced emfs as explained below:

Emf-1 : The current $\bar{I}_1$ entering at the undotted end in coil-A will induce an emf $j2\omega\bar{I}_1$ in coil-C such that the undotted end is positive.

Emf-2 : The current $\bar{I}_1 - \bar{I}_2$ entering at the dotted end in coil-B will induce an emf $j3\omega\left(\bar{I}_1 - \bar{I}_2\right)$ in coil-D such that the dotted end is positive.

Emf-3 : The current $\bar{I}_2$ entering at the dotted end in coil-C will induce an emf $j2\omega\bar{I}_2$ in coil-A such that the dotted end is positive.

Emf-4 : The current $\bar{I}_2$ entering at the dotted end in coil-D will induce an emf $j3\omega\bar{I}_2$ in coil-B such that the dotted end is positive.

The self- and mutual induced emfs in the coils are shown in Fig. 3.

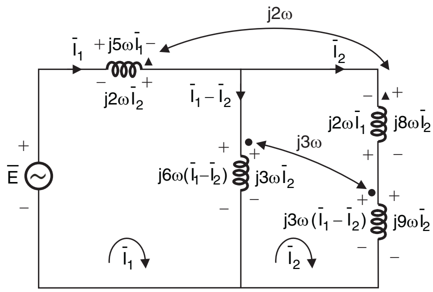


Fig. 3.

By KVL in mesh-1,

$$j5\omega\bar{I}_1 + j6\omega(\bar{I}_1 - \bar{I}_2) + j3\omega\bar{I}_2 = j2\omega\bar{I}_2 + \bar{E}$$

$$\therefore\ j11\omega\bar{I}_1 - j5\omega\bar{I}_2 = \bar{E} \quad .....(1)$$

By KVL in mesh-2,

$$j8\omega\bar{I}_2 + j9\omega\bar{I}_2 + j3\omega(\bar{I}_1 - \bar{I}_2) = j6\omega(\bar{I}_1 - \bar{I}_2) + j3\omega\bar{I}_2 + j2\omega\bar{I}_1$$

$$\therefore -j5\omega\bar{I}_1 + j17\omega\bar{I}_2 = 0 \quad .....(2)$$

On arranging equations (1) and (2) in matrix form we get,

$$\begin{bmatrix} j11\omega & -j5\omega \\ -j5\omega & j17\omega \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{E} \\ 0 \end{bmatrix}$$

Now, $\Delta_1 = \begin{vmatrix} \bar{E} & -j5\omega \\ 0 & j17\omega \end{vmatrix} = \bar{E} \times j17\omega - 0 = j17\omega\bar{E}$

$$\Delta = \begin{vmatrix} j11\omega & -j5\omega \\ -j5\omega & j17\omega \end{vmatrix} = j11\omega \times j17\omega - \left(-j5\omega\right)^2$$

$$= -187\omega^2 + 25\omega^2 = -162\omega^2$$

$$\bar{I}_1 = \frac{\Delta_1}{\Delta} = \frac{j17\omega\bar{E}}{-162\omega^2} = \frac{17}{j162\omega}\bar{E}$$

$$\therefore \frac{\bar{E}}{\bar{I}_1} = \frac{j162}{17}\omega$$

Here, $\frac{\bar{E}}{\bar{I}_1} = j\omega L_{eq}$, $\therefore\ j\omega L_{eq} = \frac{j162}{17}\omega \quad \Rightarrow \quad L_{eq} = \frac{162}{17}\,H$

RESULT

Equivalent inductance, $L_{eq} = \frac{162}{17}\,H = 9.5294\,H$

EXAMPLE 5.25

Determine the equivalent impedance of the parallel-connected impedance with magnetic coupling shown in Fig. 1.

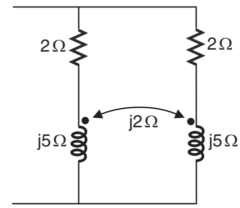


Fig. 1.

SOLUTION

Let us assume the current through parallel arms as $\bar{I}_A$ and $\bar{I}_B$ as shown in Fig. 2. Now the current enters at the dotted end in both the coils. Hence, the connection is parallel aiding.

The magnetic coupling in parallel aiding can be represented as shown in Fig. 3.

Hence, the network of Fig. 1 can be redrawn as shown in Fig. 4.

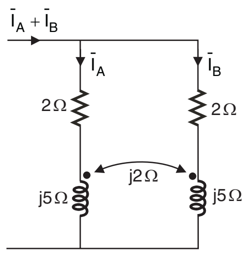


Fig. 2.

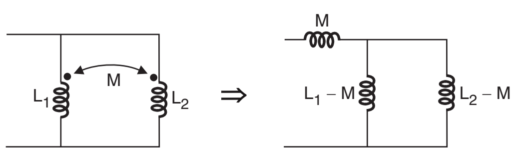


Fig. 3.

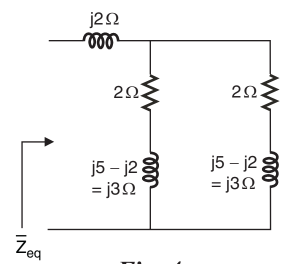


Fig. 4.

The equivalent impedance can be obtained by reducing the network of Fig. 4 to a single equivalent impedance as shown below:

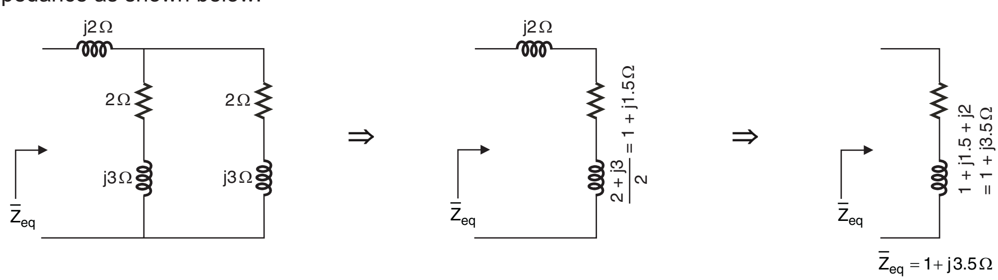


***Fig. 5 :** Equivalent impedance of the network shown in Fig. 1.*

Alternate Method

Let us connect a sinusoidal voltage source of value $\bar{E}$ as shown in Fig. 6. Let, $\bar{I}_1$ and $\bar{I}_2$ be the mesh currents as shown in Fig. 6.

$$\text{Now, Equivalent impedance, } \bar{Z}_{eq} = \frac{\bar{E}}{\bar{I}_1}$$

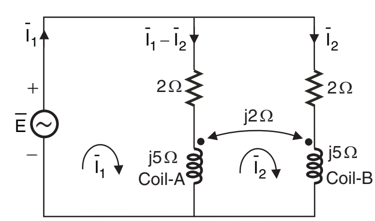


Fig. 6.

Let us name the coils as coil-A and coil-B. Now the current through coil-A is $\bar{I}_1 - \bar{I}_2$ and the current through coil-B is $\bar{I}_2$.

The current $\bar{I}_1 - \bar{I}_2$ flowing in coil-A will induce an emf $j2\left(\bar{I}_1 - \bar{I}_2\right)$ in coil-B. Since $\bar{I}_1 - \bar{I}_2$ enters at the dotted end in coil-A, the sign of this emf will be positive at the dotted end in coil-B.

The current $\bar{I}_2$ flowing in coil-B will induce an emf $j2\bar{I}_2$ in coil-A. Since $\bar{I}_2$ enters at the dotted end in coil-B, the sign of this induced emf will be positive at the dotted end in coil-A.

The self- and mutual induced emfs in the coils are shown in Fig. 7.

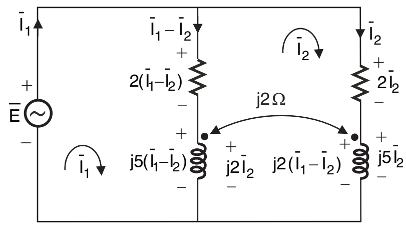


Fig. 7.

By KVL in mesh-1,

$$2(\bar{I}_1 - \bar{I}_2) + j5(\bar{I}_1 - \bar{I}_2) + j2\bar{I}_2 = \bar{E}$$

$$\therefore\ (2 + j5)\bar{I}_1 + (-2 - j3)\bar{I}_2 = \bar{E} \qquad .....(1)$$

By KVL in mesh-2,

$$2\bar{I}_2 + j5\bar{I}_2 + j2(\bar{I}_1 - \bar{I}_2) = j5(\bar{I}_1 - \bar{I}_2) + j2\bar{I}_2 + 2(\bar{I}_1 - \bar{I}_2)$$

$$\therefore (-2 - j3)\bar{I}_1 + (4 + j6)\bar{I}_2 = 0 \qquad .....(2)$$

On arranging equations (1) and (2) in matrix form, we get,

$$\begin{bmatrix} 2+j5 & -2-j3 \\ -2-j3 & 4+j6 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{E} \\ 0 \end{bmatrix}$$

Now, $\Delta_1 = \begin{vmatrix} \bar{E} & -2-j3 \\ 0 & 4+j6 \end{vmatrix} = \bar{E} \times \left(4 + j6\right) - 0 = \left(4 + j6\right)\bar{E}$

$$\Delta = \begin{vmatrix} 2+j5 & -2-j3 \\ -2-j3 & 4+j6 \end{vmatrix} = \left(2 + j5\right) \times \left(4 + j6\right) - \left(-2 - j3\right)^2 = -17 + j20$$

$$\bar{I}_1 = \frac{\Delta_1}{\Delta} = \frac{\left(4 + j6\right)\bar{E}}{-17 + j20} \quad \Rightarrow \quad \frac{\bar{E}}{\bar{I}_1} = \frac{-17 + j20}{4 + j6}$$

$$\therefore\ \bar{Z}_{eq} = \frac{\bar{E}}{\bar{I}_1} = \frac{-17 + j20}{4 + j6} = 1 + j3.5\,\Omega$$

EXAMPLE 5.26

A transformer with a primary having R_p = 100 Ω and L_p = 0.1 *H* and a secondary having R_s = 40 Ω and L_s = 0.4 *H* is connected between source voltage of 200 *V* at 159.2 *Hz* and a load resistance of 500 Ω. Determine the load current if k = 0.1.

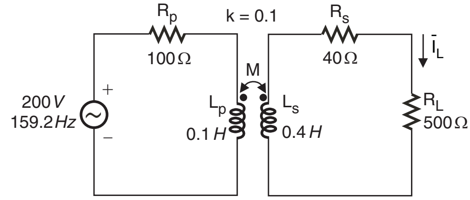


Fig. 1.

SOLUTION

The transformer connected between the source and load can be represented by the circuit shown in Fig. 1.

Let, $\bar{I}_L$ = Load current.

$$\text{Primary inductive reactance,} \quad X_p = 2\pi fL_p = 2 \times \pi \times 159.2 \times 0.1 = 100.0283\,\Omega \approx 100\,\Omega$$

$$\text{Secondary inductive reactance,} \quad X_s = 2\pi fL_s = 2 \times \pi \times 159.2 \times 0.4 = 400.1132\,\Omega \approx 400\,\Omega$$

$$\text{Mutual inductance}, \quad M = k\sqrt{L_s L_p} = 0.1\sqrt{0.1 \times 0.4} = 0.02\,H$$

$$\text{Mutual reactance,} \quad X_m = 2\pi fM = 2 \times \pi \times 159.2 \times 0.02 = 20.0057\,\Omega \approx 20\,\Omega$$

The frequency domain representation of the transformer connected between the source and load is shown in Fig. 2. Let $\bar{I}_1$ and $\bar{I}_2$ be the mesh currents. Now the current enters at the dotted end in one coil and leaves at the dotted end in the other coil, and so the fluxes are opposing. Also, the mesh currents are in the same orientation. Hence, it is group-1 coupled coil. The electrical equivalent of the circuit of Fig. 2 is shown in Fig. 3.

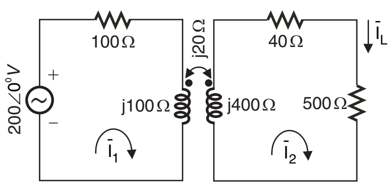


Fig. 2.

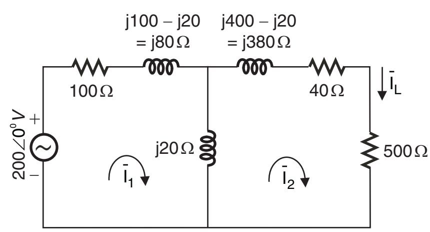


Fig. 3.

With reference to Fig. 3, the mesh basis matrix equation is,

$$\begin{bmatrix} 100 + j80 + j20 & -j20 \\ -j20 & j20 + j380 + 40 + 500 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} 200\angle 0^\circ \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 100 + j100 & -j20 \\ -j20 & 540 + j400 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} 200 \\ 0 \end{bmatrix}$$

$$\text{Now,} \quad \Delta = \begin{vmatrix} 100 + j100 & -j20 \\ -j20 & 540 + j400 \end{vmatrix} = (100 + j100) \times (540 + j400) - (-j20)^2$$

$$= 14400 + j94000$$

$$\Delta_2 = \begin{vmatrix} 100 + j100 & 200 \\ -j20 & 0 \end{vmatrix} = 0 - (-j20) \times 200 = j4000$$

$$\text{Load current,} \quad \bar{I}_L = \bar{I}_2 = \frac{\Delta_2}{\Delta} = \frac{j4000}{14400 + j94000} = 0.0416 + j0.0064\,A$$

$$= 0.0421\angle 8.7^\circ A = 42.1 \times 10^{-3}\angle 8.7^\circ A = 42.1\angle 8.7^\circ mA$$

EXAMPLE 5.27

(AU Dec'15, 16 Marks)

Find the mutual reactance X_m in the coupled coils shown in Fig. 1, if the average power in 8 Ω resistance is 100 *W*.

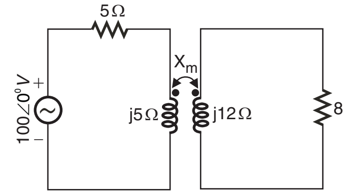


Fig. 1.

SOLUTION

Let us assume two mesh currents $\bar{I}_1$ and $\bar{I}_2$ as shown in Fig. 2. Now the current enters at the dotted end in one coil and leaves at the dotted end in the other coil. So the fluxes are opposing. Also, the mesh currents are in the same orientation. Hence, it is group-1 coupled coil. The electrical equivalent of the coupled circuit is shown in Fig. 3.

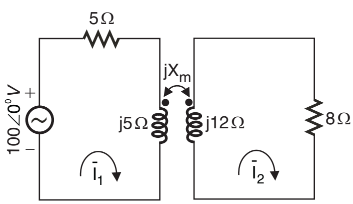


Fig. 2.

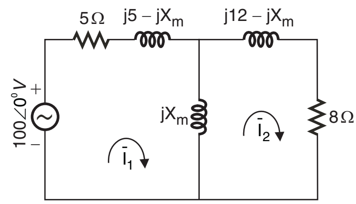


Fig. 3.

With reference to Fig. 3, the mesh basis matrix equation is,

$$\begin{bmatrix} 5+j5-jX_m+jX_m & -jX_m \\ -jX_m & jX_m+j12-jX_m+8 \end{bmatrix}\begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} 100\angle 0^\circ \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5+j5 & -jX_m \\ -jX_m & 8+j12 \end{bmatrix}\begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

$$\text{Now, } \Delta = \begin{vmatrix} 5+j5 & -jX_m \\ -jX_m & 8+j12 \end{vmatrix} = (5+j5)\times(8+j12)-(-jX_m)^2$$

$$= -20 + j100 + X_m^2 = X_m^2 - 20 + j100$$

$$\Delta_2 = \begin{vmatrix} 5+j5 & 100 \\ -jX_m & 0 \end{vmatrix} = 0-(-jX_m)\times 100 = j100X_m$$

$$\text{Now, } \bar{I}_2 = \frac{\Delta_2}{\Delta} = \frac{j100X_m}{X_m^2-20+j100} \quad .....(1)$$

Given that,

Power in 8 Ω resistance = 100 *W*

$$\text{Here, Power in } 8\,\Omega \text{ resistance} = \left|\bar{I}_2\right|^2 \times 8$$

$$\therefore \left|\bar{I}_2\right|^2 \times 8 = 100$$

Using equation (1)

$$\left|\frac{j100X_m}{X_m^2-20+j100}\right|^2 \times 8 = 100$$

$$\therefore \left(\frac{100X_m}{\sqrt{\left(X_m^2-20\right)^2+100^2}}\right)^2 \times 8 = 100 \quad \Rightarrow \quad \frac{80000X_m^2}{\left(X_m^2-20\right)^2+100^2} = 100$$

$$\therefore\ \left(X_m^2 - 20\right)^2 + 100^2 = \frac{80000X_m^2}{100} \quad \Rightarrow \quad X_m^4 + 400 - 40X_m^2 + 10000 = 800\,X_m^2$$

$$\therefore\ X_m^4 - 40X_m^2 - 800X_m^2 + 10400 = 0 \quad \Rightarrow \quad X_m^4 - 840X_m^2 + 10400 = 0$$

Let, $X_m^2 = X$

Now, $X^2 - 840X + 10400 = 0$

The roots of the quadratic are,

$$X = \frac{-(-840) \pm \sqrt{(-840)^2 - 4 \times 10400}}{2} = \frac{840 \pm 814.862}{2}$$

$$= 827.431 \quad \text{or} \quad 12.569$$

Let us take smaller value of X for realizability.

Now, $X_m^2 = X$

$$\therefore\ X_m = \sqrt{X} = \sqrt{12.569} = 3.5453\,\Omega$$

RESULT

Mutual reactance, $X_m = 3.5453\,\Omega$

EXAMPLE 5.28 *(AU June'16, 8 Marks)*

Determine the equivalent conductively coupled circuit for the magnetically coupled circuit as shown Fig. 1 and solve the mesh currents.

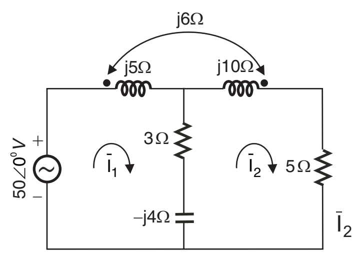


Fig. 1.

SOLUTION

Let us name the coupled coils as coil-A and coil-B as shown in Fig. 2. The current $\bar{I}_1$ entering at the dotted end in coil-A will induce an emf $j6\bar{I}_1$ in coil-B such that the dotted end is positive. The current entering at the undotted end in coil-B will induce an emf $j6\bar{I}_2$ in coil-A such that the undotted end is positive.

The self- and mutual induced emfs in the coupled coils are shown in Fig. 2.

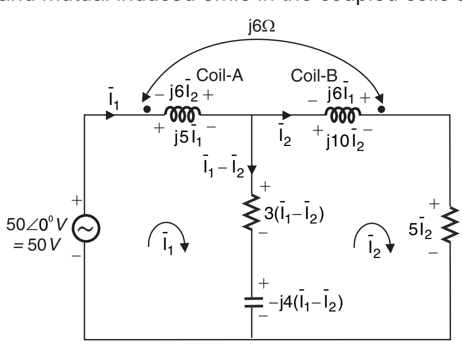


Fig. 2.

By KVL in mesh-1,

$$j5\bar{I}_1 + 3(\bar{I}_1 - \bar{I}_2) + [-j4(\bar{I}_1 - \bar{I}_2)] = 50 + j6\bar{I}_2$$

$$j5\bar{I}_1 + 3\bar{I}_1 - 3\bar{I}_2 - j4\bar{I}_1 + j4\bar{I}_2 - j6\bar{I}_2 = 50$$

$$\therefore (3+j)\,\bar{I}_1 - (3+j2)\,\bar{I}_2 = 50 \qquad .....(1)$$

By KVL in mesh-2,

$$j10\bar{I}_2 + 5\bar{I}_2 = -j4(\bar{I}_1 - \bar{I}_2) + 3(\bar{I}_1 - \bar{I}_2) + j6\bar{I}_1$$

$$j10\bar{I}_2 + 5\bar{I}_2 + j4\bar{I}_1 - j4\bar{I}_2 - 3\bar{I}_1 + 3\bar{I}_2 - j6\bar{I}_1 = 0$$

$$\therefore -(3+j2)\,\bar{I}_1 + (8+j6)\,\bar{I}_2 = 0 \qquad .....(2)$$

Using equations (1) and (2) the conductively coupled equivalent circuit can be drawn as shown in Fig. 3.

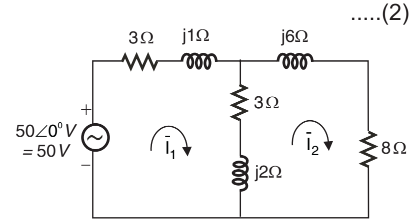


Fig. 3.

On arranging equations (1) and (2) in matrix form we get,

$$\begin{bmatrix} 3+j & -(3+j2) \\ -(3+j2) & 8+j6 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \end{bmatrix}$$

$$\text{Now, } \Delta = \begin{vmatrix} 3+j & -(3+j2) \\ -(3+j2) & 8+j6 \end{vmatrix} = (3+j)\times(8+j6) - (3+j2)^2 = 13+j14$$

$$\Delta_1 = \begin{vmatrix} 50 & -(3+j2) \\ 0 & 8+j6 \end{vmatrix} = 50\times(8+j6) - 0 = 400+j300$$

$$\Delta_2 = \begin{vmatrix} 3+j & 50 \\ -(3+j2) & 0 \end{vmatrix} = 0 - [-(3+j2)\times 50] = 150+j100$$

$$\therefore\ \bar{I}_1 = \frac{\Delta_1}{\Delta} = \frac{400+j300}{13+j14} = 25.7534 - j4.6575\ A = 26.1712\angle -10.3^\circ\ A$$

$$\bar{I}_2 = \frac{\Delta_2}{\Delta} = \frac{150+j100}{13+j14} = 9.1781 - j2.1918\ A = 9.4432\angle -13.4^\circ\ A$$

EXAMPLE 5.29

Calculate the primary and secondary current in the coupled circuit shown in Fig. 1.

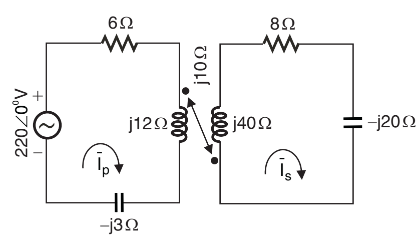


Fig. 1.

SOLUTION

The current $\bar{I}_p$ entering at the dotted end in the primary coil will induce an emf $j10\bar{I}_p$ in the secondary such that the dotted end is positive. The current $\bar{I}_s$ entering at the dotted end in the secondary coil will induce an emf $j10\bar{I}_s$ in the primary such that the dotted end is positive. The self- and mutual induced emfs are shown in Fig. 2.

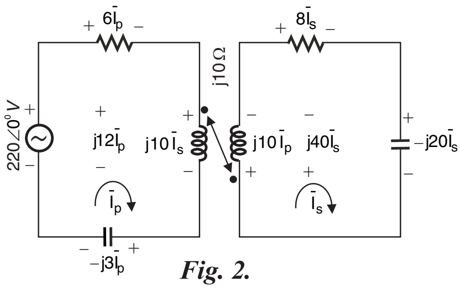


Fig. 2.

By KVL in primary we get,

$$6\bar{I}_p + j12\bar{I}_p + j10\bar{I}_s + (-j3\bar{I}_p) = 220$$

$$\therefore\ (6 + j9)\,\bar{I}_p + j10\bar{I}_s = 220 \qquad .....(1)$$

By KVL in secondary we get,

$$j10\bar{I}_p + j40\bar{I}_s + 8\bar{I}_s + (-j20\bar{I}_s) = 0$$

$$\therefore\ j10\bar{I}_p + (8 + j20)\,\bar{I}_s = 0 \qquad .....(2)$$

On arranging equations (1) and (2) in matrix form we get,

$$\begin{bmatrix} 6+j9 & j10 \\ j10 & 8+j20 \end{bmatrix} \begin{bmatrix} \bar{I}_p \\ \bar{I}_s \end{bmatrix} = \begin{bmatrix} 220 \\ 0 \end{bmatrix}$$

Alternatively, the mesh basis matrix equation can be obtained from the electrical equivalent of the coupled coils shown in Fig. 3. (Here it is group-II coupled coil)

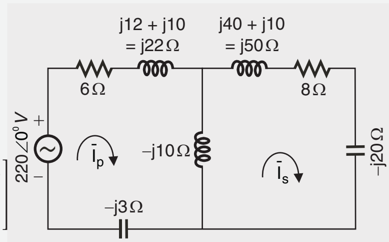


Fig. 3.

With reference to Fig. 3, we can write,

$$\begin{bmatrix} 6+j22-j10-j3 & -(-j10) \\ -(-j10) & -j10+j50+8-j20 \end{bmatrix} \begin{bmatrix} \bar{I}_p \\ \bar{I}_s \end{bmatrix} = \begin{bmatrix} 220\angle 0^\circ \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6+j9 & j10 \\ j10 & 8+j20 \end{bmatrix} \begin{bmatrix} \bar{I}_p \\ \bar{I}_s \end{bmatrix} = \begin{bmatrix} 220 \\ 0 \end{bmatrix}$$

Now, $$\Delta = \begin{vmatrix} 6+j9 & j10 \\ j10 & 8+j20 \end{vmatrix} = \big(6+j9\big)\times\big(8+j20\big)-\big(j10\big)^2 = -32+j192$$

$$\Delta_p = \begin{vmatrix} 220 & j10 \\ 0 & 8+j20 \end{vmatrix} = 220\times\big(8+j20\big)-0 = 1760+j4400$$

$$\Delta_s = \begin{vmatrix} 6+j9 & 220 \\ j10 & 0 \end{vmatrix} = 0-j10\times 220 = -j2200$$

$$\bar{I}_p = \frac{\Delta_p}{\Delta} = \frac{1760+j4400}{-32+j192} = 20.8108-j12.6351\,A = 24.3462\angle-31.3^\circ\ A$$

$$\bar{I}_s = \frac{\Delta_s}{\Delta} = \frac{-j2200}{-32+j192} = -11.1486+j1.8581\,A = 11.3024\angle 170.5^\circ\ A$$

EXAMPLE 5.30

A voltage of 115 V at a frequency of 10 kHz is applied to the primary of the coupled circuit shown in Fig. 1. Determine the total impedance referred to primary and the currents in primary and secondary.

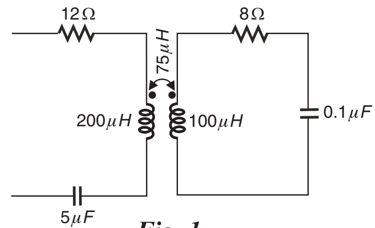


Fig. 1.

SOLUTION

Given that, $f = 10\,kHz$, $L_p = 200\,\mu H$, $C_p = 5\,\mu F$

$M = 75\,\mu H$, $L_s = 100\,\mu H$, $C_s = 0.1\,\mu F$

Primary inductive reactance $= j2\pi f L_p = j2\pi \times 10 \times 10^3 \times 200 \times 10^{-6}$

$= j12.5664\,\Omega$

$$\text{Primary capacitive reactance} = \frac{1}{j2\pi f C_p} = \frac{1}{j2\pi \times 10 \times 10^3 \times 5 \times 10^{-6}} = -j3.1831\,\Omega$$

Mutual reactance $= j2\pi f M = j2\pi \times 10 \times 10^3 \times 75 \times 10^{-6} = j4.7124\,\Omega$

Secondary inductive reactance $= j2\pi f L_s = j2\pi \times 10 \times 10^3 \times 100 \times 10^{-6} = j6.2832\,\Omega$

$$\text{Secondary capacitive reactance} = \frac{1}{j2\pi f C_s} = \frac{1}{j2\pi \times 10 \times 10^3 \times 0.1 \times 10^{-6}} = -j159.1549\,\Omega$$

The frequency domain equivalent of the coupled circuit is shown in Fig. 2. Let $\bar{I}_p$ and $\bar{I}_s$ be the primary and secondary currents as shown in Fig. 2. The electrical equivalent of the coupled circuit is shown in Fig. 3. (Here it is group-I coupled circuit).

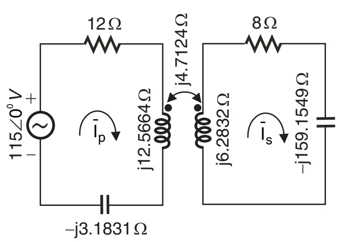


Fig. 2.

⇒

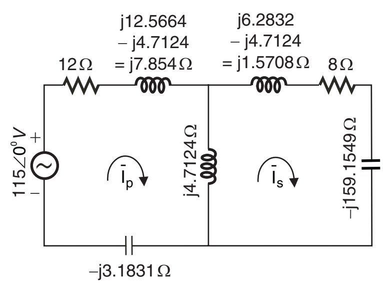


Fig. 3.

With reference to Fig. 3, the mesh basis matrix equation can be obtained as shown below:

$$\begin{bmatrix} 12 + j7.854 + j4.7124 - j3.1831 & -j4.7124 \\ -j4.7124 & j4.7124 + j1.5708 + 8 - j159.1549 \end{bmatrix} \begin{bmatrix} \bar{I}_p \\ \bar{I}_s \end{bmatrix} = \begin{bmatrix} 115\angle 0^\circ \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 12 + j9.3833 & -j4.7124 \\ -j4.7124 & 8 - j152.8717 \end{bmatrix} \begin{bmatrix} \bar{I}_p \\ \bar{I}_s \end{bmatrix} = \begin{bmatrix} 115 \\ 0 \end{bmatrix}$$

$$\text{Now, } \Delta = \begin{vmatrix} 12 + j9.3833 & -j4.7124 \\ -j4.7124 & 8 - j152.8717 \end{vmatrix} = (12 + j9.3833) \times (8 - j152.8717) - (-j4.7124)^2$$

$$= 1552.6477 - j1759.394$$

$$\Delta_p = \begin{vmatrix} 115 & -j4.7124 \\ 0 & 8 - j152.8717 \end{vmatrix} = 115 \times (8 - j152.8717) - 0$$

$$= 920 - j17580.2455$$

$$\Delta_s = \begin{vmatrix} 12+j9.3833 & 115 \\ -j4.7124 & 0 \end{vmatrix} = 0-\left(-j4.7124\right)\times 115$$
$$= j541.926$$

$$\bar{I}_p = \frac{\Delta_p}{\Delta} = \frac{920-j17580.2455}{1552.6477-j1759.394} = 5.8769-j4.6634\,A = 7.5023\angle-38.4^\circ\,A$$

$$\bar{I}_s = \frac{\Delta_s}{\Delta} = \frac{j541.926}{1552.6477-j1759.394} = -0.1732+j0.1528\,A = 0.231\angle 138.6^\circ\,A$$

$$\text{Total impedance referred to primary} = \frac{115\angle 0^\circ}{\bar{I}_p}$$
$$= \frac{115\angle 0^\circ}{7.5023\angle-38.4^\circ} = 15.3286\angle 38.4^\circ\,\Omega$$

RESULT

Primary current, $\bar{I}_p = 7.5023\angle-38.4^\circ\,A$

Secondary current, $\bar{I}_s = 0.231\angle 138.6^\circ\,A$

Total impedance referred to primary $= 15.3286\angle 38.4^\circ\,A$

EXAMPLE 5.31

In the coupled circuit shown in Fig. 1. Determine the voltage ratio $\frac{\bar{V}_2}{\bar{V}_1}$, which will make the current $\bar{I}_1$ equal to zero.

