Circuit Theory and Networks

Fourth Edition

WBUT-2014

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Dedicated to our families

Wife, *Lipika*

and

My Parents

S P Ghosh

Wife, *Indira*

Daughters, *Amrita* and *Ananya*

A K Chakraborty

Contents

Preface

Brief Introduction to the Subject

Circuit Theory and Networks is a gateway course to all engineering subjects; Electrical Engineering, Electronics and Communication Engineering, Computer Science and Engineering, Information Technology, and Instrumentation Engineering in particular. Almost all engineering systems use electric circuits as components. To understand the operation of these systems, the knowledge of Circuit Theory is very essential. Also, the subject of Circuit Theory provides the background for understanding the behaviour of many other electrical and electronic devices. The present book has been written keeping this in view and aims to provide requisite information of circuit theory and networks, starting from the fundamentals. It is an acceptable fact that students do not automatically acquire conceptual understanding; the concepts must be explained and the students given a chance to grapple with them. Our presentation, based on years of teaching this course, blends conceptual understanding with analytical skills.

Objective

This book has been written as per the syllabus of Circuit Theory and Networks (EE 301) of the **West Bengal University of Technology (WBUT)**. Although there are several books on Circuit Theory, there is hardly any book meeting the entire syllabi of WBUT. This text works well in our self-paced course, where students can rely on it as their primary learning resource. Nonetheless, completeness and clarity are equally advantageous when the book is used in a more traditional classroom setting. Cognizance of the present standard of the students and the difficulties of the teachers have been given due thought. The conceptual examples and practice problems and a variety of conceptual and multiple-choice questions at the end of each chapter give students a chance to check and enhance their conceptual understanding.

Scope

This book is mainly written for the third-semester students under WBUT. However, as this course mitigates a definite percentage in every competitive examination of engineering professionals, namely, IES, UPSC, GATE, etc., we have written this book to help students see that a relatively small number of basic concepts are applied to a wide variety of situations.

Salient Features

- Chapter organization and terminology in agreement with syllabus structure: EE-301 for EEE/EE/PE/ICE undergraduate engineering students, Sem. 3; EC 301 for ECE undergraduate engineering students, Sem. 3; EE (EI) 301 for AEIE undergraduate engineering students, Sem. 3; and BME (EC) 301 for BME undergraduate engineering students, Sem. 3
- Detailed coverage of different types of systems and networks
- Every theorem has been explained in the following sequence:
  Statement > Proof > Points to be Noted
- Varied pedagogy including MCQs, Solved Examples, Exercises, Short-Answer-Type Questions, and Solved Exercises to explain involved concepts:
  - Illustrations: 700
  - MCQs: 400
  - Exercises: 150
  - Solved Exercises: 90
  - Short-Answer-Type Questions: 105
  - Solved Examples: 180
- Use the APP to
  - **Study** Important topics on-the-go
  - **Revise** Quick pointers to ace the examination
  - **Test** Questions with answers that test concepts learnt

Organization

This book has a total of ten chapters. **Chapter 1** provides information about the basic characteristics of different types of systems. **Chapter 2** deals with the basic circuit theory concepts, laws and techniques for circuit analysis. **Chapter 3** introduces the new topic on magnetically coupled circuits. It deals with different models in coupled circuits and analysis of the same. **Chapter 4** discusses the application of graph-theory concepts in circuit analysis. In this chapter, the application of a mathematical tool like graph theory has been presented with the help of a large number of practical examples. **Chapter 5** is devoted to various network theorems necessary for simplified analysis of electrical problems. This chapter is very important for examination purposes as several questions from this chapter are asked in exams.

Chapter 6 introduces a new method of circuit analysis—Laplace Transform method. Starting from the very fundamental concept of Laplace transform, its applications in various complicated circuit problems have been discussed in detail in this chapter. **Chapter 7** deals with the concepts of the two-port network which has vast applications in many fields like transmission lines, filters and attenuators. **Chapter 8** is divided into two parts. *Part I* presents the fundamentals of Fourier series and its applications for circuit analysis. *Part II* discusses Fourier transforms and their applications. **Chapter 9** is devoted to operational amplifiers and active filters. **Chapter 10** talks about analysis of various resonances and their combinations in detail.

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Feedback

Criticism and suggestions for improvement shall be gratefully acknowledged. Readers may contact **A K Chakraborty** at akcalll@yahoo.co.in and **S P Ghosh** at ghosh_shankar@rediffmail.com

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ROADMAPS TO THE SYLLABUS

Circuit Theory and Networks

The text is suitable for the following paper codes and disciplines:

- **EE 301 Electric Circuit Theory (for EE, EEE, PE & ICE)**
- **EC 301 Circuit Theory and Networks (for ECE)**
- **EE (EI) 301 Circuit Theory and Networks (for AEIE)**
- **BME (EC) 301 Circuit Theory and Networks (for Biomedical Engineering)**

ROADMAP TO THE SYLLABUS
(EE 301)

Module 1
Introduction: Continuous and discrete; Fixed and time-varying; Linear and nonlinear; Lumped and distributed; Passive and active networks and systems; Independent and dependent sources; Step, ramp, impulse, sinusoidal, square and sawtooth signals.

GO TO **Chapter 1:** *Introduction to Different Types of Systems*

Module 2
Coupled Circuits: Magnetic coupling; Polarity of coils; Polarity of induced voltage; Concept of Self- and mutual inductances; Coefficient of coupling; Modeling of coupled circuits; Solutions of problems.

GO TO **Chapter 3:** *Magnetically Coupled Circuits*

Module 3
Laplace Transforms: Impulse; Step and sinusoidal response of *RL*, *RC* and *RLC* circuits; Transient analysis of different electrical circuits with and without initial conditions; Concept of convolution theorem and its applications; Solutions of problems with dc and ac sources.

GO TO **Chapter 6:** *Laplace Transform and its Applications*

Module 4
Fourier Method of Waveform Analysis: Fourier series and Fourier transform (in continuous domain only); Application in circuit analysis; Solutions of problems.

GO TO **Chapter 8:** *Fourier Series and Fourier Transform*

Module 5

Network Equations and Theorems: Formulation of network equations; Source transformation; Loop-variable analysis; Node-variable analysis; Superposition, Thevenin's; Norton's and maximum power transfer theorems; Millman's theorem and its application in three-phase unbalanced circuit analysis; Solution of problems with dc and ac sources.

GO TO

Chapter 4: *Network Topology (Graph Theory)*
Chapter 5: *Network Theorems*

Module 6

Graph Theory and Network Equations: Concept of tree, branch, tree link; Incidence matrix; Tie-set matrix and loop currents; Cut-set matrix and node pair potentials; Duality; Solutions of problems.

GO TO

Chapter 4: *Network Topology (Graph Theory)*

Module 7

Two-port Network Analysis: Open-circuit impedance and short-circuit admittance parameter; Transmission parameters; Hybrid parameters and their inter-relations; Driving-point impedance and admittance; Solutions of problems.

GO TO

Chapter 7: *Two-Port Network*

Module 8

Filter Circuits: Analysis and synthesis of low-pass, high-pass, band-pass, band-reject, all-pass filters (first- and second-order only) using operational amplifier; Solutions of problems.

GO TO

Chapter 9: *Filter Circuits*

(EC 301)
APPLIED ELECTRONICS AND INSTRUMENTATION ENGINEERING

Module 1

Resonant Circuits: Series and parallel resonance; Impedance and admittance characteristics; Quality factor; Half-power points; bandwidth; Phasor diagrams; Transform diagrams; Practical resonant and series circuits; Solutions of problems.

Mesh Current Network Analysis: Kirchhoff's voltage law; Formulation of mesh equations; Solutions of mesh equations by Cramer's rule and matrix method; Driving-point impedance; Transfer impedance; Solutions of problems with dc and ac sources.

GO TO

Chapter 10: *Resonance*
Chapter 2: *Introduction to Circuit Theory Concepts*

Module 2

Node Voltage Network Analysis: Kirchhoff's current law; Formulation of node equations and solutions; Driving-point admittance; Transfer admittance; Solutions of problems with dc and ac sources.

Network Theorems: Definition and implication of superposition theorem; Thevenin's theorem; Norton's theorem; Reciprocity theorem; Compensation theorem; Maximum power transfer theorem; Millman's theorem; Star-delta transformations; Solutions and problems with dc and ac sources.

GO TO

Chapter 5: *Network Theorems*
Chapter 2: *Introduction to Circuit Theory Concepts*

Module 3

Graph of Network: Concept of tree and branch; Tree link; Junctions; Incident matrix; Tie-set matrix; Determination of loop current and node voltages.

Coupled Circuits: Magnetic coupling; Polarity of coils; Polarity of induced voltage; Concept of self- and mutual inductance; Coefficient of coupling; Solutions of problems.

Circuit Transients: dc transients in *RL* and *RC* circuits with and without initial charge; *RLC* circuits; ac transients in sinusoidal *RL*; *RC* and *RLC* circuits; Solutions of problems.

GO TO

Chapter 4: *Network Topology (Graph Theory)*
Chapter 3: *Magnetically Coupled Circuits*
Chapter 6: *Laplace Transform and its Applications*

Module 4

Laplace Transform: Concept of complex frequency; Transform of $f(t)$ into $F(s)$; Transform of step; Exponential surge, overdamped surge, critically damped surge; Damped and un-damped sine functions; Properties of Laplace transform; Linearity; Real differentiation; Real integration; Initial value and final value theorems; Inverse Laplace transform, application in circuit analysis; Partial fraction expansion; Heaviside's expansion theorem; Laplace transform and Inverse Laplace transform; Solutions of problems.

Two-port Networks: Relationship of two-port network variables; Short-circuit admittance parameters; Open-circuit impedance parameters; Transmission parameters; Relationship between parameter sets; Network functions for ladder and general networks.

GO TO

Chapter 6: *Laplace Transform and its Applications*
Chapter 7: *Two-Port Network*

ROADMAP TO THE SYLLABUS
[EE (EI) 301]

Module 1

Introduction: Continuous and discrete; Fixed and time-varying; Linear and nonlinear; Lumped and distributed; Passive and active networks and systems; Independent and dependent sources; Step, ramp, impulse, sinusoidal, square; sawtooth signals.

Coupled Circuits: Magnetic coupling; Polarity of coils; Polarity of induced voltage; Concept of Self- and mutual inductance; Coefficient of coupling; Modeling of coupled circuits; Solutions of problems.

Resonant Circuits: Series and parallel resonance; Impedance and admittance characteristics; Quality factor; Half-power points; Bandwidth; Resonant voltage rise; Transform diagrams; Solutions of problems.

GO TO

Chapter 1: *Introduction to Different Types of Systems*
Chapter 3: *Magnetically Coupled Circuits*
Chapter 10: *Resonance*

Module 2

Laplace Transforms: Concept of complex frequency; Transformation of step, exponential, overdamped surges; Critically damped surge; Damped sine and undamped sine functions; Properties of Laplace transforms; Linearity; Real differentiation; Real integration; Initial value and final value theorems; Inverse Laplace transforms and applications in circuit analysis; Partial fraction expansion; Heaviside's expansion theorem; Impulse; Step and sinusoidal response of *RL*, *RC* and *RLC* circuits; Transient analysis of different electrical circuits with and without initial conditions; Concept of convolution theorem and its application; Solutions of problems with dc and ac sources.

GO TO

Chapter 6: *Laplace Transform and its Applications*

Module 3

Network Equations: Kirchhoff's voltage law and current law; Formulation of network equations; Source transformation; Loop-variable analysis; Node-variable analysis.

Network Theorems: Superposition, Thevenin's, Norton's and maximum power transfer theorems; Millman's theorem and its application in three-phase unbalanced circuit analysis; Solutions of problems with dc and ac sources.

Graph of Network: Concept of tree, branch, tree link, junctions; Incident matrix; Tie-set matrix and loop currents; Cut-set matrix and node pair potentials; Duality; Solutions of problems.

GO TO

Chapter 2: *Introduction to Circuit Theory Concepts*
Chapter 5: *Network Theorems*
Chapter 4: *Network Topology (Graph Theory)*

Module 4

Two-port Network Analysis: Open-circuit impedance and short-circuit admittance parameter; Transmission parameters; Hybrid parameters and their inter-relations; Driving-point impedance and admittance; Solutions of problems with dc and ac sources.

Circuit Transients: dc transient in *RL* and *RC* circuits with and without initial charge; *RLC* circuits; ac transients in sinusoidal *RL*, *RC* and *RLC* circuits; Solutions of problems.

Filter Circuits: Analysis of low-pass, high-pass, band-pass, band-reject, all-pass filters (first- and second-order only) using operational amplifier; Solutions of problems.

GO TO

Chapter 7: *Two-Port Network*
Chapter 6: *Laplace Transform and its Applications*
Chapter 9: *Filter Circuits*

ROADMAP TO THE SYLLABUS
BME (EC) 301

Module 1

Resonant Circuits: Series and parallel resonance; Impedance and admittance characteristics; Quality factor; Half-power points; Bandwidth; Resonant voltage rise; Transform diagrams; Solutions of problems.

GO TO **Chapter 10:** *Resonance*

Module 2

Mesh Current Network Analysis: Kirchhoff's voltage laws; Formulation of mesh equations; Solution of mesh equations by Cramer's rule and matrix method; Driving-point admittance; Transfer impedance; Solutions of problems with dc and ac sources.

GO TO **Chapter 2:** *Introduction to Circuit Theory Concepts*

Module 3

Node Voltage Network Analysis: Kirchhoff's current law; Formulation of node equations and solutions; Driving-point admittance; Transfer admittance; Solutions of problems with dc and ac sources.

GO TO **Chapter 2:** *Introduction to Circuit Theory Concepts*

Module 4

Network Theorems: Definition and implications of Superposition, Thevenin's, Norton's, Reciprocity, Compensation, maximum power transfer, and Millman's theorems; Star-Delta transformations; Solutions and problems with dc and ac sources.

GO TO **Chapter 5:** *Network Theorems*

Module 5

Graph of Networks: Concept of tree, branch, tree link, junctions, incidence matrix, tie-set matrix; cut-set matrix; Determination of loop current and node voltages.

GO TO **Chapter 4:** *Network Topology (Graph Theory)*

Module 6

Coupled Circuits: Magnetic coupling; Polarity of coils; Polarity of induced voltage; Concept of self- and mutual inductances; Coefficient of coupling; Solutions of problems.

GO TO **Chapter 3:** *Magnetically Coupled Circuits*

Module 7

Circuit Transients: dc transients in *RL* and *RC* circuits with and without initial charge; *RLC* circuits; ac transients in sinusoidal *RL*, *RC* and *RLC* circuits; Solutions of problems.

GO TO **Chapter 3:** *Magnetically Coupled Circuits*

Module 8

Laplace Transforms: Concept of complex frequency; Transformation of $f(t)$ into $f(s)$; Transformation of step, exponential, overdamped surge, critically damped surge; Damped and undamped sine functions; Properties of Laplace transforms; Linearity; Real differentiation; Real integration; Initial value and final value theorems; Inverse Laplace transform; Applications of circuit analysis; Partial fractions expansion; Heaviside's expansion theorem; Solutions of problems.

GO TO **Chapter 6:** *Laplace Transform and its Applications*

Module 9

SPICE: Introduction; Model statement; Elementary dc and small-signal analysis.

CHAPTER

1

Introduction to Different Types of Systems

1.1 INTRODUCTION

An electrical network is one of the many important physical systems. In order to understand the basic characteristics of an electric network, we must first know the different concepts of systems. In this chapter, the different types of systems have been discussed.

1.2 CONCEPTS OF SIGNALS AND SYSTEMS

1.2.1 Signals

A signal is defined as a function of one or more variables, which provides information on the nature of a physical phenomenon.

When the function depends on a single variable, the signal is said to be one-dimensional, for example, a speech signal whose amplitude varies with time, depending on the spoken word and who speaks it.

When the function depends on two or more variables, the signal is said to be multidimensional, for example, an image (2-D signal).

1.2.2 Systems

A system is an entity that takes an input signal and produces an output signal. It is a combination and interconnection of several components to perform a desired task.

The system responds to one or more input quantities, called input signals or excitation, to produce one or more output quantities, called output signals or response.

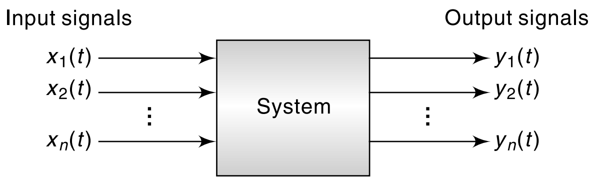


Figure 1.1 *Block diagram representation of a system*

1.3 DIFFERENT TYPES OF SYSTEMS

1. Continuous and Discrete Time Systems
2. Fixed and Time-varying Systems
3. Linear and Non-linear Systems
4. Lumped and Distributed Systems
5. Instantaneous and Dynamic Systems
6. Active and Passive Systems
7. Causal and Non-causal Systems
8. Stable and Unstable Systems
9. Invertible and Non-invertible Systems

1.3.1 Continuous and Discrete Time Systems

Signals are represented mathematically as functions of one or more independent variables. We classify signals as being either ***continuous-time*** (functions of a real-valued variable) or ***discrete-time*** (functions of an integer-valued variable).

In other words, a continuous-time signal has a value defined for each point in time and a discrete-time signal is defined only at discrete points in time.

To signify the difference, we (usually) use round parenthesis around the argument for continuous time signals, e.g., $x(t)$ and square brackets for discrete-time signals, e.g., $x[n]$. We will also use the notation x_n for discrete-time signals.

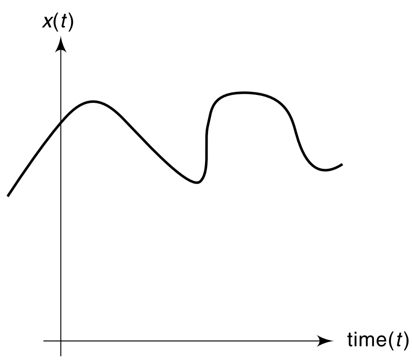


Figure 1.2(a) *Continuous-time signal*

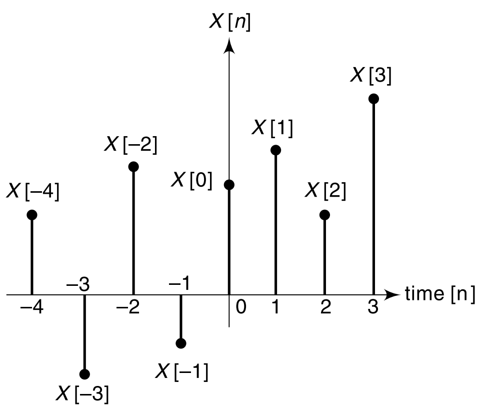


Figure 1.2(b) *Discrete-time signal*

A continuous-time system is a system which accepts only continuous-time signal to produce continuous-time internal and output signals. On the other hand, a discrete-time system is a system that transforms discrete-time input(s) into discrete-time output(s).

The examples given below are common in our daily life.

Continuous-time systems

(i) Atmospheric pressure as a function of altitude

(ii) Electric circuits composed of resistors, inductors, capacitors driven by continuous-time sources.

Discrete-time systems

(i) Weekly stock market index

(ii) Balance in a bank account from month to month.

The sequence of values of the discrete-time signal shown in Fig. 1.2(b) defined at discrete points in time are called ***samples*** and the spacing between them is called the ***sample spacing***. For equal sample spacing, the sequence of values are expressed as a function of the signed integer n as $x[n]$, where n is termed as a ***sequence of samples*** or ***sequence***, in short.

1.3.2 Time-Invariant (Fixed) and Time-Varying Systems

A system is time-invariant or fixed if the behaviour and characteristics of the system do not change with time. Otherwise, the system is time-varying.

Mathematically, if the input $x(t)$ gives the output $y(t)$, then the system is time-invariant if the input $x(t - T)$ gives the output $y(t - T)$ for any delay T. Hence, a time-shift of the input gives the same time-shift of the output.

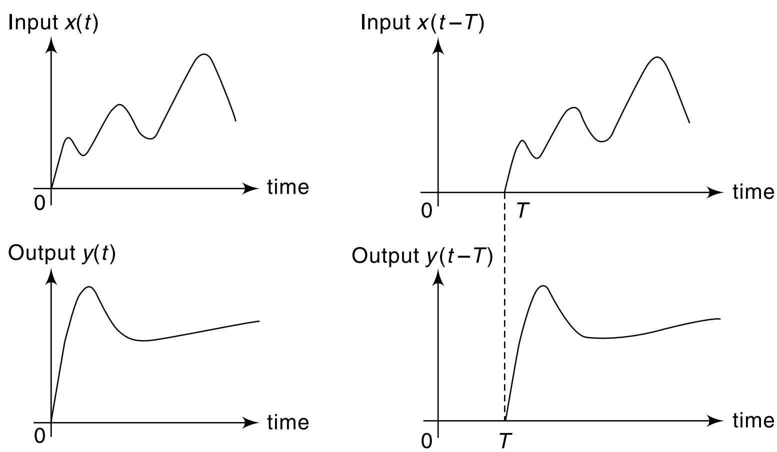


Figure 1.3 *Time-invariant system*

Whether a system is time-invariant or time-varying can be seen in the differential equation (or difference equation) describing it. ***Time-invariant systems are modeled with constant coefficient equations.*** A constant coefficient differential (or difference) equation means that the parameters of the system are not changing over time and an input now will give the same result as the input later.

Example 1.1 *A continuous system is modeled by the equation $y(t) = tx(t) + 4$, and a discrete-time system is modeled by $y[n] = x^2[n]$. Are these systems time-invariant?*

Solution *For continuous-time system:*

For input $x(t) = x_1(t)$, output $y_1(t) = tx_1(t) + 4$ (i)

For input $x(t) = x_1(t - T)$, output, $y_2(t) = tx_1(t - T) + 4$ (ii)

From the condition of time-invariance, the output should be,

$$y_1(t - T) = (t - T)x_1(t - T) + 4 \quad \text{(iii)}$$

From Eqs (ii) and (iii), $y_2(t) \neq y_1(t - T)$

Hence, the system is not time-invariant.

For discrete-time system:

For input $x_1[n]$, output $y_1[n] = x_1^2[n]$

For input $x_1[n - n_0]$, output $= x_1^2[n - n_0]$

From the condition of time-invariance, the shifted output $y_1[n - n_0] = x_1^2[n - n_0]$

Hence, the system is time-invariant.

1.3.3 Linear and Non-Linear Systems

A system, in continuous-time or discrete-time, is said to be linear, if it obeys the ***properties of superposition, i.e., additivity and homogeneity (or scaling),*** while a system is non-linear if it does not obey at least any one of these properties.

The superposition principle says that the output to a linear combination of input signals is the same linear combination of the corresponding output signals. Mathematically, the linearity condition is based on two properties.

1. *Additivity* If the input signals $x_1(t)$ and $x_2(t)$ correspond to the output signals $y_1(t)$ and $y_2(t)$, respectively, then the input signal $\{x_1(t) + x_2(t)\}$ should correspond to the output signal $\{y_1(t) + y_2(t)\}$.

2. *Homogeneity* If the input signal $x_1(t)$ corresponds to the output signal $y_1(t)$, then the input signal $a_1x_1(t)$ should correspond to the output signal $a_1y_1(t)$ for any constants a_1.

Combining these two properties, the condition for a linear system can be written as, if the input signals $x_1(t)$ and $x_2(t)$ correspond to the output signals $y_1(t)$ and $y_2(t)$, respectively, then the input signal $a_1x_1(t) + a_2x_2(t)$ should correspond to the output signal $a_1y_1(t) + a_2y_2(t)$ for any constants a_1 and a_2.

Example 1.2 Check whether the systems with the input-output relationship given below are linear.

(a) $y(t) = mx(t) + c$, (b) $y(t) = tx(t)$

Solution (a) For an input $x_1(t)$, output, $y_1(t) = mx_1(t) + c$

For an input $x_2(t)$, output, $y_2(t) = mx_2(t) + c$

For an input $\{x_1(t) + x_2(t)\}$, output, $y_3(t) = m\{x_1(t) + x_2(t)\} + c$ (i)

From the condition of linearity, the output should be

$$\{y_1(t) + y_2(t)\} = m\{x_1(t) + x_2(t)\} + 2c \quad \text{(ii)}$$

From Eqs (i) and (ii), we conclude that the *system is non-linear.*

(b) For an input $x_1(t)$, output, $y_1(t) = tx_1(t)$

For an input $x_2(t)$, output, $y_2(t) = tx_2(t)$

For an input $\{k_1x_1(t) + k_2x_2(t)\}$, output, $y_3(t) = t\{k_1x_1(t) + k_2x_2(t)\}$ (i)

where, k_1 and k_2 are any arbitrary constants.

From the condition of linearity, the output should be

$$\{k_1y_1(t) + k_2y_2(t)\} = k_1tx_1(t) + k_2tx_2(t) = t\{k_1x_1(t) + k_2x_2(t)\} \quad \text{(ii)}$$

From Eqs (i) and (ii), we conclude that the *system is linear.*

1.3.4 Lumped and Distributed Systems

All physical systems contain distributed parameters because of the physical size of the system components. For example, the resistance of a resistor is distributed throughout its volume.

However, if the ***size of the system components is very small with respect to the wavelength*** of the highest frequency present in the signals associated with it, then the system components behave as if it all were occurring at a point. This system is said to be ***lumped-parameter system.***

Distributed parameter systems are modeled as given below.

1. By partial differential equations if they are continuous-time systems
2. By partial difference equations if they are discrete-time systems.

Lumped parameter systems are modeled with ordinary differential or difference equations.

Example 1.3 Consider an electric power system of frequency 50 Hz. The wavelength of the signal is obtained as,

$$n\lambda = C \Rightarrow \lambda = \frac{C}{n} = \frac{3\times10^5}{50} = 6000 \text{ km}$$

Thus, the electrical system inside a room can be treated as a lumped-parameter system, but will be treated as distributed system for a long-distance transmission line.

1.3.5 Instantaneous (Static or Memoryless) and Dynamic Systems

An instantaneous or static or memoryless system is a system where the output at any specific time depends on the input at that time only. On the other hand, a dynamic system is one whose output depends on the past or future values of the input in addition to the present time.

A static system has no memory. Physically, it contains no energy-storage elements, whereas a dynamic system has one or more energy-storage element(s).

Example 1.4 An electrical circuit containing resistance R has the v–i relationship as, $v(t) = Ri(t)$, and so the system is static. But an electrical circuit containing capacitor C has the v–i relationship as, $v(t) = \frac{1}{C}\int_0^t i(t)dt$, and so the system is dynamic system.

1.3.6 Active and Passive Systems

A system having no source of energy is known as a passive system, for example, electric circuits containing resistance, capacitance, inductance, diodes, etc.

A system having source of energy together with other passive elements is known as an active system, for example, electric circuits containing voltage source or current source or op-amp, etc.

1.3.7 Causal and Non-causal Systems

A system is said to be causal if the output of the system depends only on the input at the present time and/or in the past, but not the future value of the input. Thus, a causal system is ***nonanticipative,*** i.e., output cannot come before the input.

On the other hand, the output of a non-causal system depends on the future values of the input.

Example 1.5 The moving-average system described by

$$y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n-2]\}$$

is causal, but the moving-average system described by

$$y[n] = \frac{1}{3}\{x[n+1] + x[n] + x[n-1]\}$$

is non-causal, since the output depends on the future value of the input $x[n+1]$.

It is obvious that the idea of future inputs does not have any physical meaning if we take time as our independent variable and for that reason all real-time systems are causal. However, for the case of image processing, the independent variable may be the pixels to the left and right (the "future") of the current position on the image, and thus, we can have a non-causal system.

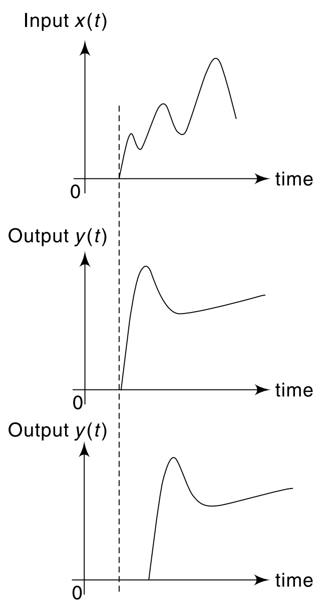


Figure 1.4(a) *Causal systems*

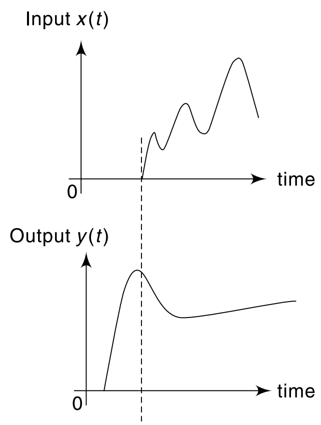


Figure 1.4(b) *Non-causal systems*

1.3.8 Stable and Unstable Systems

A stable system is one where the output does not diverge as long as the input does not diverge. A bounded input produces a bounded output. For this reason, this type of system is known as ***bounded input-bounded output (BIBO)*** stable system.

Mathematically, a stable system must have the following property:

If $x(t)$ be the input and $y(t)$ be the output, then the output must satisfy the condition.

$$|y(t)| \leq M_y < \infty; \quad \text{for all } t$$

whenever the input satisfy the condition

$$|x(t)| \leq M_x < \infty; \quad \text{for all } t$$

where, M_x and M_y both represent a set of finite positive numbers.

If these conditions are not met, i.e., the output of the system grows without limit (diverges) from a bounded input, then the system is **unstable**.

1.3.9 Invertible and Non-invertible Systems

A system is referred to as an invertible system if

(i) distinct inputs lead to distinct output, and

(ii) the input can be recovered from the output.

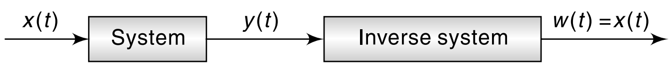


Figure. 1.5 *Invertible system*

The property of invariability is important in the design of communication systems. When a transmitted signal propagates through a communication channel, it becomes distorted due to the physical characteristics of the channel. An equalizer is connected in cascade with the channel in the receiver to compensate this distortion. By designing the equalizer to be inverse of the channel, the transmitted signal is restored.

1.4 DIFFERENT TYPES OF SIGNALS

Signals can be classified into different categories, as given below.

1. Continuous-time and discrete-time signals
2. Periodic and non-periodic signals
3. Odd and even signals

1.4.1 Continuous-time and Discrete-time Signals

Signals are represented mathematically as function of one or more independent variables. We classify signals as being either continuous-time (functions of a real-valued variable) or ***discrete-time*** (functions of an integer-valued variable).

In other words, a continuous-time signal has a value defined for each point in time and a discrete-time signal is defined only at discrete points in time.

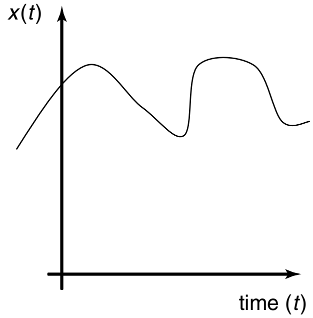


Figure 1.6(a) *Continuous-time signal*

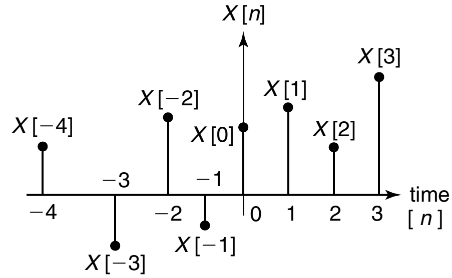


Figure 1.6(b) *Discrete-time signal*

To signify the difference, we (usually) use round parenthesis around the argument for continuous time signals, e.g., $x(t)$ and square brackets for discrete-time signals, e.g., $x[n]$. We will also use the notation x_n for discrete-time signals.

The sequences of values of the discrete-time signal shown in Fig. 1.6(b) defined at discrete points in time are called ***samples***, and the spacing between them is called the ***sample spacing***. For equal sample spacing, the sequences of values are expressed as a function of the signed integer n as $x[n]$, where n is termed as a ***sequence of samples*** or ***sequence***, in short.

1.4.2 Periodic and Non-Periodic Signals

A signal $f(t)$ is said to be periodic if

$$f(t) = f(t \pm nT) \tag{1.1}$$

where n is a positive integer and 'T' is the period. Thus, a periodic signal repeats itself every T seconds. Some periodic signals are shown in Fig. 1.7.

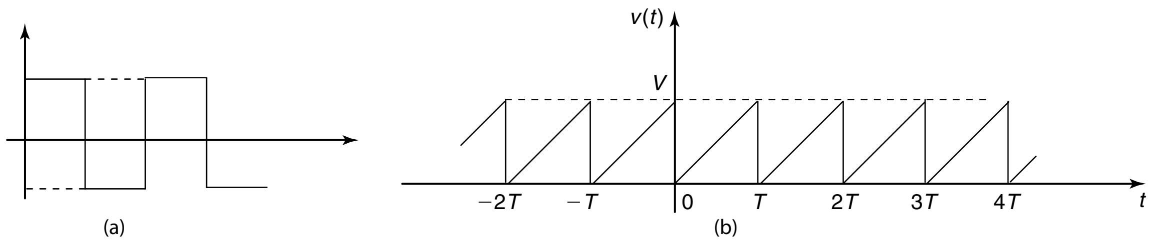


Figure 1.7 *Periodic signals*

A signal not satisfying the above condition of Eq. (1.1) is called a non-periodic signal. Examples of some non-periodic signals are e', t, etc.

1.4.3 Odd and Even Signals

A signal $f(t)$ is said to be odd if

$$f(t) = -f(-t) \tag{1.2}$$

Some examples of odd signals are sine functions, triangular functions and square function, as shown in Fig. 1.8.

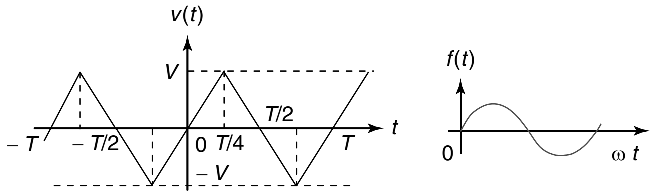


Figure 1.8 *Odd signals*

A signal $f(t)$ is said to be even if

$$f(t) = f(-t) \tag{1.3}$$

Some examples of even signals are shown in Fig. 1.5.

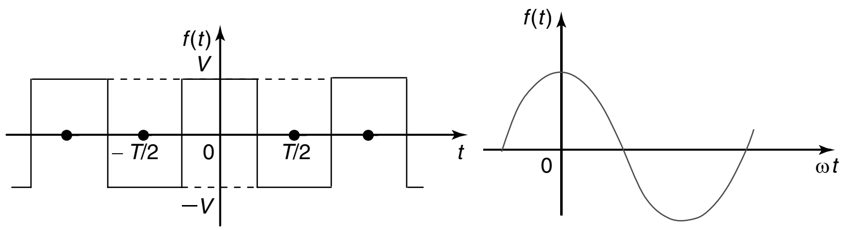


Figure 1.9 *Even signals*

Decomposition of a Signal into Odd and Even Components For any function $f(t)$, let the odd component be denoted by $f_0(t)$ and the even component by $f_e(t)$, so that,

$$f(t) = f_0(t) + f_e(t) \tag{1.4}$$

$$\therefore \quad f(-t) = f_0(-t) + f_e(-t) = f_0(t) + f_e(t) \tag{1.5}$$

[By Eq. (1.2 and (1.3)]

By addition and subtraction of Eqs (1.4) and (1.5), we get,

$$f_e(t) = \frac{1}{2}\left[f(t) + f(-t)\right] \tag{1.6}$$

$$f_0(t) = \frac{1}{2}\left[f(t) - f(-t)\right] \tag{1.7}$$

By these two equations, we can decompose a signal into its odd and even components.

Example 1.6 Decompose the following signal into its odd and even components.

Solution To find the even and odd components we need the folded signal, i.e., $f(-t)$, as shown in Fig. 1.10(b).

By point-by-point addition and subtraction, we get the even and odd components as shown in Fig. 1.10 (c) and Fig. 1.10 (d).

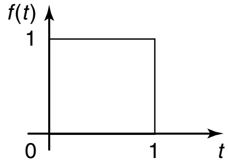


Figure 1.10(a) *Signal of Ex. 1.1*

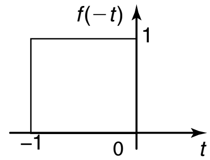
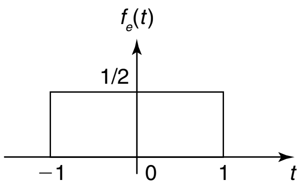


Fig. 1.10(b) *Folded Signal of Fig. 1.10(a)*



Figυρε 1.10(c) *Even component of signal of Fig. 1.10(a)*

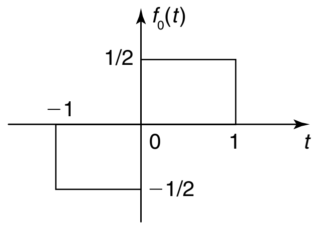


Figure 1.10(d) *Odd component of signal of signal of Fig. 1.10(a)*

1.5 SOME STANDARD SIGNALS

There are some standard signals which can be generated easily in the laboratory. Some of these standard signals are discussed below.

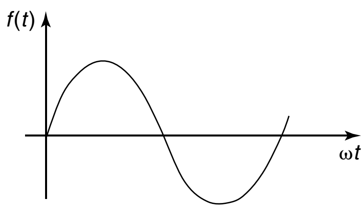


Figure 1.11(a) sin ωt

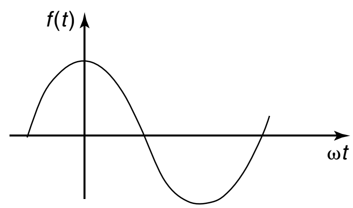


Figure 1.11(b) cos ωt

Sinusoidal Signal A sinusoid is a signal that has the form of a sine or cosine function.

We consider a sinusoidal voltage, $v(t) = V_m \sin \omega t$

where V_m is the amplitude,

ω_t is the argument of the sinusoid,

ω is the angular frequency of the sinusoid in rad/s $= 2\pi\phi = \dfrac{2\pi}{T}$,

and T is the time period of the sinusoid.

As the sinusoid is periodic, it repeats itself; such that

$$v(t) = v(t + T) = V_m \sin \omega\left(t + \frac{2\pi}{\omega}\right) = V_m \sin(\omega t + \pi) = V_m \sin \omega t$$

A shifted sinusoid can be written as, $v(t) = V_m \sin(\omega t + \phi)$ where ϕ is the phase of the sinusoid.

Thus, we see that, $-\sin \omega t = \sin(\omega t \pm 180°)$

$$-\cos \omega t = \cos(\omega t \pm 180°)$$

$$\pm\cos \omega t = \sin(\omega t \pm 90°)$$

$$\mp\sin \omega t = \cos(\omega t \pm 90°)$$

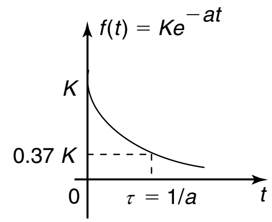


Figure 1.12 *Exponential signal*

Exponential Signal An exponential signal is a function of time defined as

$$f(t) = 0,\ t < 0$$

$$Ke^{-at},\ t \geq 0$$

where K and a are some real constants. The reciprocal of a has the dimension of time and is known as time constant, $\left(\tau = \frac{1}{a}\right)$. This is the time to reach 63.2% of the total change from the initial to final value.

Square Wave Signal A square wave is a type of waveform where the signal has only two levels. The signal switches between these levels at regular intervals and the switch is instant.

An ideal square wave signal is shown in the Fig. 1.13

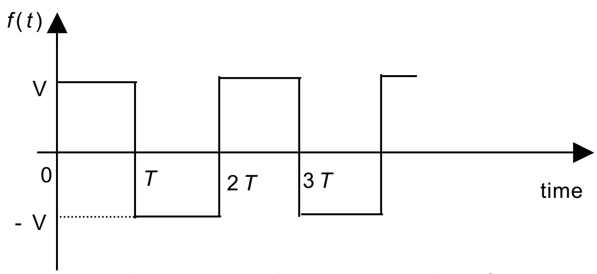


Figure 1.13 *Square Wave Signal*

Thus, square wave is a special kind of non-sinusoidal periodic signal with a time period T. Square waves are universally encountered in digital switching circuits and are naturally generated by binary logic devices.

Saw Tooth Wave

The **saw-tooth wave** is a kind of non-sinusoidal waveform. Since the wave has some resemblance to the teeth on the blade of a saw, it is named so.

In general, a saw-tooth wave rises upward and then drops sharply. However, a saw-tooth wave may also ramp downward and then rise sharply. This type of wave is known as a *reverse saw-tooth wave* or *inverse saw-tooth wave*.

A typical saw-tooth wave is shown in Fig. 1.14

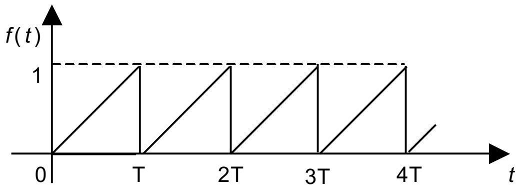


Figure 1.14 *Saw-tooth Wave*

Saw-tooth signal is used in many applications, such as in PMW modulator or oscilloscope sweep circuitry.

1.6 SINGULARITY SIGNALS

(a) Step signal,
(b) Ramp signal, and
(c) Impulse signal,

Figure 1.15 *(a) Unit step; (b) Step function of magnitude K*

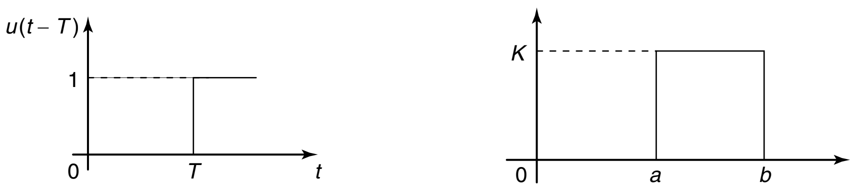


Figure 1.15 *(c) Shifted unit step function; (d) Gate function*

1.6.1 Step Signal

This function is also known as Heaviside unit function. It is defined as given below,

$$f(t) = u(t) = 1 \quad \text{for } t > 0$$
$$= 0 \quad \text{for } t < 0$$

and is undefined at $t = 0$.

A *step function of magnitude K* is defined as

$$f(t) = Ku(t) = K \quad \text{for } t > 0$$

$$= 0 \quad \text{for t} < 0$$

and in undefined at $t = 0$.

A *shifted or delayed unit step function* is defined as

$$f(t) = u(t - T) = 1 \quad \text{for } t > T$$

$$= 0 \quad \text{for } t < T$$

and is undefined at $t = T$.

Another function, called gate function, can be obtained from step function as follows.

Therefore, $g(t) = Ku(t - a) - Ku(t - b)$

1.6.2 Ramp Signal

A unit ramp function is defined as

$$f(t) = r(t) = t \quad \text{for } t \geq 0$$

$$= 0 \quad \text{for } t < 0$$

A ramp function of any slope K is defined as

$$f(t) = Kr(t) = Kt \quad \text{for } t \geq 0$$

$$= 0 \quad \text{for } t < 0$$

A shifted unit ramp function is defined as

$$f(t) = r(t - T) = t \quad \text{for } t \geq T$$

$$= 0 \quad \text{for } t < T$$

1.6.3 Impulse Signal

This function is also known as *Dirac Delta function*, denoted by d(t). This is a function of a real variable t, such that the function is zero everywhere except at the instant $t = 0$. Physically, it is a very sharp pulse of infinitesimally small width and very large magnitude, the area under the curve being unity.

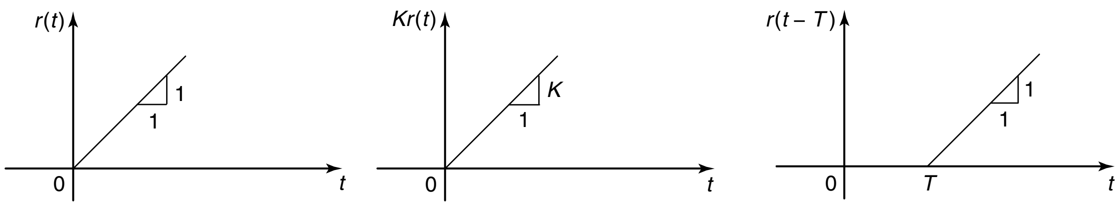


Figure 1.16 *(a) Unit ramp function; (b) Ramp function; (c) Shifted unit ramp function*

Consider a gate function as shown in Fig. 1.15.

The function is compressed along the time-axis and stretched a long the y-axis, keeping the area under the pulse as unity. As $a \to 0$, the value of $\frac{1}{a} \to \propto$ and the resulting function is known as impulse.

It is defined as $\delta(t) = 0$ for $t \neq 0$

$$\text{and} \quad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Also, $\delta(t) = \lim_{a \to} \frac{1}{a}[u(t) - u(t-a)] = \frac{d}{dt}[u(t)]$

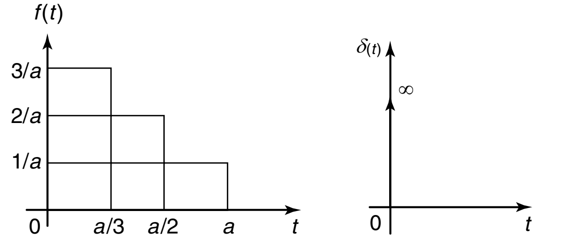


Figure 1.17 *(a) Generation of impulse function from gate function; (b) Impulse signal*

SOLVED PROBLEMS

1.1 Check whether the system defined by,

$$y(t) = \sin[x(t)]$$

is time-invariant.

Solution: For input $x(t) = x_1(t)$, output $y_1(t) = \sin[x_1(t)]$ ...(i)

For input $x(t) = x_1(t - T)$, output, $y_2(t) = \sin[x_1(t-T)]$ ...(ii)

From the condition of time-invariance, the output should be,

$$y_1(t-T) = \sin[x_1(t-T)] \quad \text{...(iii)}$$

From equations (ii) and (iii), $y_2(t) = y_1(t - T)$

Hence, the system is time-invariant.

1.2 Consider a system S with input $x[n]$ and output $y[n]$ related by,

$$y[n] = x[n]\{g[n] + g[n-1]\}$$

(a) If $g[n] = 1$, for all n, show that S is time-invariant.

(b) If $g[n] = n$, show that S is not time-invariant.

(c) If $g[n] = 1 + (-1)^n$, show that S is time-invariant.

Solution:

(a) If $g[n] = 1$, for all n, then $y[n] = x[n]\{1+1\} = 2x[n]$

For input $x[n] = x_1[n]$, output $y_1[n] = 2x_1[n]$ ...(i)

For input $x[n] = x_1[n - n_0]$, output, $y_2[n] = 2x_1[n - n_0]$ ...(ii)

From the condition of time-invariance, the output should be,

$$y_1[n-n_0] = 2x_1[n-n_0] \qquad \ldots\text{(iii)}$$

From equations (ii) and (iii), $y_2[n] = y_1[n - n_0]$
Hence, the system is time-invariant.

(b) If $g[n] = n$, then $y[n] = x[n]\{n+n-1\} = (2n-1)x[n]$

For input $x[n] = x_1[n]$, output $y_1[n] = (2n-1)x_1[n]$ ...(i)

For input $x[n] = x_1[n - n_0]$, output, $y_2[n] = (2n-1)x_1[n-n_0]$ ...(ii)

From the condition of time-invariance, the output should be,

$$y_1[n-n_0] = \{2(n-n_0)-1\}x_1[n-n_0] \qquad \ldots\text{(iii)}$$

From equations (ii) and (iii), $y_2[n] \neq y_1[n - n_0]$
Hence, the system is not time-invariant.

(c) If $g[n] = 1+ (-1)^n$, then $y[n] = x[n]\{1+(-1)^n+1+(-1)^{n-1}\} = 2x[n]$

This relation is same as that of part (a). Hence the system is time-invariant.

1.3 Consider the system S whose input and output are related by,

$$y(t) = x^2(t)$$

Check whether S is linear.

Solution: For an input $x_1(t)$, output, $y_1(t) = x_1^2(t)$
For an input $x_2(t)$, output, $y_2(t) = x_2^2(t)$

For an input $\{k_1x_1(t) + k_2x_2(t)\}$, output, $y_3(t) = [k_1x_1(t)+k_2x_2(t)]^2$ ...(i)

where, k_1 and k_2 are any arbitrary constants.
From the condition of linearity, the output should be

$$\{k_1y_1(t) + k_2y_2(t)\} = k_1x_1^2(t) + k_2x_2^2(t) \qquad \ldots\text{(ii)}$$

From equations (i) and (ii), we conclude that the *system is not linear.*

1.4 Consider the following discrete-time system with input-output relationships as given,

$$y[n] = \mathrm{Re}\{x[n]\}$$

Check whether the system is linear.

Solution: Let, the input be, $x_1[n] = r[n] + js[n]$

Therefore, the output is, $y_1[n] = \mathrm{Re}\{x_1[n]\} = \mathrm{Re}\{r[n]+js[n]\} = r[n]$

Now we consider scaling of the input $x_1[n]$ by a complex number, say, $(a+jb)$, i.e. the input is,

$$x_2[n] = (a+jb)x_1[n] = (a+jb)\{r[n]+js[n]\} = \{ar[n]-bs[n]\}+j\{br[n]+as[n]\}$$

Corresponding output is,

$$y_2[n] = \mathrm{Re}\{x_2[n]\} = \mathrm{Re}\{ar[n]-bs[n]\}+j\{br[n]+as[n]\} = ar[n]-bs[n]$$

But the scaled output for linear system is,

$$(a+jb)y_1[n] = ar[n] + jbr[n]$$

As he two outputs are not same, the system is not linear.

1.5 Consider a discrete-time system whose output $y[n]$ is the average of the three most recent values of the input signal, $x[n]$, given as,

$$y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n-2]\}$$

Show that the system is BIBO stable.

Solution: Let us assume that, $|x[n]| < M_x < \infty$ for all n,

$$\begin{aligned}
\therefore \quad |y[n]| &= \frac{1}{3}|x[n] + x[n-1] + x[n-2]| \\
&\le \frac{1}{3}\{|x[n]| + |x[n-1]| + |x[n-2]|\} \\
&\le \frac{1}{3}[M_x + M_x + M_x] \\
&\le M_x
\end{aligned}$$

Hence, the absolute value of the output signal $y[n]$ is always less than the maximum absolute value of the input signal $x[n]$ for all n; which shows that the system is stable.

1.6 Determine whether the following continuous-time systems are stable:

(a) $y(t) = tx(t)$ (b) $y(t) = x(t)\sin 100\pi t$

Solution: Here, let the input be bounded.

(a) $y(t) = tx(t)$

As $t \to \infty$, $y(t) \to \infty$ [since $x(t)$ is multiplied by t]

Hence, the system is an unstable system.

(b) $y(t) = x(t)\sin 100\pi t$

Here $x(t)$ is multiplied by $\sin 100\pi t$. We know that the value of sine varies between –1 and 1. Hence $y(t)$ is bounded as long as $x(t)$ is bounded. Hence the system is stable.

1.7 Determine whether the following continuous-time systems are causal or non-causal:

(a) $y(t) = x(t)\cos(t+1)$ (b) $y(t) = x(2t)$

(c) $y(t) = x(-t)$ (d) $\dfrac{dy(t)}{dt} + 10y(t) + 5 = x(t)$

(e) $y(t) = \int_{-\infty}^{t} x(t)dt$

Solution:

(a) $y(t) = x(t)\cos(t+1)$

Here, $y(t)$ depends on the present input $x(t)$. A cosine function can be evaluated at $(t + 1)$. Therefore, the system is causal.

(b) $y(t) = x(2t)$

Here, if $t = 5$, then $y(5) = x(10)$

Thus, the output $y(t)$ depends on the future input. Therefore, the system is non-causal.

(c) $y(t) = x(-t)$

Here, if $t = -3$, then $y(-3) = x(3)$

Thus, the output $y(t)$ depends on the future input. Therefore, the system is non-causal.

(d) $\dfrac{dy(t)}{dt} + 10y(t) + 5 = x(t)$

Here, $y(t)$ depends upon the present value of $x(t)$. Therefore, the system is causal.

(e) $y(t) = \int\limits_{-\infty}^{t} x(t)dt$

Here, $y(t)$ depends upon the present and the past values of $x(t)$, but not on the future value. Therefore, the system is causal.

1.8 Determine whether the following systems are invertible:

(a) $y(t) = 10x(t)$ (b) $y(t) = x^2(t)$

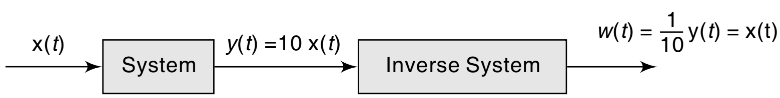
(c) $y(t) = x(t-n)$ (d) $y(t) = x(2t)$

Solution:

(a) $y(t) = 10x(t)$

For this system, the inverse system will be,

$$w(t) = \frac{1}{10}y(t)$$



Therefore, the system is an *invertible* system.

(b) $y(t) = x^2(t)$

The inverse system would be,

$$w(t) = \sqrt{y(t)} = \sqrt{x^2(t)} = \pm x(t)$$

Here, two outputs are possible: $= x(t)$ or $-x(t)$. This implies that there is no unique output for unique input. Therefore, the system is a *non-invertible* system.

(c) $y(t) = x(t-n)$

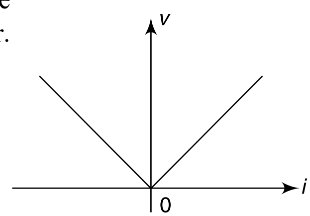
Here, the output is the delayed input, by 'n' samples. Clearly, the system is *invertible*. These can be another system for which the output is the advanced input by 'n' samples. The inverse system is,

$$w(t) = y(t+n)$$

(d) $y(t) = x(2t)$

Here, the input is compressed by a factor 2. Hence, there can be another system which will expand the input by the same factor. Hence the system is *invertible*. The inverse system is,

$$w(t) = y\left(\frac{t}{2}\right)$$



1.9 Determine whether the following systems are static or dynamic:

(a) $y(t) = e^{x(t)}$ (b) $y(t) = \frac{d}{dt}x(t)$

Solution:

(a) $y(t) = e^{x(t)}$

Here, the output depends on present input only. Hence the system is a *static system.*

(b) $y(t) = \frac{d}{dt}x(t)$

Here, the output depends on differentiation of the input. Calculation of differentiation depends on the present as well as past values. Therefore, the system is a *dynamic system*.

MULTIPLE-CHOICE QUESTIONS

1.1 The output $y(t)$ and the input $x(t)$ of a system are related by the equation $y(t) = mx(t) + c$, where m and c are constants. The system is
(a) linear
(b) non-linear
(c) may be linear or non-linear depending on $y(t)$ and $x(t)$
(d) none of the above

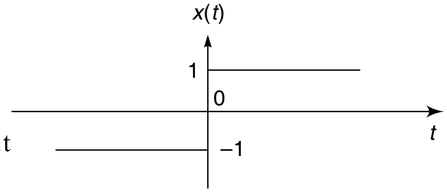
1.2 If the impulse response is realizable by delaying it appropriately and is bounded for bounded excitation, then the system is said to be
(a) causal and stable (b) causal but not stable
(c) non-causal but stable (d) non-causal, not stable

1.3 In a linear circuit, when the ac input is doubled, the ac output becomes
(a) one fourth (b) half (c) two times (d) four times

1.4 A circuit having an e.m.f. source or any energy source is
(a) active circuit (b) passive circuit (c) unilateral circuit (d) bilateral circuit

1.5 A network is said to be linear if and only if
(a) a response is proportional to the excitation function
(b) the principle of superposition applies
(c) the principle of homogeneity applies
(d) both the principles (b) and (c).

1.6 Consider the following data.
1. Input applied for $t < t_0$ 2. Input
applied for $t \geq t_0$



3. State of the network at $t = t_0$ 4. State of the network at $t < t_0$

Among these, those needed for determining the response of a linear network for $t > t_0$ would include

(a) 1, 3 and 4 (b) 2, 3 and 4 (c) 2 and 3 (d) 2 and 4.

1.7 An excitation is applied to a system at $t = T$ and its response is zero for $-\infty < t < T$. Such a system is

(a) non-causal system (b) stable system (c) causal system (d) unstable system.

1.8 The elements which are not capable of delivering energy by its own are known as

(a) unilateral elements (b) non-linear elements
(c) passive elements (d) active elements.

1.9 The v–i characteristic of an element is shown in the given figure. The element is

(a) non-linear, active, non-bilateral
(b) linear, active, non-bilateral
(c) non-linear, passive, non-bilateral
(d) non-linear, active, bilateral

1.10 What is the input-output relation of the causal moving-average system (discrete time)?

(a) $y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n-2]\}$ (b) $y[n] = \frac{1}{3}\{x[n-1] + x[n] + x[n+1]\}$

(c) $y[n] = \frac{1}{3}\left\{x[n] + (x[n])^2 + (x[n])^{1/2}\right\}$ (d) $y[n] = \frac{1}{3}\{x[n] + x[n+1] + x[n+2]\}$

1.11 Which one of the following is a linear system?

(a) $y(t) = 2u(t)$ (b) $y(t) = 2u(t) + 5$ (c) $y(t) = 2u^2(t)$ (d) $y(t) = 2u^2(t) + 5$

1.12 A function $f(\ .\ )$ is linear under the conditions (s)

(a) $f(x_1 + x_2) = f(x_1) + f(x_2)$ only.

(b) $f(kx) = kf(x)$ only.

(c) $f(x_1 + x_2) = f(x_1) + f(x_2)$ and $f(kx) = kf(x)$.

(d) $f(x_1 + x_2) = f(x_1) + f(x_2)$ or $f(kx) = kf(x)$.

1.13 The v-i characteristic of a resistor is $i = 2v^2$. The resistor is

(a) linear, passive, bilateral (b) non-linear, passive, bilateral
(c) non-linear, active, bilateral (d) non-linear, active, unilateral

1.14 The system $y(t) = tx(t) + 4$ is

(a) non-linear, time-varying and unstable. (b) linear, time-varying and unstable.
(c) non-linear, time-invariant and unstable. (d) non-linear, time-varying and stable.

1.15 The following is true

(a) A finite signal is always bounded.
(b) A bonded signal always possesses finite energy.
(c) A bounded signal is always zero outside the interval [$-t_0$, t_0] for some t_0.
(d) A bounded signal is always finite.

1.16 The function $x(t)$ is shown in the figure. Even and odd parts of a unit-step function $u(t)$ are respectively,

(a) $\frac{1}{2}, \frac{1}{2}x(t)$ (b) $-\frac{1}{2}, \frac{1}{2}x(t)$

(c) $\frac{1}{2}, -\frac{1}{2}x(t)$ (d) $-\frac{1}{2}, -\frac{1}{2}x(t)$

1.17 The input and output of a continuous time system are respectively denoted by $x(t)$ and $y(t)$. Which of the following description corresponds to a causal system?

(a) $y(t) = x(t-2) + x(t-4)$ (b) $y(t) = (t-4)x(t+1)$

(c) $y(t) = (t+4)x(t-1)$ (d) $y(t) = (t+5)x(t+5)$

1.18 The impulse response $h(t)$ of a linear time-invariant continuous time system is described by $h(t) = \exp(\alpha t)\, u(t) + \exp(\beta t) u(-t)$, where, $u(t)$ denotes the unit step function, and α and β are real constants. The system is stable if

(a) α is positive and β is positive (b) α is negative and β is negative

(c) α is positive and β is negative (d) α is negative and β is positive

1.19 Which of the following represent a stable system?

1. Impulse response of the system decreases exponentially.
2. Area within the impulse response if finite.
3. Eigen values of the system are positive and real.
4. Roots of the characteristic equation of the system are real and negative.

Select the correct answer using the codes given below.

(a) 1 and 4 (b) 1 and 3 (c) 2, 3 and 4 (d) 1, 2 and 4

EXERCISES

1.1 A discrete-time system is modeled by,

$$y[n] = x^2[n]$$

Is this system time-invariant?

1.2 Consider the systems S whose input and output are related by,

(a) $y(t) = t\,x(t)$

(b) $y(t) = x(t)\, x(t-1)$

(c) $y(t) = x^2(t)$

(d) $y = mx + c$

Check whether S is linear.

1.3 Consider the following discrete-time systems with input-output relationships as given

(a) $y[n] = 2x[n] + 3$

(b) $y[n] = n\,x[n]$

Check whether the systems are linear.

SHORT-ANSWER TYPE QUESTIONS

1.1 What is a system? What are the different types of systems? Give their definitions.

1.2 Define the following and give examples of each.

(a) Continuous and discrete system.

(b) Time-invariant and time-varying system.

(c) Lumped and distributed system.

(d) Instantaneous (Static or Memoryless) and dynamic system.
(e) Causal and non-causal system.
(f) Active and passive system.

1.5 (a) What are the conditions for a system to be a linear system?
(b) Give the conditions for a BIBO stability of a system.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

1.1 (b)	1.2 (a)	1.3 (c)	1.4 (a)	1.5 (d)	1.6 (c)	1.7 (c)
1.8 (c)	1.9 (b)	1.10 (a)	1.11 (a)	1.12 (c)	1.13 (b)	1.14 (a)
1.15 (b)	1.16 (a)	1.17 (a)	1.18 (d)	1.19 (b)		

CHAPTER

2

Introduction to Circuit-Theory Concepts

2.1 INTRODUCTION

The fundamental theory on which many branches of electrical engineering, such as electric power, electric machines, control, electronics, computers, communications and instrumentation are built is the electric circuit theory. Thus, it is essential to have a proper grounding with electric circuit theory as the base. An electric circuit is the interconnection of electrical elements.

2.2 SOME BASIC TERMINOLOGIES OF ELECTRIC CIRCUITS

2.2.1 Concept of Electric Charge

The most basic quantity in an electric circuit is the *electric charge q*. Electric charge is a fundamental conserved property of some subatomic particles, which determines their electromagnetic interaction. Electrically charged matter is influenced by, and produces, electromagnetic fields.

It is known that an atom consists of a positively charged nucleus surrounded by negatively charged electrons. In a neutral atom, the total charge of the nucleus is equal to the total charge of the electrons. When electrons are removed from a substance, the substance becomes positively charged and if excess electrons are given to a substance, it becomes negatively charged.

The SI unit of charge is Coulomb (C). The charge of an electron is 1.602×10^{-19}C. Thus, one Coulomb charge is defined as the charge possessed by $\left(\dfrac{1}{1.602 \times 10^{-19}}\right)$electrons.

$\therefore$ 1 Coulomb charge = charge of 6.24×10^{18} electrons

The total electric charge of an isolated system remains constant regardless of changes within the system itself. This is known as the law of *conservation of charge*. The law of *conservation of charge* states that charge can neither be created nor destroyed.

The electric charge of a macroscopic object is the sum of the electric charges of its constituent particles. Often, the net electric charge is zero, because it is favourable for the number of electrons in every atom to equal the number of protons (or, more generally, for the number of anions, or negatively charged atoms, in every molecule to equal the number of cations, or positively charged atoms). When the net electric charge is non-zero and motionless, the phenomenon is known as *static electricity*. Even when the net charge is zero, it can be distributed non-uniformly due to an external electric field, or due to molecular motion; in such cases the material is said to be polarised. The charge due to the polarisation is known as *bound charge*, while the excess charge brought from outside is called *free charge*. The motion of charged particles (e.g., of electrons in metals) in a particular direction is said to constitute an *electric current*

2.2.2 Conductors, Insulators and Semiconductors

In some materials, there is a large number of free electrons or loosely bound valence-band electrons present. These electrons are easily knocked out of their orbit and easily constitute a large current. Such materials are known as *conductors*. Almost all metals and some liquids are good conductors.

In some materials, no free electrons are available; the valence-band electrons are tightly bound to the nucleus. Such materials are known as *insulators*. Examples of some insulators include glass, mica, plastics, etc.

In between the limits of these two major categories is a third general class of materials called *semiconductors*; where there are no such free electrons present, but free electrons can easily be created by adding some impurities. Examples of some insulators include germanium, silicon, etc. For example, germanium, a semiconductor, has approximately one trillion times (1×10^{12}) the conductivity of glass, an insulator, but has only about one thirty-millionth (3×10^{-8}) part of the conductivity of copper, a conductor.

2.2.3 Concept of Electric Current

The phenomenon of transferring electric charge from one point in a circuit to another is described by the term *electric current*. Electric current is defined as the rate of flow of electric charges or electrons through a cross-sectional area. By convention, the electric current flows in the opposite direction to the electrons.

If Q amount of charges flow through an area in time t, then the current is given as,

$$I = \frac{Q}{t} \tag{2.1}$$

or in differential form,

$$i = \frac{dq}{dt} \tag{2.2}$$

and the charge transferred between time t_0 and t is given by

$$q = \int_{t_0}^{t} i dt \tag{2.3}$$

As Q is expressed in Coulomb, the unit of electric current is Coulomb per second and it is given the name Ampere (A). Thus,

1 A current = flow of 6.24×10^{18} electrons per second through an area

2.2.4 Current Density

Current density at any point is a vector whose magnitude is the electric current per unit cross-sectional area and whose direction is normal to the cross-sectional are, i.e., $\vec{J} = \frac{I}{A}\hat{n}$. Its unit is Ampere per square meter (A/m^2).

2.2.5 Concept of Electric Potential and Potential Difference

To move an electron in a conductor in a particular direction, or to create a current, requires some work or energy. This work is done by the *potential* or the *potential difference*. This is also known as *voltage difference* or *voltage* (with reference to a selected point such as *earth*). The unit of potential is volt.

The potential of a point is 1Volt if 1Joule work is done in bringing 1Coulomb charge from infinity to that point.

The voltage V_{ab} between two points a and b is the *energy* (or work) ***w*** required to move a unit positive charge from ***a*** to ***b***. [Unit of voltage is *volt* (V).]

$$V_{ab} = \frac{dw}{dq} \tag{2.4}$$

The potential difference between two points is 1Volt if 1Joule work is done to displace 1 Coulomb charge from one point to the other.

2.2.6 Drift Velocity

Electric current is the number of coulombs of charge which pass a point in the circuit per unit time. Because of its definition, it is often confused with the quantity drift velocity. **Drift velocity** refers to the average distance traveled by a charge carrier per unit time. Like the velocity of any object, the drift velocity of an electron is the distance to time ratio. The path of a typical electron through a wire could be described as a rather chaotic, zigzag path characterised by collisions with fixed atoms. Each collision results in a change in direction of the electron.

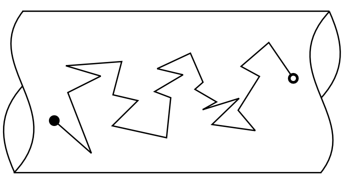
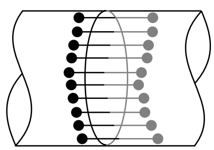


Figure 2.1 (a) *Typical path of an electron*



A high current results from
many charge carries passing
through a given cross section
of wine on a circuit

Figure 2.1(b) *Current is constituted by flow of many charge carriers through a cross section*

The net effect of these collisions results in slow drifting of the electrons with a constant average drift velocity. The drift velocity is defined as the vector average velocity of the charge carriers moving under the influence of electric field.

Mathematically, if n number of charge carriers (electrons) with charge Q each passes through an area A with drift velocity v, then the current is given by, $I = nQvA$.

2.2.7 Concept of Electromotive Force (EMF)

The phenomenon of electric current depends on the presence of free electrons. If a material is having a large number of free electrons, these electrons will always move in random directions as shown in Fig. 2.1 (a). If an external effort is applied to the material, it is possible to drift all the electrons in a definite direction as shown in Fig. 2.1 (b). Such an external factor is known as *electromotive force* (emf). In other words, the voltage or potential of an electrical energy source is known as emf.

When we say something as electrical energy source, we mean that the energy is converted from non-electrical form (such as, mechanical, chemical, tidal, etc) into electrical form. Please note that *emf is not a force, but it is the energy or work done.*

2.2.8 Electric Circuits and Networks

Any combination and interconnection of network elements like resistor or inductor or capacitor or electrical energy sources are known as 'networks'. However, a closed energised network is known as 'circuit'. A network need not contain an energy source; but a circuit must contain an energy source. Therefore, it can be stated that all circuits are networks, but all networks are not circuits.

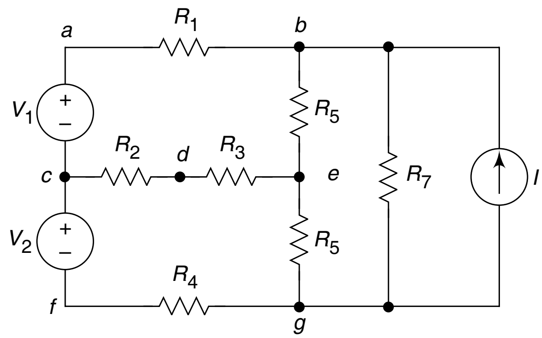


Figure 2.2 *Circuit illustrating terminologies*

2.2.9 Loop and Mesh

A loop or mesh denote a closed path obtained by starting at a node and returning back to the same node through a set of connected circuit elements without passing through any intermediate node more than once. However, the difference between mesh and loop is that a mesh does not contain any other loop within it, i.e., mesh is the smallest loop. In Fig. 2.2, some loops are *a-b-e-d-c-a, a-b-e-g-f-c-a, c-d-e-b-g-f-c*, etc; and some meshes are: *a-b-e-d-c-a, c-d-e-g-f-c, g-e-b-g* (through R_7) and *g-e-b-g* (through I).

2.2.10 Node and Branch

A node is a point in a circuit where two or more circuit elements join. A node is said to be an essential node if it joins three or more elements. Examples of nodes for Fig. 2.2 are *a, b, c, d, e, f* and *g* and examples of some essential node of Fig. 2.2 are *b, c, e* and *g*.

A branch is a path that connects two nodes. Those paths that connect essential nodes without passing through an essential node are known as essential branches. Examples of branches of Fig. 2.2 are: V_1, R_1, R_2, R_3, V_2, R_4, R_5, R_6, R_7 and I and some essential branches of Fig. 2.2 are *c-a-b, c-d-e, c-f-g, b-e, e-g, b-g* (through R_7), and *b-g* (through I).

2.3 DIFFERENT NOTATIONS

C	Capacitance	Farad, F
E	Voltage source	Volt, V
e	Instantaneous value of E	Volt, V
G	Conductance	Siemens, S
I	Current	Ampere, A
i	Instantaneous current	Ampere, A
k	Coefficient	Unit less
L	Inductance	Henry, H
M	Mutual inductance	Henry, H
N	Number of turns	Unit less
P	Power	Watt, W
Q	Charge	Coulomb, C
q	Instantaneous charge	Coulomb, C
R	Resistance	Ohm, Ω
τ	Time constant	Second
t	Instantaneous time	Second
V	Voltage drop	Volt, V
v	Instantaneous V	Volt, V
W	Energy	Joule, J
ϕ	Magnetic flux	Weber, Wb
Ψ	Magnetic linkage	Weber, Wb
φ	Instantaneous Ψ	Weber, Wb

2.4 BASIC CIRCUIT ELEMENTS

(i) Active and Passive Elements Electric Circuits consist of two basic types of elements. These are the *active elements* and the *passive elements*.

An ***active element*** is capable of generating electrical energy. (In electrical engineering, generating or producing electrical energy actually refers to conversion of electrical energy from a non-electrical form to electrical form. Similarly, energy loss would mean that electrical energy is converted to a non-useful form of energy and not actually lost. *Principle of Conservation of Mass and Energy*).

Examples of active elements are *voltage source* (such as a battery or generator) and *current source*. Most sources are independent of other circuit variables, but some elements are *dependent* (modeling elements such as transistors and operational amplifiers would require dependent sources).

Active elements may be ***ideal*** voltage sources or current sources. In such cases, the particular generated voltage (or current) would be independent of the connected circuit.

A ***passive element*** is one which does not generate electricity but either consumes it or stores it. *Resistors, Inductors* and *Capacitors* are simple passive elements. Diodes, transistors etc. are also passive elements.

Passive elements may either be *linear* or *non-linear*. Linear elements obey a straight-line law. For example, a linear resistor has a linear *voltage* vs *current* relationship which passes through the origin ($V = R.I$). A linear inductor has a linear *flux* vs *current* relationship which passes through the origin ($\phi = k\,I$) and a linear capacitor has a linear *charge* vs *voltage* relationship which passes through the origin ($q = CV$). [R, k and C are constants].

Resistors, inductors and capacitors may be linear or non-linear, while diodes and transistors are always nonlinear.

(ii) Linear Element A circuit/network element is linear if the relation between current and voltage involves a constant coefficient.

Examples Voltage-current relationship of resistor, inductor and capacitor (both with zero initial conditions) are linear $\left(v = ri, v = L\dfrac{di}{dt}, v = \dfrac{1}{c}\int idt\right)$ Hence, the elements are linear.

Diode and transistors are non-linear devices having non-linear characteristics.

(iii) Bilateral System In a bilateral system, the same relationship between current and voltage exists for current flowing in either direction. On the other hand, a unilateral system has different current-voltage relationships for the two possible directions of current, as in diode.

2.5 PASSIVE CIRCUIT ELEMENTS

2.5.1 Electrical Resistance

Electrical resistance is a measure of the degree to which an object opposes an electric current through it.

The SI unit of electrical resistance is ohm (Ω). Its reciprocal quantity is **electrical conductance** measured in Siemens. Electrical resistance shares some conceptual parallels with the mechanical notion of friction.

The resistance of an object determines the amount of current through the object for a given voltage across the object.

$$I = \frac{V}{R} \tag{2.5}$$

where, R is the resistance of the object, measured in ohm equivalent to $J.s/C^2$

V is the voltage across the object, measured in volt
I is the current through the object, measured in ampere

For a wide variety of materials and conditions, the electrical resistance does not depend on the amount of current through or the amount of voltage across the object, meaning that the resistance R is constant.

Resistance of a Conductor

DC Resistance As long as the current density is totally uniform in the conductor, the DC resistance R of a conductor of regular cross section can be computed as

$$R = \rho \frac{l}{A} \tag{2.6}$$

where, l is the length of the conductor, measured in meter,
A is the cross-sectional area, measured in square meter,
ρ (Greek: rho) is the electrical resistivity (also called *specific electrical resistance*) of the material, measured in ohm metre. Resistivity is a measure of the material's ability to oppose the flow of electric current.

For practical reasons, almost any connections to a real conductor will almost certainly mean the current density is not totally uniform. However, this formula still provides a good approximation for long thin conductors such as wires.

AC Resistance If a wire conducts high-frequency alternating current then the effective cross-sectional area of the wire is reduced. This is because of the skin effect.

This formula applies to isolated conductors. In a conductor close to others, the actual resistance is higher because of the proximity effect.

Resistor A **resistor** is a two-terminal electrical or electronic component that resists an electric current by producing a voltage drop between its terminals in accordance with Ohm's law:

$$R = \frac{V}{I} \tag{2.7}$$

The *electrical resistance* is equal to the voltage drop across the resistor divided by the current through the resistor. Resistors are used as part of electrical networks and electronic circuits.

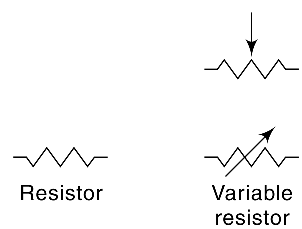


Figure 2.3 *Resistor symbols*

Energy in a Resistor Instantaneous power absorbed in the resistor,

$$p = vi = iR \times i = i^2 R \text{ (in Watt)} \tag{2.8}$$

Therefore, the energy converted into heat energy is given by,

$$W = \int_0^t p\,dt = \int_0^t i^2 R\,dt = i^2 Rt \text{ (in Joule)} \tag{2.9}$$

Series and Parallel Arrangements of Resistors Resistors in a parallel configuration each have the same potential difference (voltage). To find their total equivalent resistance (R_{eq}):

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n} \tag{2.10}$$

The parallel, property can be represented in equations by two vertical lines "||" (as in geometry) to simplify equations. For two resistors,

$$R_{eq} = R_1 || R_2 = \frac{R_1 R_2}{R_1 + R_2} \tag{2.11}$$

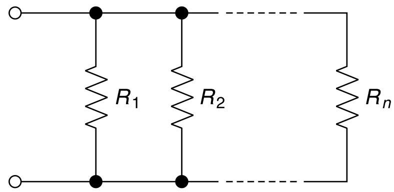


Figure 2.4 *Parallel arrangement of resistors*

The current through resistors in series stays the same, but the voltage across each resistor can be different. The sum of the potential differences (voltage) is equal to the total voltage. To find their total resistance:

$$R_{eq} = R_1 + R_2 + \ldots + R_n \tag{2.12}$$

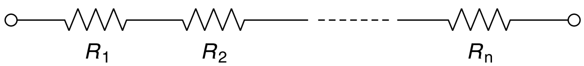


Figure 2.5 *Series arrangement of resistors*

A resistor network that is a combination of parallel and series can sometimes be broken up into smaller parts. For instance,

$$R_{eq} = (R_1 || R_2) + R_3 = \frac{R_1 R_2}{R_1 + R_2} + R_3 \tag{2.13}$$

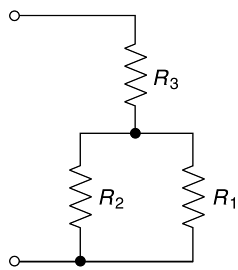


Figure 2.6 *Series-parallel arrangement of resistors*

Current Division by Parallel Resistances When a total current I_P is passed through parallel connected resistances R_1 and R_2, the voltage V_P which appears across the parallel circuit is:

$$V_P = I_P R_P = I_P R_1 R_2 / (R_1 + R_2)$$

The currents I_1 and I_2 which pass through the respective resistances R_1 and R_2 are:

$$I_1 = V_P / R_1 = I_P R_P / R_1 = I_P R_2 / (R_1 + R_2)$$
$$I_2 = V_P / R_2 = I_P R_P / R_2 = I_P R_1 / (R_1 + R_2)$$

In general terms, for resistances R_1, R_2, R_3, …, R_n (with conductances G_1, G_2, G_3, …, G_n) connected in parallel:

$$V_P = I_P R_P = I_P / G_P = I_P / (G_1 + G_2 + G_3 + \ldots)$$
$$I_n = V_P / R_n = V_P G_n = I_P G_n / G_P = I_P G_n / (G_1 + G_2 + G_3 + \ldots)$$

where $G_n = 1/R_n$ and I_n is the current through n^{th} resistance R_n

Note that the highest current passes through the highest conductance (with the lowest resistance).

2.5.2 Capacitance

Capacitance is a measure of the amount of electric charge stored (or separated) for a given electric potential. The most common form of charge storage device is a two-plate capacitor. If the charges on the plates are $+Q$ and $-Q$, and V gives the voltage difference between the plates, then the capacitance is given by

$$C = \frac{O}{V} \tag{2.14}$$

The SI unit of capacitance is Farad; 1 Farad = 1 Coulomb per volt.

The capacitance of the majority of capacitors used in electronic circuits is several orders of magnitude smaller than the farad. The most common units of capacitance in use today are milli-farad (mF), microfarad (μF), the nano-farad (nF) and the pico-farad (pF)

The capacitance can be calculated if the geometry of the conductors and the dielectric properties of the insulator between the conductors are known. For example, the capacitance of a *parallel-plate* capacitor constructed of two parallel plates of area A separated by a distance d is approximately equal to the following:

$$C = \varepsilon \frac{A}{d} \tag{2.15}$$

where

C is the capacitance in farad, F

ε is the permittivity of the insulator used (or ε_0 for a vacuum)

A is the area of each plate, measured in square meter

d is the separation between the plates, measured in meter

The equation is a good approximation if d is small compared to the other dimensions of the plates.

Capacitor A **capacitor** is an electrical device that can store energy in the electric field between a pair of closely-spaced conductors. When current is applied to the capacitor, electric charges of equal magnitude, but opposite polarity, build up on each conductor.

Capacitors are used in electrical circuits as energy-storage devices. They can also be used to differentiate between high-frequency and low-frequency signals and this makes them useful in electronic filters.

Capacitors are occasionally referred to as **condensers**. This is now considered an antiquated term.

Properties of Capacitance The relation between charge and voltage in a capacitor is written as,

$$Q = CV \tag{2.16}$$

The current, $$i = \frac{dQ}{dt} = C\frac{dV}{dt} + V\frac{dC}{dt}$$

In most physical cases, the capacitance is constant with time.

$$\therefore \quad i = C\frac{dW}{dt} \tag{2.17}$$

$$\therefore \quad dV = \frac{1}{C} i dt$$

Taking integration on both sides,

$$\int_0^{v_c} dV = \frac{1}{C}\int_0^t i dt$$

or $$v_c(t) = \frac{1}{C}\int_0^t i(t)dt + v_c(0)$$

where, $v_c(0)$ is the initial voltage across the capacitor. For zero initial voltage,

$$v_c = \frac{1}{C}\int_0^t i\,dt \tag{2.18}$$

From equation (2.17), it is clear that for an abrupt change of voltage across the capacitor, the current becomes infinite. Also, from equation (2.18), it is observed that for a finite change of current in zero time the integral must be zero.

Therefore, *the voltage acorss a capacitor cannot change instantaneously.*

Explanation of Initial Voltage $v_c(0)$ It is possible that this capacitor might have been used in some other circuit earlier, where it absorbed some energy and then it was disconnected. Because of its non-dissipative nature, the energy was stored within the capacitor. Now, as this capacitor is connected to a circuit, it gets some path to release its stored energy. Here, this stored energy is represented by the initial voltage $v_c(0)$.

Energy Stored in Capacitors The energy (measured in Joule) stored in a capacitor is equal to the *work* done to charge it. Consider a capacitance C, holding a charge $+q$ on one plate and $-q$ on the other. Moving a small element of charge dq from one plate to the other against the potential difference $V = q/C$ requires the work dW.

$$dW = \frac{q}{C}dq \tag{2.19}$$

where, W is the work measured in Joule
q is the charge measured in Coulomb
C is the capacitance, measured in Farad

We can find the energy stored in a capacitance by integrating this equation. Starting with an uncharged capacitance ($q = 0$) and moving charge from one plate to the other until the plates have charge $+Q$ and $-Q$ requires the work W.

$$W_{\text{charging}} = \int_0^Q \frac{q}{C}dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = W_{\text{stored}} \tag{2.20}$$

Combining this with the Eq. (2.15) for the capacitance of a flat-plate capacitor, we get

$$W_{\text{stored}} = \frac{1}{2}CV^2 = \frac{1}{2}\varepsilon\frac{A}{d}V^2 \tag{2.21}$$

where W is the energy measured in Joule,
C is the capacitance, measured in Farad,
V is the voltage measured in Volt.

Series or Parallel Arrangements of Capacitors Capacitors in a parallel configuration each have the same potential difference (voltage). Their total capacitance (C_{eq}) is given by

$$C_{eq} = C_1 + C_2 + \ldots + C_n$$

The reason for putting capacitors in parallel is to increase the total amount of charge stored. In other words, increasing the capacitance also increases the amount of energy that can be stored. Its expression is

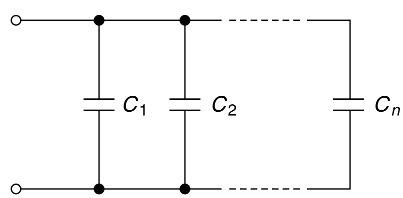


Figure 2.7 *Parallel arrangement of capacitors*

$$E_{\text{stored}} = \frac{1}{2}CV^2 \tag{2.22}$$

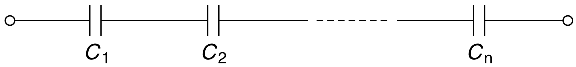


Figure 2.8 *Series arrangement of capacitors*

The current through capacitors in series stays the same, but the voltage across each capacitor can be different. The sum of the potential differences (voltage) is equal to the total voltage. Their total capacitance is given by

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n} \tag{2.23}$$

In parallel, the effective area of the combined capacitor has increased, increasing the overall capacitance. In series, the distance between the plates has effectively been increased, reducing the overall capacitance.

Voltage Division by Capacitances

In series connection When a total voltage $\boldsymbol{E_S}$ is applied to series connected capacitances $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$, the charge $\boldsymbol{Q_S}$ which accumulates in the series circuit is:

$$\boldsymbol{Q_S = i_S\,dt = E_S C_S = E_S C_1 C_2/(C_1 + C_2)}$$

The voltages $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ which appear across the respective capacitances $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$ are

$$\boldsymbol{V_1 = i_S\,dt/C_1 = E_S C_S/C_1 = E_S C_2/(C_1 + C_2)}$$
$$\boldsymbol{V_2 = i_S\,dt/C_2 = E_S C_S/C_2 = E_S C_1/(C_1 + C_2)}$$

In general terms, for capacitances $\boldsymbol{C_1}$, $\boldsymbol{C_2}$, $\boldsymbol{C_3}$, ... connected in series

$$\boldsymbol{Q_S = i_S\,dt = E_S C_S = E_S/(1/C_S) = E_S/(1/C_1 + 1/C_2 + 1/C_3 + \ldots)}$$
$$\boldsymbol{V_n = i_S\,dt/C_n = E_S C_S/C_n = E_S/C_n(1/C_S) = E_S/C_n(1/C_1 + 1/C_2 + 1/C_3 + \ldots)}$$

Note that the highest voltage appears across the lowest capacitance.

In parallel connection When a voltage $\boldsymbol{E_P}$ is applied to parallel connected capacitances $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$, the charge $\boldsymbol{Q_P}$ which accumulates in the parallel circuit is

$$\boldsymbol{Q_P = i_P dt = E_P C_P = E_P(C_1 + C_2)}$$

The charges $\boldsymbol{Q_1}$ and $\boldsymbol{Q_2}$ which accumulate in the respective capacitances $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$ are:

$$\boldsymbol{Q_1 = i_1 dt = E_P C_1 = Q_P C_1/C_P = Q_P C_1/(C_1 + C_2)}$$
$$\boldsymbol{Q_2 = i_2 dt = E_P C_2 = Q_P C_2/C_P = Q_P C_2/(C_1 + C_2)}$$

In general terms, for capacitances $\boldsymbol{C_1}$, $\boldsymbol{C_2}$, $\boldsymbol{C_3}$, … connected in parallel:

$$\boldsymbol{Q_P = i_P dt = E_P C_P = E_P(C_1 + C_2 + C_3 + \ldots)}$$
$$\boldsymbol{Q_n = i_n dt = E_P C_n = Q_P C_n/C_P = Q_P C_n/(C_1 + C_2 + C_3 + \ldots)}$$

Note that the highest charge accumulates in the highest capacitance.

2.5.3 Inductance

Inductance is the property by virtue of which a circuit opposes the changes in the value of a time-varying current flowing through it. Inductance causes opposition only to varying currents and does not cause any opposition to steady or direct current.

An electric current i flowing around a circuit produces a magnetic field and hence a magnetic flux Φ through the circuit. The ratio of the magnetic flux to the current is called the **inductance**, or more accurately **self-inductance** of the circuit. It is customary to use the symbol L for inductance, possibly in honour of the physicist Heinrich Lenz. The quantitative definition of the inductance is, therefore,

$$L = \frac{\phi}{i} \tag{2.24}$$

It follows that the SI unit for inductance is Webbers per ampere. In honour of Joseph Henry, the unit of inductance has been given the name **Henry (H)**: 1 H = 1 Wb/A.

In the above definition, the magnetic flux φ is that caused by the current flowing through the circuit concerned. There may, however, be contributions from other circuits. Consider, for example, two circuits C_1, C_2, carrying the currents i_1, i_2. The magnetic fluxes Φ_1 and Φ_2 in C_1 and C_2, respectively, are given by

$$\Phi_1 = L_{11}i_1 + L_{12}i_2,$$
$$\Phi_2 = L_{21}i_1 + L_{22}i_2$$

According to the above definition, L_{11} an L_{22} are the self-inductances of C_1 and C_2, respectively. It can be shown (see below) that the other two coefficients are equal: $L_{12} = L_{21} = M$, where M is called the **mutual inductance** of the pair of circuits.

Inductor An **inductor** is a passive electrical device employed in electrical circuits for its property of inductance.

Properties of Inductance The equation relating inductance and flux linkages can be rearranged as follows.

$$\lambda = Li \tag{2.25}$$

Taking the time derivative of both sides of the equation yields

$$\frac{d\lambda}{dt} = L\frac{di}{dt} + i\frac{dL}{dt}$$

In most physical cases, the inductance is constant with time and so

$$\frac{d\lambda}{dt} = L\frac{di}{dt}$$

By Faraday's Law of Induction,we have

$$\frac{d\lambda}{dt} = -E = v$$

where E is the Electromotive force (emf) and v is the induced voltage. Note that the emf is opposite to the induced voltage. Thus

$$v = L\frac{di}{dt} \tag{2.26}$$

or $$i(t) = \frac{1}{L}\int_0^t v(t)dt + i(0)$$

where $i(0)$ is the initial current. When initial current is zero,

$$i(t) = \frac{1}{L}\int_0^t v(t)\,dt \tag{2.27}$$

These equations together state that, for a steady applied voltage v, the current changes in a linear manner, at a *rate* proportional to the applied voltage, but inversely proportional to the inductance. Conversely, if the current through the inductor is changing at a constant rate, the induced voltage is constant.

From equation (2.26), it is clear that for an abrupt change in current, the voltage across the inductor becomes infinite. Also, from equation (2.27), it is observed that for a finite change in voltage in zero time the integral must be zero.

Therefore, *the current through an inductor cannot change instantaneously.*

Explanation of Initial Current i(0) It is possible that this inductor might have been used in some other circuit earlier, where it absorbed some energy and then it was disconnected. Because of its non-dissipative nature, the energy was stored within the inductor core. Now, as this inductor is connected to a circuit, it gets some path to release its stored energy. Here, this stored energy is represented by the initial current $i(0)$.

Series and Parallel Arrangement of Inductors Inductors *in a parallel* configuration each have the same potential difference (voltage). To find their total equivalent inductance (L_{eq}):

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_n} \tag{2.28}$$

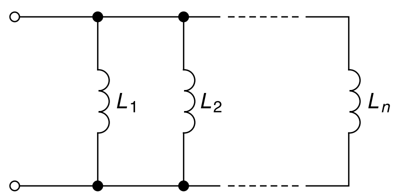


Figure 2.9 *Parallel arrangement of inductors*

The current through inductors *in series* stays the same, but the voltage across each inductor can be different. The sum of the potential differences (voltage) is equal to the total voltage. To find their total inductance:

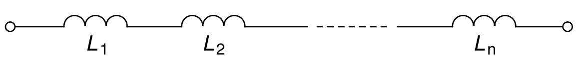


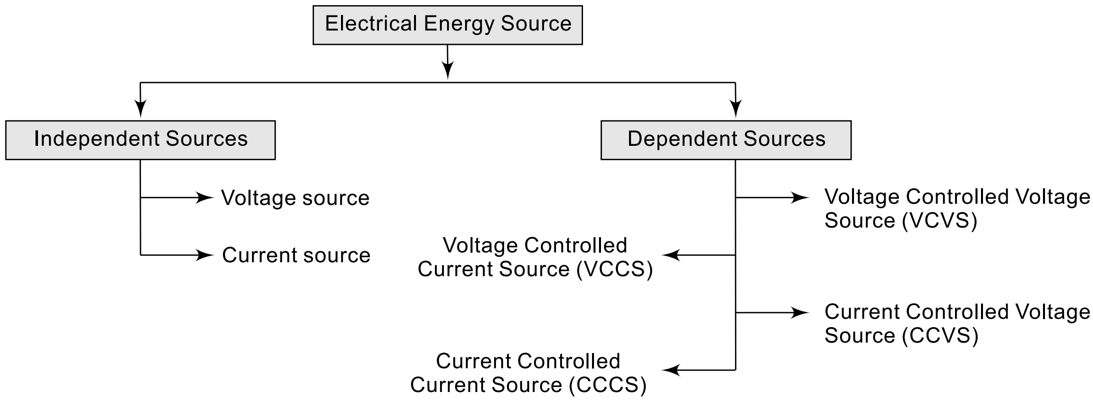
Figure 2.10 *Series arrangement of inductors*

$$L_{eq} = L_1 + L_2 + \ldots + L_n \tag{2.29}$$

These simple relationships hold true only when there is no mutual coupling of magnetic fields between individual inductors.

2.6 TYPES OF ELECTRICAL ENERGY SOURCES

Energy source is defined as the device that generates electrical energy. They are classified according to the current voltage characteristics. The classification is given below.



Independent Voltage Source An ideal voltage source has the following features.

(i) It is a voltage generator whose output voltage remains absolutely constant whatever be the value of the output current.
(ii) It has zero internal resistance so that voltage drop in the source is zero.
(iii) The power drawn by the source is zero.

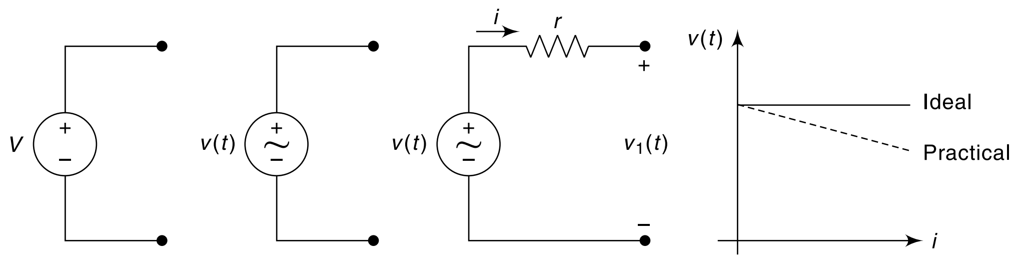


Figure 2.11 *Independent voltage sources and their characteristics*

In practical, the voltage does not remain constant, but falls slightly. This is taken care of by connecting a small resistance (r) in series with the ideal source. In this case, the terminal voltage will be,

$$v_1(t) = v(t) - ir$$

i.e., it will decrease with increase in current i.

An ideal voltage source is not practically possible. No voltage source can maintain its terminal voltage constant even when its terminals are short-circuited. The terminal voltage of a practical voltage source decreases as the load current increases. The v-i characteristics of an ideal and practical voltage source are shown in Fig. 2.14. A dc or ac generator or batteries are some examples of independent voltage sources. A lead-acid battery and a dry-cell are some examples of constant voltage source which can produce constant terminal voltage within a specified range of output current.

Independent Current Source An ideal current source has the following features.

(i) It produces a constant current irrespective of the value of the voltage across it.
(ii) It has infinity resistance.
(iii) It is capable of supplying infinity power.

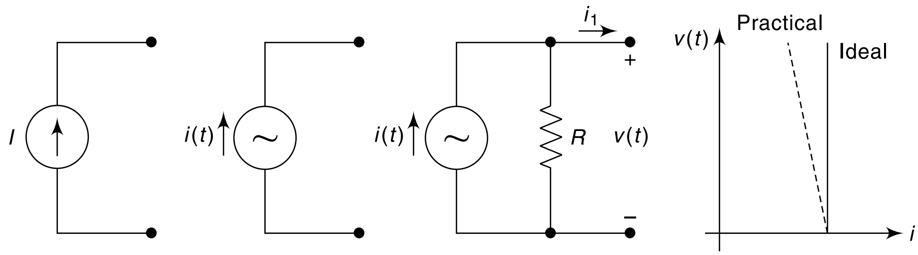


Figure 2.12 *Independent current sources and their characteristics*

In practice, the output current does not remain constant but decreases with increase in voltage. So, a practical current source is represented by an ideal current source in parallel with a high resistance (R) and the output current becomes,

$$i_1(t) = i(t) - \frac{v(t)}{R}$$

Similar to voltage sources, an ideal current source is not practically possible. No current source can maintain constant current even when its terminals are open-circuited. The output current of a practical current source decreases as the output voltage increases. The v-i characteristics of an ideal and practical current source are shown in Fig. 2.15. A solar cell, which can produce constant current within a specified range of output voltage, is an example of independent current source. A natural lightning can be considered to be an ideal current source. When a natural lightning strikes the top of a conductor, the resistance to the ground path is ideally zero. But, when the lightning strikes a non-conducting element (like the top of a tree) a large voltage is developed across the element which is flashed out immediately.

Dependent Sources In dependent sources (also referred as controlled sources), the source voltage or current is not fixed, but is dependent on a voltage or current at some other location in the circuit. Thus, there are four types of dependent sources.

(a) Voltage Controlled Voltage Source (VCVS)
(b) Current Controlled Voltage Source (CCVS)
(c) Voltage Controlled Current Source (VCCS)
(d) Current Controlled Current Source (CCCS)

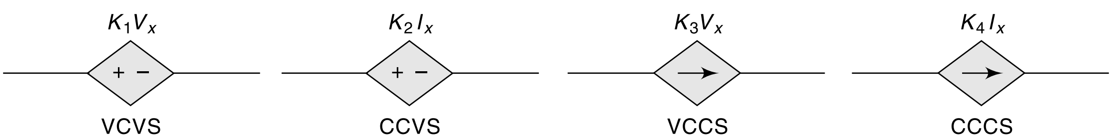


Figure 2.13 *Symbols of dependent sources*

Dependent sources are unilateral, because for a voltage controlled voltage source, say, $v_2 = kv_1$, the output voltage v_2 is controlled by the input voltage v_1, but the output current i_2 has no influence on the input v_1.

Application in electronic systems that uses either the transistors or vacuum tubes needs dependent sources.

2.7 FUNDAMENTAL LAWS

The fundamental laws that govern electric circuits are the Ohm's law and Kirchhoff's laws.

2.7.1 Ohm's Law

Ohm's law states that the voltage $v(t)$ across a resistor R is directly proportional to the current $i(t)$ flowing through it.

$$v(t) \propto i(t)$$

or $$v(t) = R \cdot i(t)$$

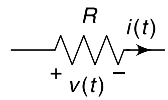


Figure 2.14(a) *An Electric Circuit*

Definition of Ohm's Law

Physical states (temperature, material, etc.) of a conductor remaining constant, the current flowing through a conductor is directly proportional to the potential difference across the two ends of the conductor.

This general statement of Ohm's law can be extended to cover inductances and capacitors as well under alternating current conditions and transient conditions. This is then known as the *Generalized Ohm's Law* This may be stated as

$$v(t) = Z(p) \cdot i(t) \text{ , where } p = d/dt = \text{differential operator}$$

$Z(p)$ is known as the impedance function of the circuit, and the above equation is the differential equation governing the behaviour of the circuit.

For a resistor, $Z(p) = R$

For an inductor $Z(p) = L\,p$

For a capacitor, $Z(p) = \dfrac{1}{Cp}$

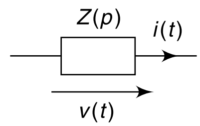


Figure 2.14(b) *Circuit Showing Impedance functin*

In the particular case of alternating current, $p = j\omega$, so that the equation governing circuit behaviour may be written as

$$V = Z(j\omega)\ .\ I, \text{ and}$$

For a resistor, $Z(j\omega)) = R$

For an inductor, $Z(j\omega) = j\omega L$

For a capacitor, $Z(j\omega) = \dfrac{1}{j\omega C}$

2.7.2 Kirchhoff's Current Law (KCL)

Kirchhoff's current law is based on the principle of conservation of charge. This requires that the algebraic sum of the charges within a system cannot change. Thus, the total rate of change of charge must add up to zero. Rate of change of charge is current.

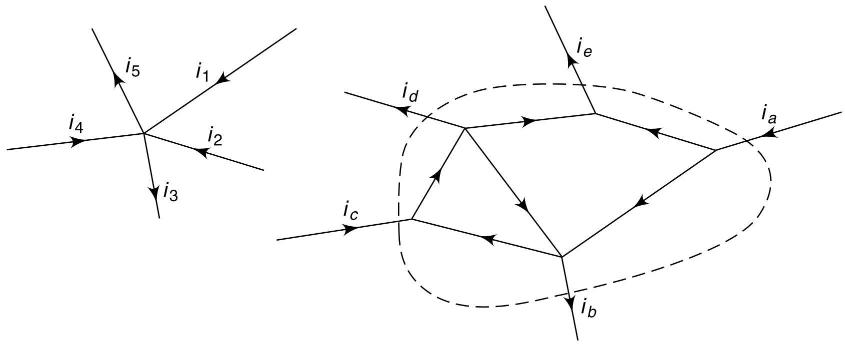


Figure 2.15 *Illustration of KCL*

This gives us our basic Kirchhoff's current law as the algebraic sum of the currents meeting at a point is zero,. i.e., at a node, $\sum I_n = 0$, where I_n are the currents in the branches meeting at the node.

This is also sometimes stated as the sum of the currents entering a node is equal to the sum of the currents leaving the node.

The theorem is applicable not only to a node, but to a closed system.

$i_1 + i_2 - i_3 + i_4 - i_5 = 0$. Also for the closed boundary, $i_a - i_b + i_c - i_d - i_e = 0$.

2.7.3 Kirchhoff's Voltage Law (KVL)

Kirchhoff's voltage law is based on the principle of conservation of energy. This requires that the total work done in taking a unit positive charge around a closed path and ending up at the original point is zero.

This gives us our basic Kirchhoff's voltage law as the algebraic sum of the potential differences taken round a closed loop is zero. i. e., around a loop, $\Sigma V_n = 0$, where V_n are the voltages across the branches in the loop.

$$v_a + v_b + v_c + v_d - v_e = 0$$

This is also sometimes stated as the sum of the emfs taken around a closed loop is equal to the sum of the voltage drops around the loop.

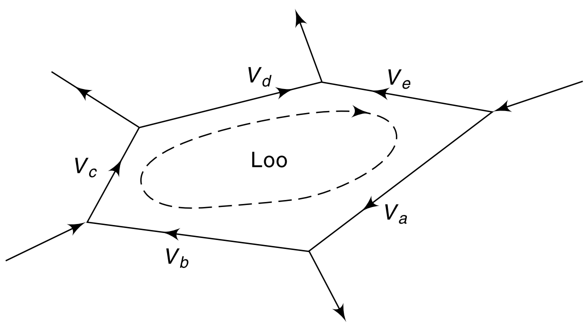


Figure 2.16 *Illustration of KVL*

Although all circuits could be solved using only Ohm's law and Kirchhoff's laws, the calculations would be tedious. Various network theorems have been formulated to simplify these calculations.

- ***Sign Conventions for applying Kirchhoff's Laws***
  1. When tracing through a voltage source from positive to negative terminal, the voltage should be given a positive sign.
  2. When tracing through a voltage source from negative to positive terminal, the voltage should be given a negative sign.
  3. When tracing through a resistance in the direction of current flow, the voltage should be given a positive sign.
  4. When tracing through a resistance in a direction opposite to the direction of current flow, the voltage should be given a negative sign.

2.8 SOURCE TRANSFORMATION

Transformation of several voltage (or current) sources into a single voltage (or current) source and a voltage source into a current source or vice-versa is known as source transformation. This makes circuit analysis easier.

There are some rules of source transformation.

Rule (1) Several voltage sources $\{V_1(t), V_2(t), \ldots, V_n(t)\}$ connected in series will be replaced by a single voltage source of value $V = V_1(t) + V_2(t) + \ldots + V_n(t)$. Similarly, a number of current sources $\{I_1(t), I_2(t), \ldots, I_n(t)\}$ connected in parallel is replaced by a single current source of value $I(t) = I_1(t) + I_2(t) + \ldots + I_n(t)$.

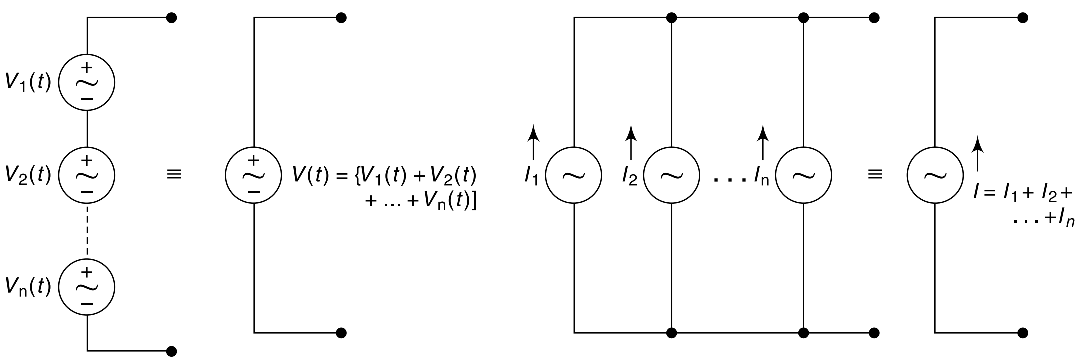


Figure 2.17 *Source transformation technique: Rule (1)*

Rule (2) A number of voltage sources $V_1(t)$, $V_2(t)$, …, $V_n(t)$ in parallel will result in a single voltage source, $V(t) = V_1(t) = V_2(t) = \ldots = V_n(t)$.

Therefore, *voltage sources should not be connected in parallel unless they have identical potential,* as paralleling of sources with non-similar potential waveforms will result in heavy current, which may damage the equipment.

Similarly, a number of current sources $I_1(t)$, $I_2(t)$, …, $I_n(t)$ in series will result in a single current source of value $I(t) = I_1(t) = I_2(t) = \ldots = I_n(t)$ and thus, *current sources cannot be connected in series if they are not identical.*

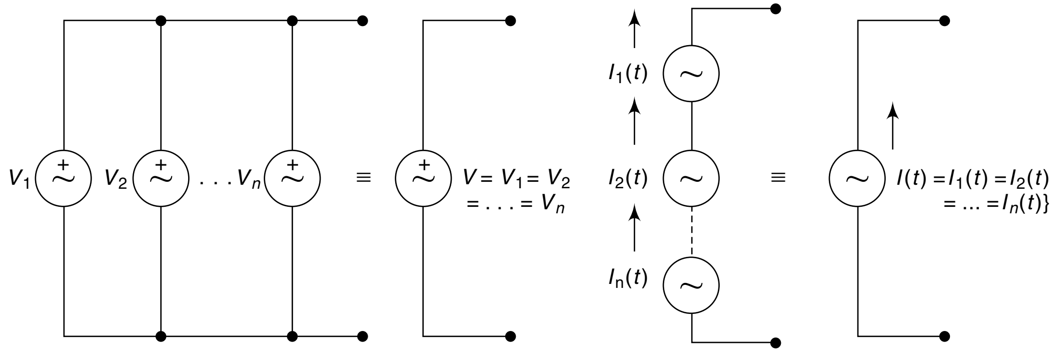


Figure 2.18 *Source transformation technique: Rule (2)*

Rule (3) As far as the computations in the remainder of the network are concerned, a resistor in parallel with an ideal voltage source and a resistor in series with an ideal current source may be ignored.

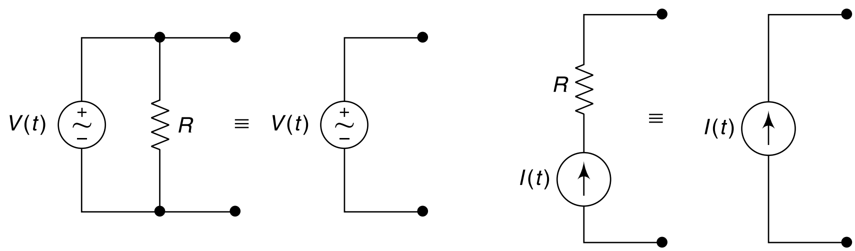


Figure 2.19 *Source transformation technique: Rule (3)*

Rule (4) A voltage source $V(t)$ in series with a resistor R can be converted into a current source $I(t)$ in parallel with the same resistor R, where, $I(t) = \dfrac{V(t)}{R}$.

Similarly, a voltage source $V(t)$ in series with a capacitor C may be converted into a current source $I(t)$ in parallel with C, where, $I(t) = C\dfrac{dV(t)}{dt}$; and a voltage source $V(t)$ in series with an inductor L may be converted into a current source $I(t)$ in parallel with L, where, $I(t) = \dfrac{1}{L}\int V(t)dt$

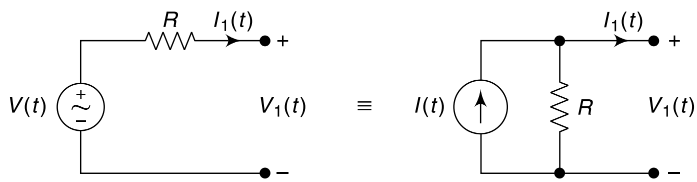


Figure 2.20 *Source transformation technique: Rule (4)*

2.9 NETWORK ANALYSIS TECHNIQUES

Network analysis is the determination of the response output of a network when the input excitation is given. There are two techniques of network analysis.

1. Nodal Analysis
2. Loop or Mesh Analysis

Nodal Analysis It is based on Kirchhoff's current law (KCL). In this method, the unknown variables are the node voltages. It is generally used when the circuit contains several current sources.

Steps

- If there is N number of nodes in a network, all nodes are labeled. One node is treated as *datum or reference node* (zero potential) and the other node voltages are treated as unknowns to be determined with respect to this reference.
- KCL is written at each node in terms of node voltages.
  - KCL is applied at $N - 1$ of the N nodes of the circuit using assumed current directions, as necessary. This will create $N - 1$ linearly independent equations, known as *node equations*.
  - In a circuit with independent voltage sources, if two nodes of interest are separated by a voltage source instead of a resistor or current source, then the concept of *supernode* is used that creates constraint equations.
  - The current is computed based on voltage difference between two nodes. The current in any branch is obtained via ohm's law as,

$$i = \frac{V_{mm}}{R} = \frac{V_m - V_m}{R}, \text{ for D.C.}$$

$$l = \frac{V_{mm}}{Z} = \frac{V_m - V_m}{Z}, \text{ for A.C.}$$

where, $V_m > V_n$ and current flows from node m to n.

- Solution of the $N - 1$ simultaneous equations (by Gaussian elimination or matrix method) gives the unknown node voltages.

For the network shown in Figure 2.21, apply Kirchhoff's current law and write the node equations.

Example 2.1 Let node voltages are E_1, E_2 and E_3 at nodes 1, 2 and 3 respectively.

At node 1,

$$I_1 = I_3 + I_4$$

$$I_1 = \frac{E_1}{R_1} + \frac{(E_1 - E_2)}{R_2}$$

$$I_1 = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)E_1 - \frac{E_2}{R_2} \quad \text{(i)}$$

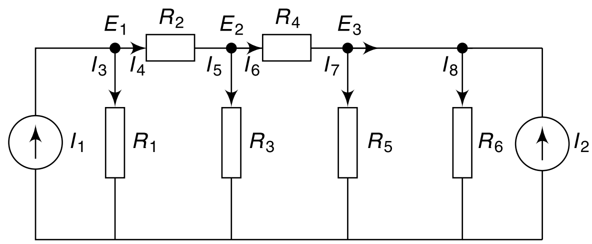


Figure 2.21 *Network explaining node analysis technique*

At node-2,

$$I_4 = I_5 + I_6$$

$$\frac{(E_1 - E_2)}{R_2} = \frac{(E_2 - E_3)}{R_4} + \frac{E_2}{R_3} \quad 0 = -\frac{E_1}{R_2} + E_2\left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4}\right) - \frac{E_3}{R_4} \qquad \text{(ii)}$$

At node-3,

$$I_6 = I_7 + I_8 - I_2$$

$$\frac{(E_2 - E_3)}{R_4} = \frac{E_3}{R_5} + \frac{E_3}{R_6} - I_2$$

$$I_2 = -\frac{E_2}{R_4} + E_3\left(\frac{1}{R_4} + \frac{1}{R_5} + \frac{1}{R_6}\right) \qquad \text{(iii)}$$

Given the other values, solution of Equations (i), (ii), and (iii) gives the values of E_1, E_2 and E_3.

Concept of Supernode This concept is used when a circuit contains voltage sources. A supernode is formed by enclosing a dependent or independent voltage source connected between two non-reference nodes and any elements connected in parallel with it. This concept is necessary for nodal analysis with voltage source, because the current through a voltage source is unknown. We consider the following two cases.

Case 1 *When a voltage source is connected between the reference node and a non-reference node:* In this case, the voltage of the non-reference node is taken equal to the voltage of the voltage source. For the circuit shown in Fig. 2.22(a),

$$V_1 = 5 \text{ V} \qquad \text{(i)}$$

Case 2 *When a voltage source is connected between two non-reference nodes:*
In this case, a supernode is considered enclosing the non-reference nodes. Both KCL and KVL is written for the supernode.

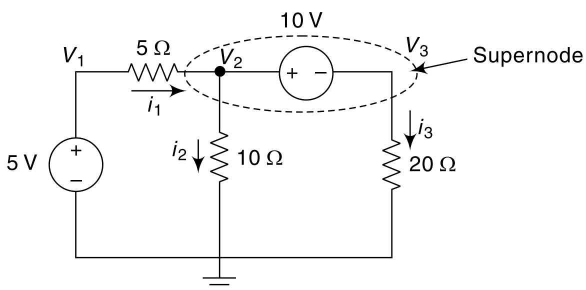


Figure 2.22(a) *Circuit with supernode*

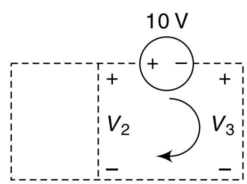


Figure 2.22(b) *KVL with supernode*

For this example, nodes 2 and 3 are forming the supernode.

By KCL at the supernode, $i_1 = i_2 + i_3$

or
$$\frac{V_1 - V_2}{5} = \frac{V_2 - 0}{10} + \frac{V_3 - 0}{20} \qquad \text{(ii)}$$

To apply KVL to the supernode, the circuit is drawn as shown in Fig. 2.22(b). By KVL,

$$10 + V_3 - V_2 = 0 \qquad \text{(iii)}$$

Solving equations (i), (ii) and (iii), the node voltages are obtained, $V_1 = 5$ V, $V_2 = 4.2857$ V, $V_3 = -5.7143$ V.

Properties of Supernode

(i) It provides the constraint equations.
(ii) Both KCL and KVL are written for supernode.
(iii) A supernode does not have any voltage of its own.

Loop or Mesh Analysis It is based on Kirchhoff's voltage law (KVL). In this method, the unknown variables are the loop currents. It is generally used when the circuit contains several voltage sources.

Steps

- If there is 'N' number of loops/meshes in a network, all loops are labeled.
- KVL is written at each loop/mesh in terms of loop/mesh currents. *Loop currents* are those currents flowing in a loop; they are used to define *branch currents*.
  - For N independent loops, total N equations are written using KVL around each loop. These equations are known as *loop/mesh equations*.
  - The concept of *supermesh* is used in case a circuit contains current source that provides the constraint equations.
- Solution of the N simultaneous equations gives the required loop/mesh currents.

Write the mesh equations for the circuit shown in Figure 2.23.

Example 2.2 Two meshes are labeled as mesh-1 and mesh-2.
Applying KVL for mesh-1,

$$V_s = R_1 I_1 + R_2(I_1 - I_2) \tag{i}$$

By constraint equation,

$$I_2 = -I_s \tag{ii}$$

Solving the equations, we get I_1 and I_2.

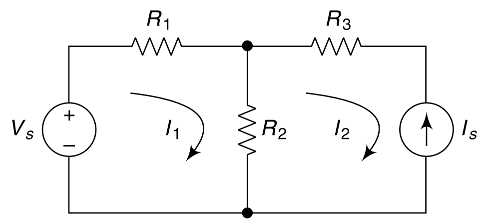


Figure 2.23 *Circuit explaining loop analysis technique*

Concept of Supermesh This concept is used when a circuit contains current sources. A supermesh is formed by excluding the branch containing a dependent or independent current source connected in common to two meshes and any elements connected in series with it. This concept is necessary for loop analysis with current source, because the voltage drop across a current source is unknown. We consider the following two cases.

Case-1 *When a current source is in one mesh:*

In this case, the mesh current is taken equal to the current of the current source. For example, for the circuit shown in Fig. 2.24,

$$i_2 = -10 \text{ A}$$

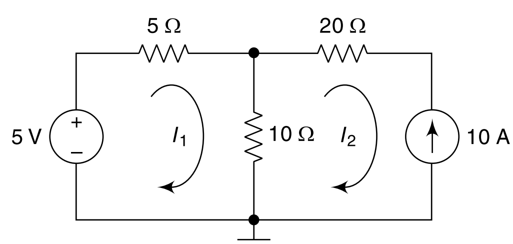


Figure 2.24 *Current source in one mesh*

Case-2 *When a current source is connected between two meshes:*

In this case, a supermesh is considered excluding the branch with the current source and any elements connected in series with it. Both KCL and KVL is written for the supermesh. For example, consider the circuit shown in Fig. 2.25. Supermesh is formed by excluding the branch with 3A current source.

By KVL for the supermesh,

$$2(i_1 - i_2) + 4(i_3 - i_2) + 8i_3 = 6 \quad \text{(i)}$$

By KCL at any one node of the omitted branch (say, *X*),

$$i_1 = 3 + i_3 \quad \text{(ii)}$$

Also by KVL for second mesh,

$$2i_2 + 4(i_2 - i_3) + 2(i_2 - i_1) = 0 \quad \text{(iii)}$$

Solving equations (i), (ii) and (iii), the mesh currents are obtained, $i_1 = 3.437\text{A}$, $i_2 = 1.1052\text{A}$, $i_3 = 0.4737\text{A}$.

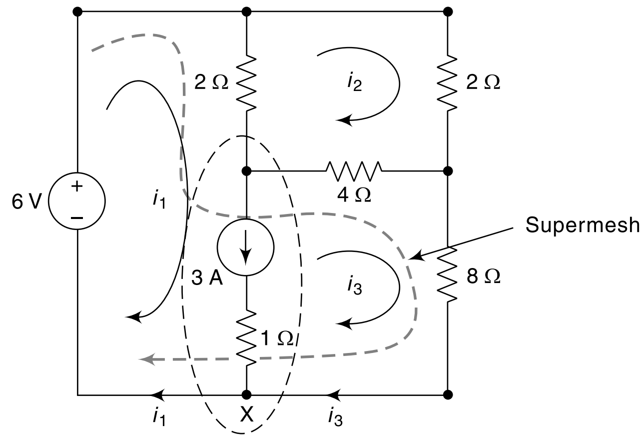


Figure 2.25 *Current source connected between two meswhes*

Properties of Supermesh

(i) It provides the constraint equations.
(ii) Both KCL and KVL are written for supermesh.
(iii) A supermesh does not have any current of its own.

Comparison of Loop and Node Analysis In any network having *N* nodes and *B* branches, there are 2*B* unknowns, i.e., *B*–branch currents and *B*–branch voltages. These unknowns can be determined either by loop analysis or nodal analysis.

The choice of the method depends on two factors given below.

1. Nature of the network The mesh-method is generally used for circuits having many series-connected elements, voltage sources, or supermeshes. On the other hand, nodal analysis is more suitable for circuits having many parallel-connected elements, current sources, or supernodes.

The main factor for selecting any one method is the *minimum number of equations*. If a circuit is having fewer nodes than meshes, then nodal analysis is used, while if a circuit with fewer meshes than nodes, then loop method is used.

2. Requirement of the problem If node voltages are required, nodal analysis is used. If branch/mesh currents are required, loop analysis is used.

However, there are some particular circuits, where only one method can be applied. For example, in analyzing transistor circuits, mesh method is the only possible method; while for op-amp circuits and for non-planar networks, node method is the only possible method.

2.10 DUALITY

Duality is a transformation in which currents and voltages are interchanged. Two phenomena are said to be dual if they are described by equations of the same mathematical form.

There are a number of similarities and analogies between the two circuit analysis techniques based on loop-current method and node voltage method. The principal quantities and concepts involved in these two methods based on KVL and KCL are dual of each other with voltage variables substituted by current variables, independent loop by independent node-pair, etc.

This similarity is termed as 'principle of duality'.

Some dual relations are:

$$v = Ri \qquad i = Gv$$

$$v = L\frac{di}{dt} \qquad i = C\frac{dv}{dt}$$

$$v = \frac{1}{C}\int i\,dt \qquad i = \frac{1}{L}\int v\,dt$$

Thus, the circuit elements (R, L, C) have some dual relationship. Duality also appears as relation between two networks. For example, an RLC series circuit with voltage excitation is dual of an RLC parallel circuit with current excitation.

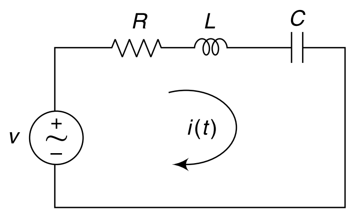


Figure 26(a) *Series RLC Circuit*

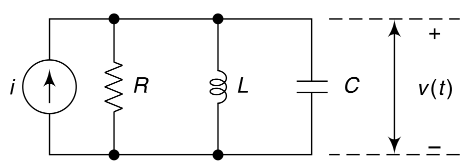


Figure 2.26(b) *Parallel RLC Circuit*

For series circuit, $v = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt$

For parallel circuit, $i = Gv + C\frac{dv}{dt} + \frac{1}{L}\int v\,dt$

Dual Quantities and Concepts

Sl No.	*Quantity/Concept*	*Dual*
1	Current	Voltage
2	Resistance	Conductance
3	Inductance	Capacitance
4	Impedance	Admittance
5	Reactance	Susceptance
6	Branch current	Branch voltage
7	Mesh or Loop	Node or Node-pair
8	Mesh Current or Loop Current	Node Voltage or Node-pair Voltage
9	Link	Tree Branch
10	Link Current	Tree Branch Voltage
11	Tree Branch Current	Link Voltage
12	Tie-set	Cut-set
13	Short-circuit	Open-circuit
14	Parallel Paths	Series Paths

Construction of Dual of a Network

1. A dot is placed inside each independent loop of the given network. These dots correspond to the non-reference nodes of the dual network.
2. A dot is placed outside the network. This dot corresponds to the datum node.
3. All internal dots are connected by dashed lines crossing the common branches and placing the elements which are duals of the elements the original network.
4. All internal dots are connected to the external dot by dashed lines crossing all external branches and placing dual elements of the external branch.

Conventions for Reference Polarities of Voltage Source and Reference Directions of Current Source

(i) A clockwise current in a loop corresponds to a positive polarity (with respect to reference node) at the dual independent node.

(ii) A voltage rise in the direction of a clockwise loop current corresponds to a current flowing towards the dual independent nodes.

Finally, the dual construction can be checked by writing mesh equations and node equations of two networks.

Example 2.3 Draw the dual of the network shown in figure.

Solution Following the steps, dual network is drawn.

Therefore, the dual network becomes as shown in Fig. 2.27.

By KVL to the original network,

$$I_1(3 + 4) - I_2(4) = 100$$

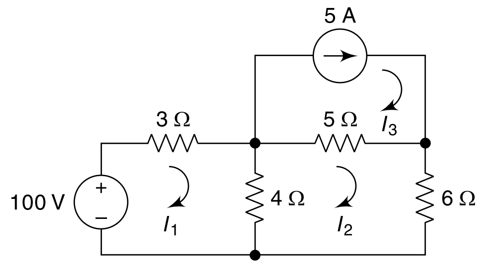


Figure 2.27 *Circuit of example*

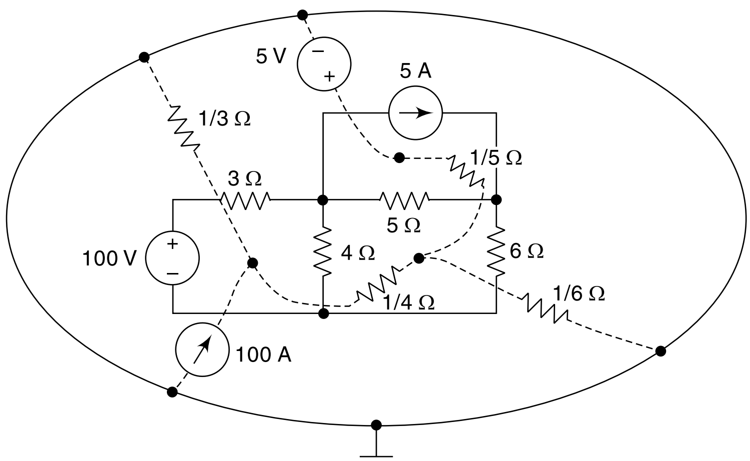


Figure 2.28 *Figure explaining drawing dual of network of Fig. 2.27*

$-I_1(4) + I_2(4 + 5 + 6) - 5I_3 = 0$
$I_3 = 5$

The dual equations will be,

$V_1(3 + 4) - V_2(4) = 100$
$-V_1I(4) + V_2(4 + 5 + 6) - 5V_3 = 0$
$V_3 = 5$

These equations satisfy the dual network.

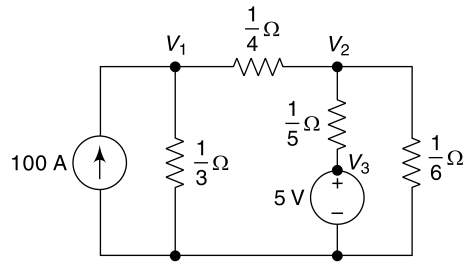


Figure 2.29 *Dual of network of Fig. 2.26*

2.11 STAR-DELTA CONVERSION TECHNIQUE

The ***Y*-Δ transform**, also written ***Y*-delta**, **Wye-delta**, Kennelly's **delta-star transformation**, **star-mesh transformation**, ***T*-Π** or ***T*-pi transform**, is a mathematical technique to simplify the analysis of an electrical network. The name derives from the shapes of the circuit diagrams, which look respectively like the letter Y and the Greek capital letter Δ.

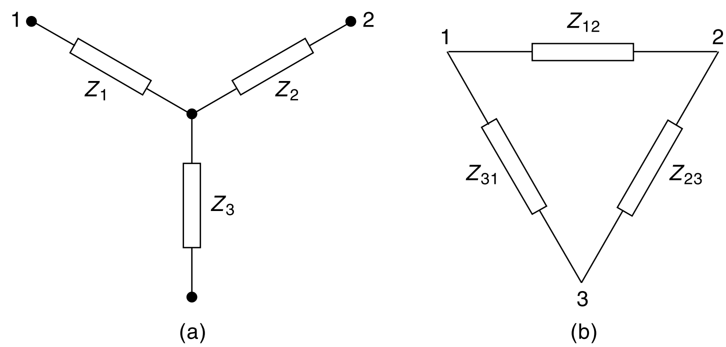


Figure 2.30 *(a) Star connection (b) Delta connection*

The transformation is used to establish equivalence for networks with three terminals. For equivalence, the impedance between any pair of terminals must be the same for both networks.

For the star connection, the impedance between terminals 1 and 2 is $Z_1 + Z_2$.

For delta connection, the the impedance between terminals 1 and 2 is

$$Z_{12} \,||\, (Z_{23} + Z_{31}) = \frac{Z_{12}(Z_{23} + Z_{31})}{Z_{12} + Z_{23} + Z_{31}}$$

As the impedance between terminals 1 and 2 should be same, therefore,

$$Z_1 + Z_2 = \frac{Z_{12}(Z_{23} + Z_{31})}{Z_{12} + Z_{23} + Z_{31}} \tag{i}$$

Similarly, for terminals 2 and 3 we get,

$$Z_2 + Z_3 = \frac{Z_{23}(Z_{31} + Z_{12})}{Z_{23} + Z_{31} + Z_{12}} \tag{ii}$$

$$Z_3 + Z_1 = \frac{Z_{31}(Z_{12} + Z_{23})}{Z_{31} + Z_{12} + Z_{23}} \tag{iii}$$

Delta to Star Conversion

In this case, Z_1, Z_2, and Z_3 are to be written in terms of Z_{12}, Z_{23}, and Z_{31}.

By (i) – (ii) + (iii), we get
$$\boxed{Z_1 = \frac{Z_{12}Z_{31}}{Z_{12} + Z_{23} + Z_{31}}} \tag{iv}$$

Similarly we get,
$$\boxed{Z_2 = \frac{Z_{23}Z_{12}}{Z_{12} + Z_{23} + Z_{31}}} \tag{v}$$

and
$$\boxed{Z_3 = \frac{Z_{31}Z_{23}}{Z_{12} + Z_{23} + Z_{31}}} \tag{vi}$$

Star to Delta Conversion

In this case, Z_{12}, Z_{23}, and Z_{31} are to be written in terms of Z_1, Z_2, and Z_3.
Let $Z = Z_1Z_2 + Z_2Z_3 + Z_3Z_1$. Then from Eq. (iv) to Eq. (vi), we get

$$Z = \frac{Z_{12}Z_{23}^2Z_{31}}{(Z_{12} + Z_{23} + Z_{31})^2} + \frac{Z_{12}Z_{23}Z_{31}^2}{(Z_{12} + Z_{23} + Z_{31})^2} + \frac{Z_{12}^2Z_{23}Z_{31}}{(Z_{12} + Z_{23} + Z_{31})^2} = \frac{Z_{12}Z_{23}Z_{31}}{Z_{12} + Z_{23} + Z_{31}} \tag{vii}$$

From Eq. (vii) and Eq. (iv), we get $Z = Z_{12}Z_3$ $\quad \Rightarrow Z_{12} = \dfrac{Z}{Z_3}$

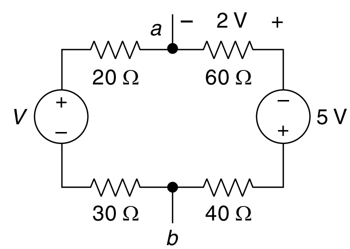
Therefore,
$$\boxed{Z_{12} = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3} = Z_1 + Z_2 + \frac{Z_1Z_2}{Z_3}}$$

Similarly,
$$\boxed{Z_{23} = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1} = Z_2 + Z_3 + \frac{Z_2Z_3}{Z_1}}$$

and
$$\boxed{Z_{31} = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2} = Z_3 + Z_1 + \frac{Z_3Z_1}{Z_2}}$$

SOLVED PROBLEMS

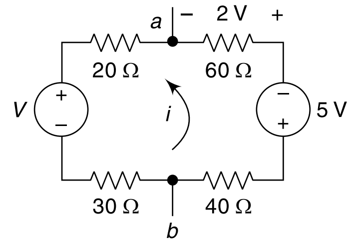
2.1 Find the values of V, V_{ab} and the power delivered by the 5V source. All values of resistances are in ohm.



Solution

Current, $\quad i = \dfrac{2}{60} = \dfrac{1}{30}$ A

By KVL,



$$20i + 2 + 5 + v + 70i = 0$$

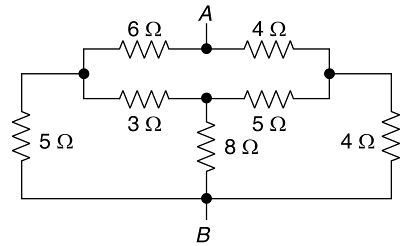
$$v = -7 - 90i = -7 - 90 \times \frac{1}{30} = -10 \text{ V}$$

$$\therefore \quad v_{ab} = 20i + v + 30i = 50i - 10$$

$$= 50 \times \frac{1}{30} - 10 = -8.33 \text{ V}$$

Power drawn by the 5V source = – (Power taken source) = $-5 \times \frac{1}{30} = -0.166$ W

2.2 Find the equivalent resistance between the terminals A and B of the circuit shown below.

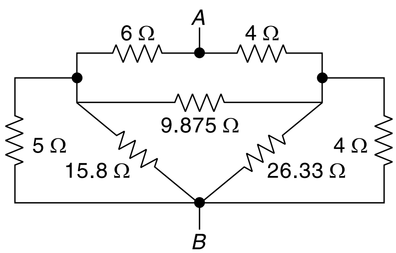


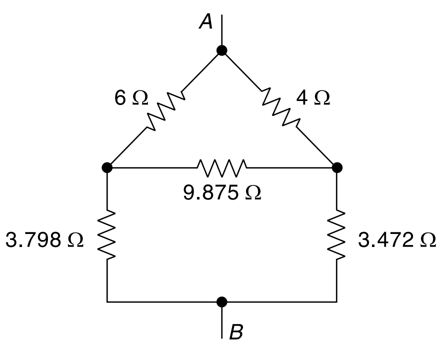
Solution Converting star into delta,

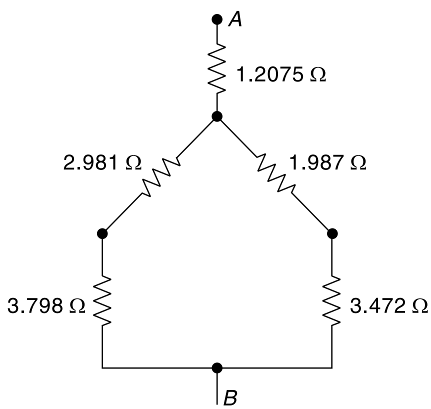
$$r_{12} = \left(r_1 + r_2 + \frac{r_1 r_2}{r_3} \right) = 8 + \frac{15}{8} = 9.875\ \Omega$$

$$r_{23} = \left(r_2 + r_3 + \frac{r_2 r_3}{r_1} \right) = 13 + \frac{40}{3} = 26.33\ \Omega$$

$$r_{31} = \left(r_3 + r_1 + \frac{r_3 r_1}{r_2} \right) = 11 + \frac{24}{5} = 15.8\ \Omega$$







Combining the parallel connections of 5 Ω and 15.8 Ω and 4 Ω and 26.33 Ω, we have the reduced circuit.

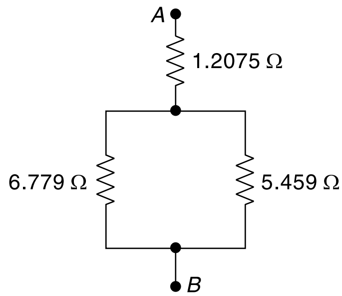
Again, converting the delta made of 6 Ω, 4 Ω and 9.875 Ω into equivalent star,

$$r_1 = \frac{r_{12}r_{31}}{r_1 + r_2 + r_3}$$

$$= \frac{6 \times 4}{19.875} = 1.2075\ \Omega$$

$$r_2 = \frac{4 \times 9.875}{19.875} = 1.987\ \Omega$$

$$r_3 = \frac{6 \times 9.875}{19.875} = 2.981\ \Omega$$

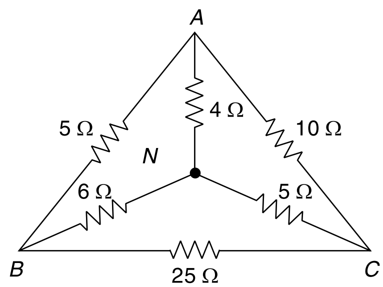


So, the given circuit becomes as shown in figure.

$$\therefore \qquad R_{AB} = 1.2075 + \frac{6.779 \times 5.459}{6.779 + 5.459} = 4.23\ \Omega \qquad \textit{Ans.}$$

2.3 Find the equivalent resistance between
(i) *A* and *B*,
(ii) *B* and *C*,
(iii) *C* and *A*, and
(iv) *A* and *N* of the circuit shown.

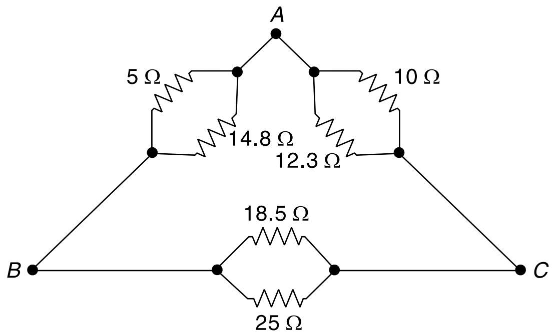
Solution Converting the star into delta,



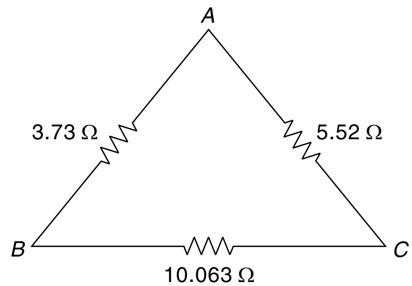
$$r_{12} = \left(r_1 + r_2 + \frac{r_1 r_2}{r_3}\right) = 4 + 5 + \frac{5 \times 4}{6} = 12.33\ \Omega$$

$$r_{23} = \left(r_2 + r_3 + \frac{r_2 r_3}{r_1}\right) = 5 + 6 + \frac{5 \times 6}{4} = 18.5\ \Omega$$

$$r_{31} = \left(r_3 + r_1 + \frac{r_3 r_1}{r_2}\right) = 6 + 4 + \frac{6 \times 4}{5} = 14.8\ \Omega$$



The circuit becomes, as shown in below figure.



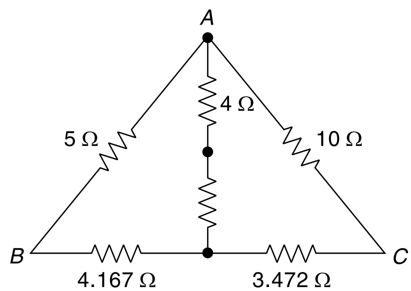
(i) Equivalent resistance between A and B,

$$R_{AB} = \frac{3.73 \times (10.06 + 5.52)}{3.70 + 10.06 + 5.52} = 3.035\ \Omega. \qquad \textit{Ans.}$$

(ii) $$R_{BC} = \frac{10.06 \times (3.73 + 5.52)}{10.06 + 3.73 + 5.52} = 4.82\ \Omega \qquad \textit{Ans.}$$

(iii) $$R_{CA} = \frac{5.52 \times (10.06 + 3.73)}{6.52 + 10.06 + 3.73} = 3.94\ \Omega \qquad \textit{Ans.}$$

(iv) Converting the delta into equivalent star,

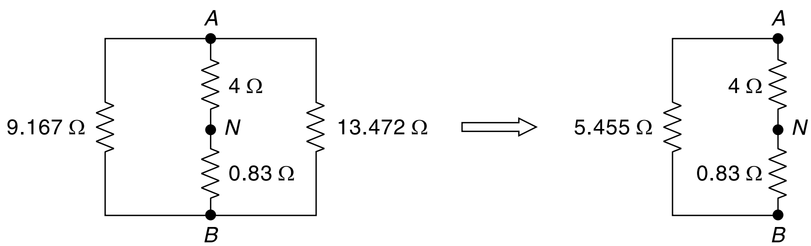


$$r_1 = \frac{5 \times 6}{5 + 6 + 25} = 0.83\ \Omega$$

$$r_2 = \frac{25 \times 6}{5 + 6 + 25} = 4.167\ \Omega$$

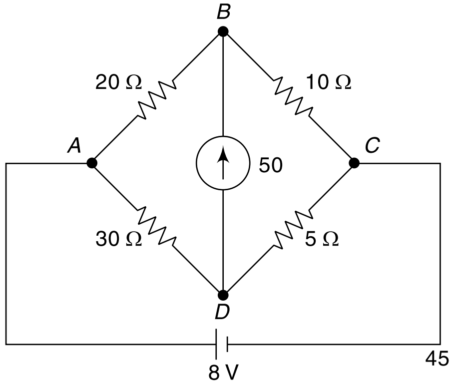
$$r_3 = \frac{25 \times 5}{5 + 6 + 25} = 3.472\ \Omega$$

The circuit becomes:

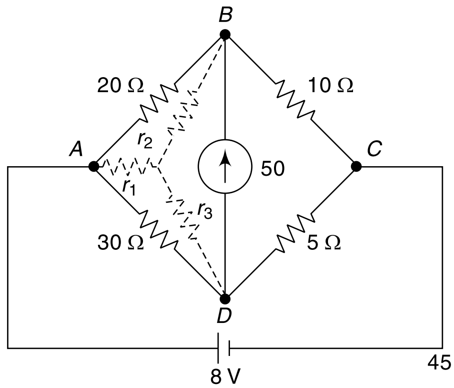


$$\therefore \qquad R_{AN} = \frac{4 \times 6.288}{4 + 6.288} = 2.4448\ \Omega \qquad \textit{Ans.}$$

2.4 Find the current through the galvanometer using delta-star conversion.



Solution Converting the delta consisting of 20 Ω, 30 Ω and 50 Ω, we get,

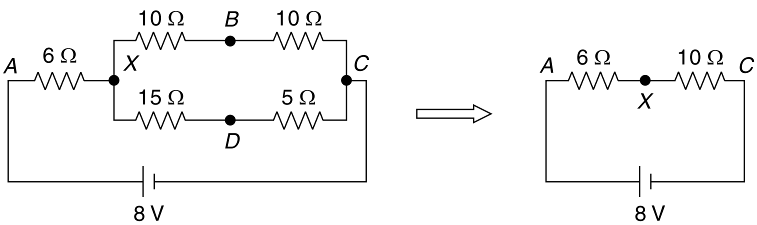


$$r_1 = \frac{20 \times 30}{20 + 30 + 50} = 6\ \Omega$$

$$r_2 = \frac{20 \times 50}{20 + 30 + 50} = 10\ \Omega$$

$$r_3 = \frac{30 \times 50}{20 + 30 + 50} = 15\ \Omega$$

$\therefore$ $R_{AC} = 16\ \Omega$



Main current $i = \dfrac{8}{16} = 0.5$ A

Now, to calculate potential difference between the points B and D ;

$$V_{XC} = 10 \times 0.5 = 5 \text{ V}$$

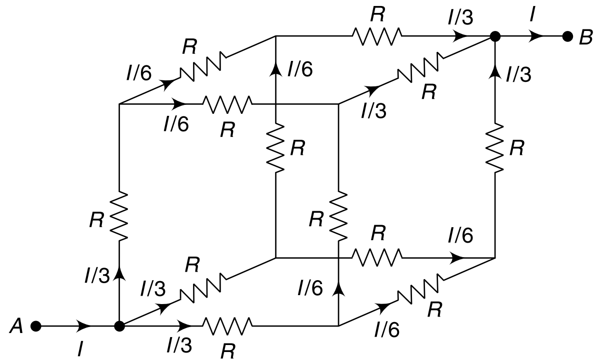
$\therefore$ $V_{BD} = (10 \times 0.25 - 5 \times 0.25) = 1.25$ V

$\therefore$ Currant through the galvanometer, (50 (Ω)

$$i_G = \frac{1.25}{50} = 0.025 \text{ A}$$ *Ans.*

2.5 Twelve similar conductors each of R resistance form a cubical frame. Find the resistance across two opposite corners of the cube.

Solution The configuration is shown in the figure.



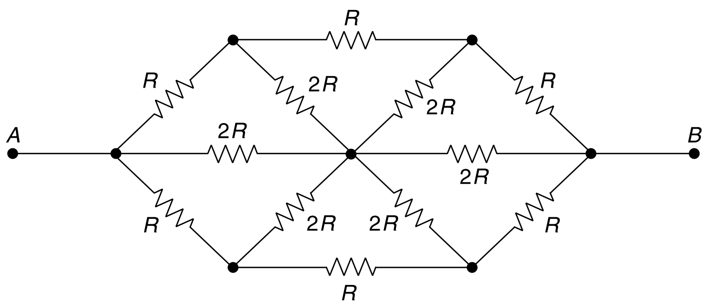
The current distribution is shown.

So, the total voltage drop between two opposite corners A and B for a total current of I is,

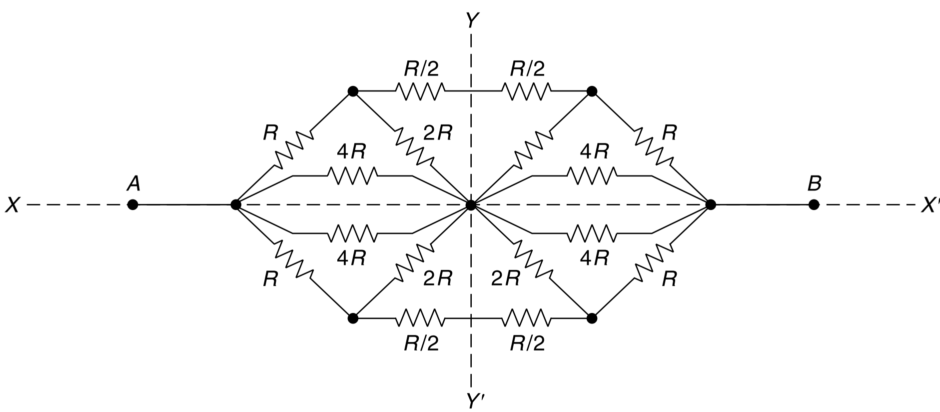
$$V_{AB} = R.\frac{I}{3} + R.\frac{I}{6} + R.\frac{I}{3} = R \times \frac{5}{6} I$$

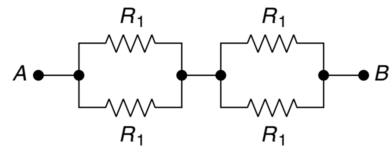
Equivalent resistance, $R_{AC} = \dfrac{V_{AB}}{I} = \dfrac{5}{6} R$ *Ans.*

2.6 A regular hexagon is formed from 6 wires of R ohm each. The corners are joined to the centre by six more wires of $2R$ ohms each. Calculate the resistance of the hexagon between any two nodes diametrically opposite.



Solution The hexagon can be redrawn as shown.





The hexagon is symmetrical about XX'

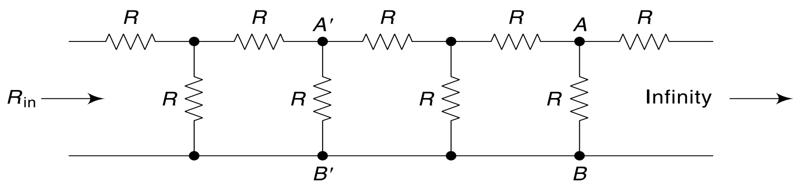
Equivalent resistance of the second quadrant,

$$R_1 = (2R \,||\, R/2 + R) \,||\, 4R = \frac{28}{27}R$$

So, the figure is modified as,

$$\therefore \qquad R_{AB} = (R_1 \,\|\, R_1) + (R_1 \,\|\, R_1) = R_1 = \frac{28}{27}R \qquad \textit{Ans.}$$

2.7 Find the input resistance of the infinite section resistive network shown below.



Solution Let the equivalent resistance be R_{in}.

The network can be terminated at $A' B'$ instead of AB.

$$R_{A'B'} = [R + (R_{in}) \,||\, (R)]$$

By assumption,

$$R_{in} = R + \frac{R_{in}R}{R + R_{in}} = \frac{2R_{in}R + R^2}{R + R_{in}}$$

$$\Rightarrow \qquad RR_{in} + R_{in}^2 = 2RR_{in} + R^2$$

$$\Rightarrow \qquad R_{in}^2 - RR_{in} - R^2 = 0$$

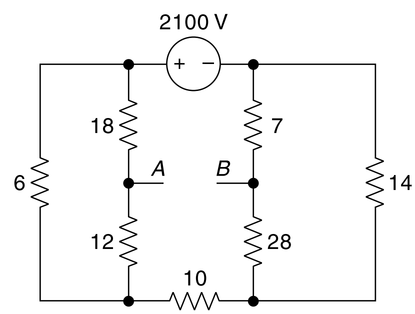
$$\Rightarrow \qquad R_{in} = \frac{R \pm \sqrt{R^2 + 4R^2}}{2} = \frac{R}{2}[1 \pm \sqrt{5}]$$

Taking positive sign, $R_{in} = \left(\dfrac{\sqrt{5}+1}{2}\right)R$

2.8 In the network shown, calculate the power input to each of the following elements when it is connected across A and B.

(a) a resistance R_{AB} of 59 Ω.

(b) a voltage source of –160 V.



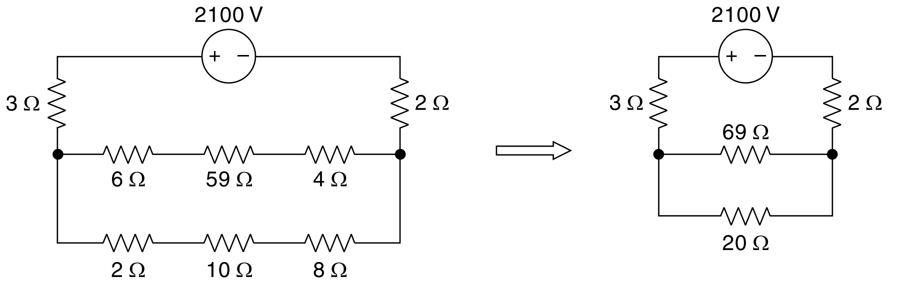
Solution

(a) Converting the two deltas into star,

$$r_1 = \frac{18 \times 6}{18 + 12 + 6} = 3\,\Omega, \quad r_2 = \frac{6 \times 12}{36} = 2\,\Omega,$$

$$r_3 = \frac{18 \times 12}{36} = 6\,\Omega$$

and $\quad r_1^1 = \dfrac{14 \times 7}{49} = 2\,\Omega, \; r_2^1 = \dfrac{28 \times 14}{49} = 8\,\Omega, \; r_3^1 = 4\,\Omega$



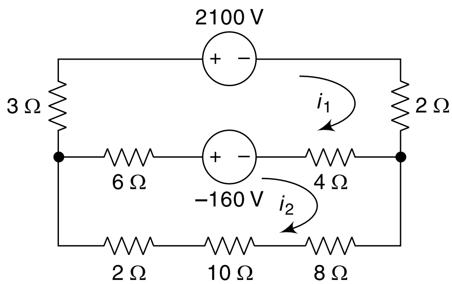
$$\therefore \qquad R_{\text{eq}} = \left(3 + \frac{69 \times 20}{69 + 20} + 2\right) = 20.5\ \Omega$$

Main current, $i = \dfrac{2100}{20.5} = 102.41$ A

Current in 59Ω resistor, $i_{59} = 102.41 \times \dfrac{20}{89} = 23.01$ A

Power input, $P_i = (i_{59})^2 \times 59 = (23 \cdot 01)^2 \times 59 = 31248 \cdot 189 \text{ W} = 31 \text{ kW}$ *Ans.*

(b)



By KVL,

$$15i_1 - 10i_2 + 2260 = 0$$

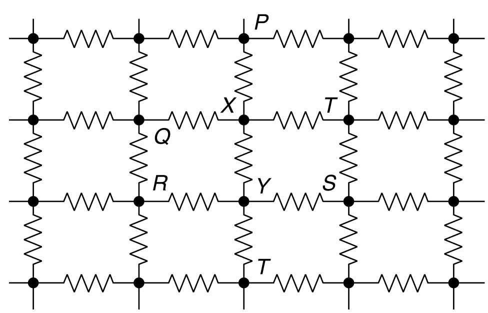
and $\quad 30i_2 - 10i_1 - 160 = 0$

Solving, $\quad i_1 = -206.285$ A

$$i_2 = -63.43 \text{ A}$$

$\therefore$ Power input, $P_i = v \times i = -160 \times (i_1 - i_2) = -160 \times (-206 \cdot 285 + 63 \cdot 43) = 17.37$ kW

2.9 The two-dimensional network of the figure consists of an infinite number of square meshes, each side of which has a resistance of R. Find the effective resistance between two adjacent nodes such as X and Y.



Solution Let the current flowing into the circuit at node X be I. Since the infinite network is symmetrical about X, the current I in going from X to infinity, is divided equally along the branches XQ, XT, XP and XY.

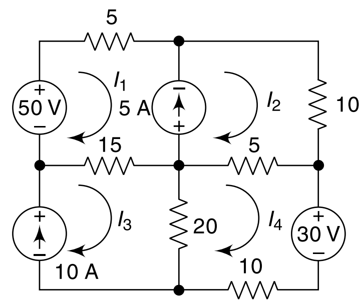
The current I then returns from infinity and is taken from the network at node Y. Again, by symmetry, the currents flowing along RY, XY, SY and TY are each $I/4$.

Hence, the total current flowing along XY is $\frac{I}{4}+\frac{I}{4}=\frac{I}{2}$. So, the voltage between X and Y,

$$V_{XY}=\frac{I}{2}\times R$$

So, the effective resistance between X and Y, $R_{XY}=\frac{V_{XY}}{I}=\frac{R}{2}$ *Ans.*

2.10 Use loop current analysis to find currents in all branches of the network of figure. Also, find the power delivered by 5A current source. All resistances are in ohm.



Solution By KVL,

$$5i_1+10i_2+5(i_2-i_4)+15(i_1-i_3)=50$$

or, $$20i_1+15i_2-15i_3-5i_4=50 \quad \text{(i)}$$

and, $$5(i_4-i_2)+30+10i_4+20(i_4-i_3)=0$$

or, $$-5i_2-20i_3+35i_4=-30 \quad \text{(ii)}$$

By constraint equations,

$$(i_2-i_1)=5 \quad \text{(iii)}$$

and $$i_3=10 \quad \text{(iv)}$$

From Equation (i) and Equation (ii),

$$20(i_2-5)+15i_2-15\times10-5i_4=50$$

or, $$35i_2-5i_4=300 \Rightarrow 7i_2-i_4=60$$

and, $$-5i_2+35i_4=170 \Rightarrow -i_2+7i_4=34$$

Solving $$i_4=6.02083\text{ A}$$

$$i_2=4.4583\text{ A},\ i_2=9.4583\text{ A}$$

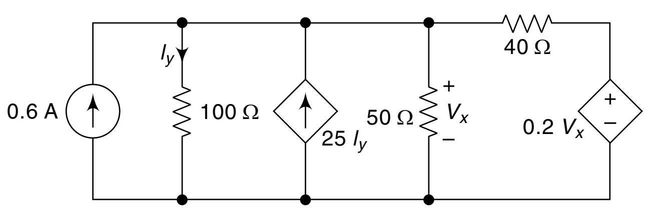
and $$i_3=10A$$

Power delivered by 5A current source = $v\times i=110.83\times5=554.16$ W

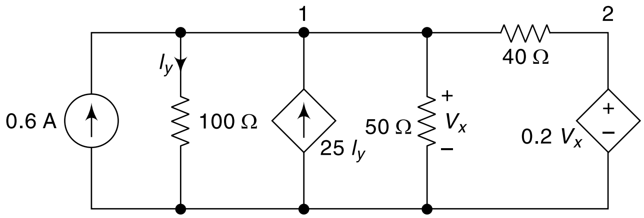
[To calculate the voltage across the 5A current source, v, writing KVL for Mesh (1),

$5i_1+v+15(i_1-i_3)=50 \Rightarrow v=50-20i_1+15i_3=200-20\times4.4583=110.83$ V]

2.11 For the circuit, find the voltage V_x using nodal analysis.



Solution



By KCL at node (1),

$$-0.6 + I_y + \frac{V_x}{50} - 25I_y + \frac{v_1 - v_2}{40} = 0 \qquad \text{(i)}$$

By KCL at node (2),

$$v_2 = 0.2\,V_x \qquad \text{(ii)}$$

and other constraint equation,

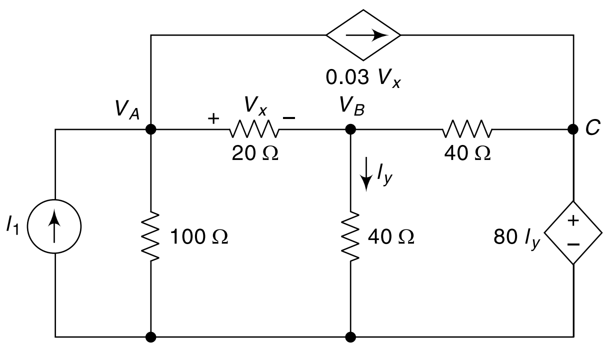
$$I_y = \frac{V_x}{100} \quad \text{and} \quad v_1 = V_x \qquad \text{(iii)}$$

From Equation (i), $-0{\cdot}6 + \frac{V_x}{100} + \frac{V_x}{50} - 25\frac{V_x}{100} + \frac{v_1 - v_2}{40} = 0$

$\Rightarrow$ $$-120 + 2V_x + 4V_x - 50V_x + 5V_x - 5 \times 0.2V_x = 0$$

$\Rightarrow$ $$V_x = \frac{120}{-40} = -3\text{ V}$$ *Ans.*

2.12 Use nodal analysis to find the voltages V_A, V_B and V_x in the circuit, in which $I_1 = 0.4$ A.



Solution By KCL at node (A),

$$-0.4 + \frac{V_A}{100} + \frac{V_A - V_B}{20} + 0.03V_x = 0 \qquad \text{(i)}$$

By KCL at node (B),

$$\frac{V_B - V_A}{20} + \frac{V_B}{40} + \frac{V_B - V_C}{40} = 0 \qquad \text{(ii)}$$

Constraint equations,

$$I_y = \frac{V_B}{40} \qquad \text{(iii)}$$

and $\qquad V_C = 80I_y \qquad$ (iv)

and $\qquad (V_A - V_B) = V_x \qquad$ (v)

From Equation (i),

$$-0.4 + \frac{V_A}{100} + \frac{V_A}{20} - \frac{V_B}{20} + 0.03V_A - 0.03V_B = 0$$

$\Rightarrow \qquad (9V_A - 8V_B) = 40 \qquad$ (vi)

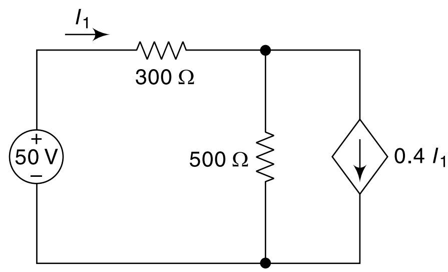
From Equation (ii),

$$V_A = V_B \quad \text{[by Eq. (iii) and Eq. (iv)]}$$

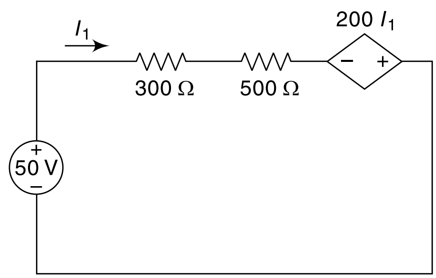
Thus, solving Equation (vi) $V_A = V_B = 40$ V

$\therefore \qquad V_x = (V_A - V_B) = 0 \qquad$ *Ans.*

2.13 For the circuit, use loop analysis to find I_1 and the power absorbed by the 500 Ω resistor.



Solution Converting the dependent current source into dependent voltage source,
By KVL,



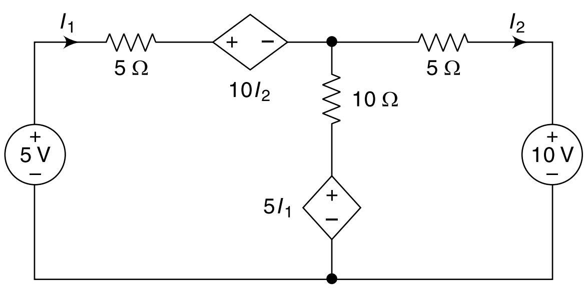
$$800I_1 - 200I_1 = 50 \Rightarrow I_1 = \frac{50}{600} = 0.083 \text{ A} \qquad \textit{Ans.}$$

Power absorbed by the 500 resistor

$$= I_1^2 R = \left(\frac{50}{600}\right)^2 \times 500$$

$$= \frac{500}{144} = 3.47 \text{ W} \qquad \textit{Ans.}$$

2.14 Determine the currents in all the branches of the network.



Solution By KVL, for Mesh (1)

$$5I_1 + 10I_2 + 10(I_1 - I_2) + 5I_1 = 5$$

$\Rightarrow$ $$20I_1 = 5 \quad \Rightarrow \quad I_1 = 0.25 \text{ A}$$

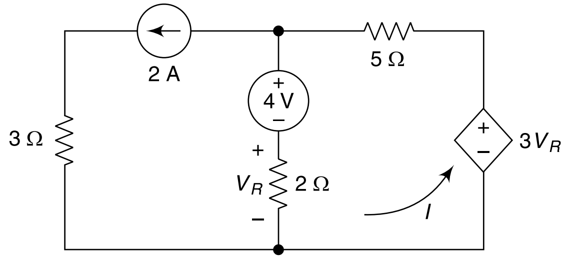
By KVL for Mesh (2),

$$5I_2 + 10 - 5I_1 + (I_2 - I_1) \times 10 = 0$$

$$15I_2 = 15I_1 - 10 = (3.75 - 10) = -6.25$$

$\therefore$ $$I_2 = -0.416 \text{ A} \qquad \textit{Ans.}$$

2.15 Obtain the current I in the network shown.



Solution By KVL for the second mesh

$$-3V_R + 5I + 4 + V_R = 0$$

or, $$-2V_R + 5I + 4 = 0 \tag{i}$$

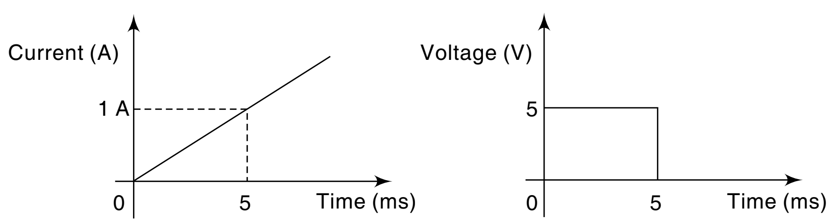
Also, $V_R = 2 \times (I - 2)$, putting this in Equation (i),

$$-2 \times 2(I - 2) + 5I + 4 = 0$$

$\Rightarrow$ $$-4I + 8 + 5I + 4 = 0$$

$\Rightarrow$ $$I = -12 \text{ A} \qquad \textit{Ans.}$$

2.16 The current and voltage profile of an element vs. time has been shown in figure. Determine the element and find its value.



Solution Here, the voltage is not proportional to the current; therefore, the element is not a resistance.

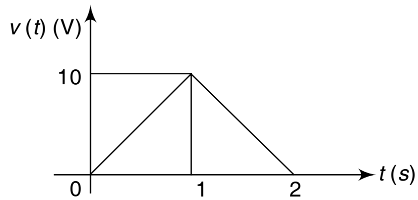
Also, at t = 5ms, $i \neq 0$, but the voltage suddenly drops to zero value, i.e., the element acts as short circuit. As the voltage across a capacitor cannot change instantaneously, the element is not a capacitor.

Now, the current is zero at $t = 0$ and the voltage is zero at t = 5ms. Therefore, we conclude that the element is an inductor.

From the figure, $\frac{di}{dt} = \frac{1}{5\times 10^{-3}} = 200\ A/s$ and $v = 5$ V.

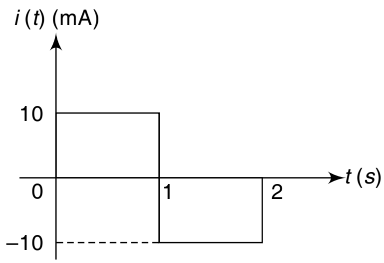
$\because$ $v = L\frac{di}{dt} \Rightarrow L = \frac{v}{di/dt} = \frac{5}{200} = 25mH$ *Ans.*

2.17 The voltage across a capacitor of value $C = 1$ mF is shown in the figure. Find the current waveform.



Solution Here, the voltage can be expressed as,

$$\begin{aligned} v(t) &= 0 && t \le 0 \\ &= 10t && 0 \le t \le 1 \\ &= 20 - 10t && 1 \le t \le 2 \\ &= 0 && t \ge 2 \end{aligned}$$



Current waveform of the capacitor

Since, $i(t) = C\frac{dv(t)}{dt}$, we have the current given as,

$$\begin{aligned} i(t) &= 0 && t < 0 \\ &= 10^{-2} && 0 < t < 1 \\ &= -10^{-2} && 1 < t < 2 \\ &= 0 && t > 2 \end{aligned}$$

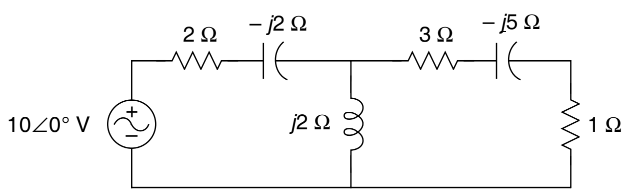
This means that the current waveform consists of two sharp positive and negative pulses of magnitude 10 mA as shown in the figure.

2.18 For the circuit shown in the figure,

(a) Determine the KVL equations

(b) Find the two loop currents I_1 and I_2

(c) Find the power supplied by the source and the power dissipated in each resistor



Solution

(a) KVL Equations:

$$\left.\begin{aligned} &2I_1 - j2I_2 = 10 \\ \text{and} \quad &-j2I_1 + (4 - j3)I_2 = 0 \end{aligned}\right\} Ans.$$

(b) Solving for the currents,

$$I_1 = \frac{\begin{vmatrix} 10 & -j2 \\ 0 & (4-j3) \end{vmatrix}}{\begin{vmatrix} 2 & -j2 \\ -j2 & (4-j3) \end{vmatrix}} = \frac{40 - j30}{12 - j6} = 3.73\angle -10.3^\circ \text{ A} \qquad Ans.$$

and $$I_2 = \frac{\begin{vmatrix} 2 & 10 \\ -j2 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & -j2 \\ -j2 & (4-j3) \end{vmatrix}} = \frac{j20}{12 - j6} = 1.49\angle 116.56^\circ \text{ A} \qquad Ans.$$

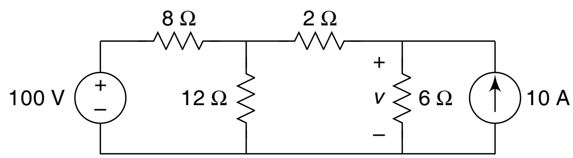
(c) Power supplied by the source,

$$P_s = VI_1 \cos\phi_1 = 10 \times 3.73 \cos(-10.3^\circ) = 36.7 \text{ Watt} \qquad Ans.$$

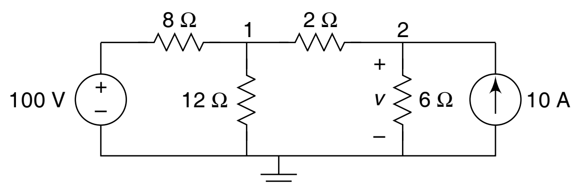
Power dissipated in resistors,

$$\left.\begin{aligned} P_{2\Omega} &= |I_1|^2 \times 2 = 27.78\ W \\ P_{3\Omega} &= |I_2|^2 \times 3 = 6.67\ W \\ P_{1\Omega} &= |I_2|^2 \times 1 = 2.22\ W \end{aligned}\right\} \qquad Ans.$$

2.19 For the circuit shown below, determine the voltage 'v' using nodal analysis.



Solution



Let the node voltages be V_1 and V_2. Here, $V_2 = v$
By KCL,

$$\frac{V_1 - 100}{8} + \frac{V_1}{12} + \frac{V_1 - V_2}{2} = 0 \quad \Rightarrow \quad 17V_1 - 12v = 300 \qquad \text{...(i)}$$

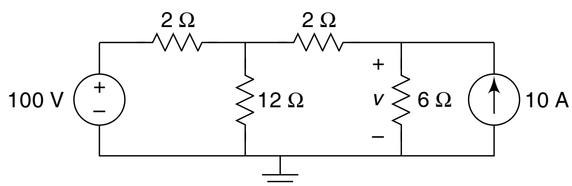
and

$$\frac{V_2 - V_1}{2} + \frac{V_2}{6} - 10 = 0 \quad \Rightarrow \quad -3V_1 + 4v = 60 \qquad \text{...(ii)}$$

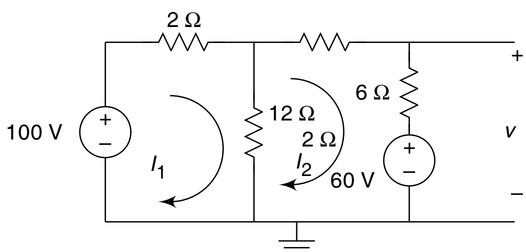
Solving Eq (i) and (ii), we get,

$$v = \frac{\begin{vmatrix} 17 & 300 \\ -3 & 60 \end{vmatrix}}{\begin{vmatrix} 17 & -12 \\ -3 & 4 \end{vmatrix}} = \frac{1920}{32} = 60 \text{ Volt} \qquad \textit{Ans.}$$

2.20 Determine the voltage v in the network in the Figure using nodal analysis.



Solution Converting the current source into equivalent voltage source, we get the following circuit.



By KVL,

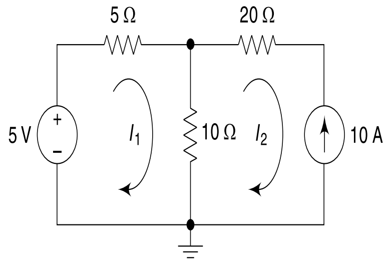
$$\begin{aligned} 14I_1 - 12I_2 &= 100 \\ -12I_1 + 20I_2 &= -60 \end{aligned}$$

Solving for I_2,

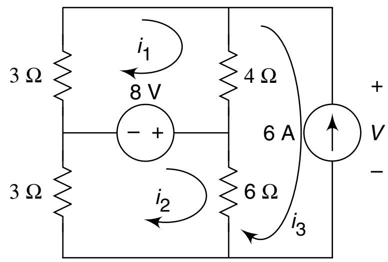
$$I_2 = \frac{\begin{vmatrix} 14 & 100 \\ -12 & -60 \end{vmatrix}}{\begin{vmatrix} 14 & -12 \\ -12 & 20 \end{vmatrix}} = \frac{-840 + 1200}{280 - 144} = \frac{360}{136} = 2.64 \text{ A}$$

$$\therefore \qquad v = (6I_2 + 60) = 75.88 \text{ Volt} \qquad \textit{Ans.}$$

2.21 Determine the voltage V using source transformation and simplification in the figure.



Solution By KVL,



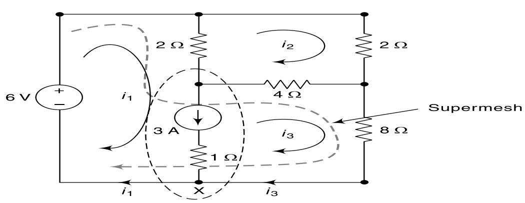
$$4(i_1+6)+8+3i_1=0 \quad \Rightarrow \quad i_1=-\frac{32}{7}\text{ A}$$

and $$9i_2+36-8=0 \quad \Rightarrow \quad i_2=-\frac{28}{9}\text{ A}$$

Thus, the voltage is,

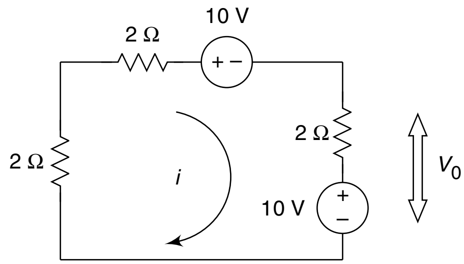
$$V=4(i_1+6)+6(i_2+6)=4\left(-\frac{32}{7}+6\right)+6\left(-\frac{28}{9}+6\right)=23.05\text{ Volt} \qquad \textit{Ans.}$$

2.22 Convert the current sources into the equivalent voltage source given in the figure and hence find the voltage V_0.



Solution Converting the current sources into voltage sources, we get the following circuit.

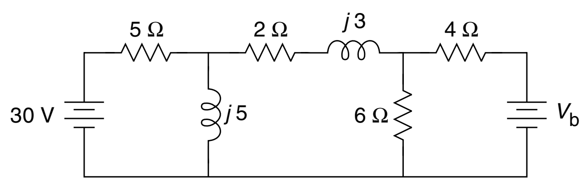
$$\therefore \qquad i=-\frac{20}{6}=-\frac{10}{3}\text{ A}$$



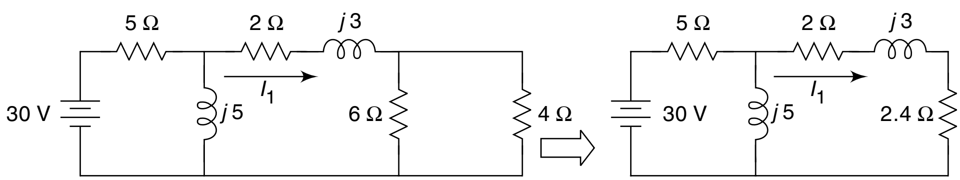
$$\therefore \quad V_0 = 2i + 10 = 2 \times \left(-\frac{10}{3} + 10\right)$$

$$= \frac{10}{3} = 3.33 \text{ Volt} \qquad \textit{Ans.}$$

2.23 In the network shown, determine the voltage V_b which result in a zero current through the $(2 + j3)\ \Omega$ impedance in a branch.



Solution When the 30 V source is acting alone, let the current through the branch $(2 + j3)\ \Omega$ be I_1.



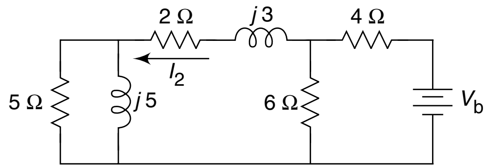
Impedance,

$$Z = 5 + \frac{j5 \times (4.4 + j3)}{4.4 + j8} = \left(\frac{7 + j62}{4.4 + j8}\right) \Omega$$

$$\therefore \quad I = \frac{30}{Z} = \frac{30(4.4 + j8)}{7 + j62}$$

$$\therefore \quad I_1 = I \times \frac{j5}{4.4 + j8} = \frac{30(4.4 + j8)}{7 + j62} \times \left(\frac{j5}{4.4 + j8}\right) = \frac{j150}{7 + j62}\ A$$

When the V_b source is acting alone, let the current through the branch $(2 + j3)\,\Omega$ be I_2.



Impedance,

$$Z = 4 + \frac{6 \times (4.5 + j5.5)}{10.5 + j5.5} = \left(\frac{69 + j55}{10.5 + j5.5}\right)\Omega$$

$$\therefore \quad I' = \frac{V_b}{Z} = \frac{V_b(10.5 + j5.5)}{69 + j55}$$

$$\therefore \quad I_2 = I' \times \frac{6}{10.5 + j5.5} = \frac{V_b(10.5 + j5.5)}{69 + j55} \times \left(\frac{6}{10.5 + j5.5}\right) = \frac{6V_b}{69 + j55}\text{ A}$$

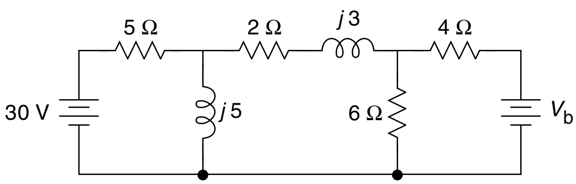
Current through the branch $(2 + j3)\,\Omega$ will be zero, if

$$I_1 = I_2$$

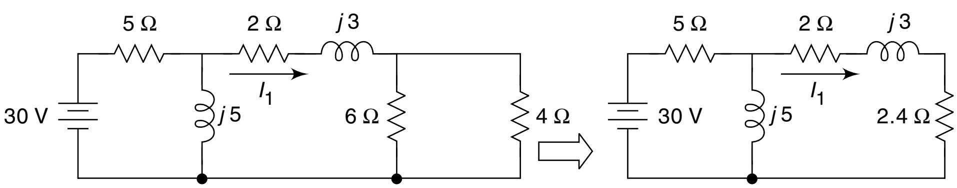
$$\Rightarrow \quad \frac{j150}{7 + j62} = \frac{6V_b}{69 + j55}$$

$$\Rightarrow \quad V_b = (25 + j25)\text{ Volt} = 35.35\angle 45°\text{ Volt} \qquad \textit{Ans.}$$

2.24 Determine the current through the impedance $(2 + j3)\Omega$ in the circuit shown in figure, where, $V_b = 20\angle° \,(V)$.



Solution When the 30 V source is acting alone, let the current through the branch $(2 + j3)\Omega$ be I_1.



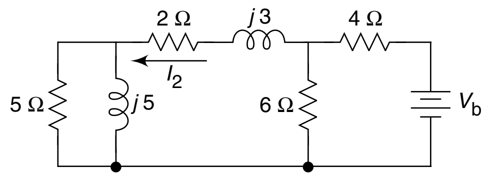
Impedance,

$$Z = 5 + \frac{j5 \times (4.4 + j3)}{4.4 + j8} = \left(\frac{7 + j62}{4.4 + j8}\right)\Omega$$

$$\therefore \quad I = \frac{30}{Z} = \frac{30(4.4 + j8)}{7 + j62}$$

$$\therefore \quad I_1 = I \times \frac{j5}{4.4+j8} = \frac{30(4.4+j8)}{7+j62} \times \left(\frac{j5}{4.4+j8}\right) = \frac{j150}{7+j62} = 2.4\angle 6.44° = (2.38 + j0.27)\ \text{A}$$

When 20 V (V_b) source is acting alone, let the current through the branch $(2 + j3)\,\Omega$ be I_2.



Impedance,

$$Z = 4 + \frac{6 \times (4.5 + j5.5)}{10.5 + j5.5} = \left(\frac{69 + j55}{10.5 + j5.5}\right)\Omega$$

$$\therefore \quad I' = \frac{V_b}{Z} = \frac{20(10.5 + j5.5)}{69 + j55}$$

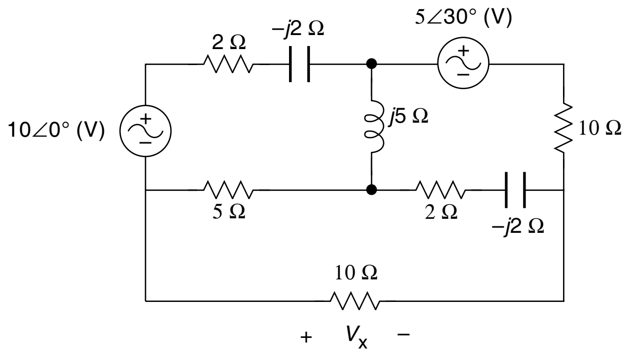
$$\therefore \quad I_2 = I' \times \frac{6}{10.5+j5.5} = \frac{20(10.5+j5.5)}{69+j55} \times \left(\frac{6}{10.5+j5.5}\right) = \frac{120}{69+j55} = 1.36\angle -38.56°$$

$$= (1.06 - j0.85)\ A$$

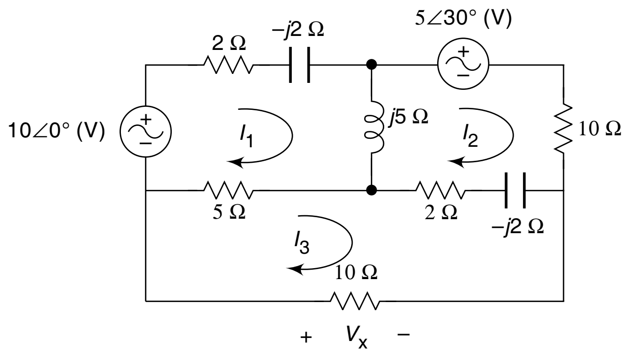
Total current through the branch $(2 + j3)\,\Omega$ is,

$$I = (I_1 - I_2) = (2.38 + j0.27) - (1.06 - j0.85) = (1.32 + j1.12) = 1.73\angle 40.31° \qquad \textit{Ans.}$$

2.25 Write the loop equations of the circuit and find the voltage V_x.



Solution By KVL for the three meshes, we get,



$$(7+j3)I_1 - j5I_2 - 5I_3 = 10 \qquad \ldots(i)$$

$$-j5I_1 + (12+j3)5I_2 - (2-j2)I_3 = -(4.33+j2.5) \qquad \ldots(ii)$$

$$-5I_1 - (2-j2)I_2 + (17-j2)I_3 = 0 \qquad \ldots(iii)$$

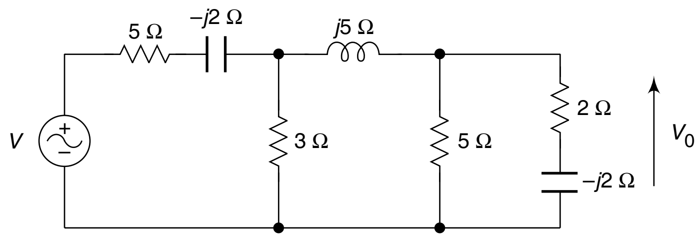
Solving for I_3 from equations (*i*), (*ii*) and (*iii*), we get,

$$I_3 = \frac{\begin{vmatrix} (7+j3) & -j5 & 10 \\ -j5 & (12+j3) & -(4.33+j2.5) \\ -5 & -(2-j2) & 0 \end{vmatrix}}{\begin{vmatrix} (7+j3) & -j5 & -5 \\ -j5 & (12+j3) & -(2-j2) \\ -5 & -(2-j2) & (17-j2) \end{vmatrix}} = 0.435\angle -194.15^\circ \quad \text{(A)}$$

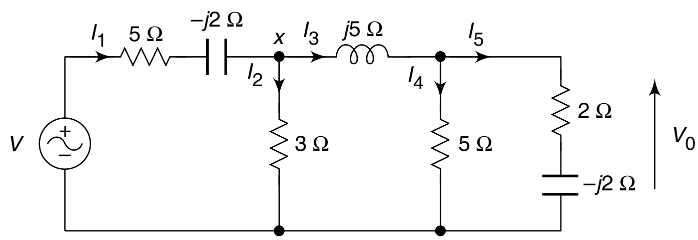
Therefore, the required voltage is,

$$V_x = 10 \times I_3 = 10 \times 0.435\angle -194.15^\circ = 4.35\angle -194.15^\circ \quad \text{(V)} \qquad \textit{Ans.}$$

2.26 For the network shown, find the value of the voltage V which results in the output voltage V_0 = 5 Volt.



Solution For V_0 = 5 V, current in $(2-j2)\ \Omega$ branch is,



$$I_5 = \frac{5}{2-j2}$$

Also, $$I_4 = \frac{5}{5} = 1\text{ A}$$

$$\therefore \qquad I_3 = (I_4 + I_5) = \left(1 + \frac{5}{2 - j2}\right) = \left(\frac{7 - j2}{2 - j2}\right) \text{A}$$

Voltage at node x is,

$$V_x = 5 + I_3 \times j5 = 5 + \left(\frac{7 - j2}{2 - j2}\right) \times j5 = \left(\frac{20 + j25}{2 - j2}\right)$$

$$\therefore \qquad I_2 = \frac{V_x}{3} = \left(\frac{20/3 + j\,25/3}{2 - j2}\right)$$

$$\therefore \qquad I_1 = (I_2 + I_3) = \left(\frac{20/3 + j\,25/3}{2 - j2}\right) + \left(\frac{7 - j2}{2 - j2}\right) = \left(\frac{13.67 + j6.33}{2 - j2}\right)$$

Now, by KVL for the left mesh, we get,

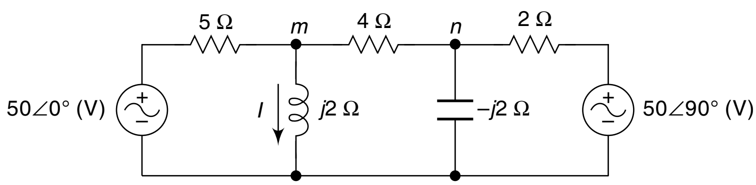
$$V = V_x + I_1(5 - j2) = \left(\frac{20 + j25}{2 - j2}\right) + \left(\frac{13.67 + j6.33}{2 - j2}\right) \times (5 - j2)$$

$$= \frac{101 + j29.33}{2 - j2}$$

$$= \frac{105.17\angle 16.19^\circ}{2.83\angle -45^\circ}$$

$$= 37.18\angle 61.19^\circ \quad \text{(V)} \qquad \textit{Ans.}$$

2.27 (a) Determine the voltages of node 'm' and 'n' with respect to the reference in the circuit shown.
(b) Find the current 'I' using node voltage method.



Solution

(a) By KCL at node (m), we get,

$$\frac{V_m - 50}{5} + \frac{V_m}{j2} + \frac{V_m - V_n}{4} = 0$$

$$\Rightarrow \qquad (10 + j9)V_m - j5V_n = j200 \tag{i}$$

By KCL at node (n), we get,

$$\frac{V_n - V_m}{4} + \frac{V_n}{-j2} + \frac{V_n - j50}{2} = 0$$

$$\Rightarrow \qquad j1V_m + (2 - j3)V_n = 100 \tag{ii}$$

Solving for V_m and V_n from equations (*i*) and (*ii*), we get,

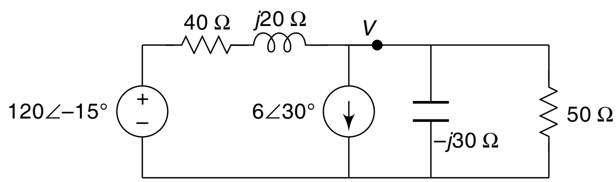
$$V_m = \frac{\begin{vmatrix} j200 & -j5 \\ 100 & (2-j3) \end{vmatrix}}{\begin{vmatrix} (10+j9) & -j5 \\ j1 & (2-j3) \end{vmatrix}} = \frac{600-j900}{42-j12} = 24.76\angle -40.36^\circ \quad \text{(V)} \qquad \textit{Ans.}$$

$$V_n = \frac{\begin{vmatrix} (10+j9) & j200 \\ j1 & 100 \end{vmatrix}}{\begin{vmatrix} (10+j9) & -j5 \\ j1 & (2-j3) \end{vmatrix}} = \frac{1200+j900}{42-j12} = 34.34\angle 52.81^\circ \quad \text{(V)} \qquad \textit{Ans.}$$

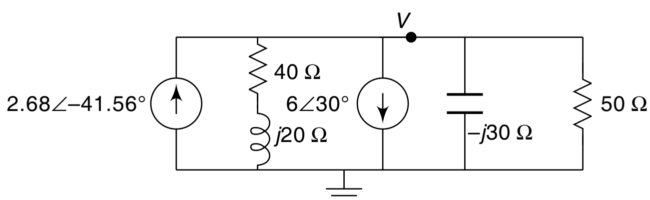
(b) Therefore, the required current is,

$$I = \frac{V_m}{j2} = \frac{24.76\angle -40.36^\circ}{2\angle 90^\circ} = 12.38\angle -130.36^\circ \quad \text{(A)} \qquad \textit{Ans.}$$

2.28 Use Node voltage method to find V in the circuit.



Solution Converting the voltage source into current source, we get the circuit shown below.



By KCL,

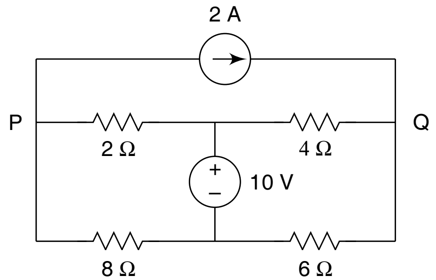
$$\frac{V}{40+j20} + \frac{V}{-j30} + \frac{V}{50} = 2.68\angle -41.56^\circ - 6\angle 30^\circ$$

$$\Rightarrow \quad V[0.022\angle 26.56^\circ + j0.033 + 0.02] = 2 - j1.78 - 5.196 - j3$$

$$\Rightarrow \qquad V = \frac{-3.196 - j4.78}{0.02 + j0.01 + j0.033 + 0.02} = \frac{-3.196 - j4.78}{0.04 + j0.043}$$

$$\Rightarrow \qquad V = -97.62\angle 8.94^\circ \text{ Volt} \qquad \textit{Ans.}$$

2.29 Using source transformation and simplification, determine the voltage between the points P and Q shown in the figure.



Solution By KCL,

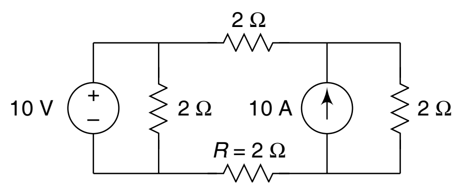
At Node-1, $$\frac{V_P - 10}{2} + \frac{V_P}{8} + 2 = 0 \Rightarrow V_P = 4.8 \text{ V}$$

At Node-2, $$\frac{V_Q - 10}{4} + \frac{V_Q}{6} - 2 = 0 \Rightarrow V_Q = 10.8 \text{ V}$$

Therefore, the voltage between the points P and Q is,

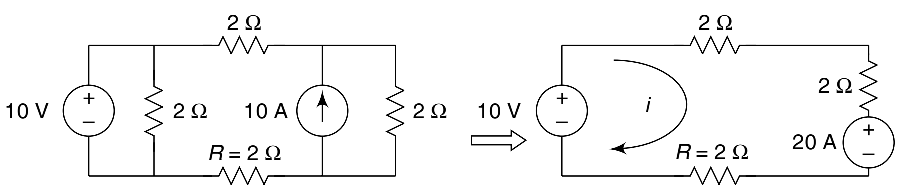
$$(V_P - V_Q) = (4.8 - 10.8) = -6 \text{ Volt} \quad \textit{Ans.}$$

2.30 Find the voltage across the resistor $R = 2\,\Omega$ in the figure.



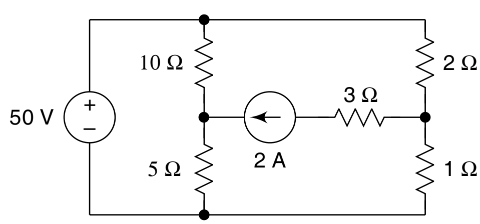
Solution Since the $2\,\Omega$ resistor is in parallel with the 10 V voltage source, it may be ignored. Also, converting the current source into equivalent voltage source, we get the simplified circuit as shown in the figure.

$$\therefore \quad i = \frac{-10}{6} = -\frac{5}{3} \text{ A}$$



Voltage across $R = 2\,\Omega$ resistor, is, $V = i \times 2 = -\frac{5}{3} \times 2 = -\frac{10}{3} = -3.33$ Volt *Ans.*

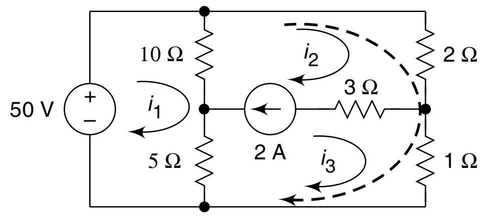
2.31 Find the current through the 5 Ω resistor in the figure using mesh analysis.



Solution

By KVL for the first mesh,

$$15i_1 - 10i_2 - 5i_3 = 50 \Rightarrow 3i_1 - 2i_2 - i_3 = 10 \quad \text{(i)}$$



By KVL for the supermesh,

$$2i_2 + i_3 + 5(i_3 - i_1) + 10(i_2 - i_1) = 0 \quad \Rightarrow \quad -15i_1 + 12i_2 + 6i_3 = 0 \qquad \text{(ii)}$$

Also, the constraint equation is that, $(i_2 - i_3) = 2 \quad \Rightarrow \quad i_2 = (2 + i_3)$

Putting this value of i_2 in equations (i) and (ii), we get,

$$3i_1 - 3i_3 = 14$$
$$-15i_1 + 18i_3 = -24$$

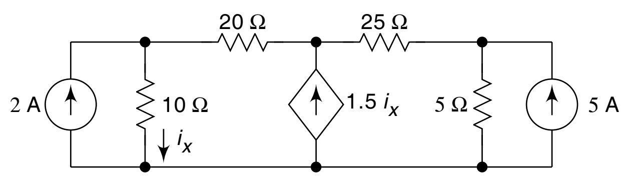
Solving these two equations,

$$i_1 = 20\text{ A} \qquad \text{and} \qquad i_2 = \frac{46}{3}\text{A} = 15.33\text{ A}$$

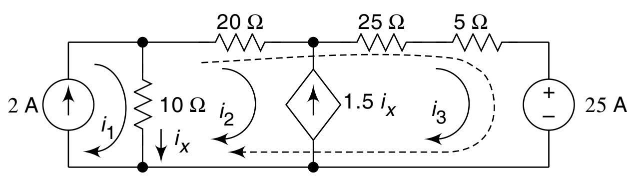
$\therefore$ Current through the 5 Ω resistor is,

$$i = (i_1 - i_3) = \frac{14}{3} = 4.67\text{ A} \qquad \textit{Ans.}$$

2.32 Use mesh analysis to find the current i_x



Solution We convert the 5 A current source into its equivalent voltage source.



From the first loop, we get, $i_1 = 2$ A

BY KVL for the supermesh as shown by the dotted line, we get,

$$20i_2 + 30i_3 + 25 + 10 \times (i_2 - i_1) = 0$$

Putting the value of i_1,

$$20i_2 + 30i_3 + 25 + 10 \times (i_2 - 2) = 0 \quad \Rightarrow \quad 6i_2 + 6i_3 + 1 = 0 \qquad \text{(i)}$$

Also by KCL we get, the following constraint equations.

$$i_x = (i_1 - i_2) = (2 - i_2) \quad \Rightarrow \quad i_2 = (2 - i_x)$$

and

$$1.5i_x = (i_3 - i_2) \quad \Rightarrow \quad i_3 = (1.5i_x + i_2) = (1.5i_x + 2 - i_x) = (2 + 0.5i_x)$$

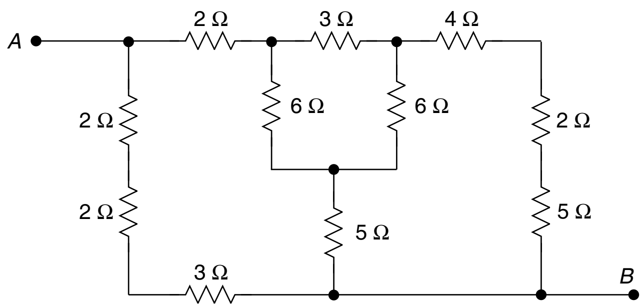
Putting the values of i_2 and i_3 in equation (i), we get,

$$6i_2 + 6i_3 + 1 = 0$$

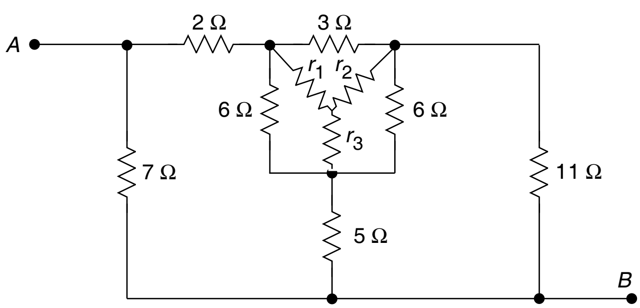
$$\Rightarrow \quad 6 \times (2 - i_x) + 6 \times (2 + 0.5i_x) = -1$$

$$\Rightarrow \quad i_x = \frac{25}{3} = 8.33 \text{ A} \qquad \textit{Ans.}$$

2.33 Calculate the effective resistance between the points A and B in the circuit shown in figure



Solution The 2 Ω, 2 Ω and 3 Ω resistances are in series and 4 Ω, 2 Ω and 5 Ω resistances are also in series. The reduced circuit is shown in the figure.

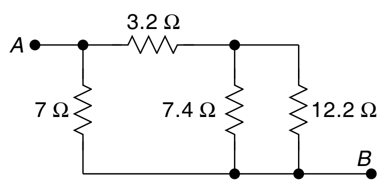


Converting the delta consisting of the resistances of 6 Ω, 6 Ω and 3 Ω into equivalent star the circuit is reduced as shown in the figure.

$$r_1 = \frac{6 \times 3}{6 + 3 + 6} = 1.2 \ \Omega$$

$$r_2 = \frac{6 \times 3}{6 + 3 + 6} = 1.2 \ \Omega$$

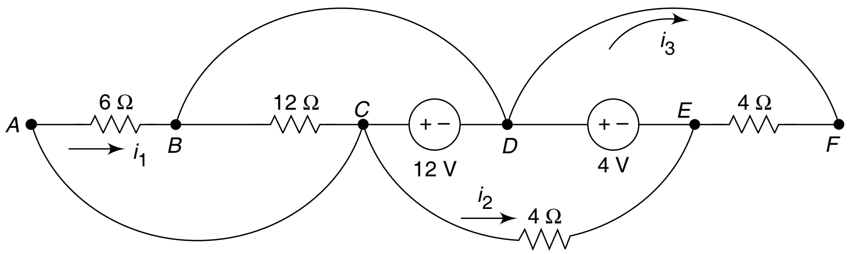
$$r_3 = \frac{6 \times 6}{6 + 3 + 6} = 2.4 \ \Omega$$



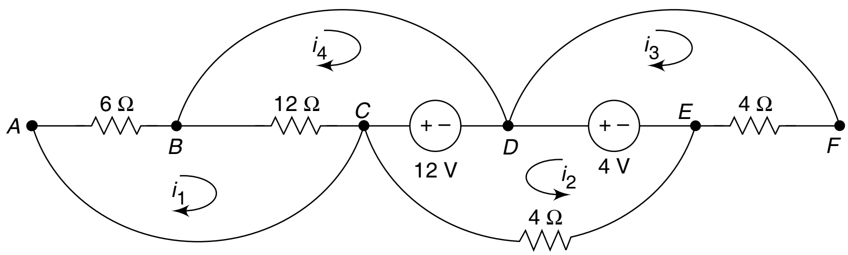
The equivalent resistance between terminals A and B is given as,

$$R_{AB} = 7 \| \left[3.2 + \frac{7.4 \times 12.2}{7.4 + 12.2}\right] = 7 \| [3.2 + 4.6061] = \frac{7 \times 7.8061}{7 + 7.8061} = 3.69\ \Omega \qquad Ans.$$

2.34 Find the currents i_1, i_2 and i_3 and powers delivered by the sources of the network shown in the figure.



Solution We consider the four meshes and the mesh currents as shown in the figure below.



By KVL for the meshes, we get,

$$18i_1 - 12i_4 = 0 \quad \Rightarrow \quad 3i_1 = 2i_4 \quad \Rightarrow \quad i_4 = \frac{3}{2}i_1$$

$$-12i_1 + 12i_4 = 12 \quad \Rightarrow \quad 12i_1 = 12i_4 - 12 \quad = 12\left(\frac{3}{2}i_1\right) - 12 = 18i_1 - 12 \quad \Rightarrow \quad i_1 = 2A$$

$\therefore$

$$i_4 = 3A$$

$$4i_2 = 16 \quad \Rightarrow \quad i_2 = 4A$$

$$4i_3 = 4 \quad \Rightarrow \quad i_3 = 1A$$

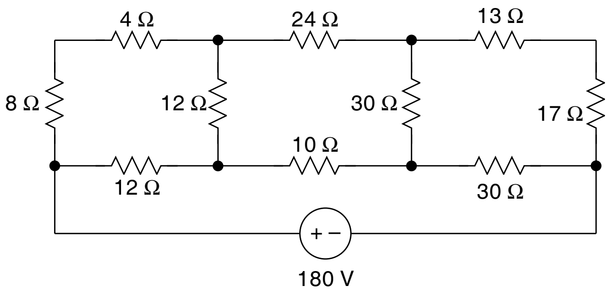
Therefore, the required currents are,

$$i_1 = 2A; \quad i_2 = 4A; \quad i_3 = 1A$$

Power delivered by 12 V source $= 12 \times (i_4 + i_2) = 12 \times 7 = 84$ W *Ans.*

Power delivered by 4 V source $= 4 \times (i_2 + i_3) = 4 \times 5 = 20$ W *Ans.*

2.35 Determine the current through $10\,\Omega$ resistance in the network shown in the figure by using star-delta conversion.



Solution The resistances 8 Ω and 4 Ω are in series and resistances 13 Ω and 17 Ω are also in series. The reduced circuit is shown in Figure (i).

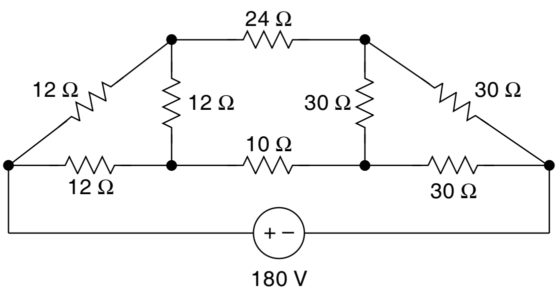


Figure (i)

There are two deltas in the circuit, one consisting of the resistances 12 Ω each and the other consisting of the resistances 30 Ω each. We convert the deltas into their equivalent star and the reduced circuit is shown in Figure (ii).

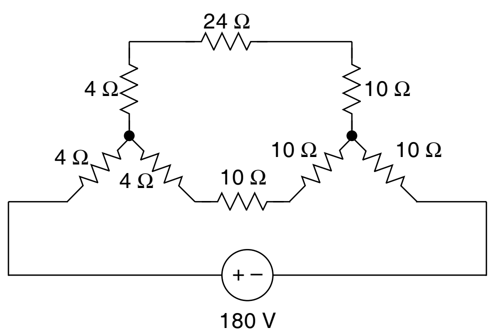


Figure (ii)

Equivalent resistances in star are,

$$R = \frac{12 \times 12}{12 + 12 + 12} = 4\,\Omega$$

$$R' = \frac{30 \times 30}{30 + 30 + 30} = 10\,\Omega$$

From Fig (ii), further modified circuit is shown in Figure (iii).

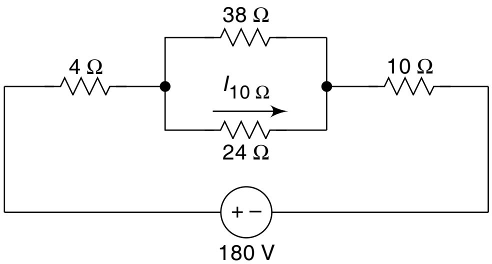


Figure (iii)

Therefore, the current through the 10Ω resistance is the current through the 24Ω resistance branch in Figure (iii). This is given as,

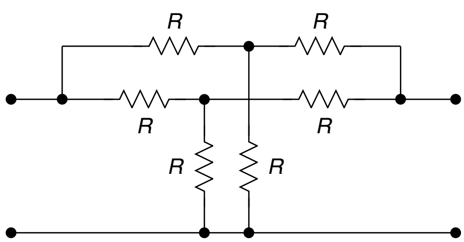
Total current,

$$I = \frac{180}{4 + \dfrac{38 \times 24}{38 + 24} + 10} = 6.27\text{A}$$

$\therefore$

$$\begin{aligned} I_{10\Omega} &= I \times \frac{38}{38+24} \\ &= 6.27 \times \frac{38}{38+24} \\ &= 3.8426\text{A} \qquad \textit{Ans.} \end{aligned}$$

2.36 Find the equivalent π network for the circuit shown in the figure.



Solution We convert the outer star into its equivalent delta with each resistance equal to $R' = R + R + \dfrac{R \times R}{R} = 3R$. The reduced circuit is shown in Figure (i).

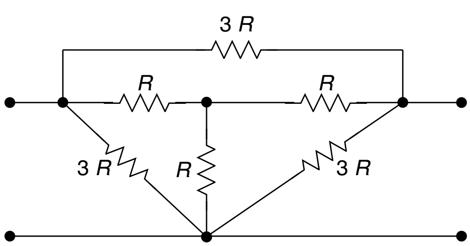
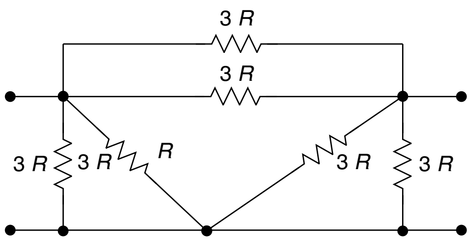
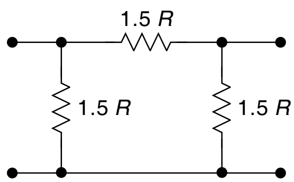


Figure (i)

Now, we convert the inner star into its equivalent delta with each resistance equal to $R'' = R + R + \dfrac{R \times R}{R} = 3R$. The reduced circuit is shown in Figure (ii).



Combining all the parallel resistances the equivalent π network is shown in Figure (iii).



2.37 The element of a 500 watt electric iron is designed for use on a 200 volt supply. What value of resistance is needed to be connected in series in order that the iron can be operated from 240 volt supply?

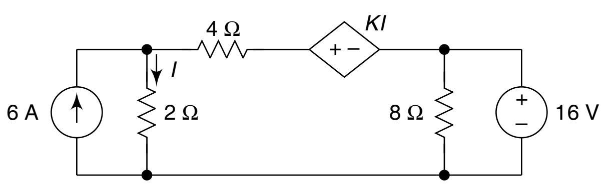
Solution Since the iron is rated for 500 W, 200 V, the resistance of the iron coil is,

$$R = \frac{200^2}{500} = 80\,\Omega$$

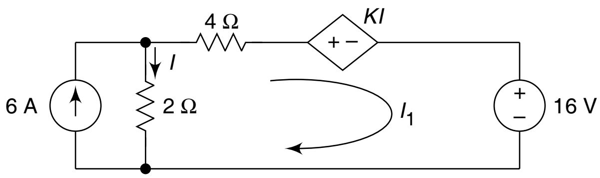
When an external resistance R_x is connected in series with the iron, the total resistance in the circuit is $R_T = (R + R_x)$. If this is connected to a 240 V supply, the power equation becomes,

$$P = \frac{V^2}{R_T} \quad \Rightarrow \quad 500 = \frac{240^2}{80 + R_x} \quad \Rightarrow \quad R_x = 35.2\,\Omega \qquad \textit{Ans.}$$

2.38 Find the value of the constant 'K' in the circuit shown in the figure, such that the power dissipated in $2\,\Omega$ resistor does not exceed 50 W.



Solution Here, the $8\,\Omega$ resistance in parallel with the 16 V source can be ignored. Converting the dependent voltage source into its equivalent current source, we get the following circuit.



By KVL for the right mesh, we get,

$$4I_1 - KI + 16 + 2\times(I_1 - 6) = 0 \quad \Rightarrow \quad I_1 = \left(\frac{KI-4}{6}\right)$$

Also,
$$I = (6 - I_1) = 6 - \left(\frac{KI-4}{6}\right) = \frac{40-KI}{6} \quad \Rightarrow \quad I = \frac{40}{6+K}$$

Now, the power dissipated in 2Ω resistance is 50 W.

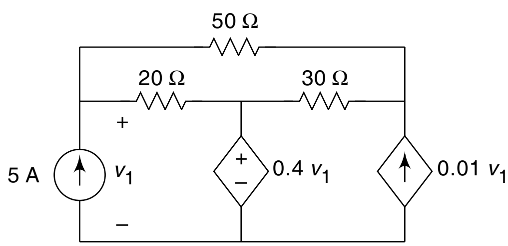
$\therefore$
$$P_{2\Omega} = I^2 \times 2$$

$$50 = \left(\frac{40}{6+K}\right)^2 \times 2$$

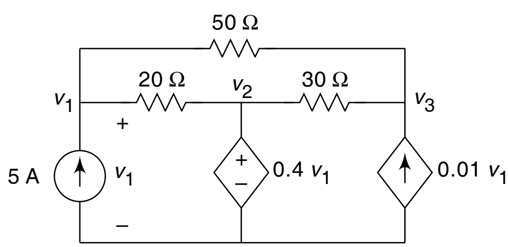
$\Rightarrow$
$$\frac{40}{6+K} = 5$$

$\Rightarrow$
$$K = 2 \qquad Ans.$$

2.39 Use nodal analysis to determine v_1 and power being supplied by the dependent current source in the circuit shown in the figure.



Solution We first label the circuit as shown in the figure below.



By KCL at node 1,

$$\frac{v_1 - v_3}{50} + \frac{v_1 - v_2}{20} = 5$$
$$7v_1 - 5v_2 - 2v_3 = 500 \qquad \text{(i)}$$

By KCL at node 3,

$$\frac{v_3 - v_1}{50} + \frac{v_3 - v_2}{30} = 0.01v_1$$
$$9v_1 + 10v_2 - 16v_3 = 0 \quad \text{(ii)}$$

Also, by constraint equation,

$$v_2 = 0.4v_1$$

Putting this value in (*i*) and (*ii*), we get,

$$7v_1 - 5(0.4v_1) - 2v_3 = 500 \quad \Rightarrow \quad 5v_1 - 2v_3 = 500 \quad \text{(iii)}$$

$$9v_1 + 10\,(0.4v_1) - 16v_3 = 0 \quad \Rightarrow \quad 13v_1 = 16v_3 \quad \text{(iv)}$$

From (iii), putting the value of v_3, we get,

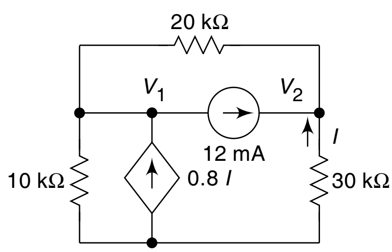
$$5v_1 - 2\left(\frac{13}{16}\right)v_1 = 500 \quad \Rightarrow \quad v_1 = \frac{500 \times 16}{54} = 148.148 \text{ V} \qquad \textit{Ans.}$$

$$\therefore \qquad v_3 = \frac{13}{16} \times v_1 = \frac{13}{16} \times 148.148 = 120.37 \text{ V}$$

$\therefore$ Power supplied by the dependent current source is,

$$P = v_3 \times 0.01v_1 = 120.37 \times 0.01 \times 148.148 = 178.32 \text{ W} \qquad \textit{Ans.}$$

2.40 Calculate the node voltages in the circuit shown in the figure.



Solution By KCL for the two nodes, we get,
At node 1,

$$\frac{V_1}{10 \times 10^3} - 0.8I + 12 \times 10^{-3} + \frac{V_1 - V_2}{20 \times 10^3} = 0$$

$$\Rightarrow \qquad 3V_1 - V_2 = 16 \times 10^3\, I - 240 \quad \text{(i)}$$

At node 2,

$$-12 \times 10^{-3} + \frac{V_2 - V_1}{20 \times 10^3} + \frac{V_2}{30 \times 10^3} = 0 \quad -3V_1 + 5V_2 = 720 \quad \text{(ii)}$$

Also, $$I = -\frac{V_2}{30 \times 10^3}$$

Putting this in (*i*), we get,

$$3V_1 - V_2 = 16 \times 10^3\left(-\frac{V_2}{30 \times 10^3}\right) - 240 = -\frac{8}{15}V_2 - 240$$

$$\Rightarrow \qquad -45V_1 + 7V_2 = 3600 \qquad \text{(iii)}$$

Solving (i) and (iii), we get,

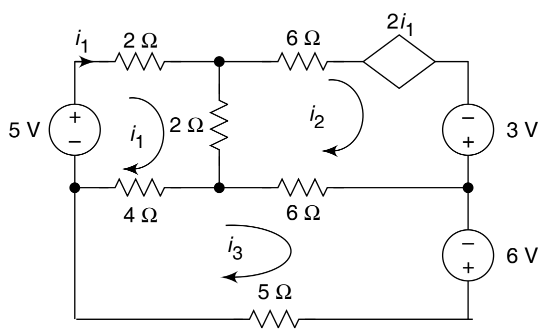
$$V_1 = \frac{\begin{vmatrix} 720 & 5 \\ 3600 & 7 \end{vmatrix}}{\begin{vmatrix} -3 & 5 \\ -45 & 7 \end{vmatrix}} = -\frac{12960}{204} = -63.53 \text{ V} \qquad \textit{Ans.}$$

$$V_2 = \frac{\begin{vmatrix} -3 & 720 \\ -45 & 3600 \end{vmatrix}}{\begin{vmatrix} -3 & 5 \\ -45 & 7 \end{vmatrix}} = -\frac{21600}{204} = 105.88 \text{ V} \qquad \textit{Ans.}$$

2.41 Draw a circuit and its dual if the mesh equations of the circuit are given as,

$$\begin{aligned} 8i_1 - 2i_2 - 4i_3 &= 5 \\ 14i_2 - 6i_3 &= 3 \\ -4i_1 - 6i_2 + 15i_3 &= 6 \end{aligned}$$

Solution The circuit satisfying the mesh equations is shown in the figure below.

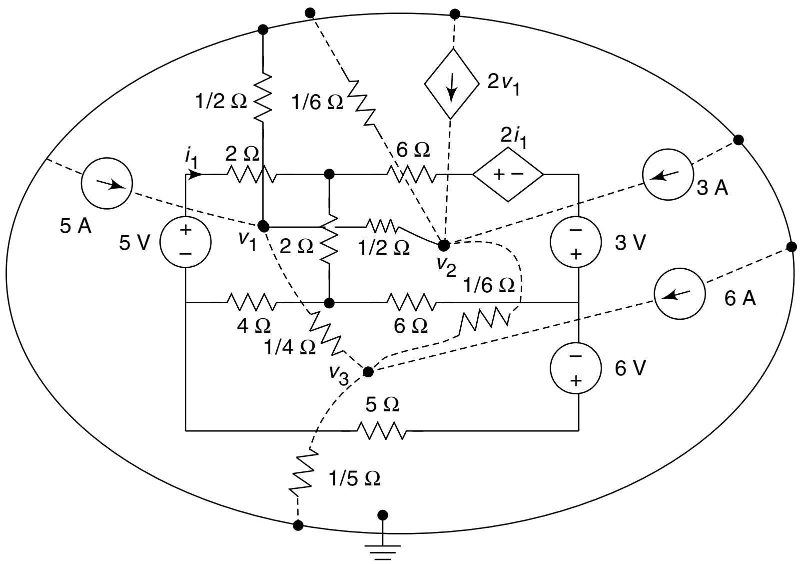


The dual equations will be

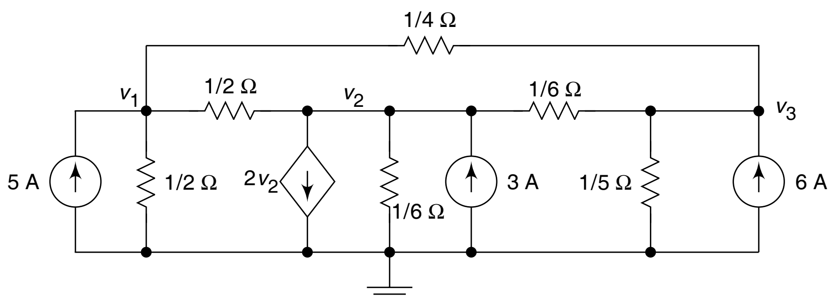
$$\begin{aligned} 8v_1 - 2v_2 - 4v_3 &= 5 \\ 14v_2 - 6v_3 &= 3 \\ -4v_1 - 6v_2 + 15v_3 &= 6 \end{aligned}$$

Here, v_1, v_2, and v_3 are the node voltages. In the dual circuit, resistances will be replaced by conductances and voltage sources by the current sources.

Following the procedure mentioned in Art. 2.10, we construct the dual circuit as shown below.

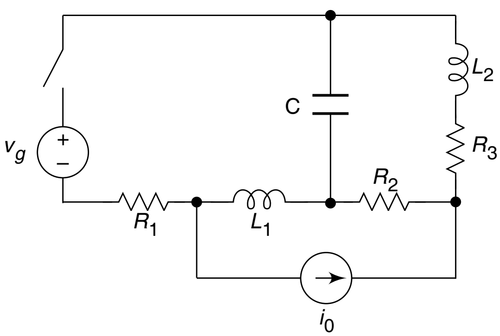


Therefore, the dual circuit is shown below.

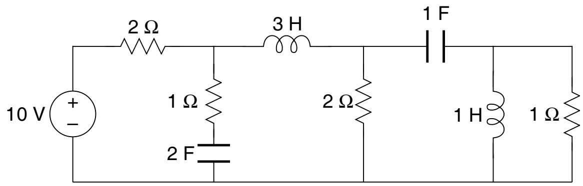


2.42 Draw the dual of the circuit shown in the figure.

(a)

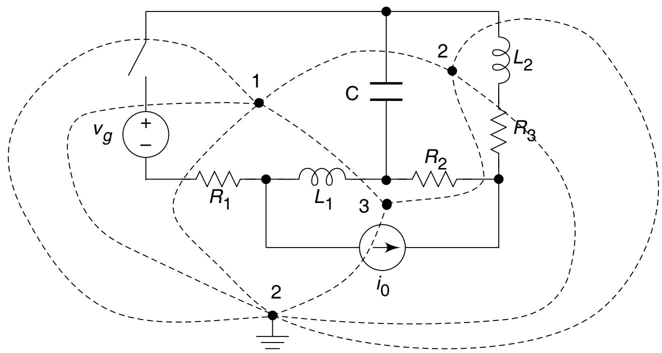


(b)

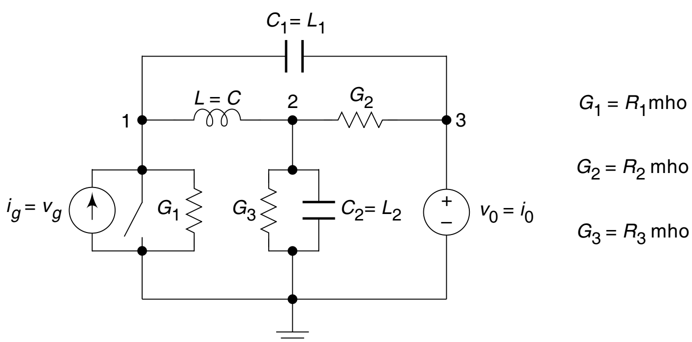


Solution

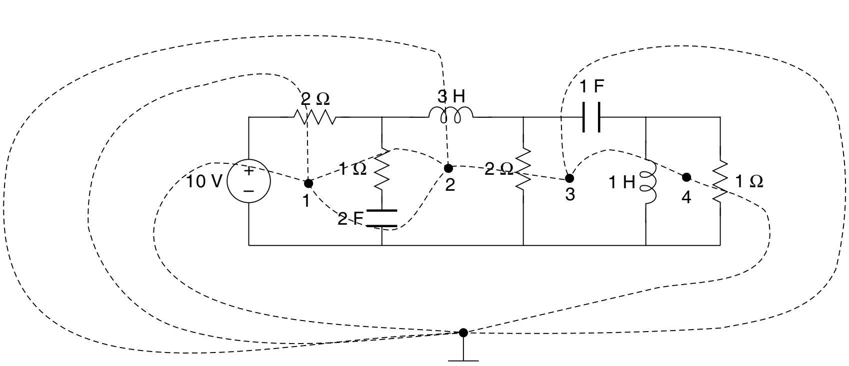
(a) The dual network is drawn as shown below.



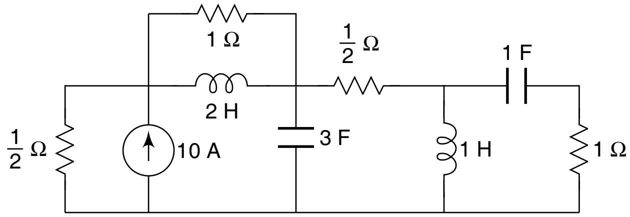
The final dual circuit becomes as shown below.



(b) The dual network is drawn as shown below.



The final dual network is shown below.



MULTIPLE-CHOICE QUESTIONS

2.1 Find the odd one from the following elements.
(a) Inductor (b) Capacitor (c) Resistor (d) Transistor

2.2 Kirchhoff's laws are valid for
(a) linear circuits only.
(b) passive time-invariant circuits.
(c) non-linear circuits only.
(d) both linear and non-linear circuits.

2.3 Kirchhoff's laws are applicable to
(a) d.c. circuits.
(b) circuits with sinusoidal excitation only.
(c) circuits with d.c. and sinusoidal excitation only
(d) circuits with any excitation.

2.4 Kirchhoff's law fails in case of
(a) linear networks.
(b) non-linear networks.
(c) dual networks.
(d) distributed parameter networks.

2.5 KCL is a consequence of law of conservation of
(a) energy (b) charge (c) flux (d) all of the above.

2.6 A component that opposes the change in circuit current is
(a) resistance (b) capacitance (c) inductance (d) conductance.

2.7 A component that opposes the change in circuit voltage is
(a) resistance (b) capacitance (c) inductance (d) conductance

2.8 For a d.c. voltage, an inductor
(a) is virtually a short-circuit.
(b) is an open-circuit.
(c) depends on polarity.
(d) depends on voltage value.

2.9 A network N' is a dual of a network N if
(a) both of them have same mesh equations.
(b) both of them have same node equations.
(c) mesh equations of one of them are node equations of the other.
(d) none of the above.

2.10 A connected planar network has 4 nodes and 5 elements. The number of meshes in its dual network is
(a) 4 (b) 3 (c) 2 (d) 1.

2.11 Two networks can be dual when
(a) their nodal equations are the same.
(b) the loop equations of one network are the nodal equations of the other.
(c) their loop equations are the same.
(d) none of these.

2.12 The internal impedance of an ideal current source is
(a) zero (b) infinite (c) both (a) and (b) (d) none of these.

2.13 The internal impedance of an ideal voltage source is
(a) zero (b) infinite (c) both (a) and (b) (d) none of these.

2.14 The internal impedance of a dependent voltage source is
(a) zero (b) infinity (c) fraction of ohm (d) any unknown value.

2.15 An ideal voltage source will charge an ideal capacitor
(a) in infinite time (b) exponentially (c) instantaneously (d) none of the above.

2.16 A practical current source is usually represented by
(a) a resistance in series with an ideal current source.
(b) a resistance in parallel with an ideal current source.
(c) a resistance in series with an ideal voltage source.
(d) none of the above.

2.17 Energy stored in a capacitor is

(a) $\frac{1}{4}CV^2$ (b) $\frac{1}{2}CV^2$ (c) $\int_0^\infty \frac{1}{2}C$ (d) 0

2.18 The node method of circuit analysis is based on
(a) KVL and Ohm's law (b) KCL and KVL
(c) KCL, KVL and Ohm's law (d) KCL and Ohm's law

2.19 The loop method of circuit analysis is based on
(a) KVL and Ohm's law. (b) KCL and KVL.
(c) KCL, KVL and Ohm's law. (d) KCL and Ohm's law.

2.20 Two wires A and B of the same material and length L and $2L$ have radius r and $2r$, respectively. The ratio of their specific resistance will be
(a) 1 : 1 (b) 1 : 2 (b) 1 : 4 (d) 1 : 8

2.21 There are two wires A and B. A is 20 times longer than B and has half the cross-section of that of B. If the resistance of B is 1 Ω, the resistance of A will be

(a) 40 Ω (b) $\frac{1}{40}$ Ω (c) 20 Ω (d) 10 Ω

2.22 The resistance between the opposite faces of 1 m cube is found to be 1 Ω. If its length is increased to 2 m, with its volume remaining the same, then its resistance between the opposite faces along its length is
(a) 2 Ω (b) 4 Ω (c) 1 Ω (d) 8 Ω
(e) $\frac{1}{2}$ Ω

2.23 A wire of length l and of circular cross-section of radius r has a resistance R ohm. Another wire of same material and cross-sectional radius $2r$ will have the same resistance R if the length is
(a) $2l$ (b) $l/2$ (c) $4l$ (d) l^2

2.24 Two resistances of equal value, when connected in parallel give an equivalent resistance of R. If these resistances are connected in series, the equivalent resistance will be

(a) R (b) $4R$ (c) $2R$ (d) $\frac{R}{2}$

2.25 A series arrangement of 'n' identical resistances is changed into a parallel arrangement. The new total resistance will become____times the original resistance.

(a) $\frac{1}{n}$ (b) $\frac{1}{n^2}$ (c) $\frac{1}{n^3}$ (d) $\frac{1}{n^4}$

2.26 If a two-terminal network element in a circuit has voltage and current variables that follow the associated reference directions and its power is negative, which of the following is true?
(a) The element is supplying energy to the rest of the circuit.
(b) The element is receiving energy from the rest of the circuit.
(c) Either (a) or (b) could be true.

2.27 If an ideal voltage source and an ideal current source are connected in parallel, what are the properties of the combination?
(a) The same as a voltage source.
(b) The same as a current source.
(c) Different from either a voltage source or a current source.

2.28 If an ideal voltage source and an ideal current source are connected in series, what are the properties of the combination?
(a) The same as a voltage source.
(b) The same as a current source.
(c) Different from either a voltage source or a current source.

2.29 When ideal voltage sources are connected in series, which of the following is true?
(a) The voltages add, independent of whether the individual sources are constant valued or have outputs that are functions of time.
(b) The connection violates KVL; thus it is not permitted.
(c) Neither is true.

2.30 When ideal arbitrary voltage sources are connected in parallel, which of the following is true?
(a) The voltages add, independent of whether the individual sources are constant valued or have outputs that are functions of time.
(b) The connection violates KVL; thus it is not permitted.
(c) Neither is true.

2.31 When ideal arbitrary current sources are connected in series, which of the following is true?
(a) The currents add, independent of whether the individual sources are constant valued or have outputs that are functions of time.
(b) The connection violates KCL; thus it is not permitted.
(c) Neither is true.

2.32 When ideal current sources are connected in parallel, which of the following is true?
(a) The currents add, independent of whether the individual sources are constant valued or have outputs that are functions of time.
(b) The connection violates KCL; thus it is not permitted.
(c) Neither is true.

2.33 In a network containing only independent current sources and resistors, if the values of all resistors are doubled, the values of the node voltages
(a) are doubled. (b) remain the same.
(c) are halved. (d) change in some other way.

2.34 In a network containing only independent current sources and resistors, if the values of all the current sources are doubled, the values of the node voltages
(a) are doubled. (b) remain the same.
(c) are halved. (d) change in some other way.

2.35 In a network containing only independent voltage sources and resistors, if the values of all the voltage sources are doubled, the values of the mesh currents
(a) are doubled. (b) remain the same.
(c) are halved. (d) change in some other way.

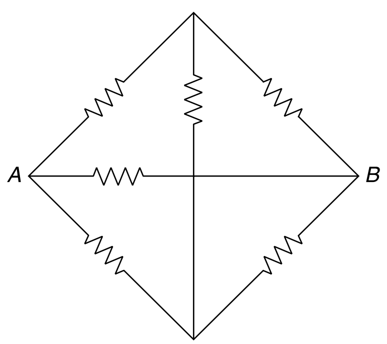
2.36 In a network containing only independent voltage sources and resistors, if the values of all the resistors are doubled, the values of the mesh currents
(a) are doubled. (b) remain the same.
(c) are halved. (d) change in some other way

2.37 If the same constant value of current is added to all the independent current sources in a network, the node voltages
(a) will all have a constant value added. (b) will remain the same.
(c) will all have a constant value subtracted. (d) will change in some other way.

2.38 If the same constant value of voltage is added to each of the independent voltage sources in an arbitrary network containing only resistors are independent voltage sources, the mesh currents
(a) will all have a constant value added. (b) will remain the same.
(c) will all have a constant value subtracted. (d) will change in some other way.

2.39 Two resistors R_1 and R_2 give combined resistance of 4.5 Ω when in series and 1 Ω when in parallel, the resistances are
(a) 2 Ω and 2.5 Ω (b) 1 Ω and 3.5 Ω (c) 1.5 Ω and 3.5 Ω (d) 4 Ω and 0.5 Ω.

2.40 When all the resistances in the circuit are of 1Ω each, the equivalent resistance across the points *A* and *B* will be



(a) 1 Ω (b) 0.5 Ω (c) 2 Ω (d) 1.5 Ω.

2.41 Energy expanded or heat generated in joules when a current of *I* flows through a conductor *R* for *t* second is given by

(a) I^2Rt (b) IRt (c) IR^2t (d) IRt^2

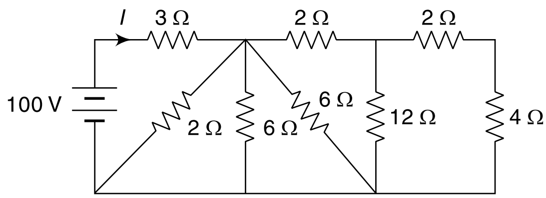
2.42 A 2 Ω resistance having a current of 2A will dissipate the power of

(a) 2 W (b) 4 W (c) 8 W (d) 8 J

2.43 The ratio of resistances of a 100 W, 220 V lamp to that of a 100 W, 110 V lamp will be, at the respective voltages

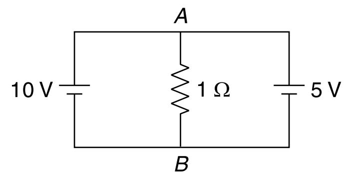
(a) 4 (b) 2 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

2.44 For the circuit shown in figure, the value of current I is



(a) 10 A (b) 15 A
(c) 20 A (d) 25 A

2.45. The current in the 1 Ω resistor is

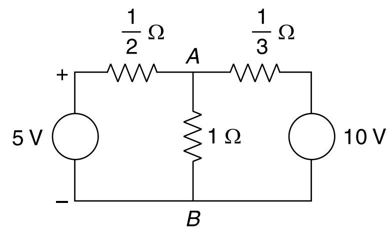


(a) 5 A (b) 10 A
(c) 15 A (d) zero.

2.46 The current in a 5 Ω resistor branch in a linear network is 5 A. If this branch is replaced by a resistor of 10 Ω, the current in this branch will be

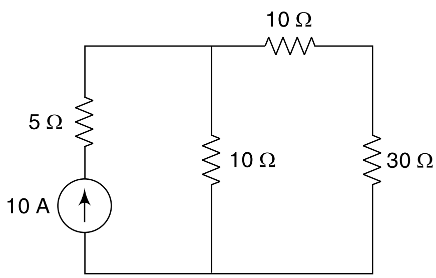
(a) 5 A (b) 10 A
(c) less than 4 A (d) none of these.

2.47 The potential of the point A in the given network is



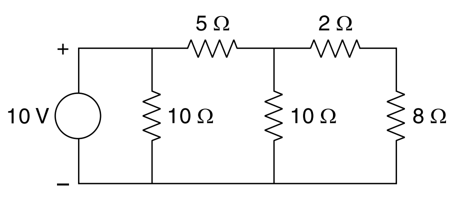
(a) 6 V
(b) 7 V
(c) 8 V
(d) none of these.

2.48 The current through 30 Ω branch in the given circuit is



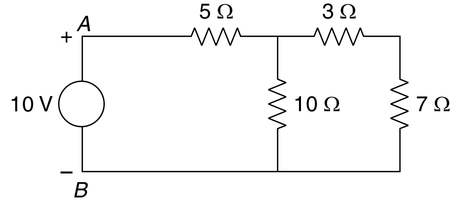
(a) 2.5 A (b) 2.25 A (c) 2 A (d) 10 A

2.49 The current through 8 Ω branch is



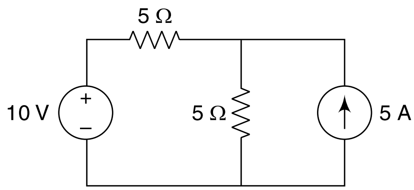
(a) 1 A
(b) 0.5 A
(c) 1.5 A
(d) none of these.

2.50 If the current in the 7 Ω resistor branch is 0.5 A as shown in the figure and now if the source is connected in series with 7 Ω branch and the terminals *AB* are shorted, the current in the 5 Ω resistor is



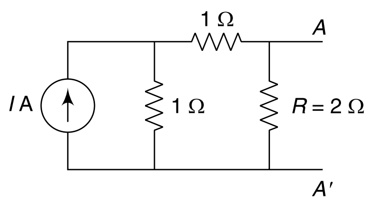
(a) 1 A (b) 0.5 A
(c) 9.75 A (d) none of these.

2.51 The voltage across 5 A source in the given circuit is



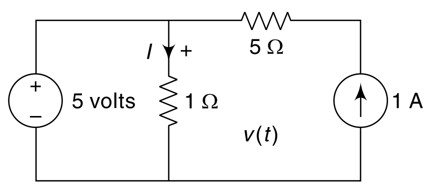
(a) 25 V (b) 15 V (c) 17.5 V (d) 20 V.

2.52 In the circuit shown in figure, current *I* flows through the resistance *R*. If a battery of e.m.f. 2 V and internal resistance of 1 Ω is connected between the terminals *A* and *A′* with positive terminal connected to *A′*, the current through *R* would be



(a) 2 A (b) 1.66 A
(c) 1 A (d) 1.5 A.

2.53 The circuit shown in the figure is linear and time-invariant. The sources are ideal. The voltage across the 1 Ω resistor and the current through it will be

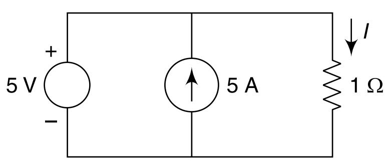


(a) –5 V and –5 A (b) 1 V and 1 A
(c) 1 1 and 6 A (d) 5 V and 5 A.

2.54 The number of 2 μF, 400 V capacitors needed to obtain a capacitance value of 1.5 μF rated for 1600 V is

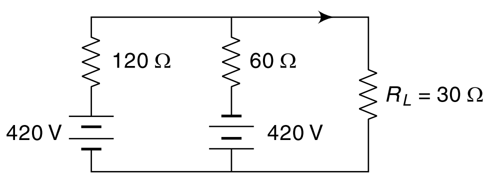
(a) 12 (b) 8 (c) 6 (d) 4.

2.55 The value of the current *I* flowing in the 1 Ω resistor in the circuit, shown in figure will be



(a) 10 A (b) 6 A (c) 5 A (d) zero.

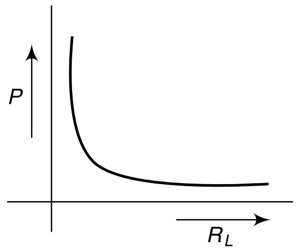
2.56 In the circuit shown in figure, the current *I* through R_L is



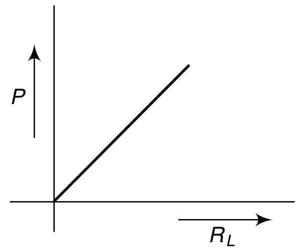
(a) 2 A (b) zero (c) –2 A (d) –6 A.

2.57 A voltage source with an internal resistance R_S, supplies power to a load R_L. The power delivered to the load varies with R_L as

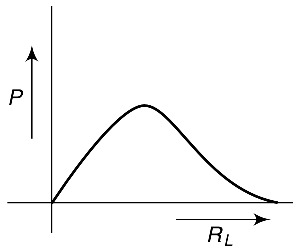
(a)



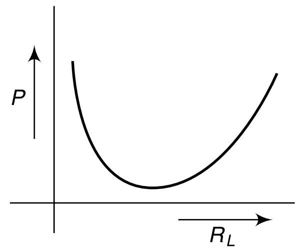
(b)



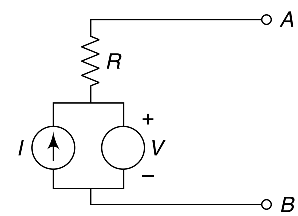
(c)



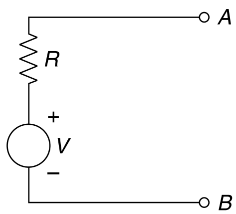
(d)



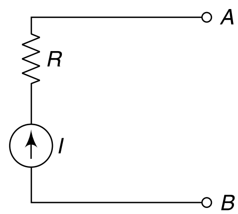
2.58 A simple equivalent circuit of the 2-terminal network shown in the figure is



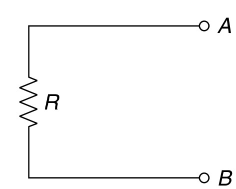
(a)



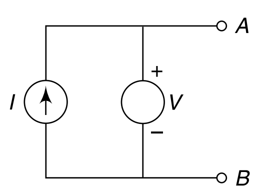
(b)



(c)

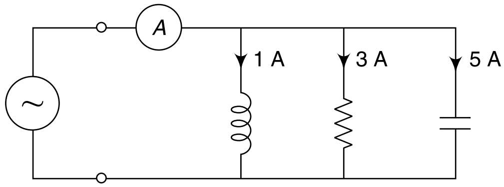


(d)



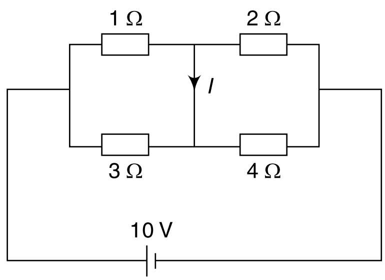
2.59 Two condensers of 20 μF and 40 μF capacitances are connected in series across a 90 V supply. After charging, they are removed from the supply and are connected in parallel with positive terminals connected together and similarly the negative terminals. Then the voltage across them will be
(a) 90 V (b) 60 V (c) 40 V (d) 20 V.

2.60 The current read by the ammeter A in the a.c. circuit shown in the given figure is



(a) 9 A (b) 5 A (c) 3 A (d) 1 A

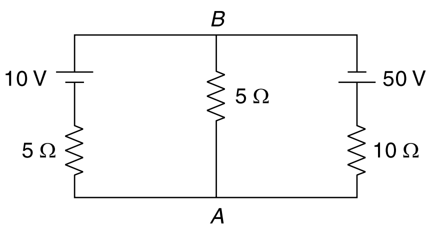
2.61 In the circuit shown in the given figure, current I is



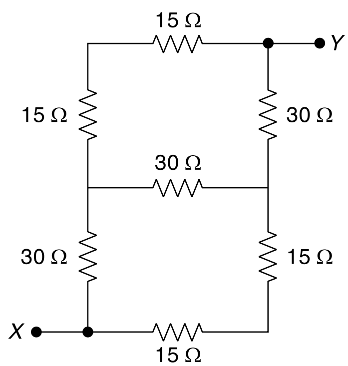
(a) $-\frac{2}{5}$ A (b) $\frac{24}{5}$ A

(c) $\frac{18}{5}$ A (d) $\frac{2}{5}$ A

2.62 For the circuit shown in the given figure, the voltage V_{AB} is
(a) 6 V (b) 10 V
(c) 25 V (d) 40 V

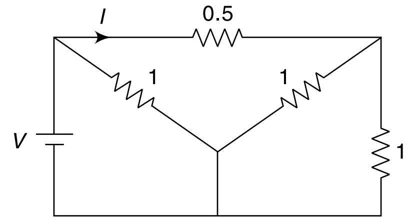


2.63 The equivalent resistance between the terminal points X and Y in the circuit shown is



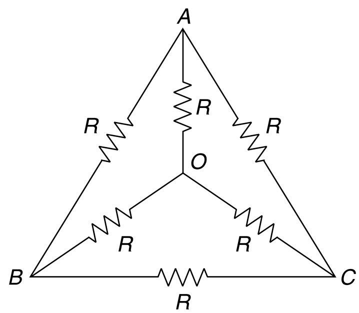
(a) 15 Ω (b) 45 Ω (c) 55 Ω (d) 30 Ω

2.64 In the circuit shown in the figure, if $I = 2$, then the value of the battery voltage V will be



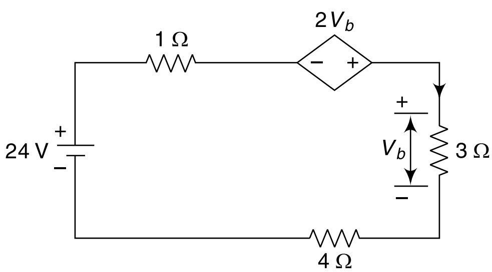
(a) 5 V (b) 3 V (c) 2 V (d) 1 V

2.65 The effective resistance between the terminals A and B in the circuit shown in the figure is



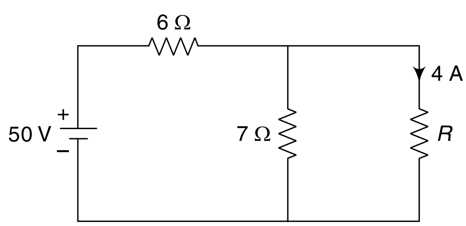
(a) R (b) $R - 1$ (c) $R/2$ (d) $\frac{6}{11}R$

2.66 The current in the given circuit with a dependent source is



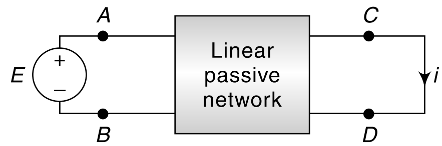
(a) 10 A (b) 12 A (c) 14 A (d) 16 A

2.67 The value of the resistance R shown in the given figure is



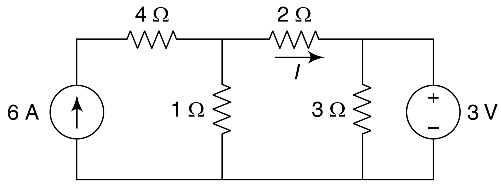
(a) 3.5 Ω (b) 2.5 Ω (c) 1 Ω (d) 4.5 Ω.

2.68 For the circuit shown in the given figure, when the voltage E is 10 V, the current i is 1 A. If the applied voltage across terminal C–D is 100 V, the short circuit current flowing through the terminal A–B will be



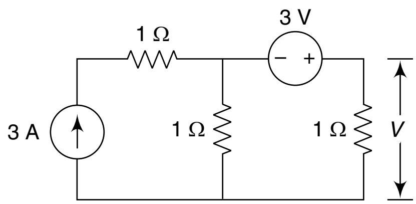
(a) 0.1 A (b) 1 A
(c) 10 A (d) 100 A.

2.69 For the circuit shown in the given figure, the current I is given by



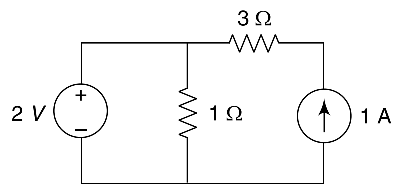
(a) 3 A (b) 2 A
(c) 1 A (d) zero.

2.70 The value of V in the circuit shown in the given figure is



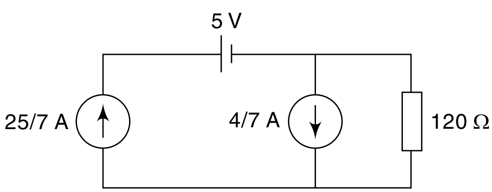
(a) 1 V (b) 2 V (c) 3 V (d) 4 V

2.71 For the circuit given in figure, the power delivered by the 2 volt source is given by



(a) 4 W (b) 2 W
(c) –2 W (d) –4 W

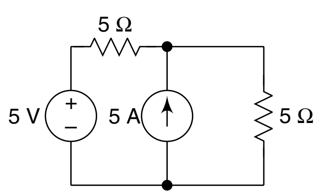
2.72 The current through 120 Ω resistor in the circuit shown in the figure is



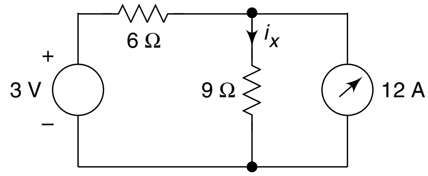
(a) 1 A (b) 2 A (c) 3 A (d) 4 A

2.73 The voltage across 5 A current source in the circuit shown in the figure is

(a) 25 V (b) 15 V
(c) 17.5 V (d) 20 V



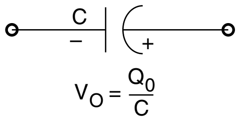
2.74 The current i_x in the network is



(a) 1A (b) $\frac{1}{2}$A

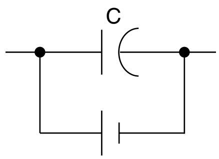
(c) $\frac{1}{3}$A (d) $\frac{4}{5}$A

2.75 The equivalent circuit of the capacitor shown is

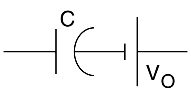


(a)

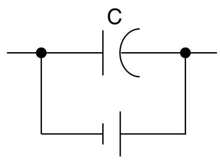
(b)



(c)



(d)



2.76 If a network has seven nodes and five independent loops, the number of branches in the network is
(a) 7 (b) 5 (c) 11 (d) 12

2.77 An electric circuit with 10 branches and 7 nodes will have
(a) 3 loop equations (b) 4 loop equations
(c) 7 loop equations (d) 10 loop equations.

2.78 A circuit having an emf source or any energy source is a/an
(a) active circuit (b) passive circuit
(c) unilateral circuit (d) bilateral circuit

2.79 If there are b branches and n nodes, the number of KVL equations required will be
(a) b (b) $b - n$ (c) $n - 1$ (d) $b - n + 1$

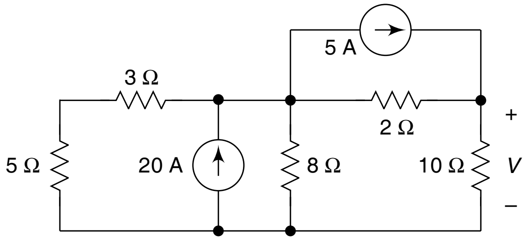
2.80 If the number of branches is 'B', the number of nodes is 'N' and the number of dependent loops is 'L', then the number of independent node equations will be
(a) $N + L - 1$ (b) $B - 1$ (c) $B - N$ (d) $N - 1$

2.81 A network has 10 nodes and 17 branches in all. The number of different node pair voltages would be
(a) 7 (b) 9 (c) 10 (d) 45

2.82 The elements which are not capable of delivering energy by their own are known as
(a) unilateral elements (b) non-linear elements
(c) passive elements (d) active elements

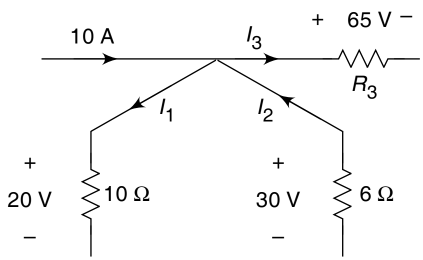
2.83 Four resistors of equal value when connected in series across a supply dissipate 25 W. If the same resistors are now connected in parallel across the same supply, what is the power dissipated?
(a) 75 W (b) 100 W (c) 200 W (d) 400 W.

2.84 What is the voltage V in the circuit shown in the figure?



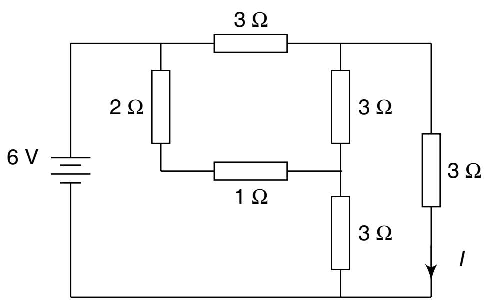
(a) 185V (b) 115V (c) 85V (d) 55V

2.85 A part of an electrical network has the configuration shown in the figure. The voltage drops across the resistances are 20 V, 30 V and 65 V with respective polarities shown in the figure. Which one of the following gives the correct value of the resistance R_3?



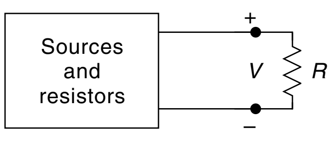
(a) 13Ω (b) 5Ω (c) 65Ω (d) 63/17 Ω

2.86 What is the current I in the circuit given in the figure?



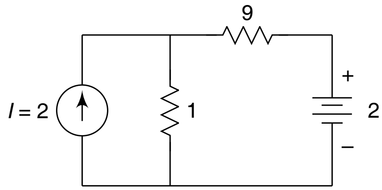
(a) 0 (b) 1A (c) 2A (d) 3A

2.87 In the circuit given when R is infinite, $V = 4$ Volt and when $R = 0$, the current through R is 4 A. If $R = 3\,\Omega$, what is the current through it?



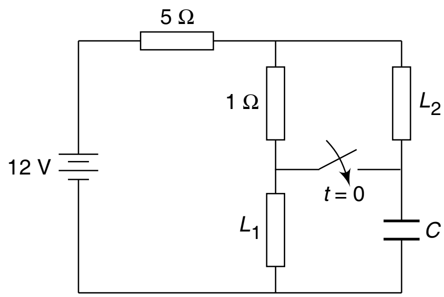
(a) 4 A (b) 3 A (c) 2 A (d) 1 A

2.88 For the circuit given, what is the current delivered by the battery?



(a) 3.0 A (b) 2.0 A (c) 1.0 A (d) 0.0 A

2.89 The circuit shown in the figure is in steady state with the switch open. At $t = 0$, the switch is closed. What is the current through the 1 Ω resistor, $i(0+)$?



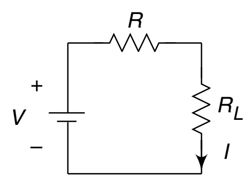
(a) 0 (b) 1.33 A (c) 1.66 A (d) 2 A

2.90 A 2-terminal network is one of the *R-L-C* elements. The element is connected to an ac supply. The current through the element is *I*. When a capacitor is inserted in series between the source and the element, then current through the element becomes 2*I*. The element

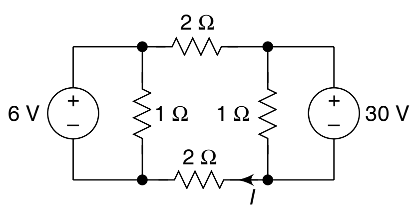
(a) is a resistor (b) is an inductor
(c) is a capacitor (d) cannot be a single element

2.91 For the circuit shown in the figure, if the current $I = 3$ A and 1.5 A for $R_L = 0$ and 2 Ω respectively, then what is the value of I for $R_L = 1$ Ω?

(a) 0.5 A (b) 1.0 A
(c) 2.0 A (d) 3.0 A

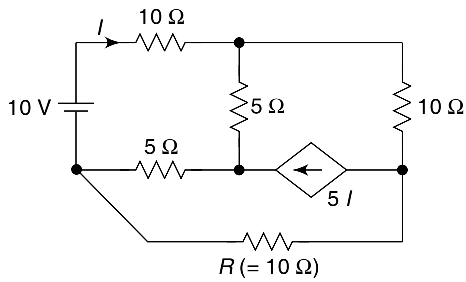


2.92 What is the value of current I in the circuit shown in the figure?



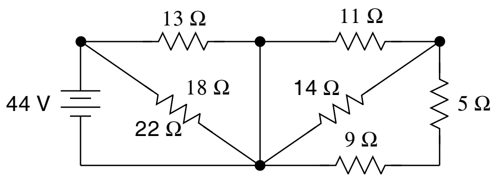
(a) 1A (b) –3A (c) –6 A (d) 9 A

2.93 In the circuit shown, current through R is



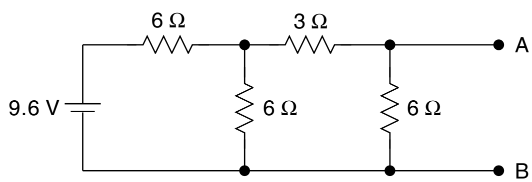
(a) 0 (b) 0.5 A (c) 2.5 A (d) 3.33 A

2.94 Referring to the circuit shown in figure, the current in the 18 Ω resistor is



(a) 2 A (b) 1.5 A (c) 1 A (d) 0.5 A

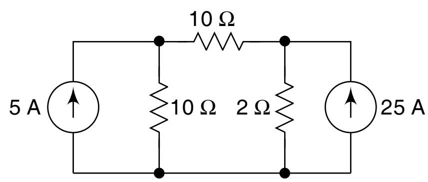
2.95



An ideal ammeter is connected between terminals *A* and *B* of the network shown above. The current through the ammeter is

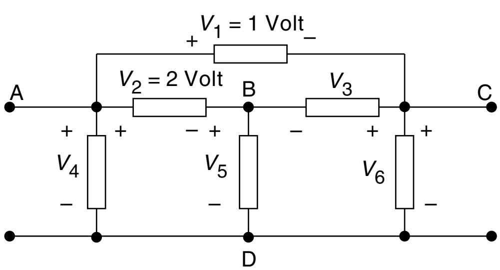
(a) 0.8 A (b) 1.6 A (c) 0 A (d) 3.2 A

2.96 For the network shown in the figure, the current in the 2 Ω resistor would be



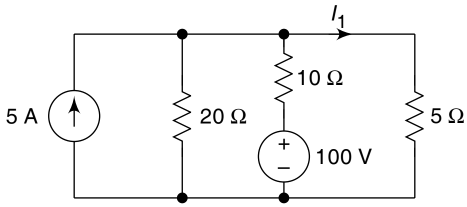
(a) 5 A (b) 20 A (c) 25 A (d) 30 A

2.97



The branch voltages are marked with proper polarity for the network shown in the figure. The value of V_5 is

(a) 3V (b) 2V (c) 1V (d) 0V

2.98 

The current I_1 through the 5 Ω resistor in the network shown in the figure, is

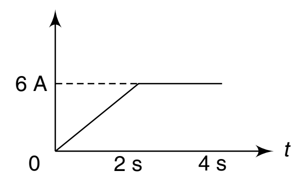
(a) 8.58 A (b) 7.54 A (c) 11.66 A (d) 15 A

2.99 A lamp rated at 10 watts, 50 volts is proposed to be used in a 110 Volt system. The wattage and resistance of the resistor to be connected in series with the lamp should be

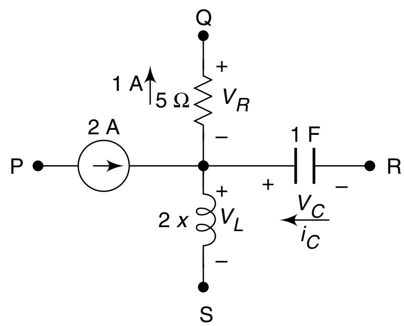
(a) 15 watts, 350 ohms (b) 10 watts, 250 ohms
(c) 12 watts, 300 ohms (d) 15 watts, 250 ohms

2.100 The figure shows the waveform of the current passing through an inductor of resistance 1 Ω and inductance 2 H. The energy absorbed by the inductor in the first four seconds is

(a) 144 J (b) 98 J
(c) 132 J (d) 168 J

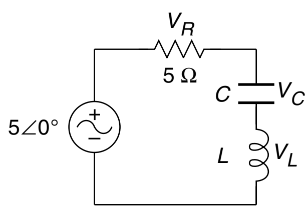


2.101 A segment of a circuit is shown in the figure. $V_R = 5\,V$, $V_C = 4\sin 2t$. The voltage V_L is given by



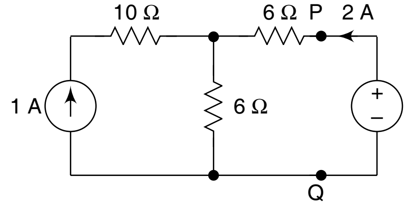
(a) $3 - 8\cos 2t$ (b) $32\sin 2t$ (c) $16\sin 2t$ (d) $16\cos 2t$

2.102 In the circuit of the figure, the magnitudes of V_L and V_C are twice that of V_R. The inductance of the coil is



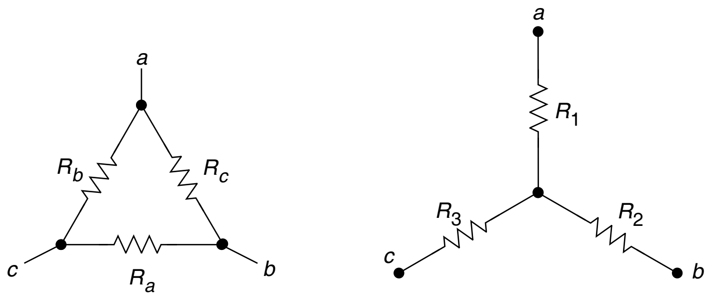
(a) 2.14 mH (b) 5.30 H (c) 3.18 mH (d) 1.32 H

2.103 In the figure, the value of the source voltage is



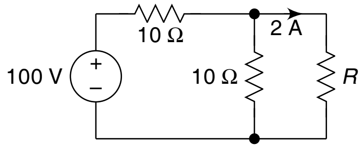
(a) 12V (b) 24V (c) 30V (d) 44V

2.104 In figure, R_a, R_b and R_c are 20 Ω, 10 Ω and 10 Ω respectively. The resistances R_1, R_2 and R_3 in Ω of an equivalent star-connection are



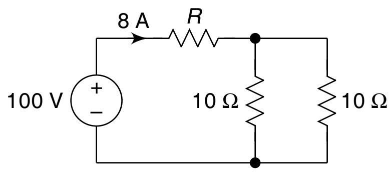
(a) 2.5, 5, 5 (b) 5, 2.5, 5 (c) 5, 5, 2.5 (d) 2.5, 5, 2.5

2.105 In figure, the value of resistance R in Ω is



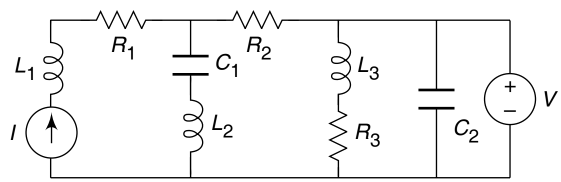
(a) 10 (b) 20 (c) 30 (d) 40

2.106 In the figure given, the value of R is



(a) 2.5 Ω (b) 5.0 Ω (c) 7.5 Ω (d) 10.0 Ω

2.107 In the circuit shown in the figure, the current source $I = 1$ A, voltage source $V = 5$ V, $R_1 = R_2 = R_3 = 1\,\Omega$, $L_1 = L_2 = L_3 = 1$ H, $C_1 = C_2 = 1$ F. The current (in A) through R_3 and the voltage source V respectively will be

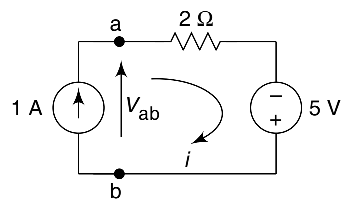


(a) 1, 4 (b) 5, 1 (c) 5, 2 (d) 5, 4

2.108 A 3V dc supply with an internal resistance of 2 Ω supplies a passive non-linear resistance characterized by the relation $V_{NL} = I^2_{NL}$. The power dissipated in the non-linear resistance is

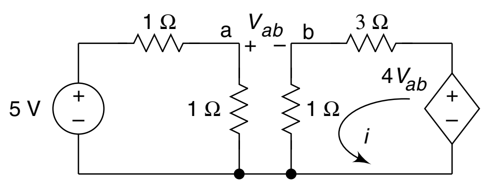
(a) 1.0 W (b) 1.5 W (c) 2.5 W (d) 3.0 W

2.109 Assuming ideal elements in the circuit shown below, the voltage V_{ab} will be



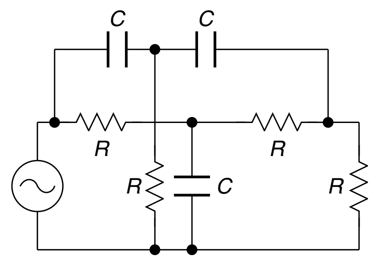
(a) –3 V (b) 0V (c) 3V (d) 5V

2.110 In the circuit shown in the figure, the value of the current i will be given by



(a) 0.31 A (b) 1.25 A (c) 1.75 A (d) 2.5 A

2.111 The minimum number of equations required to analyze the circuit shown in the figure is



(a) 3 (b) 4 (c) 6 (d) 7

2.112 Twelve 1 Ω resistances are used as edges to form a cube. The resistance between two diagonally opposite corners of the cube is

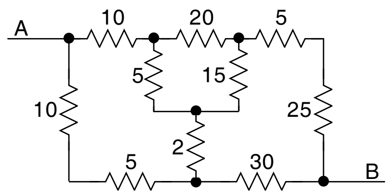
(a) $\frac{5}{6}\Omega$ (b) 1 Ω (c) $\frac{6}{5}\Omega$ (d) $\frac{3}{2}\Omega$

2.113 A two-terminal black box contains one of the R, L, C elements. The black box is connected to a 220 volt ac supply. The current through the source is I. When a capacitance of 0.1 F is inserted in series between the source and the box, then current through the source is $2I$. The element is

(a) a resistance
(b) an inductance
(c) a capacitance of 0.5 F
(d) not readily identifiable from the given data

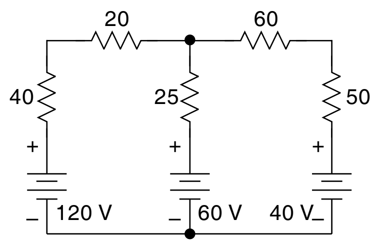
EXERCISES

2.1 Find R_{AB} in the network shown below. All resistance values are in ohms.



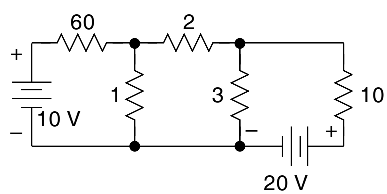
[23.52 Ω]

2.2 Use loop current analysis to find the current in each battery in the network shown. All resistance values are in ohms.



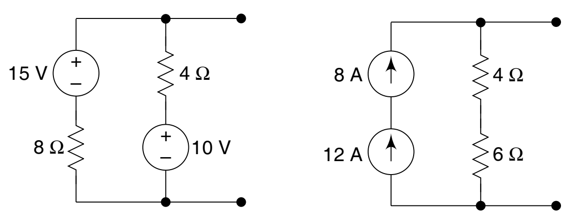
[0.793 A, 0.408 A, 0.295 A]

2.3 Find the current through 2 Ω resistance in the network shown below. Use loop current method.



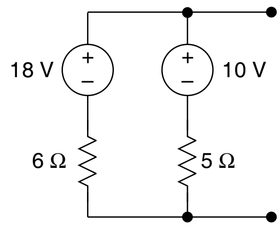
[–0.841A]

2.4 Convert the circuits shown in figure to a single voltage source in series with a single resistor.



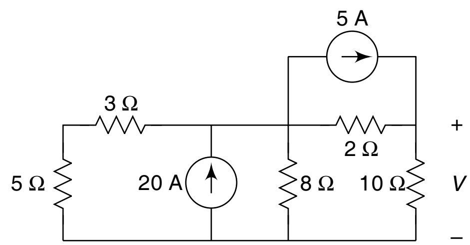
$\left[V = \dfrac{5}{3}\text{ Volt, } R = \dfrac{8}{3}\ \Omega\right]$ $\qquad V = 104$ Volt, $R = 10\ \Omega$

2.5 Convert the circuit shown in the figure to a single current source in parallel in with a single resistor.



[$I = 1$ A, $R = 2.73\,\Omega$]

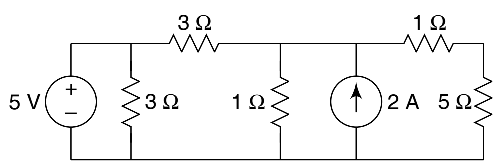
2.6 Determine the voltage V in the circuit, using the source transformation technique and/ or any other method.



[$V = 56.25$ V]

2.7 Find the current flowing through $5\,\Omega$ resistor using source transformation technique.

$$\left[\frac{11}{27}\text{ A}\right]$$



2.8 Reduce the network shown in Figure (a) to a form shown in the Figure (b) using successive source transformations.

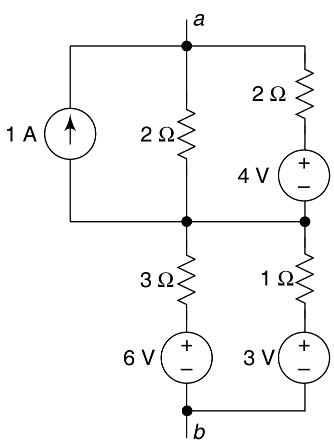


Figure (a)

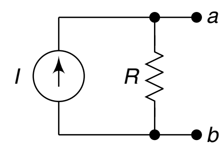
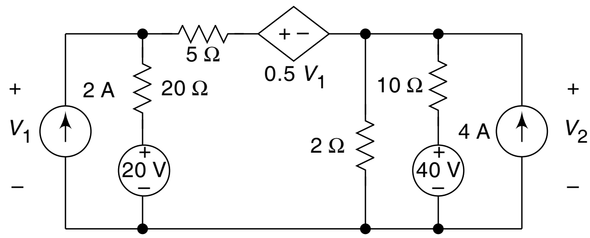


Figure (b)

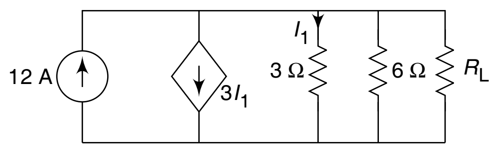
[$I = 2.14$ A, $R = 1.75\,\Omega$]

2.9 For the circuit of the figure, apply source transformation and then find V_1 and V_2 by nodal analysis.



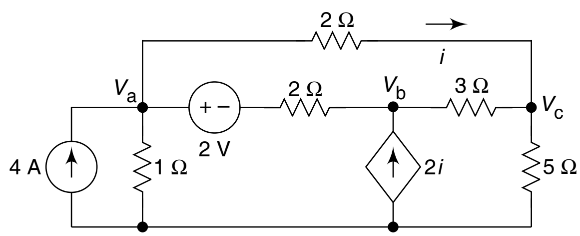
[$V_1 = 40$ V, $V_2 = 15$ V]

2.10 In the circuit shown in figure if $I_1 = 2$ A, determine R_L and the power delivered in it.



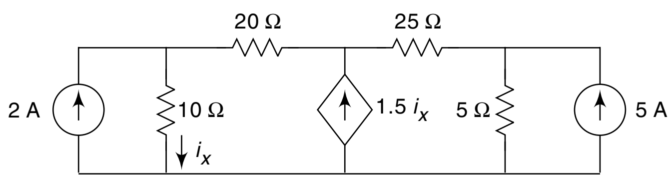
[2 Ω; 18 W]

2.11 Find the node voltages V_a, V_b and V_c using nodal analysis.



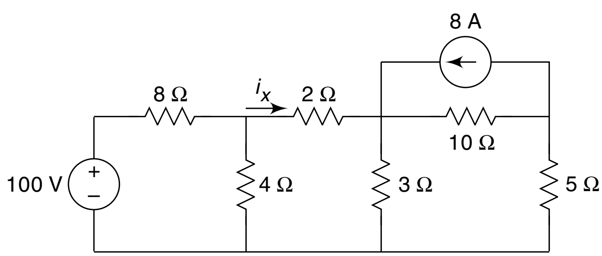
[4.3 V; 3.9 V; 3.3 V]

2.12 Use mesh analysis to find the current i_x in the circuit shown in the figure.



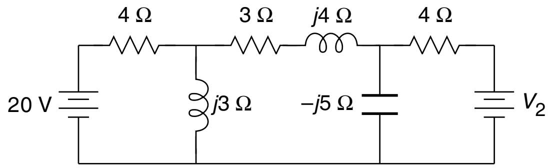
[8.33 A]

2.13 Use mesh analysis to find the current i_x in the circuit shown in the figure.



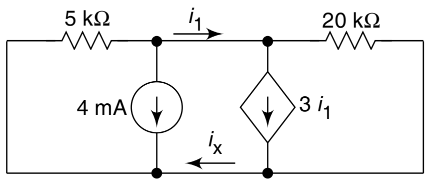
[2.79 A]

2.14 Determine the value of V_2, such that the current through $(3 + j4)\,\Omega$ impedance is zero.



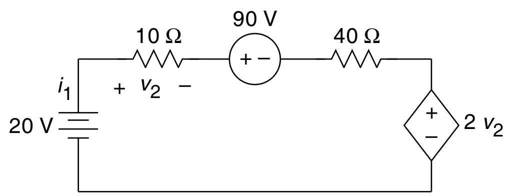
[80.43∠119.55° (V)]

2.15 Find the current i_x in the circuit shown in the figure.



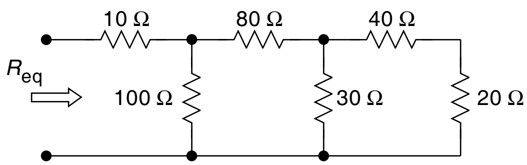
[0.571 mA]

2.16 Find the current i_1 in the circuit shown in the figure.



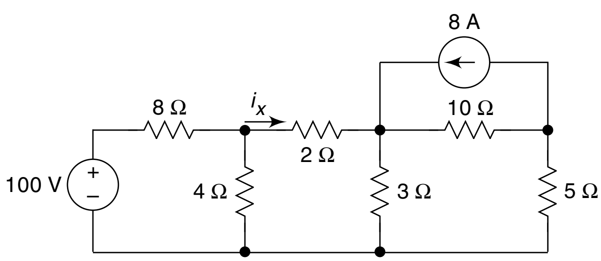
[–1 A]

2.17 Find the equivalent resistance between the terminals A and B for the circuit shown in figure.



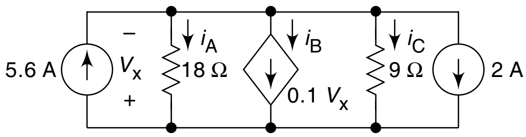
[60 Ω]

2.18 Using mesh analysis, find the current i_x in the circuit shown in the figure.



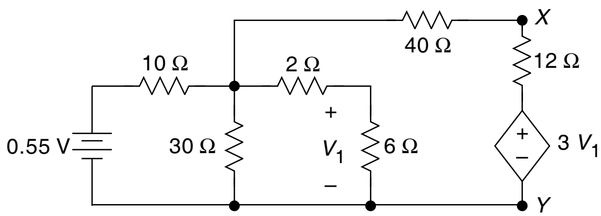
[2.79A]

2.19 For the circuit shown in the figure, find the currents i_A, i_B, and i_C.



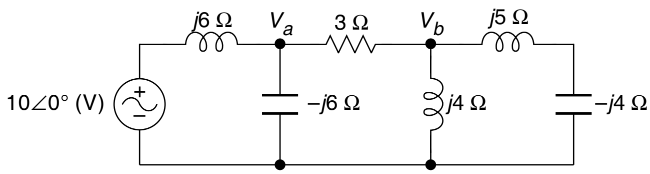
[3 A, −5.4 A, 6 A]

2.20 Use nodal analysis to find the voltage V_{xy} in the circuit shown in the figure below.



[−0.257 V]

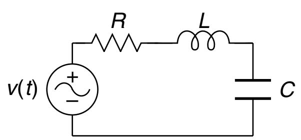
2.21 Determine V_a and V_b in the circuit shown in the figure.



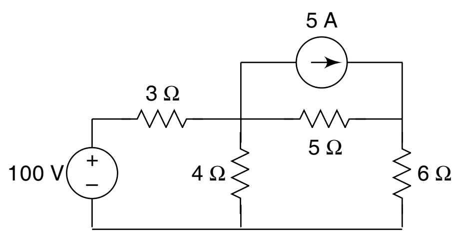
[5.17∠−75° V;1.33V]

2.22 Construct the dual of the networks shown below.

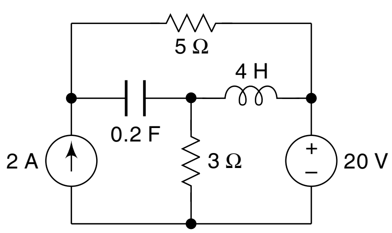
(a)



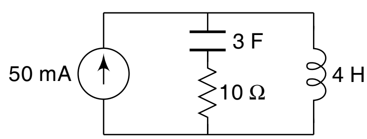
(b)



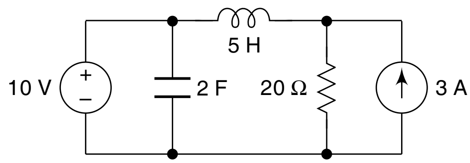
(c)



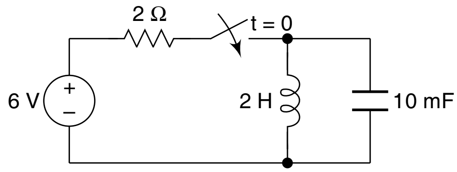
(d)



(e)



(f)



2.23 Draw a circuit and its dual if the mesh equations of the circuit are

(a) $8i_1 - 2i_2 - 4i_3 = 6;\ 7i_2 - 5i_3 = -3;\ -4i_1 - 5i_2 + 9i_3 = 5$

(b) $4i_1 - i_2 - i_3 = -4;\ -i_1 + 6i_2 - 5i_3 = 6;\ -i_1 - 5i_2 + 8i_3 = 2$

2.24 Draw a circuit and its dual if the node equations of the circuit are

(a) $4v_1 - v_2 - v_3 = -4;\ -v_1 + 6v_2 - 5v_3 = 6;\ -5v_2 + 8v_3 = 2$

(b) $4v_1 - v_2 - v_3 = 5;\ 3v_2 - v_3 = -3;\ -v_1 - v_2 + 2v_3 = 6$

(c) $6v_1 - 2v_2 - v_3 = 5;\ -2v_1 + 8v_2 - 3v_3 = -4;\ -v_1 - 3v_2 + 9v_3 = 0$

SHORT-ANSWER TYPE QUESTIONS

2.1 Define the following terms:

(a) Electric charge

(b) Electric current

(c) Current density

(d) Electric potential and potential difference

(e) Drift Velocity

(f) EMF

2.2 Why should the current in different cross-sections of a cable be constant even though the cross-sectional area is different at different places? Is current a scalar or vector quantity?

2.3 Define an electrical network. “All circuits are networks, but all networks are not circuits.”Justify this statement.

2.4 Explain linearity conditions of elements in detail.

2.5 Differentiate between unilateral and bilateral elements. Give examples.

2.6 (a) State the basic assumptions for circuit analysis.

(b) Briefly mention the different source transformation techniques.

(c) Discuss the properties of an ideal current source and ideal voltage source.

(d) Explain how a voltage source can be converted into an equivalent current source and vice-versa.

2.7 Explain the properties of basic elements R, L and C in networks.

2.8 What is electrical resistance? Explain the factors that affect the resistance.

2.9 Define capacitance. Derive an expression of the energy stored in a capacitor.

2.10 Define self-inductance of a coil. Derive an expression of the energy stored in an inductor.

2.11 (a) Explain why a capacitor is considered as a linear circuit element.

(b) Explain why an inductor is considered as a linear circuit element.

2.12 Explain why
   (a) the current through an inductor cannot change instantaneously
   (b) the voltage across a capacitor cannot change instantaneously
2.13 Discuss the characteristics of ideal and practical sources (voltage and current). What is loading of sources? Explain.

Or,

Draw the *V-I* characteristics for voltage and current source for ideal and actual cases.

Or,

Draw the symbol and characteristics of ideal and practical voltage and current sources.
2.14 Give one practical example each of an ideal voltage source and an ideal current source.
2.15 Explain voltage source to current source transformation. Define *V*-shift in the source transformation.
2.16 Establish the conditions for equivalence of practical voltage and current sources.
2.17 Give a brief introduction to the dependent (controlled) sources.
2.18 (a) State Kirchhoff's voltage and current laws.
   (b) Give a brief comparison of the loop method and node method of circuit analysis.
   (c) Comment briefly on the choice between loop and nodal methods of analysing a network.
2.19 Explain 'duality' in electrical engineering. How can you draw the dual of a network?
2.20 State the steps followed in finding the dual of a network.
2.21 Elaborate the statement: "A voltage impulse causes a current to be established in an inductance in zero time." What is the value of this current? Is it a violation of the fact that current in an inductance cannot change instantaneously?

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

2.1 (d)	2.2 (d)	2.3 (d)	2.4 (d)	2.5 (b)	2.6 (c)	2.7 (b)
2.8 (a)	2.9 (c)	2.10 (b)	2.11 (b)	2.12 (b)	2.13 (a)	2.14 (d)
2.15 (c)	2.16 (b)	2.17 (b)	2.18 (d)	2.19 (a)	2.20 (a)	2.21 (a)
2.22 (b)	2.23 (c)	2.24 (b)	2.25 (b)	2.26 (a)	2.27 (a)	2.28 (a)
2.29 (a)	2.30 (b)	2.31 (b)	2.32 (a)	2.33 (a)	2.34 (a)	2.35 (a)
2.36 (c)	2.37 (d)	2.38 (d)	2.39 (c)	2.40 (b)	2.41 (a)	2.42 (c)
2.43 (a)	2.44 (d)	2.45 (d)	2.46 (c)	2.47 (c)	2.48 (c)	2.49 (b)
2.50 (b)	2.51 (c)	2.52 (d)	2.53 (d)	2.54 (a)	2.55 (c)	2.56 (c)
2.57 (c)	2.58 (a)	2.59 (c)	2.60 (b)	2.61 (b)	2.62 (a)	2.63 (d)
2.64 (c)	2.65 (c)	2.66 (b)	2.67 (a)	2.68 (c)	2.69 (c)	2.70 (c)
2.71 (b)	2.72 (c)	2.73 (b)	2.74 (a)	2.75 (a)	2.76 (c)	2.77 (b)
2.78 (a)	2.79 (d)	2.80 (d)	2.81 (d)	2.82 (c)	2.83 (d)	2.84 (d)
2.85 (b)	2.86 (b)	2.87 (d)	2.88 (d)	2.89 (a)	2.90 (b)	2.91 (c)
2.92 (c)	2.93 (a)	2.94 (c)	2.95 (a)	2.96 (c)	2.97 (d)	2.98 (a)
2.99 (c)	2.100 (a)	2.101 (b)	2.102 (c)	2.103 (c)	2.104 (a)	2.105 (b)
2.106 (c)	2.107 (d)	2.108 (a)	2.109 (a)	2.110 (b)	2.111 (b)	2.112 (a)
2.113 (b)						

CHAPTER 3 Magnetically Coupled Circuits

3.1 INTRODUCTION

The circuits we have considered so far may be termed as conductively coupled in the sense that one coil affects the adjacent coils by current conduction. But when two or more coils are very close to each other, then the current in one coil will affect the e.m.f. induced in other coils and these coils are said to be mutually coupled or magnetically coupled coils.

In this chapter, we will first discuss the concepts of magnetic coupling and dot conventions required to write KVL equations with correct polarities. Then we will learn the theoretical aspects of transformers and tuned circuits.

3.2 SELF-INDUCTANCE

Consider a coil consisting of N turns and carrying current I in the counterclockwise direction, as shown in Fig. 3.1. If the current is steady, then the magnetic flux through the loop will remain constant. However, suppose the current I changes with time, then according to Faraday's law, an induced e.m.f will arise to oppose the change. The induced current will flow clockwise if $\frac{dI}{dt} < 0$, and counterclockwise if $\frac{dI}{dt} > 0$. The property of the loop in which its own magnetic field opposes any change in current is called 'self-inductance', and the e.m.f generated is called the self-induced e.m.f or back e.m.f, which we denote as ε_L. All current-carrying loops exhibit this property. In particular, an inductor is a circuit element which has a large self-inductance. Mathematically, the self-induced e.m.f can be written as

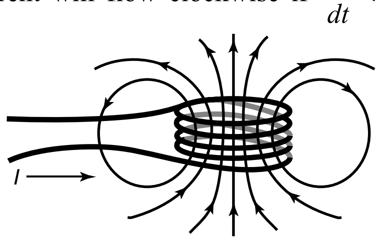


Figure 3.1 *Magnetic flux through the current loop*

$$\varepsilon_L = -N\frac{d\phi_B}{dt} = -N\frac{d}{dt}\iint \vec{B}.d\vec{A}$$

and is related to the self-inductance L by,

$$\varepsilon_L = -L\frac{dI}{dt}$$

The two expressions can be combined to yield

$$L = \frac{N\phi_B}{I}$$

Physically, the inductance L is a measure of an inductor's 'resistance' to the change of current; the larger the value of L, the lower the rate of change of current.

Example 3.1 Self-inductance of a solenoid:

Compute the self-inductance of a solenoid with turns N, length l, and radius R with a current I flowing through each turn, as shown in Fig. 3.2.

Solution: Ignoring edge effects and applying Ampere's law, the magnetic field inside a solenoid is given by,

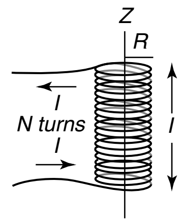


Figure 3.2 *Solenoid*

$$\vec{B} = \frac{\mu_0 NI}{l}\hat{k} = \mu_0 nI\hat{k}$$

where, $n = \dfrac{N}{l}$ is the number of turns per unit length. The magnetic flux through each turn is,

$$\phi = BA = \mu_0 nI.\left(\pi R^2\right) = \mu_0 nI\pi R^2$$

Thus, the self-inductance is,

$$L = \frac{N\phi}{I} = \mu_0 n^2 \pi R^2 l$$

We see that L depends only on the geometrical factors (n, R and l) and is independent of the current I.

3.3 COUPLED INDUCTOR

When the magnetic flux produced by an inductor links another inductor, these inductors are said to be coupled. Coupling is often undesired but in many cases, this coupling is intentional and is the basis of the transformer. When inductors are coupled, there exists a mutual inductance that relates the current in one inductor to the flux linkage in the other inductor. Thus, there are three inductances defined for coupled inductors:

L_{11} – the self inductance of inductor 1

L_{22} – the self inductance of inductor 2

$L_{12} = L_{21}$ – the mutual inductance associated with both inductors

When either side of the transformer is a tuned circuit, the amount of mutual inductance between the two windings determines the shape of the frequency response curve. Although no boundaries are defined, this is often referred to as loose-, critical-, and over-coupling. When two tuned circuits are loosely coupled through mutual inductance, the bandwidth will be narrow. As the amount of mutual inductance increases, the bandwidth continues to grow. When the mutual inductance is increased beyond a critical point, the peak in the response curve begins to drop, and the center frequency will be attenuated more strongly than its direct sidebands. This is known as over-coupling.

3.4 MUTUAL INDUCTANCE

Mutual inductance is the ability of one inductor to induce an e.m.f. across another inductor placed very close to it.

When two coils are placed very close to each other, the magnetic flux caused by current in one coil links with the other coil and induces some voltage in the second coil. This phenomenon is known as *mutual inductance.*

Suppose two coils are placed near each other, as shown in Fig. 3.3.

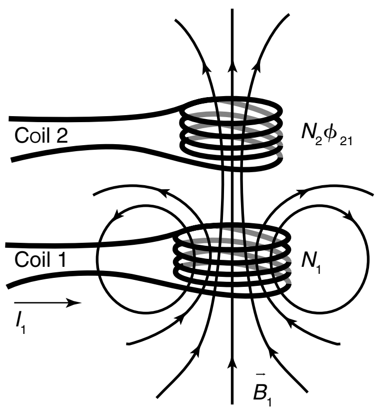


Figure 3.3 *Changing current in coil 1 produces changing magnetic flux in coil 2*

The first coil has N_1 turns and carries a current I_1 which gives rise to a magnetic field $\vec{B}_1$. Since the two coils are close to each other, some of the magnetic field lines through coil 1 will also pass through coil 2. Let f_{21} denote the magnetic flux through one turn of coil 2 due to I_1. Now, by varying I_1 with time, there will be an induced e.m.f. associated with the changing magnetic flux in the second coil:

$$\varepsilon_{21} = N_2 \frac{d\phi_{21}}{dt} = \frac{d}{dt} \iint\limits_{\text{coil 2}} \vec{B}_1 . d\vec{A}_2$$

The time rate of change of magnetic flux f_{21} in coil 2 is proportional to the time rate of change of the current in coil 1and thus the voltage can be written as,

$$\varepsilon_{21} = N_2 \frac{d\phi_{21}}{dt} = N_2 \frac{d\phi_{21}}{dI_1} \times \frac{dI_1}{dt} = M_{21} \frac{dI_1}{dt}$$

where $M_{21} = \dfrac{N_2\phi_{21}}{I_1}$ is called the mutual inductance.

The mutual inductance M_{21} depends only on the geometrical properties of the two coils such as the number of turns and the radii of the two coils.
In a similar manner, suppose instead there is a current I_2 in the second coil and it is varying with time [Fig. (b)]. Then the induced e.m.f. in coil 1 becomes

$$\varepsilon_{21} = N_2 \frac{d\phi_{21}}{dt} = \frac{d}{dt} \iint_{\text{coil 2}} \vec{B}_1.d\vec{A}_2$$

and a voltage is induced in coil 1.

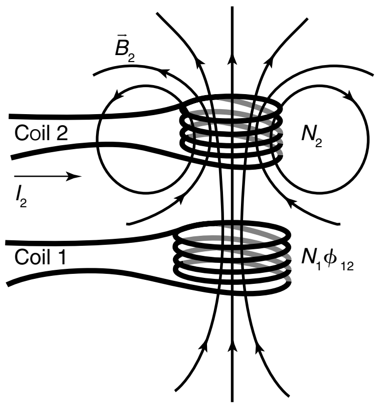


Figure 3.4 *Changing current in coil 2 produces changing magnetic flux in coil 1*

This changing flux in coil 1 is proportional to the changing current in coil 2,

$$\varepsilon_{21} = N_2 \frac{d\phi_{21}}{dt} = N_2 \frac{d\phi_{21}}{dI_1} \times \frac{dI_1}{dt} = M_{21} \frac{dI_1}{dt}$$

where $M_{12} = \dfrac{N_1\phi_{12}}{I_2}$ is another mutual inductance.

Using the *reciprocity theorem* which combines Ampere's law and the Biot-Savart law, it can be shown that the two mutual inductances are.

$$M_{12} \equiv M_{21} \equiv M$$

3.5 MUTUAL INDUCTANCE BETWEEN TWO COUPLED INDUCTORS

Let, L_1, L_2 – two inductors placed very close to each other.
$v_2(t)$ – open circuit voltage induced in L_2 by a current i_1(t) in L_1
$v_1(t)$ – open circuit voltage induced in L_1 by a current i_2(t) in L_2

So, when only i_1(t) is flowing, the magnetic flux emerging from L_1 is given as,

$$\phi_1 = \phi_{11}(\text{Linking with } L_1) + \phi_{12}(\text{Linking with } L_2)$$

$$\therefore\ v_1 = N_1 \frac{d\phi_1}{dt} = N_1 \frac{d\phi_1}{di_1}\frac{di_1}{dt} = L_1 \frac{di_1}{dt}$$

$$\text{where, } L_1 = N_1 \frac{d\phi_1}{di_1}\ \phi$$

$$\text{and } v_2 = N_2 \frac{d\phi_{12}}{dt} = N_2 \frac{d\phi_{12}}{di_1}\frac{di_1}{dt} = M_{21}\frac{di_1}{dt}$$

where, $M_{21} = N_2 \dfrac{d\phi_{12}}{di_1}$ = Mutual Inductance of coil L_2 with respect to coil L_1

Now, when only i_2(t) is flowing, the magnetic flux emerging from L_2 is given as,

$$\phi_2 = \phi_{21}\left(\text{Linking with } L_1\right) + \phi_{22}\left(\text{Linking with } L_2\right)$$

$$\therefore\ v_2 = N_2 \frac{d\phi_2}{dt} = N_2 \frac{d\phi_2}{di_2}\frac{di_2}{dt} = L_2 \frac{di_2}{dt}$$

$$\text{where, } L_2 = N_2 \frac{d\phi_2}{di_2}$$

$$\text{and } v_1 = N_1 \frac{d\phi_{21}}{dt} = N_1 \frac{d\phi_{21}}{di_2}\frac{di_2}{dt} = M_{12}\frac{di_2}{dt}$$

where, $M_{12} = N_1 \dfrac{d\phi_{21}}{di_2}$ = Mutual Inductance of coil L_1 with respect to coil L_2

3.6 DOT CONVENTION

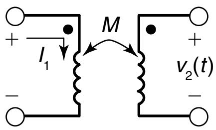
Mutual inductance is a positive quantity; but the sign of e.m.f. induced by it depends on the direction of winding of the coils.

In circuit analysis, the **dot convention** is a convention used to denote the voltage polarity of the mutual inductance of two components. Two conventions are:

1. If a current enters the dotted terminal of one coil, then the polarity of the e.m.f. induced in the second coil will be positive at the dotted terminal of the second coil.
2. If a current leaves the dotted terminal of one coil, then the polarity of the e.m.f. induced in the second coil will be negative at the dotted terminal of the second coil.

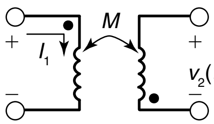
Following these conventions, we find the four possible combinations:

Combination (1):



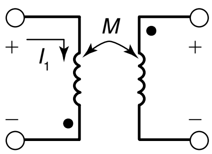
$$v_2(t) = M\frac{di_1(t)}{dt}$$

Combination (2):



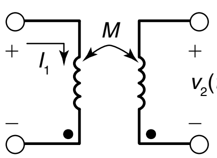
$$v_2(t) = -M\frac{di_1(t)}{dt}$$

Combination (3):



$$v_2(t) = -M\frac{di_1(t)}{dt}$$

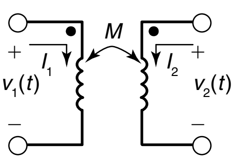
Combination (4):



$$v_2(t) = M\frac{di_1(t)}{dt}$$

If we assume the current flowing in both the coils, then we have the following combinations:

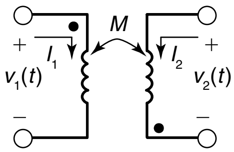
Combination (1):



$$v_1(t) = L_1\frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt}$$

$$v_2(t) = L_2\frac{di_2(t)}{dt} + M\frac{di_1(t)}{dt}$$

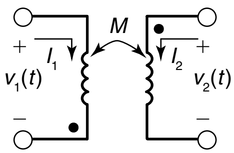
Combination (2):



$$v_1(t) = L_1 \frac{di_1(t)}{dt} - M \frac{di_2(t)}{dt}$$

$$v_2(t) = L_2 \frac{di_2(t)}{dt} - M \frac{di_1(t)}{dt}$$

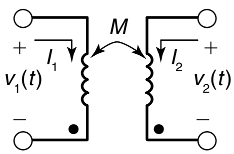
Combination (3):



$$v_1(t) = L_1 \frac{di_1(t)}{dt} - M \frac{di_2(t)}{dt}$$

$$v_2(t) = L_2 \frac{di_2(t)}{dt} - M \frac{di_1(t)}{dt}$$

Combination (4):



$$v_1(t) = L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt}$$

$$v_2(t) = L_2 \frac{di_2(t)}{dt} + M \frac{di_1(t)}{dt}$$

Also, for series connection of inductors, as shown,

Figure 3.5 *(a) Total inductance* = $(L_1 + L_2 + 2M)$ *(b)Total inductance* = $(L_1 + L_2 - 2M)$

3.7 COEFFICIENT OF COUPLING

Coefficient of coupling between two coupled coils is defined as the ratio of the flux linking to the other coil to the total flux.

$$\therefore \qquad k = \frac{\phi_{21}}{\phi_1} = \frac{\phi_{12}}{\phi_2}$$

k attains a maximum value of unity when $\phi_{21} = \phi_1$ and $\phi_{12} = \phi_2$.

Now, the mutual inductance between two coils is,

$$M = \frac{N_1\phi_{12}}{I_2} = \frac{N_2\phi_{21}}{I_1}$$

$$\therefore \qquad M^2 = \frac{N_1\phi_{12}}{I_2} \times \frac{N_2\phi_{21}}{I_1}$$

$$= \frac{N_1 k \phi_1}{I_2} \times \frac{N_2 \phi_2}{I_1} \quad \left\{\because k = \frac{\phi_{21}}{\phi_1} = \frac{\phi_{12}}{\phi_2}\right\}$$

$$= k^2 \left(\frac{N_1 \phi_1}{I_1}\right) \times \left(\frac{N_2 \phi_2}{I_2}\right)$$

$$= k^2 L_1 L_2$$

where, L_1 and L_2 are the self-inductances of the coils.

$$\therefore \qquad \boxed{k = \frac{M}{\sqrt{L_1 L_2}}}$$

3.7.1 Determination of Co-efficient of Coupling from Energy Calculations in Coupled Circuits

To find the energy stored in the coupled circuit, we consider two cases:

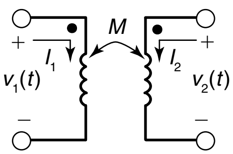


Figure 3.6 *Coupled Circuit*

Case (1): We assume i_2 = 0 and let i_1 increase from 0 to I_1

$\therefore$ Power in L_1, $p_1(t) = v_1(t) i_1(t) = L_1 \dfrac{di_1}{dt} i_1$.

$\therefore$ Energy stored in the circuit,

Case (2): We assume i_1 = 0 and let i_2 increase from 0 to I_2.

$\therefore$ Power in L_2, $p_2(t) = v_2(t) i_2(t) = L_2 \dfrac{di_2}{dt} i_2$

and Power in L_1, $p_1(t) = v_1(t) i_1(t) = M_{12} \dfrac{di_2}{dt} I_1$

$\therefore$ Energy stored in the circuit, $w_2 = \displaystyle\int_{t_1}^{t_2} (p_1 + p_2)\, dt = \int_0^{I_2} (L_2 i_2 di_2 + M_{12} I_1 di_2) = \frac{1}{2} L_2 I_2^2 + M_{12} I_1 I_2$

From case (1) and case (2), total energy stored in the coupled circuit when both i_1 and i_2 have reached constant values of I_1 and I_2 is,

$$W = (w_1 + w_2) = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{12} I_1 I_2 \qquad \text{(A)}$$

Now, if we reverse the order in which the current reach their final values (i.e. first i_2 increases from 0 to I_2 with i_1 = 0 and then i_1 reaches from 0 to I_1 with i_2 = I_2), then the total energy will be,

$$W = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{21} I_1 I_2 \tag{B}$$

From (A) and (B), we get,

$$\boxed{M_{12} = M_{21} = M}$$

$\therefore$ Total Energy, $W = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2$

and for any instantaneous values,

$$w(t) = \frac{1}{2}L_1 i_1^2(t) + \frac{1}{2}L_2 i_2^2(t) + M i_1(t) i_2(t)$$

If the dotted terminals are in opposite sides, then

$$W = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 - M I_1 I_2$$

In general,

$$\boxed{W = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 \pm M I_1 I_2}$$

To find the limiting value of *M*:
Energy stored cannot be negative.

$$\therefore \quad \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 - M I_1 I_2 \geq 0$$

$$\Rightarrow \quad \frac{1}{2}\left(L_1 I_1^2 + L_2 I_2^2 - 2\sqrt{L_1}\sqrt{L_2} I_1 I_2\right) + \sqrt{L_1}\sqrt{L_2} I_1 I_2 - M I_1 I_2 \geq 0$$

$$\Rightarrow \quad \frac{1}{2}\left(\sqrt{L_1} I_1 - \sqrt{L_2} I_2\right)^2 + \left(\sqrt{L_1}\sqrt{L_2} I_1 I_2 - M I_1 I_2\right) \geq 0$$

$$\Rightarrow \quad \frac{1}{2}\left(\sqrt{L_1} I_1 - \sqrt{L_2} I_2\right)^2 + \left(\sqrt{L_1 L_2} - M\right) I_1 I_2 \geq 0$$

The squared term is never negative.

$$\therefore \quad \sqrt{L_1 L_2} - M \geq 0 \quad \Rightarrow \quad \boxed{M \leq \sqrt{L_1 L_2}}$$

Therefore, the *maximum possible value of the mutual inductance is the geometric mean of the self-inductances of the two coils.*

The degree to which the mutual inductance approaches its maximum value is given by *co-efficient of coupling (k)*, defined as,

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

So, $0 \le k \le 1$ or, $0 \le M \le \sqrt{L_1 L_2}$

NB:

(i) For $k = 1$, the coils are called *perfectly coupled coils*.
(ii) For $k \le 0.5$, the coils are called *loosely coupled coils*.
(iii) For $k \ge 0.5$, the coils are called *tightly coupled coils*.

3.8 INDUCTIVE COUPLING

When two coils are connected in series or parallel, mutual inductance exists between them. Depending upon the type of connection, the voltage equation will be different.

3.8.1 Series Coupling

When two coils of self-inductances L_1 and L_2 are connected in series, two types of connection are possible:

1. Series Aiding Connection

In this connection, the two coils are connected in series such a way that their induced e.m.f.'s are of same polarities.

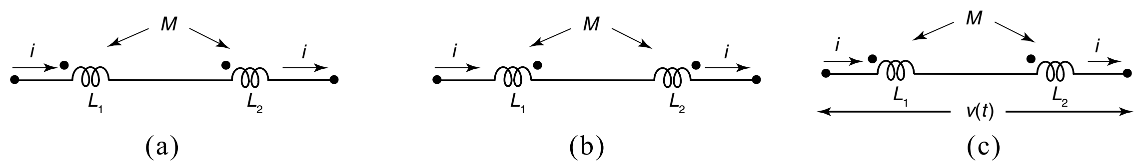


Figure 3.7 *Series Aiding Connections*

Here, total inductance = $(L_1 + L_2 + 2M)$

Derivation:

By KVL,

$$v(t) = L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + 2M \frac{di}{dt} = (L_1 + L_2 + 2M)\frac{di}{dt}$$

$\therefore \ L_{eq} = (L_1 + L_2 + 2M)$

2. Series Opposing Connection

In this connection, the two coils are connected in series such a way that their induced e.m.f.'s are of opposite polarities.

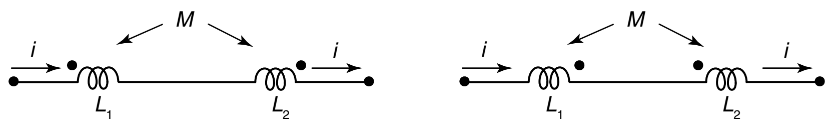


Figure 3.8 *Series Opposing Connections*

Here, total inductance = $(L_1 + L_2 - 2M)$

3.8.2 Parallel Coupling

When two coils of self-inductances L_1 and L_2 are connected in parallel, two types of connection are possible:

1. Parallel Aiding Connection

In this connection, the two coils are connected in parallel such a way that their induced e.m.f.'s are of same polarities.

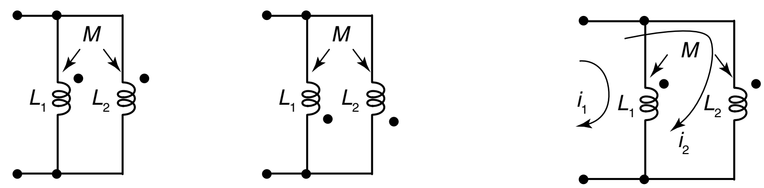


Figure 3.9 *Parallel Aiding Connections*

Here, total inductance $= \dfrac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$

Derivation

By KVL,

$$L_1\frac{di_1}{dt} + M\frac{di_2}{dt} = v(t)$$

$$\text{and}\quad M\frac{di_1}{dt} + L_2\frac{di_2}{dt} = v(t)$$

In sinusoidally steady state,

$$j\omega L_1 I_1 + Mj\omega I_2 = V$$

$$\text{and}\quad j\omega M I_1 + j\omega L_2 I_2 = V$$

Solving for $I\text{-}_1$ and I_2, we get,

$$I_1 = \frac{\begin{vmatrix} V & j\omega M \\ V & j\omega L_2 \end{vmatrix}}{\begin{vmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{vmatrix}} = \frac{j\omega(L_2 - M)V}{\omega^2\left(M^2 - L_1L_2\right)}$$

and $$I_2 = \frac{\begin{vmatrix} j\omega L_1 & V \\ j\omega M & V \end{vmatrix}}{\begin{vmatrix} j\omega L_1 & j\omega M \\ j\omega M & j\omega L_2 \end{vmatrix}} = \frac{j\omega(L_1 - M)V}{\omega^2 (M^2 - L_1 L_2)}$$

$\therefore$ Total current, $$I = (I_1 + I_2) = \frac{j\omega(L_1 + L_2 - 2M)V}{\omega^2 (M^2 - L_1 L_2)}$$

$\therefore$ Input Impedance, $$Z = \frac{V}{I} = \frac{\omega^2 (M^2 - L_1 L_2)}{j\omega(L_1 + L_2 - 2M)} = j\omega\left[\frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}\right]$$

Thus, the equivalent inductance is, $$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

2. Parallel Opposing Connection

In this connection, the two coils are connected in parallel such a way that their induced e.m.f.'s are of opposite polarities.

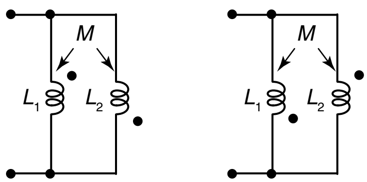


Figure 3.10 *Parallel Opposing Connections*

Here, total inductance $$= \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$

It can be derived in the same way as done for parallel opposing connection.

3.9 LINEAR TRANSFORMER

A transformer is a four-terminal device comprising of two (or more) magnetically coupled coils. It is composed of two coils:

– a primary coil of resistance R_1 and self-inductance L_1

– a secondary coil of resistance R_2 and self-inductance L_2

A transformer is said to be linear if the coils are wound on a magnetically linear material for which the magnetic permeability is a constant. Some of linear materials are air, plastic, Bakelite, wood, etc.

Circuit representation of a linear transformer is shown in Fig. 3.11.

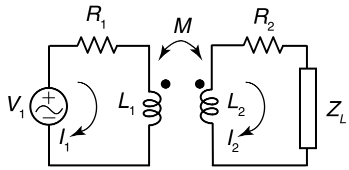


Figure 3.11 *Circuit Representation of Linear Transformer*

Calculation of Input and Reflected Impedances

By KVL for the two meshes,

$$V_1 = (R_1 + j\omega L_1) I_1 - j\omega M I_2 \tag{i}$$

$$0 = -j\omega M I_1 + (R_2 + j\omega L_2 + Z_L) I_2 \tag{ii}$$

From (ii), $I_2 = \dfrac{j\omega M I_1}{R_2 + j\omega L_2 + Z_L}$

Putting this value in (i),

$$V_1 = (R_1 + j\omega L_1) I_1 - \frac{j\omega M \times j\omega M I_1}{R_2 + j\omega L_2 + Z_L} = (R_1 + j\omega L_1) I_1 + \frac{\omega^2 M^2 I_1}{R_2 + j\omega L_2 + Z_L}$$

$\therefore$ Input Impedance, $\boxed{Z_{in} = \dfrac{V_1}{I_1} = (R_1 + j\omega L_1) + \dfrac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L}}$

Here, $(R_1 + j\omega L_1)$ = Impedance of Primary Winding

and, $\boxed{Z_R = \dfrac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L}}$

where,

Z_R = Impedance due to coupling between primary and secondary, known as Reflected Impedance

NB: The input impedance and reflected impedance value do not change with the position of dots on the winding, as the same result is obtained by replacing M by $-M$.

3.10 DETERMINATION OF EQUIVALENT *T* AND π CIRCUIT OF LINEAR TRANSFORMER (Conductively EQuivalent Circuit of a Magnetically Coupled Circuit)

A linear transformer can be replaced by an equivalent T or π network. A linear transformer with a source in the primary and a load in the secondary is shown in Fig. 3.10. If we separate the resistances from the transformer, there remains only a pair of mutually coupled inductors, as shown in Fig. 3.12.

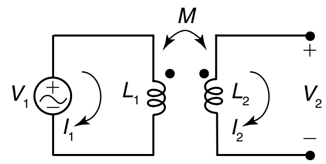


Figure 3.12 *Circuit Representation of Linear Transformer without resistances*

By KVL for the two meshes,

$$v_1 = L_1\frac{di_1}{dt} + M\frac{di_2}{dt} \text{ and } v_2 = M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

or,

$$V_1 = j\omega L_1 I_1 + j\omega M I_2 = j\omega(L_1 - M)I_1 + j\omega M(I_1 + I_2)$$

and

$$V_2 = j\omega M I_1 + j\omega L_2 I_2 = j\omega M(I_2 + I_1) + j\omega(L_2 - M)I_2$$

Equivalent T Circuit

The above two equations can be written as,

$$V_1 = j\omega L_1 I_1 + j\omega M I_2 = j\omega(L_1 - M)I_1 + j\omega M(I_1 + I_2)$$

and

$$V_2 = j\omega M I_1 + j\omega L_2 I_2 = j\omega M(I_2 + I_1) + j\omega(L_2 - M)I_2$$

Therefore, the equivalent *T* network for the linear transformer is shown in Fig. 3.13.

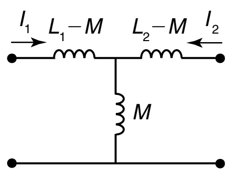


Figure 3.13 *Equivalent* T *network of Linear Transformer*

Note that, if the dots of any one of the windings are placed in opposite end of the coil, the mutual term becomes negative and the equivalent circuit can be obtained by replacing *M* by –*M*. In that case, the three inductances are: $L_1 + M$, $-M$, and $L_2 + M$.

Equivalent π Circuit

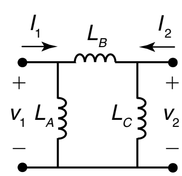


Figure 3.14 *Equivalent* p *network of Linear Transformer*

Using the concept of T-π conversion or, star-delta conversion, we get the equivalent π circuit of a linear transformer as follows.

The three inductances of the equivalent p circuit are:

$$L_A = \frac{(L_1 - M)M + M(L_2 - M) + (L_1 - M)(L_2 - M)}{(L_2 - M)} = \frac{L_1L_2 - M^2}{L_2 - M}$$

Similarly, $$L_B = \frac{L_1L_2 - M^2}{M} \text{ and } L_C = \frac{L_1L_2 - M^2}{L_1 - M}$$

Here also, if any one dot changes its location on the winding, the sign of M will change and in that case, the three inductances will be:

$$L_A = \frac{L_1L_2 - M^2}{L_2 + M},\ L_B = -\frac{L_1L_2 - M^2}{M} \text{ and } L_C = \frac{L_1L_2 - M^2}{L_1 + M}$$

3.11 IDEAL TRANSFORMER

A transformer is said to be ideal; if it has the following properties:

1. Primary and secondary coils are lossless (i.e. $R_1 = R_2 = 0$).
2. Primary and secondary coils have very large reactances compared to any connected impedance (i.e. L_1, L_2, $M \to \infty$)
3. Coupling between primary and secondary coils is perfect, i.e. $k = 1$ or the leakage flux is zero.

An ideal transformer is a useful approximation of a very tightly coupled transformer ($k \approx 1$) in which both the primary and secondary inductive reactances are extremely large compared to the load impedance.

Calculation of Input Impedance for Ideal Transformer

The circuit symbol of an ideal transformer is shown in Fig. 3.15.

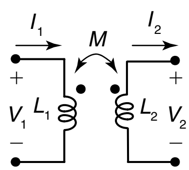


Figure 3.15 *Circuit Symbol of Ideal Transformer*

By KVL,

$$V_1 = j\omega L_1 I_1 - j\omega M I_2 \tag{i}$$

$$0 = -j\omega M I_1 + (j\omega L_2 + Z_L) I_2 \tag{ii}$$

From (ii), $$I_2 = \frac{j\omega M}{j\omega L_2 + Z_L} I_1$$

Putting this in (i), we get,

$$V_1 = j\omega L_1 I_1 - j\omega M I_2 = j\omega L_1 I_1 - j\omega M \frac{j\omega M}{j\omega L_2 + Z_L} I_1$$

$$= \left(\frac{-\omega^2 L_1 L_2 + j\omega L_1 Z_L + \omega^2 M^2}{j\omega L_2 + Z_L} \right) I_1$$

$$= \left(\frac{-\omega^2 L_1 L_2 + j\omega L_1 Z_L + \omega^2 L_1 L_2}{j\omega L_2 + Z_L} \right) I_1 \quad \left[\because k = 1, \therefore \quad M = \sqrt{L_1 L_2} \right]$$

$$= \left(\frac{j\omega L_1 Z_L}{j\omega L_2 + Z_L} \right) I_1$$

$\therefore$ Input Impedance,

$$Z_{\text{in}} = \frac{V_1}{I_1} = \frac{j\omega L_1 Z_L}{j\omega L_2 + Z_L} \approx \frac{j\omega L_1 Z_L}{j\omega L_2} \quad [\because L_2 >> Z_L; \text{ for ideal transformer}]$$

$$= Z_L \left(\frac{L_1}{L_2} \right) = Z_L \left(\frac{N_1}{N_2} \right)^2 \quad \left[\because L \propto N^2 \right]$$

$$\Rightarrow \qquad \boxed{Z_{in} = Z_L \left(\frac{N_1}{N_2} \right)^2 = \frac{Z_L}{n^2}}$$

where, $n = \dfrac{N_2}{N_1}$ is the turns ratio. Thus, the *load impedance is approximately transferred as the square of turns ratio*. This input impedance is also known as the *reflected impedance* as the load impedance is reflected to the primary side.

This property of an ideal transformer to transform a given impedance into another impedance is used in *impedance matching*, which is very useful in different applications involving maximum power transfer.

Calculation of Voltage and Current Transformation Ratio for Ideal Transformer

From (ii), $\qquad I_1 = \dfrac{j\omega L_2 + Z_L}{j\omega M} I_2$

Putting this in (i), we get,

$$V_1 = j\omega L_1 I_1 - j\omega M I_2 = j\omega L_1 \left(\frac{j\omega L_2 + Z_L}{j\omega M} \right) I_2 - j\omega M I_2$$

$$= \left(\frac{-\omega^2 L_1 L_2 + j\omega L_1 Z_L + \omega^2 M^2}{j\omega M}\right) I_2$$

$$= \left(\frac{-\omega^2 L_1 L_2 + j\omega L_1 Z_L + \omega^2 L_1 L_2}{j\omega M}\right) I_2 \quad \left[\because k = 1, \therefore \quad M = \sqrt{L_1 L_2}\right]$$

$$= Z_L \left(\frac{L_1}{M}\right) I_2$$

$$= Z_L \left(\frac{L_1}{\sqrt{L_1 L_2}}\right) I_2$$

$$V_1 = I_2 Z_L \sqrt{\frac{L_1}{L_2}}$$

$$= V_2 \sqrt{\frac{L_1}{L_2}}$$

$\therefore$ Voltage Transformation Ratio, $\boxed{\frac{V_2}{V_1} = \sqrt{\frac{L_2}{L_1}} = \frac{N_2}{N_1} = n}$

where, n is the turns ratio. Depending upon the value of the turns ratio, three types of transformer are obtained:

Case (I): $n > 1$

In this case, the secondary voltage is greater than the primary voltage and the transformer is termed as *step-up transformer*.

Case (II): $n < 1$

In this case, the secondary voltage is less than the primary voltage and the transformer is termed as *step-down transformer*.

Case (III): $n = 1$

In this case, the secondary voltage is equal to the primary voltage and the transformer is termed as *isolation transformer*.

Also,

$$I_1 = \frac{j\omega L_2 + Z_L}{j\omega M} I_2 \approx \frac{j\omega L_2}{j\omega M} I_2 \quad [\because L_2 >> Z_L;\ \text{for ideal transformer}]$$

$$= \frac{L_2}{\sqrt{L_1 L_2}} I_2$$

$$= \sqrt{\frac{L_2}{L_1}} I_2$$

$\therefore$ Current Transformation Ratio, $\boxed{\frac{I_2}{I_1} = \sqrt{\frac{L_1}{L_2}} = \frac{N_1}{N_2} = \frac{1}{n}}$

where, n is the turns ratio. Thus, the ratio of the primary current to the secondary current is the turns ratio. It must be noted that if any one dot changes its location on the winding, the current ratio will become the negative of the turns ratio.

3.12 TUNED COUPLED CIRCUITS

When a capacitor is introduced in the primary and/or in the secondary circuit of a transformer, the circuit becomes selective and the transformer is termed as a *tuned coupled circuit* or a *tuned transformer*.

Tuned circuits are used for amplification of signals in the radio frequency (RF) range, such as in broadcasting receivers.

Tuned circuits are of two types:

1. Single Tuned Circuit, and
2. Double Tuned Circuit

1. Single Tuned Coupled Circuit

In this circuit, a capacitor is introduced only in the secondary so that only the secondary is tuned. The circuit is shown in Fig. 3.16.

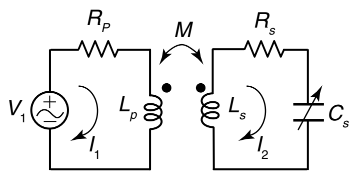


Figure 3.16 *Single Tuned Coupled Circuit*

Let,

R_p – Total resistance in the primary = (internal resistance of the source + resistance of primary coil

R_s – Total resistance in the secondary

L_p, L_s – Leakage inductance of primary and secondary, respectively

C_s – Variable capacitor for tuning, connected across the secondary

M – Mutual inductance between primary and secondary

V_1 – Source voltage

V_0 – Output voltage

By KVL for the two meshes,

$$V_1 = (R_p + j\omega L_p)\, I_1 - j\omega M I_2$$

$$0 = -j\omega M I_1 + \left(R_s + j\omega L_s - \frac{j}{\omega C_s}\right) I_2$$

Solving for I_2,

$$I_2 = \frac{\begin{vmatrix} (R_p + j\omega L_p) & V_1 \\ -j\omega M & 0 \end{vmatrix}}{\begin{vmatrix} (R_p + j\omega L_p) & -j\omega M \\ -j\omega M & \left\{R_s + j\left(\omega L_s - \dfrac{1}{\omega C_s}\right)\right\} \end{vmatrix}} = \frac{V_1 (j\omega M)}{R_p\left[R_s + j\left(\omega L_s - \dfrac{1}{\omega C_s}\right)\right] + \omega^2 M^2}$$

Therefore, the output voltage is,

$$V_0 = \frac{I_2}{j\omega C_s} = \frac{V_1 M / C_s}{R_p\left[R_s + j\left(\omega L_s - \dfrac{1}{\omega C_s}\right)\right] + \omega^2 M^2}$$

∴ Voltage Amplification,

$$A = \frac{V_0}{V_1} = \frac{M / C_s}{R_p\left[R_s + j\left(\omega L_s - \dfrac{1}{\omega C_s}\right)\right] + \omega^2 M^2}$$

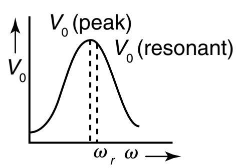


Figure 3.17 *Frequency Response of Single Tuned Circuit*

This shows that the output voltage depends upon M. Fig. 3.17 shows the variation of the output voltage against w.

For a constant value of M, tuning is obtained by varying C_s. For resonance,

$$\omega_r L_s = \frac{1}{\omega_r C_s}$$

$$\Rightarrow \qquad \omega_r = \sqrt{\frac{1}{L_s C_s}}$$

At this resonant frequency, the output voltage (or amplification) is given as,

$$|A|_{res} = \frac{M/C_s}{R_p R_s + \omega_r^2 M^2} \text{ or } |V_0|_{res} = \frac{V_1 M/C_s}{R_p R_s + \omega_r^2 M^2} \text{ or}$$

For maximum output voltage at resonance,

$$\frac{d|V_0|_{res}}{dM}\left(\text{or } \frac{d|A|_{res}}{dM}\right) = 0$$

$$\Rightarrow \qquad \frac{V_1/C_s}{R_p R_s + \omega_r^2 M^2} - \frac{2\omega_r^2 M^2 V_1/C_s}{\left(R_p R_s + \omega_r^2 M^2\right)^2} = 0$$

$$\Rightarrow \qquad M = \frac{\sqrt{R_p R_s}}{\omega_r}$$

Under this condition, the coefficient of coupling is,

$$k_C = \frac{M}{\sqrt{L_p L_s}} = \frac{\sqrt{R_p R_s}}{\omega_r \sqrt{L_p L_s}} = \frac{1}{\sqrt{Q_1 Q_2}}$$

where, Q_1 and Q_2 are the quality factors of uncoupled primary and secondary circuits, respectively. Here, k_C is known as the critical coefficient of coupling.

Substituting the value of M, the maximum output voltage is,

$$|V_0|_{res,\max} = \frac{V_1}{2\omega_r C_s \sqrt{R_p R_s}} = \frac{V_1}{2k_C}\sqrt{\frac{L_s}{L_p}}$$

Fig. 3.18 shows the variation of the output voltage against w for different values of k.

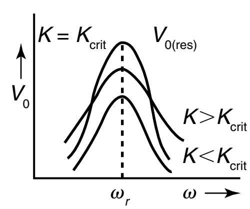


Figure 3.18 *Frequency Response of Single Tuned Circuit for Different Values of k*

Now, the secondary impedance reflected to the primary side in the form of coupled impedance is,

$$\frac{\omega^2 M^2}{R_s + j\left(\omega L_s - \frac{1}{\omega C_c}\right)} \quad \text{or} \quad \frac{\omega^2 M^2}{R_s} \text{ at resonance}$$

At resonance, the total primary resistance becomes,

$$R_1 = \left(R_p + \frac{\omega^2 M^2}{R_s}\right)$$

∴ Effective quality factor of the primary is,

$$Q_e = \frac{\omega_r L_p}{R_1} = \frac{\omega_r L_p}{\left(R_p + \frac{\omega^2 M^2}{R_s}\right)} = \frac{Q_1}{\left(1 + \frac{\omega^2 M^2}{R_p R_s}\right)}$$

∴ Bandwidth of the primary is,

$$BW = \frac{\omega_r}{Q_e} = \omega_r \left(\frac{\omega^2 M^2 + R_p R_s}{Q_1 R_p R_s}\right)$$

2. Double Tuned Coupled Circuit

In a double tuned circuit, both the primary and secondary of the coupled coils are tuned using variable capacitors. The circuit is shown in Fig. 19.

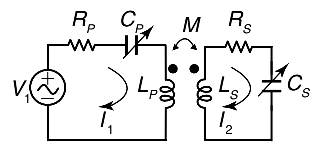


Figure 19 *Double Tuned Coupled Circuit*

Let,

R_p – Total resistance in the primary = (internal resistance of the source + resistance of primary coil

R_s – Total resistance in the secondary

L_p, L_s – Leakage inductance of primary and secondary, respectively

C_s – Variable capacitor for tuning, connected across the secondary

M – Mutual inductance between primary and secondary

V_1 – Source voltage

V_0 – Output voltage

By KVL for the two meshes,

$$V_1 = \left(R_p + j\omega L_p - \frac{j}{\omega C_p}\right) I_1 - j\omega M I_2$$

$$0 = -j\omega M I_1 + \left(R_s + j\omega L_s - \frac{j}{\omega C_s}\right) I_2$$

Solving for I_2,

$$I_2 = \frac{\begin{vmatrix} \left(R_p + j\omega L_p - \frac{j}{\omega C_p}\right) & V_1 \\ -j\omega M & 0 \end{vmatrix}}{\begin{vmatrix} \left(R_p + j\omega L_p - \frac{j}{\omega C_p}\right) & -j\omega M \\ -j\omega M & \left\{R_s + j\omega L_s - \frac{j}{\omega C_s}\right\} \end{vmatrix}} = \frac{V_1 (j\omega M)}{\left(R_p + j\omega L_p - \frac{j}{\omega C_p}\right)\left(R_s + j\omega L_s - \frac{j}{\omega C_s}\right) + \omega^2 M^2}$$

Therefore, the output voltage is,

$$V_0 = \frac{I_2}{j\omega C_s} = \frac{V_1 M / C_s}{\left(R_p + j\omega L_p - \frac{j}{\omega C_p}\right)\left(R_s + j\omega L_s - \frac{j}{\omega C_s}\right) + \omega^2 M^2}$$

$\therefore$ Voltage Amplification,

$$\boxed{A = \frac{V_0}{V_1} = \frac{M}{C_s\left[\left(R_p + j\omega L_p - \frac{j}{\omega C_p}\right)\left(R_s + j\omega L_s - \frac{j}{\omega C_s}\right) + \omega^2 M^2\right]}}$$

At resonant frequency,

$$\omega = \omega_r \text{ and } \quad \frac{1}{\sqrt{L_p C_p}} = \frac{1}{\sqrt{L_s C_s}}$$

At this resonant frequency, the output voltage (or amplification) is given as,

$$|A|_{res} = \frac{M / C_s}{R_p R_s + \omega_r^2 M^2} \quad \text{or} \quad |V_0|_{res} = \frac{V_1 M / C_s}{R_p R_s + \omega_r^2 M^2}$$

For maximum output voltage at resonance,

$$\frac{d|V_0|_{res}}{dM} \left(or \ \frac{d|A|_{res}}{dM}\right) = 0$$

$$\Rightarrow \qquad \frac{V_1/C_s}{R_pR_s+\omega_r^2M^2}-\frac{2\omega_r^2M^2V_1/C_s}{\left(R_pR_s+\omega_r^2M^2\right)^2}=0$$

$$\Rightarrow \qquad M=\frac{\sqrt{R_pR_s}}{\omega_r}$$

Under this condition, the coefficient of coupling is,

$$k_C=\frac{M}{\sqrt{L_pL_s}}=\frac{\sqrt{R_pR_s}}{\omega_r\sqrt{L_pL_s}}=\frac{1}{\sqrt{Q_1Q_2}}$$

where, Q_1 and Q_2 are the quality factors of uncoupled primary and secondary circuits, respectively. Here, k_C is known as the critical coefficient of coupling.

Substituting the value of M, the maximum output voltage is,

$$\left|V_0\right|_{res,\max}=\frac{V_1}{2\omega_rC_s\sqrt{R_pR_s}}=\frac{V_1}{2k_C}\sqrt{\frac{L_s}{L_p}}$$

At resonance, the total resistance referred to primary is,

$$R_1=\left(R_p+\frac{\omega^2M^2}{R_s}\right)$$

The coupled impedance is the maximum at resonance. It becomes inductive below resonance and capacitive above resonance.

$\therefore$ Effective quality factor of the primary is,

$$Q_e=\frac{\omega_rL_p}{R_1}=\frac{\omega_rL_p}{\left(R_p+\dfrac{\omega^2M^2}{R_s}\right)}$$

$\therefore$ Bandwidth of the primary is,

$$BW=\frac{\omega_r}{Q_e}=\omega_r\left(\frac{\omega^2M^2+R_pR_s}{Q_1R_pR_s}\right)$$

All these results are similar to those for a single-tuned circuit with $k \le k_C$. The variation of the output voltage against w is shown in Fig. 3.20.

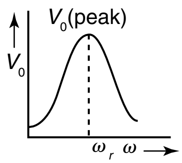


Figure 3.21 *Frequency Response of Double Tuned Circuit*

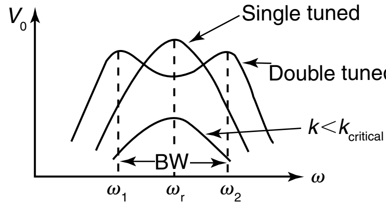


Figure 3.21 *Frequency Response of Double Tuned Circuit for Different Values of k*

The variation of the secondary current or output voltage for different values of coefficient of coupling (k) is shown in Fig. 3.21.

❖ **When Coefficient of Coupling is small (k <<)**
- ✓ The effect of coupled impedance is negligible.
- ✓ The variation of V_0 (or I_2) is similar to that for the series resonant curve of the primary circuit.
- ✓ The secondary current is small and the variation with frequency has a peaky nature than the resonance curve of the secondary circuit.

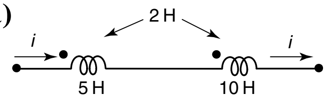
❖ **When Coefficient of Coupling is increased gradually**
- ✓ The effect of coupled impedance increases.
- ✓ The total impedance of the primary circuit is increased.
- ✓ The magnitude of the primary current is reduced and the curve of the primary circuit becomes broader.
- ✓ The secondary current-peak becomes higher and the curve of the secondary current becomes broader.

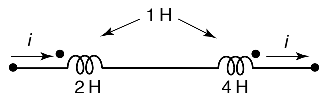
❖ **When $k = k_C$**
- ✓ In this condition, the resistance which the secondary circuit couples into the primary at resonance is equal to the primary resistance.
- ✓ The secondary current will be maximum.
- ✓ The curve of the secondary current will be broader and flat-topped.
- ✓ The curve of the primary current will have two peaks.

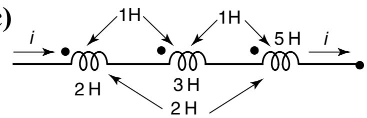
❖ **When $k > k_C$**
- ✓ The double peaks of the primary current become more prominent; peaks being separated from each other.
- ✓ The magnitude of the primary current at peaks becomes smaller as the value of k is increased.
- ✓ The curve of the secondary current will also have two peaks.

SOLVED PROBLEMS

1. Find the effective value of the inductance for the following connections:

(a)


(b)


(c)


Sol:

(a) This is a series aiding connection. The effective inductance is,

$$\therefore \qquad L_{eq} = (L_1 + L_2 + 2M) = 5 + 10 + 2 \times 2 = 19 \ \ H \quad Ans$$

(b) This is a series opposing connection. The effective inductance is,

$$\therefore \qquad L_{eq} = (L_1 + L_2 - 2M) = 2 + 4 - 2 \times 1 = 4 \ \ H \quad Ans$$

(c) Since the coils are magnetically coupled in series aiding or they assist each other, therefore,

Effective inductance for coil 1 is: $L_{1\text{eff}} = (L_1 + M_{12} + M_{13}) = (2 + 1 + 2) = 5 \ \ H$

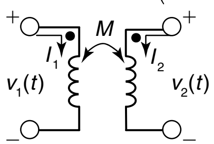
Effective inductance for coil 2 is: $L_{2\text{eff}} = (L_2 + M_{12} + M_{23}) = (3 + 1 + 1) = 5 \ \ H$

Effective inductance for coil 3 is: $L_{3\text{eff}} = (L_3 + M_{13} + M_{23}) = (5 + 2 + 1) = 8 \ \ H$

Total effective inductance is,

$$L_{\text{eff}} = (L_{1\text{eff}} + L_{2\text{eff}} + L_{34\text{ff}}) = L_1 + L_2 + L_3 + +2(M_{12} + M_{23} + M_{13}) = 18 \ \ H \quad Ans$$

2. For the circuit shown in the figure, if $L_1 = 0.4 \ H, \ L_2 = 2.5 \ H, \ k = 0.6,$ **and** $i_1 = 4i_2 = 20\cos\left(500t - 20^0\right)$ mA, **evaluate the following quantities at t = 0:**



(a) i_2

(b) v_1 **and**

(c) the total energy stored in the system.

Sol: Here, $i_1 = 20\cos(500t - 20°)$ mA, and $i_2 = 5\cos(500t - 20°)$ mA,

(a) At t = 0, $i_2 = 5\cos(-20°) = 4.7$ mA *Ans*

(b) Mutual inductance, $M = k\sqrt{L_1 L_2} = 0.6\sqrt{(0.4)(2.5)} = 0.6 \text{ H}$

Now,
$$v_1(t) = L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt}$$
$$= 0.4\frac{d}{dt}[20\cos\,(500t - 20°)] + 0.6\frac{d}{dt}[5\cos(500t - 20°)]$$
$$= -0.4 \times 20 \times 500 \sin\,(500t - 20°) - 0.6 \times 5 \times 500 \sin(500t - 20°)$$
$$= (-4000 - 1500)\sin\,(500t - 20°)$$
$$= -5500\sin(500t - 20°)$$

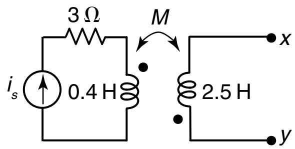
At $t = 0$, $v_1(0) = -5500\sin(-20°) = 1.881\ V$ *Ans*

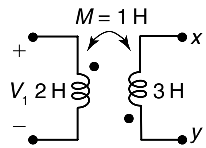
(c) At $t = 0$, $i_1 = 20\cos(-20°) = 18.8$ mA

$\therefore$ Total energy stored in the system,

$$W = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + MI_1 I_2 = \frac{1}{2}(0.4)\times(18.8)^2 + \frac{1}{2}(2.5)\times(4.7)^2 + 0.6\times 18.8\times 4.7\ \mu\text{J}$$

$$= 151.32\ \mu\text{J}\quad Ans$$

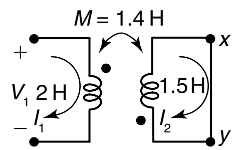




3. (a) If i_s = 2 cos 10t (A). find the total energy stored in the passive network at t = 0 for k = 0.6 and terminals x and y left open-circuited,

(b) Determine the amount of energy stored after 0.5 s, when the primary side of the circuit shown in figure is connected to a dc source of 15V and the secondary is short-circuited. Given: L_1 = 2H, L_2 = 3H and M = 1H.

Sol:



(a) When x and y are open-circuited:

The current in the second coil $I_2 = 0$.

The energy stored is,

$$W = \frac{1}{2}L_1 I_1^2 = \frac{1}{2}\times 0.4\times 2^2 = 0.8\ \text{J}\quad Ans$$

(b) When x and y are short-circuited:

Applying KVL for the two meshes,

For mesh1: $V_1 = L_1\dfrac{di_1}{dt} - M\dfrac{di_2}{dt}$

For mesh 2: $0 = -M\dfrac{di_1}{dt} + L_2\dfrac{di_2}{dt} \Rightarrow \dfrac{di_2}{dt} = \dfrac{M}{L_2}\dfrac{di_1}{dt}$

Substituting this in first equation,

$$V_1 = L_1\frac{di_1}{dt} - M\frac{M}{L_2}\frac{di_1}{dt} = \left(\frac{L_1 L_2 - M^2}{L_2}\right)\frac{di_1}{dt} = L_{\text{eq}}\frac{di_1}{dt}$$

where, $L_{\text{eq}} = \dfrac{L_1 L_2 - M^2}{L_2} = \dfrac{2\times 3 - 1^2}{3} = \dfrac{5}{3} = 1.67$

or, $\quad 15 = 1.67\dfrac{di_1}{dt}$

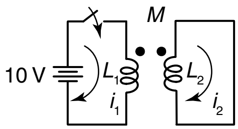
or, $\quad \dfrac{di_1}{dt} = 9$

$\Rightarrow \quad i_1 = 9t$

At t = 0.5 s, $i_1 = 9t = 9 \times 0.5 = 4.5$ A
Thus, the total energy stored in the system,

$$W = \frac{1}{2}L_{eq}I_1^2 = \frac{1}{2}\times\frac{5}{3}\times(4.5)^2 = 16.875 \text{ J} \quad Ans$$

4. In the circuit shown in the figure, L_1 = 1 H, L_2 = 2 H, M = 1.2 H. Find an expression for the energy stored t second after the switch is closed.



Sol: Applying KVL for the two loops, we get,

$$L_1\frac{di_1}{dt} - M\frac{di_2}{dt} = 10$$

$$-M\frac{di_1}{dt} + L_2\frac{di_2}{dt} = 0$$

Taking Laplace transform,

$$sL_1I_1(s) - sMI_2(s) = \frac{10}{s}$$

$$-sMI_1(s) + sL_2I_2(s) = 0$$

Solving for I_{-1}-(s) and I_2(s),

$$I_1(s) = \frac{\begin{vmatrix} 10/s & -sM \\ 0 & sL_2 \end{vmatrix}}{\begin{vmatrix} sL_1 & -sM \\ -sM & sL_2 \end{vmatrix}} = \frac{10L_2}{s^2\left(L_1L_2 - M^2\right)}$$

$$I_2(s) = \frac{\begin{vmatrix} sL_1 & 10/s \\ sM & 0 \end{vmatrix}}{\begin{vmatrix} sL_1 & -sM \\ -sM & sL_2 \end{vmatrix}} = \frac{10M}{s^2\left(L_1L_2 - M^2\right)}$$

Taking inverse Laplace transform,

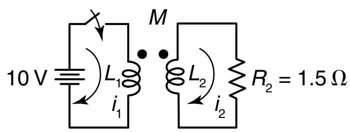
$$i_1(t) = \left(\frac{10L_2}{L_1L_2 - M^2}\right)t = \left(\frac{10\times 2}{1\times 2 - 1.2^2}\right)t = 35.71t$$

$$i_2(t) = \left(\frac{10M}{L_1L_2 - M^2}\right)t = \left(\frac{10\times 1.2}{1\times 2 - 1.2^2}\right)t = 21.43t$$

Therefore, at any time t, the energy stored is given as,

$$E = \frac{1}{2}L_1 i_1^2 - \frac{1}{2}L_2 i_2^2 = \frac{1}{2}\times 1\times(35.71t)^2 - \frac{1}{2}\times 2\times(21.43t)^2 = 178.57t^2 \text{ J} \quad Ans$$

5. Find the voltage $v(t)$ across 1.5 Ω resistance in the network shown in the figure when a 10V source is switched on. The primary and secondary self inductances are $L_1 = L_2 = 1$ H and $M = 0.5$ H.



Sol: Applying KVL for the two loops,
Applying KVL for the two loops, we get,

$$L_1\frac{di_1}{dt} - M\frac{di_2}{dt} = 10$$

$$-M\frac{di_1}{dt} + L_2\frac{di_2}{dt} + R_2 i_2 = 0$$

Taking Laplace transform,

$$sL_1I_1(s) - sMI_2(s) = \frac{10}{s} \qquad \Rightarrow sI_1(s) - 0.5sI_2(s) = \frac{10}{s}$$

$$-sMI_1(s) + (sL_2 + R_2)I_2(s) = 0 \Rightarrow -0.5sI_1(s) + (s+1.5)I_2(s) = 0$$

Solving for I_2(s),

$$I_2(s) = \frac{\begin{vmatrix} s & 10/s \\ -0.5s & 0 \end{vmatrix}}{\begin{vmatrix} s & -0.5s \\ -0.5s & (s+1.5) \end{vmatrix}} = \frac{5}{s(s+1.5) - 0.25s^2} = \frac{5}{0.75s^2 + 1.5s} = \frac{20/3}{s(s+2)} = \frac{10}{3}\left(\frac{1}{s} - \frac{1}{s+2}\right)$$

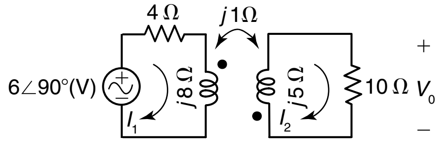
Taking inverse Laplace transform,

$$i_2(t) = \frac{10}{3}(1 - e^{-2t})$$

Therefore, the voltage across the 1.5 Ω resistance is,

$$v(t) = R_2 \times i_2(t) = 1.5\times\frac{10}{3}\left(1 - e^{-2t}\right) = 5\left(1 - e^{-2t}\right) \ (V) \quad Ans$$

6. Determine the voltage V_0 in the circuit.
Sol:
For coil 1, by KVL,



$$4I_1 + j8I_1 + j1I_2 = j6$$

or $$(4+j8)I_1 + j1I_2 = j6 \quad \text{(i)}$$

For coil 2, by KVL,

$$10I_2 + j5I_2 + j1I_1 = 0$$

or $$j1I_1 + (10+j5)I_2 = 0 \quad \text{(ii)}$$

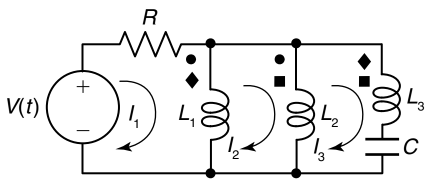
Solving (*i*) and (*ii*),

$$I_2 = \frac{\begin{vmatrix} (4+j8) & j6 \\ j1 & 0 \end{vmatrix}}{\begin{vmatrix} (4+j8) & j1 \\ j1 & (10+j5) \end{vmatrix}} = \frac{6}{j100+1}$$

Therefore, the voltage V_0 is,

$$V_0 = 10I_2 = \frac{60}{100+j1} = 0.6\angle -90°\,(V) \quad Ans$$

7. Write the loop equations for the circuit shown in the figure.



Sol:

We apply dot conventions for the circuit. By KVL for the three meshes, we get,
For mesh 1,

$$R_1I_1 + L_1\frac{d}{dt}(I_1 - I_2) + M_{12}\frac{d}{dt}(I_2 - I_3) + M_{13}\frac{dI_3}{dt} = V(t) \quad \text{(i)}$$

For mesh 2,

$$L_1\frac{d}{dt}(I_2 - I_1) - M_{21}\frac{d}{dt}(I_2 - I_1) + L_2\frac{d}{dt}(I_2 - I_3) - M_{12}\frac{d}{dt}(I_2 - I_3)$$

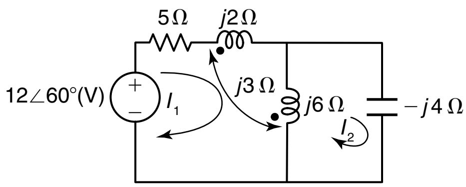
$$+M_{23}\frac{dI_3}{dt} - M_{13}\frac{dI_3}{dt} = 0 \quad \text{(ii)}$$

For mesh 3,

$$L_3 \frac{dI_3}{dt} + +L_2 \frac{d}{dt}(I_3 - I_2) - M_{23}\frac{dI_3}{dt} - M_{32}\frac{d}{dt}(I_3 - I_2)$$

$$+M_{31}\frac{d}{dt}(I_1 - I_2) + M_{21}\frac{d}{dt}(I_1 - I_2) + \frac{1}{C}\int_0^t I_3 dt = 0 \qquad \text{(iii)}$$

8. For the circuit shown in figure, determine the phasor currents I_1 and I_2.



Sol:
By KVL for mesh 1,

$$-12\angle 60° + I_1(5 + j2 + j6) - j6I_2 + j3I_2$$

or, $$I_1(5 + j8) - j3I_2 = 12\angle 60° \qquad \text{(i)}$$

For mesh 2,

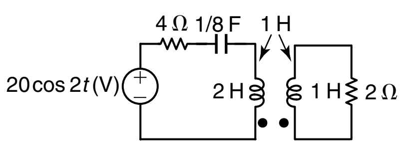
$$I_2(j6 - j4) - j6I_1 + j3I_1 = 0$$

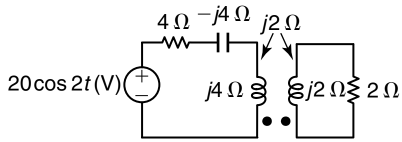
or, $$-j3I_1 + j2I_2 = 0 \qquad \text{(ii)}$$

Solving (*i*) and (*ii*),

$$\left.\begin{aligned} I_1 &= \frac{\begin{vmatrix} 12\angle 60^0 & -j3 \\ 0 & j2 \end{vmatrix}}{\begin{vmatrix} (5+j8) & -j3 \\ -j3 & j2 \end{vmatrix}} = \frac{24\angle 150°}{j10 - 16 + 9} = \frac{24\angle 150°}{-7 + j10} = \frac{24\angle 150°}{12.206\angle 125°} = 1.966\angle 25° \ (A) \\ I_2 &= \frac{\begin{vmatrix} (5+j8) & 12\angle 60° \\ -j3 & 0 \end{vmatrix}}{\begin{vmatrix} (5+j8) & -j3 \\ -j3 & j2 \end{vmatrix}} = \frac{36\angle 150°}{j10 - 16 + 9} = \frac{36\angle 150°}{-7 + j10} = \frac{36\angle 150°}{12.206\angle 125°} = 2.949\angle 25° \ (A) \end{aligned}\right\} Ans$$

9. Determine the coupling co-efficient and the energy stored in the coupled circuits at t = 1.5 s.





Sol: In phasor domain, the reactances become:

$$\frac{1}{8}\ F \Rightarrow \frac{-j}{2\times 1/8} = -j4\ \ \Omega$$

$$2\ H \Rightarrow j2 \times 2\ j4\ \Omega$$

$$1\ H \Rightarrow j2 \times 1 = j2\ \Omega$$

$\therefore$ Coefficient of coupling, $k = \dfrac{M}{\sqrt{L_1 L_2}} = \dfrac{1}{\sqrt{2\times 1}} = 0.707$ *Ans*

By KVL for the two meshes,
For mesh 1,

$$I_1\left(4 - j4 + j4\right) - j2I_2 = 20\angle 0°$$

or, $$4I_1 - j2I_2 = 20 \qquad \text{(i)}$$

For mesh 2,

$$I_2\left(2 + j2\right) - j2I_1 = 0$$

or, $$-j2I_1 + \left(2 + j2\right)I_2 = 0 \qquad \text{(ii)}$$

Solving (*i*) and (*ii*),

$$I_1 = \frac{\begin{vmatrix} 20 & -j2 \\ 0 & (2+j2) \end{vmatrix}}{\begin{vmatrix} 4 & -j2 \\ -j2 & (2+j2) \end{vmatrix}} = \frac{40(1+j)}{12+j8} = 3.922\angle 11.31°\ \ (A)$$

$$I_2 = \frac{\begin{vmatrix} 4 & 20 \\ -j2 & 0 \end{vmatrix}}{\begin{vmatrix} 4 & -j2 \\ -j2 & (2+j2) \end{vmatrix}} = \frac{j40}{12+j8} = 2.773\angle 56.31°\ \ (A)$$

$\therefore$ In time domain, the current are,

$$i_1 = 3.922\cos\left(2t + 11.31°\right)\ \ (A) \text{ and } i_2 = 2.773\cos\left(2t + 56.31°\right)\ \ (A)$$

At t = 1.5 s, $2t$ = 3 rad = 171.89°

$$\therefore \qquad i_1 = 3.922\cos\left(171.89° + 11.31°\right) = -3.916\ \ (A) \text{ and}$$

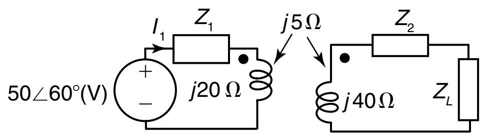
$$\therefore \qquad i_2 = 2.773\cos\left(171.89° + 56.31°\right) = -1.845\ \ (A)$$

$\therefore$ Energy stored in the coupled circuit,

$$W = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2 = \frac{1}{2}\times 2\times(-3.916)^2 + \frac{1}{2}\times 1\times(-1.848)^2 + 1\times(-3.916)\times(-1.848)$$

$= 24.29$ J *Ans*

10. For a linear transformer shown in figure, with Z_1 = (60 – *j*100) Ω, Z_2 = (30 + *j*40) Ω, and Z_L = (80 + *j*60) Ω,; find the input impedance and the current I_1.



Sol:

Input impedance,

$$Z_{in} = \frac{V_1}{I_1} = Z_1 + j20 + \frac{\omega^2 M^2}{j40 + Z_2 + Z_L}$$

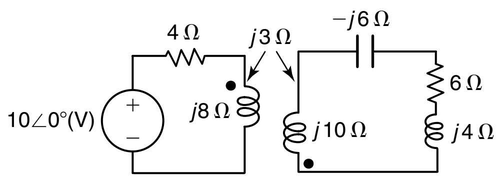
$$= (60 - j100) + j20 + \frac{5^2}{j40 + 30 + j40 + 80 + j60}$$

$$= 60.09 - j80.11$$

$$= 100.14\angle -53.1^\circ\ (\Omega) \quad Ans$$

$$\therefore \text{Current,} \quad I_1 = \frac{V_1}{Z_{in}} = \frac{50\angle 60^\circ}{100.14\angle -53.1^\circ} = 0.5\angle 113.1^\circ\,(A) \quad Ans$$

11. Find the input impedance of the circuit; and current from the voltage source.



Sol:

Input impedance,

$$Z_{in} = \frac{V_1}{I_1} = 4 + j8 + \frac{3^2}{6 + j4 - j6 + j10}$$

$$= \frac{-31 + j80}{6 + j8}$$

$$= 8.579\angle 58.05^\circ\ (\Omega) \quad Ans$$

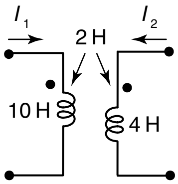
$$\therefore \text{Current,} \quad I_1 = \frac{V_1}{Z_{in}} = \frac{10\angle 0^\circ}{8.579\angle 58.05^\circ} = 1.165\angle -58.05^\circ\ (A) \quad Ans$$

12. Determine the *T*-equivalent and *π*-equivalent circuits of the linear transformer shown in figure.

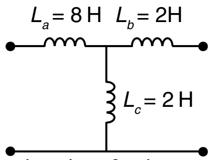
Sol:

Here, L_1 = 10 H, L_2 = 4 H, M = 2 H

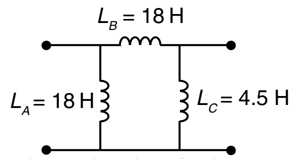
Inductances of the T-equivalent circuit are:



$$\left.\begin{aligned} L_a &= L_1 - M = 10 - 2 = 8 \ \ H \\ L_b &= L_2 - M = 4 - 2 = 2 \ \ H \\ L_c &= M = 2 \ \ H \end{aligned}\right\} \ Ans$$



T-equivalent circuit of Linear Transformer

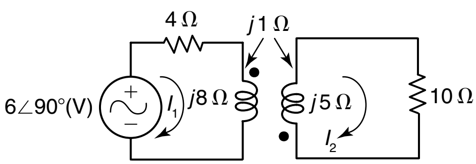


π-equivalent circuit of Linear Transformer

Inductances of the π-equivalent circuit are:

$$\left.\begin{aligned} L_A &= \frac{L_1L_2 - M^2}{L_2 - M} = \frac{10 \times 4 - 2^2}{4-2} = 18 \ \ H \\ L_B &= \frac{L_1L_2 - M^2}{M} = \frac{10 \times 4 - 2^2}{2} = 18 \ \ H \\ L_C &= \frac{L_1L_2 - M^2}{L_1 - M} = \frac{10 \times 4 - 2^2}{10-2} = 4.5 \ \ H \end{aligned}\right\} \ Ans$$

13. Determine the currents I_1 and I_2 in the circuit shown in the figure using *T*-equivalent circuit for the linear transformer.

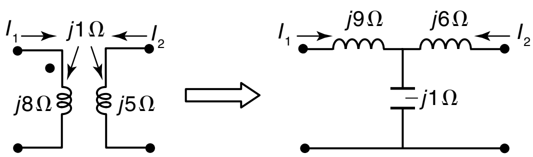


Sol:

Using the *T*-equivalent circuit for the linear transformer, the inductances are:

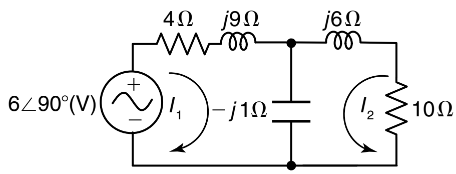
$$\begin{aligned} L_a &= L_1 + M = 8 + 1 = 9 \ \ H \\ L_b &= L_2 + M = 5 + 1 = 6 \ \ H \\ L_c &= -M = -1 \ \ H \end{aligned}$$

(**NB:** Negative inductance in design is implemented by a capacitive reactance.)



T-equivalent network of Linear Transformer

Now inserting the *T*-equivalent circuit in the original circuit, the modified circuit involving no mutual coupling is shown below.



By KVL,

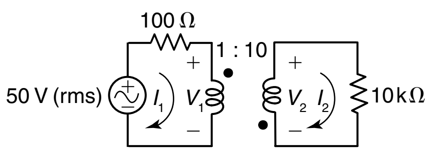
$$j6 = I_1\left(4 + j9 - j1\right) + I_2\left(-j1\right) \Rightarrow \left(4 + j8\right)I_1 + \left(-j1\right)I_2 = j6$$
$$0 = I_1\left(-j1\right) + I_2\left(10 + j6 - j1\right) \Rightarrow \left(-j1\right)I_1 + \left(10 + j5\right)I_2 = 0$$

Solving for I_1 and I_2, we get,

$$\left.\begin{aligned} I_1 &= \frac{\begin{vmatrix} j6 & -j1 \\ 0 & (10+j5) \end{vmatrix}}{\begin{vmatrix} (4+j8) & -j1 \\ -j1 & (10+j5) \end{vmatrix}} = \frac{-30 + j60}{j100 + 1} = 30\left(\frac{-1 + j2}{j100 + 1}\right) = 0.67\angle 27.14° \ \ (A) \\ I_2 &= \frac{\begin{vmatrix} (4+j8) & j6 \\ -j1 & 0 \end{vmatrix}}{\begin{vmatrix} (4+j8) & -j1 \\ -j1 & (10+j5) \end{vmatrix}} = \frac{-6}{j100 + 1} = 0.06\angle -90° \ \ (A) \end{aligned}\right\} \ Ans$$

[**NB:** See solved problem no.4.]

14. For the ideal transformer shown in the figure, determine the average power dissipated in the 10 kΩ resistor.



Sol:

Here, the turns ratio, n = 10, load impedance, Z_L = kΩ

$$\therefore \text{ Input Impedance, } Z_{\text{in}} = \frac{Z_L}{n^2} = \frac{10 \times 10^3}{10^2} = 100 \ \ \Omega$$

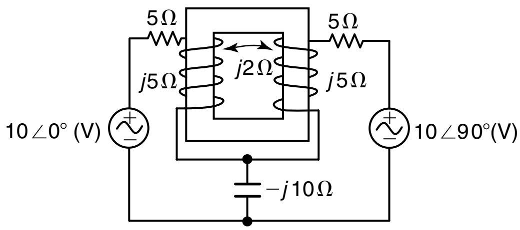
$\therefore$ Primary current, $I_1 = \dfrac{50}{100+100} = 0.25$ A

$\therefore$ Secondary current, $I_2 = \dfrac{I_1}{n} = \dfrac{0.25}{10} = 0.025$ A

$\therefore$ Average Power dissipated in 10 kΩ resistor is,

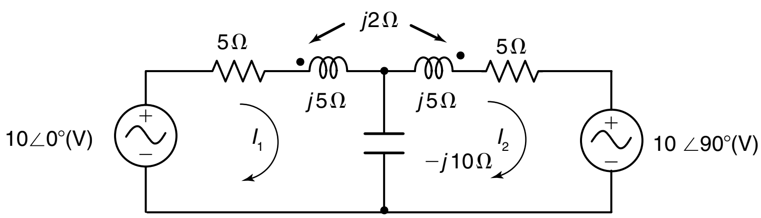
$$P = |I_2|^2 \times 100 \times 10^3 = (0.025)^2 \times 100 \times 10^3 = 6.25 \text{ W} \quad Ans$$

15. Obtain the dotted equivalent circuit for the coupled circuit shown in the figure and use it to find the voltage across the capacitor of –*j*10 Ω reactance.



Sol:

By dot convention, if we assume that the current is entering into the top of the left coil and place a dot at this terminal, then the flux direction for this current will be upward. Now, by Lenz's law, the flux at the right coil must be upward directed to oppose the first flux. In that case, the current will leave the winding by the top terminal, where, another dot is placed. Thus, the dotted equivalent circuit will be as shown in figure below.



By KVL for the two meshes, we get,

$$I_1(5 + j5 - j10) - I_2(-j10) - j2I_2 = 10 \quad \Rightarrow \quad (5 - j5)I_1 + (j8)I_2 = j6$$

$$I_1(-j10) + I_2(5 + j5 - j10) - j2I_1 = j10 \quad \Rightarrow \quad (j8)I_1 + (5 - j5)I_2 = j10$$

Solving for I_1 and I_2, we get,

$$I_1 = \frac{\begin{vmatrix} 10 & j8 \\ j10 & (5-j5) \end{vmatrix}}{\begin{vmatrix} (5-j5) & j8 \\ j8 & (5-j5) \end{vmatrix}} = \frac{130 - j50}{64 - j50}$$

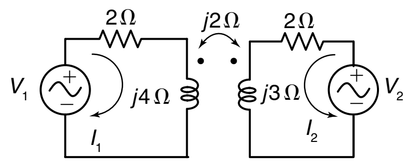
$$I_2 = \frac{\begin{vmatrix} (5-j5) & 10 \\ j8 & j100 \end{vmatrix}}{\begin{vmatrix} (5-j5) & j8 \\ j8 & (5-j5) \end{vmatrix}} = \frac{50-j30}{64-j50}$$

$\therefore$ $$(I_1 - I_2) = \frac{80-j20}{64-j50} = 1.015\angle 23.96° \ (A)$$

$\therefore$ Voltage across the capacitor is,

$$V_c = -j10 \times (I_1 - I_2) = -j10 \times \left(\frac{80-j20}{64-j50}\right) = 10.15\angle -66.04° \ (V) \ \ Ans$$

16. Find V_2 in the circuit given in the figure such that $I_1 = 0$.



Sol: By KVL for the two meshes, we get,

$$(2+j4)I_1 + j2I_2 = V_1 = 5 \quad \text{(i)}$$

$$j2I_1 + (1+j3)I_2 = V \quad \text{(ii)}$$

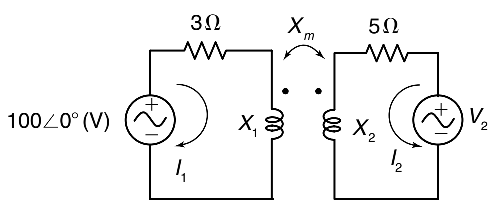
Solving for I_1,

$$I_1 = \frac{\begin{vmatrix} 5 & j2 \\ V_2 & (1+j3) \end{vmatrix}}{\begin{vmatrix} (2+j4) & j2 \\ j2 & (1+j3) \end{vmatrix}} = \frac{5 + j(15-2V_2)}{-6+j10}$$

For the condition, $I_1 = 0$,

$\therefore$ $$5 + j(15-2V_2) = 0 \Rightarrow V_2 = 7.9\angle -18.43° (V) \ \ Ans$$

17. In the circuit of the figure, calculate the current I_2 for which I_1 will be zero. Also, calculate the value of V_2 for this condition. Assume: $X_1 = X_2 = 15\ \Omega\ X_m = 100$.



Sol: By KVL for the two meshes, we get,

$$(3 + j15)I_1 + j10I_2 = 100 \qquad \text{(i)}$$

$$j10I_1 + (5 + j15)I_2 = V_2 \qquad \text{(ii)}$$

Solving for I_1,

$$I_1 = \frac{\begin{vmatrix} 100 & j10 \\ V_2 & (5 + j15) \end{vmatrix}}{\begin{vmatrix} (3 + j15) & j10 \\ j10 & (5 + j15) \end{vmatrix}} = \frac{500 - j(10V_2 - 1500)}{-110 + j150}$$

For the condition, $I_1 = 0$,

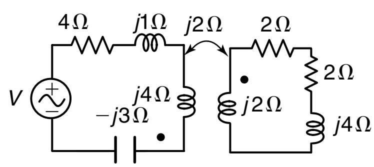
$$500 - j(10V_2 - 1500) = 0$$

$$\Rightarrow \qquad V_2 = (150 - j50) = 158.11\angle -18.43°(V) \quad \textit{Ans}$$

With this value of V_2, solving equations (*i*) and (*ii*) for I_2, we get,

$$I_2 = \frac{\begin{vmatrix} (3 + j15) & 100 \\ V_2 & (150 - j50) \end{vmatrix}}{\begin{vmatrix} (3 + j15) & j10 \\ j10 & (5 + j15) \end{vmatrix}} = \frac{700 - j400}{-110 + j150} = 4.33\angle -156° \,(A) \quad \textit{Ans}$$

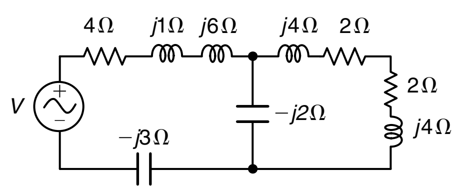
18. Find the conductively equivalent circuit for the network shown in the figure.



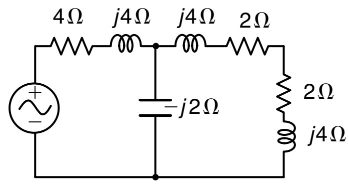
Sol: Here, using the *T*-equivalent circuit for the linear transformer, the inductances are:

$$L_a = L_1 + M = 4 + 2 = 6 \ \ H$$
$$L_b = L_2 + M = 2 + 2 = 4 \ \ H$$
$$L_c = -M = -2 \ \ H$$

Therefore, the conductively equivalent circuit is shown in the figure below.



Simplifying, the conductively equivalent circuit is shown in the figure below.



19. Three similar coils are wound on a long common core in such a way that the voltage of mutual inductance between each set of coils is positive. The self-inductance of each coil is 0.2 H. The effective inductance of first two in series is 0.6 H and all of them in series is 1 H. When the terminals of first two coils are interchanged the effective inductance of three coils in series becomes 0.5 H. Determine co-efficient of coupling between each set of coils.

Sol:

Here, self-inductance of each coil, L = 0.2H,

Let the mutual inductance between first and second coil is M_1, between second and third coil is M_2 and between third and first coil is M_3.

Effective inductance of first coil, $L_{\text{eff}1} = L + M_1 + M_3$

Effective inductance of second coil, $L_{\text{eff}2} = L + M_1 + M_2$

Effective inductance of first coil, $L_{\text{eff}3} = L + M_2 + M_3$

Therefore, effective inductance of first two in series is,

$$L_{\text{eff}1} + L_{\text{eff}2} = L + M_1 + M_3 + L + M_1 + M_2 = 2L + 2M_1 + M_2 + M_3$$

From the given value,

$$2L + 2M_1 + M_2 + M_3 = 0.6$$

or, $$2M_1 + M_2 + M_3 = (0.6 - 2 \times 0.2) = 0.2 \qquad \text{(i)}$$

When all the three coils are connected in series, the effective inductance becomes,

$$L_{\text{eff}1} + L_{\text{eff}2} + L_{\text{eff}3} = 3L + 2(M_1 + M_2 + M_3)$$

From the given value,

$$3L + 2(M_1 + M_2 + M_3) = 1$$

or, $$M_1 + M_2 + M_3 = \frac{(1 - 3 \times 0.2)}{2} = 0.2 \qquad \text{(ii)}$$

After interchanging the terminals of the first two coils, the effective inductances are as given below.

Effective inductance of first coil, $L'_{\text{eff}1} = L - M_1 + M_3$

Effective inductance of second coil, $L'_{\text{eff}2} = L - M_1 - M_2$

Effective inductance of first coil, $L'_{\text{eff}3} = L - M_2 + M_3$

After interchanging the effective inductance of three coils in series is,

$$L'_{\text{eff}1} + L'_{\text{eff}2} + L'_{\text{eff}3} = 3L - 2M_1 - 2M_2 + 2M_3$$

From the given value,

$$3L - 2M_1 - 2M_2 + 2M_3 = 0.5$$

or, $$M_1 + M_2 - M_3 = \frac{3 \times 0.2 - 0.5}{2} = 0.05 \qquad \text{(iii)}$$

Solving equations (*i*), (*ii*), and (*iii*), we get,

$$M_1 = \frac{\begin{vmatrix} 0.2 & 1 & 1 \\ 0.2 & 1 & 1 \\ 0.05 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{0.2 \times (-2) + 1 \times (0.25) + 1 \times (0.15)}{2 \times (-2) + 1 \times 2 + 1 \times 0} = \frac{0}{-2} = 0$$

$$M_2 = \frac{\begin{vmatrix} 2 & 0.2 & 1 \\ 1 & 0.2 & 1 \\ 1 & 0.05 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{0.2 \times (-0.25) + 1 \times (0.15) + 0.2 \times 0}{2 \times (-2) + 1 \times 2 + 1 \times 0} = \frac{-0.25}{-2} = 0.125$$

$$M_3 = \frac{\begin{vmatrix} 2 & 1 & 0.2 \\ 1 & 1 & 0.2 \\ 1 & 1 & 0.05 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{0.2 \times (-0.15) + 1 \times (0.15) + 0.2 \times 0}{2 \times (-2) + 1 \times 2 + 1 \times 0} = \frac{-0.15}{-2} = 0.075$$

Thus, the coefficients of coupling are as given below.

$$\left.\begin{aligned} k_1 &= \frac{M_1}{\sqrt{L \times L}} = \frac{M_1}{L} = \frac{0}{0.2} = 0 \\ k_2 &= \frac{M_2}{\sqrt{L \times L}} = \frac{M_2}{L} = \frac{0.125}{0.2} = 0.625 \\ k_3 &= \frac{M_3}{\sqrt{L \times L}} = \frac{M_3}{L} = \frac{0.075}{0.2} = 0.375 \end{aligned}\right\} \quad Ans$$

20. The combined inductance of two coils connected in series is 0.6H and 0.1H depending on the relative directions of the currents in the coils. If one of the coils when isolated has a self inductance of 0.2H, calculate mutual inductance and coefficient of coupling.

Sol: The combined inductance of two coils connected in series, depending on the relative directions of the currents in the coils, is given. It is known that the combined inductance in series aiding condition is more than that in series opposing condition. If the self-inductances are L_1 and L_2 and mutual inductance M, then,

$$L_1 + L_2 + 2M = 0.6 \tag{i}$$

and,
$$L_1 + L_2 - 2M = 0.1 \tag{ii}$$

Also, it is given that L_1 = 0.2H. Putting this value in (*i*) and (*ii*),

$$\therefore \quad L_2 + 2M = 0.4 \tag{iii}$$

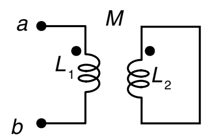
$$L_2 - 2M = -0.1 \tag{iv}$$

Solving (*iii*) and (*iv*), we get,

$$L_2 = 0.15H;\ M = 0.125H \quad Ans$$

$\therefore$ Coefficient of coupling is, $k = \dfrac{M}{\sqrt{L_1 L_2}} = \dfrac{0.125}{\sqrt{0.2 \times 0.15}} = 0.721 \quad Ans$

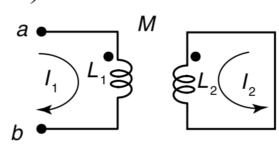
21. Show that the equivalent inductance of the circuit shown in the figure as seen from the terminals *a* and *b* is $L_{eq} = \left(L_1 - \dfrac{M^2}{L_2}\right)$ irrespective of the polarity of the coils.



Sol: Two cases may appear depending upon the polarity of the coils:

Case (1): Both currents entering the dotted terminals

In this case, by KVL for the two meshes, we get,



$$L_1 \frac{dI_1}{dt} + M\frac{dI_2}{dt} = V \tag{i}$$

$$L_2 \frac{dI_2}{dt} + M\frac{dI_1}{dt} = 0 \Rightarrow \frac{dI_2}{dt} = -\frac{M}{L_2}\frac{dI_1}{dt} \tag{ii}$$

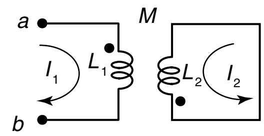
Substituting the value of $\dfrac{dI_2}{dt}$ in (*i*), we get,

$$L_1 \frac{dI_1}{dt} + M\left(-\frac{M}{L_2}\frac{dI_1}{dt}\right) = V$$

$$\left[L_1 - \frac{M^2}{L_2}\right]\frac{dI_1}{dt} = V \tag{iii}$$

Thus the effective inductance with respect to the terminals a and b is,

$$L_{\text{eff}} = \left[L_1 - \frac{M^2}{L_2}\right]$$



Case (2): One current entering the dot and other current leaving the dot

In this case, by KVL for the two meshes, we get,

$$L_1 \frac{dI_1}{dt} - M\frac{dI_2}{dt} = V \tag{i}$$

$$L_2 \frac{dI_2}{dt} - M\frac{dI_1}{dt} = 0 \Rightarrow \frac{dI_2}{dt} = \frac{M}{L_2}\frac{dI_1}{dt} \tag{ii}$$

Substituting the value of $\frac{dI_2}{dt}$ in (iv), we get,

$$L_1 \frac{dI_1}{dt} - M\left(\frac{M}{L_2}\frac{dI_1}{dt}\right) = V$$

$$\left[L_1 - \frac{M^2}{L_2}\right]\frac{dI_1}{dt} = V \tag{iii}$$

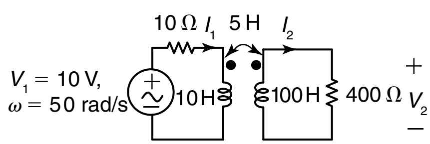
Thus the effective inductance with respect to the terminals a and b is,

$$L_{\text{eff}} = \left[L_1 - \frac{M^2}{L_2}\right]$$

So, for the given circuit, the effective inductance with respect to the terminals a and b is,

$L_{\text{eff}} = \left[L_1 - \frac{M^2}{L_2}\right]$ irrespective of the polarity of the coils.

22. For the coupled circuit, find the ratio of output voltage to the source voltage.



Sol: By KVL for the meshes, we get,

$$(10 + j50 \times 10)I_1 - j50 \times 5I_2 = 10 \Rightarrow (10 + j500)I_1 - j250I_2 = 10 \tag{i}$$

and,
$$-j50 \times 5I_1 + (400 + j50 \times 100)I_2 = 0 \Rightarrow -j250I_1 + (400 + j5000)I_2 = 0 \tag{ii}$$

Solving for I_2 from (i) and (ii), we get,

$$I_2 = \frac{\begin{vmatrix} (10+j500) & 10 \\ -j250 & 0 \end{vmatrix}}{\begin{vmatrix} (10+j500) & -j250 \\ -j250 & (400+j5000) \end{vmatrix}} = \frac{j2500}{-2433500 + j250000} = \frac{2500\angle 90°}{2.446\times 10^6 \angle 174.13°}$$

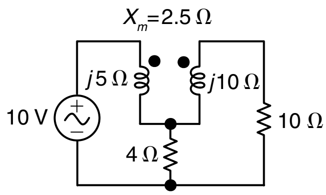
$$= 1.022\times 10^{-3}\angle -84.13°\,(A)$$

$$\therefore \quad V_2 = 400 \times I_2 = 0.409\angle -84.13°\,(V)$$

$\therefore$ Ratio of the output voltage to the source voltage is,

$$\frac{V_2}{V_1} = \frac{0.409\angle -84.13°}{10\angle 0°} = 40.9\times 10^{-3}\angle -84.13° \quad Ans$$

23. The figure shows a network with mutual coupling. (a) Find the current in the 10 ohm resistance. Assume that inductors have negligible resistance. (b) If the direction of winding of one of the coil is reversed, find the current in the 10 ohm resistance.

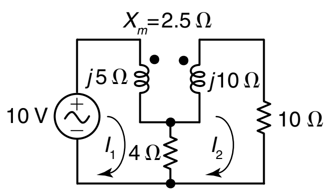


Sol:

(a) We consider the two loop currents as I_1 and I_2 as shown in figure below.
Applying KVL for the two meshes, we get,

$$(4+j5)I_1 - (4+j2.5)I_2 = 10 \quad \text{(i)}$$

$$-(4+j2.5)I_1 + (14+j10)I_2 = 0 \quad \text{(ii)}$$



Solving (*i*) and (*ii*), we get the current through the 10 ohm resistance as,

$$I_2 = \frac{\begin{vmatrix} (4+j5) & 10 \\ -(4+j2.5) & 0 \end{vmatrix}}{\begin{vmatrix} (4+j5) & -(4+j2.5) \\ -(4+j2.5) & (4+j5) \end{vmatrix}} = \frac{10(4+j2.5)}{(4+j5)(4+j5) - (4+j2.5)^2} = 0.523\angle -60.4°\,(A) \quad Ans$$

(b) If the direction of winding of one of the coil is reversed, the sign of mutual inductance will be positive. Here, the KVL equations will become,

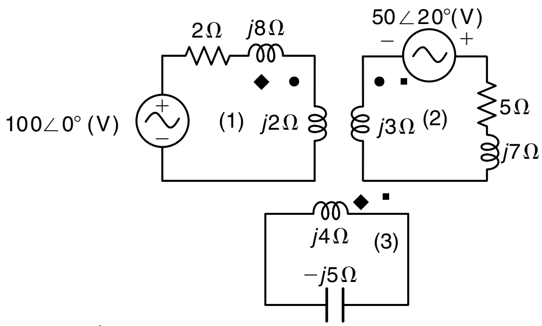
$$(4 + j5)I_1 - (4 - j2.5)I_2 = 10 \tag{iii}$$

$$-(4 - j2.5)I_1 + (14 + j10)I_2 = 0 \tag{iv}$$

Solving (*iii*) and (*iv*), we get the current through the 10 ohm resistance as,

$$I_2 = \frac{\begin{vmatrix} (4+j5) & 10 \\ -(4-j2.5) & 0 \end{vmatrix}}{\begin{vmatrix} (4+j5) & -(4-j2.5) \\ -(4-j2.5) & (4+j5) \end{vmatrix}} = \frac{10(4+j2.5)}{(4+j5)(4+j5)-(4-j2.5)^2} = 0.362\angle -123.65°(A) \quad \textit{Ans}$$

24. Find the current flowing through the capacitor in the network shown in the figure. Take k = 1. Also find reactive power in loop 3.



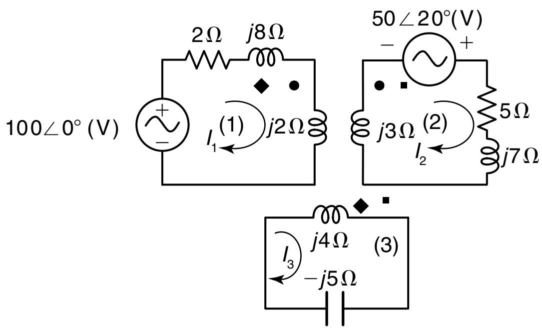
Sol: The mutual reactances are written as,

$$\omega M_{12} = 1 \times \sqrt{2 \times 3} = 2.45\ \Omega$$

$$\omega M_{13} = 1 \times \sqrt{2 \times 4} = 2.83\ \Omega$$

$$\omega M_{23} = 1 \times \sqrt{3 \times 4} = 3.46\ \Omega$$

For the three loops, by applying KVL we get,



$$(2+j10)I_1+(-j2.45)I_2+(-j2.83)I_3=100\angle 0° \quad \text{(i)}$$

$$-j2.45I_1+(5+j10)I_2+j3.46I_3=50\angle 20° \quad \text{(ii)}$$

$$-j2.83I_1+j3.46I_2+(-j1)I_3=0 \quad \text{(iii)}$$

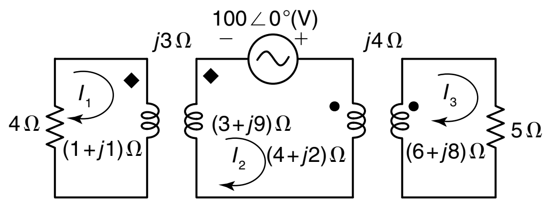
Solving (*i*), (*ii*) and (*iii*), for I_3 we get,

$$I_3=\frac{\begin{vmatrix}(2+j10) & -j2.45 & 100\angle 0° \\ -j2.45 & (5+j10) & 50\angle 20° \\ -j2.83 & j3.46 & 0\end{vmatrix}}{\begin{vmatrix}(2+j10) & -j2.45 & -j2.83 \\ -j2.45 & (5+j10) & j3.46 \\ -j2.83 & j3.46 & -j1\end{vmatrix}}=6.13\angle 49.44°(A) \quad Ans$$

Reactive power in loop 3 is,

$$Q=I_3^2\times(4-5)=6.13^2\times(-1)=-37.58 \text{ VAR} = 37.58 \text{ VAR (capacitive)}$$

25. Find the current I_1 in the network shown in the figure.



Sol:

Considering the dots as marked, we write the KVL equations as,

$$(5+j1)I_1-j3I_2=0 \quad \text{(i)}$$

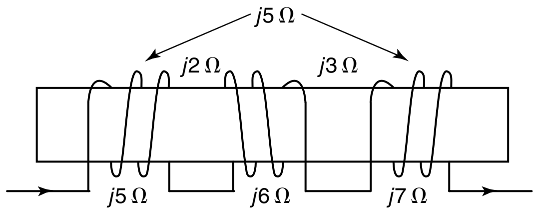
$$-3I_1+(7+j11)I_2-j4I_3=100\angle 0° \quad \text{(ii)}$$

$$-j4I_2+(11+j8)I_3=0 \quad \text{(iii)}$$

Solving (*i*), (*ii*) and (*iii*) for I_1,

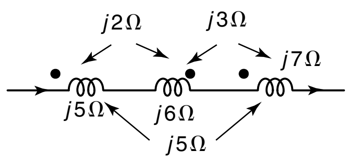
$$I_1=\frac{\begin{vmatrix}0 & -j3 & 0 \\ 100\angle 0^0 & (7+j11) & -j4 \\ 0 & -j4 & (11+j8)\end{vmatrix}}{\begin{vmatrix}(5+j1) & -j3 & 0 \\ -j3 & (7+j11) & -j4 \\ 0 & -j4 & (11+j8)\end{vmatrix}}=\frac{-2400+j3300}{-133+j946}=4.27\angle 28°(A \quad Ans)$$

26. Draw the dotted equivalent circuit for the circuit shown in the figure and find the equivalent inductive reactance.



Sol:

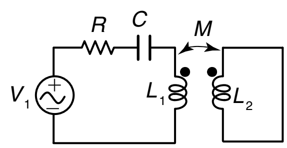
The dotted equivalent circuit is shown below.



∴ Equivalent inductive reactance,

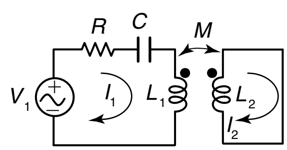
$$X_L = (j5 + j6 + j7) - j2 - j3 + j5 - j2 - j3 + j5 = j18\,\Omega \quad Ans$$

27. Determine the resonance frequency and the *Q*-factor of the circuit shown in the figure.



Data: *R* = 10 Ω, *C* = 3μF, L_1 = 40 mH, L_2 = 10 mH and *M* = 10 mH.

Sol: Applying KVL for the two loops,



$$I_1\left(R + j\omega L_1 + \frac{1}{j\omega C}\right) - j\omega M I_2 = V_1 \tag{i}$$

$$-j\omega M I_1 + j\omega L_2 I_2 = 0 \tag{ii}$$

From (*ii*), we get, $I_2 = I_1\left(\frac{M}{L_2}\right)$. Substituting this value in (*i*), we get,

$$I_1\left(R + j\omega L_1 + \frac{1}{j\omega C}\right) - j\omega M\left(\frac{M}{L_2}\right)I_1 = V_1$$

$$\Rightarrow \qquad \frac{V_1}{I_1} = R + j\omega L_1 + \frac{1}{j\omega C} - j\omega\frac{M^2}{L_2} = R + j\left(\omega L_1 - \frac{1}{\omega C} - \omega\frac{M^2}{L_2}\right)$$

For resonance to occur, the input impedance must be resistive, i.e.

$$\left(\omega_0 L_1 - \frac{1}{\omega_0 C} - \omega_0\frac{M^2}{L_2}\right) = 0$$

$$\Rightarrow \qquad \omega_0\left(L_1 - \frac{M^2}{L_2}\right) = \frac{1}{\omega_0 C}$$

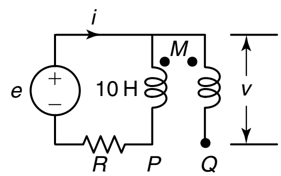
$$\Rightarrow \qquad \omega_0 = \sqrt{\frac{1}{C\left(L_1 - \dfrac{M^2}{L_2}\right)}} = \sqrt{\frac{L_2}{C}\frac{1}{\left(L_1L_2 - M^2\right)}} \quad Ans$$

Substituting the values of the components, the resonance frequency is,

$$\omega_0 = \sqrt{\frac{L_2}{C}\frac{1}{\left(L_1L_2 - M^2\right)}} = \sqrt{\frac{10\times10^{-3}}{3\times10^{-6}}\frac{1}{\left[40\times10^{-3}\times10\times10^{-3} - \left(10\times10^{-3}\right)^2\right]}} = 3333.33 \text{ rad/s}$$

$$\text{or,} \qquad f_0 = \frac{\omega_0}{2\pi} = \frac{3333.33}{2\pi} = 530.52\text{ Hz} \quad Ans$$

28. Given the following two sets of values of different quantities in the circuit of the figure. At two different instants of time. Find the values of v_{pq} at these instants.



$$t = t_1,\ e = 20\ V,\ i = i_1 = 1\ A,\ v = v_1 = 20\ V$$
$$t = t_2,\ e = 10\ V,\ i = i_2 = 0.3\ A,\ v = v_2 = 14\ V$$

Sol: At $t = t_1$, the KVL equations are,

$$20 = 10\frac{di_1}{dt} + R\times1 \text{ and } 20 = M\frac{di_1}{dt}$$

$$\Rightarrow \qquad \frac{20}{M} = \frac{20-R}{10} \qquad \text{(i)}$$

At $t = t_2$, the KVL equations are,

$$10 = 10\frac{di_2}{dt} + R \times 0.3 \text{ and } 14 = M\frac{di_2}{dt}$$

$$\Rightarrow \qquad \frac{14}{M} = \frac{10 - 0.3R}{10} \qquad \text{(ii)}$$

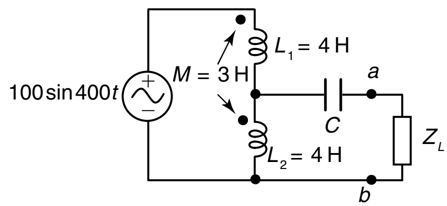
Solving (*i*) and (*ii*), we get, $R = 10\,\Omega,\ M = 20\,H$

$$\therefore \qquad \frac{di_1}{dt} = \frac{20}{M} = 1;\ \frac{di_2}{dt} = \frac{14}{M} = 0.7$$

At $t = t_1$, $v_{pq} = -10\frac{di_1}{dt} + 20 = 10\text{ V}$ *Ans*

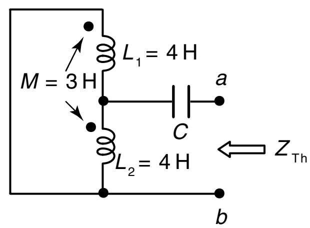
At $t = t_2$, $v_{pq} = -10\frac{di_2}{dt} + 14 = 7\text{ V}$ *Ans*

29. Find the value of C required in the circuit shown in the figure if the voltage across Z_L is to be independent of the value of Z_L.



Sol: We first find the Thevenin's equivalent circuit across the terminals *a* and Thevenin equivalent voltage with terminals *a-b* open-circuited is obtained as,

$$V_{Th} = V \times \frac{1}{2} = 50\sin 400\,t$$



($\because$ two inductors of same value are connected, the voltage division rule is applied.) To find the Thevenin's equivalent impedance, we have the circuit as shown in figure below.

$$Z_{Th} = j\omega\left(\frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}\right) - \frac{j}{\omega C}$$

$$= j\omega\left(\frac{4 \times 4 - 3^2}{4 + 4 - 2 \times 3}\right) - \frac{j}{\omega C}$$

$$= \left(j400 \times 3.5 - \frac{j}{400 \times C}\right)$$

$$= \left(j1400 - \frac{j}{400 \times C}\right)$$

$\therefore$ Current through the load impedance,

$$I_L = \frac{V_{Th}}{Z_{Th} + Z_L} = \frac{50\sin 400t}{\left(j1400 - \dfrac{j}{400 \times C}\right) + Z_L}$$

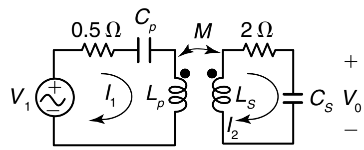
$\therefore$ Voltage across the load impedance,

$$V_L = I_L \times Z_L = \frac{50\sin 400t \times Z_L}{\left(j1400 - \dfrac{j}{400 \times C}\right) + Z_L}$$

This current will be independent of Z_L, if

$$j1400 - \frac{j}{400 \times C} = 0 \Rightarrow C = \frac{1}{1400 \times 400} = 1.7857\ \mu\text{F} \quad Ans$$

30. For the double tuned circuit shown in figure, both the primary and the secondary are tuned to the same frequency of $\frac{10^5}{2\pi}$ Hz. By varying the coupling coefficient *k*, the maximum output voltage across the coupling capacitor is 50 V. Determine the supply voltage. Given: L_s = 10 μH.



Sol: Here, $\omega = 10^5$ rad/s

From the condition for maximum output voltage, we get,

$$\omega M = \sqrt{R_p R_s} = \sqrt{0.5 \times 2} = 1\ \Omega$$

Also, for the resonant frequency, $\dfrac{1}{\omega C_s} = \omega L_s = 10^5 \times 10 \times 10^{-6} = 1\ \Omega$

By KVL,

$$0.5I_1 - j1I_2 = V_1$$

$$-j1I_1 + 2I_2 = 0$$

Solving for I_2,

$$I_2 = \frac{\begin{vmatrix} 0.5 & V_1 \\ -j1 & 0 \end{vmatrix}}{\begin{vmatrix} 0.5 & -j1 \\ -j1 & 2 \end{vmatrix}} = \frac{jV_1}{1+1} = j0.5\,V$$

or, $\quad |I_2| = 0.5\ V_1$

Output voltage is, $|V_0| = \dfrac{I_2}{\omega C_s} \Rightarrow 50 = \dfrac{0.5V_1}{1} \Rightarrow V_1 = 100\ \text{Volt} \quad Ans$

SUMMARY

1. When the magnetic flux produced by an inductor links another inductor, these inductors are said to be coupled.
2. When two coils are placed very close to each other, the magnetic flux caused by current in one coil links with the other coil and induces some voltage in the second coil. This phenomenon is known as *mutual inductance.*
3. For two coils of inductances L_1 and L_2, the mutual inductance is given as, $M = k\sqrt{L_1 L_2}$, where, k is the *coefficient of coupling*.
4. To denote the correct voltage polarity of the mutual inductance of two colis, *dot convention* is used. As per dot convention, if the directions of currents through the two coils are in the same sense with respect to the dotted terminals, the sign of the mutual;ly induced emf is the same as the self-induecd emf.
5. Energy stored in two mutually coupled coils is, $W = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 \pm M I_1 I_2$.
6. The equivalent inductances of two coupled coils of self-inductances L_1 and L_2 when connected in series is,

$$L_{eq} = (L_1 + L_2 + 2M) \quad \textit{for series aiding connection}$$
$$= (L_1 + L_2 - 2M) \quad \textit{for series opposing connection}$$

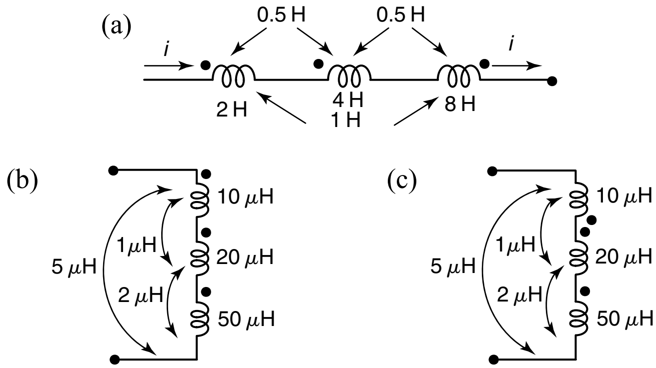
7. The equivalent inductances of two coupled coils of self-inductances L_1 and L_2 when connected in parallel is,

$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M} \quad \textit{for parallel aiding connection}$$
$$= \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M} \quad \textit{for parallel opposing connection}$$

8. A *transformer* is a four-terminal device comprising of two (or more) magnetically coupled coils, called *primary* and s*econdary*.
9. A transformer is said to be *linear* if the coils are wound on a magnetically linear material (such as, air, plastic, Bakelite, wood, etc) for which the magnetic permeability is a constant. A linear transformer can be represented by an equivalent T or p network.
10. A transformer is said to be *ideal*; if its primary and secondary coils are lossless (i.e. $R_1 = R_2 = 0$), have very large reactances compared to any connected impedance (i.e. L_1, L_2, $M \rightarrow \infty$) and their coupling is perfect, (i.e. $k = 1$).
11. When a capacitor is introduced in the primary and/or in the secondary circuit of a transformer, the circuit becomes selective and the transformer is termed as a *tuned coupled circuit* or a *tuned transformer.* There are two types of tuned circuits, single-tuned circuit and double-tuned circuit.

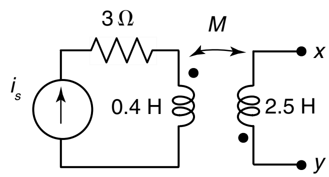
EXERCISES

1. Find the effective value of the inductance for the following connections:

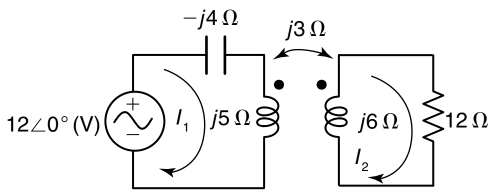


[12 H; 96 μH; 72 μH]

2. If $i_s = 2 \cos 10t$ (A), find the total energy stored in the passive network at $t = 0$ for $k = 0.6$ and terminals x and y is short-circuited. [0.512 J]

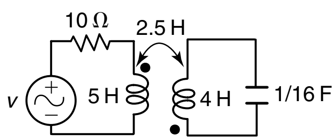


3. Calculate the phasor currents I_1 and I_2.



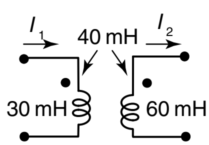
[13.01∠–49.39° (A); 2.91 ∠14.04° (A)]

4. Determine the coupling coefficient. Calculate the energy stored in the coupled circuits at time $t = 1$s if, $v = 60 \cos (4t + 30°)$ (V).

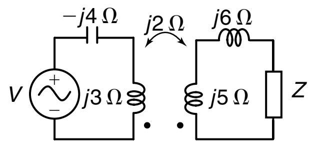


[0.56; 20.73 J]

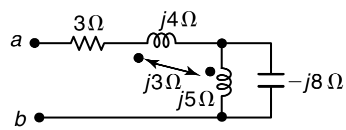
5. Find the T and p equivalent of the linear transformer shown in figure below.



6. Find the conductively equivalent circuit for the network shown in figure.

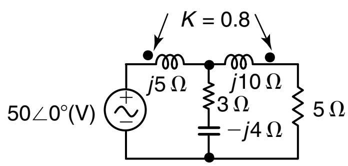


7. A coil of 800 μH is magnetically coupled to another coil of 200 μH. The coefficient of coupling between two coils is 0.05. Calculate the effective inductance if two coils are connected in:
   (i) series aiding, (ii) series opposing, (iii) parallel aiding, and (iv) parallel opposing.
   [1040 μH, 960 μH, 166.25 μH, 153.46 μH]
8. Two coils each with a series connection of L = 300 μH and C = 1000 pF are magnetically coupled with M = 60 μH. An emf of 10Volt at $\frac{1}{2}$ MHz is injected into the circuit. Determine:
   (i) current in other circuit if its terminals are shorted.
   (ii) the coefficient of coupling. [49.18 ∠90° mA; 0.2]
9. For the coupled circuit, find the input impedance at terminals a and b. [(3 + j36.33) Ω]

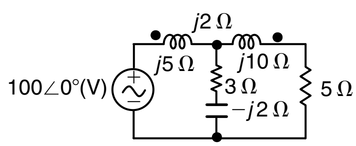


10. In the coupled circuit, find the voltage across the 5 Ω resistor.

(a)

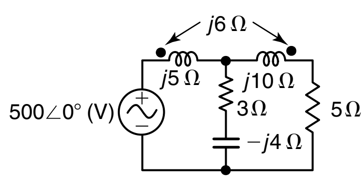


(b)



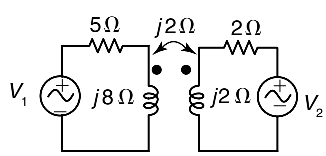
[43.05∠–24.94° (V); 30.7∠–100.6° (V)]

11. For the given magnetically coupled circuit, obtain the conductively coupled circuit.



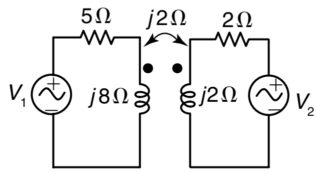
[–j1 Ω; (5 + j4)Ω; (3 + j2)Ω]

12. For the coupled circuit, find the ratio which will result zero current I_1.



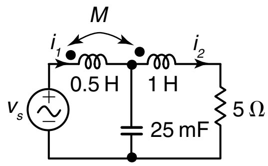
[(1 – j1)]

13. In the coupled circuit shown in the figure, find V_2 for which $I_1 = 0$. What voltage appears at 8 Ω inductive reactance under this condition?



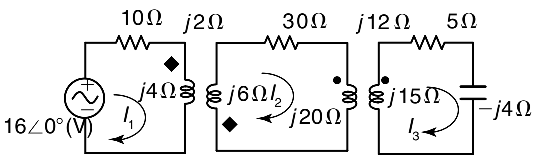
[141.42∠–45° (V)]

14. If $M = 0.2$H and $v_s = 12 \cos 10t$ V in the circuit of figure, find i_1 and i_2.



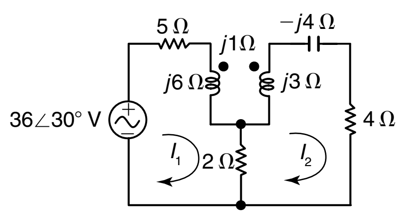
$$\begin{bmatrix} 5.068\cos(10t + 52.54°) \\ 2.719\cos(10t - 100.89°) \end{bmatrix}$$

15. Find the currents I_1, I_2 and I_3 in the circuit shown in figure.



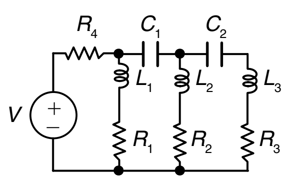
[1.47∠–21.4° (A); 0.077∠–134.85° (A); 0.077∠–110.41° (A)]

16. Calculate the power absorbed by the 4Ω resistor in the circuit shown in figure.



[3.67 Watt]

17. Write the loop equations for the network shown in figure.



SHORT-ANSWER TYPE QUESTIONS

1. What is meant by self and mutual inductances? Explain. Also give their units.

   or,

   Explain what is meant by self inductance and mutual inductance. Define the units in which each is measured.
2. Define "self inductance" and "mutual inductance". Derive an expression for the mutual inductance between two magnetically coupled coils having self inductances L_1 and L_2, respectively.
3. Obtain an expression for the mutual inductance of two co-axial solenoids of wires closely wound one upon the other.
4. Explain the concept of mutual inductance. Define coefficient of coupling and derive the expression between self inductances of two coils, mutual inductance between them and the coefficient of coupling.

   or,

   Prove that the coefficient of mutual inductance M between two coils of self inductances L_1 and L_2 is given by $M/\sqrt{L_1 L_2}$.
5. Explain the dot convention used in magnetically coupled circuits (mutual inductances) with the help of suitable examples.
6. Two coils are magnetically coupled to each other. Show that the maximum possible value of the mutual inductance is the geometric mean of the self-inductances of the two coils. What is the energy stored in the systems?
7. Two coils of self inductances L_1 and L_2 are placed side by side so that the mutual inductance between them is M. If they are connected in series addition derive the expression for the net inductance of the coils.
8. Two coils of self inductances L_1 and L_2 are mutually coupled. Derive the expression for the net inductance of the coils if they are connected in:
   (a) series aiding;
   (b) series opposing;
   (c) parallel aiding;
   (d) parallel opposing.
9. What is a linear transformer? Derive then expressions for input impedance and the reflected impedance from the secondary to the primary circuit of a linear transformer.
10. What are the properties of an ideal transformer? Obtain the input impedance of an ideal transformer. How can the turns ratio of a transformer be adjusted for maximum power transfer to the load? What is '*impedance matching*'?
11. Determine the voltage and current transformation ratio of an ideal transformer.
12. What is a tuned circuit? For a single tuned circuit, determine the maximum value of the output voltage and the effective bandwidth and Q factor.
13. For the mutually coupled circuit shown in Figure., show that the secondary current and voltage E_2 will have its largest value if the following relationship holds true:

$$\frac{1}{\omega C_2} = \omega\left(\frac{R_2}{R_1} L_1 + L_2\right) \approx \omega L_2$$

Hence prove that the maximum value of E_2 is obtained when $\omega M = \sqrt{R_1 R_2}$.

14. An inductively coupled doubly tuned circuit has both circuits tuned to the same frequency with the same Q. Define and determine the value of the critical coupling and obtain the bandwidth for this critical coupling.

CHAPTER

4

Network Topology (Graph Theory)

4.1 INTRODUCTION

The word topology refers to the science of place. In mathematics, topology is a branch of geometry in which figures are considered perfectly elastic.

Network Topology refers to the properties that relate to the geometry of the network (circuit). These properties remain unchanged even if the circuit is bent into any other shape provided that no parts are cut and no new connections are made.

In electrical engineering, solution of network analysis problems involves finding the current through and voltage across different circuit elements. Different laws (like, Ohm's law, Kirchhoff's laws, etc.) have been postulated for simplifying the solution method. However, it is sometimes found that the algebraic equations written by different laws are not independent. On the other hand, the equations formed by network topology method are all independent.

Network topology method has many other merits and can be listed as follows.

1. The graph theory or network topology deals with those properties of networks which do not change with the change in the shape of the networks.
2. All the equations (KCL and KVL) formed by graph theory concept are independent equations.
3. The graph theory concept eases the solution method for solving networks with a large number of nodes and branches.

In this chapter, we will discuss the fundamentals of graph theory (network topology) and their applications for solving network analysis problems.

4.2 GRAPH OF A NETWORK

A linear graph (or simply a graph) is defined as a collection of points called nodes, and line segment called branches, the nodes being joined together by the branches.

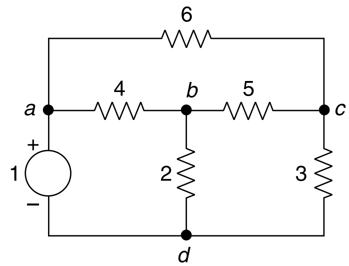


Figure 4.1(a) *Circuit*

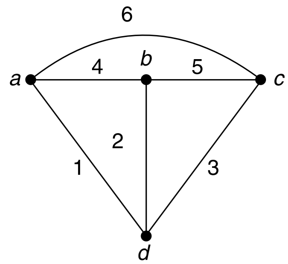


Figure 4.1(b) *Graph of the Circuit*

While drawing graph of a given network, the following rules are to be noted.

(i) All passive elements between the nodes are represented by lines.

(ii) The independent current sources and voltage sources are represented by their internal impedances (i.e., current sources by open circuit and voltage sources by short circuit) if they are accompanied by passive element, viz., a shunt admittance in a current source and a series impedance in a voltage source.

(iii) If the sources are not accompanied by passive elements, an arbitrary impedance (say resistance R) or admittance is assumed to accompany the sources and finally, we find the results by letting the impedance $R \rightarrow 0$ or $R \rightarrow \propto$ as the case may be for the current or voltage sources.

4.3 TERMINOLOGY

In order to discuss the more involved methods of circuit analysis, we must define a few basic terms necessary for a clear, concise description of important circuit features.

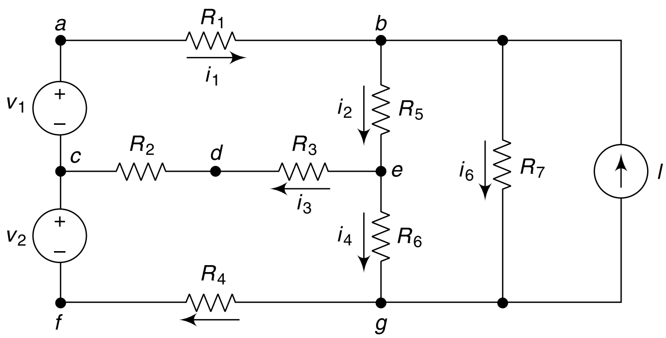


Figure 4.2 *Circuit illustrating terminologies*

(a) ***Node*** A node is a point in a circuit where two or more circuit elements join.

Example a, b, c, d, e, f and g

(b) ***Essential Node*** A node that joins three or more elements.

Example b, c, e and g

(c) ***Branch*** A branch is a path that connects two nodes.

Example v_1, R_1, R_2, R_3, v_2, R_4, R_5, R_6, R_7 and I

(d) ***Essential branch*** Those paths that connect essential nodes without passing through an essential node.

Example c–a–b, c–d–e, c–f–g, b–e, e–g, b–g (through R_7), and b–g (through I)

(e) ***Loop*** A loop is a complete path, i.e., it starts at a selected node, traces a set of connected basic circuit elements and returns to the original starting node without passing through any intermediate node more than once.

Example *abedca*, *abegfca*, *cdebgfc*, etc.

(f) ***Mesh*** A mesh is a special type of loop, i.e., it does not contain any other loops within it.

Example *abedca*, *cdegfc*, *gebg* (through R_7) and *gebg* (through I)

(g) ***Oriented Graph*** A graph whose branches are oriented is called a directed or oriented graph.

(h) ***Rank of Graph*** The rank of a graph is $(n-1)$ where n is the number of nodes or vertices of the graph.

(i) ***Planar and Non-planar Graph*** A graph is planar if it can be drawn in a plane such that no two branches intersect at a point which is not a node.

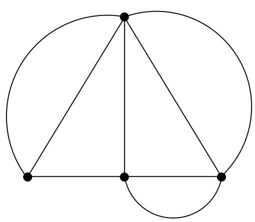


Figure 4.3(a) *Planar graph*

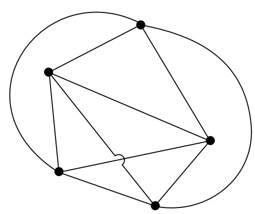


Figure 4.3(b) *Non-planar graph*

(j) ***Subgraph*** A subgraph is a subset of the branches and nodes of a graph. The subgraph is said to be proper if it consists of strictly less than all the branches and nodes of the graph.

(k) ***Path*** A path is a particular sub graph where only two branches are incident at every node except the internal nodes (i.e., starting and finishing nodes). At the internal nodes, only one branch is incident.

In the example in the Fig. 4.3 (c), branches 2, 3, and 4, together with all the four nodes, constitute a path. A graph is connected if there exists a path between any pair of vertices. Otherwise, the graph is disconnected.

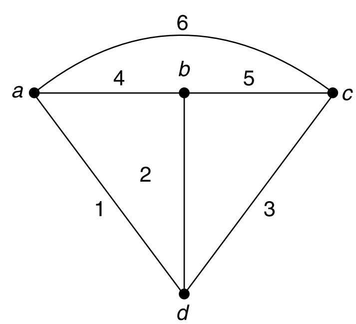


Figure 4.3(c)

4.4 CONCEPT OF TREE

For a given connected graph of a network, a connected subgraph is known as a tree of the graph if the subgraph has all the nodes of the graph without containing any loop.

Twigs The branches of tree are called twigs or tree-branches. The number of branches or twigs, in any selected tree is always one less than the number of nodes, i.e.,

Twigs = $(n - 1)$, where n is the number of nodes of the graph.

For this case, twigs = $(4 - 1) = 3$ twigs. These are shown by solid lines in Fig. 4.4 (b).

Links and Co-tree If a graph for a network is known and a particular tree is specified, ***the remaining branches*** are referred to as the ***links***. The ***collection of links*** is called a ***co-tree***. So, co-tree is the complement of a tree. These are shown by dotted lines in Fig. 4.4(b).

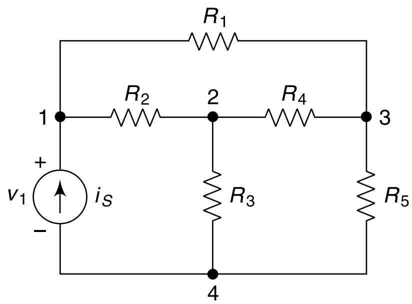


Figure 4.4(a) *Circuit*

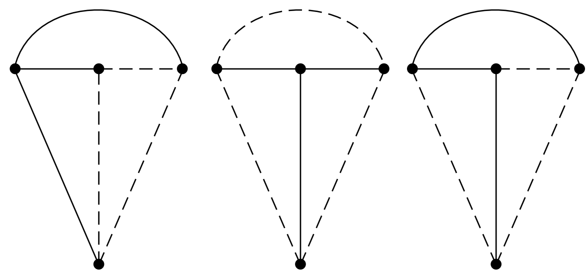


Figure 4.4(b) *Trees and links of circuit of Fig. 4.4(a)*

The branches of a co-tree may or may not be connected, whereas the branches of a tree are always connected.

To summarize,

Number of nodes in a graph = n

Number of independent voltages = $n - 1$

Number of tree-branches = $n - 1$

Number of links = L = (Total number of branches) – (Number of tree-branches)

$= b - (n - 1) = b - n + 1$

Total number of branches = $b = L + (n - 1)$

Properties of a Tree

1. In a tree, there exists one and only one path between any pairs of nodes.
2. Every connected graph has at least one tree.
3. A tree contains all the nodes of the graph.
4. There is no closed path in a tree and hence, tree is circuitless.
5. The rank of a tree is $(n - 1)$.

4.5 INCIDENCE MATRIX [A_a]

The incidence matrix symbolically describes a network. It also facilitates the testing and identification of the independent variables. Incidence matrix is a matrix which represents a graph **uniquely**.

For a given graph with n nodes and b branches, the complete incidence matrix A_a is a rectangular matrix of order $n \times b$, whose elements have the following values.

Number of columns in $[A_a]$ = Number of branches = b

Number of rows in $[A_a]$ = Number of nodes = n

A_{ij} = 1, if branch j is associated with node i and oriented away from node i.
= –1, if branch j is associated with node i and oriented towards node i.
= 0, if branch j is not associated with node i.

This matrix tells us which branches are incident at which nodes and what are the orientations relative to the nodes.

Example

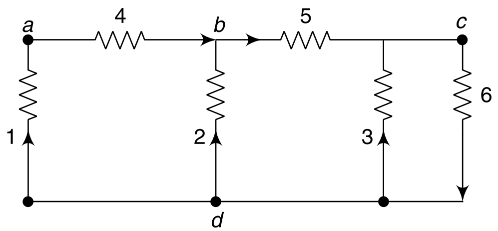


Figure 4.5(a) *Network*

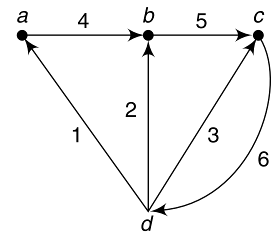


Figure 4.5(b) *Graph of network*

Incidence matrix A_a

		Branches						
		1	2	3	4	5	6	
Nodes	a	–1	0	0	+1	0	0	Reduced Incidence Matrix A
	b	0	–1	0	–1	+1	0	
	c	0	0	–1	0	–1	+1	
Reference Node	d	+1	+1	+1	0	0	–1	

4.5.1 Incidence Matrix and KCL

For the graph, shown in Fig. 4.6, Kirchhoff's current law for the branch currents (i_1, i_2, …, i_6) gives the equations,

$$i_1 + i_2 + i_6 = 0$$
$$-i_1 + i_3 - i_5 = 0$$
$$-i_2 - i_3 + i_4 = 0$$
$$-i_4 + i_5 - i_6 = 0$$

In matrix form, these equations can be represented as,

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_{b1} \\ i_{b2} \\ i_{b3} \\ i_{b4} \\ i_{b5} \\ i_{b6} \end{bmatrix} = 0$$

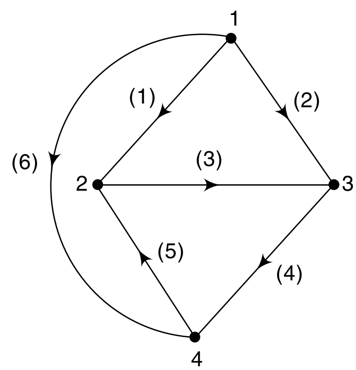


Figure 4.6 *Graph illustrating incidence matrix and KLC*

or $\boxed{A_a I_b = 0}$

where, A_a is the complete incidence matrix of the graph.

Reduced Incidence Matrix [A] The matrix obtained from A_a by eliminating one of the rows is called Reduced Incidence Matrix. In other words, suppression of the datum node (reference node) from the incidence matrix results in reduced incidence matrix.

For the graph shown in Fig. 4.6, reduced incidence matrix is given as,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}$$

4.5.2 Incidence Matrix and KVL

For the graph shown in Fig. 4.6, the branch voltages (v_{b1}, v_{b2}, $\ldots v_{b6}$) can be represented in terms of the node voltages (v_{n1}, v_{n2}, v_{n3}, v_{n4}) as,

$$v_{b1} = (v_{n1} - v_{n2}),\quad v_{b2} = (v_{n1} - v_{n3}),\quad v_{b3} = (v_{n2} - v_{n3}),\quad v_{b4} = (v_{n3} - v_{n4}),$$
$$v_{b5} = (-v_{n1} + v_{n4}),\ v_{b6} = (v_{n1} - v_{n4}),$$

Thus, the Kirchhoff's voltage law in matrix form can be written as,

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} V_{n_1} \\ V_{n_2} \\ V_{n_3} \\ V_{n_4} \end{bmatrix} = \begin{bmatrix} v_{b1} \\ v_{b2} \\ v_{b3} \\ v_{b4} \\ v_{b5} \\ v_{b6} \end{bmatrix}$$

or $$\boxed{A_a^T V_n = V_b}$$

4.5.3 Properties of Complete Incidence Matrix

(i) The sum of the entries in any column is zero.
(ii) The determinant of the incidence matrix of a closed loop is zero.
(iii) The rank of incidence matrix of a connected graph is (n–1).

4.6 NUMBER OF POSSIBLE TREES OF A GRAPH

The number of possible trees of a graph, = det $\{[A] \times [A]^T\}$
where, A is the reduced incidence matrix obtained by eliminating any one row of the complete incidence matrix A_a, and $[A]^T$ is the transpose of the matrix $[A]$.

Example For the graph shown in Fig. 4.6, the complete incidence matrix is,

$$A_a = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix}$$

So, reduced incidence matrix is,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}$$

Thus, the number of possible trees of the graph of Fig. 4.6

$$= \det\left\{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right\} = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 16$$

4.7 TIE-SET MATRIX AND LOOP CURRENTS

Tie-Set A ***tie-set*** is a set of branches contained in a loop such that each loop contains one link or chord and the remainder are tree branches.

Consider the graph and the tree as shown. This selected tree will result in ***three fundamental loops*** as we connect each link, in turn to the tree.

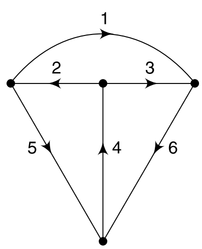


Figure 4.7(a) *Graph*

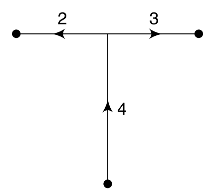


Figure 4.7(b) *Tree of the graph*

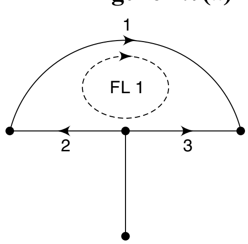


Figure 4.7(c) *Loop-1*

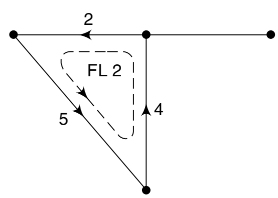


Figure 4.7(d) *Loop-2*

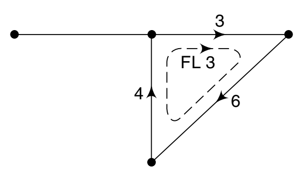


Figure 4.7(e) *Loop-3*

Fundamental Loop 1 (FL1): Connecting link 1 to the tree.
Fundamental Loop 2 (FL2): Connecting link 5 to the tree.
Fundamental Loop 3 (FL3): Connecting link 6 to the tree.
These sets of branches (1, 2, 3), (2, 4, 5) and (3, 4, 6) form three tie-sets.

4.7.1 Tie-Set Matrix or Loop Incidence Matrix or Circuit Matrix (B_a)

For a given graph having *n* nodes and *b* branches, tie-set matrix is a rectangular matrix with *b* columns and as many rows as there are loops. Its elements have the following values:

B_{ij} = 1, if branch *j* is in loop *i* and their orientations coincide (i.e., loop current and branch current flow in the same direction);
= –1, if branch *j* is in loop *i* and their orientations do not coincide;
= 0, if branch *j* is not in loop *i*.

Example For the graph shown in Fig. 4.8(a) and tree selected in Fig. 4.8(b), the tie-set matrix is written as follows. The entries in the Tie-set schedule are given as +1 or –1 if the branch current is in the same direction as the link current or not. If the branch current does not depend on the link current, then entry is zero.

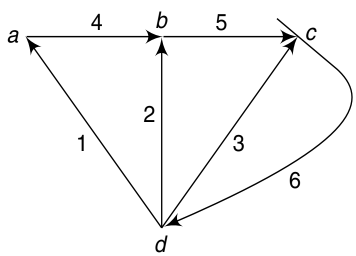


Figure 4.8(a) *Graph*

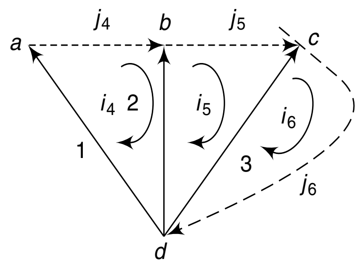


Figure 4.8(b) *Formation of loops*

Links (*j*)	Branch no. (*i*) 1	2	3	4	5	6
4	1	–1	0	1	0	0
5	0	1	–1	0	1	0
6	0	0	1	0	0	1

4.7.2 Tie-Set Matrix and KVL

For the graph shown in Fig. 4.7(a) and three loops shown in Fig. 4.7(c), (d) and (e), three fundamental mesh KVL equations can be written as follows.
For Fundamental Loop 1 (FL1): $v_{b1} - v_{b3} + v_{b2} = 0$
For Fundamental Loop 2 (FL2): $v_{b2} + v_{b4} + v_{b5} = 0$
For Fundamental Loop 3 (FL3): $v_{b3} + v_{b6} + v_{b4} = 0$

These equations in matrix form is written as,

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{b1} \\ v_{b2} \\ v_{b3} \\ v_{b4} \\ v_{b5} \\ v_{b6} \end{bmatrix} = 0$$

or $\boxed{B_a V_b = 0}$

4.7.3 Tie-Set Matrix and KCL

For the graph shown in Fig. 4.7(a) and three loops shown in Fig. 4.7(c), (d) and (e), the branch currents ($i_{b1}, i_{b2}, \ldots, i_{b6}$) can be represented in terms of the loop currents (I_{L1}, I_{L2}, I_{L3}) as,

$$i_{b1} = I_{L1},\quad i_{b2} = (I_{L1} + I_{L2}),\quad i_{b3} = (-I_{L1} + I_{L3}),\quad i_{b4} = (I_{L2} + I_{L3}),\quad i_{b5} = I_{L2},\ i_{b6} = I_{L3}$$

In matrix form, these equations can be written as,

$$\begin{bmatrix} i_{b1} \\ i_{b2} \\ i_{b3} \\ i_{b4} \\ i_{b5} \\ i_{b6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{L1} \\ I_{L2} \\ I_{L3} \end{bmatrix}$$

or $\boxed{I_b = B_a^T I_L}$

4.8 CUT-SET MATRIX AND NODE-PAIR POTENTIAL

Cut-Set A cut-set is a minimum set of elements that when cut, or removed, separates the graph into two groups of nodes. A cut-set is a **minimum set of branches** of a connected graph, such that the removal of these branches from the graph **reduces the rank of the graph by one.**

In other words, for a given connected graph (G), a set of branches (C) is defined as a cut-set if and only if:

(i) the removal of all the branches of C results in an unconnected graph.
(ii) the removal of all but one of the branches of C leaves the graph still connected.

Example Consider the graph shown in Fig. 4.9(a). The rank of the graph is 3.
The removal of branches 1 and 3 reduces the graph into two connected subgraphs as shown in Fig. 4.9(b).

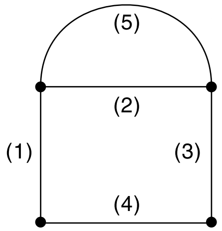


Figure 4.9(a) *Graph*

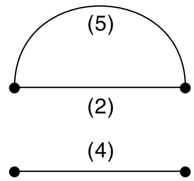


Figure 4.9(b) *Subgraphs with removal of 1 and 3*

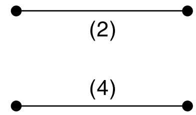


Figure 4.9(c) *Subgraphs with removal of 1, 3 and 5*

The rank of the graph of Fig. 4.9(a) = (4 – 1) = 3
The rank of the graph of Fig. 4.9(b) = Addition of the ranks of the subgraphs = (1 + 1) = 2
So, branches [1, 3] may be a cut-set.
Also, removal of the branches 1, 3 and 5 reduces the graph into two connected subgraphs as shown in Fig. 4.9(c) and the rank becomes 2. So, [1, 3, 5] may also be a cut-set.
As cut-set is the minimum set of branches and [1, 3] is a subset of [1, 3, 5], so [1, 3] is the cut-set, [1, 3, 5] is not a cut-set.

4.8.1 Fundamental Cut-Set

A fundamental cut-set (FCS) is a cut-set that cuts or contains **one and only one tree branch**. Therefore, for a given tree, the number of fundamental cut-sets will be equal to the number of twigs.

4.8.2 Procedure for Finding the Fundamental Cut-sets

1. First, select a tree of the given graph.
2. Focus on a tree branch (b_k).
3. Check whether removing this tree branch (b_k) from the tree disconnects the tree into two separate parts.
4. All the links which go from one part of this disconnected tree to the other, together with the tree branch (b_k) forms a fundamental cut-set.

Following this procedure, the fundamental cut-sets for the graph of Fig. 4.10 will be

f-cut-set – 1: [1, 2, 6];

f-cut-set – 2: [2, 3, 5, 6];

f-cut-set – 3: [4, 5, 6]

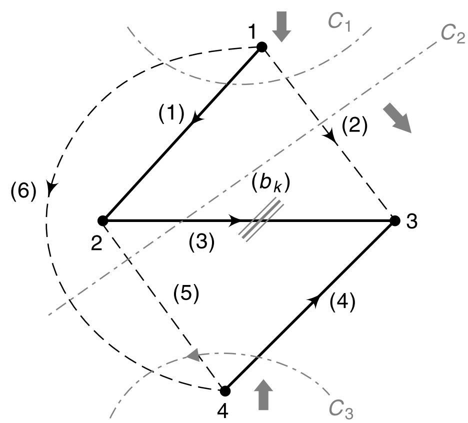


Figure 4.10 *Graph illustrating fundamental cut-set*

4.8.3 Properties of Cut-Set

1. A cut-set divides the set of nodes into two subsets.
2. Each fundamental cut-set contains one tree-branch, the remaining elements being links.
3. Each branch of the cut-set has one of its terminals incident at a node in one subset and its other terminal at a node in the other subset.

4. A cut-set is oriented by selecting an orientation from one of the two parts to the other. Generally, the direction of cut-set is chosen same as the direction of the tree branch.

4.8.4 Cut-Set Matrix (Q_C)

For a given graph, a cut-set matrix (Q_C) is defined as a rectangular matrix whose rows correspond to cut-sets and columns correspond to the branches of the graph. Its elements have the following values:

$$Q_{ij} = 1, \text{ if branch } j \text{ is in the cut-set } i \text{ and the orientations coincide.}$$
$$= -1, \text{ if branch } j \text{ is in the cut-set } i \text{ and the orientations do not coincide.}$$
$$= 0, \text{ if branch } j \text{ is not in the cut-set } i.$$

Example For the graph shown in Fig. 4.10, fundamental cut-sets have been identified as follows.

f-cut-set – 1: [1, 2, 6];
f-cut-set – 2: [2, 3, 5, 6];
f-cut-set – 3: [4, 5, 6]

So, the cut-set matrix is written as,

Branch no:

f-cut-sets	1	2	3	4	5	6
1	1	1	0	0	0	1
2	0	1	1	0	1	1
3	0	0	0	1	–1	–1

4.8.5 Cut-Set Matrix and KVL

By cut-set schedule, the branch voltages can be expressed in terms of the tree-branch voltages.

A cut-set consists of *one and only one* branch of the tree together with any links which must be cut to divide the network into two parts. A set of fundamental cut-sets includes those cut-sets which are obtained by applying cut-set division for each of the branches of the network tree.

Consider the following graph.

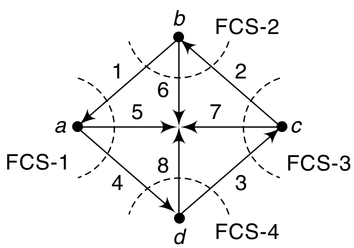


Figure 4.11(a) *Graph*

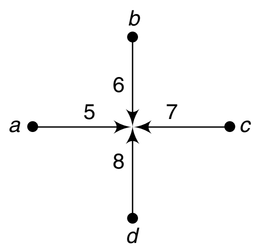


Figure 4.11(b) *Tree*

Applying cut-sets at nodes *a*, *b*, *c*, *d*, which are the fundamental cut-sets (FCS), we can write the cut-set schedule as follows.