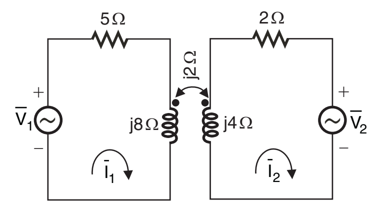


Fig. 1.

SOLUTION

The given coupled coils are group-I coupled coils. The electrical equivalent of the coupled coils is shown in Fig. 2.

With reference to Fig. 2, the mesh basis matrix equation is obtained as shown below :

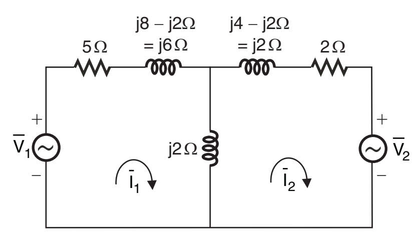


Fig. 2.

$$\begin{bmatrix} 5+j6+j2 & -j2 \\ -j2 & j2+j2+2 \end{bmatrix}\begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{V}_1 \\ -\bar{V}_2 \end{bmatrix}$$

$$\begin{bmatrix} 5+j8 & -j2 \\ -j2 & 2+j4 \end{bmatrix}\begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} \bar{V}_1 \\ -\bar{V}_2 \end{bmatrix}$$

We know that, $\bar{I}_1 = \frac{\Delta_1}{\Delta}$

For, $\bar{I}_1 = 0, \quad \Delta_1 = 0$

$$\Delta_1 = \begin{vmatrix} \bar{V}_1 & -j2 \\ -\bar{V}_2 & 2+j4 \end{vmatrix} = \bar{V}_1\times\left(2+j4\right)-\left(-\bar{V}_2\right)\times\left(-j2\right)$$
$$= \bar{V}_1\left(2+j4\right)-j2\bar{V}_2$$

Put, $\Delta_1 = 0$

$$\therefore\ \bar{V}_1\left(2+j4\right)-j2\bar{V}_2 = 0$$
$$-j2\bar{V}_2 = -\bar{V}_1\left(2+j4\right)$$

$$\therefore \frac{\bar{V}_2}{\bar{V}_1} = \frac{2+j4}{j2} = 2-j = 2.2361\angle -26.6^\circ$$

RESULT

The ratio $\frac{\bar{V}_2}{\bar{V}_1}$ to make $\bar{I}_1$ as zero $= 2.2361\angle -26.6^\circ$

EXAMPLE 5.32 *(AU Dec'14, 10 Marks)*

For the coupled circuit shown in Fig. 1, find the value of $\bar{V}_2$ so that the current $\bar{I}_1 = 0$.

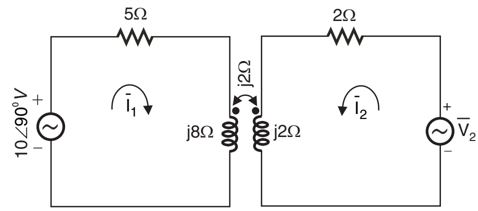


Fig. 1.

SOLUTION

The self- and mutual induced emfs for the given direction of mesh currents are shown in Fig. 2.

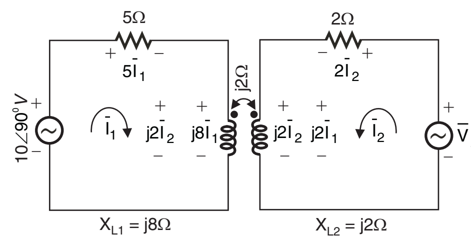


Fig. 2.

With reference to Fig. 2, the mesh basis matrix equation can be written as shown below:

By applying KVL in mesh-1,

$$5\bar{I}_1 + j8\bar{I}_1 + j2\bar{I}_2 = 10\angle 90^\circ$$

$$(5+j8)\bar{I}_1 + j2\bar{I}_2 = j10 \qquad .....(1)$$

By applying KVL in mesh-2,

$$2\bar{I}_2 + j2\bar{I}_2 + j2\bar{I}_1 = \bar{V}_2$$

$$j2\bar{I}_1 + (2+j2)\bar{I}_2 = \bar{V}_2 \qquad .....(2)$$

By arranging equations (1) and (2) in matrix form, we get,

$$\begin{bmatrix} 5+j8 & +j2 \\ +j2 & 2+j2 \end{bmatrix}\begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} j10 \\ \bar{V}_2 \end{bmatrix}$$

We know that, $\bar{I}_1 = \frac{\Delta_1}{\Delta}$

For $\bar{I}_1 = 0, \quad \Delta_1 = 0$

$$\therefore \Delta_1 = \begin{vmatrix} j10 & j2 \\ \bar{V}_2 & 2+j2 \end{vmatrix} = 0$$

$$j10 \times (2+j2) - \bar{V}_2 \times j2 = 0$$

$$\therefore \bar{V}_2 = \frac{j10 \times (2+j2)}{j2} = 10 + j10\, V$$

$$= 14.1421\angle 45^\circ\, V$$

EXAMPLE 5.33 *(AU May'17 & Dec'16, 8 Marks)*

In the coupled circuit shown in Fig. 1, determine the voltage across 12 Ω resistor.

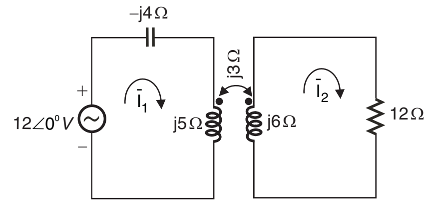


Fig. 1.

SOLUTION

The given coupled coils are group-1 coupled coils. The electrical equivalent of the coupled coils is shown in Fig. 2.

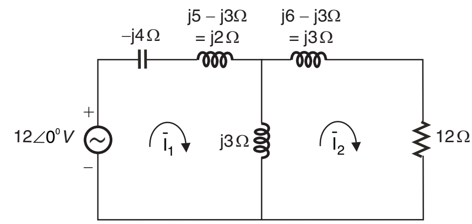


Fig. 2.

With reference to Fig. 2, the mesh basis matrix equation is obtained as shown below:

$$\begin{bmatrix} -j4+j2+j3 & -j3 \\ -j3 & j3+j3+12 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} j & -j3 \\ -j3 & 12+j6 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \end{bmatrix}$$

$$\text{Now, } \Delta = \begin{vmatrix} j & -j3 \\ -j3 & 12+j6 \end{vmatrix} = j \times (12+j6) - (-j3)^2 = 3+j12$$

$$\Delta_2 = \begin{vmatrix} j & 12 \\ -j3 & 0 \end{vmatrix} = 0 - (-j3) \times 12 = j36$$

$$\therefore\ \bar{I}_2 = \frac{\Delta_2}{\Delta} = \frac{j36}{3+j12} = 2.8235 + j0.7059\ A$$

$\therefore$ Voltage across 12Ω resistor = $12 \times \bar{I}_2 = 12 \times (2.8235 + j0.7059)$

$$= 33.882 + j8.4708 = 34.9248\angle 14^\circ\ V$$

EXAMPLE 5.34

In the coupled circuit shown in Fig. 1, determine the voltage across 10 Ω resistor.

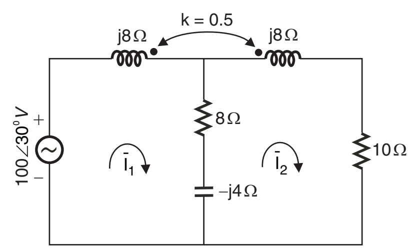


Fig. 1.

SOLUTION

Given that, k = 0.5 , X_{L1} = 8 Ω and X_{L2} = 8 Ω

$\therefore$ Mutual reactance, $X_m = k\sqrt{X_{L1}\,X_{L2}} = 0.5\sqrt{8 \times 8} = 4\Omega$

Let us name the coupled coils as coil-A and coil-B as shown in Fig. 2. The current $\bar{I}_1$ entering at the undotted end in coil-A will induce an emf $jX_m\bar{I}_1$ in coil-B such that the undotted end is positive.

The current $\bar{I}_2$ entering at the dotted end in coil-B will induce an emf $jX_m\bar{I}_2$ in coil-A such that the dotted end is positive.

The self- and mutual induced emfs in the coupled coils are shown in Fig. 2.

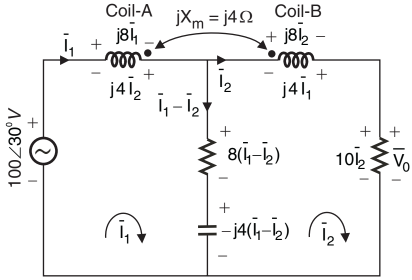


Fig. 2.

$$100\angle 30^\circ = 100\cos 30^\circ + j100\sin 30^\circ = 86.6025 + j50$$

Let, $\bar{V}_0$ be the voltage across the $10\,\Omega$ resistor as shown in Fig. 2.

Now, $\bar{V}_0 = \bar{I}_2 \times 10 = 10\,\bar{I}_2$

By KVL in mesh-1,

$$j8\bar{I}_1 + 8(\bar{I}_1 - \bar{I}_2) + [-j4(\bar{I}_1 - \bar{I}_2)] = j4\bar{I}_2 + 86.6025 + j50$$

$$\therefore\ (8 + j4)\,\bar{I}_1 - 8\bar{I}_2 = 86.6025 + j50 \qquad .....(1)$$

By KVL in mesh-2,

$$j8\bar{I}_2 + 10\bar{I}_2 = [-j4(\bar{I}_1 - \bar{I}_2)] + 8(\bar{I}_1 - \bar{I}_2) + j4\bar{I}_1$$

$$\therefore -8\bar{I}_1 + (18 + j4)\,\bar{I}_2 = 0 \qquad .....(2)$$

On arranging equations (1) and (2) in matrix form, we get,

$$\begin{bmatrix} 8+j4 & -8 \\ -8 & 18+j4 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} 86.6025 + j50 \\ 0 \end{bmatrix}$$

$$\text{Now, } \Delta = \begin{vmatrix} 8+j4 & -8 \\ -8 & 18+j4 \end{vmatrix} = \left(8+j4\right)\times\left(18+j4\right) - \left(-8\right)^2 = 64 + j104$$

$$\Delta_2 = \begin{vmatrix} 8+j4 & 86.6025 + j50 \\ -8 & 0 \end{vmatrix} = 0 - \left(-8\right)\times\left(86.6025 + j50\right) = 692.82 + j400$$

$$\bar{I}_2 = \frac{\Delta_2}{\Delta} = \frac{692.82 + j400}{64 + j104} = 5.7632 - j3.1152\ A$$

$$\bar{V}_0 = 10\bar{I}_2 = 10 \times \left(5.7632 - j3.1152\right) = 57.632 - j31.152\ V = 65.5126\angle -28.4^\circ\ V$$

RESULT

The voltage across $10\,\Omega$ resistance $= 65.5126\angle{-28.4^\circ}\ V$

EXAMPLE 5.35

In the coupled circuit shown in Fig. 1, determine the voltage across $j5\,\Omega$ reactance.

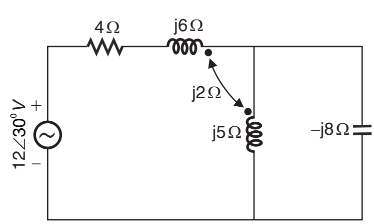


Fig. 1.

SOLUTION

Let $\bar{I}_1$ and $\bar{I}_2$ be the mesh currents as shown in Fig. 2. Now $\bar{I}_1$ is the current through j6 reactance and $\bar{I}_1 - \bar{I}_2$ is the current through the $j5\,\Omega$ reactance.

The $\bar{I}_1$ entering at the undotted end in j6 Ω reactance will induce an emf $j2\bar{I}_1$ in the j5 Ω reactance such that the undotted end is positive.

The $\bar{I}_1 - \bar{I}_2$ entering at the dotted end in j5 Ω reactance will induce an emf $j2\left(\bar{I}_1 - \bar{I}_2\right)$ in j6 Ω reactance such that the dotted end is positive.

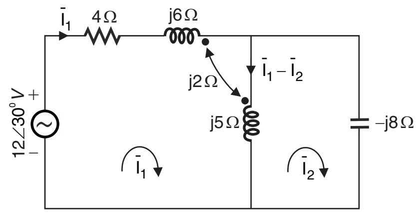


Fig. 2.

The self- and mutual induced emfs are shown in Fig. 3.

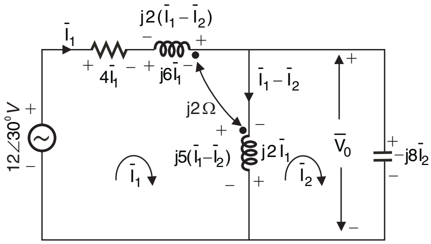


Fig. 3.

$12\angle 30^\circ = 12 \cos 30^\circ + j12 \sin 30^\circ$
$= 10.3923 + j6$

Let $\bar{V}_0$ be the voltage across j5 Ω reactance. With reference to Fig. 3, by KVL, we can write,

$$\bar{V}_0 + j2\,\bar{I}_1 = j5\left(\bar{I}_1 - \bar{I}_2\right)$$

$$\therefore\ \bar{V}_0 = j5\bar{I}_1 - j5\bar{I}_2 - j2\bar{I}_1$$

$$= j3\bar{I}_1 - j5\bar{I}_2$$

By KVL in mesh-1,

$$4\bar{I}_1 + j6\bar{I}_1 + j5(\bar{I}_1 - \bar{I}_2) = j2(\bar{I}_1 - \bar{I}_2) + j2\bar{I}_1 + 10.3923 + j6$$

$$\therefore\ (4 + j7)\,\bar{I}_1 + -j3\bar{I}_2 = 10.3923 + j6 \qquad .....(1)$$

By KVL in mesh-2,

$$j2\bar{I}_1 + (-j8\bar{I}_2) = j5(\bar{I}_1 - \bar{I}_2)$$

$$\therefore -j3\bar{I}_1 - j3\bar{I}_2 = 0 \qquad .....(2)$$

On arranging equations (1) and (2) in matrix form, we get,

$$\begin{bmatrix} 4+j7 & -j3 \\ -j3 & -j3 \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{bmatrix} = \begin{bmatrix} 10.3923 + j6 \\ 0 \end{bmatrix}$$

Now, $$\Delta = \begin{vmatrix} 4+j7 & -j3 \\ -j3 & -j3 \end{vmatrix} = \left[(4+j7) \times (-j3)\right] - (-j3)^2 = 30 - j12$$

$$\Delta_1 = \begin{vmatrix} 10.3923 + j6 & -j3 \\ 0 & -j3 \end{vmatrix} = (10.3923 + j6) \times (-j3) - 0 = 18 - j31.1769$$

$$\Delta_2 = \begin{vmatrix} 4+j7 & 10.3923 + j6 \\ -j3 & 0 \end{vmatrix} = 0 - (-j3) \times (10.3923 + j6) = -18 + j31.1769$$

$$\bar{I}_1 = \frac{\Delta_1}{\Delta} = \frac{18 - j31.1769}{30 - j12} = 0.8756 - j0.689\ A$$

$$\bar{I}_2 = \frac{\Delta_2}{\Delta} = \frac{-18 + j31.1769}{30 - j12} = -0.8756 + j0.689\ A$$

Now,

$$\overline{V}_0 = j3\bar{I}_1 - j5\bar{I}_2$$

$$= j3(0.8756 - j0.689) - j5(-0.8756 + j0.689)$$

$$= 5.512 + j7.0048 = 8.9134\angle 51.8^\circ\ V$$

Since $-j8\,\Omega$ reactance is parallel to $j5\,\Omega$ reactance, the voltage $\overline{V}_0$ is the same as the voltage across $j8\,\Omega$ reactance.

$$\therefore\ \overline{V}_0 = -j8\bar{I}_2$$

$$= -j8 \times (-0.8756 + j0.689)$$

$$= 5.512 + j7.0048 = 8.9134\angle 51.8^\circ\ V$$

RESULT

Voltage across $j5\,\Omega$ reactance = $8.9134\angle 51.8^\circ\ V$

5.10 Summary of Important Concepts

1. Resonance is a circuit condition at which an RLC circuit behaves as a purely resistive circuit. Resonance in a series RLC circuit is called series resonance and in a parallel RLC circuit is called parallel resonance.
2. Resonance in an RLC circuit can be achieved by varying the frequency of the exciting sinusoidal source.
3. The frequency at which resonance occurs is called resonance frequency.
4. In a series RLC circuit, at resonance, the inductive reactance cancels the capacitive reactance and so the total reactance is zero.
5. In a series RLC circuit, the angular resonant frequency, $\omega_r = \frac{1}{\sqrt{LC}}$, and the resonant frequency in *Hz* is, $f_r = \frac{1}{2\pi\sqrt{LC}}$.
6. In a series RLC circuit, at resonance, the impedance is minimum and equal to R and so current is maximum.
7. For $\omega < \omega_r$, a series RLC circuit behaves as a capacitive (RC) circuit and for $\omega > \omega_r$, it behaves as an inductive (RL) circuit.
8. Quality factor (Q-factor) is defined as the ratio of maximum energy stored to the energy dissipated in one period.
9. In an RLC series circuit, the Q-factor at resonance is a measure of voltage magnification at resonance.
10. In an RLC series circuit, when the inductor stores energy the capacitor discharges energy and vice-versa.
11. At resonance, the sum of energy stored in the inductor and capacitor is maximum.

12. In an RLC series circuit, for $\omega < \omega_r$, the energy stored in the capacitor is maximum and for $\omega > \omega_r$, the energy stored in the inductor is maximum.

13. The various expressions for Q-factor of an RLC series circuit are,

$$Q_r = \frac{\omega_r L}{R} \quad ; \quad Q_r = \frac{1}{\omega_r CR} \quad ; \quad Q_r = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$\text{When } \omega \le \omega_r,\ Q = \frac{1}{\omega CR} \quad ; \quad \text{When } \omega \ge \omega_r,\ Q = \frac{\omega L}{R}$$

14. In an RLC series circuit, bandwidth is defined as the range of frequencies over which power is greater than or equal to 1/2 times the maximum power.

15. Alternatively, bandwidth is the range of frequencies over which current is greater than or equal to $1/\sqrt{2}$ times the maximum current.

16. Bandwidth is given by the difference between the cut-off frequencies.

17. The various equations for bandwidth, β of RLC series circuit are,

$$\beta = \frac{R}{L} \text{ in } rad/s \quad ; \quad \beta = \frac{\omega_r}{Q_r} \text{ in } rad/s \quad ; \quad \text{Bandwidth in } Hz = \frac{\beta}{2\pi}$$

18. Half-power frequencies (or cut-off frequencies) are frequencies at which power is 1/2 times the maximum power $\left(\text{or current is } 1/\sqrt{2} \text{ times the maximum current}\right)$.

19. The various equations for half-power (or cut-off) frequencies of RLC series circuit are,

$$\omega_l = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \text{ in } rad/s \quad ; \quad \omega_h = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \text{ in } rad/s$$

$$\omega_l = \omega_r\left[-\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \text{ in } rad/s \quad ; \quad \omega_h = \omega_r\left[\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \text{ in } rad/s$$

$$f_l = \frac{\omega_l}{2\pi} \text{ in } Hz \quad ; \quad f_h = \frac{\omega_h}{2\pi} \text{ in } Hz$$

20. Resonant frequency is given by the geometric mean of two half-power frequencies.

$$\therefore\ \omega_r = \sqrt{\omega_l \omega_h}$$

21. In an RLC series circuit, at half-power frequencies, the absolute value of total reactance is equal to resistance.

$$\therefore\ \omega L - \frac{1}{\omega C} = \pm R \quad ; \quad \omega_h L - \frac{1}{\omega_h C} = R \quad ; \quad \omega_l L - \frac{1}{\omega_l C} = -R$$

22. Selectivity is defined as the ratio of bandwidth and resonant frequency.

$$\therefore\ \text{Selectivity} = \frac{\beta}{\omega_r}$$

23. Alternatively, selectivity is the inverse of quality factor (Q-factor).

$$\therefore\ \text{Selectivity} = \frac{1}{Q_r}$$

24. The RLC circuit is highly selective in selecting a particular frequency if the selectivity is low, which in turn demands smaller bandwidth and large value of Q-factor.
25. In an RLC series circuit, the magnitude of voltage across inductor V_{Lr} and capacitor V_{Cr} at resonance is Q_r times supply voltage, V where Q_r is Q-factor at resonance.

$$\therefore \; V_{Lr} = V_{Cr} = Q_r V$$

26. In a parallel RLC circuit, at resonance, the inductive susceptance cancels the capacitive susceptance and so the total susceptance is zero.
27. In a parallel RLC circuit, at resonance, the admittance is minimum and equal to conductance of the circuit and so the current is also minimum.
28. The current is maximum in series resonance and minimum in parallel resonance, and so parallel resonance is also called anti-resonance.
29. In a parallel RLC circuit, the effective resistance of the circuit at resonance is called dynamic resistance.
30. For $\omega < \omega_r$, the parallel RLC circuit behaves as an inductive circuit and for $\omega > \omega_r$, behaves as a capacitive circuit.
31. The frequency of resonance and dynamic resistance for the popular four parallel combinations of R, L and C are given below:

Case i : *R, L and C are in parallel.*

$$\omega_r = \frac{1}{2\pi\sqrt{LC}} \quad ; \quad R_{dynamic} = R$$

Case ii : *A branch with R_1 and L in series is parallel with another branch with R_2 and C in series.*

$$\omega_r = \frac{1}{\sqrt{LC}}\sqrt{\frac{L - CR_1^2}{L - CR_2^2}} \quad ; \quad R_{dynamic} = \frac{1}{\dfrac{R_1}{R_1^2 + X_{Lr}^2} + \dfrac{R_2}{R_2^2 + X_{Cr}^2}}$$

$$\text{where, } X_{Lr} = \omega_r L \quad ; \quad X_{Cr} = \frac{1}{\omega_r C}$$

Condition for resonance at all frequency is, $R_1 = R_2 = \sqrt{\dfrac{L}{C}}$

Case iii : *A branch with R_1 and L in series is parallel with C.*

$$\omega_r = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{CR_1^2}{L}} \quad ; \quad R_{dynamic} = \frac{L}{R_1 C}$$

Case iv : *A branch with R_2 and C in series is parallel with L.*

$$\omega_r = \frac{1}{\sqrt{LC}}\sqrt{\frac{L}{L - CR_2^2}} \quad ; \quad R_{dynamic} = \frac{L}{R_2 C}$$

32. In an RLC parallel circuit excited by a current source, the Q-factor at resonance is a measure of current magnification at resonance.

33. In an RLC parallel circuit for $\omega < \omega_r$, the energy stored in the inductor is maximum and for $\omega > \omega_r$, the energy stored in the capacitor is maximum.

34. The various expressions for Q-factor of RLC parallel circuit are,

$$Q_r = \omega_r CR \quad ; \quad Q_r = \frac{R}{\omega_r L} \quad ; \quad Q_r = R\sqrt{\frac{C}{L}}$$

$$\text{When } \omega \leq \omega_r,\ Q = \frac{R}{\omega L} \quad ; \quad \text{When } \omega \geq \omega_r,\ Q = \omega CR$$

35. In an RLC parallel circuit excited by a voltage source, the power is minimum at resonance, and when excited by a current source, the power is maximum at resonance.

36. In an RLC parallel circuit excited by a voltage source, bandwidth is defined as the range of frequencies over which power is less than or equal to twice the minimum power. Alternatively, bandwidth is the range of frequencies over which current is less than or equal to $\sqrt{2}$ times the minimum current.

37. In an RLC parallel circuit excited by a current source, bandwidth is defined as the range of frequencies over which power is greater than or equal to 1/2 times the maximum power. Alternatively, bandwidth is the range of frequencies over which voltage is greater than or equal to $1/\sqrt{2}$ times the maximum voltage.

38. The various equations for bandwidth, β of RLC parallel circuit when excited by either a voltage or current source are,

$$\beta = \frac{1}{RC} \text{ in } \textit{rad/s} \quad ; \quad \beta = \frac{\omega_r}{Q_r} \text{ in } \textit{rad/s} \quad ; \quad \text{Bandwidth in } \textit{Hz} = \frac{\beta}{2\pi}$$

39. The various equations for half-power (or cut-off) frequencies of an RLC parallel circuit are,

$$\omega_l = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \text{ in } \textit{rad/s} \quad ; \quad \omega_h = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \text{ in } \textit{rad/s}$$

$$\omega_l = \omega_r\left[-\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \text{ in } \textit{rad/s} \quad ; \quad \omega_h = \omega_r\left[\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \text{ in } \textit{rad/s}$$

$$f_l = \frac{\omega_l}{2\pi} \text{ in } \textit{Hz} \quad ; \quad f_h = \frac{\omega_h}{2\pi} \text{ in } \textit{Hz}$$

40. In an RLC parallel circuit at half-power frequencies, the absolute value of total susceptance is equal to conductance.

$$\therefore\ \omega C - \frac{1}{\omega L} = \pm R \quad ; \quad \omega_h C - \frac{1}{\omega_h L} = \frac{1}{R} \quad ; \quad \omega_l C - \frac{1}{\omega_l L} = -\frac{1}{R}$$

41. The connection between two or more coils via magnetic flux is called magnetic coupling.

42. The connection between two or more coils via electric current is called conductive coupling.

43. In magnetically coupled elements, the power/energy transfer occurs through magnetic flux.

44. In conductively coupled elements, the power/energy transfer occurs through current flow.

45. The induced emf in a coil due to change in flux in the same coil is called self-induced emf and the direction of self-induced emf will oppose the current through the coil.

46. The induced emf in a coil due to change in flux in a nearby coil is called mutual induced emf and the direction of mutual induced emf depends on flux/current in the nearby coil.

47. The self-inductance L of a coil with N turns carrying a current of i amperes and developing a flux of ϕ webers is given by,

$$L = \frac{N\phi}{i}.$$

48. The coefficient of coupling k is defined as the fraction of the total flux produced by one coil linking the other coil.

49. In coupled coils, when k = 1, the coils are said to be tightly coupled or closely coupled, and when k = 0, the coils are said to be magnetically isolated, and when k is very low, the coils are said to be loosely coupled.

50. The mutual inductance M between two coils with self-inductances L_1 and L_2 and coefficient of coupling k is given by,

$$M = k\sqrt{L_1 L_2}.$$

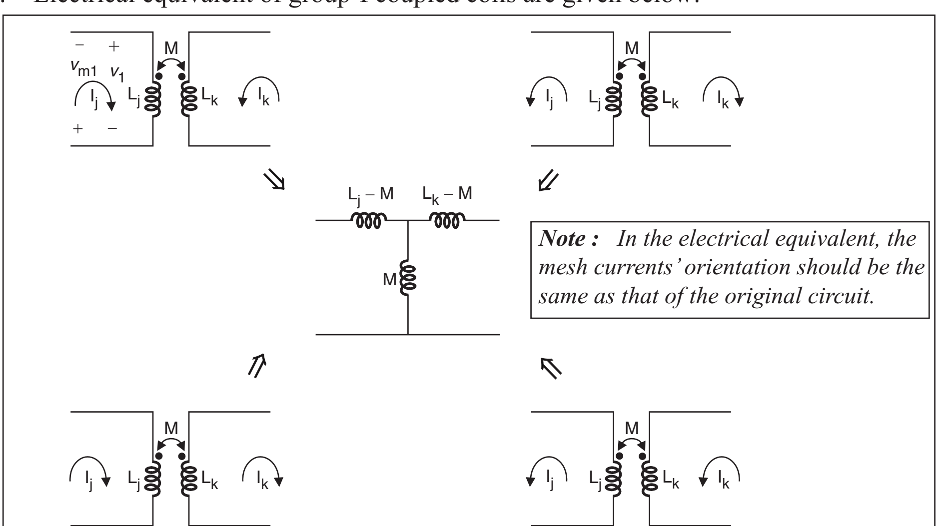
51. In coupled coils, the maximum value of mutual inductance M is possible when coefficient of coupling k is equal to one.

52. In coupled coils, if the fluxes produced by the two coils aid each other then the sign/polarity of self- and mutual induced emfs will be the same.

53. In coupled coils, if the fluxes produced by two coils oppose each other then the sign/polarity of self- and mutual induced emfs will be opposite.

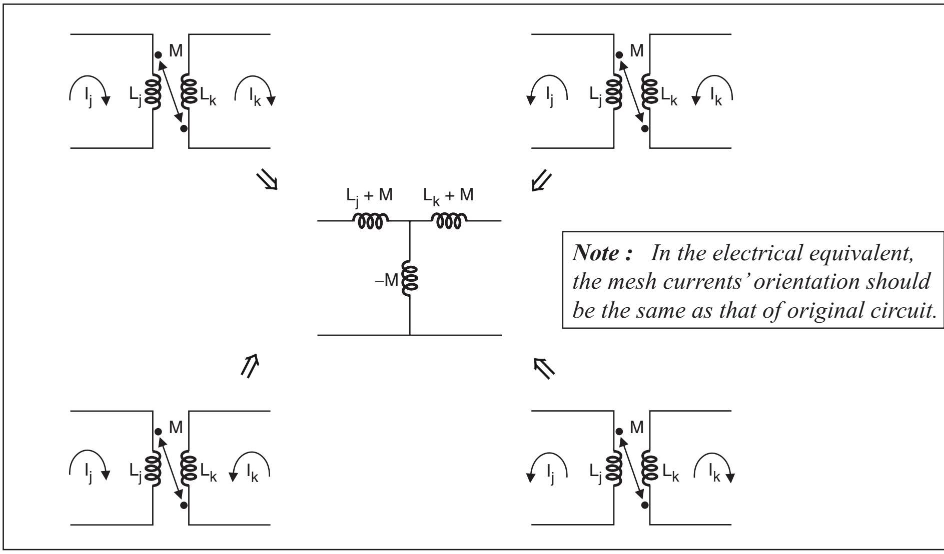
54. Dot rule : In coupled coils, current entering at the dotted terminal of one coil induces an emf in the second coil, which is positive at the dotted terminal of the second coil. [Conversely, current entering at the undotted terminal of one coil induces an emf in the second coil, which is positive at the undotted terminal of the second coil].