		1	2	3	4	5	6	7	8
FCS-1 →	*a*	–1	0	0	1	1	0	0	0
FCS-2 →	*b*	1	–1	0	0	0	1	0	0
FCS-3 →	*c*	0	1	1	0	0	0	1	0
FCS-4 →	*d*	0	0	–1	–1	0	0	0	1

The tree-branch voltages are $[v_{t5}, v_{t6}, v_{t7}, v_{t8}]$ and the branch voltages are $[V_{b1}, V_{b2}, \ldots V_{b8}]$ and the relationship between tree-branch voltages and branch voltages are:

$$\begin{array}{ll} V_{b1} = -v_{t5} + v_{t6} & V_{b5} = v_{t5} \\ V_{b2} = -v_{t6} + v_{t7} & V_{b6} = v_{t6} \\ V_{b3} = v_{t7} - v_{t8} & V_{b7} = v_{t7} \\ V_{b4} = v_{t5} - v_{t8} & V_{b8} = v_{t8} \end{array}$$

The above equations can be related by using the cut-set schedule as:

$$\begin{bmatrix} V_{b1} \\ V_{b2} \\ V_{b3} \\ V_{b4} \\ V_{b5} \\ V_{b6} \\ V_{b7} \\ V_{b8} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{t5} \\ v_{t6} \\ v_{t7} \\ v_{t8} \end{bmatrix}$$

or $$\boxed{V_b = Q_c^T V_t}$$

4.8.6 Cut-Set Matrix and KCL

For the graph of Fig. 4.11, writing Kirchhoff's current laws for the nodes, the branch currents can be expressed as,

$$\begin{array}{ll} \text{Node } a: & -i_{b1} + i_{b4} + i_{b5} = 0 \\ \text{Node } b: & i_{b1} - i_{b2} + i_{b6} = 0 \\ \text{Node } c: & i_{b2} + i_{b3} + i_{b7} = 0 \\ \text{Node } d: & -i_{b3} - i_{b4} + i_{b8} = 0 \end{array}$$

In matrix form, they can be written as,

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_{b1} \\ i_{b2} \\ i_{b3} \\ i_{b4} \\ i_{b5} \\ i_{b6} \\ i_{b7} \\ i_{b8} \end{bmatrix} = 0$$

or $$\boxed{Q_c I_b = 0}$$

There is a cut-set matrix for a given tree. If a graph contains more than one tree, there will be as many numbers of cut-set matrices as the number of tree of the graph.

To summarize, KVL and KCL equations in three matrix forms are given below.

Matrix	*KCL*	*KVL*
Incidence Matrix (A_a)	$A_a \times I_b = 0$	$V_b = A_a^T \times V_n$
Tie-Set Matrix (B_a)	$I_b = B_a^T \times I_L$	$B_a \times V_b = 0$
Cut-Set Matrix (Q_C)	$Q_C \times I_b = 0$	$V_b = Q_C^T \times V_t$

4.9 FORMULATION OF NETWORK EQUILIBRIUM EQUATIONS

The network equilibrium equations are a set of equations that completely and uniquely determine the state of a network at any instant of time. These equations are written in terms of suitably chosen current variables or voltage variables.

These equations will be unique if the number of independent variables be equal to the number of independent equations.

Number of Independent Variables or Equations $= b - (n - 1)$; for loop method of analysis
$= (n - 1)$; for node method of analysis.

The equations for a network can be formed in either of the two methods as given below.

1. Through a set of voltage law equations in which the currents are the independent variables (Loop-Basis Method);
2. Through a set of current law equations in which the node-pair voltages are the independent variables (Node-Basis Method).

4.9.1 Formulation of Network Equations on Loop Basis

Steps

1. Draw the ***directed graph*** of the network selecting the direction of assumed current flow to coincide for current sources.
2. Select a tree of the graph.
3. Place all ***voltage sources in the tree*** and all ***current sources in the co-tree***.
4. Place all control-voltage branches for voltage-controlled dependent sources in the tree and all control-current branches for current-controlled dependent sources in the co-tree, if possible.
5. Add one link to the tree, creating a fundamental loop, and write a KVL equation for this fundamental loop (FL). Repeat for each additional link until L ($= b - n + 1$) mesh equations are obtained in the form $B_a \times V_b = 0$.
6. The current sources in the cotree, if present, will provide the constraint equations.
7. The KCL equations are obtained by representing the branch currents in terms of loop currents in the form $I_b = B_a^T \times I_L$.
8. For each branch, the relationship between the voltage and current is obtained from Ohm's law ($V = RI$).
9. Finally, the equilibrium equations are obtained in terms of loop currents by suitable substitution of the equations obtained in Steps 5 to 8.

4.9.2 Formulation of Network Equations on Node Basis

Steps

1. Draw a directed graph of the circuit under considerations, selecting the directions of assumed current flow to coincide for current sources.
2. Select the tree of the graph so that current ***sources are in the co-tree and the voltage sources are within the tree, if possible***. Also, if possible, select the tree so that ***at least two branches of the tree are incident at the reference node***.
3. Identify $(n - 1)$ fundamental cut-sets (FCS) and draw the FCS lines.
4. Write the $(n - 1)$ FCS KCL equations in the form $\boldsymbol{A_a \times I_b = 0}$ **or** $\boldsymbol{Q_C \times I_b = 0}$.
5. Obtain each of the branch currents in terms of node voltages in the form $\boldsymbol{V_b = A_a^T \times V_n}$ or, $\boldsymbol{V_b = Q_C^T \times V_t}$.
6. For each branch, the relationship between the voltage and current is obtained from Ohm's law ($\boldsymbol{V = RI}$).
7. Substitute the equations of step 6 into the KVL equations of step 5 and finally into the KCL equations of step 4, thus obtaining the $(n - 1)$ independent node voltage equations.

4.9.3 Generalized Equations in Matrix Forms for Circuits Having Sources

A general branch consisting of a voltage source V_s and a current source I_s is shown in Fig. 4.12.

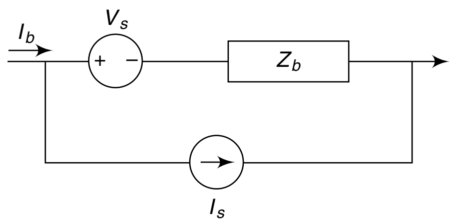


Figure 4.12

Here, the branch current is $(I_b + I_s)$ and the branch voltage is $(V_b + V_s)$.

Without sources, the KCL and KVL equations are:

$$\left.\begin{aligned} A_a \times I_b &= 0 \quad (4.1)\\ I_b &= B_a^T \times I_L \quad (4.2)\\ Q_C \times I_b &= 0 \quad (4.3)\end{aligned}\right\} \text{KCL}$$

and

$$\left.\begin{aligned} V_b &= A_a^T \times V_n \quad (4.4)\\ B_a \times V_b &= 0 \quad (4.5)\\ V_b &= Q_C^T \times V_t \quad (4.6)\end{aligned}\right\} \text{KVL}$$

With the sources, the KCL and KVL equations are modified as,

$$A_a I_b + A_a I_s = 0 \tag{4.7}$$
$$I_b + I_s = B_a^T I_L \tag{4.8}$$
$$Q_c I_b + Q_c I_s = 0 \tag{4.9}$$

and

$$V_b + V_s = A_a^T V_n \tag{4.10}$$
$$B_a V_b + B_a V_s = 0 \tag{4.11}$$
$$V_b + V_s = Q_c^T V_t \tag{4.12}$$

The branch voltage-current relations for the passive network elements are written in matrix form as,

$$V_b = Z_b I_b \tag{4.13}$$

and

$$I_b = Y_b V_b \tag{4.14}$$

where, $\boldsymbol{Z_b}$ is the *branch impedance matrix* and $\boldsymbol{Y_b}$ is the *branch admittance matrix*, both of the order $b \times b$. On the basis of these equations the general equations can be written in terms of three matrices as follows.

Node Equations From equation (4.7),

$$A_a I_s = -A_a I_b = -A_a Y_b V_b = -A_a Y_b (A_a^T V_n - V_s) \qquad \text{\{by equation (4.10)\}}$$

or $$A_a Y_b A_a^T V_n = A_a Y_b V_s - A_a I_s = A_a [Y_b V_s - I_s]$$

or $$\boxed{YV_n = A_a[Y_b V_s - I_s]}$$

where, $\boldsymbol{Y}$ is called the *nodal admittance matrix* of the order of $(n - 1) \times (n - 1)$. The above equation represents a set of $(n - 1)$ number of equations, known as *node equations*.

In case of node analysis, one node is taken as the datum node and potential of that node is zero. Consequently, the complete incident matrix becomes reduced incidence matrix. Thus, the node equations become

$$\boxed{Y V_n = A [Y_b V_s - I_s]}$$

where, $Y = AY_b A^T$ is called the *nodal admittance matrix* of the order of $(n - 1) \times (n - 1)$. The above equation represents a set of $(n - 1)$ number of equations, known as *node equations*.

Mesh Equations From equation (4.11),

$$B_a V_s = -B_a V_b = -B_a Z_b I_b = -B_a Z_b (B_a^T I_L - I_s) \qquad \text{\{by equation (4.8)\}}$$

or $$B_a Z_b B_a^T I_L = B_a [Z_b I_s - V_s]$$

or $$\boxed{ZL_L = B_a[Z_b I_s - V_s]}$$

where, $\boldsymbol{Z}$ is the *loop-impedance matrix* of the order of $(b - n + 1) \times (b - n + 1)$. The above equation represents a set of $(b - n + 1)$ number of equations, known as *mesh* or *loop equations*.

Cut-set Equations From equation (4.8),

$$Q_c I_s = -Q_c I_b = -Q_c Y_b V_b = -Q_c Y_b (Q_c^T V_t - V_s) \qquad \text{\{by equation (4.12)\}}$$

or $$Q_c Y_b Q_c^T V_t = Q_c [Y_b V_s - I_s]$$

or $$\boxed{Y_c V_t = Q_c[Y_b V_s - I_s]}$$

where, Y_c is the *cut-set admittance matrix* of the order of $(n - 1) \times (n - 1)$ and the set of $(n - 1)$ equations represented by the above equation is known as *cut-set equations*.

4.10 SOLUTION OF EQUILIBRIUM EQUATIONS

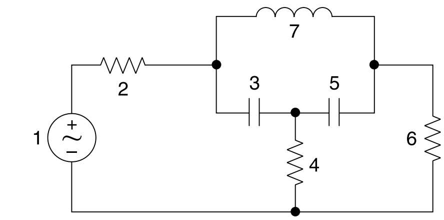
There are two methods of solving equilibrium equations given as follows.

(i) *Elimination method*: by eliminating variables until an equation with a single variable is achieved, and then by the method of substitution.

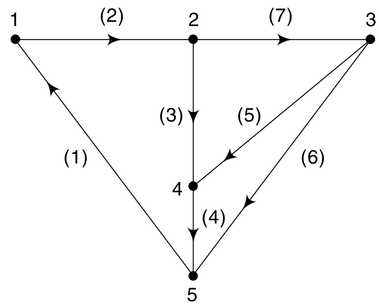
(ii) *Determinant method*: by the method known as Cramer's rule.

SOLVED PROBLEMS

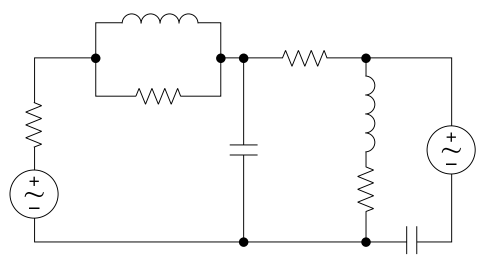
4.1 Draw the graph of the network shown in the figure.



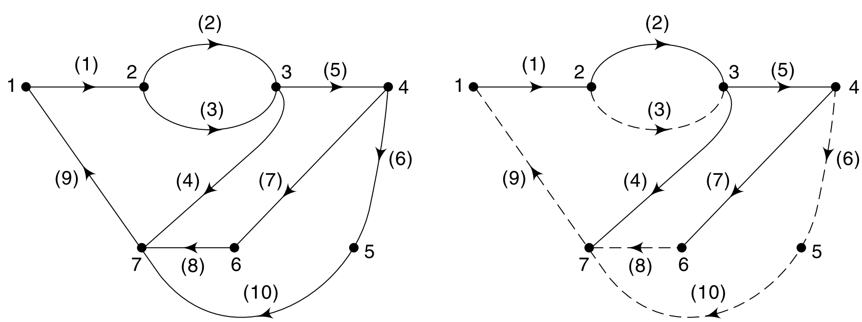
Solution The graph of the network is shown below.



4.2 From the figure, make the graph and find one tree. How many mesh currents are required for solving the network? Find the number of possible trees.



Solution The graph of the network is shown below. One tree of the graph is shown.



Graph of the network *Tree of the graph*

The complete incidence matrix is obtained as,

$A_a =$

Nodes	*Branches*									
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
1	1	0	0	0	0	0	0	0	–1	0
2	–1	1	1	0	0	0	0	0	0	0
3	0	–1	–1	1	1	0	0	0	0	0
4	0	0	0	0	–1	1	1	0	0	0
5	0	0	0	0	0	–1	0	0	0	1
6	0	0	0	0	0	0	–1	1	0	0
7	0	0	0	–1	0	0	0	–1	1	–1

Reduced incidence matrix becomes,

$A =$

Nodes	*Branches*									
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
1	1	0	0	0	0	0	0	0	–1	0
2	–1	1	1	0	0	0	0	0	0	0
3	0	–1	–1	1	1	0	0	0	0	0
4	0	0	0	0	–1	1	1	0	0	0
5	0	0	0	0	0	–1	0	0	0	1
6	0	0	0	0	0	0	–1	1	0	0

Hence the number of possible trees is,

$$n = \det\left\{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}\right\}$$

$$= \det\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -2 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

$\Rightarrow \qquad n = 12 \qquad$ *Ans.*

4.3 Branch current and loop current relations are expressed in matrix form as,

$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}$$

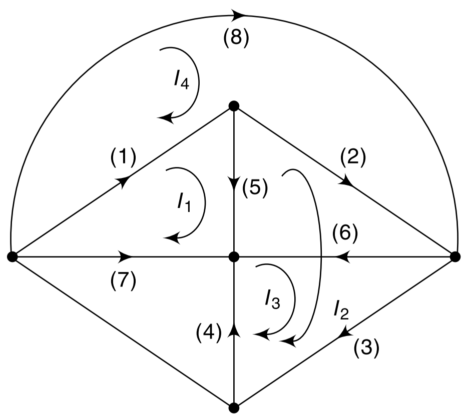
Draw the oriented graph.

Solution We know that, $[I_b]=[B_a]^T[I_L]$. So, the tie-set matrix, here, is,

$B_a =$

Loop or Link Currents	*Branches* 1	2	3	4	5	6	7	8
1	1	0	0	0	1	0	–1	0
2	0	1	1	1	–1	0	0	0
3	0	0	1	1	0	–1	0	0
4	–1	–1	0	0	0	0	0	1

So, the graph consists of four loops and eight branches. Loop 1 consists of branch 1, 5 and 7. The orientations are given following the sign +1 or –1. Following the procedure, the complete oriented graph is shown below.



4.4 The fundamental cut-set matrix is given as,

Twigs				*Links*		
1	2	3	4	5	6	7
1	0	0	0	–1	0	0
0	1	0	0	1	0	1
0	0	1	0	0	1	1
0	0	0	1	0	1	0

Draw the oriented graph of the network.

Solution The graph has seven branches and four fundamental cut-sets:

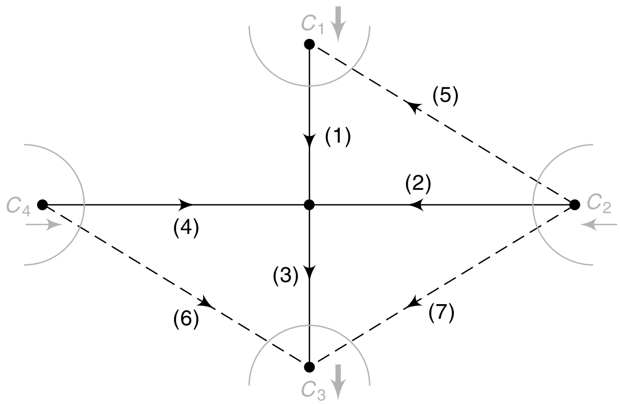
Cut-set-1: [1, 5]

Cut-set-2: [2, 5, 7]

Cut-set-3: [3, 6, 7]

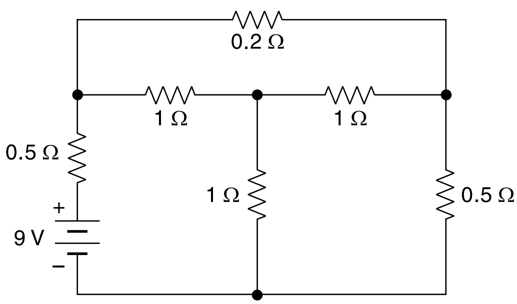
Cut-set-4: [4, 6]

So, the oriented graph is as shown in figure.

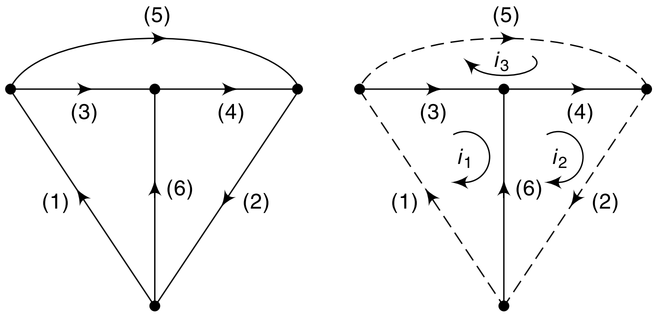


4.5 (a) For the network of the figure, draw the graph and write a tie-set schedule. Using the tie-set schedule obtain the loop equations and find the currents in all branches.

(b) For the network of (a), write a cut-set schedule, obtain nodal equations and find branch currents.



Solution The graph and one tree are shown in figure.



The tie-set matrix,

$$B_a = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix}$$

Branch impedance matrix is,

$$Z_b = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$[B_a][Z_b] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0.5 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0.2 & 0 \end{bmatrix}$$

$$\therefore \qquad [B_a][Z_b][B_a]^T = \begin{bmatrix} 0.5 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0.5 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2.5 & -1 & -1 \\ -1 & 2.5 & -1 \\ -1 & -1 & 2.2 \end{bmatrix}$$

Now,
$$-[B_a][V_s] = -\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}$$

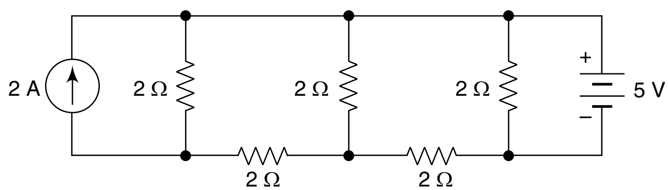
So, the loop equations are,

$$\begin{bmatrix} 2.5 & -1 & -1 \\ -1 & 2.5 & -1 \\ -1 & -1 & 2.2 \end{bmatrix} \times \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}$$

Solving three equations,

$$i_1 = 8.9 \text{ A}, \ i_2 = 6.33 \text{ A}, \ i_3 = 6.92 \text{ A} \qquad \textit{Ans.}$$

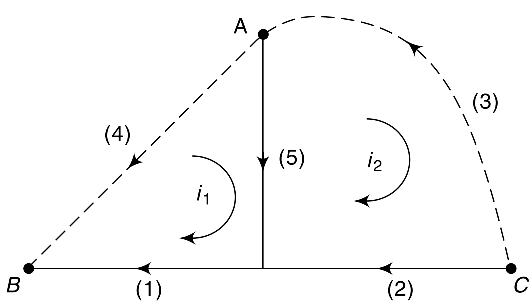
4.6 The figure shows a d.c. network. (a) Draw a graph of the network. Which elements are not included in the graph and why? (b) Write a loop incidence matrix and use it to obtain loop equations. (c) Find branch currents.



Solution

(a) The graph is shown below.

The 2 Ω resistor in parallel with voltage source and the 2 A current source have not been included in the graph. This is because of the reason that passive elements in parallel with a voltage source are not included in graph and the current source in parallel with a passive element is open-circuited while drawing graph.



(b) The tie-set matrix for the tree chosen is,

$$B_a = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix}$$

Branch impedance matrix is,

$$Z_b = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$B_a Z_b B_a^T = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & -2 & 2 \\ 0 & 2 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$$

Now,

$$B_a Z_b I_s - B_a V_s = \begin{bmatrix} 2 & 0 & 0 & -2 & 2 \\ 0 & 2 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

So, the loop equations are,

$$\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

Solving these equations, $i_1 = 0.3$ A, $i_2 = -1.1$ A *Ans.*

(c) Putting these values, the branch voltages are

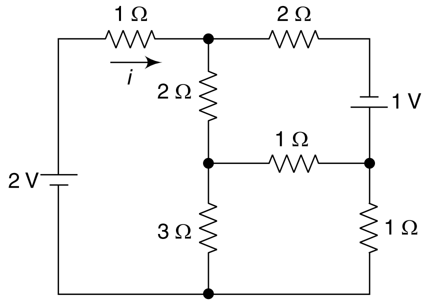
$V_1 = 2 \times i_1 = 0.6$ V, $V_2 = 2 \times i_2 = -2.2$ V, $V_3 = -5$ V, $V_4 = -2 \times i_1 + 4 = 3.4$ V, $V_5 = 2.8$ V *Ans.*

Thus, the branch currents are

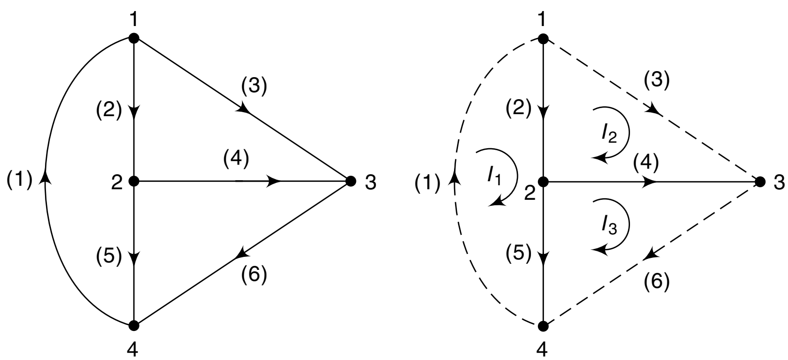
$$I_{AB} = \frac{3.4}{2} = 1.7\text{ A},\ I_{AD} = \frac{2.8}{2} = 1.4\text{ A},\ I_{AC} = \frac{5}{2} = 2.5\text{ A},\ I_{DB} = \frac{0.6}{2} = 0.3\text{ A},\ I_{DC} = \frac{2.2}{2} = 1.1\text{ A}$$

So, the current supplied by the battery = (1.7 + 1.4 + 2.5 – 2) = 4.6 A *Ans.*

4.7 For the network shown in the figure, draw the oriented graph and obtain the tie-set matrix. Use this matrix to calculate *i*.



Solution The oriented graph and any one tree are shown.



The tie-set matrix is given as,

$$B_a = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

The branch impedance matrix,

$$Z_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \qquad B_a Z_b = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 0 \\ 0 & -2 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 \end{bmatrix}$$

$$\therefore \qquad B_a Z_b B_a^T = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 0 \\ 0 & -2 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -2 & -3 \\ -2 & 5 & -1 \\ -3 & -1 & 5 \end{bmatrix}$$

Now, $$-B_a V_s = -\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

So, the loop equations become,

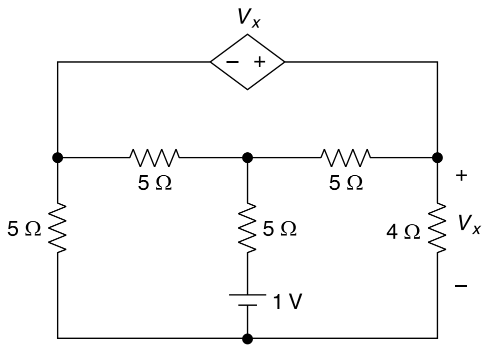
$$\begin{bmatrix} 6 & -2 & -3 \\ -2 & 5 & -1 \\ -3 & -1 & 5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Solving for I_1,

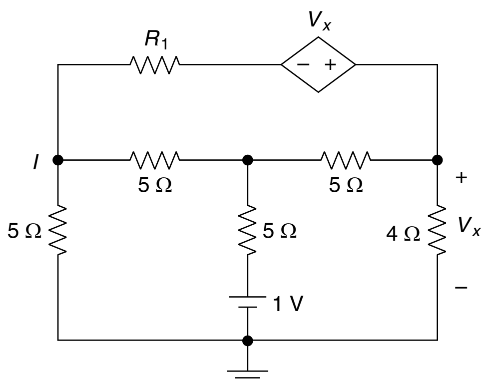
$$I_1 = \frac{\begin{vmatrix} 2 & -2 & -3 \\ 1 & 5 & -1 \\ 0 & -1 & 5 \end{vmatrix}}{\begin{vmatrix} 6 & -2 & -3 \\ -2 & 5 & -1 \\ -3 & -1 & 5 \end{vmatrix}} = 0.91 \text{ A}$$

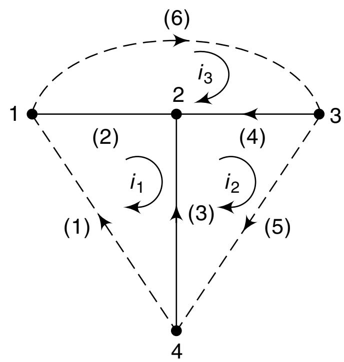
$$\therefore \qquad i_1 = 0.91 \text{ A}$$

4.8 The circuit of the figure contains a voltage controlled voltage source. For this circuit, draw the oriented graph. By selecting a proper tree obtain the tie-set matrix and hence calculate the voltage, V_x.



Solution Since the controlled voltage source is not accompanied by any passive element, we will consider a resistance R_1 in series with the controlled voltage source, and finally let $R_1 \rightarrow 0$.





The graph of the network is shown with one tree.
The tie-set matrix is,

$$B_a = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The branch impedance matrix,

$$Z_b = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_1 \end{bmatrix}$$

$$\therefore \qquad B_a Z_b = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_1 \end{bmatrix} = \begin{bmatrix} 5 & 5 & -5 & 0 & 0 & 0 \\ 0 & 0 & 5 & -5 & 4 & 0 \\ 0 & -5 & 0 & 5 & 0 & R_1 \end{bmatrix}$$

$$\therefore \qquad B_a Z_b B_a^T = \begin{bmatrix} 5 & 5 & -5 & 0 & 0 & 0 \\ 0 & 0 & 5 & -5 & 4 & 0 \\ 0 & -5 & 0 & 5 & 0 & R_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 & -5 & -5 \\ -5 & 14 & -5 \\ -5 & -5 & (10+R_1) \end{bmatrix}$$

Now, $$-B_a V_s = -\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -V_x \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \\ -V_x \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ V_x \end{bmatrix}$$

So, the loop equations become,

$$\begin{bmatrix} 15 & -5 & -5 \\ -5 & 14 & -5 \\ -5 & -5 & (10+R_1) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ V_x \end{bmatrix}$$

With $R_1 \to 0$ and $V_x = 4I_2$, the equations reduce to,

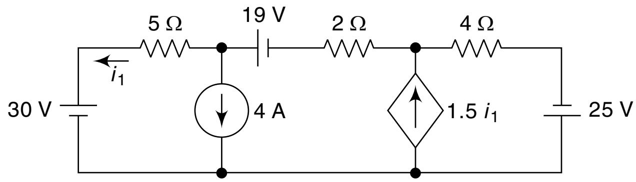
$$\begin{bmatrix} 15 & -5 & -5 \\ -5 & 14 & -5 \\ -5 & -9 & 10 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Solving for I_2,

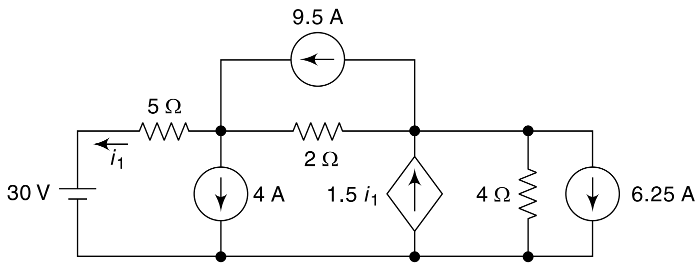
$$I_2 = \frac{\begin{vmatrix} 15 & -1 & -5 \\ -5 & 1 & -5 \\ -5 & 0 & 10 \end{vmatrix}}{\begin{vmatrix} 15 & -5 & -5 \\ -5 & 14 & -5 \\ -5 & -9 & 10 \end{vmatrix}} = \frac{1}{19}\ \text{A}$$

$$\therefore \qquad V_x = 4 \times I_2 = 4 \times \frac{1}{19} = \frac{4}{19}\text{ V} \qquad \textit{Ans.}$$

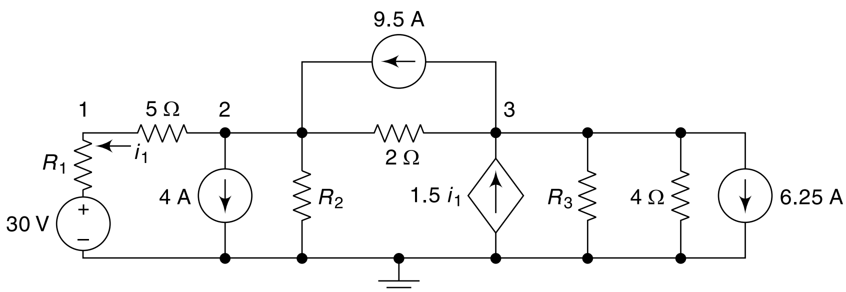
4.9 Determine the current i_1 in the circuit using nodal analysis method and graph theory concepts.



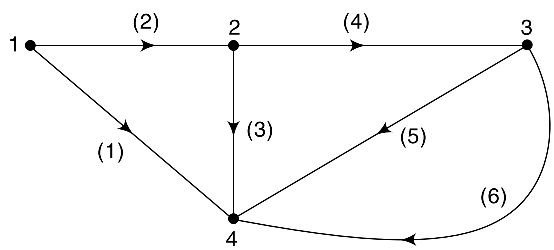
Solution By source transformation technique, we convert the 19 V and 25 V voltage sources into current sources.



Since the 30 V voltage source, the 4 A current source, and controlled current source are not accompanied by the passive elements, we consider three resistors R_1, R_2 and R_3 and finally let, $R_1 \to 0$, $R_2 \to \infty$, and $R_3 \to \infty$.



The graph of the network is shown.



The complete incidence matrix is,

$$A_a = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

Reduced Incidence matrix is,

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}$$

Branch admittance matrix is,

$$Y_b = \begin{bmatrix} G_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & G_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \quad \text{where, } G_1 = \frac{1}{R_1},\ G_2 = \frac{1}{R_2},\ G_3 = \frac{1}{R_3}$$

$$\therefore \qquad AY_b = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} G_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & G_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} G_1 & -\frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & G_2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{2} & G_3 & \frac{1}{4} \end{bmatrix}$$

$$\therefore \qquad AY_bA^T = \begin{bmatrix} G_1 & -\frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & G_2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{2} & G_3 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(G_1 + \frac{1}{5}\right) & -\frac{1}{5} & 0 \\ -\frac{1}{5} & \left(G_1 + \frac{1}{5} + \frac{1}{2}\right) & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \left(G_3 + \frac{1}{2} + \frac{1}{4}\right) \end{bmatrix}$$

$$AY_bV_s - AI_s = -AI_s = -\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -30G_1 \\ 0 \\ 4 \\ -9.5 \\ -1.5i_1 \\ 6.25 \end{bmatrix} \quad \{\because \text{We made } V_s = 0\}$$

Now,

$$= -\begin{bmatrix} -30G_1 \\ -5.5 \\ (15.75 - 1.5\, i_1) \end{bmatrix}$$

Thus, node equations are,

$$\begin{bmatrix} \left(G_1 + \frac{1}{5}\right) & -\frac{1}{5} & 0 \\ -\frac{1}{5} & \left(G_2 + \frac{7}{10}\right) & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \left(G_3 + \frac{3}{4}\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 30G_1 \\ 5.5 \\ 1.5\, i_1 - 15.75 \end{bmatrix}$$

With $R_1 \to 0$, $G_1 \to \propto$, $R_2 \to \propto$, $G_2 \to 0$, $R_3 \to \propto$, $G_3 \to 0$ the equations become:

$$\left(G_1 + \frac{1}{5}\right)V_1 - \frac{1}{5}V_2 = 30G_1$$

$$-\frac{1}{5}V_1 + \left(G_2 + \frac{7}{10}\right)V_2 - \frac{1}{2}V_3 = 5.5$$

$$-\frac{1}{2}V_2 + \frac{3}{4}V_3 = (1.5i_1 - 15.75)$$

or

$$V_1 = 30 \quad \text{(i)}$$

$$-\frac{1}{5}V_1 + \frac{7}{10}V_2 - \frac{1}{2}V_3 = 5.5 \quad \Rightarrow \quad 7V_2 - 5V_3 = 115 \quad \text{(ii)}$$

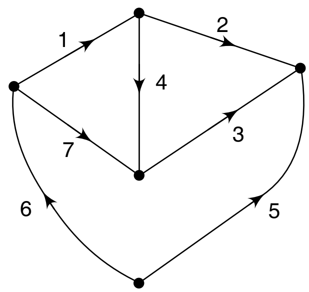
$$-\frac{1}{2}V_2 + \frac{3}{4}V_3 = \left[1.5\left(\frac{V_2 - V_1}{5}\right) - 15.75\right] \quad \Rightarrow \quad 16V_2 - 15V_3 = 495 \quad \text{(iii)}$$

Solving equations (i), (ii), and (iii), we get,

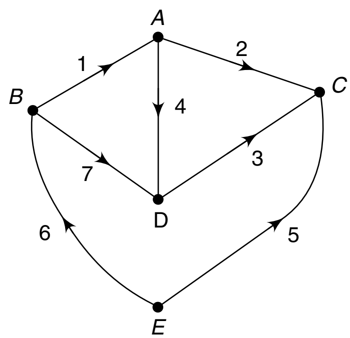
$$V_2 = -30 \text{ V}, \quad V_3 = 65 \text{ V}$$

Hence, the current, $i_1 = \left(\dfrac{V_2 - V_1}{5}\right) = \dfrac{-30 - 30}{5} = -12 \text{ A}$ *Ans.*

4.10 Write the complete incidence matrix for the graph shown in the figure.



Solution We first label the nodes as shown in the figure.

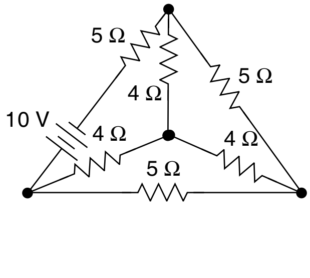


The complete incidence matrix is given as,

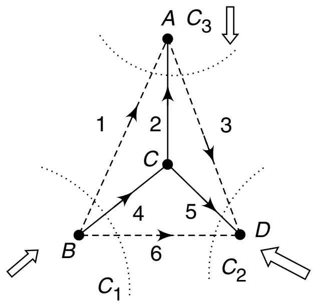
$A_a =$

	1	*2*	*3*	*4*	*5*	*6*	*7*
A	–1	1	0	1	0	0	0
B	1	0	0	0	0	–1	1
C	0	–1	–1	0	–1	0	0
D	0	0	1	–1	0	0	–1
E	0	0	0	0	1	1	0

4.11 Write down the incidence matrix and cutset matrices for the network shown.



Solution The graph and a suitable tree for the network are shown in the figure.



The complete incidence matrix is given as,

		1	2	3	4	5	6
	A	–1	–1	1	0	0	0
A_a =	B	1	0	0	1	0	1
	C	0	1	0	–1	1	0
	D	0	0	–1	0	–1	–1

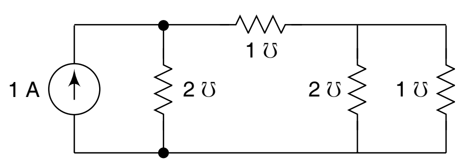
The fundamental cutsets are identified as,

f-cutset-1: [1, 4, 6]
f-cutset-2: [3, 5, 6]
f-cutset-3: [1, 2, 3]

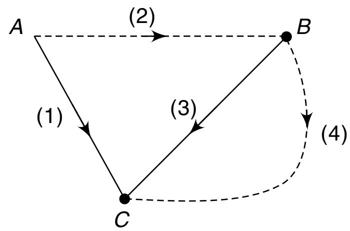
The fundamental cutset matrix is given as,

		1	2	3	4	5	6
	C_1	–1	0	0	1	0	1
Q =	C_2	0	0	–1	0	–1	–1
	C_3	–1	–1	1	0	0	0

4.12 For the network shown in the figure, give fundamental cutset matrix and hence find KCL equations.



Solution The graph and one tree are shown for the network.
The fundamental cutsets are identified as
f-cutset-1: [1, 2]
f-cutset-2: [2, 3, 4]



The fundamental cutset matrix is given as,

		1	2	3	4
$Q_a =$	C_1	1	1	0	0
	C_2	0	–1	1	1

The KCL equations in terms of the cutset matrix is given as,

$$[Q][Y_b][Q^T][Vt] = -[Q][I_S]$$

Here,

$$[Q][Y_b][Q^T] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}$$

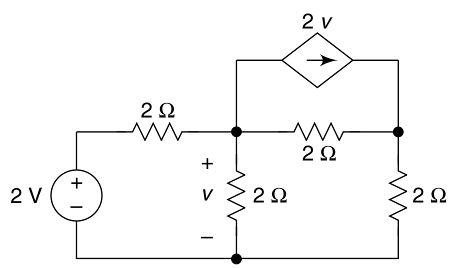
$$-[Q][I_S] = -\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus, the KCL equations are

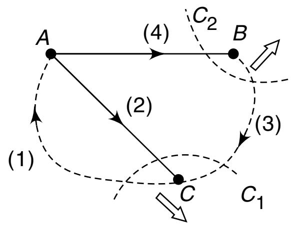
$$\begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} V_{t1} \\ V_{t3} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \textit{Ans.}$$

4.13 For the network shown in the figure, draw the oriented graph, select a suitable tree and obtain the fundamental cutset matrix. Determine the node equations and find v.

Solution The oriented graph of the network is shown in the figure. Since we have to find v, we take branch (2) in the twig and a possible tree is selected.



The fundamental cutsets are identified as



f-cutset-1: [1, 2, 3];
f-cutset-2: [3, 4];

The fundamental cutset matrix is given as,

$Q_a =$		1	2	3	4
	C_1	–1	1	1	0
	C_2	0	0	–1	1

The node equations are given as,

$$[Q][Y_b][Q^T][Vt] = [Q] \times \{[Y_b][V_S] - [I_S]\}$$

Here,

$$[Q][Y_b][Q^T] = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

$$[Q] \times \{[Y_b][V_s] - [I_S]\} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2v \end{bmatrix} \right\} = \begin{bmatrix} 1 \\ -2v \end{bmatrix}$$

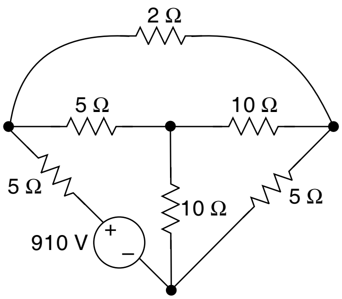
Thus, the KCL equations are

$$\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} V_{t2} \\ V_{t4} \end{bmatrix} = \begin{bmatrix} 1 \\ -2v \end{bmatrix}$$

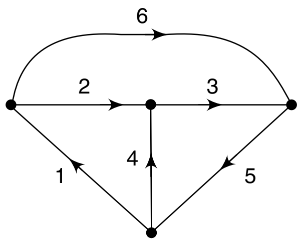
Here, $V_{t2} = v$. Putting this in the KCL equations and solving we get,

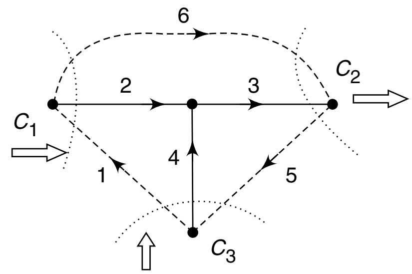
$$v = \frac{4}{9} V \qquad Ans.$$

4.14 For the resistive network, write a cutset schedule and equilibrium equations on voltage basis. Hence obtain values of branch voltages and branch currents.



Solution The graph of the network is shown in the figure. A suitable tree is shown.





The fundamental cutsets are identified as,

f-cutset-1: [1, 2, 6];
f-cutset-2: [3, 5, 6];
f-cutset-3: [1, 4, 5]

The fundamental cutset matrix is given as,

$Q =$		*1*	*2*	*3*	*4*	*5*	*6*
	C_1	–1	1	0	0	0	1
	C_2	0	0	1	0	–1	1
	C_3	1	0	0	1	–1	0

The node equations are given as,

$$[Q][Y_b][Q^T][Vt] = [Q] \times \{[Y_b][V_S] - [I_S]\} = [Q]\,[Y_b][V_S] \quad \{\text{since } I_S = 0 \text{ here}\}$$

Here,

$$[Q][Y_b][Q^T] = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1/5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9 & 0.5 & -0.2 \\ 0.5 & 0.8 & 0.2 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$$

$$[Q][Y_b][V_s] = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1/5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} -910 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 182 \\ 0 \\ 0 \end{bmatrix}$$

Thus, the KCL equations are

$$= \begin{bmatrix} 0.9 & 0.5 & -0.2 \\ 0.5 & 0.8 & 0.2 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} V_{t2} \\ V_{t3} \\ V_{t4} \end{bmatrix} = \begin{bmatrix} 182 \\ 0 \\ 0 \end{bmatrix}$$

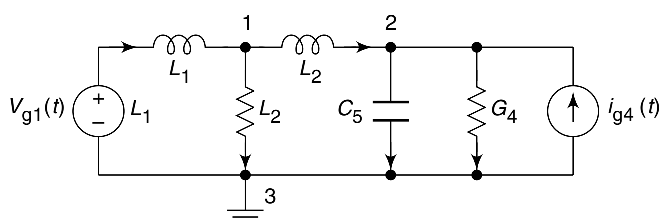
Solving by Cramer's rule, we get the tree-branch voltages as,

$$V_{t2} = 143 \text{ V}; \quad V_{t3} = -14.3 \text{ V}; \quad V_{t4} = -300 \text{ V} \qquad \textit{Ans.}$$

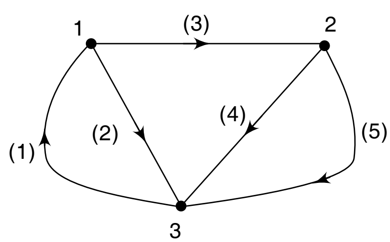
4.15 Using topological method, obtain node equations and node voltages in *s* domain for the network shown in the figure, when

$$L_1 = L_2 = 1H, \quad C_5 = 1F, \quad G_3 = G_4 = 1\Omega, \quad V_{g1}(t) = 2u(t) \quad \text{and} \quad i_{g4}(t) = 2\,\delta(t),$$

where, $u(t)$ is unit step function and $\delta(t)$ is the unit impulse function.



Solution The graph of the network is shown in the figure.



The incidence matrix is given as,

		1	2	3	4	5
$A_a =$	1	–1	1	1	0	0
	2	0	0	–1	1	1
	3	1	–1	0	–1	–1

Reduced incidence matrix is,

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}$$

Branch admittance matrix is,

$$Y_b = \begin{bmatrix} 1/s & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/s & 0 & 0 \\ 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \qquad A\,Y_b = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/s & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/s & 0 & 0 \\ 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/s & 1 & 1/s & 0 & 0 \\ 0 & 0 & -1/s & s & 1 \end{bmatrix}$$

$$\therefore \qquad A\,Y_b A^T = \begin{bmatrix} -1/s & 1 & 1/s & 0 & 0 \\ 0 & 0 & -1/s & s & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (2/s+1) & -1/s \\ -1/s & (s+1+1/s) \end{bmatrix}$$

Now,

$$A\,Y_b V_s - A\,I_s = \begin{bmatrix} -1/s & 1 & 1/s & 0 & 0 \\ 0 & 0 & -1/s & s & 1 \end{bmatrix} \begin{bmatrix} 2/s^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2/s^2 \\ 2 \end{bmatrix}$$

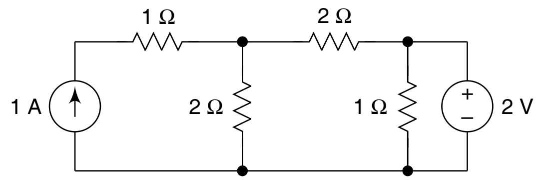
Thus, node equations are,

$$\begin{bmatrix} (2/s+1) & -1/s \\ -1/s & (s+1+1/s) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 2/s^2 \\ 2 \end{bmatrix}$$

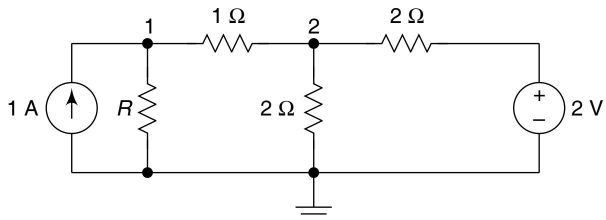
Solving by Cramer's rule, we get the voltages as,

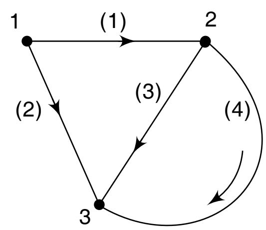
$$V_1 = \frac{2\left(2s^2+s+1\right)}{s(s+1)\left(s^2+2s+1\right)} \quad \text{and} \quad V_2 = \frac{2\left(s^3+s^2+1\right)}{s(s+1)\left(s^2+2s+1\right)}$$

4.16 Determine the currents in all branches of the network shown in the figure using node analysis method. Use graph theory method.



Solution Here, the 1 Ω resistance in parallel with the 2 V voltage source can be ignored. Also, there is no passive element in parallel with 1 A current source. We assume a resistance R in parallel with 1 A current source and finally let $R \to \infty$. Therefore, the graph of the network is shown in the figure.





The complete incidence matrix is,

$$A_a = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

Reduced incidence matrix is,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix}$$

Branch admittance matrix is,

$$Y_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/R & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

$$\therefore \qquad A\,Y_b = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/R & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1/R & 0 & 0 \\ -1 & 0 & 1/2 & 1/2 \end{bmatrix}$$

$$\therefore \qquad AY_b A^T = \begin{bmatrix} 1 & 1/R & 0 & 0 \\ -1 & 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (1+1/R) & -1 \\ -1 & 2 \end{bmatrix}$$

Now,

$$AY_bV_s - AI_s = \begin{bmatrix} 1 & 1/R & 0 & 0 \\ -1 & 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, node equations are,

$$\begin{bmatrix} (1+1/R) & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

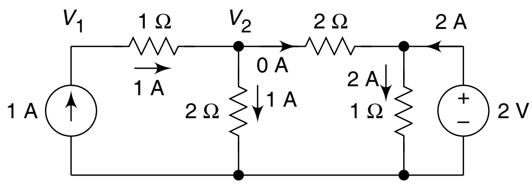
With $R \to \infty$, the equations become

$$V_1 - V_2 = 1$$
$$-V_1 + 2V_2 = 1$$

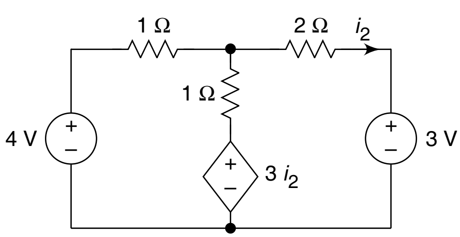
Solving equations, we get,

$$V_1 = 3\text{V}, \qquad V_2 = 2\text{V}$$

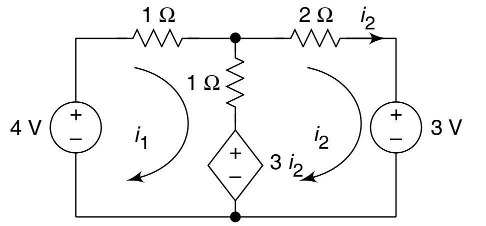
Hence, the currents in different branches are shown in the figure.



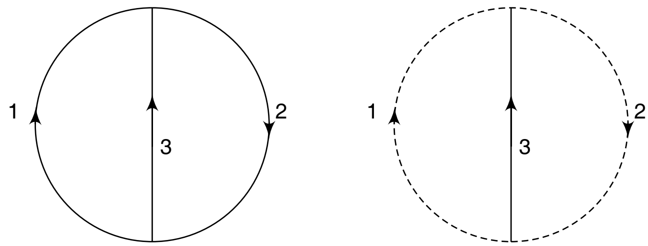
4.17 Consider the network shown in the figure using loop method of analysis, determine currents in all the branches, indicating their directions. Use graph theory method.



Solution



The graph of the network is shown below. Also, the tree is selected as shown.



For the selected tree, the tie-set matrix is given as,

$$B_a = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

The branch impedance matrix is,

$$Z_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \qquad B_a Z_b = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

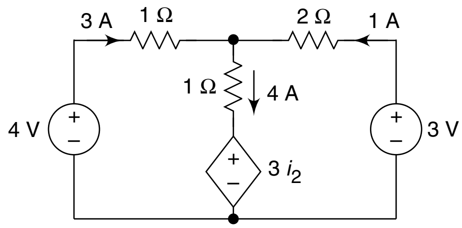
$$\therefore \qquad B_a Z_b B_a^T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

Now,

$$-B_a V_s = -\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \\ -3i_2 \end{bmatrix} = \begin{bmatrix} 4-3i_2 \\ -3+3i_2 \end{bmatrix}$$

So, the loop equations become,

$$\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 4-3i_2 \\ -3+3i_2 \end{bmatrix}$$



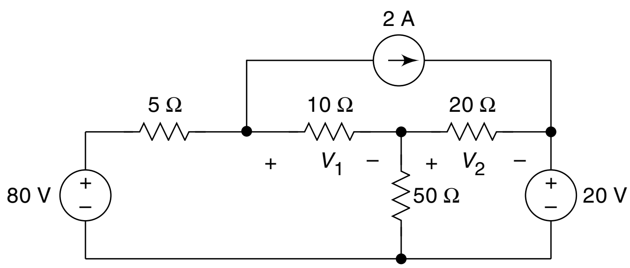
These equations reduce to,

$$\begin{aligned} 2i_1 - i_2 &= 4 - 3i_2 \quad \Rightarrow \quad i_1 + i_2 = 2 \\ -i_1 + 3i_2 &= -3 + 3i_2 \quad \Rightarrow \quad i_1 = 3A \end{aligned}$$

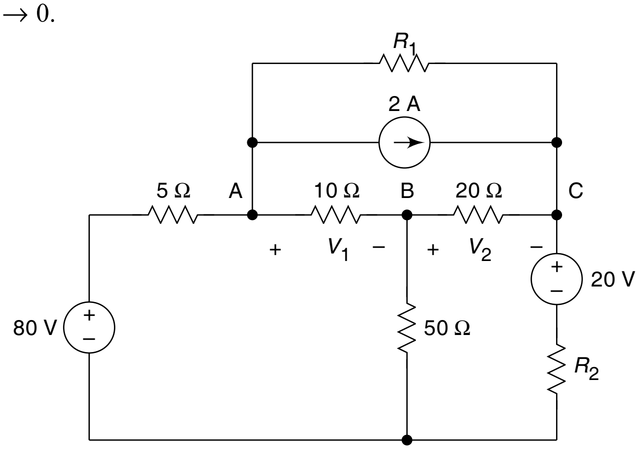
$\therefore$ $$i_2 = -1A$$

Thus, the branch currents are shown with their directions.

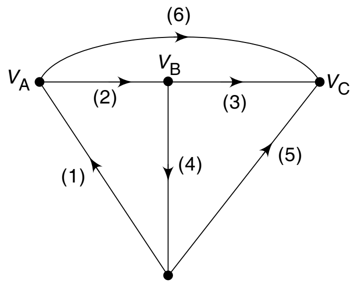
4.18 For the circuit shown in the figure construct a tree in which $10\,\Omega$ and $20\,\Omega$ are in tree branches. Using node analysis, solve for V_1 and V_2.

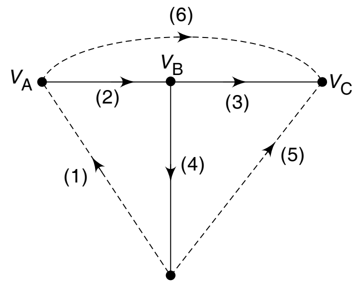


Solution Here, we have on e current source without parallel resistance and one voltage source without series resistance. Therefore, we connect a parallel resistance R_1 in parallel with the 2 A current source and a series resistance R_2 in series with the 20 V voltage source. Finally, we will let $R_1 \to \infty$ and $R_2 \to 0$.



Now, we construct the graph of the network as shown below. A tree, in which 10 Ω and 20 Ω are in tree branches, is selected.





The complete incidence matrix is,

$$A_a = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Reduced incidence matrix is,

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

Branch admittance matrix is,

$$Y_b = \begin{bmatrix} 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.05 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.02 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{R_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{R_1} \end{bmatrix}$$

$$\therefore \qquad AY_b = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.05 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.02 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{R_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{R_1} \end{bmatrix}$$

$$= \begin{bmatrix} -0.2 & 0.1 & 0 & 0 & 0 & \frac{1}{R_1} \\ 0 & -0.1 & 0.05 & 0.02 & 0 & 0 \\ 0 & 0 & -0.05 & 0 & -\frac{1}{R_2} & -\frac{1}{R_1} \end{bmatrix}$$

$$\therefore \qquad AY_bA^T = \begin{bmatrix} -0.2 & 0.1 & 0 & 0 & 0 & \frac{1}{R_1} \\ 0 & -0.1 & 0.05 & 0.02 & 0 & 0 \\ 0 & 0 & -0.05 & 0 & -\frac{1}{R_2} & -\frac{1}{R_1} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(0.3+\frac{1}{R_1}\right) & -0.1 & -\frac{1}{R_1} \\ -0.1 & 0.17 & -0.05 \\ -\frac{1}{R_1} & -0.05 & \left(0.05+\frac{1}{R_2}+\frac{1}{R_1}\right) \end{bmatrix}$$

Now,

$$AY_bV_s - AI_s = \begin{bmatrix} -0.2 & 0.1 & 0 & 0 & 0 & \frac{1}{R_1} \\ 0 & -0.1 & 0.05 & 0.02 & 0 & 0 \\ 0 & 0 & -0.05 & 0 & -\frac{1}{R_2} & -\frac{1}{R_1} \end{bmatrix} \begin{bmatrix} -80 \\ 0 \\ 0 \\ 0 \\ -20 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 14 \\ 0 \\ \left(\frac{20}{R_2}+2\right) \end{bmatrix}$$

Thus, node equations are,

$$\begin{bmatrix} \left(0.3+\frac{1}{R_1}\right) & -0.1 & -\frac{1}{R_1} \\ -0.1 & 0.17 & -0.05 \\ -\frac{1}{R_1} & -0.05 & \left(0.05+\frac{1}{R_2}+\frac{1}{R_1}\right) \end{bmatrix} \begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ \left(\frac{20}{R_2}+2\right) \end{bmatrix}$$

With $R_1 \to \infty$, the equations become

$$0.3V_A - 0.1V_B = 14$$
$$-0.1V_A + 0.17V_B - 0.05V_C = 0$$
$$-0.05V_B + \left(0.05+\frac{1}{R_2}\right)V_C = \left(\frac{20}{R_2}+2\right)$$

Solving equations, we get,

$$V_A = \frac{\begin{vmatrix} 14 & -0.1 & 0 \\ 0 & 0.17 & -0.05 \\ \left(\frac{20}{R_2}+2\right) & -0.05 & \left(0.05+\frac{1}{R_2}\right) \end{vmatrix}}{\begin{vmatrix} 0.3 & -0.1 & 0 \\ -0.1 & 0.17 & -0.05 \\ 0 & -0.05 & \left(0.05+\frac{1}{R_2}\right) \end{vmatrix}} = \frac{14\left[0.17(0.05R_2+1)-0.0025R_2\right]+0.005(20+2R_2)}{0.3\left[0.17(0.05R_2+1)-0.0025R_2\right]-0.01(0.05R_2+1)}$$

With $R_2 = 0$,

$$V_A = \frac{2.48}{0.041} = 60.49 \text{ V}$$

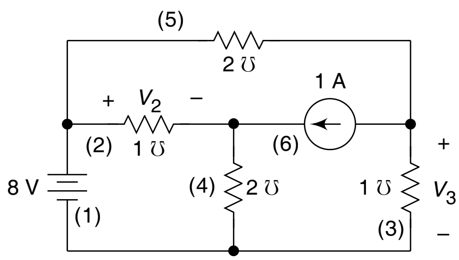
Similarly, with $R_2 = 0$, we get,

$$V_B = 41.47\text{V}$$
$$V_C = 20\text{V}$$

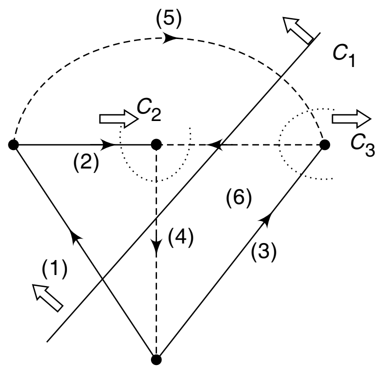
$\therefore$ $V_1 = (V_A - V_B) = (60.49 - 41.47) = 19.02 \text{ V}$ *Ans.*

and $V_2 = (V_B - V_C) = (41.420) = 21.47 \text{ V}$ *Ans.*

4.19 In the following circuit, determine the voltages V_2 and V_3 using cutset analysis. Select the circuit elements (1), (2) and (3) in the tree.



Solution The graph and tree are shown in the figure. Hence, there is no series impedance with voltage source and parallel admittance with current source. We consider two resistances R_1 and R_2 in series with the voltage source and in parallel with the current source, respectively. Finally, we will let $R_1 \to 0$, $R_2 \to \infty$.



Three fundamental cutsets are:

f-cutset-1: [1, 4, 5, 6];
f-cutset-2: [2, 4, 6];
f-cutset-3: [3, 5, 6]

The fundamental cutset matrix is given as,

$$Q = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline C_1 & 1 & 0 & 0 & -1 & -1 & 1 \\ C_2 & 0 & 1 & 0 & 1 & 0 & -1 \\ C_3 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}$$

The node equations are given as,

$$[Q][Y_b][Q^T][Vt] = [Q] \times \{[Y_b][V_S] - [I_S]\}$$

Here,

$$[Q][Y_b][Q^T] = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/R_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (4 + 1/R_1 + 1/R_2) & -(2 + 1/R_2) & -(2 + 1/R_2) \\ -(2 + 1/R_2) & -(3 + 1/R_2) & 1/R_2 \\ -(2 + 1/R_2) & 1/R_2 & (3 + 1/R_2) \end{bmatrix}$$

$$[Q][Y_b][V_s] = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/R_2 \end{bmatrix} \begin{bmatrix} -8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -8/R_1 \\ 0 \\ 0 \end{bmatrix}$$

$$-[Q][I_s] = -\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Thus, the KCL equations are

$$= \begin{bmatrix} (4 + 1/R_1 + 1/R_2) & -(2 + 1/R_2) & -(2 + 1/R_2) \\ -(2 + 1/R_2) & -(3 + 1/R_2) & 1/R_2 \\ -(2 + 1/R_2) & 1/R_2 & (3 + 1/R_2) \end{bmatrix} \begin{bmatrix} V_{t1} \\ V_{t2} \\ V_{t3} \end{bmatrix} = \begin{bmatrix} -(8/R_1 + 1) \\ 1 \\ 1 \end{bmatrix}$$

When $R_1 \to 0$, $R_2 \to \infty$, the equations reduce to the form as given below.

$$V_{t1} = -8 \text{ volts}$$

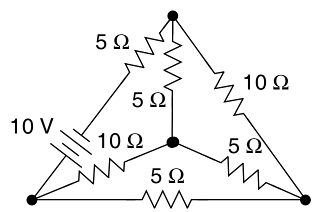
$$-2V_{t1} - 3V_{t2} = 1$$
$$-2V_{t1} + 3V_{t3} = 1$$

Solving the last two equations, $V_{t2} = 5$ volts; $V_{t3} = -5$ volts

Therefore,

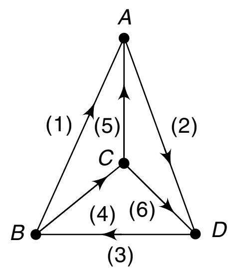
$$V_2 = V_{t2} = 5 \qquad \textit{Ans.}$$
$$V_3 = -V_{t3} = 5 \qquad \textit{Ans.}$$

4.20 For the network shown in the figure, write the tie-set matrix and determine the loop currents and branch currents.



Solution The graph and a suitable tree for the network are shown in the figure.
The tie-set matrix is given as,

$$B_a = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$



The branch impedance matrix is given as,

$$Z_b = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\therefore \qquad B_a Z_b = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 & 10 & -5 & 0 \\ 0 & 10 & 0 & 0 & 5 & -5 \\ 0 & 0 & 5 & -10 & 0 & 5 \end{bmatrix}$$

$$\therefore \quad B_a Z_b B_a^{\,T} = \begin{bmatrix} 5 & 0 & 0 & 10 & -5 & 0 \\ 0 & 10 & 0 & 0 & 5 & -5 \\ 0 & 0 & 5 & -10 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 20 & -5 & -10 \\ -5 & 20 & -5 \\ -10 & -5 & 15 \end{bmatrix}$$

$$-B_a V_s = -\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

Thus, the loop equations are given as,

$$\begin{bmatrix} 20 & -5 & -10 \\ -5 & 20 & -5 \\ -10 & -5 & 15 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

Solving by Cramer's rule, we get the loop currents as,

$$I_1 = \frac{\begin{vmatrix} 10 & -5 & -10 \\ 0 & 20 & -5 \\ 0 & -5 & 15 \end{vmatrix}}{\begin{vmatrix} 20 & -5 & -10 \\ -5 & 20 & -5 \\ -10 & -5 & 15 \end{vmatrix}} = \frac{2750}{2625} = 1.047\,\text{A}$$

$$I_2 = \frac{\begin{vmatrix} 20 & 10 & -10 \\ -5 & 0 & -5 \\ -10 & 0 & 15 \end{vmatrix}}{\begin{vmatrix} 20 & -5 & -10 \\ -5 & 20 & -5 \\ -10 & -5 & 15 \end{vmatrix}} = \frac{1250}{2625} = 0.476\,\text{A}$$

$$I_3 = \frac{\begin{vmatrix} 20 & -5 & 10 \\ -5 & 20 & 0 \\ -10 & -5 & 0 \end{vmatrix}}{\begin{vmatrix} 20 & -5 & -10 \\ -5 & 20 & -5 \\ -10 & -5 & 15 \end{vmatrix}} = \frac{2250}{2625} = 0.857\,\text{A}$$

Also, the branch currents are given as, $I_b = B_a{}^T I_L$

$$\therefore \quad \begin{bmatrix} I_{b1} \\ I_{b2} \\ I_{b3} \\ I_{b4} \\ I_{b5} \\ I_{b6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1.047 \\ 0.476 \\ 0.857 \end{bmatrix}$$

$$\left.\begin{aligned} \therefore \quad I_{b4} &= (1.047 - 0.857) = 0.19\text{ A} \\ \therefore \quad I_{b5} &= (-1.047 + 0.476) = -0.571\text{A} \\ \therefore \quad I_{b6} &= (-0.476 + 0.857) = 0.381\text{ A} \end{aligned}\right\} \qquad \textit{Ans.}$$

MULTIPLE-CHOICE QUESTIONS

4.1 The number of links for a graph having n nodes and b branches are
(a) $b - n + 1$ (b) $n - b + 1$ (c) $b + n - 1$ (d) $b + n$

4.2 The tree branches of a graph are called
(a) chords (b) links (c) twigs (d) co-tree

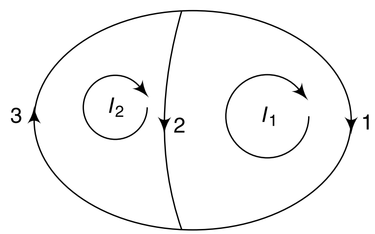
4.3 The tie-set matrix gives the relation between
(a) branch currents and link currents. (b) branch voltages and link currents.
(c) branch currents and link voltages. (d) none of these.

4.4 The graph of a network has six branches with three tree branches. The minimum number of equations required for the solution of the network is
(a) 2 (b) 3 (c) 4 (d) 5

4.5 For a connected planar graph of v vertices and e edges, the number of meshes is
(a) $(e - v + 1)$ (b) $(e + v + 1)$ (c) $(e + v - 1)$

4.6 The number of chords of a tree of a connected graph G of v vertices and e edges is
(a) $(v - 1)$ (b) $(e - v + 1)$ (c) $(e - v - 1)$

4.7 The table meant for the oriented graph represents



Link or Loop Current	← Branch →		
	1	2	3
i_1	+1	−1	0
i_2	0	+1	+1

(a) tie-set matrix (b) cut-set matrix (c) incidence matrix (d) none of the above.

4.8 The reduced incidence matrix of a circuit is given by

$$A_i = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

The set of branches forming a tree are

(a) 1, 2 and 3 (b) 2, 3 and 5 (c) 1, 2 and 4 (d) 1, 2 and 6.

4.9 Relative to a given fixed tree of a network

1. link currents form an independent set.
2. branch currents form an independent set.
3. link voltages form an independent set.
4. branch voltages form an independent set.

Of these statements

(a) 1, 2, 3 and 4 are correct. (b) 1, 2 and 3 are correct.
(c) 2, 3 and 4 are correct (d) 1, 3 and 4 are correct

4.10 For a given network the incidence matrix is given as

$$\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The series branches in the graph are

(a) 3 and 4 (b) 6 and 7 (c) 2 and 3 (d) none of the above.

4.11 For a given network the incidence matrix is given as

$$\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The parallel branches in the graph are

(a) 1 and 2 (b) 2 and 3 (c) 6 and 7 (d) none of the above.

4.12 For a given network the incidence matrix is given as

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The series branches in the graph are

(a) 3 and 4 (b) 3 and 5 (c) 3 and 6 (d) none of the above.

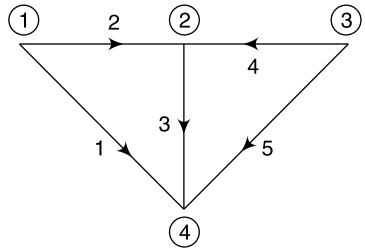
4.13 For a given network the incidence matrix is given as

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The parallel branches in the graph are

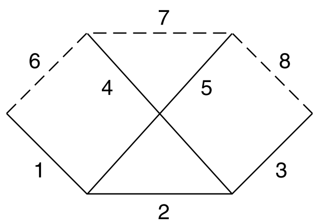
(a) 3 and 5 (b) 4 and 5 (c) 3 and 6 (d) none of the above.

4.14 Which one of the following represents the total number of trees in the graph given in the figure?



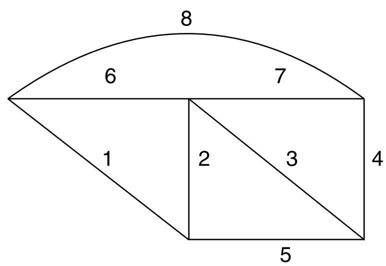
(a) 4 (b) 5 (c) 6 (d) 8

4.15 In the graph and the tree shown in the given figure, the fundamental cut-set for the branch 2 is



(a) 2, 1, 5 (b) 2, 6, 7, 8 (c) 2, 1, 3, 4, 5 (d) 2, 3, 4

4.16 In the graph shown in the figure, one possible tree is formed by the branches 4, 5, 6, 7. Then one possible fundamental cut set is



(a) 1, 2, 3, 8 (b) 1, 2, 5, 6 (c) 1, 5, 6, 8 (d) 1, 2, 3, 7, 8

4.17 Which one of the following statements is correct?

A tree in a network is a connected graph containing

(a) all the nodes only

(b) all the branches only

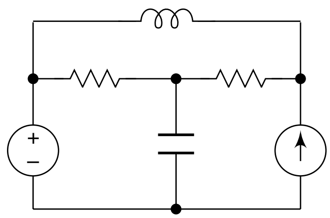
(c) all the branches and nodes
(d) all the nodes but no close path

4.18

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ A = & \begin{bmatrix} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

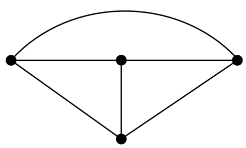
For the reduced incidence matrix given, which is the set of branches forming a tree?
(a) 1, 2, 3 (b) 2, 4, 6 (c) 2, 3, 5 (d) 1, 4, 6

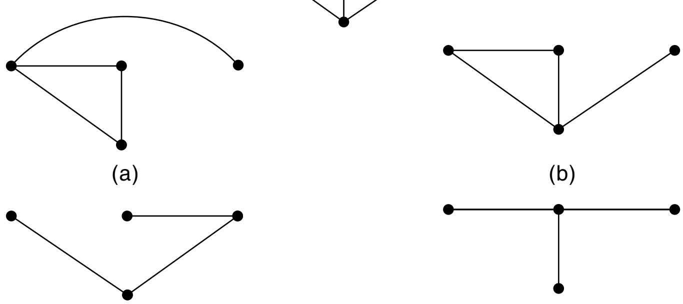
4.19 The number of chords in the graph of the given circuit will be



(a) 3 (b) 4 (c) 5 (d) 6

4.20 Consider the network graph shown in the figure. Which one of the following is NOT a 'tree' of this graph?

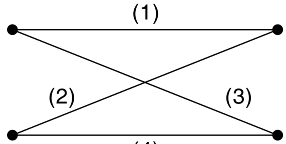




(a) (b)

(c) (d)

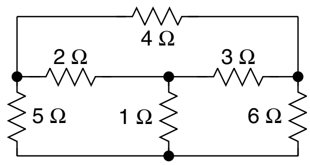
4.21 In the following graph, the number of trees (P) and the number of cut-sets (Q) are



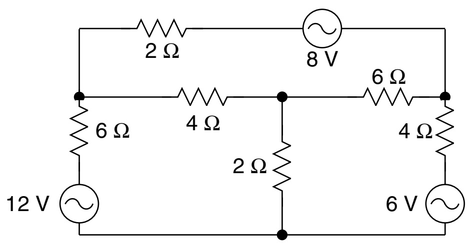
(a) $P = 2, Q = 2$ (b) $P = 2, Q = 6$ (c) $P = 4, Q = 6$ (d) $P = 4, Q = 10$

EXERCISES

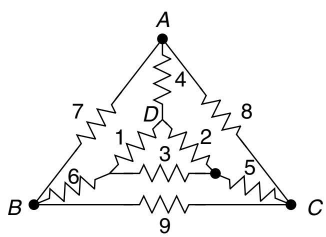
4.1 For the network shown in the figure, draw the graph and a possible tree. Show the links and write the tie-set matrix. Write the equations of the branch currents in terms of loop currents.



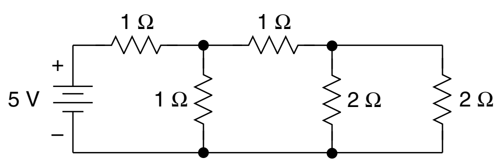
4.2 Find out the currents through and voltage across all branches of the network shown in the figure with the help of its tie-set schedule.



4.3 Find a tree from the graph of the network shown in the figure. Make the tie-set matrix and write the equations containing branch currents and loop currents. All the values are in ohms.

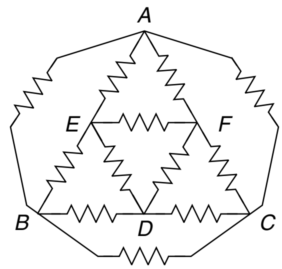


4.4 Draw the graph of the circuit shown in the figure and select a suitable tree to write tie-set matrix. Then find the three loop currents.

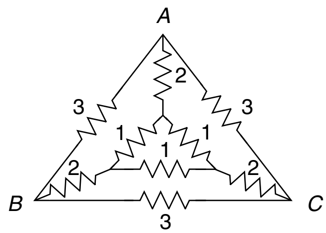


$$[i_1 = 3\text{A}, i_2 = 1\text{A}, i_3 = 0.5\text{A}]$$

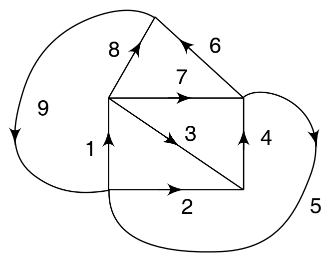
4.5 For the given network of the figure, draw the graph and a tree. Write the cutsets and the cutset matrix of the tree. Write the equations of link branch voltages in terms of tree branch voltages.



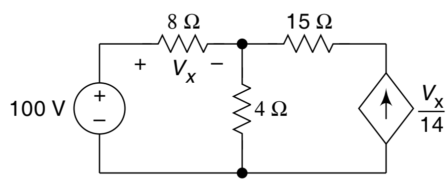
4.6 For the given network of the figure, draw the graph and a tree. Write the cutsets and the cutset matrix of the tree. Write the equations of link branch voltages in terms of tree branch voltages. All the values are in ohms.



4.7 The linear oriented graph is given in the figure Considering a tree, mark all the fundamental cutsets and form the cutset matrix.



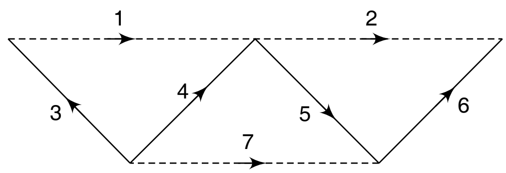
4.8 For the network shown, determine



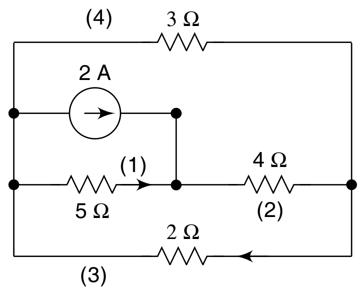
(a) tie-set matrix,
(b) loop impedance matrix, and
(c) loop currents.

[7A, 4A]

4.9 Select the (i) fundamental loops, and (ii) fundamental cutsets corresponding to a tree of the network graph which is shown by solid lines in the figure. Hence write the KCL and KVL equations for the network in matrix form.



4.10 Draw the graph of the network. Select a tree with tree branches of elements (1) and (2) and write the equilibrium equation taking tree branch voltages as variables.



4.11 The incidence matrix of a network graph is given below. Draw the oriented graph.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix}$$

SHORT-ANSWER TYPE QUESTIONS

4.1 Give the topological description of networks.

4.2 (a) Define the following terms:

(i) Graph of a network (iv) Planar and non-planar graph
(ii) Oriented graph (v) Subgraph
(iii) Rank of graph (vi) Path

(b) State the advantages offered by the graph theory as applied to electric circuit problems.

4.3 What is meant by graph? How does a graph help in circuit analysis?

4.4 (a) Define a tree of a graph of a network. Mention some basic properties of a 'Tree'. How can you calculate the number of possible trees of a given graph?

(b) Define the followings

(i) Twigs. (ii) Cotree (iii) Links or chords

4.5 Show that the number of links for a graph having n nodes and b branches is $b - n + 1$.

4.6 Show that for a network graph with P separate parts, n nodes and b branches, the number of chords C is given as, $C = b - n + P$.

4.7 Explain with illustrative examples the meaning of the following terms:
(a) Incidence matrix
(b) Tie-set matrix
(c) Cut-set matrix

4.8 (a) Explain what is meant by incidence matrix of a graph and indicate how the values of the incidence matrix elements are obtained.
(b) List the properties of an incidence matrix.
(c) How can you determine the number of possible trees of a graph with this matrix?

4.9 Show that the determinant of the complete incidence matrix of a closed loop is zero.

4.10 (a) Explain the term 'tie-set' and 'tie-set matrix' of a network with an illustrative example.
(b) Show that the matrix equation,

$$I_b = B^T I_L$$

where, B is the tie-set matrix and I_b and I_L represent branch current and loop current matrix respectively.
(c) Write the tie-set schedule and formulate the equilibrium equation on loop current basis.

4.11 (a) Define cut-set in a network graph. How can you find out a fundamental cut-set? Mention some properties of a cut-set.
(b) Define cut-set matrix with an illustrative example and show that the matrix equation $QI_b = 0$, where Q is the cut-set matrix and I_b represents the branch current matrix of the graph.
(c) Briefly discuss the relation between branch voltage matrix and node voltage matrix in terms of cut-sets.

4.12 Prove that in a linear graph, every cut-set has an even number of branches in common with every loop.

4.13 (a) Write notes on network equilibrium equation.
(b) Establish that the independent loop equations of a network can be formulated from the tie-set matrix of its graph, with an illustrative example.
(c) Establish the formulation of node equations of a network from the cut-set matrix.

4.14 Using the topological properties of a network graph, describe the step-by-step procedure of analysing a network by node voltage method.

4.15 Using the topological properties of a network graph, describe the step-by-step procedure of analysing a network by loop current method.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

4.1 (a)	4.2 (c)	4.3 (a)	4.4 (b)	4.5 (a)	4.6 (b)	4.7 (a)
4.8 (a)	4.9 (b)	4.10 (d)	4.11 (c)	4.12 (c)	4.13 (b)	4.14 (d)
4.15 (b)	4.16 (d)	4.17 (d)	4.18 (a)	4.19 (a)	4.20 (b)	4.21 (c)

CHAPTER

5
Network Theorems

5.1 INTRODUCTION

A *theorem* is a relatively simple rule used to solve a problem, derived from a more intensive analysis using fundamental rules of mathematics. At least hypothetically, any problem in mathematics can be solved just by using the simple rules of arithmetic, but human beings are not as consistent or as fast as a digital computer. We need some shortcut methods in order to avoid procedural errors.

In electric network analysis, the fundamental rules are Ohm's Law and Kirchhoff's Laws. While these humble laws may be applied to analyse any circuit configuration, for complex circuits, it is sometimes necessary to simplify the network to find current or voltage in a particular branch without solving the entire circuit. For this purpose, there are some 'shortcut' methods of analysis, known as *Network Theorem*. As with any theorem of geometry or algebra, the network theorems are also derived from fundamental rules.

5.2 NETWORK THEOREMS

In this chapter, we will discuss the following network theorems:

1. Substitution Theorem
2. Superposition Theorem
3. Reciprocity Theorem
4. Thevenin's Theorem
5. Norton's Theorem
6. Maximum Power Transfer Theorem
7. Millman's Theorem

5.2.1 Superposition Theorem

Statement This theorem states that in a linear bilateral network, the current at any point (or voltage between any two points) due to the simultaneous action of a number of independent sources in the network is equal to the summation of the component currents (or voltages). A component current (or voltage) is defined as that due to one source acting alone in the network with all the remaining sources removed.

Proof

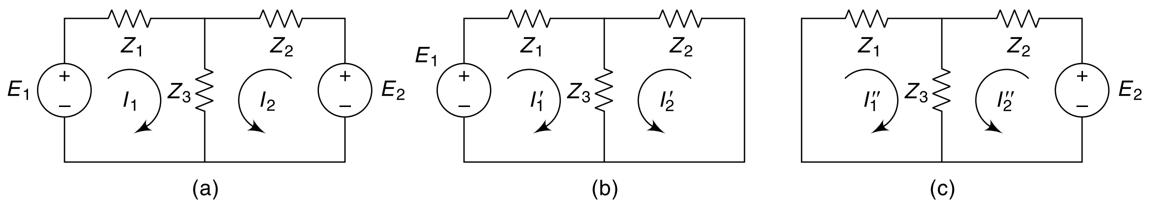


Figure 5.1 *Proof of Superposition Theorem*

Using KVL for the above network, as shown in Fig. 5.1(a)

$$E_1 = I_1(Z_1 + Z_3) + I_2 Z_3$$

$$E_2 = I_1 Z_3 + I_2(Z_2 + Z_3)$$

Solving above two equations,

$$I_1 = \frac{Z_2 + Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_1 - \frac{Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_2$$

$$I_2 = \frac{-Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_1 + \frac{Z_1 + Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_2$$

Making E_2 inoperative then the circuit diagram becomes as shown in Fig. 5.1(b)
Then the KVL equations are,

$$E_1 = I_1'(Z_1 + Z_3) + I_2' Z_3$$

$$0 = I_1' Z_3 + I_2'(Z_2 + Z_3)$$

Solving above two equations,

$$I_1' = \frac{Z_2 + Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_1$$

$$I_2' = \frac{-Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_1$$

Making E_1 inoperative then the circuit diagram becomes as shown in Fig. 5.1(c)
Then the KVL equations are,

$$0 = I_1''(Z_1 + Z_3) + I_2'' Z_3$$

$$E_2 = I_1'' Z_3 + I_2''(Z_2 + Z_3)$$

Solving above two equations,

$$I_1'' = \frac{-Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_2$$

$$I_2'' = \frac{Z_2 + Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_2$$

So, $\quad I_1 = I_1' + I_1''\,,\ \ I_2 = I_2' + I_2''$ (Proved)

If an excitation $e_1(t)$ alone gives a response $r_1(t)$, and an excitation $e_2(t)$ alone gives a response $r_2(t)$, then, by superposition theorem, the excitation $e_1(t)$ and the excitation $e_2(t)$ together would give a response $r(t) = r_1(t) + r_2(t)$.

The superposition theorem can even be stated in a more general manner, where the superposition occurs with scaling.

Thus an excitation of $k_1\ e_1(t)$ and an excitation of $k_2\ e_2(t)$ occurring together would give a response of $k_1\ r_1(t) + k_2\ r_2(t)$.

Steps to Apply Superposition Theorem

1. Only one source is considered to act alone. The other sources are replaced by their internal impedances, i.e., ideal independent voltage sources are short-circuited and ideal independent current sources are open-circuited. All dependent sources will act normally.
2. Using any suitable network analysis technique, the current through or the voltage across the desired element is found out due to the source under consideration.
3. The above steps are repeated considering all the independent sources one by one.
4. The total response (current or voltage) is obtained by taking the algebraic sum of all the responses.

Points to be noted

(i) This theorem is valid for all types of linear circuits having time-varying or time-invariant elements.
(ii) This theorem is used to find the current or voltage in a branch when the circuit has a large number of independent sources.
(iii) This theorem is not valid for power relationship.
(iv) This theorem is not applicable to circuits containing only dependent sources. With dependent sources, superposition can be used only when the controlling functions are external to the network containing sources, so that the controls are unchanged when the sources act at a time.
(v) This theorem is not applicable for circuits with non-linear elements.
(vi) This theorem is not useful for circuit with only one independent source.

5.2.2 Thevenin's Theorem

Statement A linear active bilateral network can be replaced at any two of its terminals, by an equivalent voltage source (Thevenin's Voltage source), V_{oc}, in series with an equivalent Impedance (Thevenin's impedance), Z_{th}.

Here, V_{oc} is the open circuit voltage between the two terminals under the action of all sources and initial conditions, and Z_{th} is the impedance obtained across the terminals with all sources removed by their internal impedance and initial conditions reduced to zero.

Proof We consider a linear active circuit of Fig. 5.2(a). An external current source is applied through the terminals *a–b* where we have access to the circuit.

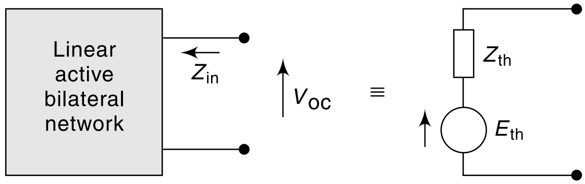


Figure 5.2 *Illustration of Thevenin's Theorem*

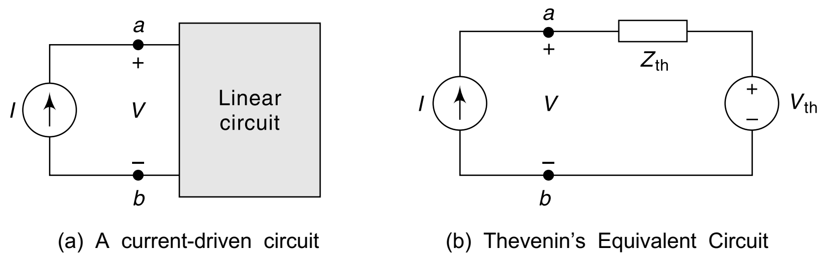


Figure 5.3

We have to prove that the v–i relation at terminals a–b of Fig. 5.3(a) is identical with that of the Thevenin's Equivalent circuit of Fig. 5.3(b).

For simplicity, we assume that the circuit contains two independent voltage sources V_{s1} and V_{s2} and two independent current sources I_{s1} and I_{s2}.

Considering the contribution due to each independent source including the external one, the voltage at a–b, V, is, by Superposition theorem,

$$V = K_0\, I + K_1\, V_{s1} + K_2\, V_{s2} + K_3\, I_{s1} + K_4\, I_{s2}$$

where, K_0, K_1, K_2, K_3, K_4 are constants.

or $$V = K_0\, I + P_0 \tag{5.1}$$

where, $P_0 = K_1\, V_{s1} + K_2\, V_{s2} + K_3\, I_{s1} + K_4\, I_{s2}$ = Total Contribution due to internal independent sources

To evaluate the constants K_0 and P_0 of equation (5.1), two conditions are to be noted.

(i) *When the terminals a and b are open-circuited*

$$I = 0, \text{ and } V = V_{oc} = V_{th}$$

From equation (5.1), $V_{th} = V_{oc} = P_0 \quad \Rightarrow \quad \boxed{V_{th} = P_0}$

(ii) *When all the internal sources are turned off*

$P_0 = 0$ and the equivalent impedance is Z_{th}.

From equation (5.1), $V = K_0 I$

or $$\frac{V}{I} = K_0 = Z_{th} \quad \Rightarrow \quad \boxed{K_0 = Z_{th}}$$

Thus, substituting the values of K_0 and P_0, the v–i relation becomes,

$$\boxed{V = Z_{th} I + V_{th}}$$

This represents the v–i relationship of Fig. 5.5(b). So, Thevenin's theorem is proved.

Points to be noted

(i) This theorem is very useful for replacement of a large portion of a network with a small equivalent circuit. This is useful for calculating the load resistance in impedance-matching problems.
(ii) This theorem is applicable to any linear, bilateral, active network.
(iii) To apply this theorem, the load branch should not be magnetically coupled to any other branch in the circuit and the load should not contain any dependent source.
(iv) This theorem is inapplicable to non-linear and unilateral networks.
(v) This theorem is inapplicable for active load.

5.2.3 Norton's Theorem

Statement A linear active bilateral network can be replaced at any two of its terminals, by an equivalent current source (Norton's current source), I_{sc}, in parallel with an equivalent admittance (Norton's admittance), Y_N.

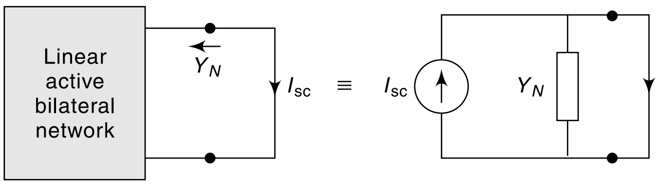


Figure 5.4 *Illustration of Norton's Theorem*

Here, I_{sc} is the short circuit current flowing from one terminal to the other under the action of all sources and initial conditions, and Y_N is the admittance obtained across the terminals with all sources removed by their internal impedance and initial conditions reduced to zero.

Proof We consider a linear active circuit of Fig. 5.5(a). An external voltage source is applied through the terminals *a–b* where we have access to the circuit.

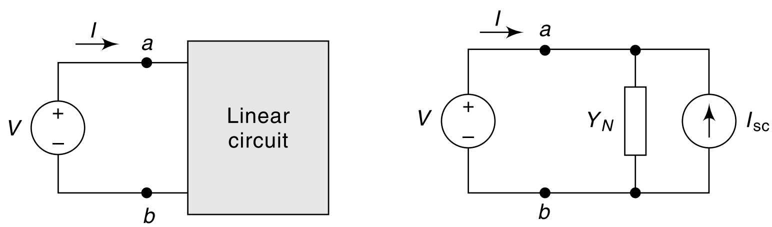


Figure 5.5 *(a) A Voltage-driven Circuit (b) Norton's Equivalent Circuit*

We have to prove that the *v–i* relation at terminals *a-b* of Fig. 5.5(a) is identical with that of the Norton's Equivalent circuit of Fig. 5.5(b).

For simplicity, we assume that the circuit contains two independent voltage sources V_{s1} and V_{s2} and two independent current sources I_{s1} and I_{s2}.

Considering the contribution due to each independent source including the external one, the current entering at a, I, is, by Superposition theorem,

$$I = K_0\ V + K_1\ V_{s1} + K_2\ V_{s2} + K_3\ I_{s1} I_{s1} + K_4\ I_{s2}$$

where, K_0, K_1, K_2, K_3, K_4 are constants.

or $$I = K_0\ V + P_0 \tag{5.2}$$

where, $P_0 = K_1\ V_{s1} + K_2\ V_{s2} + K_3\ I_{s1} + K_4\ I_{s2}$

= Total contribution due to internal independent sources

To evaluate the constants K_0 and P_0 of equation (5.2), two conditions are:

(iii) *When the terminals a and b are short-circuited*

$$V = 0, \text{ and } I = -I_{sc} = I_N$$

From equation (5.2), $-I_{sc} = P_0 \quad \Rightarrow \quad \boxed{I_{sc} = -P_0}$

(iv) *When all the internal sources are turned off*

$P_0 = 0$ and the equivalent admittance is Y_N.

From equation (5.2), $I = K_0 V$

or $$\frac{I}{V} = K_0 = Y_N \quad \Rightarrow \quad \boxed{K_0 = Y_N}$$

Thus, substituting the values of K_0 and P_0, the v–i relation becomes,

$$\boxed{I = VY_N - I_N}$$

This represents the v–i relationship of Fig. 5.5(b). So, Norton's theorem is proved.

Points to be noted

(i) This theorem is very useful for replacement of a large portion of a network with a small equivalent circuit. This is useful for calculating the load resistance in impedance-matching problems.

(ii) This theorem is applicable to any linear, bilateral, active network.

(iii) To apply this theorem, the load branch should not be magnetically coupled to any other branch in the circuit and the load should not contain any dependent source.

(iv) This theorem is inapplicable to non-linear and unilateral networks.

(v) This theorem is inapplicable for active load.

Steps for Determination of Thevenin's/Norton's Equivalent Circuit

1. The portion of the network across which the Thevenin's or Norton's equivalent circuit is to be found out is removed from the network.
2. (a) The open circuit voltage (V_{oc} or V_{th}) is calculated keeping all the sources at their normal values.

   (b) The short circuit current (I_{sc} or I_N) flowing from one terminal to the other is calculated keeping all the sources at their normal values.
3. Calculation of Z_{th} or Y_N.

   Case I. *When circuit contains only independent sources, the following points are to be noted.*

   All voltage sources are short-circuited.

   All current sources are open-circuited.

Equivalent impedance or admittance is calculated looking back to the circuit with respect to the two terminals.

Case II. *When circuit contains both dependent and independent sources,the following points are to be noted.*

Open circuit voltage (V_{oc}) is calculated with all sources alive.

Short circuit current (I_{sc}) is calculated with all sources alive.

Thevenin's impedance is obtained as, $Z_{th} = \dfrac{V_{oc}}{I_{sc}} = \dfrac{1}{Y_N}$

Case III. *When circuit contains only dependent sources the following points are to be noted.*

In this case, $V_{oc} = 0$.

We connect a test voltage (or current) source at the terminals *a* and *b* and the current flowing through *a–b* (voltage drop between the terminals *a–b*) is calculated.

Thevenin's impedance is obtained as, $Z_{th} = \dfrac{V_{test}}{I_{test}} = \dfrac{1}{Y_N}$

4. Finally, Thevenin's equivalent circuit is obtained by placing V_{oc} in series with Z_{th} and Norton's equivalent is obtained by placing I_{sc} in parallel with Y_N.

5.2.4 Maximum Power Transfer Theorem

Statement Maximum power is absorbed by one network from another connected to it at two terminals, when the impedance of one is the complex conjugate of the other.

This means that for maximum active power to be delivered to the load, load impedance must correspond to the conjugate of the source impedance (or in the case of direct quantities, be equal to the source impedance).

The statement and proof of this theorem are discussed for four different cases:

Case I : Purely Resistive Circuit with Variable Load Resistance
Case II : Load Impedance with Variable Resistance and Variable Reactance
Case III : Load Impedance with Variable Resistance and Fixed Reactance
Case IV : Load Impedance with Fixed Ratio, i.e., with Variable Magnitude but Fixed Angle

Case I. *Purely Resistive Circuit with Variable Load Resistance.*

In this case, the statement of this theorem is given as, *maximum power will be delivered from a source to a load when the load resistance is equal to the source resistance.*

Proof Let V be the voltage source, R_S the internal resistance of the source and R_L the load resistance.

∴ current,

∴ power delivered to the load is,

$$P = |I|^2 R_L = \frac{V^2 R_L}{(R_S + R_L)^2} \tag{5.3}$$

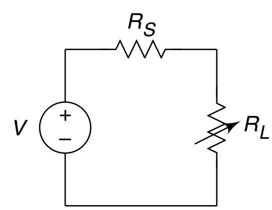


Figure 5.6(a) *Purely resistive circuit with varible load resistance*

For maximum power,

$$\frac{\partial P}{\partial R_L} = 0$$

$$\Rightarrow \qquad V^2\left[\frac{(R_L + R_S) - 2R_L(R_S + R_L)}{(R_S + R_L)^4}\right] = 0$$

$$\therefore \qquad R_S = R_L$$

i.e., the load resistance is equal to the source resistance.

Putting $R_L = R_S$ in Eq. (5.3), the maximum power transferred will be $P_{\max} = \frac{V^2}{4R_L} = \frac{(V/2)^2}{R_L}$ and thus, the efficiency will be 50%. This case arises in a purely dc circuit.

Case II. *Load Impedance with Variable Resistance and Variable Reactance.*
In this case, the statement of the theorem is given as, *maximum power will be delivered from a source to a load when the load impedance is the complex conjugate of the source impedance.*

Proof Let V be the voltage source, $(R_S + jX_S)$ the internal impedance of the source and $(R_L + jX_L)$ the load impedance.

$$\therefore \qquad \text{current, } I = \frac{V}{Z_S + Z_L} = \frac{V}{(R_S + R_L) + j(X_S + X_L)} \tag{5.4}$$

Power delivered to the load is,

$$P = |I|^2 R_L = \frac{V^2 R_L}{(R_S + R_L)^2 + (X_S + X_L)^2} \tag{5.5}$$

where, $\qquad Z_S = R_S + jX_S\ ,\ Z_L = R_L + jX_L$

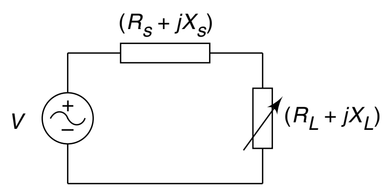


Figure 5.6(b) *Load Impedance with Variable Resistance and Variable Reactance*

For maximum power, $\frac{\partial P}{\partial X_L}$ must be zero.

Now, $$\frac{\partial P}{\partial X_L} = \frac{-2(V)^2 R_L (X_L + X_S)}{\left[(R_L + R_S)^2 + (X_L + X_S)^2\right]^2} = 0$$

From which, $\qquad X_L + X_S = 0$ or $X_L = -X_S$

i.e., the reactance of the load impedance is of opposite sign to the reactance of the source impedance.

Putting $\qquad X_L = -X_S$ in equation (5.5) $P = \frac{V^2 R_L}{(R_L + R_S)^2}$

For maximum power, $\dfrac{\partial P}{\partial R_L} = \dfrac{V^2(R_L+R_S)^2 - 2V^2 R_L(R_L+R_S)}{(R_L+R_S)^4} = 0$

or, $V^2(R_L+R_S) - 2V^2 R_L = 0$ or $R_L = R_S$

The maximum power transferred will be $P_{\max} = \dfrac{V^2}{4R_L} = \dfrac{(V/2)^2}{R_L}$ and thus, the efficiency will be 50%.

Case III. *Load Impedance with Variable Resistance and Fixed Reactance.*
Maximum power transfer in this case takes place under certain conditions as obtained below.
Here, the current,

$$I = \frac{V}{Z_S + Z_L} = \frac{V}{(R_S+R_L) + j(X_S+X_L)} \tag{5.6}$$

Power delivered to the load is,

$$P = |I|^2 R_L = \frac{V^2 R_L}{(R_S+R_L)^2 + (X_S+X_L)^2} \tag{5.7}$$

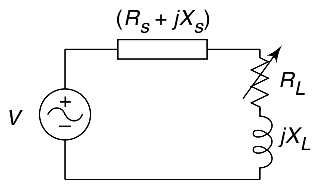


Figure 5.6(c) *Load Impedance with Variable Resistance and Fixed Reactance*

where, $Z_S = R_S + jX_S$, $Z_L = R_L + jX_L$

For maximum power,

$$\frac{\partial P}{\partial R_L} = 0$$

$$V^2\left[\frac{(R_S+R_L)^2 + (X_S+X_L)^2 - 2R_L(R_S+R_L)}{\left[(R_S+R_L)^2+(X_S+X_L)^2\right]^2}\right] = 0$$

$\Rightarrow$ $$R_L^2 = R_S^2 + (X_S+X_L)^2$$

$\Rightarrow$ $$R_L = \sqrt{R_S^2 + (X_S+X_L)^2}$$

Case (*a*) If the source impedance is purely resistive, i.e., $X_S = 0$, the condition for maximum power transfer becomes,

$$R_L = \sqrt{R_S^2 + X_L^2}$$

Case (*b*) If the load impedance is purely resistive, i.e., $X_L = 0$, the condition for maximum power transfer becomes,

$$R_L = \sqrt{R_S^2 + X_S^2} = |Z_S|$$

i.e., the load resistance is equal to the source impedance.

Case IV. *Load Impedance with Fixed Ratio, i.e., with Variable Magnitude but Fixed Angle.* In this case, the statement of the theorem is given as, *maximum power is delivered from a source to a load when the magnitude of the load impedance is equal to the magnitude of the source impedance.*

Proof Let the angle of the load impedance be, ϕ.

$$\therefore \qquad Z_L = |Z_L|\cos\phi + j|Z_L|\sin\phi$$

$\therefore$ Power delivered to the load is,

$$P = \frac{V^2|Z_L|\cos\phi}{\left(R_S + |Z_L|\cos\phi\right)^2 + \left(X_S + |Z_L|\sin\phi\right)^2}$$

For maximum power transfer,

$$\frac{dP}{d|Z_L|} = 0$$

$$\Rightarrow \qquad \frac{d}{d|Z_L|}\left[\frac{V^2|Z_L|\cos\phi}{\left(R_S + |Z_L|\cos\phi\right)^2 + \left(X_S + |Z_L|\sin\phi\right)^2}\right] = 0$$

Simplifying we get,

$$|Z_L|^2 = R_S^2 + X_S^2 = |Z_S|^2$$

$$\therefore \qquad |Z_L| = |Z_S|$$

This case arises in a transformer where the turns ratio is varied for maximum power transfer.

Points to be noted

(i) It is to be noted that when maximum power is being transferred, only half the applied voltage is available to the load and the other half drops across the source. Also, under these conditions, half the power supplied is wasted as dissipation in the source.

Thus, the useful maximum power will be less than the theoretical maximum power derived and will depend on the voltage required to be maintained at the load.

(ii) For circuits having a resistive load being supplied from a source with only an internal resistance (the case for d.c.), the maximum power will be transferred to the load when the load resistance is equal to the source resistance.

Concept of Internal Resistance of Voltage and Current Sources A **voltage source** is any device or system that produces an electromotive force between its terminals. An example of a primary source is a common battery. Similarly, a **current source** is an electrical or electronic device that delivers electric current. Examples of current sources are a large voltage source in series with a large resistor (however, this type of current source has very poor efficiency), an active current source involving transistors, high voltage current source like Van de Graff generator, etc. A current source is the dual of a voltage source.

In circuit theory, an **ideal voltage source** is a circuit element where the voltage across it is independent of the current through it. It only exists in mathematical models of circuits. The internal resistance of an ideal voltage source is zero; it is able to supply any amount of current. The current through an ideal voltage source is completely determined by the external circuit. When connected to an open circuit, there is zero current and thus zero power. When connected to a load resistance, the current through the source approaches infinity as the load resistance approaches zero (a short circuit). Thus, an ideal voltage source can supply unlimited power.

Similarly, an **independent current source** with zero current is identical to an ideal open circuit. For this reason, the internal resistance of an ideal current source is infinite. The voltage across an ideal current source is completely determined by the circuit it is connected to. When connected to a short circuit, there is zero voltage and thus zero power delivered. When connected to a load resistance, the voltage across the source approaches infinity as the load resistance approaches infinity (an open circuit). Thus, an ideal current source can supply unlimited power forever and so represents an unlimited source of energy. Connecting an ideal open circuit to an ideal non-zero current source is not valid in circuit analysis as the circuit equation would be paradoxical, e.g., 3 = 0.

However, no real voltage source is ideal; all have a non-zero effective internal resistance, and none can supply unlimited current. *The internal resistance of a real voltage source is effectively modeled in linear circuit analysis by combining a non-zero resistance in series with an ideal voltage source.* Similarly, no real current source is ideal (no unlimited energy sources exist) and all have a finite internal resistance (none can supply unlimited voltage). *The internal resistance of a physical current source is effectively modeled in circuit analysis by combining a non-zero resistance in parallel with an ideal current source.*

5.2.5 Millman's Theorem

Consider a number of admittances Y_1, Y_2, Y_3 …Y_p… Y_q,…Y_n are connected together at a common point S. If the voltages of the free ends of the admittances with respect to a common reference N are known to be V_{1N}, V_{2N}, V_{3N} …V_{pN}…V_{qN},…V_{nN}, then Millman's theorem gives the voltage of the common point S with respect to the reference N, as follows.

Applying Kirchhoff's Current law at node S,

$$\sum_{p=1}^{n} I_p = 0, I_p = Y_p(V_{pN} - V_{sN})$$

or

$$\sum_{p=1}^{n} Y_p(V_{pN} - V_{sN}) = 0$$

or

$$\sum_{p=1}^{n} Y_p V_{pN} = V_{sN} \sum_{p=1}^{n} Y_p$$

$$\boxed{\Rightarrow \quad V_{sN} = \frac{\sum_{p=1}^{n} Y_p V_{pN}}{\sum_{p=1}^{n} Y_p}}$$

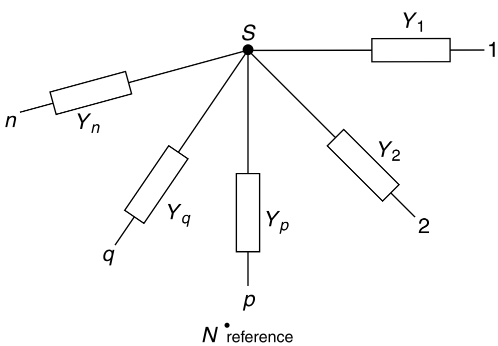


Figure 5.7 *Illustration of Millman's Theorem*

An extension of the Millman's theorem is the *equivalent generator theorem.*

Statement

(I) This theorem states that if several ideal voltage sources (V_1, V_2, …) in series with impedances (Z_1, Z_2,…) are connected in parallel , then the circuit may be replaced by a single ideal voltage source (V) in series with an impedance (Z) such that

$$V = \frac{\sum_{i=1}^{n} V_i Y_i}{\sum_{i=1}^{n} Y_i} \quad \text{and} \quad Z = \frac{1}{\sum_{i=1}^{n} Y_i}$$

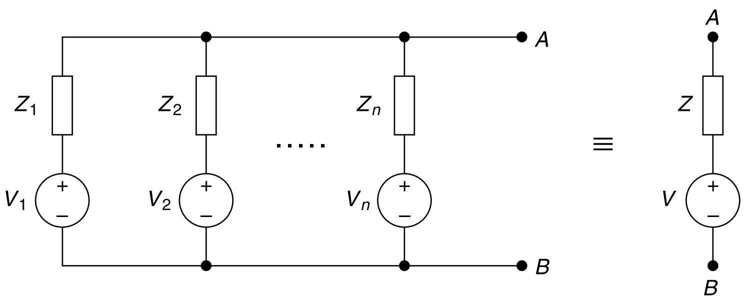


Figure 5.8 *Voltage Source Equivalent using Millman's Theorem*

(II) If several ideal current sources (I_1, I_2,…) in parallel with impedances (Z_1, Z_2…) are connected in series, they may be replaced by a single ideal current source (I) in parallel with an impedance (Z) such that

$$I = \frac{\sum_{i=1}^{n} I_i / Y_i}{\sum_{i=1}^{n} 1/Y_i} \quad \text{and,} \quad Y = \frac{1}{\sum_{i=1}^{n} 1/Y_i} \quad \text{or,} \quad Z = \sum_{i=1}^{n} Z_i$$

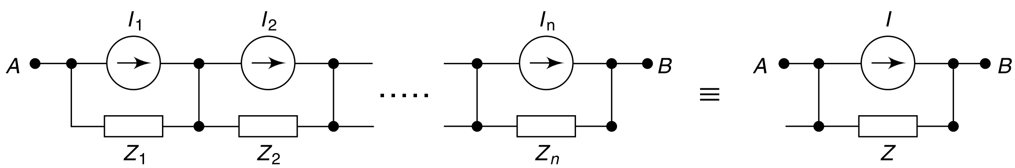


Figure 5.9 *Current Source Equivalent using Millman's Theorem*

Proof

(I) Using Superposition theorem, the short circuit current through A–B considering only one source acting alone and replacing other sources by their internal impedances, (i.e., short circuit for ideal voltage sources),

$$I_{sc1} = V_1 Y_1$$

$$I_{sc2} = V_2 Y_2$$

$$I_{scn} = V_n Y_n$$

Total short circuit current through A–B, $I_{sc} = (I_{sc1} + I_{sc2} + \dots + I_{sc\ n})$

$$= V_1\ Y_1 + V_2\ Y_2 + \dots + V_n\ Y_n$$

$$= \sum_{i=1}^{n} V_i Y_i \tag{5.8}$$

Impedance looking back from A–B with all the sources removed,

$$Z = \frac{1}{Y_1 + Y_2 + \dots + Y_n} = \frac{1}{\sum_{i=1}^{n} Y_i} \tag{5.9}$$

Thus, by Thevenin's theorem, the equivalent voltage is,

$$V = I_{sc} \cdot Z = \frac{\sum_{i=1}^{n} V_i Y_i}{\sum_{i=1}^{n} Y_i} \tag{5.10}$$

Form equations (5.8), (5.9) and (5.10), Millman's Theorem is proved.

(II) Using Superposition theorem, the short circuit current through A–B considering only one source acting alone and replacing other sources by their internal impedances, (i.e., open circuit for ideal current sources),

$$I_{sc1} = \frac{I_1 Z_1}{\sum_{i=1}^{n} Z_i};\ I_{sc2} = \frac{I_2 Z_2}{\sum_{i=1}^{n} Z_i};\ \dots\ I_{sc\,n} = \frac{I_n Z_n}{\sum_{i=1}^{n} Z_i}$$

Total short circuit current, $I_{sc} = I = (I_{sc1} + I_{sc2} + \dots + I_{sc\ n})$

$$I = \frac{\sum_{i=1}^{n} I_i Z_i}{\sum_{i=1}^{n} Z_i} \tag{5.11}$$

Impedance looking back from A–B with all the sources removed,

$$Z = \sum_{i=1}^{n} Z_i \tag{5.12}$$

From equation (5.11) and (5.12), Millman's theorem is proved.

Points to be noted

(i) This theorem provides the equivalent circuits which are either Thevenin or Norton equivalent circuits.

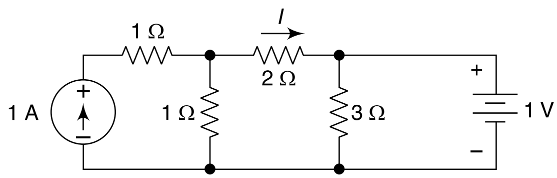
(ii) This theorem is applicable only to independent voltage sources with their internal series impedances connected directly in parallel or independent current sources with their internal series admittances connected directly in series.

(iii) This theorem is not applicable to circuits where impedances or dependent sources are present between the independent sources.
(iv) This theorem is not useful for circuits with less than two independent sources.

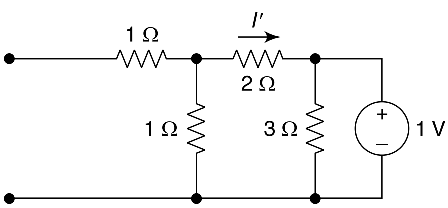
SOLVED PROBLEMS

Superposition Theorem

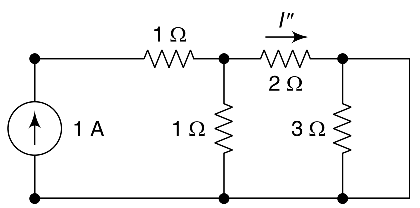
5.1



Find the current I in the circuit shown in the figure. using superposition theorem.
Solution



(i) Voltage Source acting alone



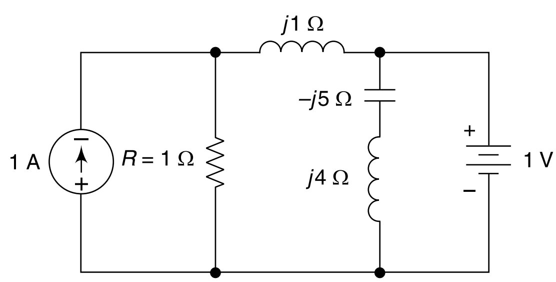
(ii) Current Source acting alone

For Figure (i) $I' = -\dfrac{1}{3}$ A

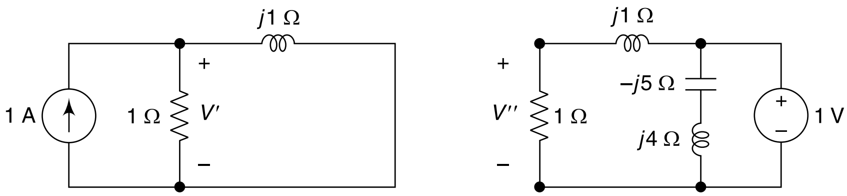
For Figure (ii) $I'' = 1 \times \dfrac{1}{1+2} = \dfrac{1}{3}$ A

By superposition, $I = (I' + I'') = -\dfrac{1}{3} + \dfrac{1}{3} = 0$ *Ans.*

5.2



Calculate the voltage V across the resistor R by using superposition theorem.
Solution



(i) Circuit with current source acting alone *(ii) Circuit with voltage source acting alone*

For Figure (i), $V' = \dfrac{j}{1+j}$

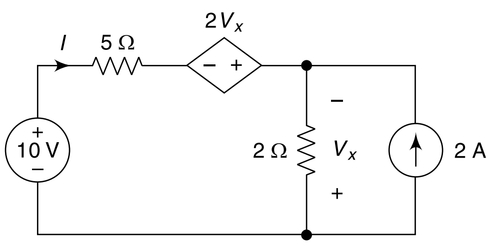
For Figure (ii),current through the resistor $I'' = \dfrac{1}{1+j}$

$\therefore \qquad V'' = I'' \times 1 = \dfrac{1}{1+j}$

So, by superposition theorem

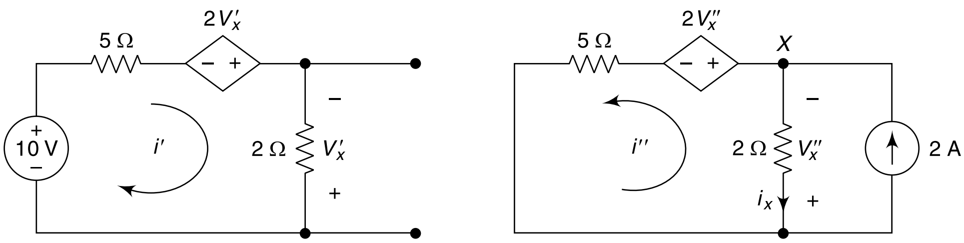
$$V = (V' + V'') = \frac{j}{1+j} + \frac{1}{1+j} = 1\text{ V}$$

5.3



Use superposition theorem on the circuit shown in figure to find I.

Solution



(i) Voltage source acting alone *(ii) Current source acting alone*

For Fig. (i), by KVL, $5i' - 2vx' + 2i' = 10$ with $v_x' = -2i'$

$\Rightarrow \qquad 7i' + 4i' = 10$

$\Rightarrow \qquad i' = 10/11\text{ A}$

For Fig (ii), by KCL at node (x)

$$2 = i_x + i'' = -\frac{v_x''}{2} + i'' \qquad \text{(i)}$$

But loop analysis in the left loop gives,

$$5i'' + 3v_x'' = 0$$

or $$i'' = -\frac{3}{5}v_x''$$

From (i), $$2 = -\frac{v_x''}{2} - \frac{3}{5}v_x''$$

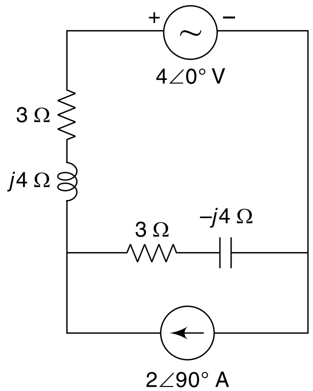
$\Rightarrow$ $$v_x'' = -\frac{20}{11}$$

$\therefore$ $$i'' = -\frac{3}{5}\times\left(-\frac{20}{11}\right) = \frac{12}{11}\text{ A}$$

So, by superposition theorem total current

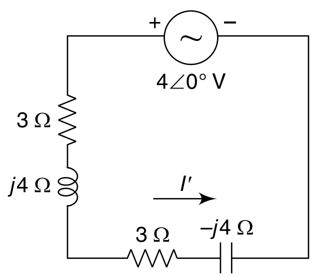
$$I = (i' - i'') = \left(\frac{10}{11} - \frac{12}{11}\right) = -\frac{2}{11}\text{ A}$$

5.4 *Determine the current in the capacitor branch by superposition theorem.*

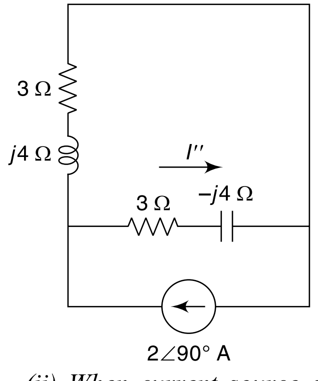


Solution *When the voltage source is acting alone:*
Here, the current in the capacitor branch is,

$$I' = \frac{4\angle 0°}{(3 + j4) + (3 - j4)} = \frac{2}{3}\angle 0°\text{ A}$$



(i) When voltage source acting alone



(ii) When current source acting alone

When the current source is acting alone:
Here, the current in the capacitor branch is,

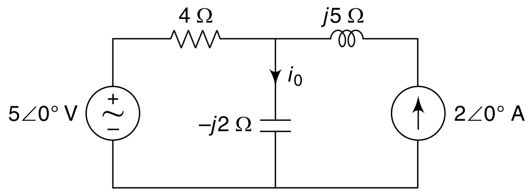
$$I'' = 2\angle 90^\circ \times \frac{(3+j4)}{(3+j4)+(3-j4)} = \left(-\frac{4}{3}+j1\right)\text{A}$$

$\therefore$ Total current when both the sources are acting simultaneously, is

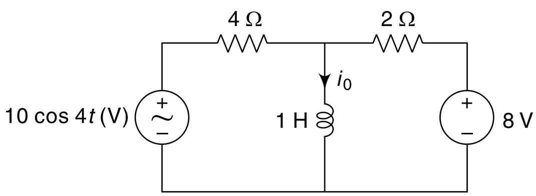
$$I = (I' + I'') = \left(\frac{2}{3} - \frac{4}{3} + j1\right) = \left(-\frac{2}{3} + j1\right) = 1.2\angle 123.7^\circ \text{ A} \qquad \textit{Ans.}$$

5.5 Find the current i_0 using superposition theorem.

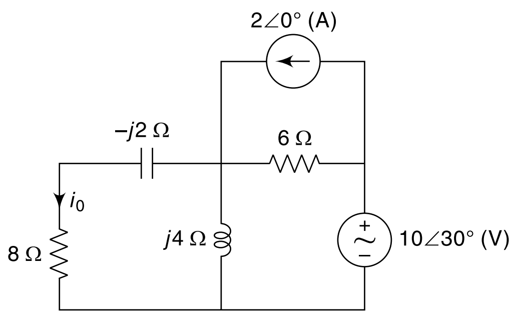
(a)



(b)



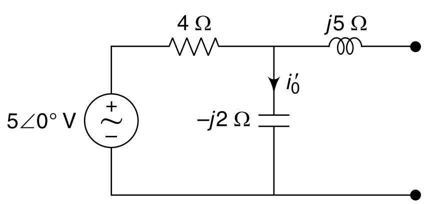
(c)



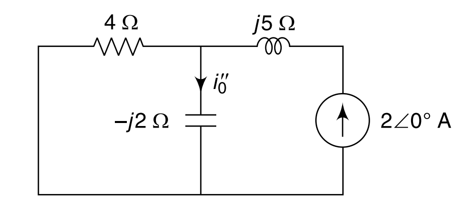
Solution

(a) *When voltage source is acting alone*

The current in this case is, $i_0' = \dfrac{5}{4-j2} = \left(1 + j\dfrac{1}{2}\right)\text{A}$



(i) Voltage source acting alone



(ii) Current source acting alone

When current source is acting alone

In this case, the current is, $i_0'' = 2\angle 0° \times \dfrac{4}{4 - j2} = \left(\dfrac{8}{5} + j\dfrac{14}{5}\right)$ A

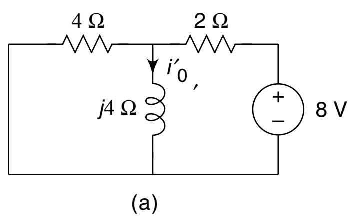
$\therefore$ By superposition theorem, total current is,

$$i_0 = (i_0' + i_0'') = \left(1 + \frac{8}{5}\right) + j\left(\frac{1}{2} + \frac{4}{5}\right) = 2.9\angle 26.56° \text{ A} \qquad \textit{Ans.}$$

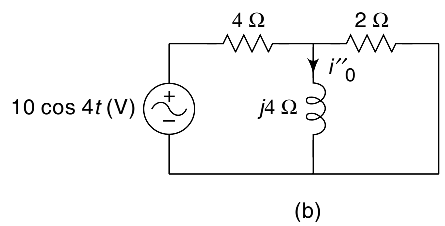
(b) *When dc source is acting alone*

Equivalent impedance, $Z = \left(\dfrac{j4 \times 4}{4 + j4} + 2\right) = \dfrac{2 + j6}{1 + j}$

$\therefore$ main current, $I = \dfrac{8}{Z} = \dfrac{8(1 + j)}{2 + j6} = \dfrac{4(1 + j)}{1 + j3}$



(i) dc source acting alone



(ii) ac source acting alone

$\therefore$ the current, $i_0' = I \times \dfrac{4}{4 + j4} = \dfrac{4(1 + j)}{1 + j3} \times \dfrac{4}{4 + j4} = \left(\dfrac{2}{5} - j\dfrac{6}{5}\right)$ A

When ac source is acting alone

Equivalent impedance, $Z = 4 + \left(\dfrac{j4 \times 2}{2 + j4}\right) = \dfrac{4 + j6}{1 + j2}$

$\therefore$ main current, $I = \dfrac{10\angle 0°}{Z} = 10\angle 0° \dfrac{(1 + j2)}{4 + j6} = \dfrac{10 + j20}{4 + j6}$

$\therefore$ the current, $i_0'' = I \times \dfrac{2}{2 + j4} = \dfrac{10(1 + j2)}{4 + j6} \times \dfrac{1}{1 + j2} = \left(\dfrac{10}{13} - j\dfrac{15}{13}\right)$ A

$\therefore$ by superposition theorem, total current is,

$$i_0 = (i_0 + i_0'') = \left(\frac{2}{5} + \frac{10}{13}\right) - j\left(\frac{6}{5} + \frac{15}{13}\right) = 2.63\angle -63.58° \text{ A} \qquad \textit{Ans.}$$

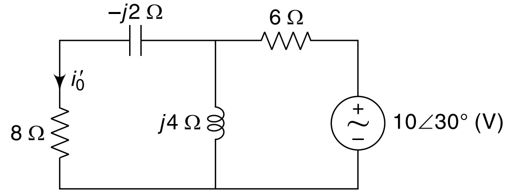
(c) *When the voltage source is acting alone*

Equivalent impedance,

$$Z = \frac{j4(8 - j2)}{8 + j2} + 6 = \frac{28 + j22}{4 + j}$$

$\therefore$ main current,

$$I = \frac{10\angle 30°(4 + j)}{28 + j22}$$

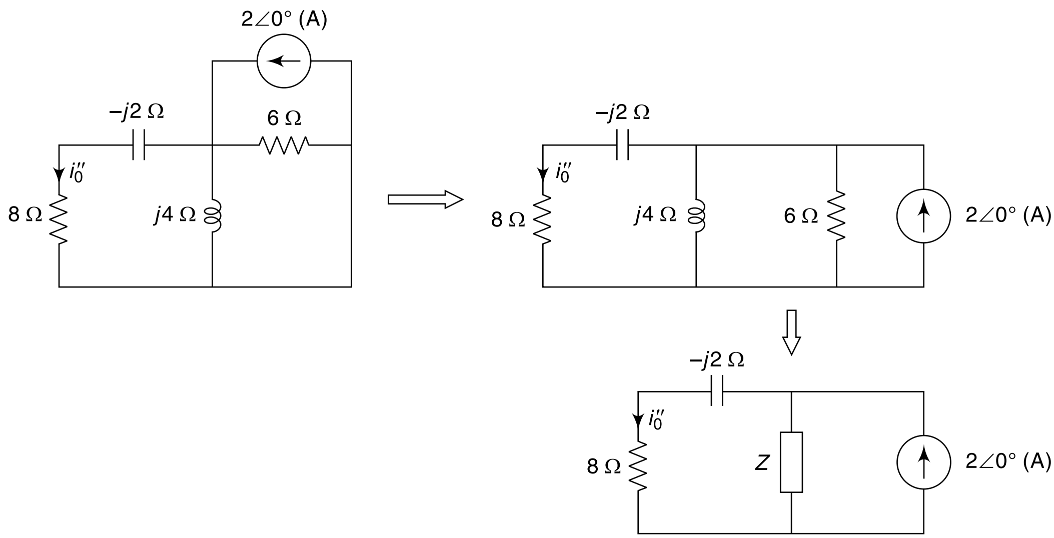


(i) Voltage source acting alone

$$= \frac{(8.66 + j5)(4 + j)}{28 + j22}$$

$\therefore$ The current, $i_0' = I \times \dfrac{8 - j2}{8 + j2} = \dfrac{8.66 + j5}{56 + j44} = 0.14\angle -8.16^\circ$ A

When current source is acting alone



Since $\qquad Z = \dfrac{j4 \times 6}{6 + j4} = \dfrac{j12}{3 + j2}$

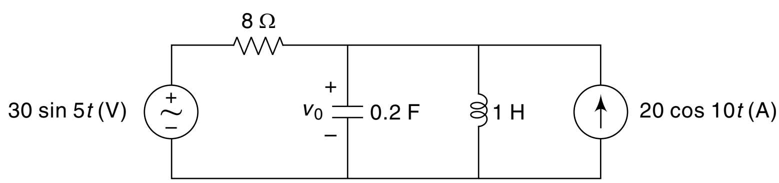
$\therefore$ The current, $i_0'' = 2\angle 0^\circ \times \dfrac{Z}{8 - j2 + Z}$

$$= \frac{j12}{12 + j11} = 0.73\angle 47.49^\circ \text{ A}$$

$\therefore$ By superposition theorem, total current is,

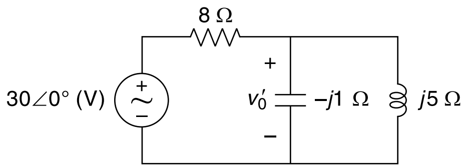
$$i_0 = (i_0' + i_0'') = \left(0.14\angle -8.16^\circ + 0.73\angle 47.49^\circ\right) = \left(0.631 + j0.518\right) = 0.81\angle 39.38^\circ \text{ A} \qquad \textit{Ans.}$$

5.6 Find v_0 using Superposition Theorem.

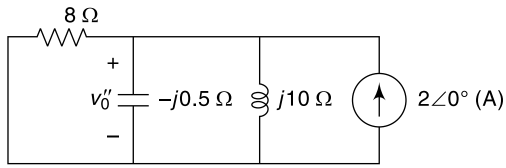


Solution *When voltage source is acting alone:*

Here, $X_C = \dfrac{-j}{5 \times 0.2} = -j1\,\Omega$ and $X_L = j \times 5 \times 1 = j5\,\Omega$



(i) Voltage source acting alone



(ii) Current soure acting alone

By KCL,

$$-\frac{30 - v_0'}{8} + \frac{v_0'}{-j1} + \frac{v_0'}{j5} = 0 \Rightarrow v_0' = \frac{30}{8(0.125 + j0.8)} = 4.631\angle -81.12^\circ \text{ (V)}$$

When current source is acting alone,

$$X_C = \frac{-j}{10 \times 0.2} = -j0.5\ \Omega \quad \text{and} \quad X_L = j \times 10 \times 1 = j10\ \Omega$$

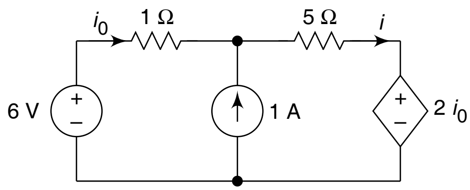
By KCL,

$$2 = v_0''\left(\frac{1}{8} + \frac{1}{j10} + \frac{1}{-j0.5}\right) \Rightarrow v_0'' = \frac{2}{0.125 + j1.9} = 1.051\angle -86.24^\circ \text{ (V)}$$

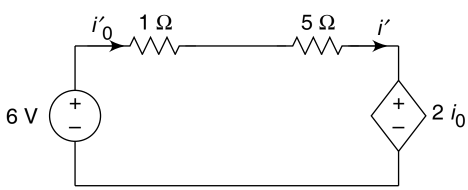
By superposition theorem, when all sources are acting simultaneously, the voltage is,

$$v_0 = (v_0' + v_0'') = 4.631 \sin(5t - 81.12^\circ) + 1.051 \cos(10t - 86.24^\circ) \text{ (V)} \qquad \textit{Ans.}$$

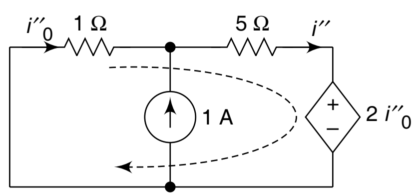
5.7 Find i_0 and i from the circuit of the figure using superposition theorem.



Solution When 6 V source is acting alone
The circuit is shown.



(i) 6 V source acting along



(ii) 1 A source acting alone

Here, $i_0' = i'$

By KVL, $6'i + 2'i = 6 \quad \Rightarrow \quad i' = i_0' = \frac{6}{8} = \frac{3}{4} A = 0.75\text{A}$

When 1 A source is acting alone

By KCL, we get,

$$1 = i'' - i_0'' \qquad \Rightarrow \qquad i'' = 1 + i_0''$$

By KVL for the supermesh,

$$1 \times i_0'' + 5i'' + 2i_0'' = 0$$

or $$3i_0'' + 5i'' = 0$$

or $$3i_0'' + 5(1 + i_0'') = 0$$

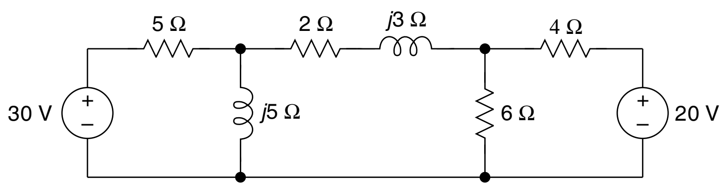
or $$i_0'' = -\frac{5}{4} = -1.25\,\text{A}$$

$\therefore$ $$i'' = 1 - 1.25 = -0.25\,\text{A}$$

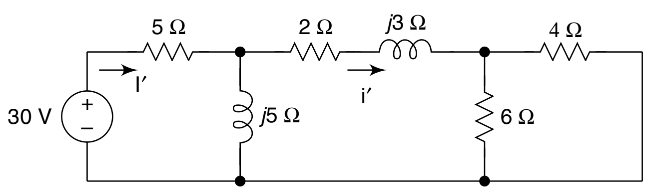
By superposition theorem, the total currents when both the sources are acting simultaneously, is given as,

$$\left.\begin{aligned} i &= (i'' + i'') = (0.75 - 0.25) = 0.5\text{A} \\ i_0 &= (i_0'' + i_0'') = (0.75 - 1.25) = -0.5\text{A} \end{aligned}\right\} \quad \textit{Ans.}$$

5.8 Using superposition theorem, calculate the current through the $(2 + j3)\,\Omega$ impedance branch of the circuit shown in the figure below.



Solution **Case I.** 30 V source is acting alone

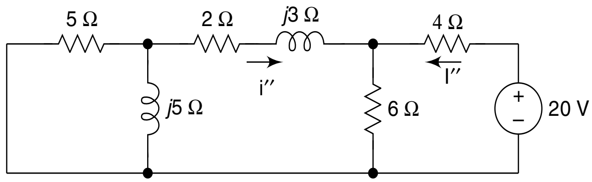


Impedance, $Z = 5 + \dfrac{(4.4 + j3) \times j5}{4.4 + j3 + j5} = (6.32 + j2.6)\,\Omega$

$\therefore$ $I' = \dfrac{30}{Z} = \dfrac{30}{6.32 + j2.6} = (4.06 - j1.67)\,\text{A}$

$i' = I' \times \dfrac{j5}{4.4 + j3 + j5} = (2.39 + j0.27)\,\text{A}$

Case II. 20 V source is acting alone



Impedance, $Z = 4 + \frac{(4.5 + j5.5) \times 6}{4.5 + j5.5 + 6} = (7.31 + j1.41)\ \Omega$

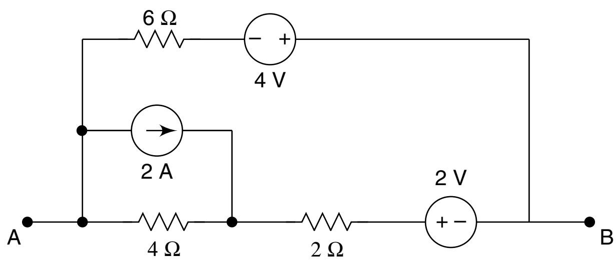
$$\therefore \qquad I'' = \frac{20}{Z} = \frac{20}{7.31 + j1.41} = (2.64 - 0.509)\,\text{A}$$

$$i'' = -I'' \times \frac{6}{4.5 + j5.5 + 6} = -(1.064 - j0.848)\ \text{A}$$

By superposition theorem, total current flowing through the $(2 + j3)$ impedance is,

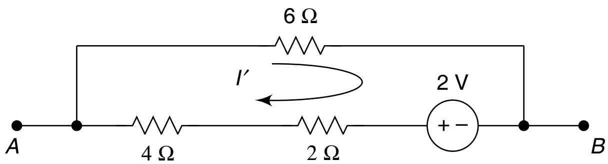
$i = (i' + i'') = (2.39 + j0.27) - (1.064 - j0.848) = (1.325 + j1.117)\ A = 1.733\angle 40.14^\circ\ \text{A}$ *Ans.*

5.9 Using Superposition theorem, find V_{AB}.



Solution We consider three cases:

Case I. 2 V source is acting alone
The circuit is shown below.



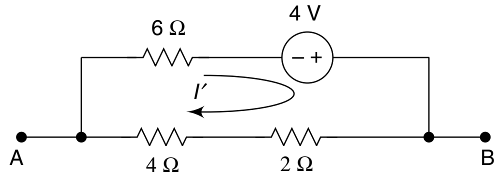
For this circuit, the current in the loop is obtained as,

$$I' = \frac{2}{12} = \frac{1}{6}\text{A}$$

$\therefore$ voltage between A and B is, $V'_{AB} = I' \times 6 = \frac{1}{6} \times 6 = 1\,\text{V}$

Case II. 4 V source is acting alone

The circuit is shown below.



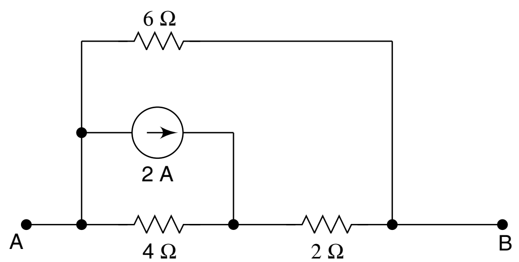
In this circuit, the loop current is obtained as,

$$I'' = \frac{4}{12} = \frac{1}{3} \text{ A}$$

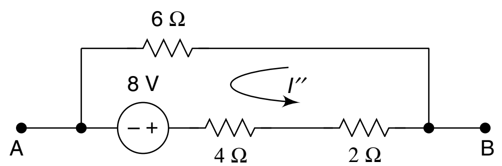
$\therefore$ voltage between A and B is, $V''_{AB} = -I'' \times 6 = -\frac{1}{3} \times 6 = -2$ V

Case III. 2 A source is acting alone

The circuit is shown below.



We convert the current source into its equivalent voltage source as shown in the figure.



The loop current is,

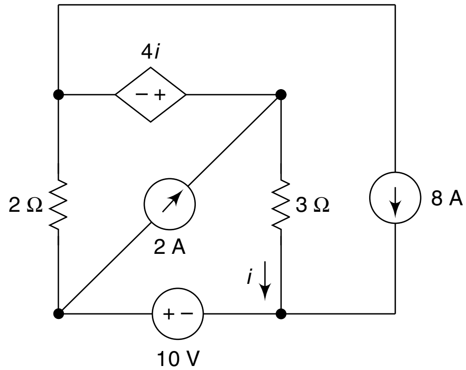
$$I''' = \frac{8}{12} = \frac{2}{3} \text{ A}$$

$\therefore$ voltage between A and B is, $V'''_{AB} = -I''' \times 6 = -\frac{2}{3} \times 6 = -4$ V

$\therefore$ voltage between A and B when all the sources are acting simultaneously is given by superposition theorem as,

$$V_{AB} = V'_{AB} + V''_{AB} + V'''_{AB} = (1 - 2 - 4) = -5 \text{ V}$$

5.10 Find the current i in the circuit shown in the figure using superposition theorem.

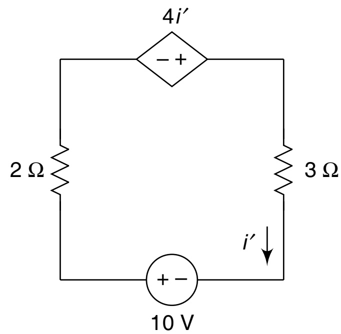


Solution We consider the three cases:

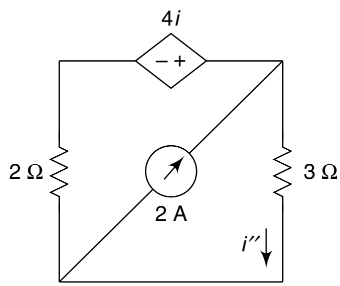
Case I. When the 10 V source is acting alone
The circuit is shown in the figure below.
By KVL for the loop, we get,

$$-4i' + 3i' - 10 + 2i' = 0$$

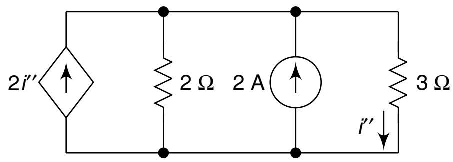
$$\Rightarrow \qquad i' = 10\,\text{A}$$



Case II. When the 2 A source is acting alone
The circuit is shown in the figure below.



We convert the dependent voltage source into its equivalent dependent current source as shown in the figure.



The total current $(2+2i'')$ is divided into two paths, resistors $2\,\Omega$ and $3\,\Omega$.

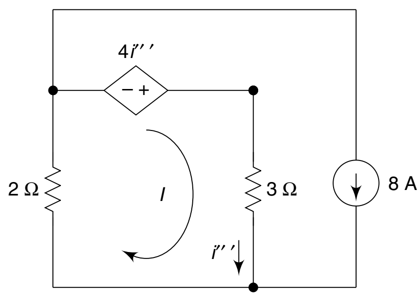
$\therefore$ By current divider rule, current through the 3Ω resistor is,

$$i'' = \left(\frac{2}{2+3}\right)\times(2+2i'')$$

$\Rightarrow$ $$i'' = 4\text{ A}$$

Case III. When the 8 A source is acting alone

The circuit is shown in the figure below.



By KVL for the loop, we get,

$$-4i''' + 3(I-8) + 2I = 0$$

where $$i''' = (I-8) \quad or, \quad I = (i'''+8)$$

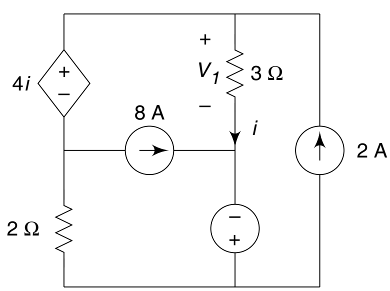
$\Rightarrow$ $$-4i''' + 3i''+2(i'''+8) = 0$$

$\Rightarrow$ $$i''' = -16\text{A}$$

$\therefore$ current when all the sources are acting simultaneously is given by the superposition theorem as,

$$i = (i' + i'' + i''') = (10+4-16) = -2\text{ A} \qquad \textit{Ans.}$$

5.11 Using superposition theorem, determine V_1, the voltage across the 3 ohm resistor in the figure.

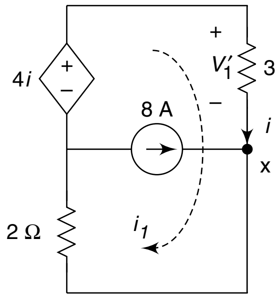


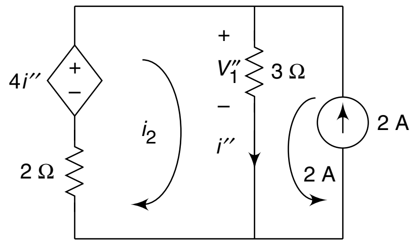
Solution **Case I.** When the 8 A current source is acting alone

By KVL for the supermesh, $3i' + 2i_1 - 4i' = 0 \quad \Rightarrow \quad i_1 = \frac{1}{2}i'$

By KCL at node x, $i_1 = (8 + i') \quad \Rightarrow \quad \frac{1}{2}i' = 8 + i' \quad \Rightarrow \quad i' = -16$ A

$\therefore$ $$V_1' = 3i' = 3 \times (-16) = -48 \text{ Volt}$$





Case II. When the 2 A current source is acting alone

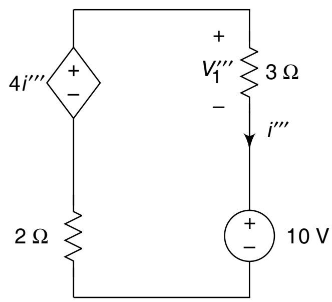
By KVL, $3(i_2 + 2) + 2i_2 - 4i'' = 0 \quad \Rightarrow \quad 5i_2 + 6 - 4i'' = 0$

Now, $$i'' = (i_2 + 2)$$

$\therefore$ $$5i_2 + 6 - 4(i_2 + 2) = 0 \quad \Rightarrow \quad i_2 = 2 \text{ A}$$

$\therefore$ $$i'' = (i_2 + 2) = (2 + 2) = 4 \text{ A}$$

$\therefore$ $$V_1'' = 3i'' = 3 \times 4 = 12 \text{ Volt}$$



Case III. When the 10 V voltage source is acting alone

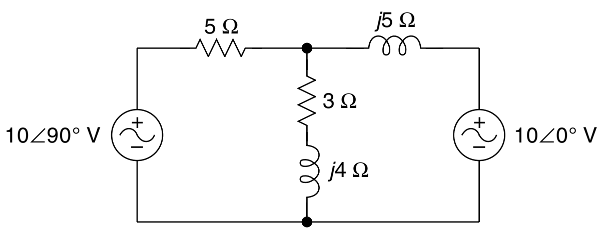
By KVL, $3i''' - 10 + 2i''' - 4i''' = 0 \quad \Rightarrow \quad i''' = 10$ A

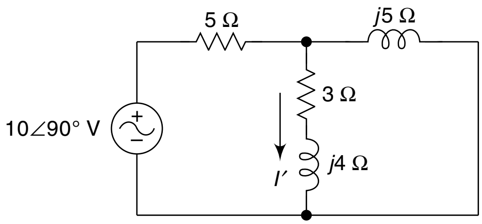
$\therefore$ $$V_1''' = 10 \times 3 = 30 \text{ Volt}$$

When all the sources are acting simultaneously, by superposition theorem the voltage is given as,

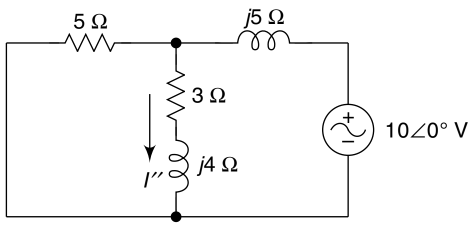
$$V_1 = (V_1' + V_1'' + V_1''') = (-48 + 12 + 30) = -6 \text{ Volt} \qquad \textit{Ans.}$$

5.12 For the network shown in the figure, calculate current throughout the impedance $(3 + j4)$ ohm using superposition theorem.





Solution When 10∠90° V is acting alone



$$\text{Main current, } I = \frac{10\angle 90^\circ}{5 + \dfrac{(3+j4)j5}{3+j4+j5}} = \frac{j10(3+j9)}{-5+j60}$$

$$\therefore \qquad I' = I \times \frac{j5}{3+j9} = \frac{j10\times j5}{-5+j60} = \frac{-10}{-1+j12}$$

When 10∠0° V is acting alone

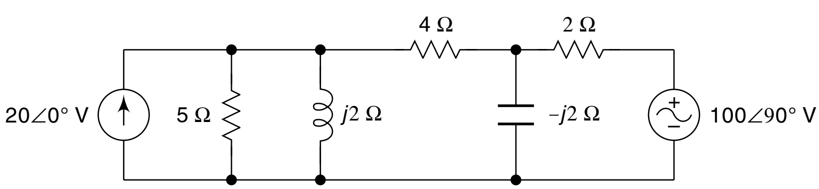
$$\text{Main current, } I = \frac{10\angle 0^\circ}{j5 + \dfrac{(3+j4)5}{3+j4+5}} = \frac{10(8+j4)}{-5+j60}$$

$$\therefore \qquad I'' = I \times \frac{5}{8+j4} = \frac{10\times 5}{-5+j60} = \frac{10}{-1+j12}$$

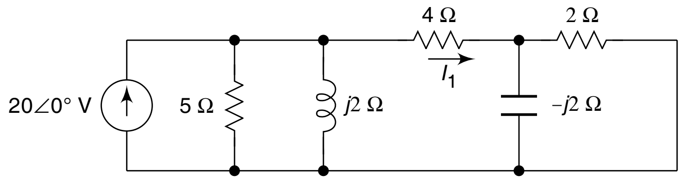
When both the sources are acting simultaneously, by superposition theorem, the total current flowing through the impedance $(3+j4)$ is,

$$I = (I' + I'') = \frac{-10}{-1+j12} + \frac{10}{-1+j12} = 0\text{ A} \qquad \textit{Ans.}$$

5.13 Using superposition theorem, determine the current in the 4Ω resistor in the network shown in the figure.



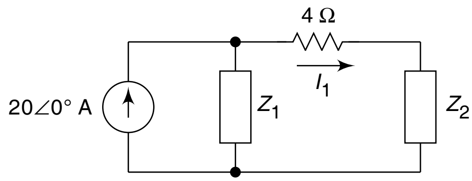
Solution **Case I.** When the $20\angle 0°$ A source is acting alone
The circuit is shown in the figure below.



Reducing the parallel combination, the simplified circuit is shown in the figure below.

$$Z_1 = \frac{5\times j2}{5+j2} = 1.857\angle 68.2° = (0.69 + j1.72)\,\Omega$$

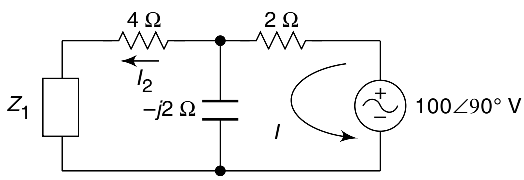
$$Z_2 = \frac{2\times(-j2)}{2-j2} = (1-j1)\,\Omega = 1.414\angle -45°\,\Omega$$



By current division rule, the current through the 4 Ω resistor is,

$$I_1 = 20\angle 0° \times \frac{Z_1}{Z_1+4+Z_2} = 20\angle 0^0 \times \frac{1.857\angle 68.2^0}{0.69+j1.72+4+1-j1}$$
$$= 6.48\angle 61° = (3.14 + j5.66)\text{A}$$

Case II. When $100\angle 90°$ V source is acting alone
Here, the current source is open-circuited. Combining the parallel connection of 5 Ω and $j2\,\Omega$, the simplified circuit is shown in the figure below.



By KVL for the two loops, we get,

$$(4 + 0.69 + j1.72 - j2)\,I_2 + j2I = 0 \quad\Rightarrow\quad (4.69 - j0.28)I_2 + j2I = 0 \qquad \text{(i)}$$

and,

$$j2I_2 + (2 - j2)I = 100\angle 90° = j100 \qquad \text{(ii)}$$

Solving (i) and (ii), we get,

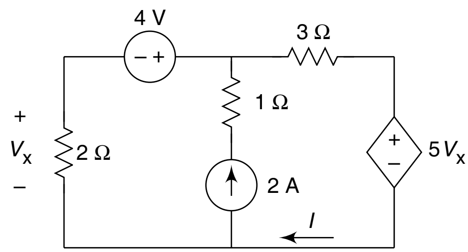
$$I_2 = \frac{\begin{vmatrix} 0 & j2 \\ j100 & (2-j2) \end{vmatrix}}{\begin{vmatrix} (4.69-0.28) & j2 \\ j2 & (2-j2) \end{vmatrix}} = \frac{200}{-12.83+j9.93} = 12.33\angle 37.75^\circ\,(A) = (9.75+j7.55)\text{ A}$$

By superposition theorem, when both the sources are acting simultaneously, the current through the $4\,\Omega$ resistor is,

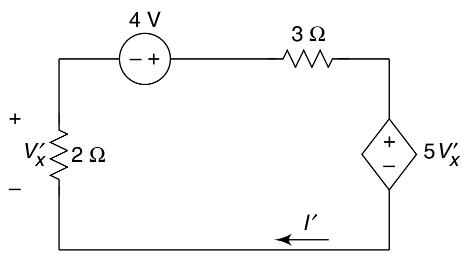
$$I = I_1 - I_2 = (3.14 + j5.66) - (9.75 + j7.55) = (-6.61 - j1.9) = 6.89\angle -163.67\text{ A}$$

The direction of the current is from right to left. *Ans.*

5.14 Find I in the figure using superposition theorem.



Solution When the 4 V voltage source is acting alone
The circuit is shown.



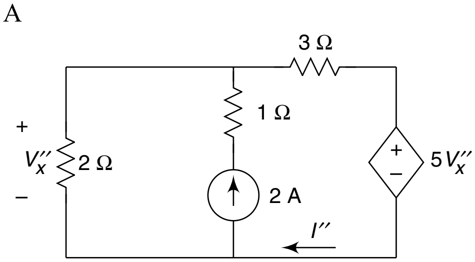
Here, by KVL,

$$-4 + 3I' + 5V_x' - V_x' = 0$$

or, $$3I' + 4\times(-2I') = 4 \quad [\because V_x' = -2I']$$

or, $$I' = -\frac{4}{5}\,A = -0.8\text{ A}$$

When the 2 A current source is acting alone
The circuit is shown.
By KCL,



$$2 = \frac{V_x''}{2} + \frac{V_x'' - 5V_x''}{3} \quad \Rightarrow \quad V_x'' = -\frac{12}{5} = -2.4\text{ V}$$

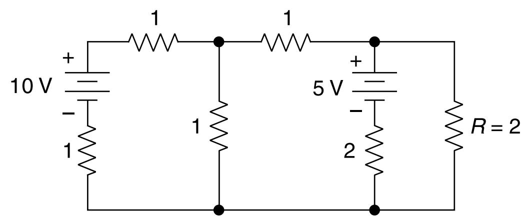
$$\therefore \qquad I'' = \frac{V_x'' - 5V_x''}{3} = \frac{-\frac{12}{5} - 5\times\left(-\frac{12}{5}\right)}{3} = \frac{16}{5} = 3.2 \text{ A}$$

When both the sources are acting simultaneously, the current by superposition theorem is given as,

$$I = (I' + I'') = (-0.8 + 3.2) = 2.4 \text{ A} \qquad \textit{Ans.}$$

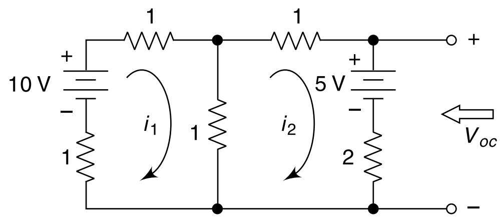
Thevenin's and Norton's Theorems

5.15



Draw the Thevenin's equivalent of the circuit in figure and find the load current, i. All values are in ohm.

Solution Open circuiting the terminals,



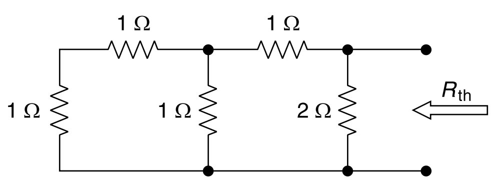
By KVL for two meshes,

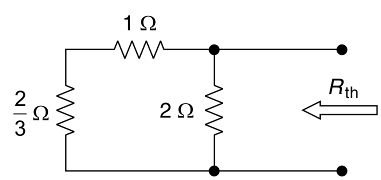
$$3i_1 - i_2 = 10$$

and $\quad -i_1 + 4i_2 = -5$

Solving, $\quad i_1 = 5/11 \quad$ and $\quad i_2 = -5/11$

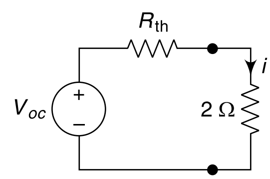
$$\therefore \qquad V_{oc} = (5 + 2i_2) = \left(5 - \frac{10}{11}\right) = \frac{45}{11} \text{ V}$$



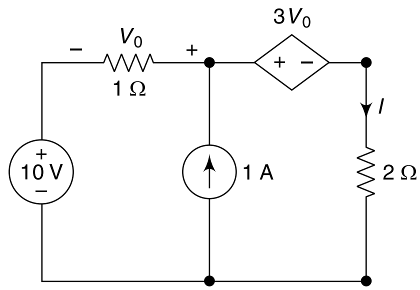


Equivalent resistance, $R_{th} = \dfrac{\frac{5}{3}\times 2}{5/3 + 2} = \dfrac{10}{11}\ \Omega$

So, the load current is, $i = \dfrac{V_{oc}}{R_{th} + 2} = \dfrac{45/11}{10/11 + 2} = \dfrac{45}{32} = 1.40625 \text{ A} \quad \textit{Ans.}$

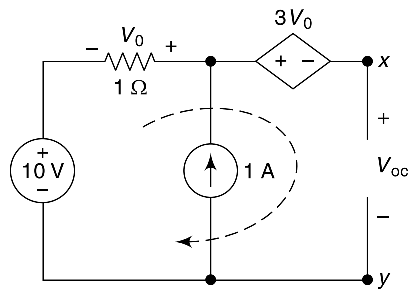


5.16



Find I in the given figure, using Thevenin's theorem.

Solution Removing the 2 Ω resistor,



By KVL for the supermesh,

$$-10 - v_0 + 3v_0 + v_{0c} = 0$$

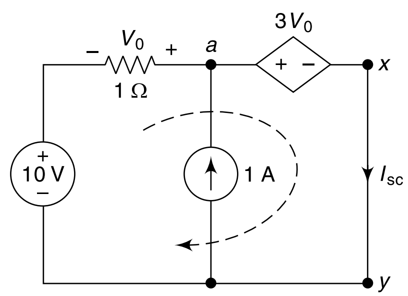
$$\Rightarrow \qquad v_{0c} = 10 - 2v_0$$

But, due to open-circuit, 1A source will circulate through 1 Ω resistor.

$$\therefore \qquad v_0 = 1 \times 1 = 1 \text{ V}$$

$$\therefore \qquad V_{0c} = (10 - 2) = 8 \text{ V}$$

Let's short circuit the terminals x-y,



By KVL,

$$-10 - v_0 + 3v_0 = 0$$

or $\qquad v_0 = 5$

But, by KCL at node (a),

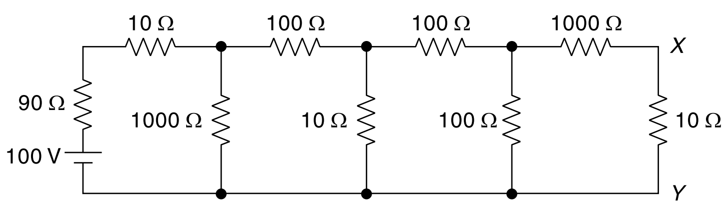
$$\frac{v_0}{1} = 1 - I_{sc}$$

$$I_{sc} = (1 - v_0) = -4 \text{ A (e.g. current is flowing from } y \text{ to } x)$$

$$\therefore \qquad R_{th} = \frac{V_{oc}}{I_{sc}} = \frac{8}{4} = 2\ \Omega$$

So, the current through 2 Ω resistor, $I = \dfrac{8}{2+2} = 2$ A *Ans.*

5.17

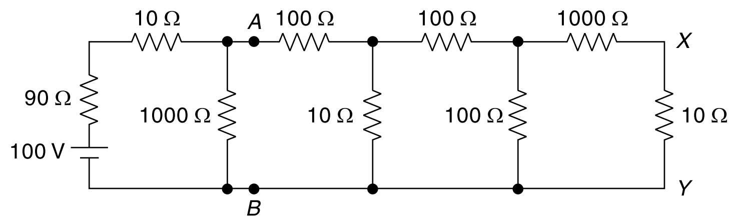


By the iterative use of Thevenin's theorem, reduce the circuit shown in figure to a single emf acting in series with a single resistor. Hence, calculate the current in the 10 Ω resistor connected across XY.

Solution Consider the section of the network to the left of *A–B*. By use of Thevenin's theorem, this portion is reduced to the form of Fig. (ii).

$$\therefore \qquad R_{\text{th}} = \frac{1000 \times 100}{1000 + 100} = \frac{1000}{11}\ \Omega$$

$$\therefore \qquad V_{\text{th}} = \frac{100 \times 1000}{1100} = \frac{1000}{11}\ \text{V}$$

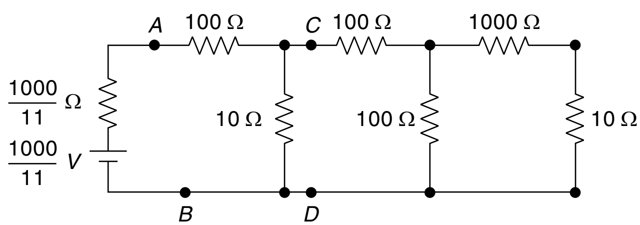


(i)

Applying Thevenin's Theorem to the section left of *CD* of Fig. (ii),

$$\therefore \qquad R_{th} = \frac{(2100/11) \times 10}{(2100/11) + 10} = \frac{2100}{221}\ \Omega$$

$$\therefore \qquad V_{th} = \frac{(1000/11) \times 10}{(2100/11) + 10} = \frac{1000}{221}\ \text{V}$$

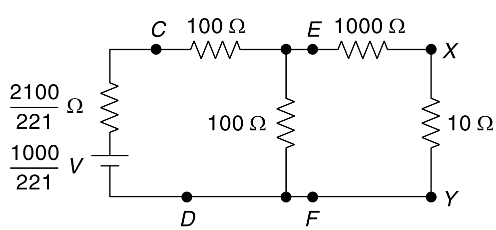


(ii)

Applying Thevenin's Theorem to the section left of *EF* of Fig. (iii),

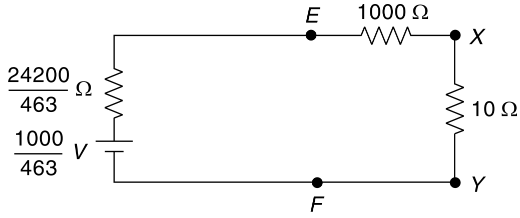
$$\therefore \qquad R_{\text{th}} = \frac{(24200/221) \times 100}{(24200/221) + 100} = \frac{24200}{463}\ \Omega$$

$$\therefore \qquad V_{\text{th}} = \frac{(1000/221) \times 100}{(2100/221) + 200} = \frac{1000}{463}\ \text{V}$$



(iii)

Section left to XY is put as in Fig. (iv).



(iv)

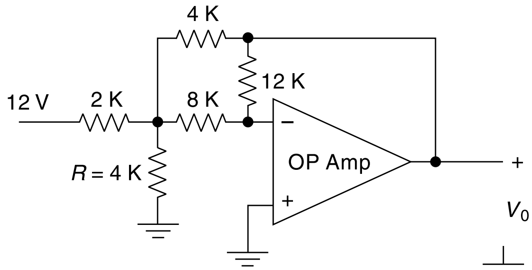
$$\therefore \qquad R_{th} = \frac{24200}{463} + 1000 = \frac{487200}{463}\ \Omega$$

$$\therefore \qquad V_{th} = \frac{1000}{463}\ \text{V}$$

Hence, the current in 10 Ω resistor is,

$$I = \frac{(1000/463)}{(487200/463)+10} = 0.002\ \text{A} \qquad \textit{Ans.}$$

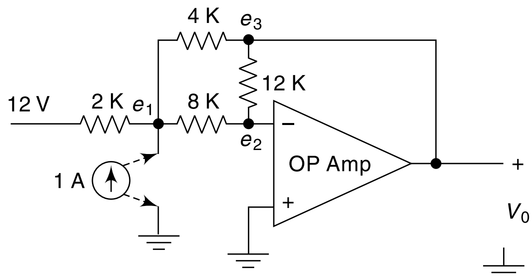
5.18



In the Operational-amplifier circuit shown in figure, find I, in the $R = 4$ kΩ resistor, using Thevenin's theorem.

Solution Open-circuiting the 4 kΩ resistor,

Here, $e_2 = 0,\ e_3 = V_0$



$$\frac{e_1 - 12}{2\times10^3} + \frac{e_1 - V_0}{4\times10^3} + \frac{e_1}{8\times10^3} = 0$$

$$\Rightarrow \qquad 7e_1 = (48 + 2V_0) \qquad \text{(i)}$$

$$\frac{0-e_1}{8\times10^3} + \frac{0-V_0}{12\times10^3} = 0$$

$$\Rightarrow \qquad V_0 = -\frac{3}{2}e_1 \qquad \text{(ii)}$$

From equation (i) and equation (ii),

$$\Rightarrow \qquad e_1 = 4.8\ \text{V} = e_{oc}$$

Now, we connect a 1 A current source at the place of 4 kΩ resistor.
By KCL at node (1),

$$\frac{e_1}{2\times10^3} + \frac{e_1 - V_0}{4\times10^3} + \frac{e_1}{8\times10^3} = 1$$

$$\Rightarrow \qquad 7e_1 = 8000 + 2V_0$$

By KCL at node (2),

$$V_0 = -\frac{3}{2}e_1$$

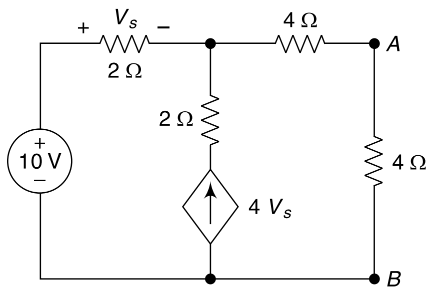
$$\Rightarrow \qquad 7e_1 = 8000 + 2\left(-\frac{3}{2}e_1\right)$$

$$\Rightarrow \qquad e_1 = 800 \text{ V}$$

$$\therefore \qquad R_{th} = \frac{e_1}{1} = 800\ \Omega$$

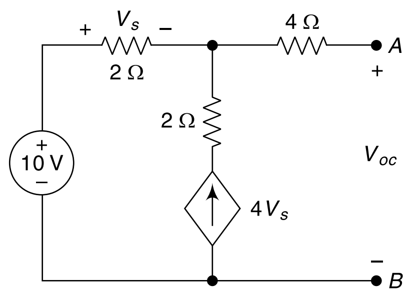
$$\therefore \qquad i = \frac{4.8}{4000+800} = \frac{4.8}{4.8\times10^3} = 1 \text{ mA} \qquad \textit{Ans.}$$

5.19



Find Thevenin's equivalent about *AB* for the circuit shown in figure.

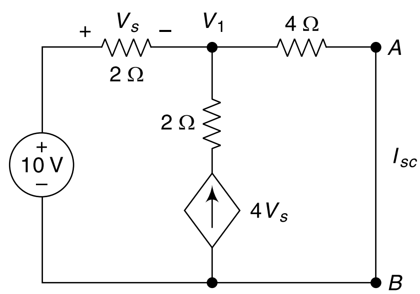
Solution Open-circuiting The 4 Ω resistor, by KCL,



$$\frac{V_{oc}-10}{2} = 4v_s = 4(10 - V_{oc})$$

$$\Rightarrow \qquad V_{oc} = 10 \text{ V}$$

Short-circuiting the terminals *AB*, by KCL,



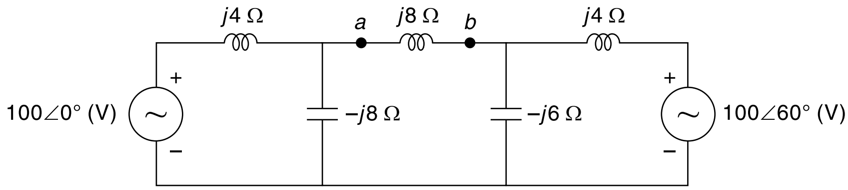
$$\frac{V_1-10}{2} + \frac{V_1}{4} = 4v_s = 4(10 - V_1)$$

$$V_1 = \frac{180}{19} = 9.47 \text{ V}$$

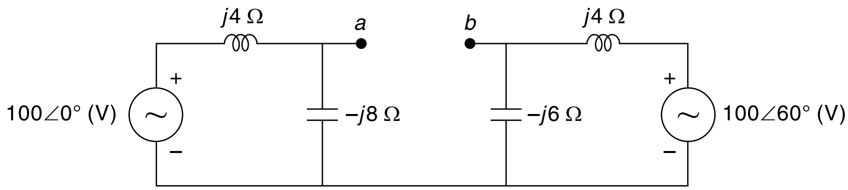
$$\therefore \qquad I_{sc} = \frac{9.47}{4} = 2.368 \text{ A}$$

$$\therefore \qquad R_{th} = \frac{V_{th}}{I_{sc}} = 4.22\ \Omega \qquad \textit{Ans.}$$

5.20 In the network, determine the steady current in the 8 Ω inductor using Thevenin's theorem.



Solution With *a–b* open-circuited,



$$V_a = \frac{100\angle 0^\circ}{j4 - j8}(-j8) = 200\angle 0^\circ \text{ V}$$

$$V_b = \frac{100\angle 60^\circ}{j4 - j6}(-j6) = 300\angle 60^\circ \text{ V}$$

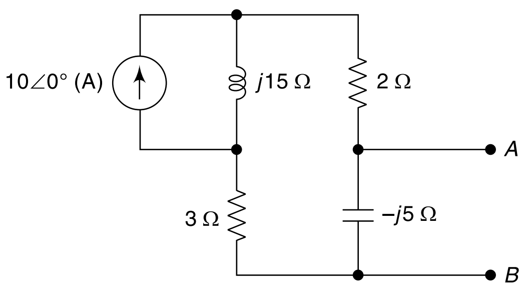
$$\therefore \quad V_{\text{th}} = (V_a - V_b) = 200\angle 0^\circ - 300\angle 60^\circ = (50 - j259.81) \text{ V}$$

$$\therefore \quad Z_{\text{th}} = \frac{(j4)(-j8)}{j4 - j8} + \frac{(j4)(-j6)}{j4 - j6} = j20\ \Omega$$

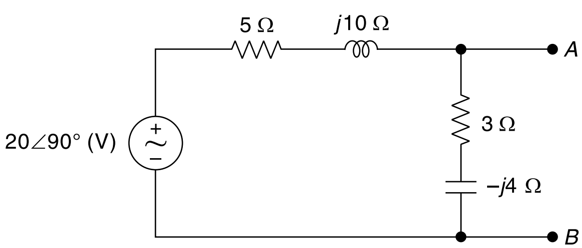
$\therefore$ Current in the 8 Ω inductor, $i = \dfrac{V_{\text{th}}}{Z_{\text{th}} + Z_L} = \dfrac{(50 - j259.81)}{j20 + j8} = 9.45\angle{-169.1^\circ}$ A *Ans.*

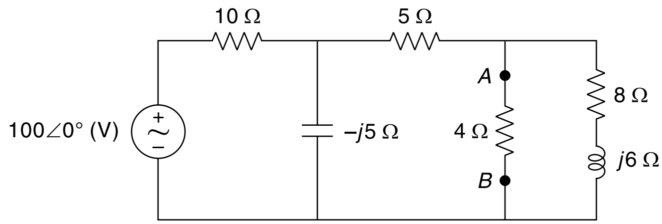
5.21 Obtain Thevenin's equivalent circuit with respect to terminals *A–B* in the networks shown below.

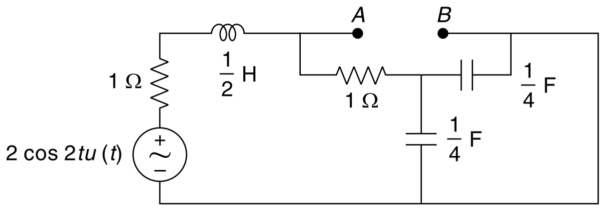
(a)

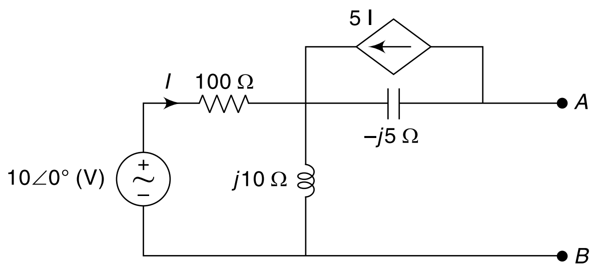


(b)



(c) 

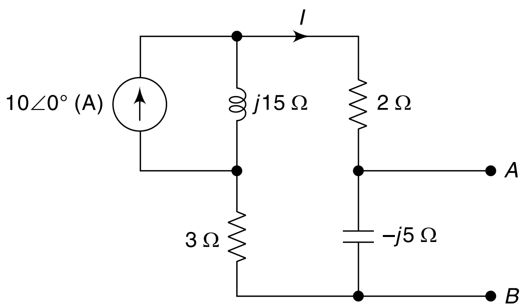
(d) 

(e) 

Solution

(a) With *A–B* open, the current is,

$$I = \frac{10\angle 0^\circ}{5 - j5 + j15} \times j15 = \frac{150\angle 90^\circ}{5 + j10}$$

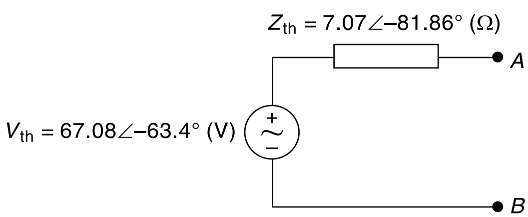


Thevenin voltage,

$$\begin{aligned} V_{\text{th}} &= V_{AB} = I \times (-j5) \\ &= \frac{150\angle 90^\circ}{5 + j10} \times (5\angle -90^\circ) \\ &= 67.08\angle -63.4^\circ \text{ V} \qquad \textit{Ans.} \end{aligned}$$

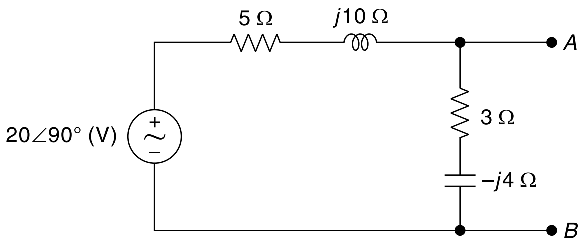
Thevenin impedance,

$$Z_{\text{th}} = Z_{AB} = \frac{-j5 \times (5 + j15)}{-j5 + 5 + j15} = 7.07\angle -81.86^\circ\ \Omega \qquad \textit{Ans.}$$



Thus, the Thevenin's equivalent circuit is shown in the figure.

(b) Here, Thevenin voltage,

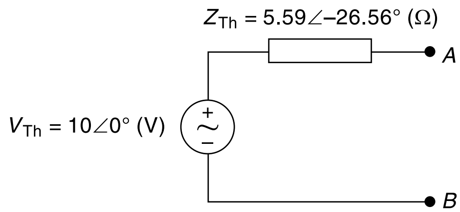


$$V_{th} = \frac{20\angle 90^\circ}{5 + j10 + 3 - j4} \times (3 - j4)$$

$$= \frac{j120(3 - j4)}{8 + j6}$$

$$V_{th} = \frac{50\angle 36.87^\circ}{5\angle 36.87^\circ} = 10\angle 0^\circ \text{ (V)}$$

Thevenin impedance,



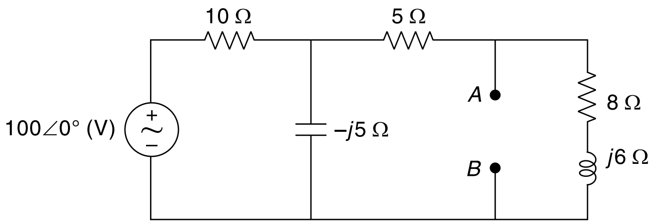
$$Z_{th} = \frac{(5 + j10) \times (3 - j4)}{(5 + j10) + (3 - j4)}$$

$$= \frac{11.8\angle 63.43^\circ \times 5\angle -53.13^\circ}{10\angle 36.87^\circ}$$

$$= 5.59\angle -26.56 \text{ (Ω)}$$

Thus, the Thevenin's equivalent circuit is shown in the figure. *Ans.*

(c)



Here, with *A*–*B* open, equivalent impedance,

$$Z = 10 + \frac{-j5 \times (13 + j6)}{-j5 + (13 + j6)} = \frac{160 - j55}{13 + j1}\ \Omega = 12.98\angle -23.37^\circ \text{ (Ω)}$$

$\therefore$ Main current, $I = \dfrac{100\angle 0^\circ}{Z} = \dfrac{100\angle 0^\circ}{12.98\angle -23.37^\circ} = 7.7\angle 23.37^\circ \text{ (A)}$

$\therefore$ Thevenin voltage,

$$V_{th} = I \times \left(\frac{-j5}{-j5 + 5 + 8 + j6}\right) \times (8 + j6)$$

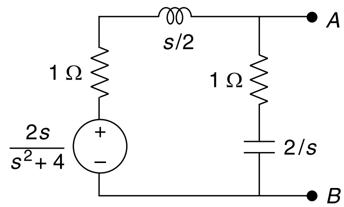
$$= 7.7\angle 23.37^\circ \times \left(\frac{-j5}{13 + j1}\right) \times (8 + j6) = 29.553\angle -34.16^\circ \text{ (V)} \qquad \textit{Ans.}$$

$\therefore$ Thevenin impedance, $Z_{th} = \left[\dfrac{10 \times (-j5)}{10 - j5} + 5\right] || (8 + j6) = 5.33\angle -0.5^\circ \text{ (Ω)}$ *Ans.*

(d) The circuit is redrawn as shown considering two capacitors in parallel.

$$\because \qquad C_{eq} = (C_1 + C_2) = \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2} \text{ F}$$

Thevenin voltage is given as,



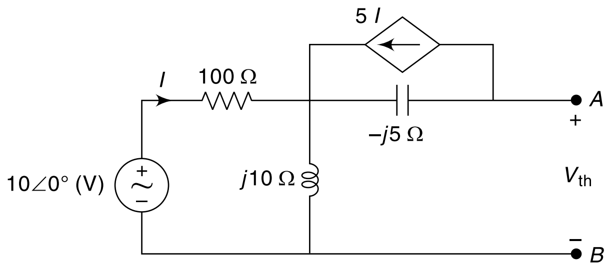
$$V_{\text{th}}(s) = \frac{2s}{s^2+4} \times \frac{(1+2/s)}{(1+2/s+1+s/2)}$$

$$= \frac{4s}{(s^2+4)(s+2)} \text{ (V)}$$

$\therefore$ Thevenin impedance, $Z_{\text{th}}(s) = (1+2/s)\,||\,(1+s/2) = 1\,\Omega$ *Ans.*

(e) *To find* V_{th}

With *A–B* open, current of the dependent source can flow through the capacitor only.

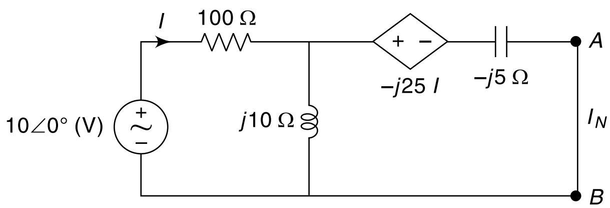


$$\therefore\ I = \frac{10\angle 0^\circ}{100+j10} = 0.09995\angle -5.7^\circ \text{ (A)}$$

$\therefore$ Thevenin voltage,

$$V_{\text{th}} = V_{AB} = (I \times j10) - \{5I \times (-j5)\} = j35I = j35 \times 0.09995\angle -5.7^\circ = 3.48\angle 84.3^\circ \text{ (V)} \quad \textit{Ans.}$$

To find I_N



Converting the dependent current source into voltage source, by KVL,

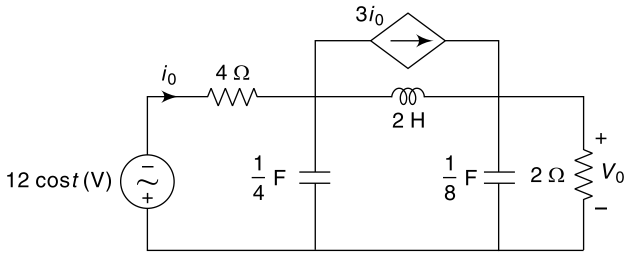
$$10\angle 0^\circ = (100+j10)\,I - j10I_N$$

and $$-(-j25I) = -j10I + I_N\,(j10 - j5)$$

Solving for I_N, $I_N = 0.6\angle 31^\circ$ (A) *Ans.*

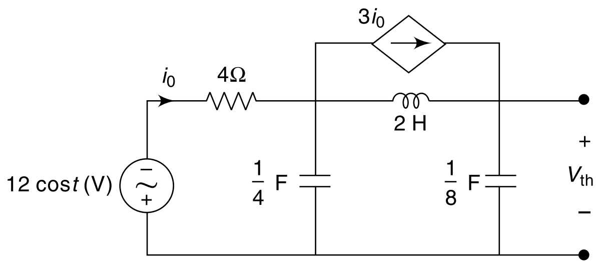
$$\therefore \text{ Thevenin impedance, } Z_{\text{th}} = \frac{V_{\text{th}}}{I_N} = \frac{3.48\angle 84.3^\circ}{0.6\angle 31^\circ} = 5.8\angle 53.3^\circ\ (\Omega) \quad \textit{Ans.}$$

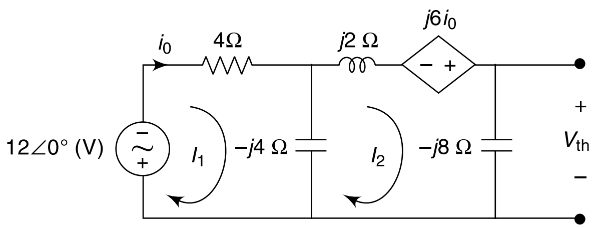
5.22 Find V_0 using Thevenin's theorem.



Solution *To find* V_{th}

Removing the 2 Ω resistor and open circuiting the terminals and then converting the dependent current source into dependent voltage source, we redraw the circuit as follows.





By KVL for the two loops, (here $i_0 = I_1$)

$$(4 - j4)\, I_1 + j4I_2 = -12$$

$$-j2I_1 + (-j6)\, I_2 = 0$$

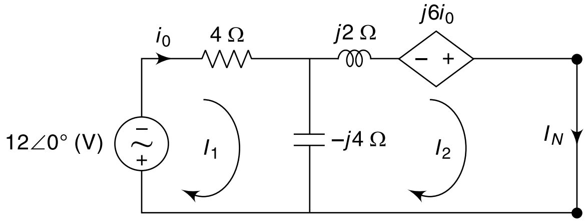
Solving for I_2,

$$I_2 = \frac{\begin{vmatrix} (4-j4) & -12 \\ -j2 & 0 \end{vmatrix}}{\begin{vmatrix} (4-j4) & j4 \\ -j2 & -j6 \end{vmatrix}} = \frac{-j24}{-j24-24-8} = \frac{j3}{4+j3} = 0.6\angle 53.13° \text{ (A)}$$

Therefore, Thevenin voltage is, $V_{th} = I_2 \times (-j8) = \dfrac{24}{4+j3} = 4.8\angle -36.87° \text{ (V)}$

To find I_N

Removing the 2 Ω resistor and short circuiting the terminals and then converting the dependent current source into dependent voltage source, we redraw the circuit as follows.



By KVL for the two loops,

$$(4-j4)\,I_1 + j4I_2 = -12$$

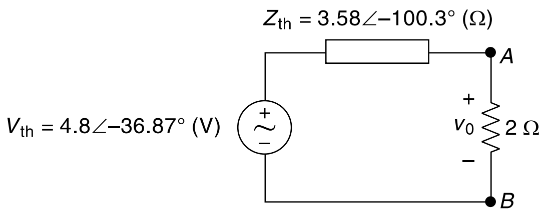
$$-j2I_1 + (-j2)\,I_2 = 0$$

Solving for I_2,

$$I_2 = I_N = \frac{\begin{vmatrix} (4-j4) & -12 \\ -j2 & 0 \end{vmatrix}}{\begin{vmatrix} (4-j4) & j4 \\ -j2 & -j2 \end{vmatrix}} = \frac{-j24}{-8-j8-8} = \frac{j3}{2+j} = 1.341\angle 63.435^\circ\ \text{(A)}$$

Therefore, Thevenin impedance is, $Z_{\text{th}} = \dfrac{V_{\text{th}}}{I_N} = \dfrac{4.8\angle -36.87^\circ}{1.341\angle 63.435^\circ} = 3.58\angle -100.3^\circ\ (\Omega)$

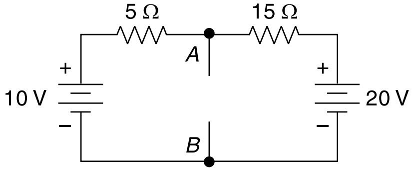
Thus, Thevenin's equivalent circuit becomes as shown.



Thus, the required voltage,

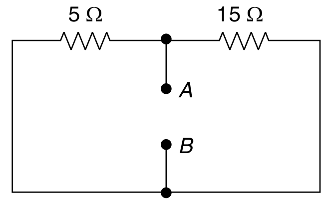
$$v_0 = \left(\frac{V_{\text{th}}}{Z_{\text{th}}+2}\right)\times 2 = \left(\frac{4.8\angle -36.87^\circ}{3.58\angle -100.3^\circ + 2}\right)\times 2 = 1.27\angle 32^\circ\ \text{(V)} \qquad \textit{Ans.}$$

5.23



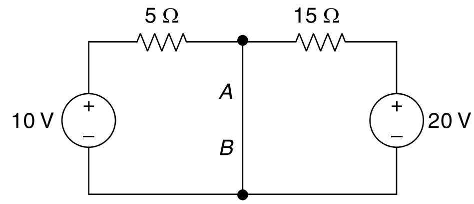
Obtain the Norton's equivalent circuit with respect to the terminals *AB* for the network shown in figure.

Solution Removing the sources,



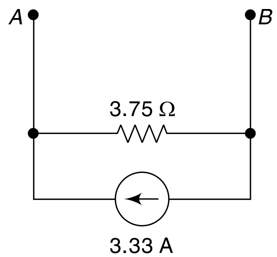
$$\therefore \qquad Z_{eq} = \frac{5 \times 15}{5 + 15} = \frac{75}{20} = 3.75\ \Omega$$

Short-circuiting *AB*,

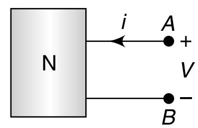


$$I_{sc} = \frac{10}{5} + \frac{20}{15} = 3.33\ \text{A}$$

So, Norton's equivalent circuit is shown.



5.24 For the one port shown in figure determine the Norton's equivalent at the terminals *AB*, if the *v*–*i* characteristic is given by, $16v = 80 - 2i$.



Solution *v*–*i* characteristic is given as,

$$16v = 80 - 2i \Rightarrow \frac{V}{5} + \frac{i}{40} = 1$$

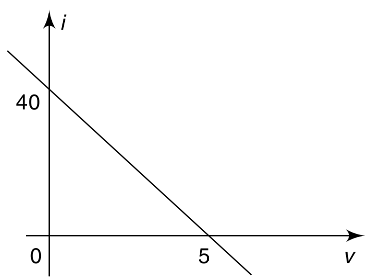
Thus, short circuit current,

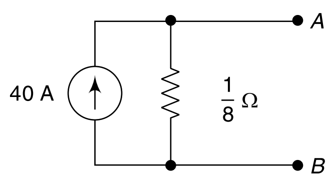
$$I_{sc} = 40\ \text{A} \ \ (\text{where } v = 0)$$

and open-circuit voltage,

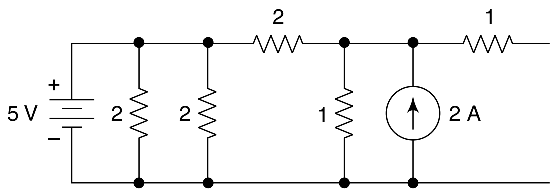
$$V_{oc} = 5\ \text{V} \ \ (\text{whare } i = 0)$$

$$\therefore \qquad R_n = \frac{V_{oc}}{I_{sc}} = \frac{5}{40} = \frac{1}{8}\ \Omega$$



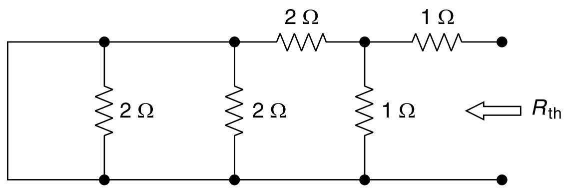


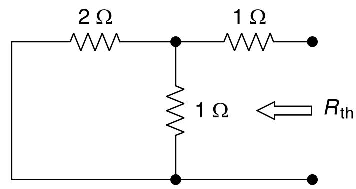
Norton's equivalent circuit is shown accordingly.

5.25 

Find both Thevenin's and Norton's equivalent circuit for the network shown in figure. All values are in ohm.

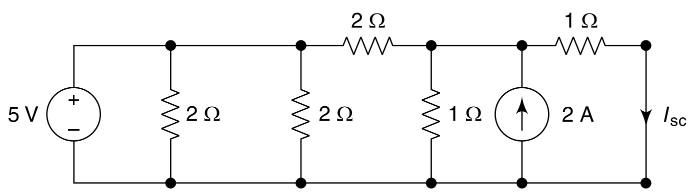
Solution Removing the sources,



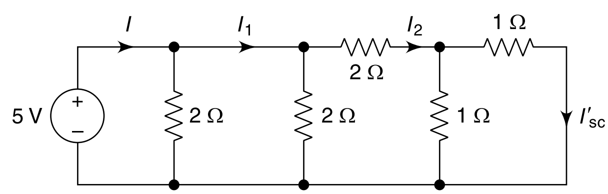


$$\Rightarrow \qquad R_N = R_{th} = \left(\frac{2}{3}+1\right) = \frac{5}{3}\ \Omega$$

Short-circuiting the terminals,



By superposition theorem, when 5 V source is acting alone,



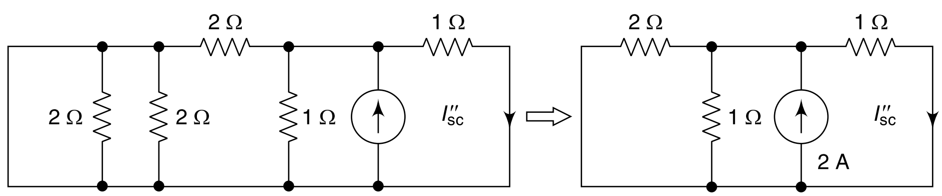
$$I = \frac{5}{5/7} = 7\text{ A}$$

$$\therefore \qquad I_1 = 4.5\text{ A}$$

$$\therefore \qquad I_2 = 2A$$

$$\therefore \qquad I'_{sc} = 1\text{ A}$$

and when 2 A source is acting alone,



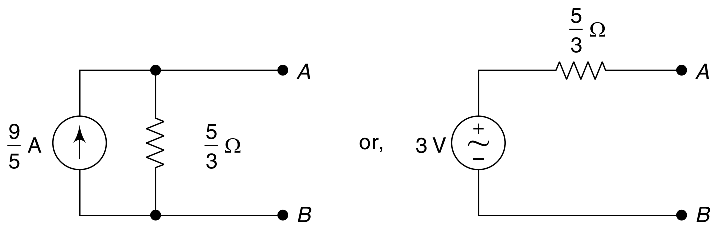
$$\therefore \qquad I''_{sc} = 2 \times \frac{2/3}{2/3+1} = \frac{4}{5}\text{ A}$$

$\therefore$ Total $I_{sc} = (I'_{sc} + I''_{sc}) = \left(1 + \frac{4}{5}\right) = \frac{9}{5}$ A

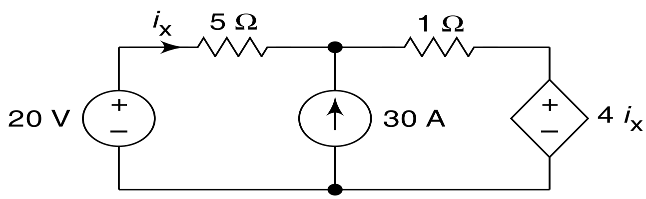
$\therefore$

$$V_{th} = I_{sc} \times R_{th} = \frac{9}{5} \times \frac{5}{3} = 3 \text{ V}$$

The circuits are shown accordingly.



5.26 Replace the circuit in the figure with the Thevenin's equivalent circuit across *A* and *B*.



Solution By KVL for the left-hand side loop,

$$1 \times 10^3 \times I + \frac{V_0}{10^4} = 10 \times 10^{-3} \qquad \text{(i)}$$

In the right-hand side loop, the dependent current source current will circulate in the resistor. By KVL,

$$V_0 = 30 \times 10^3 \times (-75I) = -225 \times 10^4 I \qquad \text{(ii)}$$

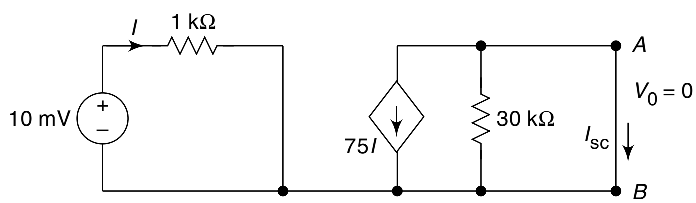
Substituting the value of *I* from (ii) in (i), we get,

$$1 \times 10^3 \times \left(-\frac{V_0}{225 \times 10^4}\right) + \frac{V_0}{10^4} = 10 \times 10^{-3}$$

$\Rightarrow$ $$-4.44 \times 10^{-4} V_0 + 1 \times 10^{-4} V_0 = 10 \times 10^{-3}$$

$\Rightarrow$ $$V_0 = -\frac{10 \times 10^{-3}}{3.44 \times 10^{-4}} = -29 \text{ V}$$

Now, short circuiting the terminals *A* and *B*, we get by KVL to left-hand side loop,



$$1 \times 10^3 \times I + 0 = 10 \times 10^{-3} \quad \Rightarrow \quad I = 1 \times 10^{-5}\,\text{A}$$

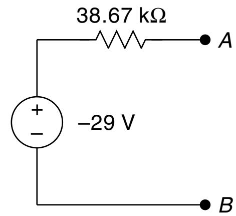
Also, from right-hand side loop on short circuit,

$$I_{sc} = -75I = -75 \times 1 \times 10^{-5} = -75 \times 10^{-5}\,\text{A}$$

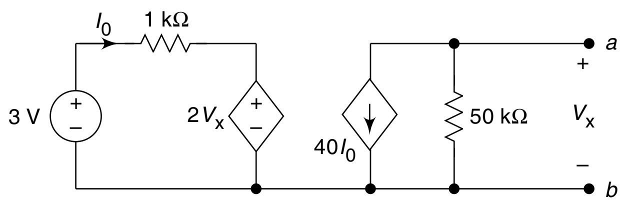
Thus, Thevenin equivalent impedance is given as,

$$Z_{Th} = \frac{V_{oc}}{I_{sc}} = \frac{-29}{-75 \times 10^{-5}} = 38.67\ \text{k}\Omega$$

Thevenin's equivalent circuit is shown in the figure.



5.27 Find the Thevenin's equivalent between terminals a and b of the circuit shown in the figure.



Solution By KVL for the right-hand side mesh,

$$V_{oc} = V_x = (-40I_0) \times 50 = -2000I_0 \qquad \text{(i)}$$

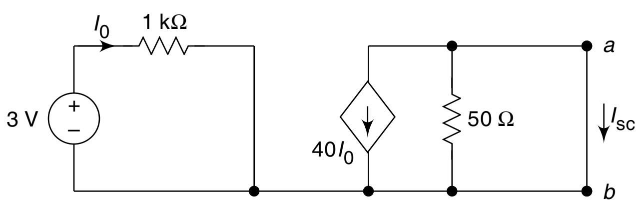
From the left-hand side loop,

$$I_0 = \frac{3 - 2V_x}{1000} = \frac{3 - 2V_{oc}}{1000} \qquad \text{(ii)}$$

From (i) and (ii), we get,

$$V_{oc} = -2000\left(\frac{3 - 2V_{oc}}{1000}\right) \quad \Rightarrow \quad V_{oc} = 2\ \text{V}$$

To determine the Thevenin's impedance, we short circuit the terminals a and b.

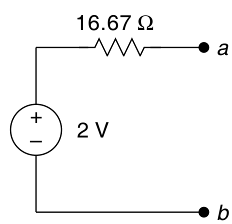


Here,

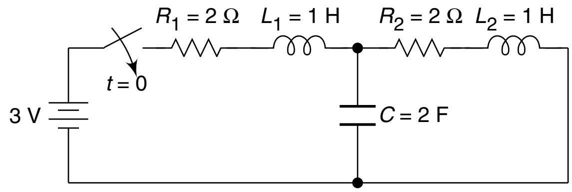
$$I_{sc} = -40I_0 = -40 \times \left(\frac{3}{1000}\right) = -0.12\ \text{A}$$

$$\therefore \qquad R_{Th} = \frac{V_{oc}}{I_{sc}} = \frac{2}{0.12} = 16.67\,\Omega$$

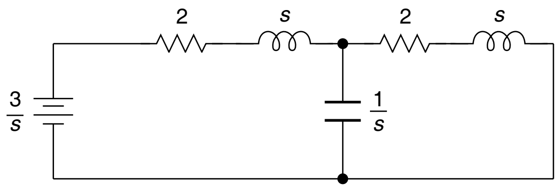
Thevenin's equivalent circuit is shown in the figure.



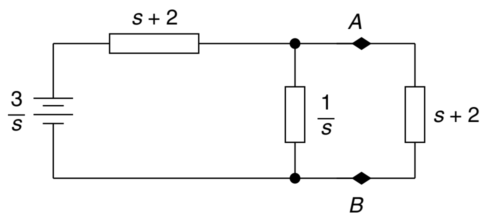
5.28 In the network shown in the figure, the switch is closed at time $t = 0$. Assuming all the initial currents and voltages as zero, find the current through the inductor L_2 by the use of Norton's theorem.



Solution The network for $t > 0$ in Laplace domain is shown in the figure below.



The equivalent network reduces to one as shown below.



To find the current in L_2, we have to find Thevenin's equivalent circuit across the terminals A and B. The impedance between terminals A and B is given as,

$$Z_{Th} = Z_{AB} = \frac{(s+2)\times\frac{1}{s}}{s+2+\frac{1}{s}} = \frac{(s+2)}{s^2+2s+1} = \frac{(s+2)}{(s+1)^2}$$

Short circuit current flowing from A to B is given as,

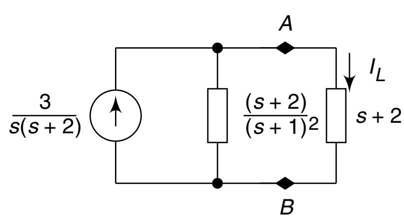
$$I_{sc} = \frac{3/s}{s+2} = \frac{3}{s(s+2)}$$

Therefore, the Norton's equivalent circuit is shown in the figure below.

Hence the current,

$$I_L = \frac{3}{s(s+2)} \times \frac{(s+2)}{(s+1)^2} \times \frac{1}{\frac{(s+2)}{(s+1)^2}+(s+2)}$$

$$= \frac{3}{s(s+2)(s^2+2s+2)}$$



By partial fraction expansion,

$$I_L = \frac{3}{s(s+2)(s^2+2s+2)} = \frac{k_1}{s} + \frac{k_2}{s+2} + \frac{k_3}{s+1+j1} + \frac{k_3^*}{s+1-j1}$$

Where,

$$k_1 = \left. \frac{3}{(s+2)(s^2+2s+2)} \right|_{s=0} = \frac{3}{4}$$

$$k_2 = \left. \frac{3}{s(s^2+2s+2)} \right|_{s=-2} = -\frac{3}{4}$$

$$k_3 = \left. \frac{3}{s(s+2)(s+1-j1)} \right|_{s=-1-j1} = j\frac{3}{4}$$

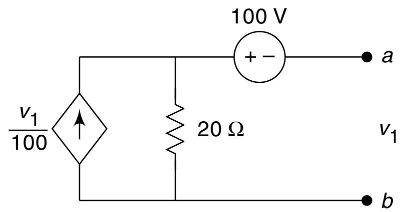
$$k_3^* = -j\frac{3}{4}$$

$$\therefore \qquad I_L = \frac{3/4}{s} - \frac{3/4}{s+2} + \frac{j3/4}{s+1+j1} - \frac{j3/4}{s+1-j1} = \frac{3/4}{s} - \frac{3/4}{s+2} - \frac{3/2}{(s+1)^2+1}$$

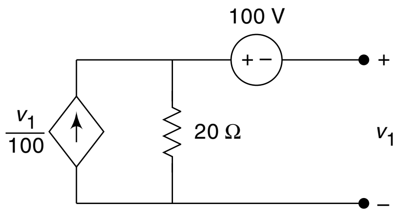
Taking inverse Laplace transform, we get the required current as,

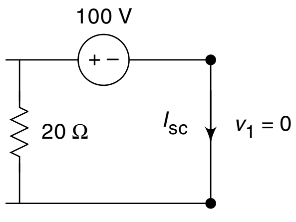
$$i(t) = \frac{3}{4} - \frac{3}{4}e^{-2t} - \frac{3}{2}e^{-t}\sin t \qquad \textit{Ans.}$$

5.29 The following circuit has a dependent current source and an independent voltage source. Find the Thevenin equivalent network of the circuit across the terminals a and b.



Solution





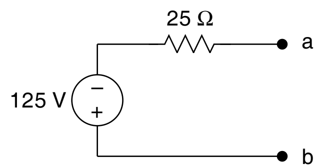
With open circuit, $v_1 = v_{oc}$. By KCL,

$$-\frac{v_{oc}}{100}+\frac{100+v_{oc}}{20}=0 \quad\Rightarrow\quad -v_{oc}+500+5v_{oc}=0 \quad\Rightarrow\quad v_{oc}=-125\ \text{Volt}$$

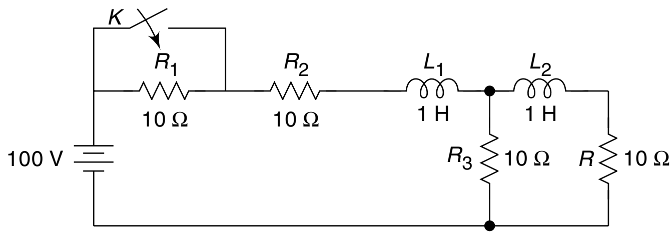
With short-circuit, $v_1 = 0$ and the dependent current source is open, so that, $I_{sc} = -5$ A

Thus, Thevenin impedance, $R_{Th}=\frac{v_{oc}}{I_{sc}}=\frac{-125}{-5}=25\,\Omega$

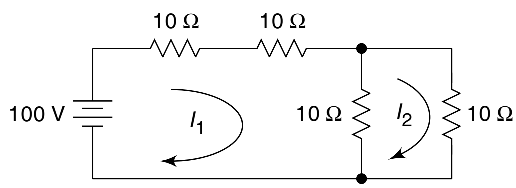
So, the Thevenin's equivalent circuit is shown in the figure below.



5.30 In the network of figure, the switch K is closed at time $t = 0$, a steady state having previously existed. Obtain the current in the resistor R using Thevenin's theorem.



Solution When the switch K is opened, under steady state condition, two inductors behave as short circuits. Therefore, the initial currents flowing through the inductors can be found out by writing the KVL equations for the circuit at $t = 0-$.



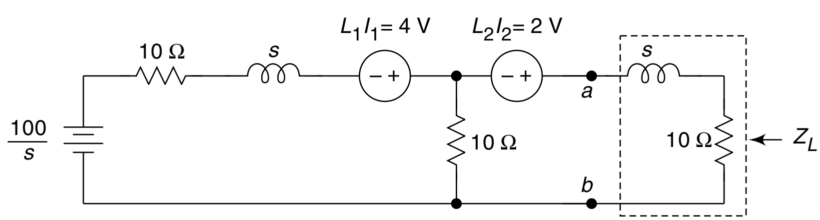
(a) Circuit at t = 0–

By KVL for the two meshes,

$$\begin{aligned} 30I_1-10I_2&=100\\ -10I_1+20I_2&=0 \end{aligned}$$

Solving, $I_1 = 4A, \quad I_2 = 2A$

Hence, the transform network for $t > 0$ is shown in Fig (b).



(b) Transform network for t > 0

Thevenin equivalent impedance with respect to the terminals a and b is given as,

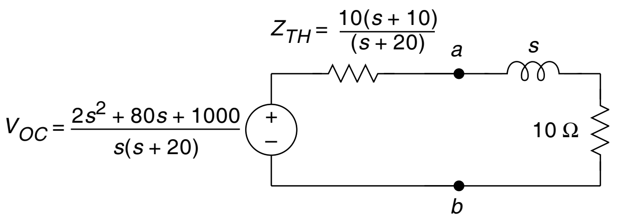
$$Z_{Th}=\frac{(s+10)\times 10}{s+10+10}=\frac{10(s+10)}{(s+20)}$$

To find the open circuit voltage across the terminals a and b, we have the current flowing in the left mesh,

$$I(s)=\frac{100/s+4}{s+10+10}=\frac{4s+100}{s(s+20)}$$

$$\therefore \qquad V_{OC}(s)=I(s)\times 10+2=\frac{4s+100}{s(s+20)}\times 10+2=\frac{2s^2+80s+1000}{s(s+20)}$$

Therefore, the Thevenin's equivalent circuit is shown in Fig. (c).



(c) Thevenin's equivalent circuit

Hence, the current through the resistor R = 10 Ω is given as,

$$I_L(s)=\frac{V_{OC}(s)}{Z_{Th}+R}=\frac{2s^2+80s+1000}{s(s+20)\left[\frac{10(s+10)}{(s+20)}+(s+10)\right]}=\frac{2s^2+80s+1000}{s(s+10)(s+30)}$$

By partial fraction expansion, let,

$$I_L(s)=\frac{2s^2+80s+1000}{s(s+10)(s+30)}=\frac{K_1}{s}+\frac{K_2}{s+10}+\frac{K_3}{s+30}$$

$$\therefore \qquad K_1=s\left[\frac{2s^2+80s+1000}{s(s+10)(s+30)}\right]_{s=0}=\frac{10}{3}$$

$$\therefore \qquad K_2=(s+10)\left[\frac{2s^2+80s+1000}{s(s+10)(s+30)}\right]_{s=-10}=-2$$

$$\therefore \qquad K_3=(s+30)\left[\frac{2s^2+80s+1000}{s(s+10)(s+30)}\right]_{s=-30}=\frac{2}{3}$$

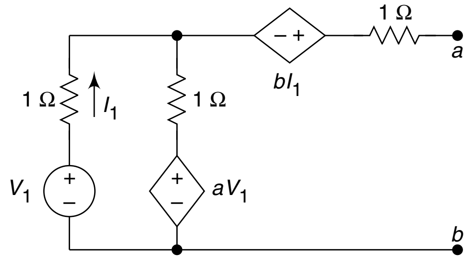
$$I_L(s)=\frac{10/3}{3}-\frac{2}{s+10}+\frac{2/3}{s+30}$$

Taking inverse Laplace transform, we get,

$$i_L(t) = \frac{10}{3} - 2e^{-10t} + \frac{2}{3}e^{-30t} = 3.33 - 2e^{-10t} + 0.67e^{-30t} \qquad \textit{Ans.}$$

5.31 For the network shown in the figure, show that the Thevenin equivalent at the terminals *a-b* is represented by,

$$V_{Th} = \frac{V_1}{2}(1 + a + b - ab) \qquad \text{and} \qquad Z_{Th} = \frac{3-b}{2}$$



Solution When the terminals *a-b* are open-circuited no current will flow through the right side 1 Ω resistor. By KVL for the left mesh,

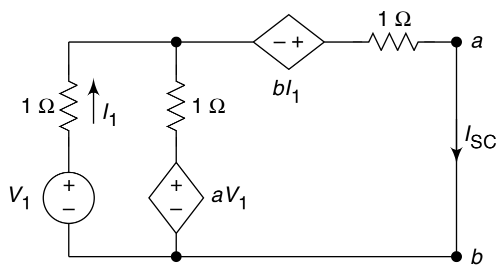
$$2I_1 + aV_1 = V_1 \quad \Rightarrow \quad I_1 = \frac{V_1}{2}(1-a)$$

$$\therefore \qquad V_{Th} = 1 \times I_1 + aV_1 + bI_1 = 1 \times \frac{V_1}{2}(1-a) + aV_1 + +b\frac{V_1}{2}(1-a) = \frac{V_1}{2}(1 - a + 2a + b - ab)$$

$$\therefore \qquad V_{Th} = \frac{V_1}{2}(1 + a + b - ab) \quad \text{(Proved)}$$

To find the Thevenin impedance we have to find the short-circuit current flowing through the terminals *a-b*.

By KVL for the two meshes, we get,



$$2I_1 - I_{SC} = V_1(1-a) \tag{i}$$

and,
$$1 \times (I_{SC} - I_1) - bI_1 + 1 \times I_{SC} = aV_1 \quad \Rightarrow \quad -(1+b)I_1 + 2I_{SC} = aV_1 \tag{ii}$$

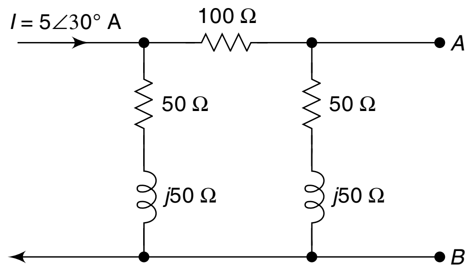
Solving (*i*) and (*ii*), we get,

$$I_{SC} = \frac{\begin{vmatrix} 2 & V_1(1-a) \\ -(1+b) & aV_1 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ -(1+b) & 2 \end{vmatrix}} = \frac{2aV_1 + V_1(1-a+b-ab)}{4-1-b} = \frac{V_1(1+a+b-ab)}{3-b}$$

Therefore, the Thevenin impedance is,

$$Z_{Th} = \frac{V_{Th}}{I_{SC}} = \frac{\frac{V_1}{2}(1+a+b-ab)}{\frac{V_1(1+a+b-ab)}{3-b}} = \frac{3-b}{2} \quad \text{(Proved)}$$

5.32 Find the Thevenin equivalent circuit for the network shown in figure at terminals *A*-*B*.



Solution When the terminals *A* and *B* are open-circuited, the current flowing through the right branch $(50 + j50)\ \Omega$ is,

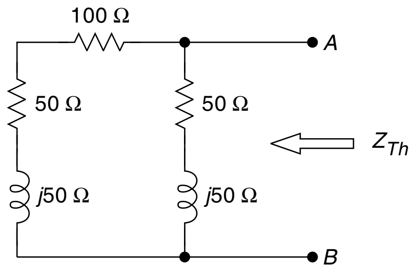
$$I = 5\angle 30^\circ \times \frac{50 + j50}{100 + 50 + j50 + 50 + j50}$$
$$= 5\angle 30^\circ \times \left(\frac{50 + j50}{200 + j100}\right)$$
$$= 5\angle 30^\circ \times \left(\frac{1 + j}{4 + j2}\right)$$

Therefore, Thevenin voltage is,

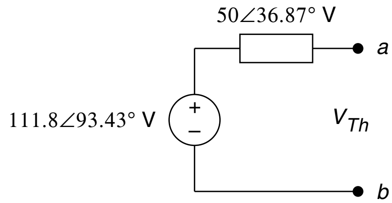
$$V_{Th} = I \times (50 + j50) = 5\angle 30^\circ \times \left(\frac{1+j}{4+j2}\right) \times (50 + j50) = 111.8\angle 93.43^\circ \text{ V}$$

Thevenin impedance is given as,

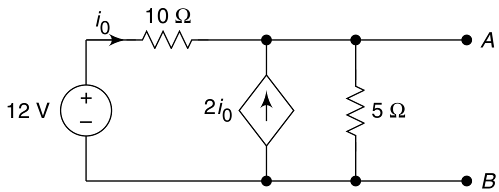
$$Z_{Th} = (150 + j50) \| (50 + j50)$$
$$= \frac{(150 + j50) \times (50 + j50)}{(150 + j50) + (50 + j50)}$$
$$= 50\angle 36.87^\circ\ \Omega$$



Thus, Thevenin equivalent circuit is shown in the figure below.

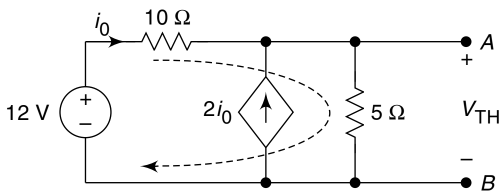


5.33 Find Thevenin's equivalent circuit across the terminals *A* and *B* for the network shown in the figure.



Solution The circuit has both dependent and independent sources. We find V_{Th} and I_{SC} and then taking the ratio we get Z_{Th}.

To find V_{Th}: By KVL for the supermesh shown,



(a)

$$10i_0 + V_{Th} - 12 = 0$$

$$\Rightarrow \qquad V_{Th} = 12 - 10i_0 \qquad \text{(i)}$$

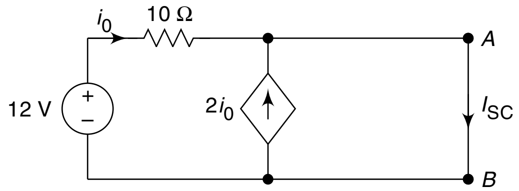
By KCL at node *A*,

$$-i_0 - 2i_0 + \frac{V_{Th}}{5} = 0$$

$$\Rightarrow \qquad V_{Th} = 15i_0 \qquad \text{(ii)}$$

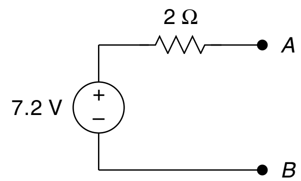
From (i) and (ii) we get,

$$V_{Th} = 7.2\text{ V} \qquad Ans.$$



(b)

To find I_{SC} When the terminals A and B are shorted, no current flows through the 5 Ω resistance. The circuit is shown in Fig. (c).



(c)

By KVL for the supermesh,

$$10i_0 = 12 \quad \Rightarrow \quad i_0 = 1.2\text{ A}$$

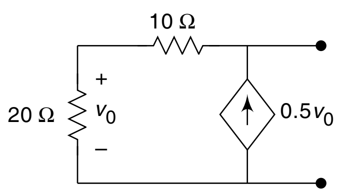
By KCL at node A,

$$I_{SC} = 3i_0 = 3.6\text{ A} \qquad Ans.$$

Therefore, the Thevenin impedance is given as,

$$Z_{Th} = \frac{V_{Th}}{I_{SC}} = \frac{7.2}{3.6} = 2\ \Omega \qquad Ans.$$

5.34 Find Thevenin's equivalent circuit for the network shown in the figure.



Solution This circuit does not have any independent source; it has only a dependent current source. Therefore, the Thevenin equivalent voltage will be zero.

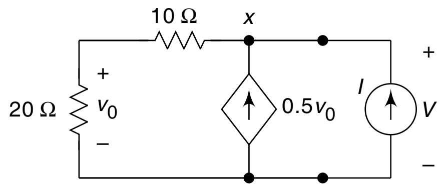
$\therefore \qquad V_{Th} = 0 \qquad Ans.$

To find Thevenin equivalent impedance, we connect a test current source of value I. Let the voltage across this test source be V.

By KCL at node x,

$$\frac{V-v_0}{10}-0.5v_0-I=0$$

$$\Rightarrow \qquad V=10I+6v_0 \qquad \text{(i)}$$



(a)

Also, $$v_0=20\times(0.5v_0+I)=10v_0+20I$$

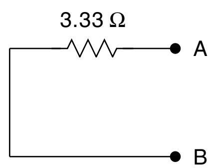
$$\Rightarrow \qquad v_0=-\frac{20}{9}I \qquad \text{(ii)}$$

Putting the value of v_0 in (i) from (ii),

$$V=10I+6\times\left(-\frac{20}{9}I\right)=-\frac{10}{3}I$$

$$\therefore \qquad Z_{Th}=\frac{V}{I}=-\frac{10}{3}=-3.33\ \Omega \qquad \textit{Ans.}$$

The Thevenin equivalent circuit is shown in Fig. (b).



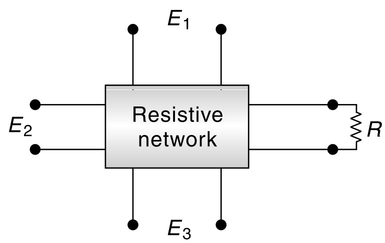
(b)

Maximum Power Transfer Theorem

5.35 In the network shown, the power dissipated in R when E_1, E_2 or E_3 acting alone is
(a) 20 W, 80 W, and 5 W respectively.
(b) 30 W, 270 W, and 120 W respectively.

Calculate the maximum power that R can dissipate due to the simultaneous action of all the sources. Calculate both for (a) and (b).

What will be the minimum power dissipated in R when all the sources are acting simultaneously?



Solution Current for E_1 at R, $i_1=\pm\sqrt{\frac{P_1}{R}}$

Current for E_2 at R, $i_2=\pm\sqrt{\frac{P_2}{R}}$

Current for E_3 at R, $i_3=\pm\sqrt{\frac{P_3}{R}}$

$\therefore$ Total current flow for simultaneous action of all the three sources is,

$$i=\pm i_1\pm i_2\pm i_3=\pm\sqrt{\frac{P_1}{R}}\pm\sqrt{\frac{P_2}{R}}\pm\sqrt{\frac{P_3}{R}}$$

$$\therefore \qquad \text{Power, } P = i^2 R = \left[\pm\sqrt{\frac{P_1}{R}} \pm \sqrt{\frac{P_2}{R}} \pm \sqrt{\frac{P_3}{R}}\right]^2 R = [\pm\sqrt{P_1} \pm \sqrt{P_2} \pm \sqrt{P_3}]^2$$

- For maximum power,

$$P_{\text{max}} = [\sqrt{P_1} + \sqrt{P_2} + \sqrt{P_3}]^2$$

(a) $P_{\text{max}} = [\sqrt{20} + \sqrt{80} + \sqrt{5}]^2 = [2\sqrt{5} + 4\sqrt{5} + \sqrt{5}]^2 = 49 \times 5 = 245 \text{ W}$ *Ans.*

(b) $P_{\text{max}} = [\sqrt{30} + \sqrt{270} + \sqrt{120}]^2 = [4\sqrt{5} - 3\sqrt{5}]^2 = 1080 \text{ W}$ *Ans.*

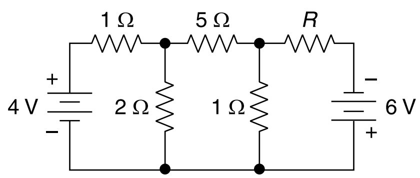
- For minimum power,

(a) $P_{\text{min}} = [-\sqrt{20} + \sqrt{80} - \sqrt{5}]^2 = [4\sqrt{5} - 3\sqrt{5}]^2 = 5 \text{ W}$ *Ans.*

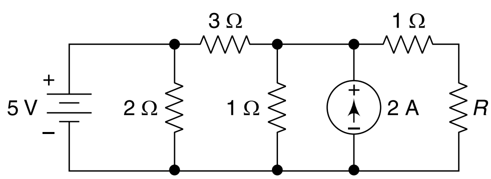
(b) $P_{\text{min}} = [-\sqrt{30} + \sqrt{270} - \sqrt{120}]^2 = [-\sqrt{30} + 3\sqrt{30} - 2\sqrt{3}]^2 = 0 \text{ W}$ *Ans.*

5.36 Find the value of R in the circuit of the figure such that maximum power transfer takes place. What is the amount of this power?

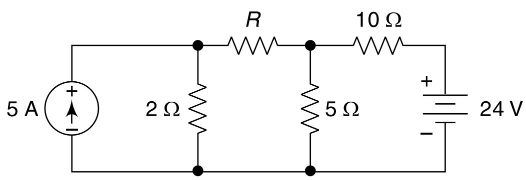
(a)



(b)



(c)



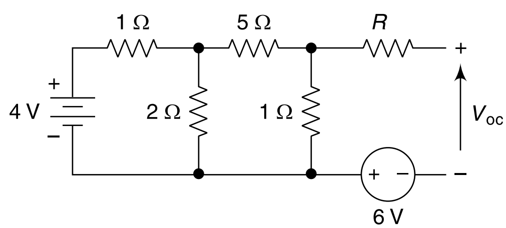
Solution

(a) Removing the résistance R,

$\therefore \qquad 3i_1 - 2i_2 = 4$

and $\qquad -2i_1 + 8i_2 = 0$

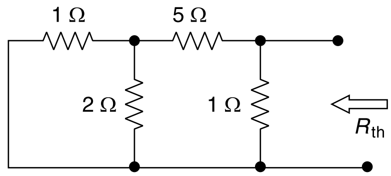
Solving, $\qquad i_2 = \frac{2}{5}$ A



$\therefore \qquad 1 \times i_2 + 6 = V_{oc}$

$\Rightarrow \qquad V_{oc} = \left(6 + \frac{2}{5}\right) = \frac{32}{5}$ V

Also, to find the R_{th},

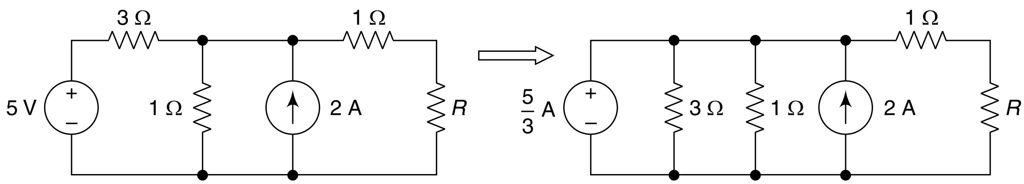


$$R_{th} = \left[\left(\frac{1\times 2}{1+2}\right)+5\right]||[1] = \left(\frac{2}{3}+5\right)||1 = \frac{\frac{17}{3}\times 1}{\frac{17}{3}+1} = \frac{17}{20}\ \Omega$$

$\therefore$ For maximum power transfer, $R = R_{th} = \dfrac{17}{20} = 0.85\ \Omega$ *Ans.*

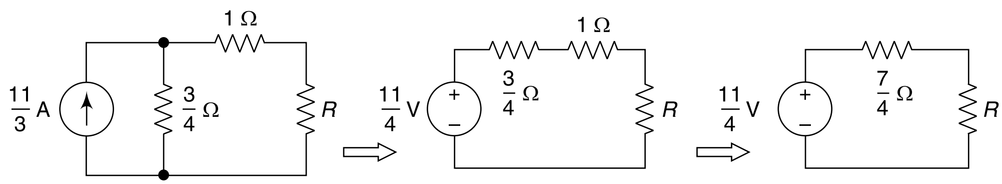
$\therefore$ Maximum power $P_{max} = \dfrac{V_{oc}^2}{4R} = 12\ \text{W}$ *Ans.*

(b) In the network, 2 Ω resistor is connected in parallel with an ideal voltage source of 5 V; hence this resistance can be removed without affecting the current flows in the other branches.



Converting the voltage source into current source,

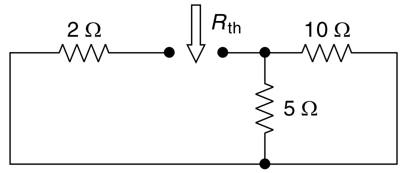
$$\left(\frac{5}{3}+2\right)\text{A} = \frac{11}{3}\ \text{A}$$



For maximum power transfer, $R = \dfrac{7}{4}\ \Omega$

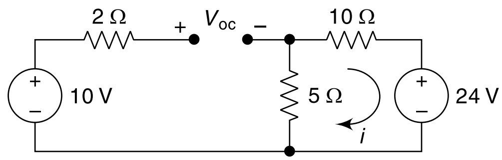
Maximum Power, $P_{max} = \dfrac{\left(\frac{11}{4}\right)^2}{4\times 7/4} = 1.08\ \text{W}$ *Ans.*

(c) *To find R_{th}*



$$R_{th} = \frac{10 \times 5}{10+5} + 2 = 5.33\ \Omega \qquad Ans.$$

To find V_{oc}

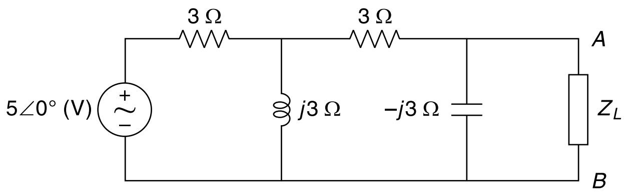


$$i = -\frac{24}{15} = -1.6\text{ A}$$

$\therefore$ $$V_{oc} = 5i + 10 = -8 + 10 = 2\text{ V}$$

$\therefore$ $$P_{max} = \frac{4}{4 \times 5.33} = 0.188\text{ W} \qquad Ans.$$

5.37 In the network shown, find the value of Z_L to which the maximum power can be delivered. Hence, find the value of the maximum power.



Solution With respect to terminals A and B, the Thevenin voltage is,

$$V_{th} = \frac{5\angle 0^\circ}{3 + \dfrac{j3(3-j3)}{3-j3+j3}} \times \left(\frac{j3}{3+j3-j3}\right) = \frac{45\angle 0^\circ}{18+j9} = 2.236\angle -26.56^\circ\text{ (V)}$$

and Thevenin impedance,

$$Z_{th} = \frac{\left(3 + \dfrac{3 \times j3}{3+j3}\right) \times (-j3)}{3 + \dfrac{3 \times j3}{3+j3} - j3} = 3\angle -53.12^\circ\ \Omega = (1.8 - j2.4)\ \Omega$$

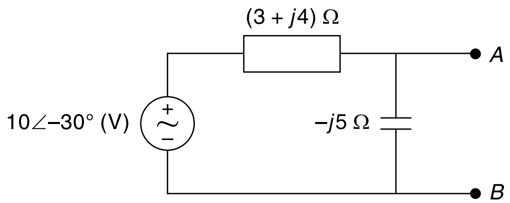
For maximum power transfer, $Z_L = Z_{th}^* = (1.8 + j2.4)\ \Omega$ *Ans.*

$\therefore$ Current, $$I = \frac{2.236\angle -26.56^\circ}{1.8 \times 2} = 0.621\angle -26.56^\circ\text{ A}$$

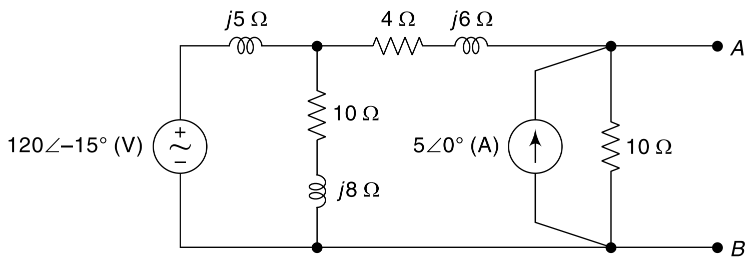
The value of the maximum power is, $$P_{max} = \frac{(V_{th})^2}{4R} = \frac{(2.236)^2}{4 \times 1.8} = 0.694\text{ W} \qquad Ans.$$

5.38 A loudspeaker is connected across terminals A and B of the network. What should its impedance be to obtain maximum power dissipation in it?

(a)



(b)



Solution

(a) Equivalent impedance with respect to the terminals A and B is,

$$Z_{th} = \frac{(3+j4)(-j5)}{3+j4-j5} = 7.9\angle -18.43°\ \Omega = (7.5 - j2.5)\ \Omega$$

For maximum power transfer, $Z_L = Z_{th}^* = (7.5 + j2.4)\ \Omega$ *Ans.*

(b) Equivalent impedance with respect to the terminals A and B is,

$$Z_{th} = \left[\frac{(10+j8)j5}{10+j8+j5} + 4 + j6\right] || \ 10 = \left(\frac{-40 + j50 + 40 + j52 + j60 - 78}{10 + j13}\right) || 10$$

$$= 6.14\angle 30°\ \Omega = (5.316 + j3.07)\ \Omega$$

For maximum power transfer, $Z_L = Z_{th}^* = 6.14\angle -30°\ \Omega = (5.316 - j3.07)\ \Omega$

5.39 Two inductors each of 1 Ω reactance and negligible resistance are connected in series across a 2 V a. c. source. Find the value of resistance which should be connected across one of the inductors for maximum power dissipation. Also, find the maximum power.

Solution Here, $Z = \dfrac{R \times j1}{R + j1} + j1 = \dfrac{-1 + j2R}{R + j1}$

$\therefore$ Current $I = \dfrac{2\angle 0°}{Z} = \dfrac{2\angle 0° \times (R + j1)}{-1 + j2R}$

$\therefore$ Current through the resistance, $I_R = I \times \dfrac{j1}{R + j1} = \dfrac{j2}{-1 + j2R}$

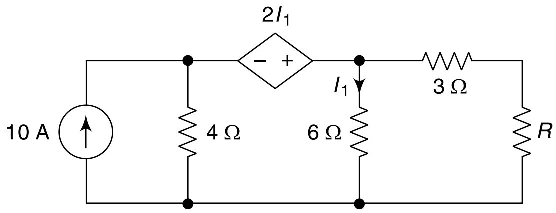
$\therefore$ Power, $P = |I|^2\ R = \dfrac{4R}{1 + 4R^2}$

For maximum power, $\dfrac{dP}{dR} = 0 \Rightarrow \dfrac{(1+4R^2)\times 4 - 4R\times 8R}{(1+4R^2)^2} = 0$

$\Rightarrow \qquad R = 0.5\ \Omega \qquad$ *Ans.*

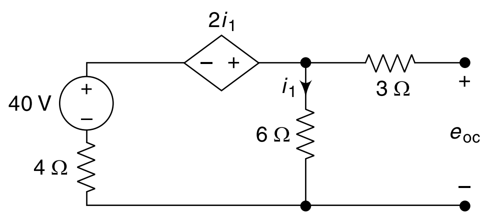
$\therefore$ Maximum Power, $P_{max} = \dfrac{4\times 0.4}{1+4\times(0.5)^2} = 1\text{ W} \qquad$ *Ans.*

5.40



In the network shown, calculate the maximum power that may be dissipated in the external resistor R.

Solution Transforming the current source into voltage source,



By KVL,

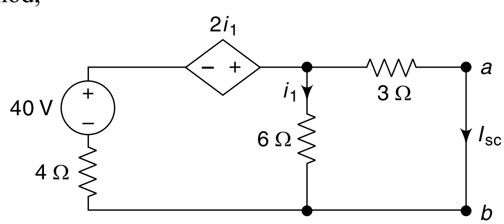
$$6i_1 + 4i_1 - 40 - 2i_1 = 0$$

$\Rightarrow \qquad i_1 = 5\text{ A}$

$\therefore \qquad e_{oc} = 6i_1 = 30\text{ V}$

For maximum power, $R = R_{eq}$

Shorting the terminals a–b and solving by loop method,

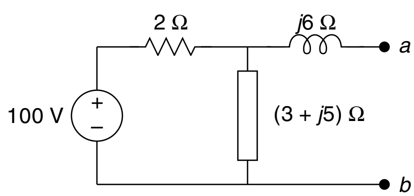


$$I_{sc} = 5\text{ A}$$

$\therefore \qquad R_{th} = \dfrac{30}{5} = 6\ \Omega$

$\therefore \qquad P_{max} = \dfrac{(30)^2}{4\times 6} = \dfrac{900}{24} = 37.5\text{ W}$

5.41 Find the Thevenin's equivalent between the points a and b for the circuit given in the figure. What should be the value of impedance connected between a and b for maximum power to be transferred from the sources? Obtain the amount of the maximum power.



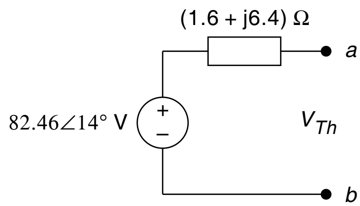
Solution Here the current I is,

$$I = \frac{100}{2+3+j5} = \frac{100}{5+j5} = (10 - j10)\text{ A}$$

$$\therefore \qquad V_{Th} = I \times (3 + j5) = (10 - j10) \times (3 + j5)$$
$$= (80 + j20)$$
$$= 82.46\angle 14^\circ \text{ Volt} \qquad \textit{Ans.}$$

$$\therefore \qquad Z_{Th} = j6 + \frac{2 \times (3 + j5)}{2 + 3 + j5} = (1.6 + j6.4)\ \Omega \qquad \textit{Ans.}$$

Thevenin's equivalent circuit is shown.

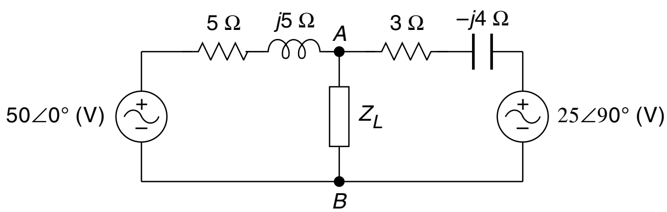


For maximum power transfer, the impedance should be complex conjugate of Thevenin impedance.

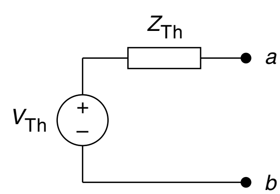
$$\therefore \qquad Z_L = (1.6 - j6.4)\ \Omega \qquad \textit{Ans.}$$

Amount of the maximum power is, $P_{\max} = \dfrac{V_{Th}^{\ 2}}{4R} = \dfrac{(82.46)^2}{4 \times 1.6} = 1062.5$ Watt $\qquad$ *Ans.*

5.42 In the network in the figure, two voltage sources act on the load impedance connected to the terminal A and B. If the load is variable in both reactance and resistance, for what load, Z_L will receive maximum power? What is the value of maximum power?



Solution



Here, $\qquad V_1 = 50\angle 0^\circ = 50$ V; $\quad$ and $\quad V_2 = 25\angle 90^\circ = j25$ V

Current in the circuit, $\quad I = \dfrac{50 - j25}{5 + j5 + 3 - j4} = \dfrac{50 - j25}{8 + j1}$ A

Thevenin voltage,

$$V_{Th} = 50 - I \times (5 + j5) = 50 - \left(\frac{50 - j25}{8 + j1}\right) \times (5 + j5) = \frac{25 - j75}{8 + j1}$$
$$= 9.8\angle -78.7^\circ$$
$$= (1.923 - j9.615) \quad \text{Volt} \qquad \textit{Ans.}$$

Thevenin impedance, $Z_{Th} = \dfrac{(5+j5)\times(3-j4)}{(5+j5)+(3-j4)} = \dfrac{35-j5}{8+j1} = (4.23 - j\,1.154)\ \Omega$ *Ans.*

Thus, the Thevenin's equivalent circuit is shown.

For maximum power transfer to the load, $Z_L = Z_m{}^* = (4.23 + j1.154)\ \Omega$ *Ans.*

The value of the maximum power is, $P_{\max} = \dfrac{V_{Th}{}^2}{4R} = \dfrac{9.8^2}{4\times 4.23} = 5.676$ Watt *Ans.*

5.43 A network has two output terminals. The open circuit voltage at these terminals is 260 V. The current flowing through the terminals is 20 A when the terminals are short circuited. Also, the current is 13 A when a coil of 11 ohm reactance and negligible resistance is connected across the terminals. Find the impedance components of the equivalent circuit feeding the terminals. What value of load impedance will give maximum power transfer and what is the value of this power?

Solution Here, $V_{Th} = 260\,\text{V};\quad I_{SC} = 20\,\text{A}$

Let the Thevenin impedance across the terminals is, $Z = (R + jX)$

$$\therefore \qquad Z = \frac{V_{Th}}{I_{SC}} = \frac{260}{20} = 13\ \Omega$$

$$\therefore \qquad R^2 + X^2 = 169 \qquad \text{(i)}$$

When 11 ohm reactance is connected across the terminals, the current is 13 A.

$$\therefore \qquad \left|\frac{260}{R + j(X+11)}\right| = 13 \quad \Rightarrow \quad \left|R + j(X+11)\right| = \frac{260}{13} = 20$$

$$\therefore \qquad R^2 + (X+11)^2 = 400 \qquad \text{(ii)}$$

Solving (*i*) and (*ii*), we get,

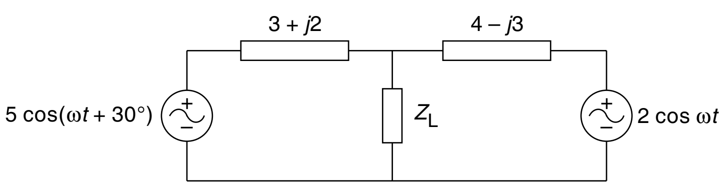
$$R = 12\ \Omega \quad X = 5\ \Omega$$

Therefore, Thevenin impedance, $Z_{Th} = (12 + j5)\,\Omega$ *Ans.*

For maximum power transfer, $Z_L = Z_{Th}{}^* = (12 - j5)\,\Omega$ *Ans.*

Value of maximum power, $P_{\max} = \dfrac{V_{Th}{}^2}{4R} = \dfrac{(260)^2}{4\times 12} = 1408.33\ \text{W}$ *Ans.*

5.44 What should be the value of Z_L for maximum power to be delivered in the circuit shown in the figure?



Solution In this circuit, when the voltage sources are replaced by their internal impedances; i.e., when they are short-circuited, the equivalent Thevenin impedance with respect to the load terminals is given by,

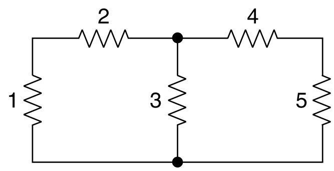
$$Z_{Th} = (3+j2)\,||\,(4-j3) = \frac{(3+j2)\times(4-j3)}{(3+j2)+(4-j3)} = \frac{18-j1}{7-j1} = \left(\frac{127}{50}+j\frac{11}{50}\right)\Omega$$
$$= (2.54 + j0.22)\ \Omega$$
$$= 2.55\angle 4.95^\circ \quad (\Omega)$$

For maximum power to be delivered, the load impedance should be complex conjugate of the Thevenin impedance, so that,

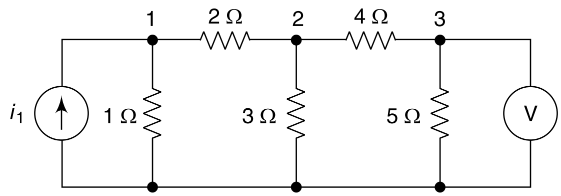
$$Z_L = Z_{Th}{}^* = (2.54 - j0.22)\Omega = 2.55\angle -4.95^\circ\ (\Omega) \qquad \textit{Ans.}$$

Reciprocity Theorem

5.45 Verify the Reciprocity Theorem for the network shown in the figure using current source and a voltmeter. All the values are in ohm.



Solution Using a current source and a voltmeter,
Let, e_1, e_2 be node voltages, v_1 be the voltmeter reading.



By KCL,

At node (1) $\Rightarrow$ $3e_1 - e_2 - 2i_1 = 0$ (i)

At node (2) $\Rightarrow$ $-6e_1 + 13e_2 - 3v_1 = 0$ (ii)

At node (3) $9v_1 = 5e_2$ (iii)

From (ii) $\Rightarrow$ $-6e_1 + 13\times\frac{9}{5}v_1 - 3v_1 = 0$

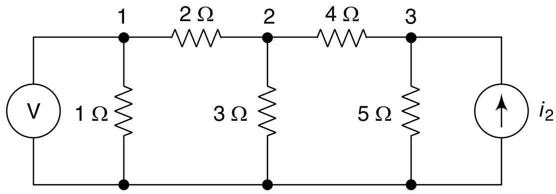
$\Rightarrow$ $-6e_1 + \left(\frac{117}{5} - 3\right)v_1 = 0$

$\Rightarrow$ $6e_1 + \frac{102}{5}v_1 \Rightarrow e_1 = \frac{17}{5}v_1$

From (i) $\Rightarrow$ $3\times\frac{17}{5}v_1 - \frac{9}{5}v_1 = 2i$

$\Rightarrow$ $\left(\frac{i_1}{v_1}\right) = \left(\frac{21}{5}\right)$ (A)

Interchanging the positions of the current source and the voltmeter,
Now, let v_2 be the voltmeter reading



By KCL,

At node (1) $\Rightarrow$ $3v_2 = e_2$ (iv)

At node (2) $\Rightarrow$ $-6v_2 + 13e_2 - 3e_3 = 0$

$\Rightarrow$ $-6v_2 + 13 \times 3v_2 - 3e_3 = 0$

$\Rightarrow$ $e_3 = 11v_2$ (v)

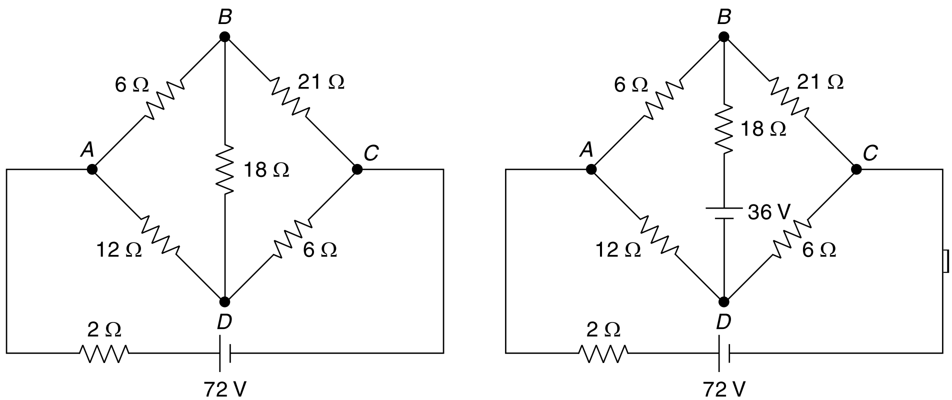
At node (3) $\Rightarrow$ $5e_3 - 5e_2 + 4e_3 - 20i_2 = 0$

$\Rightarrow$ $20i_2 = 9e_3 - 5e_2 = 9 \times 11v_2 - 5 \times 3v_2 = 84v_2$

$\Rightarrow$ $\left(\frac{i_2}{v_2}\right) = \left(\frac{21}{5}\right)$ (B)

From equations (A) and (B), Reciprocity theorem is proved.

5.46 Solve the network shown in Figure (a) and hence find the current in the 2 Ω resistor in Figure (b) when an emf of 36 V is added in the branch *BD* as shown in Figure 7(b). All values are in ohm.

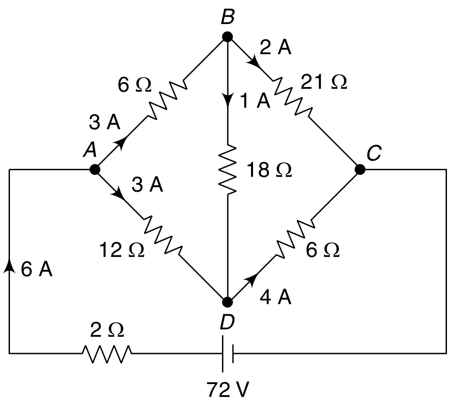


(a) *(b)*

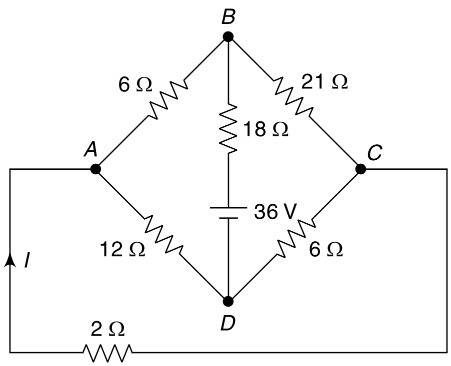
Solution

- Solve by any method of network analysis.
- We consider the 36 V source acting alone.

When 72 V sourer is acting alone, by network analysis,
The current in 2 Ω resistor = 6 A and in 18 Ω resistor = 1 A



(a)



(b)

By Reciprocity theorem,

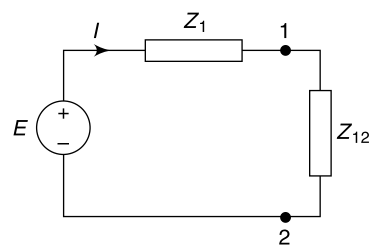
$$\frac{72}{1} = \frac{36}{I} \quad \Rightarrow \quad I = 0\cdot 5 \text{ A}$$

[Here, I = Current in 2 Ω resistor when 36 V source is acting alone]

∴ Current in 2Ω resistor for simultaneous action of two sources

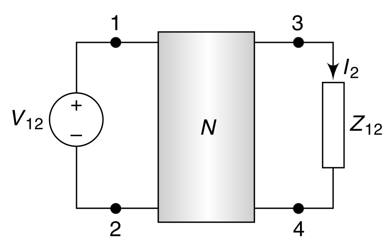
$$I = (6 - 0.5) = 5.5 \text{ A}$$

5.47 An e.m.f. source E, having negligible internal impedance is connected in series with an impedance Z_1 to the input terminals 1–2 of a linear, bilateral four terminal network. It produces a current I_2 in impedance Z_L connected across the output terminals 3–4. The emf source is now transferred so as to act, in series with Z_2, between terminal 3–4. Z_1 is disconnected and the input terminals 1–2 are short circuited. The short-circuited current traversing terminals 1–2 is then I_1. Prove that the impedance looking into terminals 1–2 under the first condition is,



$$Z_{12} = \frac{Z_1 I_2}{I_1 - I_2}$$

Solution Let the impedance looking into terminals 1–2 be Z_{12}. Thus the network becomes:



$$\therefore \qquad I = \frac{E}{Z_1 + Z_{12}}$$

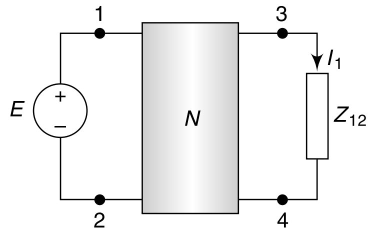
∴ Voltage across 1–2, $V_{12} = \dfrac{E \times Z_{12}}{Z_1 + Z_{12}}$

So, the circuit becomes as shown.

The given network is linear and bilateral and according to the reciprocity theorem, if the source E is put across terminals 1–2, the response current flowing through Z_2 will be I_1 as shown.

Now, if a voltage equal to V_{12} is applied instead of E, the current flowing through Z_2 will be,

$$\frac{I_1}{E}\times V_{12}=\frac{I_1}{E}\times\frac{E\times Z_{12}}{Z_1+Z_{12}}=I_1\times\frac{Z_{12}}{Z_1+Z_{12}}$$

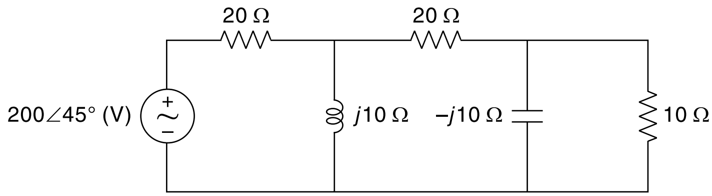


But, this current is equal to I_2.

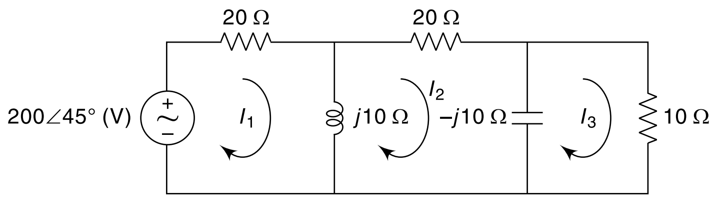
$$\therefore \qquad I_2=I_1\frac{Z_{12}}{Z_1+Z_{12}}$$

$$\Rightarrow \qquad Z_{12}=\left(\frac{Z_1I_2}{I_1-I_2}\right) \quad \text{(Proved)}$$

5.48 Verify the reciprocity theorem for the ladder network shown in figure.



Solution Let, the three loop currents be I_1, I_2, and I_3. By KVL for the three loops,



$$(20+j10)I_1-j10I_2=200\angle 45^\circ$$

$$-j10I_1+20I_2+j10I_3=0$$

$$j10I_2+(10-j10)I_3=0$$

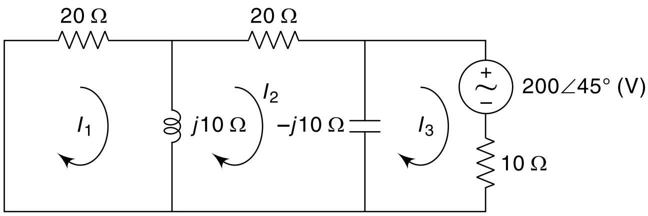
Solving for I_3,

$$I_3=\frac{\begin{vmatrix}(20+j10) & -j10 & 200\angle 45^\circ\\ -j10 & 20 & 0\\ 0 & j10 & 0\end{vmatrix}}{\begin{vmatrix}(20+j10) & -j10 & 0\\ -j10 & 20 & j10\\ 0 & j10 & (10-j10)\end{vmatrix}}=\frac{200\angle 45^\circ\times 100}{(20+j10)(200-j200+100)-j10(j100+100)}$$

$$=2.169\angle 57.53^\circ\ (\text{A})$$

Now by interchanging the positions of the voltage source and the response current, we get,

By KVL,



$$(20+j10)I_1-j10I_2=0$$

$$-j10I_1 + 20I_2 + j10I_3 = 0$$

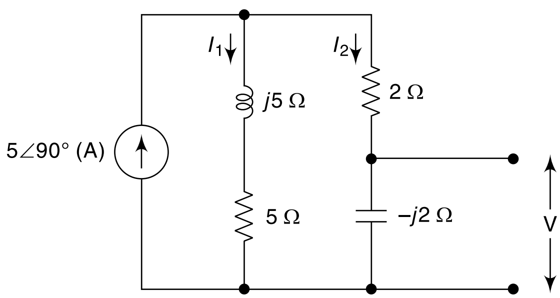
$$j10I_2 + (10 - j10)I_3 = 200\angle 45°$$

Solving for I_1,

$$I_1 = \frac{\begin{vmatrix} 0 & -j10 & 0 \\ 0 & 20 & 0 \\ 200\angle 45° & j10 & 0 \end{vmatrix}}{\begin{vmatrix} (20+j10) & -j10 & 0 \\ -j10 & 20 & j10 \\ 0 & j10 & (10-j10) \end{vmatrix}} = 2.169\angle 57.53° \text{ (A)}$$

Since the currents in both the cases are the same, reciprocity theorem is verified.

5.49 In this circuit, find voltage V. Interchange the current source and resulting voltage V and show that the reciprocity theorem is verified.



Solution Here, the current $I_2 = 5\angle 90° \times \dfrac{5 + j5}{5 + j5 + 2 - j2} = 4.64\angle 111.8°$ (A)

$\therefore$ The voltage, $V = I_2 \times Z_C = 4.64\angle 111.8° \times (-j2) = 9.28\angle 21.8°$ (V)

Now, interchanging the positions of the current source and the finding the resulting voltage, we get,

$$I_1 = 5\angle 90° \times \frac{-j2}{-j2 + 5 + 2 + j5}$$

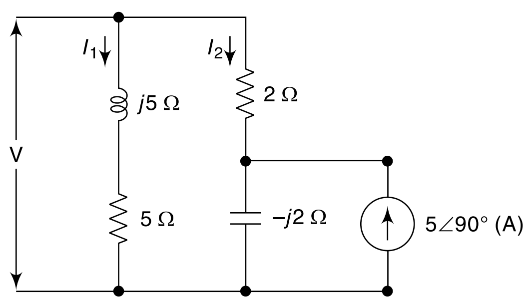
$$= 1.31\angle{-23.2°} \text{ (A)}$$

$\therefore$ The voltage,

$$V = 1.31\angle{-23.2°} \times (5 + j5)$$

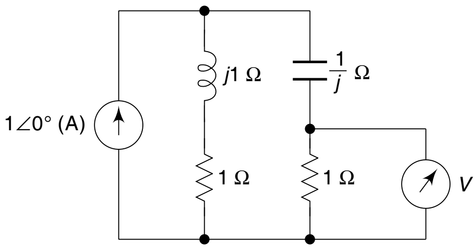
$$= 1.31\angle{-23.2°} \times 7.075\angle 45°$$

$$= 9.28\angle 21.8° \text{ (V)}$$



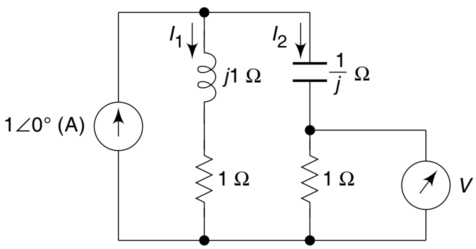
As V is same as obtained before interchanging the position of the current source, reciprocity theorem is verified.

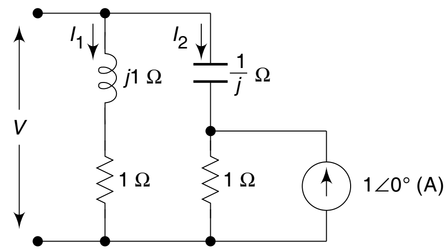
5.50 In the given circuit of the figure, find the reading of the voltmeter V. Interchange the current source and voltmeter and verify the reciprocity theorem.



Solution Here, the current $I_2 = 1\angle 0^\circ \times \dfrac{1+j1}{1+j1+1-j1} = 0.707\angle 45^\circ \text{ (A)}$

$\therefore$ the voltage, $V = I_2 \times Z_C = 0.707\angle 45^\circ \times (1) = 0.707\angle 45^\circ \text{ (V)}$





Now, interchanging the positions of the current source and the finding the resulting voltage, we get,

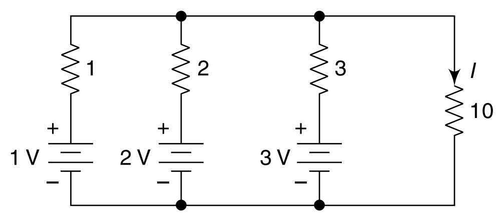
$$I_1 = 1\angle 0^\circ \times \frac{1}{1-j1+j1+1} = 0.5\angle 0^\circ \text{ (A)}$$

$\therefore$ the voltage, $V = 0.5\angle 0^\circ \times (1+j1) = 0.5\angle -23.2^\circ \times \sqrt{2}\angle 45^\circ = 0.707\angle 45^\circ \text{ (V)}$

As 'V' is same as obtained before interchanging the position of the current source, reciprocity theorem is verified.

Millman's Theorem

5.51 Find the load current using Millman's theorem. All values are in ohm.



Solution Here, $E_1 = 1$ V, $E_2 = 2$ V, $E_3 = 3$ V

$Z_1 = 1$ ℧, $Z_2 = 2$ ℧, $Z_3 = 3$ ℧

$\therefore$ $Y_1 = 1$ ℧, $Y_2 = 0.5$ ℧, $Y_3 = \dfrac{1}{3}$ ℧

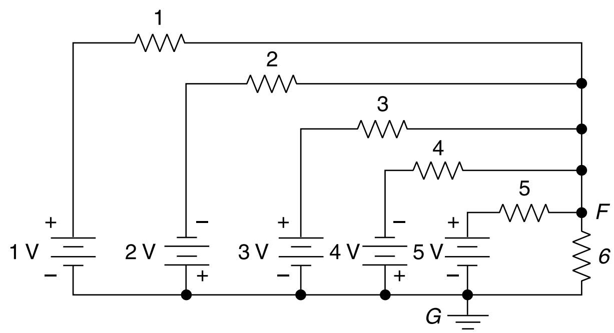
By Millman's theorem, the equivalent circuit is shown.

$$\therefore \quad E = \frac{\sum_{i=1}^{3} E_i Y_i}{\sum_{i=1}^{3} Y_i} = \frac{1\times 1 + 2\times 0.5 + 3\times \frac{1}{3}}{1 + 0.5 + \frac{1}{3}} = \frac{3}{\frac{11}{6}} = \frac{18}{11} \text{ V}$$

and $$Z = \frac{1}{\sum_{i=1}^{3} Y_i} = \frac{6}{11}\ \Omega$$

$\therefore$ $$I = \frac{E}{Z+10} = \frac{\frac{18}{11}}{\frac{6}{11}+10} = \frac{18}{116} = \frac{9}{58}\ \text{A} \qquad \textit{Ans.}$$

5.52 Obtain the potential of node F with respect to node G in the circuit of the figure. All values are in ohm.



Solution By Millman's theorem, equivalent voltage is,

$$V = \frac{\sum_{i=1}^{5} E_i Y_i}{\sum_{i=1}^{5} Y_i} = \frac{1\times 1 - 2\times 1/2 + 3\times 1/3 - 4\times 1/4 + 5\times 1/5}{1 + 1/2 + 1/3 + 1/4 + 1/5} = \frac{60}{137}\ \text{V}$$

Equivalent impedance, $$Z = \frac{1}{\sum_{i=1}^{5} Y_i} = \frac{1}{1 + 1/2 + 1/3 + 1/4 + 1/5} = \frac{60}{137}\ \Omega$$

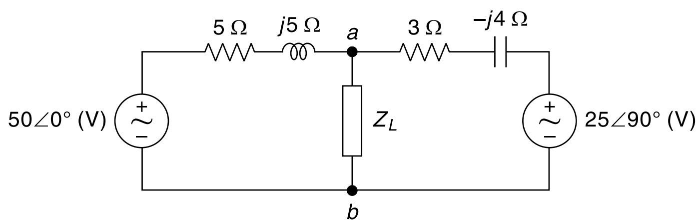
Therefore, the current through the $6\ \Omega$ resistance is,

$$I = \frac{V}{Z+6} = \frac{60/137}{60/137+6} = \frac{60}{882}\ \text{A}$$

Hence, the voltage between the points F and G is,

$$V_{FG} = 6\times I = 6\times\frac{60}{882} = \frac{60}{147} \quad \text{Volt}$$

5.53 In the network, two voltage sources act on the load impedance connected to terminals a, b. If the load is variable in both reactance and resistance, what load Z_L will receive the maximum power? What is the value of the maximum power? Use Millman's theorem.



Solution $V_1 = 50\angle 0° = 50$ V; $Z_1 = (5 + j5)\ \Omega$; $Y_1 = \dfrac{1}{Z_1} = \dfrac{1}{(5+j5)} = (0.1 - j0.1)$ ℧

$V_2 = 25\angle 90° = j25$ V; $Z_2 = (3 - j4)\ \Omega$; $Y_2 = \dfrac{1}{Z_2} = \dfrac{1}{(3-j4)} = (0.12 + j0.16)$ ℧

∴ Millman voltage source,

$$V_m = \frac{V_1Y_1 + V_2Y_2}{Y_1 + Y_2} = \frac{50(0.1 - j0.1) + j25(0.12 + j0.16)}{(0.1 - j0.1) + (0.12 + j0.16)} = 9.807\angle{-78.65°}\ \text{(V)}$$

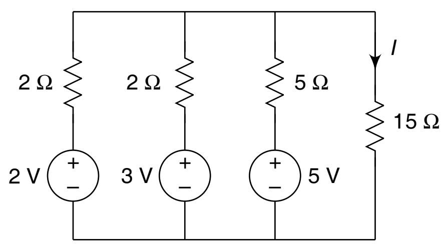
∴ Millman impedance,

$$Z_m = \frac{1}{Y_1 + Y_2} = \frac{1}{0.22 - j0.06} = 4.385\angle{-15.25°} = (4.23 - j1.15)\ \Omega$$

For maximum power transfer to the load, $Z_L = Z_m^* = (4.23 + j1.15)\ \Omega$ *Ans.*

∴ Maximum power, $P_{\max} = \dfrac{V_m^2}{4R_L} = \dfrac{(9.807)^2}{4 \times 4.23} = 5.68$ W *Ans.*

5.54 Calculate the load current I in the circuit in the figure by Millman's theorem.



Solution By Millman's theorem, equivalent voltage,

$$V = \frac{\sum EY}{\sum Y} = \frac{\frac{2}{2} + \frac{3}{2} + \frac{5}{5}}{\frac{1}{2} + \frac{1}{2} + \frac{1}{5}} = \frac{35}{12} = 2.91667 \text{ Volt}$$

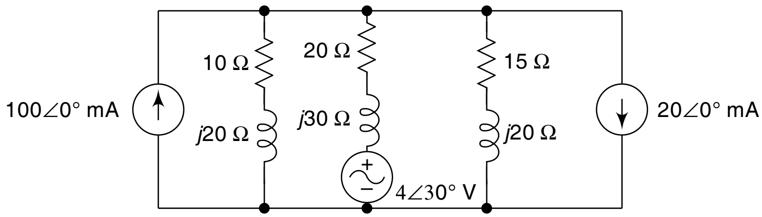
and equivalent impedance,

$$Z = \frac{1}{\sum Y} = \frac{1}{\frac{1}{2} + \frac{1}{2} + \frac{1}{5}} = \frac{10}{12} = 0.833\ \Omega$$

Therefore the current through the load resistance,

$$I = \frac{V}{Z + 15} = \frac{2.91667}{0.833 + 15} = 0.184 \text{ A} \qquad \textit{Ans.}$$

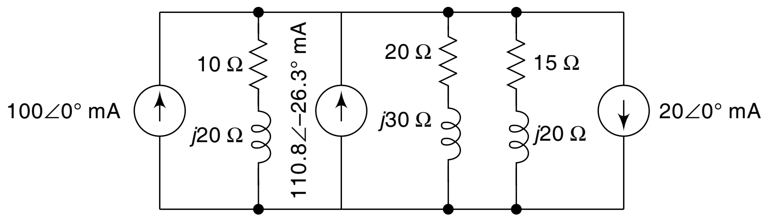
5.55 Use Millman's theorem to obtain an equivalent current source for the circuit shown in the figure. Also obtain the equivalent voltage source.



Solution We convert the voltage source into equivalent current source as,

$$I = \frac{V}{Z} = \frac{4\angle 30^\circ}{20 + j30} = 0.1108\angle -26.3^\circ (\text{A})$$

The modified circuit is shown in figure.



Total equivalent current source is,

$$I = 100\angle 0^\circ + 110.8\angle -26.3^\circ - 20\angle 0^\circ = (89.33 - j49.1) = 101.93\angle -28.8^\circ (\text{mA}) \qquad \textit{Ans.}$$

Total equivalent impedance is obtained as,

$$\frac{1}{Z} = \frac{1}{10 + j20} + \frac{1}{20 + j30} + \frac{1}{15 + j20} = (0.059 - j0.095)$$

$$\Rightarrow \qquad Z = (4.73 + j7.57)\,\Omega \qquad \textit{Ans.}$$

Equivalent voltage source is obtained as,

$$V = 101.93\angle -28.8^\circ \times 10^{-3} \times (4.73 + j7.57) = 0.9\angle 29.2^\circ (\text{V}) \qquad \textit{Ans.}$$

5.56 A symmetrical 440V, 3-phase system supplies a star-connected load. The branch impedances are $Z_R = 10\angle 30^\circ\ \Omega$, $Z_Y = 12\angle 45^\circ\ \Omega$, $Z_B = 15\angle 40^\circ\ \Omega$. Assuming the neutral of the supply to be earthed, calculate the voltage to earth of the star point. Assume the phase sequence RYB.

Solution Here, line voltages are, $V_{RY} = 400\angle 0^\circ$; $V_{YB} = 400\angle -120^\circ$;
$V_{BR} = 400\angle +120^\circ$

∴ Phase voltages are

$$V_R = \frac{400}{\sqrt{3}}\angle -30^\circ = 254\angle -30^\circ \text{ V};$$
$$V_Y = 254\angle -120^\circ - 30^\circ = 254\angle -150^\circ \text{ V};$$
$$V_B = 254\angle +120^\circ - 30^\circ = 254\angle 90^\circ \text{ V}$$

Phase admittances are,

$$Y_R = \frac{1}{Z_R} = \frac{1}{10\angle 30^\circ} = 0.1\angle -30^\circ \text{ (S)}$$
$$Y_Y = \frac{1}{Z_Y} = \frac{1}{12\angle 45^\circ} = 0.0833\angle -45^\circ \text{ (S)}$$
$$Y_B = \frac{1}{Z_B} = \frac{1}{15\angle 40^\circ} = 0.0667\angle -40^\circ \text{ (S)}$$

By Millman's theorem, the voltage of the load star point with respect to earth is,

$$V_{N'N} = \frac{V_R Y_R + V_Y Y_Y + V_B Y_B}{Y_R + Y_Y + Y_B}$$
$$= \frac{254\angle -30^\circ \times 0.1\angle -30^\circ + 254\angle -150^\circ \times 0.0833\angle -45^\circ + 254\angle 90^\circ \times 0.0667\angle -40^\circ}{0.1\angle -30^\circ + 0.0833\angle -45^\circ + 0.0667\angle -40^\circ}$$
$$= 18.59\angle -11.9^\circ \ (V) \qquad \textit{Ans.}$$

MULTIPLE-CHOICE QUESTIONS

5.1 Millman's theorem yields
(a) equivalent voltage source. (b) equivalent voltage or current source.
(c) equivalent resistance. (d) equivalent impedance.

5.2 The superposition theorem is applicable to
(a) current only. (b) voltage only.
(c) both current and voltage. (d)current, voltage and power.

5.3 Superposition theorem is not applicable for
(a) voltage calculations. (b) bilateral elements
(c) power calculations. (d) passive elements.

5.4 Thevenin's theorem can be applied to calculate the current in
(a) any load. (b) a passive load only.
(c) a linear load only. (d) a bilateral load only.

5.5 Norton's equivalent circuit consists of
(a) voltage source in parallel with impedance.
(b) voltage source in series with impedance.
(c) current source in parallel with impedance.
(d) current source in series with impedance.

5.6 The superposition theorem is applicable to
(a) linear responses only. (b) linear and non-linear responses.
(c) linear, non-linear and time-variant responses.

5.7 When a source is delivering maximum power to a load, the efficiency of the circuit
(a) is always 50%. (b) depends on the circuit parameters.
(c) is always 75%. (d) none of these.

5.8 Maximum power transfer occurs at a
(a) 100% efficiency. (b) 50% efficiency.
(c) 25% efficiency. (d) 75% efficiency.

5.9 Which of the following statements is true?
(a) A Norton's equivalent is a series circuit.
(b) A Thevenin's equivalent circuit is a parallel circuit.
(c) R-L circuit is dual pair.
(d) L-C circuit is a dual pair.

5.10 For a linear network containing generators and impedances, the ratio of the voltage to the current produced in other loop is the same as the ratio of voltage and current obtained if the position of the voltage source and the ammeter measuring the current are interchanged. This network theorem is known as
(a) Millman's theorem. (b) Norton's theorem.
(c) Tellegen's theorem. (d) Reciprocity theorem.

5.11 Under conditions of maximum power transfer from an ac source to a variable load
(a) the load impedance must also be inductive, if the generator impedance is inductive.
(b) the sum of the source and load impedance is zero.
(c) the sum of the source reactance and load reactance is zero.
(d) the load impedance has the same phase angle as the generator impedance.

5.12 Consider the following statements
The transfer impedances and admittances of a network remain constant when the position of excitation and response are interchanged if the network
1. is linear
2. consists of bilateral elements
3. has high impedance or admittance as the case may be.
4. is resonant.
Out of above these statements
(a) 1 and 2 are correct. (b) 1, 3 and 4 are correct.
(c) 2 and 4 are correct. (d) 1, 2, 3 and 4 are correct.

5.13 In a linear network, the ratio of voltage excitation to current response is unaltered when the position of excitation and response are interchanged. This assumption stems from the
(a) principle of duality. (b) reciprocity theorem.
(c) principle of superposition. (d) equivalence theorem.

5.14 If all the elements in a particular network are linear, then the superposition theorem hold when the excitation is
(a) dc only (b) ac only (c) either ac or dc (d) an impulse.

5.15 An a.c source of voltage E_s and an internal impedance of $Z_s = (R_s + jX_s)$ is connected to a load of impedance$Z_L = (R_L + jX_L)$. Consider the following conditions in this regard
1. $X_L = X_s$, if only X_L is varied.
2. $X_L = X_s$, if only X_S is varied.
3. $R_L = \sqrt{R_S^2 + (X_S + X_L)^2}$, if only R_L is varied.
4. $|Z_L| = |Z_S|$ if the magnitude of Z_L is varied, keeping the phase angle fixed.

Among these conditions, those which are to be satisfied for maximum power transfer from the source to the load would include

(a) 2 and 3 (b) 1 and 3 (c) 1, 2 and 4 (d) 2, 3 and 4

5.16 Reciprocity theorem is applicable to a network

1. which contains R, L and C as elements.
2. which is initially relaxed system.
3. which has both independent and dependent sources.

Tick out the correct combination from the combination given above

(a) 1 and 2 (b) 1 and 3 (c) 2 and 3 (d) 1, 2 and 3.

5.17 Reciprocity theorem is applicable to

(a) circuits with one independent source
(b) circuits with only one independent source and no dependent source
(c) circuits with any number of independent sources
(d) circuits with any number of sources.

5.18 Substitution theorem is applicable for a network which has

1. unique solution.
2. one or two non-linear elements.
3. one non-linear or time-varying element.

Choose the correct combination from the combination given above

(a) 1 and 2 (b) 1 and 3 (c) 2 and 3 (d) 1, 2 and 3.

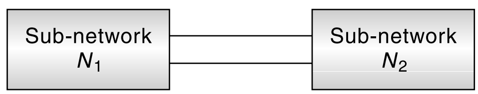
5.19 Substitution theorem applies to

(a) linear networks. (b) non-linear networks.
(c) linear time-invariant networks. (d) any networks.

5.20 Which of the following theorems is applicable for both linear and non-linear circuits?

(a) Superposition (b) Thevenin (c) Norton (d) none of these.

5.21 A network is composed of two sub-networks N_1 and N_2 as shown in the given figure. If the sub-network N_1 contains only linear, bilateral, time-invariant elements, then it can be replaced by its Thevenin equivalent even if the sub-network N_2 contains



(a) a two-terminal element which is non-linear
(b) a non-linear inductance mutually coupled to an element in N_1
(c) an element which is linear, but mutually coupled to some element in N_1
(d) a dependent source the value of which depends upon the voltage or current in some element in N_1.

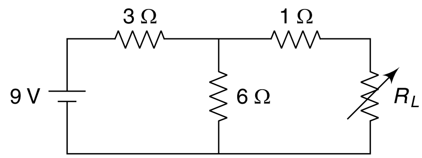
5.22 A certain network consists of two ideal identical voltage sources and a large number of ideal resistors. The power consumed in one of the resistors is 4W when either of the two sources is active and the other is replaced by a short-circuit. The power consumed by the same resistor when both the sources are active would be

(a) zero or 16 W (b) 4 W or 8 W (c) zero or 8 W (d) 8 W or 16 W.

5.23 If a network has all linear elements except for a few non-linear ones, then superposition theorem

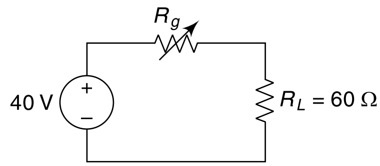
(a) cannot hold at all.
(b) always holds.
(c) may hold on careful selection of element values, source waveform and response.
(d) holds in case of direct current excitations.

5.24 The maximum power that can be dissipated in the load in the circuit shown in figure is



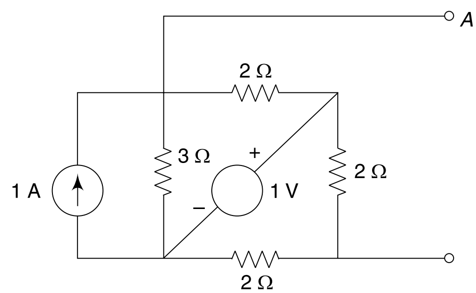
(a) 3 W (b) 6 W (c) 6.75 W (d) 13.5 W

5.25 If R_g in the circuit shown in figure is variable between 20 Ω and 80 Ω, then the maximum power transferred to the load R_L will be



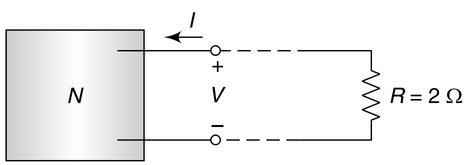
(a) 15 W (b) 13.33 W
(c) 6.67 W (d) 2.4 W

5.26 Thevenin impedance across the terminals *AB* of the given network is



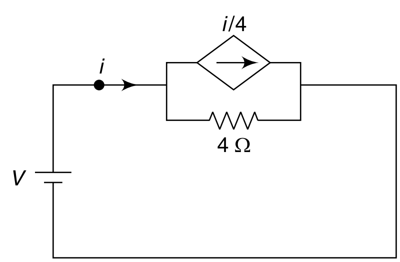
(a) $\frac{10}{3}$ Ω (b) $\frac{20}{9}$ Ω (c) $\frac{13}{4}$ Ω (d) $\frac{11}{5}$ Ω

5.27 The *V–I* relation for the network shown in the given box is $V = 4I - 9$.
If now a resistor $R = 2\ \Omega$ is connected across it, then the value of *I* will be



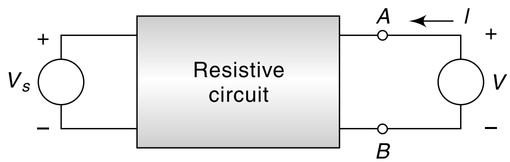
(a) –4.5 A (b) –1.5 A (c) 1.5 A (d) 4.5 A

5.28 In the network shown in the figure, the effective resistance faced by the voltage source is



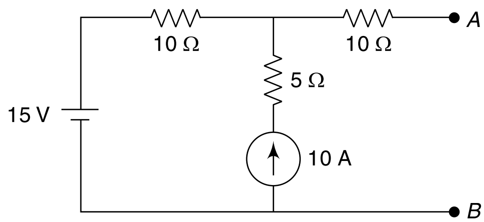
(a) 4 Ω (b) 3 Ω (c) 2 Ω (d) 1 Ω

5.29 For the network shown in the figure, if $Vs = V_1$ and $V = 0$, then $I = -5$ A and if $V_s = 0$ then $I = \frac{1}{2}$ A. The values of I_{SC} and R_1 of the Norton's equivalent across AB would be respectively



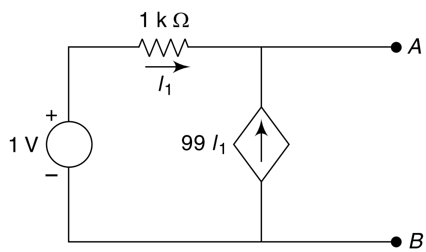
(a) –5 A and 2 Ω (b) 10 A and 0.5 Ω (c) 5 A and 2 Ω (d) 2.5 A and 5 Ω

5.30 In the network shown in the given figure, the Thevenin source and the impedance across terminals A–B will be respectively



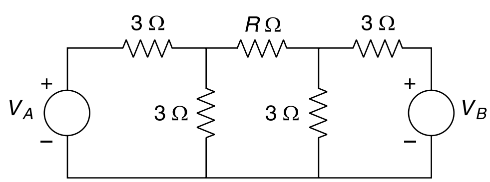
(a) 15 V and 13.33 Ω (b) 50 V and 15 Ω (c) 115 V and 20 Ω (d) 100 V and 25 Ω

5.31 Which one of the following combination of open-circuit voltage and Thevenin's equivalent resistance represents the Thevenin's equivalent of the circuit shown in the given figure?



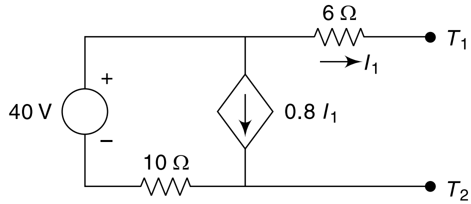
(a) 1 V, 10 Ω (b) 1 V, 1 kΩ (c) 1 mV, 1 kΩ (d) 1 mV, 10 Ω

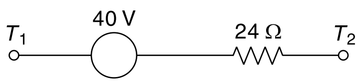
5.32 For the circuit shown in the given figure, the current through R, when $V_A = 0$ and $V_B = 15$ V is I ampere. Now, if both V_A and V_A are increased by 15 V, then the current through R will be

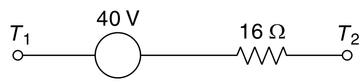


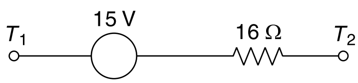
(a) I ampere (b) $\frac{I}{2}$ ampere (c) $3I$ ampere (d) $\frac{I}{3}$ ampere

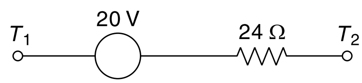
5.33 Thevenin's equivalent circuit of the network shown in the given figure, between terminals T_1 and T_2 is



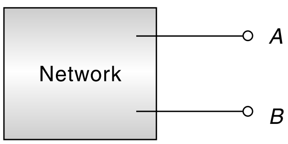
(a) 

(b) 

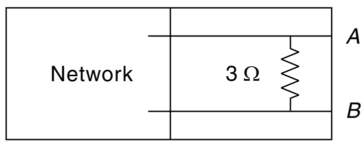
(c) 

(d) 

5.34 The Thevenin equivalent of the network shown in Figure (a) is 10 V in series with a resistance of 2 Ω. If now, resistance of 3 Ω is connected across AB in Figure (b), the Thevenin equivalent of the modified network across AB will be



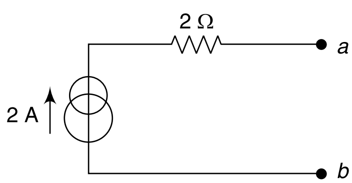
(a)



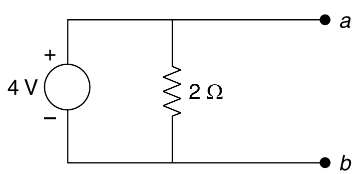
(b)

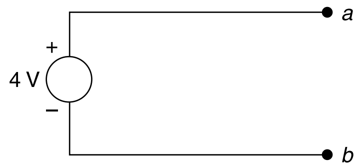
(a) 10 V in series with 1.2 Ω resistance
(b) 6 V in series with 1.2 Ω resistance
(c) 10 V in series with 5 Ω resistance
(d) 6 V in series with 5 Ω resistance

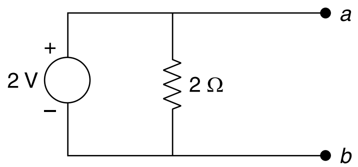
5.35 A d.c. current source is connected as shown in Figure below.



The Thevenin's equivalent of the network at terminals a–b will be

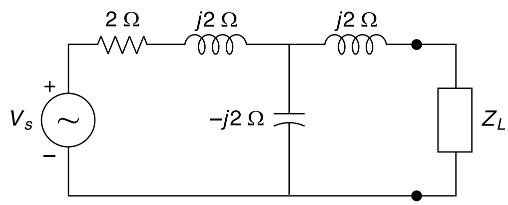
(a) 

(b) 

(c) 

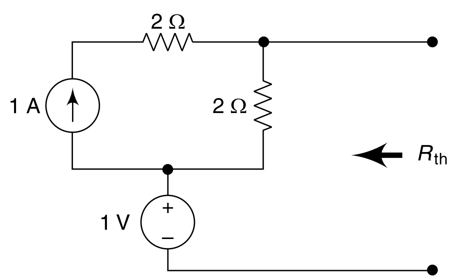
(d) is NOT feasible

5.36 Which one of the following impedance values of load will cause maximum power to be transferred to the load for the network shown in the given figure?



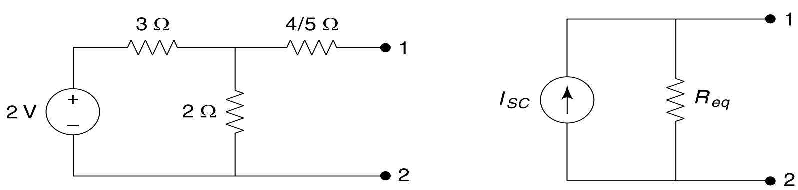
(a) $(2 + j2)$ (b) $(2 - j2)$ (c) $-j2$ (d) 2

5.37 The Thevenin's equivalent resistance R_{th} for the given network is



(a) 1 Ω (b) 2 Ω (c) 4 Ω (d) infinity

5.38 The Norton's equivalent of circuit shown in Figure (a) is drawn in the circuit shown in Figure (b). The values of I_{SC} and R_{eq} in Figure (b) are respectively



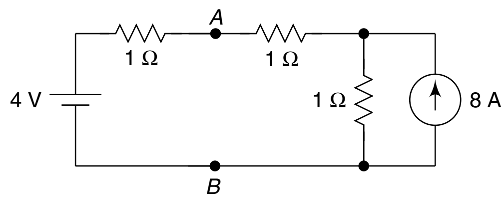
(a) (b)

(a) $\frac{5}{2}$ A and 2 Ω (b) $\frac{2}{5}$ A and 1 Ω (c) $\frac{4}{5}$ A and $\frac{12}{5}$ Ω (d) $\frac{2}{5}$ A and 2 Ω

5.39 For the circuit shown in the figure, the current flowing through 1 Ω resistor is adjusted to zero by varying the value of R. What is the value of R?

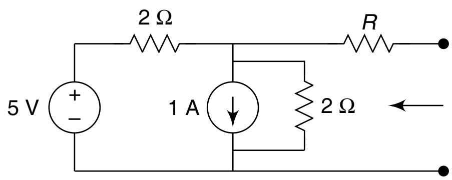
(a) 2Ω (b) 3Ω (c) 4Ω (d) 6Ω

5.40 What is the Thevenin's equivalent between A and B for the circuit shown in the figure?



(a) 12V, 3/2Ω (b) 4V, 3/2Ω (c) 16/3V, 2/3Ω (d) 16/3V, 3Ω

5.41 If Thevenin's equivalent resistance of the circuit shown in the figure seen from the open terminals is 2 Ω, then the value of 'R' will be



(a) 4Ω (b) 2Ω (c) 1Ω (d) zero

5.42

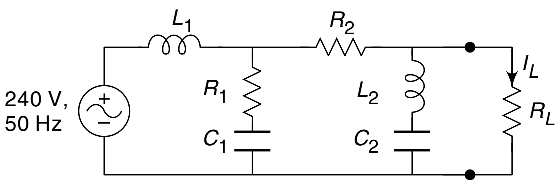


Figure I

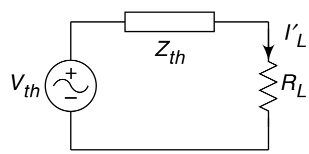


Figure II

Thevenin's equivalent of the network shown in Figure-I would correspond to the network shown in Figure-II, if one or more of the following conditions are met:

1. $I'_L = I_L$
2. The equivalence is valid only if the frequency of V_{th} is maintained at 50 Hz
3. $I'_L = 2I_L$, if the voltage V_{th} is doubled.

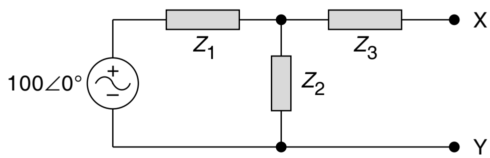
The correct set of conditions would include

(a) 1, 2 and 3 (b) 1 and 2
(c) 2 and 3 (d) 1 and 3

5.43 Thevenin's theorem is not applicable for circuits with

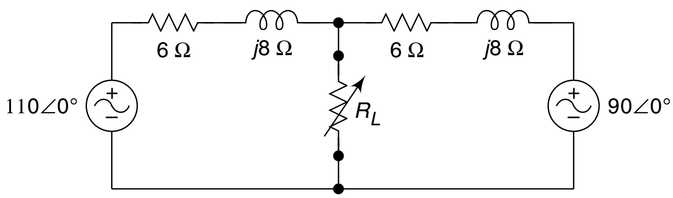
(a) passive load (b) active load
(c) bilateral load (d) none of these

5.44 In the figure, $Z_1 = 10\angle{-60°}$, $Z_2 = 10\angle 60°$, $Z_3 = 50\angle 53.13°$. Thevenin impedance seen from X-Y is



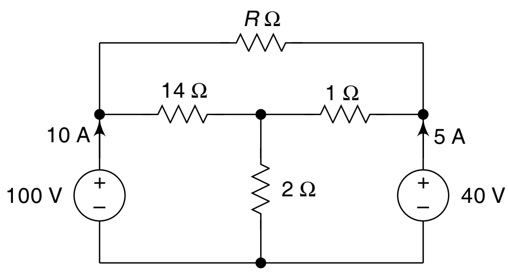
(a) $56\angle 45°$ (b) $60\angle 30°$ (c) $70\angle 30°$ (d) $34.4\angle 65°$

5.45 Two ac sources feed a common variable resistive load as shown in the figure. Under the maximum power transfer condition, the power absorbed by the load resistance R_L is



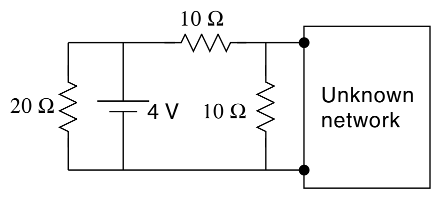
(a) 2200 Ω (b) 1250 Ω (c) 1000 Ω (d) 625 Ω

5.46 In the figure, the value of R is



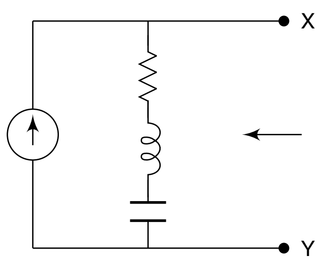
(a) 10 Ω (b) 18 Ω (c) 24 Ω (d) 12 Ω

5.47 In the given figure, the Thevenin's equivalent pair (voltage, impedance), as seen at the terminals P-Q, is given by



(a) (2 V, 5 Ω) (b) (2 V, 7.5 Ω)
(c) (4 V, 5 Ω) (d) (4 V, 7.5 Ω)

5.48 In the figure, the current source is $1\angle 0°$ A, $R = 1\,\Omega$, the impedances are $Z_C = -j\,\Omega$, and $Z_L = 2j\,\Omega$. The Thevenin equivalent looking into the circuit across X-Y is



(a) $\sqrt{2}\angle 0\,\text{V},\ (1+2j)\,\Omega$ (b) $2\angle 45°\,\text{V},\ (1-2j)\,\Omega$

(c) $2\angle 45°\,\text{V},\ (1+j)\,\Omega$ (d) $\sqrt{2}\angle 45°\,\text{V},\ (1+j)\,\Omega$

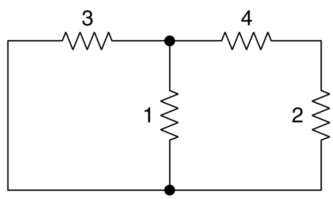
5.49 A source of angular frequency 1 rad/s has a source impedance consisting of $1\,\Omega$ resistance in series with 1 H inductance. The load that will obtain the maximum power transfer is

(a) $1\,\Omega$ resistance
(b) $1\,\Omega$ resistance in parallel with 1 H inductance
(c) $1\,\Omega$ resistance in series with 1 F capacitor
(d) $1\,\Omega$ resistance in parallel with 1 F capacitor

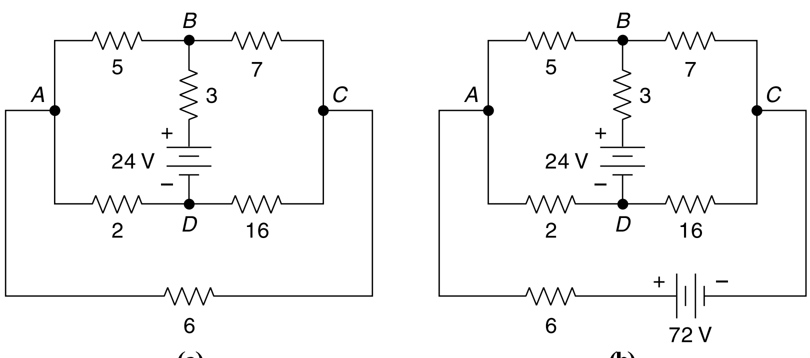
EXERCISES

Reciprocity Theorem

5.1 In the network shown in figure below, verify the Reciprocity Theorem using a voltage source and an ammeter. What are the methods of verifying the Reciprocity Theorem? All values are in ohm.

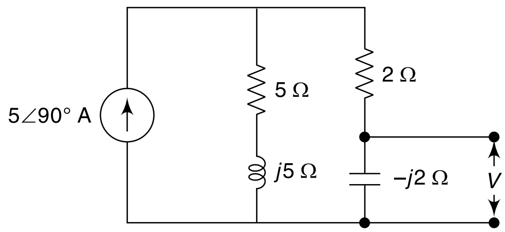


5.2 Find the current in the 6 Ω resistor and the source current in Figure (a). Hence, determine the current in the 3 Ω resistor when an emf of 72 V is added in series with the 6 Ω resistor as shown in Figure (b). [0.5 A, 6 A]

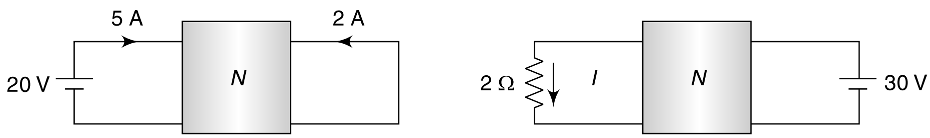


(a) **(b)**

5.3 In this circuit, find the voltage V. Interchange the current source and resulting voltage V and show that the reciprocity theorem is verified. [$9.28\angle 21.8°$ (V)]



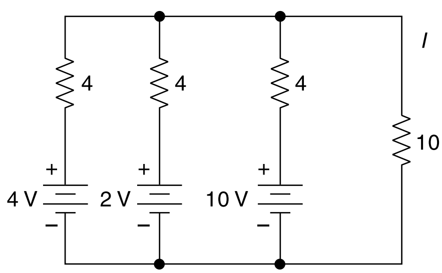
5.4 Two sets of measurements are made on a linear passive resistive network in Figure (a) and (b). Find the current through the 2Ω resistor. [2 A]



(a) (b)

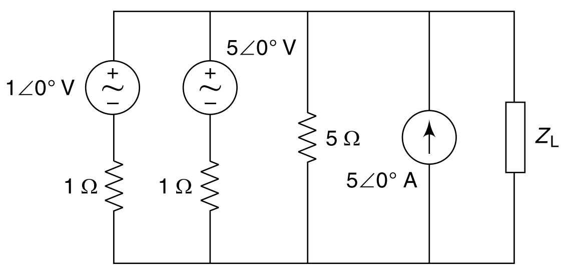
Millman's Theorem

5.5 Find the load current using Millman's theorem. All values are in ohm. [1.176 A]

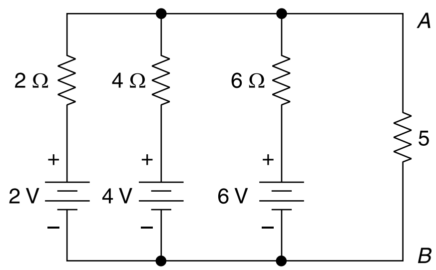


5.6 Using Millman's theorem, find the current in the load impedance, Z_L =(2 + j4) Ω

[1.06∠–58.46° (A)]

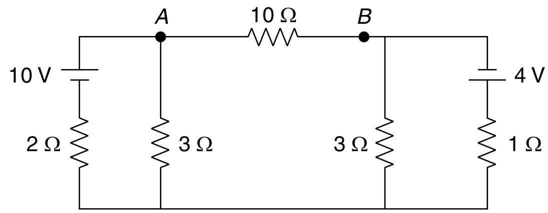


5.7 Determine the current through the branch AB using Millman's theorem. $\left[\frac{36}{67}\text{ A}\right]$



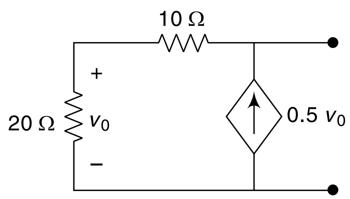
Thevenin's and Norton's Theorems

5.8 Determine the Thevenin equivalent circuit with respect to the terminals *A* and *B* for the circuit shown in the figure and hence the current flowing through 10 Ω resistor. [0.75 A]

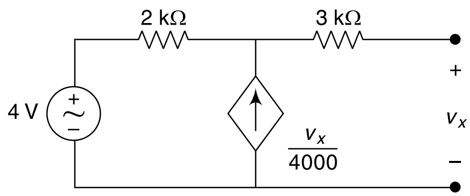


5.9 Find the Thevenin equivalent circuit for the following networks

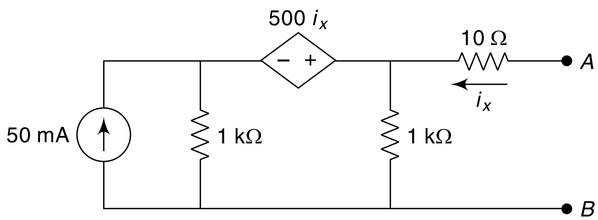
(i)



(ii)

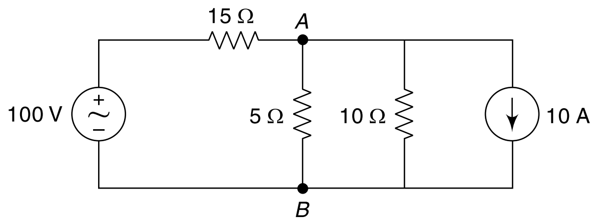


(iii)

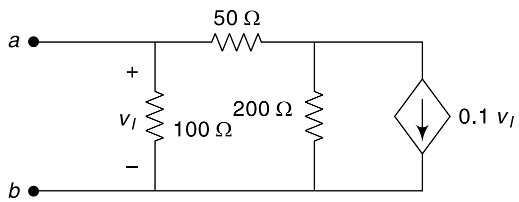


[(i) 0; – 0.33 Ω (ii) 8 V; 10 kΩ (iii) 25 V; 350 Ω]

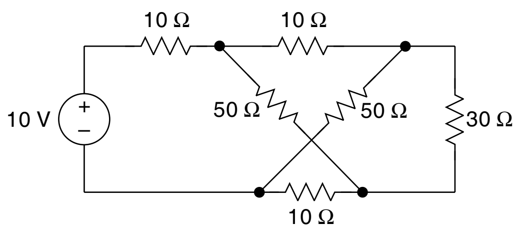
5.10 Determine the current in the branch *AB* for the circuit shown in figure by using Thevenin's theorem. [1.818 A]



5.11 Find Norton's equivalent at terminals *a*–*b*. [0 A; 10.64 Ω]

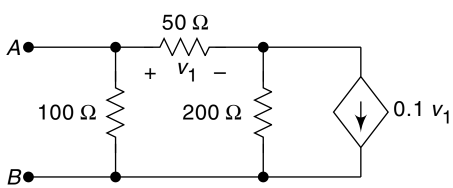


5.12 Use Thevenin's theorem to find the current supplied by the battery.
[$R_{Th} = 33.34\ \Omega$; $V_{Th} = 10$ V; $i = 0.3$ A]

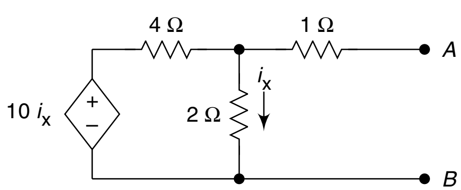


5.13 Find the Thevenin equivalent circuit with respect to the terminals *A* and *B*.

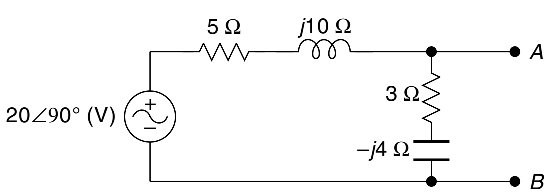
(a)



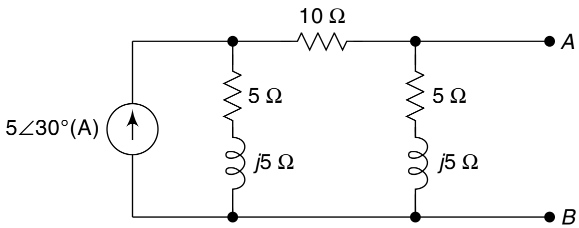
(b)



(c)

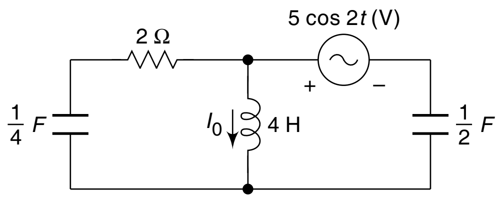


(d)



[(a) $V_{\text{Th}} = 0$; $R_{\text{Th}} = 10.64\,\Omega$; (b) $V_{\text{Th}} = 0$; $R_{\text{Th}} = 1\,\Omega$;
(c) $V_{\text{Th}} = 10\angle 0°$ (V); $Z_{\text{Th}} = 5.59\angle{-26.6°}$ (Ω);
(d) $V_{\text{Th}} = 11.18\angle 93.44°$ (V); $Z_{\text{Th}} = 5\angle 36.87°$ (Ω)]

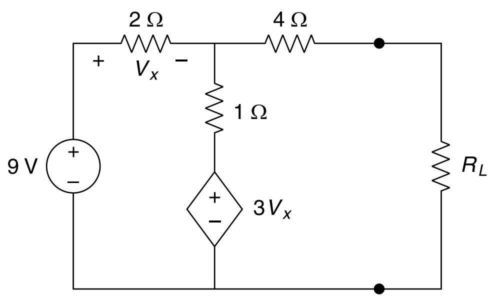
5.14 Compute I_0 using Norton's theorem.



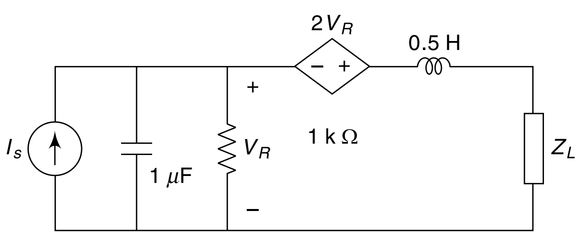
$$\left[I_0 = 0.542\cos\left(2t - 77.47°\right)(\text{A})\right]$$

Maximum Power Transfer Theorem

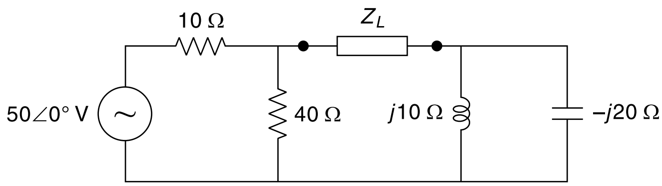
5.15 Determine the value of the resistor R_L that will draw maximum power from the rest of the circuit. What is the maximum power? [4.22 Ω, 2.901 W]



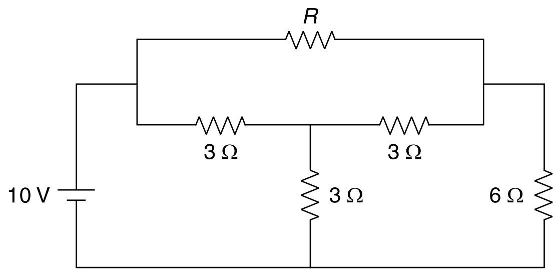
5.16 The circuit operates in the sinusoidal steady state with $\omega = 1000$ rad/s and $I_s = 1\angle 0°$ A(rms). Find the value of the load impedance for maximum average power transfer. Also, find the average power absorbed by the load under this condition. [$(1500 + j1000)\,\Omega$; 83.33 W]



5.17 Determine Z_L so that the maximum power is absorbed by it. [40∠0° V; (8 – j20) Ω, 50 W]

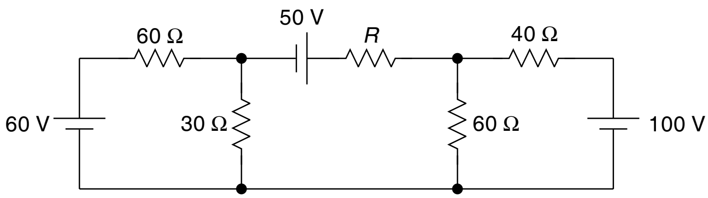


5.18 Determine the value of R such that the 6Ω resistor consumes the maximum power. [R = –18 Ω]

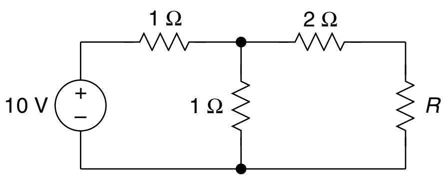


5.19 Find the value of the resistance R for maximum power to be transferred to it. Also, find the maximum power.

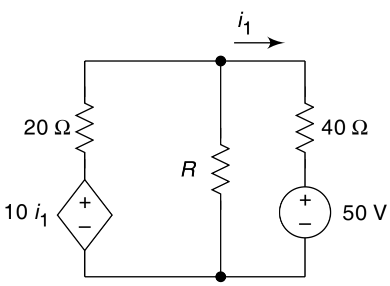
(a)



(b)



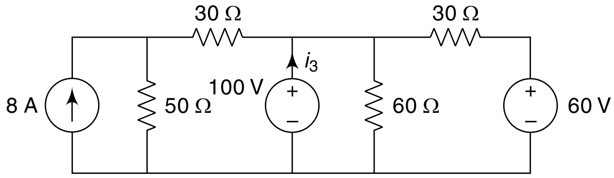
(c)



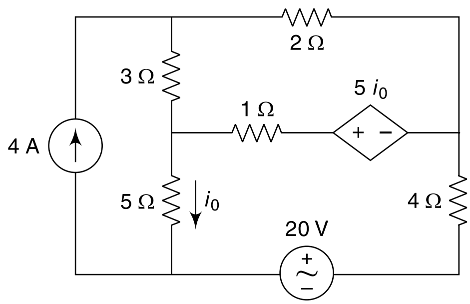
[(a) 44 Ω, 0.568 W; (b) 4.5 Ω, 1.39 W; (c) 16 Ω]

Superposition Theorem

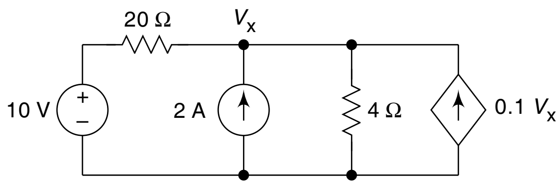
5.20 Apply superposition theorem to the circuit to find i_3. [–0.75 A]



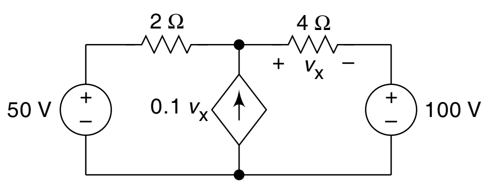
5.21 Find the current i_0 using superposition theorem. [–0.4706 A]



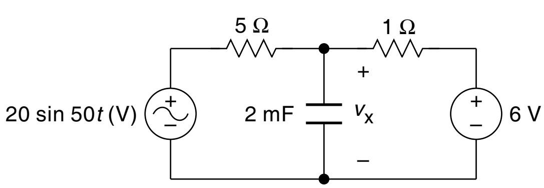
5.22 Use superposition theorem to find the voltage V_x. [12.5 V]



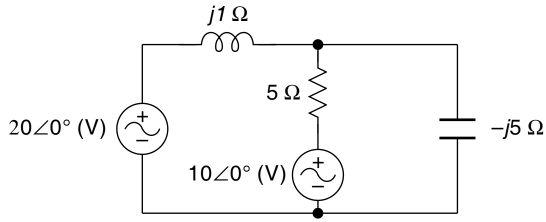
5.23 Determine the voltage v_x in the circuit using superposition theorem. [–38.5 V]



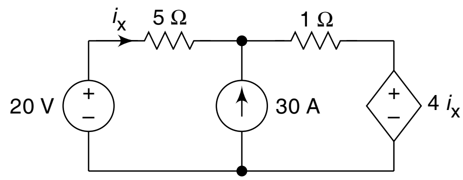
5.24 Use superposition theorem to find the voltage v_x. $\left[5+2.56\sin\left(500t-39.8^\circ\right)(\text{V})\right]$



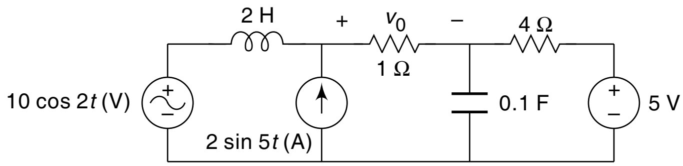
5.25 Find the current through the capacitor using superposition theorem. [4.86∠80.8° (A)]



5.26 Find the current i_x by the superposition theorem. [5 A]



5.27 Find v_0 using superposition theorem.



$$\left[v_0 = -1 + 2.498\cos\left(2t - 30.79^\circ\right) + 2.328\sin\left(5t + 10^\circ\right)\text{ V}\right]$$

SHORT-ANSWER TYPE QUESTIONS

5.1 State and explain substitution theorem.

5.2 State and explain superposition theorem. Give a proof for a general n-mesh network indicating the conditions under which it is applicable.

5.3 State reciprocity theorem as applied to a network and give a proof of the same for a general network. Mention two networks where this theorem is not applicable.

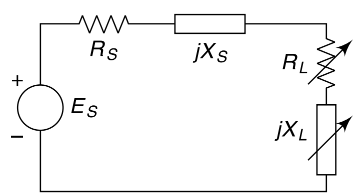
5.4 State Thevenin's theorem and give a proof of the same. Mention one example of a network where this network is not applicable.

5.5 (a) State Norton's theorem as applied to a network and give a proof of the same.
(b) What is 'Dual Network'? Mention the procedure for drawing the dual of a given network.

5.6 State and prove maximum power transfer theorem.

or

In the circuit, the source emf E_S, resistance R_S and reactance jX_S are fixed but both the load resistance R_L and reactance jX_L are variable. Show that maximum power is consumed in the load when $X_L = -X_S$ and $R_L = R_S$.



Prove that the load impedance which absorbs the maximum power from a source is the conjugate of the impedance of the source.

5.7 State and prove the following theorem
 (a) Tellegen's theorem.
 (b) Millman's theorem.
 (c) Compensation theorem.

5.8 State and explain clearly Thevenin's theorem as applied in ac circuits.

5.9 State and explain Thevenin's theorem, specify the types of network to which it is applicable. Also, state the theorem which is the dual of the above theorem.

5.10 State maximum power transfer theorem for all the various kinds of networks and loads.

5.11 State maximum power transfer theorem. Derive conditions for maximum power transfer for a resistive network and resistive load.

5.12 Prove the condition for maximum power transfer for an ac circuit.

5.13 A source with internal impedance $R_S + jX_S$ delivers power to a variable load impedance $R_L + j0$. Show that the condition for maximum power in the load is $R_L^2 = R_S^2 + X_S^2$.

5.14 State maximum power transfer theorem and verify that only 50% of the total power supplied by the source can be transferred to load.

Or,

State and explain maximum power transfer theorem. Derive the expression for efficiency for maximum power transfer.

5.15 Derive the condition for maximum power transfer for
 (a) Load impedance with variable resistance and variable reactance
 (b) Load impedance with variable resistance and fixed reactance

5.16 State and clearly prove, with the help of a suitable example, the maximum power transfer theorem as applicable to *RLC* circuits excited from the sinusoidal energy source. Hence explain clearly the concept and its significance in impedance matching.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

5.1 (b)	5.2 (c)	5.3 (c)	5.4 (a)	5.5 (c)	5.6 (a)	5.7 (a)
5.8 (b)	5.9 (d)	5.10 (d)	5.11 (c)	5.12 (a)	5.13 (b)	5.14 (c)
5.15 (d)	5.16 (a)	5.17 (b)	5.18 (b)	5.19 (d)	5.20 (d)	5.21 (a)
5.22 (a)	5.23 (a)	5.24 (a)	5.25 (c)	5.26 (d)	5.27 (b)	5.28 (d)
5.29 (c)	5.30 (c)	5.31 (b)	5.32 (a)	5.33 (a)	5.34 (b)	5.35 (d)
5.36 (d)	5.37 (b)	5.38 (d)	5.39 (b)	5.40 (c)	5.41 (c)	5.42 (a)
5.43 (b)	5.44 (a)	5.45 (d)	5.46 (d)	5.47 (a)	5.48 (d)	5.49 (c)

CHAPTER

6
Laplace Transform and its Applications

6.1 INTRODUCTION

Classical methods of solving differential equations become quite cumbersome when used for networks involving higher order differential equations. In such cases, Laplace Transform method is used.

The classical methods consist of three steps:

(i) determination of complementary function,
(ii) determination of particular integral, and
(iii) determination of arbitrary constants.

But, these methods become difficult for the equations containing derivatives; and transform methods prove to be superior.

The Laplace transform is an integral that transforms a time function into a new function of a complex variable. The term Laplace comes from the name of the French mathematician Pierre Simon Laplace (1749–1827). The transformation method is a very effective tool for solving integro-differential equations.

Laplace transformation is also a very powerful tool for network analysis. Any linear circuit consisting of linear circuit elements can be solved by the knowledge of Laplace transformation.

In this chapter, we will first discuss the basics of Laplace transformation and then apply this transform method to study the transient behaviour of electric circuits.

6.2 ADVANTAGES OF LAPLACE TRANSFORM METHOD

Laplace transforms methods offer the following advantages over the classical methods.

1. It gives complete solution.

2. Initial conditions are automatically considered in the transformed equations.
3. Much less time is involved in solving differential equations.
4. It gives systematic and routine solutions for differential equations.

6.3 DEFINITION OF LAPLACE TRANSFORM

Let $f(t)$ be a function of time which is zero for $t < 0$ and which is arbitrarily defined for $t > 0$, subject to some mild conditions. Then the Laplace Transform of the function $f(t)$, denoted by $F(s)$ is defined as,

$$\mathcal{L}[f(t)] = F(s) = \int_{0_-}^{\infty} f(t)e^{-st}\,dt$$

Thus, the operator $\mathcal{L}[\ ]$ transforms $f(t)$, which is in time domain, into $F(s)$, which is in the complex frequency domain, or simply the s-domain, where,

s = Complex frequency (unit is in Hz) = $(\sigma + j\omega)$

where, σ = Real part of s = neper frequency and ω = Imaginary part of s = radian frequency.

NB: The *lower limit of the integration should be 0– instead of* 0_+ *or simple 0.* If $f(t)$ is continuous at $t = 0$, then the value of $f(0)$ is well-defined. But, if $f(t)$ is not continuous at $t = 0$, then the meaning of $f(0)$ becomes ambiguous. To consider the effect of "instantaneous energy transfer" we must use 0– as the lower limit to include the impulses at $t = 0$. The use of 0 will exclude the existence of any impulses at the origin.

So, we use 0– as the lower limit.

6.4 BASIC THEOREMS OF LAPLACE TRANSFORM

1. Linearity Theorem If Laplace transform of the functions $f_1(t)$ and $f_2(t)$ are $F_1(s)$ and $F_2(s)$ respectively, then Laplace transform of the functions $[K_1 f_1(t) + K_2 f_2(t)]$ will be $[K_1 F_1(s) + K_2 F_2(s)]$.

$$\mathcal{L}[K_1 f_1(t) + K_2 f_2(t)] = [K_1 F_1(s) + K_2 F_2(s)]$$

where, K_1 and K_2 are constants.

2. Scaling Theorem If Laplace transform of $f(t)$ is $F(s)$, then

$$\mathcal{L}[f(Kt)] = \frac{1}{K}F\left(\frac{s}{K}\right), \text{ where } K \text{ is a constant and } K > 0.$$

3. Time Differentiation Theorem If Laplace transform of $f(t)$ is $F(s)$, then,

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0_-)$$

4. Frequency Differentiation Theorem If Laplace transform of $f(t)$ is $F(s)$, then,

$$L[tf(t)] = -\frac{dF(s)}{ds}$$

5. Time Integration Theorem If Laplace transform of $f(t)$ is $F(s)$, then,

$$\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$$

In general, for n^{th} order integration,

$$\mathcal{L}\left[\int_0^{t_1}\int_0^{t_2}\ldots\int_0^{t_n} f(t)\, dt_1\, dt_2 \ldots dt_n\right] = \frac{F(s)}{s^n}$$

6. Shifting Theorem The shifting may be done with respect to time or frequency.

(a) Time Shifting Theorem

If Laplace transform of $f(t)$ is $F(s)$, then

$$\mathcal{L}[f(t \pm a)] = e^{\pm as}F(s)$$

(b) Frequency Shifting Theorem

If Laplace transform of $f(t)$ is $F(s)$, then

$$\mathcal{L}[e^{\mp at}\, f(t)] = F(s \pm a)$$

7. Initial Value Theorem If the Laplace Transform of $f(t)$ is $F(s)$ and the first derivative of $f(t)$ is Laplace transformable, then, the initial value of $f(t)$ is,

$$f(0^+) = \underset{t\to 0}{\text{Lt}}\, f(t) = \underset{s\to\infty}{\text{Lt}}\,[sF(s)]$$

Proof

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = \int_{0\_}^{\infty}\left[\frac{df(t)}{dt}\right]e^{-st}dt$$

or

$$sF(s) - f(0\_) = \int_{0\_}^{\infty}\left[\frac{df(t)}{dt}\right]e^{-st}dt \qquad \text{[by time differentiation theorem]}$$

Taking limit $s \to \infty$,

$$\underset{s\to\infty}{\text{Lt}}\,[sF(s) - f(0\_)] = \underset{s\to\infty}{\text{Lt}}\int_{0\_}^{\infty}\left[\frac{df(t)}{dt}\right]e^{-st}dt$$

or

$$\underset{s\to\infty}{\text{Lt}}\,[sF(s)] - f(0\_) = \underset{s\to\infty}{\text{Lt}}\left[\int_{0\_}^{0^+} e^0\frac{df(t)}{dt}dt + \int_{0^+}^{\infty} e^{-st}\frac{df(t)}{dt}dt\right]$$

or

$$\underset{s\to\infty}{\text{Lt}}\,[sF(s)] - f(0\_) = \underset{s\to\infty}{\text{Lt}}\left[\int_{0\_}^{0^+} e^0\frac{df(t)}{dt}dt\right] \qquad \text{[as } s \text{ is not a function of time } t\text{]}$$

or

$$\underset{s\to\infty}{\text{Lt}}\,[sF(s)] - f(0\_) = \underset{s\to\infty}{\text{Lt}}\int_{0\_}^{0^+} df(t) = f(0^+) - f(0\_)$$

or

$$\boxed{f(0^+) = \underset{s\to\infty}{\text{Lt}}\,[sF(s)]}$$

8. Final Value Theorem If a function $f(t)$ and its derivatives are Laplace transformable, then the final value of $f(t)$ is,

$$f(\infty) = \operatorname*{Lt}_{t\to\infty} f(t) = \operatorname*{Lt}_{s\to 0} [sF(s)]$$

Proof
$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = \int_{0_-}^{\infty} \left[\frac{df(t)}{dt}\right] e^{-st} dt$$

or
$$sF(s) - f(0_-) = \int_{0_-}^{\infty} \left[\frac{df(t)}{dt}\right] e^{-st} dt \qquad \text{[by time differentiation theorem]}$$

Taking limit $s \to 0$,

$$\operatorname*{Lt}_{s\to 0}\left[sF(s) - f(0_-)\right] = \operatorname*{Lt}_{s\to 0} \int_{0_-}^{\infty} \left[\frac{df(t)}{dt}\right] e^{-st} dt = \int_{0_-}^{\infty} \left[\frac{df(t)}{dt}\right] dt = \operatorname*{Lt}_{t\to\infty} \int_{0_-}^{t} \left(\frac{df(t)}{dt}\right) dt$$

or
$$\operatorname*{Lt}_{s\to 0}\left[sF(s) - f(0_-)\right] = \operatorname*{Lt}_{t\to\infty}\left[f(t) - f(0_-)\right]$$

or
$$\operatorname*{Lt}_{s\to 0}\left[sF(s)\right] - f(0_-) = \operatorname*{Lt}_{t\to\infty}\left[f(t)\right] - f(0_-)$$

or
$$\boxed{\operatorname*{Lt}_{t\to\infty}\left[f(t)\right] = \operatorname*{Lt}_{s\to 0}\left[sF(s)\right]}$$

This theorem is only applicable if the value of the function $f(t)$ is finite as t becomes infinity, i.e., $F(s)$ has all poles lying in the left half of s-plane or at most one simple pole at the origin.

6.5 LAPLACE TRANSFORM OF SOME BASIC FUNCTIONS

1. Exponential Function

$$f(t) = e^{at}$$

By definition of Laplace transform,

$$F(s) = L[f(t)] = \int_{0-}^{\infty} e^{at} \cdot e^{-st} dt = \int_{0-}^{\infty} e^{(a-s)t} dt = \left[\frac{e^{(a-s)t}}{(a-s)}\right]_{0-}^{\infty} = \left(0 - \frac{1}{(a-s)}\right) = \frac{1}{(s-a)}$$

Similarly, for $f(t) = e^{-at}$, $F(s) = \dfrac{1}{s+a}$

2. Unit Step Function or, Heaviside Unit Function

$$\begin{aligned} f(t) = u(t) &= 1 \text{ for } t > 0 \\ &= 0 \text{ for } t < 0 \end{aligned}$$

and is undefined for $t = 0$.

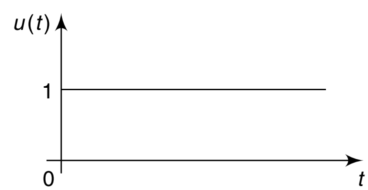


Figure 6.1(a) *Unit step function*

$$F(s) = L[f(t)] = \int_{0-}^{\infty} u(t) \cdot e^{-st} dt = \int_{0-}^{\infty} 1 \cdot e^{-st} dt = \left[\frac{e^{-st}}{-s} \right]_{0-}^{\infty}$$

$$= 0 - \frac{1}{-s} = \frac{1}{s}$$

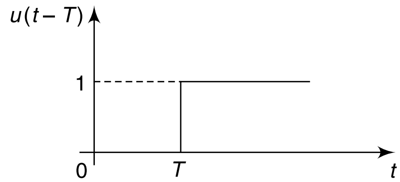


Figure 6.1(b) *Shifted unit step function*

Also, the Laplace transform of step function of magnitude K is

$$L[Ku(t)] = \frac{K}{s}$$

Similarly, the Laplace transform of the shifted unit step function $u(t - T)$ is,

$$\mathcal{L}[u(t-T)] = \frac{e^{-sT}}{s} \qquad \{\text{by differentiation theorem}\}$$

Another function, called *gate function* can be obtained from step function as follows.

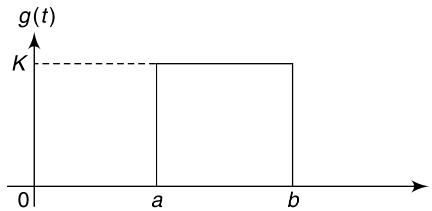


Figure 6.2 *Gate function*

Therefore, $g(t) = Ku(t-a) - Ku(t-b)$ and, $L[g(t)] = \frac{K}{s}(e^{-as} - e^{-bs})$

3. The Sine Function

$$f(t) = \sin \omega t = \frac{1}{2j}[e^{j\omega t} - e^{-j\omega t}]$$

$$F(s) = L[f(t)] = \int_{0-}^{\infty} \left[\frac{1}{2j}[e^{j\omega t} - e^{-j\omega t}] \right] \cdot e^{-st}\, dt = \frac{1}{2j} \int_{0-}^{\infty} [e^{(j\omega - s)t} - e^{-(j\omega + s)t}]\, dt$$

$$= \frac{1}{2j} \left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right] = \frac{\omega}{s^2 + \omega^2}$$

4. The Cosine Function

$$f(t) = \cos \omega t = \frac{1}{2}[e^{j\omega t} + e^{-j\omega t}]$$

$$F(s) = L[f(t)] = \int_{0-}^{\infty} \left[\frac{1}{2j}[e^{j\omega t} - e^{-j\omega t}] \right] \cdot e^{-st}\, dt$$

$$= \frac{1}{2} \int_{0-}^{\infty} [e^{(j\omega - s)t} + e^{-(j\omega + s)t}]\, dt = \frac{1}{2} \left[\frac{1}{s - j\omega} + \frac{1}{s + j\omega} \right] = \frac{s}{s^2 + \omega^2}$$

5. The Hyperbolic Sine Function

$$f(t) = \sinh at = \frac{1}{2}[e^{at} - e^{-at}]$$

$$F(s) = L[f(t)] = \int_{0-}^{\infty}\left[\frac{1}{2j}[e^{at} - e^{-at}]\right]\cdot e^{-st}\,dt$$

$$= \frac{1}{2}\int_{0-}^{\infty}[e^{(a-s)t} - e^{-(a+s)t}]\,dt = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2 - a^2}$$

6. The Hyperbolic Cosine Function

$$f(t) = \cosh at = \frac{1}{2}[e^{at} + e^{-at}]$$

$$F(s) = L[f(t)] = \int_{0-}^{\infty}\left[\frac{1}{2}[e^{at} + e^{-at}]\right]\cdot e^{-st}\,dt$$

$$= \frac{1}{2}\int_{0-}^{\infty}[e^{(a-s)t} + e^{-(a+s)t}]\,dt = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{a}{s^2 - a^2}$$

7. The Damped Sinusoidal Function

$$f(t) = e^{-at}\cdot\sin\omega t = e^{-at}\cdot\left\{\frac{1}{2j}[e^{j\omega t} - e^{-j\omega t}]\right\} = \left\{\frac{1}{2j}[e^{-(a-j\omega)t} - e^{-(a+j\omega)t}]\right\}$$

$$F(s) = L[f(t)] = \int_{0-}^{\infty}\left[\frac{1}{2j}[e^{-(a-j\omega)t} - e^{-(a+j\omega)t}]\right]\cdot e^{-st}dt$$

$$= \frac{1}{2j}\int_{0-}^{\infty}[e^{-(s+a-j\omega)t} - e^{-(s+a+j\omega)t}]\cdot dt$$

$$= \frac{1}{2j}\left[\frac{1}{\{(s+a)-j\omega\}} - \frac{1}{\{(s+a)+j\omega\}}\right] = \frac{\omega}{(s+a)^2 + \omega^2}$$

8. The Damped Cosine Function

$$f(t) = e^{-at}\cdot\cos\omega t = e^{-at}\cdot\left\{\frac{1}{2}[e^{j\omega t} + e^{-j\omega t}]\right\} = \left\{\frac{1}{2}[e^{-(a-j\omega)t} + e^{-(a+j\omega)t}]\right\}$$

$$F(s) = L[f(t)] = \int_{0-}^{\infty}\left[\frac{1}{2}[e^{-(a-j\omega)t} + e^{-(a+j\omega)t}]\right]\cdot e^{-st}dt$$

$$= \frac{1}{2}\int_{0-}^{\infty}[e^{-(s+a-j\omega)t} + e^{-(s+a+j\omega)t}]\cdot dt$$

$$= \frac{1}{2}\left[\frac{1}{\{(s+a)-j\omega\}} + \frac{1}{\{(s+a)+j\omega\}}\right] = \frac{(s+a)}{(s+a)^2 + \omega^2}$$

9. The Ramp Function

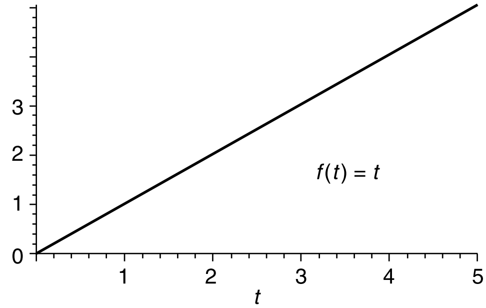


Figure 6.3 *Ramp function*

$$f(t) = t^n$$

$$F(s) = L[f(t)] = L[t^n] = \int_{0-}^{\infty} t^n \cdot e^{-st} dt$$

Integrating by parts, let,

$$u = t^n \quad \text{and} \quad dv = e^{-st}\, dt$$

then $\quad du = nt^{n-1} \quad$ and $\quad v = \int e^{-st} dt = -\dfrac{e^{-st}}{s}$

Now, $\quad F(s) = \mathcal{L}[f(t)] = \int_{0-}^{\infty} udv = uv|_{0-}^{\infty} - \int_{0-}^{\infty} vdu = -\dfrac{t^n}{s}[e^{-st}]_{0-}^{\infty} + \dfrac{n}{s}\int_{0-}^{\infty} t^{(n-1)} e^{-st} dt$

$$= \frac{n}{s} \cdot \int t^{(n-1)} e^{-st} dt = \frac{n}{s} L[t^{(n-1)}] = \frac{n}{s} \cdot \frac{(n-1)}{s} L[t^{(n-2)}]$$

$$= \frac{n}{s} \cdot \frac{(n-1)}{s} \cdot \frac{(n-2)}{s} \cdots\cdots \frac{2}{s} \cdot \frac{1}{s} L[t^0] = \frac{\angle n}{s^n} L[u(t)]$$

$$= \frac{\angle n}{s^n} \cdot \frac{1}{s} = \frac{\angle n}{s^{n+1}}$$

For $n = 1$, $\mathcal{L}[t] = \dfrac{1}{s^2}$

For $n = 2$, $\mathcal{L}[t^2] = \dfrac{\angle 2}{s^3}$

10. Impulse Function or Dirac Delta Function [$\delta(t)$]

It is a function of a real variable t, such that the function is zero everywhere except at the instant $t = 0$. Physically, it is a very sharp pulse of infinitesimally small width and very large magnitude, the area under the curve being unity.

Consider a gate function as shown in Fig. 6.4.

The function is compressed along the time-axis and stretched along the y-axis, keeping area under the pulse unity. As $a \to 0$, the value of $\dfrac{1}{a} \to \infty$ and the resulting function is known as impulse.

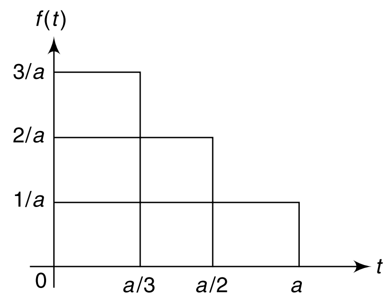


Figure 6.4 *Generation of impulse function from gate function*

It is defined as, $\delta(t) = 0$ for $t \neq 0$

and $\quad \int_{-\infty}^{\infty} \delta(t) dt = 1$

Also, $\quad \delta(t) = \lim\limits_{a \to 0} \dfrac{1}{a}[u(t) - u(t-a)]$

The Laplace transform of the impulse function is obtained as,

$$\mathcal{L}[\delta(t)] = \lim_{a \to 0} L\left\{\frac{1}{a}[u(t) - u(t-a)]\right\}$$

$$= \lim_{a \to 0} \frac{1}{a}\left[\frac{1}{s} - \frac{e^{-as}}{s}\right]$$

$$= \lim_{a \to 0} \frac{1 - e^{-as}}{as}$$

$$= \lim_{a \to 0} \frac{se^{-as}}{s} \qquad \text{[by L'Hospital's rule]}$$

$$= 1$$

6.6 LAPLACE TRANSFORM TABLE

Table 6.1 *Standard Laplace Transforms*

Sl. No.	*Functions [f(t)] in Time(t) Domain*	*Laplace Transform [F(s)] in Frequency(s) Domain*
Definition	If $f(t)$ is Laplace transformable	then $L[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-St}dt$
1	$U(t)$ (unit step function)	$\frac{1}{s}$
2	$U(t - T)$ (unit step function shifted/delayed by T)	$\frac{e^{-sT}}{s}$
3	$\delta(t)$ (unit impulse)	1
4	e^{at} (exponential function)	$\frac{1}{s-a}$
5	e^{-at} (exponential function)	$\frac{1}{s+a}$
6	$\sin \omega t$ (sine function)	$\frac{\omega}{s^2+\omega^2}$
7	$\cos \omega t$ (cosine function)	$\frac{s}{s^2+\omega^2}$
8	t^n (n = 1, 2, 3, ……) (ramp function)	$\frac{n!}{s^{n+1}}$
9	t (unit ramp function)	$\frac{1}{s^2}$
10	$e^{-at} \sin \omega t$ (damped sine function)	$\frac{\omega}{(s+a)^2+\omega^2}$

(*Contd*)

(*Contd*)

11	$e^{-at}\cos\omega t$ (damped cosine function)	$\dfrac{(s+a)}{(s+a)^2+\omega^2}$
12	$e^{-at}t^n$ (damped ramp function)	$\dfrac{n!}{(s+a)^{n+1}}$
13	$\dfrac{d}{dt}f(t)$ (Differentiation theorem)	$sF(s)-f(0-)$
14	$\int_0^t f(t)dt$ (Integration theorem)	$\dfrac{F(s)}{s}+\dfrac{f(0-)}{s}$
15	$\sinh\omega t$ (hyperbolic sine function)	$\dfrac{\omega}{s^2-\omega^2}$
16	$\cosh\omega t$ (hyperbolic cosine function)	$\dfrac{s}{s^2-\omega^2}$
17	$e^{-at}\sinh\omega t$ (damped hyperbolic sine function)	$\dfrac{\omega}{(s+a)^2-\omega^2}$
18	$e^{-at}\cosh\omega t$ (damped hyperbolic cosine function)	$\dfrac{(s+a)}{(s+a)^2-\omega^2}$
19	Initial value theorem	$\underset{t\to 0}{\text{Lt}}\, f(t)=\underset{s\to\infty}{\text{Lt}}\, sF(s)$
20	Final value theorem	$\underset{t\to\infty}{\text{Lt}}\, f(t)=\underset{s\to 0}{\text{Lt}}\, sF(s)$
21	Shifting theorem $f(t\pm a)$	$e^{\pm as}F(s)$

6.6.1 Other Important Laplace Transforms

1	$\delta(t)$	1
2	$\delta(t-a)$	e^{-as}
3	$\delta(t-a)\,g(t)$	$e^{-as}\,g(a)$ *Note:* $g(a)$ NOT $G(a)$
4	$\dfrac{\omega_n}{\sqrt{1-\zeta^2}}e^{-z\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}\,t$	$\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $(\zeta<1)$
5	$1-\dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-z^2}\,t+\theta)$, where $\theta=\cos^{-1}\zeta$	$\dfrac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$ $(\zeta<1)$

6.7 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

If $f(t)$ is periodic with time period T (> 0), so that $f(t + T) = f(t)$, then the Laplace transform of the function is equal to $\left(\dfrac{1}{1-e^{-Ts}}\right)$ times the Laplace transform of the first cycle.

$$\therefore \qquad \mathcal{L}[f(t)] = F(s) = F_1(s)\left[\frac{1}{1-e^{-Ts}}\right]$$

Proof

Let $f(t)$ be the periodic function, and

T be the time period,

Let $f_1(t), f_2(t), \ldots, f_n(t)$ be the functions representing the first, second, …, n^{th} cycle, respectively

$$\therefore \qquad f(t) = f_1(t) + f_2(t) + \ldots + f_n(t) + \ldots$$
$$= f_1(t) + f_1(t - T) + f_1(t - 2T) + \ldots$$

Taking Laplace transform,

$$\mathcal{L}[f(t)] = F(s) = L[f_1(t)] + L[f_1(t-T)] + L[f_1(t-2T)] + \ldots$$
$$= F_1(s) + e^{-Ts}F_1(s) + e^{-2Ts}F_1(s) + \ldots$$
$$= F_1(s)[1 + e^{-Ts} + e^{-2Ts} + e^{-3Ts} + \ldots]$$

Therefore,
$$\boxed{F(s) = F_1(s)\left[\frac{1}{1-e^{-Ts}}\right]}$$

Example 6.1 Find the Laplace transform of the square wave.

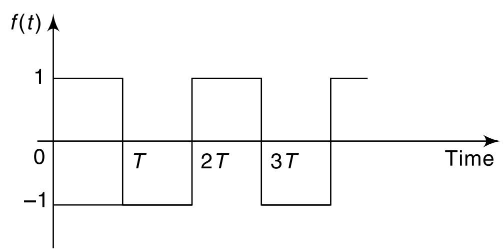


Figure 6.5(a) *Square wave of Example 6.1*

Solution The first cycle is shown below.

It can be written as,

$$f_1(t) = u(t) - 2u(t - T) + u(t - 2T)$$

Taking Laplace transform of the first cycle,

$$F_1(s) = \frac{1}{s} - \frac{2e^{-Ts}}{s} + \frac{e^{-2Ts}}{s} = \frac{1}{s}(1 - e^{-Ts})^2$$

By the theory of time periodicity, the Laplace transform of the square wave is given as,

$$F(s) = \frac{1}{s}(1 - e^{-Ts})^2 \times \frac{1}{1 - e^{-2Ts}}$$

(Since time period of the square wave is $2T$)

$$= \frac{1}{s}\left(\frac{1 - e^{-Ts}}{1 + e^{-Ts}}\right) = \frac{1}{s}\tanh\left(\frac{Ts}{2}\right)$$

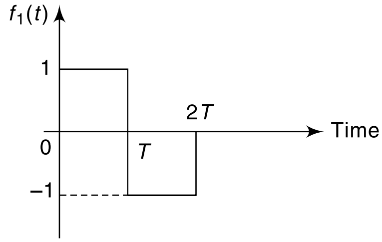


Figure 6.5(b) *First cycle of the square wave of Example 5.1*

6.8 SINGULARITY FUNCTIONS AND WAVEFORM SYNTHESIS

In order to synthesise any signal, there are some standard singularity functions which can be realised in the laboratory. Other signals can be written in terms of these singularity functions. Those singularity functions are

1. Step Function,
2. Ramp Function,
3. Impulse Function, and
4. Unit Doublet Function.

1. Step Function This function is also known as *Heaviside unit function*. It is defined as given below.

$$f(t) = u(t) = 1 \quad \text{for } t > 0$$
$$= 0 \quad \text{for } t < 0$$

and is undefined at $t = 0$.

A *step function of magnitude K* is defined as,

$$f(t) = Ku(t) = K \quad \text{for } t > 0$$
$$= 0 \quad \text{for } t < 0$$

and is undefined at $t = 0$.

A *shifted or delayed unit step function* is defined as,

$$f(t) = u(t - T) = 1 \quad \text{for } t > T$$
$$= 0 \quad \text{for } t < T$$

and is undefined at $t = T$.

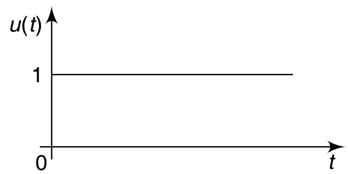


Figure 6.6(a) *Unit Step Function*

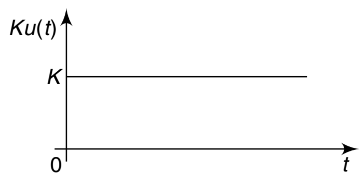


Figure 6.6(b) *Step function of magnitude K*

The Laplace transform of a unit step function is given as,

$$F(s) = L[f(t)] = \int_{0-}^{\infty} u(t).e^{-st}dt = \int_{0-}^{\infty} 1.e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_{0-}^{\infty}$$

$$= 0 - \frac{1}{-s} = \frac{1}{s}$$

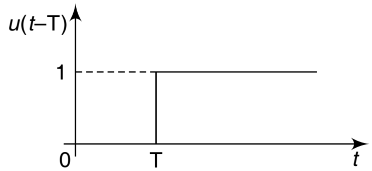


Figure 6.6(c) *Shifted Unit Step Function*

Also, the Laplace transform of step function of magnitude K is $L[Ku(t)] = \frac{K}{s}$

Similarly, the Laplace transform of the shifted unit step function $u(t - T)$ is,

$$L[u(t-T)] = \frac{e^{-sT}}{s} \quad \text{\{by differentiation theorem\}}$$

Another function, called *gate function* can be obtained from step function as follows.

Therefore, $$g(t) = Ku(t-a) - Ku(t-b)$$

$$L[g(t)] = \frac{K}{s}\left(e^{-as} - e^{-bs}\right)$$

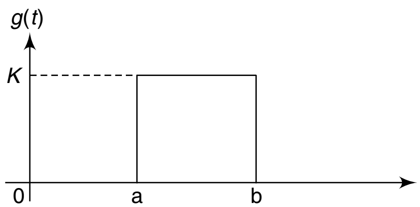


Figure 6.7 *Gate Function*

2. Ramp Function A unit ramp function is defined as,

$$f(t) = r(t) = t \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$

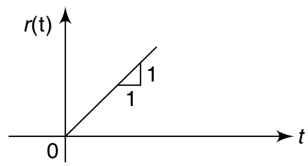


Figure 6.8(a) *Unit Ramp Function*

A ramp function of any slope K is defined as,

$$f(t) = Kr(t) = Kt \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$

A shifted unit ramp function is defined as,

$$f(t) = r(t-T) = t \quad \text{for } t \geq T$$
$$= 0 \quad \text{for } t < T$$

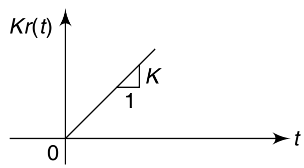


Figure 6.8(b) *Ramp Function*

The Laplace transform of a unit ramp function is,

$$L[r(t)] = \int_{0-}^{\infty} r(t).e^{-st}dt = \int_{0-}^{\infty} te^{-st}dt$$

Integrating by parts, let,

$$u = t \quad \text{and} \quad dv = e^{-st}dt$$

then $$du = dt \quad \text{and} \quad v = \int e^{-st}dt = -\frac{e^{-st}}{s}$$

Now, $$L[r(t)] = \int_{0-}^{\infty} u dv = uv|_{0-}^{\infty} - \int_{0-}^{\infty} v du = \left[-\frac{t}{s}\left(e^{-st}\right)\right]_{0-}^{\infty} + \frac{1}{s}\int_{0-}^{\infty} e^{-st} dt$$

$$= \frac{1}{s}\int_{0-}^{\infty} e^{-st} dt$$

$$= \frac{1}{s^2}$$

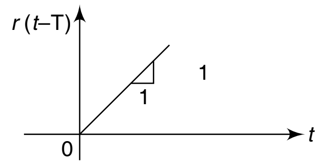


Figure 6.8(c) *Shifted Unit Ramp Function*

Similarly, Laplace transform of a ramp of slope K is,

$$L[Kr(t)] = \frac{K}{s^2}$$

and Laplace transform of a shifted ramp function is,

$$L[Kr(t-T)] = \frac{Ke^{-Ts}}{s^2}$$

3. Impulse Function This function is also known as *Dirac Delta function*, denoted by $\delta(t)$. This is a function of a real variable t, such that the function is zero everywhere except at the instant $t = 0$. Physically, it is a very sharp pulse of infinitesimally small width and very large magnitude, the area under the curve being unity.

Consider a gate function as shown in Fig. 6.9.

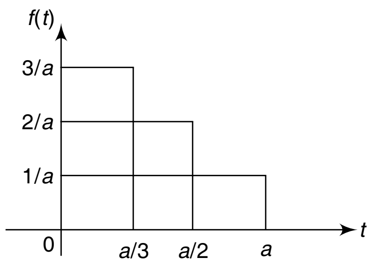


Figure 6.9 *Generation of impulse function from gate function*

The function is compressed along the time-axis and stretched along the y-axis, keeping area under the pulse unity. As $a \to 0$, the value of $\frac{1}{a} \to \infty$ and the resulting function is known as impulse.

It is defined as,

$$\delta(t) = 0 \quad \textit{for } t \neq 0$$

and $$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Also, $$\delta(t) = \lim_{a \longrightarrow 0} \frac{1}{a}[u(t) - u(t-a)]$$

The Laplace transform of the impulse function is obtained as,

$$L[\delta(t)] = \lim_{a \to o} L\left\{\frac{1}{a}[u(t) - u(t-a)]\right\}$$
$$= \lim_{a \to o} \frac{1}{a}\left[\frac{1}{s} - \frac{e^{-as}}{s}\right]$$
$$= \lim_{a \to o} \frac{1 - e^{-as}}{as}$$
$$= \lim_{a \to o} \frac{se^{-as}}{s} \qquad [by \quad L'Hospital's \quad rule]$$
$$= 1$$

4. Unit Doublet Function The derivative of unit impulse function with respect to time at any instant of time is known as unit doublet function. It is defined as,

$$\frac{d}{dt}[\delta(t-T)] = \delta(t-T) = 0 \qquad \qquad for\ t \neq 0$$
$$= +\infty \quad and \quad -\infty \qquad for\ t = T$$

The name of the function is given as doublet because it can be obtained from the function shown in Fig.6.10 (a) with $a \to 0$.

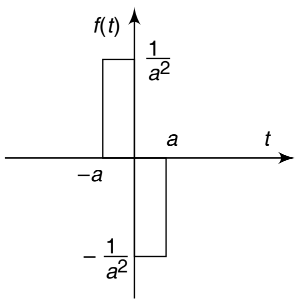


Figure 6.10(a) *Generation of Unit Doublet Function with a → 0*

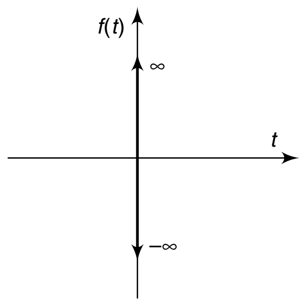


Figure 6.10(b) *Unit Doublet Function*

The Laplace transform of a unit doublet function is obtained as,

$$L[\delta'(t-T)] = L\left[\frac{d}{dt}\,\delta(t-T)\right] = sL[\delta(t-T)] = se^{-Ts}$$

6.9 INVERSE LAPLACE TRANSFORM

Let, $F(s)$ have the general form of

$$F(s) = \frac{N(s)}{D(s)}$$

where, $N(s)$ is the numerator polynomial and $D(s)$ is the denominator polynomial. The roots of $N(s) = 0$ are called the *zeros of F(s)* while the roots of $D(s) = 0$ are the *poles of F(s)*.

For example, for the function $F(s) = \dfrac{s-1}{s(s-2)(s-3)}$, the zero is at $s = 1$ and the poles are at $s = 0$, 2 and 3.

We use Partial Fraction Expansion to break $F(s)$ down into simple terms. Thus, there are two steps to find inverse Laplace transform as given below.

I. Decomposition of $F(s)$ into simple terms using Partial Fraction Expansion.
II. Evaluation of the inverse of each term comparing with the standard forms of Laplace transforms.

We consider the following three cases given below.

I. Simple Poles

Let
$$F(s) = \frac{N(s)}{(s+p_1)(s+p_2)(s+p_3)\ldots(s+p_n)}$$

where, $s = -p_1, -p_2 -p_3, \ldots, -p_n$ are the simple poles, and $p_i \neq p_j$ for all $i \neq j$ (i.e. poles are distinct) Assuming that the degree of $N(s)$ is less than the degree of $D(s)$,

$$F(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \frac{k_3}{s+p_3} + \ldots + \frac{k_n}{s+p_n} \tag{1}$$

where, expansion co-efficients $k_1, k_2, k_3, \ldots, k_n$ are known as the residues of $F(s)$. These can be found out by Residue method explained below.

Multiplying both sides of Eq. (1), by $(s + p_1)$,

$$(s+p_1)F(s) = k_1 + \frac{(s+p_1)k_2}{s+p_2} + \frac{(s+p_1)k_3}{s+p_3} + \ldots + \frac{(s+p_1)k_n}{s+p_n}$$

Putting
$$s = -p_1 \quad \Rightarrow \quad (s+p_1)\,F(s)\big|_{s=-p_1} = k_1$$

In general, $\boxed{k_i = (s+p_i)\,F(s)\big|_{s=-p_i}}$. This is known as Heaviside's Theorem.

Once, the values of k_i are known, the inverse Laplace is obtained as,

$$f(t) = (k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + k_3 e^{-p_3 t} + \ldots + k_n e^{-p_n t})u(t)$$

Example 6.2 Find the inverse Laplace transform of the function,

$$F(s) = \frac{2s+1}{(s+1)(s+2)(s+3)}.$$

Solution Let
$$F(s) = \frac{2s+1}{(s+1)(s+2)(s+3)} = \frac{k_1}{s+1} + \frac{k_2}{s+2} + \frac{k_3}{s+3}$$

$$\therefore \quad k_1 = (s+1)F(s)\big|_{s=-1} = \left.\frac{2s+1}{(s+2)(s+3)}\right|_{s=-1} = -\frac{1}{2}$$

$$\therefore \qquad k_2 = (s+2)F(s)|_{s=-2} = \left.\frac{2s+1}{(s+1)(s+3)}\right|_{s=-2} = 3$$

$$\therefore \qquad k_3 = (s+3)F(s)|_{s=-3} = \left.\frac{2s+1}{(s+1)(s+2)}\right|_{s=-3} = -\frac{5}{2}$$

$$\therefore \qquad F(s) = -\frac{1}{2(s+1)} + \frac{3}{s+2} - \frac{5}{2(s+3)}$$

Thus, the inverse Laplace transform is given as,

$$f(t) = -\frac{1}{2}e^{-t} + 3e^{-2t} - \frac{5}{2}e^{-3t}$$

II. Repeated Poles

Suppose, $F(s)$ has n repeated poles at $s = -p$.

$$\therefore \qquad F(s) = \frac{k_n}{(s+p)^n} + \frac{k_{n-1}}{(s+p)^{n-1}} + \frac{k_{n-2}}{(s+p)^{n-2}} + \ldots + \frac{k_2}{(s+p)^2} + \frac{k_1}{(s+p)} + F_1(s)$$

where, $F_1(s)$ is the remaining part of $F(s)$ that does not have a pole at $s = -p$.
We find,

$$\therefore \qquad k_n = (s+p)^n F(s)|_{s=-p}$$

To find k_{n-1}, k_{n-2},…, k_{n-m}, the procedure is,

$$k_{n-1} = \left.\frac{d}{ds}[(s+p)^n F(s)]\right|_{s=-p}$$

$$k_{n-1} = \left.\frac{1}{2!}\frac{d^2}{ds^2}[(s+p)^n F(s)]\right|_{s=-p}$$

In general, $k_{n-m} = \left.\dfrac{1}{m!}\dfrac{d^m}{ds^m}[(s+p)^n F(s)]\right|_{s=-p}$, where, $m = 1, 2, \ldots, (n-1)$.

Once, the values of k_1, k_2, …, k_n are known, the inverse Laplace is obtained as,

$$\boxed{f(t) = \left(k_1 e^{-pt} + k_2 t e^{-pt} + \frac{k_3}{3!}t^2 e^{-pt} + \ldots + \frac{k_n}{(n-1)!}t^{n-1}e^{-pt}\right)u(t) + f_1(t)}$$

Example 6.3 Find the inverse Laplace transform of the function $F(s) = \dfrac{12}{(s+2)^2(s+4)}$.

Solution Let $F(s) = \dfrac{12}{(s+2)^2(s+4)} = \dfrac{k_1}{(s+2)^2} + \dfrac{k_2}{s+2} + \dfrac{k_3}{s+4}$

By residue method,

$$k_1 = (s+2)^2 F(s)\big|_{s=-2} = \left.\frac{12}{(s+4)}\right|_{s=-2} = 6$$

$$\therefore \quad k_2 = \frac{d}{ds}[(s+2)^2 F(s)]\big|_{s=-2} = \frac{d}{ds}\left.\left[\frac{12}{(s+4)}\right]\right|_{s=-2} = -3$$

$$k_3 = (s+4)F(s)\big|_{s=-4} = \left.\frac{12}{(s+2)^2}\right|_{s=-4} = 3$$

Thus, $F(s) = \dfrac{6}{(s+2)^2} - \dfrac{3}{s+2} + \dfrac{3}{s+4}$

Taking inverse Laplace transform, $f(t) = 3e^{-4t} - 3e^{-2t} + 6te^{-2t}$

III. Complex Poles

Since $N(s)$ and $D(s)$ always have real co-efficients and as the complex roots of polynomials with real co-efficients occur in conjugate form, $F(s)$ may have the general form,

$$F(s) = \frac{A_1 s + A_2}{s^2 + as + b} + F_1(s) = \frac{k_1}{s+\alpha - j\beta} + \frac{k_2}{s+\alpha + j\beta} + F_1(s)$$

where, $F_1(s)$ is the remaining part of $F(s)$ that does not have this pair of complex poles.

Let $\quad (s^2 + as + b) = (s^2 + 2\alpha s + \alpha^2 + \beta^2) = (s+\alpha)^2 + \beta^2$

$$\therefore \quad s_{1,2} = (-\alpha \pm j\beta) = -\frac{a}{2} \pm j\sqrt{b - \frac{a^2}{4}}$$

Thus, the coefficients are,

$$\boxed{k_1 = (s - s_1)F(s)\big|_{s=s_1} \quad \text{and} \quad k_2 = k_1^* = \text{Complex conjugate of } k_1}$$

Example 6.4 Find the inverse Laplace transform of the function $F(s) = \dfrac{2s+1}{(s+1)(s^2+2s+5)}$.

Solution Let $F(s) = \dfrac{2s+1}{(s+1)(s^2+2s+5)} = \dfrac{A}{s+1} + \dfrac{k_1}{s+1-j2} + \dfrac{k_2}{s+1+j2}$

$$\therefore \quad A = (s+1)F(s)\big|_{s=-1} = \left.\frac{2s+1}{s^2+2s+5}\right|_{s=-1} = -\frac{1}{4}$$

$$k_1 = (s+1-j2)F(s)\big|_{s=(-1+j2)} = \left.\frac{2s+1}{(s+1)(s+1+j2)}\right|_{s=(-1+j2)} = \left(\frac{1}{8} - j\frac{1}{2}\right)$$

$$\therefore \qquad k_2 = k_1^* = \left(\frac{1}{8} + j\frac{1}{2}\right)$$

$$\therefore \qquad F(s) = -\frac{1}{4}\left(\frac{1}{s+1}\right) + \frac{\frac{1}{8} - j\frac{1}{2}}{s+1-j2} + \frac{\frac{1}{8} + j\frac{1}{2}}{s+1+j2}$$

Taking inverse Laplace transform,

$$f(t) = -\frac{1}{4}[e^{-t} - e^{-t}\cos 2t] + e^{-t}\sin 2t = -\frac{1}{2}e^{-t}\sin^2 t + e^{-t}\sin 2t$$

6.10 APPLICATIONS OF LAPLACE TRANSFORM

1. Solving Integro-Differential Equations and Simultaneous Differential Equations
2. Transient Analysis of Electrical Circuits.

6.10.1 Solving Integro Differential Equations and Simultaneous Differential Equations

An *integro-differential equation* is an integral equation in which various derivatives of the unknown function can also be present. Using the Laplace transform of integrals and derivatives, an integro-differential equation can be solved.

Similarly, it is easier with the Laplace transform method to solve *simultaneous differential equations* by transforming both equations and then solving the two equations in the s-domain and finally obtaining the inverse to get the solution in the time domain.

Example 6.5 (Integro-Differential Equation) Solve the equation for the response $i(t)$, given that

$$\frac{di}{dt} + 2i + 5\int_0^t i\,dt = u(t) \text{ and } i(0) = 0.$$

Solution Let $\mathcal{L}[i(t)] = I(s)$.

$$\therefore \qquad \mathcal{L}\left[\frac{di}{dt}\right] = sI(s) - i(0) = sI(s) - 0 = sI(s)$$

Taking Laplace transform on both sides of the given equation,

$$sI(s) + 2I(s) + 5\frac{I(s)}{s} = \frac{1}{s}$$

or

$$I(s) = \frac{1}{s^2 + 2s + 5} = \frac{1}{2}\,\frac{2}{(s+1)^2 + (2)^2}$$

Taking inverse Laplace transform, we get

$$i(t) = \frac{1}{2}e^{-t}\sin 2t\,(A),\, t > 0$$

Example 6.6 (Integro-Differential Equation) Solve the initial value problem for *y(t)* when $\frac{d^2y}{dt^2} + y(t) = 3\sin 2t$ and $y(0) = 1,\ y'(0) = -2$.

Solution Let $\mathcal{L}[y(t)] = Y(s)$.

$$\therefore \quad \mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 2$$

or $\quad Y(s) = \dfrac{s}{s^2+1} - \dfrac{2}{s^2+4}$

Taking inverse Laplace transform, we get, $y(t) = (\cos t - \sin 2t)$

Example 6.7 (Simultaneous Differential Equations) Find the solution of the system: $\frac{dx}{dt} - 6x + 3y = 8e^t$ and $\frac{dy}{dt} - 2x - y = 4e^t$ with initial conditions $x(0) = -1$, $y(0) = 0$.

Solution Taking Laplace transform,

$$(s-6)X + 3Y = \frac{-s+9}{s-1} \tag{i}$$

$$-2X + (s-1)Y = \frac{4}{s-1} \tag{ii}$$

Solving for X and Y,

$$X = \frac{-s+7}{(s-1)(s-4)} = -\frac{2}{s-1} + \frac{1}{s-4}$$

$$Y = \frac{2}{(s-1)(s-4)} = \frac{-2/3}{s-1} + \frac{2/3}{s-4}$$

Taking inverse Laplace transform,

$$x(t) = -2e^t + e^{4t} \quad \text{and} \quad y(t) = -\frac{2}{3}e^t + \frac{2}{3}e^{4t}$$

Example 6.8 (Simultaneous Differential Equations) Solve for $x(t)$ and $y(t)$, given that $x(0) = 4$, $y(0) = 3$ and

$$\frac{dx}{dt} + x + 4y = 10 \quad \text{and} \quad x - \frac{dy}{dt} - y = 0$$

Solution Following the same procedures, as in Ex (6.7), we get,

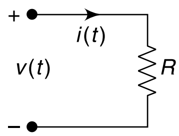
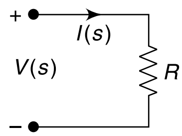
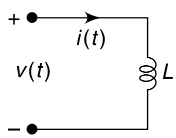
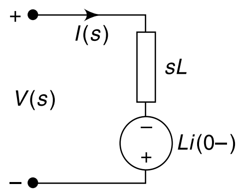
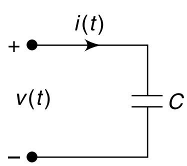
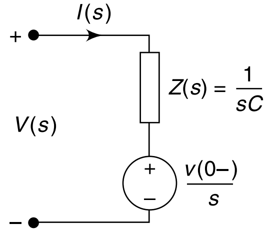
$$X = \frac{4s^2 + 2s + 10}{s(s^2 + 3)} \quad \text{and} \quad Y = \frac{3s^2 + s + 10}{s(s^2 + 3)}$$

Taking inverse Laplace transform, we get the desired results.

6.11 APPLICATION OF LAPLACE TRANSFORM METHOD TO CIRCUIT ANALYSIS

We now apply the mathematical tool for the analysis of electric circuits.

6.11.1 Transform Impedance of Network Elements

Element	*Time Domain*	*s-Domain*
1. Resistor (R)	$v(t) = Ri(t)$ 	$V(s) = RI(s)$ 
2. Inductor (L)	$v(t) = L\dfrac{di(t)}{dt}$ $i(t) = \dfrac{1}{L}\int_{-\infty}^{t} v(t)dt$ 	$V(s) = L[sI(s) - i(0\_)]$ $I(s) = \dfrac{1}{L}\left[\dfrac{V(s)}{s} + \dfrac{i(0-)}{s}\right]$ 
3. Capacitor (C)	$i(t) = C\dfrac{dv(t)}{dt}$ $v(t) = \dfrac{1}{C}\int_{-\infty}^{t} i(t)dt$ 	$I(s) = sCV(s) - Cv(0\_)$ $V(s) = \dfrac{I(s)}{Cs} + \dfrac{v(0\_)}{s}$ 

6.12 TRANSIENT ANALYSIS OF ELECTRIC CIRCUITS USING LAPLACE TRANSFORM

In electrical engineering, a **transient response** or **natural response** is the electrical response of a system to a change from equilibrium.

The condition prevailing in an electric circuit between two steady-state conditions is known as the *transient state*; it lasts for a very short time. The currents and voltages during the transient state are called *transients*.

In general, transient phenomena occur whenever

(i) a circuit is suddenly connected or disconnected to/from the supply,

(ii) there is a sudden change in the applied voltage from one finite value to another,

(iii) a circuit is short-circuited.

A simple example would be the output of a 5 volt DC power supply when it is turned on: the transient response is from the time the switch is turned on and the output is a steady 5 volt. At this point the power supply reaches its steady-state response of a constant 5 volt.

The transient response is not necessarily tied to "on/off" events but to any event that affects the equilibrium of the system. If in an RC circuit the resistor or capacitor is replaced with a variable resistor or variable capacitor (or both) then the transient response is the response to a change in the resistor or capacitor.

The transient currents are not caused by any part of the supply voltage, but are entirely associated with the changes in the stored energy in capacitor and inductors. As there is no energy stored in resistors, there are *no transients in purely resistive circuits*.

Although transients last for a very short time, their study is very important because.

(i) They indicate what dangerous rises in voltage or current may happen in individual sections of a circuit.

(ii) They indicate how signals are distored in waveform or amplitude as they pass through amplifiers, filters, or other circuit elements.

We consider the transient analysis for the following circuits subject to step input, impulse input and sinusoidal input:

1. *RL* Series Circuit,
2. *RC* Series Circuit,
3. *RLC* Series Circuit, and
4. *RLC* Parallel Circuit.

6.12.1 *RL* Series Circuit

1. RL Series Circuit with Step Input We consider an *RL* series circuit as shown in the figure.

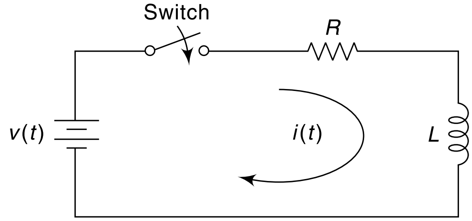


Figure 6.11 *R-L series circuit*

If the switch is closed at time $t = 0$, the voltage across the RL combination would be $v(t)$ which is a step of magnitude V [or $Vu(t)$] and not a constant as is the supply voltage V.

$$v(t) = 0, \text{ for } t \le 0$$
$$= V, \text{ for } t \ge 0$$

Thus the differential equation governing the behaviour of the circuit would be

$$Ri(t) + \mathcal{L}\frac{di(t)}{dt} = Vu(t)$$

Taking Laplace transform, we get

$$RI(s) + \mathcal{L}[sI(s) - i(0-)] = \frac{V}{s}$$

or,
$$I(s) = \frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)} + \frac{i(0-)}{s+\frac{R}{L}} = \frac{V}{R}\left(\frac{1}{s} - \frac{1}{s+\frac{R}{L}}\right) + \frac{i(0-)}{s+\frac{R}{L}}$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right) + i(0-)\, e^{-\left(\frac{R}{L}\right)t} = \frac{V}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right) \text{ with } i(0-) = 0.$$

The transient part of the current response, $i_{tr} = [i(t) - i_s] = -\frac{V}{R}e^{-\frac{R}{L}t}$

From the current equation at $t = \tau = \frac{L}{R}$, $i = \frac{V}{R}(1 - e^{-1}) = 0.63\frac{V}{R} = 0.63 i_s$

When the switch is first closed, the voltage across the inductor will immediately jump to battery voltage (acting as though it were an open-circuit) and decay down to zero over time (eventually acting as though it were a short-circuit). Voltage across the inductor is determined by calculating how much voltage is being dropped across R, given the current through the inductor, and subtracting that voltage value from the battery. When the switch is first closed, the current is zero, then it increases over time until it is equal to the battery voltage divided by the series resistance. This behavior is precisely opposite that of the series resistor-capacitor circuit, where current started at a maximum and capacitor voltage at zero.

The steady state part of the current response, $i_s = \frac{V}{R}$

The variation of the current is shown in Figure 6.12.

The quantity $\tau = \frac{L}{R}$ is known as the Time-constant of the circuit and it is defined as follows.

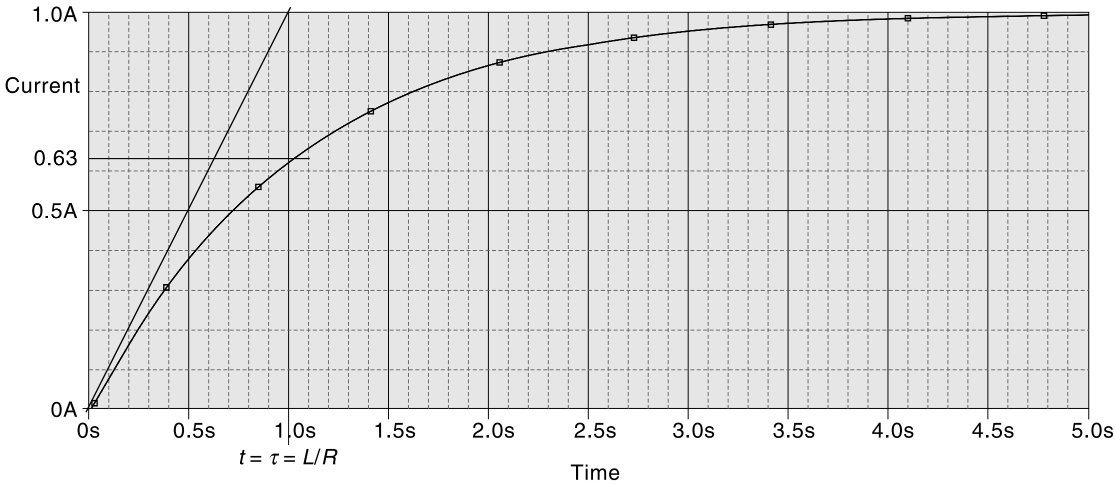


Figure 6.12 *Variation of current with time RL series circuit with step input*

Definitions of Time-constant (τ)

1. It is the time taken for the current to reach 63% of its final value. Thus, it is a measure of the rapidity with which the steady state is reached.
   Also, at $t = 5\tau$, $i = 0.993 i_s$; the transient is therefore, said to be practically disappeared in five time constants.
2. The tangent to the equation $i = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$ at $t = 0$, intersects the straight line, $i = \frac{V}{R}$ at $t = \tau = \frac{L}{R}$. Thus, time-constant is the time in which steady state would be reached if the current increases at the initial rate.

Physically, time-constant represents the speed of the response of a circuit. A low value of time-constant represents a fast response and a high value of time-constant represents a sluggish response.

Calculations of the Voltage Across Elements

Voltage across the resistor, $V_R = Ri(t) = V\left(1 - e^{-\frac{R}{L}t}\right)$

Voltage across the inductor, $V_L = L\frac{di(t)}{dt} = L\frac{d}{dt}\left[\frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)\right] = Ve^{-\frac{R}{L}t}$

2. RL Series Circuit with Impulse Input By KVL, the mesh equation becomes,

$$Ri(t) + L\frac{di(t)}{dt} = V\delta(t)$$

Taking Laplace transform,

$$RI(s) + sLI(s) = V \text{ with } i(0-) = 0$$

or $$I(s) = \frac{V}{L}\left(\frac{1}{s + R/L}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{L} e^{-\frac{R}{L}t}$$

Here, the plot of the current is shown in Figure 6.13.

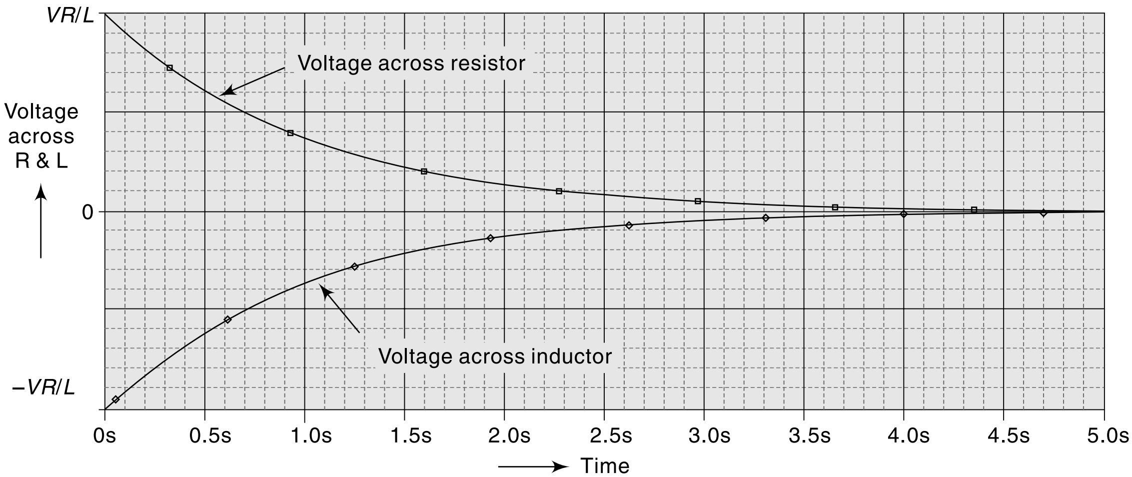


Figure 6.13 *Variation of voltages with time in RL series circuit with impulse input*

Voltage across the resistor, $V_R = Ri(t) = \frac{VR}{L} e^{-\frac{R}{L}t}$

Voltage across the inductor, $V_L = L\frac{di(t)}{dt} = L\frac{d}{dt}\left(\frac{V}{L} e^{-\frac{R}{L}t}\right) = -\frac{VR}{L} e^{-\frac{R}{L}t}$

3. RL Series Circuit with Sinusoidal Input Here, the input voltage is given as, $v(t) = V \sin \omega t$

By KVL,

$$Ri(t) + L\frac{di(t)}{dt} = V \sin \omega t, \text{ with } i(0-) = 0$$

or $$I(s)[R + sL] = \frac{V\omega}{s^2 + \omega^2}$$

or $$I(s) = \frac{\frac{V\omega}{L}}{(s^2 + \omega^2)\left(s + \frac{R}{L}\right)} = \frac{V\omega}{L}\left\{\frac{1}{(s + j\omega)(s - j\omega)\left(s + \frac{R}{L}\right)}\right\}$$

$$= \frac{V\omega}{L}\left[\frac{A_1}{s-j\omega}+\frac{A_2}{s+j\omega}+\frac{A_3}{s+\frac{R}{L}}\right]$$

where, $$A_1 = \left\{(s-j\omega)\frac{1}{(s+j\omega)(s-j\omega)\left(s+\frac{R}{L}\right)}\right\}_{s=j\omega} = \frac{L}{2j\omega(R+j\omega L)}$$

$$A_2 = \left\{(s+j\omega)\frac{1}{(s+j\omega)(s-j\omega)\left(s+\frac{R}{L}\right)}\right\}_{s=-j\omega} = -\frac{L}{2j\omega(R-j\omega L)}$$

and $$A_3 = \left\{\left(s+\frac{R}{L}\right)\frac{1}{(s+j\omega)(s-j\omega)\left(s+\frac{R}{L}\right)}\right\}_{s=-\frac{R}{L}} = \frac{L^2}{(R^2+\omega^2L^2)}$$

$\therefore$ $$I(s) = \frac{V\omega}{L}\left[\frac{L}{2j\omega(R+j\omega L)(s-j\omega)}-\frac{L}{2j\omega(R-j\omega L)(s+j\omega)}+\frac{L^2}{(R^2+\omega^2L^2)\left(s+\frac{R}{L}\right)}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{V\omega}{L}\left[\frac{Le^{j\omega t}}{2j\omega(R+j\omega L)}-\frac{Le^{-j\omega t}}{2j\omega(R-j\omega L)}+\frac{L^2e^{-\frac{R}{L}t}}{R^2+\omega^2L^2}\right]$$

$$= \frac{V}{2j}\left[\frac{e^{j\omega t}}{R+j\omega L}-\frac{e^{-j\omega t}}{R-j\omega L}\right]+V\omega L\frac{e^{-\frac{R}{L}t}}{R^2+\omega^2L^2}$$

Let, $(R+j\omega L)=Ze^{j\theta}$ and $(R-j\omega L)=Ze^{-j\theta}$ so that, $Z=\sqrt{(R^2+\omega^2L^2)}$ and $\theta=\tan^{-1}\left(\frac{\omega L}{R}\right)$

Putting these values,

$$i(t) = \frac{V}{2j}\left[\frac{e^{j\omega t}}{Ze^{j\theta}}-\frac{e^{-j\omega t}}{Ze^{-j\theta}}\right]+V\omega L\frac{e^{-\frac{R}{L}t}}{Z^2}$$

$$= \frac{V}{Z}\left[\frac{e^{j(\omega t-\theta)}-e^{-j(\omega t-\theta)}}{2j}\right]+\frac{V\omega L}{Z^2}e^{-\frac{R}{L}t}$$

or, finally, the current is,

$$\boxed{i(t)=\frac{V}{Z}\sin(\omega t-\theta)+\frac{V\omega L}{Z^2}e^{-\frac{R}{L}t}}$$

From this result, it is clear that the current in *RL* series circuit lags behind the voltage by an angle, $\theta=\tan^{-1}\left(\frac{\omega L}{R}\right)$. If the resistance R = 0, then $\theta = 90°$ as is the case for a perfect inductor.

6.12.2 *RC* Series Circuit

1. RC Series Circuit with Step Input We consider an *RC* series circuit as shown in Figure 6.14.

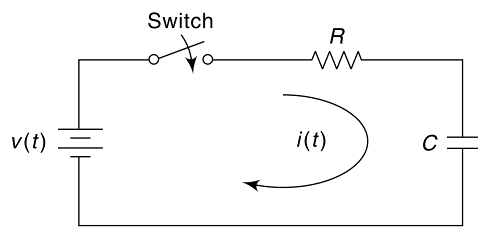


Figure 6.14 *RC series circuit*

By KVL, $Ri(t)+\frac{1}{C}\int_0^t i(t)\,dt = Vu(t)$

Taking Laplace transform,

$$RI(s)+\frac{1}{C}\left[\frac{I(s)}{s}+\frac{q(0-)}{s}\right]=\frac{V}{s}$$

or
$$I(s)\left[R+\frac{1}{Cs}\right]=\frac{V}{s}-\frac{q(0-)}{Cs}$$

or
$$I(s)=\frac{V-\frac{q(0-)}{C}}{s(R+1/Cs)}=\frac{1}{R}\frac{V-\frac{q(0-)}{C}}{(s+1/RC)}$$

Taking inverse Laplace transform,

$$i(t) = \left[\frac{V}{R}-\frac{q(0-)}{RC}\right]e^{-\frac{t}{RC}};\ \text{for } t \geq 0$$

$$= \frac{V}{R}e^{-\frac{t}{RC}};\ \text{if } q(0-) = 0$$

The steady state part of the current response, $i_s = 0$

The transient part of the current response, $i_{tr} = [i(t) - i_s] = \frac{V}{R}e^{-\frac{t}{RC}}$

From the current equation at $t = \tau = RC$, $i=\frac{V}{R}e^{-1}=0.37\frac{V}{R}$

When the switch is first closed, the voltage across the capacitor (which was fully discharged) is zero volt; thus, it first behaves as though it were a short-circuit. Over time, the capacitor voltage will rise to equal battery voltage, ending in a condition where the capacitor behaves as an open-circuit. Current through the circuit is determined by the difference in voltage between the battery and the capacitor, divided by the resistance. As the capacitor voltage approaches the battery voltage (V), the current approaches zero. Once the capacitor voltage has reached V, the current will be exactly zero.

The variation of current in the circuit is shown in Figure 6.15.

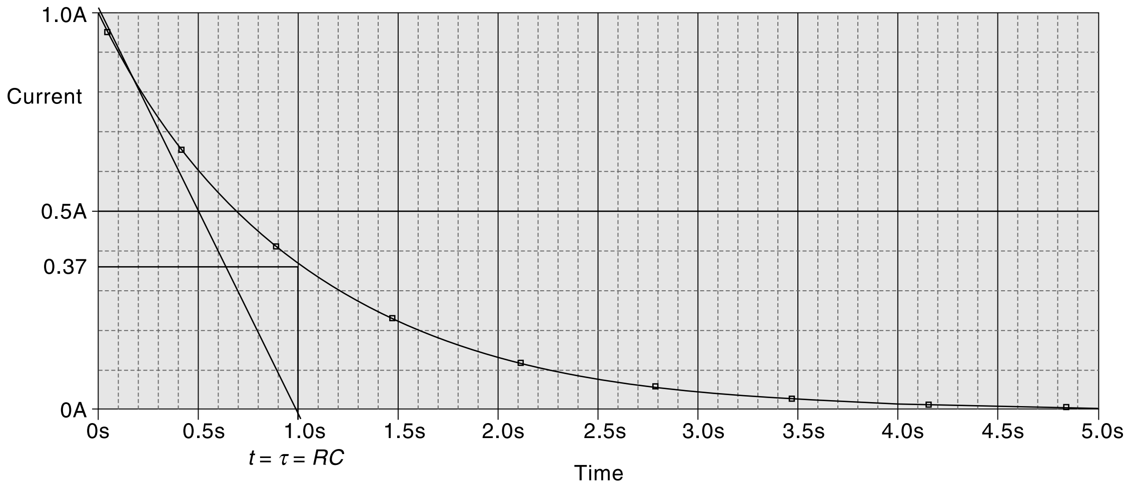


Figure 6.15 *Variation of current with time in RC series circuit with step input*

The quantity $\tau = RC$ is known as the Time-constant of the circuit and it is defined as follows.

Definitions of Time-constant (τ)

1. It is the time in which the current decays to 37% of its initial value.

   Also, at $t = 5\tau$, $i = 0.07\dfrac{V}{R}$; the transient is therefore, said to be practically disappeared in five time constants.

2. The tangent to the equation $i = \dfrac{V}{R}e^{-\frac{t}{RC}}$ at $t = 0$, intersects the time axis at $t = \tau = RC$.

Thus, time-constant is the time in which the current would reach the steady state zero value if the current decays at the initial rate.

Physically, time-constant represents the speed of the response of a circuit. A low value of time-constant represents a fast response and a high value of time-constant represents a sluggish response.

Calculations of the Voltage Across Elements

Voltage across the resistor, $V_R = Ri(t) = Ve^{-\frac{t}{RC}}$

Voltage across the capacitor, $V_C = \dfrac{1}{C}\int_0^t i(t)dt = \dfrac{1}{C}\int_0^t \dfrac{V}{R}e^{\frac{-t}{RC}}dt = V\left(1 - e^{\frac{-t}{RC}}\right)$

2. RC Series Circuit with Impulse Input With zero initial condition, $q(0-) = 0$, KVL equation becomes,

$$Ri(t) + \frac{1}{C}\int_0^t i(t)dt = V\delta(t)$$

$$RI(s) + \frac{I(s)}{Cs} = V$$

or, $$I(s)=\frac{V}{R+\frac{1}{Cs}}=\frac{V}{R}\left(\frac{s}{s+\frac{1}{RC}}\right)=\frac{V}{R}\left[1-\frac{\frac{1}{RC}}{s+\frac{1}{RC}}\right]$$

Taking inverse Laplace transform,

$$i(t)=\frac{V}{R}\left[\delta(t)-\frac{1}{RC}e^{\frac{-t}{RC}}\right];\text{ for } t\geq 0$$

Voltage across the resistor, $V_R=Ri(t)=V\left[\delta(t)-\frac{1}{RC}e^{\frac{-t}{RC}}\right]$

Voltage across the capacitor, $V_C=\{V\delta(t)-V_R\}=\frac{V}{RC}e^{\frac{-t}{RC}}$

These variations of the voltages are shown in Figure 6.16.

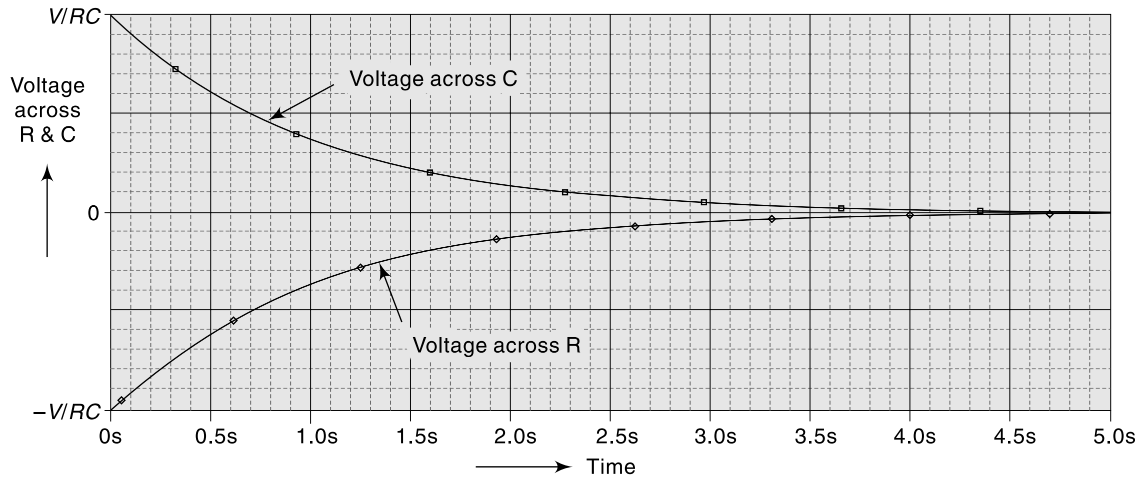


Figure 6.16 *Variation of voltages with time in RC series circuit with impulse input*

3. RC Series Circuit with Sinusoidal Input Here, the input voltage is given as, $v(t) = V\sin\omega t$

By KVL,

$$Ri(t)+\frac{1}{C}\int_0^t i(t)dt=V\sin\omega t\text{, with } q(0-)=0$$

or $$I(s)\left[R+\frac{1}{Cs}\right]=\frac{V\omega}{s^2+\omega^2}$$

or $$I(s)=\frac{V\omega Cs}{(s^2+\omega^2)(1+sRC)}=\frac{V\omega}{R}\left\{\frac{s}{(s+j\omega)(s-j\omega)\left(s+\frac{1}{RC}\right)}\right\}$$

$$=\frac{V\omega}{L}\left[\frac{A_1}{s-j\omega}+\frac{A_2}{s+j\omega}+\frac{A_3}{s+\frac{1}{RC}}\right]$$

where, $$A_1 = \left\{(s - j\omega)\frac{1}{(s + j\omega)(s - j\omega)\left(s + \frac{1}{RC}\right)}\right\}_{s = j\omega} = \frac{RC}{2(1 + j\omega RC)}$$

$$A_2 = \left\{(s + j\omega)\frac{1}{(s + j\omega)(s - j\omega)\left(s + \frac{1}{RC}\right)}\right\}_{s = -j\omega} = -\frac{RC}{2(1 - j\omega RC)}$$

and $$A_3 = \left\{\left(s + \frac{1}{RC}\right)\frac{1}{(s + j\omega)(s - j\omega)\left(s + \frac{1}{RC}\right)}\right\}_{s = -\frac{1}{RC}} = \frac{-\frac{1}{RC}}{\left(\omega^2 + \frac{1}{R^2C^2}\right)}$$

$$\therefore \qquad I(s) = \frac{V\omega}{R}\left[\frac{RC}{2(1 + j\omega RC)(s - j\omega)} - \frac{RC}{2(1 - j\omega RC)(s + j\omega)} + \frac{-\frac{1}{RC}}{\left(\omega^2 + \frac{1}{R^2C^2}\right)\left(s + \frac{1}{RC}\right)}\right]$$

Taking inverse Laplace transform,

$$i(t) = V\omega C\left[\frac{e^{j\omega t}}{2(1 + j\omega RC)} - \frac{e^{-j\omega t}}{2(1 - j\omega RC)}\right] - \frac{Ve^{\frac{-t}{RC}}}{\omega RC\left(R^2 + \frac{1}{\omega^2C^2}\right)}$$

$$= \frac{V}{2j}\left[\frac{e^{j\omega t}}{R + \frac{1}{j\omega C}} - \frac{e^{-j\omega t}}{R - \frac{1}{j\omega C}}\right] - \frac{Ve^{\frac{-t}{RC}}}{\omega RC\left(R^2 + \frac{1}{\omega^2C^2}\right)}$$

Let, $\left(R + \frac{1}{j\omega C}\right) = \left(R - \frac{j}{\omega C}\right) = Ze^{-j\theta}$ and $\left(R - \frac{1}{j\omega C}\right) = \left(R + \frac{j}{\omega C}\right) = Ze^{j\theta}$

so that, $Z = \sqrt{\left(R^2 + \frac{1}{\omega^2C^2}\right)}$ and $\theta = \tan^{-1}\left(\frac{1}{\omega RC}\right)$

Putting these values,

$$i(t) = \frac{V}{2j}\left[\frac{e^{j\omega t}}{Ze^{-j\theta}} - \frac{e^{-j\omega t}}{Ze^{j\theta}}\right] - \frac{V}{\omega CZ^2}e^{\frac{-t}{RC}}$$

$$= \frac{V}{Z}\left[\frac{e^{j(\omega t + \theta)} - e^{-j(\omega t + \theta)}}{2j}\right] - \frac{V}{\omega CZ^2}e^{\frac{-t}{RC}}$$

or, finally, the current is,

$$\boxed{i(t) = \frac{V}{Z}\sin(\omega t + \theta) - \frac{V}{\omega C Z^2} e^{\frac{-t}{RC}}}$$

From this result, it is clear that the current in *RC* series circuit leads the voltage by an angle, $\theta = \tan^{-1}\left(\frac{1}{\omega RC}\right)$. If the resistance $R = 0$, then $\theta = 90°$ as is the case for a perfect capacitor.

6.12.3 *RLC* Series Circuit

1. RLC Series Circuit with Step Input With zero initial conditions, the Kirchhoff's voltage law equation becomes,

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(t)dt = Vu(t)$$

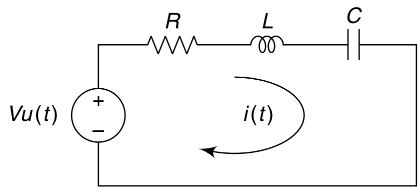


Figure 6.17 *RLC series circuit*

or $$RI(s) + sLI(s) + \frac{1}{Cs}I(s) = \frac{V}{s}$$

or $$I(s) = \frac{\frac{V}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{6.1}$$

The roots of the denominator polynomial of equation are,

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

or $$s_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \quad \text{and,} \quad s_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Let $$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \xi\omega_0 = \frac{R}{2L} \quad \text{i.e.} \quad \xi = \frac{R}{2}\sqrt{\frac{C}{L}} = \text{Damping Ratio}$$

Then, $$s_1 = -\xi\omega_0 + \omega_0\sqrt{\xi^2 - 1} \quad \text{and} \quad s_2 = -\xi\omega_0 - \omega_0\sqrt{\xi^2 - 1}$$

So, $$I(s) = \frac{\frac{V}{L}}{(s - s_1)(s - s_2)} = \frac{A}{s - s_1} + \frac{B}{s - s_2}$$

$\therefore$ $$A = (s - s_1)\left.\frac{\frac{V}{L}}{(s - s_1)(s - s_2)}\right|_{s = s_1} = \frac{\frac{V}{L}}{(s_1 - s_2)} = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}}$$

and, therefore $$B = (s - s_2)\left.\frac{\frac{V}{L}}{(s - s_1)(s - s_2)}\right|_{s = s_2} = \frac{\frac{V}{L}}{(s_2 - s_1)} = -\frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}}$$

Putting these values of A and B, we get,

$$I(s) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}}\left[\frac{1}{s - s_1} - \frac{1}{s - s_2}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}}[e^{s_1 t} - e^{s_2 t}] = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}[e^{(\omega_0\sqrt{\xi^2-1})t} - e^{-(\omega_0\sqrt{\xi^2-1})t}]$$

Depending upon the values of R, L and C, three cases may appear:

(a) $\frac{R}{2L} > \frac{1}{\sqrt{LC}}$ (Overdamped condition)

(b) $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$ (Underdamped condition)

(c) $\frac{R}{2L} = \frac{1}{\sqrt{LC}}$ (Critically Damped condition)

A. Overdamped Condition The condition is, $\frac{R}{2L} > \frac{1}{\sqrt{LC}}$ or, $\xi > 1$ or $Q < \frac{1}{2}$

$$\left(\text{Since, Quality Factor, } Q = \frac{\omega_0 L}{R} \quad \text{and} \quad \omega_0 = \frac{1}{\sqrt{LC}}\right)$$

Under this condition, the current becomes,

$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}[e^{(\omega_0\sqrt{\xi^2-1})t} - e^{-(\omega_0\sqrt{\xi^2-1})t}] = \frac{V}{\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}\sinh(\omega_0\sqrt{\xi^2 - 1})t$$

The graphical plot for the current is shown in Figure 6.18.

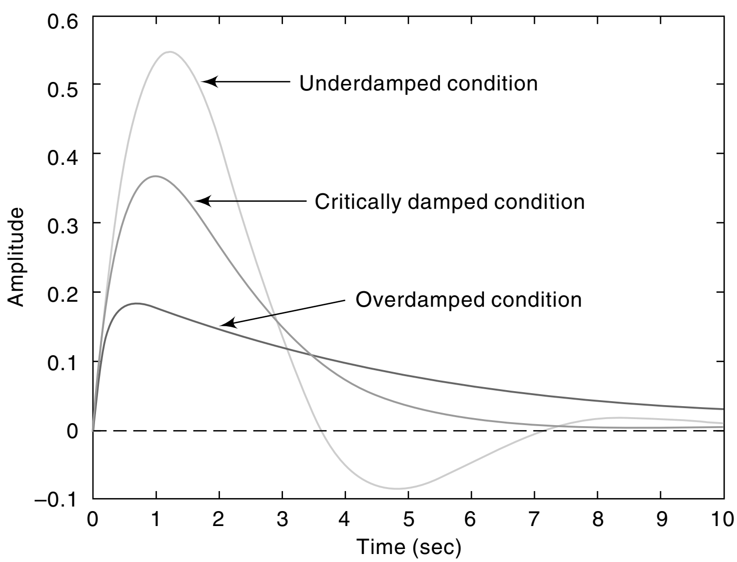


Figure 6.18 *Current response in RLC series circuit for three different damping conditions*

B. Critically Damped Condition The condition is, $\dfrac{R}{2L} = \dfrac{1}{\sqrt{LC}}$ or, $\xi = 1$ or $Q = \dfrac{1}{2}$

From equation (6.1),

$$I(s) = \frac{\frac{V}{L}}{s^2 + 2\omega_0 s + \omega_0^2} = \frac{V}{L}\left(\frac{1}{(s+\omega_0)^2}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{L} t e^{-\omega_0 t}$$

The graphical plot for the current is shown in Figure 6.13.

C. Underdamped Condition The condition is, $\dfrac{R}{2L} < \dfrac{1}{\sqrt{LC}}$ or, $\xi < 1$ or $Q > \dfrac{1}{2}$

So, the current becomes,

$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}\left[e^{(\omega_0\sqrt{\xi^2-1})t} - e^{-(\omega_0\sqrt{\xi^2-1})t}\right]$$

$$= \frac{V}{\omega_0 L\sqrt{1-\xi^2}} e^{-\xi\omega_0 t}\left[\frac{e^{\left(j\omega_0\sqrt{1-\xi^2}\right)t} - e^{-\left(j\omega_0\sqrt{1-\xi^2}\right)t}}{2j}\right]$$

$$= \frac{V}{\omega_0 L\sqrt{1-\xi^2}} e^{-\xi\omega_0 t} \sin\left(\omega_0\sqrt{1-\xi^2}\right)t$$

So, the circuit is oscillatory. When $R = 0$, $\xi = 0$, the oscillations are undamped or sustained. The frequency of the undamped oscillation (ω_0) is known as *undamped natural frequency*.

2. RLC Series Circuit with Impulse Input With zero initial conditions, the Kirchhoff's voltage law equation becomes,

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_0^t i(t)dt = V\delta(t)$$

or $$RI(s) + sLI(s) + \frac{1}{Cs}I(s) = V$$

or $$I(s) = \frac{\left(\frac{V}{L}\right)s}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \qquad (6.2)$$

The roots of the denominator polynomial of equation are,

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

or $$s_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \quad \text{and,} \quad s_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Let $$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \xi\omega_0 = \frac{R}{2L} \quad \text{i.e.} \quad \xi = \frac{R}{2}\sqrt{\frac{C}{L}} = \text{Damping Ratio}$$

Then, $$s_1 = -\xi\omega_0 + \omega_0\sqrt{\xi^2 - 1} \quad \text{and} \quad s_2 = -\xi\omega_0 - \omega_0\sqrt{\xi^2 - 1}$$

So, $$I(s) = \frac{\left(\frac{V}{L}\right)}{(s - s_1)(s - s_2)} = \frac{A}{s - s_1} + \frac{B}{s - s_2}$$

$\therefore$ $$A = (s - s_1)\left.\frac{\left(\frac{V}{L}\right)s}{(s - s_1)(s - s_2)}\right|_{s = s_1} = \frac{\left(\frac{V}{L}\right)s_1}{(s_1 - s_2)} = \frac{Vs_1}{2\omega_0 L\sqrt{\xi^2 - 1}}$$

and, therefore $$B = (s - s_2)\left.\frac{\left(\frac{V}{L}\right)s_2}{(s - s_1)(s - s_2)}\right|_{s = s_2} = \frac{\left(\frac{V}{L}\right)s_2}{(s_2 - s_1)} = -\frac{Vs_2}{2\omega_0 L\sqrt{\xi^2 - 1}}$$

Putting these values of A and B, we get,

$$I(s) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}}\left[\frac{s_1}{s - s_1} - \frac{s_2}{s - s_2}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}}[s_1 e^{s_1 t} - s_2 e^{s_2 t}] = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}[s_1 e^{(\omega_0\sqrt{\xi^2 - 1})t} - s_2 e^{-(\omega_0\sqrt{\xi^2 - 1})t}]$$

$$= \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}[(-\xi\omega_0 + \omega_0\sqrt{\xi^2 - 1})e^{(\omega_0\sqrt{\xi^2 - 1})t}$$

$$- (-\xi\omega_0 - \omega_0\sqrt{\xi^2 - 1})e^{-(\omega_0\sqrt{\xi^2 - 1})t}]$$

Three cases are considered:

(A) $\frac{R}{2L} > \frac{1}{\sqrt{LC}}$ (Overdamped condition)

(B) $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$ (Underdamped condition)

(C) $\frac{R}{2L} = \frac{1}{\sqrt{LC}}$ (Critically Damped condition)

A. Overdamped Condition Here, $\xi > 1$

The current becomes,

$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}[(-\xi\omega_0 + \omega_0\sqrt{\xi^2 - 1})e^{(\omega_0\sqrt{\xi^2-1})t}$$

$$- (-\xi\omega_0 - \omega_0\sqrt{\xi^2 - 1})e^{-(\omega_0\sqrt{\xi^2-1})t}]$$

$$= \frac{V}{L\sqrt{\xi^2 - 1}}[\sqrt{\xi^2 - 1}\cosh(\omega_0\sqrt{\xi^2 - 1})t - \xi\sinh(\omega_0\sqrt{\xi^2 - 1})t]$$

B. Critically Damped Condition The condition is, $\xi = 1$

From equation (6.2),

$$I(s) = \frac{(V/L)s}{s^2 + 2\omega_0 s + \omega_0^2} = \frac{V}{L}\left(\frac{s}{(s + \omega_0)^2}\right) = \frac{V}{L}\left[\frac{A}{(s + \omega_0)^2} + \frac{B}{s + \omega_0}\right]$$

where $$A = (s + \omega_0)^2 \frac{s}{(s + \omega_0)^2}\bigg|_{s=-\omega_0} = -\omega_0$$

and $$B = \frac{d}{ds}\left[(s + \omega_0)^2 \frac{s}{(s + \omega_0)^2}\right]_{s=-\omega_0} = 1$$

So, $$I(s) = \frac{V}{L}\left[\frac{1}{s + \omega_0} - \frac{\omega_0}{(s + \omega_0)^2}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{L}[1 - \omega_0 t]e^{-\omega_0 t}$$

C. Underdamped Condition The condition is, $\xi < 1$

So, the current becomes,

$$i(t) = \frac{V}{2\omega_0 L\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}[(\omega_0\sqrt{\xi^2 - 1})\, \{e^{(\omega_0\sqrt{\xi^2-1})t}$$

$$+ e^{-(\omega_0\sqrt{\xi^2-1})t}\} - \xi\omega_0\, \{e^{(\omega_0\sqrt{\xi^2-1})t} - e^{-(\omega_0\sqrt{\xi^2-1})t}\}]$$

$$= \frac{V}{2\omega_0 Lj\sqrt{1-\xi^2}} e^{-\xi\omega_0 t}[(j\omega_0\sqrt{1-\xi^2})\, \{e^{(j\omega_0\sqrt{1-\xi^2})t} + e^{-(j\omega_0\sqrt{1-\xi^2})t}\}$$

$$- \xi\omega_0\, \{e^{(j\omega_0\sqrt{1-\xi^2})t} - e^{-(j\omega_0\sqrt{1-\xi^2})t}\}]$$

$$i(t) = \frac{V}{L\sqrt{1-\xi^2}} e^{-\xi\omega_0 t}[\sqrt{1-\xi^2}\cos(\omega_0\sqrt{1-\xi^2})t - \xi\sin(\omega_0\sqrt{1-\xi^2})t]$$

$$= \frac{V}{L\sqrt{1-\xi^2}} e^{-\xi\omega_0 t}\cos\{(\omega_0\sqrt{1-\xi^2})t + \theta\}, \text{ where } \theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

3. RLC Series Circuit with Sinusoidal Input Sinusoidal voltage $v(t) = V_m \sin(\omega t + \theta)$ is applied to a series *RLC* circuit at time $t = 0$. We want to find the complete solution for the current $i(t)$ using Laplace transform method.

$$v(t) = V_m \sin(\omega t + \theta)$$

By KVL,

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(t)dt = V_m \sin(\omega t + \theta)$$

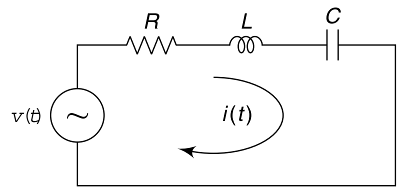


Figure 6.19 *RLC series circuit with sinusoidal input*

Taking Laplace transform with zero initial conditions,

$$I(s)\left[R + sL + \frac{1}{Cs}\right] = V_m \frac{(s\sin\theta + \omega\cos\theta)}{s^2 + \omega^2}$$

$$\text{or} \quad I(s) = \frac{V_m s\,(s\sin\theta + \omega\cos\theta)}{L(s^2+\omega^2)\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

$$= \frac{V_m}{L}\,\frac{s(s\sin\theta + \omega\cos\theta)}{(s + j\omega)\,(s + j\omega)\,(s - s_1)\,(s - s_2)}$$

where, s_1, s_2 are the roots of the quadratic equation:

$$\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

Thus, $s_1 = -\dfrac{R}{2L} + \sqrt{\dfrac{R^2}{4L^2} - \dfrac{1}{LC}}$ and, $s_2 = -\dfrac{R}{2L} - \sqrt{\dfrac{R^2}{4L^2} - \dfrac{1}{LC}}$

Now, let $\dfrac{s(s \sin\theta + \omega\cos\theta)}{(s + j\omega)(s - j\omega)(s - s_1)(s - s_2)} = \dfrac{K_1}{s - s_1} + \dfrac{K_2}{s - s_2} + \dfrac{K_3}{s + j\omega} + \dfrac{K_4}{s - j\omega}$

So, by residue method, multiplying by $(s - s_1)$ and putting $s = s_1$,

$$K_1 = \frac{s_1(s_1 \sin\theta + \omega\cos\theta)}{(s_1 + j\omega)(s_1 - j\omega)(s_1 - s_2)} \quad \text{and} \quad K_2 = \frac{s_2(s_2 \sin\theta + \omega\cos\theta)}{(s_2 + j\omega)(s_2 - j\omega)(s_2 - s_1)}$$

Similarly, multiplying by $(s + j\omega)$ and putting $s = -j\omega$,

$$K_3 = \frac{-j\omega(-j\omega\sin\theta + \omega\cos\theta)}{(-j\omega - j\omega)(-j\omega - s_1)(-j\omega - s_2)} = \frac{\omega(\cos\theta - j\sin\theta)}{2(s_1 + j\omega)(s_2 + j\omega)}$$

and, $$K_4 = \frac{j\omega(-\omega\sin\theta + \omega\cos\theta)}{(j\omega + j\omega)(j\omega - s_1)(j\omega - s_2)} = \frac{\omega(\cos\theta + j\sin\theta)}{2(s_1 - j\omega)(s_2 - j\omega)}$$

Hence the current response becomes,

$$i(t) = \frac{V_m}{L}[K_1 e^{s_1 t} + K_2 e^{s_2 t}] + \frac{V}{L}[K_3 e^{-j\omega t} + K_4 e^{j\omega t}] = I_{tr} + I_{ss}$$

Thus, the *transient part of the total current* is,

$$\boxed{I_{tr} = \frac{V_m}{L}\left[\frac{s_1(s_1\sin\theta + \omega\cos\theta)}{(s_2^2 + \omega^2)\sqrt{\dfrac{R^2}{L^2} - \dfrac{4}{LC}}} e^{s_1 t} - \frac{s_2(s_2\sin\theta + \omega\cos\theta)}{(s_2^2 + \omega^2)\sqrt{\dfrac{R^2}{L^2} - \dfrac{4}{LC}}} e^{s_2 t}\right]}$$

The steady-state part of the total current is obtained as follows.

$$I_{ss} = \frac{V_m}{2L}\left[\frac{\omega e^{-j\theta} e^{-j\omega t}}{(s_1 + j\omega)(s_2 + j\omega)} + \frac{\omega e^{j\theta} e^{j\omega t}}{(s_1 - j\omega)(s_2 - j\omega)}\right]$$

$$= \frac{V_m\omega}{2L}\left[\frac{e^{-j(\omega t + \theta)}}{(s_1 + j\omega)(s_2 + j\omega)} + \frac{e^{j(\omega t + \theta)}}{(s_1 - j\omega)(s_2 - j\omega)}\right]$$

$$= \frac{V_m\omega}{2L(s_1^2 + \omega^2)(s_2^2 + \omega^2)}[e^{-j(\omega t + \theta)}(s_1 s_2 - \omega^2 - j\omega s_1 - j\omega s_2)]$$

$$= \frac{V_m\omega}{2L(s_1^2 + \omega^2)(s_1^2 + \omega^2)}[(s_1 s_2 - \omega^2)\, 2\cos(\omega t + \theta) - (\omega s_1 + \omega s_2)\, 2\sin(\omega t + \theta)]$$

$$= \frac{V_m\omega}{L}\frac{1}{(s_1^2 + \omega^2)(s_2^2 + \omega^2)}\left[\left(\frac{1}{LC} - \omega^2\right)\cos(\omega t + \theta) - \left(-\frac{\omega R}{L}\right)\sin(\omega t + \theta)\right]$$

or $$I_{ss} = \frac{V_m \omega}{L} \frac{\left[\frac{\omega R}{L}\sin(\omega t+\theta) - \left(\omega^2 - \frac{1}{LC}\right)\cos(\omega t+\theta)\right]}{(s_1^2+\omega^2)(s_2^2+\omega^2)}$$

$$= \frac{V_m \omega}{L} \frac{1}{(s_1^2+\omega^2)(s_2^2+\omega^2)} \sin\left\{\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right\}$$

$$\times \frac{\omega}{L}\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

or, $$\boxed{I_{ss} = \frac{V_m}{\sqrt{R^2+\left(\omega L - \frac{1}{\omega C}\right)^2}} \sin\left\{\omega t + \theta - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right\}}$$

This gives the steady-state current of the series *RLC* circuit to a sinusoidal voltage.

6.12.4 *RLC* Parallel Circuit

1. RLC Parallel Circuit with Step Current Input With zero initial conditions, the Kirchhoff's current law equation becomes,

$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)dt = Iu(t)$$

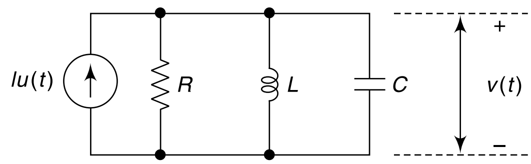


Figure 6.20 *RLC parallel circuit*

or $$\frac{V(s)}{R} + sCV(s) + \frac{1}{sL}V(s) = \frac{I}{s}$$

or $$V(s) = \frac{I/C}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \tag{6.3}$$

The roots of the denominator polynomial of equation are,

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

or $$s_1 = -\frac{1}{2RC} + \sqrt{\frac{1}{4R^2C^2} - \frac{1}{LC}} \quad \text{and,} \quad s_2 = -\frac{1}{2RC} - \sqrt{\frac{1}{4R^2C^2} - \frac{1}{LC}}$$

Let $$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \xi\omega_0 = \frac{1}{2RC} \quad \text{i.e. } \xi = \frac{1}{2R}\sqrt{\frac{L}{C}} = \text{Damping Ratio}$$

Then, $s_1 = -\xi\omega_0 + \omega_0\sqrt{\xi^2 - 1}$ and $s_2 = -\xi\omega_0 - \omega_0\sqrt{\xi^2 - 1}$

So, $$V(s) = \frac{I/C}{(s - s_1)(s - s_2)} = \frac{A}{s - s_1} + \frac{B}{s - s_2}$$

$\therefore$ $$A = (s - s_1)\frac{I/C}{(s - s_1)(s - s_2)}\bigg|_{s = s_1} = \frac{I/C}{(s_1 - s_2)} = \frac{I}{2\omega_0 C\sqrt{\xi^2 - 1}}$$

and, therefore, $$B = (s - s_2)\frac{I/C}{(s - s_2)(s - s_2)}\bigg|_{s = s_2} = \frac{I/C}{(s_2 - s_1)} = -\frac{I}{2\omega_0 C\sqrt{\xi^2 - 1}}$$

Putting these values of A and B, we get,

$$V(s) = \frac{I}{2\omega_0 C\sqrt{\xi^2 - 1}}\left[\frac{1}{s - s_1} - \frac{1}{s - s_2}\right]$$

Taking inverse Laplace transform,

$$v(t) = \frac{I}{2\omega_0 C\sqrt{\xi^2 - 1}}[e^{s_1 t} - e^{s_2 t}] = \frac{I}{2\omega_0 C\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}[e^{(\omega_0\sqrt{\xi^2 - 1})t} - e^{-(\omega_0\sqrt{\xi^2 - 1})t}]$$

Depending upon the values of R, L and C, three cases may appear:

(a) $\frac{1}{2RC} > \frac{1}{\sqrt{LC}}$ (Overdamped condition)

(b) $\frac{1}{2RC} < \frac{1}{\sqrt{LC}}$ (Underdamped condition)

(c) $\frac{1}{2RC} = \frac{1}{\sqrt{LC}}$ (Critically Damped condition)

A. Overdamped Condition The condition is, $\frac{1}{2RC} > \frac{1}{\sqrt{LC}}$ or, $\xi > 1$ or $Q < \frac{1}{2}$

$$\left(\text{Since, Quality Factor, } Q = \frac{1}{\omega_0 RC} \text{ and } \omega_0 = \frac{1}{\sqrt{LC}}\right)$$

Under this condition, the current becomes,

$$v(t) = \frac{I}{2\omega_0 C\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}[e^{(\omega_0\sqrt{\xi^2 - 1})t} - e^{-(\omega_0\sqrt{\xi^2 - 1})t}] = \frac{I}{\omega_0 C\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}\sinh\left(\omega_0\sqrt{\xi^2 - 1}\right)t$$

The graphical plot for the voltage is shown in Figure 6.21.

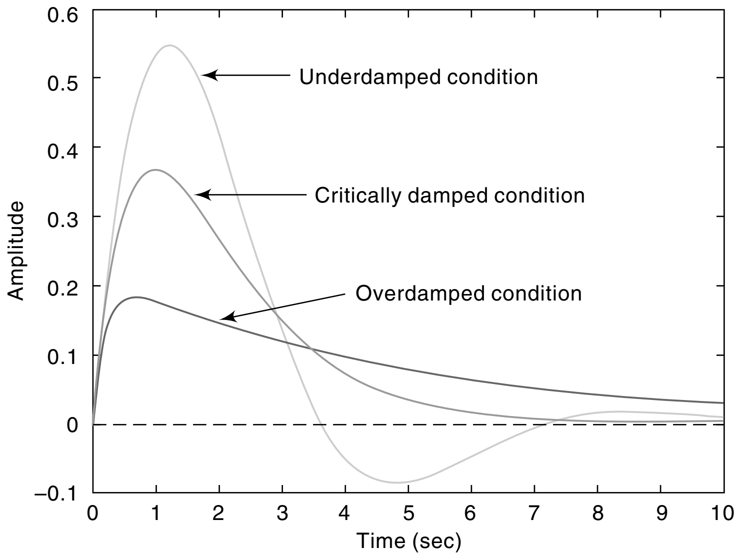


Figure 6.21 *Voltage response in RLC parallel circuit for three different damping conditions*

B. Critically Damped Condition The condition is, $\dfrac{1}{2RC} = \dfrac{1}{\sqrt{LC}}$ or, $\xi = 1$ or $Q = \dfrac{1}{2}$

From equation (6.3),

$$V(s) = \frac{I/C}{s^2 + 2\omega_0 s + \omega_0{}^2} = \frac{I}{C}\left(\frac{1}{(s+\omega_0)^2}\right)$$

Taking inverse Laplace transform,

$$v(t) = \frac{I}{C} t e^{-\omega_0 t}$$

The graphical plot for the voltage is shown in Fig. 6.21.

C. Underdamped Condition The condition is, $\dfrac{1}{2RC} < \dfrac{1}{\sqrt{LC}}$ or, $\xi < 1$ or $Q > \dfrac{1}{2}$

So, the voltage becomes,

$$v(t) = \frac{I}{2\omega_0 C\sqrt{\xi^2 - 1}} e^{-\xi\omega_0 t}\,[e^{(\omega_0\sqrt{\xi^2-1})t} - e^{-(\omega_0\sqrt{\xi^2-1})t}]$$

$$= \frac{I}{\omega_0 C\sqrt{1-\xi^2}} e^{-\xi\omega_0 t}\left[\frac{e^{(j\omega_0\sqrt{1-\xi^2})t} - e^{-(j\omega_0\sqrt{1-\xi^2})t}}{2j}\right]$$

$$= \frac{I}{\omega_0 C\sqrt{1-\xi^2}} e^{-\xi\omega_0 t} \sin\,(\omega_0\sqrt{1-\xi^2})t$$

Similarly we can find out the impulse response and sinusoidal response of a parallel *RLC* circuit using Laplace transform method as for the series *RLC* circuit.

6.12.5 Response with Pulse Input Voltage

1. RC Series Circuit If a voltage pulse of width as shown in Fig. 6.22 is applied to an *RC* series circuit, then by KVL,

$$Ri(t) + \frac{1}{C}\int i(t)dt = v(t)$$

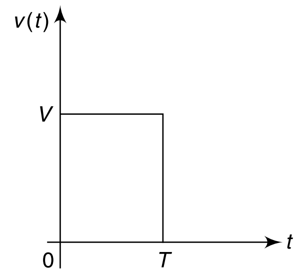


Figure 6.22 *Pulse voltage*

Taking Laplace transform with zero initial condition,

$$RI(s) + \frac{1}{Cs}I(s) = \frac{V}{s} - \frac{Ve^{-sT}}{s}$$

or $$I(s) = \frac{V}{R}\frac{1 - e^{-sT}}{s + 1/RC}$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{R}[e^{-t/RC} - e^{-(t-T)/RC}]$$

Hence the voltage across the resistance is given as,

$$v_R(t) = Ri(t) = V[e^{-t/RC} - e^{-(t-T)/RC}]$$

and the voltage across the capacitor is given as,

$$v_c(t) = V - v_R(t) = V[e^{-t/RC} + e^{-(t-T)/RC}]$$

To plot the two voltages with varying time, we have the following observations:

(i) At $t = 0$, all the voltage appears across the resistance R and thus,

$$v_R = V \quad \text{and} \quad v_C = 0$$

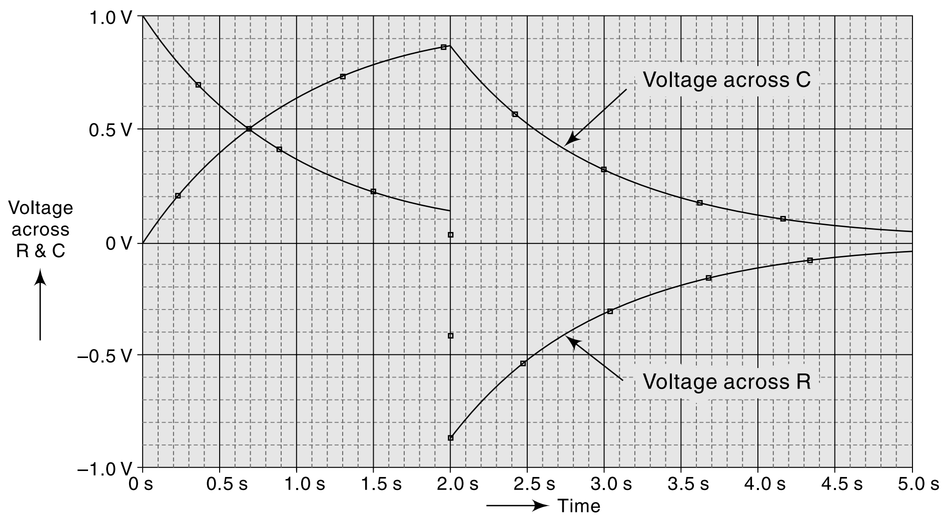


Figure 6.23 *Voltage response of RC series circuit with pulse input*

(ii) As the time increases, the voltage v_C grows and the voltage v_R decays exponentially, with time-constant $\tau = RC$.

(iii) At $t = T$, voltage across the network drops abruptly to zero from V. Again this entire drop is instantaneously felt across the resistance R.

(iv) For time $t > T$, total voltage across the circuit is zero. So, at any instant of time t, $v_R(t) + v_c(t) = 0$ and both v_R and v_C asymptotically approach zero.

Case (1) If Time-constant (τ = RC) << Pulse-width (T) The voltage across the resistance v_R will consist of two trigger pulses one positive and the other negative, of height V at the points where the voltage across the network changes abruptly (i.e., $t = 0$ and T).

In this case, the voltage across capacitor attains the steady state very quickly, i.e. $v_c = V$.

$$\therefore \qquad v_R = Ri = RC\frac{dv_C}{dt} \approx RC\frac{dV}{dt} \quad \text{or,} \quad \boxed{v_R = RC\frac{dV}{dt}}$$

Thus, the voltage v_R is the differentiation of the input voltage and hence the circuit acts as a *Differentiator.*

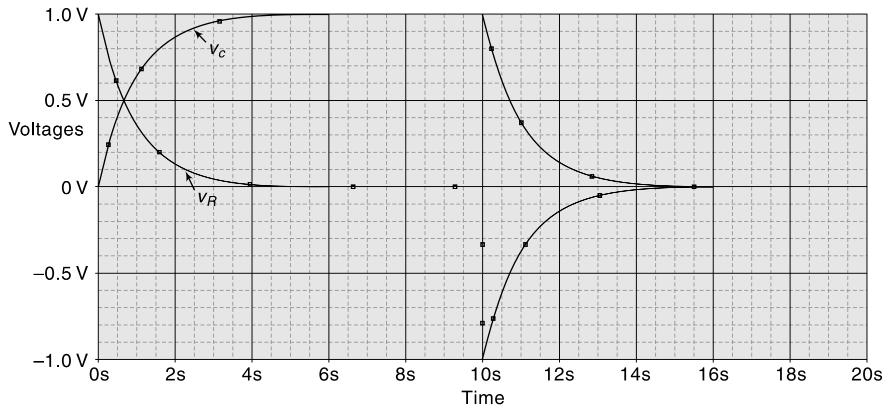


Figure 6.24 *Voltage response of RC series circuit (RC << T) with pulse input*

Case (2) If Time-constant (τ = RC) >> Pulse-width (T) In this case, the voltage across the capacitor varies with time almost linearly and the value is far from the steady state value V; i.e. $v_R = V$.

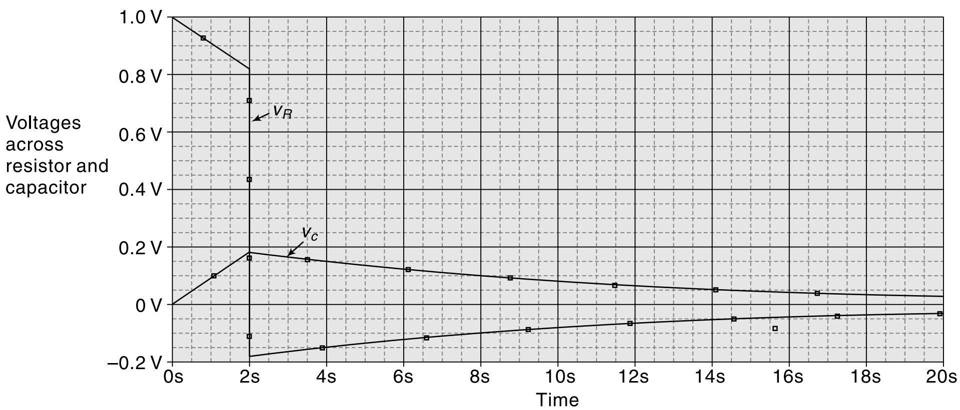


Figure 6.25 *Voltage response of RC series circuit (RC >> T) with pulse input*

$$\therefore \qquad v_C = \frac{1}{C}\int_0^t i\,dt = \frac{1}{C}\int_0^t \frac{v_R}{R}\,dt \approx \frac{1}{RC}\int_0^t V dt \quad \text{or,} \quad \boxed{\therefore\ v_C \approx \frac{1}{RC}\int_0^t V dt}$$

Thus, the voltage v_C is the integration of the input voltage and hence the circuit acts as an *Integrator*.

2. RL Series Circuit If a similar pulse voltage is applied to an *RL* series circuit, then the KVL equation will be,

$$Ri(t) + L\frac{di}{dt} = v(t)$$

Taking Laplace transform with zero initial condition,

$$RI(s) + sLI(s) = \frac{V}{s} - \frac{Ve^{-sT}}{s}$$

or
$$I(s) = \frac{V}{L}\left[\frac{1}{s(s+R/L)} - \frac{e^{-sT}}{s(s+R/L)}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{R}\left[\left(1-e^{-\frac{R}{L}t}\right)u(t) - \left(1-e^{-\frac{R}{L}(t-T)}\right)u(t-T)\right]$$

The variation of the two voltages is shown in Figure 6.26.

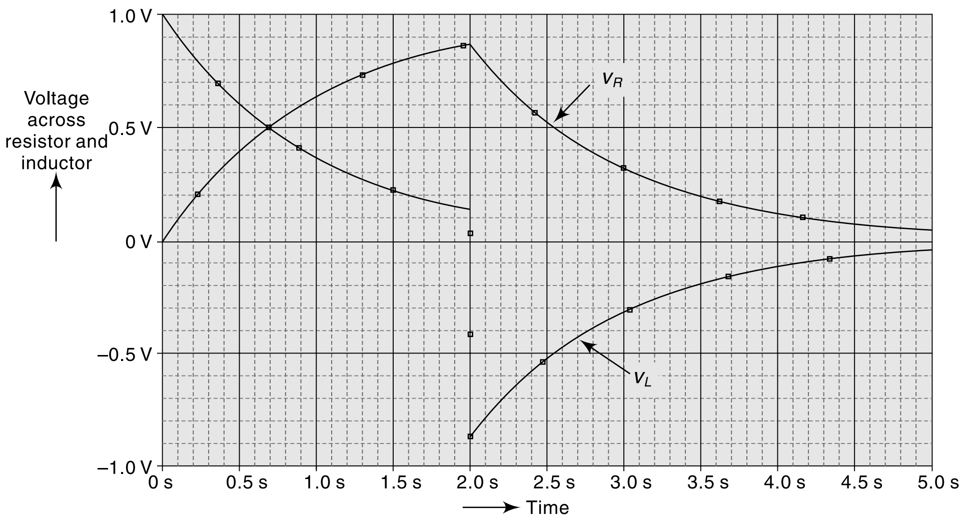


Figure 6.26 *Voltage response of RL series circuit with pulse input*

Case (1) If Time-constant (τ = L/R) << Pulse-width (T) In this case, the voltage across resistor attains the steady state very quickly, i.e. $v_R = V$.

$$\therefore \qquad v_L = L\frac{di}{dt} = L\frac{d}{dt}\left(\frac{v_R}{R}\right) = L\frac{d}{dt}\left(\frac{V}{R}\right) \approx \frac{L}{R}\frac{dV}{dt} \quad \text{or,} \quad \boxed{v_L = \frac{L}{R}\frac{dV}{dt}}$$

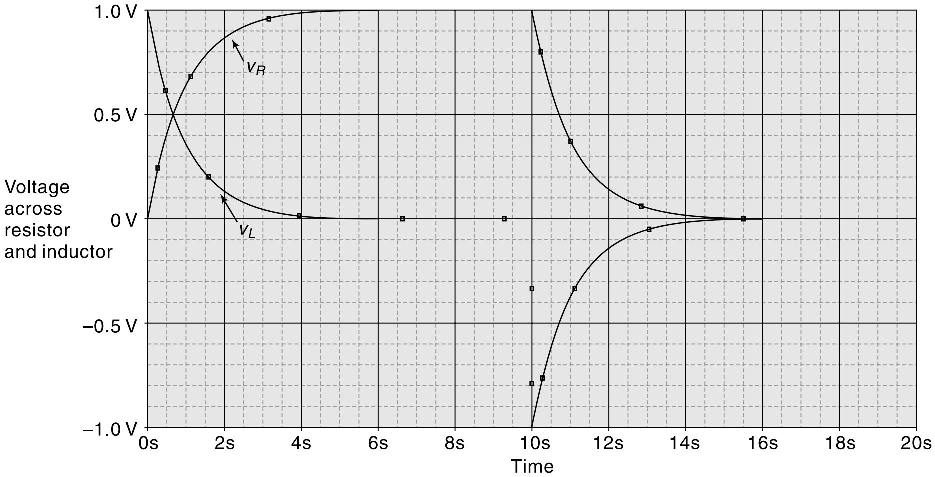


Figure 6.27 *Voltage response of RL series circuit ($L/R << T$) with pulse input*

Thus, the voltage v_L is the differentiation of the input voltage and hence the circuit acts as a *Differentiator.*

Case (2) If Time-constant ($\tau = L/R$) >> Pulse-width (T) In this case, the voltage across the resistor varies with time almost linearly and the value is far from the steady state value V; i.e. $v_L = V$.

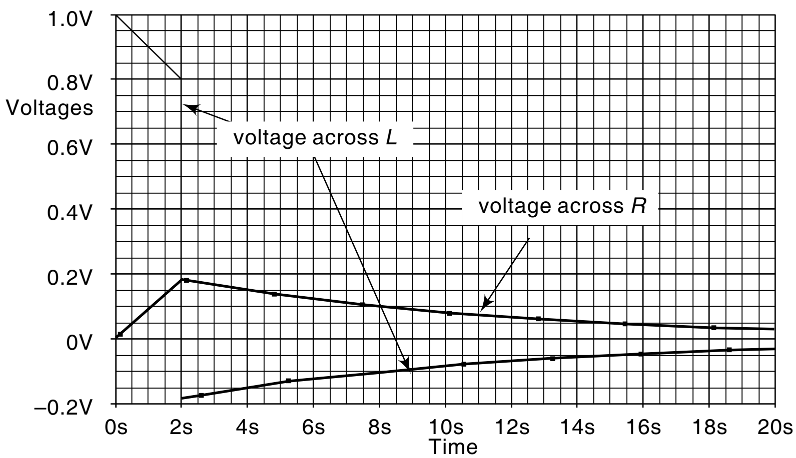


Figure 6.28 *Voltage response of RL series circuit ($L/R >> T$) with pulse input*

$$\therefore \qquad v_R = Ri = R\frac{1}{L}\int_0^t v_L dt \approx \frac{R}{L}\int_0^t V dt \text{ or, } \boxed{\therefore v_R \approx \frac{R}{L}\int_0^t V dt}$$

Thus, the voltage v_R is the integration of the input voltage and hence the circuit acts as an *Integrator*.

6.13 STEPS FOR CIRCUIT ANALYSIS USING LAPLACE TRANSFORM METHOD

1. All circuit elements are transformed from time-domain to Laplace domain with initial conditions.
2. Excitation function is transformed into Laplace domain.
3. The circuit is solved using different circuit analysis techniques, such as, mesh analysis, node analysis, etc.
4. Time domain solution is obtained by taking inverse Laplace transform of the solution.

6.14 CONCEPT OF CONVOLUTION THEOREM

6.14.1 Convolution Integral

If $h(t)$ is the impulse response of a linear network, then the response of the same network $y(t)$ subject to any arbitrary input $w(t)$ is given by the convolution integral as,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)w(t-\tau)d\tau = \int_{-\infty}^{\infty} w(\tau)h(t-\tau)d\tau$$

Thus, if the impulse response of any linear time-invariant system is known, we can obtain the zero-state response of the system to any other type of input.

6.14.2 Convolution Theorem

If $f_1(t)$ and $f_2(t)$ are two functions of time which are zero for $t < 0$, and if their Laplace transforms are $F_1(s)$ and $F_2(s)$, respectively, then the convolution theorem states that the Laplace transform of the convolution of $f_1(t)$ and $f_2(t)$ is given by the product $F_1(s)\ F_2(s)$.