55. Electrical equivalent of group-I coupled coils are given below:

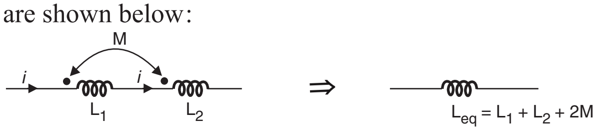


Note : *In the electrical equivalent, the mesh currents' orientation should be the same as that of the original circuit.*

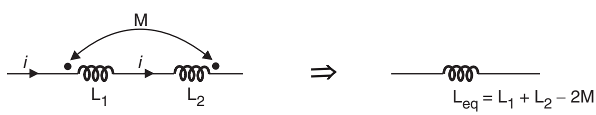
56. Electrical equivalent of group-II coupled coils are given below:



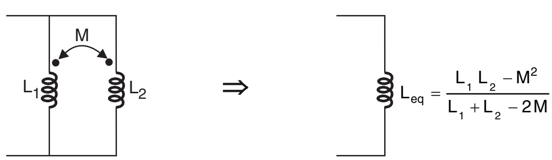
57. In series connection of coupled coils, if the sign/polarity of self- and mutual induced emfs are the same then the connection is called series aiding. The series aiding connection and its equivalent are shown below:



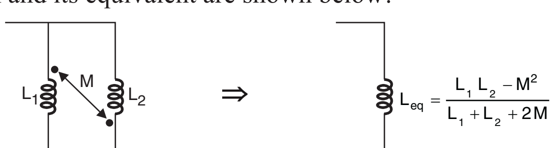
58. In series connection of coupled coils, if the sign/polarity of the self-and mutual emfs are opposite then the connection is called series opposing. The series opposing connection and its equivalent are shown below:



59. In parallel connection of coupled coils, if the sign/polarity of self-and mutual induced emfs are the same then the connection is called parallel aiding. The parallel aiding connection and its equivalent are shown below:



60. In parallel connection of coupled coils, if the sign/polarity of self-and mutual induced emfs are opposite then the connection is called parallel opposing. The parallel opposing connection and its equivalent are shown below:



61. In a coupled coil, if a capacitor is added to the secondary coil in order to resonate the secondary circuit then it is called a single tuned coupled coil.
62. In a tuned coupled circuit, the critical coupling k_C is the value of coupling coefficient k at which the output voltage is maximum.
63. The value of mutual inductance at critical coupling k_C is called optimum mutual inductance M_{opt}.

5.11 Short-answer Questions

Q5.1 What is resonance ?

Resonance is a circuit condition at which an RLC circuit behaves as a purely resistive circuit.

Q5.2 Write the expressions for resonant frequency and current at resonance of a RLC series circuit.

Angular resonant frequency, $\omega_r = \dfrac{1}{\sqrt{LC}}$ in *rad/s*

Resonant frequency, $f_r = \dfrac{1}{2\pi\sqrt{LC}}$ in *Hz*

Current at frequency, $I_r = \dfrac{V}{R}$

Q5.3 Define the frequency response of RLC series circuit. ***(AU May'15, 2 Marks)***

The variation of current with frequency is called frequency response, which is shown in Fig. Q5.3.

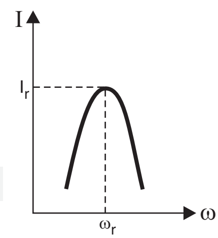


Fig. Q5.3.

Q5.4 Define quality factor. ***(AU June'14 & Dec'14, 2 Marks)***

Quality factor is defined as the ratio of maximum energy stored to the energy dissipated in one period.

$$\text{Quality factor, } Q = 2\pi \times \frac{\text{Maximum energy stored}}{\text{Energy dissipated in one period}}$$

Q5.5 Write the expressions for quality factor of a series RLC circuit.

Quality factor at resonance, $Q_r = \dfrac{\omega_r L}{R}$

$$\boxed{\text{Alternatively, } Q_r = \frac{1}{\omega_r CR} \quad ; \quad Q_r = \frac{1}{R}\sqrt{\frac{L}{C}} \quad ; \quad Q_r = \frac{\omega_r}{\beta}}$$

When, $\omega \le \omega_r$, $\quad Q = \dfrac{1}{\omega CR}$

When, $\omega \ge \omega_r$, $\quad Q = \dfrac{\omega L}{R}$

Q5.6 Determine the quality factor of a coil for the series resonant circuit consisting of R = 10 ohm, L = 0.1H, and C = 10 microfarad. ***(AU June'14, 2 Marks)***

$$\text{Quality factor at resonant, } Q_r = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10} \times \sqrt{\frac{0.1}{10 \times 10^{-6}}} = 10$$

Q5.7 Define bandwidth of an RLC series circuit.

The bandwidth β of an RLC series circuit is defined as the range of frequencies over which current is greater than or equal to $1/\sqrt{2}$ times the maximum current. The bandwidth is shown in Fig. Q5.7.

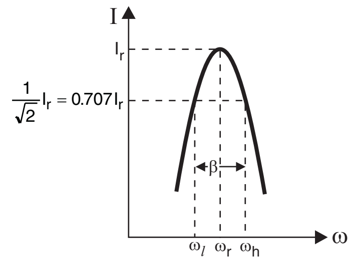


Fig. Q5.7.

Q5.8 What are half-power frequencies ?

In RLC circuits, the frequencies at which power is half the maximum/minimum power are called half-power frequencies.

Q5.9 Write the expression for half-power frequencies of an RLC series circuit.

$$\text{Lower cut-off frequency, } f_l = \frac{1}{2\pi}\left[-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right] \text{ in } Hz$$

$$\text{Higher cut-off frequency, } f_h = \frac{1}{2\pi}\left[\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right] \text{ in } Hz$$

$$\boxed{\text{Alternatively, } f_l = f_r\left[-\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \; ; \; f_h = f_r\left[\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right]}$$

Q5.10 Write the expression for impedance of an RLC series circuit at half-power frequencies.

At half-power frequencies in an RLC series circuit, the total reactance is equal to resistance.

$$\text{i.e., at } \omega = \omega_h, \quad \omega L - \frac{1}{\omega C} = R$$

$$\therefore \text{ At } \omega = \omega_h, \quad Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{R^2 + R^2} = \sqrt{2R^2} = \sqrt{2}\, R$$

$$\boxed{\begin{aligned} \text{Alternatively, } \quad Z &= \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \\ &= \sqrt{2\left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{2}\left(\omega L - \frac{1}{\omega C}\right) \end{aligned}}$$

Q5.11 Write the expression for bandwidth of an RLC series circuit.

$$\text{Bandwidth, } \beta = \frac{R}{L} \text{ in } rad/s \qquad \boxed{\text{Alternatively, } \beta = \frac{\omega_r}{Q_r} \text{ in } rad/s}$$

$$\text{Bandwidth in } Hz = \frac{R}{2\pi L} \qquad \boxed{\text{Alternatively, Bandwidth in } Hz = \frac{f_r}{Q_r}}$$

Q5.12 How is resonant frequency related to half-power frequencies in an RLC series/parallel circuit?

Resonant frequency is given by the geometric mean of the two half-power frequencies.

$$\text{i.e., } \omega_r = \sqrt{\omega_h\, \omega_l} \quad \text{or} \quad f_r = \sqrt{f_h\, f_l}$$

(AU Dec'15, 2 Marks)

Q5.13 Define selectivity.

Selectivity is defined as the ratio of bandwidth and resonant frequency.

$$\text{Selectivity} = \frac{\beta}{\omega_r} \qquad \boxed{\text{Alternatively, Selectivity} = \frac{1}{Q_r}}$$

Q5.14 Write the characteristics of series resonance.

- At resonance, impedance is minimum and equal to resistance, therefore current is maximum.
- Below resonant frequency, the circuit behaves as a capacitive circuit and above resonant frequency, the circuit behaves as an inductive circuit.
- At resonance, the magnitude of voltage across inductance and capacitance will be Q times the supply voltage but they are in-phase opposition.

Q5.15 An RLC series circuit has $R = 10\Omega$ and $X_C = 62.833\Omega$. Find the value of L for resonance at 50 Hz.

At resonance, $X_L = X_C$, $\therefore$ $X_L = 62.833\,\Omega$

Since, $X_L = 2\pi fL$, Inductance, $L = \frac{X_L}{2\pi f} = \frac{62.833}{2\pi \times 50} = 0.2\,H$

Q5.16 Determine the quality factor of an RLC series circuit with $R = 5\ \Omega$, $L = 0.01\ H$ and $C = 100\ \mu F$.

$$\text{Quality factor at resonance, } Q_r = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{5}\sqrt{\frac{0.01}{100 \times 10^{-6}}} = 2$$

Q5.17 The impedance and quality factor of an RLC series circuit at $\omega = 10^7$ rad/s are $100 + j0\Omega$ and 100 respectively. Find the values of R, L and C.

Since the impedance is resistive, the circuit will be in resonance. Therefore, $\omega_r = 10^7$ *rad/s*.

At $\omega = \omega_r$, $Z = R$, $\therefore$ Resistance, $R = 100\,\Omega$

We know that, $Q_r = \frac{\omega_r L}{R}$

$$\therefore \text{ Inductance, } L = \frac{Q_r R}{\omega_r} = \frac{100 \times 100}{10^7} = 1\,mH$$

We know that, $\omega_r = \frac{1}{\sqrt{LC}} \Rightarrow \omega_r^2 = \frac{1}{LC}$

$$\therefore \text{ Capacitance, } C = \frac{1}{\omega_r^2 L} = \frac{1}{(10^7)^2 \times 1 \times 10^{-3}} = 1 \times 10^{-11}\,F = 10 \times 10^{-12}\,F = 10\,pF$$

Q5.18 An RLC series circuit with $R = 10\Omega$, $X_L = 20\Omega$ and $X_C = 20\Omega$ is excited by a sinusoidal source of voltage 200 V. What will be the voltage across inductance ?

Since $X_L = X_C$, the circuit will be in resonance. At resonance, voltage across inductance is Q_r times the supply voltage.

$$\text{Quality factor at resonance, } Q_r = \frac{\omega_r L}{R} = \frac{X_L}{R} = \frac{20}{10} = 2$$

$\therefore$ Voltage across inductance = $Q_r V = 2 \times 200 = 400\,V$

Q5.19 An RLC series circuit excited by a 10 V sinusoidal source resonates at a frequency of 50 Hz. If the bandwidth is 5 Hz, what will be the voltage across the capacitor ?

$$\text{Quality factor at resonance, } Q_r = \frac{\omega_r}{\beta} = \frac{f_r}{\beta} = \frac{50}{5} = 10$$

$\therefore$ Voltage across capacitor = $Q_r V = 10 \times 10 = 100\,V$

Q5.20 What is anti-resonance ?

In an RLC parallel circuit, the current is minimum at resonance, whereas in series resonance, the current is maximum. Therefore, parallel resonance is called anti-resonance.

Q5.21 Write the expression for resonant frequency for the RLC network shown in Fig. Q5.21. What happens when $R_1 = R_2 = R$ and $L = CR^2$?

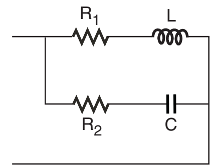


Fig. Q5.21.

Resonant frequency, $f_r = \dfrac{1}{2\pi\sqrt{LC}}\sqrt{\dfrac{L - CR_1^2}{L - CR_2^2}}$

When, $R_1 = R_2 = R$ and $L = CR^2$ the circuit will resonate at all frequencies.

Q5.22 Find the resonant frequency in Hz of an RLC circuit with L = 100 mH and C = 0.1 mF.

(AU June'16, 2 Marks)

Resonant frequency, $f_r = \dfrac{1}{2\pi\sqrt{LC}}$

$$= \frac{1}{2\pi\sqrt{100 \times 10^{-3} \times 0.1 \times 10^{-6}}} = 1591.5\ Hz$$

Q5.23 A coil of resistance 2.2 Ω and an inductance 0.01 H is connected in series with a capacitor across 220 V mains. Find the value of capacitance such that maximum current flows in the circuit at a frequency of 190 Hz. Also find the maximum current. ***(AU Dec'14, 2 Marks)***

Given that, $R = 2.2\ \Omega$; $L = 0.01\ H$ and supply voltage, $V = 220\ V$

The current will be maximum only at resonance. Therefore, the resonance frequency is 190 Hz.

Resonant frequency, $f_r = \dfrac{1}{2\pi\sqrt{LC}}$

$$\therefore\ C = \frac{1}{4\pi^2 f_r^2 L} = \frac{1}{4\pi^2 \times 190^2 \times 0.01} = 7.0167 \times 10^{-5}\ F = 70.167 \times 10^{-6}\ F = 70.167\ \mu F$$

Current at resonance, (Maximum current) $I_r = \dfrac{V}{R} = \dfrac{220}{2.2} = 100\ A$

Q5.24 Draw the frequency response of an RLC parallel circuit.

The variation of current with frequency is called frequency response, which is shown in Fig. Q5.24.

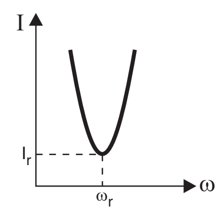


Fig. Q5.24.

Q5.25 Write the expressions for quality factor of a parallel RLC circuit.

Quality factor at resonance, $Q_r = \dfrac{R}{\omega_r L}$

$$\boxed{\text{Alternatively, } Q_r = \omega_r CR\ ;\quad Q_r = R\sqrt{\frac{C}{L}}\ ;\quad Q_r = \frac{\omega_r}{\beta}}$$

When, $\omega \le \omega_r$, $\quad Q = \dfrac{R}{\omega L}$

When, $\omega \ge \omega_r$, $\quad Q = \omega CR$

Q5.26 Write the expressions for bandwidth of an RLC parallel circuit.

Bandwidth, $\beta = \dfrac{1}{RC}$ in *rad/s* $\quad \boxed{\text{Alternatively, } \beta = \dfrac{\omega_r}{Q_r} \text{ in } rad/s}$

Bandwidth in $Hz = \dfrac{1}{2\pi RC}$ $\quad \boxed{\text{Alternatively, Bandwidth in } Hz = \dfrac{f_r}{Q_r}}$

Q5.27 Write the expression for half-power frequencies of an RLC parallel circuit.

Lower cut-off frequency, $f_l = \frac{1}{2\pi}\left[-\frac{1}{RC}+\sqrt{\left(\frac{1}{2RC}\right)^2+\frac{1}{LC}}\right]$ in *Hz*

Higher cut-off frequency, $f_h = \frac{1}{2\pi}\left[\frac{1}{RC}+\sqrt{\left(\frac{1}{2RC}\right)^2+\frac{1}{LC}}\right]$ in *Hz*

Alternatively, $f_l = f_r\left[-\frac{1}{2Q_r}+\sqrt{1+\frac{1}{4Q_r^2}}\right]$; $f_h = f_r\left[\frac{1}{2Q_r}+\sqrt{1+\frac{1}{4Q_r^2}}\right]$

Q5.28 Write the expression for admittance of an RLC parallel circuit at half-power frequencies.

At half-power frequencies of an RLC parallel circuit, the total susceptance is equal to conductance.

i.e., at $\omega = \omega_h$, $\omega C - \frac{1}{\omega L} = G$

$\therefore$ At $\omega = \omega_h$, $Y = \sqrt{G^2+\left(\omega C-\frac{1}{\omega L}\right)^2} = \sqrt{G^2+G^2} = \sqrt{2G^2} = \sqrt{2}\,G$

$$\text{Alternatively,} \quad Y = \sqrt{G^2+\left(\omega C-\frac{1}{\omega L}\right)^2} = \sqrt{\left(\omega C-\frac{1}{\omega L}\right)^2+\left(\omega C-\frac{1}{\omega L}\right)^2}$$

$$= \sqrt{2\left(\omega C-\frac{1}{\omega L}\right)^2} = \sqrt{2}\left(\omega C-\frac{1}{\omega L}\right)$$

Q5.29 Write the characteristics of parallel resonance.

- At resonance, admittance is minimum and equal to conductance, therefore, current is minimum.
- Below resonant frequency, the circuit behaves as an inductive circuit and above resonant frequency, the circuit behaves as a capacitive circuit.
- At resonance, the magnitude of current through inductance and capacitance will be Q times the current supplied by the source but they are in-phase opposition.

Q5.30 What is dynamic resistance? Write the expression for dynamic resistance of an RL circuit parallel with C.

The resistance of an RLC parallel circuit at resonance is called dynamic resistance.

For an RL circuit parallel with C, the dynamic resistance is given by,

$$R_{\text{dynamic}} = \frac{L}{RC}$$

Q5.31 In Fig. Q5.31, the source voltage and current are in-phase. Find the value of C.

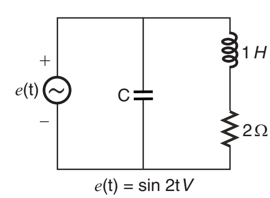


Fig. Q5.31.

Since the source voltage and current are in-phase, the circuit will be in resonance.

Given that, $e(t) = \sin 2t$

The standard form of sinusoidal voltage is,

$e(t) = E_m \sin\omega t$

On comparing the above two equations of e(t), we get,

$\omega_r = 2$ rad/s

For the RLC circuit shown in Fig. Q8.30, $\omega_r = \frac{1}{\sqrt{LC}}\sqrt{1-\frac{CR^2}{L}}$

$$\therefore\ \omega_r^2 = \frac{1}{LC}\left(1-\frac{CR^2}{L}\right) \Rightarrow \omega_r^2 + \frac{R^2}{L^2} = \frac{1}{LC} \Rightarrow LC = \frac{1}{\omega_r^2 + \frac{R^2}{L^2}}$$

$$\therefore\ C = \frac{1}{L\left(\omega_r^2 + \frac{R^2}{L^2}\right)} = \frac{1}{1\times\left(2^2\times\frac{2^2}{1^2}\right)} = \frac{1}{16} = 0.0625\,F = 62.5\times10^{-3}F = 62.5\,mF$$

Q5.32 For the RLC circuit shown in Fig. Q5.32, find the resonant frequency.

$$\text{Impedance}, \bar{Z} = 10 + \frac{\left(j4\omega + \frac{1}{j\omega}\right)\left(\frac{1}{j\omega}\right)}{\left(j4\omega + \frac{1}{j\omega}\right) + \frac{1}{j\omega}}$$

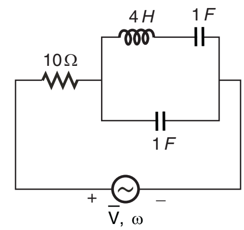


Fig. Q5.32.

At resonance, the imaginary part of impedance should be zero. Therefore, the numerator of the imaginary part should be zero.

Therefore, at $\omega = \omega_r$,

$$\left(j4\omega_r + \frac{1}{j\omega_r}\right)\frac{1}{j\omega_r} = 0 \Rightarrow j4\omega_r + \frac{1}{j\omega_r} = 0 \Rightarrow j4\omega_r = -\frac{1}{j\omega_r}$$

$$\therefore\ j^2\omega_r^2 = -\frac{1}{4} \Rightarrow -\omega_r^2 = -\frac{1}{4} \Rightarrow \omega_r = \sqrt{\frac{1}{4}} \Rightarrow \omega_r = \frac{1}{2} = 0.5\,rad/s$$

Q5.33 An RLC parallel circuit with G = 10℧, B_L = 20℧ and B_C = 20℧ draws a current of 5 A when excited by a sinusoidal source. Determine the current through inductance.

Since $B_L = B_C$, the circuit will be in resonance. At resonance, the current through inductance will be Q_r times the current drawn from the source.

$$\therefore\ \text{Quality factor at resonance}, Q_r = \frac{R}{\omega_r L} = \frac{B_L}{G} = \frac{20}{10} = 2$$

$$\text{Current through inductance} = Q_r I = 2\times5 = 10\,A$$

Q5.34 What are coupled circuits?

The coupled circuits refer to circuits involving elements with magnetic coupling. If the flux produced by an element of a circuit links other elements of the same circuit or a nearby circuit then the elements are said to have magnetic coupling.

Q5.35 What are coupled coils?

When two or more coils are linked by magnetic flux, the coils are called coupled coils (or coupled coils are group of two or more coils linked by magnetic flux).

Q5.36 Define self-inductance. ***(AU May'17, 2 Marks)***

When permeability is constant, the self-inductance of a coil is defined as the ratio of flux linkage and current. (The flux linkage is the product of flux and number of turns.)

$$\therefore\ \text{Self inductance}, L = \frac{N\phi}{i}$$

Q5.37 Define mutual inductance. ***(AU May'15&'17, 2 Marks)***

When permeability is constant, the mutual inductance between two coupled coils is defined as the ratio of flux linkage in one coil due to common flux and current through another coil. (The flux linkage in one coil due to common flux is the product of the flux linking both the coils and the number of turns of the coil.)

$$\therefore \text{ Mutual inductance, } M = \frac{N_1\phi_{21}}{i_2} \text{ or } M = \frac{N_2\phi_{12}}{i_1}$$

where, $\phi_{12} = \phi_{21}$ = Common flux.

N_1, N_2 = Number of turns in coil 1 and 2.

i_1, i_2 = Currents in coil 1 and 2.

Q5.38 Define coefficient of coupling. ***(AU June'16 & Dec'15, 2 Marks)***

In coupled coils, the coefficient of coupling is defined as the fraction of the total flux produced by one coil linking another coil.

$$\therefore \text{ Coefficient of coupling, } k = \frac{\phi_{12}}{\phi_1} \text{ or } k = \frac{\phi_{21}}{\phi_2}$$

where, ϕ_{12} = Flux produced by coil-1 linking coil-2.

ϕ_{21} = Flux produced by coil-2 linking coil-1.

ϕ_1 = Flux produced by coil-1.

ϕ_2 = Flux produced by coil-2.

Q5.39 Write the expression which relates self- and mutual inductance.

Mutual inductance between two coils linked by magnetic flux is given by,

$$\text{Mutual inductance, } M = k\sqrt{L_1 L_2}$$

where, L_1 = Self-inductance of coil-1.

L_2 = Self-inductance of coil-2.

k = Coefficient of coupling.

Q5.40 What is dot convention? Why it is required?

The sign or polarity of mutual induced emf depends on the winding sense (or coil orientation) and current through the coil. The winding sense is decided by the manufacturer and to inform the user about the winding sense, a dot is placed at one end/terminal of each coil. When current enters at the dotted end in one coil, the mutual induced emf in the other coil is positive at the dotted end. Hence, the dot convention is required to predict the sign of mutual induced emf.

Q5.41 State dot rule for coupled coils.

The dot rule states that in coupled coils, current entering at the dotted terminal of one coil induces an emf in the second coil which is positive at the dotted terminal of the second coil.

Conversely, current entering at the undotted terminal of one coil induces an emf in the second coil which is positive at the undotted terminal of the second coil.

Q5.42 Draw the electrical equivalent of the coupled coils shown in Figs. Q5.42.1 and Q5.42.2.

The electrical equivalent of the coupled coils of Figs Q5.42.1 and Q5.42.2 is shown in Figs Q5.42.3 and Q5.42.4, respectively.

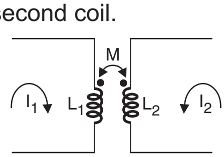


Fig. Q5.42.1.

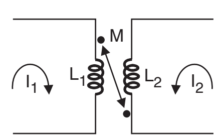


Fig. Q5.42.2.

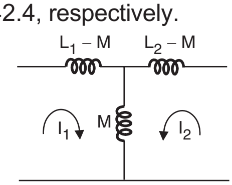


Fig. Q5.42.3.

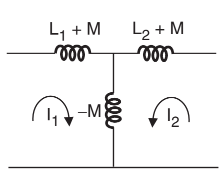


Fig. Q5.42.4.

Q5.43 Write the expression for the equivalent inductance of two coupled coils connected in series.

In series aiding, equivalent inductance, $L_{eq} = L_1 + L_2 + 2M$.

In series opposing, equivalent reactance, $L_{eq} = L_1 + L_2 - 2M$.

Q5.44 Write the expression for the equivalent inductance of two coupled coils connected in parallel.

In parallel aiding, equivalent inductance, $L_{eq} = \dfrac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$

In parallel opposing, equivalent inductance, $L_{eq} = \dfrac{L_1L_2 - M^2}{L_1 + L_2 + 2M}$

Q5.45 Determine the equivalent inductance of the circuit shown in Fig. Q5.45, if the coefficient of coupling(k) between the two coils is 0.6.

(AU Dec'14, 2 Marks)

Mutual inductance, $M = k\sqrt{L_1L_2} = 0.6 \times \sqrt{33 \times 47} = 23.6296\ mH$

Equivalent inductance, $L_{eq} = L_1 + L_2 + 2M$

$= 33 + 47 + 2 \times 23.6296 = 127.2592\ mH$

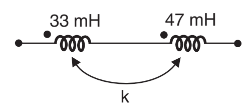


Fig. 5.45.

Q5.46 Determine the equivalent inductance of the series-connected inductances shown in Fig. Q5.46.

The equivalent inductance of series-connected inductances can be determined as shown below :

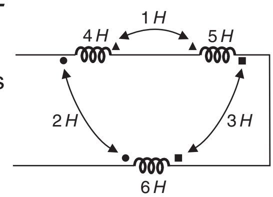
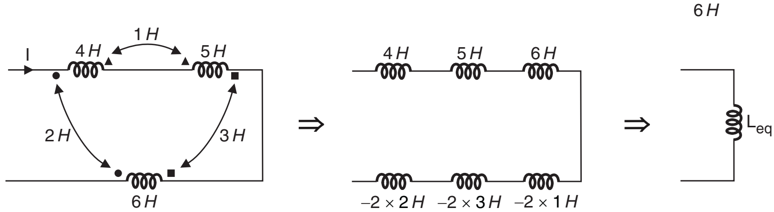


Fig. Q5.46.



Equivalent inductance, $L_{eq} = 4 + 5 + 6 - 2 \times 1 - 2 \times 3 - 2 \times 2 = 3H$

Q5.47 What is tuned coupled circuit?

In a coupled circuit, when capacitors are added to the primary and secondary of coupled coils to resonate the coils to achieve maximum power transfer condition, the coupled circuit is called a tuned coupled circuit.

Q5.48 Why and where are tuned coupled circuits employed?

Tuned coupled circuits are mainly used to transfer energy from a weak source to a load or employed for maximum power transfer from one circuit to another. This is possible only when both the coils work at resonance condition.

Q5.49 What is a single tuned circuit?

In a coupled circuit, when a capacitor is added to the secondary coil to resonate the secondary, the coupled circuit is called a single tuned coupled circuit.

Q5.50 Draw the frequency response of a single tuned coupled circuit.

The frequency response of a single tuned coupled circuit for various value of k is shown in Fig. Q5.50.

Here, $\overline{V}_0$ is the output voltage.

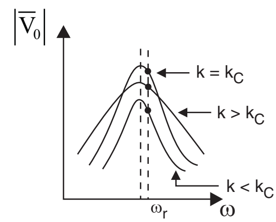


***Fig. Q5.50 :** $|V_0|$ Vs ω for different values of k.*

Q5.51 Define critical coefficient of coupling and critical mutual inductance.

In tuned coupled circuits, the critical coefficient of coupling is defined as the value of coupling coefficient when the output voltage at resonance is maximum.

In tuned coupled circuits, the optimum or critical mutual inductance is defined as the value of mutual inductance when the output voltage at resonance is maximum.

Q5.52 Write the expression for k_C and M_C.

$$\text{Critical coefficient of coupling, } k_C = \frac{M_C}{\sqrt{L_1 L_2}} \quad \text{or} \quad \frac{1}{\omega_r}\sqrt{\frac{R_1 R_2}{L_1 L_2}} \quad \text{or} \quad \frac{1}{\sqrt{Q_1 Q_2}}$$

$$\text{Critical mutual-inductance , } M_C = k\sqrt{L_1 L_2} \quad \text{or} \quad \frac{\sqrt{R_1 R_2}}{\omega_r}$$

5.12 Exercises

I. Fill in the Blanks With Appropriate Words

1. The ________ is the circuit condition at which the circuit behaves as a resistive circuit.
2. In a series RLC circuit, when $\omega > \omega_r$, the total reactance is ________ and when $\omega < \omega_r$, the total reactance is ________ .
3. In an RLC series circuit, ________ is maximum and ________ is minimum at resonance.
4. When $\omega < \omega_r$, the current ________ and when $\omega > \omega_r$, the current ________ in an RLC series circuit.
5. In an RLC parallel circuit, ________ and ________ are minimum at resonance.
6. In a series RLC circuit, when $\omega < \omega_r$, the energy stored in the inductor is ________ than the energy stored in the capacitor.
7. The ________ is the ratio of energy stored and energy dissipated.
8. Resonant frequency is given by the ________ of the two half-power frequencies.
9. The ________ is the range of frequencies over which power is greater than or equal to 1/2 times the maximum power.
10. The impedance of an RLC series circuit at half-power frequency is ________ .
11. The admittance of an RLC parallel circuit at half-power frequency is ________ .
12. When excited by a voltage source, a series RLC circuit magnifies ________ and a parallel RLC circuit magnifies ________ at resonance.
13. At resonance in a series RLC circuit, the ________ across inductance and capacitance is ________ times supply voltage.

14. At resonance in a parallel RLC circuit, the ________ in inductance and capacitance is ________ times the current drawn from the source.
15. The ________ is the ratio of bandwidth and resonant frequency.
16. In a series RLC circuit, ________ across inductance will be maximum at a frequency ________ than resonant frequency.
17. The coils linked by magnetic flux are called ________ .
18. The ________ emf is the emf induced due to change in flux in the same coil.
19. The ________ emf is the emf induced due to change in flux in a nearby coil.
20. When all the flux produced by one coil links another coil then the coils are said to be ________ coupled.
21. In coupled coils, when the value of ________ is very low, the coils are said to be ________ coupled.
22. In coupled coils, the self-inductances of two coils are 20 *mH* and 5 *mH*. The maximum possible value of mutual inductance is ________ .
23. The equivalent inductance of series aiding connection is ________ than equivalent inductance of series opposing connection.
24. The equivalent inductance of parallel opposing connection is ________ than equivalent inductance of parallel aiding connection.
25. In ________ tuned coupled circuit, a capacitor is added in the secondary circuit.
26. In ________ tuned coupled circuit, capacitors are added to both the primary and secondary.
27. In ________ tuned circuit, the primary and secondary are tuned to different frequencies.
28. In tuned circuits ________ is varied to maximise the output voltage.
29. At critical coupling, the output voltage is ________ .
30. When $M = M_C$, the output ________ is maximum.
31. The frequency response of ________ tuned circuit exhibits ________ when $k > k_C$.