Mathematically, if the convolution of $f_1(t)$ and $f_2(t)$ is written as,

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau)f_2(t-\tau)\, d\tau = \int_0^t f_1(t-\tau)f_2(\tau)\, d\tau = f_2(t) * f_1(t)$$

where, τ is a dummy variable for time t, then the convolution theorem is written as,

$$L[f_1(t) * f_2(t)] = F_1(s)F_2(s)$$

Proof By the definition of convolution,

$$L[f_1(t) * f_2(t)] = L\left[\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right] = \int_0^{\infty}\left[\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right]e^{-st}dt \qquad \text{(i)}$$

Also, by the definition of a shifted unit step function, using dummy variable,

$$\begin{aligned} u(t-\tau) &= 1; \quad \text{for } \tau \le t \\ &= 0; \quad \text{for } \tau > t \end{aligned}$$

$$\therefore \qquad \int_0^t f_1(t-\tau)f_2(\tau)\, d\tau = \int_0^{\infty} f_1(t-\tau)\, u(t-\tau) f_2(\tau)\, d\tau$$

Putting this in (i), we get,

$$L[f_1(t) * f_2(t)] = \int_0^\infty \left[\int_0^\infty f_1(t-\tau)\, u(t-\tau) f_2(\tau) d\tau \right] e^{-st} dt \qquad \text{(ii)}$$

Now, let $(t - \tau) = x \therefore dt = dx$,

t	0	$\propto$
x	$-\tau$	$\propto$

From equation (ii), we get,

$$L[f_1(t) * f_2(t)] = \int_0^\infty \left[\int_{-\tau}^\infty f_1(x)\, u(x) f_2(\tau)\, d\tau \right] e^{-s(x+\tau)}\, dx$$

$$= \int_{-\tau}^\infty f_1(x)\, u(x)\, e^{-sx} dx \int_0^\infty f_2(\tau) e^{-s\tau}\, d\tau$$

$$= \int_0^\infty f_1(x)\, e^{-sx}\, dx \int_0^\infty f_2(\tau)\, e^{-s\tau}\, d\tau \quad \{\because u(x) = 0 \quad \text{for } x < 0\}$$

$$\boxed{\therefore L\big[[f_1(t) * f_2(t)]\big] = F_1(s) F_2(s)}$$

Thus, the convolution in time domain becomes multiplication in the frequency domain, and vise-versa.

6.14.3 Application of Convolution Theorem

The convolution theorem is used to find the response of a linear system to any arbitrary excitation if the impulse response of the system is known.

We know that the transfer function is defined as the ratio of response transform to excitation transform with zero initial conditions. Thus,

$$\text{Transfer Function} = \left.\frac{\text{Laplace transform of Response}}{\text{Laplace transform of Excitation}}\right|_{\text{all initial conditions reduced to zero}}$$

or $$H(s) = \left.\frac{Y(s)}{W(s)}\right|_{IC=0}$$

Thus, $$Y(s) = H(s) W(s)$$

Here, $W(s) = L[w(t)]$, is the input Laplace transform and $Y(s) = L[y(t)]$, is the output Laplace transform.

Now, if the input is an impulse function, then $w(t) = \delta(t)$ or $W(s) = 1$

$\therefore$ $$Y(s) = H(s)W(s) = H(s)$$

Taking inverse Laplace transform,

$$y(t) = h(t)$$

Thus, $h(t)$ is the impulse response of the system. If this impulse response of the system is known, we can find out the response of the system due to any arbitrary input *w(t)* from the following relation:

$$Y(s) = H(s)W(s)$$

or
$$y(t) = h(t) * w(t) = \int_0^t h(\tau)w(t-\tau)d\tau = \int_0^t h(t-\tau)w(\tau)d\tau$$

Example 6.9 Find the convolution integral when $f_1(t) = e^{-at}$ and $f_2(t) = t$.

Solution Here, the convolution integral is given as,

$$f_1(t) * f_2(t) = \int_0^t e^{-a(t-\tau)}\,\tau d\tau = e^{-at}\int_0^t \tau e^{a\tau} d\tau$$

$$= e^{-at}\left[\frac{\tau e^{a\tau}}{a} - \int 1\cdot\frac{e^{a\tau}}{a}d\tau\right]_0^t$$

$$= e^{-at}\left[\frac{\tau e^{a\tau}}{a} - \frac{\tau e^{a\tau}}{a^2}\right]_0^t$$

$$= e^{-at}\left[\frac{te^{at}}{a} - \frac{e^{at}}{a^2} + \frac{1}{a^2}\right]$$

$$= \frac{1}{a^2}[at - 1 + e^{-at}] \qquad \textit{Ans.}$$

SOLVED PROBLEMS

6.1 (a) Find the initial value of the function whose Laplace Transform is,

$$V(s) = A\cdot\frac{(s+a)\sin\theta + b\cos\theta}{(s+a)^2 + b^2}$$

Check the result by solving it for $v(t)$.

(b) Find the final value of the function whose Laplace Transform is, $I(s) = \dfrac{s+6}{s(s+3)}$

Solution

(a) By initial value theorem,

$$V(0+) = \lim_{s\to\infty} sV(s)$$

$$= \lim_{s\to\infty} SA\frac{(s+a)\sin\theta + b\cos\theta}{(s+a)^2 + b^2}$$

$$= \lim_{s\to\infty} A \frac{\left(1+\frac{a}{s}\right)\sin\theta + \frac{b}{s}\cos\theta}{\left(1+\frac{a}{s}\right)^2 + \left(\frac{b}{s}\right)^2}$$

$= A \sin\theta$ *Ans.*

In order to check this result, we find *v(t)* and then put $t = 0$.

$$v(t) = L^{-1}\left[A\frac{(s+a)\sin\theta + b\cos\theta}{(s+a)^2+b^2}\right]$$

$$= AL^{-1}\left[\frac{(s+a)\sin\theta}{(s+a)^2+b^2} + \frac{b\cos\theta}{(s+a)^2+b^2}\right]$$

$$= A[\sin\theta\, e^{-at}\cos bt + \cos\theta\, e^{-at}\sin bt]$$

$$= Ae^{-at}\sin(bt+\theta)$$

At $t = 0$, $v(0+) = Ae^0 \sin(0+\theta) = A\sin\theta$ [Checked]

(b) By final value theorem,

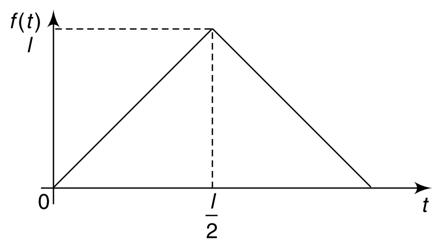
$$I(\infty) = \lim_{s\to 0} sI(s) = \lim_{s\to 0} s\frac{s+6}{s(s+3)} = \lim_{s\to 0}\frac{s+6}{(s+3)} = 2 \qquad \textit{Ans.}$$

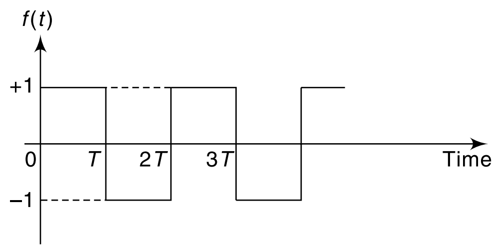
For checking it, $i(t) = L^{-1}\left[\frac{s+6}{s(s+3)}\right] = L^{-1}\left[\frac{2}{s} - \frac{1}{s+3}\right] = 2 - e^{-3t}$

At $t = \infty$, $i(\infty) = 2 - e^{-\infty} = 2$ [Checked]

6.2 (a) Obtain the Laplace Transform of square wave of unit amplitude and periodic time $2T$, as shown.

(b) Find the Laplace Transform of the following function:





Solution

(a) The equation of the square wave is,

$$f(t) = u(t) - u(t-T) - u(t-T) + u(t-2T) + u(t-2T) - u(t-3T) - \ldots$$

$$= u(t) - 2u(t-T) + 2u(t-2T) - 2u(t-3T) + \ldots$$

Taking Laplace transform,

$$F(s) = \frac{1}{s} - \frac{2e^{-Ts}}{s} + \frac{2e^{-2Ts}}{s} - \frac{2e^{-3Ts}}{s} + \dots$$

$$= \frac{1}{s}[1 - 2e^{-Ts}(1 - e^{-Ts} + e^{-2Ts} - e^{-3Ts} + \dots)]$$

$$= \frac{1}{s}\left[1 - \frac{2e^{-Ts}}{1+e^{-Ts}}\right] \qquad \left\{\because \text{ sum of G.P. series} = \frac{1}{1-e^{-Ts}}\right\}$$

$$= \frac{1}{s}\left[\frac{1-e^{-Ts}}{1+e^{-Ts}}\right]$$

$$F(s) = \frac{1}{s}\tanh\left(\frac{Ts}{2}\right) \qquad \textit{Ans.}$$

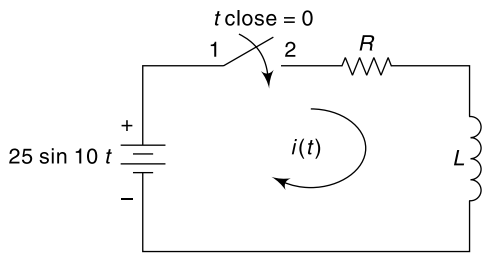
(b) The equation can be written as,

$$f(t) = 2r(t) - 4r\left(t - \frac{1}{2}\right) + 2r(t-1)$$

Taking Laplace transform,

$$F(s) = 2\frac{1}{s^2} - \frac{4e^{-\frac{1}{2}s}}{s^2} + \frac{2e^{-s}}{s^2} = \frac{2}{s^2}[1 - 2e^{-s/2} + e^{-s}] = \frac{2}{s^2}[1 - 2e^{-s/2}]^2$$

6.3 A sinusoidal voltage 25 sin $10t$ is applied at time $t = 0$ to a circuit as shown in the figure. Find the current $i(t)$, by Laplace transform method. $R = 5\Omega$ and $L = 1H$.



Solution By KVL, $RI(s) + sLI(s) = 25\dfrac{10}{s^2+100}$

with zero initial condition.

$$I(s) = \frac{250}{(s+5)(s^2+100)} = \frac{250}{(s+5)(s+j10)(s-j10)}$$

$$= 250\left[\frac{A_1}{s+5} + \frac{A_2}{s+j10} + \frac{A_3}{s-j10}\right]$$

$$A_1 = (s+5)\left.\frac{1}{(s+5)(s^2+100)}\right|_{s=-5} = \frac{1}{125}$$

where, $$A_2 = (s+j10)\left.\frac{1}{(s+5)(s+j10)(s-j10)}\right|_{s=-j10} = -\frac{1}{j20(5-j10)} = -\frac{1}{100(2+j)}$$

$$A_3 = (s-j10)\left.\frac{1}{(s+5)(s+j10)(s-j10)}\right|_{s=j10} = \frac{1}{100(-2+j)}$$

Substituting these,

$$I(s) = 250\left[\frac{A_1}{s+5} + \frac{A_2}{s+j10} + \frac{A_3}{s-j10}\right]$$

Taking inverse Laplace transform,

$$i(t) = 250[A_1 e^{-5t} + A_2 e^{-j10t} + A_3 e^{j10t}]$$

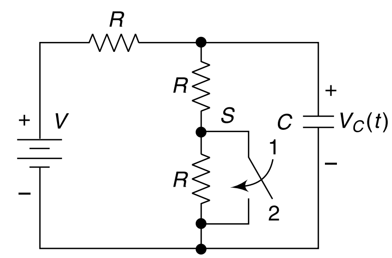
$$= 2e^{-5t} + 250\left\{-\frac{1}{100(2+j)}e^{-j10t} + \frac{1}{100(-2+j)}e^{j10t}\right\}$$

$$= 2e^{-5t} - \frac{5}{2}\left\{\frac{(2-j)e^{-j10t}}{5} - \frac{(-2-j)e^{j10t}}{5}\right\}$$

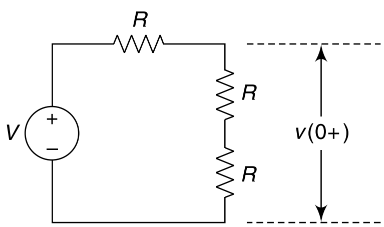
$$= 2e^{-5t} - \frac{1}{2}\{2e^{-j10t} - je^{-j10t} + 2e^{j10t} + je^{j10t}\}$$

or $$i(t) = 2e^{-5t} - 2\cos 10t + \sin 10t\ (A)$$

6.4 The circuit of the figure is initially in the steady state. The switch S is closed at $t = 0$.
(a) Find $V_c(t)$
(b) Determine the final value of $V_c(t)$ and verify it from the final value theorem of Laplace Transform.



Solution At steady-state before closing the switch, the capacitor becomes open-circuited. So, the circuit becomes as shown above.



$$v(0+) = \frac{2}{3}\text{ V}$$

For $t > 0$, by KVL,

$$RI_1 + R(I_1 - I_2) = \frac{V}{s} \Rightarrow 2RI_1 - RI_2 = \frac{V}{s} \quad \text{(i)}$$

and $$\frac{1}{Cs}I_2 + R(I_2 - I_1) = -\frac{2V}{3s} \Rightarrow -RI_1 + \left(R + \frac{1}{Cs}\right)I_2 = -\frac{2V}{3s} \quad \text{(ii)}$$

Solving equations (i) and (ii),

$$I_2 = \frac{\begin{vmatrix} 2R & V/s \\ -R & -2V/3s \end{vmatrix}}{\begin{vmatrix} 2R & -R \\ -R & (R+1/Cs) \end{vmatrix}} = \frac{-\dfrac{4VR}{3s} + \dfrac{VR}{s}}{2R(R+1/Cs) - R^2} = -\frac{V}{3s}\left(\frac{Cs}{2+RCs}\right)$$

$$\therefore \qquad V_C(s) = I_2 \times \frac{1}{Cs} + \frac{2V}{3s} = -\frac{V}{3s\,(2+RCs)} + \frac{2V}{3s} = \frac{V}{3s}\left[2 - \frac{1}{RCs+2}\right]$$

$$= \frac{V}{2s} + \frac{V}{6}\left(\frac{1}{s+2/RC}\right)$$

Taking inverse Laplace transform,

$$v_C(s) = \frac{V}{2} + \frac{V}{6}e^{-2t/RC} \text{ (Volt)}, \quad t > 0 \qquad \textit{Ans.}$$

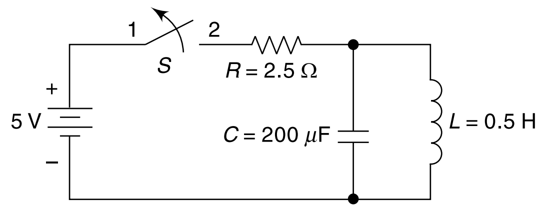
Thus, the final value of the voltage,

$$v_C(\infty) = \lim_{t\to\infty} v_C(t) = \frac{V}{2} \qquad \textit{Ans.}$$

By final value theorem,

$$v_C(\infty) = \lim_{s\to 0} SV_C(s) = \lim_{s\to 0}\left(\frac{V}{2} + \frac{Vs}{(s+2/RC)}\right) = \frac{V}{2} \qquad \text{(Proved)}$$

6.5 In the network shown in the figure, the switch S is closed and a steady state is attained. At $t = 0$, the switch is opened. Determine the current through the inductor for $t > 0$.



Solution When the switch S is closed and the steady-state exists, the current through the inductor is,

$$i(0-) = \frac{V}{R} = \frac{5}{2.5} = 2 \text{ A}$$

The voltage across the capacitor, $V_C(t) = 0$ as it is shorted.
For $t > 0$, the switch is opened. By KVL,

$$L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt = 0$$

Taking Laplace transform,

$$L[sI(s) - i(0-)] + \frac{I(s)}{Cs} = 0$$

or

$$I(s)\left[sL + \frac{1}{Cs}\right] = Li(0-)$$

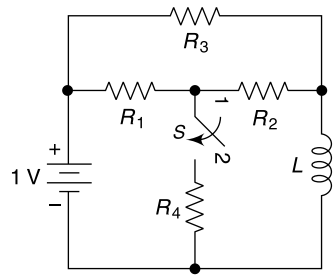
Putting the values,

$$I(s) = 2\frac{s}{s^2 + 10^4}$$

Taking inverse Laplace transform,

$$i(t) = 2\cos 100t \quad (A);\ t \geq 0 \qquad \textit{Ans.}$$

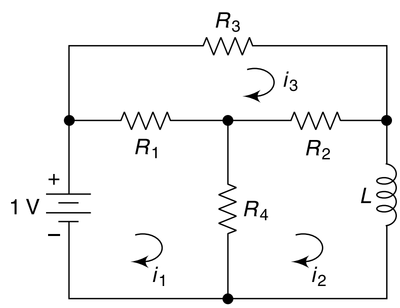
6.6 The circuit shown in the figure is initially in the steady state with the switch S open. At $t = 0$, the switch S is closed. Obtain the current through the inductor for $t > 0$. Take $R_1 = R_2 = R_4 = 1\Omega$ and $R_3 = 2\Omega$ and $L = 1H$.



Solution When the switch S is open and steady state exists, the current through the inductor is,

$$i_2(0-) = \frac{1}{R_3\,(R_1 + R_2)/R_3 + R_1 + R_2} = 1\text{ A}$$

After S is closed, for $t > 0$, by KVL,



$$2i_1 - i_2 - i_3 = 1$$

$$-i_1 + 2i_2 + \frac{di_2}{dt} - i_3 = 0$$

$$-i_1 - i_2 + 4i_3 = 0$$

Taking Laplace transform,

$$2I_1(s) - I_2(s) - I_3(s) = \frac{1}{s}$$

$$-I_1(s) + I_2(s)[s + 2] - I_3(s) = i_2(0-) = 1$$

$$-I_1(s) - I_2(s) + 4I_3(s) = 0$$

By Cramer's Rule,

$$I_2(s) = \frac{\begin{vmatrix} 2 & 1/s & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & -1 \\ -1 & (s+2) & -1 \\ -1 & -1 & 4 \end{vmatrix}} = \frac{\frac{5}{6}}{s} + \frac{\frac{1}{6}}{s + \frac{6}{7}}$$

Taking inverse Laplace transform,

$$i_2(t) = \frac{5}{6} + \frac{1}{6}e^{-6t/7} \quad (A); \quad t > 0 \qquad \textit{Ans.}$$

6.7 A series R-L-C circuit with $R = 3\Omega$, $L = 1H$ and $C = 0.5$ F is excited with a unit step voltage. Obtain an expression for the current, using Laplace transform. Assume that the circuit is relaxed initially.

Solution By KVL,

$$RI(s) + sLI(s) - Li(0-) + \frac{1}{sC}I(s) + \frac{Q(0-)}{sC} = \frac{1}{s}$$

Since the circuit is initially relaxed,

$$\therefore \qquad i(0-) = 0 \quad \text{and} \quad Q(0-) = 0$$

Putting the values,

$$I(s)\left[3 + s + \frac{2}{s}\right] = \frac{1}{s}$$

or $$I(s) = \frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)} = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

where, $$A_1 = \left.\frac{1}{s+2}\right|_{s=-1} = 1 \quad \text{and} \quad A_2 = \left.\frac{1}{s+1}\right|_{s=-2} = -1$$

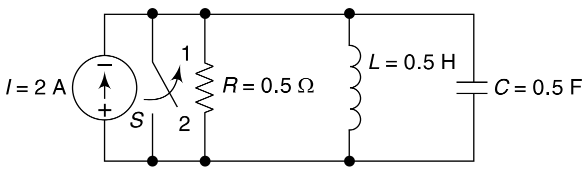
∴ $$I(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

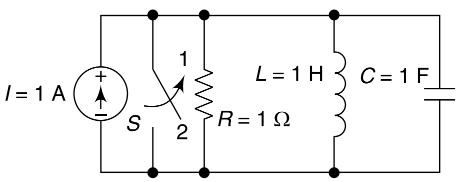
Taking inverse Laplace transform,

$$i(t) = e^{-t} + e^{-2t}\ (\text{A})$$

$$= 2e^{3t/2}\sinh\left(\frac{t}{2}\right)(\text{A}) \qquad \textit{Ans.}$$

6.8 The switch S in the figure is opened at $t = 0$. Determine the voltage $v(t)$, for $t > 0$. What is the nature of the response?

(a) 

(b) 

Solution

(a) By KVL,

$$\frac{v(t)}{R} + i(0-) + \frac{1}{L}\int_0^t v\,dt + C\frac{dv}{dt} = I$$

Taking Laplace transform,

$$V(s)\left[\frac{1}{R} + \frac{1}{sL} + sC\right] = \frac{I}{s}$$

Putting the values,

$$V(s)\left[2 + \frac{2}{s} + \frac{s}{2}\right] = \frac{2}{s}$$

or $$V(s) = \frac{4}{s^2+4s+4} = \frac{4}{(s+2)^2}$$

Taking inverse Laplace transform,

$$v(t) = 4te^{-2t}\ (V),\ t > 0 \qquad \textit{Ans.}$$

The response is *critically damped* ($\because\ \xi = 1$) *Ans.*

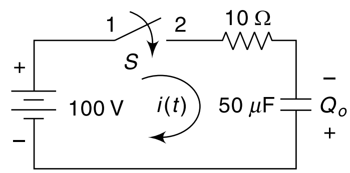
(b) Proceeding in the same way as Prob. 6.8(a),

$$V(s)=\frac{1}{s^2+s+1}=\frac{(\sqrt{3}/2)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\times\frac{2}{\sqrt{3}}$$

$$\Rightarrow \qquad v(t)=\frac{2}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\ \text{(V)};\quad t>0 \qquad \textit{Ans.}$$

The response is under-damped ($\because\ \xi < 1$) *Ans.*

6.9 In the *R-C* series circuit of figure, the capacitor has an initial charge of 2.5 mC. At t = 0, the switch is closed and a constant voltage source of V = 100 V is applied. Use the Laplace transform method to find the current $i(t)$ in the circuit.



Solution By KVL, after the switch is closed,

$$Ri(t)+\frac{1}{C}\left[Q(0-)+\int_0^t i(t)dt\right]=V$$

Taking Laplace transform,

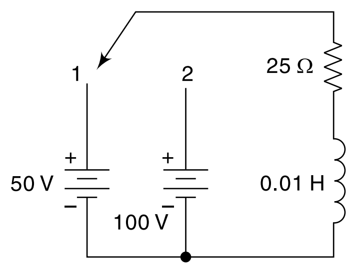
$$10I(s)+\frac{I(s)}{50\times10^{-6}s}-\frac{2.5\times10^{-3}}{50\times10^{-6}s}=\frac{100}{s}$$

or $$I(s)=\frac{15}{s+2\times10^3}$$

Taking inverse Laplace transform,

$$i(t)=15e^{-2\times10^3 t}\ \text{(A)};\quad t>0 \qquad \textit{Ans.}$$

6.10 In the R-L circuit as shown, the switch is in position-1 long enough to establish steady state condition and at t = 0 it is switched to position-2. Find the resulting current, $i(t)$.



Solution When the switch is in position 1, steady-state exists and the initial current through the inductor is,

$$i(0-)=\frac{50}{25}=2\ \text{A}$$

After the switch is moved to position 2, the KVL gives, in Laplace transform,

$$25I(s)+0.01sI(s)-0.01\times2=\frac{100}{s}$$

or $$I(s)=\frac{10^4}{s(s+2500)}-\frac{2}{s+2500}=\frac{A_1}{s}+\frac{A_2}{s+2500}-\frac{2}{s+2500}$$

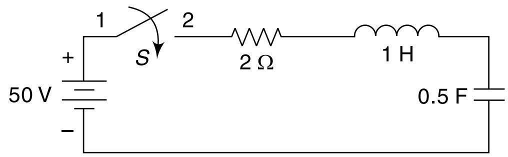
where, $$A_1=\left.\frac{10^4}{(s+2500)}\right|_{s=0}=4 \text{ and } A_2=\left.\frac{10^4}{s}\right|_{s=-2500}=-4$$

$\therefore$ $$I(s) = \frac{4}{s} - \frac{4}{s+2500} - \frac{2}{s+2500} = \frac{4}{s} - \frac{6}{s+2500}$$

Taking inverse Laplace transform,

$$i(t) = 4 - 6e^{-2500t} \text{ (A)}; \quad t > 0 \qquad \textit{Ans.}$$

6.11 In the series *RLC* circuit as shown, there is no initial charge on the capacitor. If the switch is closed at $t = 0$, determine the resulting current at $i(t)$.



Solution By KVL, for $t > 0$,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t idt = V \quad [\because i(0-) = 0]$$

Taking Laplace transform,

$$RI(s) + sLI(s) + \frac{I(s)}{Cs} = \frac{V}{s}$$

Putting the values,

$$2I(s) + sI(s) + 2\frac{I(s)}{s} = \frac{50}{s}$$

or $$I(s) = \frac{50}{s^2 + 2s + s} = \frac{50}{(s+1+j)(s+1-j)} = \frac{50}{(s+1)^2 + 1}$$

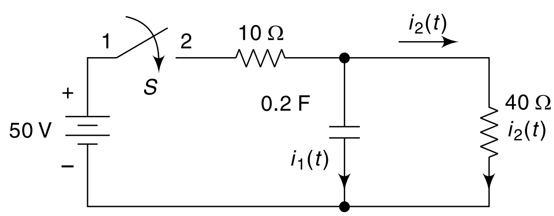
By Partial Fraction Expansion,

$$I(s) = \frac{j25}{s+1+j} - \frac{j25}{s+1-j}$$

Taking inverse Laplace transform,

$$i(t) = j25\,[e^{(-1-j)t} - e^{(-1+j)t}] = 50e^{-t}\sin t \text{ (A)}; \quad t > 0 \qquad \textit{Ans.}$$

6.12 In the two-mesh network shown in the figure, there is no initial charge on the capacitor. Find the loop currents $i_1(t)$ and $i_2(t)$ which result when the switch is closed at $t = 0$.



Solution Writing two mesh equations,

$$10i_1(t) + \frac{1}{0.2}\int_{0-}^{t} i_1(t)dt + 10i_2(t) = 50$$

and $$50i_2(t) + 10i_1(t) = 50$$

Taking Laplace transform,

$$10I_1(s) + \frac{I_1(s)}{0.2s} + 10I_2(s) = \frac{50}{s} \Rightarrow I_1(s)\left[10 + \frac{5}{s}\right] + 10I_2(s) = \frac{50}{s}$$

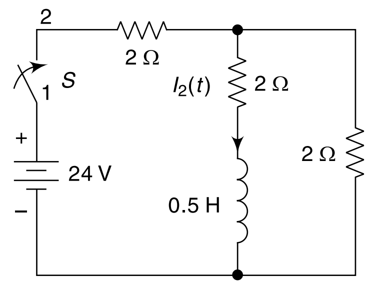
and $$10I_1(s) + 50I_2(s) = \frac{50}{s}$$

Solving, $$I_1(s) = \frac{5}{s + 0.625} \quad \text{and} \quad I_2(s) = \frac{1}{s} - \frac{1}{s + 0.625}$$

Taking inverse Laplace transform,

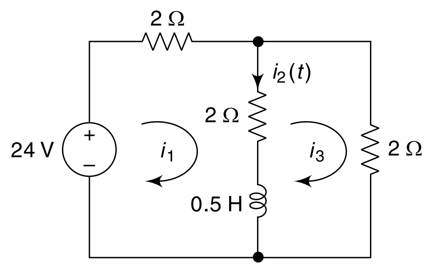
$$i_1(t) = 5e^{-0.625t}\ \text{(A)} \quad \text{and} \quad i_2(t) = 1 - e^{-0.625t}\ \text{(A)},\ t > 0$$

6.13 Find using Final value theorem, the steady state value of $I_2(t)$ in the circuit shown in figure below. Switch S is closed at $t = 0$. The inductor is initially de-energized.



Solution Circuit for $t > 0$ is,

By KVL, in Laplace transform,



$$I_1(s)[2 + 2 + 0.5s] - [2 + 0.5s]I_3(s) = \frac{24}{s}$$

or $$I_1(s)[s + 8] - [s + 4]I_3(s) = \frac{48}{s} \qquad \text{(i)}$$

and $$-I_1(s)[2 + 0.5s] + [4 + 0.5s]I_3(s) = 0$$

or $$-I_1(s)[s + 4] + [s + 8]I_3(s) = 0 \qquad \text{(ii)}$$

Solving equations (i) and (ii),

$$I_1(s) = \frac{\begin{vmatrix} 48/s & -(s+4) \\ 0 & s+8 \end{vmatrix}}{\begin{vmatrix} s+8 & -(s+4) \\ -(s+4) & s+8 \end{vmatrix}} = \frac{6(s+8)}{s(s+8)}$$

and

$$I_3(s) = \frac{\begin{vmatrix} s+8 & 48/s \\ -(s+4) & 0 \end{vmatrix}}{\begin{vmatrix} s+8 & -(s+4) \\ -(s+4) & s+8 \end{vmatrix}} = \frac{6(s+4)}{s(s+6)}$$

$\therefore \quad I_2(s) = I_1(s) - I_3(s) = \dfrac{6(s+8)}{s(s+6)} - \dfrac{6(s+4)}{s(s+6)} = \dfrac{24}{s(s+6)}$

$\therefore \quad$ Final value of the current, $i_2(\infty) = \lim_{s\to 0} sI_2(s) = \lim_{s\to 0} \dfrac{24}{s+6} = 4$ A *Ans.*

6.14 In a series LC circuit, the supply voltage being $v = V_m \cos(t)$, find $i(t)$ with zero initial conditions. Assume $L = 1$H, $C = 1$F.

Solution By KVL, for $t > 0$,

$$I(s)\left[sL + \frac{1}{Cs}\right] = \frac{sV_m}{s^2+1}$$

or
$$I(s) = \frac{sV_m}{(s^2+1)\left(s+\dfrac{1}{s}\right)} = V_m\left[\frac{s^2}{(s^2+1)^2}\right] = V_m\left[\frac{s^2}{(s+j)(s-j)(s+j)(s-j)}\right]$$

$$= V_m\left[\frac{s^2}{(s+j)^2(s-j)^2}\right]$$

$$= V_m\left[\frac{K_1}{(s-j)^2} + \frac{K_1^*}{(s+j)^2} + \frac{K_2}{(s-j)} + \frac{K_2^*}{(s+j)}\right]$$

$$K_1 = I(s)\times(s-j)^2\big|_{s=j} = \frac{1}{4}$$

where,
$$K_2 = \frac{1}{(2-1)!}\frac{d}{ds}(s-j)^2 I(s)\bigg|_{s=j} = \frac{(s+j)^2\, 2s - s^2\times 2(s+j)}{(s+j)^4} = -\frac{j}{4}$$

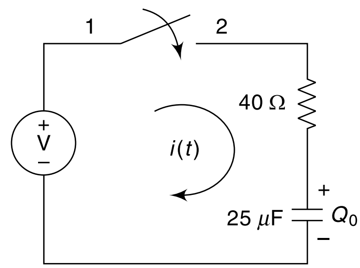
$\therefore \quad K_1^* = \dfrac{1}{4}$; and $K_2^* = \dfrac{j}{4}$

Thus
$$I(s) = \frac{V_m}{4}\left[\frac{1}{(s-j)^2} + \frac{1}{(s+j)^2} - \frac{j}{(s-j)} + \frac{j}{(s+j)}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{V_m}{4}[te^{jt} + te^{-jt} - je^{jt} + je^{-jt}] = \frac{V_m}{4}[t\cos t + \sin t]\ (\text{A});\quad t > 0 \qquad \textit{Ans.}$$

6.15 The series RC circuit of figure has a sinusoidal voltage source, $v = 180 \sin(2000t + \phi)$ (V) and an initial charge on the capacitor $Q_0 = 1.25$ mC with polarity as shown. Determine the current if the switch is closed at a time corresponding to $\phi = 90^\circ$. What is the current at time $t = 0$?



Solution By KVL, for $t > 0$,

$$40i(t) + \frac{1}{25\times10^{-6}}\left[1.25\times10^{-3} + \int_0^t i(t)dt\right] = 180\cos 2000t$$

Taking Laplace transform,

$$40I(s)+\frac{1.25\times10^{-3}}{25\times10^{-6}s}+\frac{4\times10^4}{s}I(s)=\frac{180s}{s^2+4\times10^6}$$

$$\Rightarrow \qquad I(s)=\frac{4.5s^2}{(s^2+4\times10^6)(s+10^3)}-\frac{1.25}{s+10^3}$$

Applying Heaviside expansion formula to find the first term on the right hand side, we have,

$$P(s)=4.5s^2,$$
$$Q(s)=s^3\times10^3s^2+4\times10^6s+4\times10^9,$$
$$Q'(s)=3s^2+2\times10^3s+4\times10^6,$$
$$a_1=-j2\times10^3;\ a_2=j2\times10^3 \quad \text{and} \quad a_{31}=-10^3$$

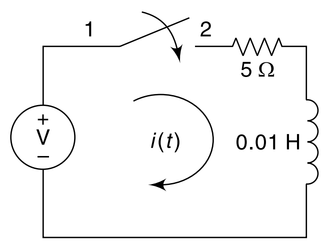
Then,
$$i(t)=\frac{P(-j2\times10^3)}{Q'(-j2\times10^3)}e^{-j2\times10^3t}+\frac{P(j2\times10^3)}{Q'(j2\times10^3)}e^{j2\times10^3t}+\frac{P(-10^3)}{Q'(-10^3)}e^{-10^3t}-1.25e^{-10^3t}$$

$$=(1.8-j0.9)\,e^{-j2\times10^3t}+(1.8+j0.9)\,e^{j2\times10^3t}-0.35\,e^{-10^3t}$$

$$=-1.8\sin 2000t+3.6\cos 2000t-0.35\,e^{-10^3t}$$

$$=4.02\sin(2000t+116.6°)-0.35\,e^{-10^3t}\ \text{(A)};\quad t>0$$

6.16 In the RL circuit of Figure, the source is $v=100\sin(500t+\phi)$. Determine the resulting current if the switch is closed at a time corresponding to $\phi=0$.



Solution By KVL,

$$RI(s)+sLI(s)-Li(0-)=V(s)$$

or
$$5I(s)+0.01sI(s)=\frac{100\times500}{s^2+25\times10^4}\quad [\because i(0-)]=0$$

or
$$I(s)=\frac{5\times10^6}{(s^2+25\times10^4)(s+500)}$$

By Partial Fraction Expansion,

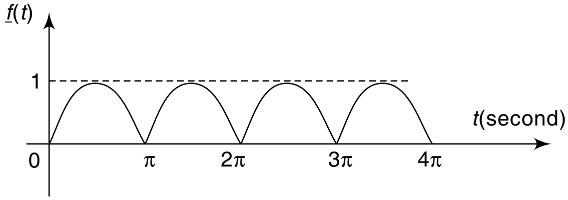
$$I(s)=5\left(\frac{-1+j}{s+j500}\right)+5\left(\frac{-1-j}{s-j500}\right)+\frac{10}{s+500}$$

Taking inverse Laplace transform,

$$i(t)=10\sin 500t-10\cos 500t+10e^{-500t}$$

$$=14.14\sin(500t-45°)+10\,e^{-500t}\ \text{(A)};\quad t>0$$

6.17 Determine the Laplace transform of the following periodic waveform.



Solution Let, for the first half sine wave, the transform is $F_1(s)$.
Now, $f_1(t) = \sin tu(t) + \sin(t - \pi)\, u(t - \pi)$
Taking Laplace transform,

$$F_1(s) = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}$$

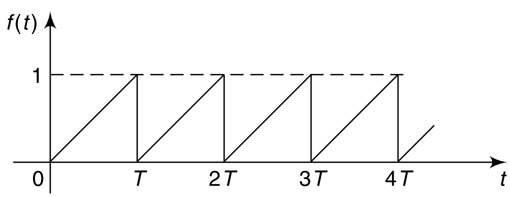
By the theory of periodicity of Laplace transform, the Laplace transform of the full periodic waveform will be,

$$F(s) = F_1(s) \times \frac{1}{1-e^{-Ts}} = \frac{1+e^{-\pi s}}{s^2+1} \times \frac{1}{1-e^{-\pi s}} \quad [\because T = \pi \text{ for the waveform given}]$$

$$= \left(\frac{1+e^{-\pi s}}{1-e^{-\pi s}}\right)\frac{1}{s^2+1}$$

$$= \frac{1}{s^2+1}\coth\left(\frac{\pi s}{2}\right) \qquad \textit{Ans.}$$

6.18 Determine the Laplace transform of the sawtooth waveform as shown below.



Solution For the first cycle,

$$f_1(t) = \frac{1}{T}r(t) - u(t-T) - \frac{1}{T}r(t-T)$$

Taking Laplace transform,

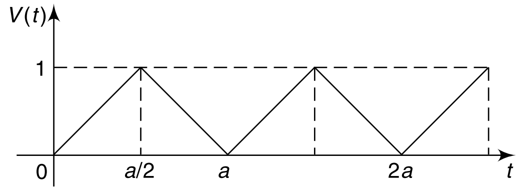
$$F_1(s) = \frac{1}{T}\frac{1}{s^2} - \frac{1}{s}e^{-Ts} - \frac{1}{T}\frac{1}{s^2}e^{-Ts} = \frac{1}{Ts^2}[1-(1+Ts)e^{-Ts}]$$

By Scalling Theorem (the theory of periodicity), Laplace transform of the given periodic function is,

$$F(s) = F_1(s) \times \frac{1}{1-e^{-Ts}} = \frac{1}{Ts^2}[1-(1+Ts)e^{-Ts}] \times \frac{1}{1-e^{-Ts}}$$

$$= \frac{1}{Ts^2} - \frac{e^{-Ts}}{s(1-e^{-Ts})} \qquad \textit{Ans.}$$

6.19 Find the Laplace transform of the waveform shown in figure.



Solution Here, $v_1(t) = \frac{2}{a}r(t) - \frac{4}{a}r\left(t - \frac{a}{2}\right) + \frac{2}{a}r(t-a)$

Taking Laplace transform,

$$V_1(s) = \frac{2}{a}\frac{1}{s^2} - \frac{4}{a}\frac{e^{-as/2}}{s^2} + \frac{2}{a}\frac{e^{-as}}{s^2}$$

$$= \frac{2}{as^2}(1 - 2e^{-as/2} + e^{-as})$$

$$= \frac{2}{as^2}(1 - e^{-as/2})^2$$

By Scalling Theorem (the theory of periodicity), Laplace transform of the given periodic function is,

$$V(s) = V_1(s) \times \frac{1}{1-e^{-Ts}} = \frac{2}{as^2}(1-e^{-as/2})^2 \times \frac{1}{1-e^{-as}}$$

$$= \frac{2}{as^2}\left(\frac{1-e^{-as/2}}{1+e^{-as/2}}\right)$$

$$= \frac{2}{as^2}\tanh\left(\frac{as}{4}\right) \qquad \textit{Ans.}$$

6.20 The unit step response of a network is given by $(1 - e^{-bt})$. Determine the unit impulse response *h(t)* of this network.

Solution Here, the input is, $w(t) = u(t) \Rightarrow W(s) = \frac{1}{s}$

and the output is $y(t) = (1 - e^{-bt}) \Rightarrow Y(s) = \frac{1}{s} - \frac{1}{s+b} = \frac{b}{s(s+b)}$

By convolution theorem,

$$Y(s) = H(s)W(s)$$

$$\Rightarrow \qquad \frac{b}{s(s+b)} = H(s)\frac{1}{s}$$

$$\Rightarrow \qquad H(s) = \frac{b}{(s+b)}$$

Taking inverse Laplace transform, the impulse response is,

$$h(t) = be^{-bt} \qquad \textit{Ans.}$$

6.21 The unit impulse response of current of a circuit having R =1 ohm and C = 1 F in series is given by $[\delta(t) - \exp(-t)u(t)]$. Find the current expression when the circuit is driven by the voltage given as, $[1 - \exp(-2t)]u(t)$.

Solution Here, the impulse response is, $h(t) = [\delta(t) - \exp(-t)\, u(t)] \;\Rightarrow\; H(s) = 1 - \dfrac{1}{s+1} = \dfrac{s}{s+1}$.

The input is, $w(t) = [1 - \exp(-2t)]u(t) \;\Rightarrow\; W(s) = \dfrac{1}{s} - \dfrac{1}{s+2} = \dfrac{2}{s(s+2)}$

By convolution theorem, the output is given by,

$$Y(s) = H(s)W(s) = \frac{s}{s+1} \times \frac{2}{s(s+2)} = \frac{2}{(s+1)\,(s+2)} = \frac{2}{s+1} - \frac{2}{s+2}$$

Taking inverse Laplace transform,

$$y(t) = (2e^{-t} - 2e^{-2t}) \qquad \textit{Ans.}$$

6.22 The response of a network to an impulse is $h(t) = 0.18(e^{-0.32t} - e^{-2.1t})$. Find the response of the network to a step function using convolution theorem.

Solution By convolution theorem,

$$Y(s) = H(s)W(s) = 0.18\left[\frac{1}{s+0.32} - \frac{1}{s+2.1}\right] \times \frac{1}{s}$$

$$= \frac{0.32}{s(s+0.32)\,(s+2.1)}$$

$$= \frac{A_1}{s} + \frac{A_2}{s+0.32} + \frac{A_3}{s+2.1}$$

$$\therefore \qquad A_1 = \left.\frac{0.32}{(s+0.32)\,(s+2.1)}\right|_{s=0} = 0.477$$

$$\therefore \qquad A_2 = \left.\frac{0.32}{s(s+2.1)}\right|_{s=-0.32} = -0.562$$

$$\therefore \qquad A_3 = \left.\frac{0.32}{s(s+0.32)}\right|_{s=-2.1} = 0.0856$$

Putting these values,

$$Y(s) = \frac{0.477}{s} - \frac{0.562}{s+0.32} + \frac{0.0856}{s+2.1}$$

Taking inverse Laplace transform,

$$y(t) = 0.477 - 0.562e^{-0.32t} + 0.0856e^{-2.1t} \qquad \textit{Ans.}$$

6.23 Find the initial and final value of the functions given as,

(a) $F(s) = \dfrac{4(s+1)}{s^2+4s+6}$ (b) $F(s) = \dfrac{5s^3-1600}{s(s^3+18s^2+90s+800)}$

Solution

(a) By initial value theorem, the initial value of the functions is given as,

$$f(0+) = \lim_{s\to\infty}[sF(s)] = \lim_{s\to\infty}\left[s \times \frac{4(s+1)}{s^2+4s+6}\right] = \lim_{s\to\infty}\left[\frac{4+\frac{4}{s}}{1+\frac{4}{s}+\frac{6}{s^2}}\right] = 4 \qquad \textit{Ans.}$$

By final value theorem, the final value of the function is given as,

$$f(\infty) = \lim_{s\to 0}[sF(s)] = \lim_{s\to 0}\left[s \times \frac{4(s+1)}{s^2+4s+6}\right] = 0 \qquad \textit{Ans.}$$

(b) By initial value theorem, the initial value of the functions is given as,

$$f(0+) = \lim_{s\to\infty}[sF(s)] = \lim_{s\to\infty}\left[s \times \frac{5s^3-1600}{s(s^3+18s^2+90s+800)}\right]$$

$$= \lim_{s\to\infty}\left[\frac{5s^3-1600}{(s^3+18s^2+90s+800)}\right]$$

$$= \lim_{s\to\infty}\left[\frac{5-\frac{1600}{s^3}}{1+\frac{18}{s}+\frac{90}{s^2}+\frac{800}{s^3}}\right]$$

$$= 5 \qquad \textit{Ans.}$$

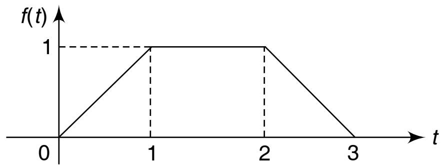
By final value theorem, the final value of the function is given as,

$$f(\infty) = \lim_{s\to 0}[sF(s)] = \lim_{s\to 0}\left[s \times \frac{5s^3-1600}{s(s^3+18s^2+90s+800)}\right]$$

$$= \lim_{s\to 0}\left[\frac{5s^3-1600}{(s^3+18s^2+90s+800)}\right]$$

$$= \frac{-1600}{800} = -2 \qquad \textit{Ans.}$$

6.24 Express the function in terms of the standard signals and find its Laplace transform.

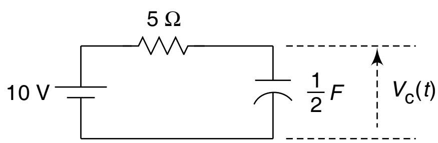


Solution The function can be written as the summation of some ramp functions as given below.

$$f(t) = r(t) - r(t-1) - r(t-2) + r(t-3)$$

$$\therefore \qquad F(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$

6.25 Find the current $i(t)$ flowing through the circuit if the circuit is initially relaxed. Find the voltage across the capacitor $v_c(t)$ also. What is the value of the steady state current?



Solution By KVL,

$$\left(5 + \frac{1}{s/2}\right) I(s) = \frac{10}{s} \qquad \Rightarrow \qquad I(s)(5s+2) = 10$$

$$\therefore \qquad I(s) = \frac{10}{5s+2} = \frac{2}{s + 2/5}$$

Taking inverse Laplace transform, the current in the circuit,

$$i(t) = 2e^{-2t/5} \quad (\text{A}) \qquad \textit{Ans.}$$

Voltage across the capacitor is,

$$V_C(s) = I(s) \times \frac{1}{\frac{1}{2}s} = \frac{2}{s} \times \frac{2}{s + 2/5} = \frac{4}{s(s+2/5)} = 10\left(\frac{1}{s} - \frac{1}{s+2/5}\right)$$

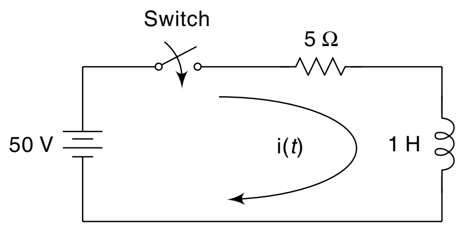
Taking inverse Laplace transform,

$$V_C(t) = 10\left[1 - e^{-2t/5}\right] \quad (\text{V}) \qquad \textit{Ans.}$$

From the current expression, as $t \to \infty$, $i(t) \to 0$. So, the steady state value of the current is,

$$I_{ss} = 0 \qquad \textit{Ans.}$$

6.26 The series *RL* circuit shown in the figure is excited by a dc voltage of 50 V. Assume the initial current flowing through the inductor to be 5 A and find the current $i(t)$ for $t > 0$. Use Laplace transform method.



Solution Applying KVL for the loop, we get,

$$Ri(t) + L\frac{di(t)}{dt} = 50$$

Taking Laplace transform,

$$RI(s) + L[sI(s) - i(0-)] = \frac{50}{s}$$

$$\Rightarrow \quad 5I(s) + sI(s) = \frac{50}{s} + 5$$

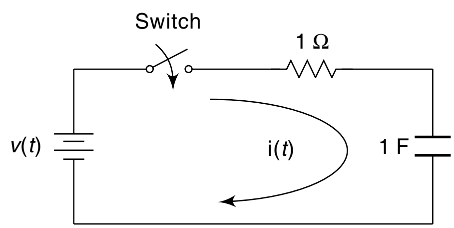
$$\Rightarrow \quad (s+5)\,I(s) = \frac{50}{s} + 5$$

$$\Rightarrow \quad I(s) = \frac{50}{s(s+5)} + \frac{5}{(s+5)} = 50\left[\frac{1}{s} - \frac{1}{s+5}\right] + \frac{5}{(s+5)}$$

Taking inverse Laplace transform, we get,

$$i(t) = 50\left(1 - e^{-5t}\right) + 5e^{-5t} = \left(50 - 45e^{-5t}\right)(\text{A}) \qquad t > 0 \qquad \textit{Ans.}$$

6.27 Find the current $i(t)$ for the circuit shown in the figure, if the voltage source is $v(t) = 5e^{-2t}u(t)$ and $v_C(0-) = 0$.



Solution Applying KVL for the loop,

$$Ri(t) + \frac{1}{C}\int_0^t i(t)dt = v(t) = 5e^{-2t}\,u(t)$$

Taking Laplace transform,

$$RI(s) + \frac{1}{C}\left[\frac{I(s)}{s} + \frac{q(0-)}{s}\right] = \frac{5}{s+2}$$

$$I(s)\left[1 + \frac{1}{s}\right] = \frac{5}{s+2} \quad (\text{since } v_C(0-) = 0)$$

$$I(s)=\frac{5s}{(s+1)(s+2)}=\frac{10}{s+2}-\frac{5}{s+1}$$

Taking inverse Laplace transform,

$$i(t)=10e^{-2t}-5e^{-t};\text{ for } t\geq 0 \qquad \textit{Ans.}$$

6.28 Determine the current $i(t)$ in a series RLC circuit consisting of $R = 5\,\Omega$, $L = 1$ H and $C = ¼$ F when the source voltage is given as: (a) ramp voltage $12r(t-2)$ and (b) step voltage $3u(t-3)$. Assume that the circuit is initially relaxed.

Solution Applying KVL for the series RLC circuit we get,

$$Ri(t)+L\frac{di(t)}{dt}+\frac{1}{C}\int i(t)dt=v(t)$$

$$\Rightarrow \quad 5i(t)+\frac{di(t)}{dt}+\frac{1}{1/4}\int i(t)dt=v(t)$$

(a) When $v(t)=12r(t-2)$

$$5i(t)+\frac{di(t)}{dt}+\frac{1}{1/4}\int i(t)dt=v(t)$$

Taking Laplace transform,

$$\left(5+s+\frac{4}{s}\right)I(s)=\frac{12}{s^2}e^{-2s}$$

$$\Rightarrow \quad I(s)=\frac{12e^{-2s}}{s\left(s^2+5s+4\right)}=12e^{-2s}\left[\frac{1}{s(s+1)(s+4)}\right]=12e^{-2s}\left[\frac{1/4}{s}-\frac{1/3}{s+1}+\frac{1/12}{s+4}\right]$$

$$=\frac{3e^{-2s}}{s}-\frac{4e^{-2s}}{s+1}+\frac{e^{-2s}}{s+4}$$

Taking inverse Laplace transform, we, get,

$$i(t)=3u(t-2)-4e^{-(t-2)}+e^{-4(t-2)} \qquad \textit{Ans.}$$

(b) When $v(t)=3u(t-3)$

$$5i(t)+\frac{di(t)}{dt}+\frac{1}{1/4}\int i(t)dt=v(t)$$

Taking Laplace transform,

$$\left(5+s+\frac{4}{s}\right)I(s)=\frac{3}{s}e^{-3s}$$

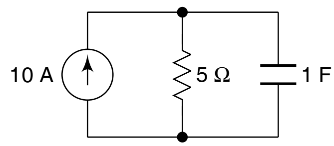
$$\Rightarrow \quad I(s)=\frac{3e^{-2s}}{\left(s^2+5s+4\right)}=3e^{-2s}\left[\frac{1}{(s+1)(s+4)}\right]=3e^{-2s}\left[\frac{1/3}{s+4}-\frac{1/3}{s+1}\right]$$

$$=\frac{e^{-2s}}{s+4}-\frac{e^{-2s}}{s+1}$$

Taking inverse Laplace transform, we, get,

$$i(t) = e^{-(t-3)} + e^{-4(t-3)} \qquad Ans.$$

6.29 For the *RC* parallel circuit shown in the figure, determine the voltage across the capacitor using Laplace transform method. Assume the capacitor to be initially relaxed.



Solution Applying KCL at the upper node,

$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} = i(t) = 10$$

Taking Laplace transform and putting the values of *R* and *C*,

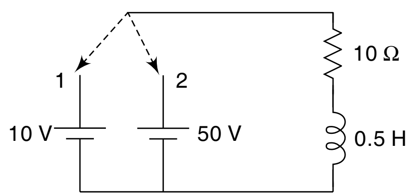
$$\frac{V(s)}{5} + sV(s) = \frac{10}{s}$$

$$\Rightarrow \qquad V(s) = \frac{10}{s\left(s+\frac{1}{5}\right)} = 50\left[\frac{1}{s} - \frac{1}{s+\frac{1}{5}}\right]$$

Taking inverse Laplace transform, we get,

$$v(t) = 50\left(1 - e^{-t/5}\right) \ \ (\text{V}) \qquad Ans.$$

6.30 The circuit was in steady state with the switch in position 1. Find the current $i(t)$ for $t > 0$ if the switch is moved from position 1 to 2 at $t = 0$.



Solution When the switch is in position 1, steady-state exists and the initial current through the inductor is,

$$i(0-) = \frac{10}{10} = 1\,A$$

After the switch is moved to position 2, the KVL gives, in Laplace transform,

$$10I(s) + 0.5sI(s) - 0.5 \times 1 = \frac{50}{s}$$

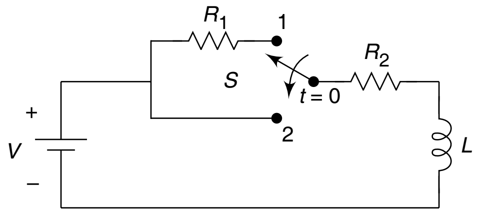
or,

$$I(s) = \frac{100}{s(s+20)} + \frac{1}{s+20} = 5\left[\frac{1}{s} - \frac{1}{s+20}\right] + \frac{1}{s+20}$$

Taking inverse Laplace transform,

$$i(t) = 5 - 4e^{-20t} \quad (A);\ t > 0; \qquad Ans.$$

6.31 (a) In the circuit shown in the figure, The switch S has been thrown to position 1 for a long period of time. Find the complete expression for the current after throwing the switch S to 2 which removes R_1 from the circuit.



(b) If the values of V, R_1, R_2 and L be 10 V, 1 ohm, 2 ohms and 1 H respectively, calculate

(i) steady state current

(ii) the energy stored in the inductance at steady state period

(iii) time constant of the circuit for both the positions of the switch S

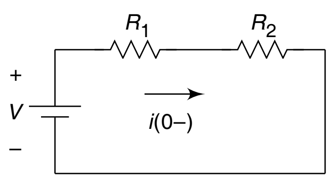
Also calculate the voltage across the resistor R_2 and inductor L, at 0.05 second after the switch S has been thrown to position 2.

Solution

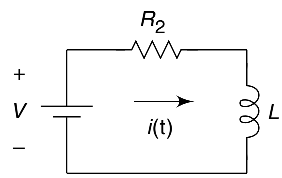
(a) For $t < 0$, as the circuit was in steady state with the switch in position 1, the circuit becomes as shown below.

$$\therefore \qquad i(0-) = \frac{V}{R_1 + R_2}$$

For $t > 0$, the circuit becomes as shown.



Circuit fort < 0



Circuit fort > 0

By KVL,

$$R_2 I(s) + sLI(s) - Li(0-) = \frac{V}{s}$$

$$\Rightarrow \qquad [R_2 + sL]I(s) = \frac{V}{s} + \frac{VL}{R_1 + R_2}$$

$$\Rightarrow \qquad I(s) = \frac{V}{R_2}\left[\frac{1}{s(s + R_2/L)}\right] + \frac{V}{R_1 + R_2}\left(\frac{1}{(s + R_2/L)}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{V}{R_2}\left(1 - e^{-(R_2/L)t}\right) + \frac{V}{R_1 + R_2} e^{-(R_2/L)t} \qquad (\text{A}), t > 0 \qquad \textit{Ans.}$$

(b) $V = 10$ V, $R_1 = 1$ ohm, $R_2 = 2$ ohm and $L = 1$H

(i) Steady state current, $I_{ss} = \frac{V}{R_2} = \frac{10}{2} = 5$ A *Ans.*

(ii) Energy stored in the inductance at steady state period,

$$W = \frac{1}{2}LI^2 = \frac{1}{2} \times 1 \times 5^2 = 12.5 \text{ watts} \qquad \textit{Ans.}$$

(iii) Time constant of the circuit for switch in position 1 is,

$$\tau_1 = \frac{L}{R_1 + R_2} = \frac{1}{1+2} = 0.33 \text{ s} \qquad \textit{Ans.}$$

Time constant of the circuit for switch in position 2 is,

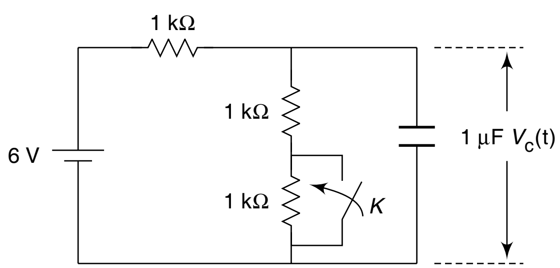
$$\tau_2 = \frac{L}{R_2} = \frac{1}{2} = 0.5 \text{ s} \qquad \textit{Ans.}$$

For $t = 0.05$, voltage across the resistor,

$$V_{R_2} = i \times R_2 = \left[5\left(1 - e^{-2t}\right) + \frac{20}{3}e^{-2t}\right]\Bigg|_{t=0.05} \times 2 = 7 \text{ V} \qquad \textit{Ans.}$$

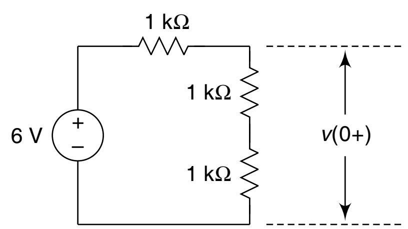
and voltage across the inductor, $V_L = (10 - 7) = 3$ V *Ans.*

6.32 The circuit given in figure is initially at steady state with the switch 'K' open. If the switch is closed at time $t = 0$, find the voltage '$V_C(t)$' across the capacitor.



Solution At steady-state before closing the switch, the capacitor becomes open-circuited. So, the circuit becomes as shown above.

$$v(0-) = \frac{2}{3} \times 6 = 4 \text{ V}$$



For $t > 0$, by KVL,

$$1\times10^3\times I_1 + 1\times10^3\times(I_1 - I_2) = \frac{6}{s} \quad\Rightarrow\quad 2000I_1 - 1000I_2 = \frac{6}{s} \qquad \text{(i)}$$

and
$$\frac{10^6}{s}I_2 + 1\times10^3\times(I_2 - I_1) = -\frac{4}{s} \quad\Rightarrow\quad -1000I_1 + \left(1000+\frac{10^6}{s}\right)I_2 = -\frac{4}{s} \qquad \text{(ii)}$$

Solving equations (i) and (ii),

$$I_2 = \frac{\begin{vmatrix} 2000 & 6/s \\ -1000 & -4/s \end{vmatrix}}{\begin{vmatrix} 2000 & -1000 \\ -1000 & \left(1000+10^6/Cs\right) \end{vmatrix}} = -\frac{2}{1000}\left(\frac{1}{s+2000}\right)$$

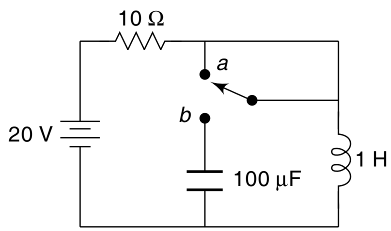
$$\therefore \qquad V_c(s) = I_2\times\frac{10^6}{s}+\frac{4}{s} = -\frac{2000}{s(s+2000)}+\frac{4}{s}$$

$$= \frac{3}{s}+\frac{1}{s+2000}$$

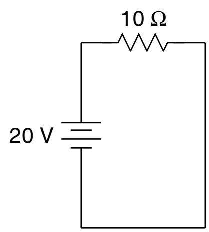
Taking inverse Laplace transform,

$$v_c(t) = 3 + e^{-2000t}\ (Volt), \quad t > 0 \qquad Ans.$$

6.33 In the circuit in the figure, the steady state exists when the switch S is in position a for a considerable period of time. Find the current response after throwing the switch from position a to b. What will be the steady state value of the current?



Solution When the switch is in position a, steady-state exists and the initial current through the inductor is,

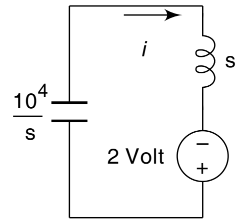


$$i(0-) = \frac{20}{10} = 2\,\text{A}$$

After the switch is moved to position b, the KVL gives, in Laplace transform,

$$\frac{1}{100\times10^{-6}s}I(s) + 1sI(s) - 1\times2 = 0$$

or,
$$I(s) = \frac{2}{s+\dfrac{10^4}{s}} = \frac{2s}{\left(s^2+10^4\right)}$$



Taking inverse Laplace transform,

$$i(t) = 2\cos 100t \quad (A); t > 0; \qquad Ans.$$

- The **steady state current will oscillate sinusoidally** following the relation $i(t) = 2\cos 100t$ with peak magnitude of 2 A and frequency of 100 rad/s or 16.9 Hz.

6.34 A dc voltage applied to a coil of inductance L and resistance R is suddenly changed from V_1 to V_2.
(a) Find an expression for current in the circuit.
(b) If $R = 10\,\Omega$, $L = 1\,\text{H}$, $V_1 = 100\,\text{V}$, and $V_2 = 200\,\text{V}$, find current at $t = 0.5\,\text{s}$.
(c) If $R = 10\,\Omega$, $L = 1\text{H}$, $V_1 = 200\,\text{V}$, and $V_2 = 100\,\text{V}$, find current at $t = 0.5\,\text{s}$.

Solution Here, initial current in the circuit, $i(0-) = \dfrac{V_1}{R}$

(a) After changing the voltage, the KVL equations is,

$$Ri(t) + L\frac{di(t)}{dt} = V_2 u(t)$$

Taking Laplace transform,

$$RI(s) + L\left[sI(s) - i(0-)\right] = \frac{V_2}{s}$$

$$\Rightarrow \qquad I(s)\left[R + sL\right] = \frac{V_2}{s} + \frac{V_1 L}{R}$$

$$\Rightarrow \qquad I(s) = \frac{V_2}{s(R + sL)} + \frac{V_1 L}{R}\left(\frac{1}{R + sL}\right) = \frac{V_2/L}{s(s + R/L)} + \left(\frac{V_1/R}{s + R/L}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{V_2}{R}\left[1 - e^{-(R/L)t}\right] + \frac{V_1}{R}e^{-(R/L)t} = \frac{V_2}{R} + \left(\frac{V_1}{R} - \frac{V_2}{R}\right)e^{-(R/L)t} \qquad Ans.$$

(b) If $R = 10\,\Omega$, $L = 1\,\text{H}$, $V_1 = 100\,\text{V}$, and $V_2 = 200\,\text{V}$, we get the current at $t = 0.5\,\text{s}$ as,

$$i(t) = \frac{200}{10} + \left(\frac{100}{10} - \frac{200}{10}\right)e^{-(10/1)\times 0.5} = 20 - 10e^{-5} = 19.93\,\text{A} \qquad Ans.$$

(c) If $R = 10\,\Omega$, $L = 1\,\text{H}$, $V_1 = 200\,\text{V}$, and $V_2 = 100\,\text{V}$, we get the current at $t = 0.5\,\text{s}$ as,

$$i(t) = \frac{100}{10} + \left(\frac{200}{10} - \frac{100}{10}\right)e^{-(10/1)\times 0.5} = 10 + 10e^{-5} = 10.07\,\text{A} \qquad Ans.$$

6.35 A $50\,\mu\text{F}$ capacitor and $20000\,\Omega$ resistor are connected in series across a 100 V battery at $t = 0$. At $t = 0.5\,\text{s}$, the battery voltage is suddenly increased to 150 V. Find the charge on capacitor at $t = 0.75\,\text{s}$.
Solution When the circuit is connected to 100 V supply, the equation of voltage across the capacitor is,

$$v_C = E\left(1 - e^{-t/RC}\right) = 100\left(1 - e^{-t/20000\times 50\times 10^{-6}}\right) = 100\left(1 - e^{-t}\right)$$

At $t = 0.5$ s, the voltage across the capacitor is, $v_C = 100\left(1 - e^{-0.5}\right) = 39.347$ V

$\therefore$ At $t = 0.5$ s, charge on the capacitor is, $q = Cv_C = 50 \times 10^{-6} \times 39.347 = 1967.35 \times 10^{-6}\, C$

This charge is the initial charge q_0 when the battery voltage is suddenly increased to 150V. When the circuit is connected to 150 V, the KVL equation becomes,

$$Ri(t) + \frac{1}{C}\int_0^t i(t)\,dt = Vu(t)$$

Taking Laplace transform,

$$RI(s) + \frac{1}{C}\left[\frac{I(s)}{s} + \frac{q_0}{s}\right] = \frac{V}{s}$$

$$\Rightarrow \qquad I(s) = \frac{\frac{V}{R} - \frac{q_0}{RC}}{s + \frac{1}{RC}}$$

Taking inverse Laplace transform,

$$i(t) = \left[\frac{V}{R} - \frac{q_0}{RC}\right]e^{-t/RC}$$

Therefore, the voltage across the capacitor,

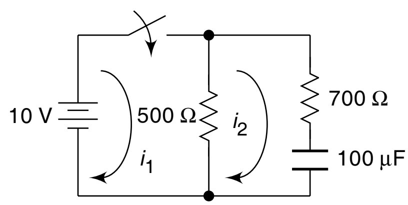
$$V_C = \frac{1}{C}\int_0^t i(t)\,dt = \frac{1}{C}\int_0^t \left(\frac{V}{R} - \frac{q_0}{RC}\right)e^{-t/RC}\,dt = V\left(1 - e^{-t/RC}\right) + \frac{q_0}{RC}e^{-t/RC}$$

Substituting the values, the voltage across the capacitor at $t = 0.75$ s i.e., 0.25 second after changing the battery voltage,

$$V_C = V\left(1 - e^{-t/RC}\right) + \frac{q_0}{RC}e^{-t/RC} = 150\left(1 - e^{-0.25}\right) + \frac{1967.35 \times 10^{-6}}{1}e^{-0.25} = 63.82\text{ V}$$

$\therefore$ charge on the capacitor is, $q = Cv_C = 50 \times 10^{-6} \times 63.82 = 3.19 \times 10^{-3}\, C$ *Ans.*

6.36 For the circuit shown in the figure, find an expression for the current supplied by the source. How much time will it take for the current to reach 25 mA? Assume the circuit to be initially relaxed.



Solution Applying KVL for the two meshes, we get,

$$500i_1 - 500i_2 = 10$$

$$-500i_1 + 1200i_2 + \frac{1}{100 \times 10^{-6}}\int_0^t i_2\,dt = 0$$

Taking Laplace transform,

$$500I_1(s) - 500I_2(s) = \frac{10}{s}$$

$$-500I_1(s) + \left(1200 + \frac{10^4}{s}\right)I_2(s) = 0$$

Solving for $I_1(s)$, we get,

$$I_1(s) = \frac{\begin{vmatrix} 10/s & -500 \\ 0 & \left(1200 + \dfrac{10^4}{s}\right) \end{vmatrix}}{\begin{vmatrix} 500 & -500 \\ -500 & \left(1200 + \dfrac{10^4}{s}\right) \end{vmatrix}} = \frac{24s + 200}{s\left(700s + 10^4\right)} = \frac{24}{700s + 10^4} + \frac{200}{s\left(700s + 10^4\right)}$$

$$= \frac{24}{700}\left(\frac{1}{s + 100/7}\right) + \frac{2}{7}\left(\frac{1}{s\left(s + 100/7\right)}\right) = \frac{1}{50}\left(\frac{1}{s}\right) + \frac{1}{70}\left(\frac{1}{s + 100/7}\right)$$

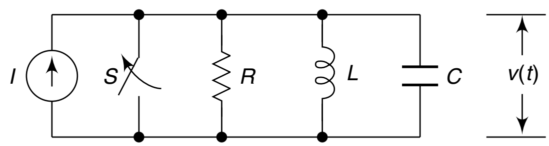
Taking inverse Laplace transform,

$$i_1(t) = \frac{1}{50} + \frac{1}{70}e^{-100t/7} \quad (A)$$

For the current to be 25 mA, we get,

$$25 \times 10^{-3} = \frac{1}{50} + \frac{1}{70}e^{-100t/7} \quad \Rightarrow \quad t = 0.0735 \text{ second} \qquad \textit{Ans.}$$

6.37 The figure shows a parallel *RLC* circuit fed from a dc current source through a switch. The circuit elements are $R = 400\,\Omega$, $L = 25$ mH, $C = 25$ nF. The source current is 24 mA. The switch which has been in closed position for a long time is opened at $t = 0$.



(a) What is the initial value of current i_L (i.e., at $t = 0$)?
(b) What is the initial value of voltage across L at $t = 0$?
(c) What is the expression for current through inductance, capacitance and resistance?
(d) What is the final value of i_L?
(e) What happens to $i_L(t)$ if R is increased from 400 to 625 Ω? Assume that initial energy is zero.

Solution Applying KCL for the node, we get,

$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v\,dt + C\frac{dv}{dt} = I$$

Taking Laplace transform,

$$V(s)\left[\frac{1}{R}+\frac{1}{sL}+sC\right]=\frac{I}{s}$$

$$\Rightarrow \qquad V(s)=\frac{I}{C\left(s^2+\dfrac{s}{RC}+\dfrac{1}{LC}\right)}$$

Substituting the values,

$$V(s)=\frac{24\times 10^{-3}}{25\times 10^{-9}\left(s^2+\dfrac{s}{400\times 25\times 10^{-9}}+\dfrac{1}{25\times 10^{-3}\times 25\times 10^{-9}}\right)}=\frac{24\times 10^{6}}{25\left(s^2+10^5 s+16\times 10^8\right)}$$

$$=\frac{24\times 10^6}{25\left(s+2\times 10^4\right)\left(s+8\times 10^4\right)}=\frac{16}{s+2\times 10^4}-\frac{16}{s+8\times 10^4}$$

Taking inverse Laplace transform,

$$v(t)=16e^{-2\times 10^4 t}-16e^{-8\times 10^4 t}\ \text{(V)}$$

Also, the current through the inductor,

$$I_L(s)=\frac{V(s)}{sL}=\frac{V(s)}{25\times 10^{-3}s}=\frac{24\times 10^6}{25\times 10^{-3}\times 25s\left(s+2\times 10^4\right)\left(s+8\times 10^4\right)}$$

$$=\frac{384\times 10^5}{s\left(s+2\times 10^4\right)\left(s+8\times 10^4\right)}$$

$$=\frac{24}{1000s}-\frac{32}{1000\left(s+2\times 10^4\right)}+\frac{8}{1000\left(s+8\times 10^4\right)}$$

Taking inverse Laplace transform,

$$i_L(t)=24-32e^{-2\times 10^4 t}+8e^{-8\times 10^4 t}\ \text{(mA)}$$

(a) At $t = 0$, we get, $i_L(0)=0$ *Ans.*

(b) At $t = 0$, we get, $v(0)=0$

(c) Current through inductance

$$i_L(t)=24-32e^{-2\times 10^4 t}+8e^{-8\times 10^4 t}\ \text{(mA)}\ \textit{Ans.}$$

Current through capacitance,

$$i_C(t)=C\frac{dv(t)}{dt}=25\times 10^{-9}\frac{d}{dt}\left[16e^{-2\times 10^4 t}-16e^{-8\times 10^4 t}\right]=32e^{-8\times 10^4 t}-8e^{-2\times 10^4 t}\quad \text{(mA)} \qquad \textit{Ans.}$$

Current through resistance,

$$i_R(t)=\frac{v(t)}{R}=\frac{1}{400}\times\left[16e^{-2\times10^4t}-16e^{-8\times10^4t}\right]=\frac{1}{25}\left[e^{-2\times10^4t}-16e^{-8\times10^4t}\right]\ \text{(A)} \qquad \textit{Ans.}$$

(d) At $t=\infty$, the final value of i_L is, $i_L(\infty)=24-32e^{-\infty}+8e^{-\infty}=24(\text{mA})$ *Ans.*

(e) Putting the value of resistance $R = 625\,\Omega$ in the expression of i_L, we get,

$$I_L(s)=\frac{V(s)}{sL}=\frac{I}{sLC\left(s^2+\frac{s}{RC}+\frac{1}{LC}\right)}$$

$$=\frac{24\times10^{-3}}{s\times25\times10^{-3}\times25\times10^{-9}\left(s^2+\frac{s}{625\times25\times10^{-9}}+\frac{1}{25\times10^{-3}\times25\times10^{-9}}\right)}$$

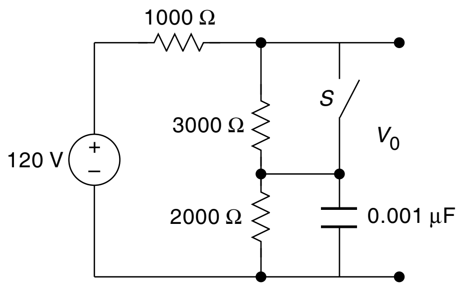
$$=\frac{384\times10^5}{s\left(s^2+64\times10^3\,s+16\times10^8\right)}$$

Taking inverse Laplace transform and simplifying, we get,

$$i_L(t)=106.67\times10^9\,e^{-32\times10^3t}\sin\left(14.4\times10^3\right)t \quad \text{(A)} \qquad \textit{Ans.}$$

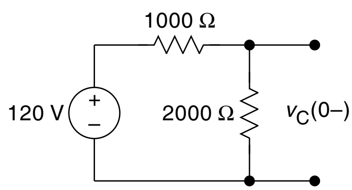
Here, with R = 400 Ω, the circuit was in overdamped condition. As the value of resistance is increased to 625 Ω, the circuit becomes underdamped.

6.38 In the network of the figure, the switch S has been closed for a long time. The switch is suddenly opened at t = 0 and reclosed at t = 20 μs. Find expression for voltage V_0 for $t \le 20$ μs and $t > 20$ μs.

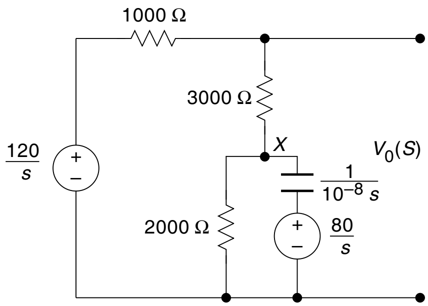


Solution With the switch closed, the initial voltage across the capacitor is,

$$v_C(0-)=\frac{120}{1000+2000}\times2000=80\,\text{V}$$



After the switch is opened, the transformed circuit is shown in the figure below.



Applying KCL at node X, we get,

$$\frac{V_X(s)-\frac{120}{s}}{4000}+\frac{V_X}{2000}+\frac{V_X-\frac{80}{s}}{\frac{1}{10^{-8}s}}=0$$

$$\Rightarrow \qquad V_X\left[\frac{1}{4000}+\frac{1}{2000}+18^{-8}s\right]=\frac{120}{4000s}+80\times 100^{-8}=\frac{0.03}{s}+80\times 100^{-8}$$

$$\Rightarrow \qquad V_X=\frac{0.03}{s\left(0.00075+10^{-8}s\right)}+\frac{80\times 100^{-8}}{0.00075+10^{-8}s}$$

$$=\frac{40}{s}-\frac{40}{s+0.075\times 10^6}+\frac{80}{s+0.075\times 10^6}=\frac{40}{s}+\frac{40}{s+0.075\times 10^6}$$

Taking inverse Laplace transform,

$$V_X(t)=40+40e^{-0.075\times 10^6 t}$$

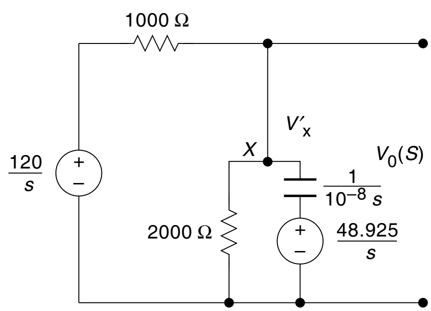
Therefore, the desired voltage is,

$$V_0(t)=V_X(t)+\frac{120-V_X(t)}{4000}\times 3000=100+10e^{-0.075\times 10^6 t} \text{ for } 0\le t\le 20\ \mu\text{s} \qquad \textit{Ans.}$$

At t = 20 μs, the voltage of node X is,

$$V_X=40+40e^{-0.075\times 10^6\times 20\times 10^{-6}}=48.925\text{ V}$$

When the switch is reclosed at t = 20 μs, the voltage across the capacitor will be 48.925 V. After reclosing the switch, the transformed circuit is shown in the figure below. Now let the voltage of node X be V_X'. Applying KCL at node X, we get,



$$\frac{V_X'(s)-\frac{120}{s}}{1000}+\frac{V_X'(s)}{2000}+\frac{V_X'(s)-\frac{48.925}{s}}{\frac{1}{10^{-8}s}}=0$$

$$\Rightarrow \qquad V_X'(s)\left[\frac{1}{1000}+\frac{1}{2000}+10^{-8}s\right]=\frac{120}{1000s}+48.925\times10^{-8}$$

$$\Rightarrow \qquad V_X'(s)\left[0.0015+10^{-8}s\right]=\frac{0.12}{s}+48.925\times10^{-8}$$

$$\Rightarrow \qquad V_X'(s)=\frac{0.12}{s\left(0.0015+10^{-8}s\right)}+\frac{48.925\times10^{-8}}{0.0015+10^{-8}s}$$

$$=\frac{0.12}{s\left(s+0.15\times10^{6}\right)}+\frac{48.925}{s+0.15\times10^{6}}$$

$$\Rightarrow \qquad V_X'(s)=\frac{80}{s}-\frac{31.075}{s+0.15\times10^{6}}$$

Taking inverse Laplace transform, we get,

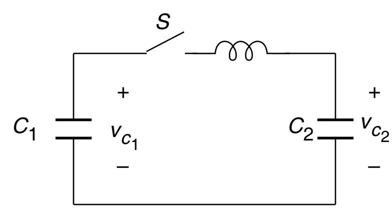
$$V_X'(t)=80-31.075e^{-0.15\times10^{6}t}$$

In this case, the output voltage V_0 is equal to $V_X'(t)$. Since time t is to be counted from the instant the switch is reclosed, t is replaced by $\left(t-20\times10^{-6}\right)$.

$$\therefore \qquad V_0(t)=80-31.075e^{-0.15\times10^{6}\left(t-20\times10^{-6}\right)} \text{ for } t>20\ \mu\text{s} \qquad \textit{Ans.}$$

6.39 In the circuit of the figure the switch S is closed at $t = 0$ and opened again at $t = \pi$ second. Prior to closing the switch at $t = 0$, $v_C = 10$ V while L and C_2 do not have any stored energy. Find the voltages v_{C_1} and v_{C_2} at $t = \pi$ second. $C_1 = C_2 = 1\,\text{F}$; $L = 2\,\text{H}$

Solution After closing the switch, applying KVL in the circuit, we get,



$$L\frac{di(t)}{dt}+\frac{1}{C_1}\int_{-\infty}^{t}i(t)dt+\frac{1}{C_2}\int_{-\infty}^{t}i(t)dt=0$$

Since initial voltage across C_1 is 10 V, we get,

$$2sI(s)+\frac{I(s)}{s}-\frac{10}{s}+\frac{I(s)}{s}=0$$

$$\Rightarrow \qquad I(s)\left[2s+\frac{2}{s}\right]=\frac{10}{s}$$

$$\Rightarrow \qquad I(s)=\frac{5}{s^2+1}$$

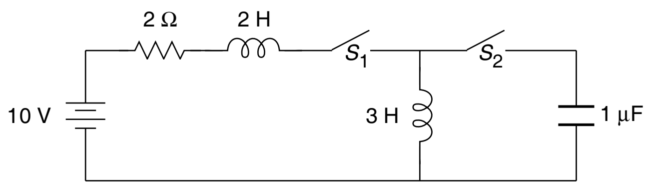
Taking inverse Laplace transform, we get,

$$i(t)=5\sin t$$

$$\therefore \qquad v_{C_1}(t)=-10+\frac{1}{C_1}\int_0^{\pi}5\sin t\,dt=-10+\left|-5csot\right|_0^{\pi}=0 \qquad \textit{Ans.}$$

$$\therefore \qquad v_{C_2}(t) = \frac{1}{C_2}\int_0^{\pi} 5\sin t\, dt = \left|-5csot\right|_0^{\pi} = 10\text{ V} \qquad \textit{Ans.}$$

6.40 The network shown in the figure is in steady state with S_1 closed and S_2 open. At $t = t_1$, S_1 is opened and S_2 is closed. Find current through capacitor for $t \geq t_1$.



Solution When switch S_1 is closed and S_2 is opened, the initial current through the 3 H inductor is, $i(0-) = \frac{10}{2} = 5\text{A}$. Initial voltage across the capacitor is zero.

When switch S_1 is opened and S_2 is closed, the current through the capacitor is given by the KVL equation as,

$$3\frac{di(t)}{dt} + \frac{1}{1\times 10^{-6}}\int_0^t i(t)dt = 0$$

Taking Laplace transform,

$$3sI(s) - 3\times 5 + \frac{1}{10^{-6}}\frac{I(s)}{s} = 0$$

$$\Rightarrow \qquad I(s) = \frac{15s}{3s^2 + 10^6} = \frac{5s}{s^2 + \frac{10^6}{3}}$$

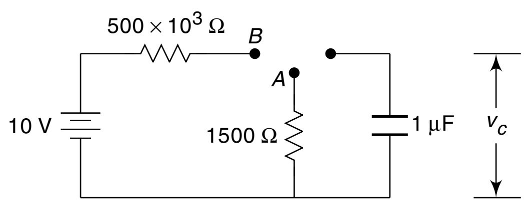
Taking inverse Laplace transform we get,

$$i(t) = 5\cos\left[\left(\sqrt{\frac{10^6}{3}}\right)t\right] = 5\cos(577.35t)$$

Since the switch is closed at $t = t_1$, the time will be shifted by $(t - t_1)$ so that the current through the capacitor is given as,

$$i(t) = 5\cos\left[577.35(t - t_1)\right] \quad \text{for} \quad t \geq t_1 \qquad \textit{Ans.}$$

6.41 The switch in figure has been in position A for a long time. At $t = 0$ it is moved to B and at $t = 1$ second, it is moved to A again. Find voltage across capacitor after a further lapse of 1 millisecond.



Solution As the switch is in position A for a long time, initial charge across the capacitor is zero.

When the switch is moved to position B, the current in the circuit is obtained from the KVL equation as,

$$I(s)\left(500\times 10^3+\frac{1}{1\times 10^{-6}s}\right)=\frac{10}{s}$$

$$\Rightarrow \qquad I(s)=\frac{10}{500\times 10^3(s+2)}=\frac{1}{5\times 10^4}\left(\frac{1}{s+2}\right)$$

Taking inverse Laplace transform,

$$i(t)=\frac{1}{5\times 10^4}e^{-2t} \quad (\text{A})$$

Therefore, voltage across the capacitor is,

$$v_C(t)=10-\left(500\times 10^3\right)i(t)=10-\left(500\times 10^3\right)\frac{1}{5\times 10^4}e^{-2t}=10\left(1-e^{-2t}\right) \quad (\text{V})$$

Therefore, voltage across the capacitor at t = 1second is,

$$v_C(t)=10\left(1-e^{-2}\right)=8.65 \quad (\text{V})$$

At t = 1second, the switch is moved to position A, so that the KVL equation becomes,

$$\frac{1}{1\times 10^{-6}}\int_0^t i(t)\,dt+1500i(t)=0$$

Taking Laplace transform,

$$I(s)\left[1500+\frac{1}{1\times 10^{-6}s}\right]=\frac{8.65}{s} \quad \text{since initial voltage across the capacitor is 8.65 V.}$$

$$\Rightarrow \qquad I(s)=\frac{8.65}{1500}\left(\frac{1}{s+666.67}\right)$$

Taking inverse Laplace transform,

$$i(t)=\frac{8.65}{1500}e^{-666.67t} \quad (\text{A})$$

Hence, the voltage across the capacitor is,

$$v_C(t)=1500i(t)=1500\times\frac{8.65}{1500}e^{-666.67t}=8.65e^{-666.67t}$$

At t = 1ms, the voltage is, $v_C=8.65e^{-666.67\times 10\times -3}=4.44$ V *Ans.*

6.42 Determine the Laplace transform of $f(t)=\dfrac{2-2e^{-t}}{t}$.

Solution

$$f(t)=\frac{2-2e^{-t}}{t}=\frac{2}{t}\left(1-e^{-t}\right)=\frac{2}{te^t}\left(e^t-1\right)=\frac{2}{te^t}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}+\ldots-1\right) \text{ (Expanding } e^t\text{)}$$

$$\therefore\ f(t)=\frac{2}{te^{t}}\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}+\dots\right)$$

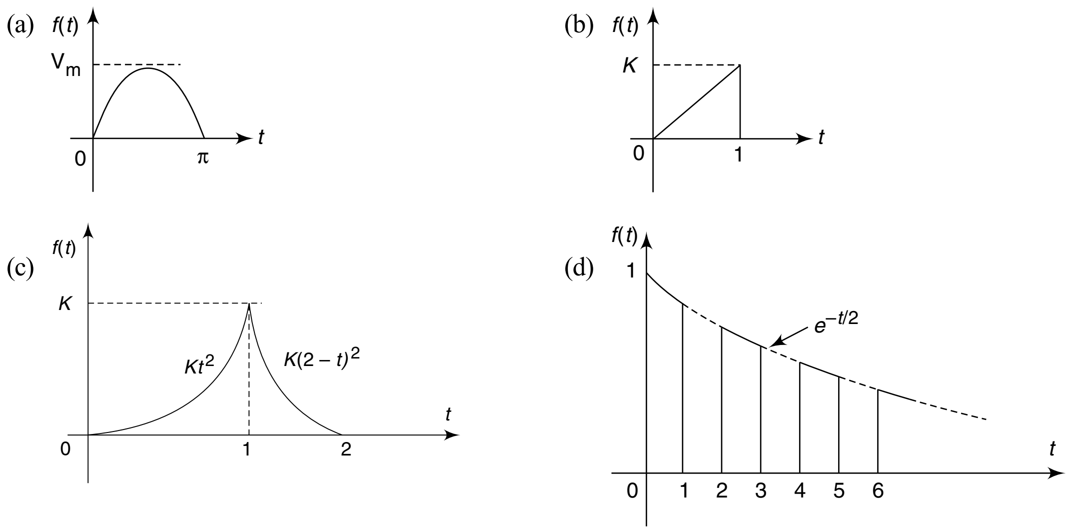
$$=2\left(e^{-t}+\frac{1}{2!}te^{-t}+\frac{1}{3!}t^2e^{-t}+\frac{1}{4!}t^3e^{-t}+\frac{1}{5!}t^4e^{-t}+\dots\right)$$

Taking Laplace transform of each term, we get,

$$F(s)=2\left[\frac{1}{s+1}+\frac{1}{2!}\frac{1!}{(s+1)^2}+\frac{1}{3!}\frac{2!}{(s+1)^3}+\frac{1}{4!}\frac{3!}{(s+1)^4}+\dots\right]$$

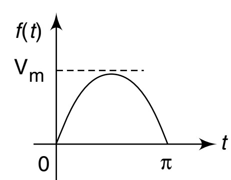
$$=2\left[\frac{1}{s+1}+\left(\frac{1}{2}\right)\left(\frac{1}{(s+1)^2}\right)+\left(\frac{1}{3}\right)\left(\frac{1}{(s+1)^3}\right)+\left(\frac{1}{4}\right)\left(\frac{1}{(s+1)^4}\right)+\dots\right] \qquad Ans.$$

6.43 Express the following functions in terms of singularity functions and find their Laplace transform:



Solution

(a) Here, the signal can be expressed in terms of step signal as,



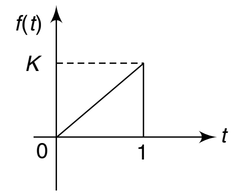
$$f(t)=V_m\sin\omega t\,u(t)+V_m\sin\omega\left(t-\frac{T}{2}\right)u\left(t-\frac{T}{2}\right)$$

$$=V_m\sin\omega t\,u(t)+V_m\sin\omega(t-\pi)\,u(t-\pi)$$

Taking Laplace transform of individual terms, we get the Laplace transform of the functions as,

$$F(s)=\frac{V_m}{s^2+1}+\frac{V_m e^{-\pi s}}{s^2+1}=\frac{V_m}{s^2+1}\left(1+e^{-\pi s}\right) \qquad Ans.$$

(b) Here, the signal starts with a straight line of slope K passing through origin and then comes to zero at $t = 1$.Hence the signal can be expressed in terms of ramp and step signals as,



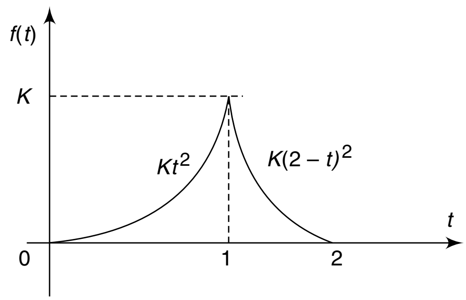
$$f(t) = Kr(t) - Kr(t-1) - Ku(t-1) \quad Ans.$$

Taking Laplace transform of individual terms, we get the Laplace transform of the functions as,

$$F(s) = \frac{K}{s^2} - \frac{Ke^{-s}}{s^2} - \frac{Ke^{-s}}{s} = \frac{K}{s^2}\left[1-(1+s)e^{-s}\right] \qquad Ans.$$

(c) The function can be written as,

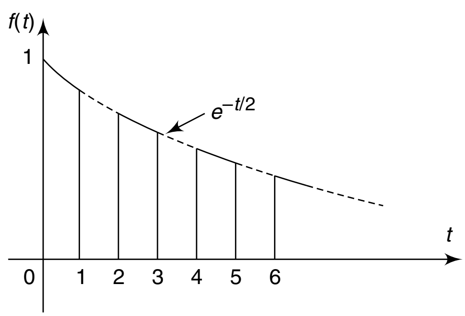
$$\begin{aligned} f(t) &= Kt^2u(t) - Kt^2u(t-1) + K(2-t)^2u(t-1) - K(2-t)^2u(t-2) \\ &= Kt^2u(t) - K\left[t^2-(2-t)^2\right]u(t-1) - K(2-t)^2u(t-2) \\ &= Kt^2u(t) - K4(t-1)\,u(t-1) - K(2-t)^2u(t-2) \end{aligned}$$



Taking Laplace transform of individual terms, we get the Laplace transform of the functions as,

$$F(s) = K\left[\frac{2}{s^3} - \frac{4e^{-s}}{s^2} - \frac{2e^{-2s}}{s^3}\right] = K\frac{2}{s^3}\left[1-2se^{-s}-e^{-2s}\right] \qquad Ans.$$

(d)



The function can be written as,

$$\begin{aligned} f(t) &= e^{-t/2}\left[u(t)-u(t-1)\right] + e^{-t/2}\left[u(t-2)-u(t-3)\right] + e^{-t/2}\left[u(t-4)-u(t-5)\right] + \ldots \\ &= e^{-t/2}\left[\left[u(t)-u(t-1)\right] + u(t-2) - u(t-3) + u(t-4) - u(t-5) + \ldots\right] \end{aligned}$$

Taking Laplace transform of individual terms, we get the Laplace transform of the functions as,

$$F(s)=\frac{1}{s+1/2}-\frac{e^{-(s+1/2)}}{s+1/2}+\frac{e^{-2(s+1/2)}}{s+1/2}-\frac{e^{-3(s+1/2)}}{s+1/2}+\ldots$$

$$=\left(\frac{1}{s+1/2}\right)\left[1-e^{-(s+1/2)}+e^{-2(s+1/2)}-e^{-3(s+1/2)}+\ldots\right]$$

$$=\left(\frac{1}{s+1/2}\right)\left(\frac{1}{1+e^{-(s+1/2)}}\right) \qquad \textit{Ans.}$$

6.44 A pulse voltage of width a and magnitude 10 V is applied at time $t = 0$ to a series RL circuit consisting of a resistance $R = 4\,\Omega$ and an inductor $L = 2$ H. Find the current $i(t)$. Assume zero current through the inductor L before application of the voltage pulse.

Solution The pulse voltage can be written as,.

$$v(t)=10u(t)-10u(t-a)$$

Applying KVL for the RL series circuit with the pulse voltage,

$$Ri(t)+L\frac{di(t)}{dt}=v(t)$$

Taking Laplace transform,

$$RI(s)+L\left[sI(s)-i(0-)\right]=V(s)$$

With zero initial current, substituting the values we get,

$$4I(s)+2sI(s)=\frac{10}{s}\left(1-e^{-as}\right)$$

$$\Rightarrow \qquad I(s)=\frac{10}{s}\left(\frac{1-e^{-as}}{2s+4}\right)=\frac{5\left(1-e^{-as}\right)}{s(s+2)}=\frac{5}{2}\left(1-e^{-as}\right)\left[\frac{1}{s}-\frac{1}{s+2}\right]$$

$$=\frac{5}{2}\left[\frac{1}{s}-\frac{1}{s+2}-\frac{e^{-as}}{s}+\frac{e^{-as}}{s+2}\right]$$

Taking inverse Laplace transform,

$$i(t)=\frac{5}{2}\left[\left(1-e^{-2t}\right)u(t)-\left(1-e^{-2(t-a)}\right)u(t-a)\right] \qquad \textit{Ans.}$$

6.45 A voltage pulse of width b and magnitude 10 V is applied at time t to a series RC circuit consisting of a resistor $R = 1\,\Omega$ and a capacitor $C = \frac{1}{4}$ F. Find the current $i(t)$. Assume zero charge across the capacitor C before application of the voltage pulse.

Solution The pulse voltage can be written as,.

$$v(t)=10u(t)-10u(t-b)$$

Applying KVL for the RC series circuit with the pulse voltage,

$$Ri(t)+\frac{1}{C}\int_{-\infty}^{t}i(t)dt=v(t)$$

Taking Laplace transform,

$$RI(s)+\frac{1}{C}\left[\frac{I(s)}{s}+\frac{v(0-)}{s}\right]=V(s)$$

With zero initial voltage, substituting the values we get,

$$4I(s)+\frac{I(s)}{s1/4}=\frac{10}{s}\left(1-e^{-bs}\right)$$

$$\Rightarrow \qquad I(s)=10\left(\frac{1-e^{-bs}}{s+4}\right)=\frac{10\left(1-e^{-bs}\right)}{s+4}=\left[\frac{10}{s+4}-\frac{10e^{-bs}}{s+4}\right]$$

Taking inverse Laplace transform,

$$i(t)=10\left[e^{-4t}u(t)-e^{-4(t-b)}u(t-b)\right] \qquad \textit{Ans.}$$

6.46 Find the response current of a series *RL* circuit consisting of a resistor $R = 3\,\Omega$ and an inductor $L =$ 1 H when each of the following driving force voltage is applied:

(a) unit ramp voltage $r(t-2)$,

(b) unit impulse voltage $\delta(t-2)$,

(c) unit step voltage $u(t-2)$,

(d) unit doublet voltage $\delta'(t-2)$, and

(e) pulse of width a and magnitude 1 V beginning at time $t = 2$ seconds.

Solution

(a) Unit ramp voltage $r(t-2)$

Applying KVL to *RL* series circuit,

$$Ri + L\frac{di}{dt}=v(t)=r(t-2)$$

Taking Laplace transform,

$$(R+sL)I(s)=\frac{1}{s^2}e^{-2s}$$

$$I(s)=\frac{e^{-2s}}{s^2(sL+R)}$$

Substituting the values,

$$I(s)=\frac{e^{-2s}}{s^2(s+3)}=e^{-2s}\left[\frac{K_1}{s^2}+\frac{K_2}{s}+\frac{K_3}{s+3}\right]$$

$$\therefore \qquad K_1=\left.\frac{1}{s+3}\right|_{s=0}=\frac{1}{3}$$

$$\therefore \qquad K_2=\left.\frac{d}{ds}\left[\frac{1}{s+3}\right]\right|_{s=0}=-\left.\frac{1}{(s+3)^2}\right|_{s=0}=-\frac{1}{9}$$

$$\therefore \qquad K_3 = \frac{1}{s^2}\bigg|_{s=-3} = \frac{1}{9}$$

$$\therefore \qquad I(s) = e^{-2s}\left[\frac{1/3}{s^2} + \frac{-1/9}{s} + \frac{1/9}{s+3}\right]$$

Taking inverse Laplace transform,

$$i(t) = -\frac{1}{9}u(t-2) + \frac{1}{3}r(t-2) + \frac{1}{9}e^{-3(t-2)}u(t-2) \qquad \textit{Ans.}$$

(b) Unit impulse voltage $\delta(t-2)$:

In this case,

$$Ri + L\frac{di}{dt} = v(t) = \delta(t-2)$$

Taking Laplace transform,

$$(R+sL)I(s) = e^{-2s}$$

$$I(s) = \frac{e^{-2s}}{(sL+R)} = \frac{e^{-2s}}{(s+3)}$$

Taking inverse Laplace transform,

$$i(t) = e^{-3(t-2)}u(t-2) \qquad \textit{Ans.}$$

(c) Unit step voltage $u(t-2)$

In this case,

$$Ri + L\frac{di}{dt} = v(t) = u(t-2)$$

Taking Laplace transform,

$$(R+sL)I(s) = \frac{e^{-2s}}{s}$$

$$I(s) = \frac{e^{-2s}}{(sL+R)} = \frac{e^{-2s}}{s(s+3)} = \frac{1}{3}e^{-2s}\left[\frac{1}{s} - \frac{1}{(s+3)}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{3}u(t-2) - \frac{1}{3}e^{-3(t-2)}u(t-2) \qquad \textit{Ans.}$$

(d) Unit doublet voltage $\delta'(t-2)$:

In this case,

$$Ri + L\frac{di}{dt} = v(t) = \delta'(t-2)$$

Taking Laplace transform,

$$(R+sL)\, I(s) = se^{-2s}$$

$$I(s) = \frac{e^{-2s}}{(sL+R)} = \frac{se^{-2s}}{(s+3)} = e^{-2s}\left[1 - \frac{3}{(s+3)}\right]$$

Taking inverse Laplace transform,

$$i(t) = \delta(t-2)\,u(t-2) - 3e^{-3(t-2)}u(t-2) \qquad \textit{Ans.}$$

(e) Pulse of width a and magnitude 1V beginning at time $t = 2$ seconds
In this case,

$$Ri + L\frac{di}{dt} = v(t) = u(t+2) - u(t-2-a)$$

Taking Laplace transform,

$$(R+sL)\,I(s) = \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-(2+a)s}$$

$$I(s) = \frac{e^{-2s} - e^{-(2+a)s}}{s(sL+R)} = \left[e^{-2s} - e^{-(2+a)s}\right]\frac{1}{s(s+3)} = \left[e^{-2s} - e^{-(2+a)s}\right]\frac{1}{3}$$

$$= \frac{1}{3}\left[\frac{e^{-2s}}{s} - \frac{e^{-(2+a)s}}{s} - \frac{e^{-2s}}{s+3} + \frac{e^{-(2+a)s}}{s+3}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{3}\,u(t-2) - u(t-2-a) - e^{-3(t-2)}u(t-2) + e^{-3(t-2-a)}\,u(t-2-a) \qquad \textit{Ans.}$$

6.47 Find the response current of a series RC circuit consisting of a resistor $R = 2\,\Omega$ and a capacitor C = $\frac{1}{4}$F when each of the following driving force voltage is applied:

(a) ramp voltage $2r(t-3)$,

(b) impulse voltage $2\delta(t-3)$,

(c) step voltage $2u(t-3)$, and

(d) doublet voltage $2\delta'(t-3)$.

Solution

(a) ramp voltage $2r(t-3)$:

Applying KVL to RC series circuit,

$$Ri + \frac{1}{C}\int_{-\infty}^{t} i\,dt = v(t) = 2r(t-3)$$

Taking Laplace transform,

$$\left(R + \frac{1}{Cs}\right)I(s) = \frac{2}{s^2}e^{-3s}$$

$$I(s) = \frac{2e^{-3s}}{s^2\left(R + \frac{1}{Cs}\right)} = \frac{2e^{-3s}}{s^2\left(2 + \frac{4}{s}\right)} = \frac{e^{-3s}}{s(s+2)} = \frac{1}{2}e^{-3s}\left[\frac{1}{s} - \frac{1}{s+2}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{2}u(t-3) - \frac{1}{2}e^{-2(t-3)}u(t-3) \qquad \textit{Ans.}$$

(b) Impulse voltage $2\delta(t-3)$

In this case,

$$Ri + \frac{1}{C}\int_{-\infty}^{t} i\,dt = v(t) = 2\delta(t-3)$$

Taking Laplace transform,

$$\left(R + \frac{1}{Cs}\right)I(s) = 2e^{-3s}$$

$$I(s) = \frac{2e^{-3s}}{\left(R + \frac{1}{Cs}\right)} = \frac{2e^{-3s}}{\left(2 + \frac{4}{s}\right)} = \frac{se^{-3s}}{(s+2)} = e^{-3s}\left[1 - \frac{2}{s+2}\right]$$

Taking inverse Laplace transform,

$$i(t) = \delta(t-3)\,u(t-3) - 2e^{-2(t-3)}\,u(t-3) \qquad \textit{Ans.}$$

(c) Step voltage $2u(t-3)$

In this case,

$$Ri + \frac{1}{C}\int_{-\infty}^{t} i\,dt = v(t) = 2u(t-3)$$

Taking Laplace transform,

$$\left(R + \frac{1}{Cs}\right)I(s) = 2e^{-3s}\frac{1}{s}$$

$$I(s) = \frac{2e^{-3s}}{s\left(R + \frac{1}{Cs}\right)} = \frac{2e^{-3s}}{s\left(2 + \frac{4}{s}\right)} = \frac{e^{-3s}}{(s+2)}$$

Taking inverse Laplace transform,

$$i(t) = e^{-2(t-3)}u(t-3) \qquad \textit{Ans.}$$

(d) Doublet voltage $2\delta'(t-3)$

In this case,

$$Ri + \frac{1}{C}\int_{-\infty}^{t} i\,dt = v(t) = 2\delta'(t-3)$$

Taking Laplace transform,

$$\left(R + \frac{1}{Cs}\right)I(s) = 2se^{-3s}$$

$$I(s) = \frac{2se^{-3s}}{\left(R + \frac{1}{Cs}\right)} = \frac{2se^{-3s}}{\left(2 + \frac{4}{s}\right)} = \frac{s^2e^{-3s}}{(s+2)} = e^{-3s}\left[s - 2 + \frac{4}{s+2}\right]$$

Taking inverse Laplace transform,

$$i(t) = \delta'(t-3) - 2\delta(t-3) + 4e^{-2(t-3)}u(t-3) \quad t \geq 3 \quad \textit{Ans.}$$

6.48 Find the response current of a series *RLC* circuit consisting of a resistor $R = 2\,\Omega$, an inductor $L = 1$ H and a capacitor $C = \frac{1}{4}$ F when each of the following driving force voltage is applied:

(a) ramp voltage $12r(t-2)$,

(b) step voltage $3u(t-3)$,

(c) impulse voltage $3\delta(t-1)$, and

(d) doublet voltage $2\delta'(t-3)$.

Solution Applying KVL for the series *RLC* circuit we get,

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt = v(t)$$

$$\Rightarrow \quad 5i(t) + \frac{di(t)}{dt} + \frac{1}{1/4}\int i(t)dt = v(t)$$

(a) Ramp voltage $12r(t-2)$

When $v(t) = 12r(t-2)$

$$5i(t) + \frac{di(t)}{dt} + \frac{1}{1/4}\int i(t)dt = v(t)$$

Taking Laplace transform,

$$\left(5 + s + \frac{4}{s}\right)I(s) = \frac{12}{s^2}e^{-2s}$$

$$\Rightarrow \quad I(s) = \frac{12e^{-2s}}{s(s^2+5s+4)} = 12e^{-2s}\left[\frac{1}{s(s+1)(s+4)}\right] = 12e^{-2s}\left[\frac{1/4}{s} - \frac{1/3}{s+1} + \frac{1/12}{s+4}\right]$$

$$= \frac{3e^{-2s}}{s} - \frac{4e^{-2s}}{s+1} + \frac{e^{-2s}}{s+4}$$

Taking inverse Laplace transform, we, get,

$$i(t) = 3u(t-2) - 4e^{-(t-2)} + e^{-4(t-2)} \quad t \geq 2 \quad \textit{Ans.}$$

(b) Step voltage $3u(t-3)$

When $v(t) = 3u(t-3)$

$$5i(t) + \frac{di(t)}{dt} + \frac{1}{1/4}\int i(t)dt = v(t)$$

Taking Laplace transform,

$$\left(5 + s + \frac{4}{s}\right)I(s) = \frac{3}{s}e^{-3s}$$

$$\Rightarrow \qquad I(s) = \frac{3e^{-2s}}{(s^2+5s+4)} = 3e^{-2s}\left[\frac{1}{(s+1)(s+4)}\right] = 3e^{-2s}\left[\frac{1/3}{s+4} - \frac{1/3}{s+1}\right]$$

$$= \frac{e^{-2s}}{s+4} - \frac{e^{-2s}}{s+1}$$

Taking inverse Laplace transform, we, get,

$$i(t) = e^{-(t-3)} + e^{-4(t-3)} \quad t \geq 3 \quad \textit{Ans.}$$

(c) Impulse voltage $3\delta(t-1)$

When $v(t) = 3\delta(t-1)$

$$5i(t) + \frac{di(t)}{dt} + \frac{1}{1/4}\int i(t)dt = v(t)$$

Taking Laplace transform,

$$\left(5 + s + \frac{4}{s}\right) I(s) = 3e^{-s}$$

$$\Rightarrow \qquad I(s) = \frac{3se^{-s}}{(s^2+5s+4)} = 3e^{-s}\left[\frac{s}{(s+1)(s+4)}\right] = 3e^{-s}\left[\frac{K_1}{s+1} + \frac{K_2}{s+4}\right]$$

$$\therefore \qquad K_1 = \left[\frac{s}{s+4}\right]_{s=-1} = \frac{-1}{3}$$

$$\therefore \qquad K_2 = \left[\frac{s}{s+1}\right]_{s=-4} = \frac{4}{3}$$

$$\therefore \qquad I(s) = 3e^{-s}\left[\frac{-\frac{1}{3}}{s+1} + \frac{\frac{4}{3}}{s+4}\right] = \frac{4e^{-s}}{s+4} - \frac{e^{-s}}{s+1}$$

Taking inverse Laplace transform, we, get,

$$i(t) = 4e^{-4(t-1)} - e^{-(t-1)} \quad t \geq 1 \quad \textit{Ans.}$$

(d) Doublet voltage $2\delta'(t-3)$

When $v(t) = 2\delta'(t-3)$

$$5i(t) + \frac{di(t)}{dt} + \frac{1}{1/4}\int i(t)\,dt = v(t)$$

Taking Laplace transform,

$$\left(5 + s + \frac{4}{s}\right) I(s) = 2se^{-3s}$$

$$\Rightarrow \qquad I(s) = \frac{2s^2 e^{-3s}}{\left(s^2 + 5s + 4\right)} = e^{-3s}\left[2 - \frac{10s + 8}{s^2 + 5s + 4}\right]$$

Let,
$$\frac{10s + 8}{s^2 + 5s + 4} = \frac{K_1}{s + 1} + \frac{K_2}{s + 4}$$

$$\therefore \qquad K_1 = \left[\frac{10s+8}{s+4}\right]_{s=-1} = \frac{-2}{3}$$

$$\therefore \qquad K_2 = \left[\frac{10s+8}{s+1}\right]_{s=-4} = \frac{32}{3}$$

$$\therefore \qquad I(s) = 2e^{-3s} - e^{-3s}\left[\frac{-\frac{2}{3}}{s+1} + \frac{\frac{32}{3}}{s+4}\right] = 2e^{-3s} + \frac{2}{3}\frac{e^{-3s}}{s+1} - \frac{32}{3}\frac{e^{-3s}}{s+4}$$

Taking inverse Laplace transform, we, get,

$$i(t) = 2\delta(t - 3) + \frac{2}{3}e^{-(t-3)} - \frac{32}{3}e^{-4(t-3)} \quad t \ge 3 \quad \textit{Ans.}$$

6.49 A voltage pulse of magnitude 6 V and duration 3 seconds to 6 seconds is applied to a series *RL* circuit consisting of $R = 6\,\Omega$ and $L = 2$ H. Obtain the current $i(t)$. Also, calculate the voltage across L and R.

Solution Applying KVL for the series *RL* circuit,

$$Ri + L\frac{di}{dt} = v(t) = 6\left[u(t - 3) - u(t - 6)\right]$$

Taking Laplace transform,

$$(R + sL)I(s) = \frac{6}{s}\left[e^{-3s} - e^{-6s}\right]$$

$$\Rightarrow \qquad I(s) = \frac{6}{s}\left[\frac{e^{-3s} - e^{-6s}}{R + sL}\right] = \frac{6}{s}\left[\frac{e^{-3s} - e^{-6s}}{6 + 2s}\right] = \frac{3}{s}\left[\frac{e^{-3s} - e^{-6s}}{s + 3}\right] = \left[e^{-3s} - e^{-6s}\right]\left[\frac{1}{s} - \frac{1}{s+3}\right]$$

Taking inverse Laplace transform,

$$i(t) = \left(1 - e^{-3(t-3)}\right)u(t - 3) - \left(1 - e^{-3(t-6)}\right)u(t - 6) \quad \textit{Ans.}$$

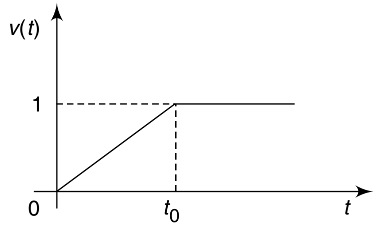
Voltage across inductor,

$$v_L = L\frac{di}{dt} = 2\left[\left\{-(-3)e^{-3(t-3)}\right\}u(t-3) - \left\{-(-3)e^{-3(t-6)}\right\}u(t-6)\right]$$
$$= 6e^{-3(t-3)}u(t-3) - 6e^{-3(t-6)}u(t - 6) \qquad \textit{Ans.}$$

Voltage across resistor,

$$v_R = Ri = 6i = 6\left[\left(1 - e^{-3(t-3)}\right)u(t-3) - \left(1 - e^{-3(t-6)}\right)u(t-6)\right] \qquad \textit{Ans.}$$

6.50 Voltage having waveform of truncated ramp as shown in the figure is applied to an *RL* series circuit consisting of a resistor $R = 3\,\Omega$ and inductor $L = 1$ H. The rise time $t_0 = 2\,\mu$s. Find the current $i(t)$.



Solution The applied voltage can be synthesised in terms of two ramp functions as,

$$v(t) = \frac{1}{t_0} r(t) - \frac{1}{t_0} r(t - t_0)$$

Applying KVL for the series *RL* circuit,

$$Ri + L\frac{di}{dt} = v(t) = \frac{1}{t_0} r(t) - \frac{1}{t_0} r(t - t_0)$$

Taking Laplace transform,

$$(R + sL)I(s) = \frac{1}{t_0}\left[\frac{1}{s^2} - \frac{1}{s^2} e^{-t_0 s}\right]$$

$$\Rightarrow \qquad I(s) = \frac{1}{t_0}\left[\frac{1}{s^2} - \frac{1}{s^2} e^{-t_0 s}\right]\left(\frac{1}{R + sL}\right) = \frac{1}{t_0}\left(1 - e^{-t_0 s}\right)\frac{1}{s^2(s+3)}$$

Let $$\frac{1}{s^2(s+3)} = \frac{K_1}{s^2} + \frac{K_2}{s} + \frac{K_3}{s+3}$$

$$\therefore \qquad K_1 = \left[\frac{1}{s+3}\right]_{s=0} = \frac{1}{3}$$

$$\therefore \qquad K_2 = \left[\frac{d}{ds}\frac{1}{s+3}\right]_{s=0} = -\frac{1}{9}$$

$$\therefore \qquad K_3 = \left[\frac{1}{s^2}\right]_{s=-3} = \frac{1}{9}$$

$$\therefore \qquad I(s) = \frac{1}{t_0}\left(1 - e^{-t_0 s}\right)\left[\frac{1}{3}\left(\frac{1}{s^2}\right) - \frac{1}{9}\left(\frac{1}{s}\right) + \frac{1}{9}\left(\frac{1}{s+3}\right)\right]$$

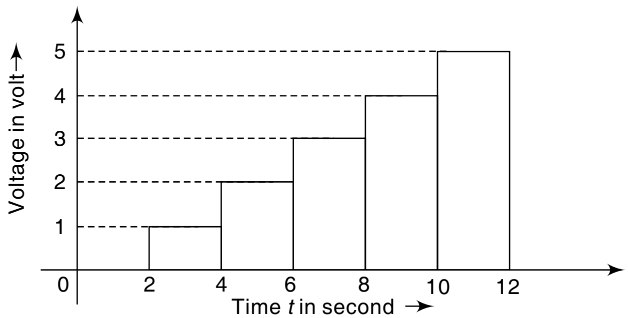
$$= \frac{1}{t_0}\left(1 - e^{-t_0 s}\right)\left[-\frac{1}{9}\left(\frac{1}{s}\right) + \frac{1}{3}\left(\frac{1}{s^2}\right) + \frac{1}{9}\left(\frac{1}{s+3}\right)\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{t_0}\left[-\frac{1}{9} + \frac{1}{3}r(t) + \frac{1}{9}e^{-3t}\right]u(t) - \frac{1}{t_0}\left[-\frac{1}{9} + \frac{1}{3}r(t - t_0) + \frac{1}{9}e^{-3(t-t_0)}\right]u(t - t_0) \qquad \textit{Ans.}$$

where, $t_0 = 2\,\mu$s.

6.51 The figure shows a staircase voltage waveform. Assuming that the staircase is not repeated, express its equation in terms of step functions. If this voltage is applied to a series *RL* circuit with $R = 2$ ohms and $L = 1$ H, find an expression for the resulting current $i(t)$; $i(0+) = 0$.



Solution Here, the applied voltage is a combination of several shifted step functions and can be written as,

$$v(t) = u(t-2) + u(t-4) + u(t-6) + u(t-8) + u(t-10) - 5u(t-12)$$

Taking Laplace transform,

$$V(s) = \frac{1}{s}\left[e^{-2s} + e^{-4s} + e^{-6s} + e^{-8s} + e^{-10s} - 5e^{-12s}\right]$$

If this voltage is applied to *RL* series circuit, applying KVL we get,

$$Ri(t) + L\frac{di(t)}{dt} = v(t)$$

Taking Laplace transform,

$$(R + sL)I(s) = V(s) = \frac{1}{s}\left[e^{-2s} + e^{-4s} + e^{-6s} + e^{-8s} + e^{-10s} - 5e^{-12s}\right]$$

$$\Rightarrow \qquad I(s) = \frac{1}{s(s+2)}\left[e^{-2s} + e^{-4s} + e^{-6s} + e^{-8s} + e^{-10s} - 5e^{-12s}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{1}{s+2}\right]\left[e^{-2s} + e^{-4s} + e^{-6s} + e^{-8s} + e^{-10s} - 5e^{-12s}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{2}\left[1 - e^{-2(t-2)}\right]u(t-2) + \frac{1}{2}\left[1 - e^{-2(t-4)}\right]u(t-4) + \frac{1}{2}\left[1 - e^{-2(t-6)}\right]u(t-6) + \frac{1}{2}$$

$$\left[1 - e^{-2(t-8)}\right]u(t-8) + \frac{1}{2}\left[1 - e^{-2(t-10)}\right]u(t-10) - \frac{5}{2}\left[1 - e^{-2(t-12)}\right]u(t-12) \qquad \textit{Ans.}$$

MULTIPLE-CHOICE QUESTIONS

6.1 The condition for over-damped response of an *RLC* series circuit is

(a) $\dfrac{R^2}{4L^2} = \dfrac{1}{LC}$ (b) $\dfrac{R^2}{4L^2} > \dfrac{1}{LC}$ (c) $\dfrac{R^2}{4L^2} < \dfrac{1}{LC}$ (d) $\dfrac{R^2}{4L^2} \le \dfrac{1}{LC}$

6.2 Transient current in an RLC circuit is oscillatory when

(a) $R = 2\sqrt{\dfrac{L}{C}}$ (b) $R > 2\sqrt{\dfrac{L}{C}}$ (c) $R < 2\sqrt{\dfrac{L}{C}}$ (d) $R = 0$.

6.3 Laplace transform analysis gives
(a) time domain response only (b) frequency domain response only
(c) both (a) and (b) (d) None of these.

6.4 A function $f(t)$ is shifted by a then it is correctly represented as
(a) $f(t-a)u(t)$ (b) $f(t)u(t-a)$ (c) $f(t-a)u(t-a)$ (d) $f(t-a)(t-a)$

6.5 Laplace transform of a delayed unit impulse function $\delta_s(t) = \delta(t-1)$ is
(a) unity. (b) zero. (c) e^{-s}. (d) s.

6.6 The condition for under damped response of an *RLC* series circuit is

(a) $\dfrac{R^2}{4L^2} = \dfrac{1}{LC}$ (b) $\dfrac{R^2}{4L^2} > \dfrac{1}{LC}$ (c) $\dfrac{R^2}{4L^2} < \dfrac{1}{LC}$ (d) $\dfrac{R^2}{4L^2} \le \dfrac{1}{LC}$

6.7 The value of the impulse function $\delta(t)$ at $t = 0$ is
(a) 0 (b) $\propto$ (c) 1 (d) indeterminate

6.8 The value of the ramp function at $t = +\infty$ is
(a) infinity (b) unity (c) zero (d) indeterminate

6.9 The value of the ramp function at $t = -\infty$ is
(a) 0 (b) ∞ (c) $-\infty$ (d) 1

6.10 The value of the impulse function $\delta(t)$ for $t > 0$ is
(a) zero (b) unity
(c) k, where k is a constant (d) infinity.

6.11 The free response of *RL* and *RC* series networks having a time constant τ is of the form

(a) $A + Be^{-\frac{t}{\tau}}$ (b) $Ae^{-\frac{t}{\tau}}$ (c) $Ae^{-\frac{t}{\tau}} + Be^{-\frac{t}{\tau}}$ (d) $(A + Bt)e^{-\frac{t}{\tau}}$

6.12 In the complex frequency $s = \sigma + j\omega$, ω has the units of rad/s and σ has the units of
(a) Hz (b) neper/s (c) rad/s (d) rad

6.13 Time constant of a series RC circuit is
(a) C/R (b) R/C (c) RC (d) $1/RC$

6.14 Time constant of a series RL circuit is
(a) L/R (b) R/L (c) LR (d) $1/LR$

6.15 A coil with a certain number of turns has a specified time constant. If the number of turns is doubled, its time constant would
(a) remain unaffected (b) become doubled (c) become four-fold (d) get halved.

6.16 An RLC series circuit has $R = 1\Omega$, L = 1 H and $C = 1$F. Damping ratio of the circuit will be
(a) more than unity (b) unity (c) 0.5 (d) zero

6.17 A step function voltage is applied to an *RLC* series circuit having $R = 2\Omega$, $L = 1$H and $C = 1$F. The transient current response of the circuit would be
(a) over-damped
(b) critically damped
(c) under damped
(d) over, under or critically damped depending upon magnitude of the step voltage.

6.18 For an *RC* circuit comprising a capacitor $C = 2\ \mu$F in series with a resistance $R = 1$ MΩ period 6 seconds will be equal to
(a) one time constant (b) two time constants
(c) three time constants (d) four time constants

6.19 A series *RL* circuit with $R = 100$ ohm; $L = 50$H, is supplied to a d.c. source of 100V. The time taken for the current to rise 70% of its steady state value is
(a) 0.3s (b) 0.6s
(c) 2.4s (d) 70% of time required to reach steady state.

6.20 If *f(t)* and its first derivative are Laplace transformable then the initial value of *f(t)* is given by

(a) $\underset{t\to 0}{\text{Lt}}\ f(t) = \underset{s\to 0}{\text{Lt}}\ sF(s)$ (b) $\underset{t\to 0}{\text{Lt}}\ f(t) = \underset{s\to\infty}{\text{Lt}}\ \dfrac{F(s)}{s}$

(c) $\underset{t\to 0}{\text{Lt}}\ f(t) = \underset{s\to 0}{\text{Lt}}\ \dfrac{F(s)}{s}$ (d) $\underset{t\to 0}{\text{Lt}}\ f(t) = \underset{s\to\infty}{\text{Lt}}\ sF(s)$

6.21 If *f(t)* and its first derivative are Laplace transformable then the final value of *f(t)* is given by

(a) $\underset{t\to\infty}{\text{Lt}}\ f(t) = \underset{s\to 0}{\text{Lt}}\ sF(s)$ (b) $\underset{t\to\infty}{\text{Lt}}\ f(t) = \underset{s\to\infty}{\text{Lt}}\ \dfrac{F(s)}{s}$

(c) $\underset{t\to\infty}{\text{Lt}}\ f(t) = \underset{s\to 0}{\text{Lt}}\ \dfrac{F(s)}{s}$ (d) $\underset{t\to\infty}{\text{Lt}}\ f(t) = \underset{s\to\infty}{\text{Lim}}\ sF(s)$

6.22 At $t = 0^+$ with zero initial condition which of the following will act as short circuit?
(a) Inductor (b) Capacitor (c) Resistor (d) None of these

6.23 At $t = 0^+$ with zero initial condition which of the following will act as open circuit?
(a) Inductor (b) Capacitor (c) Resistor (d) None of these

6.24 A capacitor at time $t = 0^+$ with zero initial charge acts as a
(a) short circuit (b) open circuit (c) current source (d) voltage source.

6.25 A series RC circuit is suddenly connected to a dc voltage of *V* volt. The current in the series circuit just after the switch is closed is equal to

(a) zero (b) $\dfrac{V}{RC}$ (c) $\dfrac{VC}{R}$ (d) $\dfrac{V}{R}$

6.26 A series LC circuit is suddenly connected to a dc voltage of *V* volt. The current in the series circuit just after the switch is closed is equal to

(a) $\dfrac{V}{L}$ (b) $\dfrac{V}{C}$ (c) zero (d) $\dfrac{V}{LC}$

6.27 The steady state current in the *RC* series circuit, on the application of step voltage of magnitude *E* will be

(a) zero (b) $\dfrac{E}{R}$ (c) $\dfrac{E}{R}e^{-t/CR}$ (d) $\dfrac{E}{RC}e^{-t}$

6.28 A 10 Ω resistor, a 1H inductor and 1F capacitor are connected in parallel. The combination is driven by a unit step current. Under steady state conditions, the source current flows through
(a) resistor (b) inductor (c) capacitor only (d) All of three elements.

6.29 When a unit impulse voltage is applied to an inductor of 1H, the energy supplied by the source is
(a) ∞ (b) 1 Joule (c) $\frac{1}{2}$ Joule (d) 0.

6.30 Which of the following conditions are necessary for validity of Initial Value Theorem: $\lim\limits_{s\to\infty} sF(s) = \lim\limits_{t\to 0} f(t)$?
(a) $f(t)$ and its derivative $f'(t)$ must have Laplace transform.
(b) If the Laplace transform of $f(t)$ is $F(s)$, then Lim $sF(s)$ must exist.
(c) Only $f(t)$ must have Laplace transform.
(d) (a) and (b) both.

6.31 Inverse Laplace transform of $\dfrac{1}{s-a}$ is
(a) sin at (b) cos at (c) e^{at} (d) e^{-at}

6.32 The impulse response of an RL circuit is a
(a) rising exponential function. (b) decaying exponential function.
(c) step function. (d) parabolic function.

6.33. Laplace transform of the output response of a linear system is the system transfer function when the input is
(a) a step signal. (b) a ramp signal. (c) an impulse signal. (d) a sinusoidal signal.

6.34 An initially relaxed RC series network with $R = 2M\Omega$ and $C = 1\mu$F is switched on to a 10V step input. The voltage across the capacitor after 2 seconds will be
(a) zero (b) 3.68 V (c) 6.32 V (d) 10 V

6.35 For $V(s) = \dfrac{(s+2)}{s(s+1)}$, the initial and final values of $v(t)$ will be respectively
(a) 1 and 1 (b) 2 and 2 (c) 2 and 1 (d) 1 and 2.

6.36 The Laplace transform of the function $i(t)$ is: $I(s)$ $\dfrac{10s+4}{s(s+1)(s^2+4s+5)}$. Its final value will be
(a) 4/5 (b) 5/4 (c) 4 (d) 5

6.37 An initially relaxed 100 mH inductor is switched 'ON' at t = 1 second to an ideal 2A dc current source. The voltage across the inductor would be
(a) zero (b) $0.2\delta(t)$ V (c) $0.2\delta(t-1)$ V (d) $0.2tu(t-1)$ V

6.38 If the unit step response of a network is $(1 - e^{-\alpha t})$, then its unit impulse response will be
(a) $\alpha e^{-\alpha t}$ (b) $\frac{1}{\alpha}e^{-t/\alpha}$ (c) $\frac{1}{\alpha}e^{-t/\alpha}$ (d) $(1-\alpha)\, e^{-\alpha t}$

6.39 The response of an initially relaxed system to a unit ramp excitation is $(1 + e^{-t})$. Its step response will be
(a) $\frac{1}{2}t^2 - e^{-t}$ (b) $1 - e^{-t}$ (c) $-e^{-t}$ (d) t.

6.40 A series circuit containing R, L and C is excited by a step voltage input. The voltage across the capacitance exhibits oscillations. Damping coefficient (ratio) of this circuit is given by

(a) $\xi = \dfrac{R}{2\sqrt{LC}}$ (b) $\xi = \dfrac{R}{LC}$ (c) $\xi = \dfrac{R}{2\sqrt{C/L}}$ (d) $\xi = \dfrac{R}{2\sqrt{L/C}}$

6.41 Consider the following statements:
A unit impulse $\delta(t)$ is mathematically defined as

1. $\delta(t) = 0, t \neq 0$ 2. $\int_{0+}^{\infty} \delta(t)\,dt = 1$ 3. $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

Of these statements
(a) 1, 2 and 3 are correct. (b) 1 and 2 are correct.
(c) 2 and 3 are correct. (d) 1 and 3 are correct.

6.42. With symbols having their usual meanings, the Laplace transform of $u(t - a)$ is

(a) $\dfrac{1}{s}$ (b) $\dfrac{1}{s-a}$ (c) $\dfrac{e^{-as}}{s}$ (d) $\dfrac{e^{as}}{s}$

6.43 Two coils having equal resistances but different inductances are connected in series. The time constant of the series combination is the
(a) sum of the time constants of the individual coils.
(b) average of the time constants of the individual coils.
(c) geometric mean of the time constants of the individual coils.
(d) product of the time constants of the individual coils.

6.44 If the step response of an initially relaxed circuit is known then the ramp response can be obtained by
(a) integrating the step response. (b) differentiating the step response.
(c) integrating the step response twice. (d) differentiating the step response twice.

6.45 If a capacitor is energized by a symmetrical square wave current source, then the steady state voltage across the capacitor will be a
(a) square wave (b) triangular wave (c) step function (d) impulse function.

6.46 A square wave is fed to an RC circuit, then
(a) voltage across R is square and across C is not square.
(b) voltage across C is not square and across R is not square.
(c) voltage across both R and C is square.
(d) voltage across both R and C is not square.

6.47 A step voltage is applied to an under-damped series RLC circuit with variable R. Which of the following statements correctly describe the behaviour of the circuit?
1. If R is increased, the steady state voltage across C will be reduced
2. If R is increased, the frequency of transient oscillation across C will be reduced.
3. If R is reduced, the transient oscillation will die down faster.
4. If R is reduced to zero, the peak amplitude of the voltage across C will be double the input step voltage.

Select the correct answer using the codes given below.
Codes: (a) 1 and 2 (b) 2 and 3 (c) 2 and 4 (d) 1, 3 and 4.

6.48 The number of turns of a coil having a time constant T is doubled. Then the new time constant will be
(i) T (b) $2T$ (c) $4T$ (d) $T/2$

6.49 The response of a network is of the form ke^{st},where $s = \sigma + j\omega$, then σ is known as
(a) radian frequency (b) neper frequency
(c) complex frequency (d) None of these.

6.50 In Laplace transform the variable 's' equals $(\sigma + j\omega)$. Which of the following represent the true nature of σ?
1. σ has a damping effect.
2. σ is responsible for convergence of integral $\int_0^\infty f(t)e^{-st}dt$.
3. σ has a value less than zero.

Select the correct answer using the coeds given below.
Codes: (a) 1, 2 and 3 (b) 1 and 2 (c) 2 and 3 (d) 1 and 3.

6.51 Laplace transform of $t^n e^{-at}$ is
(a) $\dfrac{n}{(s-a)^{n+1}}$ (b) $\dfrac{n!}{(s+a)^{n+1}}$ (c) $\dfrac{n!}{(s-a)^{n}}$ (d) $\dfrac{n!}{(s-a)^{n+1}}$

6.52 $\dfrac{s}{(s^2+\omega^2)}$ is the Laplace transform of
(a) sin ωt (b) cos ωt (c) cosh ωt (d) sinh ωt

6.53 Consider the following statements regarding an RC differentiating network.
1. For an applied rectangular pulse, the output is spiky in nature for $RC \ll$ pulse duration.
2. The output is a ramp for rectangular input pulse.
3. The output has zero average for all inputs.

Of these statements:
(a) 1, 2 and 3 are correct. (b) 1 and 2 are correct.
(c) 2 and 3 are correct (d) 1 and 3 are correct.

6.54 The Laplace transform method enables one to find the response in
(a) the transient state only.
(b) the steady state only.
(c) both transient and steady states.
(d) the transient state provided sinusoidal forcing functions do not exist.

6.55 The convolution of a function $f(t)$ with the unit impulse function $\delta(t)$ is
(a) $\delta(t)$ (b) $f(t)\delta(t)$ (c) $f(t)$ (d) $f(\tau)\delta(t)$

6.56 The d.c. gain of a system represented by the transfer function $\dfrac{25}{(s+2)(s+3)}$ is
(a) 25 (b) 25/6 (c) 5 (d) 10

6.57 Consider the following statements
The impulse response of a linear network can be used to determine the
1. step response. 2. response of the sinusoidal input.
3. elements of the network uniquely. 4. interconnection of network elements.
6. Which of these statements are correct?
(a) 1 and 2 (b) 2 and 3 (c) 3 and 4 (d) 1 and 4.

6.58 Double integration of a unit step function would lead to
(a) an impulse (b) a parabola (c) a ramp (d) zero.

6.59 Which of the following integrals represents the convolution of two functions $f_1(t)$ and $f_2(t)$?

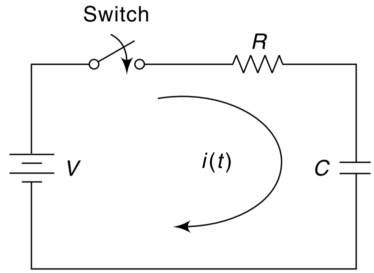
(a) $\int_0^t f_1(t) f_2(\tau - t) d\tau$ (b) $\int_0^t f_1(t-\tau) f_2(\tau) d\tau$

(c) $\int_0^t f_1(t-\tau) f_2(t) dt$ (d) $\int_0^t f_1(\tau - t) f_2(\tau) dt$

6.60 If $F(s) = \frac{1}{s}\frac{(s+1)}{(s+k)}$ and $f(t)$ as $t \to \infty$ is $\frac{1}{2}$, then the value of k is

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) ∞

6.61 The transient response of the initially relaxed network shown in the figure is



(a) $i(t) = \frac{V}{R} e^{-t/RC}$ (b) $i(t) = \frac{V}{R} e^{t/RC}$

(c) $i(t) = \frac{V}{R}(1 - e^{-t/RC})$ (d) $i(t) = \frac{V}{R}(1 + e^{-t/RC})$

6.62 A first order linear system is initially relaxed. For a unit step signal $u(t)$, the response is $v_1(t) = (1 - e^{-3t})$ for $t > 0$. If a signal $3u(t) + \delta(t)$ is applied to the same initially relaxed system, the response will be

(a) $(3 - 6e^{-3t})u(t)$ (b) $(3 - 3e^{-3t})u(t)$ (c) $3u(t)$ (d) $(3 + 3e^{-3t})\,u(t)$

6.63 A unit impulse input to a linear network has a response $R(t)$ and a unit step input to the same network has response $S(t)$. The response $R(t)$

(a) equals $\frac{dS(t)}{dt}$ (b) equals the integral of $S(t)$

(c) is the reciprocal of $S(t)$ (d) has no relation with $S(t)$

6.64 The response of an initially relaxed linear circuit to a signal V_S is $e^{-2t}u(t)$. If the signal is changed to $\left(V_S + 2\frac{dV_S}{dt}\right)$, the response would be

(a) $-4e^{-2t}\,u(t)$ (b) $-3e^{-2t}\,u(t)$ (c) $4e^{-2t}\,u(t)$ (d) $5e^{-2t}\,u(t)$

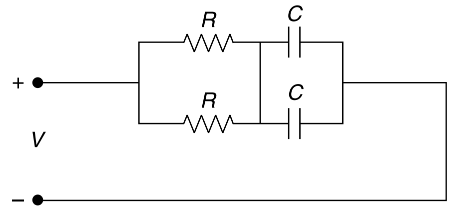
6.65 The impulse response of a circuit is given by $h(t)=\frac{1}{L}e^{-\frac{R}{L}t}u(t)$. Its step response is given as

(a) $\left(1-e^{-\frac{R}{L}t}\right)u(t)$ (b) $\frac{1}{R}\left(1-e^{-\frac{R}{L}t}\right)u(t)$ (c) $\frac{L}{R}\left(1-e^{-\frac{R}{L}t}\right)u(t)$ (d) None of these.

6.66 The time constant of the network shown in the figure is

(a) CR (b) $2CR$

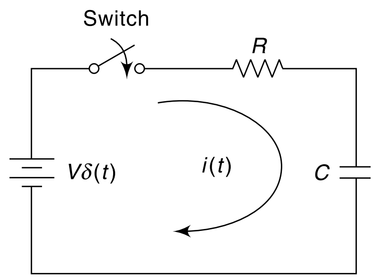
(c) $\frac{CR}{4}$ (d) $\frac{CR}{2}$



6.67 Non-linear system cannot be analyzed by Laplace transform because

(a) it has no zero initial conditions.
(b) superposition law cannot be applied.
(c) non-linearity is generally not well defined.
(d) All of the above.

6.68 In the circuit shown in figure, the response *i(t)* is



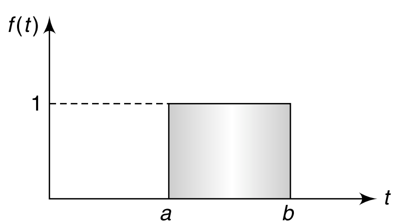
(a) $\frac{V}{R}\exp\left(-\frac{t}{RC}\right)$ (b) $\frac{V}{R}\delta(t)$

(c) $\frac{V}{R}\left[\delta(t)-\frac{1}{RC}\exp\left(-\frac{t}{RC}\right)\right]$ (d) $\frac{V}{R}\left[\delta(t)-\exp\left(-\frac{t}{RC}\right)\right]$

6.69 A voltage $v(t)=6e^{-2t}$ is applied at $t = 0$ to a series RL circuit with L = 1H. If $i(t)=6[e^{-2t}-e^{-3t}]$, then R will have a value of

(a) $\frac{2}{3}\ \Omega$ (b) $1\ \Omega$ (c) $3\ \Omega$ (d) $\frac{1}{3}\Omega$

6.70 The Laplace transform of the signal described in figure is

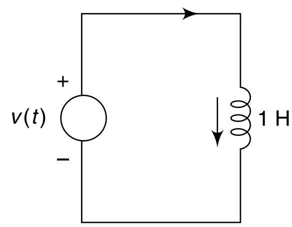


(a) $e^{-as/s}$ (b) e^{-bs}/s^2 (c) $(e^{-as}+e^{-bs})/s$ (d) $(e^{-as}-e^{-bs})/s$

6.71 If a pulse voltage *v(t)* of 4V magnitude and 2 second duration is applied to a pure inductor of 1H, with zero initial current, the current (in A) drawn at $t = 3$ second, will be

(a) zero (b) 2

(c) 4 (d) 8.

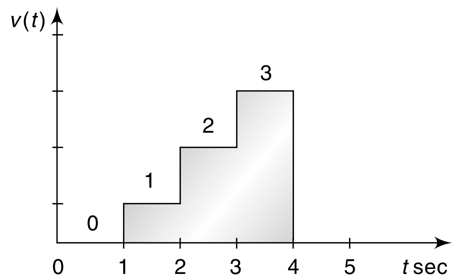


6.72 At certain current, the energy stored in an iron-cored coil is 1000J and its copper loss is 2000W. The time constant (in second) of the coil is

(a) 0.25 (b) 0.5

(c) 1.0 (d) 2.0.

6.73 Consider the voltage waveform shown in the given figure.

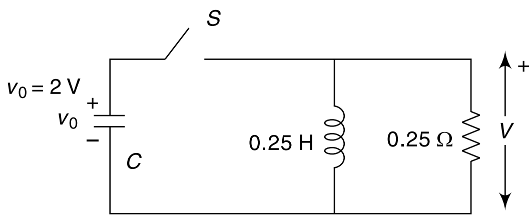


The equation for $v(t)$ is

(a) $u(t-1)+u(t-2)+u(t-3)$ (b) $u(t-1)+2u(t-2)+3u(t-3)$

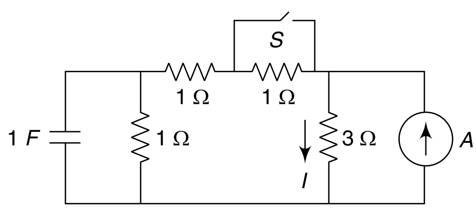
(c) $u(t)+u(t-1)+u(t-2)+u(t-4)$ (d) $u(t-1)+u(t-2)+u(t-3)-3u(t-4)$

6.74 For the circuit given in the figure $V_0 = 2$ V and inductor is initially relaxed. The switch S is closed at $t = 0$. The value of v at $t = 0+$ is



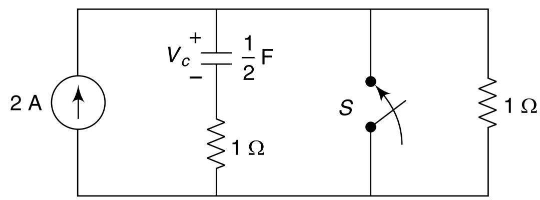
(a) 3 V (b) 2 V (c) 0.5 V (d) 0.25 V

6.75 In the circuit shown in the given figure, S is open for a long time and steady state is reached. S is closed at $t = 0$. The current I at $t = 0+$ is



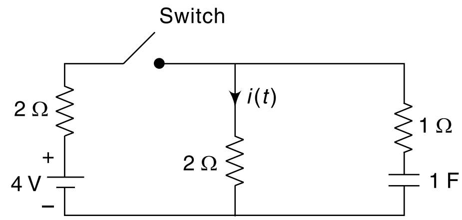
(a) 4 A (b) 3 A (c) 2 A (d) 2 A

6.76 The circuit shown in the given figure is in steady state with switch S open. The switch is closed at $t = 0$. The values of $V_C(0+)$ and $V_C(\infty)$ will be respectively



(a) 2 V, 0 V (b) 0 V, 2 V (c) 2 V, 2 V (d) 0 V, 0 V

6.77 In the circuit shown, the switch is opened at $t = 0$. Prior to that switch was closed, $i(t)$ at $t = 0+$ is



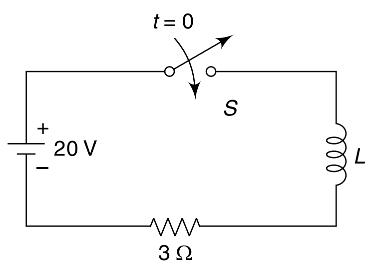
(a) $\frac{2}{3}$ A (b) $\frac{3}{2}$ A (c) $\frac{1}{3}$ A (d) 1 A.

6.78 Given the Laplace transform $\mathcal{L}[v(t)] = \int_0^\infty e^{-st}v(t)dt$, the inverse transform $v(t)$ is

(a) $\int_{\sigma-j\infty}^{\sigma+j\infty} e^{st}V(s)ds$ (b) $\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} e^{st}V(s)ds$

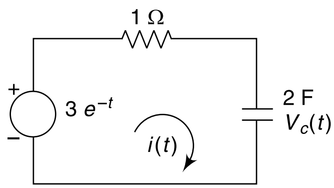
(c) $\frac{1}{2\pi j}\int_0^\infty e^{st}V(s)ds$ (d) $\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} e^{-st}V(s)ds$

6.79 In the circuit shown in the given figure, switch S is closed at $t = 0$. After some time when the current in the inductor was 6A, the rate of change of current through it was 4 A/s. The value of the inductor is



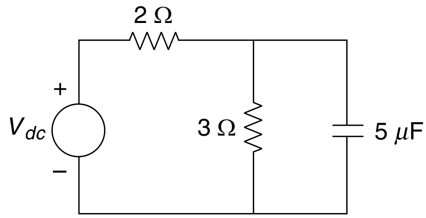
(a) indeterminate (b) 1.5 H (c) 1.0 H (d) 0.5 H

6.80 A circuit consisting of a 1Ω resistor and a 2F capacitor in series is excited from a voltage source with the voltage expressed as $3e^{-t}$, as shown in the given figure. If the $i(0–)$ and $v_c(0–)$ are both zero, then the values of $i(0+)$ and $i(\infty)$ will be respectively



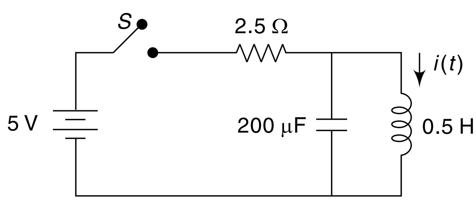
(a) 3 A and 1.5 A (b) 1.5 A and zero (c) 3 A and zero (d) 1.5 A and 3 A

6.81 The time constant associated with the capacitor charging in the circuit shown in the given figure is



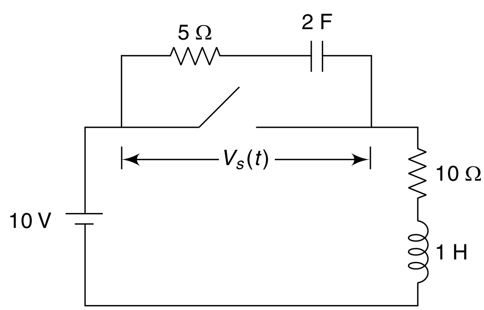
(a) 6 μs (b) 10 μs (c) 15 μs (d) 25 μs

6.82 In the network shown in the figure, the switch S is closed and a steady state is attained. If the switch is opened at t = 0, then the current $i(t)$ through the inductor will be



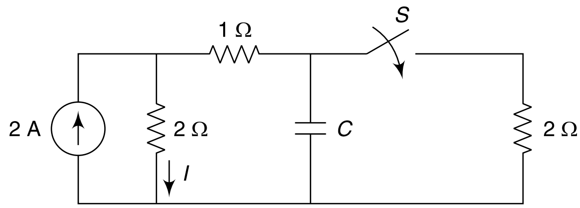
(a) cos 50t A (b) 2 A (c) 2 cos100t A (d) 2 sin 50t A

6.83 In the network shown, the switch is opened at t = 0. Prior to that, the network was in the steady state. $V_s(t)$ at $t = 0^+$ is



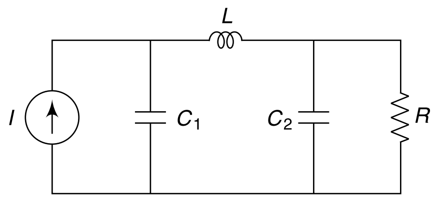
(a) 0 (b) 5 V (c) 10 V (d) 15 V

6.84 The steady state in the circuit, shown in the given figure is reached with S open. S is closed at t = 0. The current I at $t = 0^+$ is



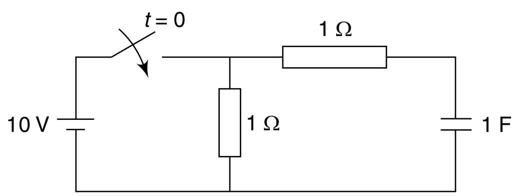
(a) 1 A (b) 2 A (c) 3 A (d) 4 A

6.85 For the circuit shown in the given figure, the current through L and the voltage across C_2 are respectively



(a) zero and RI (b) I and zero (c) Zero and zero (d) I and RI

6.86 In the circuit shown in the given figure, the switch is closed at t = 0. The current through the capacitor will decrease exponentially with a time constant

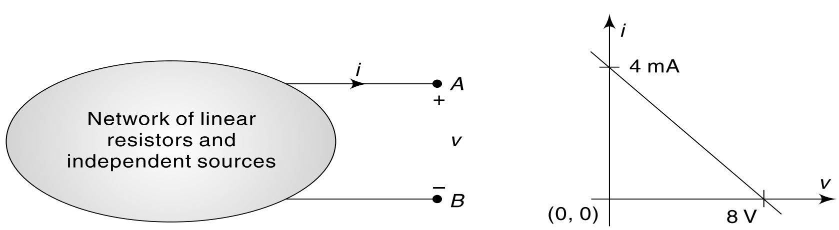


(a) 0.5 s (b) 1 s (c) 2 s (d) 10 s

6.87 The Laplace transformation of $f(t)$ is $F(s)$. Given $F(s) = \dfrac{\omega}{s^2 + \omega^2}$, the final value of $f(t)$ is

(a) infinity (b) zero (c) one (d) None of the above

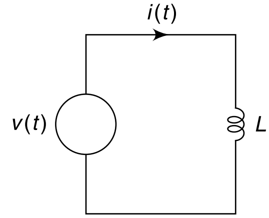
6.88 The v–i characteristics as seen from the terminal-pair (A, B) of the network of Figure (a) is shown in Figure (b). If an inductance of value 6 mH is connected across the terminal-pair (A, B), the time constant of the system will be



(a) (b)

(a) 3 μs (b) 12 s
(c) 32 s (d) unknown, unless the actual network is specified.

6.89 In the circuit shown in figure, it is desired to have a constant direct current $i(t)$ through the ideal inductor L. The nature of the voltage source $v(t)$ must be



(a) constant voltage (b) linearly increasing voltage
(c) an ideal impulse (d) exponentially increasing voltage.

6.90 The value of the integral $\int_{-\infty}^{\infty} e^{5t}\delta(t-5)dt$ is

(a) 1 (b) $(e^5 - 1)$ (c) e^{25} (d) zero.

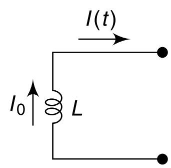
6.91 An inductor at $t = 0+$ with initial current I_0 acts as
(a) voltage source (b) current source (c) open-circuit (d) short-circuit

6.92 A capacitor at $t = 0+$ with initial charge Q_0 acts as
(a) voltage source (b) current source (c) open-circuit (d) short-circuit

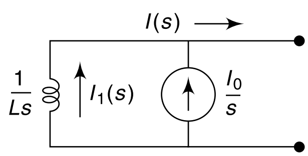
6.93 Consider the following statements
1. Current through an inductor cannot change abruptly.
2. Voltage across the capacitor cannot change abruptly.
3. Initial value of a function *f(t)* is $\lim_{s\to 0} sF(s)$
4. Final value of a function *f(t)* is $\lim_{s\to \infty} sF(s)$

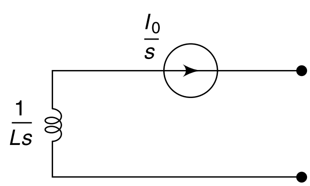
Of these statements
(a) 3 and 4 are correct (b) 1 and 4 are correct
(c) 1 and 2 are correct (d) 2 and 3 are correct.

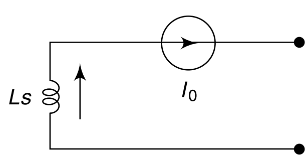
6.94 An inductor with inductance L and initial current I_0 is shown as

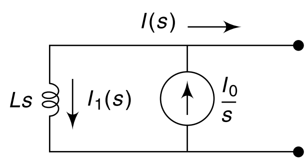


The correct admittance diagram for it is

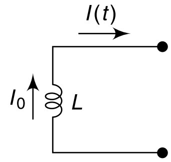
(a)


(b)


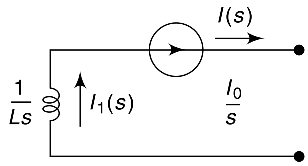
(c)


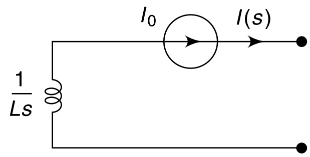
(d)


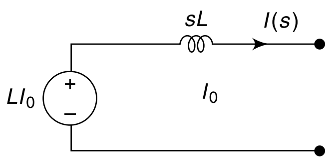
6.95 An inductor with inductance L and initial current I_0 is shown as

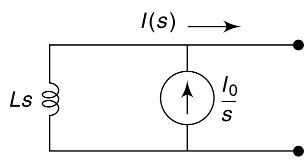


The correct impedance diagram for it is

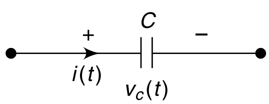
(a)


(b)


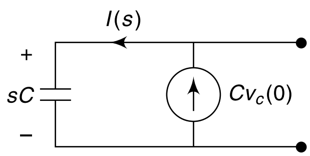
(c)


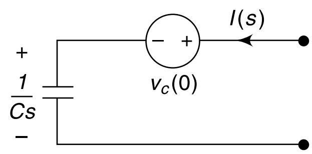
(d)


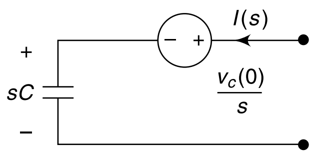
6.96 A capacitor with capacitance C and initial voltage $v_c(t)$ is shown here

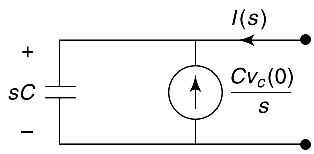


The correct admittance diagram for this circuit is

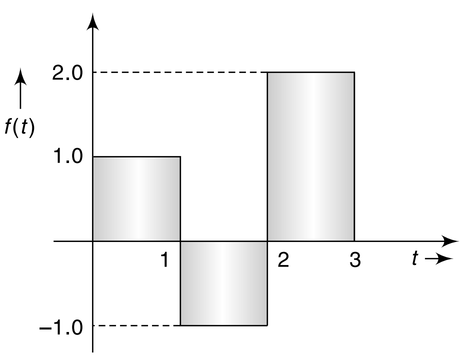
(a)


(b)


(c)


(d)


6.97

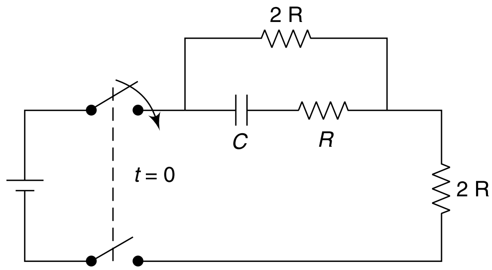


Laplace transform of $f(t)$ shown in the given figure is

(a) $F(s)=\frac{1}{s}-\frac{2}{s}e^{-s}+\frac{3}{s}e^{-s}$ (b) $F(s)=\frac{1}{s}-\frac{2}{s}e^{-s}+\frac{3}{s}e^{-2s}-\frac{2}{s}e^{-3s}$

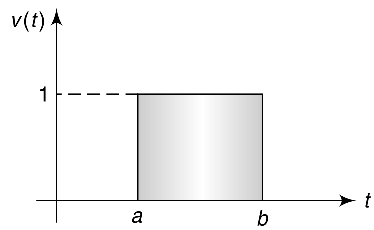
(c) $F(s)=\frac{1}{s}-\frac{e^{-s}}{s}+\frac{2}{s}e^{-2s}-\frac{2}{s}e^{-3s}$ (d) $F(s)=\frac{1}{s}+\frac{2}{s}e^{-s}-\frac{3}{s}e^{-s}$

6.98 The time constant of the circuit shown in the given figure is



(a) R C (b) R C (c) R C (d) R C

6.99 Consider the following functions for the rectangular voltage pulse shown in the given figure



(a) $v(t)=u(t-a)-u(t-b)$ (b) $v(t)=u(b-t)-u(a-t)$

(c) $v(t)=u(b-t).u(t-a)$ (d) $v(t)=u(a-t).u(t-b)$

6.100 If $F_1(s)=\dfrac{1}{s+3}$, $F_2(s)=\dfrac{2}{s^2+4}$; what is the Laplace transform of the product $F_1(s)\,F_2(s)$?

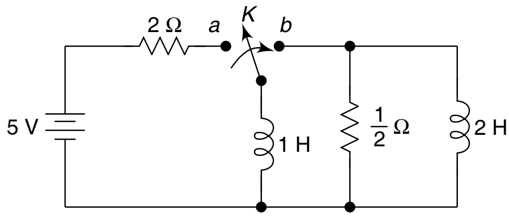
(a) $f(t)=\frac{1}{5}\left[e^{-t}+3\cos 2t-2\sin t\right]$ (b) $f(t)=\frac{1}{13}\left[2e^{-3t}+3\sin 2t-2\cos 2t\right]$

(c) $f(t)=\frac{1}{7}\left[e^{-2t}+2\sin 2t-\cos 2t\right]$ (d) $f(t)=\frac{1}{11}\left[e^{-2t}+\sin t-2\sin 2t\right]$

6.101 The impulse response of a linear network is given by e^{-2t}. Which one of the following gives its unit step response?

(a) $1-e^{-2t}$ (b) $e^{-t}-e^{-2t}$ (c) $\frac{1}{2}\left(1-e^{-2t}\right)$ (d) $\frac{1}{2}\left(e^{-t}-e^{-2t}\right)$

6.102 The network shown in the figure given above reaches a steady state with the switch K in position a. At $t = 0$, the switch is moved from a to b by a make-before-break mechanism. Assume the initial current in 2 H inductor as zero. What is the current in 1H inductor at $t = 0+$ and $t = \infty$, respectively?

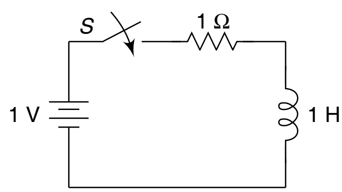


(a) 1 A and 0 A
(b) 2.5 A and 0 A
(c) 1 A and 2.5 A
(d) 2.5 A and 2.5 A

6.103 If $F(s) = \dfrac{2(s+1)}{s^2+2s+5}$, then what are the values of $f(0+)$ and $f(\infty)$ respectively?

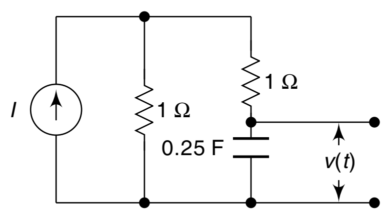
(a) 0, 2
(b) 2, 0
(c) 0, 1
(d) 2/5, 0

6.104 In the circuit shown in the figure, the switch S is closed at $t = 0$. Which one of the following gives the expression for the voltage across the inductance as a function of time?



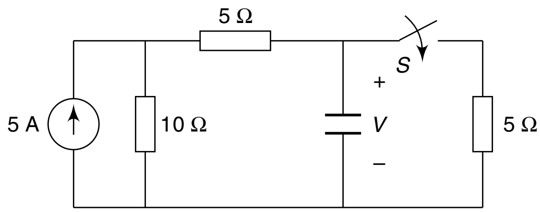
(a) $e^{-t/2}$
(b) $\dfrac{(1-e^{-t})}{2}$
(c) $(1-e^{-t})$
(d) e^{-t}

6.105 For the circuit shown in figure, the initial capacitor voltage is 2 V and I is a unit step function. Then, what is the expression for $v(t)$ for $t > 0$?



(a) $2+e^{-t}$
(b) $2-e^{-2t}$
(c) $1+e^{-2t}$
(d) $1-e^{-2t}$

6.106 In the circuit shown in the figure, steady state is reached with switch S open. Switch S is then closed at $t = 0$. What is the value of voltage V under steady state (when $t = \infty$)?



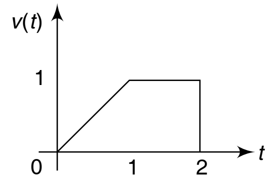
(a) 50 V (b) 12.5 V (c) 25 V (d) 0 V

6.107 If $f_1(t)$ and $f_2(t)$ have the widths (duration) T_1 and T_2 respectively, then what is the width (duration) of $f_1(t) * f_2(t)$ where * denotes convolution)?

(a) The larger of T_1 and T_2 (b) The smaller of T_1 and T_2

(c) $T_1 - T_2$ (d) $T_1 + T_2$

6.108 The Laplace transform of $v(t)$ shown in the figure is



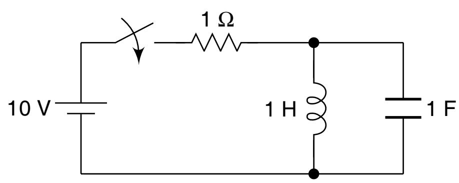
(a) $\frac{1}{s^2}\left(1-e^{-s}\right)-\frac{1}{s}e^{-2s}$ (b) $\frac{1}{s^2}\left(1-e^{s}\right)-\frac{1}{s}e^{2s}$

(c) $\frac{1}{s^2}\left(1+e^{-s}\right)+\frac{1}{s}e^{-2s}$ (d) $\frac{1}{s^2}\left(1+e^{s}\right)+\frac{1}{s}e^{2s}$

6.109 If $f(t)$ and $F(s)$ form the Laplace transform pair, then what is the Laplace transform of $f(t/t_0)$?

(a) $t_0 F(t_0 s)$ (b) $\frac{1}{t_0}F(t_0 s)$ (c) $t_0 F\left(\frac{1}{t_0}s\right)$ (d) $\frac{1}{t_0}F\left(\frac{1}{t_0}s\right)$

6.110 The switch in the circuit is closed at $t = 0$. The current through the battery at $t = 0+$ and $t = \infty$ is, respectively



(a) 10 A and 10 A (b) 0 A and 10 A

(c) 10 A and 0 A (d) 0 A and 0 A

6.111 The Laplace transform of the voltage across the capacitor of 0.5 F is

$$V(s) = \frac{s+1}{s^3+s^2+s+1}$$

Then the value of the current through the capacitor at $t = 0+$ is given by

(a) 0 A (b) 0.5 A (c) 1.0 A (d) 1 .5 A

6.112 If $u(t)$ and $\delta(t)$ are the step function and the impulse function respectively at $t = 0$, then the Laplace transform of the function $f(t) = u(t-1)\,\delta(t)$ is equal to

(a) 1 (b) $\frac{1}{s}$ (c) 0 (d) $\frac{1}{s+1}$

6.113 The step response of a system is $C(t) = 1 - 5e^{-t} + 10e^{-2t} - 6e^{-3t}$. The impulse response of the system is

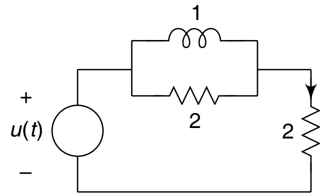
(a) $5e^{-t} - 20e^{-2t} + 18e^{-3t}$ (b) $5e^{t} + 20e^{2t} + 18e^{-3t}$

(b) $5e^{-t} + 20e^{-2t} + 18e^{-3t}$ (d) $5e^{-t} + 20e^{-2t} + 18e^{3t}$

6.114 The Laplace transform of $e^{\alpha t}\cos \alpha t$ is

(a) $\dfrac{(s+\alpha)}{(s+\alpha)^2 + \alpha^2}$ (b) $\dfrac{(s-\alpha)}{(s-\alpha)^2 + \alpha^2}$

(c) $\dfrac{(s+\alpha)}{(s-\alpha)^2 + \alpha^2}$ (d) $\dfrac{(s-\alpha)}{(s+\alpha)^2 + \alpha^2}$



6.115 For the circuit shown in the figure, the initial inductor current is 2A. The value of $i(t)$ for $t > 0$ is

(a) $0.5 - 0.75e^{-t}$ (b) $1 - e^{-t}$ (c) $0.5 - 0.25e^{-t}$ (d) $0.5 + 0.75e^{-t}$

6.116 Consider a system described by the transfer function $G(s) = \dfrac{2s+3}{s^2 + 2s + 5}$. It is subjected to an input $f(t) = 10u(t)$. The initial and final values of the response are given by

(a) 0, 2/3 (b) 1, 4 (c) 0, 6 (d) 0, 4

6.117 The impulse response of a linear time invariant system is given by

$$h(t) = 2e^{-t}u(t)$$

The unit step response is given by

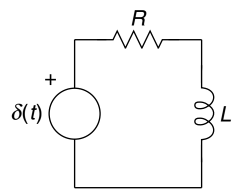
(a) $y(t) = 2\left(1 - e^{-t}\right)u(t)$ (b) $y(t) = 2\left(e^{-t} - 1\right)u(t)$

(c) $y(t) = 2\left(1 - e^{-2t}\right)u(t)$ (d) $y(t) = 2\left(2 - e^{-2t}\right)u(t)$

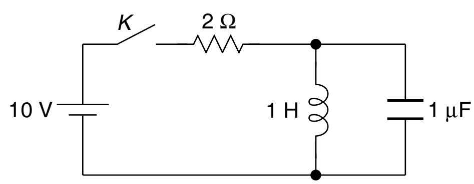
6.118 In the given circuit, if the inductor is initially relaxed, then the current in the circuit will be

(a) zero (b) $\dfrac{L}{R}\delta(t)$

(c) $\dfrac{1}{L}e^{-\frac{Rt}{L}}$ (d) $\dfrac{1}{L}\left(1 - e^{-\frac{Rt}{L}}\right)$



6.119 For the circuit shown in figure, the switch 'K' was closed for a long time till steady state conditions reached. At time $t = 0$, the switch 'K' is opened, then the current through inductor will be



(a) $5\cos 10t$ (b) $5\cos 100t$ (c) $5\cos 1000t$ (d) $5\cos 10000t$

6.120 The response of a system to a unit ramp input is $\frac{1}{2}t - \frac{1}{8}u(t) + \frac{1}{8}e^{-4t}$. Which one of the following is the unit impulse response of the system?

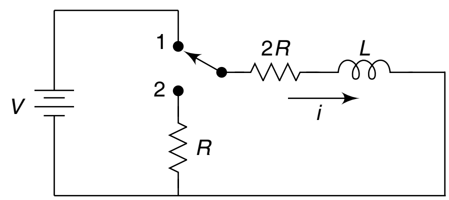
(a) $1 - e^{-4t}$ (b) $2\left(1 - e^{-4t}\right)$ (c) e^{-4t} (d) $2e^{-4t}$

6.121 The Laplace transform of current in an *RLC* series circuit with $R = 2\,\Omega$, $L = 1$ H and $C = \frac{1}{2}$ F is $I(s) = \frac{1}{s^2 + 2s + 2}$. The voltage across the inductor '*L*' will be

(a) $e^{-t}\sin t\,u(t)$ (b) $e^{-t}\cos t\,u(t)$

(c) $e^{-t}(\sin t + \cos t)\,u(t)$ (d) $e^{-t}(\cos t - \sin t)\,u(t)$

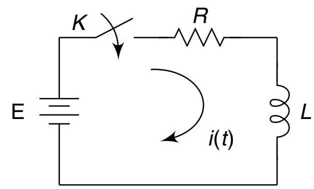
6.122 For the network shown in the figure, the initial position of switch '*S*' is '1'. After reaching steady-state, if the position of the switch is changed over to '2', the current '*i*' for $t \geq 0$ will be equal to



(a) $\frac{V}{2R}\exp\left(-\frac{Rt}{L}\right)$ (b) $\frac{V}{R}\exp\left(-\frac{2Rt}{L}\right)$

(c) $\frac{V}{R}\exp\left(-\frac{3Rt}{L}\right)$ (d) $\frac{V}{2R}\exp\left(-\frac{3Rt}{L}\right)$

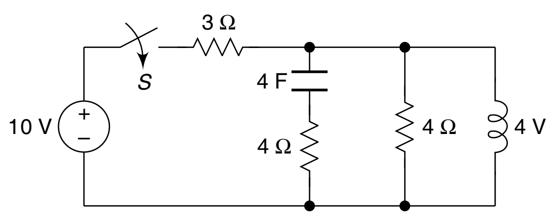
6.123 The correct value of the current $i(t)$ at any instant when *K* is switched on at $t = 0$ in the network shown in the given figure is



(a) $\frac{E}{R} + \frac{E}{R}e^{(R/L)t}$ (b) $\frac{E}{R} - \frac{E}{R}e^{(R/L)t}$

(c) $\frac{E}{R} + \frac{E}{R}e^{-(R/L)t}$ (d) $\frac{E}{R} - \frac{E}{R}e^{-(R/L)t}$

6.124 In the circuit shown in figure, the switch *S* is closed at $t = 0$. The voltage across the inductance at $t = 0+$, is



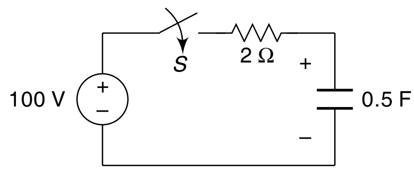
(a) 2V (b) 4V (c) –6 V (d) 8V

6.125 Consider the function, $F(s) = \dfrac{5}{s(s^2 + 3s + 2)}$ where $F(s)$ is the Laplace transform of the function $f(t)$.

The initial value of $f(t)$ is equal to

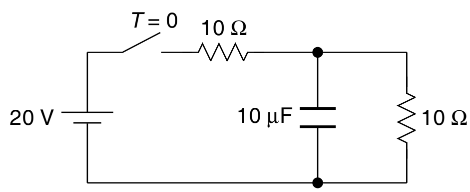
(a) 5 (b) $\frac{5}{2}$ (c) $\frac{5}{3}$ (d) 0

6.126 In the figure, the capacitor initially has a charge of 10 coulombs. The current in the circuit one second after the switch S is closed will be



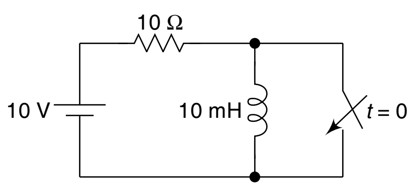
(a) 14.7 A (b) 18.5 A (c) 40.0 A (d) 50.0 A

6.127 In the figure given, the initial capacitor voltage is zero. The switch is closed at $t = 0$. The final steady-state voltage across the capacitor is



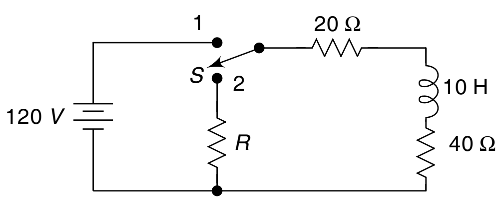
(a) 20 V (b) 10 V (c) 5 V (d) 0 V

6.128 The circuit shown in the figure is in steady state, when the switch is closed at $t = 0$. Assuming that the inductance is ideal, the current through the inductor at $t = 0+$ equals



(a) 0 A (b) 0.5 A (c) 1 A (d) 2 A

6.129 If, at $t = 0+$, the voltage across the coil is 120 V, the value of resistance R is



(a) 0 Ω (b) 20 Ω (c) 40 Ω (d) 60 Ω

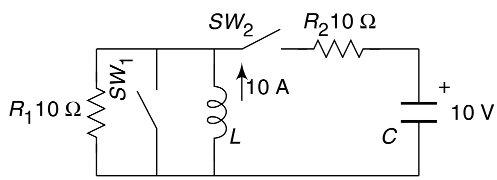
6.130 For the value obtained in Q. 129, the time taken for 95% of the stored energy to be dissipated is close to

(a) 0.10 second (b) 0.15 second (c) 0.50 second (d) 1.0 second

6.131 An ideal capacitor is charged to a voltage V_0 and connected at $t = 0$ across an ideal inductor L. (The circuit now consists of a capacitor and inductor alone). If we let $\omega_0 = \dfrac{1}{\sqrt{LC}}$, the voltage across the capacitor at time $t > 0$ is given by

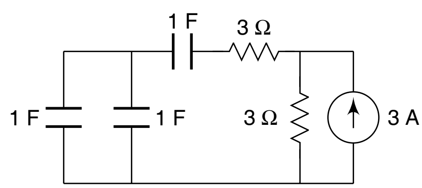
(a) V_0 (b) $V_0 \cos(\omega_0 t)$ (c) $V_0 \sin(\omega_0 t)$ (d) $V_0 e^{-\omega_0 t} \cos(\omega_0 t)$

6.132 In the circuit shown in the figure, switch SW_1 is initially CLOSED and SW_2 is OPEN. The inductor L carries a current of 10 A and the capacitor is charged to 10V with polarities as indicated. SW_2 is initially CLOSED at $t = 0–$ and SW_1 is OPENED at $t = 0$. The current through C and the voltage across L at $t = 0+$ is



(a) 55 A, 4.5 V (b) 5.5 A, 45 V (c) 45 A, 5.5 V (d) 5.5 A, 5.5 V

6.133 The time constant for the given circuit will be



(a) $\frac{1}{9}$ s (b) $\frac{1}{4}$ s (c) 4 s (d) 9 s

6.134 The Laplace transform of $i(t)$ is given by

$$I(s) = \frac{2}{s(1+s)}$$

As $t \to \infty$, the value of $i(t)$ tends to

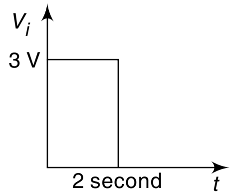
(a) 0 (b) 1 (c) 2 (d) ∞

6.135 In what range should $Re(s)$ remain so that the Laplace transform of the function $e^{(a+2)t+5}$ exits?

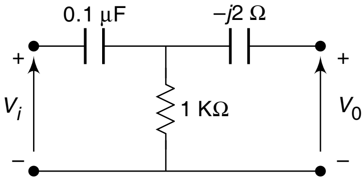
(a) $Re(s) > a + 2$ (b) $Re(s) > a + 7$

(c) $Re(s) < 2$ (d) $Re(s) > a + 5$

6.136 A square pulse of 3 V amplitude is applied to C-R circuit shown in the figure. The capacitor is initially uncharged. The output voltage V_0 at time $t = 2$ seconds is



(a)



(b)

(a) 3 V (b) –3 V (c) 4 V (d) –4 V

EXERCISES

6.1 (a) Find the initial values of the functions:

(i) $f(t) = e^{-at} \cos \omega t u(t)$ (ii) $F(s) = \dfrac{2(s+1)}{s^2 + 2s + 5}$

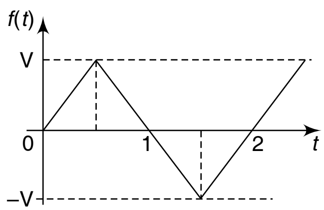
[(i) 1, (ii) 2]

(b) Find the final value of the functions:

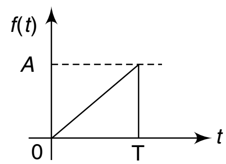
(ii) $F(s) = \dfrac{7}{s(s+3)^2}$ (ii) $F(s) = \dfrac{s-1}{(s+1)(s+2)}$

$\left[\text{(i) } \dfrac{7}{9}, \text{(ii) } 0\right]$

6.2 Obtain the Laplace transform of the following functions:



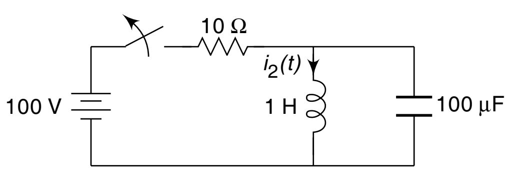
(i)



(ii)

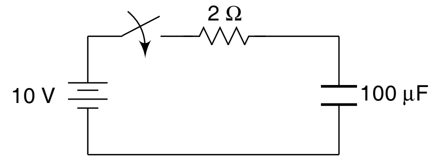
$$\left[\text{(i) } F(s) = \frac{A}{Ts^2}(1 - e^{-Ts} - se^{-Ts}) \text{ (ii) } F(s) = \frac{A}{Ts^2}(1 - e^{-Ts} - Te^{-Ts})\right]$$

6.3 In the network shown, the switch is closed and a steady state is reached in the network. At time t = 0, the switch is opened. Find an expression for the current through the inductor $i_2(t)$.



[10 cos 100t (A)]

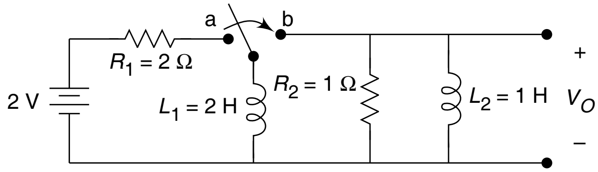
6.4 Find for the circuit shown, the current through C using Laplace transform. The switch is closed at $t = 0$ and the initial charge in the capacitor, i.e., at $t = 0$ is zero.



$$\left[5e^{-5000t} \text{ (A)}\right]$$

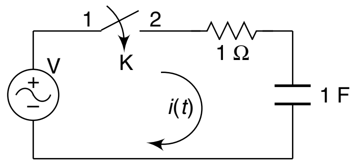
6.5 The circuit of a figure was initially in the steady state with the switch S in position a. At $t = 0$, the switch goes from a to b. Find an expression for the voltage $v_0(t)$ for $t > 0$. Take the initial current in the inductor L_2 to be zero.

$$\left[v_0(t) = \frac{1}{2}e^{-\frac{3}{2}t} \quad \text{(V)}\right]$$



6.6 In the circuit of the figure, the applied voltage is $v(t) = 10\sin(10t + \pi/6)$, $R = 1\,\Omega$, $C = 1$ F. Using Laplace Transformation, find complete solution for current $i(t)$. Switch K is closed at time $t = 0$. Assume zero charge across the capacitor before switching.

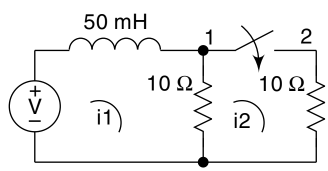
$$\left[i(t) = \frac{5}{101}(1 - 10\sqrt{3})e^{-t} + \frac{100}{\sqrt{101}}\cos(10t - 54°8')\text{(A)}\right]$$



6.7 A series RLC circuit, with $R = 5\,\Omega$, $L = 0.1$ H and $C = 500\,\mu$F, has a sinusoidal voltage source, $v = 1000 \sin 250t$. Find the resulting current if the switch is closed at $t = 0$.

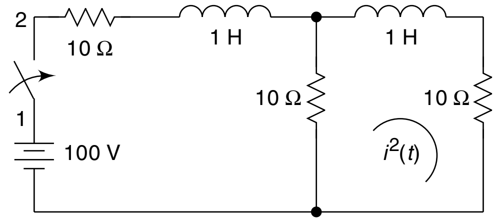
$$\left[i(t) = e^{-25t}(5.42\cos 139t + 1.89\sin 139t) + 5.65\sin(250t - 73.6^o)(A)\right]$$

6.8 The two-mesh network shown in the figure contains a sinusoidal voltage source, $v = 100\sin(200t + \phi)(\text{V})$. The switch is closed at an instant when the voltage is increasing at its maximum rate. Find the resulting mesh currents, with directions as shown in the figure.



$$[i_1(t) = 3.01^{e-100t} + 8.96\sin(200t - 63.4°)i_2(t) = 1.505e^{-100t} + 4.48\sin(200t - 63.4°)]$$

6.9 Find $i_2(t)$ for $t > 0$; assume the all initial conditions to be zero.

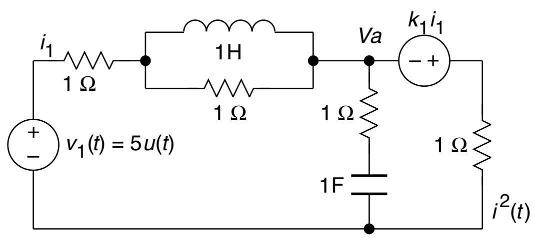


$$\left[i_2(t) = \frac{10}{3} + \frac{5}{3}e^{-30t} - 5e^{-10t} \text{ for } t > 0\right]$$

6.10 In the network shown,

(a) determine $V_a(t)$, using Laplace transform method if $k_1 = -3$.

(b) determine $i_2(t)$, using Laplace transform method if $k_1 = 3$.



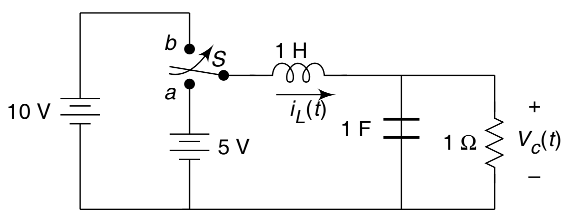
$$[(a)\ v_a(t) = 4 - e^{-0.75t}(1.5\cos 0.25t - 0.5\sin 0.25t)$$
$$(b)\ i_2(t) = -5 + 16.3375e^{-0.707t} - 1.3375e^{-0.707t}\ \text{(A)}]$$

6.11 The network shown in the figure, has reached steady state when the switch S moves from a to b.

(i) Determine initial values for $i_L(t)$ and $V_c(t)$ with switch in position b.

(ii) Determine $V_c(t)$ for $t > 0$. Sketch $V_c(t)$ as a function of time.

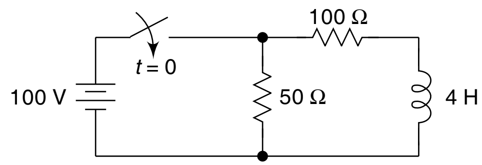
(iii) Determine damping ratio, undamped and damped natural frequencies.



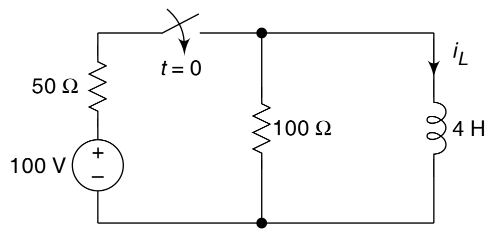
$$\left[\text{(i) 5 A, 5 V; (ii) } 15 - 10e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{20}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)(V)\right]$$

6.12 Find the source current after the switch is closed at $t = 0$. Take initial current to be zero.

$$\left[\left(3 - e^{-25t}\right)\text{(A)}\right]$$

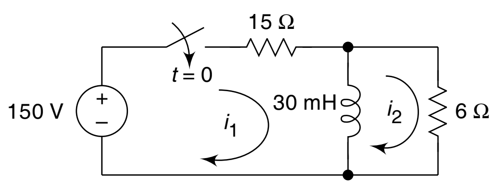


6.13 Find an expression for the current in the inductor at time t after the switch is closed. What is the final value of the current and how long will it take for the inductor current to reach 95% of its final value?

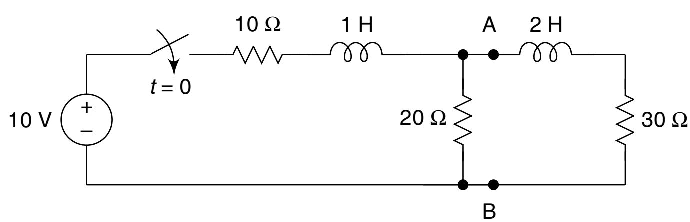


$$\left[2\left(1-e^{-50t/6}\right)(A);\ 2\text{A},\ 0.36\ \text{second}\right]$$

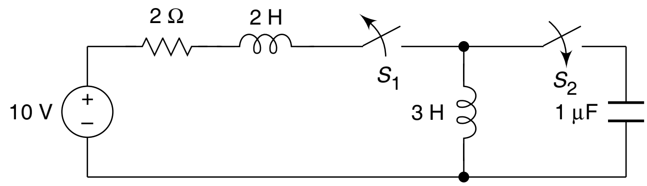
6.14 In the circuit, find the initial and final values of currents i_1 and i_2 when the switch is closed at $t = 0$. Use initial value and final value theorems. $[i_1(0) = 7.14\text{A}, i_1(\infty) = 10\text{A}; i_2(0) = 7.14\text{A}, i_2(\infty) = 0\text{A}]$



6.15 In the network shown in figure, the switch is closed at $t = 0$, prior to which the circuit is in zero state. Using Thevenin's theorem, transform the circuit to the left of points A and B into Thevenin equivalent in frequency domain and find the current in 30 Ω resistance. Convert the expression for current in time domain. $[0.1818 - 0.265e^{-13.14t} + 0.083e^{-41.86t}\ (\text{A})]$



6.16 The network shown in figure is in steady state with switch S_1 and S_2 open. At $t = t_1$, S_1 is opened and S_2 is closed. Find the current through the capacitor for $t \geq t_1$.

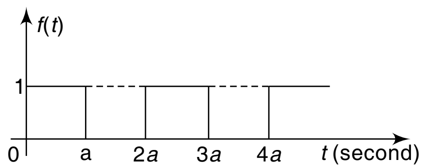


$[i(i) = 5 \cos \{0.577 \times 10^3 (t - t_1)\}$ for $t \geq t_1]$

6.17 An RC series circuit has $R = 2\ \Omega$, $C = 0.25$ F. Find the current response if the driving voltage is: (a) step voltage $2u(t-3)$, (b) ramp voltage $2r(t-3)$.

$$\left[\text{(a) } e^{-2(t-3)}\, u(t-3);\ \text{(b) } \frac{1}{2}\, u(t-3) - \frac{1}{2} e^{-2(t-3)}\, u(t-3)\right]$$

6.18 Show that the Laplace transform of the square wave is, $F(s) = \dfrac{1}{s(1+e^{-as})}$



6.19 Determine the current response of a series RL circuit with $R = 6\,\Omega$ and $L = 3$ H for each of the following driving voltages:

(a) a step voltage $2u(t-2)$

(b) a ramp voltage $2r(t-3)$

Assume that the circuit is initially relaxed.

$$\left[\text{(a) } \frac{1}{3}\left(1 - e^{-2(t-2)}\right) u(t-2);\ \text{(b) } \frac{2}{3}\left[2r(t-3) - \frac{1}{4}\, u(t-3) + \frac{1}{4} e^{-2(t-3)u(t-3)}\right]\right]$$

6.20 A series RL circuit has a resistor $R = 4\,\Omega$ and an inductor $L = 2$ H. A pulse of magnitude 10 V and duration 5 ms is applied to the circuit at $t = 3$ ms. Find $i(t)$. Assume that the circuit was initially relaxed.

$$\left[\frac{5}{2}\left[u(t - 0.003) - u(t - 0.008)\right] - \frac{5}{2}\left[e^{-2(t-0.003)}\, u(t - 0.003) - e^{-2(t-0.008)} u(t - 0.008)\right]\right]$$

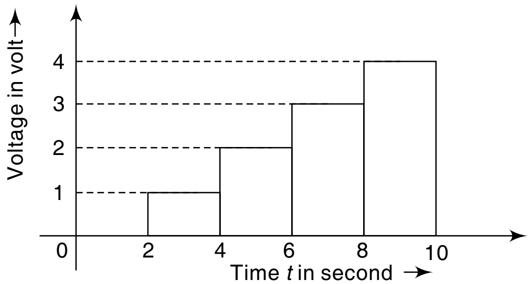
6.21 A voltage pulse of 20 V magnitude and 10 ms duration is applied to an RC circuit. Determine the current. Assume that the circuit was initially relaxed. Take $R = 10\,\Omega$ and $C = 10\ \mu$F.

$$\left[2\left[e^{-10^{-4}t} - e^{-10^{-4}(t-10^{-5})}\right] u(t - 10^{-5})\right]$$

6.22 A unit doublet voltage $\delta'(t-5)$ is applied at $t = 0$ to a series RLC circuit consisting of a resistor $R = 4\ \Omega$, $L = 1$ H and $C = \frac{1}{3}$ F. Determine $i(t)$. Assume that the circuit was initially relaxed.

$$\left[\delta(t-5) + \frac{1}{2}e^{-(t-5)}u(t-5) - \frac{9}{2}e^{-3(t-5)}u(t-5)\right]$$

6.23 Figure shows a staircase voltage waveform. Assuming that the staircase is not repeated, express its equation in terms of step functions. If this voltage is applied to a series RL circuit with $R = 4$ ohms and $L = 2$ H, find an expression for the resulting current $i(t)$; $i(0+) = 0$.



$$\left[\begin{array}{r} i(t) = \left[1 - e^{-2(t-2)}\right]u(t-2) + \left[1 - e^{-2(t-4)}\right]u(t-4) + \left[1 - e^{-2(t-6)}\right]u(t-6) + \left[1 - e^{-2(t-8)}\right]u(t-8) \\ -4\left[1 - e^{-2(t-10)}\right]u(t-10) \end{array}\right]$$

6.24 Find the current $i(t)$ in a series RLC circuit comprising resistor $R = 4\ \Omega$, inductor $L = 1$ henry and capacitor $C = \frac{1}{3}$ Farad when each of the following driving voltage is applied:

(c) ramp voltage $9r(t-2)$,

(d) step voltage $4u(t-3)$, and

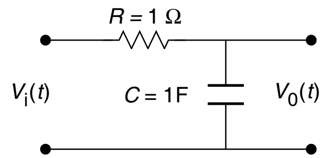
(e) impulse voltage $2\delta(t-1)$.

$$\left[\begin{array}{r} (a)\left[3 - \frac{9}{2}e^{-(t-2)} - \frac{3}{2}e^{-3(t-2)}\right]u(t-2);\quad (b)\ 2\left[e^{-(t-3)} - e^{-3(t-3)}\right]u(t-3);\quad c)\ 3e^{-3(t-1)}u \\ (t-1) - e^{-(t-1)}u(t-1) \end{array}\right]$$

6.25 Verify that the convolution between two functions $f_1(t) = 2u(t)$ *and* $f_2(t) = \exp(-3t)\,u(t)$ is $\frac{2}{3}\left[1 - \exp(-3t)\right]$; $t > 0$ where $u(t)$ is the unit step function.

6.26 Find the response of the network shown in figure when the input voltage is: (a) unit impulse, and (b) $v_i(t) = e^{-2t}$.

$$\left[\text{(a) } e^{-t};\ \text{(b) } \left(e^{-t} - e^{-2t}\right)\right]$$



SHORT-ANSWER TYPE QUESTIONS

6.1 (a) What do you understand by Complex Frequency? Give its physical significance.
(b) Define Laplace transform of a function $f(t)$. What are the advantages of Laplace transform?

or

Explain clearly the advantages of Laplace transform method over classical method of solving differential equation with constant co-efficient describing electrical network.
(c) State and deduce initial-value and final value theorems.
(d) Write notes on: Application of Laplace transform to network analysis.

6.2 Define unit-step, unit ramp and unit impulse functions and derive their Laplace transform from first principles.

6.3 (a) Find the current $i(t)$ if unit step voltage is applied to an RL circuit.

or

Derive an expression for the current response in an R–L series circuit excited with constant voltage source.
(b) Define the term 'time-constant' of a circuit. What is the physical significance of time-constant of a circuit? Find its value for R–L series circuit.

6.4 (a) Derive an expression for the decay current in an RC circuit excited by a unit step voltage. What is the time-constant of the circuit?
Also, determine the nature of the voltage response across the capacitor.
(b) Under what conditions an RC series circuit will act as (i) a Differentiator? (ii) an Integrator?

6.5 (a) Explain the terms critical resistance, damping ratio and frequency as applied to the study of RLC series circuit. How they help in simplifying the analysis of the circuit?
(b) Derive an expression for the current $i(t)$ flowing through an RLC series circuit. Explain with suitable sketches the variation of current with time under three conditions:
(I) Under damped,
(II) Critically damped,
(III) Over damped.

6.6 What do you understand by the impulse response of a network? Briefly explain its importance in network analysis.

6.7 What do you understand by transient and steady state parts of response? How can they be identified in a general solution?

or

Discuss the natural and steady state response of an electrical circuit with illustrative examples.

or

Write notes on: (a) Transient and steady state response (b) Free and forced response.

6.8 State and prove Convolution Theorem. What is the necessity of convolution theorem in circuit analysis?

6.9 What is Laplace transformation? Give reasons for its wide use in the electric circuit analysis.
6.10 Discuss the advantages of analysing the circuits using frequency domain rather than the time domain. How can the initial conditions of a circuit is incorporated using Laplace transform?
6.11 Explain why the lower limit of the Laplace transform integral $\left[\int_{0-}^{\infty} f(t)e^{-st}dt\right]$ is taken as 0– instead of 0+.
6.12 What is the Laplace transform of a function which is nonzero for $t < 0$?
6.13 Does every signal $f(t)$, such $f(t) = 0$ for $t < 0$, have a Laplace transform?
6.14 Define unit-step, unit ramp and unit impulse functions and derive their Laplace transform from first principles.
6.15 Define and sketch ramp, unit step and unit impulse functions.
6.16 Derive from the first principle the Laplace transform of a unit step function. Hence or otherwise, determine the Laplace transform of unit ramp function and unit impulse function.
6.17 Explain gate function. Obtain the equation of a gate function starting at origin and of duration T.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

6.1 (b)	6.2 (c)	6.3 (a)	6.4 (c)	6.5 (c)	6.6 (c)	6.7 (c)
6.8 (a)	6.9 (a)	6.10 (a)	6.11 (a)	6.12 (b)	6.13 (c)	6.14 (a)
6.15 (b)	6.16 (c)	6.17 (b)	6.18 (c)	6.19 (b)	6.20 (d)	6.21 (a)
6.22 (b)	6.23 (a)	6.24 (a)	6.25 (d)	6.26 (c)	6.27 (a)	6.28 (b)
6.29 (c)	6.30 (d)	6.31 (c)	6.32 (b)	6.33 (c)	6.34 (c)	6.35 (d)
6.36 (a)	6.37 (a)	6.38 (a)	6.39 (c)	6.40 (d)	6.41 (d)	6.42 (d)
6.43 (b)	6.44 (a)	6.45 (b)	6.46 (d)	6.47 (c)	6.48 (b)	6.49 (b)
6.50 (b)	6.51 (b)	6.52 (b)	6.53 (d)	6.54 (c)	6.55 (c)	6.56 (b)
6.57 (a)	6.58 (b)	6.59 (b)	6.60 (c)	6.61 (a)	562 (c)	6.63 (a)
6.64 (b)	6.65 (b)	6.66 (a)	6.67 (a)	6.68 (c)	6.69 (c)	6.70 (d)
6.71 (d)	6.72 (b)	6.73 (d)	6.74 (b)	6.75 (a)	6.76 (b)	6.77 (d)
6.78 (b)	6.79 (d)	6.80 (c)	6.81 (a)	6.82 (c)	6.83 (b)	6.84 (b)
6.85 (d)	6.86 (b)	6.87 (d)	6.88 (a)	6.89 (c)	6.90 (c)	6.91 (b)
6.92 (a)	6.93 (c)	6.94 (a)	6.95 (c)	6.96 (a)	6.97 (b)	6.98 (c)
6.99 (a)	6.100 (b)	6.101 (c)	6.102 (b)	6.103 (b)	6.104 (d)	6.105 (c)
6.106 (b)	6.107 (d)	6.108 (a)	6.109 (a)	6.110 (a)	6.111 (a)	6.112 (c)
6.113 (a)	6.114 (b)	6.115 (d)	6.116 (c)	6.117 (a)	6.118 (c)	6.119 (c)
6.120 (d)	6.121 (d)	6.122 (d)	6.123 (d)	6.124 (b)	6.125 (d)	6.126 (a)
6.127 (b)	6.128 (c)	6.129 (a)	6.130 (c)	6.131 (b)	6.132 (d)	6.133 (c)
6.134 (c)	6.135 (a)	6.136 (b)				

CHAPTER 7

Two-port Network

7.1 INTRODUCTION

A ***port*** is a pair of nodes across which a device can be connected. The voltage is measured across the pair of nodes and the current going into one node is the same as the current coming out of the other node in the pair. These pairs are entry (or exit) points of the network.

So, a network with two input terminals and two output terminals is called a ***four-terminal network*** or a ***two-port network***.

It is convenient to develop special methods for the systematic treatment of networks. In the case of a single-port linear active network, we obtained the Thevenin's equivalent circuit and the Norton's equivalent circuit. When a linear passive network is considered, it is convenient to study its behaviour relative to a pair of designated nodes.

In a two-port network, there are two voltage variables and two current variables. According to the choice of input and output port, these voltage and current variables can be arranged in different equations, giving rise to different port parameters.

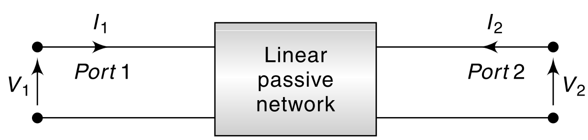


Figure 7.1 *Block diagram of a two-port network*

In this chapter, we will discuss the behaviours of two-port networks.

7.2 RELATIONSHIPS OF TWO-PORT VARIABLES

In order to describe the relationships among the port voltages and currents of an n-port network, 'n' number of linear equations is required. However, the choice of two independent and two dependent variables is dependent on the particular application.

For n-port network, the number of voltage and current variables is $2n$. The number of ways in which these $2n$ variables can be arranged in two groups of n each is $\frac{2n!}{n! \times n!} = \frac{2n!}{(n!)^2}$. So, there will be $\frac{2n!}{(n!)^2}$ types of port parameters.

For a two-port network (n = 2), there are six types of parameters as mentioned below:—

1. Open-Circuit Impedance Parameters (z-parameters),
2. Short-Circuit Admittance Parameters (y-parameters),
3. Transmission or Chain Parameters (T- parameters or $ABCD$ – parameters),
4. Inverse Transmission Parameters (T'-parameters),
5. Hybrid Parameters (h-parameters), and
6. Inverse Hybrid Parameters (g-parameters).

Note: Inverse parameters (T' and g) are not included in WBUT syllabus.

7.2.1 Open-Circuit Impedance Parameters (*z*-parameters)

The impedance parameters represent the relation between the voltages and the currents in the two-port network.

The impedance parameter matrix may be written as,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad \text{or} \quad \begin{matrix} V_1 = z_{11} I_1 + z_{12} I_2 \\ V_2 = z_{21} I_1 + z_{22} I_2 \end{matrix}$$

In this matrix equation, it is easily seen without even expanding the individual equations, that

$$z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2 = 0} = \text{Driving Point Impedance at Port-1.}$$

$$z_{12} = \left. \frac{V_1}{I_2} \right|_{I_1 = 0} = \text{Transfer Impedance}$$

$$z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2 = 0} = \text{Transfer Impedance}$$

$$z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1 = 0} = \text{Driving Point Impedance at Port-2}$$

It can be seen that the z-parameters correspond to the *driving point* and *transfer* impedances at each port with the other port having zero current (i.e. open circuit). Thus these parameters are also referred to as the open circuit parameters.

7.2.2 Short-Circuit Admittance Parameters (*y*-parameters)

The admittance parameters represent the relation between the currents and the voltages in the two-port network.

The admittance parameter matrix may be written as

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad \text{or} \quad \begin{aligned} I_1 &= y_{11}V_1 + y_{12}V_2 \\ I_2 &= y_{21}V_1 + y_{22}V_2 \end{aligned}$$

The parameters y_{11}, y_{12}, y_{21}, y_{22} can be defined in a similar manner, with either V_1 or V_2 on short circuit.

$$y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \text{Driving Point Admittance at Port-1}$$

$$y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \text{Transfer Admittance}$$

$$y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \text{Transfer Admittance}$$

$$y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \text{Driving Point Admittance at Port-2}$$

It can be seen that the y-parameters correspond to the *driving point* and *transfer* admittances at each port with the other port having zero voltage (i.e., short circuit). Thus these parameters are also referred to as the short circuit parameters.

7.2.3 Transmission Line Parameters (ABCD-parameters)

The *ABCD* parameters represent the relation between the input quantities and the output quantities in the two-port network. They are thus voltage-current pairs.

However, as the quantities are defined as an input-output relation, the output current is marked as going out rather than as coming into the port.

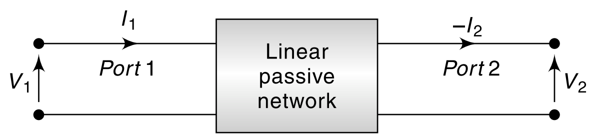


Figure 7.2 *Two-port current and voltage variables for calculation of transmission line parameters*

The transmission parameter matrix may be written as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \quad \text{or} \quad \begin{aligned} V_1 &= AV_2 - BI_2 \\ I_1 &= CV_2 - DI_2 \end{aligned}$$

The parameters A, B, C, D can be defined in a similar manner with either port 2 on short circuit or port 2 on open circuit.

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \text{Open Circuit Reverse Voltage Gain}$$

$$B = -\left.\frac{V_1}{I_2}\right|_{V_2=0} = \text{Short Circuit Transfer Impedance}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \text{Open Circuit Transfer Admittance}$$

$$D = -\left.\frac{I_1}{I_2}\right|_{V_2=0} = \text{Short Circuit Reverse Current Gain}$$

These parameters are known as transmission parameters as in a transmission line, the currents enter at one end and leaves at the other end, and we need to know a relation between the sending end quantities and the receiving end quantities.

7.2.4 Hybrid Parameters (*h*-parameters)

The hybrid parameters represent a mixed or hybrid relation between the voltages and the currents in the two-port network.

The hybrid parameter matrix may be written as

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} \quad \text{or} \quad \begin{aligned} V_1 &= h_{11}I_1 + h_{12}V_2 \\ I_2 &= h_{21}I_1 + h_{22}V_2 \end{aligned}$$

The h-parameters can be defined in a similar manner and are commonly used in some electronic circuit analysis.

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \text{Short Circuit Impedance at Port-1}$$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \text{Open Circuit Reverse Voltage Gain}$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \text{Short Circuit Current Gain}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \text{Open Circuit Output Admittance}$$

As the h-parameters are dimensionally mixed, they are also named mixed parameters. Transistor circuit models are generally represented by these parameters as the input impedance (h_{11}) and the short-circuit current gain (h_{21}) can be easily measured by making the output short-circuited.

7.3 CONDITIONS FOR RECIPROCITY AND SYMMETRY

A network is said to be reciprocal if the ratio of the response transform to the excitation transform is invariant to an interchange of the positions of the excitation and response of the network.

A two-port network will be reciprocal if the interchange of an ideal voltage source at one port with an ideal current source at the other port does not alter the ammeter reading.

A two-port network is said to be symmetrical if the input and output ports can be interchanged without altering the port voltages and currents.

1. Conditions in terms of z-parameters

Condition for Reciprocity We short circuit port 2 – 2′ and apply a voltage source V_s at port 1 – 1′.

Therefore, $V_1 = V_s$, $V_2 = 0$, $I_2 = -I_2'$

Writing the equations of z-parameters,

$$V_s = z_{11}I_1 - z_{12}I_2'$$

$$0 = z_{21}I_1 - z_{22}I_2'$$

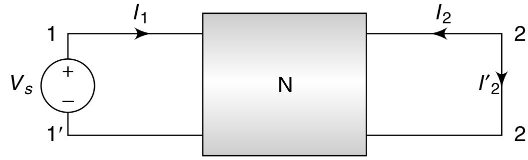


Fig. 7.3(a) *Reciprocal network*

Solving these two equations for I_2',

$$I_2' = V_s \frac{z_{21}}{z_{11}z_{22} - z_{12}z_{21}} \tag{7.1}$$

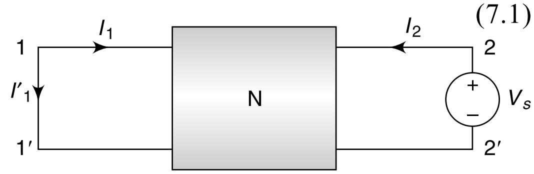


Fig. 7.3(b) *Reciprocal network*

Now, interchanging the positions of response and excitations, i.e., shorting port 1 – 1′ and applying V_s at port 2 – 2′; $V_1 = 0$, $V_2 = V_s$, $I_1 = I_1'$

Writing the equations of z-parameters,

$$0 = -z_{11}I_1' + z_{12}I_2$$

$$V_s = -z_{21}I_1' + z_{22}I_2$$

Solving these two equations for I_1',

$$I_1' = V_s \frac{z_{12}}{z_{11}z_{22} - z_{12}z_{21}} \tag{7.2}$$

For the two-port network to be reciprocal, from Eq. (7.1) and Eq. (7.2), we have the condition as,

$$\boxed{\mathbf{z_{12} = z_{21}}}$$

Condition for Symmetry

Applying a voltage V_s at port 1 – 1′ with port 2 – 2′ open, we have the equation,

$$V_s = z_{11}I_1 - z_{12}\cdot 0 = z_{11}I_1 \quad \Rightarrow \quad \left.\frac{V_s}{I_1}\right|_{I_2=0} = z_{11} \tag{7.3}$$

Now, applying a voltage V_s at port 2 – 2′ with port 1 – 1′ open, we have the equation,

$$V_s = z_{21}\cdot 0 + z_{22}I_2 = z_{22}I_2 \quad \Rightarrow \quad \left.\frac{V_s}{I_2}\right|_{I_1=0} = z_{22} \tag{7.4}$$

For the network to be symmetrical, the voltages and currents should be same. From Eq. (7.3) and Eq. (7.4), we have the condition for symmetry as,

$$\boxed{\mathbf{z_{11} = z_{22}}}$$

2. Conditions in terms of y-parameters

Condition for Reciprocity

From Fig. 7.3(a), writing the y-parameter equations,

$$\begin{aligned} I_1 &= y_{11} V_s \\ -I_2' &= y_{21} V_s \end{aligned} \quad \Rightarrow \quad -\frac{I_2'}{V_s} = y_{21} \tag{7.5}$$

From Fig. 7.3(b), writing the y-parameter equations,

$$\begin{aligned} -I_1' &= y_{12} V_s \\ I_2 &= y_{22} V_s \end{aligned} \quad \Rightarrow \quad -\frac{I_1'}{V_s} = y_{12} \tag{7.6}$$

From the principle of reciprocity, the condition for reciprocity is,

$$\boxed{\mathbf{y}_{12} = \mathbf{y}_{21}}$$

Condition for Symmetry

As already stated, a two-port network is said to be symmetric if the ports can be interchanged without changing the port voltages and currents and thus the condition of symmetry becomes,

$$\boxed{\mathbf{y}_{11} = \mathbf{y}_{22}}$$

3. Conditions in terms of ABCD-parameters

Condition for Reciprocity

From Fig. 7.3(a), writing the $ABCD$-parameter equations,

$$\begin{aligned} V_s &= A \cdot 0 - B(-I_2') = BI_2' \\ I_1 &= C \cdot 0 - D(-I_2') = DI_2' \end{aligned} \quad \Rightarrow \quad \frac{I_2'}{V_s} = \frac{1}{B} \tag{7.7}$$

From Fig. 7.3(b), writing the $ABCD$-parameter equations,

$$\begin{aligned} 0 &= AV_s - BI_2 \\ -I_1' &= CV_s - DI_2 \end{aligned} \quad \Rightarrow \quad \frac{I_1'}{V_s} = \frac{AD - BC}{B} \tag{7.8}$$

From the principle of reciprocity, the condition for reciprocity is, $\dfrac{1}{B} = \dfrac{(AD - BC)}{B}$

$$\boxed{\mathbf{(AD - BC) = 1}}$$

Condition for Symmetry

From Eq. (7.7), $I_1 = DI_2' = D\dfrac{V_s}{B}$ (7.9)

From Eq. (7.8), $I_2 = \dfrac{I_1' + CV_s}{D} = \dfrac{1}{D}\left\{V_s\left(\dfrac{AD - BC}{B}\right) + CV_s\right\} = V_s\dfrac{A}{B}$ (7.10)

From Eq. (7.9) and Eq. (7.10), we have the condition for symmetry as,

$$\boxed{\mathbf{A = D}}$$

4. Conditions in terms of h-parameters

Condition for Reciprocity

From Fig. 7.3(a), writing the h-parameter equations,

$$\left.\begin{aligned} V_s &= h_{11}I_1 + h_{12}\cdot 0 = h_{11}I_1 \\ -I_2' &= h_{21}I_1 + h_{22}\cdot 0 = h_{21}I_1 \end{aligned}\right\} \Rightarrow \frac{I_2'}{V_s} = -\frac{h_{21}}{h_{11}} \tag{7.11}$$

From Fig. 7.3(b), writing the h-parameter equations,

$$\left.\begin{aligned} 0 &= -h_{11}I_1' + h_{12}V_s \\ I_2 &= -h_{21}I_1' + h_{22}V_s \end{aligned}\right\} \Rightarrow \frac{I_1'}{V_s} = \frac{h_{12}}{h_{11}} \tag{7.12}$$

From the principle of reciprocity, the condition for reciprocity is,

$$\boxed{\mathbf{h_{12} = -h_{21}}}$$

From Eq. (7.11), $I_1 = \dfrac{V_s}{h_{11}}$ (7.13)

From Eq. (7.12), $I_2 = -h_{21}\left(\dfrac{h_{12}}{h_{11}}V_s\right) + h_{22}V_s = V_s\dfrac{h_{11}h_{22} - h_{12}h_{21}}{h_{11}}$ (7.14)

From Eq. (7.13) and Eq. (7.14), we have the condition for symmetry as,

$$\boxed{(h_{11}h_{22} - h_{12}h_{21}) = 1}$$

Table 7.1 *Conditions of Reciprocity and Symmetry in terms of different Two-Port Parameters*

Parameter	*Condition of Reciprocity*	*Condition of Symmetry*
z	$z_{12} = z_{21}$	$z_{11} = z_{22}$
y	$y_{12} = y_{21}$	$y_{11} = y_{22}$
T ($ABCD$)	$(AD - BC) = 1$	$A = D$
h	$h_{12} = -h_{21}$	$(h_{11}h_{22} - h_{12}h_{21}) = 1$

7.4 INTERRELATIONSHIPS BETWEEN TWO-PORT PARAMETERS

Each type of two-port parameter has its own utility and is suited for certain specific applications. However, it is sometime necessary to convert one set of parameters to another. It is possible through simple mathematical manipulations to convert one set to any of the remaining sets. It is discussed below.

1. z-parameters in Terms of Other Parameters

(a) In terms of y-parameters

The z-parameter equations are,

$$V_1 = z_{11}I_1 + z_{12}I_2 \tag{7.15}$$

$$V_2 = z_{21}I_1 + z_{22}I_2$$

The y-parameter equations are,

$$I_1 = y_{11}V_1 + y_{12}V_2 \qquad (7.16)$$

$$I_2 = y_{21}V_1 + y_{22}V_2$$

From Eq. (7.16), $V_2 = \dfrac{I_2}{y_{22}} - \dfrac{y_{21}}{y_{22}}V_1$; substituting this in first equation,

$$I_1 = y_{11}V_1 + y_{12}\left(\frac{I_2}{y_{22}} - \frac{y_{21}}{y_{22}}V_1\right) \quad \text{or} \quad V_1 = \frac{y_{22}}{\Delta y}I_1 - \frac{y_{12}}{\Delta y}I_2 \qquad (7.17)$$

where, $\Delta y = (y_{11}y_{22} - y_{12}y_{21})$

Substituting this value in second equation of Eq. 7.16

$$I_2 = y_{21}\left(\frac{y_{22}}{\Delta y}I_1 - \frac{y_{12}}{\Delta y}I_2\right) + y_{22}V_2 \quad \text{or,} \quad V_2 = -\frac{y_{21}}{\Delta y}I_1 + \frac{y_{11}}{\Delta y}I_2 \qquad (7.18)$$

Comparing Eqs (7.15), (7.17) and (7.18), we get,

$$\boxed{z_{11} = \frac{y_{22}}{\Delta y};\ z_{12} = -\frac{y_{12}}{\Delta y};\ z_{21} = -\frac{y_{21}}{\Delta y};\ z_{22} = \frac{y_{11}}{\Delta y}}$$

(b) In terms of transmission parameters

The Transmission parameter equations are,

$$V_1 = AV_2 - BI_2$$

$$I_1 = CV_2 - DI_2 \qquad (7.19)$$

From second equation of Eq. (7.19),

$$V_2 = \left(\frac{1}{C}\right)I_1 + \left(\frac{D}{C}\right)I_2 \qquad (7.20)$$

From first equation of Eq. (7.19),

$$V_1 = A\left[\left(\frac{1}{C}\right)I_1 + \left(\frac{D}{C}\right)I_2\right] - BI_2 = \left(\frac{A}{C}\right)I_1 + \left(\frac{AD - BC}{C}\right)I_2 \qquad (7.21)$$

Comparing Eq. (7.20) and (7.21) with Eq. (7.15), we get,

$$\boxed{z_{11} = \frac{A}{C};\ z_{12} = \frac{AD - BC}{C} = \frac{\Delta T}{C};\ z_{21} = \frac{1}{C};\ z_{22} = \frac{D}{C}}$$

(c) In terms of hybrid parameters

The hybrid parameter equations are,

$$V_1 = h_{11}I_1 + h_{12}V_2$$

$$I_2 = h_{21}I_1 + h_{22}V_2 \qquad (7.22)$$

From second equation, $V_2 = \left(-\dfrac{h_{21}}{h_{22}}\right)I_1 + \left(\dfrac{1}{h_{22}}\right)I_2 \qquad (7.23)$

From first equation, $V_1 = h_{11}I_1 + h_{12}\left[\left(-\dfrac{h_{21}}{h_{22}}\right)I_1 + \left(\dfrac{1}{h_{22}}\right)I_2\right] = \left(\dfrac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}}\right)I_1 + \left(\dfrac{h_{12}}{h_{22}}\right)I_2 \qquad (7.24)$

Comparing Eqs (7.23) and (7.24) with Eq. (7.15), we get,

$$\boxed{z_{11} = \frac{h_{11}h_{22} - h_{12}h_{21}}{h_{22}} = \frac{\Delta h}{h_{22}};\ z_{12} = \frac{h_{12}}{h_{22}};\ z_{21} = -\frac{h_{21}}{h_{22}};\ z_{22} = \frac{1}{h_{22}}}$$

Similarly, the inter-relation of the other parameter in terms of the remaining parameters is obtained by writing the remaining parameter equations in the same format as those of the other parameter; and comparing the co-efficients of the two sets of equations, a relation is obtained.

A summary of the relationships between impedance z-parameters, admittance y-parameters, hybrid h-parameters, and transmission $ABCD$-parameters is shown in Table where $\Delta z = (z_{11}z_{22} - z_{12}z_{21})$, $\Delta h = (h_{11}h_{22} - h_{12}h_{21})$, $\Delta T = (AD - BC)$, $\Delta T' = (A'D' - B'C')$, and $\Delta g = (g_{11}g_{22} - g_{12}g_{21})$.

Table 7.2 *Interrelationships between Two-Port Parameters*

	[z]	*[y]*	*[ABCD]*	*[A′B′C′D′]*	*[h]*	*[g]*
$[z]$	$\begin{matrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{matrix}$	$\begin{matrix} \frac{y_{22}}{\Delta y} & -\frac{y_{12}}{\Delta y} \\ \frac{-y_{21}}{\Delta y} & \frac{y_{11}}{\Delta y} \end{matrix}$	$\begin{matrix} \frac{A}{C} & \frac{\Delta T}{C} \\ \frac{1}{C} & \frac{D}{C} \end{matrix}$	$\begin{matrix} \frac{D'}{C'} & \frac{1}{C'} \\ \frac{\Delta T'}{C'} & \frac{A'}{C'} \end{matrix}$	$\begin{matrix} \frac{\Delta h}{h_{22}} & \frac{h_{12}}{h_{22}} \\ -\frac{h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{g_{11}} & -\frac{g_{12}}{g_{11}} \\ \frac{g_{21}}{h_{11}} & \frac{\Delta g}{g_{11}} \end{matrix}$
$[y]$	$\begin{matrix} \frac{z_{22}}{\Delta z} & -\frac{z_{12}}{\Delta z} \\ -\frac{z_{21}}{\Delta z} & \frac{z_{11}}{\Delta z} \end{matrix}$	$\begin{matrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{matrix}$	$\begin{matrix} \frac{D}{B} & -\frac{\Delta T}{B} \\ -\frac{1}{B} & \frac{A}{B} \end{matrix}$	$\begin{matrix} \frac{A'}{B'} & -\frac{1}{B'} \\ -\frac{\Delta T'}{B'} & \frac{D'}{B'} \end{matrix}$	$\begin{matrix} \frac{1}{h_{11}} & -\frac{h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta h}{h_{11}} \end{matrix}$	$\begin{matrix} \frac{\Delta g}{g_{22}} & \frac{g_{12}}{g_{22}} \\ -\frac{g_{21}}{g_{22}} & \frac{1}{g_{22}} \end{matrix}$
$[ABCD]$	$\begin{matrix} \frac{z_{11}}{z_{21}} & \frac{\Delta z}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{matrix}$	$\begin{matrix} -\frac{y_{22}}{y_{21}} & -\frac{1}{y_{21}} \\ -\frac{\Delta y}{y_{21}} & -\frac{y_{11}}{y_{21}} \end{matrix}$	$\begin{matrix} A & B \\ C & D \end{matrix}$	$\begin{matrix} \frac{D'}{\Delta T'} & \frac{B'}{\Delta T'} \\ \frac{C'}{\Delta T'} & \frac{A'}{\Delta T'} \end{matrix}$	$\begin{matrix} -\frac{\Delta h}{h_{21}} & -\frac{h_{11}}{h_{21}} \\ -\frac{h_{22}}{h_{21}} & -\frac{1}{h_{21}} \end{matrix}$	$\begin{matrix} \frac{1}{g_{21}} & -\frac{g_{22}}{g_{21}} \\ \frac{g_{11}}{g_{21}} & \frac{\Delta g}{g_{21}} \end{matrix}$
$[A'B'C'D']$	$\begin{matrix} \frac{z_{22}}{z_{12}} & \frac{\Delta z}{z_{12}} \\ \frac{1}{z_{12}} & \frac{z_{11}}{z_{12}} \end{matrix}$	$\begin{matrix} -\frac{y_{11}}{y_{12}} & -\frac{1}{y_{12}} \\ -\frac{\Delta y}{y_{12}} & -\frac{y_{22}}{y_{12}} \end{matrix}$	$\begin{matrix} \frac{D}{\Delta T} & \frac{B}{\Delta T} \\ \frac{C}{\Delta T} & \frac{A}{\Delta T} \end{matrix}$	$\begin{matrix} A' & B' \\ C' & D' \end{matrix}$	$\begin{matrix} \frac{1}{h_{22}} & \frac{h_{11}}{h_{12}} \\ \frac{h_{22}}{h_{12}} & \frac{\Delta h}{h_{12}} \end{matrix}$	$\begin{matrix} -\frac{\Delta g}{g_{12}} & -\frac{g_{22}}{g_{12}} \\ -\frac{g_{11}}{g_{12}} & -\frac{1}{g_{12}} \end{matrix}$
$[h]$	$\begin{matrix} \frac{\Delta z}{z_{22}} & \frac{z_{12}}{z_{22}} \\ -\frac{z_{21}}{z_{22}} & \frac{1}{z_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{y_{11}} & -\frac{y_{12}}{y_{11}} \\ \frac{y_{21}}{y_{11}} & \frac{\Delta y}{y_{11}} \end{matrix}$	$\begin{matrix} \frac{B}{D} & \frac{\Delta T}{D} \\ -\frac{1}{D} & \frac{C}{D} \end{matrix}$	$\begin{matrix} \frac{B'}{A'} & \frac{1}{A'} \\ -\frac{\Delta T'}{A'} & \frac{D'}{B'} \end{matrix}$	$\begin{matrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{matrix}$	$\begin{matrix} \frac{g_{22}}{\Delta g} & -\frac{g_{12}}{\Delta g} \\ -\frac{g_{211}}{\Delta g} & \frac{g_{11}}{\Delta g} \end{matrix}$
$[g]$	$\begin{matrix} \frac{1}{z_{11}} & -\frac{z_{12}}{z_{11}} \\ \frac{z_{21}}{z_{11}} & \frac{\Delta z}{z_{11}} \end{matrix}$	$\begin{matrix} \frac{\Delta y}{y_{22}} & \frac{y_{12}}{y_{22}} \\ -\frac{y_{21}}{y_{22}} & -\frac{1}{y_{22}} \end{matrix}$	$\begin{matrix} \frac{C}{A} & -\frac{\Delta T}{A} \\ \frac{1}{A} & \frac{B}{A} \end{matrix}$	$\begin{matrix} \frac{C'}{D'} & -\frac{1}{D'} \\ \frac{\Delta T'}{D'} & \frac{B'}{D'} \end{matrix}$	$\begin{matrix} \frac{h_{22}}{\Delta h} & -\frac{h_{12}}{\Delta h} \\ -\frac{h_{21}}{\Delta h} & \frac{h_{11}}{\Delta h} \end{matrix}$	$\begin{matrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{matrix}$

7.5 INTERCONNECTION OF TWO-PORT NETWORKS

In certain applications, it becomes necessary to connect the two-port networks together. The common connections are (a) series, (b) parallel and (c) cascade.

(a) Series Connection of Two-port Networks

As in the case of elements, a series connection is defined when the currents in the series elements are equal and the voltages add up to give the resultant voltage.

In the case of two-port networks, this property must be applied individually to each of the ports. Thus, if we consider 2 networks ***r*** and ***s*** connected in series

At port 1,

$$I_{r1} = I_{s1} = I_1, \text{ and } V_{r1} + V_{s1} = V_1$$

Similarly, at port 2,

$$I_{r2} = I_{s2} = I_2 \text{ and } V_{r2} + V_{s2} = V_2$$

The two networks, ***r*** and ***s*** can be connected in the following manner to be in series with each other.

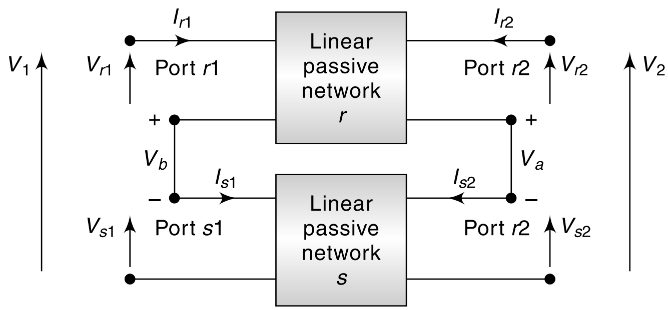


Figure 7.4 *Series connection of two-port networks*

Under these conditions,

$$V_1 = (V_{r1} + V_{s1}) = (z_{11r} + z_{11s})I_1 + (z_{12r} + z_{12s})I_2$$

$$V_2 = (V_{r2} + V_{s2}) = (z_{21r} + z_{21s})I_1 + (z_{22r} + z_{22s})I_2$$

It is seen that, the ***resultant impedance parameter matrix for the series combination is the addition of the two individual impedance matrices.***

$$[Z] = [Zr] + [Zs]$$

Note: In the interconnection of series networks, there is a strong requirement of isolation, since the ground node of upper network form the non-ground node of the lower network. For the port properties to be valid, the voltages V_a and V_b must be identically zero for the two networks r and s to be connected in series. If V_a and V_b are not zero, then by connecting the two ports there will be a circulating current and port property of the individual networks r and s will be violated.

(b) Parallel Connection of Two-port Networks

As in the case of elements, a parallel connection is defined when the voltages in the parallel elements are equal and the currents add up to give the resultant current.

In the case of two-port networks, this property must be applied individually to each of the ports. Thus, if we consider 2 networks ***r*** and ***s*** connected in parallel,

At port 1,

$$I_{r1} + I_{s1} = I_1, \text{ and } V_{r1} = V_{s1} = V_1$$

Similarly, at port 2,

$$I_{r2} + I_{s2} = I_2 \text{ and } V_{r1} = V_{s1} = V_1$$

The two networks, ***r*** and ***s*** can be connected in following manner to be in parallel with each other. Under these conditions,

$$I_1 = (I_{r1} + I_{s1}) = (y_{11r} + y_{11s})V_1 + (y_{12r} + y_{12s})V_2$$

$$I_2 = (I_{2r} + I_{2s}) = (y_{21r} + y_{21s})V_1 + (y_{22r} + y_{22s})V_2$$

It is seen that, the *resultant admittance parameter matrix for the parallel combination is the addition of the two individual admittance matrices.*

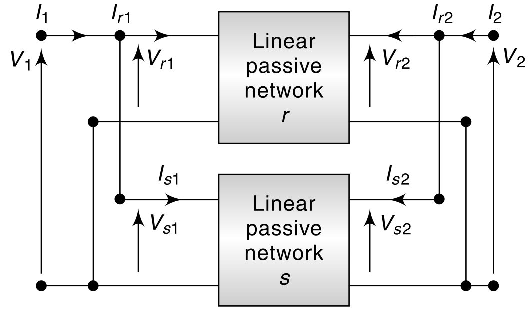


Figure 7.5 *Parallel connection of two-port networks*

$$[Y] = [Yr] + [Ys]$$

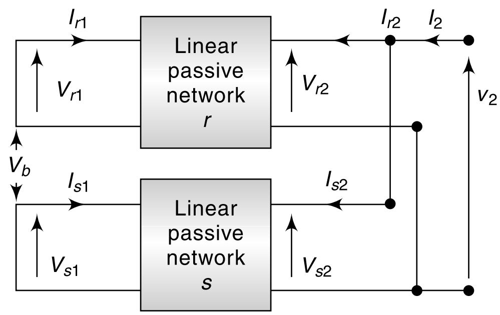


Figure 7.6(a) $V_b = 0$

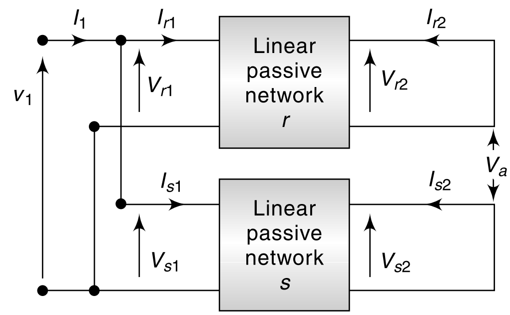


Figure 7.6(b) $V_a = 0$

Note: As in series connection, parallel connection is also possible under the condition that $V_a = V_b = 0$; otherwise they cannot be connected in parallel as that will violate the port properties.

(c) Cascade Connection of Two-port Networks

A cascade connection is defined when the output of one network becomes the input to the next network.

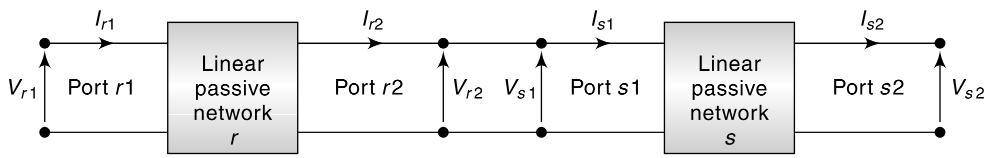


Figure 7.7 *Cascade connection of two-port network*

It can be easily seen that $I_{r2} = I_{s1}$ and $V_{r2} = V_{s1}$.

Therefore it can easily be seen that the *ABCD* parameters are the most suitable to be used for this connection.

$$\begin{bmatrix} V_{r1} \\ I_{r1} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \begin{bmatrix} V_{r2} \\ I_{r2} \end{bmatrix}, \quad \begin{bmatrix} V_{s1} \\ I_{s1} \end{bmatrix} = \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} \begin{bmatrix} V_{s2} \\ I_{s2} \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} V_{r1} \\ I_{r1} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \begin{bmatrix} V_{r2} \\ I_{r2} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \begin{bmatrix} V_{s1} \\ I_{s1} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} \begin{bmatrix} V_{s2} \\ I_{s2} \end{bmatrix}$$
$$= \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

Thus it is seen that the *overall ABCD matrix is the product of the two individual ABCD matrices.* This is a very useful property in practice, especially when analyzing transmission lines.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix}$$

7.6 TWO-PORT NETWORK FUNCTIONS

Two-port network functions are broadly divided into two groups:

1. Transfer function, and
2. Driving point functions.

7.6.1 Transfer Function

It is defined as the ratio of an output transform to an input transform, with zero initial condition and with no internal energy sources ecxcept the controlled sources.

For a two-port network, having the variables $I_1(s)$, $I_2(s)$, $V_1(s)$ and $V_2(s)$, the transfer function can take the following four forms:

Voltage Transfer Function $\quad G_{12}(s) = \dfrac{V_1(s)}{V_2(s)}; G_{21}(s) = \dfrac{V_2(s)}{V_1(s)}$

Current Transfer Function $\quad \alpha_{12}(s) = \dfrac{I_1(s)}{I_2(s)}; \alpha_{21}(s) = \dfrac{I_2(s)}{I_1(s)}$

Transfer Impedance Function $\quad Z_{12}(s) = \dfrac{V_1(s)}{V_2(s)}; Z_{21}(s) = \dfrac{V_2(s)}{V_1(s)}$

Transfer Admittance Function $\quad Y_{12}(s) = \dfrac{I_1(s)}{I_2(s)}; Y_{21}(s) = \dfrac{I_2(s)}{I_1(s)}$

Note: *(i) For a one-port network, Z(s) = 1/Y(s); but for a two-port network, in general* $Z_{12} \neq 1/Y_{12}$; $G_{12} \neq 1/\alpha_{12}$;

(ii) Z and Y functions will becomes z and y parameters under the conditions of open-circuits or short-circuits, respectively.

7.6.2 Driving Point Function

It takes two forms:

Driving Point Impedance [Z(s)] For a two-port newtork in zero state with no internal energy sourceds, the driving point impedance s the ratio of transform voltage at any port to the transform current at the same port.

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)}; Z_{22}(s) = \frac{V_2(s)}{I_2(s)}$$

Driving Point Admittance [Y(s)] For a two-port network in zero state with no internal energy sources, the driving point admittance is the ratio of transform current at any port to the transform voltage at the same port

$$Y_{11}(s) = \frac{I_1(s)}{V_1(s)}; Y_{22}(s) = \frac{I_2(s)}{V_2(s)}$$

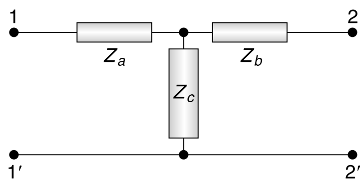
Note: *(i) Driving point impedance and admittance functions together are known as immittance function.*

(2) Z and Y functions will becomes z and y parameters under the conditions of open circuits or short circuits,

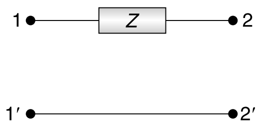
SOLVED PROBLEMS

7.1 Find the Z and Y parameter for the networks shown in figure.

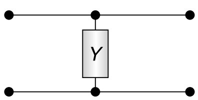
(a)



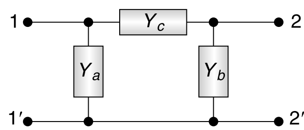
(b)



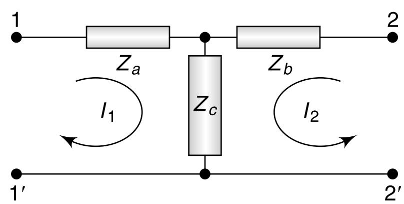
(c)



(d)



Solution



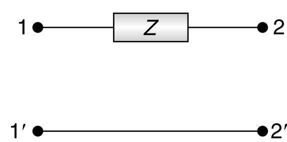
(a) By KVL, $(Z_a + Z_c)I_1 + Z_c I_2 = V_1$

and $Z_c I_1 + (Z_b + Z_c)I_2 = V_2$

Thus, the Z-parameters are:

$$z_{11} = (Z_a + Z_c),\ z_{12} = z_{21} = Z_c, \quad z_{22} = (Z_b + Z_c)$$

(b) By KCL,



$$I_1 = \frac{V_1 - V_2}{Z} = \frac{1}{Z}V_1 - \frac{1}{Z}V_2$$

and $$I_2 = \frac{V_2 - V_1}{Z} = -\frac{1}{Z}V_1 + \frac{1}{Z}V_2$$

Thus, the y-parameters are,

$$y_{11} = \frac{1}{Z} = y_{22} \quad y_{12} = y_{21} = -\frac{1}{Z}$$

Since, $\Delta y = y_{11}y_{22} - y_{12}y_{21} = 0$, the z-parameters do not exist for this network.

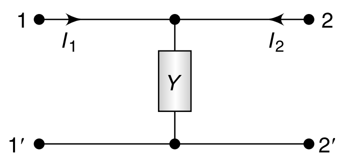
(c) By KVL,

$$V_1 = \frac{I_1 + I_2}{Y} = V_2 \quad \text{or, } V_1 = \left(\frac{1}{Y}\right)I_1 + \left(\frac{1}{Y}\right)I_2 \quad \text{and} \quad V_2 = \left(\frac{1}{Y}\right)I_1 + \left(\frac{1}{Y}\right)I_2$$

Thus, the z-parameters are,

$$z_{11} = z_{22} = \frac{1}{Y} = z_{12} = z_{21}$$

Since, $\Delta z = z_{11}z_{22} - z_{12}z_{21} = 0$, the y-parameters do not exist for this network.

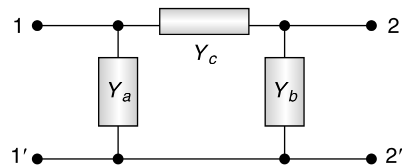


(d) By KCL,

$$I_1 = Y_aV_1 + (V_1 - V_2)Y_c = V_1(Y_a + Y_c) - V_2Y_c$$
$$I_2 = Y_bV_2 + (V_2 - V_1)Y_c = -V_1Y_c + V_2(Y_b + Y_c)$$

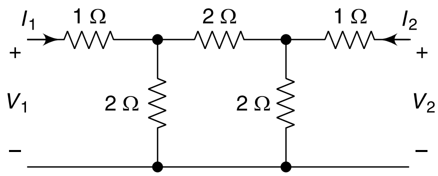
Thus, the y-parameters are:

$$y_{11} = Y_a + Y_c;\ y_{12} = y_{21} = -Y_c;\ y_{22} = Y_b + Y_c$$

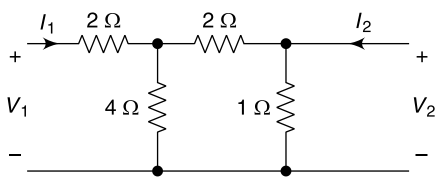


7.2 Obtain the z-parameters for the circuit shown in figure.

(a)

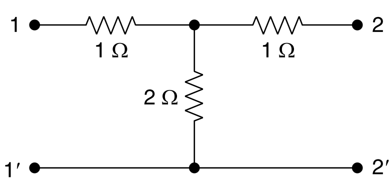


(b)

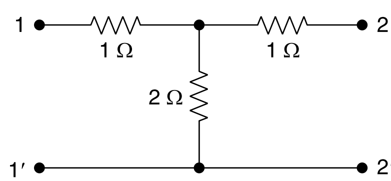


Solution

(a) The given circuit can be considered as the cascade connection of the following two networks:



(a)



(b)

From Prob. 7.1(a), $z_{11a} = z_{11b} = z_{22a} = z_{22b} = 3\Omega$

$$z_{12a} = z_{21a} = z_{12b} = z_{21b} = 2\Omega$$

So, the transmission parameters are,

$$\therefore \qquad A_a = A_b = \frac{z_{11}}{z_{21}} = \frac{3}{2}$$

$$\therefore \qquad B_a = B_b = \frac{\Delta z}{z_{21}} = \frac{9-4}{2} = \frac{5}{2}\ \Omega$$

$$\therefore \qquad C_a = C_b = \frac{1}{z_{21}} = \frac{1}{2}\ \mho$$

$$\therefore \qquad D_a = D_b = \frac{z_{22}}{z_{21}} = \frac{3}{2}$$

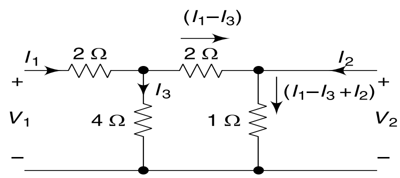
So, the transmission parameters of the resulting network are:

$$T = T_a \times T_b = \begin{bmatrix} 3/2 & 5/2 \\ 1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 3/2 & 5/2 \\ 1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 7/2 & 15/2 \\ 3/2 & 7/2 \end{bmatrix}$$

So, the z-parameters are:

$$\left.\begin{aligned} z_{11} &= \frac{A}{C} = \frac{7}{3}\ \Omega \\ z_{12} &= \frac{\Delta T}{C} = \frac{2}{3}\ \Omega \\ z_{21} &= \frac{1}{C} = \frac{2}{3}\ \Omega \\ z_{22} &= \frac{D}{C} = \frac{7}{3}\ \Omega \end{aligned}\right\}$$

(b) By KVL,



$$V_1 = 2I_1 + 4I_3$$
$$V_2 = I_1 + I_2 - I_3$$

and $\qquad 2(I_1 - I_3) + I_1 + I_2 - I_3 - 4I_3 = 0$

Eliminating I_3 from above equations,

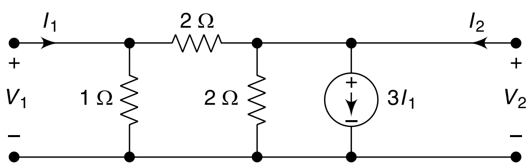
$$V_1 = \frac{26}{7} I_1 + \frac{4}{7} I_2$$

$$V_2 = \frac{4}{7} I_1 + \frac{6}{7} I_2$$

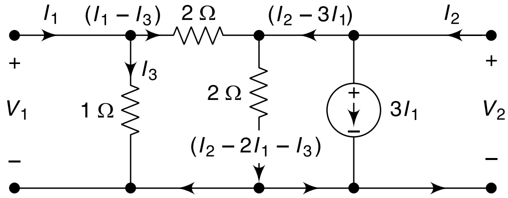
Thus, the z-parameters are:

$$[z] = \begin{bmatrix} 26/7 & 4/7 \\ 4/7 & 6/7 \end{bmatrix} \Omega$$

7.3 For the network shown, find z and y-parameters.



Solution From the figure, we can write the KVL equations,



$$V_1 = I_3 \tag{i}$$

$$V_2 = 2I_2 - 4I_1 - 2I_3 \tag{ii}$$

and, $\quad 2I_1 - 2I_3 + 2I_2 - 4I_1 - 2I_3 - I_3 = 0 \quad \Rightarrow \quad I_3 = \dfrac{2}{5}(I_2 - I_1)$

From (i), $V_1 = -\dfrac{2}{5}I_1 + \dfrac{2}{5}I_2 = -0.4I_1 + 0.4I_2$

From (ii), $V_2 = 2I_2 - 4I_1 - \dfrac{4}{5}I_2 + \dfrac{4}{5}I_1 = -3.2I_1 + 1.2I_2$

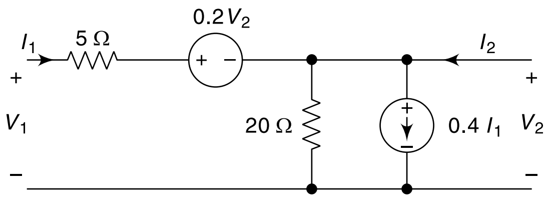
$\therefore$
$$[z] = \begin{bmatrix} -0.4 & 0.4 \\ -3.2 & 1.2 \end{bmatrix} \Omega$$

$$\Delta z = (-0.4 \times 1.2) - 0(0.4) \times (-3.2) = 0.8$$

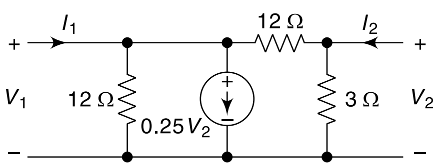
$\therefore$
$$[y] = \begin{bmatrix} 1.2/0.8 & -\dfrac{0.4}{0.8} \\ 3.2/0.8 & -\dfrac{0.4}{0.8} \end{bmatrix} \mho = \begin{bmatrix} 1.5 & -0.5 \\ 4 & -0.5 \end{bmatrix} \mho$$

7.4 Find the y-parameters for the 2-port networks shown.

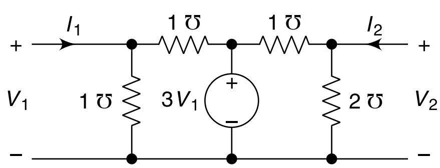
(a)



(b)



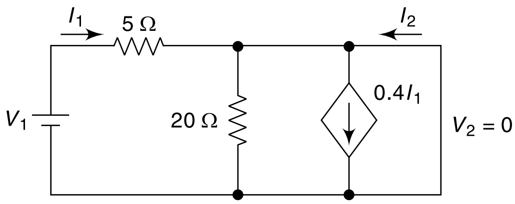
(c)

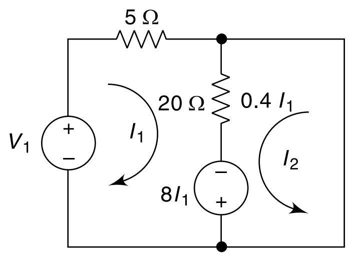


Solution

(a) We consider two cases to find out the *y*-parameters.

Case (I) Making port- 2 shorted and applying a voltage of V_1 at port- 1





By KVL,

$$17I_1 + 20I_2 = V_1$$

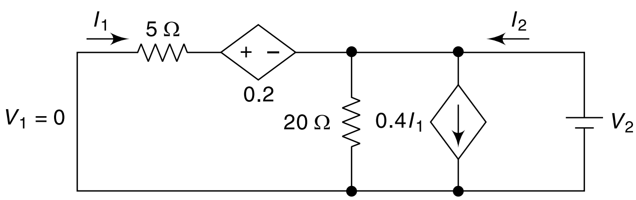
and $$12I_1 + 20I_2 = 0$$

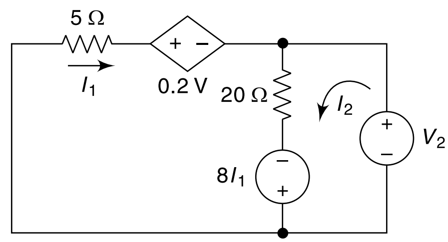
$$I_1 = \frac{\begin{vmatrix} V_1 & 20 \\ 0 & 20 \end{vmatrix}}{\begin{vmatrix} 17 & 20 \\ 12 & 20 \end{vmatrix}} = 0.2V_1 \quad \Rightarrow \quad y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = 0.2\ \mho$$

Solving,

$$I_2 = \frac{\begin{vmatrix} 17 & V_1 \\ 12 & 0 \end{vmatrix}}{\begin{vmatrix} 17 & 20 \\ 12 & 20 \end{vmatrix}} = -0.12V_1 \quad \Rightarrow \quad y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -0.12\ \mho$$

Case (II) Making port-1 shorted and applying a voltage of V_2 at port- 2





By KVL,

$$17I_1 + 20I_2 = -0.2V_2$$

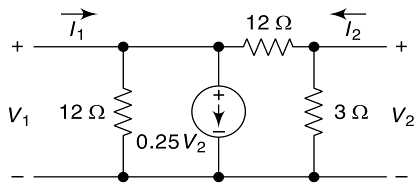
and $$12I_1 + 20I_2 = V_2$$

$$I_1 = \frac{\begin{vmatrix} -0.2V_2 & 20 \\ V_2 & 20 \end{vmatrix}}{\begin{vmatrix} 17 & 20 \\ 12 & 20 \end{vmatrix}} = -0.24V_2 \quad \Rightarrow \quad y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -0.24\ \mho$$

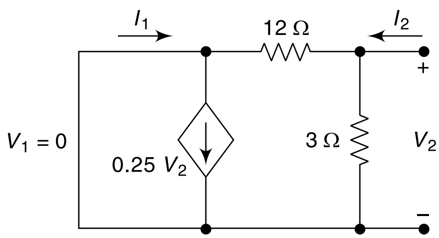
Solving,

$$I_2 = \frac{\begin{vmatrix} 17 & -0.2V_2 \\ 12 & V_2 \end{vmatrix}}{\begin{vmatrix} 17 & 20 \\ 12 & 20 \end{vmatrix}} = 0.194V_2 \Rightarrow y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = 0.194\ \mho$$

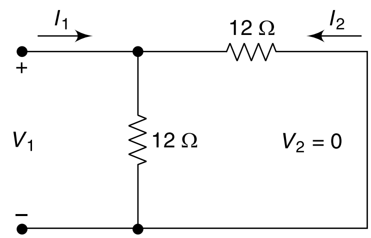
Thus, $$[y] = \begin{bmatrix} 0.2 & -0.24 \\ -0.12 & 0.194 \end{bmatrix}\mho$$



(b) We consider two cases.
Case (I) $V_1 = 0$
Case (II) $V_2 = 0$



Case (I): $V_1 = 0$



Case (II): $V_2 = 0$

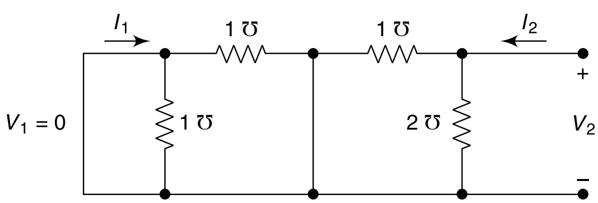
By KCL,

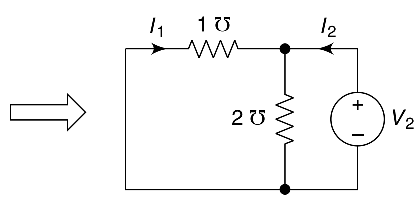
$$\left.\begin{aligned} I_1 &= y_{12}V_2|_{V_1=0} = \frac{V_2}{4} + \left(\frac{0-V_2}{12}\right) \Rightarrow y_{12} = \frac{1}{6}\mho \\ I_2 &= y_{22}V_2|_{V_1=0} = \frac{V_2}{3} + \frac{V_2}{12} \Rightarrow y_{22} = \frac{5}{12}\mho \\ I_1 &= y_{11}V_1|_{V_2=0} = \left(\frac{1}{12} + \frac{1}{12}\right)V_1 \Rightarrow y_{11} = \frac{1}{6}\mho \\ I_2 &= y_{21}V_1|_{V_2=0} = -\frac{V_1}{12} \Rightarrow y_{21} = -\frac{1}{12}\mho \end{aligned}\right\}$$

(c) For $V_1 = 0$, the circuit becomes as shown.

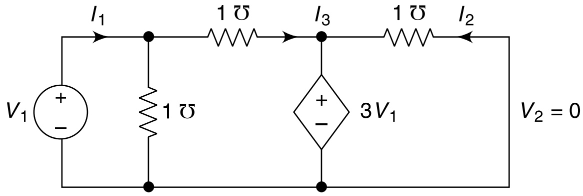
$\therefore$ $$I_2 = y_{22}V_2 = (1+2)V_2 = 3V_2 \Rightarrow y_{22} = 3\ \mho$$

Also, $$-\frac{I_1}{1} = V_2 \Rightarrow y_{12} = -1\ \mho$$





For $V_2 = 0$, the circuit becomes as shown.



$$\therefore \qquad -\frac{I_2}{1} = 3V_1 \qquad \text{(i)}$$

$$\frac{I_3}{1} + 3V_1 = V_1 \Rightarrow 2V_1 = -I_3 \qquad \text{(ii)}$$

$$I_1 = I_3 + I_4 \qquad \text{(iii)}$$

$$\text{and} \quad V_1 = \frac{I_4}{1} \qquad \text{(iv)}$$

From (i) to (iv),

$$I_1 = V_1 + I_3 = V_1 - 2V_1 = -V_1 \quad \Rightarrow \quad y_{11} = -1\,℧$$

From (i), $y_{21} = -3$ ℧

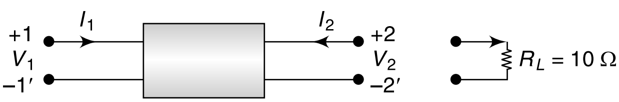
Thus, the y-parameters are:

$$[y] = \begin{bmatrix} -1 & -1 \\ -3 & 3 \end{bmatrix} ℧$$

From the interrelationship, we get the z-parameters as:

$$[z] = \begin{bmatrix} -1 & 0 \\ -1 & 1/3 \end{bmatrix} (\Omega)$$

7.5 Measurements were made on a two-port network shown in the figure.



(i) With port-2 open, a voltage of $100\angle 0°$ volt is applied to port-1, resulted in, $I_1 = 10\angle 0°$ amp and $V_2 = 25\angle 0°$ volt.

(ii) With port-1 open, a voltage of $100\angle 0°$ volt is applied to port-2, resulted in, $I_2 = 20\angle 0°$ amp and $V_1 = 50\angle 0°$ volt.

(a) Write the loop equations for the network and also find the driving point and transfer impedance.

(b) What will be the voltage across a 10 Ω resistor connected across port-2 if a $100\angle 0°$ volt source is connected across port-1.

Solution

(a) From the given data, we get the *z*-parameters as:

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{100\angle 0°}{10\angle 0°} = 10\ \Omega$$

$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{25\angle 0°}{10\angle 0°} = 2.5\ \Omega$$

$$z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{50\angle 0°}{20\angle 0°} = 2.5\ \Omega$$

$$z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{100\angle 0^\circ}{20\angle 0^\circ} = 5\ \Omega$$

So, the loop equations are:

$$\left.\begin{aligned} V_1 &= 10I_1 + 2.5I_2 \\ V_2 &= 2.5I_1 + 5I_2 \end{aligned}\right\}$$

(b) Here, $V_1 = 100\angle 0^\circ$ and $V_2 = -I_2R_L = -10I_2$

Putting these values in loop equations,

$$100 = 10I_1 + 2.5I_2 \quad \Rightarrow \quad I_1 = 10 - 0.25I_2$$

and $\quad -10I_2 = 2.5I_1 + 5I_2$

or, $\quad -10I_2 = 2.5(10 - 0.25I_2) + 5I_2$

or, $\quad -15I_2 = 25 - 0.625I_2$

or, $\quad I_2 = \dfrac{-25}{14.375} = -1.74\ \text{A}$

$\therefore$ Voltage across the resistor $= -I_2R_L = 17.4$ V

7.6 (a) The following equations give the voltages V_1 and V_2 at the two ports of a two port network, $V_1 = 5I_1 + 2I_2$, $\quad V_2 = 2I_1 + I_2$;

A load resistance of 3 Ω is connected across port-2. Calculate the input impedance.

(b) The z-parameters of a two port network are $z_{11} = 5\ \Omega$, $z_{22} = 2\ \Omega$, $z_{12} = z_{21} = 3\ \Omega$. Load resistance of 4 Ω is connected across the output port. Calculate the input impedance.

Solution

(a) From the given equations,

$$V_1 = 5I_1 + 2I_2 \tag{i}$$

$$V_2 = 2I_1 + I_2 \tag{ii}$$

At the output, $V_2 = -I_2R_L = -3I_2$

Putting this value in (ii),

$$-3I_2 = 2I_1 + I_2 \quad \Rightarrow \quad I_2 = -I_1/2$$

Putting in (i), $V_1 = 5I_1 + \left(\dfrac{-I_1}{2}\right) = 4I_1$

$\therefore$ Input impedance, $Z_{\text{in}} = \dfrac{V_1}{I_1} = 4\Omega$

(b) [Same as Prob. (a)] $Z_{\text{in}} = \dfrac{V_1}{I_1} = 3.5\Omega$

7.7 Determine the h-parameter with the following data:

(i) with the output terminals short circuited, $V_1 = 25$ V, $I_1 = 1$ A, $I_2 = 2$ A

(ii) with the input terminals open circuited, $V_1 = 10$ V, $V_2 = 50$ V, $I_2 = 2$ A

Solution The h-parameter equations are,

$$V_1 = h_{11}I_1 + h_{12}V_2$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

(a) With output short-circuited, $V_2 = 0$, given: $V_1 = 25$ V, $I_1 = 1$ A and $I_2 = 2$ A.

$$\therefore \quad \left.\begin{aligned} 25 &= h_{11} \times 1 \\ \text{and} \quad 2 &= h_{21} \times 1 \end{aligned}\right\} \Rightarrow h_{11} = 25\ \Omega, \text{ and } h_{21} = 2$$

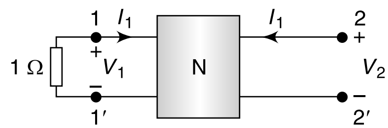
(b) With input open-circuited, $I_1 = 0$, given: $V_1 = 10$ V, $V_2 = 50$ V and $I_2 = 2$ A.

$$\therefore \quad \left.\begin{aligned} 10 &= h_{12} \times 50 \\ \text{and} \quad 2 &= h_{22} \times 50 \end{aligned}\right\} \Rightarrow h_{12} = \frac{1}{5} = 0.2 \text{ and } h_{23} = \frac{1}{25}\ \mho = 0.04\ \mho$$

Thus, the h-parameters are:

$$[h] = \begin{bmatrix} 25\ \Omega & 0.2 \\ 2 & 0.04\ \Omega^{-1} \end{bmatrix}$$

7.8 The y-parameters for a two-port network N are given as, $[y_{11} = 4\ \mho, y_{22} = 5\ \mho, y_{12} = y_{21} = 4\ \mho]$
If a resistor of 1 ohm is connected across port-1 of N, then find out the output impedance.



Solution Output impedance is given as,

$$Z_{\text{out}} = \frac{z_{11}z_{22} - z_{12}z_{21}}{z_{11} + Z_L}$$

Here, $\quad y_{11} = 4\ \Omega^{-1},\ y_{12} = y_{21} = 4\ \Omega^{-1},\ y_{22} = 5\ \Omega^{-1}$

$$\therefore \quad z_{11} = \frac{y_{22}}{\Delta y} = \frac{5}{20 - 16} = \frac{5}{4}\ \Omega$$

$$z_{12} = z_{21} = -\frac{y_{12}}{\Delta y} = -\frac{4}{4} = -1\ \Omega$$

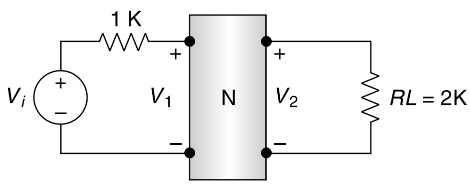
$$\text{and} \quad z_{22} = \frac{y_{11}}{\Delta y} = \frac{4}{4} = 1\ \Omega$$

Putting these values,

$$Z_{\text{out}} = \frac{z_{11}z_{22} - z_{12}z_{21}}{z_{11} + Z_L} = \frac{\frac{5}{4} \times 1 - (-1) \times (-1) + 1 \times 1}{5/4 + 1} = \frac{5}{9}\ \Omega$$

7.9 (a) The h-parameters of a two-port network are $h_{11} = 100\ \Omega$, $h_{12} = 0.0025$, $h_{21} = 20$ and $h_{22} = 1$ m$\mho$. Find V_2/V_1.

(b) The h-parameters of a two-port network are $h_{11} = 1\ \Omega$, $h_{12} = -h_{21} = 2$, $h_{22} = 1\ \mho$. The power absorbed by a load resistance of $1\ \Omega$ connected across port-2 is 100 W. The network is excited by a voltage source of generated voltage V_s and internal resistance $2\ \Omega$. Calculate the value of V_s.



Solution

(a) The h-parameter equations are:

$$V_1 = 100I_1 + 0.0025V_2 \tag{i}$$

$$I_2 = 20I_1 + 0.001V_2 \qquad \text{(ii)}$$

By KVL at the output mesh, $V_2 = -2000I_2$ (iii)

$$V_1 = 100\left[\frac{I_2 - 0.001V_2}{20}\right] + 0.0025V_2 = 5\left(-\frac{V_2}{2000}\right) - 0.005V_2 + 0.0025V_2$$

From (i),

or $$\frac{V_2}{V_1} = -200$$

(b) The h-parameter equations are:

$$V_1 = I_1 + 2V_2 \qquad \text{(i)}$$
$$I_2 = -2I_1 + V_2 \qquad \text{(ii)}$$

Since the load resistance of 1 Ω is connected across port-2,

$$\therefore \qquad \frac{V_2^2}{1} = 100 \quad \Rightarrow \quad V_2 = 10 \text{ V}$$

By KVL, $V_2 = -I_2R_L = -I_2 \quad \Rightarrow \quad I_2 = -10$ A

and $2I_1 + V_1 = V_s$ (iii)

From (ii), putting the values of I_2 and V_2,

$$-10 = -2I_1 + 10 \quad \Rightarrow \quad I_1 = 10 \text{ A}$$

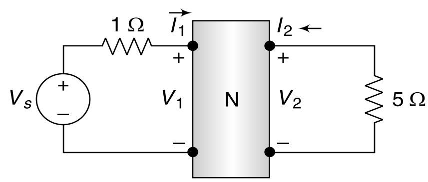
From (iii),

$$V_s = 2 \times 10 + V_1 = 20 + I_1 + 2V_2 \quad \{\text{by (i)}\}$$
$$= 20 + 10 + 2 \times 10$$

or, $V_s = 50$ V

7.10 The z-parameters for a network N are:

$$\begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix}$$



The terminal connections for the network are shown in the adjacent figure. Calculate the voltage ratio V_2/V_s, current ratio $-I_2/I_1$ and input resistance V_1/I_1.

Solution The *z*-parameter equations are:

$$V_1 = 2I_1 + I_2 \qquad \text{(i)}$$
$$V_2 = 2I_1 + 5I_2 \qquad \text{(ii)}$$

By KVL at the input and output circuits,

$$I_1 + V_1 = V_s \quad \Rightarrow \quad 3I_1 + I_2 = V_s \qquad \text{(iii)} \quad \{\text{by (i)}\}$$

and $$5I_2 + V_2 = 0 \quad \Rightarrow \quad 2I_1 + 10I_2 = 0 \qquad \text{(iv)} \quad \{\text{by(ii)}\}$$

Solving (iii) and (iv),

$$I_1 = \frac{\begin{vmatrix} V_s & 1 \\ 0 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 2 & 10 \end{vmatrix}} = \frac{10}{28}V_s \quad \text{and} \quad I_2 = \frac{\begin{vmatrix} 3 & V_s \\ 2 & 0 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 2 & 10 \end{vmatrix}} = -\frac{2}{28}V_s$$

$\therefore \qquad -\dfrac{I_2}{I_1} = \dfrac{1}{5}$

Now, $\qquad V_2 = (2I_1 + 5I_2) = \left(\dfrac{20}{28} - \dfrac{10}{28}\right)V_s = \dfrac{10}{28}V_s$

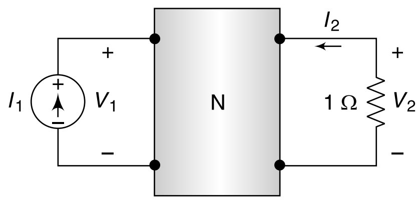
$\therefore \qquad \dfrac{V_2}{V_s} = \dfrac{5}{14}$

Again,

$$V_1 = (2I_1 + I_2) = \left(\frac{20}{28} - \frac{2}{28}\right)V_s = \frac{18}{28}V_s$$

$\therefore \qquad \dfrac{V_1}{I_1} = \dfrac{9}{14}\,\Omega$

7.11 For the two-port network in figure, terminated in a 1 Ω resistance, show that, $\dfrac{V_2}{I_1} = \dfrac{z_{21}}{1 + z_{22}}$ and $\dfrac{V_1}{I_1} = \dfrac{z_{11} + \Delta z}{1 + z_{22}}$



Solution The z-parameter equations are:

$$V_1 = z_{11}I_1 + z_{12}I_2 \qquad \text{(i)}$$
$$V_2 = z_{21}I_1 + z_{22}I_2 \qquad \text{(ii)}$$

By KVL at the output, $V_2 = -I_2 \times 1 \Rightarrow I_2 = -V_2$

$$V_2 = z_{21}I_1 + z_{22}I_2 = z_{21}I_1 + z_{22}(-V_2)$$

From (ii), or, $V_2(1 + z_{22}) = z_{21}I_1$ (iii)

or $\qquad \dfrac{V_2}{I_1} = \dfrac{z_{21}}{1 + z_{22}}$ (Proved)

From (i),

$$V_1 = z_{11}\left[\frac{V_2(1 + z_{22})}{z_{21}}\right] + z_{12}(-V_2) \quad \{\text{by (iii)}\}$$

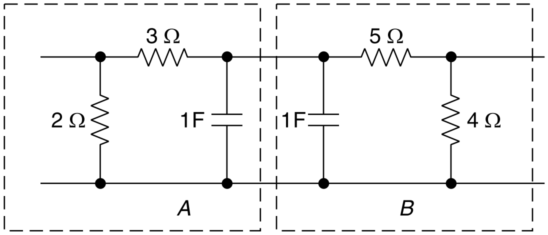
$$= V_2\left[\frac{z_{11} + z_{11}z_{22} - z_{12}z_{21}}{z_{21}}\right]$$

$$= V_2\left[\frac{z_{11} + \Delta z}{z_{21}}\right]$$

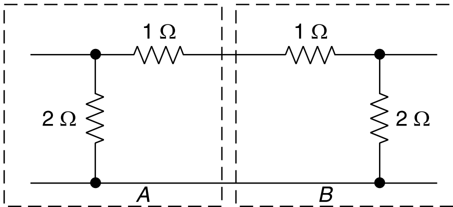
$\therefore \qquad \dfrac{V_1}{I_1} = \dfrac{V_1}{V_2} \times \dfrac{V_2}{I_1} = \dfrac{z_{11} + \Delta z}{z_{21}} \times \dfrac{z_{21}}{1 + z_{22}} = \dfrac{z_{11} + \Delta z}{1 + z_{22}}$ (Proved)

7.12 Calculate the T-parameters for the block A and B separately and then using these results, calculate the T-parameters of the whole circuit shown in the figure. Prove any formula used.

(a)



(b)



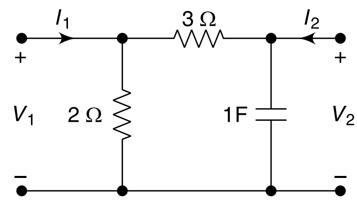
Solution

(a) We consider the given network as a cascade connection of two networks as shown.

For Block A:

Opening the port-2,

By KCL,



Block A

$$\left(\frac{1}{2}+\frac{1}{3}\right)V_1-\frac{1}{3}V_2=I_1$$

and $$-\frac{1}{3}V_1+\left(\frac{1}{3}+s\right)V_2=0$$

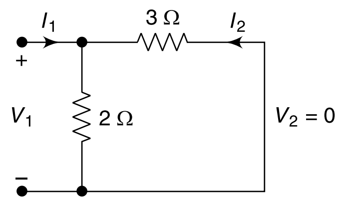
Solving for V_1 and V_2,

$$V_1=\frac{2I_1(1+3s)}{(1+5s)} \quad \text{and} \quad V_2=\frac{2I_1}{(1+5s)}$$

$$\left.\begin{aligned} \therefore \quad A_a &= \left.\frac{V_1}{V_2}\right|_{I_2=0}=(1+3s) \\ \text{and} \quad C_a &= \left.\frac{I_1}{V_2}\right|_{I_2=0}=\frac{(1+5s)}{2} \end{aligned}\right\}$$

Short-circuiting port-2,

$$\therefore \quad I_1=\frac{V_1}{2}+\frac{V_1}{3}=\frac{5}{6}V_1$$



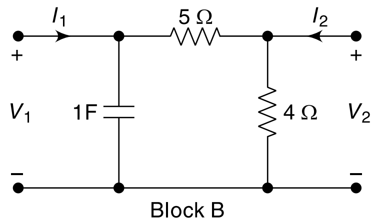
and $$V_1=-3I_2 \Rightarrow B_a=-\left.\frac{V_1}{I_2}\right|_{V_2=0}=3\Omega$$

and $$D_a=-\left.\frac{I_1}{I_2}\right|_{V_2=0}=\frac{5V_1}{6}\times\frac{3}{V_1}=\frac{5}{2}$$

For Block B:

Opening the port-2,

By KCL,



$$\left(\frac{1}{5}+s\right)V_1-\frac{1}{5}V_2=I_1$$

and $$-\frac{1}{5}V_1+\left(\frac{1}{5}+\frac{1}{4}\right)V_2=0$$

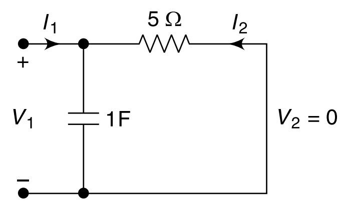
Solving for V_1 and V_2,

$$V_1=\frac{9I_1}{(1+9s)} \quad \text{and} \quad V_2=\frac{4I_1}{(1+9s)}$$

$$\left.\begin{aligned} \therefore \quad A_b &= \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{9}{4} \\ \text{and} \quad C_b &= \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{(1+9s)}{4} \end{aligned}\right\}$$

Short-circuiting port-2,

$$\therefore \quad I_1=\left(\frac{1}{5}+s\right)V_1$$



and $$V_1=-5I_2 \quad \Rightarrow \quad B_b=-\left.\frac{V_1}{I_2}\right|_{V_2=0}=5\,\Omega$$

and $$D_b=-\left.\frac{I_1}{I_2}\right|_{V_2=0}=(5s+1)$$

Since the two networks are connected in cascade, the overall transmission parameter matrix is obtained as,

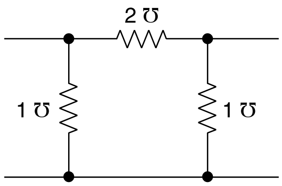
$$[T]=[T_a]\times[T_b]=\begin{bmatrix}(3s+1) & 3\\ \left(\dfrac{5s+1}{2}\right) & 5/2\end{bmatrix}\times\begin{bmatrix}9/4 & 5\\ \left(\dfrac{1+9s}{4}\right) & (5s+1)\end{bmatrix}=\begin{bmatrix}(13.5s+3) & (30s+8)\\ (11.25s+1.75) & (25s+5)\end{bmatrix}$$

(b) [Same as Prob. (a)]

Here, $$[T_a]=\begin{bmatrix}1 & 1\\ 1/2 & 3/2\end{bmatrix} \quad \text{and} \quad [T_b]=\begin{bmatrix}3/2 & 1\\ 3/2 & 1\end{bmatrix}$$

$$\therefore \quad [T]=[T_a]\times[T_b]=\begin{bmatrix}3 & 2\\ 3 & 2\end{bmatrix}$$

7.13 Two identical sections of the network shown in the figure are connected in parallel. Obtain the y-parameters of the resulting network and verify by direct calculation.



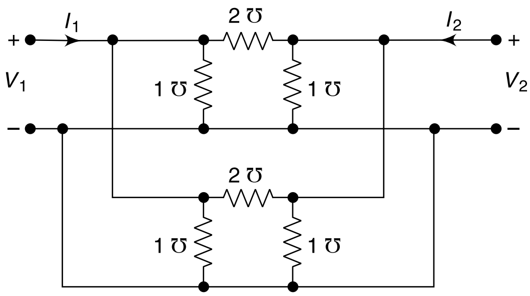
Solution For the circuit, $y_{11} = 3\ \Omega^{-1}$, $y_{12} = y_{21} = -2\ \Omega^{-1}$ and $y_{22} = 3\Omega^{-1}$

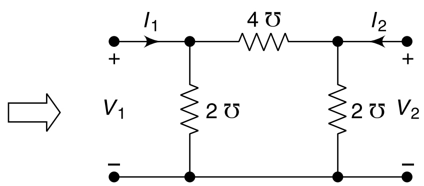
The y-parameters for the combination will be,

$$\left.\begin{aligned} y_{11} &= (y'_{11} + y''_{11}) = 6\ \Omega^{-1} \\ y_{12} = y_{21} &= (y'_{12} + y''_{12}) = -4\ \Omega^{-1} \\ y_{22} &= (y'_{22} + y''_{22}) = 6\ \Omega^{-1} \end{aligned}\right\}$$

To find the y-parameters by direct calculation, we consider the resulting network as shown.

For the entire network, $y_{11} = 4 + 2 = 6\ \Omega^{-1}$; $y_{12} = y_{21} = -4\ \Omega^{-1}$; $y_{22} = 4 + 2 = 6\ \Omega^{-1}$ (Proved)





7.14 Two networks have general $ABCD$ parameters as shown below:

Parameter	*Network-1*	*Network-2*
A	1.50	5/3
B	11Ω	4Ω
C	0.25 siemens	1 siemens
D	2.5	3.0

If the two networks are connected with their inputs and outputs in parallel, obtain the admittance matrix of the resulting network.

Solution *For network-1:*

$$y_{11} = \frac{D}{B} = \frac{2.5}{11} = \frac{5}{22}\ \Omega^{-1}$$

$$y_{12} = -\frac{AD - BC}{B} = -\frac{1.5 \times 2.5 - 11 \times 0.25}{11} = -\frac{1}{11}\ \Omega^{-1}$$

$$y_{21} = -\frac{1}{B} = -\frac{1}{11}\ \Omega^{-1}$$

$$y_{22} = \frac{A}{B} = \frac{1.5}{11} = \frac{3}{22}\ \Omega^{-1}$$

For network-2:

$$y_{11} = \frac{D}{B} = \frac{3}{4}\,\Omega^{-1}$$

$$y_{12} = -\frac{AD - BC}{B} = -\frac{1}{4}\,\Omega^{-1}$$

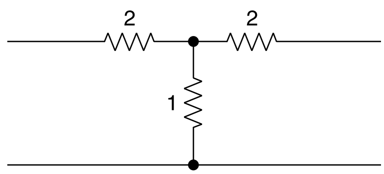
$$y_{21} = -\frac{1}{B} = -\frac{1}{4}\,\Omega^{-1}$$

$$y_{22} = \frac{A}{B} = \frac{5}{3\times 4} = \frac{5}{12}\,\Omega^{-1}$$

So, the admittance matrix of the resulting network is:

$$[y] = \begin{bmatrix} 5/22 & -1/11 \\ -1/11 & 3/22 \end{bmatrix} + \begin{bmatrix} 3/4 & -1/4 \\ -1/4 & 5/12 \end{bmatrix} = \begin{bmatrix} 43/44 & -15/44 \\ -15/44 & 73/132 \end{bmatrix}\Omega^{-1}$$

7.15 Two identical sections of figure are connected in series. Obtain the z-parameters of the resulting network and verify by direct calculation. All values are in ohm.



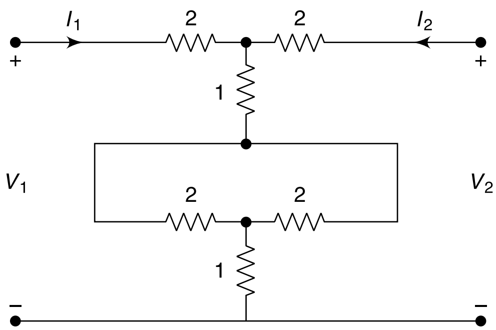
Solution The z-parameters of each section:

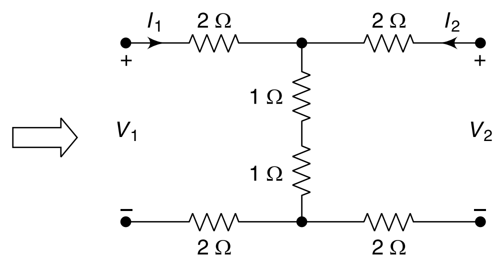
$$z_{11} = 3\,\Omega,\ z_{12} = z_{21} = 1\,\Omega,\ z_{22} = 3\,\Omega$$

So, the z-parameters of the combined series network are:

$$z_{11} = (3+3) = 6\,\Omega,\ z_{12} = z_{21} = (1+1) = 2\,\Omega,\ z_{22} = (3+3) = 6\,\Omega$$

To find the z-parameters by direct calculation, we consider the resulting network as shown.

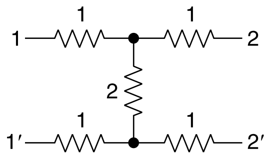




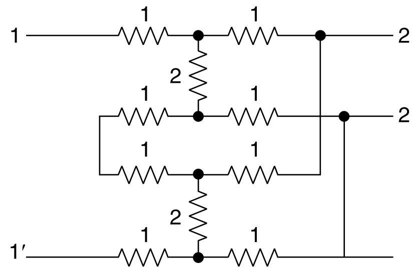
For the resulting network,

$$\left.\begin{aligned} z_{11} &= \left.\frac{V_1}{I_1}\right|_{I_2=0} = 6\,\Omega \quad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = 2\,\Omega \\ z_{22} &= \left.\frac{V_2}{I_2}\right|_{I_1=0} = 6\,\Omega \quad z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = 2\,\Omega \end{aligned}\right\}$$

7.16 (a) Find out the z- and h-parameters for the circuit shown in Fig. (a). All values are in ohm.
(b) Hence, obtain the hybrid parameters for the two-port network of Fig. (b).



(a)



(b)

Solution

(a) For Fig. (a), the z-parameters are:

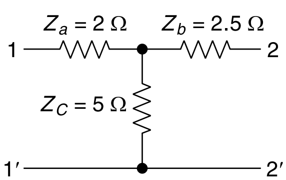
$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 4\ \Omega,\ z_{12} = z_{21} = 2\ \Omega,\ z_{11} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = 4\ \Omega$$

$$\therefore \quad \left.\begin{aligned} h_{11} &= \frac{\Delta z}{z_{12}} = \frac{16-4}{4} = 3\ \Omega \\ h_{12} &= \frac{z_{12}}{z_{22}} = \frac{2}{4} = 0.5 \\ h_{21} &= -\frac{z_{21}}{z_{22}} = -\frac{2}{4} = -0.5 \\ h_{22} &= \frac{1}{z_{12}} = \frac{1}{4} = 0.25\ \Omega^{-1} \end{aligned}\right\}$$

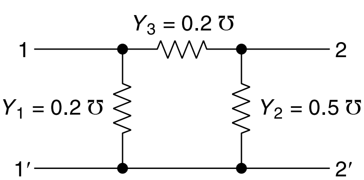
(b) The connection is series-parallel connection. For this connection, the overall h-parameters will be the sum of individual h-parameters.

$$\therefore \quad \left.\begin{aligned} h_{11} &= (3+3) = 6\Omega \\ h_{12} &= (0.5+0.5) = 1 \\ h_{21} &= (-0.5-0.5) = -1 \\ h_{22} &= (0.25+0.25) = 0.5\Omega^{-1} \end{aligned}\right\}$$

7.17 (a) Find the equivalent π-network for the T-network shown in the Fig. (a).
(b) Find the equivalent T-network for the π-network shown in the Fig. (b).



(a)



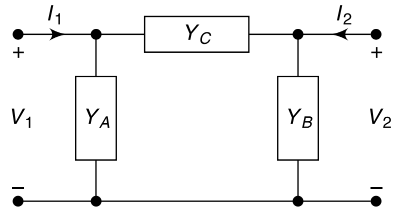
(b)

Solution

(a) Let the equivalent π-network have Y_C as the series admittance and Y_A and Y_B as the shunt admittances at port-1 and port-2, respectively.

Now, the z-parameters are given as:

$$z_{11} = (Z_A + Z_C) = 7\ \Omega,\ z_{12} = z_{21} = Z_C = 5\ \Omega,\ z_{22} = (Z_B + Z_C) = 7.5\ \Omega$$



$$\therefore \quad \Delta z = (7 \times 7.5 - 5 \times 5) = 27.5\ \Omega^2$$

$$\therefore \quad y_{11} = \frac{z_{22}}{\Delta z} = \frac{7.5}{27.5}\ \mho$$

$$y_{12} = y_{21} = -\frac{z_C}{\Delta z} = -\frac{5}{27.5}\ \mho$$

$$y_{22} = \frac{z_{11}}{\Delta z} = \frac{7}{27.5}\ \mho$$

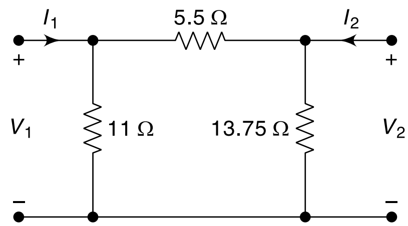
$$\therefore \quad Y_A = (y_{11} + y_{12}) = \frac{2.5}{27.5} = \frac{1}{11}\ \mho$$

$$\therefore \quad Y_B = (y_{22} + y_{12}) = \frac{2}{27.5}\ \mho$$

and $$Y_C = -y_{21} = \frac{5}{27.5} = \frac{2}{11}\ \mho$$

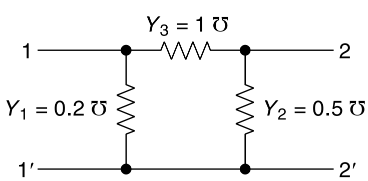
Thus, the impedances of the equivalent π-networks are:

$$\left.\begin{aligned} Z_A &= \frac{1}{Y_A} = 11\ \Omega, \\ Z_B &= \frac{1}{Y_B} = 13.75\ \Omega, \\ Z_C &= \frac{1}{Y_C} = 5.5\ \Omega \end{aligned}\right\}$$

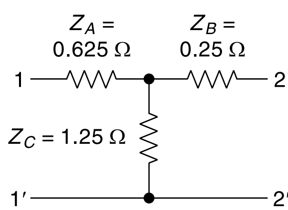


Equivalent π-network

(b)



π-network



Equivalent T-network

The y-parameters,

$$y_{11} = 1.2\ \mho,\ y_{12} = y_{21} = -1\ \mho, \text{ and } y_{22} = 1.5\ \mho$$

$$\therefore \quad \Delta y = (1.2 \times 1.5 - 1) = 0.8$$

$$\therefore \qquad z_{11} = \frac{y_{22}}{\Delta y} = \frac{1.5}{0.8}\,\Omega,\; z_{12} = z_{21} = -\frac{y_{12}}{\Delta y} = \frac{1}{0.8}\,\Omega,\; z_{22} = \frac{y_{11}}{\Delta y} = \frac{1.2}{0.8}\,\Omega$$

$$\therefore \qquad \left.\begin{aligned} Z_A &= (z_{11} - z_{12}) = \frac{0.5}{0.8} = 0.625\,\Omega \\ Z_B &= (z_{22} - z_{12}) = \frac{0.2}{0.8} = 0.25\,\Omega \\ Z_C &= z_{12} = \frac{1}{0.8} = 1.25\,\Omega \end{aligned}\right\}$$

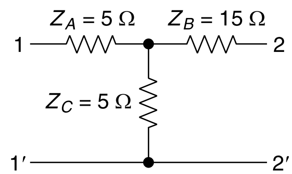
7.18 The z-parameter of a 2-port network are:

$$z_{11} = 10\,\Omega,\; z_{22} = 20\,\Omega,\; z_{12} = z_{21} = 5\,\Omega.$$

Find the *ABCD*-parameters. Also find the equivalent *T*-network.

Solution

From the inter-relationship, we get the *ABCD* parameters as:



Equivalent T-network

$$A = \frac{z_{11}}{z_{21}} = \frac{10}{5} = 2$$

$$B = \frac{z_{11}Z_{22} - Z_{12}Z_{21}}{z_{21}} = \frac{10 \times 20 - 5 \times 5}{5} = 35\,\Omega$$

$$C = \frac{1}{z_{21}} = \frac{1}{5} = 0.2\,\mho$$

$$D = \frac{z_{22}}{z_{21}} = \frac{20}{5} = 4$$

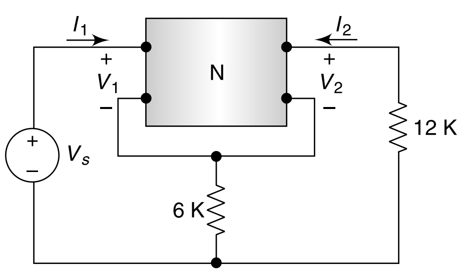
To find the equivalent *T*-network, we have the relations,

$$\left.\begin{aligned} z_{11} &= (Z_A + Z_C) = 10\,\Omega \\ z_{12} &= z_{21} = Z_C = 5\,\Omega \\ \text{and} \qquad z_{22} &= (Z_B + Z_C) = 20\,\Omega \end{aligned}\right\} \Rightarrow \; Z_A = 5\,\Omega,\, Z_B = 15\,\Omega,\, Z_C = 5\,\Omega$$

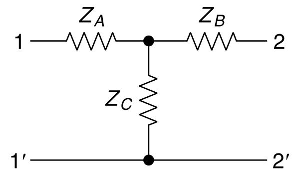
7.19 Z-parameters of the two-port network N in figure. are, $z_{11} = 4s$, $z_{12} = z_{21} = 3s$, $z_{22} = 9s$.

(a) Replace N by its T-equivalent.

(b) Use part (a) to find the input current I_1 for $V_s = \cos 1000t$.



Solution



Equivalent T-network

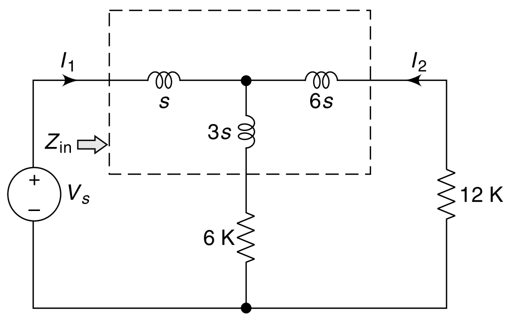
(a) The z-parameters are: $[z]=\begin{bmatrix}4s & 3s\\ 3s & 9s\end{bmatrix}(\Omega)$

Since the network is reciprocal, its T-equivalent exists. Its elements are:

$$Z_A = (z_{11}-z_{12}) = s,\ Z_B = (z_{22}-z_{21}) = 6s,$$

and $$Z_C = z_{21} = z_{12} = 3s$$

So, the equivalent circuit is shown in figure.



(b) We repeatedly combine the series and parallel elements of above figure, with resistors in kΩ and s in Krad/s to find the input impedance, Z_{in} in kΩ.

$$\therefore \qquad Z_{in} = \frac{V_s}{I_1} = s + \frac{(6s+12)\,(3s+6)}{(6s+12)+(3s+6)} = (3s+4)$$

or $$Z_{in}(j) = (3j+4) = 5\angle 36.9^\circ \text{ k}\Omega$$

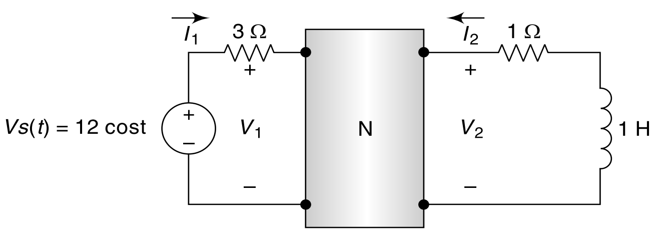
So, the current,

$$i(t) = \frac{v_s(t)}{Z_{in}(j)} = \frac{1}{5}\cos(1000t - 36.9^\circ)\ (\text{mA})$$

7.20 The z-parameters of a two-port network N are given by, $z_{11} = (2s + 1/s)$, $z_{12} = z_{21} = 2s$, $z_{22} = (2s + 4)$.

(a) Find the T-equivalent of N.

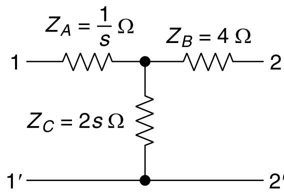
(b) The network N is connected to a source and a load as shown in figure. Replace N by its T-equivalent and then find I_1, I_2, V_1, and V_2.



Solution

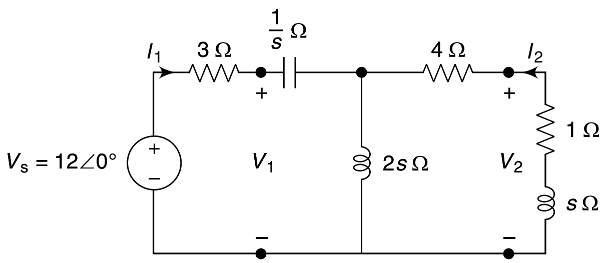
(a) To find the equivalent T-network, we have the relations,

$$\left.\begin{aligned} z_{11} &= (Z_A + Z_C) = \left(2s + \frac{1}{s}\right)\Omega \\ z_{12} &= z_{21} = Z_C = 2s\ \Omega \\ \text{and} \qquad z_{22} &= (Z_B + Z_C) = (2s+4)\ \Omega \end{aligned}\right\} \Rightarrow Z_A = \frac{1}{s}\Omega,\ Z_B = 4\Omega,\ Z_C = 2s\ \Omega$$



Equivalent T-network

(b) The equivalent circuit is shown below.



By KVL, $I_1(3+j) + I_2(j2) = 12\angle 0°$

$I_1(j2) + I_2(5+j3) = 0$

$$I_1 = \frac{\begin{vmatrix} 12\angle 0° & j2 \\ 0 & (5+j3) \end{vmatrix}}{\begin{vmatrix} (3+j) & j2 \\ j2 & (5+j3) \end{vmatrix}} = \frac{\begin{vmatrix} 12\angle 0° & 2\angle 90° \\ 0 & 5.831\angle 30.96° \end{vmatrix}}{16+j14} = 3.29\angle -10.22° (\text{A})$$

Solving,

$$\text{and} \qquad I_2 = \frac{\begin{vmatrix} (3+j) & 12\angle 0° \\ j2 & 0 \end{vmatrix}}{\begin{vmatrix} (3+j) & j2 \\ j2 & (5+j3) \end{vmatrix}} = 1.13\angle -131.19° (\text{A})$$

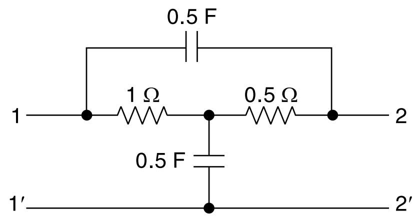
$\therefore \qquad V_1 = 12\angle 0° - I_1 \times 3 = 12 - 3.29 \times 3\angle -10.22° = 2.28 + j1.75 = 2.88\angle 37.504°\ (\text{V})$

and $\qquad V_2 = -I_2(1+j) = -1.13(1+j)\angle -131.186° = 1.59\angle 93.81°$

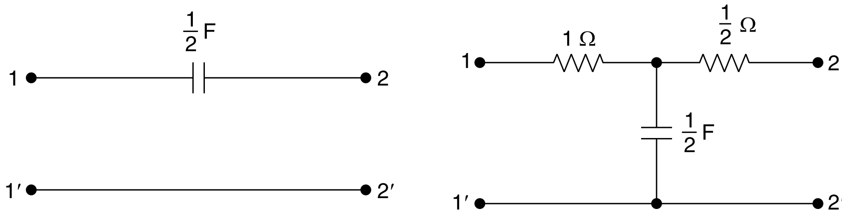
So, the currents and voltages are:

$$\left.\begin{aligned} i_1(t) &= 3.29 \cos(t - 10.2°) \text{ (A)} \\ i_2(t) &= 1.13 \cos(t - 131.2°) \text{ (A)} \\ v_1(t) &= 2.88 \cos(t + 37.5°) \text{ (A)} \\ v_2(t) &= 1.6 \cos(t + 93.8°) \text{ (A)} \end{aligned}\right\}$$

7.21 For the bridge-TRC network, find the y-parameters and its equivalent π-network.



Solution The given network is the parallel combination of the two networks:



(a) Network *(b) Network*

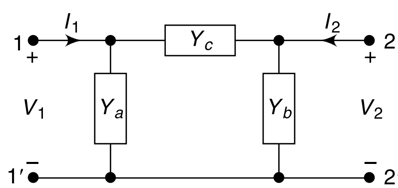
For network (a), the y-parameters are: $[y_a] = \begin{bmatrix} s/2 & -s/2 \\ -s/2 & s/2 \end{bmatrix}$ ℧

For network (b), the z-parameters are: $[z_b] = \begin{bmatrix} (1 + 2/s) & 2/s \\ 2/s & (1/2 + 2/s) \end{bmatrix}$ Ω

$$\therefore \qquad y_{11b} = \frac{z_{22b}}{\Delta z_b} = \frac{(1/2 + 2/s)}{(1 + 2/s)(1/2 + 2/s) - 4/s^2} = \frac{s+4}{s+6}$$

$$\therefore \qquad y_{12b} = y_{21b} = -\frac{z_{12b}}{\Delta z_b} = \frac{2/s}{(s+6)/2s} = \frac{4}{s+6}$$

$$\therefore \qquad y_{22b} = \frac{z_{11b}}{\Delta z_b} = \frac{(s+2)/2}{(s+6)/2s} = \frac{2(s+2)}{s+6}$$



Equivalent π network

For network (b), the y-parameters are: $[y_b] = \begin{bmatrix} \dfrac{s+4}{s+6} & \dfrac{4}{s+6} \\ \dfrac{4}{s+6} & \dfrac{2(s+2)}{s+6} \end{bmatrix}$

Thus, the overall y-parameters are:

$$[y] = [y_a]+[y_b]=\begin{bmatrix} s/2 & -s/2 \\ -s/2 & s/2 \end{bmatrix}+\begin{bmatrix} \dfrac{s+4}{s+6} & \dfrac{4}{s+6} \\ \dfrac{4}{s+6} & \dfrac{2(s+2)}{s+6} \end{bmatrix}$$

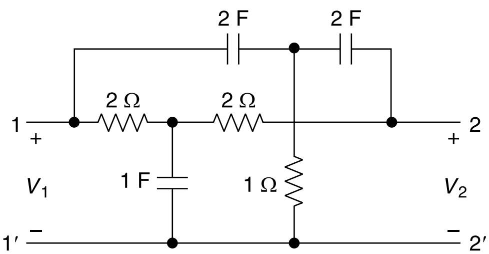
$$=\begin{bmatrix} \dfrac{s^2+8s+8}{2(s+6)} & -\dfrac{s^2+6s+8}{2(s+6)} \\ -\dfrac{s^2+6s+8}{2(s+6)} & \dfrac{s^2+10s+8}{2(s+6)} \end{bmatrix}$$

Equivalent π network can be found out from the relations:

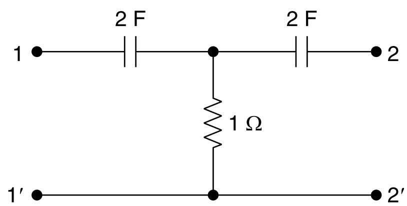
$$Y_a = (y_{11}+y_{12}) = \frac{s}{(s+6)};\ Y_b = (y_{22}+y_{12})$$

$$= \frac{2s}{(s+6)};\ Y_c = -y_{12} = -y_{21} = \frac{s^2+6s+8}{2(s+6)}$$

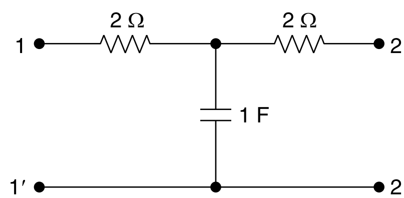
7.22 For the notch-filter network, determine the y-parameters.



Solution The given network is the parallel combination of the two networks:



(a) Network



(b) Network

For network (a), $z_{11a} = \left(\dfrac{1}{2s}+1\right) = \dfrac{1+2s}{2s};\ z_{12a} = z_{21a} = 1;\ z_{22a} = \left(\dfrac{1}{2s}+1\right) = \dfrac{1+2s}{2s}$

$\therefore$ $$\Delta z_a = \frac{1+4s}{4s^2}$$

$\therefore$ $$y_{11a} = \frac{z_{22a}}{\Delta z_a} = \frac{2s(1+2s)}{(1+4s)};\ y_{12a} = y_{21a} = -\frac{z_{12a}}{\Delta z_a}$$

$$= -\frac{4s^2}{(1+4s)};\ y_{22a} = \frac{z_{11a}}{\Delta z_a} = \frac{2s(1+2s)}{(1+4s)}$$

For network (b), $z_{11b} = (1/s+2) = \dfrac{1+2s}{s}$; $z_{12b} = z_{21b} = \dfrac{1}{s}$; $z_{22b} = (1/s+2) = \dfrac{1+2s}{s}$

$$\therefore \quad \Delta z_b = \frac{4(s+1)}{s}$$

$$\therefore \quad y_{11b} = \frac{z_{22b}}{\Delta z_b} = \frac{(1+2s)}{4(s+1)};\ y_{12b} = y_{21b} = -\frac{z_{12b}}{\Delta z_b} = -\frac{1}{4(s+1)};\ y_{22b} = \frac{z_{11b}}{\Delta z_b} = \frac{(1+2s)}{4(s+1)}$$

Thus, the overall y-parameters are,

$$y_{11} = y_{22} = (y_{11a} + y_{11b}) = \frac{2s(1+2s)}{1+4s} + \frac{(1+2s)}{4+4s} = \frac{(1+2s)\,(8s^2+12s+1)}{4(s+1)\,(4s+1)}$$

and $$y_{12} = y_{21} = (y_{12a} + y_{12b}) = -\frac{4s^2}{1+4s} - \frac{1}{4(s+1)} = -\frac{16s^3+16s^2+4s+1}{4(4s+1)\,(s+1)}$$

7.23 A network has two input terminals a, b and two output terminals c, d. The input impedance with c-d open-circuited is $(250 + j100)$ ohm and with c-d short-circuited is $(400 + j300)$ ohm. The impedance across c-d with a-b open-circuited is 200 ohm. Determine the equivalent T-network parameters.

Solution For c-d Terminals opened,

$$(Z_A + Z_B) = (250 + j100) \tag{i}$$

But, for c-d terminals shorted,

$$Z_A + \frac{Z_B Z_C}{Z_B + Z_C} = (400 + j300) \tag{ii}$$

Again, with a-b terminals opened,

$$(Z_B + Z_C) = 200 \tag{iii}$$

From (ii) and (i), we get,

$$\frac{Z_B Z_C}{Z_B + Z_C} - Z_B = 150 + j200$$

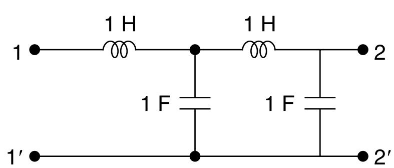
or $$Z_B Z_C - Z_B^2 - Z_B Z_C = 200(150 + j200) \qquad \{\text{by (iii)}$$

or $$Z_B^2 = 200(-150 - j200) = 10^4(1 - j2)^2$$

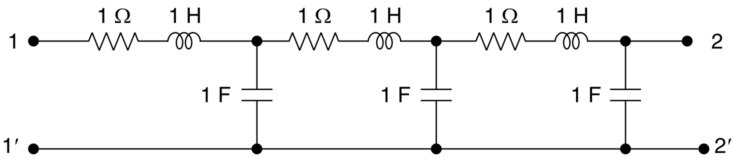
$$\left.\begin{array}{ll} \therefore & Z_B = (100 - j200)\Omega \\ \therefore & Z_A = (150 + j300)\Omega \\ \textit{and} & Z_C = (100 + j200)\Omega \end{array}\right\}$$

7.24 Find the driving point impedance at the terminals 1-1′ of the ladder network shown in figure.

(a)



(b)



Solution

(a) The driving point impedance at 1-1′ is

$$Z_{11} = s + \cfrac{1}{s + \cfrac{1}{s + \cfrac{1}{s}}} = \frac{s^4 + 3s^2 + 1}{s^2 + 2s}$$

(b) The driving point impedance at 1-1′ is,

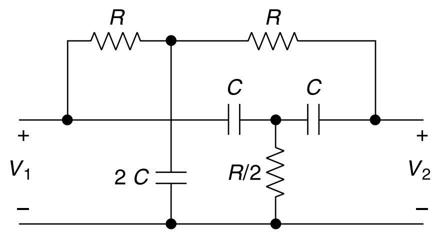
$$Z_{11} = (s+1) + \cfrac{1}{s + \cfrac{1}{(s+1) + \cfrac{1}{s + \cfrac{1}{(s+1) + \cfrac{1}{s}}}}} = \frac{s^6 + 3s^5 + 8s^4 + 11s^3 + 11s^2 + 6s + 1}{s^5 + 2s^4 + 5s^3 + 4s^2 + 3s}$$

7.25 For the Notch-filter (Twin-T) network, determine:

(a) y-parameters,

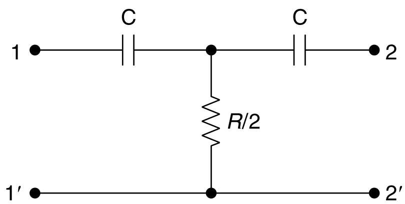
(b) the voltage ratio transfer function V_2/V_1 when no-load impedance is present, and

(c) the value of the frequency at which the output voltage is zero.

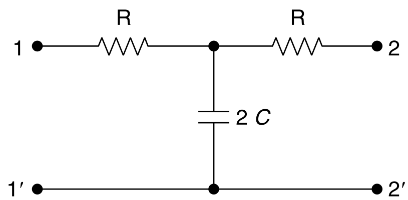


Solution

(a) The given network is the parallel combination of the two networks:



(a) Network



(b) Network

For network (a),

$$z_{11a} = \left(\frac{1}{Cs} + \frac{R}{2}\right) = \frac{2 + RCs}{2Cs};\ z_{12a} = z_{21a} = \frac{R}{2};\ z_{22a} = \left(\frac{1}{Cs} + \frac{R}{2}\right) = \frac{2 + RCs}{2Cs}$$

$$\therefore \qquad \Delta z_a = \frac{1 + RCs}{C^2 s^2}$$

$$\therefore \quad y_{11a} = \frac{z_{22a}}{\Delta z_a} = \frac{RCs(2+RCs)}{2R(1+RCs)}; \qquad y_{12a} = y_{21a} = -\frac{z_{12a}}{\Delta z_a} = -\frac{R^2C^2s^2}{2R(1+RCs)};$$

$$y_{22a} = \frac{z_{11a}}{\Delta z_a} = \frac{Cs\left(1+\frac{1}{2}Cs\right)}{(1+RCs)}$$

For network (b),

$$z_{11b} = \left(\frac{1}{2Cs} + R\right) = \frac{1+2RCs}{2Cs}; \; z_{12b} = z_{21b} = \frac{1}{2Cs}; \; z_{22b} = \left(\frac{1}{s} + 2\right) = \frac{1+2RCs}{2Cs}$$

$$\therefore \quad \Delta z_b = \frac{1+RCs}{C^2s^2}$$

$$\therefore \quad y_{11b} = \frac{z_{22b}}{\Delta z_b} = \frac{(1+2RCs)}{2R(RCs+1)}; \qquad y_{12b} = y_{21b} = -\frac{z_{12b}}{\Delta z_b} = -\frac{1}{2R(RCs+1)};$$

$$y_{22b} = \frac{z_{11b}}{\Delta z_b} = \frac{(1+2RCs)}{2R(RCs+1)}$$

Thus, the overall y-parameters are,

$$y_{11} = y_{22} = (y_{11a} + y_{11b}) = \frac{RCs(2+RCs)}{2R(1+RCs)} + \frac{(1+2RCs)}{2R(RCs+1)} = \frac{(R^2C^2s^2+4RCs+1)}{2R(RCs+1)}$$

and
$$y_{12} = y_{21} = (y_{12a} + y_{12b}) = -\frac{R^2C^2s^2}{2R(1+RCs)} - \frac{1}{2R(RCs+1)} = -\frac{R^2C^2s^2+1}{2R(RCs+1)}$$

(b) Now,
$$I_1 = y_{11}V_1 + y_{12}V_2$$
$$I_2 = y_{21}V_1 + y_{22}V_2$$

When no-load impedance is present, $I_2 = 0$,

$$\therefore \quad \frac{V_2}{V_1} = -\frac{y_{21}}{y_{22}} = \frac{R^2C^2s^2+1}{2R(RCs+1)} \times \frac{2R(RCs+1)}{(R^2C^2s^2+4RCs+1)} = \frac{R^2C^2s^2+1}{(R^2C^2s^2+4RCs+1)}$$

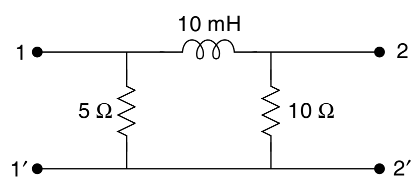
(c) For $V_2 = 0 \Rightarrow 1 + R^2C^2s^2 = 0$

Putting $s = j\omega$, $1 - \omega^2R^2C^2 = 0$

$$\therefore \quad \omega = \frac{1}{RC}$$

Thus, the notch frequency is given by, $f_N = \dfrac{1}{2\pi RC}$

7.26 Find the open circuit impedance parameters for the two-port network shown in the figure below.



Solution For this π-network, the y-parameters are given as,

$$y_{11} = \left(\frac{1}{5} + \frac{1}{0.01s}\right) = \left(0.2 + \frac{100}{s}\right);$$

$$y_{12} = y_{21} = -\frac{1}{0.01s} = -\frac{100}{s};$$

$$y_{22} = \left(\frac{1}{10} + \frac{1}{0.01s}\right) = \left(0.1 + \frac{100}{s}\right)$$

$$\therefore \qquad \Delta y = (y_{11}y_{22} - y_{12}y_{21}) = \left(0.2 + \frac{100}{s}\right) \times \left(0.1 + \frac{100}{s}\right) - \left(-\frac{100}{s}\right)^2$$

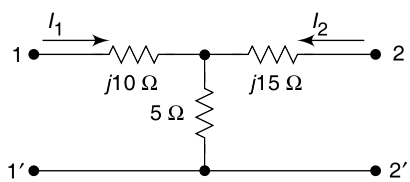
$$= 0.02 + \frac{30}{s} + \left(\frac{100}{s}\right)^2 - \left(-\frac{100}{s}\right)^2$$

$$= \left(0.02 + \frac{30}{s}\right)$$

Thus, the z-parameters are,

$$\left.\begin{aligned} z_{11} &= \frac{y_{22}}{\Delta y} = \frac{0.1 + 100/s}{0.02 + 30/s} = \frac{0.1s + 100}{0.02s + 30} = \frac{5s + 5000}{s + 1500}\ \Omega \\ z_{12} = z_{21} &= -\frac{y_{12}}{\Delta y} = -\frac{-100/s}{0.02 + 30/s} = \frac{100}{0.02s + 30} = \frac{5000}{s + 1500}\ \Omega \\ z_{22} &= \frac{y_{11}}{\Delta y} = \frac{0.2 + 100/s}{0.02 + 30/s} = \frac{0.2s + 100}{0.02s + 30} = \frac{10s + 5000}{s + 1500}\ \Omega \end{aligned}\right\} \quad \textit{Ans.}$$

7.27 Find the open-circuit impedance parameters of the circuit given in the figure. Also, find the h-parameters of the circuit.



Solution By KVL,

$$(j10 + 5)I_1 + 5I_2 = V_1 \qquad \text{(i)}$$

and

$$5I_1 + (j15 + 5)I_2 = V_2 \qquad \text{(ii)}$$

Thus, the z-parameters are:

$$z_{11} = (5 + j10)\ \Omega \quad z_{12} = z_{21} = 5\ \Omega \quad Z_{22} = (5 + j15)\ \Omega \qquad \textit{Ans.}$$

The hybrid parameter matrix may be written as

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

From Eq (ii), we get,

$$I_2 = -\frac{5}{5+j15}I_1 + \frac{V_2}{5+j15}$$

$$= -\frac{1}{1+j3}I_1 + \frac{1}{5+j15}V_2 \qquad \text{(iii)}$$

Putting this value of I_2 in Eq (i), we get,

$$(5+j10)I_1 + 5\left[-\frac{5}{5+j15}I_1 + \frac{V_2}{5+j15}\right] = V_1$$

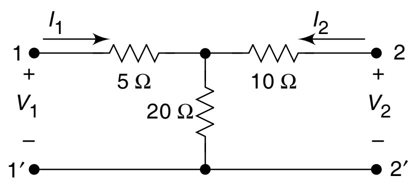
$$\Rightarrow \qquad V_1 = \frac{(5+j10)\times(5+j15)-25}{(5+j15)}I_1 + \frac{5}{5+j15}V_2$$

$$= \frac{30+j25}{1+j3}I_1 + \frac{1}{1+j3} \qquad \text{(iv)}$$

Comparing Eq (iii) and (iv) with the standard equations of h-parameters, we get,

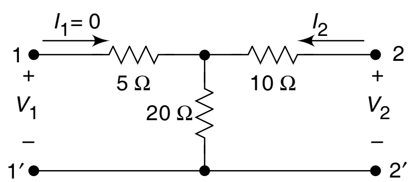
$$h_{11} = \frac{30+j25}{1+j3}\,\Omega;\quad h_{12} = \frac{1}{1+j3};\quad h_{21} = -\frac{1}{1+j3};\quad h_{22} = \frac{1}{5+j15}\mho \qquad \textit{Ans.}$$

7.28 Determine the z-parameters for the network shown in the figure.



Solution We consider two situations:

(a) When $I_1 = 0$, i. e. port-1 is open-circuited: In this case no current will flow through the 5Ω resistor.



Figure(a) *When $I_1 = 0$*

By KVL in the right mesh, we get,

$$10I_2 + 20I_2 - V_2 = 0$$

$$\therefore \qquad z_{22} = \left|\frac{V_2}{I_2}\right|_{I_1=0} = 30\ \Omega$$

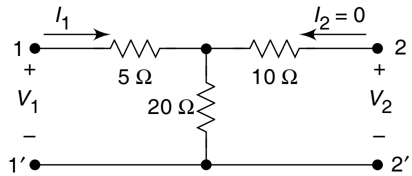
From Fig. (a), we get,

$$V_1 = 20I_2$$

$$\therefore \qquad z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = 20\ \Omega$$

(b) When $I_2 = 0$, i.e., port-2 is open-circuited: In this case no current will flow through the 10 Ω resistor.

By KVL in the left mesh, we get,



$$5I_1 + 20I_1 - V_1 = 0$$

$$\therefore \qquad z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 25\ \Omega$$

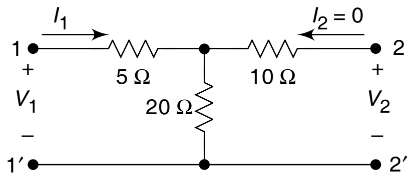


Figure (b) *When $I_2 = 0$*

From Fig. (b), we get,

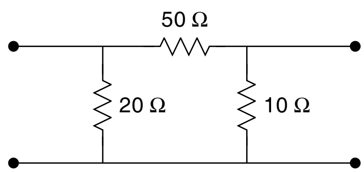
$$V_2 = 20I_1$$

$$\therefore \qquad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = 20\ \Omega$$

Therefore, the z-parameters of the network are:

$$[z] = \begin{bmatrix} 25 & 20 \\ 20 & 30 \end{bmatrix} (\Omega) \qquad \textit{Ans.}$$

7.29 Find the y-parameters for the network shown in the figure.



Solution We consider two situations:

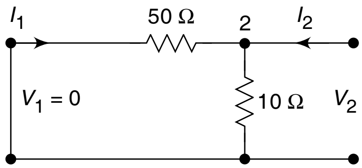
When $V_1 = 0$, i.e., port-1 is short-circuited

In this case, no current will flow through the 20 Ω resistor. The modified circuit is shown in Fig. (a).

By KCL at node 2,

$$\frac{V_2 - 0}{10} + \frac{V_2 - 0}{50} = I_2$$

$$\therefore \qquad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{1}{10} + \frac{1}{50} = 0.12\ \mho \qquad \textit{Ans.}$$



Figure(a) *When $V_1 = 0$*

Also, from Fig. 7.5 (a) we get,

$$I_1 = \frac{0 - V_2}{50}$$

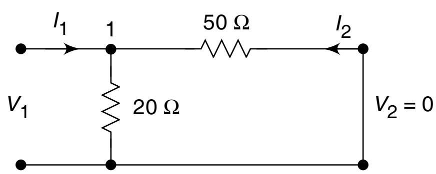
$$\therefore \qquad y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \frac{1}{50} = 0.02 \ \mho \qquad \textit{Ans.}$$

When $V_2 = 0$, i.e., port-2 is short-circuited

In this case, no current will flow through the 10 Ω resistor. The modified circuit is shown in Fig. (b). By KCL at node 1,

$$\frac{V_1 - 0}{20} + \frac{V_1 - 0}{50} = I_1$$

$$\therefore \qquad y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{1}{20} + \frac{1}{50} = 0.07 \ \mho \qquad \textit{Ans.}$$



Figure(b) *When $V_2 = 0$*

Also, from Fig. 7.5 (b) we get,

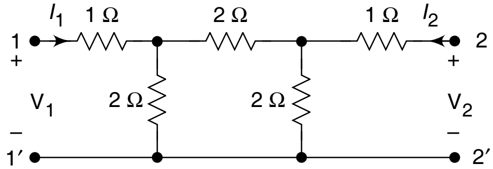
$$I_2 = \frac{0 - V_1}{50}$$

$$\therefore \qquad y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{1}{50} = 0.02 \ \mho \qquad \textit{Ans.}$$

Therefore, the y-parameters of the network are

$$[y] = \begin{bmatrix} 0.07 & 0.02 \\ 0.02 & 0.12 \end{bmatrix} \mho \qquad \textit{Ans.}$$

7.30 For the network shown in the figure, determine the *ABCD* parameters.



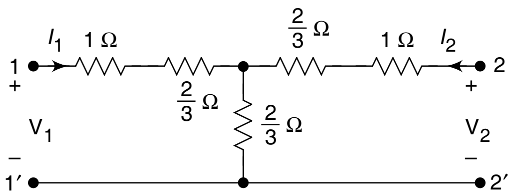
Solution The *ABCD*-parameter equations are,

$$V_1 = AV_2 - BI_2$$

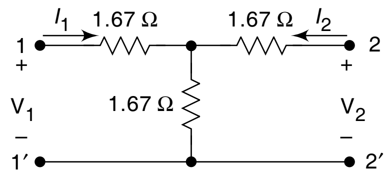
$$I_1 = CV_2 - DI_2$$

For the network shown in the figure. we convert the delta consisting of the resistances of 2 Ω each into its equivalent star so that the circuit becomes as shown in Fig. (a) and Fig. (b).

$$r_1 = r_2 = r_3 = \frac{2\times 2}{2+2+2} = \frac{2}{3}\Omega$$



(a) Modified network



(b)

To find the *ABCD* parameters, we consider two situations:

When $V_2 = 0$, i.e., port-2 is short-circuited

As shown in Fig. (c), by KVL we get,

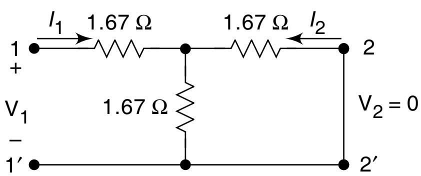
$$1.67I_1 + 0.67(I_1 + I_2) = V_1$$

or,
$$2.33I_1 + 0.67I_2 = V_1$$

and,
$$0.67(I_1 + I_2) + 1.67I_2 = 0$$

or,
$$I_1 = -\frac{2.33}{0.67}I_2 = -3.5I_2$$

$\therefore$
$$D = \left|-\frac{I_1}{I_2}\right|_{V_2=0} = 3.5$$



(c)

Putting this value in the first equations, we get,

$$2.33 \times (-3.5)I_2 + 0.67 I_2 = V_1 \quad \Rightarrow \quad B = \left| -\frac{V_1}{I_2} \right|_{V_2=0} = 7.5\ \Omega$$

When $I_2 = 0$, i. e. port-2 is open-circuited

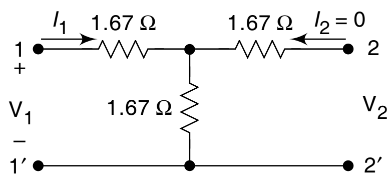
Here, no current will flow through the right side 1.67 Ω resistance. By KVL, we get,

$$V_1 = (1.67 + 0.67)I_1 = 2.33 I_1$$

and, $$V_2 = 0.67 I_1$$

∴ $$C = \left| \frac{I_1}{V_2} \right|_{I_2=0} = \frac{1}{0.67} = 1.5\ ℧$$

∴ $$A = \left| \frac{V_1}{V_2} \right|_{I_2=0} = \frac{2.33 I_1}{0.67 I_1} = 3.5$$

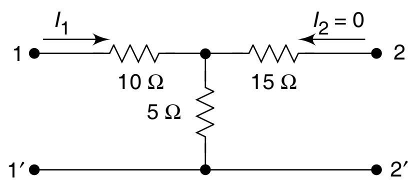


(d)

Therefore, the *ABCD* parameters of the network are

$A = 3.5$; $B = 7.5\ \Omega$; $C = 15\ ℧$; and $D = 3.5$ *Ans.*

7.31 Find the hybrid parameters for the network shown in the figure.



Solution By KVL,

$$15 I_1 + 5 I_2 = V_1 \tag{i}$$

$$5 I_1 + 20 I_2 = V_2 \tag{ii}$$

Thus, the z-parameters are

$$z_{11} = (5 + j10)\ \Omega \quad z_{12} = z_{21} = 5\ \Omega \quad Z_{22} = (5 + j15)\ \Omega \qquad \textit{Ans.}$$

The hybrid parameter equations are,

$$V_1 = h_{11}\ I_1 + h_{12}\ V_2$$

$$I_2 = h_{21}\ I_1 + h_{22}\ V_2$$

From Eq (ii), we get,

$$I_2 = -\frac{5}{20} I_1 + \frac{V_2}{20}$$

$$= -\frac{1}{4} I_1 + \frac{1}{20} V_2 \tag{iii}$$

Putting this value of I_2 in Eq (i), we get,

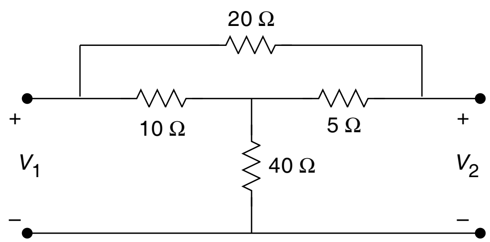
$$15I_1 + 5\left[-\frac{1}{4} I_1 + \frac{V_2}{20}\right] = V_1$$

$$\Rightarrow \qquad V_1 = \frac{55}{4} I_1 + \frac{1}{4} V_2 \tag{iv}$$

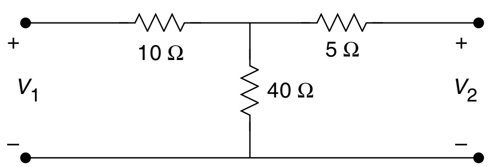
Comparing Eq (iii) and (iv) with the standard equations of h-parameters, we get,

$$h_{11} = \frac{55}{4}\,\Omega; \quad h_{12} = \frac{1}{4}; \quad h_{21} = -\frac{1}{4}; \quad h_{22} = \frac{1}{20}\,℧ \qquad \textit{Ans.}$$

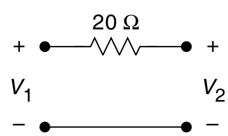
7.32 Find the y parameters for the following network:



Solution This two-port network can be considered as the parallel connection of two two-port networks as shown below.



(a)



(b)

For network (a), the z-parameters are:

$$z_{11a} = 50\ \Omega; \quad z_{12a} = z_{21a} = 40\ \Omega; \quad z_{22a} = 45\ \Omega; \qquad \therefore \Delta z = \left(50 \times 45 - 40^2\right) = 650$$

Thus, the y-parameters are

$$y_{11a} = \frac{z_{22a}}{\Delta z} = \frac{45}{650} = \frac{9}{130} \text{ mho}$$

$$y_{12a} = y_{21a} = -\frac{z_{12}}{\Delta z} = -\frac{40}{650} = -\frac{4}{65} \text{ mho}$$

$$y_{22a} = \frac{z_{11a}}{\Delta z} = \frac{50}{650} = \frac{1}{13} \text{ mho}$$

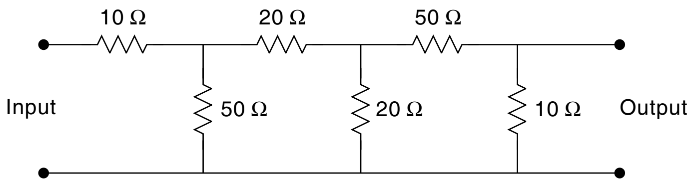
For network (b), the y-parameters are

$$y_{11b} = y_{22b} = \frac{1}{20} \text{ mho}; \quad y_{12b} = y_{21b} = -\frac{1}{20} \text{ mho}$$

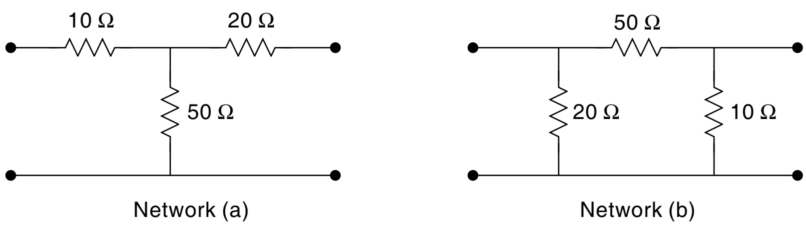
We know that for parallel connection of two two-port networks the overall y-parameters are the summation of individual y-parameters. Thus,

$$\left.\begin{aligned} y_{11} &= (y_{11a} + y_{11b}) = \left(\frac{9}{130} + \frac{1}{20}\right) = 0.119 \text{ mho} \\ y_{12} &= y_{21} = (y_{12a} + y_{12b}) = \left(-\frac{4}{65} - \frac{1}{20}\right) = -0.111 \text{ mho} \\ y_{22} &= (y_{22a} + y_{22b}) = \left(\frac{1}{13} + \frac{1}{20}\right) = 0.127 \text{ mho} \end{aligned}\right\} \quad \textit{Ans.}$$

7.33 Obtain the $ABCD$ parameters for the network shown in the figure.



Solution This two-port network can be considered as the cascade connection of two two-port networks as shown below.



For Network (a), as this is a T-network, the z-parameters are given as,

$$z_{11} = 60\ \Omega; \quad z_{12} = 50\ \Omega; \quad z_{22} = 70\ \Omega; \qquad \therefore \Delta z = (z_{11}z_{22} - z_{12}z_{21}) = (60 \times 70 - 50^2) = 1700$$

$$\therefore \qquad A_a = \frac{z_{11}}{z_{21}} = \frac{60}{50} = \frac{6}{5} \qquad\qquad B_a = \frac{\Delta z}{z_{21}} = \frac{1700}{50} = 34\ \Omega$$

$$C_a = \frac{1}{z_{21}} = \frac{1}{50} \text{ mho} \qquad\qquad D_a = \frac{z_{22}}{z_{21}} = \frac{70}{50} = \frac{7}{5}$$

For Network (b), as this is a π-network, the y-parameters are given as,

$$y_{11} = \left(\frac{1}{50} + \frac{1}{20}\right) = \frac{7}{100} \text{ mho}; \quad y_{12} = y_{21} = -\frac{1}{50} \text{ mho}; \quad y_{22} = \left(\frac{1}{50} + \frac{1}{10}\right) = \frac{3}{25} \text{ mho}$$

$$\therefore \qquad \Delta y = (y_{11}y_{22} - y_{12}y_{21}) = \frac{7}{100} \times \frac{3}{25} - \left(-\frac{1}{50}\right)^2 = \frac{1}{125}$$

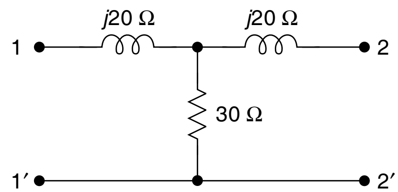
$$\therefore \quad A_b = -\frac{y_{22}}{y_{21}} = -\frac{3/25}{-1/50} = 6 \qquad B_b = -\frac{1}{y_{21}} = -\frac{1}{-1/50} = 50\ \Omega$$

$$C_b = -\frac{\Delta y}{y_{21}} = -\frac{1/125}{-1/50} = \frac{2}{5}\text{ mho} \qquad D_b = -\frac{y_{11}}{y_{21}} = -\frac{7/100}{-1/50} = \frac{7}{2}$$

For the entire network, the *ABCD* parameters are given as,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \times \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} = \begin{bmatrix} 6/5 & 34 \\ 1/50 & 7/5 \end{bmatrix} \times \begin{bmatrix} 6 & 50 \\ 2/5 & 7/5 \end{bmatrix} = \begin{bmatrix} 20.8 & 179 \\ 0.68 & 5.9 \end{bmatrix} \qquad \textit{Ans.}$$

7.34 Calculate the *ABCD* parameters of the network shown in the figure below.



Solution For this *T*-circuit, the *z*-parameters are given as,

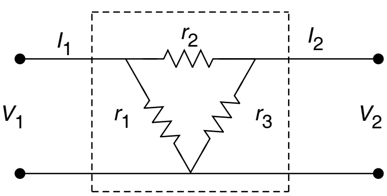
$$z_{11} = z_{22} = (30 + j20)\ \Omega$$

$$z_{12} = z_{21} = 30\ \Omega$$

$$\therefore \quad \Delta z = (z_{11}z_{22} - z_{12}\,z_{21}) = (30 + j20)^2 - 30^2 = (60 + j20)j20 = (-400 + j1200)$$

$$\left.\begin{aligned} \therefore \quad A &= \frac{z_{11}}{\Delta z} = \frac{30 + j20}{(60 + j20)j20} = \left(1 + j\frac{2}{3}\right) \\ \therefore \quad B &= \frac{\Delta z}{z_{21}} = \frac{(60 + j20)j20}{30} = \left(-\frac{40}{3} + j40\right)\Omega \\ \therefore \quad C &= \frac{1}{z_{12}} = \frac{1}{30}\ mho \\ \therefore \quad D &= \frac{z_{22}}{z_{12}} = \frac{30 + j20}{30} = \left(1 + j\frac{2}{3}\right) \end{aligned}\right\} \qquad \textit{Ans.}$$

7.35 Determine the hybrid parameters for the network in the figure shown below.



Solution For this π-network, the *y*-parameters are given as,

$$y_{11} = \left(\frac{1}{r_1} + \frac{1}{r_2}\right) = \left(\frac{r_1 + r_2}{r_1\,r_2}\right); \quad y_{12} = y_{21} = -\frac{1}{r_2}; \qquad y_{22} = \left(\frac{1}{r_2} + \frac{1}{r_3}\right) = \left(\frac{r_2 + r_3}{r_2\,r_3}\right)$$

By inter-relationship, the h-parameters are obtained as,

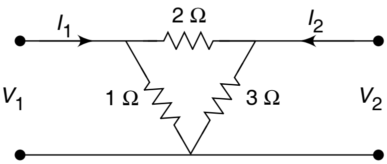
$$h_{11}=\frac{1}{y_{11}}=\left(\frac{r_1 r_2}{r_1+r_2}\right)$$

$$h_{12}=-\frac{y_{12}}{y_{11}}=-\frac{-\frac{1}{r_2}}{\left(\frac{r_1+r_2}{r_1 r_2}\right)}=\frac{r_1}{r_1+r_2}$$

$$h_{21}=\frac{y_{21}}{y_{11}}=\frac{-\frac{1}{r_2}}{\left(\frac{r_1+r_2}{r_1 r_2}\right)}=-\frac{r_1}{r_1+r_2}$$

$$h_{22}=\frac{\Delta y}{y_{11}}=\left\{\frac{(r_1+r_2)(r_2+r_3)-r_1 r_3}{r_1 r_2^2 r_3}\right\}\times\left(\frac{r_1 r_2}{r_1+r_2}\right)=\frac{(r_1+r_2)(r_2+r_3)-r_1 r_3}{r_2(r_1+r_2)}$$

7.36 Find the hybrid parameters of the circuit given in the figure.



Solution For this π-network, the y-parameters are given as,

$$y_{11}=\left(\frac{1}{1}+\frac{1}{2}\right)=\frac{3}{2};\quad y_{12}=y_{21}=-\frac{1}{2};\qquad y_{22}=\left(\frac{1}{2}+\frac{1}{3}\right)=\frac{5}{6}$$

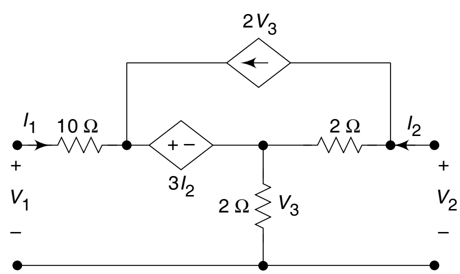
$$\therefore\qquad \Delta y=y_{11}\,y_{22}-y_{12}\,y_{21}=\frac{3}{2}\times\frac{5}{6}-\left(-\frac{1}{2}\right)^2=1$$

By inter-relationship, the h-parameters are obtained as,

$$h_{11}=\frac{1}{y_{11}}=\frac{2}{3}\,\Omega \qquad h_{12}=-\frac{y_{12}}{y_{11}}=-\frac{-1/2}{3/2}=\frac{1}{3}$$

$$h_{21}=\frac{y_{21}}{y_{11}}=\frac{-\frac{1}{2}}{3/2}=-\frac{1}{3} \qquad h_{22}=\frac{\Delta y}{y_{11}}=1\times\frac{3}{2}=\frac{3}{2}\ \mho$$

7.37 For the network shown in the figure, determine the z and y parameters.



Solution By KVL for the three meshes, we get,

$V_1 = 10I_1 + 3I_2 + 2(I_1 + I_2) \Rightarrow 12I_1 + 5I_2 = V_1$ (i)

$V_2 = 2(I_2 - 2V_3) + 2(I_1 + I_2) \Rightarrow 2I_1 + 4I_2 - 4V_3 = V_2$ (ii)

$V_3 = 2(I_1 + I_2)$ (iii)

From (ii) and (iii),

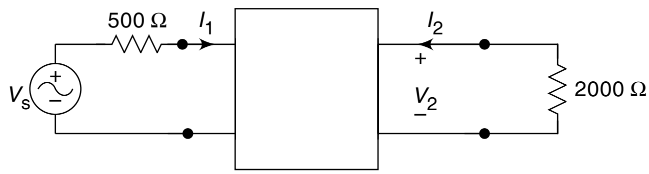
$$V_2 = 2I_1 + 4I_2 - 4(2I_1 + 2I_2) \Rightarrow V_2 = -6I_1 - 4I_2 \qquad \text{(iv)}$$

From (i) and (iv), we get,

$$z = \begin{bmatrix} 12 & 5 \\ -6 & -4 \end{bmatrix} (\Omega) \qquad \textit{Ans.}$$

$$\therefore \qquad y = [z]^{-1} = \begin{bmatrix} 2/9 & 5/18 \\ -1/3 & -2/3 \end{bmatrix} (\mho) \qquad \textit{Ans.}$$

7.38 The h-parameters of a two-port network shown in figure are $h_{11} = 1000\Omega$, $h_{12} = 0.003$, $h_{21} = 100$, and $h_{22} = 50 \times 10^{-6}$ mho. Find V_2 and z-parameters of the network if $V_s = 10^{-2}\angle 0^\circ\ (V)$.



Solution The h-parameter equations are,

$$V_1 = h_{11}I_1 + h_{12}V_2 = 1000I_1 + 0.003V_2 \qquad \text{(i)}$$

$$I_2 = h_{21}I_1 + h_{22}V_2 = 100I_1 + 50 \times 10^{-6}V_2 \qquad \text{(ii)}$$

By KVL for the two meshes,

$$V_1 = V_s - 500I_1 \qquad \text{(iii)}$$

$$V_2 = -200I_2 \qquad \text{(iv)}$$

From (i) and (iii),

$$V_s - 500I_1 = 1000I_1 + 0.003V_2$$

or,

$$10^{-2} - 1500I_1 = 0.003V_2 \qquad \text{(v)}$$

From (ii) and (iv),

$$-\frac{V_2}{2000} = 100I_1 + 50 \times 10^{-6}V_2$$

or,

$$I_1 = -5.5 \times 10^{-6}V_2 \qquad \text{(vi)}$$

From (v) and (i),

$$0.003V_2 = 10^{-2} + 1500(-5.5 \times 10^{-6}V_2)$$

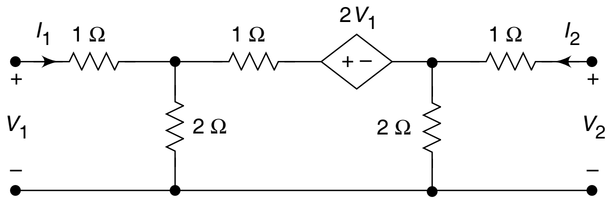
$$\Rightarrow \qquad V_2 = -1.905\ V \qquad \textit{Ans.}$$

The z-parameters are calculated as follows.

$$z_{11} = \frac{\Delta h}{h_{22}} = -500\Omega \quad z_{12} = \frac{h_{12}}{h_{22}} = 60\Omega \quad z_{21} = -\frac{h_{21}}{h_{22}} = -2 \times 10^6\,\Omega$$

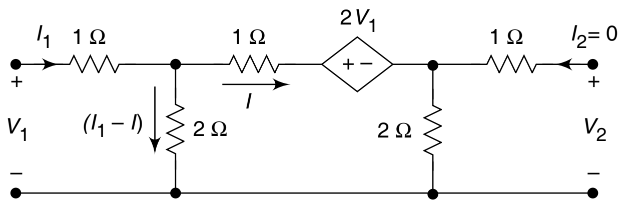
$$z_{22} = \frac{1}{h_{22}} = 20 \times 10^3\,\Omega \qquad \textit{Ans.}$$

7.39 For the two-port network shown in the figure, find the z-parameters.



Solution We consider two cases:

When $I_2 = 0$ Here, as the output port is open-circuited, no current will flow through the 1Ω resistor connected at port 2. The modified circuit is shown in Fig (a).



(a)

By KVL for the middle mesh, we get,

$$I + 2V_1 + 2I - 2 \times (I_1 - I) = 0$$

$$\Rightarrow \qquad I = \left(\frac{2}{5}I_1 - \frac{2}{5}V_1\right) \qquad \text{(i)}$$

By KVL for the left mesh, we get,

$$V_1 = I_1 + 2 \times (I_1 - I) = 3I_1 - 2I$$

$$= 3I_1 - 2 \times \left(\frac{2}{5}I_1 - \frac{2}{5}V_1\right) \qquad \text{\{by equation (i)\}}$$

or, $$V_1 = 11I_1$$

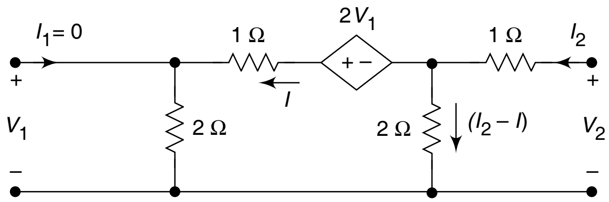
$$\therefore \qquad z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 11\ \Omega$$

Also, by KVL for the right mesh, we get,

$$V_2 = 2I = 2\times\left(\frac{2}{5}I_1 - \frac{2}{5}V_1\right) = \frac{4}{5}I_1 - \frac{4}{5}V_1 = \frac{4}{5}I_1 - \frac{4}{5}\times 11 \times I_1 = -8I_1$$

$$\therefore \qquad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = -8\ \Omega$$

When $I_1 = 0$ Here, as the output port is open-circuited, no current will flow through the 1 Ω resistor connected at port 1. The modified circuit is shown in Fig (b).



(b)

By KVL for the middle mesh, we get,

$$I - 2V_1 + 2I - 2 \times (I_2 - I) = 0$$

$$\Rightarrow \qquad I = \left(\frac{2}{5}I_2 + \frac{2}{5}V_1\right) \qquad \text{(ii)}$$

By KVL for the left mesh, we get,

$$V_1 = 2I = 2 \times \left(\frac{2}{5}I_2 + \frac{2}{5}V_1\right) = \frac{4}{5}I_2 + \frac{4}{5}V_1$$

$$\Rightarrow \qquad V_1 = 4I_2$$

$$\therefore \qquad z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=} = 4\ \Omega$$

Also, by KVL for the right mesh, we get,

$$V_2 = I_2 + 2 \times (I_2 - I) = 3I_2 - 2I$$

$$= 3I_2 - 2 \times \left(\frac{2}{5}I_2 + \frac{2}{5}V_1\right) \qquad \{by\ equation\ (ii)\}$$

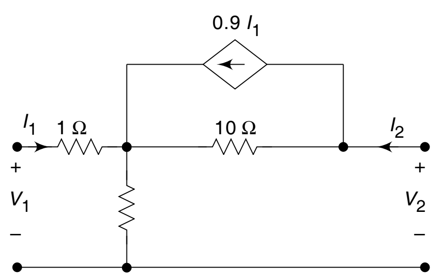
$$= \frac{11}{5}I_2 - \frac{4}{5}V_1 = \frac{11}{5}I_2 - \frac{4}{5} \times 4I_2 = -I_2$$

$$\therefore \qquad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = -1\ \Omega$$

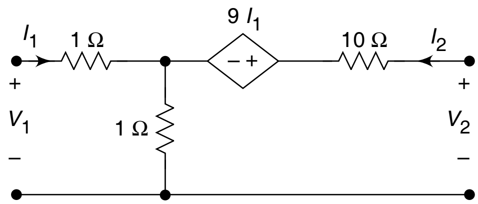
Therefore, the z-parameters of the network are,

$$[z] = \begin{bmatrix} 11 & 4 \\ -8 & -1 \end{bmatrix} (\Omega) \qquad Ans.$$

7.40 Find the z and y parameters of the network shown in the figure.



Solution We convert the dependent current source into its equivalent voltage source as shown in the figure below.



By KVL for the two meshes, we get,

$$I_1 + 1 \times (I_1 + I_2) = V_1 \Rightarrow V_1 = 2I_1 + I_2 \qquad \text{(i)}$$

and,
$$10I_2 + 9I_1 + 1 \times (I_1 + I_2) = V_2 \Rightarrow V_2 = 10I_1 + 11I_2 \qquad \text{(ii)}$$

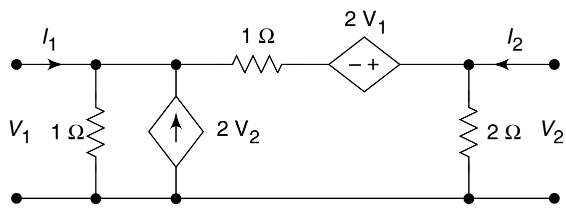
From (i) and (ii), we get the z-parameters as,

$$[z] = \begin{bmatrix} 2 & 1 \\ 10 & 11 \end{bmatrix} (\Omega) \qquad \textit{Ans.}$$

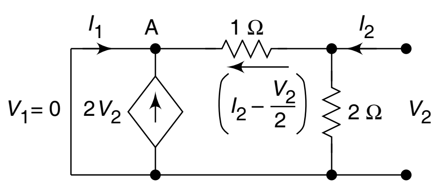
Therefore, the y-parameters are,

$$[y] = [z]^{-1} = \begin{bmatrix} 2 & 1 \\ 10 & 11 \end{bmatrix}^{-1} = \begin{bmatrix} 11/12 & -1/12 \\ -10/12 & 2/12 \end{bmatrix} \mho \qquad \textit{Ans.}$$

7.41 The network shown in the figure contains both dependent current source and dependent voltage source. For this circuit, determine the y and z parameters.



Solution We first find out the y parameters. To find the y parameters, we consider two situations:
When $V_1 = 0$ Here, port 1 is shorted and hence, the dependent voltage source is zero, i.e., short-circuited. The 1 Ω resistance in port 1 becomes redundant. The circuit is shown in Fig (a).



(a)

By KCL at node (A), we get,

$$-I_1 - 2V_2 - \left(I_2 - \frac{V_2}{2}\right) = 0 \Rightarrow I_1 + I_2 = -\frac{3V_2}{2} \qquad \text{(i)}$$

By KVL for the outer loop, we get,

$$V_2 = 1\times\left(I_2 - \frac{V_2}{2}\right) = I_2 - \frac{V_2}{2}$$

$$\Rightarrow \qquad \frac{3}{2}V_2 = I_2$$

$$\therefore \qquad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{3}{2}\,\mho$$

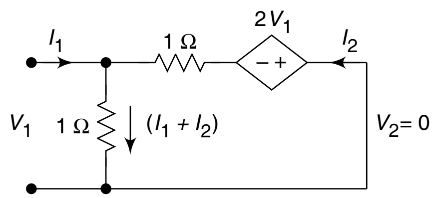
Substituting the value of I_2 in (i), we get,

$$I_1 + \frac{3}{2}V_2 = -\frac{3}{2}V_2$$

$$\Rightarrow \qquad I_1 = -3V_2$$

$$\therefore \qquad y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -3\,\mho$$

When $V_2 = 0$ Here, port 2 is shorted and hence, the dependent current source is zero, i.e., open-circuited. The 2 Ω resistance in port 2 becomes redundant. The circuit is shown in Fig (b).



(b)

By KVL for the left loop, we get,

$$V_1 = (I_1 + I_2) \qquad \text{(ii)}$$

By KVL for the outer loop, we get,

$$2V_1 + I_2 + V_1 = 0 \quad \Rightarrow \quad I_2 = -3V_1$$

$$\therefore \qquad y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -3\,\mho$$

From (ii),

$$V_1 = I_1 - 3V_1 \quad \Rightarrow \quad I_1 = 4V_1$$

$$\therefore \qquad y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = 4\,\mho$$

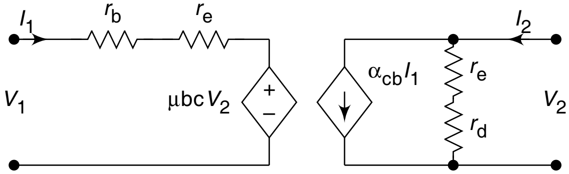
Therefore, the y parameters of the netwrok is given as,

$$[y] = \begin{bmatrix} 4 & -3 \\ -3 & \frac{3}{2} \end{bmatrix} \qquad \textit{Ans.}$$

Hence, the z parameters are given as,

$$[z]=[y]^{-1}=\begin{bmatrix} 4 & -3 \\ -3 & \frac{3}{2} \end{bmatrix}^{-1}=\begin{bmatrix} \frac{-1}{2} & -1 \\ -1 & -\frac{4}{3} \end{bmatrix}(\Omega) \qquad Ans.$$

7.42 The model of a transistor in *CE* mode is shown in the figure. Determine the *h* parameters of the model.

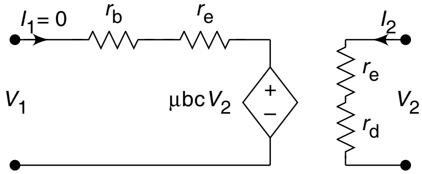


Solution The equations of *h* parameters are,

$$V_1 = h_{11} I_1 + h_{12} V_2$$
$$I_2 = h_{21} I_1 + h_{22} V_2$$

To find *h* parameters, we consider two cases:

When $I_1 = 0$ Here, the dependent current source is open-circuited. The modified circuit is shown in Fig (a).



(a)

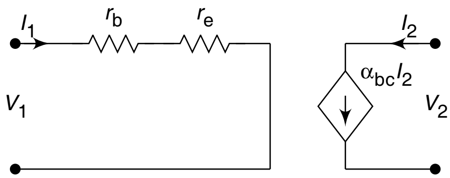
$\therefore$ $$V_1 = \mu_{bc} V_2$$

$\Rightarrow$ $$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \mu_{bc}$$

Also, $V_2 = I_2 (r_e + r_d)$

$\Rightarrow$ $$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{1}{r_e + r_d} \mho$$

When $V_2 = 0$ Here, the dependent voltage source is short-circuited. The modified circuit is shown in Fig (b).



(b)

$$\therefore \qquad V_1 = I_1(r_b + r_e)$$

$$\Rightarrow \qquad h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = (r_b + r_e)\,\Omega$$

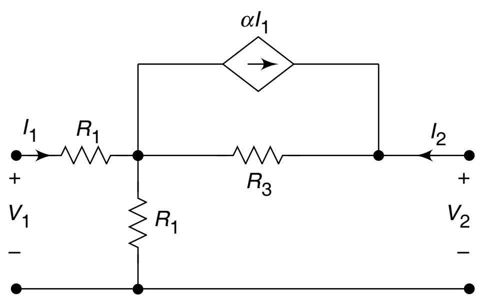
Also, $I_2 = \alpha_{cb} I_1$

$$\Rightarrow \qquad h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \alpha_{cb}$$

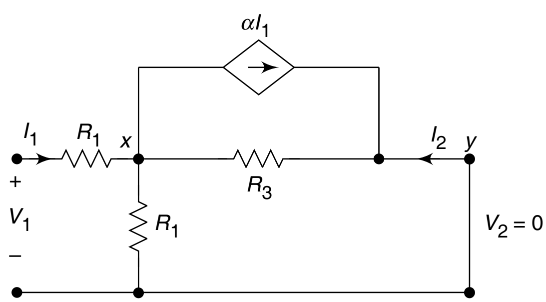
Therefore, the h parameters for the transistor model is given as,

$$[h] = \begin{bmatrix} (r_b + r_e) & \mu_{bc} \\ \alpha_{cb} & \dfrac{1}{r_e + r_d} \end{bmatrix} \qquad \textit{Ans.}$$

7.43 Find the hybrid parameters for the network of the figure (which represents a transistor).



Solution **Case (I): When $V_2 = 0$**
The circuit is modified as shown in the figure.



By KCL at node x,

$$\frac{V_x}{R_2} + \frac{V_x}{R_3} + \alpha I_1 = I_1 \qquad \Rightarrow \qquad V_x = (1-\alpha)\frac{R_2 R_3}{R_2 + R_3} I_1$$

By KVL,

$$V_1 = I_1 R_1 + V_x = I_1 R_1 + (1-\alpha)\left(\frac{R_2 R_3}{R_2 + R_3}\right) I_1$$

$$\therefore \qquad h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \left[R_1 + \frac{(1-\alpha)R_2 R_3}{R_2 + R_3}\right] \qquad \textit{Ans.}$$

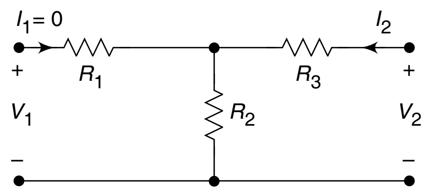
By KCL at node y,

$$\frac{0-V_x}{R_3} = I_2 + \alpha I_1 \quad \Rightarrow \quad I_2 = -\alpha I_1 - (1-\alpha)\left(\frac{R_2 R_3}{R_2 + R_3}\right) I_1 = -I_1\left(\frac{R_2 + \alpha R_3}{R_2 + R_3}\right)$$

$$\therefore \qquad h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = -\left(\frac{R_2 + \alpha R_3}{R_2 + R_3}\right) \qquad \textit{Ans.}$$

Case (II): When $I_1 = 0$

Here, the dependent current source is to be opened (since $I_1 = 0$). The circuit is modified as shown in the figure.



$$\therefore \qquad V_2 = I_2(R_2 + R_3)$$

and $\qquad V_1 = I_2 R_2$

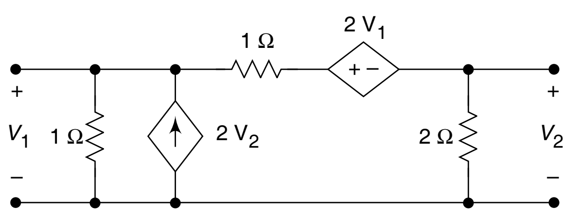
$$\therefore \qquad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{R_2}{R_2 + R_3} \qquad \textit{Ans.}$$

and $$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{1}{R_2 + R_3} \qquad \textit{Ans.}$$

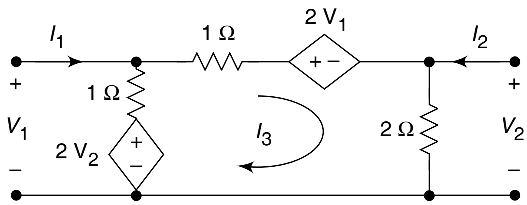
Therefore, the hybrid parameters are

$$h_{11} = \left[R_1 + \frac{(1-\alpha)R_2 R_3}{R_2 + R_3}\right]; \quad h_{12} = \frac{R_2}{R_2 + R_3}; \quad h_{21} = -\left(\frac{R_2 + \alpha R_3}{R_2 + R_3}\right); \quad h_{22} = \frac{1}{R_2 + R_3} \qquad \textit{Ans.}$$

7.44 Determine the *y* and *z* parameters for the network shown in the figure.



Solution We convert the dependent current source into equivalent dependent voltage source. The modified network is shown in the figure.



By KVL for three meshes, we get,

$$V_1 = 1 \times (I_1 - I_3) + 2V_2 \Rightarrow I_3 = I_1 + 2V_2 - V_1 \qquad \text{(i)}$$

and $$1 \times I_3 - 2V_1 + 2(I_2 + I_3) - 2V_2 + 1 \times (I_3 - I_1) = 0 \Rightarrow 2V_1 + 2V_2 = -I_1 + 2I_2 + 4I_3 \qquad \text{(ii)}$$

and, $$V_2 = 2 \times (I_2 + I_3) \qquad \text{(iii)}$$

Substituting the value of I_3 from (i) into (ii) and (iii), we get,

$$2V_1 + 2V_2 = -I_1 + 2I_2 + 4(I_1 + 2V_2 - V_1) \Rightarrow 6V_1 - 6V_2 = 3I_1 + 2I_2 \qquad \text{(iv)}$$

and, $$V_2 = 2(I_2 + I_1 + 2V_2 - V_1) \Rightarrow 2V_1 - 3V_2 = 2I_1 + 2I_2 \qquad \text{(v)}$$

By (iv) – (v), we get,

$$I_1 = 4V_1 - 3V_2 \qquad \text{(vi)}$$

Also, from (v) and (vi), we get,

$$2V_1 - 3V_2 = 2(4V_1 - 3V_2) + 2I_2$$

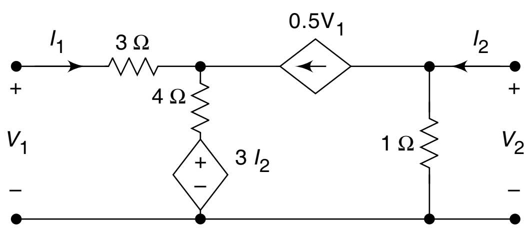
$$\Rightarrow \qquad I_2 = -3V_1 + \frac{3}{2}V_2 \qquad \text{(vii)}$$

From (vi) and (vii), we get,

$$y = \begin{bmatrix} 4 & -3 \\ -3 & \frac{3}{2} \end{bmatrix} (\text{mho}) \ \textit{Ans.}$$

$$\therefore \qquad z = [y]^{-1} = \begin{bmatrix} 4 & -3 \\ -3 & \frac{3}{2} \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{2} & -1 \\ -1 & -\frac{4}{3} \end{bmatrix} (\Omega) \qquad \textit{Ans.}$$

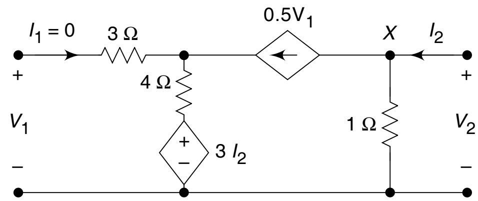
7.45 Find the *h*-parameters for the two-port network shown in the figure.



Solution To find *h* parameters, we consider two cases:

When $I_1 = 0$ Here, no current will flow through the 3 Ω resistance.

By KVL at the left mesh, we get,

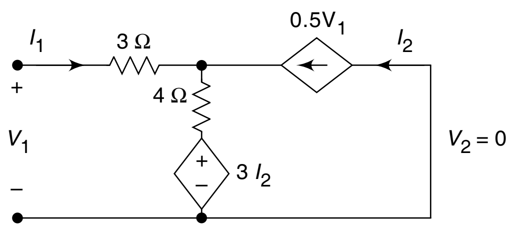


$$V_1 = 4 \times (0.5V_1) + 3I_2$$
$$= 2V_1 + 3I_2$$
$$\Rightarrow \quad V_1 = -3I_2$$

Also, by KCL at Node (X), we get,

$$I_2 = \frac{V_2}{1} + 0.5V_1 = V_2 + 0.5V_1$$
$$= V_2 + 0.5 \times (-3I_2)$$
$$\Rightarrow \quad 2.5I_2 = V_2$$
$$\therefore \quad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{1}{2.5} = 0.4 \ \mho$$
$$\therefore \quad V_1 = -3I_2 = -3 \times \left(\frac{V_2}{2.5}\right) = -1.2V_2$$
$$\therefore \quad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = -1.2$$

When $V_2 = 0$ Here, the port 2 is short circuited. The 1Ω resistance becomes redundant. The modified circuit is shown in Fig (b).



(b)

$$\therefore \quad I_2 = 0.5V_1$$
$$= 0.5 \times [3I_1 + 4I_1 + 4I_2 + 3I_2]$$
$$= 3.5I_1 + 3.5I_2$$
$$\Rightarrow \quad 2.5I_2 = -3.5I_1$$
$$\therefore \quad h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = -\frac{3.5}{2.5} = -1.4$$

Also,

$$V_1 = 3I_1 + 4I_1 + 4I_2 + 3I_2 = 7I_1 + 7I_2 = 7I_1 + 7 \times (-1.4I_1)$$
$$= -2.8I_1$$
$$\therefore \quad h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = -2.8 \ \Omega$$

Therefore, the h parameters of the network are given as,

$$[h] = \begin{bmatrix} -2.8 & -1.2 \\ -1.4 & 0.4 \end{bmatrix} \qquad \textit{Ans.}$$

MULTIPLE-CHOICE QUESTIONS

7.1 Which one of the following pairs is correctly matched?
(a) Symmetrical two-port network: $AD - BC = 1$
(b) Reciprocal two-port network: $z_{11} = z_{22}$.
(c) Inverse hybrid parameters: A, B, C, D
(d) Hybrid parameters: $(V_1, I_2) = f(I_1, V_2)$

7.2 What is the condition for reciprocity in terms of h-parameters?
(a) $h_{11} = h_{22}$ (b) $h_{12}h_{21} = h_{11}h_{22}$ (c) $h_{12} + h_{21} = 0$ (d) $h_{12} = h_{21}$

7.3 For a reciprocal network, the two-port $ABCD$ parameters are related as follows
(a) $AD - BC = 1$ (b) $AD - BC = 0$ (c) $AC - BD = 0$ (d) $AC - BD = 1$

7.4 For a symmetrical two port network
(a) $z_{11} = z_{22}$ (b) $z_{12} = z_{21}$ (c) $z_{11}z_{22} - z_{12}^2 = 0$ (d) $z_{11} = z_{22}$ and $z_{12} = z_{21}$

7.5 For a two port network to be reciprocal, it is necessary that
(a) $z_{11} = z_{22}$ and $y_{12} = y_{21}$ (b) $z_{11} = z_{22}$ and $AD - BC = 0$.
(c) $h_{21} = -h_{12}$ and $AD - BC = 0$ (d) $y_{12} = y_{21}$ and $h_{21} = -h_{12}$

7.6 A two port network is symmetrical if
(a) $z_{11}z_{22} - z_{12}z_{21} = 1$ (b) $AD - BC = 1$ (c) $h_{11}h_{22} - h_{12}h_{21} = 1$ (d) $y_{11}y_{22} - y_{12}y_{21} = 1$

7.7 A two port network is reciprocal if and only if
(a) $z_{11} = z_{22}$ (b) $BC - AD = -1$ (c) $y_{12} = -y_{21}$ (d) $h_{12} = h_{21}$

7.8 In terms of ABCD parameters, a two port network is symmetrical if and only if:
(a) $A = B$ (b) $B = C$ (c) $C = D$ (d) $D = A$

7.9 The condition for reciprocity of a two port network having different parameters are:
1. $h_{12} = -h_{21}$ 2. $g_{12} = -g_{21}$ 3. $A = D$
Choose the correct combination.
(a) 1 and 2 (b) 1 and 3 (c) 2 and 3 (d) 1, 2 and 3.

7.10 Two two-port networks with transmission parameters A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2 respectively are cascaded. The transmission parameter matrix of the cascaded network will be

(a) $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} + \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$ (b) $\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$

(c) $\begin{bmatrix} A_1A_2 & B_1B_2 \\ C_1C_2 & D_1D_2 \end{bmatrix}$ (d) $\begin{bmatrix} (A_1A_2 + C_1C_2) & (A_1A_2 - B_1D_2) \\ (C_1A_2 - D_1C_2) & (C_1C_2 + D_1D_2) \end{bmatrix}$

7.11 Consider the following statements.
For a bilateral network,
1. $A = D$ 2. $z_{12} = z_{21}$ 3. $h_{12} = -h_{21}$
Of these statements.
(a) 1, 2 and 3 are correct (b) 1 and 2 are correct
(c) 1 and 3 are correct (d) 2 and 3 are correct.

7.12 In a two port network containing linear bilateral passive circuit elements, which one of the following conditions for z parameters would hold?

(a) $z_{11} = z_{22}$ (b) $z_{12}z_{21} = z_{11}z_{22}$ (c) $z_{11}z_{12} = z_{22}z_{21}$ (d) $z_{12} = z_{21}$

7.13 The relation $AD - BC = 1$, where A, B, C and D are the elements of a transmission matrix of a network, is valid for

(a) any type of network.
(b) passive but not reciprocal network.
(c) passive and reciprocal network.
(d) both active and passive network.

7.14 When a number of 2-port networks are connected in cascade, the individual:

(a) Z_{oc} matrices are added.
(b) Y_{sc} matrices are added.
(c) chain matrices are multiplied.
(d) H-matrices are multiplied.

7.15 The h parameters h_{11} and h_{22} are related to z and y parameters as

(a) $h_{11} = z_{11}$ and $h_{22} = \dfrac{1}{z_{22}}$
(b) $h_{11} = z_{11}$ and $h_{22} = y_{22}$
(c) $h_{11} = \dfrac{\Delta z}{z_{22}}$ and $h_{22} = \dfrac{1}{z_{22}}$
(d) $h_{11} = \dfrac{1}{y_{11}}$ and $h_{22} = y_{22}$

7.16 Two two-port networks α and β having A B C D parameters as

$A_\alpha = 4 = D_\alpha$ $A_\beta = 3 = D_\beta$ $B_\alpha = 5, C_\alpha = 3$ and $B_\beta = 4, C_\beta = 2$

are connected in cascade in the order of α, β. The equivalent A parameters of the combination is

(a) 17 (b) 22 (c) 24 (d) 31.

7.17 With the usual notation, a two-port resistive network satisfies the condition $A = D = \dfrac{3}{2}B = \dfrac{4}{3}C$

The z_{11} of the network is

(a) $\dfrac{5}{3}$ (b) $\dfrac{4}{3}$ (c) $\dfrac{2}{3}$ (d) $\dfrac{1}{3}$

7.18 The reciprocal of a network function is

(a) an immittance function, if the original function is an immittance function.
(b) a transfer function, if the original function is a transfer function.
(c) never an immittance function.
(d) never a transfer function.

7.19 A two-port network is defined by the relations $I_1 = 2V_1 + V_2$, $I_2 = 2V_1 + 3V_2$. Then z_{12} is

(a) $-2\ \Omega$ (b) $-1\ \Omega$ (c) $-\dfrac{1}{2}\ \Omega$ (d) $-\dfrac{1}{4}\ \Omega$

7.20 Consider the following statements

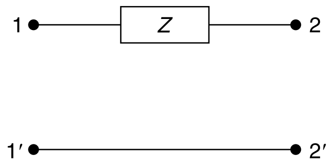
1. Transfer impedance is the reciprocal of transfer admittance.
2. One can derive transfer impedance of a network if its driving-point impedance and admittance are known.
3. Driving-point impedance is the ratio of the Laplace transform of voltage and current functions at the input.

Of these statements:

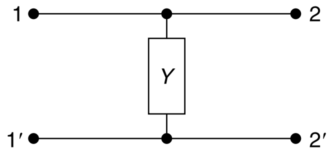
(a) 1, 2 and 3 are correct
(b) 1 and 2 are correct
(c) 2 and 3 are correct
(d) 3 alone is correct.

7.21 Consider the following statements

1. The two-port network shown below does NOT have an impedance matrix representation.



2. The two-port network shown below does NOT have an admittance matrix representation.



3. A two-port network is said to be reciprocal if it satisfies $z_{12} = z_{21}$ or an equivalent relationship.

Of these statements:

(a) 1 and 2 are correct (b) 1 and 3 are correct

(c) 1 and 3 are correct (d) None is correct.

7.22 If two two-port networks are connected in series, and if the port current requirement is satisfied, which of the following is true?

(a) The z-parameter matrices add (b) The y-parameter matrices add.

(c) The $ABCD$-parameter matrices add. (d) None of these.

7.23 If two two-port networks are connected in parallel, and if the port current requirement is satisfied, which of the following is true?

(a) The z-parameter matrices add (b) The y-parameter matrices add.

(c) The $ABCD$-parameter matrices add (d) None of these.

7.24 If two two-port networks are connected in cascade, and if the port current requirement is satisfied, which of the following is true?

(a) The z-parameter matrices add (b) The y-parameter matrices add.

(c) The $ABCD$-parameter matrices add (d) None of these.

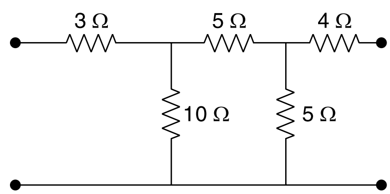
7.25 The z_{11} and z_{22} parameters of the given network are

(a) 8 Ω, 7.75 Ω

(b) 13 Ω, 9 Ω

(c) 12 Ω, 8.5 Ω

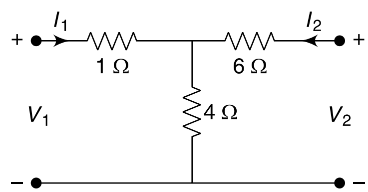
(d) None of the above.



7.26 For the network shown, the parameters h_{11} and h_{21} are

(a) 5 Ω and –2/3 Ω (b) 3.4 Ω and –2/5 Ω

(c) 3.4 Ω and –3/5 Ω (d) None of the above.



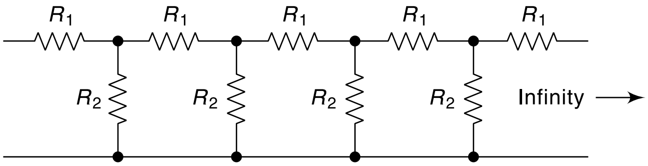
7.27 The maximum value of the transmission parameter A for a passive, reciprocal, linear two-port network is

(a) 1 (b) 2

(c) 3 (d) none of the above.

7.28. The unique feature of *ABCD* parameters as compared to *x*, *y* and *h* parameters is
(a) none
(b) short-circuit functions
(c) open-circuit functions
(d) reverse transverse functions

7.29. The driving point impedance of the infinite ladder network shown in the given figure is



(given $R_1 = 2\ \Omega$ and $R_2 = 1.5\ \Omega$)

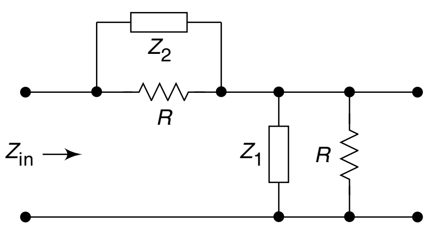
(a) $3\ \Omega$
(b) $3.5\ \Omega$
(c) $\frac{3}{3.5}\ \Omega$
(d) $\ln\left(1+\frac{3}{3.5}\right)\Omega$

7.30 A 2-port network is described by the relations:

$$V_1 = 2V_2 + 0.5I_2$$
$$I_1 = 2V_2 + I_2$$

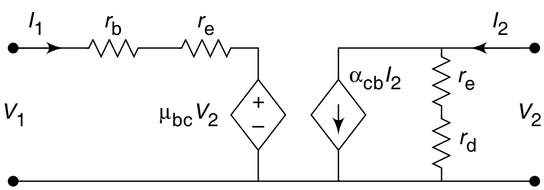
What is the value of the h_{22} parameter of the network?
(a) 1 mho
(b) $2\,\Omega$
(c) -2 mho
(d) $4\,\Omega$

7.31 What are the suitable values for Z_1 and Z_2, to make the input impedance, Z_{in}, of the network equal to R?



(a) R and R
(b) $2R$ and R
(c) $3R$ and $2R$
(d) $4R$ and $4R$

7.32



Which one of the following gives the *h*-parameter matrix for the network shown in the figure?

(a) $\begin{bmatrix} \frac{1}{r_e + r_d} & \mu_{bc} \\ \alpha_{cb} & r_b + r_e \end{bmatrix}$

(b) $\begin{bmatrix} r_b + r_e & \alpha_{cb} \\ \mu_{bc} & \frac{1}{r_e + r_d} \end{bmatrix}$

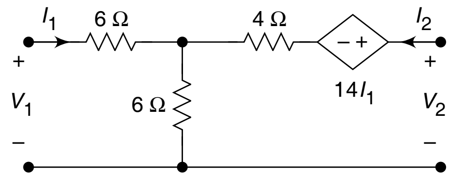
(c) $\begin{bmatrix} r_b + r_e & \mu_{bc} \\ \alpha_{cb} & \frac{1}{r_e + r_d} \end{bmatrix}$

(d) $\begin{bmatrix} \mu_{bc} & \alpha_{cb} \\ r_b + r_e & \frac{1}{r_e + r_d} \end{bmatrix}$

7.33 In a two-port network, the output short-circuit current was measured while the source voltage at the input was 1 V; the value of the output current would provide the parameter

(a) B (b) y_{12} (c) h_{21} (d) y_{21}

7.34 The y-parameter 'y_{21}' of the network shown in the figure

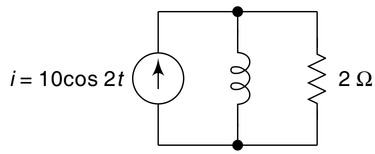


(a) is 2 mho (b) is 6 mho (c) is 3 mho (d) does not exist

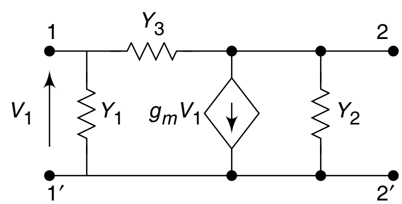
7.35 The phasor current through the inductance in the circuit shown is

(a) $\left(\dfrac{10}{\sqrt{2}}\right)\angle -45^\circ$ (b) $\left(\dfrac{10}{\sqrt{2}}\right)\angle 45^\circ$

(c) $5\angle 45^\circ$ (d) $5\angle -45^\circ$

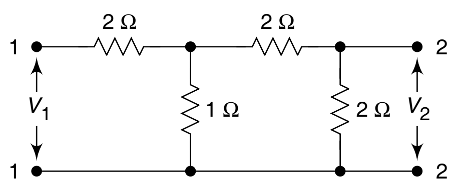


7.36 For the two-port network, the parameter y_{21} will be



(a) $Y_2 + Y_3$ (b) $g_m - Y_3$ (c) $Y_3 - g_m$ (d) $g_m + Y_2 + Y_3$

7.37 For the given two-port network, z_{21} will be

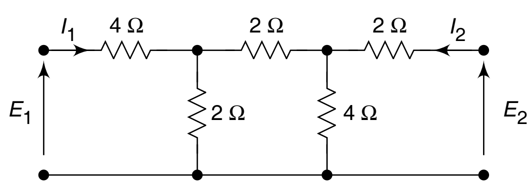


(a) 2/5 Ω (b) 3/5 Ω (c) 1/5 Ω (d) 4/5 Ω

7.38 The h-parameters for a two-port network are defined by $\begin{bmatrix} E_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ E_2 \end{bmatrix}$. For the two-port network shown in the figure, the value of h_{12} is given by

(a) 0.125 (b) 0.167

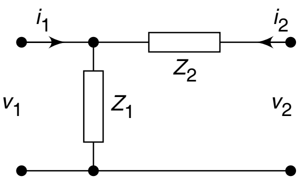
(c) 0.625 (d) 0.25



7.39 The z matrix of a two-port network as given by $\begin{bmatrix} 0.9 & 0.2 \\ 0.2 & 0.6 \end{bmatrix}$. The element y_{22} of the corresponding y matrix of the same network is given by

(a) 1.2 (b) 0.4 (c) −0.4 (d) 1.8

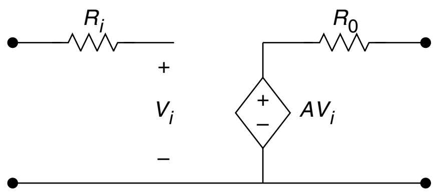
7.40 For the two-port network shown in the figure, the z-matrix is given by



(a) $\begin{bmatrix} Z_1 & Z_1+Z_2 \\ Z_1+Z_2 & Z_2 \end{bmatrix}$ (b) $\begin{bmatrix} Z_1 & Z_1 \\ Z_1+Z_2 & Z_2 \end{bmatrix}$

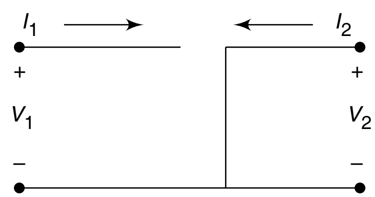
(c) $\begin{bmatrix} Z_1 & Z_2 \\ Z_2 & Z_1+Z_2 \end{bmatrix}$ (d) $\begin{bmatrix} Z_1 & Z_1 \\ Z_1 & Z_1+Z_2 \end{bmatrix}$

7.41 The parameters of the circuit shown in the figure are $R_i = 1\ \text{M}\Omega$, $R_0 = 10\ \Omega$, $A = 10^6$ V/V. If $V_i = 1\ \mu$V, then output voltage, input impedance and output impedance respectively are



(a) 1 V, ∞, 10 Ω (b) 1 V, 0, 10 Ω (c) 1 V, 0, ∞ (d) 10 V, ∞, 10 Ω

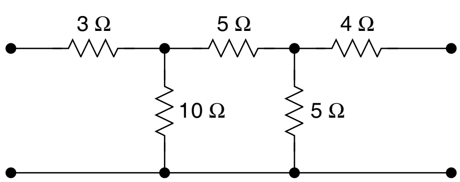
7.42 The parameter type and the matrix representation of the relevant two port parameters that describe the circuit shown are



(a) z parameters, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (b) h parameters, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

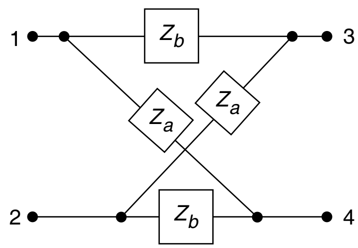
(c) h parameters, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (d) z parameters, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

7.43 The impedance parameters z_{11} and z_{12} of the two-port network in the figure are



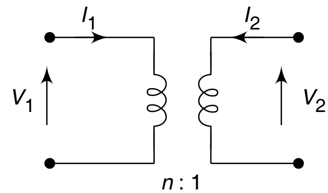
(a) $z_{11} = 2.75\ \Omega$ and $z_{12} = 0.25\ \Omega$
(b) $z_{11} = 3\ \Omega$ and $z_{12} = 0.5\ \Omega$
(c) $z_{11} = 3\ \Omega$ and $z_{12} = 0.25\ \Omega$
(d) $z_{11} = 2.25\ \Omega$ and $z_{12} = 0.5\ \Omega$

7.44 For the lattice circuit shown in the figure, $Z_a = j2\ \Omega$ and $Z_b = 2\ \Omega$. The values of the open circuit impedance parameters $z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$ are



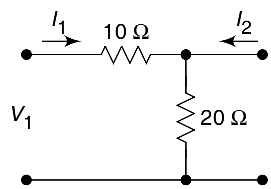
(a) $\begin{bmatrix} 1-j & 1+j \\ 1+j & 1+j \end{bmatrix}$
(b) $\begin{bmatrix} 1-j & 1+j \\ -1+j & 1-j \end{bmatrix}$
(c) $\begin{bmatrix} 1+j & 1+j \\ 1-j & 1-j \end{bmatrix}$
(d) $\begin{bmatrix} 1-j & -1+j \\ -1-j & 1-j \end{bmatrix}$

7.45 The *ABCD* parameters of an ideal $n : 1$ transformer shown in the figure are $\begin{bmatrix} n & 0 \\ 0 & X \end{bmatrix}$. The value of X will be



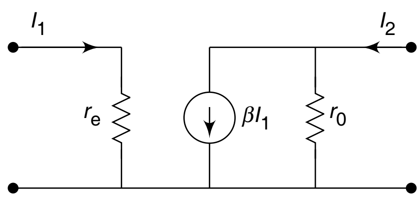
(a) n
(b) $\frac{1}{n}$
(c) n^2
(d) $\frac{1}{n^2}$

7.46 The *h*-parameters of the circuit shown in the figure are



(a) $\begin{bmatrix} 0.1 & 0.1 \\ -0.1 & 0.3 \end{bmatrix}$
(b) $\begin{bmatrix} 10 & -1 \\ 1 & 0.05 \end{bmatrix}$
(c) $\begin{bmatrix} 30 & 20 \\ 20 & 20 \end{bmatrix}$
(d) $\begin{bmatrix} 10 & 1 \\ -1 & 0.05 \end{bmatrix}$

7.47 In the two-port network shown in the figure below, z_{12} and z_{21} are, respectively,



(a) r_e and βr_0 (b) 0 and $-\beta r_0$ (c) 0 and βr_0 (d) r_e and $-\beta r_0$

7.48 A two-port network is represented by *ABCD* parameters given by

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

If port-2 is terminated by R_L, then the input impedance seen at port-1 given by

(a) $\dfrac{A + BR_L}{C + DR_L}$ (b) $\dfrac{AR_L + C}{BR_L + D}$

(c) $\dfrac{DR_L + A}{BR_L + C}$ (d) $\dfrac{B + AR_L}{D + CR_L}$

EXERCISES

7.1 Current I_1 and I_2 entering at ports 1 and 2 respectively of a two-port network are given by the following equations:

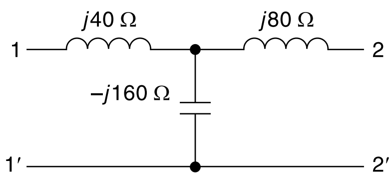
$$I_1 = 0.5V_1 - 0.2V_2$$
$$I_2 = -0.2V_1 + V_2$$

where V_1 and V_2 are the voltages at ports 1 and 2 respectively. Find the *y*, *z* and *ABCD* parameters for the network. Also find its equivalent π-network.

[$y_{11} = 0.5$ ℧; $y_{12} = -0.2$ ℧; $y_{21} = -0.2$ ℧; $y_{22} = 1$ ℧;
$z_{11} = 2.174\ \Omega$; $z_{12} = z_{21} = -0.435\ \Omega$; $z_{22} = 1.086\ \Omega$;
$A = 5$, $B = 5\ \Omega$, $C = 2.3$ ℧, $D = 2.5$; $Y_1 = 0.3$ ℧; $Y_2 = 0.2$ ℧; $Y_3 = 0.8$ ℧]

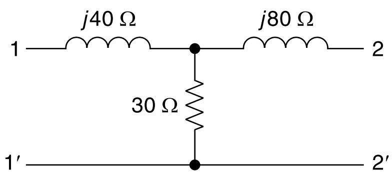
7.2 Determine the *z*-and *y*-parameters of the networks shown in figure.

(a)



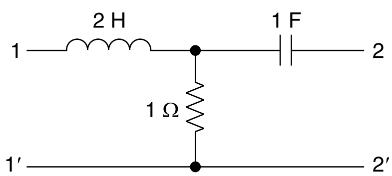
$$\left\{ z = \begin{bmatrix} -j120 & -j160 \\ -j160 & -j80 \end{bmatrix} (\Omega);\ y = \begin{bmatrix} -j/120 & j110 \\ j/110 & -j\dfrac{3}{4} \end{bmatrix} (\Omega^{-1}) \right\}$$

(b)



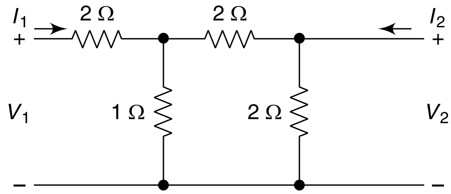
$$\left\{ z = \begin{bmatrix} (30 + j40) & j40 \\ j40 & (30 + j80) \end{bmatrix} (\Omega) \right\}$$

(c)



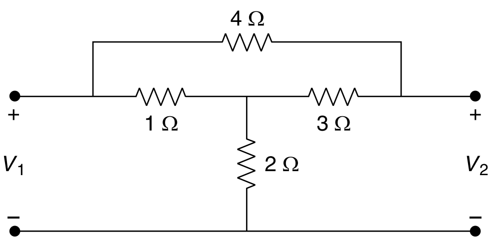
$$\left\{ z = \begin{bmatrix} (1 + 25) & 1 \\ 1 & \left(1 + \dfrac{1}{5}\right) \end{bmatrix} (\Omega) \right\}$$

7.3 Obtain the z-parameters for the circuit shown in figure and hence draw the z-parameter equivalent circuit.



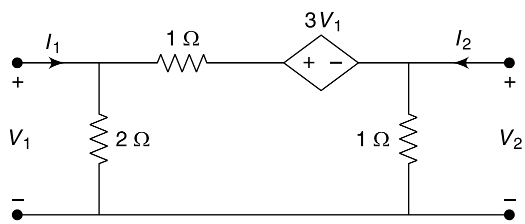
$$\left\{ z = \begin{bmatrix} \frac{14}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{6}{5} \end{bmatrix} (\Omega) \right\}$$

7.4 Find the open-circuit and short-circuit impedances of the network shown in figure.



$$\left[y = \begin{bmatrix} \frac{31}{44} & -\frac{19}{44} \\ -\frac{19}{44} & \frac{23}{44} \end{bmatrix}; \ z\text{-parameters do not exist} \right]$$

7.5 Find the z-parameters for the 2-port networks shown in figure containing a controlled source.



$$\left\{ z = \begin{bmatrix} -2 & -1 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} (\Omega) \right\}$$

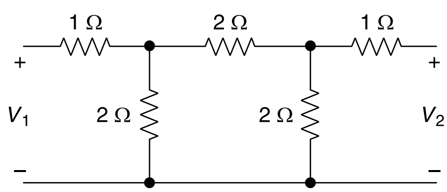
7.6 A 2-port network made up of passive linear resistors is fed at port 1 by an ideal voltage source of V volt. It is loaded at port 2 by a resistor R.

(i) With $V = 10$ volt and $R = 6\ \Omega$ currents at ports 1 and 2 were 1.44 A and 0.2 A respectively.

(ii) With $V = 15$ volt and $R = 8\ \Omega$ current at port 2 was 0.25 A.

Determine the π-equivalent circuit of the 2-port network. $\{Y_A = 0.2;\ Y_B = 0.3;\ Y_C = 0.5 \text{ (mho)}\}$

7.7 Calculate the T-parameters for the block A and B separately and then using these results calculate the T-parameters of the whole circuit shown in figure. Prove any formula used.

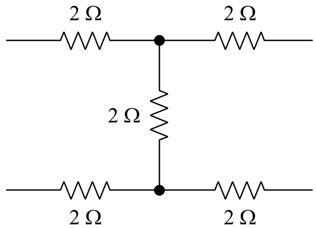


$$\left\{ A_a = A_b = D_a = D_b = \frac{3}{2} \right.$$

$$C_a = C_b = \frac{1}{2};\ B_a = B_b = \frac{5}{2};$$

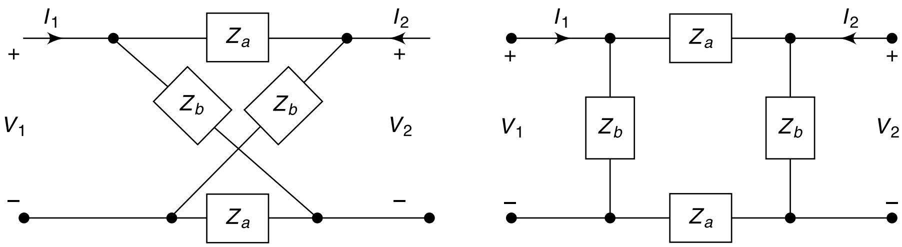
$$\left. T = \begin{bmatrix} 7/2 & 15/2 \\ 3/2 & 7/2 \end{bmatrix} \right\}$$

7.8 Find out the z-parameters of the two-port network shown in the figure.



$$\left\{ z = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix} (\Omega) \right\}$$

7.9 Find the z-parameters for the lattice network shown in the figure.



$$\left\{ z_{11} = z_{22} = \left(\frac{Z_b + Z_a}{2} \right); z_{12} = z_{21} = \left(\frac{Z_b - Z_a}{2} \right);\right.$$

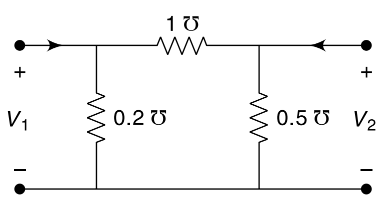
$$\left. z = \frac{1}{2Z_a + Z_b} \begin{bmatrix} 2(Z_a + Z_b) & Z_b^2 \\ Z_b^2 & 2Z_a Z_b \end{bmatrix} \right\}$$

7.10 Current I_1 and I_2 entering at port-1 and port-2 respectively of a two port network are given by the following equations: I_1=0.5V_1-0.2V_2, I_2=-0.2V_1+V_2, where V_1 and V_2 are the voltages at port-1 and port-2 respectively. Find the y, z and $ABCD$ parameters for the network. Also find the equivalent π-network.

$$\left\{ y = \begin{bmatrix} 0.5 & -0.2 \\ -0.2 & 1 \end{bmatrix} (\Omega^{-1}); Z = \begin{bmatrix} 2.174 & 0.435 \\ 0.435 & 1.087 \end{bmatrix} (\Omega), \right.$$

$$\left. T = \begin{bmatrix} 5 & 5\,\Omega \\ 2.3\,℧ & 2.5 \end{bmatrix}; Y_a = 0.3\,℧, Y_b = 0.8\,℧, Y_c = 0.2\,℧ \right\}$$

7.11 Two identical sections of the circuit shown in the figure are connected in series. Obtain the z-parameters of the combination and verify by direct calculation. $[z_{11} = z_{22} = 6\ \Omega; z_{12} = z_{21} = 4\ \Omega]$



7.12 Test results for a two-port network are

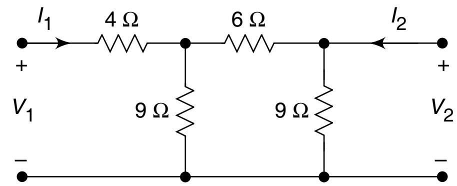
(a) port 2 open-circuited, $I_1 = 0.01\angle 0^\circ(\mathrm{A}), V_1 = 1.4\angle 45^\circ(V), V_2 = 2.3\angle -26.4^\circ(\mathrm{V})$

(b) port 1 open-circuited, $I_2 = 0.01\angle 0^\circ(\mathrm{A}), V_1 = 1\angle -90^\circ(V), V_2 = 1.5\angle -53.1^\circ(\mathrm{V})$

The source frequency in both the tests was 1000 Hz. Find z-parameters.

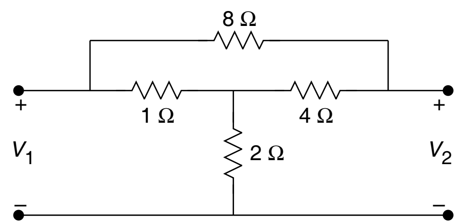
$$\left[\begin{bmatrix} 140\angle 45^\circ & 100\angle -90^\circ \\ 230\angle -26.4^\circ & 150\angle -53.1^\circ \end{bmatrix}(\Omega)\right]$$

7.13 Find the z-parameters for the network shown in the figure.



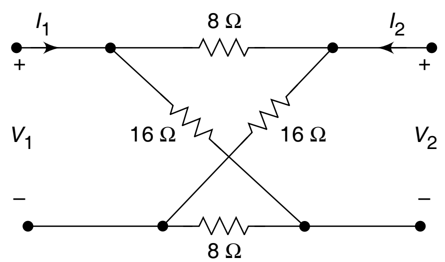
$$\left[\begin{bmatrix} 10 & 3 \\ 3 & 6 \end{bmatrix}(\Omega)\right]$$

7.14 For the network shown in the figure, find the y-parameters and also the equivalent T-network.



$$\left[\begin{bmatrix} 62/112 & -30/112 \\ -30/112 & 38/112 \end{bmatrix}, Z_a = 8/13\,\Omega, Z_b = 32/13\,\Omega, Z_c = 30/13\,\Omega\right]$$

7.15 Find the h-parameters for the network shown in the figure.



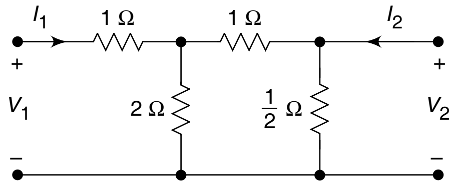
$$\left[h_{11} = \frac{32}{3}\,\Omega;\quad h_{12} = \frac{1}{3};\quad h_{21} = -\frac{1}{3};\quad h_{22} = \frac{1}{12}\,\text{mho}\right]$$

7.16 The h-parameters of a two-port network are

$$h_{11} = 35\Omega; \quad h_{12} = 2.6 \times 10^{-4}; \quad h_{21} = -0.98; \quad h_{22} = 0.3 \times 10^{-6}\,\text{mho}$$

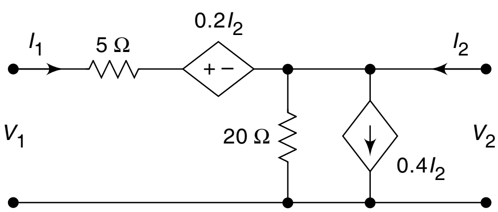
The input terminals are connected to 0.001V sinusoidal source and a 10^4 ohm resistance is connected across the output port. Find the output voltage. [0.26 V]

7.17 Find the y and z-parameters for the network shown in the figure.



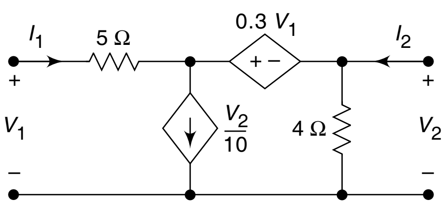
$$\left[\begin{bmatrix} 13/7 & 2/7 \\ 2/7 & 3/7 \end{bmatrix}(\Omega); \begin{bmatrix} -3/5 & -2/5 \\ -2/5 & 13/5 \end{bmatrix}(\text{mho})\right]$$

7.18 Find the y-parameters for the network shown in the figure.



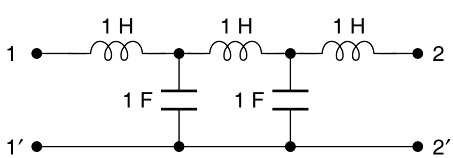
$$\left[\begin{bmatrix} 0.2 & -0.24 \\ -0.333 & 0.4833 \end{bmatrix}\mho\right]$$

7.19 Find the transmission parameters of the network shown in the figure.



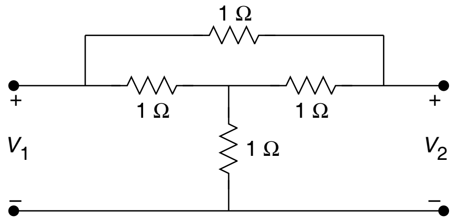
$$\left[\begin{bmatrix} \frac{55}{26} & \frac{50}{13} \\ \frac{7}{20} & 1 \end{bmatrix}\right]$$

7.20 Determine the T-parameters for the network shown in the figure using the concept of interconnection of two two-port networks.



$$\begin{bmatrix} 1+3s^2+s^4 & 3s+4s^3+s^5 \\ 2s+s^3 & 1+3s^2+s^4 \end{bmatrix}$$

7.21 Determine the y parameters of the overall network, considering two networks connected in parallel.



$$\left[y_{11} = y_{22} = \frac{5}{3}\,\mho;\, y_{12} = y_{21} = -\frac{4}{3}\,\mho \right]$$

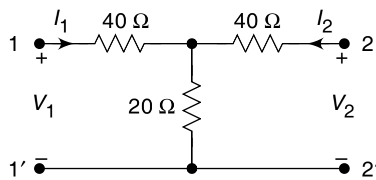
7.22 The z-parameters of a two-port network are

$$z_{11} = 50\ \Omega;\ z_{22} = 30\ \Omega;\ z_{12} = z_{21} = 20\ \Omega$$

Calculate the y-parameters and $ABCD$ parameters of the network.

$$\left[\begin{array}{l} y_{11} = 0.0273 \text{ mho};\ y_{22} = 0.0454 \text{ mho};\ y_{12} = y_{21} = -0.01818 \text{ mho}; \\ A = 2.5;\ B = 55\Omega;\ C = 0.05 \text{ mho};\ D = 1.5 \end{array} \right]$$

7.23 For the symmetrical two-port network, calculate the z-parameters and $ABCD$ parameters.



$$\left[z_{11} = z_{22} = 60\ \Omega;\ z_{12} = z_{21} = 20\ \Omega;\ A = D = 3;\ B = 160\ \Omega;\ C = 0.05 \text{ mho}; \right]$$

SHORT-ANSWER TYPE QUESTIONS

7.1 (a) Consider a linear passive two-port network and explain what are meant by (i) open-circuit impedance parameters and (ii) short-circuit admittance parameters.

(b) What are the open-circuit impedance parameters of a two-port network? How can the transmission parameters be obtained from open-circuit impedance parameters?

(c) Establish, for two-port networks, the relationship between the transmission parameters and the open-circuit parameters.

(d) Define z- and y-parameters of a typical four terminal network. Determine the relationship between the z and y parameters.

(e) Express h-parameters in terms of z-parameters for a two-port network.

(f) Derive expressions for the y-parameters in terms of $ABCD$ parameters of a two-port network.

7.2 (a) What do you understand by a reciprocal network? What is a symmetrical network?

(b) Write technical note on derivation of short-circuit admittance parameter y_{12} of a symmetrical and reciprocal two-port lattice network.

(c) How will you find the π-equivalent of a given network when its y-parameters are known?

7.3 (a) Explain what are meant by the transmission ($ABCD$) parameters of a two-port network. Derive the conditions necessary to be satisfied for the network to be (i) reciprocal and (ii) symmetrical.

Or,

Prove that for a reciprocal two-port network,

$$\Delta T = (AD - BC) = 1$$

(b) Prove that for a symmetrical two-port network,

$$\Delta h = (h_{11}h_{22} - h_{12}h_{21}) = 1$$

7.4 (a) Two two-port networks are connected in parallel. Prove that the overall y-parameters are the sum of corresponding individual y-parameters.

(b) Two two-port networks are connected in cascade. Prove that the overall transmission parameter matrix is the product of individual transmission parameter matrices.

(c) Two two-port networks are connected in series. Prove that the overall z-parameters are the sum of corresponding individual z-parameters.

7.5 What are transmission parameters? Where are they most effectively used? Establish, for two-port networks, the relationship between the transmission parameters and the open circuit impedance parameters.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

7.1 (d)	7.2 (c)	7.3 (a)	7.4 (a)	7.5 (d)	7.6 (c)	7.7 (b)
7.8 (d)	7.9 (a)	7.10 (b)	7.11 (d)	7.12 (d)	7.13 (c)	7.14 (c)
7.15 (c)	7.16 (b)	7.17 (b)	7.18 (a)	7.19 (d)	7.20 (d)	7.21 (b)
7.22 (a)	7.23 (b)	7.24 (d)	7.25 (a)	7.26 (b)	7.27 (d)	7.28 (d)
7.29 (a)	7.30 (c)	7.31 (a)	7.32 (c)	7.33 (d)	7.34 (d)	7.35 (a)
7.36 (b)	7.37 (a)	7.38 (d)	7.39 (d)	7.40 (d)	7.41 (a)	7.42 (c)
7.43 (a)	7.44 (d)	7.45 (b)	7.46 (d)	7.47 (b)	7.48 (d)	

CHAPTER

8
Fourier Series and Fourier Transform

PART I: FOURIER SERIES

8.1 INTRODUCTION

In 1807, the French mathematician Joseph Fourier (1768–1830) submitted a paper to the Academy of Sciences in Paris. In it he presented a mathematical description of problems involving heat conduction. Although the paper was at first rejected, it contained ideas that would develop into an important area of mathematics named in honour, *Fourier analysis*. One surprising ramification of Fourier's work was that many familiar functions can be expanded in infinite series and integrals involving trigonometric functions. The idea today is important in modeling many phenomena in physics and engineering.

In this chapter, in the first part, we will discuss the basic concepts of Fourier series. Then we will apply this concept to find the steady-state response of an electric circuit subject to a periodic excitation. A function of time $f(t)$ is said to be periodic if $f(t) = f(t \pm nT)$; where, n is a positive integer and 'T' is the period. Thus, a periodic function repeats itself every T second.

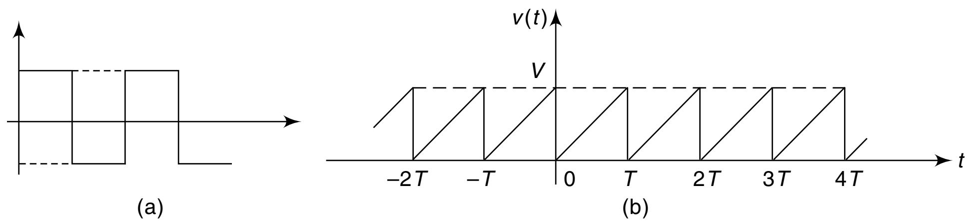


Figure 8.1 *Periodic functions*

In the second part of this chapter, we will learn about another transform method, namely Fourier transform, which is used to find the steady-state response of a network to aperiodic excitation.

8.2 DEFINITION OF FOURIER SERIES

French mathematician J.B.J. Fourier first studied the periodic function in 1822 and published his theorem which states that,

"Any arbitrary periodic function can be represented by an infinite series of sinusoids of harmonically related frequencies." This infinite series is known as Fourier series.

Thus, if $f(t)$ is a periodic function, then the Fourier series is,

$$\begin{aligned} f(t) &= a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + \ldots + a_n \cos n\omega t + \ldots \\ &\quad + b_1 \sin \omega t + b_2 \sin 2\omega t + \ldots + b_n \sin n\omega t + \ldots \\ &= a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \end{aligned}$$

where, ω is the fundamental frequency $= \dfrac{2\pi}{T}$

$n\omega$ is the n^{th} harmonic of fundamental frequency

a_0, a_n, b_n are the Fourier Co-efficients

8.3 DIRICHLET'S CONDITIONS

The conditions, under which a periodic function $f(t)$ can be expanded in a convergent Fourier series, are known as Dirichlet's conditions.

These are as follows:

(i) $f(t)$ is a single valued function.

(ii) $f(t)$ has a finite number of discontinuities in each period, T.

(iii) $f(t)$ has a finite number of maxima and minima in each period, T.

(iv) The integral, $\int_0^T |f(t)| dt$ exists and is finite or in other way, $\int_0^T [f(t)]^2 dt < \infty$.

Note: If $f(t)$ is current or voltage, $\int_0^T [f(t)]^2 dt$ represents energy which would be supplied by the source in one cycle. That means the energy in the waveform for each cycle must be finite. All physical waveforms would, of course, satisfy this criterion.

Therefore, in practical engineering problems, it is not necessary to check whether a function satisfies Dirichlet condition.

8.4 FOURIER ANALYSIS

This involves two operations:

1. The evaluation of the co-efficient a_0, a_n and b_n.

2. Truncation of the infinite series after a finite number of terms so that $f(t)$ is represented within allowable error (-Done later).

8.4.1 Evaluation of Fourier Coefficients

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \tag{8.1}$$

From (8.1),

$$\int_0^T f(t)dt = a_0 \int_0^T dt + \sum_{n=1}^{\infty} \int_0^T (a_n \cos n\omega t + b_n \sin n\omega t)dt = a_0 T$$

$$\left\{ \because \int_{t_0}^{t_0+T} \sin m\omega t dt = 0 \text{ for all } m; \quad \text{and} \quad \int_{t_0}^{t_0+T} \cos n\omega t dt = 0 \text{ for all } n; \right\}$$

$$\boxed{\therefore a_0 = \frac{1}{T}\int_0^T f(t)dt}$$

This shows hat a_0 is the average value of *f(t)* over a period; therefore, called dc value of the signal. Now from equation (8.1),

$$\int_0^T f(t) \cos k\omega t dt = \int_0^T a_0 \cos k\omega t dt + \sum_{n=1}^{\infty} \int_0^T (a_n \cos k\omega t \cos n\omega t + b_n \cos k\omega t \sin n\omega t)\, dt$$

$$= 0 + a_k \frac{T}{2} + 0$$

$$\left\{ \because \int_{t_0}^{t_0+T} \sin n\omega t \sin m\omega t dt = 0 \text{ for } m \neq n \quad \text{and} \quad \int_{t_0}^{t_0+T} \cos n\omega t \cos m\omega t dt = 0 \text{ for } n \neq m \right.$$

$$\left. = \frac{T}{2} \text{ for } n = m \qquad\qquad = \frac{T}{2} \text{ for } n = m \right\}$$

$$\therefore \quad a_k = \frac{2}{T}\int_0^T f(t) \cos k\omega t dt$$

Again from equation (8.1),

$$\int_0^T f(t) \sin k\omega t dt = \int_0^T a_0 \sin k\omega t dt + \sum_{n=1}^{\infty} \int_0^T (a_n \sin k\omega t \cos n\omega t + b_n \sin k\omega t \sin n\omega t)\, dt$$

$$= 0 + 0 + b_k \frac{T}{2}$$

$$\therefore \qquad b_k = \frac{2}{T}\int_0^T f(t) \sin k\omega t dt$$

Example 8.1 For the periodic waveform shown in the figure, find the Fourier series expansion.

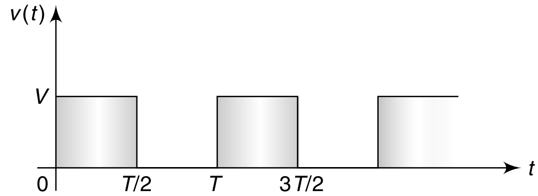


Figure 8.2 *Periodic waveform of Example (8.1)*

Solution Here, $v(t) = V$, for $0 < t < T/2$

$= 0$, for $T/2 < t < T$

$$a_0 = \frac{1}{T}\int_0^T v(t)dt = \frac{1}{T}\int_0^{T/2} Vdt = \frac{V}{2}$$

$$a_n = \frac{2}{T}\int_0^T v(t)\cos n\omega t dt = \frac{2}{T}\int_0^{T/2} V\cos\left(n\frac{2\pi}{T}\right)dt = 0$$

and, $$b_n = \frac{2}{T}\int_0^T v(t)\sin n\omega t dt = \frac{2}{T}\int_0^{T/2} V\sin\left(n\frac{2\pi}{T}\right)dt$$

$$= \frac{V}{n\pi}(1-\cos n\pi);\ n = \pm 1, \pm 2, \pm 3, \ldots$$

$= 0$; for even n

$= \frac{V}{n\pi}$; for odd n

So, the Fourier series of the square wave is given as,

$$v(t) = V\left[\frac{1}{2} + \frac{2}{\pi}\sin\omega t + \frac{2}{3\pi}\sin 3\omega t + \frac{2}{5\pi}\sin 5\omega t + \ldots\right]$$

Exponential Form of Fourier Series We have the trigonometric Fourier series,

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\omega t + b_n\sin n\omega t)$$

We know that, $\sin n\omega t = \frac{e^{jn\omega t} - e^{-jn\omega t}}{2j}$ and $\cos n\omega t = \frac{e^{jn\omega t} + e^{-jn\omega t}}{2}$

Thus,

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\frac{(e^{jn\omega t} + e^{-jn\omega t})}{2} + b_n\frac{(e^{jn\omega t} - e^{-jn\omega t})}{2j}\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\frac{1}{2}\left[\left(a_n + \frac{b_n}{j}\right)e^{jn\omega t} + \left(a_n - \frac{b_n}{j}\right)e^{-jn\omega t}\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[\left(\frac{a_n - jb_n}{2}\right)e^{jn\omega t} + \left(\frac{a_n + jb_n}{2}\right)e^{-jn\omega t}\right]$$

Let, $C_0 = a_0,\quad C_n = \left(\dfrac{a_n - jb_n}{2}\right)$ and C_n^* (or $C_{-n}) = \left(\dfrac{a_n + jb_n}{2}\right)$

Thus the series becomes,

$$f(t) = C_0 + \sum_{n=1}^{\infty} [C_n e^{jn\omega t} + C_{-n} e^{-jn\omega t}]$$

or $\boxed{f(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}}$ This is the exponential form of the Fourier series.

Now, $$C_n = \frac{a_n - jb_n}{2} = \frac{1}{2}\left[\frac{2}{T}\int_0^T f(t)\cos n\omega t dt - j\frac{2}{T}\int_0^T f(t)\sin n\omega t dt\right]$$

$$= \frac{1}{T}\int_0^T f(t)(\cos n\,\omega t - j\sin n\,\omega t)\,dt$$

Thus, $\boxed{C_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega t}dt}$ This equation is valid for both positive, negative and zero values of n.

Example 8.2 For the square wave shown in Example 8.1, find the exponential Fourier series.

Solution $f(t) = v(t) = V$, for $0 < t < T/2$
$= 0$, for $T/2 < t < T$

So, $$C_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega t}dt = \frac{1}{T}\int_0^{T/2} Ve^{-jn\omega t}dt$$

For $n = 0$, $$C_0 = \frac{1}{T}\int_0^{T/2} Vdt = \frac{V}{2}$$

For $n \neq 0$ $$C_n = \frac{1}{T}\int_0^{T/2} Ve^{-jn\omega t}dt = \frac{V}{T}\frac{1}{-jn\omega}[e^{-jn\omega T/2} - 1] = \frac{jV}{2\pi n}[e^{-jn\pi} - 1]$$

(since $\omega T = 2\pi$)

or $C_n = 0$ for even n

$= -\dfrac{jV}{\pi n}$ for odd n

Thus, the exponential Fourier series becomes,

$$v(t) = \ldots + \frac{jV}{5\pi}e^{j5\omega t} + \frac{jV}{3\pi}e^{j3\omega t} + \frac{jV}{\pi}e^{j\omega t} + \frac{V}{2} - \frac{jV}{\pi}e^{-j\omega t} - \frac{jV}{3\pi}e^{-j3\omega t} - \frac{jV}{5\pi}e^{-j5\omega t} - \ldots \text{ for odd } n$$

Amplitude and Phase Spectrum From the trigonometric Fourier series,

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos (n\omega t - \phi_n)$$

where, $A_0 = a_0$, $A_n = \sqrt{a_n^2 + b_n^2}$; $\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$

Also, for exponential form, C_n is complex and we may write it as,

$$C_n = |C_n| e^{j\phi_n} \text{ and } |C_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} = \frac{A_n}{2} \text{ and } \phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

The quantities A_n and ϕ_n are called *the amplitude and the phase of the n^{th} harmonic*, respectively.

- Variation of A_n with n (or $n\omega$) is known as *the amplitude spectrum or Frequency – spectrum.*
- Variation of ϕ_n with n (or $n\omega$) is known as *the phase- spectrum* of the signal.

As both A_n and ϕ_n occurs at discrete values of the frequency, i.e., n = 1, 2, 3, etc. these spectra are called *Line spectra*.

Since $|C_n| = \frac{A_n}{2}$; there is a scale factor of ½ for the amplitude spectrum for exponential form for the Fourier series compared to the trigonometric form for all lines except the one for $n = 0$. Also, in the case of exponential form spectral lines are drawn for both for positive and negative values of n.

Example 8.3 For the square wave shown in Example 8.1, draw the amplitude and phase spectra.

Solution From the results of Example 8.1, we have,

$$v(t) = V\left[\frac{1}{2} + \frac{2}{\pi}\sin \omega t + \frac{2}{3\pi}\sin 3\omega t + \frac{2}{5\pi}\sin 5\omega t + \dots\right]$$

Magnitudes: $V_0 = \frac{V}{2}\angle 0°$; $V_1 = \frac{2V}{\pi}\angle 90°$; $V_2 = 0$; $V_3 = \frac{2V}{3\pi}\angle 90°$ [since the cosine components are all zero, the phase angle will be $\tan^{-1}\left(\frac{b_n}{0}\right) = \tan^{-1}(\infty) = 90°$]

So, the line spectra become,

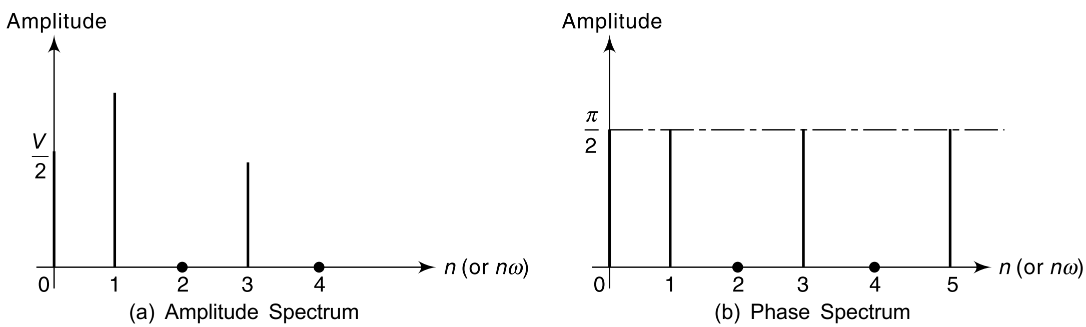


Figure 8.3 *Amplitude and phase spectra of Example 8.3*

Significance for Line Spectra: The amplitude- spectrum renders valuable information as to where to truncate the infinite series and yet maintain a good approximation to the original waveform.

Effective Value of a Periodic Function The effective (or R.M.S.) value of a periodic function $f(t)$ is defined as,

$$F_{eff}\ (F_{rms}) = \sqrt{\frac{1}{T}\int_0^T [f(t)]^2\,dt} = \sqrt{\frac{1}{T}\int_0^T \left[A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t - \phi_n)\right]^2 dt}$$

$$= \sqrt{\frac{1}{T}\left[A_0^2 T + \sum_{n=1}^{\infty} A_n^2 \frac{T}{2}\right]}$$

$$\boxed{F_{eff}\ (F_{rms}) = \sqrt{A_0^2 + \sum_{n=1}^{\infty}\left(\frac{A_n}{\sqrt{2}}\right)^2}}$$

This shows that the effective value of a periodic function is the square root of the effective values of the harmonic components and the square of the d. c. value.

Waveform Symmetry There are few methods by which the evaluation of Fourier co-efficients is simplified by symmetry consideration.

These methods reduce the amount of labour involved in finding out the co-efficients.

Now, $$a_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\left[\int_{-T/2}^{0} f(t)dt + \int_0^{T/2} f(t)dt\right]$$

Putting $t = -x$ in the first integrand and $t = x$ in the second integrand, we get

$$a_0 = \frac{1}{T}\left[\int_0^{T/2} [f(x) + f(-x)]dx\right]$$

Now, $$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t dt = \frac{2}{T}\left[\int_0^{T/2} f(t)\cos n\omega t dt + \int_{-T/2}^{0} f(t)\cos n\omega t dt\right]$$

$$= \frac{2}{T}[I_1 + I_2]$$

Since the variable t in I_1 and I_2 integrals is dummy variable, let $x = t$ in I_1 and $x = -t$ in I_2.

$\therefore$ $$a_n = \frac{2}{T}\left[\int_0^{T/2} f(x)\cos n\omega x dx - \int_0^{T/2} f(-x)\cos n\omega x(-dx)\right]$$

Thus, $$\boxed{a_n = \frac{2}{T}\int_0^{T/2} [f(x) + f(-x)]\cos n\omega x dx}$$

Similarly, $\boxed{b_n = \frac{2}{T}\int_0^{T/2} [f(x) - f(-x)]\sin n\omega x dx}$

Following symmetries are considered:

1. Odd or Rotation Symmetry,
2. Even or Mirror Symmetry,
3. Half-Wave or, Alternation Symmetry, and
4. Quarter-Wave Symmetry.

1. Odd Symmetry

A function $f(x)$ is said to be odd if,

$$f(x) = -f(-x)$$

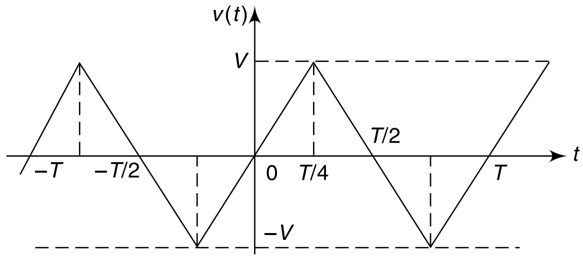


Figure 8.4 *Odd function*

Hence, for odd functions $a_0 = 0$ and $a_n = 0$ and $b_n = \frac{1}{T}\int_0^{T/2} f(x)\sin n\omega x\, dx$

Thus, the *Fourier series expansion of an odd function contains only the sine terms, the constant and the cosine terms being zero.*

2. Even Symmetry

A function $f(x)$ is said to be even, if

$$f(x) = f(-x)$$

$$\therefore \quad a_0 = \frac{2}{T}\int_0^{T/2} f(x)dx$$

$$\therefore \quad a_n = \frac{4}{T}\int_0^{T/2} f(x)\cos n\omega x dx$$

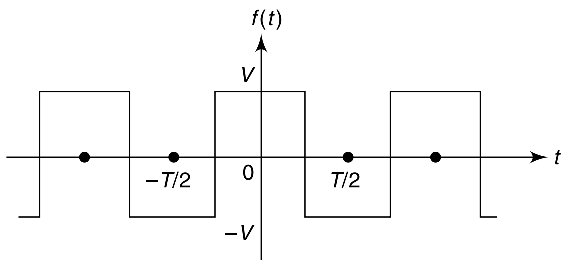


Figure 8.5 *Even function*

and $b_n = 0$

Thus, the *Fourier series expansion of an even periodic function contains only the cosine terms plus a constant, all sine terms being zero.*

3. Half –Wave or Alternation Symmetry

A periodic function $f(t)$ is said to have half wave symmetry if it satisfies the condition,

$f(t) = -f(t \pm T/2)$, where T – time period of the function

x	$T/2$	0
t	0	$-T/2$

$$\therefore \quad a_0 = \frac{1}{T}\left[\int_{-T/2}^{0} f(t)dt + \int_{0}^{T/2} f(t)dt\right] = \frac{1}{T}[I_1 + I_2]$$

For I_1, let $x = (t + T/2)$; so, $f(t) = f(x - T/2) = -f(x)$ and $dt = dx$

$$\therefore \quad I_1 = \int_{-T/2}^{0} f(t)dt = \int_{0}^{T/2} -f(x)dx = -\int_{0}^{T/2} f(x)dx$$

$$\therefore \quad a_0 = \frac{1}{T}\left[-\int_{0}^{T/2} f(x)dx + \int_{0}^{T/2} f(t)dt =\right] = \frac{1}{T}\left[\int_{0}^{T/2} f(x)dx - \int_{0}^{T/2} f(x)dx\right] = 0$$

$$\therefore \quad a_n = \frac{2}{T}\left[\int_{-T/2}^{T/2} f(t)\cos n\omega t dt\right] = \frac{2}{T}\left[\int_{-T/2}^{0} f(t)\cos n\omega t dt + \int_{0}^{T/2} f(t)\cos n\omega t dt\right] = \frac{2}{T}[I_1 + I_2]$$

Again putting $x = (t + T/2)$ and following the same procedure,

$$I_1 = \int_{-T/2}^{0} f(t)\cos n\omega t dt = \int_{0}^{T/2} -f(x)\cos n\omega(x - T/2)\,dx = \int_{0}^{T/2} -f(x)\cos(n\omega x - n\pi)\,dx$$

$$= \int_{0}^{T/2} -f(x)\cos n\pi \cos n\omega x dx = \int_{0}^{T/2} -f(t)\cos n\pi \cos n\omega t dt$$

$$a_n = \frac{2}{T}(1 - \cos n\pi)\int_{0}^{T/2} f(t)\cos n\omega t dt$$

$= 0$; for even n, and

$$= \frac{4}{T}\int_{0}^{T/2} f(t)\cos n\omega t dt, \text{ for odd } n.$$

Similarly, $\quad b_n = 0$, for even n; and

$$= \frac{4}{T}\int_{0}^{T/2} f(t)\sin n\omega t dt\,, \text{ for odd } n.$$

Thus, the Fourier series expansion of a periodic function having half-wave symmetry contains only odd harmonics, the constant term being zero.

4. Quarter–Wave Symmetry

The symmetry may be regarded as a combination of first three kinds of symmetry provided that the origin is properly chosen.

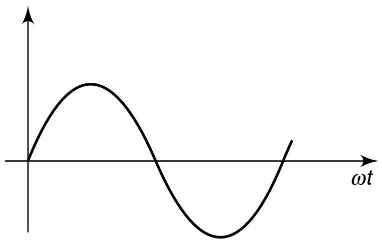


Figure 8.6(a) *Sin ωt: combination of half-wave and odd symmtery*

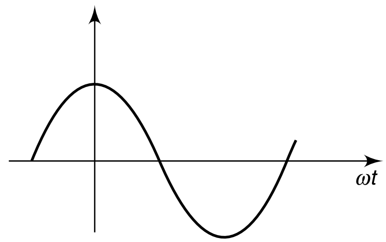


Figure 8.6(b) *Cos ωt: combination of half-wave and even symmetry*

For Figure 8.6(a), the wave has alternation and odd symmetry; thus the Fourier series consists of odd sine terms only.

$$\therefore \qquad a_0 = 0,\ a_n = 0 \text{ and } b_n = \frac{8}{T}\int_0^{T/4} f(t)\sin n\omega t\,dt,\ n \text{ being odd only.}$$

For Figure 8.6(b), the origin, having chosen one quarter cycle away, as in Figure 8.6(a), the wave has alternation and even symmetry; thus the Fourier series consists of odd cosine terms only.

$$\therefore \qquad a_0 = 0;\ b_n = 0; \text{ and } a_n = \frac{8}{T}\int_0^{T/4} f(t)\cos n\omega t dt,\ n \text{ being odd only.}$$

Note:

(i) The sum or product of two or more even functions is an even function, and with the addition of a constant, the even nature of the function is still preserved.

(ii) The sum of two or more odd functions is an odd function, but the addition of a constant removes the odd nature of the function. The product of two odd functions is an even function.

8.4.2 Truncating Fourier Series

When a periodic function is represented by a Fourier series, the series is truncated after a finite number of terms.

So, the periodic function is approximated by a trigonometric series of $(2N + 1)$ terms as,

$$S_N(t) = a_0 + \sum_{n=1}^{N}(a_n \cos n\omega t + b_n \sin n\omega t) \tag{8.2}$$

such that the co-efficients a_0, a_n and b_n are chosen to give the least mean square error.

The truncation error is,

$$e_N(t) = f(t) - S_N(t) \tag{8.3}$$

So, the mean square error/figure of merit/the cost criterion for optimal minimal error is,

$$E_N = \overline{e_N^2}(t) = \frac{1}{T}\int_0^T [e_N(t)]^2\,dt \tag{8.4}$$

where, E_N is a function of a_0 , a_n and b_n, but not of t.

Example 8.4 Show that the mean square error is a minimum if the co- efficients in the approximated trigonometric series $S_N(t)$ are the Fourier co- efficients.

Solution In order to make 'E_N' minimum, the necessary conditions are,

$$\frac{\partial E_N}{\partial a_n} = 0, \text{ for } n = 0, 1, 2, \ldots \tag{8.5a}$$

and
$$\frac{\partial E_N}{\partial b_n} = 0, \text{ for } n = 0, 1, 2, \ldots \tag{8.5b}$$

These two equations give $(2N + 1)$ equations from which $(N + 1)$ number of a_n for $n = 1, 2, \ldots, N$ and N number of b_n for $n = 1, 2, \ldots, N$ can be determined.
From Equations 8.4 and 8.5

$$\frac{\partial E_N}{\partial a_n} = \frac{2}{T}\int_0^T e_N(t)\frac{\partial e_N(t)}{\partial a_n}dt = \frac{2}{T}\int_0^T [f(t) - S_N(t)]\cos n\omega t dt = 0$$

or
$$\int_0^T f(t)\cos n\omega t dt = \int_0^T S_N(t)\cos n\omega t dt$$

$$= \int_0^T \left[a_0 + \sum_{n=1}^{N}(a_n \cos n\omega t + b_n \sin n\omega t)\right]\cos n\omega t dt$$

$$= \int_0^T a_n \cos^2 n\omega t dt = a_n \frac{T}{2}$$

or
$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t dt \quad (n = 0, 1, 2, \ldots, N)$$

Similarly, from equation 8.5(b), we get,

$$\therefore \quad b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t dt \quad (n = 0, 1, 2, \ldots, N)$$

Therefore, it is proved that a Fourier series with a finite number of terms represents the best approximation for a given periodic function by any trigonometric series with the same number of terms.

However, there is no analytical method for the evaluation of estimation of error due to truncation of infinite series; i.e., we can not predict the number of minimum terms to be retained in the series within a prescribed accuracy. The minimisation of error is done by trial and error method, using more terms until specifications are met.

Example 8.5 If $f(t)$ is approximated by $\frac{8}{\pi^2}\sin \omega t$, i.e., the first term in the Fourier Series, find the mean square error.

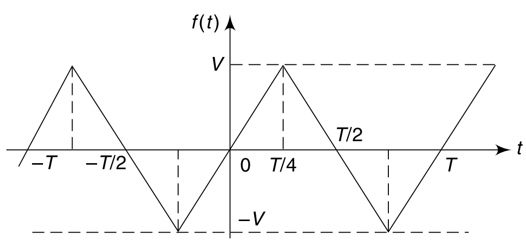


Figure 8.7 *Waveform of Example 8.5*

Solution Truncation Error, $e_N = f(t) - \frac{8}{\pi^2}\sin \omega t$

Mean Square Error, $E_N = e_N^2 = \frac{1}{T}\int_0^T e_N^2(t)dt = \frac{4}{T}\int_0^{T/4}\left[f(t) - \frac{8}{\pi^2}\sin \omega t\right]^2 dt$

(from symmetry consideration)

$$= \frac{4}{T}\int_0^{T/4}\left[\frac{4t}{T} - \frac{8}{\pi^2}\sin \omega t\right]^2 dt$$

$$= 0.0047$$

8.5 STEADY- STATE RESPONSE OF NETWORK TO PERIODIC SIGNALS

The voltage (periodic) is,

$$v(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega t - \phi_n)$$

We want to find out the steady state current, $i(t)$. Phasors corresponding to terms in right hand side are,

$$\mathbf{V}_0 = A_0 e^{j0} \quad \text{and} \quad \mathbf{V}_n = A_n e^{-j\phi_n}$$

Let, $\boldsymbol{Z}$ $(j\omega)$ = Impedance phasor of the network at any frequency ω.
So, the current phasors are,

$$\mathbf{I}_0 = \frac{\mathbf{V}_0}{\mathbf{Z}(j0)} = \frac{A_0 e^{j0}}{Z(j0)} = |I_0|e^{j0}$$

$$\mathbf{I}_n = \frac{\mathbf{V}_n}{\mathbf{Z}(j\omega)} = \frac{A_0 e^{-j\phi_n}}{Z(j\omega)} = |I_n|\, e^{-j\alpha_n}$$

By superposition principle, the net current phasor is,

$$i(t) = I_0 + I_1 + I_2 + \ldots$$

So, transforming from frequency domain to time domain,

$$i(t) = I_0 + \sum_{n=1}^{\infty} |I_n| \cos(n\omega t - \alpha_n)$$

8.5.1 Average Power Calculation

$$v(t) = V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega t - \phi_n)$$

$$i(t) = I_0 + \sum_{n=1}^{\infty} I_n \cos(n\omega t - \alpha_n)$$

Here, V_0 = DC voltage component
V_n = the amplitude of the n^{th} harmonic voltage
ϕ_n = the phase angle of the n^{th} harmonic voltage

I_0 = DC current component
I_n = the amplitude of the n^{th} harmonic current
α_n = the phase angle of the n^{th} harmonic current

Instantaneous power,

$$P(t) = v(t)\ i(t)$$

Average power,

$$P_{av} = \frac{1}{T}\int_0^T v(t)i(t) = \frac{1}{T}\int_0^T \left[\left(V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega t - \phi_n)\right)\left(I_0 + \sum_{n=1}^{\infty} I_n \cos(n\omega t - \alpha_n)\right)\right] dt$$

or
$$P_{av} = V_0 I_0 + \sum_{n=1}^{\infty}\int_0^T V_n I_n \cos(n\omega t - \phi_n)\cos(n\omega t - \alpha_n)\, dt$$

or
$$P_{av} = V_0 I_0 + \sum_{n=1}^{\infty} \frac{V_n I_n}{2} \cos(\phi_n - \alpha_n)$$

8.5.2 Steady State Current in Series Circuits

It is known that when a sinusoidal voltage is applied to a single phase series circuit, the resulting current will also be a sinusoidal. But if an alternating voltage containing various harmonics is applied to such a circuit, each harmonic voltage will produce a component current independent of the others and the resulting current will be the phasor sum of all the harmonic currents. The wave-shape of the current may altogether be different from the wave-shape of the applied voltage.

We consider the following four series circuits:

1. Purely Resistive Circuit We consider a voltage as given below be applied to a pure resistor R.

$$v = V_{1m}\sin\omega t + V_{2m}\sin 2\omega t + V_{3m}\sin 3\omega t + \ldots$$

Since the impedance offered by different harmonics is constant and equal to R, the resulting current will be as given.

$$i = \frac{v}{R} = \frac{V_{1m}}{R}\sin\omega t + \frac{V_{2m}}{R}\sin 2\omega t + \frac{V_{3m}}{R}\sin 3\omega t + \ldots$$

Therefore, the waveform of the current and voltage will be the same and the percentage harmonics in the current wave is the same as that in the voltage wave.

2. Purely Inductive Circuit We consider a voltage as given below be applied to a pure inductor L.

$$v = V_{1m}\sin\omega t + V_{2m}\sin 2\omega t + V_{3m}\sin 3\omega t + \ldots$$

The inductance reactances for different harmonics are as given.

$$\begin{aligned} X_L &= \omega L; \quad \text{for fundamental} \\ &= 2\omega L; \quad \text{for second harmonic} \\ &= 3\omega L; \quad \text{for third harmonic, and so on.} \end{aligned}$$

Hence, the current waveform is obtained by the principle of superposition considering the different harmonic components.

$$\therefore \qquad i = \frac{V_{1m}}{\omega L}\sin\left(\omega t - 90^0\right) + \frac{V_{2m}}{2\omega L}\sin\left(2\omega t - 90^0\right) + \frac{V_{3m}}{3\omega L}\sin\left(3\omega t - 90^0\right) + \ldots$$

From v and i, it is seen that the percentage harmonics in the current wave is less than that in the voltage wave. For n^{th} harmonic, the percentage harmonic in the current wave is $\frac{1}{n}$-times than in the voltage wave.

Respective RMS values of the voltage and current are given as,

$$V_{RMS} = \frac{1}{\sqrt{2}}\sqrt{V_{1m}^{\ 2} + V_{2m}^{\ 2} + V_{3m}^{\ 2} + \ldots}$$

$$I_{RMS} = \frac{1}{\sqrt{2}}\sqrt{\left(\frac{V_{1m}}{\omega L}\omega C\right)^2 + \left(\frac{V_{2m}}{2\omega L}\right)^2 + \left(\frac{V_{3m}}{3\omega L}\right)^2 + \ldots} = \frac{1}{\sqrt{2}}\sqrt{V_{1m}^{\ 2} + V_{2m}^{\ 2}/4 + V_{3m}^{\ 2}/9 + \ldots}$$

From the above discussion, we conclude that, when a complex voltage wave is applied to a pure inductor, the current wave has lesser harmonics than the applied voltage wave and thus, the current waveform will be smoother than the voltage wave.