ANSWERS

1.	resonance	11.	$\sqrt{2}\,G$ or $\sqrt{2}\,(B_C - B_L)$	21.	k, loosely
2.	inductive, capacitive	12.	voltage, current	22.	10 *mH*
3.	current, impedance	13.	voltage, Q_r	23.	greater
4.	leads, lags	14.	current, Q_r	24.	lesser
5.	current, admittance	15.	Selectivity	25.	single
6.	less	16.	voltage, greater	26.	double
7.	quality factor	17.	coupled coils	27.	stagger
8.	geometric mean	18.	self-induced	28.	coefficient of coupling
9.	bandwidth	19.	mutual induced	29.	maximum
10.	$\sqrt{2}\,R$ or $\sqrt{2}\,(X_L - X_C)$	20.	tightly or closely	30.	voltage
				31.	double, double hump

II. State Whether the Following Statements are True or False

1. At resonance, the total reactance of a circuit is zero.
2. In an RLC series circuit, the impedance is minimum at resonance but in an RLC parallel circuit, the impedance is maximum at resonance.
3. In an RLC parallel circuit, the current leads when $\omega < \omega_r$ and lags when $\omega > \omega_r$.
4. At resonance, the current is in-phase opposition with voltage.
5. At resonance, the energy stored in the inductor and capacitor will be equal.
6. In a series RLC circuit, when $\omega > \omega_r$, the inductor stores more energy than the capacitor.
7. The Q_r of an RLC series circuit will be high if the inductance is very large and the capacitance is low.
8. The Q_r of an RLC parallel circuit will be low if the capacitance is very large and the inductance is low.
9. The frequency response of an RLC circuit is symmetric with respect to resonant frequency.
10. In an RLC series circuit, at half-power frequencies the total reactance is equal to resistance.
11. In an RLC parallel circuit, at half-power frequencies the total susceptance is equal to conductance.
12. At resonance in a series RLC circuit, the voltages across inductance and capacitance are equal in magnitude.
13. At resonance in a parallel RLC circuit, the currents through inductance and capacitance are equal in magnitude.
14. In a series RLC circuit, the voltage across capacitance will be maximum at a frequency greater than resonant frequency.
15. Selectivity is directly proportional to quality factor.
16. In magnetically coupled elements, power transfer occurs through flux.
17. In conductively coupled elements, power transfer occurs through flow of current.
18. Self-induced emf is proportional to the rate of change of current in a nearby coil.
19. Mutual induced emf is proportional to the rate of change of current in the same coil.
20. Between two coils with inductances L_1 and L_2, the maximum possible value of mutual inductance is $\sqrt{L_1 L_2}$.
21. In two coils placed nearby, if the flux produced by one does not link the other then they are said to be magnetically isolated.
22. In coupled coils, the sign of self- and mutual induced emf will be the same if the fluxes are opposing.
23. In coupled coils, current entering at the dotted end in a coil will induce an emf in another coil which is positive at undotted end.
24. In coupled coils, when current enters or leaves at the dotted ends in both the coils, the sign of self- and mutual induced emf are the same.
25. In coupled coils, when current enters at the dotted end in one coil and leaves at the dotted end in another coil, sign of self and mutual induced emf are opposite.
26. Tuned circuits work at resonance condition.
27. In double tuned circuits, the primary and secondary are tuned to different frequencies.
28. In tuned circuits, k is varied to maximise output voltage.
29. In single tuned circuit, the output voltage is maximum at a frequency lesser than resonant frequency.
30. The frequency response of single tuned circuit exhibit double hump.

ANSWERS

1. True	7. True	13. True	19. False	25. True
2. True	8. False	14. False	20. True	26. True
3. False	9. False	15. False	21. True	27. False
4. False	10. True	16. True	22. False	28. True
5. True	11. True	17. True	23. False	29. True
6. True	12. True	18. False	24. True	30. False

III. Choose the Right Answer for the Following Questions

1. ***A coil with inductance 1 mH and resistance 10Ω is connected in series with a condenser and excited by a sinusoidal source of frequency 10,000 rad/s. What is the value of capacitance of the condenser for resonance?***

   a) $0.1\,\mu F$ b) $1\,\mu F$ c) $10\,\mu F$ d) $100\,\mu F$

2. ***An RLC series circuit consists of R = 16Ω, L = 13 mH and C = 41 μF. The resonance frequency is,***

   a) $207\,Hz$ b) $218\,Hz$ c) $436\,Hz$ d) $1370\,Hz$

3. ***An RLC series circuit with R = 0.4Ω, L = 0.25 mH and C = 40 μF is excited by a sinusoidal source of voltage 6 V. The value of current at resonance and quality factor respectively are,***

   a) $15\,A$, 6.25 b) $6.25\,A$, 15 c) $1.5\,A$, 2.5 d) $5\,A$, 0.625

4. ***An RLC series circuit has a Q-factor of 5 and resonates at a frequency of 1000 Hz. The cut-off frequencies are,***

   a) $965\,Hz$, $1085\,Hz$ b) $925\,Hz$, $1082\,Hz$ c) $950\,Hz$, $1054\,Hz$ d) $905\,Hz$, $1105\,Hz$

5. ***The cut-off frequencies of an RLC series circuit are 1810 Hz and 2210 Hz. The resonance frequency is,***

   a) $200\,Hz$ b) $400\,Hz$ c) $2000\,Hz$ d) $2010\,Hz$

6. ***The current through an RLC series circuit at resonance is 5 A when excited by a 200 V sinusoidal source. The total reactance at half-power frequencies is,***

   a) $20\,\Omega$ b) $40\,\Omega$ c) $60\,\Omega$ d) $80\,\Omega$

7. ***An RLC circuit has a Q-factor of 4 at resonance. The selectivity is,***

   a) 0.25 b) 2 c) 4 d) 8

8. ***An RLC series circuit with R = 5Ω, L = 2 mH and C = 4 μF is excited by a 120 V sinusoidal source. The voltage across inductance at resonance is,***

   a) $432\,V$ b) $537\,V$ c) $657\,V$ d) $777\,V$

9. An RLC parallel circuit consists of R = 5Ω, L = 0.9mH and C = 3300μF. The resonance angular frequency and dynamic resistance respectively are,

a) $580\,rad/s$, 5Ω b) $290\,rad/s$, 5Ω c) $92\,rad/s$, 5Ω d) $580\,rad/s$, 0.2Ω

10. A coil with 95mH inductance and 0.2Ω resistance is connected in parallel with a condensor to make the power factor unity when excited from a 115V, 60Hz ac supply. What is the value of the capacitance of the condensor?

a) $96\mu F$ b) $37\mu F$ c) $74\mu F$ d) $1276\mu F$

11. For the RLC parallel circuit shown in Fig. 11, the expression for angular frequency of resonance is,

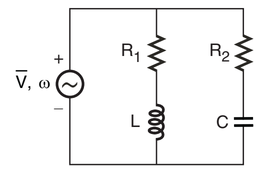


Fig. 11.

a) $\dfrac{1}{LC}\sqrt{\dfrac{L-CR_1^2}{L-CR_2^2}}$ b) $\dfrac{1}{\sqrt{LC}}\sqrt{\dfrac{L-CR_1^2}{L-CR_2^2}}$

c) $\dfrac{1}{\sqrt{LC}}\left(\dfrac{L-CR_1^2}{L-CR_2^2}\right)$ d) $\dfrac{1}{\sqrt{LC}}\sqrt{\dfrac{L-CR_2^2}{L-CR_1^2}}$

12. For the RLC parallel circuit shown in Fig. 12, the condition for resonance at all frequency is,

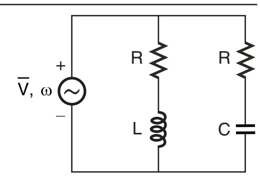


Fig. 12.

a) $R = \dfrac{L}{C}$ b) $R = \dfrac{C}{L}$

c) $R = \sqrt{\dfrac{C}{L}}$ d) $R = \sqrt{\dfrac{L}{C}}$

13. For the RLC circuit shown in Fig. 13, the expression for resonance angular frequency and dynamic resistance respectively are,

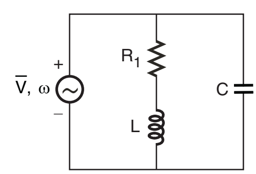


Fig. 13.

a) $\dfrac{1}{\sqrt{LC}}\sqrt{1-\dfrac{CR_1^2}{L}}\;;\;\dfrac{L}{R_1C}$ b) $\dfrac{1}{LC}\sqrt{1-\dfrac{CR_1^2}{L}}\;;\;\dfrac{L}{R_1C}$

c) $\dfrac{1}{\sqrt{LC}}\sqrt{1-\dfrac{L}{CR_1^2}}\;;\;\dfrac{R_1C}{L}$ d) $\dfrac{1}{LC}\sqrt{1-\dfrac{L}{CR_1^2}}\;;\;\dfrac{R_1C}{L}$

14. For the RLC circuit shown in Fig. 14, the expression for resonant angular frequency and dynamic resistance respectively are,

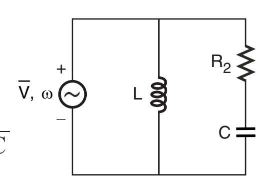


Fig. 14.

a) $\dfrac{1}{LC}\sqrt{\dfrac{L}{L-CR_2^2}}\;;\;\dfrac{L}{R_2C}$ b) $\dfrac{1}{\sqrt{LC}}\sqrt{\dfrac{L}{L-CR_2^2}}\;;\;\dfrac{L}{R_2C}$

c) $\dfrac{1}{LC}\sqrt{1-\dfrac{CR_2^2}{L}}\;;\;\dfrac{R_2C}{L}$ d) $\dfrac{1}{\sqrt{LC}}\sqrt{1-\dfrac{CR_2^2}{L}}\;;\;\dfrac{R_2C}{L}$

15. An RLC parallel circuit with R = 8Ω, L = 1mH and C = 250μF is excited by a sinusoidal source of 12V. The value of current at resonance and quality factor respectively are,

a) $4A$, 1 b) $20A$, 2 c) $1.5A$, 4 d) $0.67A$, 8

16. ***An RLC parallel circuit has a Q-factor of 3 and resonates at a frequency of 200Hz. The cut-off frequencies are,***

a) 169 *Hz*, 236 *Hz* b) 169 *Hz*, 231 *Hz* c) 164 *Hz*, 236 *Hz* d) 164 *Hz*, 231 *Hz*

---

17. ***The cut-off frequencies of an RLC parallel circuit are 543Hz and 663Hz. The resonance frequency is,***

a) 1206 *Hz* b) 663 *Hz* c) 603 *Hz* d) 600 *Hz*

---

18. ***The current through an RLC parallel circuit at resonance is 12A when excited by 120V sinusoidal source. The total susceptance at half-power frequencies is,***

a) 10 ℧ b) 0.1 ℧ c) 20 ℧ d) 0.2 ℧

---

19. ***An RLC parallel circuit with R = 20Ω, L = 2mH and C = 20μF is excited by a 60A sinusoidal current source. The current through inductance at resonance is,***

a) 30 *A* b) 60 *A* c) 120 *A* d) 240 *A*

---

20. ***For the circuit shown in Fig. 20, the angular frequency of resonance and dynamic resistance respectively are,***

a) 3500 *rad/s* , 0.02Ω b) 3200 *rad/s* , 50Ω

c) 3391 *rad/s* , 50Ω d) 4375 *rad/s* , 0.02Ω

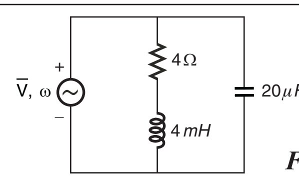


Fig. 20.

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21. ***The self-inductance of a coil with 500 turns carrying a current of 2A and developing a flux of 0.25×10^{-2} Wb is,***

a) 2.5 *H* b) 0.625 *H* c) 6.25 *mH* d) 25 *mH*

---

22. ***The coefficient of coupling k between two coils with self-inductances L_1 and L_2 and with mutual inductance M between them is given by,***

a) $k = \frac{\sqrt{L_1 L_2}}{M}$ b) $k = \frac{\sqrt{L_1 + L_2}}{M}$ c) $k = \frac{M}{\sqrt{L_1 + L_2}}$ d) $k = \frac{M}{\sqrt{L_1 L_2}}$

---

23. ***The mutual inductance between two coils having self-inductances of 40 μH and 90 μH when 40% of flux produced by coils link each other is given by,***

a) 24 μ*H* b) 52 μ*H* c) 150 μ*H* d) 325 μ*H*

---

24. ***The mutual inductance between two coils with inductances 0.2 H and 0.8 H when all the flux produced by coils link each other is,***

a) 1 *H* b) 0.6 *H* c) 0.4 *H* d) 0.3 *H*

---

25. ***The equivalent inductance of two series-connected coils with self-inductances L_1 and L_2 and mutual inductance M is given by,***

a) $L_1 L_2 \pm M$ b) $L + L_2 \pm 2M$ c) $M(L_1 \pm L_2)$ d) $2M(L_1 \pm L_2)$

---

26. ***The equivalent inductance of two parallel-connected coils with self-inductances L_1 and L_2 and mutual inductance M is given by,***

a) $\dfrac{L_1 + L_2 \pm M^2}{L_1 L_2 + M}$ b) $\dfrac{L_1 + L_2 \pm 2M}{L_1 L_2 + M^2}$ c) $\dfrac{L_1 L_2 + M^2}{L_1 + L_2 \pm M}$ d) $\dfrac{L_1 L_2 - M^2}{L_1 + L_2 \pm 2M}$

27. ***The two possible equivalent inductances of two series-connected coils with self-inductances 0.07 H and 0.09 H and mutual inductance 0.01 H are,***

a) 0.17 *H*, 0.15 *H*

b) 0.18 *H*, 0.14 *H*

c) 0.18 *H*, 0.04 *H*

d) 0.0032 *H*, 0.0004 *H*

28. ***The two possible equivalent inductances of two parallel-connected coils with self-inductances 0.06 H and 0.03 H and mutual inductance 0.02 H are,***

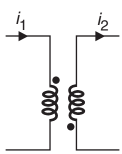
a) 0.011 *H*, 0.028 *H*

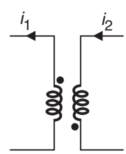
b) 0.022 *H*, 0.014 *H*

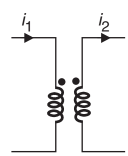
c) 0.017 *H*, 0.044 *H*

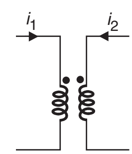
d) 0.034 *H*, 0.022 *H*

29. ***Which one of the given coupled coils represents a natural transformer?***

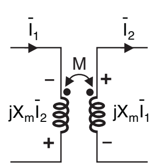
a) 

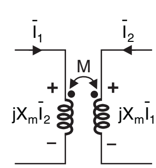
b) 

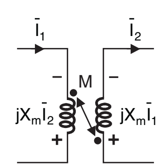
c) 

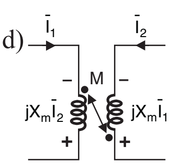
d) 

30. ***In the given coupled coils with mutual induced emf, identify the one that does not satisfy dot rule?***

a) 

b) 

c) 

d) 

31. ***In the coupled coil shown in Fig. 31, the mutual induced emf in coils 1 and 2 respectively are,***

a) $j2\bar{I}_2,\ j3\bar{I}_1$

b) $j6\bar{I}_2,\ j7\bar{I}_1$

c) $j13\bar{I}_1,\ j7\bar{I}_2$

d) $j2\bar{I}_2,\ j2\bar{I}_1$

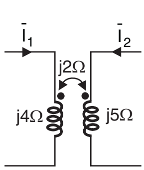


Fig. 31.

32. *The equivalent inductance of the series-connected coupled coils shown in Fig. 32 is,*

a) $1.5H$

b) $2.7H$

c) $3.5H$

d) $4.7H$

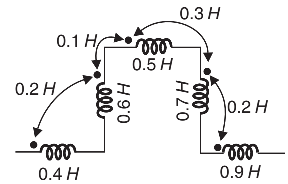


Fig. 32.

33. *The equivalent inductance of the parallel-connected coupled coils shown in Fig. 33 is,*

a) $0.5H$

b) $0.25H$

c) $1.33H$

d) $0.75H$

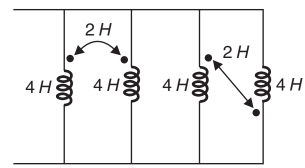


Fig. 33.

34. *In tuned coupled circuit, the critical value of mutual inductance M_C is given by,*

a) $\dfrac{\omega_r}{\sqrt{R_1 R_2}}$ b) $\dfrac{\sqrt{R_1 R_2}}{\omega_r}$ c) $\dfrac{\omega_r}{R_1 R_2}$ d) $\dfrac{R_1 R_2}{\omega_r}$

35. *In tuned coupled circuit, the critical coefficient of coupling k_C is given by,*

a) $\dfrac{1}{\sqrt{Q_1 Q_2}}$ b) $\sqrt{\dfrac{Q_1}{Q_2}}$ c) $\sqrt{\dfrac{Q_2}{Q_1}}$ d) $\dfrac{1}{Q_1 Q_2}$

ANSWERS

1. c	8. b	15. c	22. d	29. a
2. b	9. a	16. a	23. a	30. c
3. a	10. c	17. d	24. c	31. d
4. d	11. b	18. b	25. b	32. c
5. c	12. d	19. c	26. d	33. d
6. b	13. a	20. c	27. b	34. b
7. a	14. b	21. b	28. a	35. a

IV. Unsolved Problems

E5.1 ***An RLC series circuit consists of $R = 50\,\Omega$, $L = 0.16$ H and $C = 4\,\mu F$. Calculate the resonant frequency, quality factor, bandwidth and half-power frequencies.***

E5.2 ***An RLC series circuit is to be designed to produce a magnification of 5 at resonance. At higher cut-off frequency 1105 rad/s the impedance of the circuit is $21.2132\,\Omega$. Find the value of R, L and C.***

E5.3 ***An RC series circuit with $R = 50\,\Omega$ and $C = 20\,\mu F$ is connected parallel to an inductance. The parallel combination is excited by a source of 10 V, 1 kHz. Determine the value of inductance if no reactive current is taken from the supply.***

E5.4 ***For the RLC circuit shown in Fig. E5.4, determine the frequency at which the circuit resonates. Also find the quality factor, voltage across inductance and capacitance at resonance.***

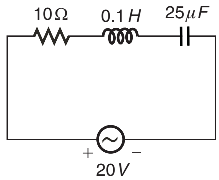


Fig. E5.4.

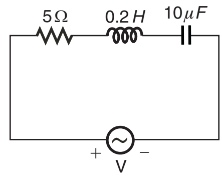


Fig. E5.5.

E5.5 ***For the RLC circuit shown in Fig. E5.5, determine the impedance a) at resonance frequency, b) at a frequency 50 Hz below resonance frequency, c) at a frequency 50 Hz above resonance frequency.***

E5.6 ***Determine the value of R_L for resonance in the network shown in Fig. E5.6. Also calculate the dynamic resistance.***

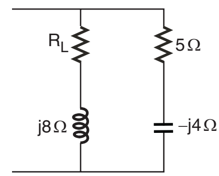


Fig. E5.6.

E5.7 ***The parameters of an RLC parallel circuit excited by a current source are, R = 20Ω, L = 5 mH and C = 10 μF. Calculate the resonant frequency, quality factor, bandwidth and cut-off frequencies.***

E5.8 ***Determine the value of R_L for resonance in the network shown in Fig. E5.8.***

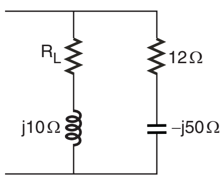


Fig. E5.8.

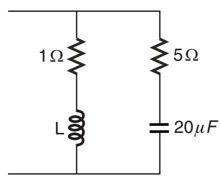


Fig. E5.9.

E5.9 ***For the RLC network shown in Fig. E5.9, determine the two possible values of inductance for the network to resonate at 5000 rad/s.***

E5.10 ***A coil of inductance 25 mH and resistance 20Ω is connected parallel to a capacitor, and this parallel combination is connected to a 200 V, 40 Hz supply. Determine the value of capacitance if no reactive current is taken from the supply.***

E5.11 ***Two coupled coils have a coefficient of coupling of 0.65. If the self-inductances of the coils are 0.02 H and 0.08 H, calculate the equivalent inductance if the two coils are connected in a) series aiding, b) series opposing, c) parallel aiding and d) parallel opposing.***

E5.12 ***Two coupled coils connected in series have an equivalent inductance of 0.725 H when connected in aiding and an equivalent inductance of 0.425 H when connected in opposing. Determine the self- and mutual inductances by taking k = 0.42.***

E5.13 ***Two coils A and B of 1000 turns and 1600 turns respectively lie in parallel plane, so that 55% of flux produced by coil-A links with coil-B. A current of 2 A in coil-A produce a flux of 0.3×10^{-4} Wb while the same current in coil-B produce a flux of 0.6×10^{-4} Wb. Determine the coefficient of coupling between the coils.***

E5.14 Determine the effective inductance of the series-connected coupled coils shown in Fig. E5.14.1, Fig. E5.14.2 and Fig. E5.14.3.

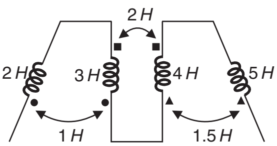


Fig. 5.14.1.

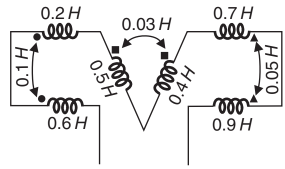


Fig. 5.14.2.

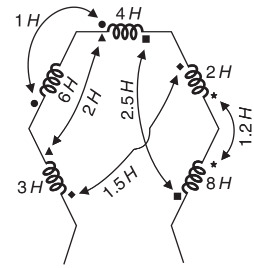


Fig. 5.14.3.

E5.15 Determine the equivalent inductance of the series-parallel-connected coupled coils shown in Fig. E5.15.1, Fig. E5.15.2 and Fig. E5.15.3.

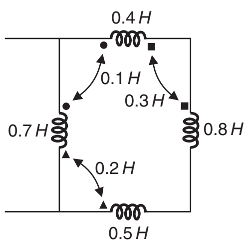


Fig. E5.15.1.

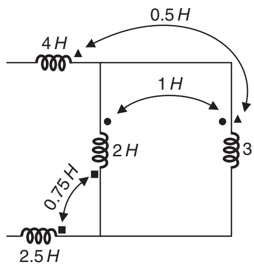


Fig. E5.15.2.

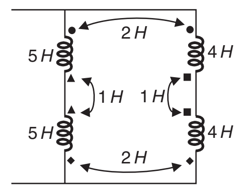


Fig. E5.15.3.

E5.16 Determine the equivalent impedance of the network with coupled coils shown in Fig. E5.16.1, Fig. E5.16.2 and Fig. E5.16.3.

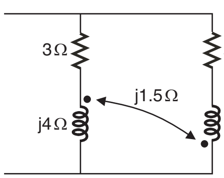


Fig. E5.16.1.

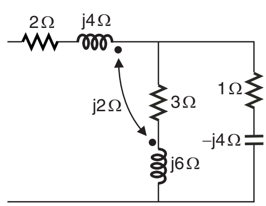


Fig. E5.16.2.

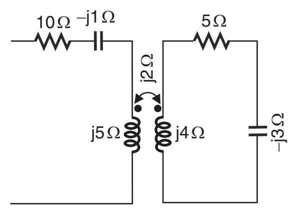


Fig. E5.16.3.

E5.17 A transformer with a primary having $R_p = 3\,\Omega$ and $L_p = 0.2\,H$ and a secondary having $R_s = 12\,\Omega$ and $L_s = 0.4\,H$ is connected between source voltage of 220 V at 50 Hz and a load of 600Ω. Determine the load current if $k = 0.5$.

E5.18 Determine the mesh currents in the coupled circuit shown in Fig. E5.18.

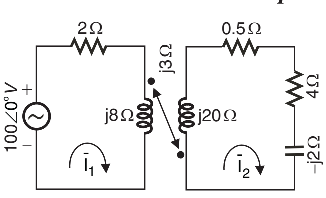


Fig. E5.18.

E5.19 ***A 10Ω load consumes 250W power when connected to a source of 220V, 50Hz through a transformer with primary impedance of $2 + j10\Omega$ and a secondary impedance of $4 + j20\Omega$. Determine the mutual-inductance and coefficient of coupling of the transformer.***

E5.20 ***In the coupled circuit shown in Fig. E5.20, determine the active and reactive power delivered to the load $\overline{Z}_L$.***

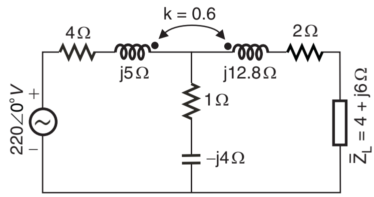


Fig. E5.20.

E5.21 ***In the circuit shown in Fig. E5.21, determine the voltage across $j6\Omega$ reactance.***

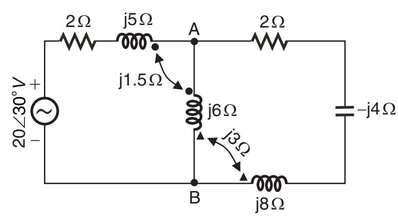


Fig. E5.21.

E5.22 ***Determine the mesh currents $\overline{I}_1$ and $\overline{I}_2$ in the circuit shown in Fig. E5.22.***

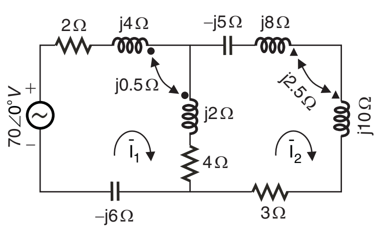


Fig. E5.22.

E5.23 ***In the single tuned coupled circuit of Fig. E5.23, determine the value of C for resonance at 1200 rad/s. Calculate the critical value of mutual inductance and coefficient of coupling. Also determine the output voltage $\overline{V}_{0,C}$ at critical coupling.***

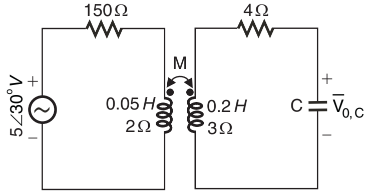


Fig. E5.23.

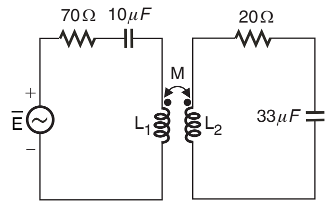


Fig. E5.24.

E5.24 ***The double tuned circuit shown in Fig. E5.24 is tuned to a frequency of 750 rad/s. Calculate the self-inductance of the two coils, critical value of mutual inductance and coefficient of coupling. Also calculate the output voltage at critical coupling if $\overline{E} = 10+0^o V$.***

E5.25 ***A double tuned circuit is tuned to a frequency of 2000 rad/s. What should be the supply voltage at critical coupling to get an output voltage of 12 V. The circuit parameters are : Q_1 = 1.5, R_1 = 4Ω , Q_2 = 2.5 and R_2 = 60Ω. Also determine M_C and k_C.***

ANSWERS

E5.1 ω_r = 1250 *rad/s* ; f_r = 198.9437 *Hz* ; Q_r = 4 ; β = 312.5 *rad/s*

Bandwidth in *Hz* = 49.7359 *Hz* ; f_h = 225.3599 *Hz* ; f_l = 175.6239 *Hz*

E5.2 R = 15Ω ; L = 75 *mH* ; C = 13.334 *μF*

E5.3 L = 51.3 *mH*

E5.4 ω_r = 632.4555 *rad/s* ; f_r = 100.6584 *Hz* ; Q_r = 6.3246

$\overline{V}_{Lr} = 126.4911\angle 90^o\ V$; $\overline{V}_{Cr} = 126.4911\angle -90^o\ V$

E5.5 $\overline{Z}_r = 5\ \Omega$; $\overline{Z}_1 = 175.9686\angle -88.4^o\ \Omega$; $\overline{Z}_2 = 106.453\angle 87.3^o\ \Omega$

E5.6 R_L = 4.2388Ω ; $R_{dynamic}$ = 5.7566Ω

E5.7 f_r = 711.7626 *Hz* ; Q_r = 0.8944 ; β = 5000 *rad/s*

Bandwidth in *Hz* = 795.7747 *Hz* ; f_h = 1213.3319 *Hz* ; f_l = 417.5329 *Hz*

E5.8 R_L = 20.7147 Ω

E5.9 L = 2.4839 *mH* or 16.1 *μH*

E5.10 C = 56.898 *μF*

E5.11 i) 0.152 *H* ii) 0.048 *H* iii) 0.01925 *H* iv) 6.0789×10^{-3} *H*

E5.12 M = 0.075 *H* ; L_1 = 0.5128 *H* ; L_2 = 0.0622 *H*

E5.13 k = 0.4919

E5.14	i) $L_{eq} = 5\,H$	ii) $L_{eq} = 2.94\,H$	iii) $L_{eq} = 26.6\,H$	
E5.15	i) $L_{eq} = 0.5667\,H$	ii) $L_{eq} = 6.8125\,H$	iii) $L_{eq} = 5.3333\,H$	
E5.16	i) $1.8884 + j1.6813\,\Omega$	ii) $4 + j0.5\,\Omega$	iii) $10.7692 + j3.8462\,\Omega$	
E5.17	$\bar{I}_L = 0.2503\angle{-6^\circ}\,A$			
E5.18	$\bar{I}_1 = 12.7852\angle{-74.3^\circ}\,A$;	$\bar{I}_2 = 2.0672\angle 119.7^\circ\,A$		
E5.19	$X_m = 5.2458\,\Omega$;	$M = 0.0167\,H$;	$k = 0.3709$	
E5.20	$P = 47.4022\ W$;	$Q = 71.1033\ VAR$		
E5.21	$\bar{V}_{AB} = 9.2183\angle 39.3^\circ\ V$			
E5.22	$\bar{I}_1 = 15.2387\angle 8^\circ\,A$;	$\bar{I}_2 = 5.3332\angle{-26.4^\circ}\,A$		
E5.23	$C = 3.4722\ \mu F$;	$M_C = 0.0272\,H$;	$k_C = 0.272$;	$\bar{V}_{0,C} = 18.3942\angle 30^\circ\,V$
E5.24	$L_1 = 0.1778\,H$;	$L_2 = 0.0539\,H$;	$M_C = 0.0499\,H$;	
	$k_C = 0.5097$;	$\bar{V}_{0,C} = 5.3992\angle 0^\circ\,V$		
E5.25	$E = 2.4762\,V$;	$M_C = 7.746\,mH$;	$k_C = 0.5164$	

USING CALCULATOR IN COMPLEX MODE

1. Addition/Subtraction/Multiplication/Division of Complex Numbers

Let, $\overline{A}_1 = -4 + j2$

$\overline{A}_2 = 3 + j5$

Choose complex mode in claculator and enter the complex numbers as shown below:

For addition **(–4 + 2 i) + (3 + 5 i)**

For subtraction **(–4 + 2 i) – (3 + 5 i)**

For multiplication **(–4 + 2 i) × (3 + 5 i)**

For division **(–4 + 2 i) ÷ (3 + 5 i)**

To perform the operation press $\boxed{=}$.

To view the real and imaginary part of the result press $\boxed{\text{SHIFT}}$ $\boxed{\text{Re} \longleftrightarrow \text{I}_\text{m}}$.

2. Polar to Rectangular Conversion

Let, $\overline{A}_1 = 5\angle -30°$

Method-1 : Choose complex mode in calculator and enter the complex number as shown below:

(5∠ – 30) $\boxed{\text{SHIFT}}$ $\boxed{\text{a + bi}}$

To perform the conversion press $\boxed{=}$.

To view the real and imaginary part press $\boxed{\text{SHIFT}}$ $\boxed{\text{Re} \longleftrightarrow \text{I}_\text{m}}$.

Method-2 : Choose complex mode in calculator and enter the complex number as shown below:

5 × cos 30 + 5i × sin 30 or 5 cos 30 + 5i sin 30

To perform the conversion press $\boxed{=}$.

To view the real and imaginary part press $\boxed{\text{SHIFT}}$ $\boxed{\text{Re} \longleftrightarrow \text{I}_\text{m}}$.

Method-3 : Choose normal computation mode in calculator and enter the complex number as shown below:

$\boxed{\text{SHIFT}}$ $\boxed{\text{Re c}}$ **(5, – 30)**

To perform the conversion press $\boxed{=}$.