3. Purely Capacitive Circuit We consider a voltage as given below be applied to a pure capacitor C.

$$v = V_{1m}\sin\omega t + V_{2m}\sin 2\omega t + V_{3m}\sin 3\omega t + \ldots$$

The capacitance reactances for different harmonics are as given.

$$\begin{aligned} X_C &= \frac{1}{\omega C}; \quad \text{for fundamental} \\ &= \frac{1}{2\omega C}; \quad \text{for second harmonic} \\ &= \frac{1}{3\omega C}; \quad \text{for third harmonic, and so on.} \end{aligned}$$

Hence, the current waveform is obtained by the principle of superposition considering the different harmonic components.

$$\therefore \qquad i = V_{1m}(\omega C)\sin\left(\omega t + 90^0\right) + V_{2m}(2\omega C)\sin\left(2\omega t + 90^0\right) + V_{3m}(3\omega C)\sin\left(3\omega t + 90^0\right) + \ldots.$$

From v and i, it is seen that the percentage harmonics in the current wave is more than that in the voltage wave. For n^{th} harmonic, the percentage harmonic in the current wave is n times than in the voltage wave.

Respective RMS values of the voltage and current are given as,

$$V_{RMS} = \frac{1}{\sqrt{2}}\sqrt{V_{1m}^{\ 2} + V_{2m}^{\ 2} + V_{3m}^{\ 2} + \ldots}$$

$$I_{RMS} = \frac{1}{\sqrt{2}}\sqrt{\left(V_{1m}\omega C\right)^2 + \left(2V_{2m}\omega C\right)^2 + \left(3V_{3m}\omega C\right)^2 + \ldots} = \frac{1}{\sqrt{2}}\sqrt{V_{1m}^{\ 2} + 4V_{2m}^{\ 2} + 9V_{3m}^{\ 2} + \ldots}$$

From the above discussion, we conclude that, when a complex voltage wave is applied to a pure capacitor, the current wave has more harmonics than the applied voltage wave and thus, the current waveform will be more distorted than the voltage wave.

4. General RLC Series Circuit We consider a voltage as given below be applied to a general *RLC* series circuit.

$$v = V_{1m}\sin\omega t + V_{2m}\sin 2\omega t + V_{3m}\sin 3\omega t + \ldots$$

The impedance for different harmonics are as given.

$$Z_1 = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}; \quad \text{for fundamental}$$

$$Z_2 = \sqrt{R^2 + \left(2\omega L - \frac{1}{2\omega C}\right)^2}; \quad \text{for second harmonic}$$

$$Z_3 = \sqrt{R^2 + \left(3\omega L - \frac{1}{3\omega C}\right)^2}; \quad \text{for third harmonic, and so on.}$$

Hence, the current waveform is obtained by the principle of superposition considering the different harmonic components.

$$\therefore \qquad i = \frac{V_{1m}}{Z_1}\sin(\omega t - \phi_1) + \frac{V_{2m}}{Z_2}\sin(2\omega t - \phi_2) + \frac{V_{3m}}{Z_3}\sin(3\omega t - \phi_3) + \ldots.$$

where, $$\phi_n = \tan^{-1}\left[\frac{1}{R}\left(n\omega L - \frac{1}{n\omega C}\right)\right].$$

8.5.3 Steps for Application of Fourier Series to Circuit Analysis

1. Fourier series of the given periodic excitation function is obtained.
2. The circuit elements are transformed from time domain to frequency domain (i.e., $R \to R$, $L \to j\omega nL$, $C \to \dfrac{1}{j\omega nC}$ for n^{th} harmonic).
3. The Fourier series of the DC and AC components of the response are calculated.
4. Using Superposition, the Fourier series of the response is obtained by summing up the individual DC and AC response components.

8.5.4 Power Spectrum

It is the distribution of the average power over the different frequency components.
Let, P_n be the average power for the n^{th} harmonic component.
Note: P_n is always positive so that only a magnitude spectrum is possible.
Another form of line spectrum for power is also possible [Fig. 8.8(b)]; obtained by assuming half of P_n to the positive frequency $n\omega$ and half to the negative frequency.

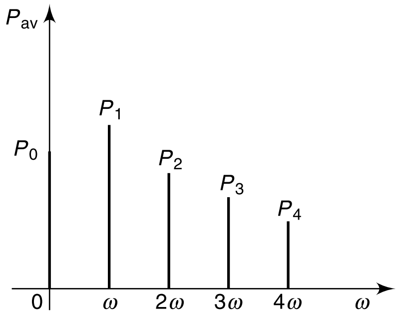


Figure 8.8(a) *Power Spectrum for positive ω*

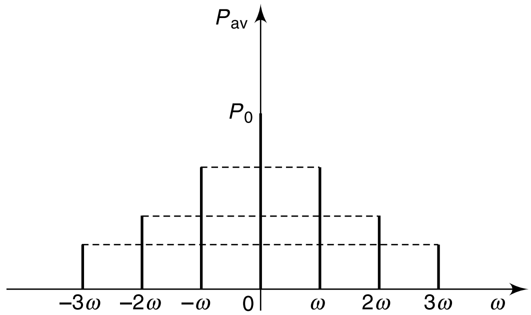


Figure 8.8(b) *Power Spectrum for both positive and negative ω*

PART II: FOURIER TRANSFORM

8.6 INTRODUCTION

The Fourier series representation of a period function describes the function in the frequency domain in terms of amplitude and phase spectra. The Fourier transform extends this frequency domain description to functions that are not periodic.

Fourier transform is a powerful tool in the study of power spectra, correlation functions, noise and other advanced problems.

8.7 DEFINITION OF FOURIER TRANSFORM

The Fourier Transform or the Fourier integral of a function f(t) is denoted by $F(j\omega)$ and is defined by,

$$F(j\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \tag{8.6}$$

and the inverse Fourier transform is defined by,

$$f(t) = \mathcal{F}^{-1}[F(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega = \int_{-\infty}^{\infty} F(j2\pi f)\, e^{j2\pi f}df \tag{8.7}$$

Equations 8.6 and 8.7 form the Fourier transform pair.

Explanation Consider the exponential Fourier Series,

$$f(t) = \sum_{-\infty}^{\infty} C_n e^{jn\omega t} \tag{8.8}$$

where, $$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega t}dt \tag{8.9}$$

If the period T becomes infinite, the function does not repeat itself and becomes aperiodic or non-periodic.
So, the interval between adjacent harmonic frequencies is,

$$\Delta\omega = (n + 1) - n\omega = \omega = \frac{2\pi}{T}$$

or $$\frac{1}{T} = \frac{\omega}{2\pi} = \frac{\Delta\omega}{2T} \tag{8.10}$$

As $T \to \infty$, $\Delta\omega \to d\omega$ and the frequency goes from a discrete variable over to a continuous variable.

$$\frac{1}{T} \to \frac{d\omega}{2\pi} \quad \text{and} \quad n\omega \to \omega \tag{8.11}$$

From 8.7 and 8.11,

$$C_nT \to \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$. This is the Fourier Transform of $f(t)$ i.e., $F(j\omega)$.

$$\boxed{F(j\omega) = F[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt}$$

So, from equation (8.8),

$$f(t) = \sum_{-\infty}^{\infty}(C_nT)e^{jn\omega t}\left(\frac{1}{T}\right) \tag{8.12}$$

As $T \to \infty$, $C_nT \to F(j\omega)$, $n\omega \to \omega$ and $\frac{1}{T} \to \frac{d\omega}{2\pi}$ and $\sum \to \int$ (summation approaches integration).

Thus, from (8.12),

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(j\omega)e^{j\omega t}d\omega}$$

Spectra Let, $F(j\omega) = |F(j\omega)|e^{j\phi(\omega)}$
The variation of $|F(j\omega)|$ with ω is referred to as the amplitude spectrum.
The variation of $\phi(\omega)$ with ω is referred to as the phase–spectrum.
Since $F(j\omega)$ is a continuous function, the corresponding amplitude and phase spectra are continuous spectra.

8.8 CONVERGENCE OF FOURIER TRANSFORM

When $f(t)$ is a single-valued function and is different from zero over an infinite interval of time, the behavior of $f(t)$ as $t \to \pm\infty$ determines the convergence of the Fourier transform.

The Fourier transform will exist, if

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

8.9 FOURIER TRANSFORM OF SOME FUNCTIONS

1. $f(t) = Ae^{-at} u(t)$, $a > 0$ Fourier transform will exist, if $a > 0$

$$\therefore \qquad F(j\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = A\int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = A\left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_0^{\infty} = \frac{A}{a+j\omega}$$

Amplitude, $|F(j\omega)| = \dfrac{A}{\sqrt{a^2+\omega^2}}$

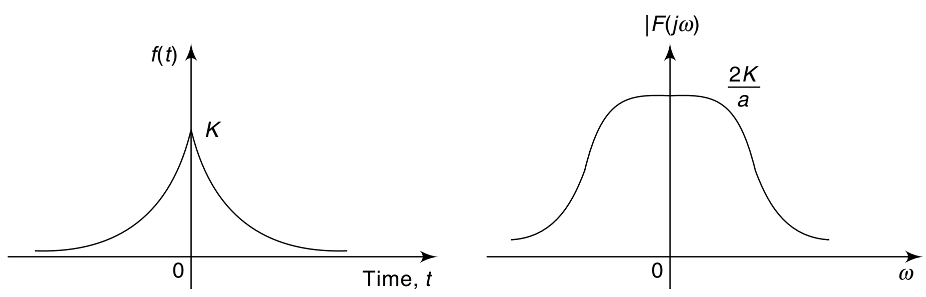
Phase, $\phi(j\omega) = -\tan^{-1}\left(\dfrac{\omega}{a}\right)$

2. $f(t) = Ke^{-a|t|}$, for all values of t

$$F(j\omega) = \mathcal{F}[Ke^{-a|t|}] = \int_{-\infty}^{\infty} Ke^{-a|t|}e^{-j\omega t}dt = \int_{-\infty}^{0} Ke^{(a-j\omega)t}dt + \int_{0}^{\infty} Ke^{-(a+j\omega)t}dt$$

$$= \frac{K}{a-j\omega} + \frac{K}{a+j\omega} = \frac{2Ka}{a^2+\omega^2}$$

Thus the Fourier transform of the double exponential function has zero phase for all values of ω and the magnitude spectrum is shown in Fig. 8.9.



(a) Double exponential function (b) Fourier transform

Figure 8.9 *Double exponential function and its Fourier transform*

NB: There are some important functions which do not have Fourier transforms in a strict sense; because they do not satisfy the Dirichlet's condition, i.e., $\int_{-\infty}^{\infty}|f(t)|dt$ in infinite (such as, the step function and sinusoidal function). However, the Fourier transform of these functions are evaluated by approximating these functions in time domain as the limiting value of another function which possesses Fourier transform.

3. Fourier transform of some constant, *K*; for all values of *t* Here, we can approximate the constant as,

$$f(t) = \underset{a \to 0}{Lt}\left[Ke^{-a|t|}\right]$$

$$\therefore \qquad \mathcal{F}[K] = \underset{a \to 0}{Lt} \int_{-\infty}^{\infty} Ke^{-a|t|}e^{-j\omega t}\,dt = \underset{a \to 0}{Lt} \frac{2Ka}{a^2+\omega^2}$$

$$\therefore \qquad \mathcal{F}[K] = 0;\ for\ \omega \neq 0$$

$$= \infty;\ for\ \omega = 0$$

[by L Hospital's rule, i.e., differentiating both numerator and denominator with respect to '*a*']
Thus, $\mathcal{F}[K]$ is an impulse function at $\omega = 0$. The strength (amplitude) of the impulse function is obtained as,

$$\int_{-\infty}^{\infty} \mathcal{F}[K]d\omega = \int_{-\infty}^{\infty} \frac{2Ka}{a^2+\omega^2}d\omega = 2\pi K$$

$$\therefore \qquad \boxed{\mathcal{F}[K] = 2\pi K\delta(\omega)}$$

Hence, Fourier transform of a constant K is an impulse of magnitude $2\pi K$ as shown in Fig. 8.10.

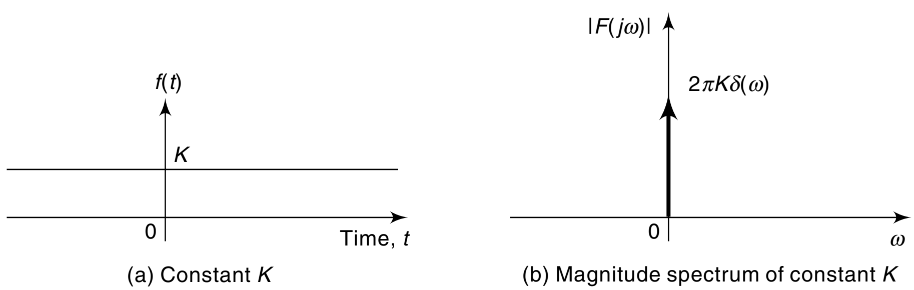


Figure 8.10 *Constant K and its magnitude*

4. Unit impulse function or Dirac Delta Function, $\delta(t)$ Some problems involve the concept of an impulse, which may be intuitively thought of as a force of very large magnitude impacting just for an instant.

$$\therefore \qquad \mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt$$

We use *shifting property of impulse function* as explained below.

The product of any arbitrary function $f(t)$ with unit impulse function $\delta(t)$ provides the function $f(t)$ to exist only at $t = 0$.

Mathematically,

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(t)\Big|_{t=0}$$

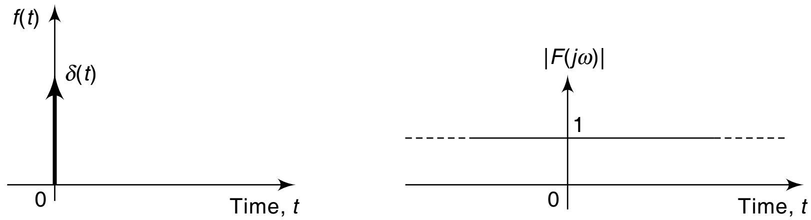
This shifting property can also be applied at any instant of time, say $t = t_0$, so that we can write,

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t)\Big|_{t=t_0} = f(t_0)$$

Using this property, we have the Fourier transform of unit impulse function as,

$$\therefore \qquad \mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = e^0 = 1$$

Thus, Fourier transform of an impulse function is unity as shown in Fig. 8.11.



(a) Impulse function

(b) Fourier transform of impulse function

Figure 8.11 *Impulse function and its Fourier transform*

5. Fourier transform of Signum Function, Sgn(*t*)

A signum function is defined as,

$$\text{Sgn}(t) = \begin{cases} +1 & \text{for } t > 0 \\ 0 & \text{for } t = 0 \\ -1 & \text{for } t < 0 \end{cases}$$

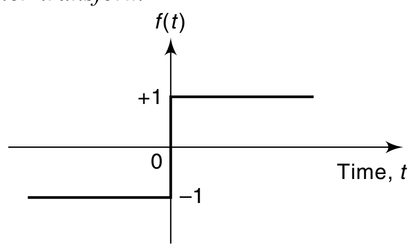


Figure 8.12(a) *Sgn (t)*

$\therefore \int_{-\infty}^{\infty} Sgn(t)dt$ is infinite, direct evaluation of Fourier transform is not possible.

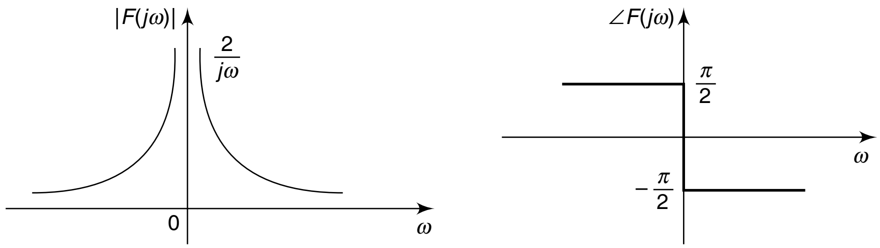
Therefore, the given function has to be expressed as limiting case of some other function and then the Fourier Transform is computed. Let, the Sgn(t) be multiplied by $e^{-a|t|}$ and $a \to 0$.

$$F[Sgn(t)] = \lim_{a \to 0} \int_{-\infty}^{\infty} e^{-a|t|} Sgn(t) e^{-j\omega t}dt = \lim_{a \to 0}\left[-\int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt\right]$$

$$= \lim_{a \to 0}\left[\frac{-1}{a-j\omega} + \frac{1}{a+j\omega}\right]$$

or, $$\boxed{F[Sgn(t)] = \frac{2}{j\omega}}$$

Figure 8.12 shows the magnitude and phase spectrum of the Signum function.



(b) Magnitude spectrum of sgn(t)   (c) Phase spectrum of sgn(t)

Figure 8.12 *Signum function and its magnitude spectrum*

6. Fourier transform of Unit Step Function, *u*(t)

$$u(\mathrm{t}) = 1 \quad \text{for } t > 0$$
$$= 0 \quad \text{for } t < 0$$

Since $\int_{-\infty}^{\infty} u(t)dt$ is infinite, direct evaluation of Fourier transform is imposable.

Let, $$u(t) = \frac{1}{2} + \frac{1}{2} Sgn(t)$$

$\therefore$ $$\mathcal{F}[u(t)] = \mathcal{F}\left[\frac{1}{2}\right] + \mathcal{F}\left[\frac{1}{2} Sgn(t)\right] = 2\pi \times \frac{1}{2}\delta(\omega) + \frac{1}{2} \times \frac{2}{j\omega}$$

or, $$\boxed{\mathcal{F}[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}}$$

Thus, the amplitude of unit step function $u(t)$ in frequency domain will be a combination of rectangular hyperbola and impulse function (of strength π at $\omega = 0$) as shown in Fig. 8.13.

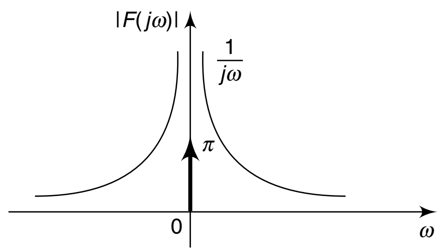


Figure 8.13 *Magnitude spectrum of unit step function*

8.10 PROPERTIES OF FOURIER TRANSFORMS

1. Linearity

If a, b, $\in$ C, then

$$F\{\alpha f(t) + \beta g(t)\} = \alpha F\{f(t)\} + \beta F\{g(t)\} = \alpha F(\omega) + \beta G(\omega)$$

provided the Fourier transforms of $f(t)$ and $g(t)$ exist.

2. Scaling

If $F\{f(t)\} = F(\omega)$ and $c \in R$, then

$$F\{cf(t)\} = \frac{1}{|c|} F\left(\frac{\omega}{c}\right)$$

3. Time shifting

If $F\{f(t)\} = F(\omega)$ and $t_0 \in R$, then

$$F\{f(t-t_0)\} = e^{-j\omega t_0} F(\omega)$$

4. Frequency shifting

If $F\{f(t)\} = F(\omega)$ and $\omega \in R$, then

$$F(\omega - \omega_0) = F\{e^{j\omega_0} f(t)\}$$

5. Symmetry

If $F\{f(t)\} = F(\omega)$, then

$$F\{F(t)\} = 2\pi f(-\omega)$$

6. Modulation

If $F\{f(t)\} = F(\omega)$ and $\omega_0 \in R$, then

$$F\{f(t)\cos(\omega_0 t)\} = \frac{1}{2}[F(\omega+\omega_0) + F(\omega-\omega_0)]$$

$$F\{f(t)\sin(\omega_0 t)\} = \frac{1}{2}[F(\omega+\omega_0) - F(\omega-\omega_0)]$$

7. Differentiation in time

Let $n \in N$ and suppose that $f^{(n)}$ is piecewise continuous. Assume that $\lim\limits_{x\to\infty} f^{(k)}(t) = 0$, then

$$F\{f^{(n)}(t)\} = (j\omega)^n F(\omega)$$

In particular

$$F\{f'(t)\} = j\omega F(\omega)$$

and

$$F\{f''(t)\} = -\omega^2 F(\omega)$$

8. Frequency differentiation

Let $n \in N$ and suppose that f is piecewise continuous. Then

$$F\{t^n f(t)\} = j^n F^{(n)}(\omega)$$

In particular

$$F\{tf(t)\} = jF'(\omega)$$

and

$$F\{t^2 f(t)\} = -F''(\omega)$$

These properties can be tabulated as follows (Table 8.1).

Table 8.1 *Properties of Fourier Transforms*

Sl No.	*Time Domain* $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}dt$	*Frequency Domain* $F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$
1	$f(t)$ real	$F(j\omega) = F^*(-j\omega)$
2	$f(t)$ even, $f(t) = f(-t)$	$F(j\omega) = F(-j\omega)$, $F(j\omega)$ is real
3	$f(t)$ odd, $f(t) = -f(-t)$	$F(j\omega) = -F(-j\omega)$, $F(j\omega)$ is imaginary,
4	$y(t) = t^n f(t)$	$Y(j\omega) = (j)^n \frac{d^n F(j\omega)}{d\omega^n}$
5	$y(t) = f(at)$	$F(j\omega) = \frac{1}{a}F\left(\frac{j\omega}{a}\right), a > 0$
6	$y(t) = f(t - t_0)$	$Y(j\omega) = e^{-j\omega t_0}F(j\omega)$
7	$y(t) = \frac{d^n f(t)}{dt^n}$	$Y(j\omega) = (j\omega)^n F(j\omega)$
8	$y(t) = \int_{-\infty}^{\infty} f(t)dt$	$Y(j\omega) = \frac{F(j\omega)}{j\omega}$
9	$y(t) = f(t)e^{j\omega_0 t}$	$Y(j\omega) = F[j(\omega - \omega_0)]$

Example 8.6 Show that when $f(t)$ is an even function of t, its Fourier transform $F(j\omega)$ is a function of ω and is real; while when $f(t)$ is an odd function of t, its Fourier transform $F(j\omega)$ is an odd function of ω and is imaginary.

Solution From the definition,

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)(\cos\omega t - j\sin\omega t)dt$$

$$= \int_{-\infty}^{\infty} f(t)\cos\omega t dt - j\int_{-\infty}^{\infty} f(t)\sin\omega t dt = P(\omega) + jQ(\omega)$$

where, $P(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t dt =$ Even function of ω, i.e., $P(\omega) = P(-\omega)$

and $Q(\omega) = \int_{-\infty}^{\infty} f(t)\sin\omega t dt =$ Odd function of ω, i.e., $Q(\omega) = -Q(-\omega)$

Now, $F(j\omega) = |F(j\omega)|\, e^{j\phi(\omega)}$

$$|F(j\omega)| = \sqrt{P^2(\omega) + Q^2(\omega)} = \text{Even function of } \omega$$

and $\quad F(j\omega) = \tan^{-1}\left[\dfrac{Q(\omega)}{P(\omega)}\right] = \text{Odd function of } \omega$

- *When f(t) is an even function*

  $f(t)$ cos ωt is an even function

  $f(t)$ sin ωt is odd function.

$\therefore \quad P(\omega) = 2\int_0^\infty f(t)\cos\omega t dt$

$Q(\omega) = 0$

so, $\quad F(j\omega) = P(\omega)$ = Even and Real (Proved)

- *When f(t) is an odd function*

  $f(t)$ cos ωt is an odd function

  $f(t)$ sin ωt is an even function

$\therefore \quad P(\omega) = 0$

and $\therefore Q(\omega) = -2\int_0^\infty f(t)\sin\omega t dt$

so, $\quad F(j\omega) = jQ(\omega)$ = Odd and Imaginary (Proved)

8.11 ENERGY DENSITY AND PARSEVAL'S THEOREM

This theorem states that the energy content (W) of a waveform (periodic or non-periodic) over the whole frequency band is,

$$W = \int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\, d\omega$$

Proof: We have,

$$W = \int_{-\infty}^{\infty} f^2(t)dt = \int_{-\infty}^{\infty} f(t)\cdot[f(t)dt]$$

$$= \int_{-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}dt\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\left[\int_{-\infty}^{\infty} f(t)e^{j\omega t}dt\right]dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\cdot F(-j\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\, d\omega$$

or $$W = \int_{-\infty}^{\infty} f^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\, d\omega \qquad \textit{Proved}$$

Note

(i) Since $|F(j\omega)|$ is an even function of ω,

$$W = \int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{\pi}\int_{0}^{\infty}|F(j\omega)|^2\, d\omega$$

(ii) Since $\omega = 2\pi f$, where f is the frequency,

$$W = \int_{-\infty}^{\infty} f^2(t)dt = \int_{-\infty}^{\infty}|F(j2\pi f)|^2\, df = 2\int_{0}^{\infty}\left|F(j2\pi f)\right|^2 df$$

The quantity $|F(j2\pi f)|^2\, df$ is the energy in an infinitesimal band of frequency df. It represents the energy density in the frequency domain and has unit of Joule/Hertz.
Total energy content within the frequency band f_1 and f_2 is,

$$W_b = 2\int_{f_1}^{f_2} F(j2\pi f)|^2\, df$$

For the integration range $-\infty$ to $+\infty$, the total energy is,

$$W_b = \int_{-f_1}^{-f_2}|F(j2\pi f)|^2\, df + \int_{f_1}^{f_2}|F(j2\pi f)|^2\, df$$

(iii) If $f(t)$ is the voltage across a 1 Ω resistance or current through the same resistance, then W_b is known as 1 Ω energy.

Example 8.7 The current in a 10 Ω resistor is $i(t) = 10e^{-2t}u(t)\,(A)$. What is the energy associated with the frequency band $0 \le \omega \le 2$ rad/s?

Solution Here, $f(t) = i(t) = 10e^{-2t}u(t)$

$$\therefore \qquad F(j\omega) = \frac{10}{2 + j\omega}$$

So, the energy associated with the given frequency band is,

$$W = \frac{10}{\pi}\int_{0}^{2}|F(j\omega)|^2\, d\omega = \frac{10}{\pi}\int_{0}^{2}\frac{100d\omega}{4+\omega^2} = \frac{10^3}{\pi}\left[\frac{1}{2}\tan^{-1}\left(\frac{\omega}{2}\right)\right]\Bigg|_{0}^{2}$$

$$= \frac{10^3}{\pi}\left[\frac{\pi}{8}\right]$$

$= 125$ Joule *Ans.*

8.12 COMPARISON BETWEEN FOURIER TRANSFORM AND LAPLACE TRANSFORM

The defining equations are,

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt \quad \text{and} \quad F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

Following are some differences and similarities:

1. Laplace Transform is one-sided in the interval $0 < t < \infty$ and Fourier Transform is double-sided in the interval $-\infty < t < \infty$. Thus, Laplace Transform is applicable for positive time function, $f(t)$, $t > 0$; while Fourier Transform is applicable for functions defined for all times.
2. Laplace Transform includes the initial conditions and is applicable for transient analysis; while Fourier Transform is only applicable for steady-state analysis.
3. For functions $f(t) = 0$ for $t < 0$ and $\int_0^{\infty} |f(t)|\,dt < \infty$, the two transforms are related as, $F(j\omega) = F(s)|_{s=j\omega}$. Thus, Laplace Transform is associated with entire s-plane, while, Fourier Transform is restricted to the imaginary ($j\omega$) axis.
4. Laplace Transform is applicable to a wider range of functions than the Fourier Transform. On the other hand, Fourier Transforms exist for signals that are not physically realizable and have no Laplace Transform.

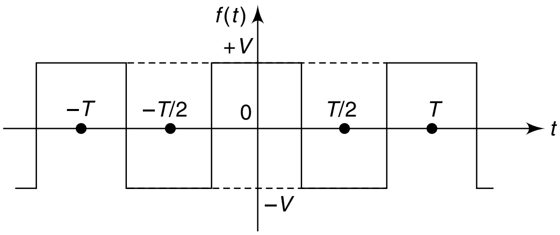
8.13 STEPS FOR APPLICATION OF FOURIER TRANSFORM TO CIRCUIT ANALYSIS

By Fourier Transform, we can find the response of a circuit due to non-periodic functions. The general procedure is described below.

1. Fourier Transform of the given excitation function is obtained.
2. Fourier Transform of the circuit elements is obtained $\left(\text{i.e., } R \to R, L \to j\omega L, C \to \dfrac{1}{j\omega C}\right)$.
3. The transfer function in Fourier Transform Domain is defined as, $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)}$ or $Y(j\omega) = H(j\omega) \cdot X(j\omega)$; where, $Y(j\omega)$ is the response transform and $X(j\omega)$ is the excitation transform.
4. Taking the inverse Fourier Transform of the product $H(j\omega) \cdot X(j\omega)$, we get the response $y(t)$.

SOLVED PROBLEMS

8.1 Determine the Fourier series for the square waveform shown below and plot the magnitude and the phase spectra.



Solution The waveform, $f(t) = V;\ 0 < t < T/4$

$= -V;\ T/4 < t < 3T/4$

$= V;\ 3T/4 < t < T$

Obviously, the given function is an even function.

$\therefore \qquad b_n = 0$

Now, $$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt = \frac{2}{T}\int_0^{T/4} V dt = -\frac{2}{T}\int_{T/4}^{T/2} V dt = 0$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t dt$$

$$= \frac{4}{T}\left[\int_0^{T/4} V\cos n\omega t dt - \int_{T/4}^{T/2} V\cos n\omega t dt\right]$$

$$= \frac{4V}{n\omega T}\left[\sin\left(\frac{n\omega T}{4}\right) - \sin\left(\frac{n\omega T}{2}\right) + \sin\left(\frac{n\omega T}{4}\right)\right]$$

$$= \frac{4V}{n2\pi}\left[2\sin\left(\frac{n\pi}{2}\right)\right] \qquad [\because \omega T = 2\pi]$$

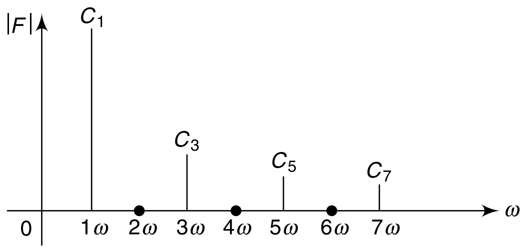
$$= \frac{4V}{n\pi}\sin\frac{n\pi}{2} = \frac{4V}{n\pi};\ \text{for } n = 1, 5, 9, \ldots\ldots\ldots$$

$$= -\frac{4V}{n\pi};\ \text{for } n = 3, 7, 11, \ldots\ldots\ldots$$

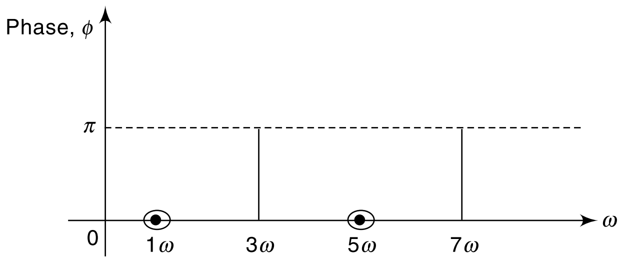
$$= 0;\ \text{for } n = 2, 4, 6, \ldots\ldots\ldots$$

So, $$f(t) = \frac{4V}{\pi}\left(\cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t - \frac{1}{7}\cos 7\omega t + \frac{1}{9}\cos 9\omega t \ldots\ldots\ldots\right)$$ *Ans.*

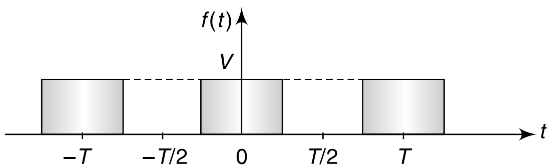
Magnitude Spectra



Phase Spectra



8.2 Find the Fourier series of the function whose periodic waveform is shown in the below figure and plot its frequency spectra.



Solution The function is even

$\therefore \quad b_n = 0$

$$\therefore \quad a_0 = \frac{2}{T}\int_0^{T/2} f(t)dt = \frac{2}{T}\int_0^{T/4} Vdt = \frac{2V}{T}\times\frac{T}{4} = \frac{V}{2}$$

$$\therefore \quad a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t dt$$

$$= \frac{4V}{T}\int_0^{T/4} f(t)\cos n\omega t dt$$

$$= \frac{4V}{T}\left[\left(\frac{\sin n\omega t}{n\omega}\right)_0^{T/4}\right] \quad [\because \omega T = 2\pi]$$

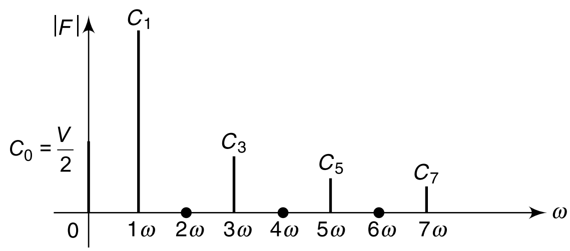
$$= \frac{4V}{n\omega T}\left[\sin\left(\frac{n\omega T}{4}\right)\right]$$

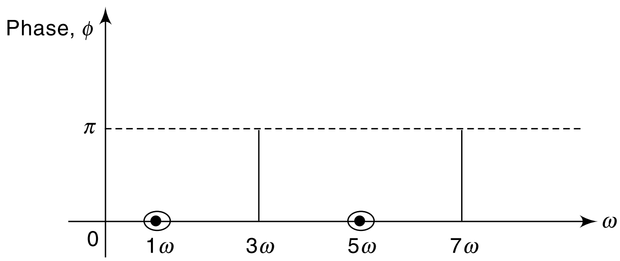
$$= \frac{4V}{2n\pi}.\sin\frac{n\pi}{2} = \frac{2V}{n\pi}\ ; n = 1, 5, 9 \ldots$$

$$= -\frac{2V}{n\pi}\ ; n = 3, 7, 11, \ldots$$

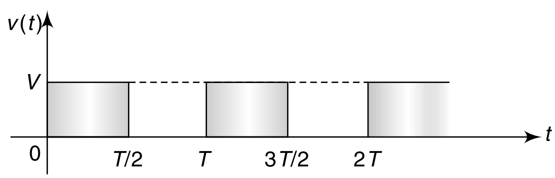
$\therefore$ $$f(t) = \frac{V}{2} + \frac{2V}{\pi}\left(\cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t - \frac{1}{7}\cos 7\omega t + \ldots\right) \qquad \textit{Ans.}$$

Line Spectra





8.3 Find the Fourier series for the train of pulses shown in the below figure and draw the amplitude and the phase spectra.



Solution Here, $v(t) = V$; for $0 < t < T/2$

$$= 0;\ \text{for } \frac{T}{2} < t < T$$

$\therefore$ $$a_0 = \frac{1}{T}\int_0^T V(t)dt = \frac{1}{T}\int_0^{T/2} Vdt = \frac{V}{2}$$

and $$a_n = \frac{2}{T}\int_0^T V(t)\cos n\omega t dt = \frac{2}{T}\int_0^{T/2} V\cos n\omega t dt$$

$$= \frac{2V}{n\omega T}\left[\sin\left(\frac{n\omega T}{2}\right)\right] = 0 \quad [\because \ \omega T = 2\pi]$$

and $$b_n = \frac{2}{T}\int_0^T V(t)\sin n\omega t dt = \frac{2V}{T}\int_0^{T/2} \sin n\omega t dt$$

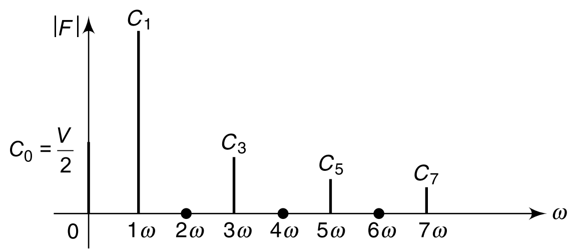
$$= \frac{2V}{n\omega T}\left[1-\cos\left(\frac{n\omega T}{2}\right)\right] = \frac{V}{n\pi}(1-\cos n\pi), \ [\because \omega T = 2\pi]$$

$$= \frac{2V}{n\pi}, \text{ for } n \text{ odd.}$$

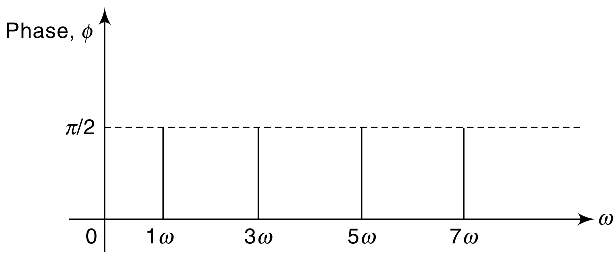
$= 0$, for n even.

$\therefore$ $$V(t) = V\left[\frac{1}{2} + \frac{2}{\pi}\sin n\omega t + \frac{2}{3\pi}\sin 3\omega t + \frac{2}{5\pi}\sin 5\omega t + \ldots\right]$$

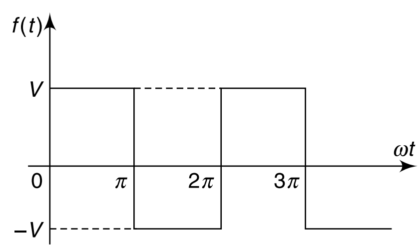
Amplitude Spectra



Phase Spectra



8.4 For the periodic function shown in the adjacent figure determine the exponential form of Fourier series and show the line spectra. Also find its trigonometric form.



Solution The function is defined as,

$f(t) = V, \qquad 0 < t < \pi, \qquad [T = 2\pi]$

$\quad = -V, \qquad \pi < t < 2\pi$

Since the function is odd, the co-efficients $\bar{C}_n$ will be purely imaginary.

$$\therefore \qquad \bar{C}_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, e^{-jn\omega t}\, dt$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} V e^{-jn\omega t}\, dt - V\int_{\pi}^{2\pi} e^{-jn\omega t}\, dt\right]; \quad \text{for} \quad n \neq 0$$

$$= \frac{1}{2\pi}\left[\frac{V}{-jn\omega} e^{-jn\omega t}\right]_0^{\pi} - \frac{V}{2\pi}\left[\frac{1}{-jn\omega}\, e^{-jn\omega t}\right]_{\pi}^{2\pi}$$

$$= \frac{V}{j2\pi n\omega}(1 - e^{-jn\omega\pi}) + \frac{V}{j2\pi n\omega}(e^{-jn\omega 2\pi} - e^{jn\omega\pi}) \qquad (\because\ T = 2\pi,\ \therefore\ \omega = 1)$$

$$= \frac{V}{j2\pi n}(1 - e^{-jn\pi}) + \frac{V}{j2\pi n}(e^{-jn2\pi} - e^{jn\pi})$$

Now, $\quad e^{-jn\pi} = \cos n\pi - j\sin n\pi = (-1)^n$

and $\quad e^{-j2n\pi} = \cos 2n\pi - j\sin 2n\pi = 1$

$$\therefore \qquad \bar{C}_n = \frac{2V}{j2n\pi}[1-(-1)^n]; \ n \neq 0$$

$$= \frac{2V}{jn\pi}; \quad \text{for } n \text{ odd;}$$

$$= 0; \quad \text{for } n \text{ even.}$$

$$C_{-n} = -\frac{2V}{jn\pi}$$

For $n = 0$, $\bar{C}_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt = \frac{1}{2\pi}\left[\int_0^{\pi} Vdt - \int_{\pi}^{2\pi} Vdt\right] = 0$

$\therefore$ Exponential form of Fourier series is,

$$f(t) = \frac{2V}{j\pi}\sum_{n-1}^{\infty}\frac{1}{n} e^{jn\omega t} \ ; \ n \text{ odd only}$$

$$= \frac{2V}{j\pi}\left[e^{j\omega t} + \frac{1}{3}e^{3j\omega t} + \frac{1}{5}e^{j5\omega t} + \frac{1}{7}e^{j7\omega t} + \ldots\right] \qquad \textit{Ans.}$$

To find Trigonometric form,

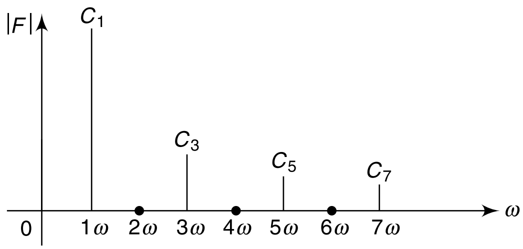
$a_0 = 0$,

$$a_n = (C_n + C_{-n}) = \frac{2V}{jn\pi} - \frac{2V}{jn\pi} = 0$$

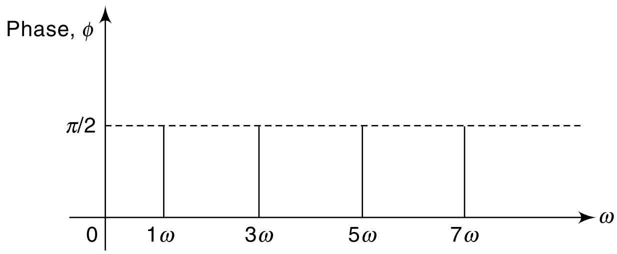
$$b_n = j(C_n - C_{-n}) = j\left[\frac{2V}{jn\pi} + \frac{2V}{jn\pi}\right] = \frac{4V}{n\pi} \qquad \text{for } n \text{ odd.}$$

$$\therefore \qquad f(t) = \frac{4V}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \ldots\right] \qquad \textit{Ans.}$$

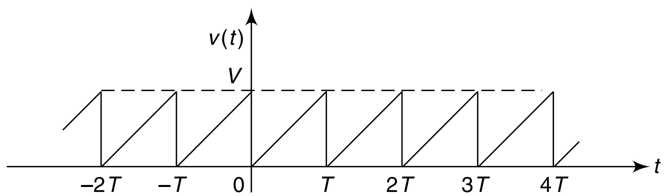
Amplitude Spectra



Phase Spectra



8.5 The waveform shown in the following figure is used as 'sweep' in radar and television circuits. Find the Fourier series and plot the line spectra.



Solution The function, $V(t) = \frac{V}{T}t\,;\ 0 < t < T$

$$\therefore \qquad \bar{C}_n = \frac{1}{T}\int_0^T \frac{V}{T} t e^{-jn\omega t}\,dt\,;\ n \neq 0$$

$$= \frac{V}{T^2}\left[\frac{te^{jn\omega t}}{-jn\omega} + \int \frac{e^{-jn\omega t}}{jn\omega}\right]_0^T$$

$$= \frac{V}{T^2}\left[\frac{Te^{-jn\omega T}}{-jn\omega} - \frac{e^{-jn\omega T}}{(jn\omega)^2}\Bigg|_0^T\right]$$

$$= \frac{V}{T^2}\left[\frac{T^2 e^{-j2\pi n}}{-j2n\pi} + \frac{(e^{-j2n\pi} - 1)}{n^2\omega^2}\right] \qquad [\because\ \omega T = 2\pi]$$

$$= \frac{jV}{2n\pi} e^{-j2n\pi} + \frac{V}{4n^2\pi^2} e^{-j2n\pi} - \frac{V}{4n^2\pi^2}$$

Since, $\quad e^{-j2n\pi} = (\cos 2n\pi - j\sin 2n\pi) = 1$

$\therefore \quad \bar{C}_n = \dfrac{jV}{2n\pi}; \quad$ for $\quad n \neq 0$

For $n = 0$, $\bar{C}_0 = \dfrac{V}{T^2}\int_0^T t\,dt = \dfrac{V}{2}$

$\therefore$ Exponential form,

$$v(t) = ... - \frac{jV}{6\pi} e^{-j3\omega t} - \frac{jV}{4\pi} e^{-j2\omega t} - \frac{jV}{2\pi} e^{-j\omega t} + \frac{V}{2} + \frac{jV}{2\pi} e^{j\omega t} + \frac{jV}{4\pi} e^{j2\omega t} + \frac{jV}{6\pi} e^{j3\omega t} + ...$$

To convert into Trigonometric form

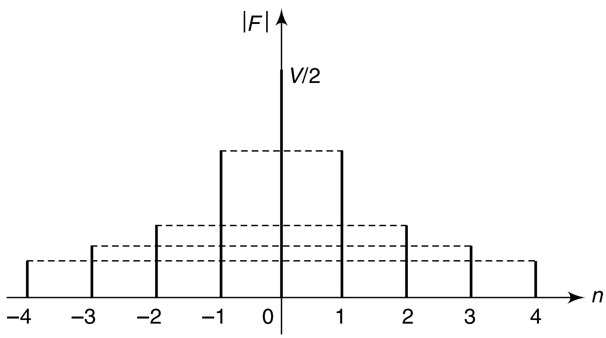
Here, $\quad \bar{C}_n = \dfrac{jV}{2n\pi}, \; \bar{C}_{-n} = -\dfrac{jV}{2n\pi}$

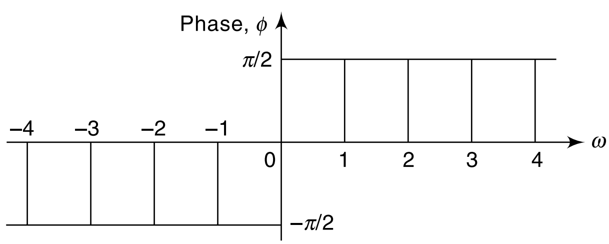
$\therefore \quad a_0 = C_0 = \dfrac{V}{2}, \quad a_n = (\bar{C}_n + \bar{C}_{-n}) = 0$

and $\quad b_n = j\left(\bar{C}_n - \bar{C}_{-n}\right) = -\dfrac{V}{n\pi}$

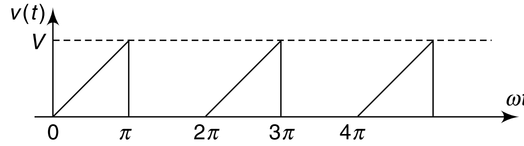
$\therefore \quad V(t) = \dfrac{V}{2} - \dfrac{V}{\pi}\left[\sin \omega t + \dfrac{1}{2}\sin 2\omega t + \dfrac{1}{3}\sin 3\omega t + ...\right] \qquad$ *Ans.*

Line Spectra





8.6 Find the trigonometric Fourier series for the waveform shown in figure and sketch the spectra.



Solution Here, $f(t) = \dfrac{V}{\pi}\omega t$; for $0 < \omega t < \pi$ and

$$= 0; \quad \text{for} \quad \pi < \omega t < 2\pi$$

$$\Rightarrow \qquad a_0 = \frac{V}{4}$$

$$\Rightarrow \qquad a_n = \frac{1}{\pi}\int_0^{\pi} \frac{V}{\pi}\omega t \cos n\omega t d(\omega t)$$

$$= -\frac{2V}{n^2\pi^2}; \text{ for } n \text{ odd.}$$

$$= 0 ; \text{ for } n \text{ even.}$$

$$\Rightarrow \qquad b_n = \frac{1}{\pi}\int_0^{\pi} \frac{V}{\pi}\omega t \sin n\omega t d(\omega t)$$

$$= -\frac{V}{n\pi}; \text{ for } n \text{ even.}$$

$$= \frac{V}{n\pi}; \text{ for } n \text{ odd.}$$

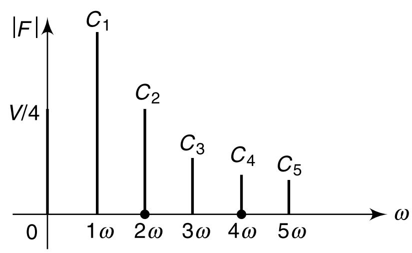
$$\therefore \quad f(t) = \frac{V}{4} - \frac{2V}{\pi^2}\left[\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t + \ldots\right] + \frac{V}{\pi}\left[\sin\omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - \ldots\right]$$

Ans.

Line Spectra

The even harmonic amplitudes are given directly by b_n coefficients, since there are no even cosine terms.

But, the odd harmonic amplitudes are given by computation



$$C_n = \frac{\sqrt{a_n^2 + b_n^2}}{2}$$

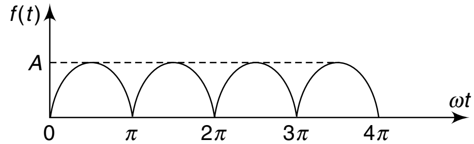
$$\therefore \qquad C_1 = \frac{1}{2}\sqrt{\left(\frac{2V}{\pi^2}\right)^2 + \left(\frac{V}{\pi}\right)^2} = \frac{(0.377)\text{V}}{2} = 0.1885 \text{ V}$$

$$C_3 = (0.054)V, \; C_5 = (0.032)V$$

and $$C_2 = \frac{V}{4\pi}, \quad C_4 = -\frac{V}{8\pi} = -0.0397 \text{ V}$$

$$= -0.08 \text{ V}$$

8.7 Find the Fourier series expansion of the rectified sine waveforms shown in the followig figure.



Solution Here, $f(t) = A \sin \omega t$; for $0 < \omega t < \pi$

$= -A \sin \omega t$; for $\pi < \omega t < 2\pi$

Since, $f(t) = f(-t) \Rightarrow$ The function is even.

$\therefore \quad b_n = 0$

$$\therefore \quad a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t d(\omega t)$$

$$= \frac{4}{2\pi}\int_0^{\pi} A\sin \omega t \cos n\omega t d(\omega t)$$

$$= \frac{A}{\pi}\int_0^{\pi} 2\sin \omega t \cos n\omega t d(\omega t)$$

$$= \frac{A}{\pi}\int_0^{\pi} [\sin(n+1)\omega t - \sin(n-1)\omega t]\, d(\omega t)$$

$$= \frac{A}{\pi}\left[\frac{-\cos(n+1)\omega t}{n+1} + \frac{\cos(n-1)\omega t}{n-1}\right]_0^{\pi}; \text{ for } n \neq 1$$

For odd n; $a_n = \dfrac{A}{\pi}\left[\left(-\dfrac{1}{n+1} + \dfrac{1}{n+1}\right) + \left(\dfrac{1}{n-1} - \dfrac{1}{n-1}\right)\right]; \; n \neq 1$

$= 0$

For even n; $a_n = \dfrac{A}{\pi}\left[\left(\dfrac{2}{n+1}\right) + \left(\dfrac{-2}{n-1}\right)\right]$

$$= \frac{A}{\pi}\left[\frac{2n - 2 - 2n - 2}{(n+1)(n+1)}\right] = -\frac{4A}{\pi(n^2-1)}$$

For $n = 1$, $a_1 = \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\cos \omega t d(\omega t)$

$$= \frac{4}{2\pi}\int_0^{\pi} A\sin \omega t \cos \omega t d(\omega t)$$

$$= \frac{A}{\pi}\int_0^{\pi} \sin 2\omega t d(\omega t)$$

$$= -\frac{A}{2\pi}[\cos 2\omega t]_0^{\pi}$$

$$= -\frac{A}{2\pi}[\cos 2\pi - 1] = 0$$

Also, $$a_0 = \frac{2}{T}\int_0^{T/2} f(t)dt = \frac{2}{2\pi}\int_0^{\pi} A\sin\omega t d(\omega t)$$

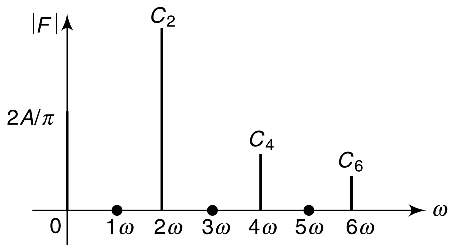
$$= -\frac{A}{\pi}[\cos\omega t]_0^{\pi} = \frac{2A}{\pi}$$

So, the Fourier series is,

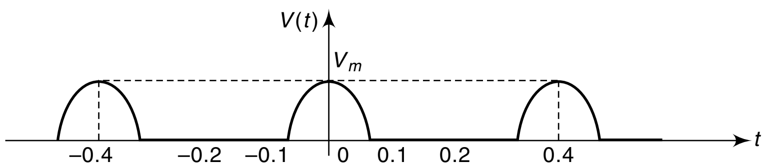
$$f(t) = \frac{2A}{\pi} - \frac{4A}{\pi}\sum_{n=2,4,6}^{\alpha}\frac{\cos n\omega t}{(n^2-1)} = \frac{2A}{\pi} - \frac{4A}{\pi}\left(\frac{1}{3}\cos 2\omega t + \frac{1}{15}\cos 4\omega t + \frac{1}{35}\cos 6\omega t + ...\right)$$

Ans.

Spectra



8.8 Determine the Fourier series of voltage response obtained at the output of a half-wave rectifier shown in the figure. Plot the discrete spectrum of the waveform.



Solution Here, time period $T = 0.4$ second;

$$f = \frac{1}{T} = 2.5 \text{ Hz};$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.4} = 5\pi \text{ rad/s}$$

The function, $v(t) = V_m \cos 5\pi t;\ 0 \le t \le 0.1$

$= 0;\quad 0.1 \le t \le 0.3$

$= V_m \cos 5\pi t;\ 0.3 \le t \le 0.4$

If the period extending from $t = -0.1$ to $t = 0.3$ is taken, it will result in fewer equations and hence, fewer integrals.

$\therefore\quad v(t) = V_m \cos 5\pi t;\ -0.1 \le t \le 0.1$

$= 0;\ 0.1 \le t \le 0.3$

$$\therefore \qquad a_0 = \frac{1}{0.4}\int_{-0.1}^{0.3} v(t)dt = \frac{1}{0.4}\left[\int_{-0.1}^{0.1} V_m \cos 5\pi dt + \int_{0.1}^{0.3}(0)dt\right] = \frac{V_m}{\pi}$$

$$\therefore \qquad a_n = \frac{2}{0.4}\int_{-0.1}^{0.3} V_m \cos 5\pi ntdt \ ; \ n \neq 1$$

$$= 5V_m \int_{-0.1}^{0.1} \cos 5\pi t \cos 5\pi ntdt$$

$$= 5V_m \int_{-0.1}^{0.1} \frac{1}{2}\left[\cos 5\pi(1+n)t + \cos 5\pi(1-n)t\right]dt$$

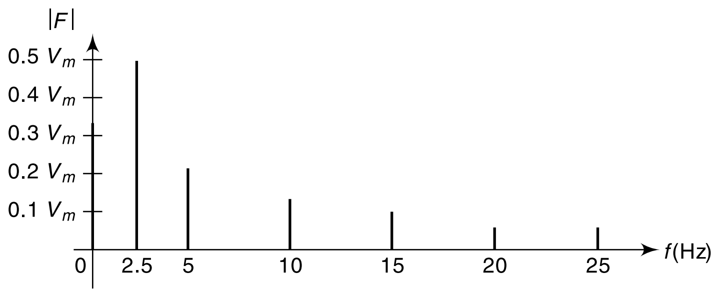
$$= \frac{2V_m}{\pi}\frac{\cos(\pi n/2)}{1-n^2} \ ; \ n \neq 1$$

For, a =1, $a_1 = 5V_m \int_{-0.1}^{0.1} \cos^2 5\pi tdt = \frac{V_m}{2}$

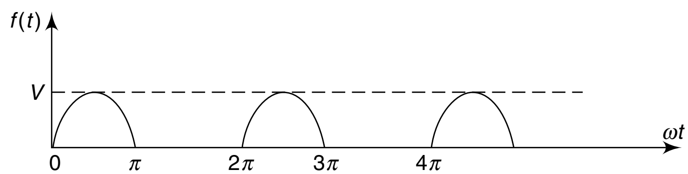
Similarly, b_n = 0 for any value of n, and the Fourier series thus contains no sine terms.

$$\therefore \qquad v(t) = \frac{V_m}{\pi} + \frac{V_m}{2}\cos 5\pi t + \frac{2V_m}{3\pi}\cos 10\pi t - \frac{2V_m}{15\pi}\cos 20\pi t + \frac{2V_m}{35\pi}\cos 30\pi t - \ldots$$

Spectra



8.9 Find the trigonometric Fourier series for the half-wave rectified sine-wave shown in the following figure. and sketch the spectrum.



Solution Here, the wave is, $f(t) = V \sin \omega t$; $0 < \omega t < \pi$

$= 0$; $\pi < \omega t < 2\pi$

$$\therefore \qquad a_0 = \frac{1}{2\pi}\int_0^{\pi} V \sin \omega t \ d(\omega t) = \frac{V}{\pi}$$

$$\therefore \quad a_n = \frac{1}{\pi}\int_0^{\pi} V \sin\omega t \cos n\omega t d(\omega t)\ ;\ n \neq 1$$

$$= \frac{V}{2\pi}\int_0^{\pi}\left[\sin(1+n)\ \omega t + \sin(1-n)\omega t\right] d\omega t$$

$$= \frac{V}{2\pi}\left[\frac{-\cos(1+n)\omega t}{1+n} - \frac{\cos(1-n)\omega t}{1-n}\right]_0^{\pi}$$

$$= \frac{V}{\pi(1-n^2)}(1+\cos n\pi)\ ;\ n \neq 1$$

$= 0$; for n odd

$$= \frac{2V}{\pi(1-n^2)}\ ;\ \text{for } n \text{ even.}$$

For $n = 1$, $a_1 = \frac{1}{\pi}\int_0^{\pi} V \sin\omega t \cos\omega t d(\omega t) = 0$

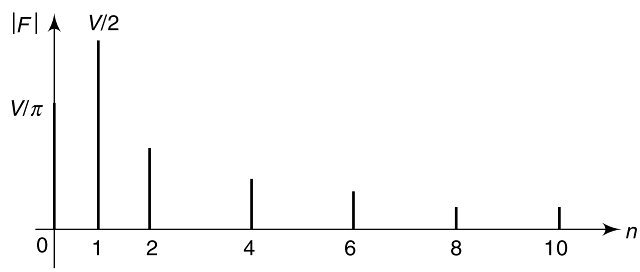
Similarly, $b_n = \frac{1}{\pi}\int_0^{\pi} V \sin\omega t \sin n\omega t d(\omega t)\ ;\ n \neq 1$

$= 0$

For $n = 1$, $b_1 = \frac{1}{\pi}\int_0^{\pi} V \sin^2 \omega t d(\omega t) = \frac{V}{2}$

So the series is,

$$f(t) = \frac{V}{\pi} + \frac{V}{2}\sin\omega t - \frac{V}{\pi}\left(\frac{2}{3}\cos 2\omega t + \frac{2}{15}\cos 4\omega t + \frac{2}{35}\cos 6\omega t + \ldots\right)$$



8.10 State and prove Parseval's theorem useful in computing the effective value of a given periodic function, $f(t)$.

Or,

A periodic function $f(\theta)$ with period 2π is expressed in Fourier series as follows:

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

Prove that,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta)]^2\,d\theta=\left(\frac{a_0}{2}\right)^2+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

Solution

$$f(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)$$

Since, $\int_{-\pi}^{\pi}\cos n\theta\sin n\theta d\theta=\int_{-\pi}^{\pi}\cos n\theta d\theta=\int_{-\pi}^{\pi}\sin n\theta d\theta=0$

$$\therefore\quad \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta)]^2\,d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\left(\frac{a_0}{2}\right)^2+\sum_{n=1}^{\infty}a_n{}^2\cos^2 n\theta+\sum_{n=1}^{\infty}b_n{}^2\sin^2\theta\right]d\theta$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{a_0}{2}\right)^2 d\theta+\frac{1}{2\pi}\sum_{n=1}^{\alpha}\frac{a_n{}^2}{2}\int_{-\pi}^{\pi}2\cos^2 n\theta d\theta+\frac{1}{2\pi}\sum_{n=1}^{\alpha}\frac{b_n{}^2}{2}\int_{-\pi}^{\pi}2\sin^2 n\theta d\theta$$

$$=\frac{1}{\pi}\left(\frac{a_0}{2}\right)^2.2\pi+\frac{a_n{}^2}{2\pi}.\frac{1}{2}\sum_{n=1}^{\alpha}\int_{-\pi}^{\pi}(1+\cos 2n\theta d\theta)+\frac{b_n{}^2}{2\pi}.\frac{1}{2}.\sum_{n=1}^{\alpha}\int_{-\pi}^{\pi}(1-\cos 2n\theta)\,d\theta$$

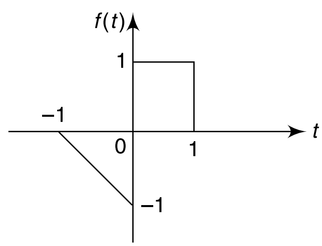
$$=\left(\frac{a_0}{2}\right)^2+\frac{1}{4\pi}\sum_{n=1}^{\alpha}\left[a_n{}^2\left\{\theta+\frac{\sin 2n\theta}{2n}\right\}_{-\pi}^{\pi}+b_n{}^2\left\{\theta-\frac{\sin 2n\theta}{2n}\right\}_{-\pi}^{\pi}\right]$$

$$=\left(\frac{a_0}{2}\right)^2+\frac{1}{4\pi}\sum_{n=1}^{\alpha}[a_n^2(2\pi)+b_n^2(2\pi)]$$

$$\therefore\quad \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta)]^2\,d\theta=\left(\frac{a_0}{2}\right)^2+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)\quad\text{[Proved]}$$

Note: *For statement and proof of this theorem consult the text earlier.*

8.11 Resolve the waveform of the adjacent figure into even and odd components and plot the two components.



Solution Let, $f_0(t)$ and $f_e(t)$ be respectively the odd and even parts of $f(t)$

$$\therefore\quad f(t)=f_e(t)+f_0(t)\qquad\text{(i)}$$

$$\therefore\quad f(-t)=f_e(-t)+f_0(-t)=f_e(t)-f_0(t)\qquad\text{(ii)}$$

Solving (i) and (ii); $f_e(t)=\frac{1}{2}[f(t)+f(-t)]$

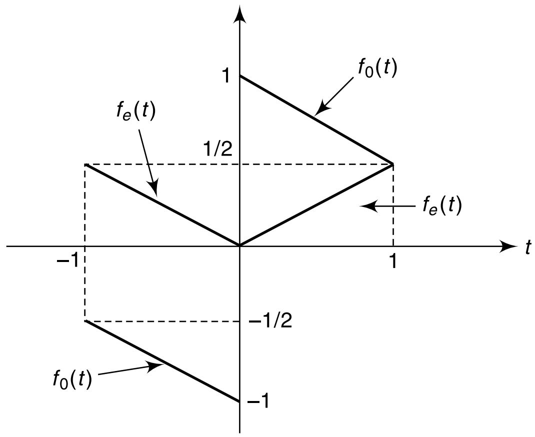
and $$f_0(t)=\frac{1}{2}[f(t)-f(-t)]$$

For the given waveform,

$$f(t) = 1;\ 0 < t < 1 \qquad \therefore\ f_e(t) = \frac{t}{2}$$

and $\quad f(-t) = (t-1);\ 0 < t < 1 \qquad$ and $\quad f_0(t) = (1 - t/2)$

Thus, the components are



8.12 If $v(t) = 10 + 6\cos(t + 45°) + 1.8\cos(2t - 10°)$ volt and $i(t) = 3 + 1.4\cos(t + 20°) + 0.5\cos 2t$ mA, calculate the average power in Watt. Determine also the effective voltage and effective current.

Solution Average Power $= \dfrac{V_{M1}I_{M1}}{2}\cos\phi_1 + \dfrac{V_{M2}I_{M2}}{2}\cos\phi_2 + \dfrac{V_{M3}I_{M3}}{2}\cos\phi_3$

$$= 10 \times 3 + \frac{6 \times 1.4}{2}\cos(45° - 20°) + \frac{1.8 \times 0.5}{2}\cos 10°$$

$$= 34.25 \text{ W}$$

$$\text{Effective Voltage} = \sqrt{10^2 + \frac{6^2 + (1.8)^2}{2}} = 12.58 \text{ V}$$

$$\text{Effective current} = \sqrt{3^2 + \frac{1}{2}(1.4^2 + 0.5^2)} = 3.178 \text{ A}$$

8.13 Determine the effective voltage, effective current, and average power supplied to a passive network if the supplied voltage is,

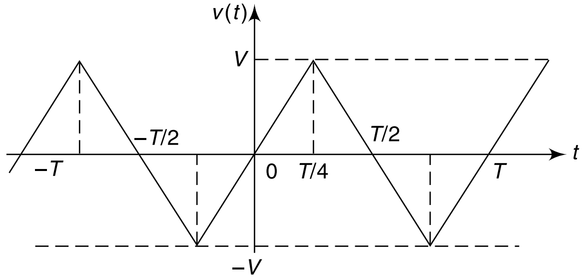
$$v(t) = 100 + 50\cos(10t + 30°) + 25\cos(30t + 60°) \text{ V}$$

and the resulting current is,

$$i(t) = 2\cos(10t + 75°) + 3\cos(30t + 78°) \text{ A}.$$

Solution Same as Prob. 8.12.

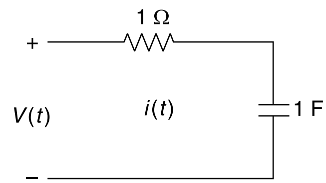
8.14 (a) Find the trigonometric Fourier series of the triangular waveform shown in the following figure.



(b) If this voltage is approximated by

$\frac{8V}{\pi^2}\sin\omega t$, find the mean-square error.

(c) If this voltage waveform is applied to the network in the below figure, then find the current i(t) and draw the magnitude and phase spectra of $i(t)$. Take $\omega_0 = 1$ radian/second for the waveform.



Solution

(a) The wave is an odd function and is having half wave symmetry.

$\therefore \qquad a_n = 0 \quad \text{and} \quad a_0 = 0$

Now, $\qquad V(t) = \frac{4V}{T}t\,;\ 0 < t < T/4$

$$= -\frac{4V}{T}t + 2V\ ;\ T/4 < t < \frac{3T}{4}$$

$\therefore \qquad b_n = \frac{8}{T}\int_0^{T/4} f(t)\sin n\omega t\,dt$; n is odd only.

$$= \frac{8}{T}\int_0^{T/4}\frac{4V}{T}t\sin n\omega t\,dt$$

$$= \frac{32V}{T^2}\left[\frac{-t\cos n\omega t}{n\omega} + \int\frac{\cos n\omega t}{n\omega}\,dt\right]_0^{T/4}$$

$$= \frac{16V}{n\pi T}\left[-\frac{T}{4}\cos\frac{n\pi}{2} + \left.\frac{\sin n\omega t}{n\omega}\right|_0^{T/4}\right]$$

$$= \frac{16}{n\pi T}\left[-\frac{T}{4}\times 0 + \frac{T}{2n\pi}\sin\frac{n\pi}{2}\right]$$

$$= \frac{8V}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \{\because\ \omega T = 2\pi\}$$

$\therefore$ $$b_n = \frac{8V}{n^2\pi^2},\ n = 1, 5, 9, \ldots$$

$$= -\frac{8V}{n^2\pi^2},\ n = 3, 7, 11, \ldots$$

Hence,

$$V(t) = \frac{8V}{\pi^2}\left(\sin\omega t - \frac{1}{3^2}\sin 3\omega t + \frac{1}{5^2}\sin 5\omega t - \frac{1}{7^2}\sin 7\omega t + \ldots\right) \qquad \textit{Ans.}$$

(b) The error is, $\varepsilon(t) = v(t) - \dfrac{8V}{\pi^2}\sin\omega t$

The main square error is,

$$E_N = \frac{1}{T}\int_0^T \varepsilon^2(t)dt$$

Since, the wave is having half–wave symmetry,

$\therefore$ $$E_N = \frac{4}{T}\int_0^{T/4} \varepsilon^2(t)dt$$

Now, $$v(t) = \frac{4V}{T}t\ ; \quad \text{for } 0 < t < T/4$$

$\therefore$ $$E_N = \frac{4}{T}\int_0^{T/4}\left[\frac{4V}{T}t - \frac{8V}{\pi^2}\sin\omega t\right]^2 dt = 0.0047\ \text{V}^2 \qquad \textit{Ans.}$$

(c) Here,

$$i(n\theta) = \frac{V_i(n\theta)}{Z(n\theta)} = \frac{V_i(n\theta)}{1 - j/n}$$

$$= \frac{nV_i(n\theta)}{\sqrt{1+n^2}}\angle\tan^{-1}(1/n)$$

$\therefore$ $$i(n\theta) = \frac{n}{\sqrt{1+n^2}} \times \frac{8V}{n^2\pi^2}\sin[nt + \tan^{-1}(1/n)]\ ; \quad \text{for } n = 1, 5, 9, \ldots$$

$$= \frac{8V}{\pi^2 n\sqrt{1+n^2}}\sin[nt + \tan^{-1}(1/n)]$$

and $$= \frac{8V}{\pi^2 n\sqrt{1+n^2}}\sin[nt + \pi + \tan^{-1}(1/n)]\ ; \text{for } n = 3, 7, 11, \ldots$$

$\therefore$ $$i_1 = \frac{8V}{\pi^2\sqrt{2}}\sin(t + 45°) = 0.707\frac{8V}{\pi^2}\sin(t + 45°)$$

$$\therefore \quad i_3 = \frac{8V}{\pi^2 3\sqrt{10}} \sin(3t + 180° + 18\cdot 44°) = 0.949 \frac{8V}{\pi^2 3^2} \sin(3t + 198\cdot 44°)$$

$$\therefore \quad i_5 = \frac{8V}{\pi^2 5\sqrt{26}} \sin(5t + 11\cdot 31°) = 0.98 \frac{8V}{\pi^2 5^2} \sin(5t + 11\cdot 31°)$$

$$\therefore \quad i(t) = \frac{8V}{\pi^2}[0.707\sin(t + 45°) + 0.105\sin(3t + 198.44°) + 0.039\sin(5t + 11.31°) + ...\} \qquad \textit{Ans.}$$

8.15 A series RL circuit with R = 10 ohm and L = 5 H contains a current

$$i(t) = 10 \sin 1000\, t + 5 \sin 3000t + 3 \sin 5000t \text{ A}$$

Find the effective voltage and the average power.

Solution Here, ω = 1000 rad/s and it contains three harmonics:

For fundamental harmonic

$$R_1 = 10\ \Omega, X_{L1} = \omega L = 1000 \times 5 = 5000\ \Omega$$

$$\therefore \quad Z_1 = R_1 + j\omega L = (10 + j5000) = 5000\angle 89.88°$$

For third harmonic

$$R_3 = 10\ \Omega,\ X_{L3} = 3\omega L = 15000\ \Omega$$

$$\therefore \quad Z_3 = (10 + j15000) = 15000\cdot 003\angle 89.96°$$

For fifth harmonic

$$R_5 = 10\ \Omega, X_{L5} = 5\omega L = 25000\ \Omega$$

$$\therefore \quad Z_5 = (10 + j25000) = 25000\cdot 001\angle 89.977°$$

$$\therefore \quad v(t) = 10\,|Z_1|\sin(1000t - 89.88°) + 5\,|Z_3|\sin(3000t - 89.96°) + 3\,|Z_5|\sin(5000t - 89.977°)$$

$$= 5000\cdot 01\sin(1000t - 89\cdot 88°) + 75000\cdot 015\sin(3000t - 89\cdot 96°)$$

$$+ 75000\cdot 003\sin(5000t - 89\cdot 977°)$$

$$\therefore \quad \text{Effective Voltage, } V = \frac{1}{\sqrt{2}}[(5000\cdot 01)^2 + (75000\cdot 015)^2 + (75000\cdot 003)^2]^{\frac{1}{2}}$$

$$= 8\cdot 291 \times 10^4 \text{ volt}$$

$$= 82\cdot 91\,\text{kV} \qquad \textit{Ans.}$$

Average power

$$P_{av} = \frac{V_{m1}I_{m1}}{2}\cos\phi_1 + \frac{V_{mL}I_{m2}}{2}\cos\phi_2 + \frac{V_{m3}I_{m3}}{2}\cos\phi_3$$

$$= \frac{5000\cdot 01 \times 10}{2}\cos 89\cdot 88° + \frac{75000\cdot 015 \times 5}{2}\cos 89\cdot 96° + \frac{75000\cdot 003 \times 3}{2}\cos 89\cdot 977°$$

$$= 691\cdot 6595 \text{ Watt} \qquad \textit{Ans.}$$

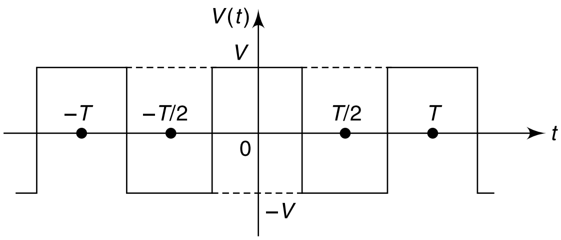
8.16 A periodic current source, $i(t) = 10 + 6\cos(100t + 45°) + 3\cos(200t - 10°) + 2.1\cos(300t + 35°)$ is the input to a parallel *RC* circuit with $R = 0.5$ ohm and $C = 0.02$ F. Calculate the steady-state response $v(t)$ of the circuit.

Solution {same as Prob. 8.15}

$$Z_1 = 0\cdot35\angle-45°;\ Z_2 = 0\cdot22\angle-63.43°;\ Z_3 = 0\cdot158\angle-71.56° \qquad \textit{Ans.}$$

$$\therefore \qquad v(t) = 5 + 2\cdot121\cos 100t + 0\cdot671\cos(200t - 73\cdot43°) + 0\cdot332\cos(300t - 36\cdot56°)$$

8.17 The square wave source, $v(t)$ shown in figure excites a series RL circuit with $R = 2$ ohm and $L = 2$ H. Determine the current response $i(t)$, taking $\omega = 1$ radian/second and $V = \dfrac{\pi}{4}$ volt.



Solution [same as Prob 8.14]

Here, from Prob 8.1

$$v(t) = \frac{4V}{\pi}\left(\cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t - \frac{1}{7}\cos 7\omega t + \ldots\right)$$

Here, $\qquad V = \dfrac{\pi}{4}$ volt

$$\therefore \qquad v(t) = \cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t - \ldots$$

$$Y(jn) = \frac{1}{2 + j2n}$$

$$\Rightarrow \qquad Y_1 = 0\cdot353\angle-45°;\ V_1 = 1\angle 0°$$

$$Y_3 = 0\cdot158\angle-71.565°;\ V_3 = \frac{1}{3}\angle-180°$$

$$Y_5 = 0\cdot098\angle-78.69°;\ V_5 = \frac{1}{5}\angle 0°$$

$$\therefore \qquad I_1 = V_1Y_1 = 0\cdot353\angle-45°$$

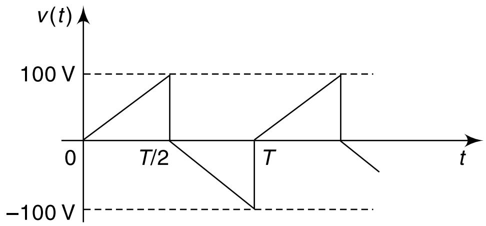
$$\therefore \qquad I_3 = V_3Y_3 = 0\cdot0527\angle 108.435°$$

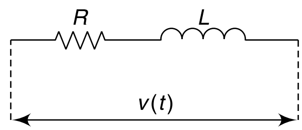
and $\qquad I_5 = V_5Y_5 = 0\cdot0196\angle-78.69°$

$$i(t) = 0\cdot353\cos(t - 45°) + 0\cdot0527\cos(3t - 251\cdot6°) + 0\cdot0196\cos(5t - 78\cdot69°) + \ldots \qquad \textit{Ans.}$$

8.18 Determine the Fourier series of repetitive waveform of figure up to 5^{th} harmonic, when time of repetition, $T = 20$ ms.

Calculate the fundamental frequency current in the circuit of the figure, where $R = 10$ ohm and $L = 0.0318$H with voltage transform of the waveform.





Solution The wave is having half wave symmetry.

$a_n = b_n = 0$; for n even ; and

For n odd,

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\, dt$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\, dt$$

and $a_0 = 0$

Now, $v(t) = \frac{200}{T}t;\ 0 \le t \le \frac{T}{2}$

$$\therefore \quad a_n = \frac{4}{T}\int_0^{T/t} \frac{200}{T} t\cos n\omega t\, dt$$

$$= \frac{800}{T^2}\left[\frac{t\sin nwt}{n\omega} - \int \frac{\sin n\omega t}{n\omega}dt\right]$$

$$= \frac{800}{T^2}\left[\frac{T}{2}\times\frac{\sin n\pi}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\Bigg|_0^{T/2}\right]$$

$$= \frac{800}{n^2\omega^2T^2}[\cos n\pi - 1]$$

$$= \frac{800}{n^2 4\pi^2}(-2)$$

$$= -\frac{400}{n^2\pi^2}$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} \frac{200}{T} t\sin n\omega t dt = \frac{200}{n\pi}$$

$$\therefore \quad v(t) = -\frac{400}{\pi^2}\left(\cos\omega t + \frac{1}{3^2}\cos 3\omega t + \frac{1}{5^2}\cos 5\omega t + ...\right) + \frac{200}{\pi}\left(\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + ...\right)$$

The fundamental frequency voltage is

$$V_f = \left(\frac{200}{\pi}\sin\omega t - \frac{400}{\pi^2}\cos\omega t\right) = \frac{400}{\pi^2}\sqrt{\left(\frac{\pi}{2}\right)^2+1} \qquad \textit{Ans.}$$

Impedance, $\quad Z = (R + j\omega L) = 10 + j\omega(0\cdot 0318)$

Current due to fundamental frequency,

$$I_f = \frac{V_f}{Z} = \frac{400}{\pi^2(10 + j0\cdot 0318\omega)}\left(\frac{\pi}{2}\sin\omega t - \cos\omega t\right)$$

or

$$I_f = \frac{400}{\pi^2}\sqrt{\left(\frac{\pi}{2}\right)^2+1} \times \frac{1}{\sqrt{(10)^2+(0\cdot 0318\omega)^2}} \angle \tan^{-1}\frac{0.0318\omega}{10}$$

Here, $\quad \omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{20\times 10^{-3}} = 100\pi \text{ rad/s}$

Patting this value,

$\therefore \quad I_f = 5\cdot 33\angle -44.9°$

$\therefore \quad I_{f(\text{rms})} = \dfrac{5\cdot 33}{\sqrt{2}}\text{ A} = 3\cdot 76\text{ A}$

8.19 An RLC series circuit with R = 25 ohm, L = 1 H, and C = 10 microfarad is energized with a voltage source,

$$V(t) = 15\sin 100t + 10\sin 200t + 5\sin 300t \text{ (V)}$$

Find the expression for the current $i(t)$. Determine the effective value of the current, and the average power consumed by the circuit.

Solution [Same as Prob. 8.16]

$$Z_1 = R + j\left(\omega L - \frac{1}{\omega C}\right) = 900\cdot 3\angle +88\cdot 4°$$

$$Z_2 = R + j\left(2\omega L - \frac{1}{2\omega C}\right) = 301.04\angle 85.2°$$

$$Z_3 = R + j\left(3\omega L - \frac{1}{3\omega C}\right) = 41\cdot 62\angle 53\cdot 1°$$

$$\therefore \quad i(t) = \frac{15}{Z_1}\sin 100t + \frac{10}{Z_2}\sin 200t + \frac{5}{Z_3}\sin 300t$$

$$= 0\cdot 0167\sin(100t + 88\cdot 4°) + 0\cdot 0332\sin(200t + 85\cdot 2°) + 0\cdot 12\sin(300t + 53\cdot 1°) + \ldots$$

Ans.

$$\therefore \quad I_{\text{rms}} = \frac{1}{\sqrt{2}}[I_1^2 + I_2^2 + I_3^2]^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}}[(0\cdot 0167)^2 + (0\cdot 0332)^2 + (0\cdot 12)^2]^{\frac{1}{2}}$$

$$= 0\cdot 088\text{ A} = 88\text{ mA} \qquad \textit{Ans.}$$

$$\therefore \quad P_{av} = \frac{15 \times 0 \cdot 0167}{2} \cos 88 \cdot 4° + \frac{10 \times 0 \cdot 0332}{2} \cos 85 \cdot 2° + \frac{5 \times 0 \cdot 12}{2} \cos 53 \cdot 1° = 0.197 \text{ W}$$

Ans.

8.20 Determine the expression for current in an impedance of $R = 10$ ohm, $L = 0.0318$ H with applied emf,

$$e(t) = 200 \sin 314t + 40 \sin (942t + 30°) + 10 \text{ V}$$

Also, calculate the rms value of voltage and current as well as the power factor of the circuit.

Solution [Same as Prob. 8.19]

$$i_1 = \frac{200 \sin 314t}{10 + j314 \times 0 \cdot 0318} = 14 \cdot 14 \sin 314t \angle -44.95°$$

$$i_2 = \frac{40 \sin (942t + 30°)}{10 + j942 \times 0 \cdot 0318} = 1 \cdot 28 \sin (942t + 30°) \angle -71 \cdot 54°$$

$$i_0 = \frac{10}{10} = 1$$

$$i(t) = 14 \cdot 14 \sin (314t - 44 \cdot 95°) + 1 \cdot 28 \sin (942t - 41.54°)$$

$$\therefore \quad V_{\text{rms}} = \sqrt{V_0^2 + \frac{V_1^2 + V_2^2}{2}}$$

$$= \sqrt{10^2 + \frac{200^2 + 40^2}{2}} = 144 \cdot 568 \text{ V} \qquad \textit{Ans.}$$

$$\therefore \quad I_{\text{rms}} = \sqrt{I_0^2 + \frac{I_1^2 + I_2^2}{2}}$$

$$= \sqrt{1^2 + \frac{14 \cdot 14^2 + 1 \cdot 28^2}{2}} = 10.089 \text{ A} \qquad \textit{Ans.}$$

$$\therefore \quad \text{Power factor} = \frac{\text{Average Power}}{\text{Apparent Power}}$$

$$= \frac{V_0 I_0 + \frac{V_1 I_1}{2} \cos \phi_1 + \frac{V_2 I_2}{2} \cos \phi_2}{V_{\text{rms}} \times I_{\text{rms}}}$$

$$= \frac{10 \times 1 + \frac{200 \times 14 \cdot 14}{2} \cos 44 \cdot 95° + \frac{40 \times 1 \cdot 28}{2} \cos 71 \cdot 54°}{144 \cdot 568 \times 10 \cdot 089}$$

$$= 0.69 \qquad \textit{Ans.}$$

8.21 In a two-element series network, voltage $v(t)$ is applied, which is given by,

$$v(t) = 50 + 50 \sin 5000t + 30 \sin 10000t + 20 \sin 20000t \text{ (V)}$$

The resulting current is given as,

$$i(t) = 11.2 \sin (5000t + 63.4°) + 10.6 \sin (10000t + 45°) + 8.97 \sin (20000t + 26.6°) \text{ (A)}$$

Determine the network elements and the power dissipated in the circuit.

Solution Power dissipated,

$$P_{av} = 50 \times 0 + \frac{50 \times 11 \cdot 2}{2} \cos 63 \cdot 4° + \frac{30 \times 10 \cdot 6}{2} \cos 45° + \frac{8 \cdot 97 \times 20}{2} \cos 26 \cdot 6° = 318 \text{ W} \qquad \textit{Ans.}$$

In the expression of current *i(t)*, the d.c. term is missing though it is present in the applied voltage, $v(t)$. Hence, in the series network, there must be a capacitor which blocks d.c. components. Again from the expression of *i(t),* we see that the current is leading by an angle less than 90°. Hence, the conclusion is the presence of a resistive element in series with the capacitor (*RC*).

Now, $$I_{eff} = \sqrt{\frac{11 \cdot 2^2 + 10 \cdot 6^2 + 8 \cdot 97^2}{2}} = 12 \cdot 6 \text{ A}$$

$\therefore$ $$P_{av} = I_{eff}^2 R \Rightarrow R = \frac{318}{(12 \cdot 6)^2} = 2\ \Omega \qquad \textit{Ans.}$$

Again, *at* $\omega = 10,000$ rad/s, $\phi = 45° = \tan^{-1}\left(\frac{1}{\omega CR}\right)$

$\Rightarrow$ $$C = \frac{1}{\omega R} = \frac{1}{20,000} = 50\ \mu\text{F} \qquad \textit{Ans.}$$

8.22 Calculate the impedance consisting of *R* and *L* and the power factor of a circuit whose expression for voltage and current are,

$$v(t) = 250 \sin 314t + 50 \sin(942t + 30°) \text{ (V)}$$

$$i(t) = 17.7 \sin(314t - 45°) + 1.583 \sin(942t - 41.6°) \text{ (A)}$$

Solution The fundamental frequency current,

$$I_1 = \frac{250 \sin 314t}{R + j\omega L} = 17 \cdot 7 \sin(314t - 45°) \qquad \text{(i)}$$

The third harmonic current,

$$I_3 = \frac{50 \sin(942t + 30°)}{R + j3\omega L} = 1 \cdot 583 \sin(942t - 41 \cdot 6°) \qquad \text{(ii)}$$

Equating the magnitudes of (i),

$$\frac{250}{\sqrt{R^2 + \omega^2 L^2}} = 17.7$$

$\Rightarrow$ $$R^2 + \omega^2 L^2 = 199.495 \qquad \text{(iii)}$$

Equating the angles of (i)

$$314t - \tan^{-1}\frac{\omega L}{R} = 314t - 45°$$

$\Rightarrow$ $$\tan^{-1}\frac{\omega L}{R} = 45° \Rightarrow \frac{\omega L}{R} = 1 \Rightarrow \omega L = R$$

Putting in (iii), $\Rightarrow (\omega L)^2 = 99 \cdot 747 \Rightarrow \omega L = 9 \cdot 987 = R$

$\therefore \quad L = \frac{9 \cdot 987}{314} = 0 \cdot 0318$

$\therefore \quad R = 9 \cdot 987\,\Omega$

$L = 0 \cdot 0318\text{ H}$ *Ans.*

$$\text{Power factor} = \frac{\text{Average Power}}{\text{Apparent Power}} = \frac{\frac{V_1 I_1}{2}\cos\phi_1 + \frac{V_3 I_3}{2}\cos\phi_3}{\sqrt{\frac{V_1^2}{2}+\frac{V_3^2}{2}} \times \sqrt{\frac{I_1^2}{2}+\frac{I_3^2}{2}}}$$

$$= \frac{\frac{250 \times 17 \cdot 7}{2}\cos 45° + \frac{50 \times 1 \cdot 583}{2}\cos 71 \cdot 6°}{\sqrt{\frac{250^2 + 50^0}{2}} \times \sqrt{\frac{17 \cdot 7^2 + 1 \cdot 583^2}{2}}}$$

$= 0.69$ *Ans.*

8.23 A square wave has a value 10 from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, zero from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, 10 from $\frac{3\pi}{2}$ to $\frac{5\pi}{2}$ and so on. Find the Fourier series expansion of the wave.

Solution For the square wave given, time period is 2π. The Fourier coefficients are evaluated as,

$$a_0 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} f(\theta)d\theta = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 10\,d\theta = 5$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(\theta)\cos nd\theta = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 10\cos n\theta d\theta = \frac{10}{n\pi}\left|\sin n\theta\right|_{-\pi/2}^{\pi/2} = \frac{20}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$\therefore \quad a_1 = \frac{20}{\pi}; \quad a_2 = 0; \quad a_3 = -\frac{20}{3\pi}; \quad a_4 = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} f(\theta)\sin nd\theta = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 10\sin n\theta d\theta = -\frac{10}{n\pi}\left|\cos n\theta\right|_{-\pi/2}^{\pi/2} = 0$$

Therefore, the Fourier series is given as,

$$v = 5 + \frac{20}{\pi}\cos\theta - \frac{20}{3\pi}\cos 3\theta + \frac{20}{5\pi}\cos 5\theta - \ldots \qquad \textit{Ans.}$$

8.24 In a linear circuit consisting of $R = 9\,\Omega$ and $L = 8$ mH, a current,

$i = 5 + 100\sin\left(1000t + 45^0\right) + 100\sin\left(3000t + 60^0\right)$ A is flowing. Find the equation of applied voltage.

Solution Here, $R = 9\,\Omega$ and $L = 8$ mH, $i = 5 + 100\sin\left(1000t + 45^0\right) + 100\sin\left(3000t + 60^0\right)$A

For DC component

Current, $I_0 = 5\text{A}, Z_0 = R = 9\,\Omega$

$\therefore \quad V_0 = I_0 \times R = 5 \times 9 = 45\text{V}$

For first harmonic component

Current, $I_1 = 100\angle 45^0$ A

Impedance, $Z_1 = R + j\omega L = 9 + j2\pi \times 1000 \times 8 \times 10^{-3} = (9 + j8) = 12.04\angle 41.63^0\ (\Omega)$

$\therefore$ $V_1 = I_1 Z_1 = 100\angle 45^0 \times 12.04\angle 41.63^0 = 1204\angle 86.63^0\ (\text{V})$

For third harmonic component

Current, $I_3 = 100\angle 60^0$ A

Impedance, $Z_{31} = R + j3\omega L = 9 + j2\pi \times 3 \times 1000 \times 8 \times 10^{-3} = (9 + j24) = 25.63\angle 69.44^0\ (\Omega)$

$\therefore$ $V_3 = I_3 Z_3 = 100\angle 60^0 \times 25.63\angle 69.44^0 = 2563\angle 129.44^0\ (\text{V})$

$\therefore$ applied voltage is given as,

$$v = 45 + 1204\sin\left(1000t + 86.63^0\right) + 2563\sin\left(3000t + 129.44^0\right) \quad (\text{V}) \qquad \textit{Ans.}$$

8.25 Calculate the impedance consisting of R and L and the power factor of a circuit whose expression for voltage and current are,

$$v(t) = 250\sin 314t + 50\sin(942t + 30^0)(\text{V})$$
$$i(t) = 17.7\sin(314t - 45^0) + 1.583\sin(942t - 41.6^0)\,(\text{A})$$

Solution The fundamental frequency current,

$$I_1 = \frac{250\sin 314t}{R + j\omega L} = 17\cdot 7\sin(314t - 45^0) \qquad \ldots\text{(i)}$$

The third harmonic current,

$$I_3 = \frac{50\sin(942t + 30^0)}{R + j3\omega L} = 1\cdot 583\sin(942t - 41\cdot 6^0) \qquad \ldots\text{(ii)}$$

Equating the magnitudes of (i),

$$\frac{250}{\sqrt{R^2 + \omega^2 L^2}} = 17.7$$

$$\Rightarrow \qquad R^2 + \omega^2 L^2 = 199.495 \qquad \ldots\text{(iii)}$$

Equating the angles of (i)

$$314t - \tan^{-1}\frac{\omega L}{R} = 314t - 45^0$$

$$\Rightarrow \qquad \tan^{-1}\frac{\omega L}{R} = 45^0 \Rightarrow \frac{\omega L}{R} = 1 \Rightarrow \omega L = R$$

Putting in (iii), $\Rightarrow (\omega L)^2 = 99\cdot 747 \Rightarrow \omega L = 9\cdot 987 = R$

$$\therefore \qquad L = \frac{9\cdot 987}{314} = 0\cdot 0318$$

$$\therefore \qquad R = 9\cdot 987\ \Omega$$

$$L = 0\cdot 0318\ \text{H} \quad \textit{Ans.}$$

$$\text{Power factor} = \frac{\text{Average Power}}{\text{Apparent Power}} = \frac{\dfrac{V_1 I_1}{2}\cos\phi_1 + \dfrac{V_3 I_3}{2}\cos\phi_3}{\sqrt{\dfrac{V_1^{\,2}}{2}+\dfrac{V_3^{\,2}}{2}} \times \sqrt{\dfrac{I_1^{\,2}}{2}+\dfrac{I_3^{\,2}}{2}}}$$

$$= \frac{\dfrac{250\times 17\cdot 7}{2}\cos 45^0 + \dfrac{50\times 1\cdot 583}{2}\cos 71\cdot 6^0}{\sqrt{\dfrac{250^2+50^0}{2}} \times \sqrt{\dfrac{17\cdot 7^2+1\cdot 583^2}{2}}}$$

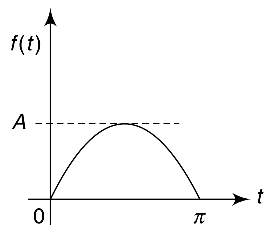
$$= 0.69 \qquad \textit{Ans.}$$

Fourier Transform

8.26 Determine the Fourier transform of one cycle of sine wave, $f(t) = A\sin\omega_0 t$.

Solution $$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$= A\int_0^T \sin\omega_0 t e^{-j\omega t}\,dt = I \quad \text{(say)}$$



$$= A\left[e^{-j\omega t}\left(-\frac{\cos\omega_0 t}{\omega_0}\right)\Bigg|_0^T - \int_0^T \frac{\cos\omega_0 t}{\omega_0}(j\omega e^{-j\omega t})dt\right]$$

$$= A\left[-\frac{1}{\omega_0}(e^{-j\omega t}\cos\omega_0 T - 1) - \frac{j\omega}{\omega_0}\left\{\int_0^T \cos\omega_0 t e^{-j\omega t}\,dt\right\}\right]$$

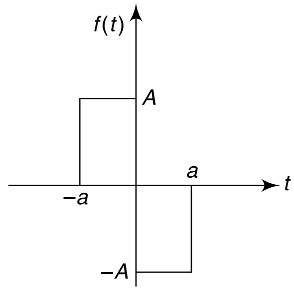
$$= A\left[\frac{1}{\omega_0}(e^{-j\omega t}+1) - \frac{j\omega}{\omega_0}\left[\left\{e^{-j\omega t}\left(\frac{\sin\omega_0 t}{\omega_0}\right)\right\}\Bigg|_0^T - \int_0^T\left(\frac{\sin\omega_0 t}{\omega_0}\right)(-j\omega)\,e^{-j\omega t}\,dt\right]\right]$$

or, $$= \frac{A}{\omega_0}(e^{-j\omega t}+1) + j\frac{A\omega}{\omega_0}\left[0 + \frac{j\omega}{\omega_0}\int_0^T \sin\omega_0 t\, e^{-j\omega t}\,dt\right] \quad [\because \cos\omega_0 T = \cos\pi = -1]$$

$$= \frac{A}{\omega_0}(e^{-j\omega t}+1) + I\frac{\omega^2}{\omega_0^2}$$

or, $$I\left[1-\frac{\omega^2}{\omega_0^2}\right] = A\omega_0(e^{-j\omega T}+1)$$

$\Rightarrow$ $$I = \frac{A\omega_0}{\omega_0^2-\omega^2}(e^{-j\omega T}+1) \qquad \textit{Ans.}$$



8.27 Find the Fourier transform of the single pulse shown in the figure. Draw the continuous magnitude and phase spectra.

Solution Here, $f(t) = A$; $-a \le t \le 0$;
$= -A$; $0 \le t \le a$
$= 0$; for all other values of t

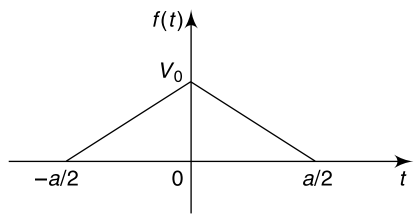
$$\therefore \qquad F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$= \int_{-a}^{0} Ae^{-j\omega t}\,dt + \int_{0}^{a} -Ae^{-j\omega t}\,dt$$

$$= A\left[\frac{je^{-j\omega t}}{\omega}\bigg|_{-a}^{0} - \frac{je^{-j\omega t}}{\omega}\bigg|_{0}^{a}\right]$$

$$= \frac{jA}{\omega}[1 - e^{+j\omega a} - e^{-j\omega a} + 1]$$

$$\Rightarrow \qquad F(j\omega) = j\frac{2A}{\omega}(1 - \cos\omega a) \qquad \textit{Ans.}$$



8.28 Find the Fourier transform of the single triangular pulse shown in the adjacent figure and draw the continuous spectra.

Solution The wave is, $f(t) = V_0\left[1 - \frac{2}{a}|t|\right]$

i.e., $$f(t) = V_0\left[1 - \frac{2}{a}t\right]; \quad \text{for} \quad t > 0$$

and $$f(t) = V_0\left[1 + \frac{2}{a}t\right]; \quad \text{for} \quad t < 0$$

$$\therefore \qquad F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} V_0\left[1 - \frac{2}{a}|t|\right]e^{-j\omega t}\,dt$$

$$= V_0\int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-j\omega t}\,dt - \frac{2V_0}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} |t|\,e^{-j\omega t}\,dt$$

$$= \frac{V_0}{-j\omega}\left[e^{-j\omega t}\Big|_{-\frac{a}{2}}^{\frac{a}{2}} - \frac{2V_0}{a}\left\{\int_{-\frac{a}{2}}^{0} -te^{-j\omega t}\,dt + \int_{0}^{\frac{a}{2}} te^{j\omega t}\,dt\right\}\right]$$

$$= \frac{V_0}{-j\omega}(e^{-j\omega\frac{a}{2}} - e^{+j\omega\frac{a}{2}}) + \frac{2V_0}{a}\int_{-\frac{a}{2}}^{0} te^{-j\omega t}dt - \frac{2V_0}{a}\int_{0}^{\frac{a}{2}} te^{-j\omega t}dt$$

$$= \frac{2V_0}{\omega}\left(\frac{e^{j\omega\frac{a}{2}} - e^{-j\omega\frac{a}{2}}}{2j}\right) + \frac{2V_0}{a}\left[\frac{te^{-j\omega t}}{-j\omega}\bigg|_{-\frac{a}{2}}^{\frac{a}{2}} - \int_{-\frac{a}{2}}^{0}\frac{e^{-j\omega t}}{-j\omega}dt\right] - \frac{2V_0}{a}\left[\frac{te^{-j\omega t}}{-j\omega}\bigg|_{0}^{\frac{a}{2}} - \int_{0}^{\frac{a}{2}}\frac{e^{-j\omega t}}{-j\omega}dt\right]$$

$$= \frac{2V_0}{\omega}\sin\left(\frac{\omega a}{2}\right) + \frac{2V_0}{2}\left[\left\{0 + \frac{a}{2}\frac{e^{+j\omega a l_2}}{-j\omega}\right\} + \frac{e^{-j\omega t}}{\omega^2}\bigg|_{-\frac{a}{2}}^{0}\right] - \frac{2V_0}{a}\left[\left\{\frac{a}{2}\frac{e^{-j\omega a l_2}}{-j\omega} - 0\right\} + \frac{e^{-j\omega t}}{\omega^2}\bigg|_{0}^{\frac{a}{2}}\right]$$

$$= \frac{2V_0}{\omega}\sin\left(\frac{\omega a}{2}\right) - \frac{V_0}{j\omega}e^{+j\omega\frac{a}{2}} + \frac{2V_0}{a\omega^2}\left(1 - e^{+j\omega\frac{a}{2}}\right) + \frac{V_0}{j\omega}e^{-j\omega a/2} - \frac{2V_0}{a\omega^2}(e^{-j\omega a/2} - 1)$$

$$= \frac{2V_0}{\omega}\sin\left(\frac{\omega a}{2}\right) + \frac{2V_0}{\omega}\left(\frac{e^{-j\omega a/2} - e^{+j\omega a/2}}{2j}\right) + \frac{2V_0}{a\omega^2}(1 - e^{+j\omega a/2} - e^{-j\omega a/2} + 1)$$

$$= \frac{2V_0}{\omega}\sin\left(\frac{\omega a}{2}\right) - \frac{2V_0}{\omega}\sin\left(\frac{\omega a}{2}\right) + \frac{2V_0}{a\omega^2}(2 - e^{-j\omega a/2} - e^{j\omega a/2})$$

$$= \frac{4V_0}{a\omega^2}\left[1 - 2\left(\frac{e^{+j\omega a/2} - e^{-j\omega a/2}}{2}\right)\right]$$

$$= \frac{4V_0}{a\omega^2}\left[1 - \cos\left(\frac{\omega a}{2}\right)\right]$$

$$= \frac{4V_0}{a\omega^2} \times 2\sin^2\left(\frac{\omega a}{4}\right)$$

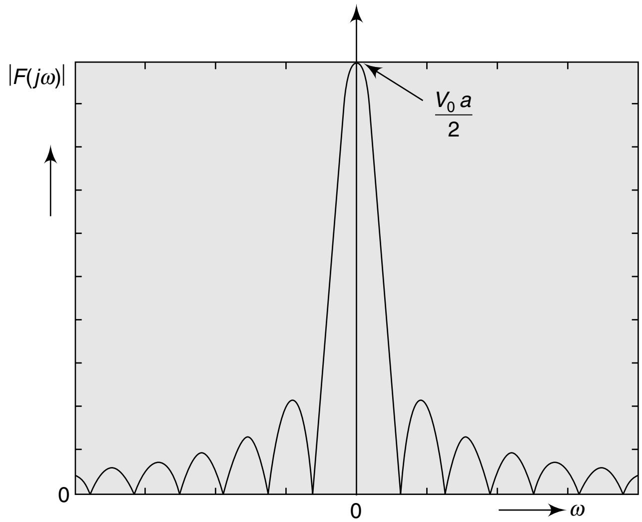
$$\therefore \qquad \boxed{F(j\omega) = \frac{8V_0}{a\omega^2}\sin^2\left(\frac{\omega a}{4}\right)}$$

Bringing it into standard form,

$$\boxed{F(j\omega) = \frac{V_0 a}{2}\frac{\sin^2\left(\frac{\omega a}{4}\right)}{\left(\frac{\omega a}{4}\right)^2}} \qquad \textit{Ans.}$$

Its continuous amplitude spectrum is shown. The first zero occurs when, $\frac{\omega a}{4} = \pi$ i.e. $\omega = \frac{4\pi}{a}$.

Spectra



8.29 Find the Fourier transform of the existing voltage,

$$v(t) = V_0 e^{-t}, \quad t \geq 0$$
$$= 0, \quad t \leq 0$$

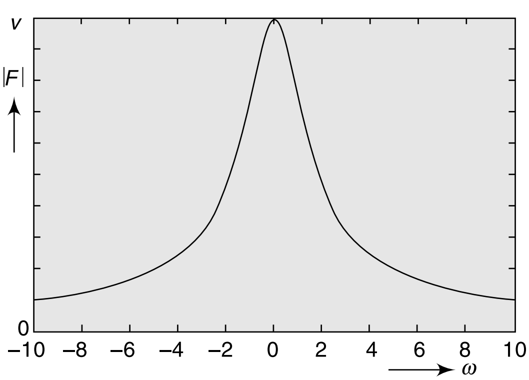
and sketch approximately its amplitude and phase spectrum.

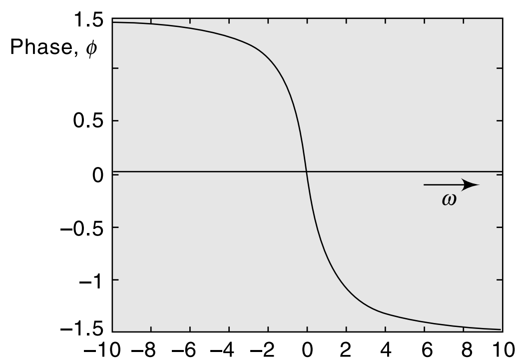
Solution $F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} V_0^{-t} e^{-j\omega t}dt = V_0 \int_{-\infty}^{\infty} e^{-(1+j\omega)t}dt = \frac{V_0}{(1+j\omega)}[e^{-(1+j\omega)t}]_{-\infty}^{\infty}$

$$= \frac{V_0}{1+j\omega}$$

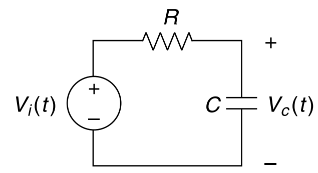
The amplitude and phase are $|F(j\omega)| = \frac{V_0}{\sqrt{1+\omega^2}}$ and $\phi(j\omega) = -\tan^{-1}(\omega)$

Spectra





8.30 In the figure, $V_i(t) = 10\ \text{sgn}(t)$ volt. Using the Fourier transform method, find $V_c(t)$ and sketch $V_c(t)$ versus time, t. Given: $R = 5$ ohm, $C = 1$F.



Solution $v_i(t) = 10\ \text{sgn}(t)$

$$V_i(j\omega) = 10 \times \frac{2}{j\omega} = \frac{20}{j\omega} \qquad V_e(j\omega) = \frac{V_i(j\omega)}{z(j\omega)} \times X_c$$

Transfer function of the circuit

$$H(j\omega) = \frac{V_c(j\omega)}{V_i(j\omega)} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC}$$

where, $V_c(j\omega)$ is the Fourier transform of $V_c(t)$

$$\therefore \qquad V_c(j\omega) = H(j\omega) \times V_i(j\omega) = \frac{V_i(j\omega)}{Z(j\omega)} \times X_C = \frac{20}{j\omega(1 + j\omega RC)} = \frac{20}{j\omega(1 + j\omega 5)}$$

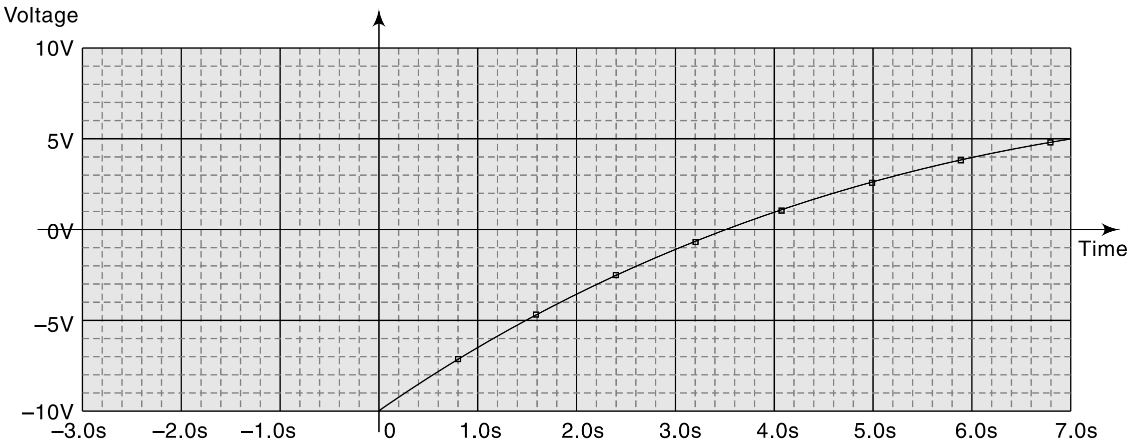
$$= \frac{20}{j\omega} - \frac{100}{1 + (j\omega 5)} = 10\left(\frac{2}{j\omega}\right) - 20\frac{1}{(1/5) + j\omega}$$

Taking inverse Laplace transform,

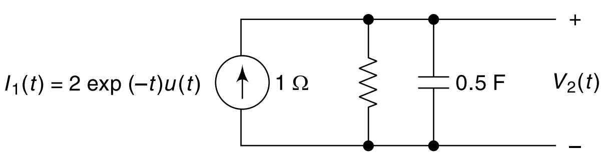
$$v_C(t) = 10\,\text{sgn}(t) - 20e^{-t/5}u(t)\ \text{V}$$

To plot this curve, we follow the following steps:

- From $-\infty < t < 0$, $v_i(t) = -10$V, $v_C(t) = -10$V;
- At $t = 0$, $v_i(t)$ jumps from -10V to 10V and thus, $v_C(t)$ approaches its final value of 10V exponentially with time-constant of 5 second.



8.31 Find the response voltage in the network shown in the below figure. Use Fourier transform method.



Solution By KCL, $i_1(t) = \dfrac{v_2(t)}{1} + \dfrac{v_2(t)}{1/j\omega\frac{1}{2}} = v_2(t) + \frac{1}{2} j\omega v_2(t)$

Given: $i_1(t) = 2e^{-t}u(t)$

Taking Fourier transform,

$$I_1(j\omega) = V_2(j\omega) + \frac{1}{2} j\omega V_2(j\omega)$$

or
$$\frac{2}{1+j\omega} = V_2(j\omega)\left[1 + \frac{1}{2} j\omega\right]$$

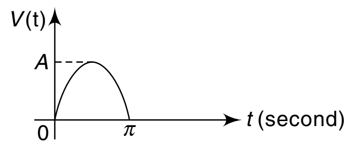
$$V_2(j\omega) = \frac{4}{(1+j\omega)(2+j\omega)} = 4\left[\frac{1}{1+j\omega} - \frac{1}{2+j\omega}\right]$$

Taking inverse Fourier transform,

$$v_2(t) = (4e^{-t} - 4e^{-2t})u(t) \qquad \textit{Ans.}$$

8.32 Find the Fourier transform of the sine pulse shown in the adjacent figure and sketch the amplitude and phase spectra.

This voltage is applied to a series RL circuit with $R = 1$ ohm and $L = 1.0$H. Determine the amplitude and phase spectra for the resulting current, $i(t)$.



Solution [from Prob. 8.23] $\Rightarrow V(j\omega) = A\left[\dfrac{1+e^{-j\omega\pi}}{1-\omega^2}\right] = \dfrac{A(1+\cos\omega\pi)}{1-\omega^2} - j\dfrac{A\sin\omega\pi}{1-\omega^2}$

$$\therefore \quad |V(j\omega)| = A\left|\frac{\sqrt{(1+\cos\omega\pi)^2 + \sin^2\omega\pi}}{1-\omega^2}\right| = A\left|\frac{\sqrt{2(1+\cos\omega\pi)}}{1-\omega^2}\right| = 2A\left|\frac{\cos\left(\frac{\omega\pi}{2}\right)}{1-\omega^2}\right|$$

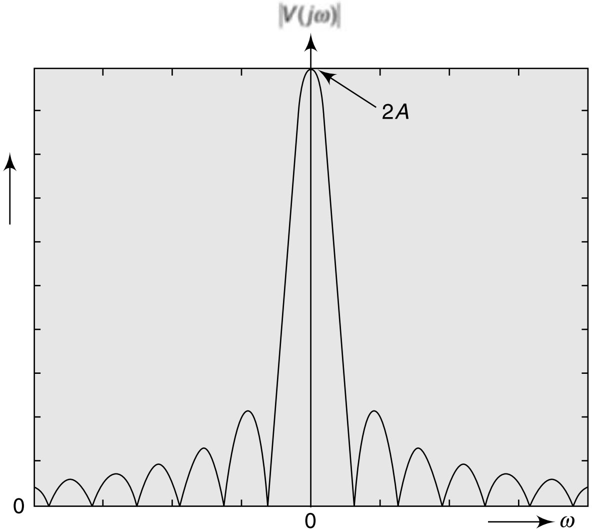
$$\therefore \quad \phi(j\omega)\ [\text{Angle of } V(j\omega)] = \tan^{-1}\left(\frac{-\sin\omega\pi}{1+\cos\omega\pi}\right) = \tan^{-1}\left(\tan\frac{-\omega\pi}{2}\right) = \frac{-\omega\pi}{2}$$

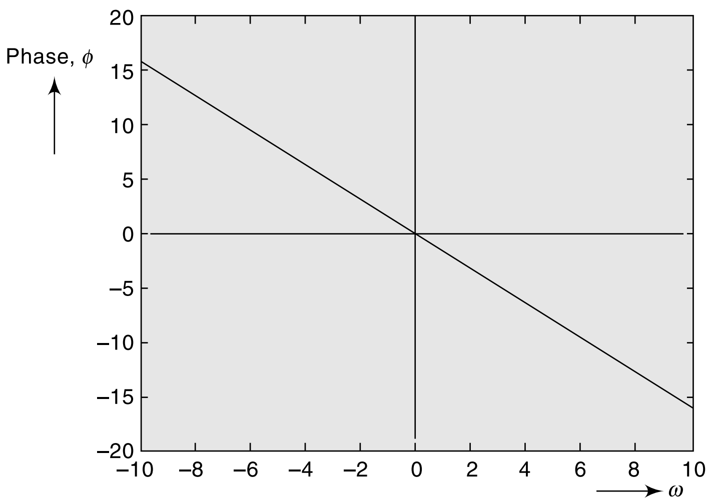
The amplitude and phase spectra are shown.

The current in the RL series circuit,

$$I(j\omega) = \frac{V(j\omega)}{R+j\omega L} = \frac{|V(j\omega)|\langle\phi(j\omega)}{\sqrt{R^2+\omega^2L^2}\angle\tan^{-1}\frac{\omega L}{R}}$$

$$= \frac{|V(j\omega)|\angle\phi(j\omega)}{\sqrt{1+\omega^2}\angle\tan^{-1}\omega}$$





$$= 2A\left|\frac{\cos \omega\pi/2}{1-\omega^2}\right| \frac{1}{\sqrt{1+\omega^2}} \angle -\frac{\omega\pi}{2} - \tan^{-1}\omega$$

$$= |I(j\omega)| \angle \theta(j\omega)$$

$\therefore$ $$|I(j\omega)| = 2A\left|\frac{\cos\left(\frac{\omega\pi}{2}\right)}{1-\omega^2}\right| \frac{1}{\sqrt{1+\omega^2}}$$

and $$\theta(j\omega) = -\frac{\omega\pi}{2} - \tan^{-1}\omega$$

8.33 The current in a 10 ohm resistor is, $i(t) = 10\,e^{-2t}u(t)$ A. Calculate the total energy W dissipated in the resistor during the time interval $t = 0$ to ∞. What is the energy W1 associated with the frequency band $0 \le \omega \le 2$ rad/s?

Solution The instantaneous power, $p(t) = i^2(t).R = 10 \times 100e^{-4t};\ t > 0$

Total energy dissipated

$$W = \int_{-\infty}^{\infty} p(t)dt = \int_{0}^{\infty} 1000e^{-4t}dt = 1000\left[\frac{e^{-4t}}{-4}\right]_0^{\infty} = -\frac{1000}{4}[0-1] = 250 \text{ Joule} \qquad \textit{Ans.}$$

The Fourier transform of $i(t)$ is,

$$I(j\omega) = \frac{10}{2 + j\omega}$$

The Energy associated,

$$W_1 = \frac{10}{\pi}\int_0^2 |I(j\omega)|^2\,d\omega \qquad \left\{\because\ 1\,\Omega \text{ Energy is, } W_{1\Omega} = \frac{1}{\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\,d\omega\right\}$$

$$= \frac{10}{\pi}\int_0^2 \frac{100}{4+\omega^2}d\omega$$

$$= \frac{1000}{\pi}\left[\frac{1}{2}\tan^{-1}\frac{\omega}{2}\right]_0^2$$

$$= \frac{500}{\pi}[\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= \frac{500}{\pi} \times \frac{\pi}{4} = 125 \text{ Joule} \qquad \textit{Ans.}$$

8.34 A voltage, $v(t) = 100e^{-25t}\,u(t)$ volt is applied to the input of an ideal low-pass filter having a cut-off frequency of 25 rad/s. Calculate the percentage of the total energy transmitted through the filter.

Solution Fourier transform of $v(t)$

$$V(j\omega) = \frac{100}{25 + j\omega}$$

$$\therefore \qquad |V(j\omega)|^2 = \frac{10^4}{625 + \omega^2}$$

Total 1 Ω energy available at the filter input is,

$$W_{i1\Omega} = \frac{1}{\pi}\int_0^{\infty} \frac{10^4\,d\omega}{625+\omega^2}$$

$$= \frac{10^4}{\pi}\int_0^{\infty}\frac{d\omega}{625+\omega^2}d\omega$$

$$= \frac{10^4}{\pi}\left[\frac{1}{25}\tan^{-1}\frac{\omega}{25}\right]_0^{\infty}$$

$$= \frac{10^4}{\pi} \times \frac{1}{25} \times \frac{\pi}{2} = 200 \text{ Joule} \qquad \textit{Ans.}$$

The 1 Ω energy available at the filter output is,

$$W_{01\Omega} = \frac{1}{\pi}\int_0^{25} |V(j\omega)|^2 \, d\omega$$

$$= \frac{10^4}{\pi}\int_0^{25} \frac{d\omega}{625+\omega^2} = \frac{10^4}{\pi}\left[\frac{1}{25} \times \tan^{-1}\frac{\omega}{25}\right]_0^{25}$$

$$= \frac{10^4}{\pi} \times \frac{1}{25} \times \frac{\pi}{4} = 100 \text{ Joule}$$

∴ Percentage of the input energy appearing at the output,

$$\frac{W_{01\Omega}}{W_{i1\Omega}} \times 100 = \frac{100}{200} \times 100\% = 50\%$$

8.35 A voltage, $v(t) = 4e^{-3t}\, u(t)$ volt is applied to the input of an ideal band-pass filter having a pass-band defined by $1 < f < 2$ Hz. Calculate the total 1 Ω energy available at the output of the filter.

Solution Let the output voltage is $v_0(t)$. The energy in $v_0(t)$ will be equal to the energy of that part of $v(t)$, having frequency components in the intervals, $1 < f < 2$ and $-2 < f < -1$.

Fourier transform of input,

$$V(j\omega) = 4\int_{-\infty}^{\infty} e^{-3t}u(t)e^{-j\omega t}\,dt = 4\int_{-\infty}^{\infty} e^{-(3+j\omega)t}u(t)dt = \frac{4}{3+j\omega}$$

So, the total 1 Ω energy in the input signal is,

$$W_{1\Omega} = \int_{-\infty}^{\infty} v^2(t)dt = 16\int_0^{\infty} e^{-6t}dt = \frac{8}{3} \text{ Joule}$$

or, $$W_{i1\Omega} = \frac{16}{\pi}\int_0^{\infty}\frac{d\omega}{9+\omega^2} = \frac{16}{\pi}\int_0^{\infty}\frac{d\omega}{9+\omega^2} = \frac{16}{\pi}\left[\frac{1}{3}\tan^{-1}\frac{\omega}{3}\right]_0^{\infty} = \frac{16}{\pi}\times\frac{1}{3}\times\frac{\pi}{2} = \frac{8}{3} \text{ Joule}$$

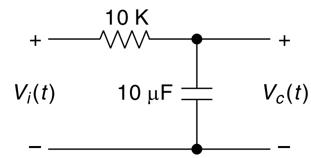
Total energy in the output is,

$$W_0 = \frac{1}{2\pi}\int_{-4\pi}^{-2\pi}\frac{16d\omega}{9+\omega^2} = \frac{16}{2\pi}\int_{0-4\pi}^{-2\pi}\frac{d\omega}{9+\omega^2} = \frac{16}{\pi}\left[\frac{1}{3}\tan^{-1}\frac{\omega}{3}\right]_{-4\pi}^{-2\pi}$$

$$= \frac{16}{\pi}\times\frac{1}{3}\times\left[\tan^{-1}\left(\frac{4\pi}{3}\right) - \tan^{-1}\left(\frac{2\pi}{3}\right)\right]$$

$= 0.358$ Joule

8.36. The voltage, $V_i(t) = 5e^{-5t}\, u(t)$ volt is applied to the input of the RC circuit shown in figure. Determine the percentage of the 1 Ω energy that is transmitted to the output.



Solution Here, the cut-off frequency,

$$\omega_c = \frac{1}{RC} = \frac{1}{10^4 \times 10 \times 10^{-6}} = 10 \text{ rad/s}$$

Fourier transform of $v_i(t)$

$$V_i(j\omega) = \frac{5}{5 + j\omega}$$

$$\therefore \qquad |V_i(j\omega)|^2 = \frac{25}{25 + \omega^2}$$

Total 1 Ω energy available at the filter input is,

$$W_{i1\Omega} = \frac{1}{\pi}\int_0^\infty \frac{25 d\omega}{25+\omega^2}\, d\omega$$

$$= \frac{25}{\pi}\int_0^\infty \frac{d\omega}{25+\omega^2}\, d\omega$$

$$= \frac{25}{\pi}\left[\frac{1}{5}\tan^{-1}\frac{\omega}{5}\right]_0^\infty$$

$$= \frac{25}{\pi} \times \frac{1}{5} \times \frac{\pi}{2} = 2.5 \text{ Joule} \qquad \textit{Ans.}$$

The 1 Ω energy available at the filter output is,

$$W_{01\Omega} = \frac{1}{\pi}\int_0^{10} |V_i(j\omega)|^2\, d\omega$$

$$= \frac{25}{\pi}\int_0^{10} \frac{d\omega}{25+\omega^2} = \frac{25}{\pi}\left[\frac{1}{5}\times\tan^{-1}\frac{\omega}{5}\right]_0^{10}$$

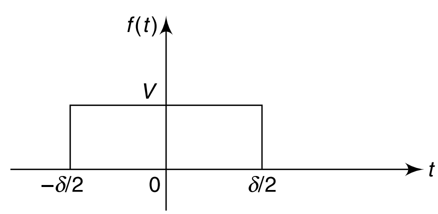
$$= \frac{25}{\pi} \times \frac{1}{5} \times 1.107 = 1.762 \text{ Joule}$$

$\therefore$ Percentage of the input energy appearing at the output,

$$\frac{W_{01\Omega}}{W_{i1\Omega}} \times 100 = \frac{1.762}{2.5} \times 100\% = 70.48\% \qquad \textit{Ans.}$$

8.37 (a) For the pulse shown in figure prove that,

$$F(j\omega) = V\delta \frac{\sin\frac{\omega\delta}{2}}{\frac{\omega\delta}{2}}$$



(b) Draw the frequency spectra of this waveform and explain how you would use this result to estimate the bandwidth required for the transmission of such a signal.

(c) Calculate the percentage of energy associated with this pulse that lies in the dominant portion of the amplitude spectrum.

Solution

(a) The pulse is, $f(t)=V, -\delta/2<t<\delta/2$

So, the Fourier transform,

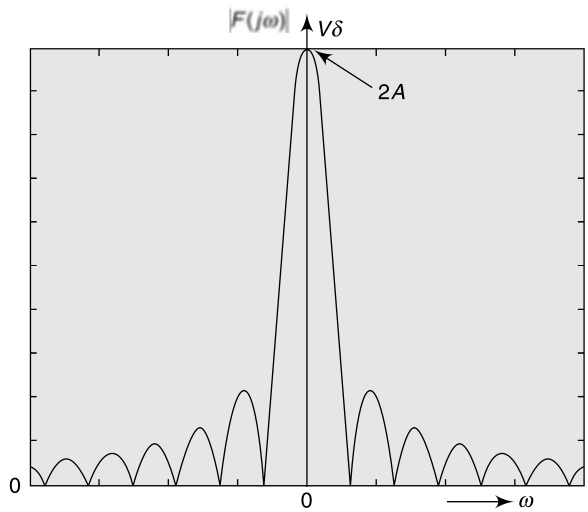
$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty} Ve^{-j\omega t}dt=V\frac{e^{j\omega\delta/2}-e^{-j\omega\delta/2}}{j\omega}$$

$$=2V\frac{\sin\left(\frac{\omega\delta}{2}\right)}{\omega}$$

$$=2V\frac{\sin\left(\frac{\omega\delta}{2}\right)}{\left(\frac{\omega\delta}{2}\right)}\times\frac{\delta}{2}$$

$$\therefore \quad F(j\omega)=V\delta\frac{\sin\left(\frac{\omega\delta}{2}\right)}{\left(\frac{\omega\delta}{2}\right)}$$

The plot of $\left|\frac{\sin x}{x}\right|$ versus x $\left(\text{here, } x=\frac{\omega\delta}{2}\right)$ is shown in the below figure.



(b) The function goes through zero when $x = \frac{\omega\delta}{2}$ is an integral multiple of π. The function is unity at $x = 0$. This form is called sampling function, and it occurs frequently in modern communication theory.

From the figure, we see that the major portion of the amplitude spectrum of the rectangular pulse spreads over the frequency range from $-\frac{2\pi}{\delta}$ to $\frac{2\pi}{\delta}$. If the pulse is carried through a transmission system, the bandwidth (BW) of the system must accommodate the major portion of the amplitude spectrum for reasonable fidelity in transmission; i.e. the cut-off frequency of the system must be at least, $\omega_C = \frac{2\pi}{\delta}$.

Thus, $\omega_C \times \delta = 2\pi \left[\text{BW} = \frac{2\pi}{\delta} \right]$

$\therefore$ Product of the bandwidth and pulse width is a constant.

(c) We know that the dominant portion of the amplitude spectrum lies in the frequency range $0 \le \omega \le \frac{2\pi}{\delta}$. The Fourier transform of the rectangular voltage pulse is,

$$V(j\omega) = V\delta \frac{\sin\left(\frac{\omega\delta}{2}\right)}{\left(\frac{\omega\delta}{2}\right)}$$

The portion of the total 1 Ω energy associated with $v(t)$ that lies in the dominant portion of the amplitude spectrum is,

$$W'_{1\Omega} = \frac{1}{\pi}\int_0^{2\pi/\delta} V^2\delta^2 \frac{\sin^2\left(\frac{\omega\delta}{2}\right)}{\left(\frac{\omega\delta}{2}\right)^2} d\omega$$

$$= \frac{2V^2\delta}{\pi}\int_0^{\pi} \frac{\sin^2 x}{x^2} dx \quad \left\{ let,\ x = \frac{\omega\delta}{2},\ \therefore dx = \frac{\delta}{2} d\omega \right\}$$

$$= \frac{2V^2\delta}{\pi}\left[\sin^2 x \left\{ -\frac{1}{x} \right\}_0^{\pi} + \int_0^{\pi} \frac{\sin 2x}{x} dx \right]$$

$$= \frac{2V^2\delta}{\pi}\left[0 + \int_0^{\pi} \frac{\sin 2x}{x} dx \right]$$

$$= \frac{4V^2\delta}{\pi}\int_0^{\pi} \frac{\sin 2x}{2x} dx$$

$$= \frac{2V^2\delta}{\pi}\int_0^{\pi}\frac{\sin\theta}{\theta}\,d\theta \quad \left[Let,\ \theta = 2x, \therefore\ dx = \frac{1}{2}d\theta\right]$$

$$= \frac{2V^2\delta}{\pi}\times 1.418$$

[The value of the integral as found from the table of sine integrals is 1.418]

$$\therefore \qquad W'_{1\Omega} = \frac{2V^2\delta}{\pi}\times 1.418$$

Total 1Ω energy for $v(t)$ is,

$$W_{1\Omega} = \int_0^{\delta} V^2 d\delta = V^2\delta$$

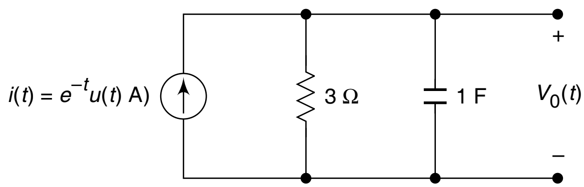
Hence the percentage of total energy contained in the dominant portion of the amplitude spectrum is,

$$\frac{W'_{1\Omega}}{W_{1\Omega}}\times 100 = \frac{2\times 1.418}{\pi}\times 100 = 90.2\% \qquad \textit{Ans.}$$

8.38 (a) Find the Fourier transform of the function,

$$f(t) = Ae^{-t/a} \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0$$

(b) Use the above transform to find the output voltage V_0 in the figure.



Solution

(a) Fourier transform of the function is,

$$I(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_0^{\infty} Ae^{-t/a}e^{-j\omega t}dt = A\int_{-\infty}^{\infty} e^{-\left(\frac{1}{a}+j\omega\right)t}dt = A\left|\frac{e^{-\left(\frac{1}{a}+j\omega\right)t}}{-\left(\frac{1}{a}+j\omega\right)}\right|_0^{\infty}$$

$$= \frac{Aa}{1+j\omega a} \qquad \textit{Ans.}$$

(b) By KCL, $I(j\omega) = \dfrac{V_0(j\omega)}{3} + \dfrac{V_0(j\omega)}{1/j\omega} = V_0(j\omega)\left[\dfrac{1+3j\omega}{3}\right]$

Here, $\quad I(j\omega) = \dfrac{1}{1+j\omega}$ (From result of (a) with A = 1 and a = 1)

or, $\quad \dfrac{1}{1+j\omega} = V_0(j\omega)\left[\dfrac{1+3j\omega}{3}\right]$

$$\therefore \qquad V_0(j\omega) = \frac{3}{(1+j\omega)(1+j3\omega)} = \left[\frac{3/2}{1/3+j\omega} - \frac{3/2}{1+j\omega}\right]$$

Taking inverse Fourier transform

$$V_0(t) = \frac{3}{2}e^{-t/3} - \frac{3}{2}e^{-t} \qquad Ans.$$

MULTIPLE-CHOICE QUESTIONS

8.1 A current consists of a fundamental component of amplitude I_1, and a third harmonic of amplitude I_3. The rms value of current will be

(a) $(I_1 + I_3)/\sqrt{2}$ (b) $(I_1 + I_3)/2\sqrt{2}$ (c) $\sqrt{I_1^2 + I_3^2}$ (d) $\sqrt{(I_1^2 + I_3^2)/2}$

8.2 The Fourier series expansion of a periodic function with half wave symmetry contains only

(a) sine terms (b) cosine terms (c) odd harmonics (d) even harmonics

8.3 A periodic function $f(t)$ is said to have a quarter wave symmetry, if it possesses

(a) even symmetry at an interval of quarter of a wave.
(b) even symmetry and half wave symmetry only
(c) even or odd symmetry without the half wave symmetry
(d) even or odd symmetry with the half wave symmetry.

8.4 If $f(t)$ is a periodic waveform with even symmetry, then its Fourier series expansion does not contain

(a) sine terms (b) cosine terms (c) odd harmonics (d) even harmonics

8.5 Periodic signal that obeys Dirichlet's condition can be represented by

(a) Fourier series (b) Fourier transform
(c) Inverse Fourier transform (d) None of these

8.6 Which of the following conditions is true for even function?

(i) $f(t) = -f(t \pm T/2)$ (b) $f(t) = -f(-t)$ (c) $f(t) = f(-t)$ (d) $f(t) = f(T)$

8.7 Which of the following conditions is true for odd function?

(a) $f(t) = -f(t \pm T/2)$ (b) $f(t) = -f(-t)$ (c) $f(t) = f(-t)$ (d) $f(t) = f(T)$

8.8 A periodic function $f(t)$ having a time period T, repeats itself after half time period $T/2$. The Fourier series of $f(t)$ would contain.

(a) cosine terms only (b) sine terms only
(c) odd harmonic terms only (d) even harmonic terms only

8.9 Which of the following statements is true for a delayed step function $u(t - T)$?

(a) It has an infinite Fourier series (b) It has no Fourier series
(c) It has a finite Fourier series (d) Its Laplace transform is $1/s$.

8.10 Which one of the following is the correct Fourier transform of the unit step signal $u(t)$?

(a) $\pi\delta(\omega)$ (b) $\dfrac{1}{j\omega}$ (c) $\dfrac{1}{j\omega} + \pi\delta(\omega)$ (d) $\dfrac{1}{j\omega} + 2\pi\delta(\omega)$

8.11 If $f(t) = -f(-t)$ and $f(t)$ satisfy the Dirichlet's conditions, then $f(t)$ can be expanded in a Fourier series containing

(a) only sine terms (b) only cosine terms
(c) cosine terms and a constant term (d) sine terms and a constant term.

8.12 Fourier transform $F(j\omega)$ of an arbitrary signal has the property:

(a) $F(j\omega) = F(-j\omega)$ (b) $F(j\omega) = -F(-j\omega)$ (c) $F(j\omega) = F^*(-j\omega)$ (d) $F(j\omega) = -F^*(-j\omega)$

8.13 The Fourier series expansion of an odd periodic function contains

(a) cosine terms (b) constant terms only (c) sine terms.

8.14 For the expansion of $f(\omega t)$ in Fourier series $a_0 + a_1 \cos \omega t + \ldots + a_n \cos n\omega t + \ldots + b_1 \sin \omega t + \ldots + b_a \sin n\omega t$ if $f(\omega t) = f(-\omega t)$ then:

(a) $a_n = 0$ (b) $b_n = 0$ for all n

(c) $a_0 = 0$ (d) $a_n = 0$ for all n except $n = 0$.

8.15 Two complex waves will have the same waveform if

(a) they contain the same harmonics.

(b) harmonics are similarly spaced with respect to the fundamental.

(c) the ratio of corresponding harmonics to their respective fundamentals is the same.

(d) all of the above.

8.16 The complex wave is symmetrical when

(a) it contains only even harmonics.

(b) it contains only odd harmonics.

(c) it contains both odd and even harmonics.

(d) phase difference between even harmonics and fundamental is either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

8.17 An even waveform when expressed in exponential Fourier series will contain:

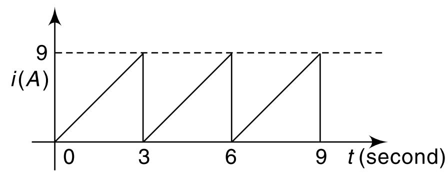
(a) only imaginary coefficient (b) only real coefficient

(c) both (a) and (b) (d) None of these.

8.18 The current waveform in a pure resistor of 10 Ω is shown in the given figure. Power dissipated in the resistor is

(a) 7.29 W (b) 52.4 W

(c) 135 W (d) 270 W.



8.19 The inverse Fourier transform of

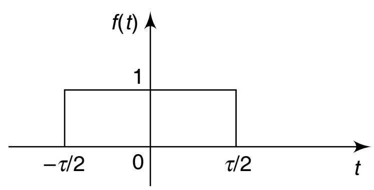
$$F(j\omega) = \int_{-\infty}^{\infty} \exp(-j\omega t) f(t)dt \text{ is:}$$

(a) $f(t) = \int_{-\infty}^{\infty} \exp(+j\omega t) F(j\omega) d\omega$ (b) $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(+j\omega t) F(j\omega) d\omega$

(c) $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-j\omega t)\, F(+j\omega) d\omega$ (d) $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-j\omega t)\, F(-j\omega)\, d\omega$

8.20 Fourier transform of the gate function as shown in the figure is

$$f(t) = 1 \quad for -\tau/2 \le t \le \tau/2$$
$$= 0 \quad \text{otherwise}$$



(where τ is the width of the gate function)

The value of $F(\omega)$ is

(i) $\dfrac{\tau \sin(\omega\tau)}{\omega\tau}$ (ii) $\dfrac{\tau \sin(2\omega\tau)}{2\omega\tau}$

(iii) $\dfrac{\tau \sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}$ (iv) $\dfrac{\tau}{2} \dfrac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)}$

8.21 The Fourier transform of Signum function is given by

(i) $j\pi f$ (ii) $\dfrac{1}{j\pi f}$ (iii) $\dfrac{1}{j\pi f}+\pi\delta(f)$ (iv) $j\pi f+\pi\delta(f)$

8.22 The Fourier transform of unit impulse function $\delta(t)$ would be

(i) $\dfrac{1}{(2+\omega^2)}$ (ii) $(2+\omega^2)$ (iii) 1 (iv) $\dfrac{1}{2}$

8.23 A ramp function
(i) has Laplace transform but not Fourier transform
(ii) has Fourier transform but not Laplace transform
(iii) have both Laplace and Fourier transform
(iv) none of these

8.24 $x(t)$ is a real valued function of a real variable with period T. Its trigonometric Fourier Series expansion contains no terms of frequency $\omega = 2\pi {(2k)}/{T}$; k = 1, 2, …Also, no sine terms are present. Then $x(t)$ satisfies the equation

(i) $x(t) = -x(t-T)$ (ii) $x(t) = x(T-t) = -x(-t)$

(iii) $x(t) = x(T-t) = -x(t - T/2)$ (iv) $x(t) = x(t-T) = x(t - T/2)$

8.25 An input voltage $v(t) = 10\sqrt{2}\cos(t+10^0) + 10\sqrt{5}\cos(2t+10^0)$ V is applied to a series combination of resistance R = 1 Ω and an inductance L = 1 H. The resulting steady-state current $i(t)$ in ampere is

(i) $10\cos(t+55^0) + 10\cos(2t+10^0+\tan^{-1}2)$

(ii) $10\cos(t+55^0) + 10\sqrt{\dfrac{3}{2}}\cos(2t+55^0)$

(iii) $10\cos(t-35^0) + 10\cos(2t+10^0-\tan^{-1}2)$

(iv) $10\cos(t-35^0) + 10\sqrt{\dfrac{3}{2}}\cos(2t-35^0)$

8.26 Choose the function $f(t)$, $-\infty < t < \infty$, for which a Fourier series cannot be defined.

(i) $3\sin(25t)$ (ii) $4\cos(20t+3) + 2\sin(710t)$

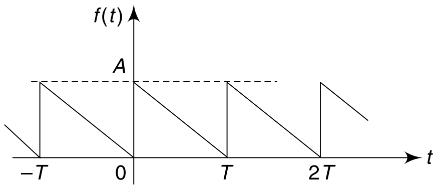
(iii) $\exp(-|t|)\sin(25t)$ (iv) 1

EXERCISES

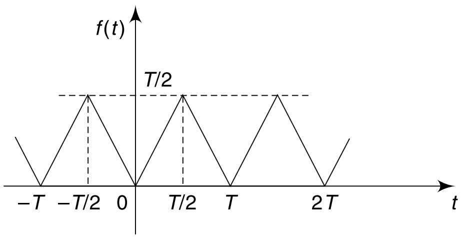
Fourier Series

8.1 Find the Fourier series expansion for the following functions and sketch the frequency spectrum.

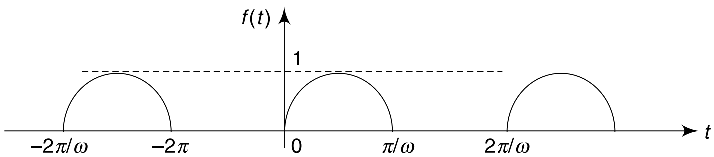
(a)



(b)



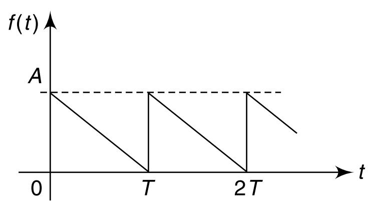
(c)



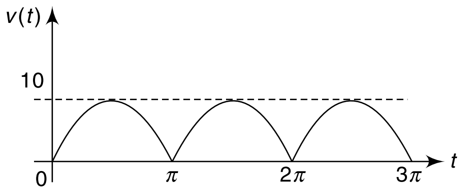
$$\left[Ans. \text{ (a) } f(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{A}{n\pi} \sin n\omega t \text{ (b) } f(t) = \frac{T}{4} - \frac{2T}{\pi^2}\left[\cos \omega t + \frac{1}{3^2}\cos 3\omega t + ..\right]\right.$$

$$\left. \text{(c) } f(t) = \frac{1}{\pi} + \frac{1}{2}\sin \omega t - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2n\omega t \right]$$

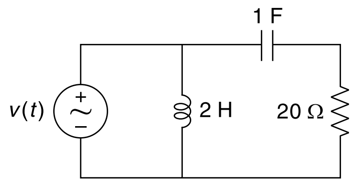
8.2 A periodic waveform as shown in the below figure feeds an RL load with $R = 10$ ohm and $L = \frac{1}{2\pi}$ H. Calculate the power at the fundamental frequency supplied to the load.



8.3 A waveform of the shape shown in the below figure (i) is applied to the network shown in the below figure (ii). Calculate the power dissipated in a 20 Ω resistor. Take $\omega = 1$ rad/s. [1.17 W]



(i)



(ii)

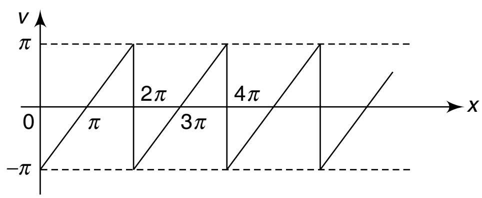
8.4 A series RLC circuit with $R = 5\ \Omega$, $L = 5$ mH, $C = 50\ \mu$F has an applied voltage

$$v(t) = 150 \sin 1000\,t + 100 \sin 2000\,t + 75 \sin 3000\,t\ (V)$$

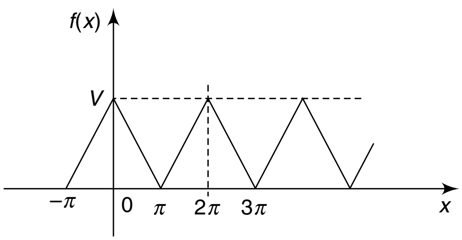
Determine the effective current and average power. [16.58 A; 1374 W]

8.5 Find the Fourier series expansion for the waveforms shown in the figure.

(a)



(b)



(a) $v = -2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \ldots\right]$

(b) $v = \frac{V}{2} + \frac{4V}{\pi^2}\cos x + \frac{4V}{(3\pi)^2}\cos 3x + \frac{4V}{(5\pi)^2}\cos 5x + \ldots$

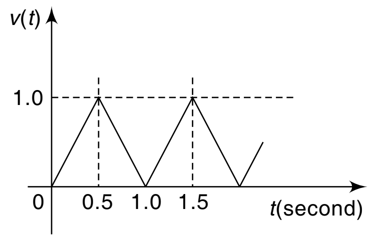
8.6 A triangular wave increases linearly from 0 to V_m during the interval 0 to π. The wave has zero value during the interval π to 2π and this cycle is repeated. Find the Fourier series representation of the wave.

$$\left[v = \frac{V_m}{4} - \frac{2V_m}{\pi^2}\left(\cos x + \frac{1}{25}\cos 5x + \ldots\right) + \frac{V_m}{\pi}\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right)\right]$$

8.7 A wave has a constant value I_m during the interval $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and $-I_m$ during the interval $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. This cycle is repeated in the next intervals. Find the Fourier series for the wave.

$$\left[i = \frac{4I_m}{\pi}\left(co\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \frac{1}{7}\cos 7\theta + \ldots\right)\right]$$

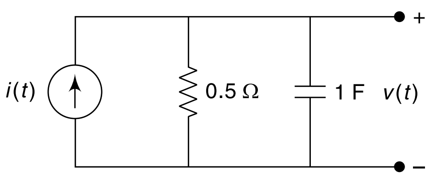
8.8 (a) Find the trigonometric Fourier series for the voltage wave shown in the figure.
(b) If this voltage is applied to a capacitor of 1 F, find the current.



Fourier Transform

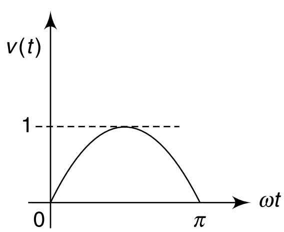
8.9 Find the Fourier transform of the following functions:
(i) $f(t) = e^{-at}\, u(t)$, $a > 0$.
(ii) $f(t) = e^{-a|t|}$, for all values of t.
(iii) $f(t) = 1$
(iv) Unit impulse function, $\delta(t)$.
(v) Signum function, sgn(t).
(vi) Unit step function, $u(t)$.

8.10 Determine the output voltage response across the capacitor to a current source excitation $i(t) = e^{-t}u(t)$, as shown in the below figure.

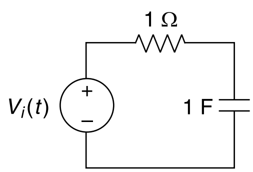


$[v(t) = e^{-t} - e^{-2t}\,(\mathrm{V})]$

8.11 Determine the response of the network shown in the below figure when a voltage having the waveform shown in figure is applied to it., by using Fourier transform method.



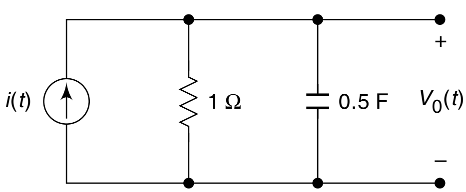
(i)



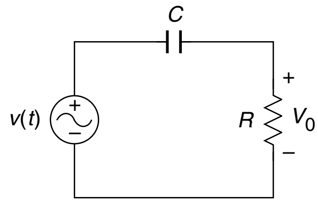
(ii)

8.12 The current source in the Figure is $i(t) = 4e^{-t}$ for t ³ 0. Find the voltage V_0 using Fourier transform method.

$$\left[v(t) = 8e^{-t} - 8e^{-2t}\,(V)\right]$$



8.13 The voltage source in the figure is an exponentially decaying pulse,



$$v(t) = 0 \quad \text{for } t < 0$$
$$= e^{-\alpha t} \quad \text{for } t \geq 0$$

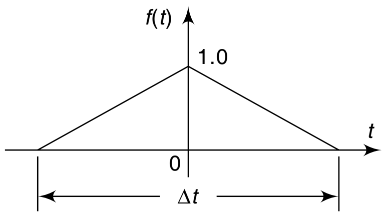
Find the output voltage V_0.

$$\left[V_0 = \left(\frac{1}{1-\alpha RC}\right) e^{-t/RC} - \frac{\alpha RC}{1-\alpha RC} e^{-\alpha t} \quad \text{for } t \geq 0. \right]$$

8.14 A cosine pulse $v = V_m \cos t$ is zero for all time except $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. Find the Fourier transform of the pulse and sketch the continuous amplitude spectrum and phase spectrum.

$$\left[\frac{2V_m}{1-\omega^2} \cos\left(\omega \frac{\pi}{2}\right) \right]$$

8.15 Find the Fourier transform of the triangular pulse shown in the figure.



$$\left[\frac{\Delta t}{2} \frac{\sin^2\left(\frac{\omega \Delta t}{4}\right)}{\left(\frac{\omega \Delta t}{4}\right)^2} \right]$$

SHORT-ANSWER TYPE QUESTIONS

8.1 (a) What are the conditions which a periodic function must satisfy to have its Fourier series expansion?

(b) Write the trigonometric form of the Fourier series for a function $f(t)$ and explain, by deriving necessary relations, how the values of various co-efficients are obtained.

or

What do you understand by Fourier series? Outline the general procedure of determining Fourier series of periodic waveform.

(c) Give the exponential form of Fourier series for a periodic function.

8.2 Derive an expression for the effective value of a non-sinusoidal periodic waveform

or

Discuss the method of computing the effective value of a non-sinusoidal periodic waveform.

8.3 (a) Explain clearly the significance of the following terms used in determining Fourier series of a given waveform:

(i) Odd symmetry or Rotation symmetry,
(ii) Even symmetry or Mirror symmetry,
(iii) Half-wave symmetry or Alternation symmetry,
(iv) Quarter-wave symmetry.

(b) Show that the Fourier series expansion of a periodic function with odd (rotation) symmetry contains only the sine terms.

(c) Show that the Fourier series expansion of a periodic function with even (mirror) symmetry contains only the cosine terms plus a constant.

(d) Show that the Fourier series expansion of a periodic function with half-wave symmetry contains only the odd harmonics.

8.4 Discuss in brief the following:

(i) Fourier series and its applications to network analysis,

(ii) Method of analyzing the complex waveform by Fourier series,

(iii) Frequency and phase spectra of periodic waveform.

(iv) Truncating Fourier series.

(v) Gibb's phenomenon.

8.5 (a) Give the definitions of a Fourier transform pair and illustrate its use in network analysis with one example.

(b) Explain clearly the difference between Fourier transform and Laplace transform and discuss briefly their importance in analyzing electrical network.

or

Define Fourier's transform. How does Fourier transform differ from (i) Fourier integral and (ii) Laplace transform?

(c) Write a brief note on the use of Fourier transform and Fourier integrals in the analysis of circuits excited by ideal sources of non-sinusoidal waveforms.

(d) Discuss the important properties of Fourier transforms.

8.6 When do we use Fourier transform?

Discuss that Fourier integral is the limit of Fourier series, as time period T of a repetitive wave approaches infinity as the limit.

or

How would you obtain Fourier integral from Fourier series?

8.7 Find the amplitude-frequency distribution of a single non-repetitive voltage pulse of duration one microsecond and explain how its frequency-bandwidth is estimated.

or

Consider a periodic voltage pulse waveform of period T (second) and width T_0 (second). Find an expression for the frequency-spectra of this waveform and explain how you would use this result to estimate the bandwidth required for the transmission of such a signal.

8.8 State and prove Parseval's theorem for a periodic function.

8.9 Show that when $f(t)$ is an even function of t, its Fourier transform $F(j\omega)$ is an even function of ω and is real; while when $f(t)$ is an odd function of t, its Fourier transform $F(j\omega)$ is an odd function of ω and is imaginary.

8.10 Explain why:

(i) When a complex voltage wave is applied to a pure capacitor, the current wave has more harmonics than the applied voltage wave.

(ii) When a complex voltage wave is applied to a pure inductor, the current wave has lesser harmonics than the applied voltage wave.

(iii) If a voltage wave containing a dc component is applied to a series RC circuit, the current wave does not contain the corresponding dc component.

8.11 Explain briefly the interrelation between Fourier series, Fourier transforms and Laplace transforms.

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

8.1 (d)	8.2 (c)	8.3 (d)	8.4 (a)	8.5 (a)	8.6 (c)	8.7 (b)
8.8 (c)	8.9 (c)	8.10 (c)	8.11 (a)	8.12 (c)	8.13 (c)	8.14 (b)
8.15 (d)	8.16 (b)	8.17 (b)	8.18 (d)	8.19 (b)	8.20 (c)	8.21 (b)
8.22 (c)	8.23 (a)	8.24 (d)	8.25 (c)	8.26 (c)		

CHAPTER

9

Filter Circuits

9.1 INTRODUCTION

Passive filters are built from passive components; resistors, capacitors, and inductors. **Active filters** also use resistors and capacitors, but the inductors are replaced by active devices capable of producing power gain. These devices can range from single transistor to integrated circuit (IC)—controlled sources such as the operational amplifier (op amp), and more exotic devices, such as the operational transconductance amplifier (OTA), the generalized impedance converter (GIC), and the frequency-dependent negative resistor (FDNR).

In this chapter, active filters with op-amp have been discussed.

9.1.1 Operational Amplifier (Op-Amp)

An operational amplifier is a direct- coupled high gain, differential-input amplifier.

With the addition of suitable external feedback components, an op-amp can be used for a variety of application, such as ac and dc signal amplification, active filters, oscilators, comparators, regulators, and others.

9.1.2 Operational Amplifier Terminals

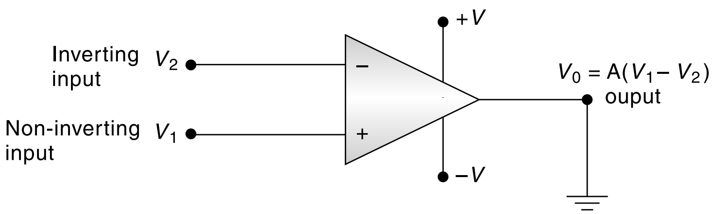


Figure 9.1 *Operational amplifier*

Op-amp has five basic terminals—

(i) Two for input signals, V_1 and V_2 – differential input terminals.
(ii) One for output signal, V_0 single-ended output.
(iii) Two for power supply, $+V$ and $-V$. (Maximum $V = \pm$ 18 V)

Note The power supply has three terminals: positive, negative and power supply common. The common terminal may or may not be wired to earth ground via the third wire of line cord. However, it has become standard practice to show power common as a ground symbol.

Use of the term 'ground' on the ground symbol is a convention which indicates that all voltage measurements are with respect to 'ground'.

9.1.3 Op-Amp Characteristics

Ideal Characteristics

(i) An infinite voltage gain
(ii) An infinite bandwidth
(iii) An infinite input impedance
(iv) Zero output impedance
(v) Perfect balance, i.e., the output is zero when equal voltages are present at the two input terminals; and
(vi) The characteristics do not change with temperature

Practical (Actual) Characteristics

(i) The gain at low- frequency is finite and very high (of the order of 10^3 to 10^6). The gain is constant upto a few hundred kHz and then decreases monotonically with the increase in frequency.
(ii) The bandwidth is finite and very high.
(iii) The input impedance lies in the range of 150 kΩ to a few hundred M Ω.
(iv) The output impedance of a practical op-amp lies between 0.75 to 100 Ω.
(v) Perfect balance is not achieved with practical op-amps.

9.2 FILTER

An electric filter is a four-terminal frequency-selective network designed generally with reactive elements to transmit freely a specified band of frequency and block or attenuate signals of frequency outside this band.

- The band of frequency transmitted through the filter is called the Pass-band.
- The band of frequency which is severely attenuated by the filter is called the attenuated on stop-band.

9.3 CLASSIFICATION OF FILTERS

This must be remembered that there is no simple hierarchical classification of filters. Filters may be classified on different bases which overlap each other in many respects.

Depending upon the type of techniques used in signal processing, filters are classified as:

(i) Analog Filters, and
(ii) Digital Filters.

Analog filters are designed to process analog signals using analog techniques, while digital filters process analog signals using digital techniques.

Depending on the type of elements used in their construction, filters are classified as:

(i) Active Filters, and
(ii) Passive Filters.

A passive filter is built with passive components such as resistors, capacitors and inductors. Active filters, on the other hand, make use of transistors or op-amps (providing voltage amplification, and signal isolation or buffering) in addition to resistors and capacitors.

Depending upon the type of elements used, the operating frequency range of the filter will be different and accordingly the filters are classified as:

(i) Low Pass Filters,
(ii) High Pass Filters,
(iii) Bans Pass Filters,
(iv) Band Stop Filters, and
(v) All Pass Filters.

1. Low-Pass Filter It is a circuit that has a constant output (or gain) from zero to a cut-off frequency, f_c and attenuation of all frequencies above f_c.

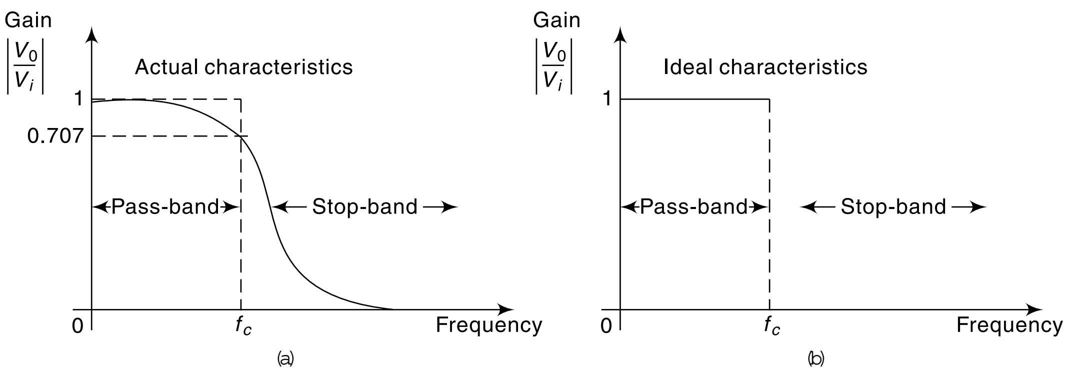


Figure 9.2 *Low-pass filter characteristics: (a) Actual (b) Ideal*

2. High-Pass Filter It is a circuit that attenuates all signals of frequency below the cut-off frequency and has a constant output (or gain) above this frequency.

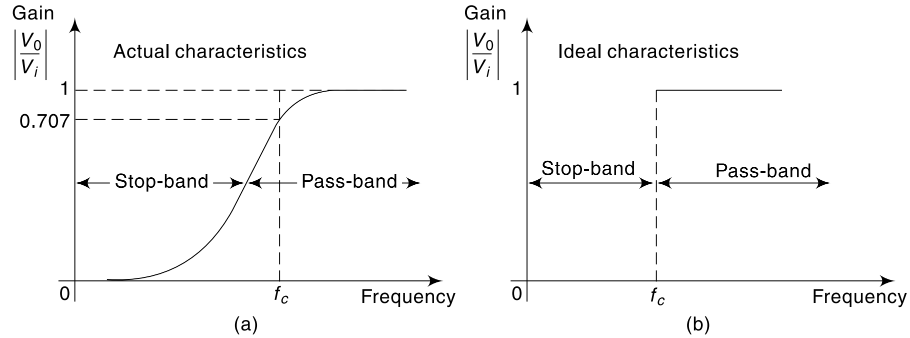


Figure 9.3 *High pass filter characteristics (a) Actual (b) Ideal*

3. Band–Pass Filter It is a circuit that passes a band of frequencies and attenuates all frequencies outside the band.

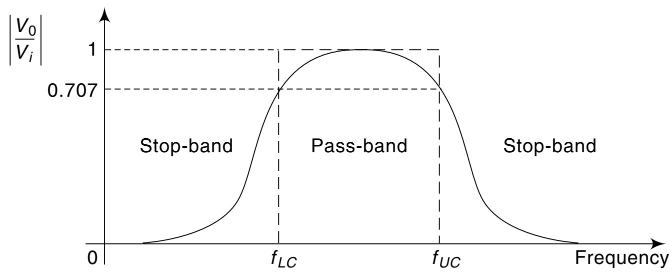


Figure 9.4 *Band pass filter characteristics*

*4. **Band-Rejection/Elimination Filter or Band Stop Filter or Notch Filter*** It rejects a specified Band of frequencies while passing all other frequencies outside the band.

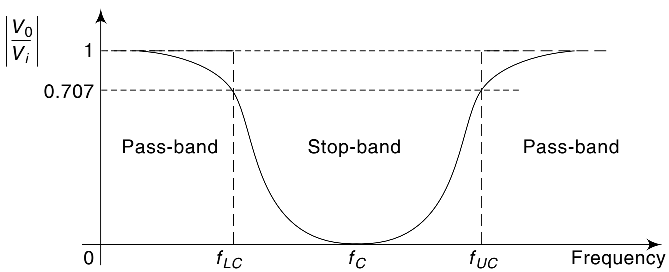


Figure 9.5 *Band reject filter characteristics*

5. All–Pass Filter It passes all frequencies equally well, i.e., output and input voltages are equal in magnitude for all frequency; with the phase–shift between the two a function of frequency.

This filter is also known as a **phase-shift filter**, **time-delay filter,** or simply the **delay equalizer.** One major application of an all-pass filter is the simulation of a lossless transmission line. The magnitude of the output voltage is the same as the input voltage but the output voltage is shifted in phase with respect to the input voltage.

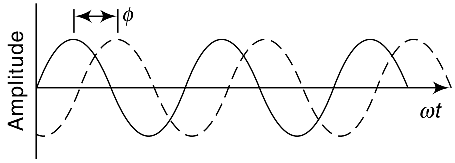


Figure 9.6 *All pass filters characteristics*

The highest frequency up to which the input and output amplitudes remain equal is dependent on the unity-gain bandwidth of the op-amp. At this frequency, however, the phase-shift between the input and output is maximum.

9.4 ADVANTAGES OF ACTIVE FILTERS OVER PASSIVE FILTERS

1. **Less Cost** Active filters are very much inexpensive than passive filters due to the variety of cheaper op-amp and the absence of costly inductors.
2. **Gain and Frequency Adjustment Flexibility** Since the op-amp is capable of providing a gain (which may also be variable), the input signal is not attenuated as it is in a passive filter. In addition, the active filter is easier to tune or adjust.
3. **No Loading Problem** Active filters provide an excellent isolation between the individual stages due to the high input impedence (ranging from a few kΩ to a several thousand MΩ) and low output impedance (ranging from less than 1 Ω to a few hundred Ω). So, the active filter does not cause loading of the source or load.
4. **Size and Weight** Active filters are small in size and less bulky (due to the absence of bulky 'L') and are rugged.
5. **Non-floating Input and Output** Active filters generally have single ended inputs and outputs which do not 'float' with respect to the system power supply or common. This property is different from that of the passive filters.

9.5 APPLICATION OF ACTIVE FILTERS

Application of active filters is given below. They are used

(i) in the field of communication and signal processing
(ii) in almost all sophisticated electronic systems, such as radio, television, telephone, radar, space satellites, biomedical equipments, and so on.

9.6 LOW-PASS ACTIVE FILTER

The circuit of Figure 9.7 is a commonly used low-pass active filter.

The filtering is done by the RC network, and the op-amp is used as a unity-gain amplifier. The resistor R_f (= R) is included for DC offset.

[At DC, the capacitive reactance is infinite and the dc resistive path to ground for both terminals should be equal.]

Here, all the voltages V_i, V_x, V_y, V_o are measured with respect to ground.

Since the input impedance of the op-amp is infinite, no current will flow into the input terminals.

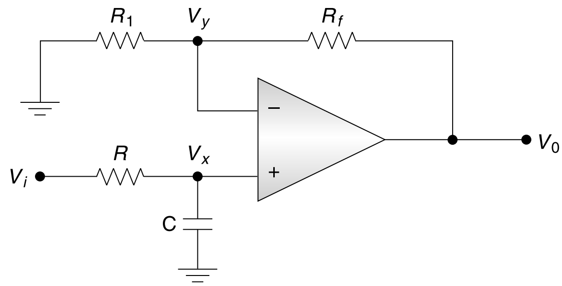


Figure 9.7 *First order low-pass active filter circuit*

$$V_y = \frac{V_0}{R_1 + R_f} \times R_1 \tag{9.1}$$

According to the voltage divider – rule, the voltage across the capacitor,

$$V_x = \frac{X_c}{R + X_c} V_i \;; \quad X_c = \frac{1}{j\omega C} = \frac{1}{j2\pi f C}$$

$$= \frac{1/j2\pi f C}{R + \dfrac{1}{j2\pi f C}} V_i$$

$$= \frac{V_i}{1 + j2\pi f RC} \tag{9.2}$$

Since the op-amp gain is infinite,

$\therefore$ $$V_x = V_y$$

or, $$\frac{V_0 R_1}{R_1 + R_f} = \frac{V_i}{1 + j2\pi f RC}$$

$\Rightarrow$ $$\frac{V_0}{V_i} = \frac{(1 + R_f/R_1)}{1 + j2\pi fRC}$$

or, $$\boxed{\frac{V_0}{V_i} = \frac{A_F}{1 + j(f/f_c)} = A_{cL}}$$

where, $A_F = \left(1 + \dfrac{R_f}{R_1}\right)$ = pass-band gain of the filter.

f = frequency of the input signal.

$f_c = \dfrac{1}{2\pi RC}$ = cut-off frequency of the filter.

A_{cL} = Closed- loop gain of the filter as a function of frequency.

The gain magnitude,

$$|A_{cL}| = \left|\frac{V_0}{V_i}\right| = \frac{A_F}{\sqrt{1 + (f/f_c)^2}} = \frac{A_F}{\sqrt{1 + \omega^2 R^2 C^2}}$$

and phase angle (in degree),

$$\phi = -\tan^{-1}\left(\frac{f}{f_c}\right) = -\tan^{-1}(\omega RC)$$

9.6.1 Operation of the Filter

The operation of the low-pass filter can be verified from the gain magnitude equation as follows:

1. At very low frequencies , i.e., $f << f_c$,

$$|A_{CL}| \cong A_F$$

2. At $f = f_C$, $|A_{CL}| = \frac{A_F}{\sqrt{2}} = 0.707\ A_F = -3\text{dB}\ A_F,\ \phi = -45°$

3. At $f > f_C$, $|A_{cL}| < A_F$

Thus, the filter has a constant gain of A_F from 0 Hz to the cut –off frequency f_c. At f_c, the gain is $0.707A_F$ and after f_c, it decreases at a constant rate with an increase in frequency.

Figure 9.8 shows that the actual response deviates from the straight dashed-line approximation at the vicinity of 'f_c'.

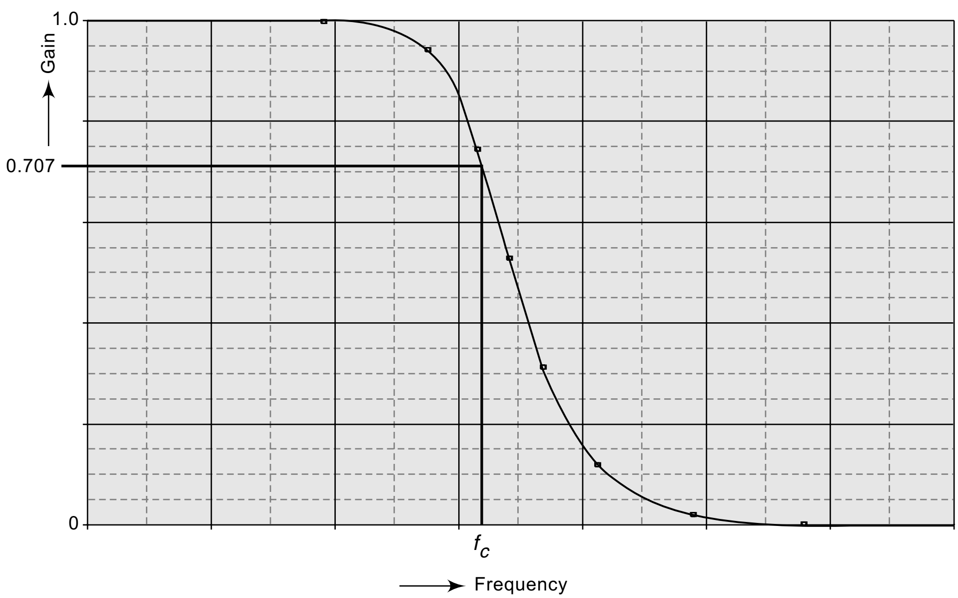


Figure 9.8 *Low pass filter characteristics*

$$\text{At } \omega = 0.1\ \omega_C,\ |A_{CL}| \cong 1(0\text{ dB})$$

$$\text{At } \omega = 10\ \omega_C,\ |A_{CL}| \cong 0.1(-20\text{ dB})$$

The table below gives the magnitude and phase angle for different values of ω between $0.1\omega_c$ and $10\omega_c$.

ω	A_{cL}	*Phase-angle (degree)*
$0.1\omega_c$	1.0	–6
$0.25\omega_c$	0.97	–14
$0.5\omega_c$	0.89	–27
ω_c	0.707	–45
$2\omega_c$	0.445	–63
$4\omega_c$	0.25	–76
$10\omega_c$	0.1	–84

9.6.2 Filter Design

A low-pass active filter can be designed by implementing the following steps:-

1. A value of the cut-off frequency ω_c (or, f_c) is chosen.
2. A value of the capacitance C is selected; usually the value is between 0.001 and 0.1µF. Mylar or tantalum capacitors are recommended for better performance.
3. The value of the resistance R is calculated from the relation,

$$R(\text{in } \Omega) = \frac{1}{\omega_C C} = \frac{1}{2\pi f_C C}$$

f_c = cut-off frequency in hertz
ω_c = cut-off frequency radian/second
C = in farad

4. Finally, the values of R_1 and R_f are selected depending on the desired pass band gain by using the relation $A_F = \left(1 + \frac{R_f}{R_1}\right)$.

9.6.3 Frequency Scaling

Once a filter is designed, there may be a need to change its cut-off frequency. The procedure used to convert an original cut-off frequency f_c to a new cut-off frequency f_c' is called '*frequency- scaling*'.

It is accomplished as follows:-

To change a cut-off frequency, multiply R or C, but not both by the ratio

$$\left(\frac{\text{Old Cut-off Frequency, } f_{\text{cold}}}{\text{New Cut-off Frequnecy, } f_{\text{cnew}}}\right)$$

Example 9.1 (a) Design a low-pass active filter at a cut-off frequency of 1kHz with a pass band gain of 2. Using the frequency scaling technique, convert this filter to a low-pass filter of cut-off frequency 1.6 kHz.

(b) Plot the frequency response of this low-pass active filter.

Solution (a) Here, f_c = 1 kHz, A_F = 2; Let, C = 0.01 µF.

$$R = \frac{1}{2\pi f_c C} = \frac{1}{2\pi \times 10^3 \times 0.01 \times 10^{-6}} = 15.9 \text{ k}\Omega$$

$$\because \quad A_F = 2 = \left(1 + \frac{R_f}{R_1}\right) \quad \Rightarrow \quad R_f = R_1 = 10 \text{ k}\Omega$$

So, the complete circuit is shown in Fig. 9.9(a).

To change the cut-off frequency from 1 kHz to 1.6 kHz, we multiply the 15.9 kΩ resistor by

$$\frac{\text{Original Cut-off frequency}}{\text{New Cut-off frequency}} = \frac{1}{1.6} = 0.625$$

$\because$ New resistor, $R = 15.9 \times 0.625 = 9.94$ kΩ

(b) To plot the frequency–response, the data are obtained from the equation,

$$\left|\frac{V_0}{V_{\text{in}}}\right| = \frac{A_F}{\sqrt{1+(f/f_c)^2}}$$

Frequency (Hz)	*Gain*	*Gain (in dB)*
10	2	6.02
100	1.99	5.98
200	1.96	5.85
700	1.64	4.29
1,000	1.41	3.01
3,000	0.63	–3.98
7,000	0.28	–10.97
10,000	0.20	–14.02
30,000	0.07	–23.53
100,000	0.02	–33.98

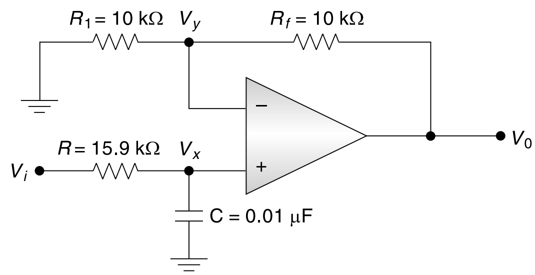


Figure 9.9(a) *Circuit of Example 9.1*

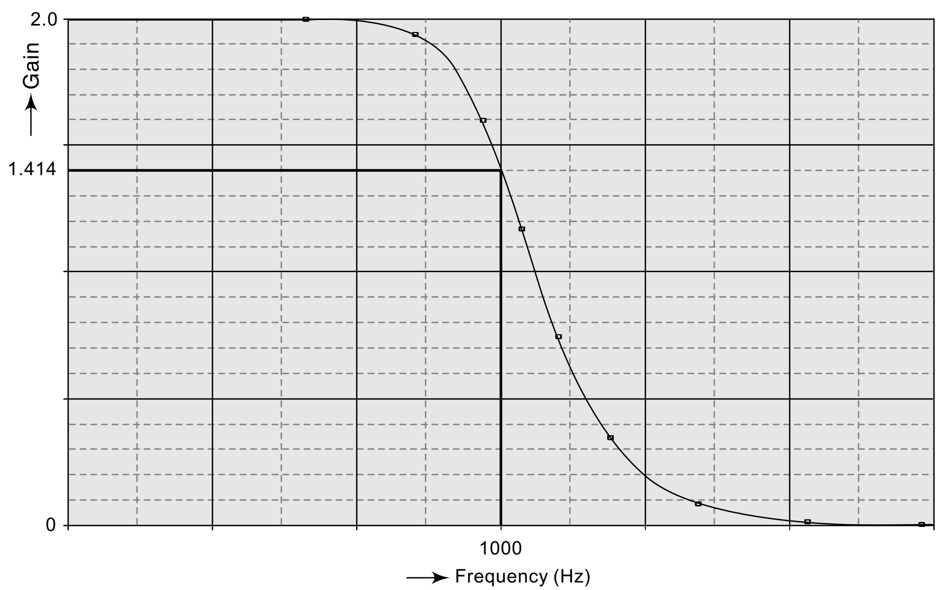


Figure 9.9(b) *Filter characteristics of Example 9.1*

9.7 HIGH-PASS ACTIVE FILTER

The circuit is shown in Fig. 9.10.

The filtering is done by the CR network and the op-amp is connected as a unity – gain follower. The feedback resistor, R_f is included to minimize dc off-set.

Here,

$$V_y = V_0 \frac{R_1}{R_1 + R_f} \tag{9.3}$$

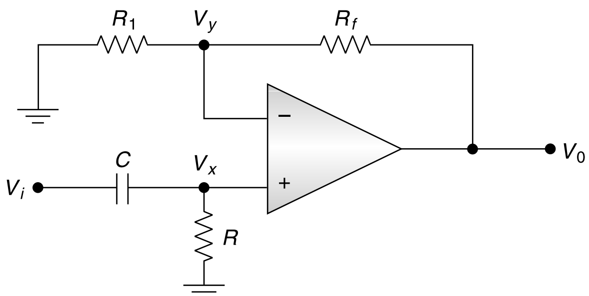


Figure 9.10 *First order high pass active filter circuit*

Voltage across the resistor R,

$$V_x = \frac{R}{R + X_c} V_i = \frac{R}{R + \dfrac{1}{j\omega C}} V_i = \frac{j\omega RC}{1 + j\omega RC} V_i \tag{9.4}$$

Since op-amp gain is infinite,

$$V_x = V_y$$

$$\Rightarrow \qquad \frac{V_0 R_1}{R_f + R_1} = \frac{j\omega RC}{1 + j\omega RC} V_i$$

$$\Rightarrow \qquad \frac{V_0}{V_i} = \left(\frac{R_f + R_1}{R_1}\right)\left(\frac{j\omega RC}{1 + j\omega RC}\right) = A_F \times \frac{j2\pi fRC}{1 + j2\pi fRC}$$

$$\boxed{\frac{\mathbf{V_0}}{\mathbf{V_i}} = \mathbf{A_F}\left[\frac{\mathbf{j(f/f_c)}}{\mathbf{1 + j(f/f_c)}}\right]}$$

where, $A_F = (1 + R_f/R_1)$ = Pass-band Gain of the filter,

f = frequency of the input signal (Hz),

$f_c = \dfrac{1}{2\pi RC}$ cut-off frequency of the filter (Hz).

The gain- magnitude,

$$\left|\frac{V_0}{V_i}\right| = \frac{A_F(f/f_c)}{\sqrt{1 + (f/f_c)^2}} = A_F \cdot \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}}$$

and phase-angle (in degree), $\phi = 90° - \tan^{-1}(f/f_c) = 90° - \tan^{-1}(\omega RC)$

9.7.1 Operation of the Filter

The operation of the high-pass filter can be verified from the gain–magnitude equation as follows:

1. At very low frequencies, i.e., $f < f_c$, $\left|\dfrac{V_0}{V_i}\right| < A_F$

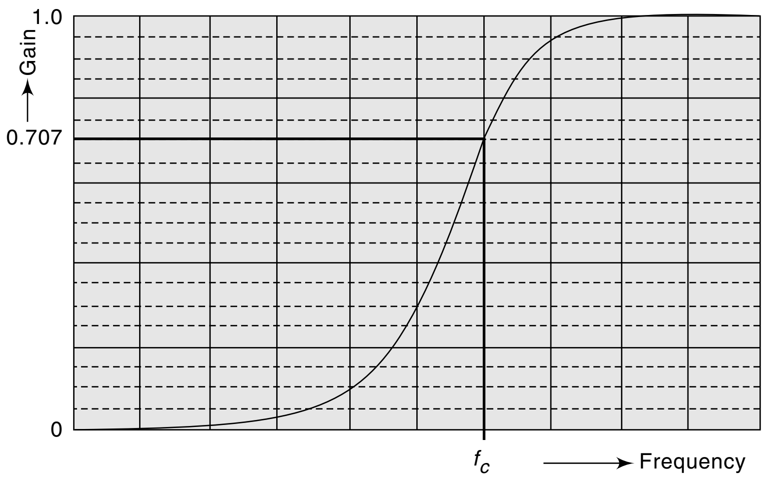


Figure 9.11 *High pass filter characteristics*

2. At $f = f_c$, $\left|\dfrac{V_0}{V_i}\right| = \dfrac{A_F}{\sqrt{2}} = 0.707\ A_F = -3$ dB, $\phi = 45°$
3. At $f >> f_c$, $\left|\dfrac{V_0}{V_i}\right| \cong A_F$

9.7.2 Filter Design

A high-pass active filter can be designed by implementing the following steps:

1. A value of the cut-off frequency, ω_c (or f_c) is chosen.
2. A value of the capacitance C, usually between 0.001 and 0.1 μF, is selected.
3. The value of the resistance R is calculated using the relation,

$$R = \frac{1}{\omega_c C} = \frac{1}{2\pi f_c C}$$

4. Finally, the values of R_1 and R_f are selected depending on the desired pass-band gain, using, the relation, $A_F = \left(1 + \dfrac{R_f}{R_1}\right)$.

Example 9.2 (a) Design a high-pass active filter of cut-off frequency 1 kHz with a pass-band gain of 2.
(b) Plot the frequency response of the filter.

Solution (a) Here, $f_c = 1\text{kHz}$, $A_F = 2$

Let, C = 0.01 μF.

$$\therefore \quad R = \frac{1}{2\pi f_c C} = \frac{1}{2\pi \times 10^3 \times 0.01 \times 10^{-6}} = 15.9\ \text{k}\Omega$$

$$\therefore \quad A_F = 2 = \left(1 + \frac{R_f}{R_1}\right) \Rightarrow R_f = R_1 = 10\ \text{k}\Omega$$

So, the complete circuit is shown in Fig. 9.12(a).

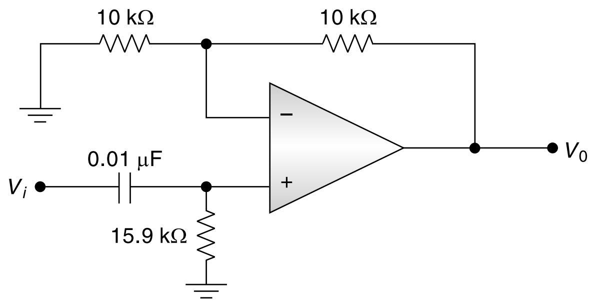


Figure 9.12(a) *Circuit of Example (9.2)*

(b) The data for the frequency response plot can be obtained by substituting the input frequency (f) values from 100 Hz to 100 kHz in the equation.

$$\frac{V_0}{V_i} = \frac{A_F(f/f_c)}{\sqrt{1+(f/f_c)^2}}$$

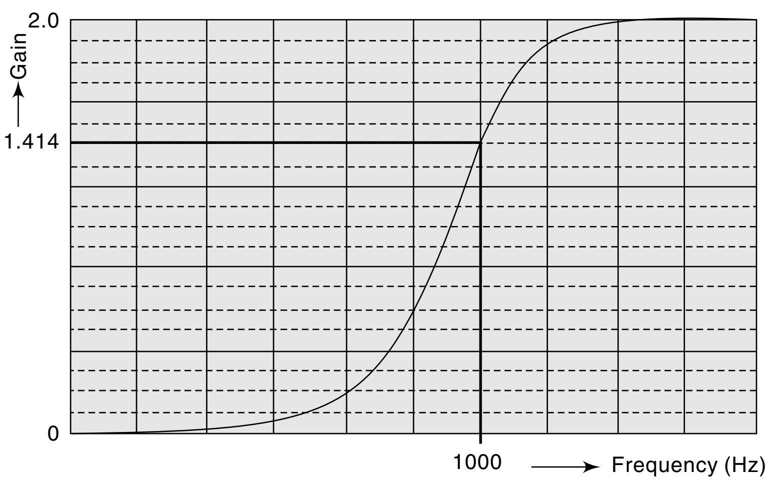


Figure 9.12(b) *Filter characteristics of Example (9.2)*

Frequency (Hz)	*Gain*	*Gain (in dB)*
100	0.20	–14.02
200	0.39	–9.13
400	0.74	–2.58
700	1.15	1.19
1,000	1.41	3.01
3,000	1.90	5.56
7,000	1.98	5.93
10,000	1.99	5.98
30,000	2	6.02
100,000	2	6.02

9.8 BAND-PASS ACTIVE FILTER

A band-pass filter has a pass-band between two cut-off frequencies f_{ce} (lower cut-off frequency) and f_{cu} (upper cut-off frequency) such that $f_{cu} > f_{cl.}$ Any input frequency outside this pass-band is attenuated.

9.8.1 Bandwidth (BW)

The range of frequency between f_{CL} and f_{CU} is called the bandwidth.

$$\mathrm{BW} = (f_{CU} - f_{CL})$$

The bandwidth is not exactly centered on the resonant frequency (f_r).
If f_{CU} and f_{CL} are known, the resonant frequency can be found from,

$$f_r = \sqrt{f_{CL} \cdot f_{CU}}$$

If 'f_r' and BW are known, cut-off frequencies are found from,

$$f_{cl} = \left(\sqrt{\left(\frac{\mathrm{BW}}{2}\right)^2 - f_r^2}\right) - \left(\frac{\mathrm{BW}}{2}\right)$$

$$f_{cu} = (f_{cl} + \mathrm{BW})$$

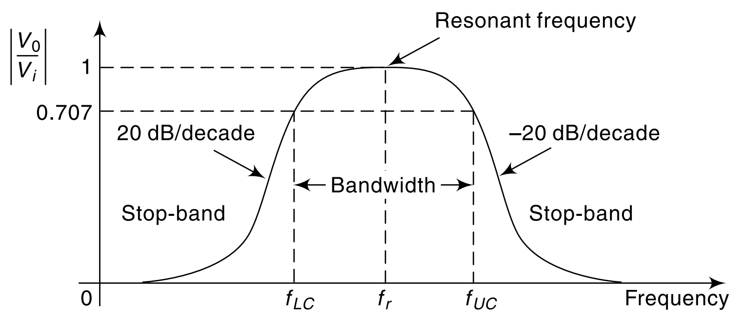


Figure 9.13 *Band pass filter characteristics*

9.8.2 Quality Factor (Q)

It is defined as the ratio of resonant frequency to bandwidth, i.e., $Q = \dfrac{f_r}{\text{BW}}$

Q is a measure of the selectivity. Higher the value of Q, the more selective is the filter, i.e., narrower is the bandwidth.

Example 9.3 A band-pass voice filter has lower and upper cut-off frequencies of 300 and 3000 Hz, respectively. Find (a) Bandwidth, (b) The resonant frequency, (c) The quality factor.

Solution (a) $\text{BW} = (f_{CU} - f_{CL}) = (3000 - 300) = 2700$ Hz *Ans.*

(b) $f_r = \sqrt{f_{CL} f_{CU}} = \sqrt{300 \times 3000} = 950$ Hz *Ans.*

(c) $Q = \dfrac{f_r}{\text{BW}} = \dfrac{950}{2700} = 0.35$ *Ans.*

[**Note** f_r is below the centre frequency $\dfrac{300 + 3000}{2} = 1650$ Hz]

Example 9.4 A band-pass filter has a resonant frequency of 950 Hz and a bandwidth of 2700 Hz. Find its lower and upper cut-off frequencies.

Solution

$$f_{CL} = \left(\sqrt{\left(\frac{\text{BW}}{2}\right)^2 + f_r^{\,2}}\right) - \left(\frac{\text{BW}}{2}\right)$$

$$= \left(\sqrt{\left(\frac{2700}{2}\right)^2 + (950)^2}\right) - \left(\frac{3700}{2}\right) = (1650 - 1350)$$

$= 300$ Hz *Ans.*

$\therefore \quad f_{cu} = (300 + 2700) = 3000$ Hz

9.8.3 Types of Band-Pass Filters

1. **Wide Band-Pass Filter** wide-band filter has a bandwidth that is two or more times the resonant frequency; i.e., $Q \le 0.5$.
   It is made by cascading a low-pass and a high-pass filter circuit.
2. **Narrow Band-Pass Filter** A narrow band filter has a quality factor, $Q > 0.5$.
   It is made by using a single op-amp and multiple feed back circuits.

Wide Band-Pass Active Filter In general, a wide-band filter ($Q \le 0.5$) is made by cascading a low-and a high-pass filter, provided the cut-off frequency of the low-pass section is greater than that for the high-pass section.

Characteristics

(i) The cut-off frequency of low-pass filter should be 10 or more times the cut-off frequency of the high-pass filter.
(ii) Each section should have the same pass band gain.
(iii) The lower cut-off frequency, f_{cl}, will be determined only by the high-pass filter.
(iv) The higher cut-off frequency, f_{cu}, will be determined only by the low-pass filter.
(v) Gain will be maximum at the resonant frequency, f_r, and equal to the pass-band gain of either filter.

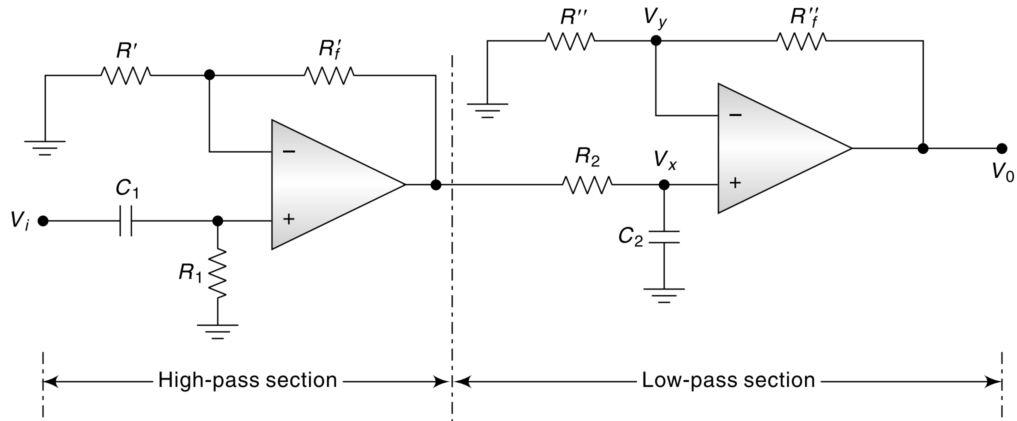


Figure 9.14(a) *Wide band-pass active filter circuit*

Frequency Response

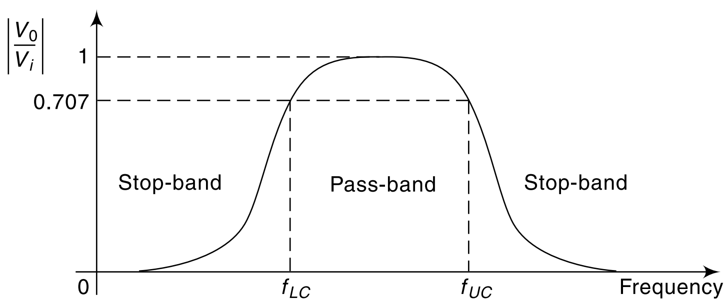


Figure 9.14(b) *Frequency response of wide band-pass active filter circuit*

Here, $f_{CL} = \dfrac{1}{2\pi R_1 C_1}$, $f_{CU} = \dfrac{1}{2\pi R_2 C_2}$

The voltage gain magnitude of the band–pass filter is equal to the product of the voltage gain magnitudes of the high-pass and the low-pass filters.

$$\boxed{\therefore \quad \left|\frac{V_0}{V_i}\right| = \frac{A_{FL} A_{FH} (f/f_{CL})}{\sqrt{[1+(f/f_{CL})^2]} \cdot \sqrt{[1+(f/f_{CU})^2]}}}$$

Where, A_{FL}, A_{FH} = Pass-band gain of low-pass and high-pass filter;
f = frequency of input signal (Hz);
f_{cl} = lower cut-off frequency (Hz);
f_{cu} = higher cut-off frequency (Hz);

At the centre frequency, $f_r\left(=\sqrt{f_{CL}f_{CU}}\right)$, the Gain is, $\left|\frac{V_0}{V_i}\right| = K = A_{FL}A_{FH}\frac{f_{CU}}{f_{CL}+f_{CU}}$

At $f = f_{CL}$,
$$\left|\frac{V_0}{V_i}\right| = \frac{A_{FL}A_{FH}(f_{CL}/f_{CL})}{\sqrt{[1+(f_{CL}/f_{CL})^2][1+(f_{CL}/f_{CU})^2]}} = \frac{A_{FL}A_{FH}}{\sqrt{(2)[1+(f_{CL}/f_{CU})^2]}}$$

$$\left|\frac{V_0}{V_i}\right| = \frac{A_{FL}A_{FH}f_{CU}}{\sqrt{2}\sqrt{f_{CL}^2+f_{CU}^2}}$$

At $f = f_{CU}$,
$$\left|\frac{V_0}{V_i}\right| = \frac{A_{FL}A_{FH}(f_{CU}/f_{CL})}{\sqrt{(2)[1+(f_{CL}/f_{CL})^2]}} = \frac{A_{FL}A_{FH}f_{CU}}{\sqrt{2}\sqrt{f_{CL}^2+f_{CU}^2}}$$

At $f = f_{CL} = f_{CU}$, Gain,
$$\left|\frac{V_0}{V_i}\right| = \frac{A_{FL}A_{FH}}{\sqrt{2}}\left[\frac{f_{CU}}{\sqrt{f_{CL}^2+f_{CU}^2}}\right]$$

$$\Rightarrow \quad \frac{V_0}{V_i} = \frac{A_{FL}A_{FH}}{2}$$

Narrow Band-pass Active Filter In general, a narrow band-pass filter is made by using multiple feedback circuit with a single op-amp.

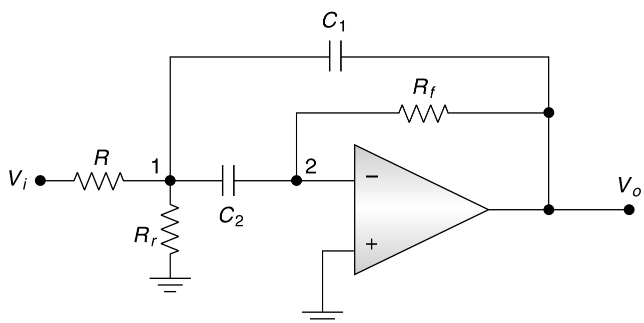


Figure 9.15 *Multiple feedback narrow BP active filter*

Compared to all other filters, it has some unique features, as given below.

(i) It has two feedback paths, hence the name 'multiple feedback filter'.
(ii) The op-amp is used in the inverting mode.
(iii) Its centre frequency can be changed without changing the gain or bandwidth.

Performance Equations

Writing KCL at (1)

$$\frac{(V_1 - V_i)}{R} + \frac{V_1 - V_0}{1/sC_1} + \frac{V_1 - 0}{1/sC_2} + \frac{V_1}{R_r} = 0$$

or $\quad (V_1 - V_i)\, R_r + (V_1 - V_0)\, sR_rRC_1 + V_1sRR_rC_2 + V_1R = 0$

or
$$V_1 = \frac{V_iR_r + V_0sRR_rC_1}{R + R_r + sRR_r(C_1 + C_2)} \tag{9.5}$$

Again, writing KCL at (2),

$$\frac{0 - V_0}{R_f} + \frac{0 - V_1}{1/sC_2} = 0$$

or $\quad V_0 = -V_1\, sR_f\, C_2$

$$= -\left[\frac{V_iR_r + V_0sRR_rC_1}{R + R_r + sRR_r(C_1 + C_2)}\right] sR_fC_2 \text{ \{by the value of } V_1 \text{ from (9.5)\}}$$

or $\quad V_0\, [R + R_r + sRR_r(C_1 + C_2) + s_2RR_rR_f\, C_1C_2] = -V_isR_rR_fC_2$

$$\therefore \quad \frac{V_0}{V_i} = -\frac{sR_rR_fC_2}{s^2RR_rR_fC_1C_2 + sRR_r(C_1 + C_2) + R + R_r}$$

So, the gain,

$$\boxed{\frac{V_0}{V_i} = -\frac{(s/RC_1)}{s^2 + s\left(\dfrac{C_1 + C_2}{R_fC_1C_2}\right) + \dfrac{R + R_r}{RR_rR_fC_1C_2}}}$$

The general transfer function is of the form,

$$\frac{V_0}{V_i} = -\frac{s\left(\dfrac{\omega_r}{Q}\right)}{s^2 + s\left(\dfrac{\omega_r}{Q}\right) + \omega_r^2} = -\frac{s(\text{BW})A_F}{s^2 + s(\text{BW}) + \omega_r^2}, \text{ where, } A_F = \text{Gain}$$

So, here, $\text{BW} = \left(\dfrac{C_1 + C_2}{R_fC_1C_2}\right) \times \dfrac{1}{2\pi} \text{(in Hz)} \;\; \{\because \omega = 2\pi f\}$

With matched capacitor, i.e., $C_1 = C_2 = C$

$$\text{BW} = \frac{1}{\pi R_fC} \quad \Rightarrow \quad R_f = \frac{Q}{\pi f_rC}$$

Also, $\quad (\text{BW})A_F = \dfrac{1}{RC_1} = \dfrac{1}{RC} \mid \text{with } C_1 = C_2 = C$

$\therefore$ $$R = \frac{1}{(BW)CA_F} = \frac{Q}{\omega_r CA_F} \Rightarrow R = \frac{Q}{2\pi f_r CA_F}$$

$$\text{BW} = \frac{1}{2\pi RCA_F}\ \text{Hz}$$

Similarly, $$\omega_r^2 = \frac{R+R_r}{RR_rR_fC^2} \Big|\ \text{with } C_1 = C_2 = C$$

or, $$4\pi^2 f_r^2 \times RR_rR_fC^2 = R + R_r$$

or $$4\pi^2 f_r^2 \times \frac{Q}{2\pi f_r CA_F} \times R_rR_fC^2 = \frac{Q}{2\pi f_r CA_F} + R_r \qquad \text{[Putting the value of } R_f]$$

or $$2\pi f_r QR_rR_fC = \frac{Q}{2\pi f_r C} + R_rA_F$$

or $$2\pi Qf_rR_r \times \frac{Q}{\pi f_r C} \times C = \frac{Q}{2\pi f_r C} + R_rA_F \qquad \text{[Putting the value of } R_f]$$

or $$R_r\left[2Q^2 - A_F\right] = \frac{Q}{2\pi f_r C}$$

$\therefore$ $$R_r = \frac{Q}{2\pi f_r C(2Q^2 - A_F)} = R\left(\frac{A_F}{2Q^2 - A_F}\right)$$

Also, $$\frac{R_f}{R} = \frac{Q}{\pi f_r C} \times \frac{2\pi frCA_F}{Q} = 2A_F$$

$\therefore$ $A_F = \dfrac{R_f}{2R}$. So, the gain is a maximum of 1 at f_r if $R_f = 2R$

However, the gain must satisfy the condition, $A_F < 2Q^2$.

So, the narrow-band-pass active filter is designed for specific values of resonant frequency f_r and Q (or, f_r and BW) by using the relations.

$$R = \frac{Q}{2\pi f_r CA_F}, \quad R_f = \frac{Q}{\pi f_r C}, \quad R_r = \frac{Q}{2\pi f_r C(2Q^2 - A_F)}, \quad A_F = \frac{R_f}{2R} \tag{9.6}$$

$$\text{BW} = \frac{f_r}{Q} = \frac{1}{2\pi RCA_F}(\text{Hz}) = \frac{0.1591}{A_F RC}(\text{H}_\text{Z}) \tag{9.7}$$

and $$f_r = \frac{1}{2\sqrt{2}RC}\sqrt{\frac{1}{A_F}\left(1 + \frac{R}{R_r}\right)} = \frac{0.1125}{RC}\sqrt{\frac{1}{A_F}\left(1 + \frac{R}{R_r}\right)} \tag{9.8}$$

Note: The resonant frequency can be changed to a frequency f_r' without changing the gain or BW, by, changing R_r to a new value R_r' = so that, $R_r' = R_r\left(\dfrac{f_r}{f_r'}\right)^2$.

Example 9.5 (a) Design a wide band-pass filter with f_{CL} = 200 Hz and f_{CU} = 1kHz, and a pass-band gain = 4.
(b) Draw the frequency response plot of this filter.
(c) Calculate the value of Q for the filter.

Solution (a) To design the low-pass section:

$$f_c = 1 \text{ kHz}$$

Let, C_2 = 0.01 μF, $R_2 = \dfrac{1}{2\pi f_c C_2} = 15.9\ \text{k}\Omega$

To design the high-pass section:

$$f_c = 200 \text{ Hz}$$

Let, C_1 = 0.05 μF, $R_1 = \dfrac{1}{2\pi f_c C_1} = 15.9\ \text{k}\Omega$

Since the band-pass gain is 4, the gain of both HP and LP sections could be set equal to 2.

$$\therefore \qquad 2 = \left(1 + \frac{R'_f}{R'}\right) = \left(1 + \frac{R''_f}{R''}\right) \quad \Rightarrow \quad R'_f = R''_f = R' = R'' = 10\ \text{k}\Omega$$

(b) The frequency response will be as shown below.

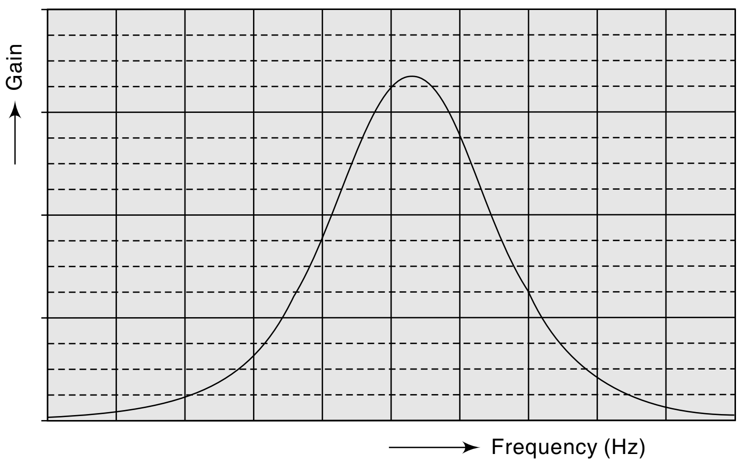


Figure 9.16 *Frequency response of Example (9.5)*

(c) Resonant frequency, $f_r = \sqrt{200 \times 1000} = 447.2\ \text{Hz}$

So, the quality factor, $Q = \dfrac{f_r}{\text{BW}} = \dfrac{447.2}{(1000 - 200)} = 0.56$

Example 9.6 (a) Design a narrow band-pass filter with resonant frequency f_r = 1 kHz, Q = 3, and A_F = 10.

(b) Change the resonant frequency to 1.5 kHz, keeping A_F and the bandwidth constant.

Solution (a) Let, $C_1 = C_2 = 0.01\ \mu\text{F}$

$$R_f = \frac{Q}{\pi f_r C} = \frac{3}{\pi \times 10^3 \times 10^{-8}} = 95.5\ \text{k}\Omega\ ;$$

$$R = \frac{Q}{2\pi f_r C A_F} = \frac{3}{2\pi \times 10^3 \times 10^{-8} \times 10} = 4.77\ \text{k}\Omega$$

$$R_r = \frac{Q}{2\pi f_r C(2Q^2 - A_F)} = \frac{4.77 \times 10}{(2 \times 9 - 10)} = 5.97\ \text{k}\Omega$$

(b) To change the resonant frequency, the resistance value will be,

$$R' = 5.97 \times 10^3 \times \left(\frac{1}{1.5}\right) = 3.98\ \text{k}\Omega$$

The frequency response is shown below.

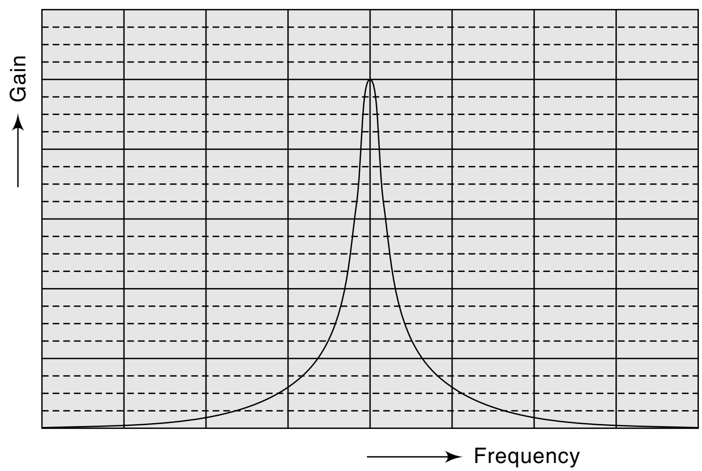


Figure 9.17 *Frequency response of Example (9.6)*

Example 9.7 A band-pass filter has the component values, R = 21.12 kΩ, R_f = 42.42 kΩ, R_r = 3.03 kΩ and $C_1 = C_2 = 0.015\ \mu\text{F}$. Find the resonant frequency and the bandwidth.

Solution Here, since $R_f = 2R$, so, $A_F = 1$.

$$\therefore \quad f_r = \frac{0.1125}{RC}\sqrt{\frac{1}{A_F}\left(1 + \frac{R}{R_r}\right)} = \frac{0.1125}{21.21 \times 10^3 \times 0.015 \times 10^{-6}}\sqrt{1 + \frac{21.21}{3.03}} \cong 1000\ \text{Hz}$$

$$BW = \frac{0.1591}{A_F RC} = \frac{0.1591}{1 \times 21.21 \times 10^3 \times 0.015 \times 10^{-6}} \cong 500 \text{ Hz}$$

9.9 BAND-REJECT (NOTCH) ACTIVE FILTER

1. It may be obtained by the parallel connection of a high-pass section with a low-pass section. The cut-off frequency of the high-pass section must be greater than that of the low-pass section.

   The outputs of HP and LP sections are fed to an adder whose output voltage V_0 will have the notch filter characteristics.

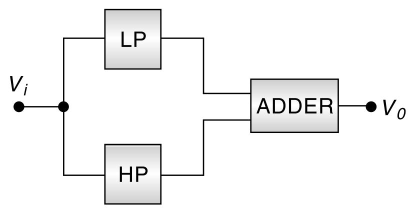


Figure 9.18(a) *Block diagram of BR filter*

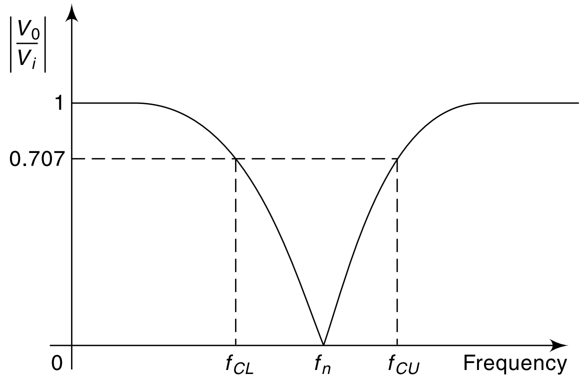


Figure 9.18(b) *Frequency response of band reject filter*

The circuit of the BR filter is shown in Fig. 9.19.

Obviously, the gain of the adder is set at unity; and thus,

$$V_0 = \left(\frac{V_0'}{R_2} + \frac{V_0''}{R_3}\right) R_4 \quad \Rightarrow \quad R_2 = R_3 = R_4$$

and $$R_{OM} = R_2 \| R_3 \| R_4$$

So, $$V_0 = A_{FH}\left[\frac{j(f/f_{CH})}{1 + j(f/f_{CH})}\right] + A_{FL}\left[\frac{1}{1 + j(f/f_{CL})}\right]$$

If $A_{FL} = A_{FH} = A$, then at the center frequency, $f_r = \sqrt{f_{CL} f_{CH}}$, the Gain is $K = A \cdot \frac{2 f_{CL}}{f_{CL} + f_{CH}}$

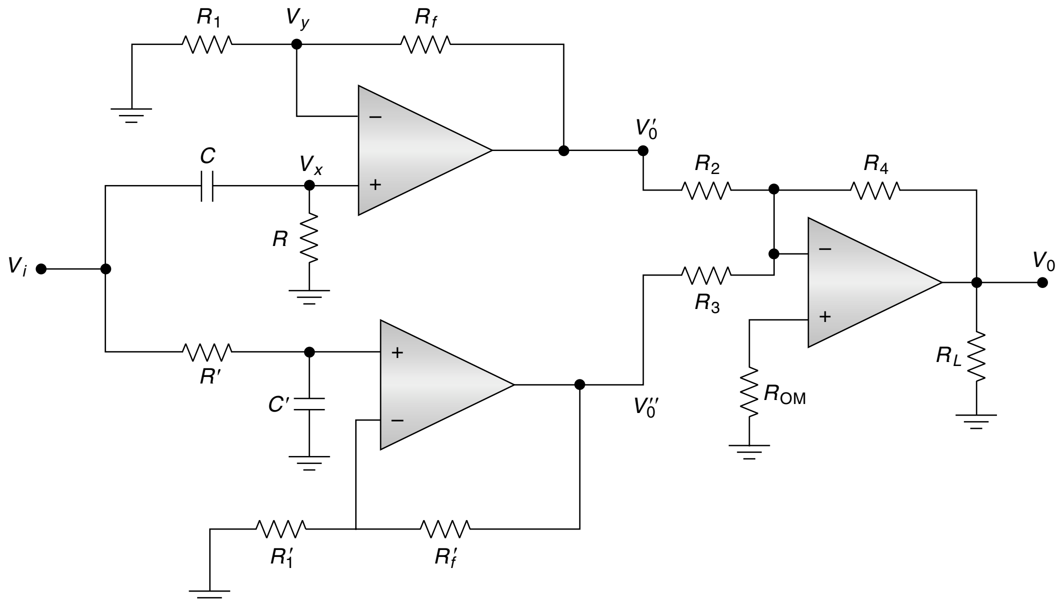


Figure 9.19 *Band reject active filter circuit using parallel connection of high pass and low pass filters*

2. Band-reject filter may also be obtained by using the multiple-feedback band-pass filter circuit with an adder. That is, the notch filter is made by a circuit that subtracts the output of a band pass filter from the original signal.

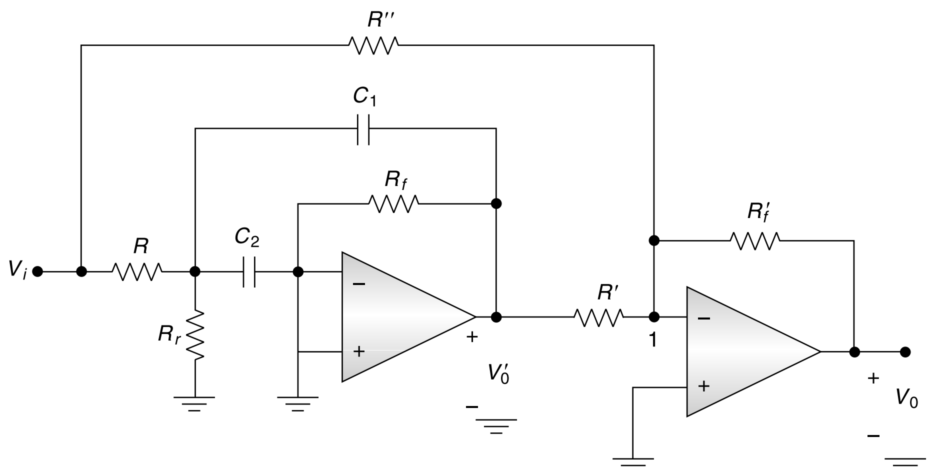


Figure 9.20 *Band reject active filter circuit using multiple feedback band pass filter with an adder*

So,
$$\frac{V_0'}{V_i} = -\frac{(s/RC_1)}{s^2 + s\left(\dfrac{C_1 + C_2}{R_f C_1 C_2}\right) + \dfrac{R + R_r}{RR_r R_f C_1 C_2}} = T(s)$$

Now, writing KCL at (1),

$$\frac{V_0'}{R'} + \frac{V_0}{R_f'} + \frac{V_i}{R''} = 0$$

$$\Rightarrow \qquad V_0 = -R_f'\left(\frac{V_1'}{R''} + \frac{V_0'}{R'}\right)$$

$$= -R_f' V_i\left[\frac{1}{R''} + \frac{T(s)}{R'}\right]$$

At notch frequency, the output is zero (ideally).

So, $\qquad T(s) = -\dfrac{R'}{R''}$

But, at ω_n (or f_n), $T(s) = -A_F$ (A_F = gain of the BP section)

With $C_1 = C_2$, Gain for BP section, $A_F = \dfrac{R_f}{2R}$

$$\boxed{\therefore A_F = \frac{R_f}{2R} = \frac{R'}{R''}}$$

So, the design equations are all those of BP section and this one.

Example 9.8 Design a notch filter having a resonant frequency, f_r = 400 Hz and Q = 10. Make the resonant frequency gain, A_F = 2.

Solution Here, f_r = 400 Hz, Q =10, A_F = 2

Let, C = 0.1 μF

$$\therefore \qquad R = \frac{Q}{2\pi frCA_F} = \frac{10}{2\pi \times 400 \times 0.1 \times 10^{-6} \times 2} = 19.89 \text{ k}\Omega \qquad \textit{Ans.}$$

$$\therefore \qquad R_f = \frac{Q}{\pi frC} = \frac{10}{\pi \times 400 \times 0.1 \times 10^{-6}} = 79.58 \text{ k}\Omega$$

$$\therefore \qquad R_r = \frac{RA_F}{2Q^2 - A_F} = \frac{19.89 \times 2 \times 10^3}{200 - 2} = 202\ \Omega$$

Let, R' = 1 kΩ (arbitrary) = R_f'

$$R'' = \frac{R'}{A_F} = 500\ \Omega \qquad \textit{Ans.}$$

9.9.1 Applications of Notch Filters

Notch filter is used where unwanted frequencies are to be attenuated while permitting the other signal frequencies to pass through.

For examples, 50 Hz, 60 Hz, or 400 Hz frequencies from power lines, ripple from a full-wave rectifiers, etc.

Example 9.9 Design an active notch filter to eliminate 120 Hz hum (noise). Take the bandwidth, BW = 12 Hz.