To view the real part press $\boxed{\text{ALPHA}}$ $\boxed{\text{E}}$.

To view the imaginary part press $\boxed{\text{ALPHA}}$ $\boxed{\text{F}}$.

3. Rectangular to Polar Conversion

Let, $\overline{A}_1 = 2 + j5$

Method-1 : Choose complex mode in calculator and enter the complex number as shown below:

(2 + 5 i) $\boxed{\text{SHIFT}}$ $\boxed{r\angle\theta}$

To perform the conversion press $\boxed{=}$.

To view the absolute value and argument press $\boxed{\text{SHIFT}}$ $\boxed{\text{Re} \longleftrightarrow \text{I}_m}$.

Method-2 : Choose complex mode in calculator.

i) To calculate the absolute value enter the complex number as shown below:

$\boxed{\text{SHIFT}}$ $\boxed{\text{Abs}}$ **(2 + 5 i)**

To view the absolute value press $\boxed{=}$.

ii) To calculate the argument enter the complex number as shown below:

$\boxed{\text{SHIFT}}$ $\boxed{\text{arg}}$ **(2 + 5 i)**

To view the argument press $\boxed{=}$.

Method-3 : Choose normal computation mode in calculator and enter the complex number as shown below:

$\boxed{\text{SHIFT}}$ $\boxed{\text{Pol}}$ **(2, 5)**

To perform the conversion press $\boxed{=}$.

To view the absolute value press $\boxed{\text{ALPHA}}$ $\boxed{\text{E}}$.

To view the argument press $\boxed{\text{ALPHA}}$ $\boxed{\text{F}}$.

Note : *The calculator treats the real and imaginary part as separate numbers, hence enclose the real and imaginary part of a complex number by parenthesis.*

Appendix-2

IMPORTANT MATHEMATICAL FORMULAE

Trigonometric Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \text{cosec}\,\theta = \frac{1}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1, \quad 1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \text{cosec}^2\theta$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$2\sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$2\sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2\cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos(\theta \pm 90^\circ) = \mp\sin\theta$$

$$\sin(\theta \pm 90^\circ) = \pm\cos\theta$$

$$\tan(\theta \pm 90^\circ) = -\cot\theta$$

$$\cos(\theta \pm 180^\circ) = -\cos\theta$$

$$\sin(\theta \pm 180^\circ) = -\sin\theta$$

$$\tan(\theta \pm 180^\circ) = \tan\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \quad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

Complex Variables

A complex number, $\overline{Z}$ may be represented as,

$$\overline{Z} = x + jy = r\angle\theta = re^{j\theta} = r(\cos\theta + j\sin\theta)$$

where, $x = \text{Re}(\overline{Z}) = r\cos\theta$; $\qquad y = \text{Im}(\overline{Z}) = r\sin\theta$

$$r = |\overline{Z}| = \sqrt{x^2 + y^2}; \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$j = \sqrt{-1}, \qquad \frac{1}{j} = -j; \qquad j^2 = -1$$

The conjugate of the complex number, $\overline{Z} = x + jy$, may be represented as,

$$\overline{Z}^* = x - jy = r\angle -\theta = re^{-j\theta} = r(\cos\theta - j\sin\theta)$$

The following relations hold good for a complex number, $\overline{Z} = x + jy$.

$$\sqrt{\overline{Z}} = \sqrt{x+jy} = \sqrt{re^{j\theta}} = \sqrt{r}\, e^{j\frac{\theta}{2}} = \sqrt{r}\angle\frac{\theta}{2}$$

$$\overline{Z}^n = (x+jy)^n = r^n e^{jn\theta} = r^n\angle n\theta, \quad \text{where n is an integer}$$

$$\overline{Z}^{1/n} = (x+jy)^{1/n} = r^{1/n}e^{j\theta/n} = r^{1/n}\angle\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right), \text{ for } k = 0, 1, 2, ...n-1$$

$$\ell n\,\overline{Z} = \ell n(re^{j\theta}) = \ell n\, r + \ell n\, e^{j\theta} = \ell n\, r + j(\theta + 2k\pi), \quad \text{where k is an integer.}$$

Demovier's theorem : $\mathbf{(e^{j\theta})^n = e^{jn\theta} = \cos n\theta + j\sin n\theta}$

Let, $\overline{Z}_1$ and $\overline{Z}_2$ be two complex numbers defined as,

$$\overline{Z}_1 = x_1 + jy_1 = r_1\angle\theta_1 = r_1 e^{j\theta_1}$$

$$\overline{Z}_2 = x_2 + jy_2 = r_1\angle\theta_2 = r_2 e^{j\theta_2}$$

Now, $\overline{Z}_1 = \overline{Z}_2$ only if $x_1 = x_2$ and $y_1 = y_2$.

$$\overline{Z}_1 \pm \overline{Z}_2 = (x_1 + x_2) \pm j(y_1 + y_2)$$

$$\overline{Z}_1\overline{Z}_2 = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1) \quad \text{or} \quad \overline{Z}_1\overline{Z}_2 = r_1r_2\, e^{j(\theta_1+\theta_2)} = r_1r_2\angle(\theta_1+\theta_2)$$

$$\frac{\overline{Z}_1}{\overline{Z}_2} = \frac{(x_1+jy_1)}{(x_2+jy_2)} \times \frac{(x_2-jy_2)}{(x_2-jy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + j\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$$

$$\text{or} \quad \frac{\overline{Z}_1}{\overline{Z}_2} = \frac{r_1}{r_2}e^{j(\theta_1-\theta_2)} = \frac{r_1}{r_2}\angle(\theta_1-\theta_2)$$

Derivatives and Integrals

Let, $U = U(x)$, $V = V(x)$, and $a = \text{constant}$.

$$\frac{d}{dx}(aU) = a\frac{dU}{dx}$$

$$\frac{d}{dx}(UV) = U\frac{dV}{dX} + V\frac{dU}{dx}$$

$$\frac{d}{dx}\left[\frac{U}{V}\right] = \frac{V\dfrac{dU}{dx} - U\dfrac{dV}{dx}}{V^2}$$

$$\int a\,dx = ax + c$$

$$\int UV = U\int V - \int\left[\int V\,dU\right] \quad \text{or} \quad \int U\,dV = U\,V - \int V\,dU$$

Appendix-3

LAPLACE TRANSFORM

The Laplace transform is used to transform a time domain function to complex frequency domain called s-domain.

In order to transform a time domain function f(t) to s-domain multiply the function by e^{-st} and then integrate from 0 to ∞. The transformed function is represented as F(s). Here $s = \sigma + j\omega$, and it is called complex frequency. This transformation was first proposed by Laplace (in the year 1780) and later adopted for circuit analysis for solving differential equations. Hence this transformation is called Laplace transform and the transformation is denoted by the script letter $\mathcal{L}$.

i.e., Symbolically the Laplace transform of f(t) is denoted as,

$$F(s) = \mathcal{L}\,[f(t)]$$

Mathematically the Laplace transform of f(t) is defined as,

$$F(s) = \int_{t=0}^{t=\infty} f(t)\, e^{-st}\, dt$$

Definition of Laplace Transform

Let f(t) be a function of t defined for all positive values of t, now the Laplace transform of f(t) denoted by $\mathcal{L}[f(t)]$ or F(s) is defined as,

$$\mathcal{L}\{f(t)\} = F(s) = \int_{t=0}^{t=\infty} f(t)\, e^{-st}\, dt$$

Definition of Inverse Laplace Transform

The s-domain function can be transformed to time domain by inverse Laplace transform.

The inverse Laplace transform of F(s) is defined as,

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{s=\sigma-j\omega}^{s=\sigma+j\omega} F(s)\, e^{st}\, ds$$

Here the path of integration is a straight line parallel to the jω-axis, such that all the poles of F(s) lie to the left of this line.

Table - A3.1 : Laplace Transform Pairs

Sl.No.	f(t)	F(s)
1.	Unit impulse, $\delta(t)$	1
2.	Unit step, $u(t)$	$1/s$
3.	t	$1/s^2$
4.	$\frac{t^{n-1}}{(n-1)!}$ $(n = 1,2,3 \ldots)$	$1/s^n$
5.	e^{-at}	$\frac{1}{s+a}$
6.	t^n $(n = 1, 2, 3 \ldots)$	$\frac{n!}{s^{n+1}}$
7.	$t\,e^{-at}$	$\frac{1}{(s+a)^2}$
8.	$\frac{1}{(n-1)!}t^{n-1}e^{-at}(n = 1,2,3 \ldots)$	$\frac{1}{(s+a)^n}$
9.	$t^n\,e^{-at}$ $(n = 1, 2, 3 \ldots)$	$\frac{n!}{(s+a)^{n+1}}$
10.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
11.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
12.	$\sinh \omega t$	$\frac{\omega}{s^2+\omega^2}$
13.	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
14.	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
15.	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Table - A3.2 : Properties of Laplace Transform

Note : $\mathcal{L}\{f(t)\} = F(s)$; $\mathcal{L}\{f_1(t)\} = F_1(s)$; $\mathcal{L}\{f_2(t)\} = F_2(s)$

Property	Time domain signal	s-domain signal
Amplitude scaling	$A\,f(t)$	$A\,F(s)$
Linearity	$a_1 f_1(t) \pm a_2 f_2(t)$	$a_1\,F_1(s) \pm a_2\,F_2(s)$
Time differentiation	$\frac{d}{dt} f(t)$	$s\,F(s) - f(0)$
	$\frac{d^n}{dt^n} f(t)$ where $n = 1, 2, 3$	$s^n\,F(s) - \sum_{K=1}^{n} s^{n-K} \left. \frac{d^{(K-1)}\,f(t)}{dt^{K-1}} \right\vert_{t=0}$
Time integration	$\int f(t)\,dt$	$\frac{F(s)}{s} + \frac{\left[\int f(t)\,dt\right]\Big\vert_{t=0}}{s}$
	$\int .....\int f(t)\,(dt)^n$ where $n = 1, 2, 3$	$\frac{F(s)}{s^n} + \sum_{K=1}^{n} \frac{1}{s^{n-K+1}} \left[\int .....\int f(t)\,(dt)^K\right]\Big\vert_{t=0}$
Frequency shifting	$e^{\pm at}\,f(t)$	$F(s \mp a)$
Time shifting	$f(t \pm \alpha)$	$e^{\pm \alpha s}\,F(s)$
Frequency differentiation	$t\,f(t)$	$-\frac{dF(s)}{ds}$
	$t^n\,f(t)$ where $n = 1, 2, 3$	$(-1)^n \frac{d^n}{ds^n} F(s)$
Frequency integration	$\frac{1}{t} f(t)$	$\int_s^{\infty} F(s)\,ds$
Time scaling	$f(at)$	$\frac{1}{\lvert a \rvert} F\left(\frac{s}{a}\right)$
Periodicity	$f(t + nT)$	$\frac{1}{1 - e^{-sT}} \int_0^T f(t)\,e^{-sT}\,dt$
Initial value theorem	$\underset{t \to 0}{\mathrm{Lt}}\ f(t) = f(0)$	$\underset{s \to \infty}{\mathrm{Lt}}\ s\,F(s)$
Final value theorem	$\underset{t \to \infty}{\mathrm{Lt}}\ f(t) = f(\infty)$	$\underset{s \to 0}{\mathrm{Lt}}\ s\,F(s)$

CRAMER'S RULE

I. Cramer's Rule for Mesh Basis Equation

The mesh basis matrix equation for resistive circuit is,

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} & \cdots & R_{1m} \\ R_{21} & R_{22} & R_{23} & \cdots & R_{2m} \\ R_{31} & R_{32} & R_{33} & \cdots & R_{3m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_{m1} & R_{m2} & R_{m3} & \cdots & R_{mm} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_m \end{bmatrix} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ \vdots \\ E_{mm} \end{bmatrix}$$

The k^{th} mesh current I_k by Cramer's rule is,

$$I_k = \frac{1}{\Delta} \sum_{j=1}^{m} \Delta_{jk} \; E_{jj}$$

where, m = Number of meshes in the circuit.

Δ_{jk} = Cofactor of R_{jk}.

E_{jj} = Sum of voltage sources in mesh-j.

Δ = Determinant of resistance matrix.

For circuit with three meshes, the mesh currents by Cramer's rule are,

$$I_1 = \frac{\Delta_{11}}{\Delta} E_{11} + \frac{\Delta_{21}}{\Delta} E_{22} + \frac{\Delta_{31}}{\Delta} E_{33}$$

$$I_2 = \frac{\Delta_{12}}{\Delta} E_{11} + \frac{\Delta_{22}}{\Delta} E_{22} + \frac{\Delta_{32}}{\Delta} E_{33}$$

$$I_3 = \frac{\Delta_{13}}{\Delta} E_{11} + \frac{\Delta_{23}}{\Delta} E_{22} + \frac{\Delta_{33}}{\Delta} E_{33}$$

The mesh currents for a circuit with three meshes using short-cut procedure for Cramer's rule are,

$$I_1 = \frac{\Delta_1}{\Delta}$$

$$I_2 = \frac{\Delta_2}{\Delta}$$

$$I_3 = \frac{\Delta_3}{\Delta}$$

where, $\Delta = \begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{vmatrix}$; $\Delta_1 = \begin{vmatrix} E_{11} & R_{12} & R_{13} \\ E_{22} & R_{22} & R_{23} \\ E_{33} & R_{32} & R_{33} \end{vmatrix}$; $\Delta_2 = \begin{vmatrix} R_{11} & E_{11} & R_{13} \\ R_{21} & E_{22} & R_{23} \\ R_{31} & E_{33} & R_{33} \end{vmatrix}$; $\Delta_3 = \begin{vmatrix} R_{11} & R_{12} & E_{11} \\ R_{21} & R_{22} & E_{22} \\ R_{31} & R_{32} & E_{33} \end{vmatrix}$

II. Cramer's Rule for Node Basis Equation

The node basis matrix equation for resistive circuit is,

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} & \cdots & G_{1n} \\ G_{21} & G_{22} & G_{23} & \cdots & G_{2n} \\ G_{31} & G_{32} & G_{33} & \cdots & G_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ G_{n1} & G_{n2} & G_{n3} & \cdots & G_{nn} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} I_{11} \\ I_{22} \\ I_{33} \\ \vdots \\ I_{nn} \end{bmatrix}$$

The k^{th} node voltage V_k by Cramer's rule is,

$$V_k = \frac{1}{\Delta'} \sum_{j=1}^{n} \Delta'_{jk} I_{jj}$$

where, n = Number of independent nodes in a circuit

Δ'_{jk} = Cofactor of G_{jk}

I_{jj} = Sum of current sources connected to node-j

Δ' = Determinant of conductance matrix

For circuit with three nodes excluding the reference node, the node voltages by Cramer's rule are,

$$V_1 = \frac{\Delta'_{11}}{\Delta'} I_{11} + \frac{\Delta'_{21}}{\Delta'} I_{22} + \frac{\Delta'_{31}}{\Delta'} I_{33}$$

$$V_2 = \frac{\Delta'_{12}}{\Delta'} I_{11} + \frac{\Delta'_{22}}{\Delta'} I_{22} + \frac{\Delta'_{32}}{\Delta'} I_{33}$$

$$V_3 = \frac{\Delta'_{13}}{\Delta'} I_{11} + \frac{\Delta'_{23}}{\Delta'} I_{22} + \frac{\Delta'_{33}}{\Delta'} I_{33}$$

The node voltages of a circuit with three nodes excluding the reference using short-cut procedure for Cramer's rule are,

$$V_1 = \frac{\Delta'_1}{\Delta'}$$

$$V_2 = \frac{\Delta'_2}{\Delta'}$$

$$V_3 = \frac{\Delta'_3}{\Delta'}$$

$$\text{where,}\quad \Delta' = \begin{vmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{vmatrix} \;;\quad \Delta'_1 = \begin{vmatrix} I_{11} & G_{12} & G_{13} \\ I_{22} & G_{22} & G_{23} \\ I_{33} & G_{32} & G_{33} \end{vmatrix} \;;$$

$$\Delta'_2 = \begin{vmatrix} G_{11} & I_{11} & G_{13} \\ G_{21} & I_{22} & G_{23} \\ G_{31} & I_{33} & G_{33} \end{vmatrix} \;;\quad \Delta'_3 = \begin{vmatrix} G_{11} & G_{12} & I_{11} \\ G_{21} & G_{22} & I_{22} \\ G_{31} & G_{32} & I_{33} \end{vmatrix}$$

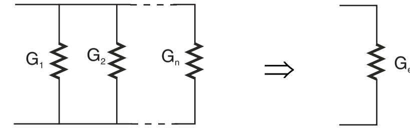
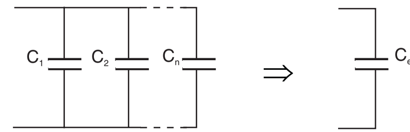
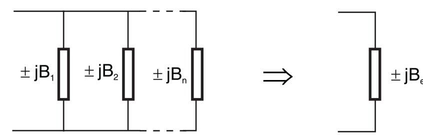
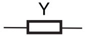
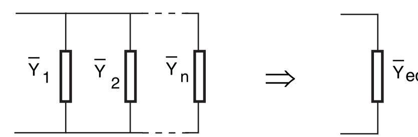
Appendix-5

EQUIVALENT OF SERIES/PARALLEL CONNECTED PARAMETERS

Summary of Equivalent of Series/Parallel-Connected Group-1 Parameters

Group-1 Parameter	Series Connection of Parameters and their equivalent	Parallel Connection of Parameters and their equivalent
Resistance R	R_1, R_2, …, R_n $\Rightarrow$ R_{eq} $R_{eq} = R_1 + R_2 + \ldots + R_n$	R_1, R_2, …, R_n $\Rightarrow$ R_{eq} $R_{eq} = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \ldots + \dfrac{1}{R_n}}$
Inductance L	L_1, L_2, …, L_3 $\Rightarrow$ L_{eq} $L_{eq} = L_1 + L_2 + \ldots + L_n$	L_1, L_2, …, L_n $\Rightarrow$ L_{eq} $L_{eq} = \dfrac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2} + \ldots + \dfrac{1}{L_n}}$
Reactance $\pm jX$	$\pm jX_1$, $\pm jX_2$, …, $\pm jX_n$ $\Rightarrow$ $\pm jX_{eq}$ $\pm jX_{eq} = \pm jX_1 \pm jX_2 \pm \ldots \pm jX_n$	$\pm jX_1$, $\pm jX_2$, …, $\pm jX_n$ $\Rightarrow$ $\pm jX_{eq}$ $\pm jX_{eq} = \dfrac{1}{\dfrac{1}{\pm jX_1} + \dfrac{1}{\pm jX_2} + \ldots + \dfrac{1}{\pm jX_n}}$
Impedance $\bar{Z}$	$\bar{Z}_1$, $\bar{Z}_2$, …, $\bar{Z}_n$ $\Rightarrow$ $\bar{Z}_{eq}$ $\bar{Z}_{eq} = \bar{Z}_1 + \bar{Z}_2 + \ldots + \bar{Z}_n$	$\bar{Z}_1$, $\bar{Z}_2$, …, $\bar{Z}_n$ $\Rightarrow$ $\bar{Z}_{eq}$ $\bar{Z}_{eq} = \dfrac{1}{\dfrac{1}{\bar{Z}_1} + \dfrac{1}{\bar{Z}_2} + \ldots + \dfrac{1}{\bar{Z}_n}}$

Summary of Equivalent of Series/Parallel-Connected Group-2 Parameters

Group-2 Parameter	Series Connection of Parameters and their equivalent	Parallel Connection of Parameters and their equivalent
Conductance 	 $G_{eq} = \frac{1}{\frac{1}{G_1}+\frac{1}{G_2}+.....+\frac{1}{G_n}}$	 $G_{eq} = G_1 + G_2 +.....+ G_n$
Capacitance 	 $C_{eq} = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}+.....+\frac{1}{C_n}}$	 $C_{eq} = C_1 + C_2 +.....+ C_n$
Susceptance 	 $\pm jB_{eq} = \frac{1}{\frac{1}{\pm jB_1}+\frac{1}{\pm jB_2}+.....+\frac{1}{\pm jB_n}}$	 $\pm jB_{eq} = \pm jB_1 \pm jB_2 \pm ..... \pm jB_n$
Admittance 	 $\bar{Y}_{eq} = \frac{1}{\frac{1}{\bar{Y}_1}+\frac{1}{\bar{Y}_2}+.....+\frac{1}{\bar{Y}_n}}$	 $\bar{Y}_{eq} = \bar{Y}_1 + \bar{Y}_2 +.....+ \bar{Y}_n$

STAR-DELTA TRANSFORMATION

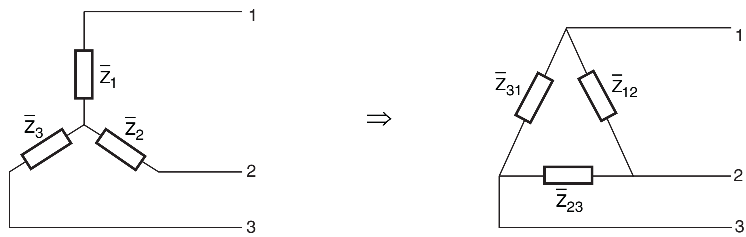


***Fig. a :** Star-connected impedances.* ***Fig. b :** Delta-connected impedances.*

***Fig. A.6 :** Star to delta transformation.*

Star to Delta Transformation

$$R_{12} = R_1 + R_2 + \frac{R_1 R_2}{R_3}$$

$$R_{23} = R_2 + R_3 + \frac{R_2 R_3}{R_1}$$

$$R_{31} = R_3 + R_1 + \frac{R_3 R_1}{R_2}$$

$$\text{Delta equivalent resistance between two terminals} = \begin{array}{c}\text{Sum of resistances}\\ \text{connected to the}\\ \text{two terminals in}\\ \text{star network}\end{array} \times \frac{\begin{array}{c}\text{Product of the resistances}\\ \text{connected to the two}\\ \text{terminals in star network}\end{array}}{\begin{array}{c}\text{The third resistance in}\\ \text{star network}\end{array}}$$

Delta to Star Transformation

$$R_1 = \frac{R_{12} R_{31}}{R_{12} + R_{23} + R_{31}}$$

$$R_2 = \frac{R_{23} R_{12}}{R_{12} + R_{23} + R_{31}}$$

$$R_3 = \frac{R_{31} R_{23}}{R_{12} + R_{23} + R_{31}}$$

$$\text{Star equivalent resistance at one terminals} = \frac{\begin{array}{c}\text{Product of resistances}\\ \text{connected to the two}\\ \text{terminals in delta network}\end{array}}{\begin{array}{c}\text{Sum of three resistances in}\\ \text{delta network}\end{array}}$$

Appendix-7

SUMMARY OF THEOREMS

S.No	Theorem	Definition
1.	**Superposition theorem**	The superposition theorem states that the response in a circuit with multiple sources is given by algebraic sum of responses due to individual sources acting alone.
2.	**Thevenin's theorem**	Thevenin's theorem states that a circuit with two terminals can be replaced by an equivalent circuit, consisting of a voltage source in series with a resistance (or impedance).
3.	**Norton's theorem**	Norton's theorem states that a circuit with two terminals can be replaced by an equivalent circuit, consisting of a current source in parallel with a resistance (or impedance).
4.	**Reciprocity theorem**	The reciprocity theorem states that, in a linear, bilateral, single source circuit, the ratio of excitation to response is constant when the position of excitation and response are interchanged.

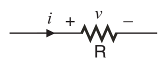
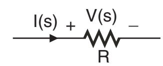
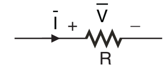
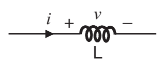
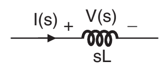
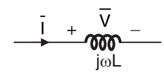
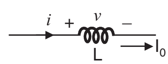
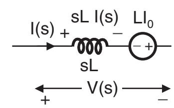
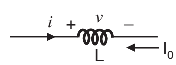
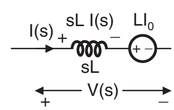
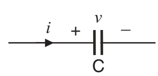
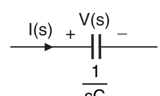
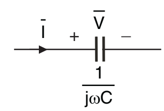
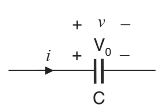
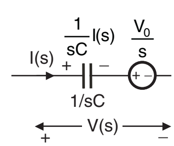
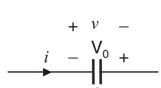
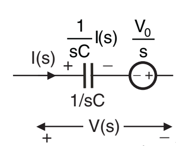
Nature of Source and Variable Element	Maximum Power Transfer Theorem
DC source with internal resistance connected to a variable resistive load.	Maximum power is transferred from source to load, when the load resistance is equal to source resistance.
AC source with internal resistance connected to a variable resistive load.	Maximum power is transferred from source to load, when the load resistance is equal to source resistance.
AC source with internal impedance connected to a variable resistive load.	Maximum power is transferred from source to load, when the load resistance is equal to magnitude of source impedance.
AC source with internal impedance connected to a load with variable resistance and variable reactance.	Maximum power is transferred from source to load, when the load impedance is equal to complex conjugate of source impedance.
AC source with internal impedance connected to a load with variable resistance and fixed reactance.	Maximum power is transferred from source to load when load resistance is equal to absolute value of the rest of the impedence of the circuit.
AC source with internal impedance connected to a load with fixed resistance and variable reactance.	Maximum power is transferred from source to load when load reactance is equal to conjugate of source reactance.

INITIAL AND FINAL CONDITIONS IN RLC CIRCUITS EXCITED BY DC SUPPLY

Element	Initial condition $t = 0^+$	Final condition $t = \infty$
R	R	R
L	O.C.	S.C.
$\xrightarrow{I_0}$ L	I_0	S.C.
$\xleftarrow{I_0}$ L	I_0	S.C.
C	S.C.	O.C.
V_0 + − C	V_0 + −	O.C.
V_0 − + C	V_0 − +	O.C.

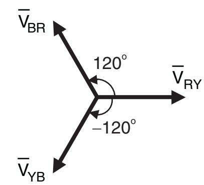
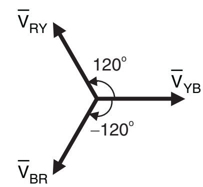
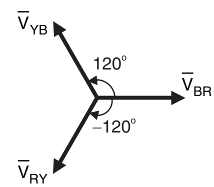
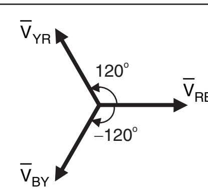
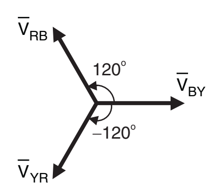
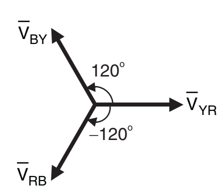
Appendix-9

R,L,C PARAMETERS AND V-I RELATIONS IN VARIOUS DOMAINS

S.No.	Parameter	Time domain	s-domain	Frequency domain
1.	Resistance, R	 $v = iR$	 $V(s) = I(s)R$	 $\bar{V} = \bar{I}R$
2.	Inductance, L	 $v = L\dfrac{di}{dt}$	 $V(s) = I(s) \times sL$	 $\bar{V} = \bar{I} \times j\omega L$
3.	Inductance, L with initial current	 $v = L\dfrac{di}{dt}$; $i(0) = I_0$	 $V(s) = I(s) \times sL - LI_0$	
		 $v = L\dfrac{di}{dt}$; $i(0) = -I_0$	 $V(s) = I(s) \times sL + LI_0$	
4.	Capacitance, C	 $v = \dfrac{1}{C}\int i\,dt$	 $V(s) = I(s) \times \dfrac{1}{sC}$	 $\bar{V} = \bar{I} \times \dfrac{1}{j\omega C}$
5.	Capacitance, C with initial voltage	 $v = \dfrac{1}{C}\int i\,dt$; $v(0) = V_0$	 $V(s) = I(s) \times \dfrac{1}{sC} + \dfrac{V_0}{s}$	
		 $v = \dfrac{1}{C}\int i\,dt$; $v(0) = -V_0$	 $V(s) = I(s) \times \dfrac{1}{sC} - \dfrac{V_0}{s}$	

Appendix-10

CHOICE OF REFERENCE PHASOR

Phase sequence	Reference phasor	Line voltages	Phasor diagram
RYB	$\overline{V}_{RY}$	$\overline{V}_{RY} = V_L\angle 0^o$ $\overline{V}_{YB} = V_L\angle -120^o$ $\overline{V}_{BR} = V_L\angle -240^o = V_L\angle 120^o$	
RYB	$\overline{V}_{YB}$	$\overline{V}_{YB} = V_L\angle 0^o$ $\overline{V}_{BR} = V_L\angle -120^o$ $\overline{V}_{RY} = V_L\angle -240^o = V_L\angle 120^o$	
RYB	$\overline{V}_{BR}$	$\overline{V}_{BR} = V_L\angle 0^o$ $\overline{V}_{RY} = V_L\angle -120^o$ $\overline{V}_{YB} = V_L\angle -240^o = V_L\angle 120^o$	
RBY	$\overline{V}_{RB}$	$\overline{V}_{RB} = V_L\angle 0^o$ $\overline{V}_{BY} = V_L\angle -120^o$ $\overline{V}_{YR} = V_L\angle -240^o = V_L\angle 120^o$	
RBY	$\overline{V}_{BY}$	$\overline{V}_{BY} = V_L\angle 0^o$ $\overline{V}_{YR} = V_L\angle -120^o$ $\overline{V}_{RB} = V_L\angle -240^o = V_L\angle 120^o$	
RBY	$\overline{V}_{YR}$	$\overline{V}_{YR} = V_L\angle 0^o$ $\overline{V}_{RB} = V_L\angle -120^o$ $\overline{V}_{BY} = V_L\angle -240^o = V_L\angle 120^o$	

Appendix-11

V - I EQUATION OF THREE-PHASE LOAD

I. Three/Four wire star-connected balanced load

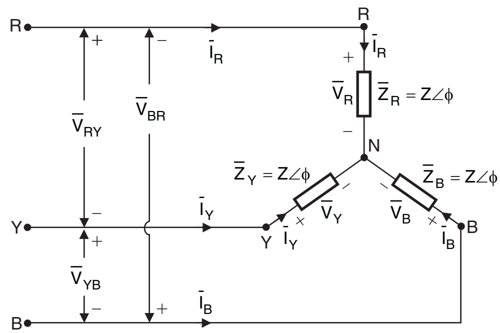


Fig. A.11.1 : Three-wire star-connected balanced load with conventional polarity of voltages and direction of currents for RYB sequence.

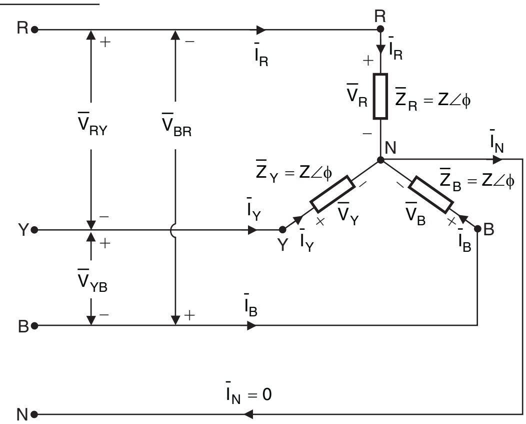


Fig. A.11.2 : Four-wire star-connected balanced load with conventional polarity of voltages and direction of currents for RYB sequencce.

Line Voltages

$$\bar{V}_{RY} = V_L\angle 0^\circ \;; \quad \bar{V}_{YB} = V_L\angle -120^\circ \;; \quad \bar{V}_{BR} = V_L\angle -240^\circ$$

V_L = Magnitude of line voltage.

Phase Voltages

$$\bar{V}_R = \frac{V_L}{\sqrt{3}}\angle(0^\circ - 30^\circ) = V\angle -30^\circ$$

$$\bar{V}_Y = \frac{V_L}{\sqrt{3}}\angle(-120^\circ - 30^\circ) = V\angle -150^\circ$$

$$\bar{V}_B = \frac{V_L}{\sqrt{3}}\angle(-240^\circ - 30^\circ) = V\angle -270^\circ$$

$V = \frac{V_L}{\sqrt{3}}$ = Magnitude of phase voltage.

Phase Currents

$$\bar{I}_R = \frac{\bar{V}_R}{\bar{Z}_R} = \frac{V\angle -30^\circ}{Z\angle\phi} = \frac{V}{Z}\angle(-30^\circ - \phi) = I\angle(-30^\circ - \phi)$$

$$\bar{I}_Y = \frac{\bar{V}_Y}{\bar{Z}_Y} = \frac{V\angle -150^\circ}{Z\angle\phi} = \frac{V}{Z}\angle(-150^\circ - \phi) = I\angle(-150^\circ - \phi)$$

$$\bar{I}_B = \frac{\bar{V}_B}{\bar{Z}_B} = \frac{V\angle -270^\circ}{Z\angle\phi} = \frac{V}{Z}\angle(-270^\circ - \phi) = I\angle(-270^\circ - \phi)$$

$I = \frac{V}{Z}$ = Magnitude of phase current.

$I_L = I$ = Magnitude of line current.

Line Currents

$$\bar{I}_R = I_L\angle(-30^\circ - \phi) \;; \quad \bar{I}_Y = I_L\angle(-150^\circ - \phi) \;; \quad \bar{I}_B = I_L\angle(-270^\circ - \phi)$$

Power

$$P = 3\,V\,I\cos\phi \quad \text{or} \quad P = \sqrt{3}\,V_L\,I_L\cos\phi$$

In star connection the line and phase currents are same.

II. Delta-connected balanced load

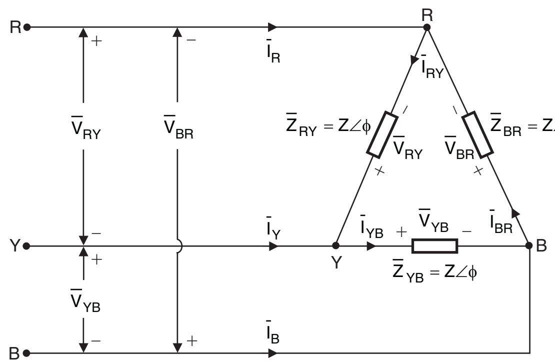


Fig. A.11.3 : Delta-connected balanced load with conventional polarity of voltages and direction of currents for RYB sequence.

Line Voltages

$$\bar{V}_{RY} = V_L\angle 0^\circ \;;\quad \bar{V}_{YB} = V_L\angle -120^\circ \;;\quad \bar{V}_{BR} = V_L\angle -240^\circ$$

V_L = Magnitude of line voltage.

Phase Voltages

$V = V_L$ = Magnitude of phase voltage.

$$\bar{V}_{RY} = V\angle 0^\circ \;;\quad \bar{V}_{YB} = V\angle -120^\circ \;;\quad \bar{V}_{BR} = V\angle -240^\circ$$

In delta connection the line and phase voltages are same.

Phase Currents

$$\bar{I}_{RY} = \frac{\bar{V}_{RY}}{\bar{Z}_{RY}} = \frac{V\angle 0^\circ}{Z\angle\phi} = \frac{V}{Z}\angle -\phi = I\angle -\phi$$

$$\bar{I}_{YB} = \frac{\bar{V}_{YB}}{\bar{Z}_{YB}} = \frac{V\angle -120^\circ}{Z\angle\phi} = \frac{V}{Z}\angle(-120^\circ - \phi) = I\angle(-120^\circ - \phi)$$

$$\bar{I}_{BR} = \frac{\bar{V}_{BR}}{\bar{Z}_{BR}} = \frac{V\angle -240^\circ}{Z\angle\phi} = \frac{V}{Z}\angle(-240^\circ - \phi) = I\angle(-240^\circ - \phi)$$

$I = \frac{V}{Z}$ = Magnitude of phase current.

Line Currents

$$\bar{I}_R = \sqrt{3}\, I\angle(-\phi - 30^\circ) = I_L\angle(-30^\circ - \phi)$$

$$\bar{I}_Y = \sqrt{3}\, I\angle(-120^\circ - \phi - 30^\circ) = I_L\angle(-150^\circ - \phi)$$

$$\bar{I}_B = \sqrt{3}\, I\angle(-240^\circ - \phi - 30^\circ) = I_L\angle(-270^\circ - \phi)$$

$I_L = \sqrt{3}\, I$ = Magnitude of line current.

Power

$$P = 3\,V\,I\cos\phi \quad \text{or} \quad P = \sqrt{3}\, V_L\, I_L \cos\phi$$

III. Three-wire star-connected unbalanced load

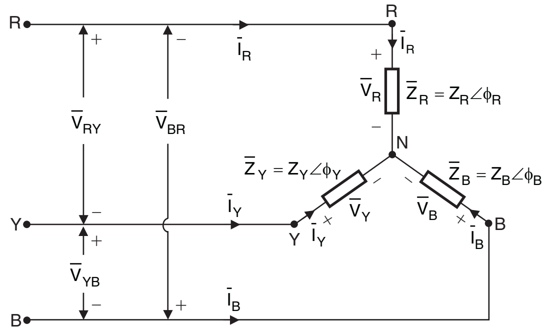


Fig. A.11.4 : Three-wire star-connected unbalanced load with conventional polarity of voltages and direction of currents for RYB sequence.

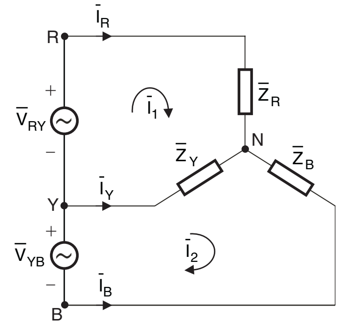


Fig. A.11.5 : Mesh analysis to solve line currents.

Line Voltages

$$\bar{V}_{RY} = V_L\angle 0^\circ \;;\quad \bar{V}_{YB} = V_L\angle -120^\circ \;;\quad \bar{V}_{BR} = V_L\angle -240^\circ$$

V_L = Magnitude of line voltage.

Line Currents

$$\bar{I}_R = \bar{I}_1 = I_R\angle\gamma_R$$

$$\bar{I}_Y = \bar{I}_2 - \bar{I}_1 = I_Y\angle\gamma_Y$$

$$\bar{I}_B = -\bar{I}_2 = I_B\angle\gamma_B$$

The mesh currents $\bar{I}_1$ and $\bar{I}_2$ are solved by mesh analysis, and the line currents are estimated from mesh currents

Phase Currents

$$\bar{I}_R = I_R\angle\gamma_R$$

$$\bar{I}_Y = I_Y\angle\gamma_Y$$

$$\bar{I}_B = I_B\angle\gamma_B$$

In star connected load the line and phase currents are same

I_R, I_Y, and I_B are magnitude of line and phase currents and γ_R, γ_Y and γ_B are phase angle of line and phase currents with respect to reference phasor

Phase Voltages

$$\bar{V}_R = \bar{I}_R\bar{Z}_R = V_R\angle\delta_R$$

$$\bar{V}_Y = \bar{I}_Y\bar{Z}_Y = V_Y\angle\delta_Y$$

$$\bar{V}_B = \bar{I}_B\bar{Z}_B = V_B\angle\delta_B$$

V_R, V_Y and V_B are magnitude of phase voltages and δ_R, δ_Y and δ_B are phase angle of phase voltages with respect to reference phasor

Power

$$P = \begin{matrix}\text{Power consumed}\\ \text{by R-phase load}\end{matrix} + \begin{matrix}\text{Power consumed}\\ \text{by Y-phase load}\end{matrix} + \begin{matrix}\text{Power consumed}\\ \text{by B-phase load}\end{matrix}$$

$$= |\bar{V}_R||\bar{I}_R|\cos\phi_1 + |\bar{V}_Y||\bar{I}_Y|\cos\phi_2 + |\bar{V}_B||\bar{I}_B|\cos\phi_3$$

$$= V_R I_R\cos\phi_1 + V_Y I_Y\cos\phi_2 + V_B I_B\cos\phi_3$$

ϕ_1 = Phase difference between $\bar{V}_R$ and $\bar{I}_R$
ϕ_2 = Phase difference between $\bar{V}_Y$ and $\bar{I}_Y$
ϕ_3 = Phase difference between $\bar{V}_B$ and $\bar{I}_B$

V. Four wire star-connected unbalanced load

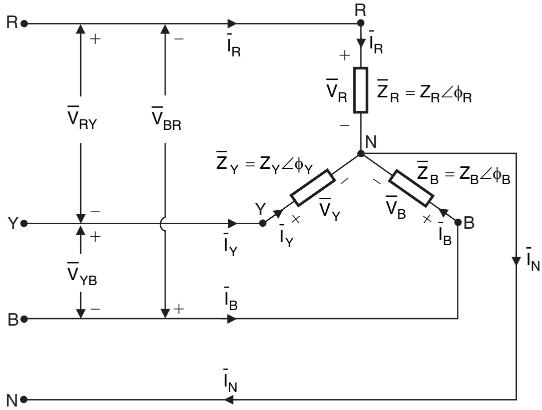


Fig. A.11.6 : Four-wire star-connected unbalanced load with conventional polarity of voltages and direction of currents.

V_L = Magnitude of line voltage.

Line Voltages

$$\bar{V}_{RY} = V_L\angle 0^\circ \;;\quad \bar{V}_{YB} = V_L\angle -120^\circ \;;\quad \bar{V}_{BR} = V_L\angle -240^\circ$$

$V = \dfrac{V_L}{\sqrt{3}}$ = Magnitude of phase current.

Phase Voltages

Since load neutral is connected to source neutral, phase voltages will be balanced even though load is unbalanced.

$$\bar{V}_R = \frac{V_L}{\sqrt{3}}\angle(0^\circ - 30^\circ) = V\angle -30^\circ$$

$$\bar{V}_Y = \frac{V_L}{\sqrt{3}}\angle(-120^\circ - 30^\circ) = V\angle -150^\circ$$

$$\bar{V}_B = \frac{V_L}{\sqrt{3}}\angle(-240^\circ - 30^\circ) = V\angle -270^\circ$$

I_R, I_Y and I_B are magnitude of R-phase, Y-phase and B-phase currents respectively.

Phase Currents

$$\bar{I}_R = \frac{\bar{V}_R}{\bar{Z}_R} = \frac{V\angle -30^\circ}{Z_R\angle\phi_R} = \frac{V}{Z_R}\angle(-30^\circ - \phi_R) = I_R\angle(-30^\circ - \phi_R)$$

$$\bar{I}_Y = \frac{\bar{V}_Y}{\bar{Z}_Y} = \frac{V\angle -150^\circ}{Z_Y\angle\phi_Y} = \frac{V}{Z_Y}\angle(-150^\circ - \phi_Y) = I_Y\angle(-150^\circ - \phi_Y)$$

$$\bar{I}_B = \frac{\bar{V}_B}{\bar{Z}_B} = \frac{V\angle -270^\circ}{Z_B\angle\phi_B} = \frac{V}{Z_B}\angle(-270^\circ - \phi_B) = I_B\angle(-270^\circ - \phi_B)$$

In star connection the line and phase currents are same.

Line Currents

$$\bar{I}_R = I_R\angle(-30^\circ - \phi_R) \;;\quad \bar{I}_Y = I_Y\angle(-150^\circ - \phi_Y) \;;\quad \bar{I}_B = I_B\angle(-270^\circ - \phi_B)$$

Power

$$P = \begin{array}{c}\text{Power consumed}\\ \text{by R-phase load}\end{array} + \begin{array}{c}\text{Power consumed}\\ \text{by Y-phase load}\end{array} + \begin{array}{c}\text{Power consumed}\\ \text{by B-phase load}\end{array}$$

$$= |\bar{V}_R||\bar{I}_R|\cos\phi_1 + |\bar{V}_Y||\bar{I}_Y|\cos\phi_2 + |\bar{V}_B||\bar{I}_B|\cos\phi_3$$

$$= VI_R\cos\phi_1 + VI_Y\cos\phi_2 + VI_B\cos\phi_3$$

$|\bar{V}_R| = |\bar{V}_Y| = |\bar{V}_B| = V_L/\sqrt{3} = V$

ϕ_1 = Phase difference between $\bar{V}_R$ and $\bar{I}_R$

ϕ_2 = Phase difference between $\bar{V}_Y$ and $\bar{I}_Y$

ϕ_3 = Phase difference between $\bar{V}_B$ and $\bar{I}_B$

III. Delta-connected unbalanced load

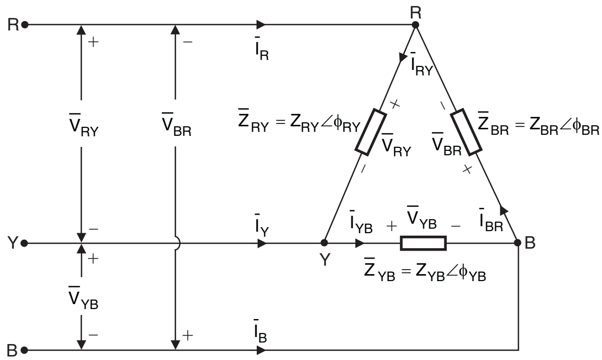


Fig. A.11.7 : Three-phase delta-connected unbalanced load with conventional polarity of voltages and direction of currents for RYB sequence.

Line Voltages

$$\bar{V}_{RY} = V_L\angle 0^o \;;\; \bar{V}_{YB} = V_L\angle -120^o \;;\; \bar{V}_{BR} = V_L\angle -240^o$$

V_L = Magnitude of line voltage.

Phase Voltages

$$\bar{V}_{RY} = V\angle 0^o \;;\; \bar{V}_{YB} = V\angle -120^o \;;\; \bar{V}_{BR} = V\angle -240^o$$

$V = V_L$ = Magnitude of phase voltage.

In delta connection the line and phase voltages are same.

Phase Currents

$$\bar{I}_{RY} = \frac{\bar{V}_{RY}}{\bar{Z}_{RY}} = I_{RY}\angle\gamma_{RY}$$

$$\bar{I}_{YB} = \frac{\bar{V}_{YB}}{\bar{Z}_{YB}} = I_{YB}\angle\gamma_{YB}$$

$$\bar{I}_{BR} = \frac{\bar{V}_{BR}}{\bar{Z}_{BR}} = I_{BR}\angle\gamma_{BR}$$

I_{RY}, I_{YB} and I_{BR} are magnitude of phase currents and γ_{RY}, γ_{YB} and γ_{BR} are phase angle of phase currents.

Line Currents

$$\bar{I}_R = \bar{I}_{RY} - \bar{I}_{BR} \;;\; \bar{I}_Y = \bar{I}_{YB} - \bar{I}_{RY} \;;\; \bar{I}_B = \bar{I}_{BR} - \bar{I}_{YB}$$

Power

$$P = \begin{array}{c}\text{Power consumed}\\\text{by R-phase load}\end{array} + \begin{array}{c}\text{Power consumed}\\\text{by Y-phase load}\end{array} + \begin{array}{c}\text{Power consumed}\\\text{by B-phase load}\end{array}$$

$$= |\bar{V}_{RY}||\bar{I}_{RY}|\cos\phi_1 + |\bar{V}_{YB}||\bar{I}_{YB}|\cos\phi_2 + |\bar{V}_{BR}||\bar{I}_{BR}|\cos\phi_3$$

$$= V_L I_{RY}\cos\phi_1 + V_L I_{YB}\cos\phi_2 + V_L I_{BR}\cos\phi_3$$

$|\bar{V}_{RY}| = |\bar{V}_{YB}| = |\bar{V}_{BR}| = V_L = V$

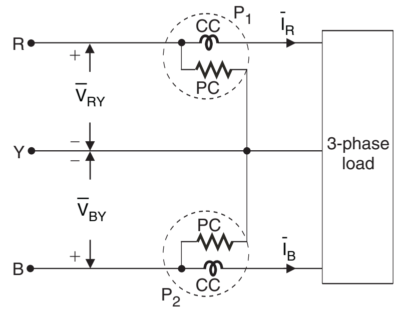
ϕ = Phase difference between $\bar{V}_{RY}$ and $\bar{I}_{RY}$

ϕ_2 = Phase difference between $\bar{V}_{YB}$ and $\bar{I}_{YB}$

ϕ_3 = Phase difference between $\bar{V}_{BR}$ and $\bar{I}_{BR}$

Appendix-12

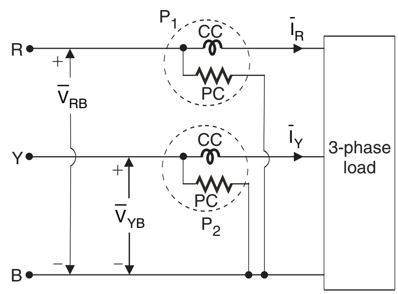
TWO WATTMETER METHOD OF POWER MEASUREMENT



$$P_1 = |\bar{V}_{RY}|\,|\bar{I}_R|\cos(\angle\bar{V}_{RY} - \angle\bar{I}_R)$$

$$P_2 = |\bar{V}_{BY}|\,|\bar{I}_B|\cos(\angle\bar{V}_{BY} - \angle\bar{I}_B)$$

***Fig. a :** Wattmeters in lines R and B.*



$$P_1 = |\bar{V}_{RB}|\,|\bar{I}_R|\cos(\angle\bar{V}_{RB} - \angle\bar{I}_R)$$

$$P_2 = |\bar{V}_{YB}|\,|\bar{I}_Y|\cos(\angle\bar{V}_{YB} - \angle\bar{I}_Y)$$

***Fig. b :** Wattmeters in lines R and Y.*

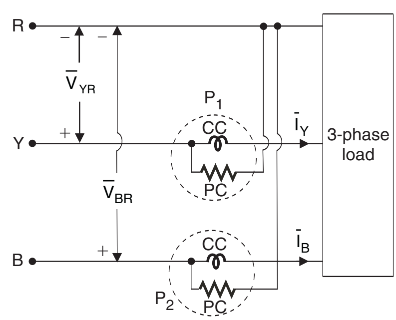


***Fig. c :** Wattmeters in lines Y and B.*

***Fig. A.12 :** Possible connections of two wattmeters for measurement of three-phase power.*

Power, $P = P_1 + P_2$

Power factor angle, $\phi = \tan^{-1}\left(\sqrt{3}\,\dfrac{P_2 - P_1}{P_1 + P_2}\right)$

Power factor, $\cos\phi = \cos\left[\tan^{-1}\left(\sqrt{3}\,\dfrac{P_2 - P_1}{P_1 + P_2}\right)\right]$

Reading of wattmeter	**pf**
Equal	Unity
If one wattmeter is zero	0.5
If one wattmeter is negative	< 0.5
Both are positive	> 0.5

Appendix-13

IMPORTANT EQUATIONS OF SERIES RESONANCE

1. Angular resonant frequency, $\omega_r = \dfrac{1}{\sqrt{LC}}$ in *rad/s*

---

2. Resonant frequency, $f_r = \dfrac{1}{2\pi\sqrt{LC}}$ in *Hz*

---

3. Q-factor at resonance, $Q_r = \dfrac{\omega_r L}{R}$; $Q_r = \dfrac{1}{\omega_r CR}$; $Q_r = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$

Note : When $\omega \le \omega_r$, $Q = \dfrac{1}{\omega CR}$; When $\omega \ge \omega_r$, $Q = \dfrac{\omega L}{R}$

---

4. Bandwidth, $\beta = \dfrac{R}{L}$ in *rad/s* ; $\beta = \dfrac{\omega_r}{Q_r}$ in *rad/s*

Bandwidth in *Hz* $= \dfrac{\beta}{2\pi}$

---

5. Half-power (or cut-off) frequencies

$$\omega_l = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \text{ in } rad/s \quad ; \quad \omega_h = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \text{ in } rad/s$$

$$\omega_l = \omega_r\left[-\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \text{ in } rad/s \quad ; \quad \omega_h = \omega_r\left[\frac{1}{2Q_r} + \sqrt{1 + \frac{1}{4Q_r^2}}\right] \text{ in } rad/s$$

$$f_l = \frac{\omega_l}{2\pi} \text{ in } Hz \quad ; \quad f_h = \frac{\omega_h}{2\pi} \text{ in } Hz$$

---

6. Total reactance at half-power frequencies

$$\omega L - \frac{1}{\omega C} = \pm R \quad ; \quad \omega_h L - \frac{1}{\omega_h C} = R \quad ; \quad \omega_l L - \frac{1}{\omega_l C} = -R$$

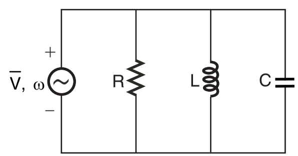
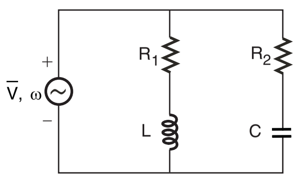
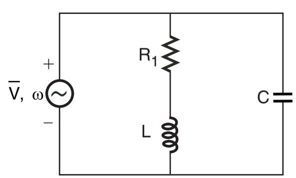
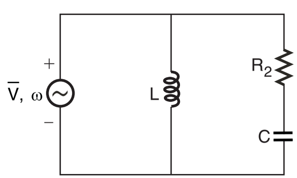
---

7. Selectivity

$$\text{Selectivity} = \frac{\beta}{\omega_r} \quad ; \quad \text{Selectivity} = \frac{1}{Q_r}$$

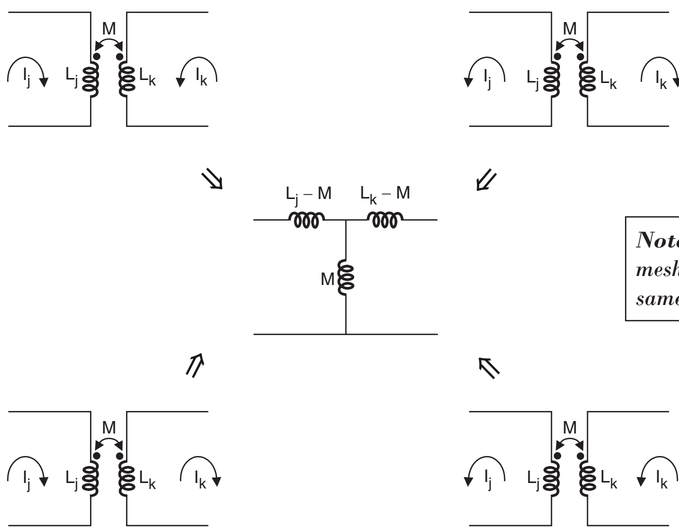
Appendix-14

PARALLEL RESONANT CIRCUITS

RLC parallel circuit	Resonant frequency ω_r in *rad/s*	Resonant frequency f_r in *Hz*	Dynamic resistance $R_{dynamic}$ in Ω
	$\omega_r = \frac{1}{\sqrt{LC}}$	$f_r = \frac{1}{2\pi\sqrt{LC}}$	$R_{dynamic} = R$
	$\omega_r = \frac{1}{\sqrt{LC}}\sqrt{\frac{L-CR_1^2}{L-CR_2^2}}$	$f_r = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{L-CR_1^2}{L-CR_2^2}}$	$R_{dynamic} = \frac{1}{\frac{R_1}{R_1^2+X_{Lr}^2}+\frac{R_2}{R_2^2+X_{Cr}^2}}$ $X_{Lr} = \omega_r L \;;\; X_{Cr} = \frac{1}{\omega_r C}$
	$\omega_r = \frac{1}{\sqrt{LC}}\sqrt{1-\frac{CR_1^2}{L}}$	$f_r = \frac{1}{2\pi\sqrt{LC}}\sqrt{1-\frac{CR_1^2}{L}}$	$R_{dynamic} = \frac{L}{R_1 C}$
	$\omega_r = \frac{1}{\sqrt{LC}}\sqrt{\frac{L}{L-CR_2^2}}$	$f_r = \frac{1}{2\pi\sqrt{LC}}\sqrt{\frac{L}{L-CR_2^2}}$	$R_{dynamic} = \frac{L}{R_2 C}$

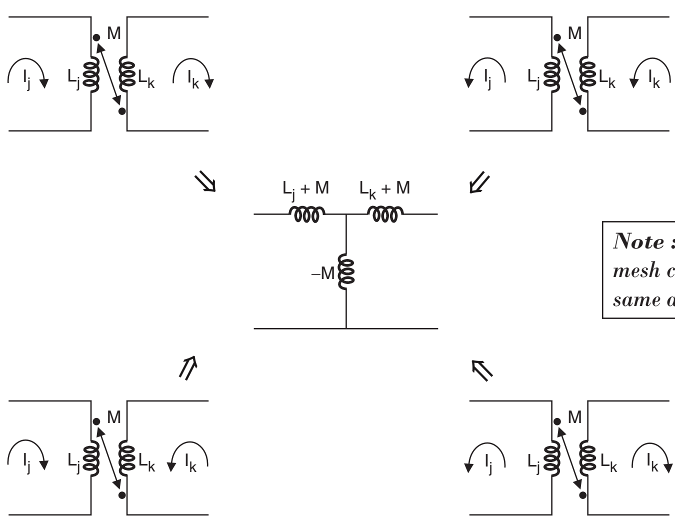
ELECTRICAL EQUIVALENT OF COUPLED COILS

Electrical equivalent of group-I coupled coils



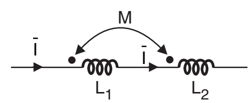
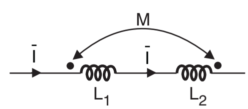
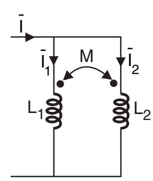
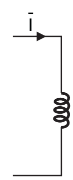
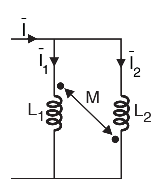
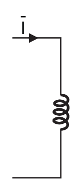
Note : *In electrical equivalent the mesh currents orientation should be same as that of original circuit.*

Electrical equivalent of group-II coupled coils



Note : *In electrical equivalent the mesh currents orientation should be same as that of original circuit.*

EQUIVALENT OF SERIES AND PARALLEL CONNECTED COUPLED COILS

Connection	Circuit	Equivalent circuit
Series Aiding		 $L_{eq} = L_1 + L_2 + 2M$
Series Opposing		 $L_{eq} = L_1 + L_2 - 2M$
Parallel Aiding		 $L_{eq} = \dfrac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$
Parallel Opposing		 $L_{eq} = \dfrac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$

B.E/ B.Tech. DEGREE EXANMINATION, MAY/ JUNE 2014

Second Semester

Electrical and Electronics Engineering

EE 6201 – CIRCUIT THEORY

(Regulation 2013)

Time: 3 hours   Maximum: 100 marks

Answer all questions

PART A – (10 × 2 = 20 marks)

1. Find the equivalent resistance of the circuit shown in Fig 1.

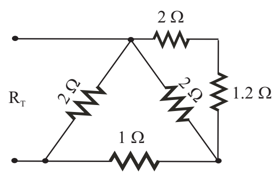


Fig. 1

Chapter 2, SA - Q2.6 ***[Page No - 2.146]***

2. Define RMS voltage.

Chapter 4, Section - 4.2.2 ***[Page No - 4.6]***

3. What is reciprocity theorem?

Chapter 2, Section - 2.6.4 ***[Page No - 2.126]***

4. Why do you short circuit the voltage source and open the current source when you find Thevenin's resistance of a network?

Usually Thevenin's resistance is obtained by network reduction technique. When this technique is applied the circuit should be converted to a network by deactivating or killing all the sources. An ideal voltage source is deactivated, when it is short circuited and ideal current source is deactivated, when it is open circuited.

5. Define quality factor in the resonant circuit.

Chapter 5, SA - Q5.4 ***[Page No - 5.102]***

6. Determine the quality factor of a coil for the series resonant circuit consisting of R = 10 ohm, L = 0.1H, and C = 10 microfarad.

Chapter 5, SA - Q5.6 ***[Page No - 5.102]***

7. Distinguish between natural and forced response.

Chapter 3, SA - Q3.3 ***[Page No - 3.97]***

8. What is the time constant for RL and RC circuit?

Chapter 3, SA - Q3.6 & 3.10 ***[Page No - 3.97 & 3.98]***

9. Write the effect of power factor in energy consumption billing.

$$\text{Energy} = \text{Power} \times \text{Time} = \text{VI} \cos \phi \times \text{Time} = \text{VI} \times \text{Power factor} \times \text{Time}$$

From the above relation it is clear that energy is directly proportional to power factor. Therefore, a high power factor will result in large energy consumption and higher value of billing.

Note: *The power factor is defined as ratio of active power and apparent power. The apparent power is the power supplied to consumer and active power is the power utilized by the consumer. If power factor is 0.8, then the consumer utilize only 80% of power and return back 20% to source. The power returned to the source increase the energy loss in transmission and so electricity boards insist for maintaining higher power factor at consumer end. In ideal case, the power factor should be unity so that the entire transmitted power is consumed which result is low transmission losses.*

10. Distinguish between unbalanced source and unbalanced load.

Chapter 4, SA - Q4.25 ***[Page No - 4.144]***

PART B – (5 × 16 = 80 marks)

11. (a) (i) Find the current I and voltage across 30Ω of the circuit shown in Fig. 11(a)(i) (8)

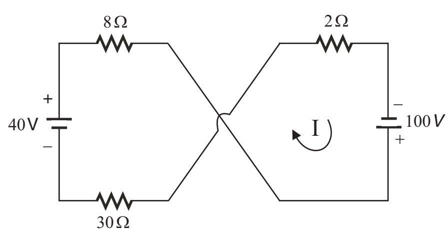


Fig. 11(a) (i).

Chapter 1, SA - Q1.25 ***[Page No - 1.144]***

(ii) Determine the current in all the resistors of the circuit shown in Fig.11(a) (ii) (8)

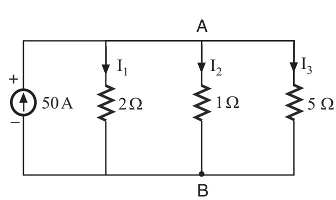


Fig. 11(a) (ii).

Chapter 1, Examplr - 1.11 ***[Page No - 1.40]***

(OR)

(b) (i) Determine the current through each resistor in the circuit shown in Fig 11.(b) (i) (6)

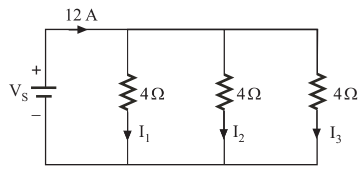


Fig. 11(b) (i).

Chapter 1, SA - Q1.26 ***[Page No - 1.144]***

(ii) When a dc voltage is applied to a capacitor, voltage across its terminals is found to build up in accordance with $v_c = 50(1 - e^{-100t})$. After 0.01 S the current flow is equal to 2 mA.

(i) Find the value of capacitance in farad.

(ii) How much energy stored in the electric field? (10)

Chapter 3, Example 3.22 ***[Page No - 3.79]***

12. (a) (i) Determine the current in the 5Ω resistor in the network shown in the Fig.12(a)(i) (8)

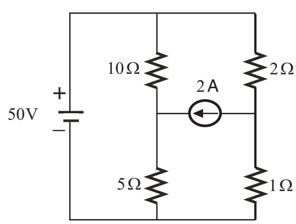


Fig. 12(a) (i).

Chapter 2, Example 2.33 ***[Page No - 2.81]***

(ii) Find out the current in the each branch of the circuit shown in Fig.12(a)(ii) (8)

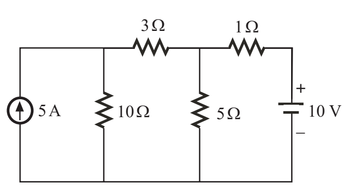


Fig. 12(a) (ii).

Chapter 1, Example 1.28 ***[Page No - 1.71]***

(OR)

(b) (i) Determine the current in each mesh of the circuit shown in Fig.12 (b) (i) (8)

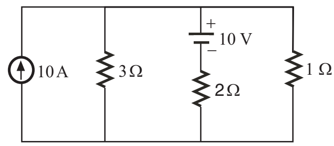


Fig. 12(b) (i).

Chapter 1, Example 1.29 ***[Page No - 1.72]***

(ii) Determine the voltages at each node of the circuit shown in Fig.12 (b) (ii) (8)

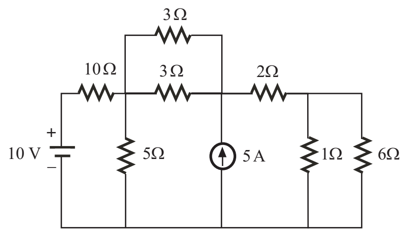


Fig. 12(b) (ii).

Chapter 1, Example 1.54 ***[Page No - 1.118]***

13. (a) For the circuit shown in the Fig. 13 (a), determine the impedance at resonant frequency, 10 Hz above resonant frequency, and 10Hz below resonant frequency. (16)

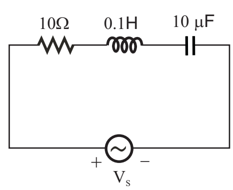


Fig. 13(a).

Chapter 5, Example 5.1 ***[Page No - 5.11]***

(OR)

(b) Explain that how to derive Q factor of parallel resonance. (16)

Chapter 5, Section 5.3.3 ***[Page No - 5.26]***

14. (a) A series RL circuit with R = 30Ω and L = 15H has a constant voltage V = 60v applied at t = 0 as shown in Fig. 14(a). Determine the current i,the voltage across resistor and the voltage across the inductor. (16)

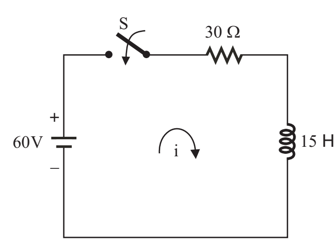


Fig. 14(a).

Chapter 3, Example 3.2 ***[Page No - 3.48]***

(OR)

(b) The circuit shown in the Fig. 14(b) consist of resistance, inductance and capacitance in series with 100V DC when the switch is closed at t = 0. Find the current transient (16)

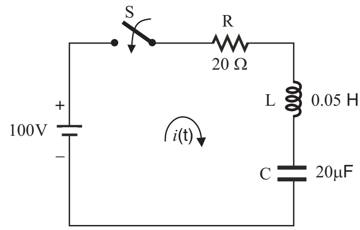


Fig. 14(b).

Chapter 3, Example 3.25 ***[Page No - 3.84]***

> ***Note :*** *For the given values of R,L and C the response will be damped oscillatory same as that of Example 3.25*

15. (a) (i) A symmetrical three-phase; three wire 440V supply to a star connected load. The impedance in each branch are $Z_R = 2 + j3\Omega$, $Z_Y = 1 - j2\Omega$, and $Z_B = 3 + j4\Omega$. Find its equivalent delta connected load. (8)

Chapter 4, Example 4.37 ***[Page No - 4.113]***

(ii) A three phase, balanced delta-connected load of $4 + j8\Omega$, is connected across a 400V, 3ϕ balanced supply. Determine the phase currents and line currents. (Phase sequence is RYB). (8)

Chapter 4, Example 4.38 ***[Page No - 4.114]***

(OR)

(b) (i) A symmetrical three-phase, three wire 400V, supply is connected to a delta-connected load. Impedance in each branch are $Z_{RY} = 10\angle 30^0\ \Omega$, $Z_{YB} = 10\angle 45^0\ \Omega$ and $Z_{BR} = 2.5\angle 60^0\ \Omega$. Find its equivalent star-connected load. (8)

Chapter 4, Example 4.36 ***[Page No - 4.113]***

(ii) A balanced star connected load having an impedance $15 + j20\Omega$ per phase is connected to 3ϕ, 440V, 50Hz. Find the line current and power absorbed by the load. (8)

Chapter 4, Example 4.23 ***[Page No - 4.94]***

B.E./B.Tech. DEGREE EXAMINATION, NOVEMBER/DECEMBER 2014

Second Semester

Electronics and Communication Engineering

EE 6201 – CIRCUIT THEORY

(Regulation 2013)

Time: 3 hours | Maximum: 100 marks

Answer all questions

PART A - (10 × 2 = 20 Marks)

1. An electrical appliance consumes 1.2 kWh in 30 minutes at 120 V. What is the current drawn by the appliance?

   Chapter 1, SA - Q1.17 ***[Page No - 1.142]***

2. Calculate the equivalent resistance between the terminals "*a*" and "*b*", in Fig. 1.

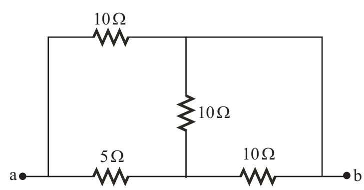


Fig. 1

   Chapter 2, SA - Q2.4 ***[Page No - 2.145]***

3. Calculate the value of I_N for the circuit shown in Fig. 2.

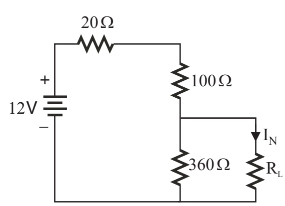


Fig. 2

   Chapter 2, SA - Q2.33 ***[Page No - 2.153]***

4. State maximum power transfer theorem for DC networks.

   Chapter 2, Section - 2.63, case (i) ***[Page No - 2.100]***

5. Calculate the total inductance of the circuit, if the coefficient of coupling (k) between the two coils is 0.6, as shown in Fig.3.

Fig. 3

   Chapter 5, SA - Q5.45 ***[Page No - 5.109]***

6. Define quality factor of a series resonant circuit.

   Chapter 5, SA - Q5.4 ***[Page No - 5.102]***

7. A coil of resistance 2.2 Ω and an inductance 0.01 H is connected in series with a capacitor across 220 V mains. Find the value of capacitance such that maximum current flows in the circuit at a frequency of 190 Hz. Also find the maximum current.

   Chapter 5, SA - Q5.23 ***[Page No - 5.105]***

8. A 50 μF capacitor is discharged through a 100 kΩ resistor. If the capacitor is initially charged to 400 V, determine the initial energy.

   Chapter 3, SA - Q3.23 ***[Page No - 3.100]***

9. Write the equations for the phasor difference between the potentials of the delta connected networks.

   Chapter 4, SA - Q4.26 ***[Page No - 4.144]***

10. Three coils, each having a resistance of 20 Ω and an inductive reactance of 15 Ω are connected in star to a 400 V, 3-phase, and 50 Hz supply. Calculate (a) the line current, (b) power factor, and (c) power supplied

    Chapter 4, SA - Q4.29 ***[Page No - 4.145]***

PART B – (5 × 16 = 80 marks)

11. (a) (i) Using node analysis, find the node voltages and the currents through all the resistors for the circuit shown in Fig. 4. (12)

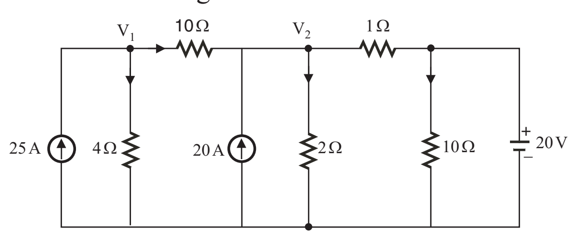


Fig. 4.

Chapter 1, Example 1.55 ***[Page No - 1.120]***

(ii) Find the equivalent resistance between the terminals '*a*' and '*b*' for the network shown in Fig. 5. (4)

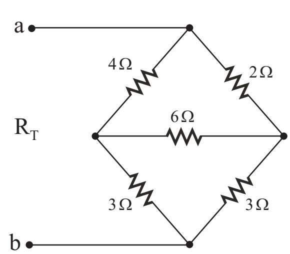


Fig. 5.

Chapter 2, Example 2.3 ***[Page No - 2.33]***

(OR)

(b) For the circuit shown in Fig. 6, find the (i) currents in different branches, (ii) current supplied by the battery, (iii) potential difference between terminals A and B. (16)

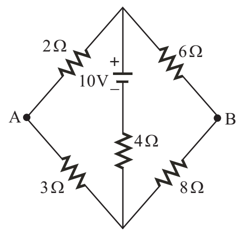


Fig. 6.

Chapter 1, Example 1.14 ***[Page No - 1.52]***

12. (a) Find the current I, through the 20 Ω resistor shown in Fig. 7 using Thevenin's theorem. (16)

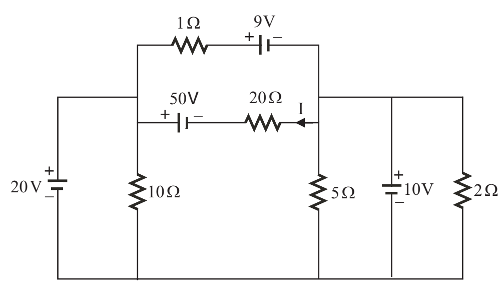


Fig. 7.

Chapter 2, Example 2.21 ***[Page No - 2.59]***

(OR)

(b) Find the current through 5Ω resistor using superposition theorem, in the circuit shown in Fig. 8. (16)

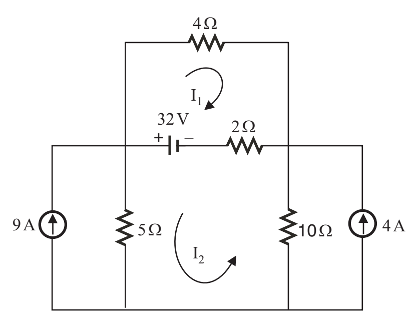


Fig. 8.

Chapter 2, Example 2.41 ***[Page No - 2.92]***

13. (a) Impedance $\overline{Z}_1$ and $\overline{Z}_2$ are parallel and this combination is in series with an impedance $\overline{Z}_3$, connected to a 100 V, 50 Hz ac supply. $\overline{Z}_1 = 5 - jX_C\,\Omega$, $\overline{Z}_2 = 5 + j0\,\Omega$, $\overline{Z}_3 = 6.25 + j1.25\,\Omega$. Determine the value of capacitance such that the total current of the circuit will be in phase with the total voltage. Find the circuit current and power. (16)

Chapter 4, Example 4.19 *[Page No - 4.58]*

(OR)

(b) The switch in the circuit shown in Fig. 9, is moved from position 1 to 2 at t = 0. Find the expression for voltage across resistance and capacitor, energy in the capacitor for t > 0. (16)

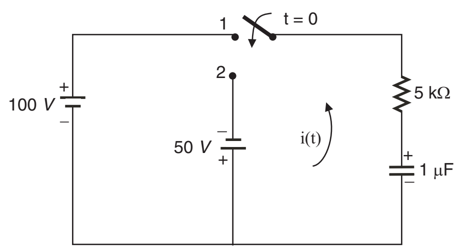


Fig. 9.

Chapter 3, Example 3.14 *[Page No - 3.66]*

14. (a) (i) For a magnetically coupled circuit, derive the expression for mutual inductance(M) in terms of L_1 and L_2. (6)

Chapter 5, Section 5.5.2 *[Page No - 5.44]*

14. (a) (ii) For the coupled circuit shown in Fig. 10, find the value of V_2 so that the current $I_1 = 0$. (10)

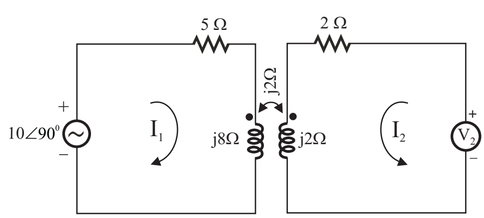


Fig. 10.

Chapter 5, Example 5.32 *[Page No - 5.92]*

(OR)

14. (b) With neat illustration describe the parallel resonant circuit and the equivalent parallel network for a series RL combination. Also derive the unity power factor f_p. (16)

Chapter 5, Section 5.3 and Example 5.12 *[Page No - 5.16 and 5.39]*

> ***Note :*** *1. Explain any one parallel RLC circuit and derive its resonance frequency [Refer Section 8.3.1, Page No - 8.17]*
> *2. The unity power factor frequency, f_p is resonance frerquency of parallel RLC circuit.*

15. (a) Show that three phase power can be measured by two wattmeters. Draw the phasor diagrams. Derive an expression for power factor interms of wattmeter readings. (16)

Chapter 4, Section 4.25 ***[Page No - 4.85]***

(OR)

(b) (i) A 400 V(line-to-line) is applied to three star-connected identical impedances each consisting of a 4 Ω resistance in series with 3 Ω inductive reactance. Find (1) line current and (2) total power supplied. (8)

Chapter 4, Example 4.23 ***[Page No - 4.94]***

15. (b) (ii) Three star-connected impedances $\overline{Z}_1 = 20 + j37.7\,\Omega$ per phase are in parallel with three delta-connected impedance $\overline{Z}_2 = 20 + j37.7\,\Omega$ per phase. The line voltage is 398 volts. Find the line current, power factor, power and reactive volt-ampere taken by the combination. (8)

Chapter 4, Example 4.39 ***[Page No - 4.115]***

B.E/ B.Tech DEGREE EXANMINATION, APRIL/ MAY 2015

Second Semester

Electronics and Communication Engineering

EE 6201 – CIRCUIT THEORY

(Regulation 2013)

Time: 3 hours | Maximum: 100 marks

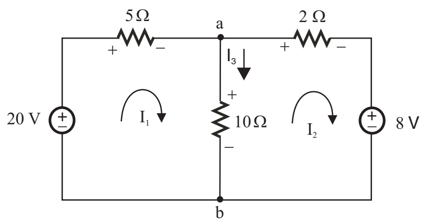
Answer all questions

PART A – (10 × 2 = 20 marks)

1. Write briefly about resistance in a circuit.

   Chapter 1, Section - 1.5.1 ***[Page No - 1.26]***

2. Obtain the current in each branch of the network shown below using Kirchhoff's Current Law.



   Chapter 1, SA - Q1.22 ***[Page No - 1.142]***

3. State maximum power transfer theorem.

   Chapter 2, SA - Q2.39 ***[Page No - 2.155]***

4. Write briefly about network reduction technique.

   Chapter 2, Section - 2.1 ***[Page No - 2.1]***

5. Define mutual inductance.

   Chapter 5, SA - Q5.37 ***[Page No - 5.107]***

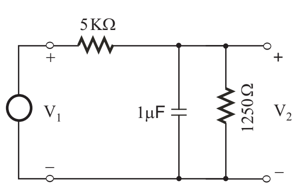
6. Write the dot rule.

   Chapter 5, Section 5.6.1 ***[Page No - 5.49]***

7. Define the frequency response of series RLC circuit.

   Chapter 5, SA - Q5.3 ***[Page No - 5.102]***

8. Find the frequency response V_2/V_1 for the two-port circuit shown below.



   Not in Regulation 2017

9. Write the distortion power factor equation of the three phase circuits.

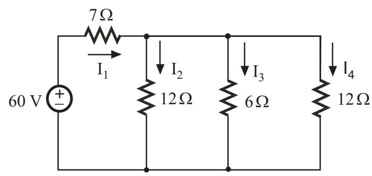
   Chapter 4, SA - Q4.27 ***[Page No - 4.145]***

10. Distinguish between unbalanced source and unbalanced load.

Chapter 4, SA - Q4.25 ***[Page No - 4.144]***

PART B – (10 × 16 = 80 marks)

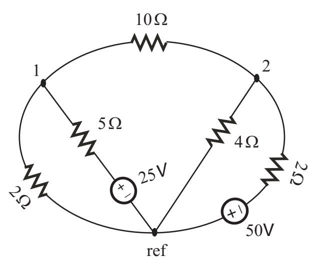
11. (a) Use branch currents in the network shown below to find the current supplied by the 60 V source. Solve the circuit by the mesh current method (16)



Chapter 1, Example 1.15 ***[Page No - 1.54]***

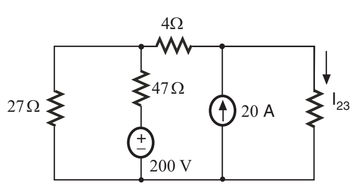
(OR)

(b) Solve the network given below by the node voltage method. (16)



Chapter 1, Example 1.56 ***[Page No - 1.22]***

12. (a) (i) Compute the current in the 23 Ω resistor of the following figure shown below by applying the superposition principle. (8)



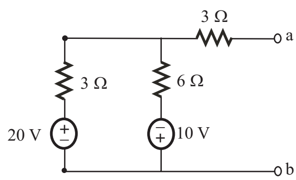
Chapter 2, Example 2.38 ***[Page No - 2.88]***

(ii) Derive the equation for transient response of RC and RL circuit for DC input. (8)

Chapter 3, Section 3.3.2 & 3.4.2 ***[Page No - 3.11 & 3.21]***

(OR)

(b) Obtain the Thevenin and Norton equivalent circuits for the active network shown below. (16)



Chapter 2, Example 2.19 ***[Page No - 2.57]***

13. (a) With neat illustration and necessary derivations, explain the linear transformer. (16)

Chapter 5, Section 5.6.1 ***[Page No - 5.49]***

(OR)

(b) Derive the mutual inductance and the coupling coefficient of the transformer with necessary illustration. (16)

Chapter 5, Section 5.5.2, 5.5.3 ***[Page No - 5.44, 5.46]***

14. (a) Explain in detail with neat illustrations the High pass and Low pass networks and derive the network parameters. (16)

Refer ***[Page No - Q.14]***

(OR)

(b) Explain the characterization of two port networks in terms of *Z*, *Y* and *h* parameters. (16)

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15. (a) Discuss in detail the three phase 3-wire circuits with star connected balanced loads. (16)

Chapter 4, Section 4.21 ***[Page No - 4.63]***

(OR)

(b) Explain in detail the phasor diagram of the voltage of the voltage and currents of a three phase unbalanced circuits. (16)

Chapter 4, Section 4.24.2 ***[Page No - 4.81]***

Note: *Explain theory from section 4.24.2 and draw phasor diagram shown in Fig.4. of Example 4.29.*

14.(a) Explain in the detail with neat illustrations the High pass and Low pass network and derive the network parameters.

SOLUTION

The low pass and high pass filters are best examples of two port network. A low pass filter will pass all low frequency signals less than a cut-off frequency. A high pass filter will pass all high frequency signals greater than a cut-off frequency.

The low pass and high pass filters can be designed using LC network and popular configurations of such filters are constant-K, T-type and Π-type filters shown in Figs.1 and 2. Here the product $\overline{Z}_1\overline{Z}_2$ is a constant independent of frequency and so such filters are called constant-K filters.

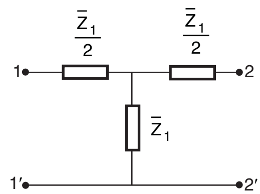

Fig.1: Constant-K T-type filter

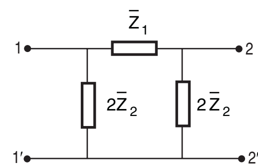

Fig.2: Constant-K Π-type filter

Low pass Filter

For a low pass filter,

$$\overline{Z}_1 = j\omega L \quad \text{and} \quad \overline{Z}_2 = \frac{1}{j\omega C}$$

$$\therefore\ \overline{Z}_1\overline{Z}_2 = j\omega L \times \frac{1}{j\omega C} = \frac{L}{C}$$

Here, $\frac{L}{C}$ is a constant independent of frequency.

Let, $\frac{L}{C} = K^2$

$$\therefore\ K = \sqrt{\frac{L}{C}}$$

Here, K is also called design impedance or load impedance of the low pass filter.

The cut-off frequency, f_c of low pass filter is given by,

$$f_c = \frac{1}{\pi\sqrt{LC}}$$

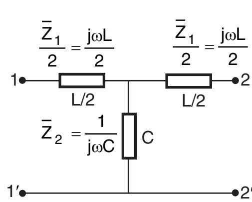

Fig.3: Constant-K T-type low pass filter

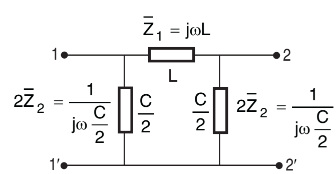

Fig.4: Constant-K Π-type low pass filter

High pass Filter

For a high pass filter,

$$\bar{Z}_1 = \frac{1}{j\omega C} \quad \text{and} \quad \bar{Z}_2 = j\omega L$$

$$\therefore \bar{Z}_1 \bar{Z}_2 = \frac{1}{j\omega C} \times j\omega L = \frac{L}{C}$$

Here, $\frac{L}{C}$ is a constant independent of frequency.

Let, $\frac{L}{C} = K^2$

$$\therefore K = \sqrt{\frac{L}{C}}$$

Here, K is also called design impedance or load impedance of the high pass filter.

The cut-off frequency, f_c of the high pass filter is given by,

$$f_c = \frac{1}{4\pi\sqrt{LC}}$$

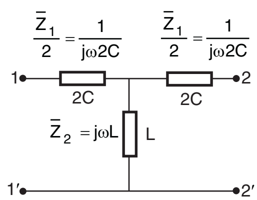

Fig.5: Constant-K T-type high pass filter

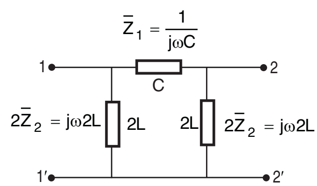

Fig.6: Constant-K Π-type high pass filter

Z - Parameters of T-type filter

Consider the T-type filter shown in Fig. 7. The equations defining Z- parameters are,

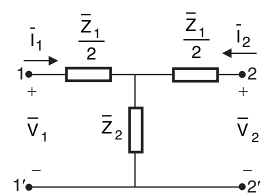

Fig.7: T-type filter

$$\bar{V}_1 = \bar{Z}_{11}\bar{I}_1 + \bar{Z}_{12}\bar{I}_2 \qquad .....(1)$$

$$\bar{V}_2 = \bar{Z}_{21}\bar{I}_1 + \bar{Z}_{22}\bar{I}_2 \qquad .....(2)$$

Let us connect a voltage source to port-1 and open circuit port-2 as shown in Fig.8.

Now, $\bar{I}_2 = 0$

Therefore,when $\bar{I}_2 = 0$, from equations (1) and (2) we get,

$$\bar{Z}_{11} = \frac{\bar{V}_1}{\bar{I}_1} \quad ; \quad \bar{Z}_{21} = \frac{\bar{V}_2}{\bar{I}_1}$$

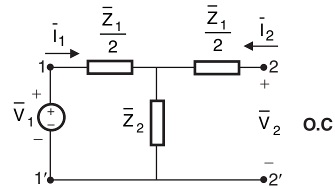

Fig.8.

⇓

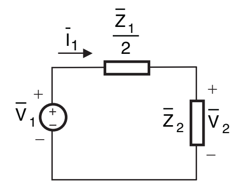

Fig.9.

From Fig.9, by Ohm's law,

$$\bar{V}_1 = \left(\frac{\bar{Z}_1}{2} + \bar{Z}_2\right)\bar{I}_1 \qquad .....(3)$$

$$\therefore \bar{Z}_{11} = \frac{\bar{V}_1}{\bar{I}_1} = \frac{\bar{Z}_1}{2} + \bar{Z}_2$$

From Fig.9, by voltage division rule,

$$\bar{V}_2 = \bar{V}_1 \times \frac{\bar{Z}_2}{\frac{\bar{Z}_1}{2} + \bar{Z}_2}$$

$$= \cancel{\left(\frac{\bar{Z}_1}{2} + \bar{Z}_2\right)}\bar{I}_1 \times \frac{\bar{Z}_2}{\cancel{\frac{\bar{Z}_1}{2} + \bar{Z}_2}} \qquad \boxed{\text{Using equation (3)}}$$

$$\therefore \bar{Z}_{11} = \frac{\bar{V}_1}{\bar{I}_1} = \bar{Z}_2$$

The T-type filter is symmetrical.

$$\therefore \bar{Z}_{22} = \bar{Z}_{11} = \frac{\bar{Z}_1}{2} + \bar{Z}_2$$

The T-type filter is reciprocal.

$$\therefore \bar{Z}_{12} = \bar{Z}_{21} = \bar{Z}_2$$

The Z- parameter matrix of constant-K, T-type filter is,

$$\mathbf{Z} = \begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} \\ \bar{Z}_{21} & \bar{Z}_{12} \end{bmatrix} = \begin{bmatrix} \frac{\bar{Z}_1}{2} + \bar{Z}_2 & \bar{Z}_2 \\ \bar{Z}_2 & \frac{\bar{Z}_1}{2} + \bar{Z}_2 \end{bmatrix}$$

The Z- parameter matrix of constant-K, T-type low pass filter is,

$$\mathbf{Z} = \begin{bmatrix} \frac{\bar{Z}_1}{2} + \bar{Z}_2 & \bar{Z}_2 \\ \bar{Z}_2 & \frac{\bar{Z}_1}{2} + \bar{Z}_2 \end{bmatrix} = \begin{bmatrix} \frac{j\omega L}{2} + \frac{1}{j\omega C} & \frac{1}{j\omega C} \\ \frac{1}{j\omega C} & \frac{j\omega L}{2} + \frac{1}{j\omega C} \end{bmatrix}$$

The Z- parameter matrix of constant-K, T-type high pass filter is,

$$\mathbf{Z} = \begin{bmatrix} \frac{\bar{Z}_1}{2} + \bar{Z}_2 & \bar{Z}_2 \\ \bar{Z}_2 & \frac{\bar{Z}_1}{2} + \bar{Z}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{j\omega 2C} + j\omega L & j\omega L \\ j\omega L & \frac{1}{j\omega 2C} + j\omega L \end{bmatrix}$$

B.E/ B.Tech DEGREE EXANMINATION, NOVEMBER/DECEMBER 2015

Second Semester

Electronics and Communication Engineering

EE 6201 – CIRCUIT THEORY

(Regulation 2013)

Time: 3 hours Maximum: 100 marks

Answer all questions

PART A – (10 × 2 = 20 marks)

1. State Kirchoff's Current law.

   Chapter 1, Section 1.4.2 ***[Page No - 1.25]***

2. Find the equivalent resistance of the circuit shown in Fig. 1.

   Chapter 2, SA - Q2.7 ***[Page No - 2.146]***

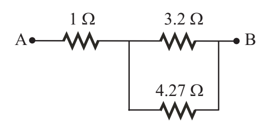

Fig. 1

3. List the applications of Thevenin's theorem.

   Chapter 2, SA - Q2.30 ***[Page No - 2.153]***

4. Two resistors of 4Ω and 6Ω are connected in parallel. If the total current is 30 A. Find the current through each resistor shown in Fig. 2.

   Chapter 2, SA - Q2.1 ***[Page No - 2.144]***

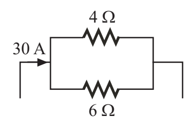

Fig. 2

5. Define selectivity.

   Chapter 5, SA - Q5.13 ***[Page No - 5.103]***

6. What is co-efficient of coupling?

   Chapter 5, SA - Q5.38 ***[Page No - 5.108]***

7. Distinguish steady state and transient state.

   Chapter 3, SA - Q3.7 ***[Page No - 3.97]***

8. What is time constant for RL and RC circuit?

   Chapter 3, SA - Q3.6 & Q3.10 ***[Page No - 3.97 & 3.98]***

9. What are the advantages of three phase system?

   Chapter 4, Section 4.18 ***[Page No - 4.59]***

10. When a 3-phase supply system is called balanced supply system?

    Chapter 4, SA - Q4.22 ***[Page No - 4.144]***

PART B – (10 × 16 = 80 marks)

11. (a) (i) Determine the magnitude and direction of the current in the 2 V battery in the circuit shown in Fig. 3. (8)

    Chapter 1, Example 1.4 ***[Page No - 1.34]***

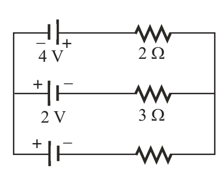

Fig. 3

(ii) Determine the power dissipation in the 4Ω resistor of the given circuit shown in Fig. 4. (8)

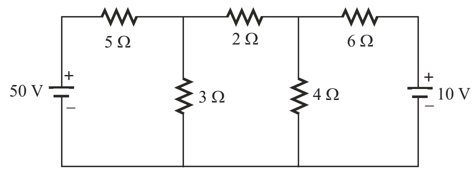

Fig. 4

Chapter 1, Example 1.21 ***[Page No - 1.60]***

(OR)

(b) Using node analysis, find the voltage V_x for the circuit shown in Fig 5. (16)

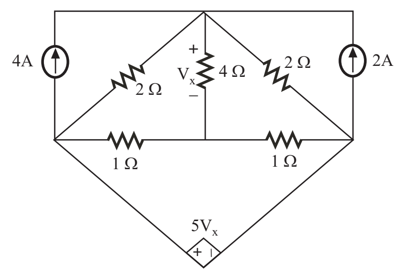

Fig. 5

Chapter 1, Example 1.62 ***[Page No - 1.133]***

12. (a) Find the Thevenin's equivalent of the network shown in Fig. 6. (16)

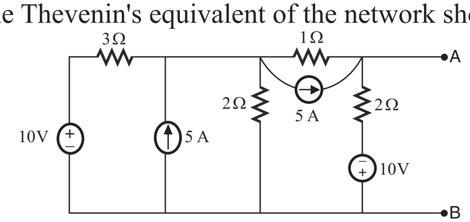

Fig. 6

Chapter 2, Example 2.22 ***[Page No - 2.61]***

(OR)

(b) Determine the value of resistance that may be connected across A and B so that maximum power is transferred from the circuit to the resistance. Also, estimate the maximum power transferred to the resistance shown in Fig. 7. (16)

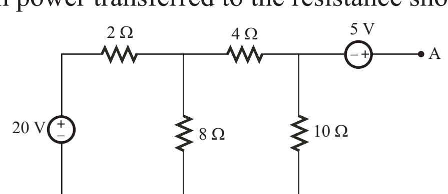

Fig. 7

Chapter 2, Example 2.48 ***[Page No - 2.109]***

13. (a) For the circuit shown in Fig. 8, determine the frequency at which the circuit resonates. Also find the quality factor, voltage across inductance and voltage across capacitance at resonance. (16)

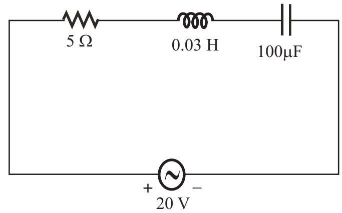

Fig. 8

Chapter 5, Example 5.3 ***[Page No - 5.13]***

(OR)

(b) Find the mutual reactance X_m in the coupled coils shown in Fig. 9. (16)

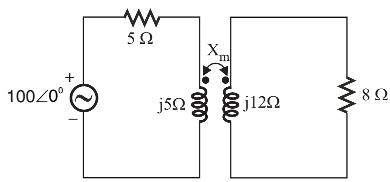

Fig. 9

Chapter 5, Example 5.27 ***[Page No - 5.86]***

14. (a) In the RL circuit shown in Fig. 10, the switch is closed to position-1 at t = 0. After t = 100 ms, the switch is changed to position-2. Find i(t) and sketch the transient. (16)

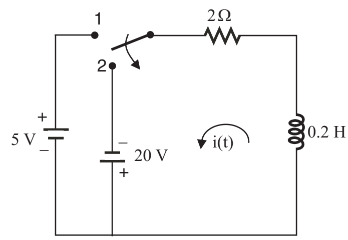

Fig. 10

Chapter 3, Example 3.5 ***[Page No - 3.52]***

(OR)

(b) (i) Determine the driving point impedance of the network shown in Fig. 11. (8)

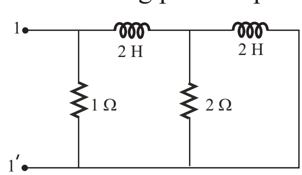

Fig. 11

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(ii) Determine the h-parameters of the two port network shown in Fig. 12. (8)

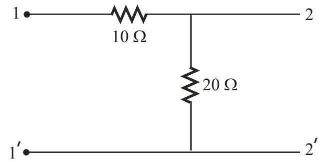

Fig. 12

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15. (a) Show that three phase power can be measured by two wattmeters. Draw the phasor diagrams. Derive an expression for power factor interms of wattmeter readings. (16)

Chapter 4, Section 4.25 ***[Page No - 4.85]***

(OR)

(b) (i) Three equal impedancces, each of 8 + j 10Ω are connected in star. This is further connected to a 440 V, 50 Hz, three phase supply. Calculate the active and reactive power and line and phase currents. (8)

Chapter 4, Example 4.23 ***[Page No - 4.94]***

(ii) Two wattmeter connected to measure the input to a balanced, three phase circuit indicate 2000 W and 500 W respectively. Find the power factor of the circuit.

(1) When both readings are positive and

(2) When the later is obtained after reversing the connections to the current coil of one instrument. (8)

Chapter 4, Example 4.41 ***[Page No - 4.117]***

B.E/ B.Tech DEGREE EXANMINATION, MAY/JUNE 2016

Second Semester

Electronics and Communication Engineering

EE 6201 – CIRCUIT THEORY

(Regulation 2013)

Time: 3 hours | Maximum: 100 marks

Answer all questions

PART A – (10 × 2 = 20 marks)

1. The resistance of two wires is 25 Ω when connected in series and 6 Ω when connected in parallel. Calculate the resistance of each wire.

   Chapter 2, SA - Q2.13 ***[Page No - 2.148]***

2. Distinguish between mesh and loop of a circuit.

   Chapter 1, SA - Q1.30 ***[Page No - 1.145]***

3. State reciprocity theorem.

   Chapter 2, Section - 2.6.4 ***[Page No - 2.126]***

4. What is the condition for the maximum power transfer in DC and AC circuits?

   Chapter 2, Table 2.3 ***[Page No - 2.105]***

5. Define co-efficient of coupling.

   Chapter 5, SA - Q5.38 ***[Page No - 5.108]***

6. In a series RLC circuit, if the value of L and C are 100 mH and 0.1 μF, find the resonance frequency in Hz.

   Chapter 5, SA - Q5.22 ***[Page No - 5.105]***

7. In a series RLC circuit, L = 2 H and C = 5 μF. Determine the value of R to give critical damping.

   Chapter 3, SA - Q3.19 ***[Page No - 3.99]***

8. Define time constant of RL circuit.

   Chapter 3, SA - Q3.6 ***[Page No - 3.97]***

9. A 3 phase 400 V is given to balanced star connected load of impedance 8 + 6j Ω. Calculate line current.

   Chapter 4, SA - Q4.28 ***[Page No - 4.145]***

10. List out the advantages of three phase system over single phase system.

    Chapter 4, Section 4.18 ***[Page No - 4.59]***

PART B – (5 × 16 = 80 marks)

11. (a) (i) Determine the current I_L in the circuit shown in Fig. 11 (a) (i). (8)

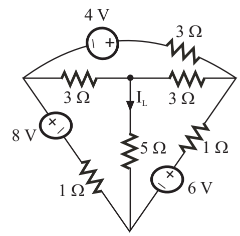

Fig. 11 (a) (i)

Chapter 1, Example 1.17 ***[Page No - 1.56]***

(ii) Calculate the voltage across A and B in the circcuit shown in Fig. 11 (a) (ii). (8)

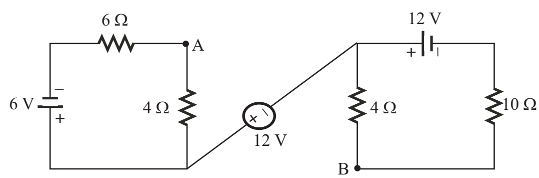

Fig. 11 (a) (ii)

Chapter 1, SA-Q1.23 ***[Page No - 1.143]***

(OR)

(b) (i) Three loads A, B, C are connected in parallel to a 240 V source. Load A takes 9.6 kW, load B takes 60 A, and load C has a resistance of 4.8 Ω. Calculate R_A and R_B, the total current, total power and equivalent resistance. (8)

Chapter 1, SA-Q2.11 ***[Page No - 2.147]***

(ii) For the circuit shown in Fig. 11 (b) (ii), determine the total current and power factor. (8)

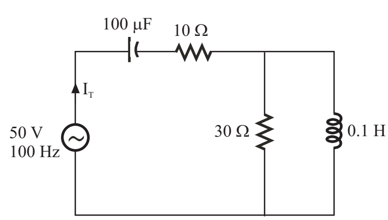

Fig. 11 (b) (ii)

Chapter 4, Example 4.5 ***[Page No - 4.41]***

12. (a) Find the voltage across 5 Ω resistor for the circuit shown in Fig. 12 (a) using source transformation technique and verify the results using mesh analysis. (16)

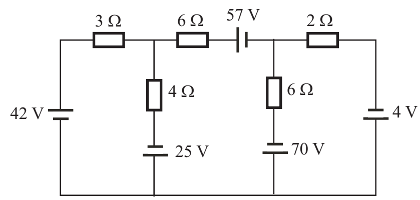

Fig. 12 (a)

Chapter 1, Example 1.25 ***[Page No -1.65]***

(OR)

(b) Obtain the Nortan's model and find the maximum power that can be transferred to the 100 Ω load resistance, in the circuit shown in Fig. 12(b). (16)

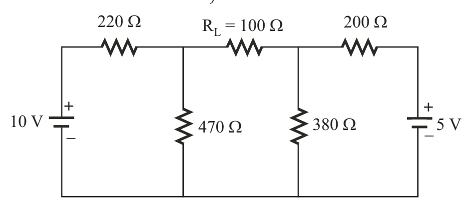

Fig. 12(b)

Chapter 2, Example 2.49 ***[Page No - 2.110]***

13. (a) Determine the resonant frequency, bandwidth and quality factor of the coil for the series resonant circuit considering R = 10 Ω, L = 0.1 H and C = 10 μF. Derive the formula used for bandwidth. (16)

Chapter 5, Section 5.2.4 and Example 5.5 ***[Page No - 5.6, 5.15]***

(OR)

(b) (i) Derive the expression for equivalent inductance of the parallel resonant circuit as shown in Fig. 13 (b) (i). (8)

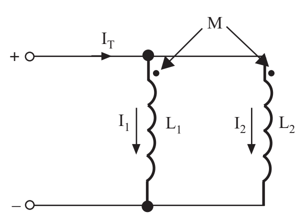

Fig. 13 (b) (i)

Chapter 5, Section 5.7 ***[Page No - 5.59]***

(ii) Write the mesh equations and obtain the conductively coupled equivalent circuit for the magnetically coupled circuit shown in Fig. 13 (b) (ii). (8)

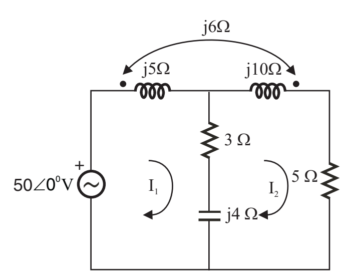

Fig. 13 (b) (ii)

Chapter 5, Example 5.28 and Section 5.6.3 ***[Page No - 5.87, 5.52]***

14. (a) A sinusoidal voltage of 10 sin 100 is connected in series with a switch and R = 10 Ω & L = 0.1 H. If the switch is closed at t = 0, determine the transient current i(t).(16)

Chapter 3, Example 3.8 ***[Page No - 3.57]***

(OR)

(b) In the circuit shown in Fig. 14(b). Determine the transient current after switch is closed at time t = 0, given that an initial charge of 100 μC is stored in the capacitor. Derive the necessary equations. (16)

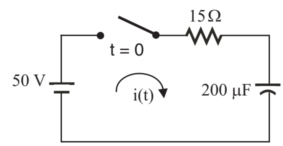

Fig. 14(b)

Chapter 3, Example 3.12 ***[Page No - 3.64]***

15. (a) Obtain the readings of two wattmeters connected to a three phase, 3 wire, 120 V system feeding a balanced Δ connected load with a load impedance of $12\angle 30^\circ\,\Omega$. Assume RYB phase sequence. Determine the phase power and compare the total power to the sum of wattmeter readings. (16)

Chapter 4, Example 4.23 ***[Page No - 4.94]***

(OR)

(b) (i) If W_1 & W_2 are the readings of two wattmeters which measures power in the three phase balanced system and if $W_1/W_2 = a$, show that the power factor of the circuit is given by (8)

$$\cos\phi = \frac{a+1}{2\sqrt{a^2 - a + 1}}$$

Chapter 4, Example 4.54 ***[Page No - 4.132]***

(ii) A symmetrical, three phase, three wire 440 V ABC system feeds a balanced Y-connected load with $Z_A = Z_B = Z_C = 10\angle 30^\circ\,\Omega$ obtain the line currents. (8)

Chapter 4, Example 4.23 ***[Page No - 4.94]***

B.E/ B.Tech DEGREE EXANMINATION, NOVEMBER/DECEMBER 2016

Second Semester

Electronics and Communication Engineering

EE 6201 – CIRCUIT THEORY

(Regulation 2013)

Time: 3 hours | Maximum: 100 marks

Answer all questions

PART A – (10 × 2 = 20 marks)

1. What are the limitations of Ohm's law?

   Chapter 1, SA - Q1.10 ***[Page No - 1.140]***

2. The equivalent resistance of four resistors joined is parallel is 30 ohms. The current flowing through them are 0.5, 0.4, 0.6 and 0.1 A. Find the value of each resistor.

   Chapter 2, SA - Q2.5 ***[Page No - 2.145]***

3. Determine the value of current I_o of the given figure. 3

   Chapter 2, SA - Q2.1 ***[Page No - 2.144]***

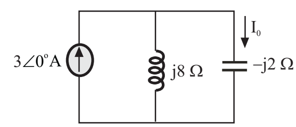

Fig. 3

4. State reciprocity theorem.

   Chapter 2, Section 2.6.4 ***[Page No - 2.126]***

5. Draw the frequency response characteristics of parallel resonant circuit.

   Chapter 5, Section 5.3.2 ***[Page No - 5.25]***

6. Determine the equivalent inductance of the circuit comprising two inductors in series opposing mode.

   Chapter 5, Section 5.7 ***[Page No - 5.58]***

7. Determine the Laplace transform of unit step function u(t) and sinusoidal function sin (ωt).

   Chapter 3, Table 3.1 ***[Page No - 3.5]***

8. A RLC series circuit has R = 10 ohms and L = 2H. What value of capacitance will make the circuit critically damped?

   Chapter 3, SA - Q3.20 ***[Page No - 3.99]***

9. What is a phase sequence of 3 phase system?

   Chapter 4, SA - Q4.21 ***[Page No - 4.144]***

10. List any two advantages of three phase system over single-phase system.

    Chapter 4, Section 4.18 ***[Page No - 4.59]***

PART B – (5 × 16 = 80 marks)

11. (a) (i) Calculate the node voltages of given circuit in fig. 11(a) (i). (8)

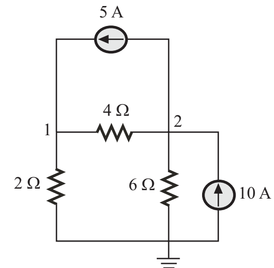

Fig. 11(a) (i)

Chapter 1, Example 1.42 ***[Page No - 1.99]***

(ii) Determine current I_o for the given circuit in Fig. 11(a) (ii) when $v_s = 12$ V. (8)

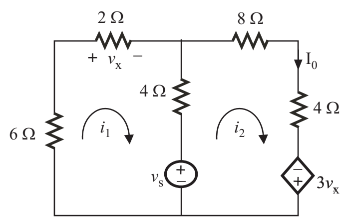

Fig. 11 (a) (ii)

Chapter 1, Example 1.37 ***[Page No - 1.85]***

(OR)

(b) (i) Using mesh analysis for the given fig. 11(b) (i), find the current I_2 and drop across 1Ω resistor. (12)

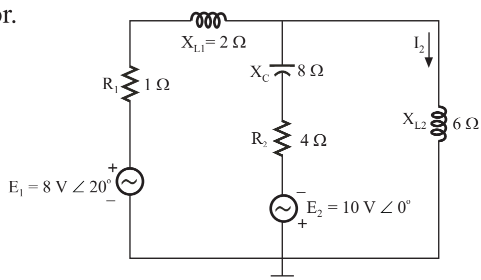

Fig. 11 (b) (i)

Chapter 1, Example 1.30 ***[Page No - 1.75]***

(ii) Find the equivalent capacitance C between terminals A and B of fig. 11(b) (ii). (4)

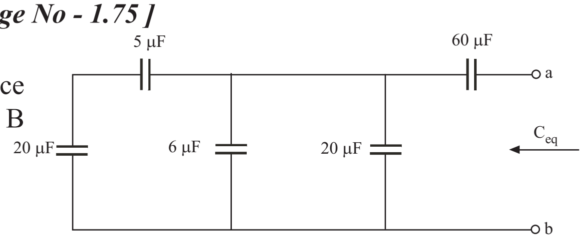

Fig. 11 (b) (ii)

Chapter 2, Example 2.11 ***[Page No - 2.44]***

12. (a) (i) Obtain the equivalent resistance R_{ab} of the circuit given in fig 12 (a) (i) and calculate the total current i. (8)

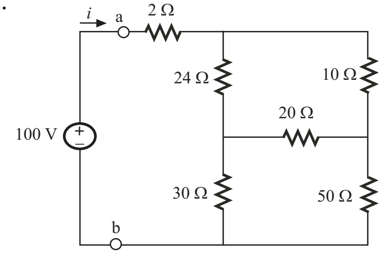

Figure 12 (a) (i)

Chapter 2, Example 2.7 ***[Page No - 2.39]***

(ii) Find the value of R_L in fig. 12(a)(ii) for maximum power to R_L and calculate the maximum power. (8)

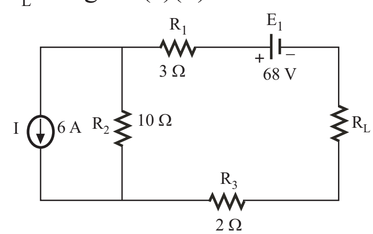

Figure 12 (a) (ii)

Chapter 2, Example 2.46 ***[Page No - 2.107]***

(OR)

(b) Apply superposition theorem to determine current i through 3Ω resistor for the given circuit in fig. 12(b). (16)

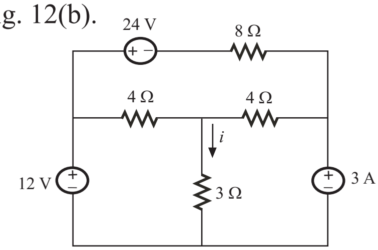

Figure 12 (b)

Chapter 2, Example 2.40 ***[Page No - 2.90]***

13. (a) For the series resonant circuit of Fig. 13(a), find I, V_R, V_L and V_C at resonance. Also,if resonant frequency is 5000Hz, determine bandwidth Q factor, half power frequencies, and power dissipated in the circuit at resonance and at the half power frequencies. Derive the expression for resonant frequency. (16)

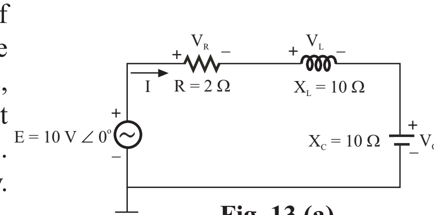

Fig. 13 (a)

Chapter 5, Example 5.2 ***[Page No - 5.12]***

(OR)

(b) (i) Obtain the conductively coupled equivalent circuit for the given circuit in Fig. 13(b)(i) and find the voltage drop across 12 Ω resistor. (8)

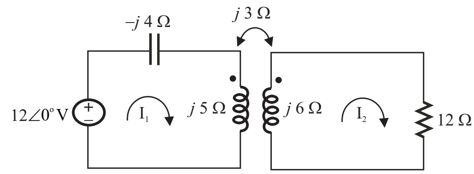

Fig. 13 (b) (i)

Chapter 5, Example 5.33 ***[Page No - 5.93]***

(ii) The number of turns in two coupled coils are 500 turns and 1500 turns respectively. When 5 A current flows in coil 1, the total flux in this coil is 0.6×10^{-3} wb and the flux linking in second coil is 0.3×10^{-3} wb. Determine L_1, L_2, M and K. (8)

Chapter 5, Example 5.20 ***[Page No - 5.75]***

14. (a) A series RL circuit with R = 50 Ω and L = 30 H has a constant voltage V = 50 volts applied at t = 0 as shown in fig.14 (a). Determine the current i, voltage across inductor. Derive the necessary expression and plot the respective curves. (16)

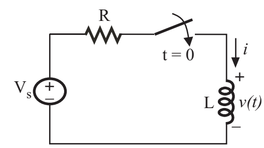

Fig. 14 (a)

Chapter 3, Example 3.2 ***[Page No - 3.48]***

(OR)

(b) (i) Determine the impedance (Z) parameter of the given two port network in Fig. 14(b)(i) (8)

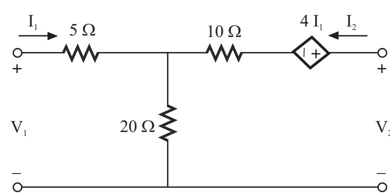

Fig. 14 (b) (i).

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(ii) Find the hybrid (h) parameter of the two port network in Fig. 14(b)(ii). (8)

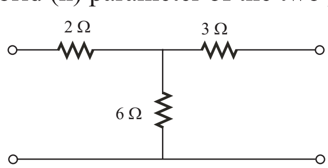

Fig. 14 (b) (ii).

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15. (a) (i) For the Δ–Δ system shown in Fig. 15(a)(i), find the phase angle θ_2 and θ_3 for the specified phase sequence. Also, find the phase current and line current in each phase of the load. (8)

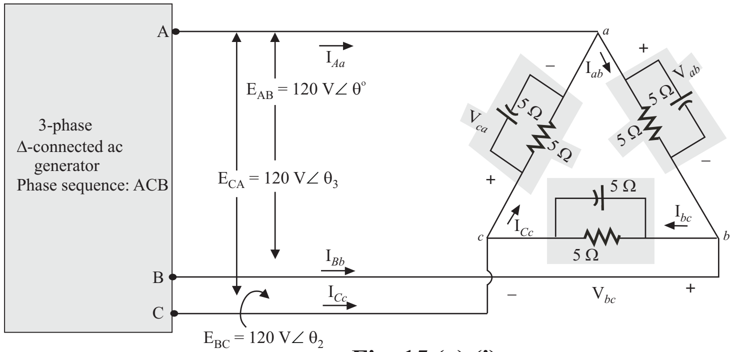


Fig. 15 (a) (i)

Chapter 4, Example 4.52 ***[Page No - 4.129]***

(ii) A 3 phase 400V supply is given to balanced star connected load of impedance (8 + 6j) ohms in each branch. Determine line current, power factor and total power. (8)

Chapter 4, Example 4.23 ***[Page No - 4.94]***

(OR)

(b) The two wattmeter produces wattemeter readings P_1 = 1560W and P_2 = 2100W when connected to a delta connected load. If the line voltage is 220 V, calculate (i) the per phase average power (ii) total reactive power, (iii) power factor and (iv) the phase impedance. Is the impedance inductive or capacitive? Justify. (16)

Chapter 4, Example 4.46 ***[Page No - 4.122]***

B.E/ B.Tech DEGREE EXANMINATION, APRIL/MAY 2017

Second Semester

Electronics and Communication Engineering

EE 6201 – CIRCUIT THEORY

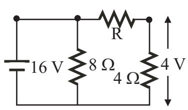
(Regulation 2013)

Time: 3 hours | Maximum: 100 marks

Answer all questions

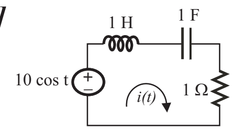
PART A – (10 × 2 = 20 marks)

1. Find 'R' in the circuit shown below.



Chapter 2, SA - Q2.8 ***[Page No - 2.146]***

2. Determine the current i(t) for the given circuit



Chapter 4, SA - Q4.3 ***[Page No - 4.140]***

3. A star connected load of 5Ω each is to be cconverted in to an equivalent delta connected load. Find the resistance be used.

Chapter 2, SA - Q2.20 ***[Page No - 2.150]***

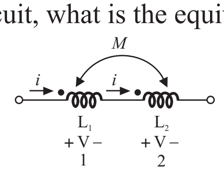
4. A load is connected to a network of the terminals to which load is connected, R_{th} = 10 ohms and V_{th} = 40 V. Calculate the maximum power supplied to the load.

Chapter 2, SA - Q2.40 ***[Page No - 2.155]***

5. Define self inductance and mutual inductance of a coil.

Chapter 5, SA - Q5.36, Q5.37 ***[Page No - 5.107]***

6. Given the circuit, what is the equivalent inductance of the system shown below.



Chapter 5, Section 5.7 ***[Page No - 5.57]***

7. Define time constant for RL circuit. Draw the transient current characteristics

Chapter 3, SA - Q3.6 & Section 3.3.2 ***[Page No - 3.97, 3.15]***

8. When a two port network is said to be reciprocal

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9. Draw the phasor diagram of line currentts and line voltages of a balanced delta connected load.

Chapter 4, Section 4.21.3 ***[Page No - 4.69, 4.71]***

10. Distinguish between unbalanced supply and unbalanced load.

Chapter 4, SA - Q4.25 ***[Page No - 4.144]***

PART B – (5 × 16 = 80 marks)

11. (a) (i) Determine the potential difference between points A and B given in fig. 11(a)(i)(8)

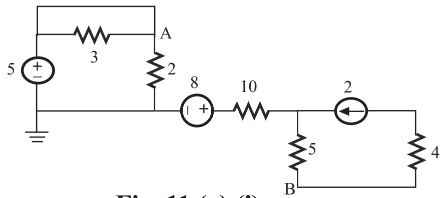

Fig. 11 (a) (i)

Chapter 1, SA - Q1.23 ***[Page No - 1.143]***

(ii) Using Mesh analysis,find the current I_on the circuit shown fig. 11(a)(ii). (8)

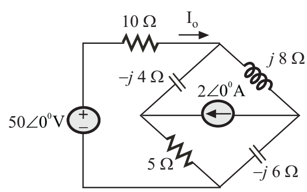

Fig. 11 (a) (ii)

Chapter 1, Example 1.33 ***[Page No - 1.79]***

(OR)

(b) (i) Determine v_x and i_x in the given fig 11 (b) (i). (10)

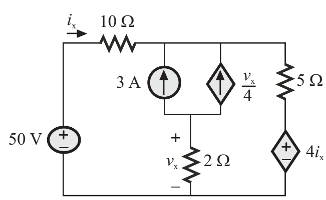

Fig 11 (b) (i)

Chapter 1, Example 1.40 ***[Page No - 1.89]***

(ii) Write the mesh equation and nodal equation for the network in fig. 11(b) (ii) by inspection method. (6)

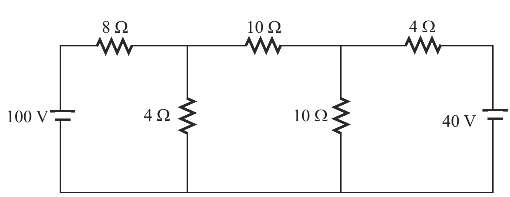

Fig. 11 (b) (ii)

Chapter 1, Example 1.48 ***[Page No - 1.108]***

12. (a) (i) Apply source transformation technique to determine current i_o in fig. 12(a)(i). (8)

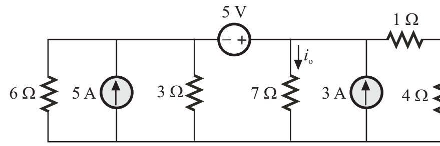

Fig. 12 (a) (i)

Chapter 2, Example 2.15 ***[Page No - 2.48]***

(ii) Find the power delivered by the 20V source using superposition theorem. (8)

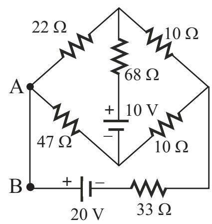

Fig. 12 (a) (ii)

Chapter 2, Example 2.34 ***[Page No - 2.82]***

(OR)

(b) Apply Norton theorem to determine current I_o for the given circuit in Fig. 12(b).(16)

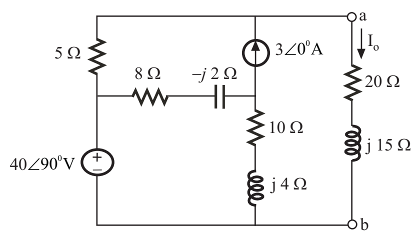

Fig. 12(b)

Chapter 2, Example 2.57 ***[Page No - 2.121]***

13. (a) (i) Derive the expression for resonant frequency and bandwidth for a series RLC resonant circuit. (8)

Chapter 5, Section 5.1, 5.2.4 ***[Page No - 5.1, 5.6]***

(ii) In the parallel RLC circuit of Fig. 13(a)(ii), let R = 8kΩ, L = 0.2 mH and C = 8 μF. Calculate ω_0, Q, half power frequencies and BW. (8)

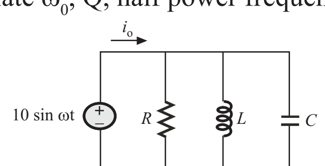

Fig. 13 (a) (ii)

Chapter 5, Example 5.8 ***[Page No - 5.36]***

(OR)

(b) (i) Find the voltage drop across 12Ω resistor for the given circuit in Fig. 13(b)(i). Also, draw the conductively coupled equivalent circuit. (8)

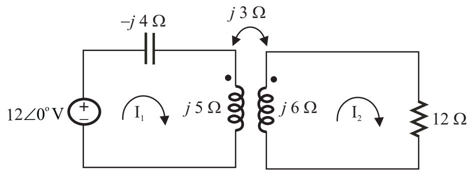

Fig. 13 (b) (i)

Chapter 5, Example 5.33 ***[Page No - 5.93]***

(ii) The number of turns in two coupled coils are 500 turns and 1500 turns respectively. When 5 A current flows in coil, the total flux in this coil is 0.6×10^{-3} wb and the flux linking in second coil is 0.3×10^{-3} wb. Determine L_1, L_2M and K. (8)

Chapter 5, Example 5.20 ***[Page No - 5.75]***

14. (a) A series RL circuit with R = 10Ω and L = 0.1H is supplied by an input voltage v(t) 10 sin 100t Volts applied at t = 0 as shown in fig. 14(a). Determine the current i, voltage across inductor. Derive the necessary expression and plot the respective curves. (16)

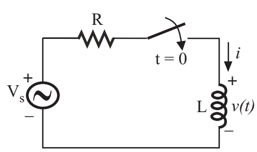

Fig. 14 (a)

Chapter 3, Example 3.8 ***[Page No - 3.57]***

(OR)

(b) Determine the impedance (Z) parameter and draw the T-equivalent circuit for the given two port network in Fig. 14 (b). Also, derive the transmission line (ABCD) parameters from Z parameter. (16)

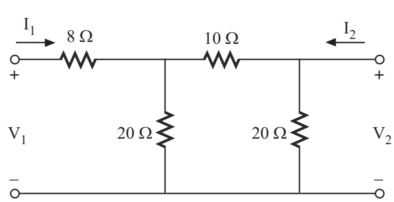

Fig. 14 (b)

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15. (a) (i) A balanced Δ-connected load having an impedance $20 - j15\Omega$ is connected to a Δ-connected, positive sequence supply $V_{ab} = 330\angle 0^{o}$ V. Calculate the phase currents of the load and the line currents. (8)

Chapter 4, Example 4.31 ***[Page No - 4.105]***

(ii) The input power to a 3ϕ load is 10kw at 0.8 pf. Two wattmeters are connected to measure power, find the individual readings of the wattmeters. (8)

Chapter 4, Example 4.42 ***[Page No - 4.118]***

(OR)

(b) For the unbalanced circuit in Fig. 15(b), determine the line currents and voltage across each load impedance. Draw the phasor diagram. (16)

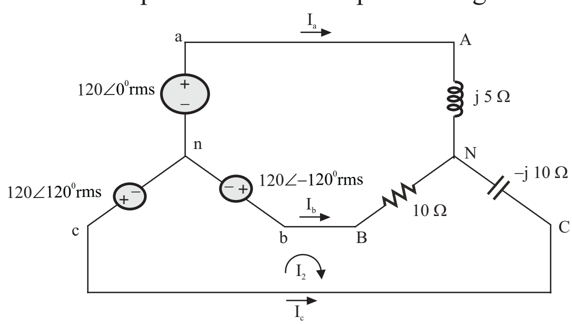

Fig. 15(b)

Chapter 4, Example 4.29 ***[Page No - 4.102]***

INDEX

F

G

H

I

J

K

L

M

N

O

P

Q

R

S

T

U

V

W

X

Y

Z