Solution Hare, f_r = 120 Hz, BW = 12Hz, $Q = \dfrac{120}{12} = 10$

The gain of the filter in the pass-band will be maximum of 1,

$AF = 1.$

Let , $C_1 = C_2 = 0.1\ \mu\text{F}$

$$R = \frac{10}{2\pi \times 120 \times 0.1 \times 10^{-6} \times 1} = 132.66\ \text{k}\Omega$$

$$R_f = 2R = 265.32\ \text{k}\Omega$$

$$R_r = \frac{R}{200 - 1} = 663.3\ \text{k}\Omega$$

Now, let $R' = R'_f = 1\ \text{k}\Omega$ (arbitrary)

So, $$R'' = \frac{R'}{A_F} = 1\ \text{k}\Omega$$

Thus the filter will pass all frequencies from (0 – 114) Hz and 126 Hz onwards.

9.10 FILTER APPROXIMATION

In the earlier sections, we saw several examples of amplitude response curves for various filter types. These always included an “ideal” curve with a rectangular shape, indicating that the boundary between the pass-band and the stop-band was abrupt and that the roll-off slope was infinitely steep. This type of response would be ideal because it would allow us to completely separate signals at different frequencies from one another. Unfortunately, such an amplitude response curve is not physically realizable. We will have to settle for the best approximation that will still meet our requirements for a given application. Deciding on the best approximation involves making a compromise between various properties of the filter’s transfer function, such as, filter order, ultimate roll-off rate, attenuation rate near the cut-off frequency, transient response, ripples, etc.

If we can define our filter requirements in terms of these parameters, we will be able to design an acceptable filter using standard design methods.

9.10.1 Butterworth Filters

The first and probably best-known filter approximation is the Butterworth or maximally-flat response. It exhibits a nearly flat pass-band with no ripple. The roll-off is smooth and monotonic, with a low-pass or high-pass roll-off rate of 20 dB/decade (6 dB/octave) for every pole. Thus, a 5^{th}-order

Butterworth low-pass filter would have an attenuation rate of 100 dB for every factor of ten increase in frequency beyond the cutoff frequency.

The general equation for a Butterworth filter's amplitude response is,

$$H(\omega) = \frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^{2n}} \tag{9.9}$$

where n is the order of the filter, and can be any positive whole number (1, 2, 3,…), and ω_0 is the -3 dB frequency of the filter.

Figure 9.21 shows the amplitude response curves for Butterworth low-pass filters of various orders.

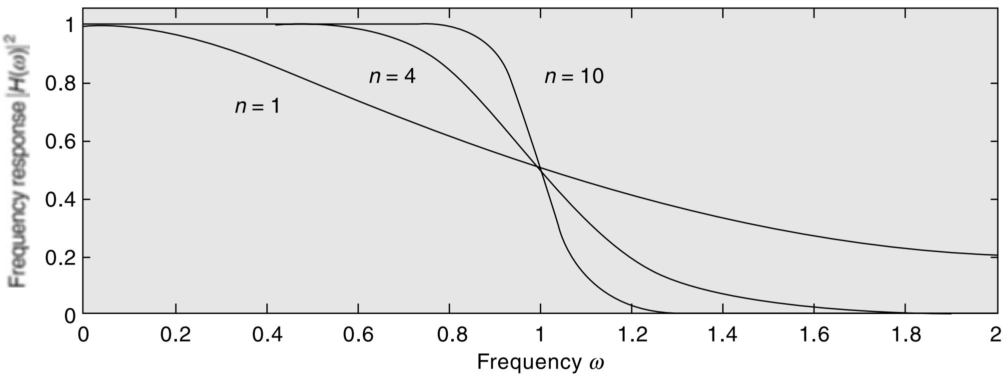


Figure 9.21 *Amplitude response curves for butterworth low-pass filters of different orders*

The coefficients for the denominators of Butterworth filters of various orders are shown in table. Table shows the denominators factored in terms of second-order polynomials. Again, all of the coefficients correspond to a corner frequency of 1 radian/s

Table 9.1 *Butterworth Polynomials*

Denominator coefficients for polynomials of the form $s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0$

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1	1									
2	1	1.414								
3	1	2.000	2.000							
4	1	2.613	3.414	2.613						
5	1	3.236	5.236	5.236	3.236					
6	1	3.864	7.464	9.142	7.464	3.864				
7	1	4.494	10.098	14.592	14.592	10.098	4.494			
8	1	5.126	13.137	21.846	25.688	21.846	13.137	5.126		
9	1	5.759	16.582	31.163	41.986	41.986	31.163	16.582	5.759	
10	1	6.392	20.432	42.802	64.882	74.233	64.882	42.802	20.432	6.392

Butterworth Quadratic Factors

n	
1	$(s + 1)$
2	$(s^2 + 1.4142s + 1)$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)$
5	$(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$
6	$(s^2 + 0.5176s + 1)(s^2 + 1.4142s + 1)(s^2 + 1.9319)$
7	$(s + 1)(s^2 + 0.4450s + 1)(s^2 + 1.2470s + 1)(s^2 + 1.8019s + 1)$
8	$(s^2 + 0.3902s + 1)(s^2 + 1.1111s + 1)(s^2 + 1.6629s + 1)(s^2 + 1.9616s + 1)$
9	$(s + 1)(s^2 + 0.3479s + 1)(s^2 + 1.0000s + 1)(s^2 + 1.5321s + 1)(s^2 + 1.8794s + 1)$
10	$(s^2 + 0.3129s + 1)(s^2 + 0.9080s + 1)(s^2 + 1.4142s + 1)(s^2 + 1.7820s + 1)(s^2 + 1.9754s + 1)$

9.10.2 Second Order Low-pass Active Filter

The circuit is shown in Figure 9.22.

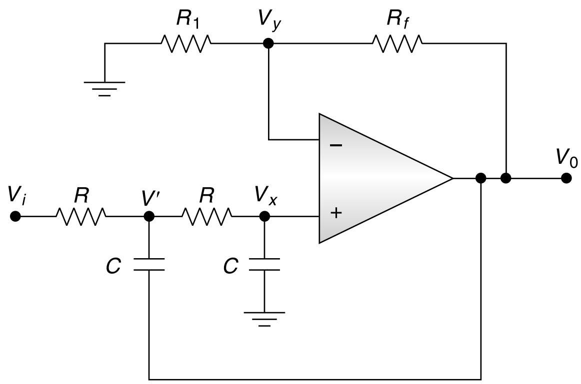


Figure 9.22 *Second order low-pass active filter*

Here, $V_y = \dfrac{V_0}{R_1 + R_f} R_1$ and $V_x = V_y$

Writing KCL at node V',

$$\frac{V' - V_i}{R} + \frac{V' - V_0}{1/sC} + \frac{V' - V_x}{R} = 0$$

or $$(V' - V_i) + (V' - V_0)sRC + (V' - V_x) = 0$$

or $$(-1)V_x + (2 + sRC)V' + (-sRC)V_0 = V_i \tag{9.10}$$

Writing KCL at node x,

$$\frac{V_x - V'}{R} + \frac{V_x}{1/sC} = 0$$

or $$(1 + sRC)V_x + (-1)V' + (0)V_0 = 0 \tag{9.11}$$

Writing KCL at node y,

$$\frac{V_x}{R_1}+\frac{V_x-V_0}{R_f}=0$$

or $$(R_1+R_f)V_x+(0)V'+(-R_1)V_0=0 \tag{9.12}$$

Solving for V_0 from equations (9.10), (9.11), and (9.12), we get,

$$V_0=\frac{\begin{vmatrix}-1 & (2+sRC) & V_i\\ (1+sRC) & -1 & 0\\ (R_1+R_f) & 0 & 0\end{vmatrix}}{\begin{vmatrix}-1 & (2+sRC) & -sRC\\ (1+sRC) & -1 & 0\\ (R_1+R_f) & 0 & -R_1\end{vmatrix}}=V_i\frac{\dfrac{(R_1+R_f)}{R_1}\times\dfrac{1}{R^2C^2}}{s^2+3sRC-sRC\left(\dfrac{(R_1+R_f)}{R_1}\right)+\dfrac{1}{R^2C^2}}$$

or, $$\frac{V_0(s)}{V_i(s)}=\frac{\dfrac{K}{R^2C^2}}{s^2+s\left(\dfrac{3-K}{RC}\right)+\left(\dfrac{1}{RC}\right)^2} \tag{9.13}$$

where, $K=\dfrac{R_1+R_f}{R_1}$ = DC gain of the amplifier.

Substituting $s=j\omega$, the transfer function is,

$$H(j\omega)=\frac{V_0(j\omega)}{V_i(j\omega)}=\frac{K}{1+j(3-K)RC\omega-R^2C^2\omega^2}$$

The magnitude of the transfer function is,

$$|H(j\omega)|=\frac{K}{\sqrt{\left[1-\left(\dfrac{\omega}{\omega_c}\right)^2\right]^2+[3-K]^2\left(\dfrac{\omega}{\omega_c}\right)^2}}\ ;\ \text{where, }\ \omega_c=\frac{1}{RC}$$

In the above equation, when $\omega\to 0$, $|H(j\omega)|=K$. Thus, the low frequency gain of the filter is K and when $\omega\to\infty$, $|H(j\omega)|=0$, i.e., high frequency gain is zero.

From the Table of the Butterworth Filter, the transfer function for second order ($n = 2$) filter is,

$$T(s)=\frac{K}{\left(\dfrac{s}{\omega_c}\right)^2+1.414\left(\dfrac{s}{\omega_c}\right)+1}=\frac{K\omega_c^2}{s^2+1.414\omega_c s+\omega_c^2} \tag{9.14}$$

where, ω_c is the cut-off frequency. Comparing equations (9.13) and (9.14), we get,

$$\boxed{\omega_c=\frac{1}{RC}\quad\text{or,}\quad f_c=\frac{1}{2\pi RC}}\quad\text{and,}\quad\boxed{K=(3-1.414)=1.586}$$

The frequency response of a second order low-pass active filter is shown in Figure 9.23. It is noted that the filter has very sharp roll-off response.

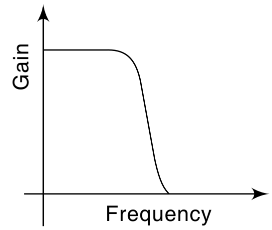


Figure 9.23 *Frequency response of the second order low-pass filter*

Filter Design

1. Choose a value of the cut-off frequency, ω_c (or f_c).
2. Select a convenient value for the capacitors C, between 100 pF and 0.1 μF.
3. Calculate the value of the resistors R from the relation,

$$R = \frac{1}{2\pi f_c C}$$

4. For minimization of dc offset, the feedback resistor is calculated from the relation, $R_f = K\ (2R) = 3.172R$.
5. Calculate the value of the resistor R_1 for the value of the gain $K = 1.586$ from the relation,

$$K = \frac{R_1 + R_f}{R_1}, \text{ i.e., } R_1 = \frac{R_f}{0.586}.$$

Example 9.10 Design a second-order low-pass filter with a gain of 11 and cut-off frequency of 20 kHz.

Solution Let us arbitrarily select $C = 200$ pF.

For a cut-off frequency of 20 kHz, we need $R = \dfrac{1}{2\pi f_c C} = \dfrac{1}{2\pi \times 20 \times 10^3 \times 200 \times 10^{-12}}$

$= 39.789$ kΩ

If we select a standard resistor of 39 kΩ for R, then the cut-off frequency is about 20.4 kHz.

The dc gain for this filter cannot be anything other than K where $K = 1.586$.

Thus, for a dc gain of 1.586, $K = 1 + R_f/R1 = 1.586$.

This in turn implies that $R_f = 0.586\ R_1$.

Imposing the dc bias-current balance condition, we obtain

$0.586\ R_1 = 1.586\ (2\ R) = 123.708$ kΩ.

Consequently, $R_1 = 211.11$ kΩ and $R_f = 123.708$ kΩ.

Let us select a standard value of 130 kΩ for R_f. Then R_1 should be about 221.8 kΩ. We need another amplifying stage to obtain the needed gain of 11. The gain of this stage should be $11/K = 6.936$. We have chosen to use non-inverting amplifier for this stage. The output amplifier resistors are calculated as,

$$6.936 = \left(1 + \frac{R_2}{R_A}\right) \text{ and for } R_A = 100 \text{ k}\Omega., R_2 = 593.6 \text{ k}\Omega.$$

Thus, the final circuit for the second order low-pass active filter becomes as shown in Fig. 9.24.

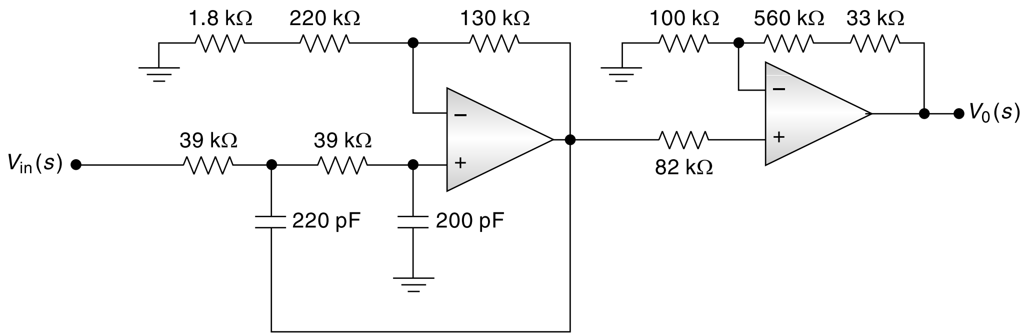


Figure 9.24 *Circuit of Example (9.10)*

9.10.3 Second Order High Pass Active Filter

The circuit is shown in Figure 9.25.

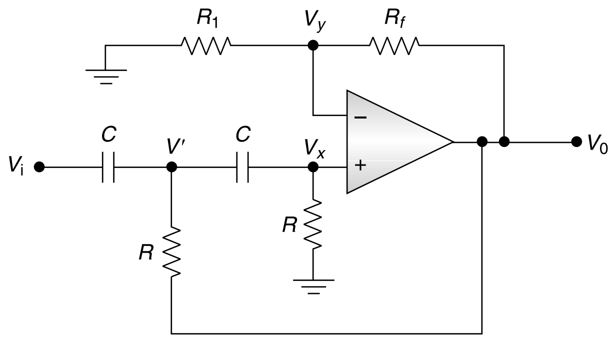


Figure 9.25 *Second order high-pass filter*

Here, $V_y = \frac{V_0}{R_1 + R_f} R_1$ and $V_x = V_y$

Writing KCL at node V',

$$\frac{V' - V_i}{1/sC} + \frac{V' - V_0}{R} + \frac{V' - V_x}{1/sC} = 0 \tag{9.15}$$

Writing KCL at node x,

$$\frac{V_x - V'}{1/sC} + \frac{V_x}{R} = 0 \tag{9.16}$$

Writing KCL at node y,

$$\frac{V_x}{R_1}+\frac{V_x-V_0}{R_f}=0 \tag{9.17}$$

Solving for V_0 from equations (9.15), (9.16), and (9.17), we get,

or,
$$\frac{V_0(s)}{V_i(s)}=\frac{Ks^2}{s^2+s\left(\frac{3-K}{RC}\right)+\left(\frac{1}{RC}\right)^2} \tag{9.18}$$

where, $K=\dfrac{R_1+R_f}{R_1}$ = DC gain of the amplifier.

Note The transfer function of the high-pass filter can also be obtained from the transfer function of the low-pass filter by the transformation $\left.\left(\dfrac{s}{\omega_c}\right)\right|_{LP}\to\left.\left(\dfrac{\omega_c}{s}\right)\right|_{HP}$

Substituting $s=j\omega$, the transfer function is,

$$H(j\omega)=\frac{V_0(j\omega)}{V_i(j\omega)}=-\frac{KR^2C^2\omega^2}{1+j(3-K)RC\omega-R^2C^2\omega^2}$$

The magnitude of the transfer function is,

$$|H(j\omega)|=\frac{K\left(\dfrac{\omega}{\omega_c}\right)^2}{\sqrt{\left[1-\left(\dfrac{\omega}{\omega_c}\right)^2\right]^2+[3-K]^2\left(\dfrac{\omega}{\omega_c}\right)^2}};\ \text{where, }\ \omega_c=\frac{1}{RC}$$

In the above equation, when $\omega\to 0$, $|H(j\omega)|=0$. Thus, the low frequency gain of the filter is zero. When $\omega\to\infty$, $|H(j\omega)|=K$, i.e., high frequency gain is K.

Here, again, comparing with Butterworth Transfer function, we get,

$$\boxed{\omega_c=\frac{1}{RC}\quad\text{or,}\quad f_c=\frac{1}{2\pi RC}}$$

$$\boxed{K=(3-1.414)=1.586}$$

The frequency response of a second order low-pass active filter is shown below. It is noted that the filter has very sharp roll-off response.

The design procedure for high-pass will be same as low-pass.

The frequency response will be a maximally flat one, i.e., having a very sharp roll-off response.

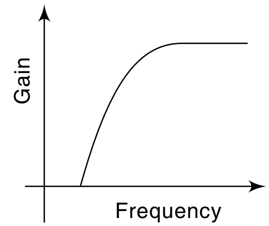


Figure 9.26 *Gain vs. frequency plot of a second-order high-pass filter*

Example 9.11 A second-order high-pass filter is given in Figure 9.27. Determine its cut-off frequency and high frequency gain. Sketch its gain vs. frequency response.

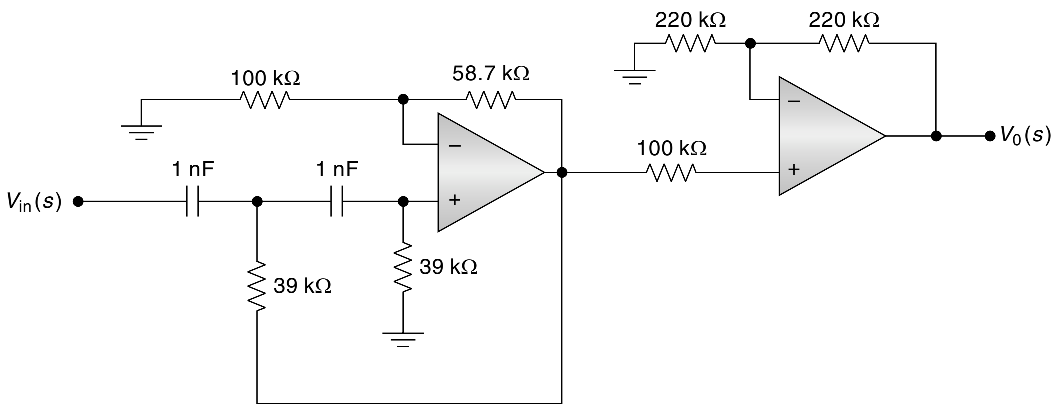


Figure 9.27 *Circuit of Example (9.11)*

Solution In the second-order filter on the left side of the figure, the gain ***K*** $= \left(1+\dfrac{58.7}{100}\right) = 1.587$.

Since it is very close to 1.586, we can assume that the filter is maximally flat and its transfer function is as given for Butterworth filters. From the given values of R and C, the cut-off frequency is,

$$\omega_c = \frac{1}{RC} = \frac{1}{39\times10^3\times1\times10^{-9}} = 25{,}641 \text{ rad/s}$$

The cut-off frequency in Hz, $f_c = \dfrac{25{,}641}{2\pi} = 4081$ Hz

The gain of the non-inverting amplifier, $A = \left(1+\dfrac{220}{220}\right) = 2$

Hence, the overall gain of the high-pass filter is,

$$A_H = 1.587 \times 2 = 3.174 \text{ or approximately 10 dB.}$$

The gain vs. frequency will be as shown in Figure 9.26.

9.10.4 Second Order Band-Pass Active Filter

It can be built by the cascade connection of a second order high-pass and a second order low-pass filter.

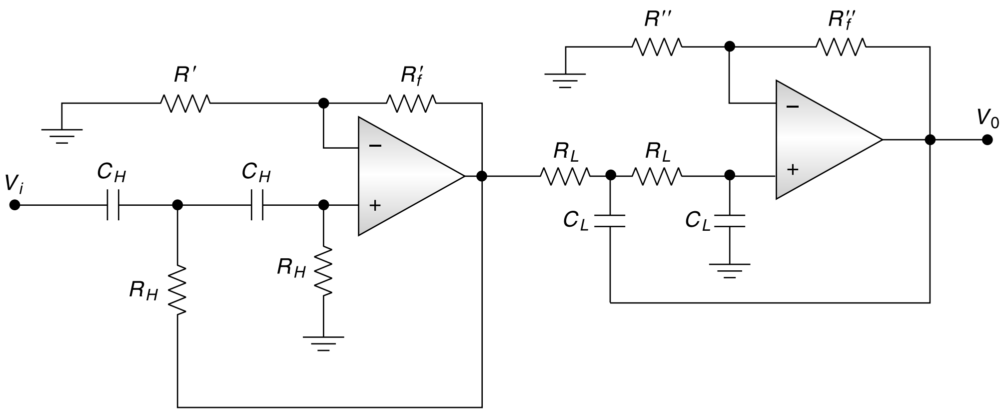


Figure 9.28 *Second-order band-pass active filter circuit*

Lower cut-off frequency, $\omega_1 = \dfrac{1}{R_H C_H}$

Upper cut-off frequency, $\omega_2 = \dfrac{1}{R_L C_L}$

Voltage gains, $K_H = \left[1 + \dfrac{R'_f}{R'}\right]$ and $K_L = \left[1 + \dfrac{R''_f}{R''}\right]$

For maximally flat response (or, Butterworth) filter, $K_H = K_L = 1.586$.

$$\therefore \qquad \frac{R'_f}{R'} = \frac{R''_f}{R''} = 0.586$$

The overall transfer function is the product of the transfer function of the high-pass and low-pass filters.

$$\therefore \qquad H(s) = \frac{K_H\left(\dfrac{s}{\omega_1}\right)}{1 + \left(\dfrac{s}{\omega_1}\right)^2 + (3 - K_H)\left(\dfrac{s}{\omega_1}\right)} \times \frac{K_L}{1 + \left(\dfrac{s}{\omega_2}\right)^2 + (3 - K_L)\left(\dfrac{s}{\omega_2}\right)}$$

Substituting the values of K_H and K_L, magnitude of the gain is,

$$|H(j\omega)| = \frac{2.5154\left(\dfrac{\omega}{\omega_1}\right)^2}{\sqrt{1 + \left(\dfrac{\omega}{\omega_2}\right)^4}\sqrt{1 + \left(\dfrac{\omega}{\omega_1}\right)^4}}$$

Note In the pass-band, the gain is 2.5154.
The frequency response is more flat near the cut-off frequencies.

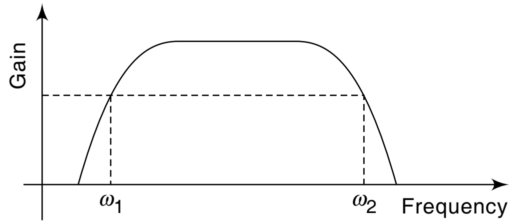


Figure 9.29 *Frequency response of second order band-pass filter*

9.10.5 Second Order Band-Reject Active Filter

It can be built by the summation of a second order high-pass and a second order low-pass filter.

The cut-off frequency of LPF, $\omega_1 = \dfrac{1}{R_L C_L}$ and the cut-off frequency of HPF, $\omega_2 = \dfrac{1}{R_H C_H}$.

The magnitude of the overall transfer function is the sum of the transfer function of the high-pass and low-pass filters,

$$|H(j\omega)| = \frac{1}{2}\left[1 + \frac{R_2}{R_1}\right]\left[\frac{K_H\left(\dfrac{\omega}{\omega_2}\right)^2}{\sqrt{1+\left(\dfrac{\omega}{\omega_2}\right)^4}} + \frac{K_L}{\sqrt{1+\left(\dfrac{\omega}{\omega_1}\right)^4}}\right]$$

where, $K_H = \left(1 + \dfrac{R'_f}{R'}\right)$ and, $K_L = \left(1 + \dfrac{R''_f}{R''}\right)$ and for Butterworth filters, $K_H = K_L = 1.586$.

The roll-off frequency response will be very smooth as shown.

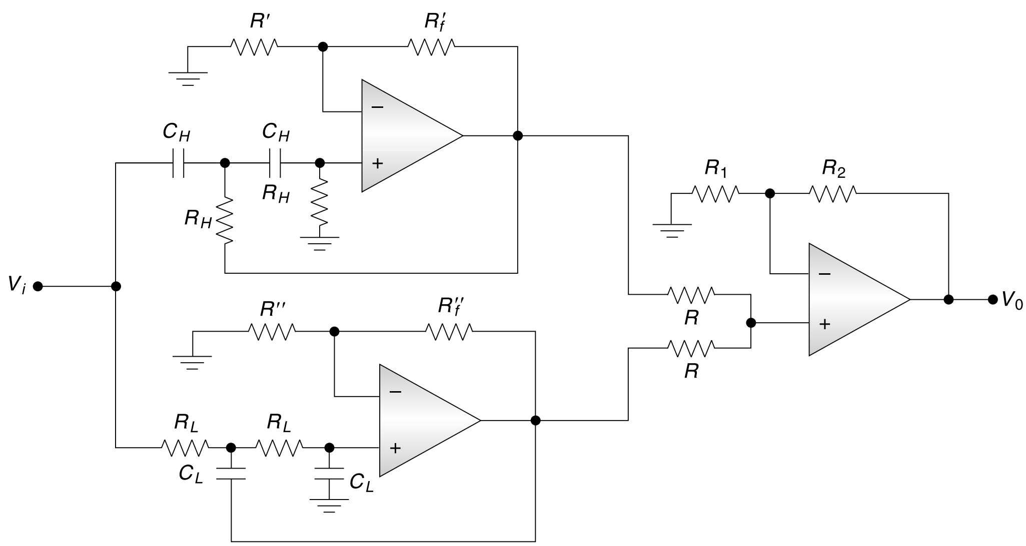


Figure 9.30 *Second-order band-reject active filter circuit*

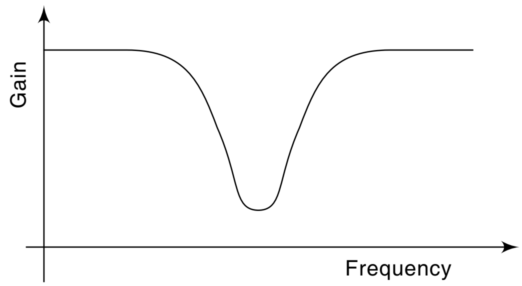


Figure 9.31 *Frequency response of second order band-reject filter*

9.11 ALL-PASS ACTIVE FILTER

This filter passes all frequency component of the input signal without attenuation and provides some phase shifts between the input and output signals.

The circuit of an active all-pass active filter with lagging output is shown in Figure 9.32.

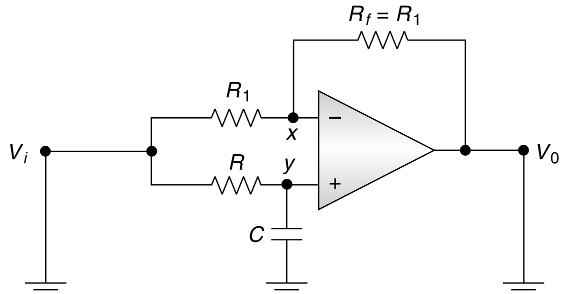


Figure 9.32 *Circuit of an all-pass active filter with lagging output*

For the circuit, by KCL at node x,

$$\frac{V_x - V_i}{R_1} + \frac{V_x - V_0}{R_1} = 0 \quad \Rightarrow \quad V_x = \frac{V_i + V_0}{2} \tag{9.19}$$

By KCL at node y,

$$\frac{V_y - V_i}{R} + \frac{V_y}{1/j\omega C} = 0 \quad \Rightarrow \quad V_y = \frac{V_i}{1 + j\omega RC} \tag{9.20}$$

Also, from Op-Amp property,

$$V_x = V_y$$

$$\Rightarrow \qquad \left(\frac{V_i + V_0}{2}\right) = \left(\frac{V_i}{1 + j\omega RC}\right)$$

$$\Rightarrow \qquad (V_i + V_0)(1 + j\omega RC) = 2V_i$$

$$\Rightarrow \qquad V_0(1 + j\omega RC) = V_i[2 - (1 + j\omega RC)] = V_i(1 - j\omega RC)$$

$$\boxed{\therefore \frac{V_0}{V_i} = \frac{1 - j\omega RC}{1 + j\omega RC}}$$

Thus, the amplitude of the gain,

$$\left|\frac{V_0}{V_i}\right| = 1 \quad \text{i.e.,} \quad |V_{\text{out}}| = |V_{\text{in}}| \text{ throughout the entire frequency range}$$

Also, the phase shift between the input and the output voltages is,

$\phi = -2 \tan^{-1} (\omega RC)$ i.e., phase-shift is a function of frequency

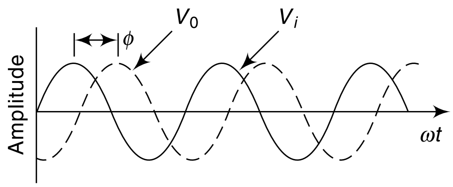


Figure 9.33 *Characteristics of all-pass filter*

By interchanging the positions of R and C in the circuit, the output can be made leading the input.

MULTIPLE-CHOICE QUESTIONS

9.1 The two input terminals of an op-amp are labeled as
(a) high and low (b) positive and negative
(c) inverting and non-inverting (d) differential and non-differential

9.2 Consider the following statements for an ideal op-amp.
1. The differential voltage across the input terminals is zero.
2. The current into the input terminals is zero.
3. The current from the output terminals is zero.
4. The input resistance is zero.
5. The output resistance is zero.

Of these statements, those which are not true are
(a) 1 and 5 (b) 3 and 4 (c) 2 and 4 (d) 1 and 4

9.3 In a series resonant circuit, to obtain a low-pass characteristic, across which element should the output voltage be taken?
(a) Resistor (b) Inductor (c) Capacitor

9.4 In a series resonant circuit, to obtain a high-pass characteristic, across which element should the output voltage be taken?
(a) Resistor (b) Inductor (c) Capacitor

9.5 In a series resonant circuit, to obtain a band-pass characteristic, across which element should the output voltage be taken?
(a) Resistor (b) Inductor (c) Capacitor

9.6 A high-pass filter circuit is basically
(a) a differentiating circuit with low time constant.
(b) a differentiating circuit with large time constant.
(c) an integrating circuit with low time constant.
(d) an integrating circuit with large time constant.

9.7 The transfer function of an electrical low-pass RC network is
(a) $\dfrac{RCs}{1+RCs}$ (b) $\dfrac{1}{1+RCs}$ (c) $\dfrac{RC}{1+RCs}$ (d) $\dfrac{s}{1+RCs}$

9.8 For a high-pass RC circuit, when subjected to a unit step input voltage, the voltage across the capacitor will be
(a) $1-e^{-t/RC}$ (b) $e^{-t/RC}$ (c) $e^{t/RC}$ (d) 1

9.9 In the magnitude plot of a low-pass filter, at what frequency does the peak of the magnitude characteristic occur?
(a) At resonant frequency (b) Below resonant frequency
(c) Above resonant frequency (d) At any frequency.

9.10 In the magnitude plot of a high-pass filter, at what frequency does the peak of the magnitude characteristic occur?
(a) At resonant frequency (b) Below resonant frequency
(c) Above resonant frequency (d) At any frequency.

9.11 In the magnitude plot of a band-pass filter, at what frequency does the peak of the magnitude characteristic occur?
(a) At resonant frequency (b) Below resonant frequency
(c) Above resonant frequency (d) At any frequency.

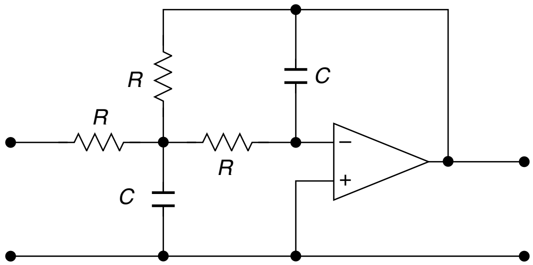
9.12 If a filter is de-normalized to a higher frequency, which of the following occurs?
(a) Inductors increase in value while capacitors decrease.
(b) Inductors decrease in value while capacitors increase.
(c) Inductors and capacitors increase in value.
(d) Inductors and capacitors decrease in value.

9.13 The transfer function $\dfrac{V_2(s)}{V_1(s)} = \dfrac{10s}{s^2+10s+100}$ is for an active
(a) low pass filter (b) band pass filter (c) high pass filter (d) all pass filter.

9.14 The transfer function $T(s) = \dfrac{s^2}{s^2+as+b}$ belongs to an active
(a) low pass filter (b) high pass filter (c) band pass filter (d) band reject filter.

9.15 The voltage-ratio transfer function of an active filter is given by $\dfrac{V_2(s)}{V_1(s)} = \dfrac{s^2+\delta}{s^2+\alpha s+\delta}$. The circuit in question is a
(a) low pass filter (b) high pass filter (c) band pass filter (d) band reject filter.

9.16



The transfer function of a second order LP filter shown in the figure is

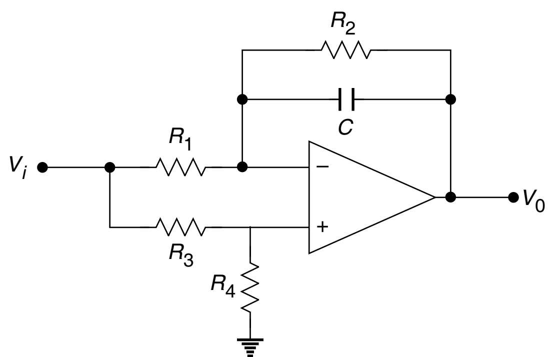
(a) $\dfrac{1}{R^2C^2s^2+3RCs+1}$ (b) $\dfrac{RCs}{R^2C^2s^2+3RCs+1}$

(c) $\dfrac{R^2C^2s^2+1}{R^2C^2s^2+3RCs+1}$ (d) $\dfrac{R^2C^2s^2}{R^2C^2s^2+3RCs+1}$

9.17 An ideal filter should have
(a) zero attenuation in the pass band
(b) infinite attenuation in the pass band
(c) zero attenuation in the attenuation band
(d) none of these.

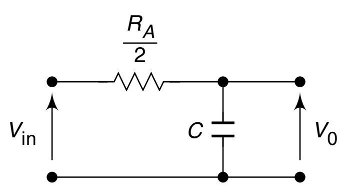
9.18 An *RLC* series circuit can act as
(a) band-pass filter (b) band-stop filter
(c) low-pass filter (d) both (a) and (b).

9.19 If $R_1 = R_2 = R_A$ and $R_3 = R_4 = R_B$, the circuit acts as a/an

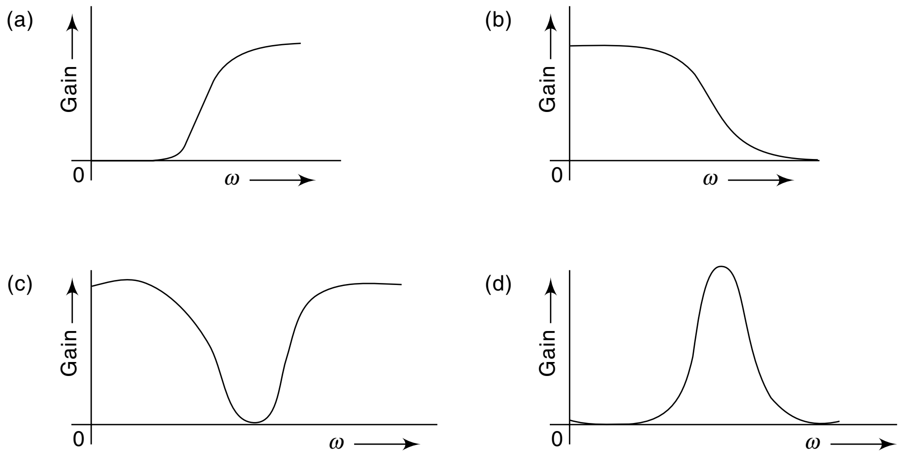


(a) all pass filter (b) band pass filter
(c) high pass filter (d) low pass filter

9.20 The output of the filter in 9.19 is given to the circuit shown in the figure.



The gain vs frequency characteristic of the output (v_0) will be



9.21 In active filter circuits, inductances are avoided mainly because they
(a) are always associated with some resistance
(b) are bulky and unstable for miniaturisation
(c) are non-linear in nature
(d) saturate quickly

9.22 The magnitude response of a normalized Butterworth low-pass filter is
(a) linear starting with values of unity at zero frequency and 0.707 at the cut-off frequency
(b) non-linear all through but with values of unity at zero frequency and 0.707 at the cut-off frequency
(c) linear up to the cut-off frequency and non-linear thereafter
(d) non-linear up to the cut-off frequency and linear thereafter

EXERCISES

9.1 Design a second order low pass active filter having a cut-off frequency of 5 kHz.
[$C = 0.03$ mF; $R = 1$ kΩ; $R_1 = 10$ kΩ; $R_2 = 5.86$ kΩ]

9.2 Design a second order band pass active filter that has a centre frequency of 1 kHz and a bandwidth of 100 Hz. Take the centre frequency gain to be 2.
[$C_1 = C_2 = 0.02$ mF; $R_1 = 40$ kΩ; $R_3 = 160$ kΩ; $R_2 = 400$ Ω]

9.3 Design a second order high pass Butterworth filter with a cut-off frequency of 200 Hz.
[$C = 0.053$ mF; $R = 1.5$ kΩ; $R_1 = 10$ kΩ; $R_2 = 5.86$ kΩ]

9.4 Design a second order band pass active filter with a centre frequency gain $A_0 = 50$. Given: $f_0 = 160$ Hz and $Q = 10$. [assuming $C_1 = C_2 = 0.1$ mF; $R_1 = 2$ kΩ; $R_3 = 200$ kΩ; $R_2 = 667$ Ω]

SHORT-ANSWER TYPE QUESTIONS

9.1 (a) What is an operational-amplifier? State the characteristics of an op-amp.
(b) What is filter? Classify them.
(c) Discuss the advantages of an active filter over a passive filter.

9.2 (a) Briefly discuss the operating principle of an active low-pass filter and derive its gain-frequency characteristics. Explain the design procedure of a low-pass active filter.
(b) Briefly discuss the operating principle of an active high-pass filter and derive its gain-frequency characteristics. Explain the design procedure of a high-pass active filter.

9.3 (a) Define the following terms with reference to a band-pass active filter: -
(i) Bandwidth,
(ii) Cut-off frequency,
(iii) Quality factor.
(b) What are the different types of band-pass filters? Give the salient features and performance equations for the following filters: -
(i) Wide Band-Pass Active Filter,
(ii) Narrow Band-Pass Active Filter.

9.4 Define Notch-frequency. Explain the operational characteristics of an active Notch filter. Where are these filters used?

ANSWERS TO MULTIPLE-CHOICE QUESTIONS

9.1 (c)	9.2 (b)	9.3 (c)	9.4 (b)	9.5 (a)	9.6 (a)	9.7 (b)
9.8 (a)	9.9 (b)	9.10 (c)	9.11 (a)	9.12 (d)	9.13 (c)	9.14 (b)
9.15 (c)	9.16 (a)	9.17 (a)	9.18 (a)	9.19 (c)	9.20 (d)	9.21 (b)
9.22 (b)						

CHAPTER 10
Resonance

10.1 INTRODUCTION

Any system having at least a pair of complex conjugate poles has a natural frequency of oscillation. If the frequency of the system driving force coincides with the natural frequency of oscillation, the system resonates and the system response becomes maximum. This phenomenon is known as '*resonance*' and the frequency at which this phenomenon occurs is known as '*resonant frequency*'.

In electrical systems, resonance occurs when the system contains at least one inductor and one capacitor. In this system, the phenomenon of cancellation of reactances when inductor and capacitor are in series or cancellation of susceptances when they are in parallel, is termed as *resonance*. The circuit under resonance is purely resistive in nature and is termed as '*resonant circuit*' or '*tuned circuit*'.

In this chapter, we consider electrical resonance in details. Electrical resonance is broadly classified into tow categories:

(a) Series Resonance, and
(b) Parallel Resonance.

10.2 SERIES RESONANCE OR VOLTAGE RESONANCE

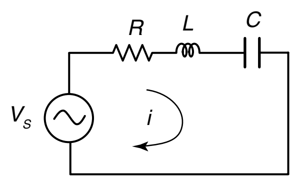


Figure 10.1 *Series* RLC *Resonant Circuit*

The basic series-resonant circuit is shown in Fig. 10.1. We want to observe how the steady state amplitude and the phase angle of the current vary with the frequency of the sinusoidal voltage source. As the frequency of the source changes, the maximum amplitude of the source voltage (V_m) is held constant.

Impedance due to capacitor: $Z_C = -\frac{j}{\omega C}$; clearly, as $\omega \to 0$, $Z_C \to \infty$ and $i \to 0$.

Impedance due to inductor: $Z_L = j\omega L$ clearly, as $\omega \to \infty$, $Z_L \to \infty$ and $i \to 0$.

Therefore, circuits containing inductors and capacitors have responses that are frequency dependent. We analyze in the following steps.

i. Current Response

Here, the supply voltage, $v_s = V_m \sin \omega t$ and the current is, $i = I_m \sin(\omega t + \phi)$. The phasor equivalents of v_s and i are **V** and **I**, respectively.

Using phasors, $$I = \frac{V}{Z} = \frac{V}{R + j\omega L - j/\omega C} = \frac{V}{R + j\left(\omega L - \frac{1}{\omega C}\right)} \qquad (10.1)$$

Thus, the current magnitude, $$|I| = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \qquad (10.2)$$

From this equation, we have the following observations:

- $|I| \to 0$ as $\omega \to 0$; and
- $|I| \to 0$ as $\omega \to \infty$.

This indicates that there must be a maximum value of the current $|I|$ for some particular value of ω. This occurs when the denominator is a minimum. i.e., when

$$\omega L = \frac{1}{\omega C} \Rightarrow \omega = \frac{1}{\sqrt{LC}} \qquad (10.3)$$

Thus, resonance occurs when the magnitudes of the inductive and capacitive reactances are equal. This frequency is termed as the *resonant frequency*, ω_0 of the series *RLC* circuit.

$$\boxed{\therefore \omega_0 = \frac{1}{\sqrt{LC}}}$$

From equation (10.1), the phase angle of the current is given by,

$$\phi = -\tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) \qquad (10.4)$$

From equation (10.4), it is clear that the phase angle for the current also depends on frequency. We have two observations:

- As $\omega \to 0$, $\phi \to \tan^{-1}\left(\dfrac{1}{\omega RC}\right)$ and in this case the current leads the voltage with the phase relationship being like that of an RC circuit.
- As $\omega \to \infty$, $\phi \to -\tan^{-1}\left(\dfrac{\omega L}{R}\right)$ and in this case the current lags the voltage with the phase relationship being like that of an RL circuit.
- At $\omega = \omega_0$, $\phi = 0$, and in this case the current and the voltage are in phase, the circuit behaving like a purely resistive circuit.

The current response and phasor diagrams are shown in Fig. 10.2.

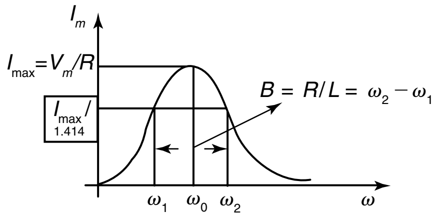


Figure 10.2 *Frequency Response of a Series – Resonant Circuit*

Phasor Diagrams The current and voltage phasor diagrams for an *RLC* series circuit are shown in Fig. 10.3.

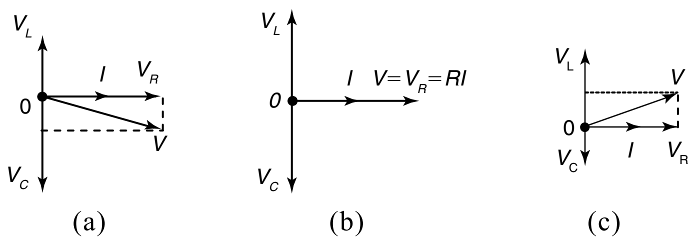


Figure 10.3 *Phasor Diagrams (a) $f < f_0$ (b) $f = f_0$ (c) $f > f_0$*

At resonant frequency, the inductive and capacitive reactances are equal so that the current and voltage are in phase. For any frequency lower than the resonant frequency, the inductive reactance is less than the capacitive reactance and hence, the circuit behaves as a capacitive circuit. Similarly, for any frequency higher than the resonant frequency, the inductive reactance is greater than the capacitive reactance and hence, the circuit behaves as an inductive circuit.

ii. Bandwidth

We define the half-power bandwidth of the *RLC* circuit as the range of frequencies (or the width of the frequency band) for which the power dissipated in R is greater than or equal to half the maximum power.

We know that the average power is,

$$P = |I|^2 R \text{ where } |I| = \frac{I_m}{\sqrt{2}} \tag{10.5}$$

Maximum power will be, $P_{max} = I_m^2 R$

Thus, the half-power points occur when

$$P = \frac{P_{max}}{2} \text{ or } |I| = \frac{I_m}{\sqrt{2}}$$

At resonance, the circuit is purely resistive, so that

$$I_m = \frac{V_m}{R} \tag{10.6}$$

Therefore, at half-power points,

$$|I| = \frac{I_m}{\sqrt{2}} = \frac{V_m}{\sqrt{2}R} \tag{10.7}$$

From equation (10.2) and (10.7), we get,

$$\frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V_m}{\sqrt{2}R}$$

To solve for the frequencies, squaring both sides and equating the denominators,

$$R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 2R^2$$

$$\Rightarrow \quad \left(\omega L - \frac{1}{\omega C}\right)^2 = R^2$$

$$\Rightarrow \quad \left(\omega L - \frac{1}{\omega C}\right) = \pm R$$

Therefore, $\omega = \pm\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$

We see that, mathematically, there are 4 possible values of ω. Taking the positive roots, the half-power frequencies are:

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \tag{10.8}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \tag{10.9}$$

By definition, the Bandwidth (BW) is given by,

$$BW = \omega_2 - \omega_1 = \frac{R}{L} \tag{10.10}$$

Also, from equations (10.8) and (10.9), we get

$$\omega_1\omega_2 = \frac{1}{LC} = \omega_0^2 \Rightarrow \omega_0 = \sqrt{\omega_1\omega_2} \tag{10.11}$$

Thus, the resonant frequency is the geometric mean of the half-power frequencies.

iii. Quality Factor or Circuit Magnification Factor (Q)

It is defined as, $Q = 2\pi \dfrac{Maximum\ \ Energy\ \ Stored}{Energy\ \ Dissipated\ \ per\ \ cycle}$

Maximum Energy stored = Electromagnetic energy in inductor or Electrostatic energy in capacitor

$$= \frac{1}{2}LI_{max}^2 \text{ or } \frac{1}{2}CV_{max}^2$$

Therefore,

$$Q = 2\pi \frac{\frac{1}{2}LI_{max}^2}{\left(I_{max}/\sqrt{2}\right)^2 R \times \frac{1}{f}} = \frac{2\pi fL}{R} = \frac{\omega L}{R}$$

or,
$$Q = \frac{\omega L}{R} = \frac{1}{\omega RC} = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{10.12}$$

Thus, Q is inversely proportional to R. Hence, for series RLC circuit, a high value of quality factor implies low losses and a low value of Q implies high losses.

Also, *quality factor* for a circuit is defined as, $Q = \dfrac{Resonant\ \ Frequency}{Bandwidth} = \dfrac{\omega_0}{\omega_2 - \omega_1}$ and the *selectivity* of the circuit is defined as the reciprocal of quality factor, i.e.

$$Selectivity = \frac{1}{Q} = \frac{BW}{\omega_0}$$

Therefore, a circuit will be highly selective if it has a high value of Q. For series RLC circuit, a high value of quality factor implies a narrow resonant peak and a low value of Q implies broad resonant peak. The variations of magnitude and phase angle of current in RLC series circuit for different values of Quality factor (Q) are shown in Fig. 10.4.

i. Voltage Across Elements

Voltage across Resistance

Since $V = RI$ and at resonance $V_R = V_m$,

Therefore,
$$I = \frac{V_m}{R} \tag{10.13}$$

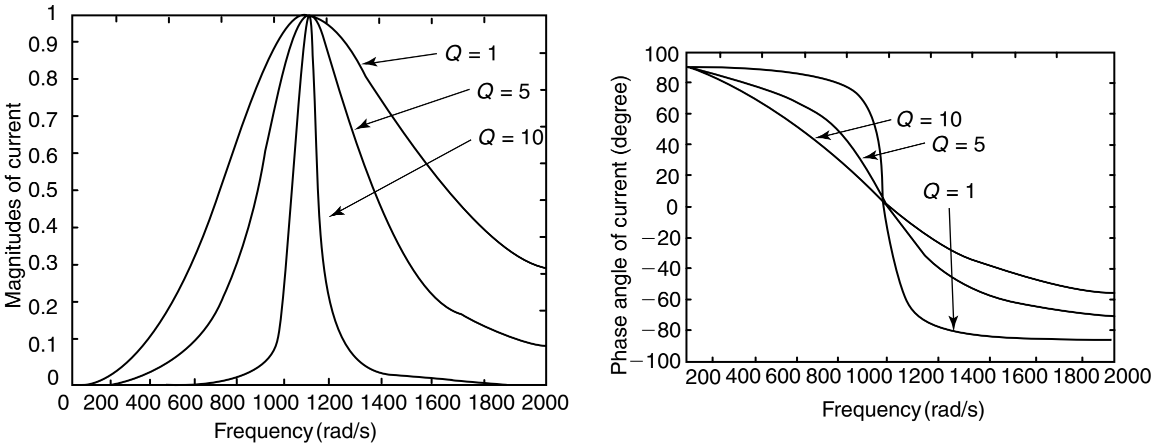


Figure 10.4 *Variation of Magnitude and Phase Angle of Current in RLC Series Circuit for Different Values of Q*

Voltage across Inductance

We know, $V_L = IZ_L = j\omega LI$, using equation (10.2), we get,

$$|V_L| = \omega L|I| = \frac{\omega L V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

At resonance, (i.e. $\omega_0 L = \frac{1}{\omega_0 C}$), $|V_L| = \frac{\omega_0 L V_m}{R} = QV_m$ (10.14)

To find the frequency at which inductor voltage will be maximum, we have,

$$\frac{d}{d\omega}\left[|V_L|\right] = 0$$

$$\Rightarrow \quad \frac{d}{d\omega}\left[\frac{\omega L V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\right] = 0$$

$$\Rightarrow \quad \omega_L = \frac{1}{\sqrt{LC - \frac{C^2 R^2}{2}}}$$

Thus, $$f_L = \left(\frac{1}{2\pi}\right)\left(\frac{1}{\sqrt{LC - \frac{C^2 R^2}{2}}}\right) \quad (10.15)$$

Voltage across Capacitance

We know, $V_C = \dfrac{I}{j\omega C}$; using equation (10.2), we get,

$$|V_C| = \frac{|I|}{\omega C} = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} \frac{1}{\omega C}$$

At resonance, (i.e. $\omega_0 L = \dfrac{1}{\omega_0 C}$),

$$|V_C| = \frac{V_m}{\omega_0 RC} = QV_m \tag{10.16}$$

To find the frequency at which capacitor voltage will be maximum, we have,

$$\frac{d}{d\omega}\left[|V_C|\right] = 0$$

$$\Rightarrow \qquad \frac{d}{d\omega}\left[\frac{V_m}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}} \frac{1}{\omega C}\right] = 0$$

$$\Rightarrow \qquad \omega_C = \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}$$

Thus
$$\boxed{f_C = \left(\frac{1}{2\pi}\right)\sqrt{\left(\frac{1}{LC} - \frac{R^2}{2L^2}\right)}} \tag{10.17}$$

From equations (10.14) and (10.16) we see that both V_L and V_C may be very large at resonance and they will add to zero (voltage across L and C are 180° out of phase). At resonant condition, the voltage across the inductor and capacitor are equal in magnitude and opposite in sign, thus canceling each other so that the entire voltage appears across the resistance. For this reason, the resonance in series *RLC* circuit is known as *voltage resonance*. Also, at resonant condition, from equations (10.14) and (10.16) we get,

$$|V_C| = |V_L| = QV_m \Rightarrow Q = \frac{|V_C|}{V_m} = \frac{|V_L|}{V_m}$$

Thus, quality factor for series circuit is also defined as the ratio of voltage across the inductor or capacitor at resonance to the supply voltage. For this reason, Q is also known as *circuit magnification factor* (here, voltage magnification).

From equations (10.15) and (10.17), it is observed that $f_L > f_C$. The variations of voltage across resistor, capacitor and inductor are shown in Fig. 10.5.

However, from equations (10.15) and (10.17), it is observed that if R is very small (or Q is very large), both f_L and f_C approach f_0. For circuits with $Q \geq 10$, the maximum voltages across R, L and C will practically occur at resonant frequency f_0.

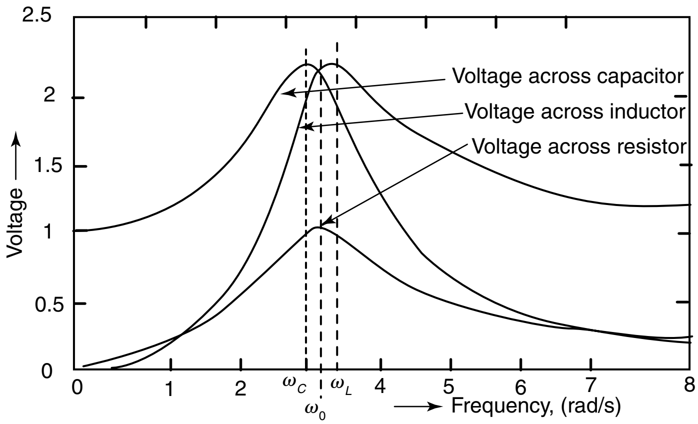


Figure 10.5 *Variation of Voltage Across Resistor, Capacitor and Inductor with Frequency*

Variation of Impedance with Frequency

The variations of the impedances with frequency are shown in Fig. 10.6.

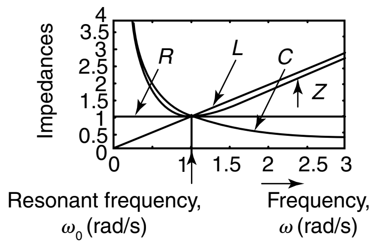


Figure 10.6 *Variations of Impedances with Frequency*

Here, R—ωL—$\dfrac{1}{\omega C}$ $\quad Z = R + j\left(\omega L - \dfrac{1}{\omega C}\right)$

i Impedance at Frequencies near Resonant Frequency

We introduce a new term, *fractional frequency deviation* or *fractional detuning*, δ, defined as,

$$\delta = \frac{\omega - \omega_0}{\omega_0} \Rightarrow \frac{\omega}{\omega_0} = (1 + \delta)$$

Thus, the impedance of the *RLC* series circuit at any frequency is given by,

$$\begin{aligned} z &= R + j\left(\omega L - \frac{1}{\omega C}\right) = R\left[1 + j\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)\right] \\ &= R\left[1 + j\left(\frac{\omega_0 L}{R}\frac{\omega}{\omega_0} - \frac{1}{\omega_0 RC}\frac{\omega_0}{\omega}\right)\right] \end{aligned}$$

At frequencies near the resonant frequency, δ <<. $\therefore$ $(2 + \delta) \approx 2$ and $(1 + \delta) \approx 1$.
$\therefore$ Impedance near resonant frequency,

$$\boxed{Z = R(1 + j2Q\delta)}$$

Note: (i) At $\omega = \omega_2$ (half-power frequency) $2Q\delta = 1$; $Z = Z = R(1 + j)$
(ii) At $\omega = \omega_1$ (half-power frequency) $2Q\delta = -1$; $Z = Z = R(1 - j)$

Example 10.1 A series *RLC* circuit consists of a resistance of 1 kΩ, an inductance of 10mH and a capacitance of 100 µF. For a supply voltage of 100 V, determine the followings:
(a) resonant frequency,
(b) maximum current in the circuit,
(c) *Q* factor of the circuit, and
(d) Half-power frequencies.

Sol:
Here, R = 1kΩ, L = 10 mH and C = 100 µF, V = 100 V
(a) Resonant frequency,

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{10\times10^{-3}\times100\times10^{-6}}} = 159.15 \text{ Hz} \quad Ans$$

(b) Maximum current in the circuit,

$$I_0 = \frac{V}{R} = \frac{100}{1\times10^3} = 0.1A \quad Ans$$

(c) *Q* factor of the circuit,

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{1\times10^3}\sqrt{\frac{10\times10^{-3}}{100\times10^{-6}}} = 0.01 \quad Ans$$

(d) To find half-power frequencies, we have,

$$(f_2 - f_1) = \frac{R}{4\pi L} = \frac{1\times10^3}{4\pi\times10\times10^{-3}} = 7957.75 \qquad \text{(i)}$$

ans

$$f_1 f_2 = \frac{1}{2\pi LC} = \frac{1}{2\pi\times10\times10^{-3}\times100\times10^{-6}} = 159154.94$$

$\therefore$

$$(f_2 + f_1) = \sqrt{(f_2 - f_1)^2 + 4f_1f_2} = \sqrt{(7957.75)^2 + 4\times159154.94} = 7997.65 \qquad \text{(ii)}$$

Adding equations (i) + (ii), f_2 = 7977.7 Hz *Ans*
Subtracting equations (ii) – (i), f1 = 19.95 Hz *Ans*

Example 10.2 A coil of resistance 2.2 Ω and inductance 0.01 H is connected in series with a capacitor across 220 V mains. Find the value of capacitance such that the maximum current flows in the circuit at a frequency of 100 Hz. Also, find the current and voltage across the capacitor.

Sol:
Here, R = 2.2 Ω, L = 0.01 H and V = 220 V, f_0 = 100 Hz

$$\because \qquad f_0 = \frac{1}{2\pi\sqrt{LC}} \Rightarrow C = \frac{1}{2\pi f_0^{\,2} L} = \frac{1}{2\pi \times 100^2 \times 0.01} = 250\,\text{ì F} \quad Ans$$

Current through the capacitor is the current flowing through the circuit at resonance, i.e.

$$I_C = I_0 = \frac{V}{R} = \frac{220}{2.2} = 100\text{ A} \quad Ans$$

Voltage across the capacitor,

$$V_C = I_0 \times \frac{j}{2\pi fC} = 100 \times \frac{j}{2\pi \times 100 \times 250 \times 10^{-6}} = 636.6\angle 90°\,(V) \quad Ans$$

Example 10.3 A series circuit is in resonance at 8 ′ 10^6 Hz and has a coil of 35 mH and 10 Ω resistor. For an applied voltage of 100 V,

(a) Find the current at resonance,
(b) Find the value of the capacitance for resonance.
(c) Find the impedance at a frequency of 8.1 MHz and also the current at this frequency.

Sol:
Here, R = 10 Ω, L = 35 μH and V = 100 V, f_0 8 × 10^6 Hz

(a) Current at resonance,

(b) $$\because \; f_0 = \frac{1}{2\pi\sqrt{LC}} \Rightarrow C = \frac{1}{2\pi f_0^{\,2} L} = \frac{1}{2\pi \times \left(8.1\times 10^6\right)^2 \times 35 \times 10^{-6}} = 11.3\text{ pF} \quad Ans$$

(c) To find impedance at 8.1 MHz (i.e. near resonant frequency), we have the expression as,

$$Z = R\left[1 + jQ\delta\left(\frac{2+\delta}{1+\delta}\right)\right]$$

Here, $$Q = \frac{\omega_0 L}{R} = \frac{2\pi \times 8 \times 10^6 \times 35 \times 10^{-6}}{10} = 176$$

$$\delta = \frac{f - f_0}{f_0} = \frac{8.1 - 8}{8} = 0.0125$$

$$\therefore \qquad Z = R\left[1 + jQ\delta\left(\frac{2+\delta}{1+\delta}\right)\right] = 10\left[1 + j176 \times 0.0125 \times \left(\frac{2+0.0125}{1+0.0125}\right)\right]$$

$$= (10 + j43.71)\,\Omega \quad Ans$$

$\therefore$ Current at this frequency is, $I = \dfrac{V}{Z} = \dfrac{100}{(10 + j43.71)} = 2.23\angle -77.11°\,(A)$ *Ans*

10.3 PARALLEL RESONANCE OR CURRENT RESONANCE OR ANTI-RESONANCE

The basic parallel-resonant circuit is shown in Fig.10.7. We want to observe how the steady state amplitude and the phase angle of the voltage vary with the frequency of the sinusoidal current source. As the frequency of the source changes, the maximum amplitude of the source current (I_m) is held constant.

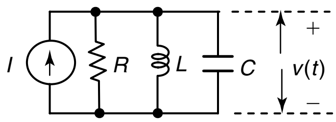


Figure 10.7 *Parallel* RLC *Resonant Circuit*

In this circuit, the impedance is given by,

$$Z = \frac{1}{\frac{1}{R} + \frac{1}{j\omega L} + j\omega C} \tag{10.18}$$

i. Voltage Response

Here, the supply current, $i_s = I_m \sin \omega t$ and the voltage is, $v = V_m \sin(\omega t + \phi)$. The phasor equivalents of i_s and v are **I** and **V**, respectively.

Using phasors, $$|V| = IZ = \frac{I_m}{\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}} \tag{10.33}$$

From this equation, we have the following observations:

- $|V| \to 0$ as $\omega \to 0$; and
- $|V| \to 0$ as $\omega \to \infty$.

This indicates that there must be a maximum value of the voltage $|V|$ for some particular value of ω. This occurs when the denominator is a minimum. i.e., when

$$\omega L = \frac{1}{\omega C} \Rightarrow \omega = \frac{1}{\sqrt{LC}} \tag{10.20}$$

Thus, resonance occurs when the magnitudes of the inductive and capacitive reactances are equal. This frequency is termed as the *resonant frequency*, w_0 of the parallel *RLC* circuit.

$$\therefore \qquad \boxed{\therefore \omega_0 = \frac{1}{\sqrt{LC}}}$$

From equation (10.19), the phase angle of the voltage is given by,

$$\phi = \tan^{-1}\left(\left(\frac{1}{\omega L} - \omega C\right)R\right) \tag{10.21}$$

From equation (10.21), it is clear that the phase angle for the voltage also depends on frequency. We have two observations:

- As $\omega \to 0$, $\phi \to 90°$ and in this case the current lags the voltage with the phase relationship being like that of an *RL* circuit.
- As $\omega \to \infty$, $\phi \to -90°$ and in this case the current leads the voltage with the phase relationship being like that of an *RC* circuit.
- At $\omega = \omega_0$, $\phi = 0$, and in this case the current and the voltage are in phase, the circuit behaving like a purely resistive circuit.

The voltage response and phasor diagrams are shown in Fig. 10.8 and 10.9.

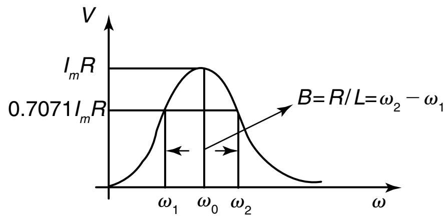
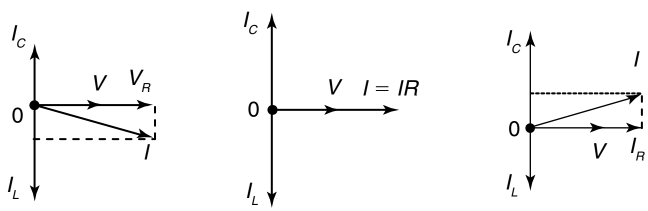


Figure 10.8 *Frequency Response of a Parallel – Resonant Circuit*

ii Phasor Diagrams

The voltage and current phasor diagrams for an *RLC* parallel circuit are shown in Fig. 10.9.



(a) (b) (c)

Figure 10.9 *Phasor Diagrams (a) $f < f_0$ (b) $f = f_0$ (c) $f > f_0$*

i. Bandwidth

We define the half-power bandwidth of the *RLC* circuit as the range of frequencies (or the width of the frequency band) for which the power dissipated in *R* is greater than or equal to half the maximum power.

We know that the average power is,

$$P = \frac{|V|^2}{R} \quad \text{where } |V| = \frac{V_m}{\sqrt{2}} = \frac{I_m R}{\sqrt{2}} \tag{22}$$

Maximum power will be, $P_{\max} = \dfrac{V_m{}^2}{R}$

Thus, the half-power points occur when

$$P = \frac{P_{\text{axm}}}{2} \quad or \quad |V| = \frac{V_m}{\sqrt{2}}$$

At resonance, the circuit is purely resistive, so that

$$V_m = I_m R V_m = I_m R \tag{10.23}$$

Therefore, at half-power points,

$$|V| = \frac{V_m}{\sqrt{2}} = \frac{I_m R}{\sqrt{2}} \tag{10.24}$$

From equations (10.19) and (10.24), we get,

$$\frac{I_m}{\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}} = \frac{I_m R}{\sqrt{2}}$$

Solve for the frequencies, in similar way as for the series circuit, we get,

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \tag{10.25}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \tag{10.26}$$

By definition, the Bandwidth (BW) is given by,

$$BW = \omega_2 - \omega_1 = \frac{1}{RC} \tag{10.27}$$

Also, from equations (10.25) and (10.26), we get

$$\omega_1 \omega_2 = \frac{1}{LC} = \omega_0^{\,2} \Rightarrow \omega_0 = \sqrt{\omega_1 \omega_2} \tag{10.28}$$

Thus, the resonant frequency is the geometric mean of the half-power frequencies.

Quality Factor or Circuit Magnification Factor (Q)

Here, $$Q = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{1/\sqrt{LC}}{1/RC} = \omega_0 RC = \frac{R}{\omega_0 L} = R\sqrt{\frac{C}{L}} \tag{10.29}$$

From (10.12) and (10.29), we see that,

$$Q_{\text{series}} = \frac{1}{Q_{\text{parallel}}}.$$

Therefore, for parallel *RLC* circuit, a high value of *Q* is achieved via a large value of *R* resulting high losses. The variations of magnitude and phase angle of voltage in *RLC* parallel circuit for different values of Quality factor (*Q*) are shown in Fig. 10.10.

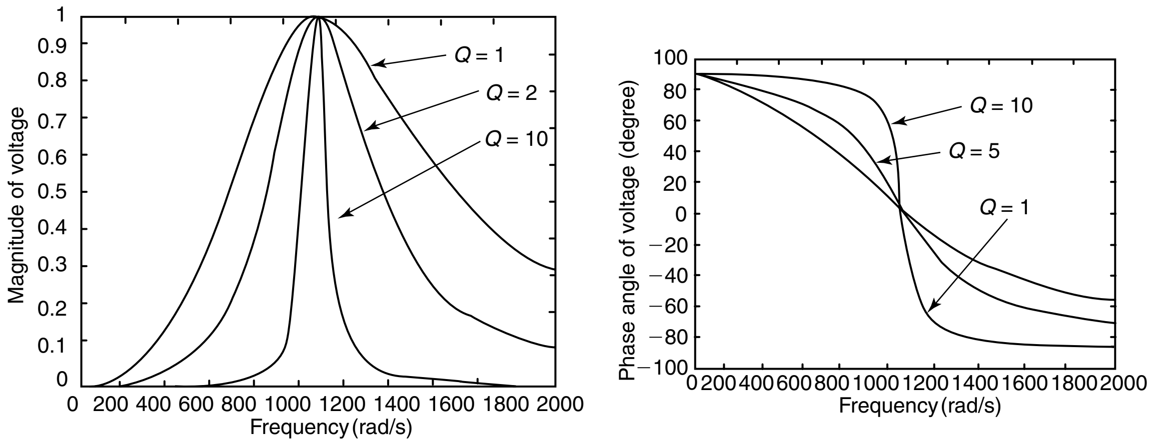


Figure 10.10 *Variations of magnitude and phase angle of voltage in* RLC *parallel circuit for different values of* Q

iii. Current Through Elements

Current through Resistance

Since $I = \dfrac{V_m}{R}$ and at resonance $I_R = I_m$, therefore,

$$V_m = I_m R \tag{10.30}$$

Current through Inductance

We know, $I_L = \dfrac{V_m}{\omega L}$, using equation (10.19), we get,

$$|I_L| = \frac{|V|}{\omega L} = \frac{I_m}{\sqrt{\dfrac{1}{R^2} + \left(\omega C - \dfrac{1}{\omega L}\right)^2}} \frac{1}{\omega L}$$

At resonance, (i.e. $\omega_0 L = \dfrac{1}{\omega_0 C}$),

$$|I_L| = \frac{RI_m}{\omega_0 L} = QI_m \tag{10.31}$$

Current through Capacitance

We know, $I_C = Vj\omega C$; using equation (10.19), we get,

$$|I_C| = |V|\omega C = \frac{I_m}{\sqrt{\dfrac{1}{R^2} + \left(\omega C - \dfrac{1}{\omega L}\right)^2}} \omega C$$

At resonance, (i.e. $\omega_0 L = \dfrac{1}{\omega_0 C}$),

$$|I_C| = I_m \omega_0 RC = QI_m \tag{10.32}$$

From equations (10.31) and (10.32) we see that both I_L and I_C may be very large at resonance and they will add to zero (current through L and C are 180° out of phase) so that the entire current will be flowing through the resistance. For this reason, the resonance in parallel RLC circuit is known as *current resonance*. Also, at resonant condition, from equations (10.31) and (10.32) we get,

$$|I_C| = |I_L| = QI_m \Rightarrow Q = \frac{|I_C|}{I_m} = \frac{|I_L|}{I_m}$$

Thus, quality factor for parallel circuit is also defined as the ratio of current through the inductor or capacitor at resonance to the source current. For this reason, Q is also known as *circuit magnification factor* (here, current magnification).

iv. Impedance at Frequencies near Resonant Frequency

The impedance of the RLC parallel circuit at any frequency is given by,

$$Z = \frac{1}{\dfrac{1}{R} + \dfrac{1}{j\omega L} + j\omega C}$$

$$= \frac{R}{1 + \dfrac{R}{j\omega L} + j\omega RC}$$

$$= \frac{R}{1 + j\left(\omega RC - \dfrac{R}{\omega L}\right)}$$

$$= \frac{R}{1 + j\left(\omega_0 RC \dfrac{\omega}{\omega_0} - \dfrac{R}{\omega_0 L}\dfrac{\omega_0}{\omega}\right)}$$

$$Z = \frac{R}{1 + jQ\left(1 + \delta - \dfrac{1}{1+\delta}\right)}$$

$$= \frac{R}{1 + jQ\delta\left(\dfrac{2+\delta}{1+\delta}\right)}$$

At frequencies near the resonant frequency, $\delta <<$. $\therefore$ $(2 + \delta) \approx 2$ and $(1 \mid \delta) \approx 1$.

$\therefore$ Impedance near resonant frequency,

$$\boxed{Z = \frac{R}{1 + jQ2\delta}}$$

Note: (i) At $\omega = \omega_2$ (half-power frequency) $2Q\delta = 1$; $Z = \ Z = \dfrac{R}{1+j}$

(ii) At $\omega = \omega_1$ (half-power frequency) $2Q\delta = -1$; $Z = \ Z = \dfrac{R}{(1-j)}$

10.4 RELATION BETWEEN DAMPING RATIO AND QUALITY FACTOR

The frequency response may be related to the natural response as discussed below.
From chapter 5, it was seen that the damping ratio is,

$$\alpha = \xi\omega_0 = \frac{R}{2L}; \ \textit{for series RLC circuit}$$

$$= \frac{1}{2RC}; \ \textit{for parallel RLC circuit}$$

From equation (10.29),

$$Q = \frac{\omega_0}{\omega_2 - \omega_1} = \omega_0 RC = \frac{\omega_0}{2\alpha}$$

$$\Rightarrow \qquad \alpha = \frac{\omega_2 - \omega_1}{2} = \frac{\omega_0}{2Q} \tag{10.33}$$

Also, from chapter 5, the damped frequency of oscillation is,

$$\omega_d = \omega_0\sqrt{1-\xi^2} = \sqrt{\omega_0^{\,2} - \alpha^2} = \omega_0\sqrt{1 - \frac{1}{4Q^2}} \tag{10.34}$$

From equation (10.34), it is seen that the damping co-efficient a is inversely proportional to the quality factor, Q. Also, the transition from underdamped to overdamped response occurs when $\omega_0^2 = \alpha^2$ (i.e. at the critical damping condition). Thus, from equation (10.33), we see that the *condition for critical damping* is,

$$Q = \frac{1}{2}$$

Similarly, from equation (10.33),

When $\quad \alpha^2 > \omega^2, \ Q < \dfrac{1}{2}$ (overdamped)

and

When $\quad \alpha^2 < \omega^2, \ Q > \dfrac{1}{2}$ (underdamped)

10.5 A MORE REALISTIC PARALLEL RESONANT CIRCUIT

A more realistic parallel-resonant circuit is shown in Fig. 10.10. It is a more realistic model because it accounts for the losses in the inductor through resistance R_L and losses in the capacitor through resistance R_C.

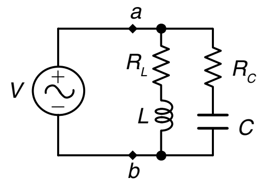


Figure 10.11 *Real Parallel Resonant Circuit*

Here,

$$Z_1 = (R_L + j\omega L)$$

$$Z_2 = \left(R_C + \frac{1}{j\omega C}\right)$$

$$Y = \frac{1}{R_L + j\omega L} + \frac{j\omega C}{1 + j\omega C R_C} = \frac{R_L - j\omega L}{R_L^2 + \omega^2 L^2} + \frac{j\omega C(1 - j\omega C R_C)}{1 + \omega^2 C^2 R_C^2}$$

$$= \left(\frac{R_L}{R_L^2 + \omega^2 L^2} + \frac{\omega^2 C^2 R_C}{1 + \omega^2 C^2 R_C^2}\right) + j\left(\frac{\omega C}{1 + \omega^2 C^2 R_C^2} - \frac{\omega L}{R_L^2 + \omega^2 L^2}\right) \tag{10.35}$$

For resonance to occur, the imaginary part of the admittance should be zero.

$$\therefore \qquad \frac{\omega C}{1 + \omega^2 C^2 R_C^2} = \frac{\omega L}{R_L^2 + \omega^2 L^2}$$

$$\Rightarrow \qquad R_L^2 C + \omega^2 L^2 C^2 = L + \omega^2 L R_C^2 C^2$$

$$\Rightarrow \qquad \omega^2 LC\left(R_C^2 - L\right) = R_L^2 C - L$$

$$\Rightarrow \qquad \omega_0 = \sqrt{\frac{1}{LC}\left(\frac{L - CR_L^2}{L - CR_C^2}\right)}$$

Thus, the resonant frewuency is given by,

$$\boxed{f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}\left(\frac{L - CR_L^2}{L - CR_C^2}\right)}}$$

We consider the following four conditions:

Case (1): $R_C = 0$

Here, the resonant frequency is,

$$\omega_0 = \sqrt{\frac{1}{LC}\left(1-\frac{CR_L^{\;2}}{L}\right)}$$

Under this condition, the total admittance becomes, from equation (10.34)

$$Y_0 = \frac{R_L}{R_L^{\;2} + \omega_0^{\;2} L^2}$$

From resonant frequency, $\omega_0^{\;2} = \dfrac{1}{LC} - \dfrac{R_L^{\;2}}{L^2} \Rightarrow \left(R_L^{\;2} + \omega_0^{\;2}L^2\right) = \dfrac{L}{C}$

Therefore, the admittance is, $Y_0 = \dfrac{R_L}{L/C}$ and the impedance under resonant condition becomes,

$$Z_0 = \frac{L}{CR_L} = \frac{1}{R_L}\left(R_L^{\;2} + \omega_0^{\;2}L^2\right) = R_L\left(1+\frac{\omega_0^{\;2}L^2}{R_L^{\;2}}\right) = R_L\left(1+Q^2\right)$$

This impedance is known as *Dynamic Resistance of the parallel tuned circuit*. It is seen that lower the value of resistance R_L, higher is the value of dynamic resistance of the parallel circuit. The current drawn from the supply at resonance is,

$$I_0 = V\frac{CR_L}{L}$$

This current is termed as the *make-up current*.

The current in the capacitor or inductor branch is called *forced oscillatory current*, given by,

$$I = V\omega_0 C$$

This type of circuit is known as a *rejector circuit*, since its impedance approaches a maxima and, therefore, the resultant current is a minima at (or near) resonant frequency.

Impedance near resonant frequency

It is obtained by the procedures as explained below.

Impedance at any frequency, ω is,

$$Z = \frac{(R_L + j\omega L)\left(-j/\omega C\right)}{R_L + j\left(\omega L - \dfrac{1}{\omega C}\right)}$$

$$= \frac{R_L\left(1+\dfrac{j\omega L}{R_L}\right)\left(-j/\omega C\right)}{R_L\left[1+\dfrac{j\omega L}{R_L}\left(1-\dfrac{1}{\omega^2 LC}\right)\right]}$$

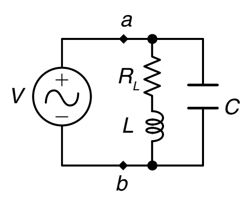


Figure 10.12 *Parallel Tuned Resonant Circuit with Lossless Capacitor*

$$= \frac{\frac{L}{CR_L}\left(1 - j\frac{R_L}{\omega L}\right)}{1 + j\frac{\omega L}{R_L}\left(1 - \frac{1}{\omega^2 LC}\right)} \tag{10.36}$$

Now, $\quad \dfrac{\omega L}{R_L} = \dfrac{\omega_0 L}{R_L} \times \dfrac{\omega}{\omega_0} = Q\dfrac{\omega}{\omega_0} = Q(1+\delta) \ \{\delta = \dfrac{\omega - \omega_0}{\omega_0}\}$

and, $\quad \dfrac{1}{\omega^2 LC} = \dfrac{\omega_0{}^2}{\omega^2} \times \dfrac{1}{\omega_0{}^2 LC} = \dfrac{1}{(1+\delta)^2} \times 1 = \dfrac{1}{(1+\delta)^2}$

{As, for high values of Q ($Q > 10$), $\because \omega_0{}^2 = \dfrac{1}{LC} \Rightarrow \dfrac{1}{\omega_0{}^2 LC} = 1$ }

Substituting these values in (10.36),

$$Z = \frac{\frac{L}{CR_L}\left(1 - j\frac{R_L}{\omega L}\right)}{1 + j\frac{\omega L}{R_L}\left(1 - \frac{1}{\omega^2 LC}\right)}$$

$$= \frac{Q^2 R_L\left[1 - \frac{j}{Q(1+\delta)}\right]}{1 + jQ\delta(1+\delta)\left[\frac{2+\delta}{(1+\delta)^2}\right]}$$

Near resonant frequency, $(2 + \delta) \approx 2$ and $(1 + \delta \approx 1)$

$$= \frac{Q^2 R_L\left[1 - \frac{j}{Q(1+\delta)}\right]}{1 + jQ\delta\left[\frac{2+\delta}{1+\delta}\right]}$$

$\therefore$ Impedance near resonant frequency,

$$\boxed{Z_0 = \frac{Q^2 R_L\left[1 - \frac{j}{Q}\right]}{1 + j2Q\delta}}$$

NB:
At resonant frequency, $\delta = 0$,

$\therefore$ Impedance at resonance frequency, $Z_0 = Q^2 R_L\left[1 - \dfrac{j}{Q}\right]$

(i) For large values of Q (i.e., $Q > 10$), at resonant frequency, the impedance is,

$$Z_0 \approx Q^2 R_L = \frac{L}{CR_L}$$

(ii) For large values of Q(i.e., $Q > 10$), for near resonant frequency, the impedance is,

$$Z_0 = \frac{Q^2 R_L}{1 + j2Q\delta} = \frac{R_d}{1 + j2Q\delta} \quad \text{[as, Dynamic Resistance, } R_d = \frac{L}{CR_L}\text{]}$$

$$\therefore \qquad \boxed{\frac{Z_0}{R_d} = \frac{1}{1 + j2Q\delta}}$$

Current in Parallel Tuned Circuit

Let,

Current delivered by the source = I_s,

Current in the inductor branch = I_L,

Current in the capacitor branch = I_C,

$\therefore$ Power delivered by the source, $P_s = I_s^2 R_d$ [R_d = Dynamic Resistance]

Power dissipated in the parallel circuit, $P = I_L^2 R_L$

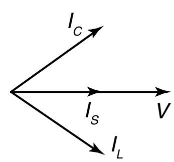


Figure 10.13 *Phasor diagram for Parallel Tune Resonant Circuit with Lossless Capacitor*

Now, $P_s = P$

$$\Rightarrow \qquad I_s^2 R_d = I_L^2 R_L \qquad (10.37)$$

$$\Rightarrow \qquad I_L^2 = I_s^2 \frac{R_d}{R_L} = I_s^2 \frac{L}{CR_L^2} \quad \left[\because R_d = \frac{L}{CR_L}\right]$$

$$\Rightarrow \qquad I_L = I_s \frac{1}{R_L}\sqrt{\frac{L}{C}}$$

Thus, the parallel tuned circuit is a current amplifier.
Voltage across the capacitor, $V_C = I_s R_d$
and current through the capacitor, $I_C = \omega_0 C V_C$

$$\therefore \qquad I_s = \frac{V_C}{R_d} = \frac{I_C}{\omega_0 C R_d}$$

From (10.37),

$$I_s^2 R_d = I_s^2 R_L$$

$$\Rightarrow \qquad \frac{I_s^{\,2}}{I_L^{\,2}} = \frac{R_L}{R_d}$$

$$\Rightarrow \qquad \frac{I_C^{\,2}}{\omega_0^{\,2} C^2 R_d^{\,2} I_L^{\,2}} = \frac{R_L}{R_d}$$

$$\Rightarrow \qquad \frac{I_C^{\,2}}{I_L^{\,2}} = \frac{R_L}{R_d} \times \omega_0^{\,2} C^2 R_d^{\,2} = \omega_0^{\,2} LC \quad \left[\because R_d = \frac{L}{CR_L}\right]$$

$$\Rightarrow \qquad \frac{I_C}{I_L} = \sqrt{LC}\omega_0 = \sqrt{LC}\left[\sqrt{\frac{1}{LC}\left(1 - \frac{CR_L^{\,2}}{L}\right)}\right] = \sqrt{\left(1 - \frac{CR_L^{\,2}}{L}\right)}$$

$$= \sqrt{1 - \frac{1}{Q^2}} \quad \left[\because Q = \frac{1}{R_L}\sqrt{\frac{L}{C}}\right]$$

$$\therefore \qquad \boxed{\frac{I_C}{I_L} = \sqrt{1 - \frac{1}{Q^2}}}$$

If R_L is very large, the currents are not equal. However, for very low value of R_L (for very high value of Q, $Q \geq 10$), both the currents are equal. Higher the value of Q, higher will be I_C and I_L and lower will be the source current, I_s. As $Q \to \propto$, both I_C and I_L tend to infinity and I_s tends to zero.

Effect of Source Resistance over Bandwidth and Quality Factor

Bandwidth for the parallel tuned circuit is affected by the source resistance as explained below

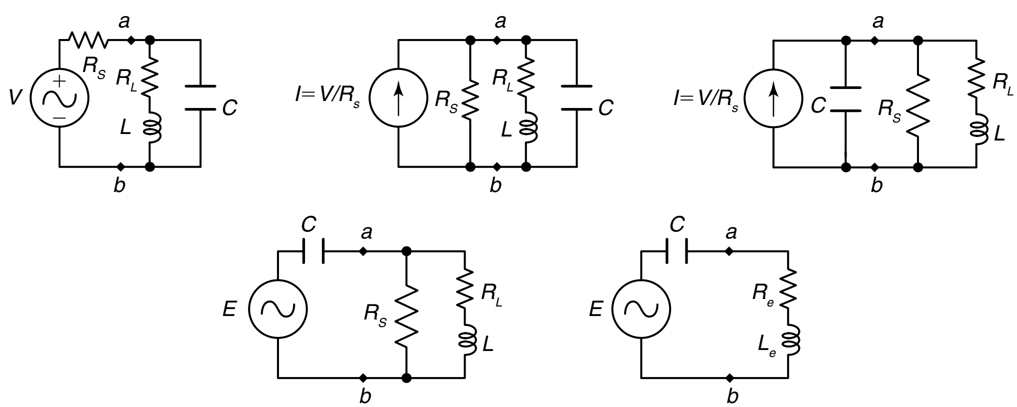


Figure 10.14 *Simplification of Real Parallel Circuit to get Series Equivalent Circuit*

By circuit reduction methods, the parallel tuned circuit is simplified as shown from Fig. 10.14 (i) to Fig. 10.14 (v),

$$Z_e = \frac{(R_L + j\omega L) R_s}{(R_L + j\omega L) + R_s}$$

$$= \frac{R_L R_s + j\omega L R_s}{(R_L + j\omega L) + R_s}$$

$$= \frac{(R_L R_s + j\omega L R_s)\{(R_L + R_s) - j\omega L\}}{\{(R_L + R_s) + j\omega L\}\{(R_L + R_s) - j\omega L\}}$$

$$= \frac{\{R_L R_s (R_L + R_s) + \omega^2 L^2 R_s\} + j\{\omega L R_s (R_L + R_s) - \omega L R_s\}}{(R_L + R_s)^2 + \omega^2 L^2}$$

$$= R_e + j\omega L_e$$

Therefore, the equivalent resistance is,

$$R_e = \frac{R_L R_s (R_L + R_s) + \omega^2 L^2 R_s}{(R_L + R_s)^2 + \omega^2 L^2}$$

and the equivalent inductance is,

$$L_e = \frac{L R_s^{\,2}}{(R_L + R_s)^2 + \omega^2 L^2}$$

Since the equivalent circuit is an RLC series circuit, the effective quality factor is,

$$Q = \frac{\omega L_e}{R_e} = \frac{\omega L R_s}{R_L (R_L + R_s) + \omega^2 L^2} = \frac{1}{\dfrac{R_L}{\omega L} + \dfrac{R_L^{\,2} + \omega^2 L^2}{\omega L R_s}} = \frac{1}{\dfrac{1}{Q} + \left(\dfrac{R_L^{\,2} + \omega^2 L^2}{R_L \omega L R_s}\right) R_L}$$

$$\left(\because Q = \frac{\omega L}{R_L}\right)$$

$$= \frac{1}{\dfrac{1}{Q} + \left(\dfrac{R_L}{\omega L}\right)\dfrac{R_d}{R_s}} = \frac{1}{\dfrac{1}{Q} + \dfrac{1}{Q}\left(\dfrac{R_d}{R_s}\right)} = \frac{Q}{1 + \dfrac{R_d}{R_s}}$$

$$\left(as,\ at\ resonance\ R_L^{\,2} + \omega^2 L^2 = \frac{L}{C}\ and\ R_d = \frac{L}{CR_L}\right)$$

Therefore,
$$\boxed{Q_e = \frac{\omega L / R_L}{\left(1 + \dfrac{L}{CR_L R_s}\right)} = \frac{Q}{\left(1 + \dfrac{R_d}{R_s}\right)}}$$

Therefore, the effective bandwidth will be,

$$(BW)_e = \frac{\omega_0}{Q_e} = \frac{1}{Q}\left(1+\frac{R_d}{R_s}\right)\omega_0$$

$$\therefore \qquad (BW)_e = \frac{\omega_0}{Q}\left(1+\frac{R_d}{R_s}\right) = \frac{\omega_0}{Q}\left[1+\frac{L}{CR_L R_s}\right]$$

Thus, the bandwidth of the circuit depends upon the circuit constants (R_L, L and C) and the source resistance (R_s). For a given resonant frequency, the circuit will be less selective and bandwidth will be large if L is large and C is small. For more selectivity, the value of L should be reduced and C increased; however, this may reduce the value of R_d, which is undesirable.

NB: *If the source resistance is infinite ($R_s \to \infty$), then the bandwidth and quality factor of this parallel tuned circuit is the same as those of the RLC series circuit.*

To obtain maximum possible value of power delivered from the source to the load, we have,

$$R_s = R_d$$

and under this condition, the bandwidth becomes,

$$BW = \frac{2}{Q}\omega_0$$

Example 10.4 A coil of inductance 1 H and 10 Ω resistance is connected in parallel with 100 μF capacitor. If the supply voltage is 200 V, find the resonant frequency and the current at resonance.

Sol:

Here, $R_L = 10\ \Omega$, $L = 1$ H, $C = 100\ \mu$F, $V = 200$ V

Resonant frequency is,

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R_L{}^2}{L^2}} = \frac{1}{2\pi}\sqrt{\frac{1}{1\times 100\times 10^{-6}}-\frac{10^2}{1^2}} = 15.84 \text{ Hz} \quad \textit{Ans}$$

At resonance, the impedance of the circuit is,

$$Z_0 = \frac{L}{CR_L} = \frac{1}{100\times 10^{-6}\times 10} = 1000\ \Omega$$

$\therefore$ Current in the circuit at resonance is, $I_0 = \dfrac{V}{Z_0} = \dfrac{200}{1000} = 0.2$ A *Ans*

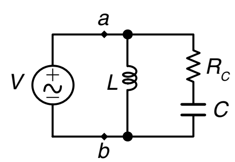


Figure 10.15 *Real Parallel Resonant Circuit with Lossless Inductor*

Case (2): $R_L = 0$:
Here, the resonant frequency is,

$$\omega_0 = \sqrt{\frac{1}{LC}\left(\frac{L}{L - CR_C^2}\right)}$$

Under this condition, the total admittance becomes, from equation (10.35)

$$Y_0 = \frac{\omega_0^2 C^2 R_C}{1 + \omega_0^2 C^2 R_C^2}$$

From resonant frequency,

$$\omega_0^2 = \frac{1}{C\left(L - CR_C^2\right)} \Rightarrow \left(1 + \omega_0^2 C^2 R_C^2\right) = \omega_0^2 LC$$

Therefore, the admittance is, $Y_0 = \dfrac{\omega_0^2 C^2 R_C}{\omega_0^2 LC} = \dfrac{CR_C}{L}$ and the impedance under resonant condition becomes,

$$Z_0 = \frac{L}{CR_C}$$

This is the *Dynamic Resistance of the parallel tuned circuit*. Here also, lower the value of resistance R_C, higher is the value of dynamic resistance of the parallel circuit. The *make-up current* i.e., current drawn from the supply at resonance is,

$$I_0 = V\frac{CR_C}{L}$$

and the *forced oscillatory current*, given by, $I = \dfrac{V}{\omega_0 L}$

This circuit is also a *rejector circuit* as the impedance approaches a maxima and the current a minima.

Approaching in the way similar to case (1), we get the equivalent quality factor and bandwidth of this resonating circuit as,

$$\boxed{Q_e = \frac{1}{\omega CR_C\left(1 + \dfrac{R_d}{R_s}\right)} = \frac{Q}{\left(1 + \dfrac{R_d}{R_s}\right)}} \quad \text{where, } Q = \frac{1}{\omega CR_C}$$

and
$$\boxed{BW = \frac{1}{Q}\left(1 + \frac{R_d}{R_s}\right)\omega_0 = \frac{1}{Q}\left[1 + \frac{L}{CR_C R_s}\right]\omega_0}$$

Case (3): $R_L = R_C = 0$:

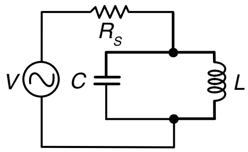


Figure 10.16 *Tuned Tank Circuit*

Here, the resonant frequency is, $\omega_0 = \sqrt{\dfrac{1}{LC}}$. This circuit with L and C in parallel is termed as *tuned tank circuit*. At resonance, with capacitive and inductive reactances equal to each other, the total impedance increases to infinity, meaning that the tank circuit draws no current from the AC power source. However, for this case, there will be some circulating current, given by,

$$I = I_C = V\omega C = V\frac{1}{\sqrt{LC}}C \quad (\text{if } \; R_s \ll)$$

$$\Rightarrow \qquad I = V\sqrt{\frac{C}{L}}$$

The quality factor and bandwidth of this circuit will be same as those of a simple parallel RLC circuit; i.e.,

$$Q = \frac{R_s}{\omega L} = \omega C R_s \quad \text{and} \quad BW = \frac{1}{R_s C}.$$

Case (4): $R_L = R_C = \sqrt{\dfrac{L}{C}}$

Here, $$R_L = R_C = \sqrt{\frac{L}{C}} = \sqrt{\frac{\omega L}{\omega C}} = \sqrt{X_L X_C}$$

Under this condition, the imaginary part of the admittance is,

$$\begin{aligned} IM(Y) &= \frac{\omega C}{1+\omega^2 C^2 R_C{}^2} - \frac{\omega L}{R_L{}^2 + \omega^2 L^2} \\ &= \frac{X_C}{X_C{}^2 + R_C{}^2} - \frac{X_L}{X_L{}^2 + R_L{}^2} \\ &= \frac{R_L{}^2 X_C + X_L{}^2 X_C - X_L X_C{}^2 + R_C{}^2 X_L}{\left(X_C{}^2 + R_C{}^2\right)\left(X_L{}^2 + R_L{}^2\right)} \end{aligned}$$

$$IM(Y) = \frac{X_L{}^2 X_C + X_L{}^2 X_C - X_L X_C{}^2 + X_L{}^2 X_C}{\left(X_C{}^2 + R_C{}^2\right)\left(X_L{}^2 + R_L{}^2\right)} = 0$$

Thus, the *resonance under this condition will occur at all frequencies.*

Example 10.5 Find C which results in resonance in the circuit shown in Fig. 10.17 when $\omega = 5000$ rad/s.

Sol:

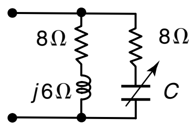


Figure 10.17 *Circuit of Example 10.5*

Let the capacitive reactance be X_C.

$$Y = \frac{1}{8+j6} + \frac{1}{8-jX_C} = \frac{8-j6}{100} + \frac{8-jX_C}{64+X_C{}^2}$$

$$= \left(\frac{8}{100} + \frac{8}{64+X_C{}^2}\right) + j\left(\frac{X_C}{64+X_C{}^2} - \frac{6}{100}\right)$$

For resonance to occur, the imaginary part of the admittance should be zero.

$$\therefore \qquad \left(\frac{X_C}{64+X_C{}^2} - \frac{6}{100}\right) = 0$$

$$\Rightarrow \qquad \frac{X_C}{64+X_C{}^2} = \frac{6}{100}$$

$$\Rightarrow \qquad 6X_C{}^2 - 100X_C + 384 = 0$$

$$\Rightarrow \qquad X_C = 10.67\ \Omega \quad or \quad 6\ \Omega$$

$$\therefore \qquad X_C = \frac{1}{\omega C}$$

$$\therefore \qquad C = \frac{1}{\omega X_C} = \frac{1}{5000 \times 10.67} = 18.75\text{ ì F} \quad \text{or} \quad \frac{1}{5000 \times 6} = 33.33\text{ ì F} \quad \textit{Ans}$$

10.6 COMPARISON OF SERIES AND PARALLEL RESONANCE

	Particulars	Series Resonance	Parallel Resonance
Difference	Circuit Impedance at resonance	Minimum	Maximum
	at resonance		
	Circuit Admittance at resonance	Maximum	Minimum
	Current in the circuit at resonance	Maximum	Minimum
	Circuit for $f > f_0$	Inductive	Capacitive
	Circuit for $f < f_0$	Capacitive	Inductive
	Amplification of	Voltage	Current
	Q-factor	$\frac{\omega_0 L}{R}$ or $\frac{1}{\omega_0 RC}$ or $\frac{1}{R}\sqrt{\frac{L}{C}}$	$\frac{R}{\omega_0 L}$ or $\omega_0 RC$ or $R\sqrt{\frac{C}{L}}$
Similarities	Power factor of the circuit at resonance	Unity	Unity
	Resonant frequency	$\omega_0 = \frac{1}{\sqrt{LC}}$	$\omega_0 = \frac{1}{\sqrt{LC}}$
	Bandwidth	$BW = \frac{f_0}{Q}$	$BW = \frac{f_0}{Q}$
	Half-power frequencies	$\omega_0 = \sqrt{\omega_1 \omega_2}$	$\omega_0 = \sqrt{\omega_1 \omega_2}$

10.7 UNIVERSAL RESONANCE CURVE

From the expressions of impedance near resonant frequency,
For series RLC circuit,

$$Z = R(1 + j2\delta Q)$$

$$Y = \frac{1}{R(1 + j2\delta Q)}$$

$$\frac{Y}{Y_0} = \frac{1}{1 + j2\delta Q} \tag{10.38}$$

$$\left(\text{where, } Y_0 = \frac{1}{R} = \text{admittance at resonance}\right)$$

For parallel *RLC* circuit,

$$Z = \frac{R}{1 + j2Q\delta}$$

or, $$\frac{Z}{R} = \frac{1}{1 + j2\delta Q} u$$

or, $$\frac{Z}{Z_0} = \frac{1}{1 + j2\delta Q} \tag{10.39}$$

$$\left(\text{where, } Z_0 = R = \text{impedance at resonance}\right)$$

For parallel tuned circuit for high values of Q,

$$Z = \frac{R_d}{1 + j2Q\delta}$$

or, $$\frac{Z}{Z_0} = \frac{1}{1 + j2Q\delta} \tag{10.40}$$

$$\left(\text{where, } Z_0 = R_d = \text{Dynamic resistance} = \text{impedance at resonance}\right)$$

From (10.38), (10.39) and (10.40), it is seen that the variations of $\frac{Y}{Y_0}$ or $\frac{Z}{Z_0}$ for high Q circuits near resonant frequency are identical for series and parallel circuit, given as,

$$H = \frac{1}{1 + j2\delta Q} = |H| \angle \phi$$

where, $$|H| = \frac{1}{\sqrt{1 + 4\delta^2 Q^2}} \quad \text{and} \quad \phi = -\tan^{-1}(2\delta Q)$$

The variation of magnitude and phase of this are shown in Fig.10.18. These curves are known as *universal resonance curve*. It gives the magnitude and phase of the quantity by which the maximum admittance or impedance of a series or parallel circuit is to be multiplied to obtain the associated admittance or impedance near resonant frequency.

NB: At half-power points, $2\delta Q = \pm\ 1$, $\therefore\ |H| = \frac{1}{\sqrt{2}} = 0.707$ and $\phi = \pm\ 45^\circ$.

10.8 APPLICATONS OF RESONANCE

Resonance is a very valuable property of reactive AC circuits, employed in a variety of applications.

One use for resonance is to establish a condition of stable frequency in circuits designed to produce AC signals. For example, when we tune a radio to a particular station, the *LC* circuits are set at resonance for that particular carrier frequency. Usually, a parallel (tank) circuit is used for this

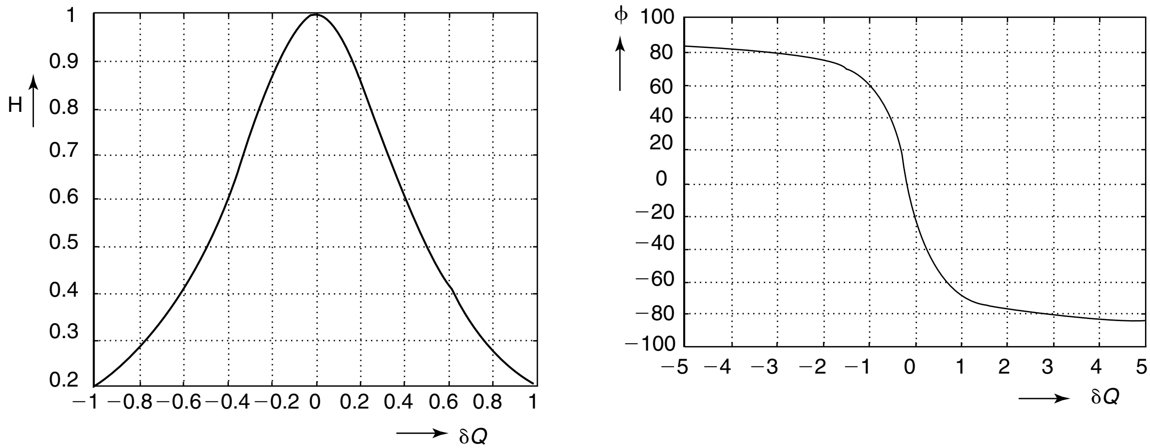


Figure 10.18 *Universal Resonance Curves*

purpose, with the capacitor and inductor directly connected together, exchanging energy between each other. Just as a pendulum can be used to stabilize the frequency of a clock mechanism's oscillations, so can a tank circuit be used to stabilize the electrical frequency of an *AC oscillator circuit*.

Another use for resonance is in applications where the effects of greatly increased or decreased impedance at a particular frequency are desired. A resonant circuit can be used to 'block' (present high impedance toward) a frequency or range of frequencies, thus acting as a 'filter' to strain certain frequencies. In fact, these particular circuits are called *filters*, discussed in the preceding chapter.

A parallel resonant circuit can alco be used as load impedance in output circuits of RF amplifiers. Due to high impedance, the gain of amplifier is maximum at resonant frequency.

Therefore, the applications of resonant effects can be summurized as follows:

1. Most common application of resonance is tuning i.e., as an **oscillator circuit**.
2. A series resonant circuit is used as **voltage amplifier**.
3. A parallel resonant circuit is used as **current amplifier**.
4. A resonant circuit is used as **filters**.
5. A realisitic parallel resonant circuit is used as **current rejector**.
6. A parallel resonant circuit is used as **load impedance in output circuits of RF amplifiers.**
7. A parallel resonant circuit can be used in **induction heating**.

In designing any mechanical systems or civil structure or electrical system, the effects of resonance must be taken into consideration. Otherwise, the oscillations of the system in certain conditions may be so large that the system may be damaged.

SOLVED PROBLEMS

1. A series *RLC* circuit has the values: *R* = 10 Ω, *L* = 0.01 H, *C* = 100 μF. Calculate resonant frequency, quality factor, bandwidth, and the half-power frequencies.

Sol:

Here, $R = 10\ \Omega$, $L = 0.01$ H, $C = 100\ \mu$F.

$$\therefore \text{ Resonant frequency, } f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.01\times 100\times 10^{-6}}} = 159.15\ \text{Hz} \quad \textit{Ans.}$$

$$\therefore \text{ Quality factor, } Q = \frac{\omega_0 L}{R} = \frac{1}{\sqrt{LC}}\times\frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10}\sqrt{\frac{0.01}{100\times 10^{-6}}} = 1 \quad \textit{Ans.}$$

$$\therefore \text{ Bandwidth, } BW = \frac{R}{L} = \frac{10}{0.01} = 1,000\ \text{rad/s} \quad \textit{Ans.}$$

To find half-power frequencies, we have,

$$(\omega_2 - \omega_1) = \frac{R}{L} = 1000$$

and

$$\omega_1\omega_2 = \frac{1}{LC} = 10^6$$

$$(\omega_2 + \omega_1) = \sqrt{(\omega_2-\omega_1)^2 + 4\omega_1\omega_2} = \sqrt{10^6 + 4\times 10^6} = 2.36\times 10^3$$

Adding equations (i) + (ii), $\omega_2 = 1.618\times 10^3$ rad/s *Ans.*

Subtracting equations (ii) – (i), $\omega_1 = 0.618\times 10^3$ rad/s *Ans.*

2. A series *RLC* circuit has the values: *R* = 100 Ω, *L* = 0.02 H, *C* = 0.02 μF. Calculate frequency of resonance. A variable frequency sinusoidal voltage of value 50V is applied to the circuit. Find the frequency at which voltage across *L* and *C* is the maximum. Also calculate voltage across *L* and *C* at frequency of resonance. Find the maximum current in the circuit.

Sol:

Here, $R = 100\ \Omega$, $L = 0.02$ H, $C = 0.02\ \mu$F.

$$\therefore \text{ Resonant frequency, } f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.02\times 0.02\times 10^{-6}}} = 7.957\ \text{kHz} \quad \textit{Ans.}$$

The frequency at which voltage across C is the maximum is given by,

$$f_C = \left(\frac{1}{2\pi}\right)\sqrt{\left(\frac{1}{LC} - \frac{R^2}{2L^2}\right)} = \left(\frac{1}{2\pi}\right)\sqrt{\left(\frac{1}{0.02\times 0.02\times 10^{-6}} - \frac{100^2}{2(0.02)^2}\right)} = 7.937\ \text{kHz} \quad \textit{Ans}$$

The frequency at which voltage across L is the maximum is given by,

$$f_L = \left(\frac{1}{2\pi}\right)\left(\frac{1}{\sqrt{LC - \frac{C^2R^2}{2}}}\right)$$

$$= \left(\frac{1}{2\pi}\right)\left(\frac{1}{\sqrt{0.02 \times 0.02 \times 10^{-6} - \frac{\left(0.02 \times 10^{-6}\right)^2 \times 100^2}{2}}}\right) = 7.977 \text{ kHz} \quad Ans$$

Voltage across the inductance or the capacitance at resonance is,

$$|V_L| = |V_C| = QV_m = \omega_0 \frac{L}{R} V_m = 2\pi f_0 \frac{L}{R} V_m = 2 \times \pi \times 7.957 \times 10^3 \times \frac{0.02}{100} \times 50 = 500 \text{ V} \quad Ans$$

$\therefore \quad V_L = 500\angle 90° \text{ V} \quad Ans$

$\therefore \quad \text{VC} = 500\angle -90° \text{ V} \quad Ans$

The maximum current in the circuit is given by,

$$I_{\max} = \frac{V_m}{R} = \frac{50}{100} = 0.5 \text{ A} \quad Ans$$

3. For a series *RLC* circuit with *R* = 2 Ω, *L* = 1 mH, *C* = 0.4 μF and a supply voltage *v*(*t*) = 20 sin ωt, find:
(a) The resonant frequency (ω_0);
(b) The half-power frequencies (ω_1 and ω_2);
(c) The quality factor and bandwidth; and
(d) The amplitude of the current at ω_0, ω_1 and ω_2.

Sol:
Here, $R = 2\ \Omega$, $L = 0.001$ H, $C = 0.4\ \mu$F

(a) $\therefore$ Resonant frequemcy, $\omega_r = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{0.001 \times 0.4 \times 10^{-6}}} = 50 \text{ krad/s} \quad Ans.$

(b) To find half-power frequencies, we have,

$$(\omega_2 - \omega_1) = \frac{R}{L} = 2000 \qquad \text{(i)}$$

and $$\omega_1\omega_2 = \frac{1}{LC} = 25 \times 10^6$$

$\therefore$ $$(\omega_2 + \omega_1) = \sqrt{(\omega_2 - \omega_1)^2 + 4\omega_1\omega_2} = \sqrt{4 \times 10^6 + 4 \times 25 \times 10^6} = 100 \times 10^3 \qquad \text{(ii)}$$

Adding equations (i) + (ii), $\omega_2 = 51$ krad/s *Ans*

Subtracting equations (ii) – (i), $\omega_1 = 49$ krad/s *Ans*

(c) $\therefore$ Quality factor, $Q = \dfrac{\omega_0 L}{R} = \dfrac{1}{\sqrt{LC}} \times \dfrac{L}{R} = \dfrac{1}{R}\sqrt{\dfrac{L}{C}} = \dfrac{1}{2}\sqrt{\dfrac{0.001}{0.4 \times 10^{-6}}} = 25 \quad Ans.$

$\therefore$ Bandwidth, $BW = \dfrac{R}{L} = \dfrac{2}{0.001} = 2 \text{ krad/s} \quad Ans.$

(b) The amplitudes of the currents are,

$$|I|_{\omega=\omega_0} = \frac{V}{R} = \frac{20}{2} = 10 \text{ A} \quad Ans.$$

$$|I|_{\omega=\omega_1=\omega_2} = \frac{|I|}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 7.071 \text{ A} \quad Ans.$$

4. A series-connected circuit has R = 4 Ω and L = 25 mH. (a) Calculate the value of C that will produce a quality factor of 50. (b) Find ω_1, ω_2 and BW. (c) Determine the average power dissipated at $\omega = \omega_0$, ω_1, ω_2. Take V_m = 100 V.

Sol:

Here, R = 4 Ω, L = 25 mH = 0.025 H, Q = 50, V_m = 100 Volt

(a)

$$\therefore \qquad Q = \frac{\omega_0 L}{R} = \frac{1}{\sqrt{LC}} \times \frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$\therefore \qquad C = \frac{L}{Q^2 R^2} = \frac{0.025}{50^2 \times 4^2} = 0.625 \text{ ì F} \quad Ans.$$

(b)

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = -\frac{4}{2 \times 0.025} + \sqrt{\left(\frac{4}{2 \times 0.025}\right)^2 + \frac{1}{0.025 \times 0.625 \times 10^{-6}}}$$

$= -80 + 8000$

$= 7920$ rad/s *Ans.*

$$\omega_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = -\frac{4}{2 \times 0.025} - \sqrt{\left(\frac{4}{2 \times 0.025}\right)^2 + \frac{1}{0.025 \times 0.625 \times 10^{-6}}}$$

$= -80 - 8000$

$= 8080$ rad/s *Ans.*

(c) Average power dissipated at resonant frequency,

$$|P_{\text{av}}|_{\omega=\omega_0} = \frac{V_{\text{rms}}^{\ 2}}{R} = \frac{\left(100/\sqrt{2}\right)^2}{2} = 1.250 \text{ kW} \quad Ans.$$

Average power dissipated at half-power frequencies,

$$|P_{\text{av}}|_{\omega=\omega_1=\omega_2} = \frac{|P_{\text{av}}|_{\omega=\omega_0}}{2} = \frac{1.25}{2} = 0.625 \text{ kW} \quad \textit{Ans.}$$

5. A series RLC circuit with R = 10 Ω, $L = \frac{100}{314}$ H, $C = \frac{1}{31,400}$ F is excited from a 10 V, 50 Hz (ω = 314 rad/s) source. Determine the rms values of voltage across (i) resistance, (ii) inductance, (iii) capacitance. Give explanation if any of the answer is more than the source voltage.

Sol:

Here, ω = 314 rad/s, R = 10 Ω, $L = \frac{100}{314}$ H $C = \frac{1}{31,400}$ F, V = 10 V

$$\therefore \ X_L = \omega L = 314 \times \frac{100}{314} = 100\,\Omega \quad \text{and} \quad X_C = \frac{1}{\omega C} = \frac{1}{314 \times \frac{1}{31,400}} = 100\,\Omega$$

Since, $X_L = X_C$, the circuit is under resonance.

At resonance, the current in the circuit, $I_0 = \frac{V}{R} = \frac{10}{10} = 1$ A

(i) $\therefore$ Voltage across resistance, $V_R = I_0 \times R = 1 \times 10 = 10$ V *Ans*

(ii) $\therefore$ Voltage across inductance, $V_L = I_0 \times \omega L = 1 \times 100 = 100$ V *Ans*

(iii) $\therefore$ Voltage across capacitance, $V_C = I_0 \times \frac{1}{\omega C} = 1 \times 100 = 100$ V *Ans*

At resonance, the voltage drops across the inductance and the capacitance will be equal in magnitude but opposite in phase and thus will nullify each other so that the supply voltage will be equal to the voltage drop across the resistance.

6. A 20 Ω resistor is connected in series with an inductor, a capacitor and an ammeter across a 25 V variable frequency supply. When the frequency is 400 Hz, the current is at its maximum value of 0.5 A and the potential difference across the capacitor is 150 V. Calculate:

i. The capacitance of the capacitor.

ii. The resistance and inductance of the inductor.

Sol:

Here, R = 20 Ω, V_m = 25 Volt, f = 400 Hz, I = 0.5 A, V_C = 150 Volt

When the current is the maximum, the circuit is in resonance and hence total reactances

$$X_L \sim X_C = 0$$

i. The capacitane value is calculated as,

$$\therefore \qquad X_C = \frac{V_C}{I} = \frac{150}{0.5} = 300\,\Omega$$

$$\Rightarrow \qquad \frac{1}{\omega C} = 300 \Rightarrow C = \frac{1}{2\pi \times 400 \times 300} = 1.325 \text{ ì F} \quad \textit{Ans.}$$

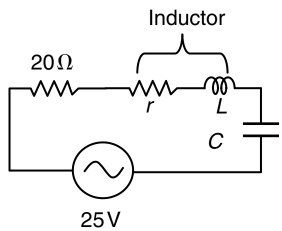
ii. $\because X_L \sim X_C = 0,$

$$\therefore \qquad X_L = X_C$$

$$\Rightarrow \qquad \omega L = 300\ \Omega$$

$$\Rightarrow \qquad L = \frac{300}{2\pi \times 400} = 0.119 \text{ H} \quad \textit{Ans.}$$

Also, at resonance, circuit resistance = circuit impedance



Let, r = resistance of the inductor

$$\text{Then} \qquad (20 + r) = \frac{25}{0.5} = 50 \Rightarrow r = 30\ \Omega \quad \textit{Ans.}$$

7. Voltages across resistance, inductance and capacitance connected in series are 3 V, 4 V and 5 V respectively. If supply voltage has 50 Hz frequency, what is the magnitude of supply voltage? Find the resonant frequency of this series *RLC* circuit.

Sol:

Here, $V_R = 3$ V; $\quad V_L = 4$ V; $\quad V_C = 5$ V, $\quad f = 50$ Hz

Supply voltage is, $V = \sqrt{V_R^2 + (V_C - V_L)^2} = \sqrt{3^2 + (5-4)^2} = \sqrt{10} = 3.162 \text{ V} \quad \textit{Ans}$

Now, voltage drop across inductance is,

$$V_L = \omega L \times I \qquad \text{(i)}$$

Voltage drop across capacitance is,

$$V_C = \frac{1}{\omega C} \times I \qquad \text{(ii)}$$

By (ii) ÷ (i), we get,

$$\frac{1}{\omega^2 LC} = \frac{V_C}{V_L} = \Rightarrow \frac{1}{LC} = \omega^2 \times \frac{V_C}{V_L} = (2\pi \times 50)^2 \times \frac{5}{4} = 123.37 \times 10^3$$

$\therefore$ Resonant frequency is,

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} = \frac{1}{2\pi}\sqrt{123.37 \times 10^3} = 55.9 \text{ Hz} \quad \textit{Ans}$$

8. A resistor and capacitor are in series with a variable inductor. When the circuit is connected to 200 V, 50 Hz supply, the maximum current obtained by varying the inductance is 0.314 A. The voltage across the capacitor, when current in the circuit is maximum, is 800 V. Find the values of the series circuit elements.

Sol:

Here, $V = 200$ V, $V_C = 800$ V, $I_0 = 0.314$ A

Resonance frequency, $f_0 = 50$ Hz

The current in series resonant circuit is maximum at resonance. It is given as,

$$I_0 = \frac{V}{R} \Rightarrow R = \frac{V}{I} = \frac{200}{0.314} = 636.95\,\Omega$$

Also, at resonance the voltage across the capacitor is,

$$V_C = QV \Rightarrow Q = \frac{V_C}{V} = \frac{800}{200} = 4$$

Now, $$Q = \frac{\omega_0 L}{R} = \frac{2\pi f_0 L}{R}$$

or, $$4 = \frac{2\pi \times 50 \times L}{636.95} \Rightarrow L = 8.1\,\text{H}$$

Also, $$Q = \frac{1}{\omega_0 RC} = \frac{1}{2\pi f_0 RC}$$

or, $$4 = \frac{1}{2\pi \times 50 \times 636.95C} \Rightarrow C = 1.25\,\text{ì F}$$

Therefore, the series elements are: $R = 636.95\,\Omega$ $L = 8.1\,\text{H}$ $C = 1.25\,\text{ì F}$ *Ans*

9. A circuit is made up of a 10 Ω resistance, a 1 μF capacitance and a 1 H inductance all connected in series. A voltage of 100 V at varying frequencies is applied to the circuit. Find the frequency (frequencies) at which the circuit would consume only 10% of the power it consumed at resonance.

Sol:

Here, $R_{-} = 10\ \Omega$, $L = 1$ H, and $C = 1\ \mu$F, $V = 100$ V

At resonance the current in the circuit is, $I_0 = \dfrac{V}{R} = \dfrac{100}{10} = 10\text{A}$

$\therefore$ Power consumed at resonance $= I_0^2 \times R = 10^2 \times 10 = 1000$ W

Let f be the frequency at which the circuit would consume only 10% of the power it consumed at resonance i.e. 100 W.

Under this condition, the circuit current is,

$$I = \sqrt{\frac{100}{R}} = \sqrt{\frac{100}{10}} = \sqrt{10}\ \text{A}$$

$\therefore$ Circuit impedance, $Z = \dfrac{V}{I} = \dfrac{100}{\sqrt{10}} = 10\sqrt{10}\ \Omega$

$\therefore$ Circuit reactance, $(X_L - X_C) = \sqrt{Z^2 - R^2} = \sqrt{\left(10\sqrt{10}\right)^2 - 10^2} = 30\ \Omega$

$$\therefore \quad \left(2\pi fL - \frac{1}{2\pi fC}\right) = 30$$

$$\Rightarrow \quad \left(2\pi f1 - \frac{1}{2\pi f1 \times 10^{-6}}\right) = 30$$

$$\Rightarrow \quad 4\pi^2 f^2 - 60\pi f - 10^6 = 0$$

$$\Rightarrow \quad f = \frac{60\pi \pm \sqrt{(60\pi)^2 + 4 \times 4\pi^2 \times 10^6}}{2 \times 4\pi^2} = 161.56\ \text{Hz} \ \text{ or } \ 156.78\ \text{Hz} \quad \textit{Ans}$$

10. A coil under test is connected in series with a variable calibrated capacitor C and sine wave generator giving a 10 V r.m.s. output at frequency of 1000 rad/s. By adjusting *C*, the current in the circuit is found to be a maximum when *C* = 10.0 μF. Further, the current falls down to 0.707 times the maximum value when *C* = 12.5 μF.

(a) Find the inductance of the coil and resistance of the coil.

(b) Find the Q of the coil at 1000 rad/s

(c) What is the maximum current in the circuit?

Sol:

Here, $V_{\text{rms}} = 10$ V, $\omega = 1000$ rad/s,

$C = 10$ μF for maximum current ($I_{\max}$)

$= 12.5$ μF for current $I_{\max}/\sqrt{2}$

(a) $\omega_0^2 = \dfrac{1}{LC} \Rightarrow L = \dfrac{1}{\omega_0^2 C} = \dfrac{1}{10^6 \times 10 \times 10^{-6}} = 0.1\ \text{H}$ *Ans.*

(b) At half-power frequency,

$\left(\omega L - \dfrac{1}{\omega C}\right) = R$; where, R is the resistance of the coil

$$\therefore \quad \left(1000 \times 0.1 - \frac{1}{1000 \times 12.5 \times 10^{-6}}\right) = R \Rightarrow R = 20\ \Omega$$

$\therefore$ Quality factor of the coil, $Q = \dfrac{\omega_0 L}{R} = \dfrac{1000 \times 0.1}{20} = 5$ *Ans.*

(c) Maximum current in the circuit $I_{\max} = \dfrac{V}{R} = \dfrac{10}{20} = 0.5\text{ A}$ *Ans.*

11. A parallel *RLC* circuit has the following values:

(a) $R = 8\text{ k}\Omega$, $L = 0.2\text{ mH}$, $C = 8\ \mu\text{F}$, $V = 10\sin\omega t$ (V)

(b) $R = 100\text{ k}\Omega$, $L = 20\text{ mH}$, $C = 5\text{ nF}$, $V = 20\sin\omega t$ (V)

Calculate (i) ω_0; (ii) ω_1 and ω_2; (iii) Q and BW; (iv) power dissipated at ω_0, ω_1 and ω_2.

Sol:

a. Here, $R = 8\text{ k}\Omega$, $L = 0.2\text{ mH}$, $C = 8\ \mu\text{F}$, $V = 10\sin\omega t$ (V)

(i) $$\therefore\ \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2\times10^{-3}\times8\times10^{-6}}} = \frac{10^5}{4} = 25\text{ krad/s} \quad \textit{Ans.}$$

(ii) $$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

$$= -\frac{1}{2\times8\times10^3\times8\times10^{-6}} + \sqrt{\left(\frac{1}{2\times8\times10^3\times8\times10^{-6}}\right)^2 + \frac{1}{0.2\times10^{-3}\times8\times10^{-6}}}$$

$$\omega_1 = 24.992\text{ krad/s} \quad \textit{Ans.}$$

(iii) $$\therefore\ Q = \omega_0 RC = 25\times10^3\times8\times10^3\times8\times10^{-6} = 1600 \quad \textit{Ans.}$$

$$\therefore\ BW = (\omega_2 - \omega_1) = \frac{\omega_0}{Q} = \frac{25\times10^3}{1600} = 15.625\text{ rad/s} \quad \textit{Ans.}$$

(iv) At $\omega = \omega_0$, $Y = \dfrac{1}{R} \Rightarrow Z = R = 8\text{ k}\Omega$

$$I = \frac{V}{Z} = \frac{10\angle-90^\circ}{8000} = 1.25\angle-90^\circ\ (\text{mA}) \quad \textit{Ans.}$$

As the entire current flows through R at resonance, the average power dissipated at $\omega = \omega_0$ is,

$$P = \frac{1}{2}|I_0|^2 R = \frac{1}{2}\left(1.25\times10^{-3}\right)^2\left(8\times10^3\right) = 6.25\text{ mW} \quad \textit{Ans.}$$

or, $$P = \frac{V_m^{\ 2}}{2R} = \frac{10^2}{2\times8\times10^3} = 6.25\text{ mW} \quad \textit{Ans.}$$

At $\omega = \omega_1 = \omega_2$, the power dissipated is,

$$P = \frac{V_m^{\ 2}}{4R} = \frac{10^2}{4\times8\times10^3} = 3.125\text{ mW} \quad \textit{Ans.}$$

b. Following the same procedures as in part a., we get,

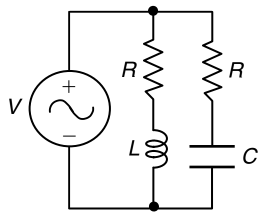
(i) ω_0 = 100 krad/s;

(ii) ω_1 = 99 krad/s, ω_2 = 101 krad/s;

(iii) Q = 50, BW = 2 rad/s;

(iv) 1 mW

12. In the circuit shown in figure, find out the value of *R* such that the impedance of the whole circuit should be independent of the frequency of the supply. If voltage = 200 V, *L* = 0.16 H and *C* = 100 μF, calculate the power loss in the circuit.



Sol:

Impedance of the inductive branch, $Z_L = (R + j\omega L)$

Impedance of the capacitive branch, $Z_C = \left(R - \dfrac{j}{\omega C}\right)$

∴ Impedance of the whole circuit,

$$Z = \frac{Z_L Z_C}{Z_L + Z_C} = \frac{(R + j\omega L)\left(R - \frac{j}{\omega C}\right)}{(R + j\omega L) + \left(R - \frac{j}{\omega C}\right)} = \frac{R^2 + \frac{L}{C} + jR\left(\omega L - \frac{1}{\omega C}\right)}{2R + j\left(\omega L - \frac{1}{\omega C}\right)}$$

In this expression, the imaginary term of the numerator is R times the imaginary term of the denominator. Thus, if the real term i.e., $\left(R^2 + \dfrac{L}{C}\right)$ is also R times the real term of the denominator i.e. $R \times 2R$, then the term consisting of w will vanish and the impedance will become independent of frequency.

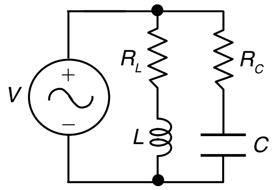
Thus, the condition is,

$$\left(R^2 + \frac{L}{C}\right) = R \times 2R = 2R^2 \Rightarrow R = \sqrt{\frac{L}{C}}$$

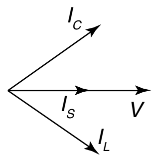
Putting the value of L and C, $R = \sqrt{\dfrac{0.16}{100 \times 10^{-6}}} = 40\ \Omega$ *Ans*

Power loss in the circuit, $P = \dfrac{V^2}{R} = \dfrac{200^2}{40} = 1000\ \text{W} = 1\ \text{kW}$ *Ans*

13. For the circuit shown in figure draw the phasor diagram. Derive the condition for the two branch currents I_L and I_C to be in quadrature.



Sol: For this resonant circuit, the phasor diagram is shown in figure.



Phase angle of the inductive branch, $\phi_L = \tan^{-1}\left(\dfrac{\omega L}{R_L}\right)$

Phase angle of the capacitive branch, $\phi_C = \tan^{-1}\left(\dfrac{1}{\omega R_C C}\right)$

For the two currents to be in quadrature, the condition is,

$$\phi_L + \phi_C = 90°$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\omega L}{R_L}\right) + \tan^{-1}\left(\frac{1}{\omega R_C C}\right) = 90°$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\dfrac{\omega L}{R_L} + \dfrac{1}{\omega R_C C}}{1 - \dfrac{\omega L}{R_L} \times \dfrac{1}{\omega R_C C}}\right] = 90°$$

$$\Rightarrow \quad \frac{\dfrac{\omega L}{R_L} + \dfrac{1}{\omega R_C C}}{1 - \dfrac{L}{R_L R_C C}} = \tan 90° = \infty$$

$$\Rightarrow \quad 1 - \frac{L}{R_L R_C C} = 0$$

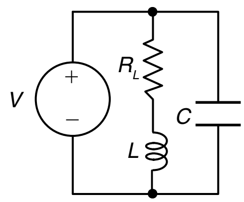
$$\Rightarrow \quad R_L R_C = \sqrt{\frac{L}{C}} \quad Ans$$

14. A coil of 10 Ω resistance and 0.1 H inductance is connected in parallel with a capacitor of 100 μF capacitance. Calculate the frequency at which the circuit will act as a non-inductive resistance of *R* ohm. Find also the value of *R*.

Sol:

Here, $R_L = 10\ \Omega$, $L = 0.1$ H, $C = 100\ \mu$F

The frequency at which the circuit will be non-inductive is the resonant frequency, given by,



$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R_L{}^2}{L}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{0.1 \times 100 \times 10^{-6} C} - \frac{10^2}{(0.1)^2}}$$

$= 47.8$ Hz *Ans.*

At resonance, $R = \dfrac{L}{CR_L} = \dfrac{0.1}{100 \times 10^{-6} \times 10} = 100\ \Omega$ *Ans.*

15. A parallel circuit has a fixed capacitor and variable inductor having constant quality factor of 4. Find the value of inductance and capacitance for the circuit impedance of 1000 Ω at resonant frequency of 2.4 MHz. What is the bandwidth of the circuit?

Sol:

Here, $Q = 4$, $Z_0 = 1000\ \Omega$, $f_0 = 2.4$ MHz

Now, impedance at resonance, is given by,

$$Z_0 = R_L\left(1 + Q^2\right) \Rightarrow R_L = \frac{Z_0}{\left(1 + Q^2\right)} = \frac{1000}{1 + 4^2} = 58.82\ \Omega$$

Also, impedance at resonance, is given by,

$$Z_0 = \frac{L}{CR_{L:}} \Rightarrow \frac{L}{C} = Z_0 \times R_{L:} = 1000 \times 58.82 = 58.82 \times 10 \qquad \text{(i)}$$

The resonant frequency is given as,

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R_L{}^2}{L^2}} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}\left(\sqrt{1 - \frac{1}{Q^2}}\right)}$$

$$\therefore \qquad 2.4 \times 10^6 = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}\sqrt{1 - \frac{1}{16}} \Rightarrow LC = 4.1227 \times 10^{-15} \qquad \text{(ii)}$$

Putting the value of C from (i) into (ii), we get,

$$L\left(\frac{L}{58.82\times10^{3}}\right)=4.1227\times10^{-15}\ \Rightarrow\ L=15.57\ \text{ì H}\quad Ans$$

Putting this value in (i), we get,

$$C=\frac{L}{58.82\times10^{3}}=\frac{15.57\times10^{-6}}{58.82\times10^{3}}=0.264\ \text{nF}\quad Ans$$

$\therefore$ Bandwidth of the circuit, $BW=\dfrac{f_0}{Q}=\dfrac{2.4\times10^{6}}{4}0.6\ \text{MHz}\quad Ans$

16. A coil resonates at 2 MHz when a 18 pF capacitor is shunted across it. When shunting capacitor is 81 pF, the resonating frequency becomes 1 MHz. Find the distributed capacitor of the coil and the self resonating frequency.

Sol:

Let the distributed capacitance of the coil be C_d (in pF).

Show when the coil is shunted with another capacitance of C, the total capacitance becomes, $(C + C_d)$.

When C = 18 pF, the resonating frequency is f_0 = 2 MHz

$$2\times10^{6}=\frac{1}{2\pi\sqrt{L\left(C+C_d\right)}}=\frac{1}{2\pi\sqrt{L\left(18+C_d\right)}}\qquad\text{(i)}$$

When C = 81 pF, the resonating frequency is f_0 = 1 MHz

$$1\times10^{6}=\frac{1}{2\pi\sqrt{L\left(C+C_d\right)}}=\frac{1}{2\pi\sqrt{L\left(81+C_d\right)}}\qquad\text{(ii)}$$

Dividing (i) by (ii), we get,

$$\sqrt{\frac{81+C_d}{18+C_d}}=2\ \Rightarrow\ C_d=3\ \text{pF}\quad Ans$$

Putting the value of C_d in (ii), we get,

$$1\times10^{6}=\frac{1}{2\pi\sqrt{L\left(18+3\right)\times10^{-12}}}\ \Rightarrow\ L=0.3\ \text{mH}$$

So, the self-resonating frequency of the coil is given as,

$$f_0=\frac{1}{2\pi\sqrt{LC_d}}=\frac{1}{2\pi\sqrt{0.3\times10^{-3}\times3\times10^{-12}}}=5.31\ \text{MHz}\quad Ans$$

17. A coil has an inductance of 250×10^{-6} H. Its reactance to resistance ratio is 170 at a frequency of 10^6 Hz. It is connected in parallel with a variable capacitor. Find:

(i) value of the capacitor to produce resonance at 10^6 Hz.

(ii) impedance of the circuit at 10^6 Hz.

(iii) impedance of the circuit at 0.99×10^6 Hz.

Sol:
Here, $L = 250 \times 10^{-6}$ H, $f_0 = 1$ MHz
From the given condition,

$$\left.\frac{X_L}{R_L}\right|_{f=1\,\text{MHZ}} = 170 \Rightarrow \frac{\omega L}{R_L} = 170 \Rightarrow R_L = \frac{2\pi f L}{170} = \frac{2\pi \times 10^6 \times 250 \times 10^{-6}}{170} = 9.24\ \Omega$$

(i) From the resonance frequency, we get,

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R_L{}^2}{L^2}}$$

$$\Rightarrow \qquad f_0^2 = \left(\frac{1}{2\pi}\right)^2\left(\frac{1}{LC} - \frac{R_L{}^2}{L^2}\right)$$

$$\Rightarrow (2\pi)^2 \times (10^6)^2 = \frac{1}{250 \times 10^{-6} \times C} - \frac{9.24^2}{\left(250 \times 10^{-6}\right)^2}$$

$$\Rightarrow \qquad C = 101.32 \text{ pF} \qquad \textit{Ans.}$$

(ii) Impedance at resonance frequency is,

$$Z_0 = \frac{L}{CR_L} = \frac{250 \times 10^{-6}}{101.31 \times 10^{-12} \times 9.24} = 267.04\ \text{k}\Omega \quad \textit{Ans}$$

(iii) The frequency, 0.99×10^6 Hz is very near to the resonance frequency.
Impedance near resonance frequency for large value of Q ($Q > 10$) is given as,

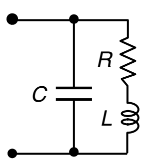
$$Z = \frac{Z_0}{1 + j2Q\delta}$$

Here, $Q = \dfrac{\omega_0 L}{R_L} = \dfrac{2\pi \times 10^6 \times 250 \times 10^{-6}}{9.24} = 170$

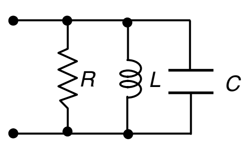
$$\delta = \frac{f - f_0}{f_0} = \frac{0.99 - 1}{1} = -0.01$$

$$\therefore \qquad Z = \frac{Z_0}{1 + j2Q\delta} = \frac{267.04 \times 10^3}{1 + j2 \times 170 \times (-0.01)} = \frac{267.04 \times 10^3}{1 - j3.4} = 75.35\angle 73.61^\circ\,(\text{k}\Omega) \quad \textit{Ans}$$

18. Show that the high-Q coil resonant circuit can be approximated as shown in figure.



High-Q Coil Resonant Circuit



Approximated Equivalent Circuit

Sol:
For the approximated circuit, the resonant frequency is given as,

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{(See section 10.3.)}$$

For the high-Q resonant circuit, the resonant frequency is given as,

$$\omega_0 = \sqrt{\frac{1}{LC}\left(1 - \frac{CR^2}{L}\right)} \qquad \text{(i)} \quad \text{(See section 10.5)}$$

Therefore, Q-factor of the inductance is,

$$Q = \frac{\omega_0 L}{R} = \frac{L}{R} \times \sqrt{\frac{1}{LC}\left(1 - \frac{CR^2}{L}\right)} = \sqrt{\frac{L}{CR^2} - 1}$$

$$\Rightarrow \qquad \frac{L}{CR^2} = 1 + Q^2$$

$$\Rightarrow \qquad \frac{CR^2}{L} = \frac{1}{1 + Q^2}$$

Putting this value in (i), we get,

$$\omega_0 = \sqrt{\frac{1}{LC}\left(1 - \frac{CR^2}{L}\right)} = \sqrt{\frac{1}{LC}\left(1 - \frac{1}{1 + Q^2}\right)}$$

For very high values of Q, the term $\left(\frac{1}{1+Q^2}\right)$ becomes negligible and we get the resonant frequency as, $\omega_0 = \sqrt{\frac{1}{LC}(1-0)} = \sqrt{\frac{1}{LC}}$ which is the same as that for the approximated equivalent circuit.

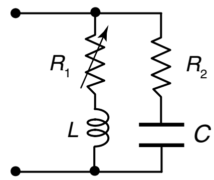
19. **A parallel resonant circuit comprising a coil of 150 nH with Q of 20 in parallel with a capacitor. What is the value of capacitor? Find also the resistance of the coil and the circuit impedance at resonance. Take f_0 = 1 MHz.**

Sol:
Here, L = 150 nH, Q = 20, $f_0 = 10^6$ Hz

$$\because \qquad f_0 = \frac{1}{2\pi\sqrt{LC}} \Rightarrow C = \frac{1}{4\pi^2 f_0^{\,2} L} = \frac{1}{4\pi^2 \times 10^{12} \times 150 \times 10^{-9}} = 0.168\text{ ì F} \quad \textit{Ans.}$$

$$\because \qquad Q_0 = \frac{\omega L}{R} \Rightarrow R = \frac{\omega L}{Q} = \frac{2\pi \times 10^6 \times 150 \times 10^{-9}}{20} = 47.1\text{ mÙ} \quad \textit{Ans.}$$

20. In a two-branch parallel circuit, calculate the resonant frequency ω_0 if R_1 = 4 Ω and R_2 = 6 Ω, C = 20 μF and L = 1 mH. If R_1 is increased, what is its maximum value for which there is a resonant frequency?



Sol:

Here, $R_1 = 4\ \Omega$, $L = 1 \times 10^{-3}$ H, $R_2 = 6\ \Omega$ and $C = 20 \times 10^{-6}$ F

$$\therefore \quad \omega_0 = \sqrt{\frac{1}{LC}\left(\frac{L - CR_2^2}{L - CR_1^2}\right)} = \sqrt{\frac{1}{20 \times 10^{-9}\,C}\left(\frac{10^{-3} - 20 \times 10^{-6} \times 36}{10^{-3} - 20 \times 10^{-6} \times 16}\right)} = 4.537.4 \text{ rad/s} \quad \textit{Ans.}$$

When R_1 is increased, resonant frequency will also increase. For $R_1^2 = \frac{L}{C}$, the resonance will occur at $\omega \to \propto$. Beyond this value of R_1, the quantity within the square root will become imaginary and no real frequency will give resonance.

$\therefore$ Maximum value of R_1 is obtained as,

$$R_1 = \sqrt{\frac{L}{C}} = \sqrt{\frac{1 \times 10^{-3}}{20 \times 10^{-6}}} = 7.071\,\Omega \quad \textit{Ans.}$$

21. A coil of inductance L and resistance R, in series with a capacitor is supplied at a constant voltage from a variable frequency source. Find the values of that frequency, in terms of R, L and ω_0 at which the circuit current would be half as much as at resonance. Hence, or otherwise, determine the bandwidth and selectivity of the circuit.

Sol:

The current at resonance is,

$$I_0 = \frac{V}{R}$$

and current at any other frequency is,

$$I = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

For this problem,

$$I = \frac{I_0}{2} \Rightarrow \frac{I_0}{I} = 2$$

$$\frac{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}{R} = 2$$

$$\Rightarrow \qquad R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 4R^2$$

$$\Rightarrow \qquad 3 = \frac{\omega^2 L^2}{R^2}\left(1 - \frac{1}{\omega^2 LC}\right)^2$$

$$\Rightarrow \qquad 3 = \frac{\omega^2 L^2}{R^2}\left(1 - \frac{\omega_0^{\,2}}{\omega^2}\right)^2 \quad \left[\because \omega_0 = \frac{1}{\sqrt{LC}}\right]$$

$$= \frac{\omega_0^{\,2} L^2}{R^2}\left(1 - \frac{\omega_0^{\,2}}{\omega^2}\right)^2 \times \frac{\omega^2}{\omega_0^{\,2}}$$

$$= Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

$$\Rightarrow \qquad = \pm\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2 = \frac{\sqrt{3}}{Q}$$

If the two frequencies are ω_2 and ω_1, ($\omega_2 > \omega_0 > \omega_1$) then,

$$\left(\frac{\omega_2}{\omega_0} - \frac{\omega_0}{\omega_2}\right) = \frac{\sqrt{3}}{Q}$$

and

$$\Rightarrow \qquad \left(\frac{\omega_0}{\omega_1} - \frac{\omega_1}{\omega_0}\right) = \frac{\sqrt{3}}{Q}$$

$$\therefore \qquad \left.\begin{aligned} \omega_1 &= \frac{-\sqrt{3}\omega_0 + \omega_0\sqrt{3 + 4Q^2}}{2Q} \\ \& \ \omega_2 &= \frac{\sqrt{3}\omega_0 + \omega_0\sqrt{3 + 4Q^2}}{2Q} \end{aligned}\right\} \ Ans.$$

Alternately,

$$\left(\omega L - \frac{1}{\omega C}\right)^2 = 3R^2$$

$$\Rightarrow \qquad \left(\omega L - \frac{1}{\omega C}\right) = \pm\sqrt{3}R$$

Therefore, $$\omega = \pm\frac{\sqrt{3}R}{2L} \pm \sqrt{\left(\frac{\sqrt{3}R}{2L}\right)^2 + \frac{1}{LC}}$$

Taking the positive roots of ω,

$$\left.\begin{aligned}\omega_1 &= -\frac{\sqrt{3}R}{2L} + \sqrt{\left(\frac{\sqrt{3}R}{2L}\right)^2 + \frac{1}{LC}} \\ \omega_2 &= \frac{\sqrt{3}R}{2L} + \sqrt{\left(\frac{\sqrt{3}R}{2L}\right)^2 + \frac{1}{LC}}\end{aligned}\right\} \quad Ans.$$

$\therefore$ Bandwidth $BW = (\omega_2 - \omega_1) = \dfrac{\sqrt{3}\omega_0}{Q} = \dfrac{\sqrt{3}R}{L}$ *Ans.*

22. A series circuit consisting of a coil and a capacitor is excited by a sinusoidal voltage source of E volt and variable frequency. The resonant frequency of the circuit is f_0 and quality factor of the circuit is Q. Calculate the frequency at which the ratio of capacitor voltage to the source voltage is maximum and the maximum value of this ratio.

Sol:

Here, $f_0 = \dfrac{1}{2\pi\sqrt{LC}}$

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

The current, $I = \dfrac{E}{R + j\left(\omega L - \dfrac{1}{\omega C}\right)}$

$\therefore$ Capacitor voltage, $V_C = I \times \dfrac{1}{j\omega C} = \dfrac{1}{j\omega C}\left[\dfrac{E}{R + j\left(\omega L - \dfrac{1}{\omega C}\right)}\right] = \dfrac{E}{j\omega RC - \omega C\left(\omega L - \dfrac{1}{\omega C}\right)}$

$$= \frac{E}{(1-\omega^2 LC) + j\omega RC}$$

$$\frac{V_C}{E} = \frac{1}{(1-\omega^2 LC) + j\omega RC}$$

This ratio will be maximum when the denominator is minimum.

$$\therefore \quad \frac{d}{d\omega}\left[\left((1-\omega^2 LC)^2 + \omega^2 R^2 C^2\right)\right] = 0$$

$$\Rightarrow \quad 2(1-\omega^2 LC)(-2\omega LC) + 2\omega R^2 C^2 = 0$$

$$\Rightarrow \quad -4\omega LC + 4\omega^3 L^2 C^2 + 2\omega R^2 C^2 = 0$$

$$\Rightarrow \quad 2\omega^2 L^2 C^2 = 2LC - R^2 C^2$$

$$\therefore \qquad \omega = \sqrt{\frac{1}{LC}\left(1 - \frac{R^2C}{2L}\right)} \quad \left(\because \omega_0 = \frac{1}{\sqrt{LC}} \text{ and } Q = \frac{1}{R}\sqrt{\frac{L}{C}}\right)$$

$$\Rightarrow \qquad \omega = \omega_0\sqrt{1 - \frac{1}{2Q^2}} \quad \textit{Ans.}$$

Putting this value, the maximum value of the ratio is,

$$\left|\frac{V_C}{E}\right|_{\max} = \frac{1}{\sqrt{\left(1-\omega^2 LC\right)^2 + \omega^2 R^2 C^2}} \qquad \text{(i)}$$

Now,

$$\left(1-\omega^2 LC\right)^2 + \omega^2 R^2C^2 = \left[1 - \left(\frac{1}{LC} - \frac{R^2}{2L^2}\right)LC\right]^2 + R^2C^2\left[\frac{1}{LC} - \frac{R^2}{2L^2}\right]$$

$$= \left(1 - 1 + \frac{R^2C^2}{2L}\right) + \frac{R^2C}{L} - \frac{R^4C^2}{2L^2}$$

$$\Rightarrow \qquad \left(1-\omega^2 LC\right)^2 + \omega^2 R^2C^2 = \left(\frac{1}{2Q^2}\right)^2 + \frac{1}{Q} - \frac{1}{2Q^4}$$

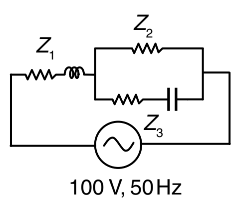
$$= \frac{1 + 4Q^3 - 2}{4Q^4} = \frac{4Q^3 - 1}{4Q^4}$$

Thus, from (i), $\left|\dfrac{V_C}{E}\right|_{\max} = \dfrac{2Q^2}{\sqrt{4Q^3 - 1}}$ *Ans.*

23. Impedances Z_2 and Z_3 in parallel are in series with impedance Z_1 across a 100 V, 50 Hz A.C. supply.

$$Z_1 = (6.25 + j1.25)\,\Omega,\ Z_2 = (5 + j0)\,\Omega,\ Z_3 = (5 - jX_C)\,\Omega$$

Determine the value of capacitance of X_C such that the total current of the circuit will be in phase with the total voltage. What is then the circuit current and power?



Sol:

Here,

$$Z_1 = (6.25 + j1.25)\ \Omega,$$
$$Z_2 = (5 + j0)\ \Omega,$$
$$Z_3 = (5 - jX_C)\ \Omega$$

Total impedance,

$$Z_T = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3} = (6.25 + j1.25) + \frac{5(5 - jX_C)}{10 - jX_C}$$

$$= \left(6.25 + \frac{250 + 5X_C^2}{100 + X_C^2}\right) - j\left(\frac{25X_C}{100 + X_C^2} - 1.25\right)$$

The total current will be in phase with the total voltage if the impedance is purely resistive.

$$\therefore \quad \left(\frac{25X_C}{100 + X_C^2} - 1.25\right) = 0$$

$$\Rightarrow \quad X_C^2 - 20X_C + 100 = 0$$

$$\Rightarrow \quad (X_C - 10)^2 = 0$$

$$\Rightarrow \quad X_C = 10$$

$$\Rightarrow \quad \frac{1}{2\pi fC} = 10$$

$$\Rightarrow \quad C = \frac{1}{2\pi \times 50 \times 10} = 318\,\text{ì F} \quad \textit{Ans.}$$

Putting $X_C = 10$, impedance, $Z_T = 6.25 + \dfrac{250 + 500}{200} = 10\,\Omega$

$$\therefore \quad \text{Current, } I = \frac{V}{Z_T} = \frac{100}{10} = 10\text{ A} \quad \textit{Ans.}$$

$$\therefore \quad \text{Power, } P = I^2 R = 10^2 \times 10 = 1\text{ kW} \quad \textit{Ans.}$$

24. In series *RLC* circuit with variable capacitance, the current is at maximum value with capacitance of 20 μF and the current reduces to 0.707 times maximum value with capacitance of 30 μF. Find the values of *R* and *L*. What is the bandwidth of the circuit if supply voltage is 20 sin (6.28 × 10^3)*t* volt?

Sol:

Here, $V_m = 20,\ \omega = 6.28 \times 10^3;\ \therefore\ f = \dfrac{\omega}{2\pi} = \dfrac{6.28 \times 10^3}{2\pi} = 1000\text{ Hz}$

We know that the maximum at resonance. At this condition the value of C is, $C = 20$ μF.

$$\therefore \quad f_0 = \frac{1}{2\pi\sqrt{LC}} \Rightarrow L = \frac{1}{C(2\pi f_0)^2} = \frac{1}{20 \times 10^{-6} \times (2\pi \times 1000)^2} = 1.2665\text{ mH} \quad \textit{Ans}$$

Here, with variable capacitance, L and f_0 remains constant. At half power frequency, the current becomes 0.707 times the current at resonance with $C = 30\ \mu$F. At this condition, the resistance of the circuit is equal to the reactance of the circuit.

$$\therefore\ R = (X_L - X_C) = \left[2\pi \times 1000 \times 1.2665 \times 10^{-3} - \frac{1}{2\pi \times 1000 \times 30 \times 10^{-6}}\right] = 2.652\ \Omega \quad Ans$$

$$\therefore \qquad \text{Bandwidth, } BW = \frac{f_0}{Q} = \frac{R}{2\pi L} = \frac{2.652}{2\pi \times 1.2665 \times 10^{-3}} = 333.26 \text{ Hz} \quad Ans$$

25. (a) Show that the sum of energy stored by the inductor and the capacitor connected in series at resonance at any instant is constant and is given by LI^2.

(b) Show that the sum of energy stored by the inductor and the capacitor in parallel *RLC* circuit at resonance at any instant is constant and is given by CV^2.

Sol:

(a) Let i and v be the instantaneous current through the inductor and the voltage across the capacitor at any instant of time, t.

Let, $i = I_m \cos \omega t$

$$\therefore \qquad \text{Energy stored in inductor, } W_L = \frac{1}{2}Li^2 = \frac{1}{2}LI_m^{\ 2}\cos^2 \omega t$$

$\therefore$ Energy stored in capacitor,

$$W_C = \frac{1}{2}\frac{q^2}{C}$$

$$= \frac{1}{2C}\left[\int_0^t i\,dt\right]^2$$

$$= \frac{1}{2C}I_m^{\ 2}\left[\int_0^t \cos \omega t\,dt\right]^2$$

$$= \frac{I_m^{\ 2}}{2C}\left[\left(\frac{\sin \omega t}{\omega}\right)_0^t\right]^2$$

$$= \frac{I_m^{\ 2}}{2C} \times \frac{1}{\omega^2}\sin^2 \omega t$$

$$= \frac{I_m^{\ 2}L}{2}\sin^2 \omega t \quad \left(\because \textit{ at resonance } \omega_0 = \frac{1}{\sqrt{LC}}\right)$$

Total energy stored at resonance,

$$W = W_L + W_C = \frac{1}{2}LI_m^{\ 2}\left(\cos^2 \omega t + \sin^2 \omega t\right) = \frac{1}{2}LI_m^{\ 2} = LI^2 \quad [\text{Proved}]$$

(b) Let, $v = V_m \cos \omega t$

$\therefore$ Energy stored in Capacitor, $W_C = \frac{1}{2}Cv^2 = \frac{1}{2}CV_m^{\ 2}\cos^2\omega t$

$\therefore$ Energy stored in Inductor,

$$W_L = \frac{1}{2}Li^2$$

$$= \frac{1}{2}L\left[\int_0^t \frac{1}{L} v dt\right]^2$$

$$= \frac{1}{2L}V_m^{\ 2}\left[\int_0^t \cos\omega t dt\right]^2$$

$$= \frac{V_m^{\ 2}}{2L}\left[\left(\frac{\sin\omega t}{\omega}\right)_0^t\right]^2$$

$$= \frac{V_m^{\ 2}}{2L}\times\frac{1}{\omega^2}\sin^2\omega t$$

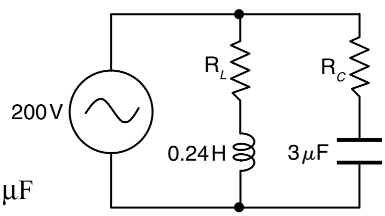
$$= \frac{1}{2}CV_m^{\ 2}\sin^2\omega t \quad \left(\because \ at \ \ resonance \ \ \omega_0 = \frac{1}{\sqrt{LC}}\right)$$

Total energy stored at resonance,

$$W = W_C + W_L = \frac{1}{2}CV_m^{\ 2}\left(\cos^2\omega t + \sin^2\omega t\right) = \frac{1}{2}CV_m^{\ 2} = CV^2 \quad [\text{Proved}]$$

26. Determine the resonant frequency, the source current and the input impedance for the circuit shown in figure for each of the following cases:

Case I	R_L = 150 Ω	R_C = 100 Ω
Case II	R_L = 150 Ω	R_C = 0 Ω
Case III	R_L = 0 Ω	R_C = 0 Ω



Sol:

Case I: Here R_L = 150 Ω, L = 0.24 H, R_C = 100 Ω, and C = 3 μF

Resonant frequency,

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}\left(\frac{L - CR_L^{\ 2}}{L - CR_C^{\ 2}}\right)} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}\left(\frac{R_L^{\ 2} - L/C}{R_C^{\ 2} - L/C}\right)}$$

$$= \frac{1}{2\pi}\sqrt{\frac{1}{0.24\times 3\times 10^{-6}}\left(\frac{150^2 - 0.24/3\times 10^{-6}}{100^2 - 0.24/3\times 10^{-6}}\right)}$$

$$= 170 \ \ Hz \quad Ans.$$

Reactances at this frequency,

$$X_L = j2\pi \times 170 \times 0.24 = j256\,\Omega$$

$$X_C = -\frac{j}{2\pi \times 170 \times 3 \times 10^{-6}} = -j312\,\Omega$$

$$\therefore \qquad I_L = \frac{200}{150 + j256} = (0.34 - j0.582)\text{ A}$$

and
$$I_C = \frac{200}{100 - j312} = (0.186 + j0.582)\text{ A}$$

$\therefore$ Total source current,

$$I = (I_L + I_C) = (0.34 - j0.582) + (0.186 + j0.582) = 0.526\text{ A} \quad \textit{Ans.}$$

$$\therefore \text{ Input impedance } = \frac{200}{0.526} = 380\,\Omega \quad \textit{Ans.}$$

Case II: Here R_L = 150 Ω, L = 0.24 H, R_C = 0, and C = 3 μF
Resonant frequency,

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}\left(1 - \frac{CR_L{}^2}{L}\right)} == \frac{1}{2\pi}\sqrt{\frac{1}{0.24 \times 3 \times 10^{-6}}\left(1 - \frac{3 \times 10^{-6} 150^2}{0.24}\right)} = 159\text{ Hz} \quad \textit{Ans.}$$

Reactances at this frequency,

$$X_L = j2\pi \times 159 \times 0.24 = j240\,\Omega$$

$$X_C = -\frac{j}{2\pi \times 159 \times 3 \times 10^{-6}} = -j334\,\Omega$$

$$\therefore \qquad I_L = \frac{200}{150 + j240} = (0.374 - j0.598)\text{ A}$$

and
$$I_C = \frac{200}{-j334} = j0.598\text{ A}$$

$$\therefore \text{ Total source current, } I = (I_L + I_C) = (0.374 - j0.598) + j0.598 = 0.374\text{A} \quad \textit{Ans.}$$

$$\therefore \text{ Input impedance } = \frac{200}{0.374} = 535\,\Omega \quad \textit{Ans.}$$

Case III: Here R_L = 0, L = 0.24 H, R_C = 0, and C = 3 μF
Resonant frequency,

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} == \frac{1}{2\pi}\sqrt{\frac{1}{0.24 \times 3 \times 10^{-6}}} = 188\text{ Hz} \quad \textit{Ans.}$$

Reactances at this frequency,

$$X_L = j2\pi \times 188 \times 0.24 = j283\,\Omega$$

$$X_C = -\frac{j}{2\pi \times 188 \times 3 \times 10^{-6}} = -j283\,\Omega$$

$$\therefore \qquad I_L = \frac{200}{j283} = -j0.706 \text{ A}$$

$$\text{and} \qquad I_C = \frac{200}{-j283} = j0.706 \text{ A}$$

$\therefore$ Total source current, $I = (I_L + I_C) = 0 \text{ A}$ *Ans.*

$\therefore$ Input impedance $= \dfrac{200}{0} = \infty$ *Ans.*

27. A voltage of v = 2000 sin ωt + 400 sin 3 ωt + 100 sin 5 ωt is applied to a series circuit having R = 10 Ω and C = 30 μF and a variable inductance. (i) Find the value of inductance so as to give resonance at 3rd harmonic frequency. (ii) What are the r.m.s. values of voltage and current with this inductance in circuit? Take ω = 300 rad/s.

Sol:

Here, ω = 300 rad/s, R = 10 Ω, $C = 30 \times 10^{-6}$ F

$\therefore$ Resonant frequency $\omega_0 = 3 \times \omega = 900$ rad/s

(i) $\because \dfrac{1}{\sqrt{LC}} = 900 \Rightarrow L = \dfrac{1}{900^2 \times 30 \times 10^{-6}} = 41.152 \text{ mH}$ *Ans.*

(ii)

For 1st Harmonic (ω = 300 rad/s):

$$R = 10\Omega,\ X_{L_1} = 300 \times 41.152 \times 10^{-3} = 12.3456\,\Omega \quad X_{C_1} = \frac{-j}{300 \times 30 \times 10^{-6}} = -j111.11\,\Omega$$

$$\therefore \qquad Z_1 = 10 + j(12.3456 - 11.11) = (10 - j98.77)\,\Omega = 99.27\angle 95.78^\circ\,(\Omega)$$

$$\therefore \qquad I_1 = \frac{2000}{Z_1} = \frac{2000}{99.27\angle 95.78^\circ} = 20.15\angle -95.78^\circ\,(\text{A})$$

For 3rd Harmonic (ω = 900 rad/s):

$$\therefore \quad R = 10\Omega,\ X_{L_3} = 900 \times 41.152 \times 10^{-3} = 37.037\,\Omega,\ X_{C_3} = \frac{-j}{900 \times 30 \times 10^{-6}} = -j37.037\,\Omega = X_L$$

$$\therefore \qquad Z_3 = 10 + j(37.037 - 37.037) = 10\,\Omega$$

$$\therefore \qquad I_3 = \frac{400}{Z_3} = \frac{400}{10\angle 0^\circ} = 40\angle 0^\circ\,(\text{A})$$

For 5th Harmonic ($\omega = 1500$ rad/s):

$$R = 10\,\Omega,\ X_{L_5} = 1500 \times 41.152 \times 10^{-3} = 61.728\,\Omega,\ X_{C_5} = \frac{-j}{1500 \times 30 \times 10^{-6}} = -j22.22\,\Omega = X_L$$

$$\therefore \qquad Z_5 = 10 + j(61.728 - 22.22) = (10 + j39.506)\,\Omega = 40.752\angle 75.795°\ (\Omega)$$

$$\therefore \qquad I_5 = \frac{100}{Z_5} = \frac{100}{40.752\angle 75.795°} = 2.454\angle -75.795°\ (\text{A})$$

$\therefore$ RMS value of the current,

$$I_{\text{rms}} = \sqrt{\frac{I_1^2 + I_3^2 + I_5^2}{2}} = \sqrt{\frac{(20.15)^2 + 40^2 + (2.454)^2}{2}} = 31.72\ (\text{A}) \quad \textit{Ans.}$$

$\therefore$ RMS value of the voltage,

$$V_{\text{rms}} = \sqrt{\frac{V_1^2 + V_3^2 + V_5^2}{2}} = \sqrt{\frac{(2000)^2 + (400)^2 + (100)^2}{2}} = 1041.63\ (\text{V}) \quad \textit{Ans.}$$

SUMMARY

1. In an electrical system, the phenomenon of cancellation of reactances when inductor and capacitor are in series or cancellation of susceptances when they are in parallel, is termed as *resonance*.
2. In series resonance, the current at resonance is the maximum, whereas is case of parallel resonance, the current at resonance is the minimum.
3. Quality factor for series resonant circuit is $\dfrac{\omega_0 L}{R}$ or $\dfrac{1}{\omega_0 RC}$ or $\dfrac{1}{R}\sqrt{\dfrac{L}{C}}$, whereas quality factor for parallel circuit is $\dfrac{R}{\omega_0 L}$ or $\omega_0 RC$ or $R\sqrt{\dfrac{C}{L}}$.
4. Series resonant circuit acts as a voltage amplifier whereas parallel resonant circuit acts as a current amplifier.
5. Under resonant condition, all circuits act as resistive so that the power factor of the circuit is unity.
6. Both for series and parallel resonant circuit, resonant frequency is $\omega_0 = \dfrac{1}{\sqrt{LC}}$ and bandwidth is $BW = \dfrac{f_0}{Q}$.
7. The concept resonant is useful in oscillator circuits, voltage amplifier, current amplifier, in RF amplifier and other filter circuits.

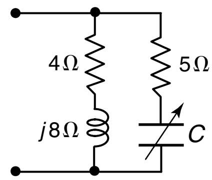
EXERCISES

1. A generator supplies a variable frequency voltage of constant amplitude 100 V(rms) to a series *RLC* circuit where $R = 5\ \Omega$, $L = 4$ mH and $C = 0.01\ \mu$F. The frequency is to be varied until maximum current flows. Predict the maximum current, the frequency at which it occurs, and the resulting voltage across the inductance and the capacitance.
[20 A(rms); 7,958 Hz; 4 kV]
2. If the bandwidth of a resonant circuit is 10 kHz and the lower half power frequency is 120 kHz, find out the value of the upper half power frequency and the quality factor of the circuit.
[130 kHz, 12.5]
3. A series *RLC* circuit consists of a 100 Ω resistor, an inductor of 0.318 H and a capacitor of unknown value. When this circuit is energized by $230\angle 0°$ V, 50Hz sinusoidal ac supply, the current was found to be $2.3\angle 0°$ A. Find (i) the value of capacitor in micro-farad (ii) the voltage across the inductor (iii) the total power consumed.
[31.86 μF; $230\angle 90°$ V (leading); 529 W]
4. For a series *RLC* circuit the inductor is variable. Source voltage is $200\sqrt{2}\sin 100\,\pi t$. Maximum current obtainable by varying the inductance is 0.314 A and the voltage across the capacitor then is 300V. Find the circuit element values.
[900 Ω; 3.332 μF; 3.04 H]
5. It is desired to design a series resonant circuit with the following specifications.
$$C = 250 \times 10^{-12}\ \text{F},\ f_0 = 600\ \text{kHz},\ BW = 20\ \text{kHz}$$
Calculate Q_0, R and L of the circuit. Also, calculate the current at 500 kHz as a fraction of the current at resonance.
[30; 35.37 Ω; 0.28 mH; 0.09 I_r]
6. A 10 mH coil is connected in series with a loss free capacitor to a variable frequency source of 20 V. The current in the circuit has the maximum value of 0.2 A at a frequency of 100 kHz. Calculate: (i) the value of capacitance (ii) the *Q*-factor of the coil (iii) the half power frequencies.
[253.3 pF, 62.83, 99.204 Kz, 100.796 Hz]
7. A series *RLC* circuit is excited from a constant voltage variable frequency source. The current in the circuit becomes maximum at a frequency $\dfrac{600}{2\pi}$ Hz and falls to half the maximum value at $\dfrac{400}{2\pi}$ Hz. If the resistance in the circuit is 3 Ω, find *L* and *C*.
[10.4 mH; 267.3 μF]
8. An *RLC* series circuit has $R = 100$ ohm, $L = 500$ mH and $C = 40\ \mu$F. Calculate:
(i) The resonant frequency;
(ii) Lower half power frequency;

(iii) Upper half power frequency;
(iv) Bandwidth;
(v) Q factor.
Also derive the expression for the above.

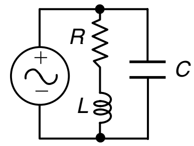
[223.6 rad/s; 22.5 rad/s; 222.5 rad/s; 200; 1.118]

9. Calculate the value of C which results in resonance for the circuit shown in figure when frequency is 1000 Hz and find Q-factor for each branch.
[31.83 mF, 2 (for R-L), 1 (for R-C)]

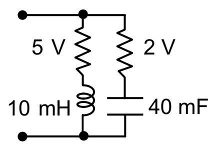


10. A capacitor is connected in parallel with a coil having L = 5.52mH and R = 10 Ω, to a 100 V, 50 Hz supply. Calculate the value of the capacitance for which the current taken from the supply is in phase with the voltage.
[53.6 μF]

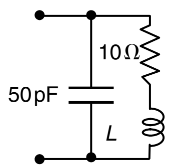
11. A parallel circuit has fixed C and variable L. Quality factor of the inductor is $Q = 4$. Find the values of L and C for the circuit impedance of $(100 + j0)$ ohm at f = 2.4 MHz. What is the bandwidth at matched condition?
[1.557 mH, 2.648 nF, 1.2 MHz]



12. Find the resonant frequency for the circuit shown in figure.
[263 Hz]



13. For a practical tank circuit shown in figure the resonance occurs at 1 MHz. Assuming a high-Q coil, find the quality factor of the high-Q coil at resonance frequency.

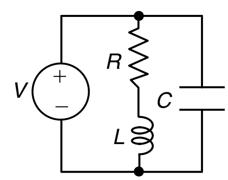


[2000]

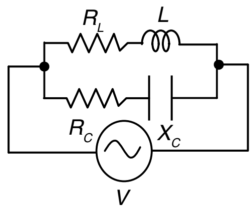
14. Calculate the impedance of the parallel tuned circuit, as shown in figure at a frequency of 500 kHz and for the bandwidth of operation equal to 20 kHz. The resistance of the coil is 5 Ω.
[3.13 kΩ]

15. A coil of resistance R and inductance L is shunted by a capacitor C.

Show that: (a) the resonant frequency is, $\omega_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$. (b) the effective resistance is L/CR. (c) the circulating current is $V\sqrt{\frac{C}{L}}$.



16. For the parallel resonant circuit, prove that:

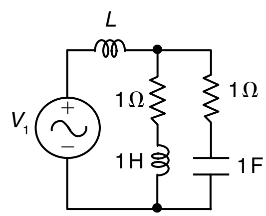


(a) The resonant frequency is

$$\frac{1}{\sqrt{LC}}\sqrt{\frac{R_L{}^2 - L/C}{R_C{}^2 - L/C}}.$$

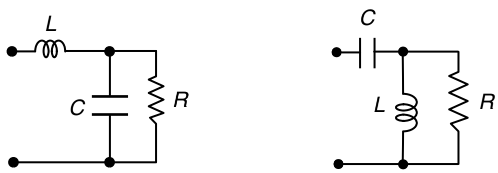
(b) The impedance is independent of frequency if $R_L = R_C = \sqrt{L/C}$.

17. What is the condition for resonance in the circuit shown in figure? For what value of L the circuit will resonate at all frequencies?



[$\omega = 0$; $L = 0$]

18. Find the resonant frequency for the circuits. Also, find the quality factor, Q. What will be the impedances at resonance?



(i) $\sqrt{\dfrac{1}{LC} - \dfrac{1}{R^2C^2}}$, $\omega_0 RC$, $\dfrac{R}{1+\omega^2 R^2 C^2}$

(ii) $\dfrac{R}{\sqrt{L\left(CR^2 - L\right)}}$, $\dfrac{R}{\omega_0 L}$, $\dfrac{L}{CR}$

QUESTIONS

1. (a) What is resonance in an ac circuit?
   (b) Discuss the effects of resonance in electrical systems.
2. Discuss briefly the phenomenon of electrical resonance in simple *RLC* circuits. Derive an expression for the condition for resonance in *RLC* circuits. Also, draw the phasor diagrams.

   Or,

   State and explain the condition of resonance in a series *RLC* circuit of an ac circuit. Draw the phasor diagram.
3. Show that $\omega_r = \sqrt{\omega_l \omega_h}$ for a series *RLC* circuit, where
   ω_r = the resonant frequency
   ω_l = lower half power frequency
   ω_h = upper half power frequency
   of the circuit.

   Or,

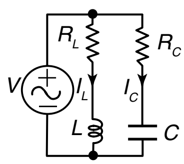
   Show that the resonant frequency ω_0 of a series *RLC* circuit is the geometric mean of ω_1 and ω_2, the lower and upper half power frequencies respectively.
4. (a) Define the terms *Q* factor and bandwidth.
   (b) Derive expression for *Q* factor for *RL* and *RC* series circuit.
   (c) Define the *Q* factor for the series resonant circuit and express it in terms of the circuit parameters.
   (d) What is the relationship between bandwidth and quality factor for a *RLC* circuit?

   Or,

   Show that: $\dfrac{f_2 - f_1}{f_0} = \dfrac{1}{Q_0}$ Or, $\quad Quality \;\; factor = \dfrac{Resonant \;\; frequency}{Bandwidth}$

   Where, f_2 and f_1 are half power frequencies, f_0 is the resonant frequency and Q_0 is the *Q* factor at resonant frequency.
   (e) Show that: (i) $f_1 f_2 = f_0^2$ and (ii) $f_1^2 + f_2^2 \geq 2f_0^2$
   where f_0 is the resonance frequency and f_1, f_2 are the half power frequencies of a series resonant circuit
5. (a) Define Selectivity and half power frequency.
   (b) Show that a circuit must have a large value of Q_0 (*Q* factor at resonance frequency) to be highly selective.
6. Prove that in a series resonant circuit, the voltage across capacitor and inductor is *Q* factor times the supply voltage.
7. In a series *RLC* circuit, the voltage across *L* and *C* at resonance may exceed even the supply voltage. Why?
8. Explain the effect of increase in *L*/*C* ratio on the following factors:
   (a) resonant frequency, (b) *Q*, (c) bandwidth of an *RLC* series circuit.

9. In an RLC series circuit, the source frequency is varied from zero to infinity. How do the values of voltage across L and C change? Draw curve showing these variations. Derive expression for maximum values of these voltages and the frequencies at which the maximum values occur.
10. (a) Describe the phenomenon of resonance in parallel circuits and explain its Q factor.
    (b) Prove that the Q factor of parallel resonant circuit is reciprocal of that for a series resonant circuit.
    (c) Find an expression for impedance at the antiresonance of a parallel tuned circuit and also sketch the variation of the impedance of the same circuit with frequency.
11. (a) Compare the properties of series and parallel resonance.
    (b) The series resonant circuit is often regarded as the acceptor circuit and the parallel resonant as the rejector circuit. Explain.
    (c) The series resonance is called voltage resonance and the parallel resonance the current resonance- why?
12. At resonance, the current is maximum in series circuit and minimum in parallel circuit. Why?
13. For the circuit shown, draw the phasor diagram. Derive the condition for the two branch currents, I_L and I_C to be in quadrature.



14. The shape of resonance curve depends on Q of the coil. Why?
15. Derive an expression for the resonant frequency of a parallel circuit consisting of inductance L and resistance R in one branch and a capacitance C in the other.
16. A coil of resistance R and inductance L is shunted by a capacitor. Show that for rejector (parallel) resonance, the effective resistance is $\dfrac{L}{CR}$. Show also that the circulating current is $V\sqrt{\dfrac{C}{L}}$ so long as the resistance is small.

    Or,

    A resistive inductive coil is connected in parallel with a condenser and the combination is connected with a sinusoidal emf. Derive the expression for current flowing through the two branches and hence find the condition for obtaining the minimum input current and the value of the maximum impedance of the circuit.
17. A series combination of a capacitance C and resistance R_C is shunted by an inductive coil having resistance R_L and inductance L and the combination is connected to an ac source. Derive the expression for the resonant frequency of the circuit. Also, show that the impedance of the circuit will be independent of frequency (or, the circuit will be under resonant condition at any frequency) if $R_C = R_L = \sqrt{\dfrac{L}{C}}$.

18. (a) Show that the sum of energy stored by the inductor and the capacitor connected in series at resonance at any instant is constant and is given by LI^2.
    (b) Show that the sum of the energy stored by the inductor and the capacitor in parallel *RLC* circuit at any instant is constant at resonance frequency and is equal to CV^2.

Circuit Theory & Networks (EE301)
Solution of 2011 WBUT Paper

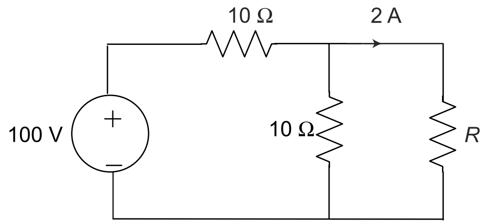
GROUP–A
(Multiple-Choice Questions)

1. Choose the correct alternatives for any *ten* of the following: **$10 \times 1 = 10$**

(i) The internal impedance of an ideal current source is

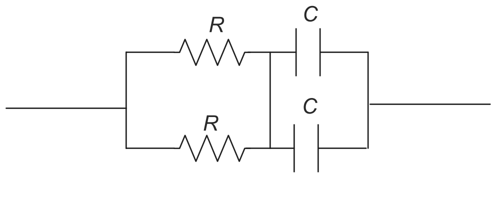
(a) zero (b) infinite (c) both (a) and (b) (d) none of these

(ii) In the figure given below, the value of the resistance R in ohms is



(a) 10 (b) 20 (c) 30 (d) 40

(iii) Time constant of the network shown below is

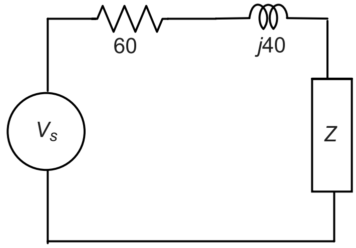


(a) CR (b) $2CR$ (c) $CR/4$ (d) $CR/2$

(iv) For a series RC circuit, when subjected to a unit step input voltage, the voltage across the capacitor will be

(a) $1-e^{-t/RC}$ (b) $e^{-t/RC}$ (c) $e^{t/RC}$ (d) 1

(v) In the figure given below, the value of the load Z which maximizes the power delivered to it is

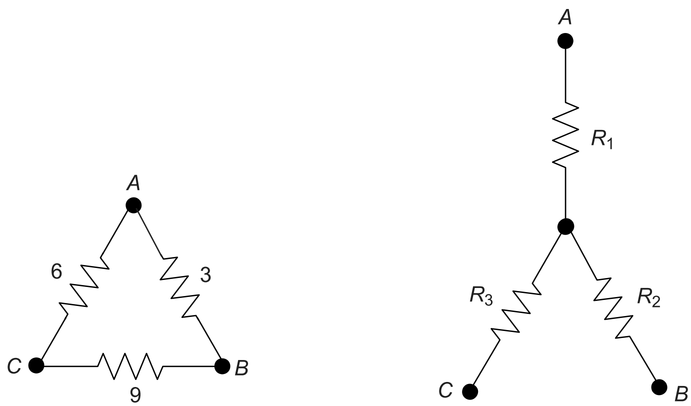


(a) $60 + j\,40$ (b) $60 - j40$ (c) 60 (d) none of these

(vi) If the unit step response of a network is $(1 - e^{-\alpha t})$, the unit impulse response will be

(a) $\alpha e^{-\alpha t}$ (b) $\dfrac{1}{\alpha e^{-\alpha t}}$ (c) $\dfrac{1}{\alpha} e^{-\alpha t}$ (d) $(1 - \alpha)e^{-\alpha t}$

(vii) The resistances R_1, R_2 and R_3 are respectively



(a) 1, 3/2 and 3 (b) 3, 3/2 and 6 (c) 9, 3 and 1 (d) 2, 1 and 9

(viii) The z-matrix of a 2-port network is given by $\begin{bmatrix} 0.9 & 0.2 \\ 0.2 & 0.6 \end{bmatrix}$. The element Y_{22} of the corresponding Y-matrix of the same network is given by

(a) 1.2 (b) 0.4 (c) −0.4 (d) 1.8

(ix) The transfer function of an electric low-pass RC network is

(a) $\dfrac{RCs}{1+RCs}$ (b) $\dfrac{1}{1+RCs}$ (c) $\dfrac{RC}{1+RCs}$ (d) $\dfrac{s}{1+RCs}$

(x) How many branches can be connected to a node?

(a) 1 (b) 2 (c) 3 (d) any number

(xi) When a number of 2-port networks are connected in cascade, the individual

(a) Z_{OC} matrices are added (b) Y_{SC} matrices are added

(c) chain matrices are multiplied (d) H matrices are multiplied

(xii) The tie-set matrix gives the relation between
(a) branch currents and link currents (b) branch voltages and link currents
(c) branch currents and link voltages (d) none of these

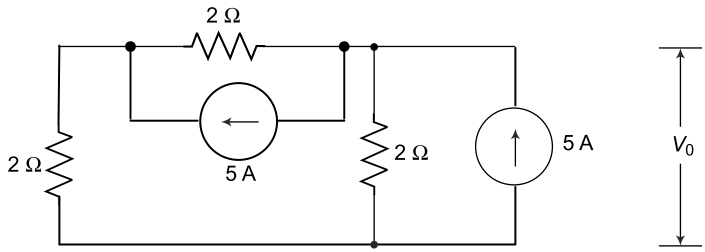
Solutions

(i) (b) infinite (ii) (b) 20 (iii) (a) CR (iv) (a) $1-e^{-t/RC}$
(v) (b) $60 - j40$ (vi) (a) $\alpha e^{-\alpha t}$ (vii) (a) 1, 3/2 and 3 (viii) (d) 1.8
(ix) (b) $\dfrac{1}{1+RCs}$ (x) (d) any number (xi) (a) Z_{OC} matrices are added
(xii) (a) branch currents and link currents

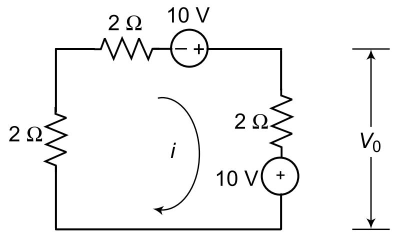
GROUP–B
(Short-Answer-Type Questions)

Answer any *three* of the following questions. **3 × 5 = 15**

2. Convert the current sources into voltage sources (equivalent) and find the voltage V_0.



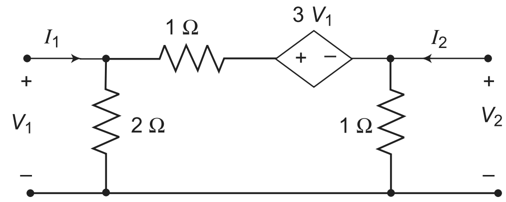
Solution: Converting the current sources into voltage sources, we get the following circuit.



$$\therefore \qquad i = -\frac{20}{6} = -\frac{10}{3}\text{ A}$$

$$\therefore \qquad V_0 = 2i + 10 = 2\times\left(-\frac{10}{3}+10\right) = \frac{10}{3} = 3.33\text{ Volt}$$ **Ans.**

3. For the network given below, determine the X-parameters.



Solution:

(NOTE: Since "X-parameters" do not exist, we find Z-parameters.)

We consider two cases:

When $I_2 = 0$: The modified circuit is shown in Fig. (a).

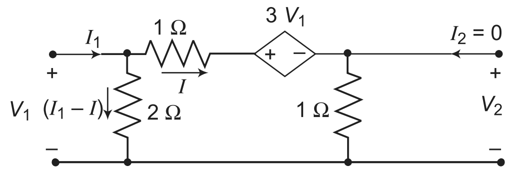


Fig. (a)

By KVL for the middle mesh, we get,

$$I + 3V_1 + I - 2 \times (I_1 - I) = 0$$

$$\Rightarrow \qquad I = \left(\frac{1}{2}I_1 - \frac{3}{4}V_1\right) \qquad \text{(i)}$$

By KVL for the left mesh, we get,

$$V_1 = 2 \times (I_1 - I) = 2I_1 - 2\left(\frac{1}{2}I_1 - \frac{3}{4}V_1\right) \quad \{\text{by Eq. (i)}\}$$

or, $\qquad V_1 = -2I_1$

$$\therefore \qquad z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = -2\ \Omega$$

Also, by KVL for the right mesh, we get,

$$V_1 = I = \left(\frac{1}{2}I_1 - \frac{3}{4}V_1\right) = \frac{1}{2}I_1 - \frac{3}{4}(-2I_1) = 2I_1$$

$$\therefore \qquad z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = 2\ \Omega$$

When $I_1 = 0$: The modified circuit is shown in Fig. (b).

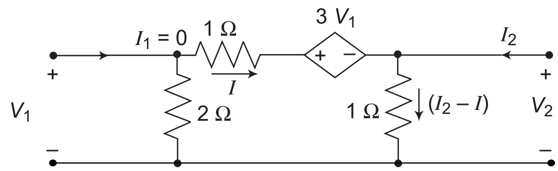


Fig. (b)

By KVL for the middle mesh, we get,

$$I - 3V_1 + 2I - 1 \times (I_2 - I) = 0$$

$$\Rightarrow \qquad I = \left(\frac{1}{4}I_2 + \frac{3}{4}V_1\right) \qquad \text{(ii)}$$

By KVL for the left mesh, we get,

$$V_1 = 2I = 2\times\left(\frac{1}{4}I_2 + \frac{3}{4}V_1\right) = \frac{1}{2}I_2 + \frac{3}{2}V_1$$

$$\Rightarrow \qquad V_1 = -I_2$$

$$\therefore \qquad z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = -1\,\Omega$$

Also, by KVL for the right mesh, we get,

$$V_2 = 1\times(I_2 - I) = I_2 - \left(\frac{1}{4}I_2 + \frac{3}{4}V_1\right) \qquad \text{\{by Eq. (ii)\}}$$

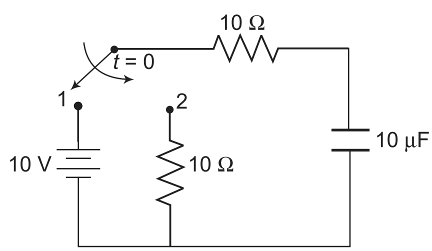
$$= \frac{3}{4}I_2 - \frac{3}{4}\times(-I_2) = 1.5I_2$$

$$\therefore \qquad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = 1.5\,\Omega$$

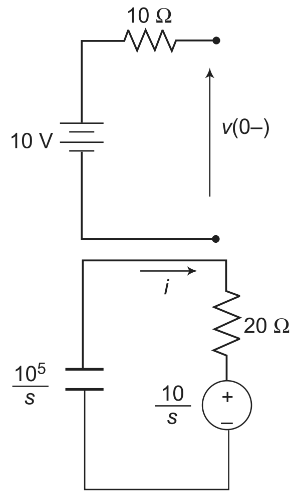
Therefore, the z-parameters of the network are,

$$[z] = \begin{bmatrix} -2 & -1 \\ 2 & 1.5 \end{bmatrix}(\Omega) \qquad \textbf{Ans.}$$

4. In the circuit given below, the switch is initially in the position 1 until the steady state is reached. At $t = 0$, the switch is moved to the position 2. Find $i(t)$, the loop current.



***Solution*:**



When the switch is in the position 1, steady state exists and the initial voltage across the capacitor is,

$$v(0-) = 10 \text{ V}$$

After the switch is moved to the position 2, the KVL gives, in Laplace transform,

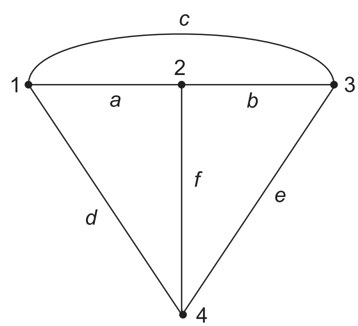
$$\frac{1}{10\times 10^{-6}s}I(s) + 20\times I(s) + \frac{10}{s} = 0$$

or, $$I(s) = \frac{0.5}{s+5000}$$

Taking inverse Laplace transform,

$$i(t) = 0.5e^{-5000t}\ (A);\ t > 0$$ **Ans.**

5. (a) Define incidence matrix. 2
   (b) For the graph shown below, find the complete incidence matrix. 3



Solution:

(a) **Incidence Matrix:** The incidence matrix symbolically describes a network. It also facilitates the testing and identification of the independent variables. Incidence matrix is a matrix which represents a graph *uniquely*.

For a given graph with 'n' nodes and 'b' branches, the complete incidence matrix A_a is a rectangular matrix of order $n \times b$, whose elements have the following values:

Number of columns in [A] = Number of branches = b

Number of rows in [A] = Number of nodes = n

A_{ij} = 1, if branch j is associated with node i and oriented away from node j.
= –1, if branch j is associated with node i and oriented towards node j.
= 0, if branch j is not associated with node i.

Example:

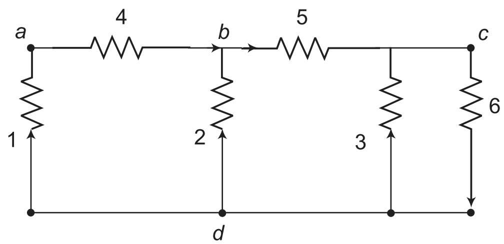


Fig. (a) *Network*

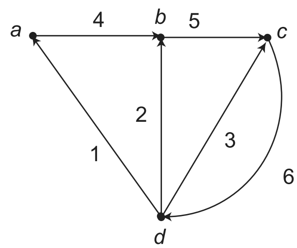
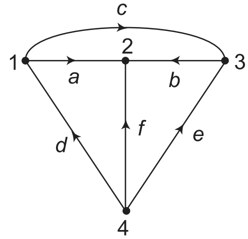


Fig. (b) *Graph of Network*

Incidence matrix A

		Branches						
		1	2	3	4	5	6	
Nodes	a	1	0	0	–1	0	0	Reduced incidence matrix A_I
	b	0	1	0	1	–1	0	
	c	0	0	1	0	1	–1	
Reference node	d	–1	–1	–1	0	0	1	

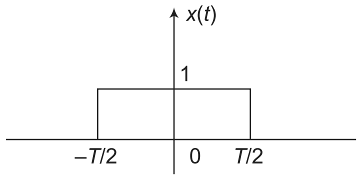
(b) The given graph does not have any orientation. We assume the orientation as shown in the figure below.



For the graph shown in the figure, the complete incidence matrix can be written as,

$$A_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

6. Find the Fourier transform for the following gate function:



Solution: The pulse is, $f(t) = 1, -\frac{T}{2} < t < \frac{T}{2}$

So, the Fourier transform is,

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} 1e^{-j\omega t}dt = \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{j\omega} = 2\frac{\sin\left(\frac{\omega T}{2}\right)}{\omega}$$

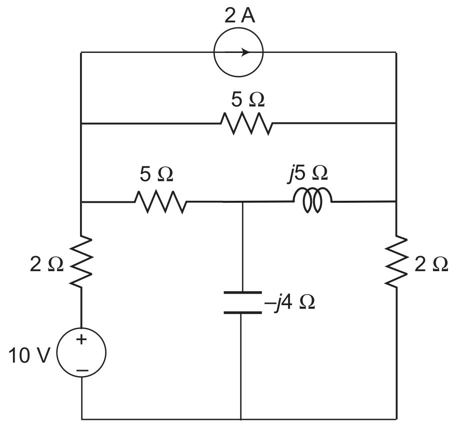
$$= 2\frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)} \times \frac{T}{2}$$

$$\therefore \qquad F(j\omega) = T\frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)}$$ **Ans.**

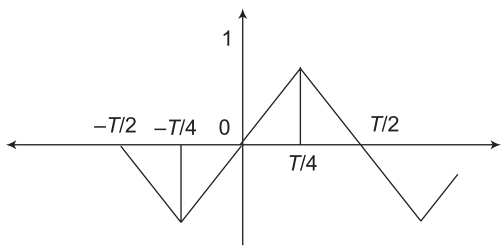
GROUP–C
(Long-Answer-Type Questions)

Answer any *three* questions. **3 × 15 = 45**

7. (a) Consider the network illustrated below, draw its graph, and determine.
   (i) Number of links
   (ii) Rank of the graph
   (iii) Total number of trees. 8

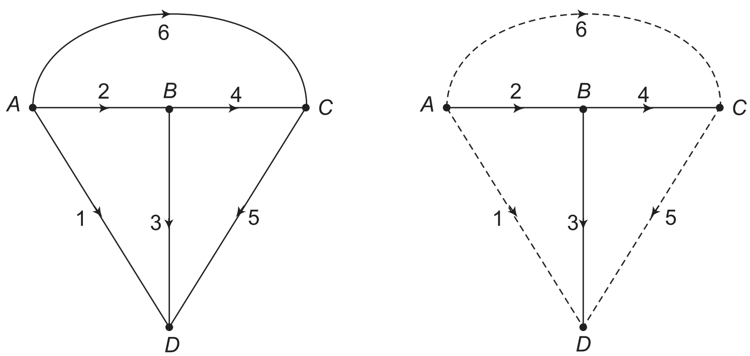


(b) Determine the Fourier series expansion for the following waveform. 7



Solution:

(a) The graph of the given graph is shown in the figure below. One tree has been shown in the figure.



(i) Number of links: For the tree shown in the figure, the number of links is 3.

(ii) Rank of the graph: Since the graph has four nodes (i.e. node A, B, C and D, $n = 4$; so, the rank of the graph is

$$(n - 1) = (4 - 1) = 3$$

(iii) Total number of trees: The number of possible trees of a graph, = det $\{[A]\times[A]^T\}$, where, A is the reduced incidence matrix. We find A for the given graph.

The complete incidence matrix is,

$$A_a = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix}$$

Reduced incidence matrix is,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix}$$

Thus, the number of possible trees of the graph is,

$$N = \det\{[A]\times[A]^T\} = \det\left\{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}\right\} = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 16$$

(b) The wave is an odd function and has half-wave symmetry.

$\therefore$ $a_n = 0$ and $a_0 = 0$

Now,

$$V(t) = \frac{4}{T}t; \quad 0 < t < \frac{T}{4}$$

$$= -\frac{4}{T}t + 2; \quad \frac{T}{4} < t < \frac{3T}{4}$$

$$\therefore \quad b_n = \frac{8}{T}\int_0^{T/4} f(t)\sin n\omega t dt; \ n \text{ is odd only.}$$

$$= \frac{8}{T}\int_0^{T/4} \frac{4}{T} t\sin n\omega t dt$$

$$= \frac{32}{T^2}\left[\frac{-t\cos n\omega t}{n\omega} + \int \frac{\cos n\omega t}{n\omega} dt\right]_0^{T/4}$$

$$= \frac{16}{n\pi T}\left[-\frac{T}{4}\cos\frac{n\pi}{2} + \frac{\sin n\omega t}{n\omega}\Bigg|_0^{T/4}\right]$$

$$= \frac{16}{n\pi T}\left[-\frac{T}{4}\times 0+\frac{T}{2n\pi}\sin\frac{n\pi}{2}\right]$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \{\because \quad \omega T = 2\pi\}$$

$$\therefore \qquad b_n = \frac{8}{n^2\pi^2},\ n = 1, 5, 9, \ldots$$

$$= -\frac{8}{n^2\pi^2},\ n = 3, 7, 11, \ldots$$

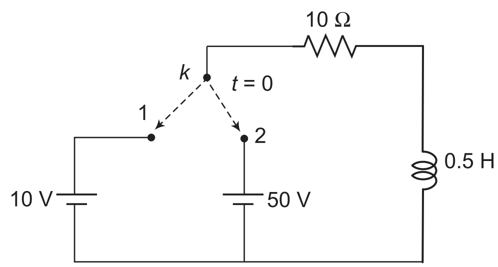
Hence,

$$\boxed{V(t) = \frac{8}{\pi^2}\left(\sin\omega t - \frac{1}{3^2}\sin 3\omega wt + \frac{1}{5^2}\sin 5\omega t - \frac{1}{7^2}\sin 7\omega t + \ldots\right)}$$ **Ans**

8. (a) State the final-value theorem. 2

(b) Find the expression for the current $i(t)$ for a series R-C circuit, if the circuit is initially relaxed. 3

(c) In the circuit shown below, determine the current $i(t)$ when the switch is changed from position 1 to position 2 at $t = 0$. Find the steady-state current using final-value theorem. 10



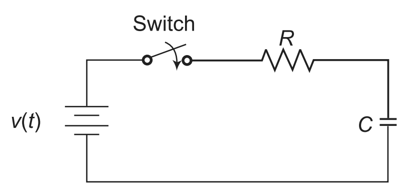
Solution:

(a) **Final-value Theorem:** If a function $f(t)$ and its derivatives are Laplace transformable then the final value of $f(t)$ is,

$$f(\infty) = \underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ [sF(s)]$$

This theorem is only applicable if the value of the function $f(t)$ is finite as t becomes infinity, i.e., $F(s)$ has all poles lying in the left half of the s-plane or at most one simple pole at the origin.

(b) **Current for an Initially Relaxed Series *R-C* Circuit:** We consider an RC series circuit as shown in figure. We assume the initial voltage (charge) across the capacitor to be zero.



By KVL, $$Ri(t) + \frac{1}{C}\int_0^t i(t)dt = Vu(t)$$

Taking Laplace transform,

$$RI(s) + \frac{I(s)}{Cs} = \frac{V}{s}$$

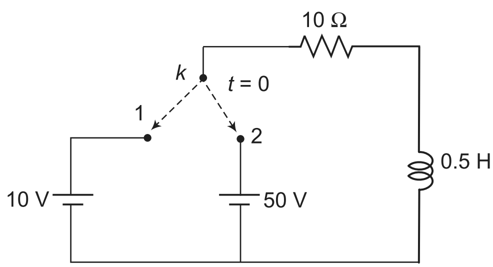
or, $$I(s)\left[R + \frac{1}{Cs}\right] = \frac{V}{s}$$

or, $$I(s) = \frac{V}{s\left(R + \frac{1}{Cs}\right)} = \frac{1}{R}\frac{V}{\left(s + \frac{1}{RC}\right)}$$

Taking inverse Laplace transform, the current is given as,

$$i(t) = \frac{V}{R}e^{-t/RC}\text{; for } t \geq 0$$

(c)



When the switch is in the position 1, steady state exists and the initial current through the inductor is,

$$i(0-) = \frac{10}{10} = 1\text{ A}$$

After the switch is moved to the position 2, the KVL gives in Laplace transform,

$$10I(s) + 0.5sI(s) - 0.5 \times 1 = \frac{50}{s}$$

or, $$I(s) = \frac{100}{s(s+20)} + \frac{1}{s+20} = 5\left[\frac{1}{s} - \frac{1}{s+20}\right] + \frac{1}{s+20}$$

Taking inverse Laplace transform,

$$i(t) = 5 - 4e^{-20t}\quad (A);\ t > 0;$$ **Ans**

Now, the Laplace transform of the current has been obtained as,

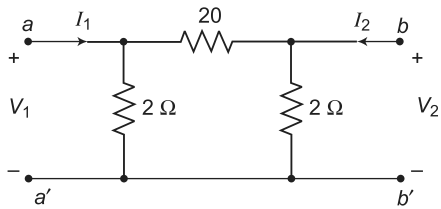
$$I(s) = \frac{100}{s(s+20)}$$

Hence, applying the final-value theorem, the steady-state value of the current is obtained as,

$$i(\infty) = \underset{t\to\infty}{\text{Lt}}\ i(t) = \underset{s\to 0}{\text{Lt}}\ [sI(s)] = \underset{s\to 0}{\text{Lt}}\left[s\frac{100}{s(s+20)}\right] = 5$$ **Ans.**

9. (a) Find the condition of reciprocity and symmetry for short-circuit parameters of a 2-port network. 4 + 4

(b) Find the transmission parameters for the circuit shown below: 7



Solution:

(a) **Condition of Reciprocity for Short-Circuit Parameters of a Two-Port Network:** A network is said to be reciprocal if the ratio of the response transform to the excitation transform is invariant to an interchange of the positions of the excitation and response of the network.

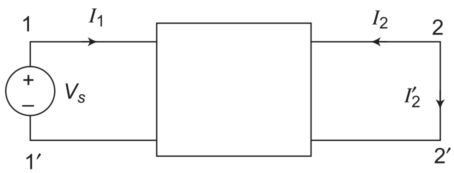


Fig. (a) *Reciprocal network*

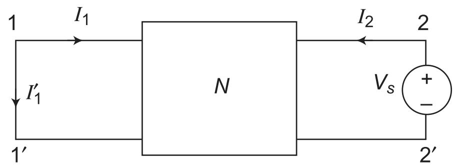


Fig. (b) *Reciprocal network*

Firstly, we short circuit port 2–2′ and apply a voltage source V_s at port 1–1′, as Fig. (a). Hence,

$$V_1 = V_s,\ V_2 = 0,\ I_2 = -I_2'$$

Writing the y-parameter equations,

$$\begin{aligned} I_1 &= y_{11}V_s \\ -I_2' &= y_{21}V_s \end{aligned} \quad \Longrightarrow \quad -\frac{I_2'}{V_s} = y_{21} \tag{1}$$

Now, interchanging the positions of response and excitations, i.e., shorting port 1–1′ and applying V_s at port 2 – 2′, as shown in Fig. (b), we have,

$$V_1 = 0,\ V_2 = V_s,\ I_1 = I_1'$$

From Fig. (b), writing the y-parameter equations,

$$\begin{aligned} -I_1' &= y_{12}V_s \\ I_2 &= y_{22}V_s \end{aligned} \quad \Longrightarrow \quad -\frac{I_1'}{V_s} = y_{12} \tag{2}$$

From the principle of reciprocity, from Eq. (1) and (2), the condition for reciprocity is obtained as,

$$\boxed{y_{12} = y_{21}}$$

Condition of Symmetry for Short-Circuit Parameters of a Two-Port Network: A two-port network is said to be symmetric if the ports can be interchanged without changing the port voltages and currents.

Applying a voltage V_s at port 1–1′ with port 2–2′ short-circuited, we have the equation,

$$I_1 = y_{11}V_s - y_{11}.0 = y_{11}V_s \longrightarrow \left.\frac{I_1}{V_s}\right|_{V_2=0} = y_{11} \tag{3}$$

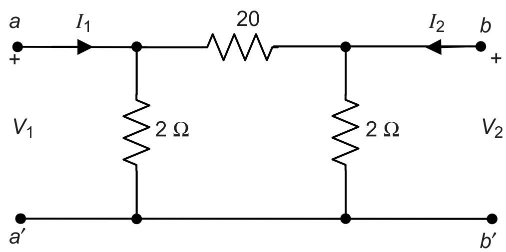
Now, applying a voltage V_s at port 2–2′ with port 1–1′ short-circuited, we have the equation,

$$I_2 = y_{12}.0 + y_{22}V_s = y_{22}V_s \longrightarrow \left.\frac{I_2}{V_s}\right|_{V_1=0} = y_{22} \tag{4}$$

For the network to be symmetrical, the voltages and currents should be same. From Eq. (3) and Eq. (4), we have the condition for symmetry as,

$$\boxed{y_{11} = y_{22}}$$

(b)



This is a π-network, the y-parameters are given as,

$$y_{11} = \left(\frac{1}{2} + \frac{1}{20}\right) = \frac{11}{20}\,\mho; \quad y_{12} = y_{21} = -\frac{1}{20}\mho; \quad y_{22} = \left(\frac{1}{2} + \frac{1}{20}\right) = \frac{11}{20}\,\mho$$

$$\therefore \qquad \Delta y = (y_{11}y_{22} - y_{12}y_{21}) = \left(\frac{11}{20}\right)^2 - \left(-\frac{1}{20}\right)^2 = \frac{120}{400} = \frac{3}{10}$$

$$\therefore \qquad A = -\frac{y_{22}}{y_{21}} = -\frac{11/20}{-1/20} = 11 \qquad B = -\frac{1}{y_{21}} = -\frac{1}{-1/20} = 20\,\Omega$$

$$C = -\frac{\Delta y}{y_{21}} = -\frac{3/10}{-1/20} = 6\,\mho \qquad D = -\frac{y_{11}}{y_{21}} = -\frac{11/20}{-1/20} = 11$$

The $ABCD$ parameters are given as,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 11 & 20 \\ 6 & 11 \end{bmatrix}$$ **Ans.**

10. (a) Differentiate between the following: 4

(i) Active filter and passive filter

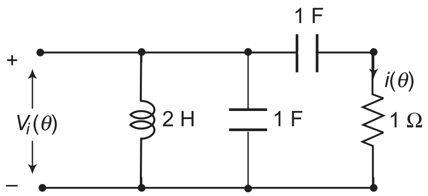
(ii) High-pass filter and low-pass filter

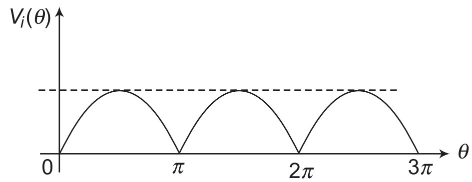
(b) The response of a network to an impulse is

$$h(t) = 0.18(e^{-0.3t} - e^{-2.1t}).$$

Find the response of the network to a step function using the convolution theorem. 6

(c) The input to the circuit shown below is a rectified sine wave as illustrated below. Determine expression of current in the 1 Ω resistance. Assume $\omega = 1$ rad/s. 5





$$V_i(\theta) = \sin\theta,\ 0 < \theta < \pi$$
$$= -\sin\theta,\ \pi < \theta < 2\pi$$

Solution:

(a) (i) **Differences between active and passive filters**

Active Filters	*Passive Filters*
1. Active filters are constructed with *active devices*, such as, op-amp, transistors, along with resistors and capacitors; no inductor is used.	1. Passive filters are constructed with the *passive components* like inductors, capacitors, resistors, etc.
2. Due to the presence of active elements, active filters can *produce power gain*.	2. Due to the absence of any active element, passive filters *cannot produce any power gain*.
3. Active filters require *dual power supply*, i.e. one external power supply apart from the signal to be filtered.	3. Passive filters do not require any *external power supply*, they operate only on the signal input.
4. As no inductor is used, active filter circuit becomes *compact*, *cheaper* and *less in weight* even at low frequencies.	4. As inductors are used, a passive filter circuit becomes *bulky* and *costly*, especially for low frequency because size of the inductor increases at lower frequency.
5. Active filters can have *high input impedances, low output impedances*. So, the active filter does not cause loading of the source or load. Due to high input impedance, load is isolated from the frequency-determining network and so variation in load does not affect the characteristic of the filter. On the other hand, due to low output impedance, active filter can drive low impedance load.	5. Passive filters have *low input impedances and high output impedances*. Due to low input impedance, the passive filter causes loading of the source. As the load is not isolated from the frequency-determining network, so variation in load may affect the characteristic of filter. Due to high output impedance, passive filter cannot drive the low impedance load.
6. *Performance of active filters at high frequencies is limited* by the gain-bandwidth product and slew rate of the amplifying elements (op-amp).	6. Since there is no restriction regarding bandwidth of op-amps, *passive filter can work well at very high frequencies*.
7. Active filters are *easier to tune and design* owing to easy adjustment of parameters like gain, pass-band, cut-off frequency, etc.	7. *Tuning and design of passive filters are difficult and time-consuming as compared to active filters.*
8. Active filters are *not suitable* in applications requiring high currents or voltages.	8. Passive filters can be used in applications involving *larger current or voltage levels*.

(ii) Differences between high-pass and low-pass filters

High Pass Filter	*Low Pass Filter*
1. A low-pass filter passes low frequency signals, and rejects signals at frequencies above the cut-off frequency.	1. A high-pass filter passes low-frequency signals, and rejects signals at frequencies below the cut-off frequency.
2. It has ideally zero output below the cut-off frequency	2. It has ideally zero output above the cut-off frequency.
3. It has ideally infinite pass band.	3. It has ideally infinite stop band.

(b) Here, impulse response is given as,

$$h(t) = 0.18(e^{-0.3t} - e^{-2.1t})$$

Taking Laplace transform,

$$H(s) = 0.18\left[\frac{1}{s+0.3} - \frac{1}{s+2.1}\right]$$

By convolution theorem,

$$\begin{aligned} Y(s) &= H(s)\ W(s) \\ &= 0.18\left[\frac{1}{s+0.3} - \frac{1}{s+2.1}\right] \times \frac{1}{s} \\ &= \frac{0.324}{s(s+0.3)(s+2.1)} \\ &= \frac{A_1}{s} + \frac{A_2}{s+0.32} + \frac{A_3}{s+2.1} \end{aligned}$$

$$\therefore \quad A_1 = \left.\frac{0.324}{(s+0.3)(s+2.1)}\right|_{s=0} = 0.514$$

$$\therefore \quad A_2 = \left.\frac{0.324}{s(s+2.1)}\right|_{s=-0.3} = -0.6$$

$$\therefore \quad A_3 = \left.\frac{0.324}{s(s+0.3)}\right|_{s=-2.1} = 0.0857$$

Putting these values,

$$Y(s) = \frac{0.514}{s} - \frac{0.6}{s+0.32} + \frac{0.0857}{s+2.1}$$

Taking inverse Laplace transform,

$$y(t) = 0.514 - 0.6e^{-0.3t} + 0.0857e^{-2.1t}$$ **Ans.**

(c) Here, $v_i(\theta) = \sin\theta$; for $0 < \theta < \pi$

$= -\sin\theta$; for $\pi < \theta < 2\pi$

where, $\theta = \omega t$.

Since, $f(t) = f(-t) \Rightarrow$ the function is even.

$$\therefore \quad b_n = 0$$

$$\therefore \quad a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\, d(\omega t)$$

$$= \frac{4}{2\pi}\int_0^{\pi} \sin\omega t \cos n\omega t\, d(\omega t)$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin\omega t \cos n\omega t\, d(\omega t)$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)\omega t - \sin(n-1)\omega t]\, d(\omega t)$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)\omega t}{n+1} + \frac{\cos(n-1)\omega t}{n-1}\right]_0^{\pi}; \text{ for } n \neq 1$$

For odd n; $$a_n = \frac{1}{\pi}\left[\left(-\frac{1}{n+1} + \frac{1}{n+1}\right) + \left(\frac{1}{n-1} - \frac{1}{n-1}\right)\right]; \quad n \neq 1$$

$$= 0$$

For even n; $$a_n = \frac{1}{\pi}\left[\left(\frac{2}{n+1}\right) + \left(\frac{-2}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\cancel{2n} - 2 - \cancel{2n} - 2}{(n+1)(n+1)}\right] = -\frac{4}{\pi(n^2-1)}$$

For $n = 1$, $$a_1 = \frac{4}{T}\int_0^{t/2} f(t)\cos\omega t\, d(\omega t)$$

$$= \frac{4}{2\pi}\int_0^{\pi} \sin\omega t \cos\omega t\, d(\omega t)$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2\omega t\, d(\omega t) = -\frac{1}{2\pi}[\cos 2\omega t]_0^{\pi}$$

$$= -\frac{1}{2\pi}[\cos 2\pi - 1] = 0$$

Also, $$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt = \frac{2}{2\pi}\int_0^{\pi} \sin\omega t\, d(\omega t) = -\frac{1}{\pi}[\cos\omega t]_0^{\pi} = \frac{2}{\pi}$$

So, the Fourier series of the input voltage is,

$$v_i(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4,6}^{\alpha} \frac{\cos n\omega t}{(n^2-1)} = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{3}\cos\omega t + \frac{1}{15}\cos 2\omega t + \frac{1}{35}\cos 3\omega t + \cdots\right)$$

The current through the 1 Ω resistance is obtained as,

$$i(t) = \frac{V_0}{Z_0} + \frac{V_2}{Z_2} + \frac{V_4}{Z_4} + \cdots$$

Here, $Z(n\omega) = 1 - \frac{j}{n\omega C} = \left(1 - \frac{j}{n}\right) = \frac{\sqrt{n^2+1}}{n} \angle -\tan^{-1}\left(\frac{1}{n}\right)$ $\quad (\because \omega = 1 \text{ and } C = 1\text{F})$

$$\therefore \quad Z_0 = 1 \quad Z_2 = \left(1 - \frac{j}{2}\right) = \frac{\sqrt{5}}{2} \angle -26.565^\circ$$

$$Z_4 = \left(1 - \frac{j}{4}\right) = \frac{\sqrt{17}}{4} \angle -10.036^\circ \quad Z_6 = \left(1 - \frac{j}{6}\right) = \frac{\sqrt{37}}{6} \angle -9.462^\circ \ldots.$$

$$\therefore \quad V_0 = \frac{2}{\pi} \qquad V_2 = \frac{4}{3\pi} \angle -180^\circ \qquad V_4 = \frac{4}{15\pi} \angle -180^\circ \qquad V_6 = \frac{4}{35\pi} \angle -180^\circ \ldots.$$

So, the current is,

$$i(t) = \frac{V_0}{Z_0} + \frac{V_2}{Z_2} + \frac{V_4}{Z_4} + \ldots$$

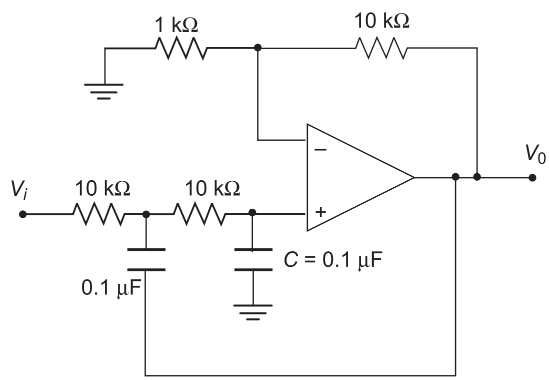
$$= \frac{2}{\pi} + \frac{4}{3\pi} \times \frac{2}{\sqrt{5}} \angle -180^\circ - 26.565^\circ + \frac{4}{15\pi} \times \frac{4}{\sqrt{17}} \angle -180^\circ - 10.036^\circ$$

$$+ \frac{4}{35\pi} \times \frac{6}{\sqrt{37}} \angle -180^\circ - 9.462^\circ + \ldots$$

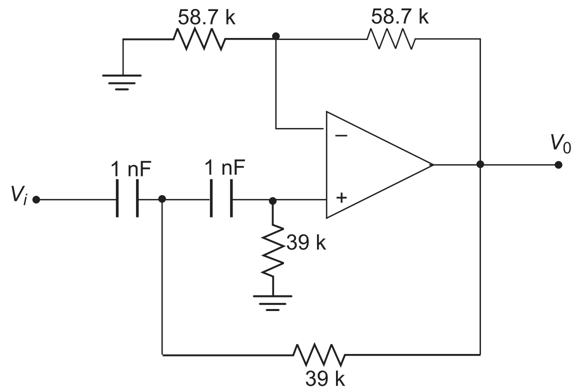
$$= \frac{2}{\pi} + \frac{4}{\pi}(0.2981 \angle -206.565^\circ + 0.0647 \angle -190.036^\circ + 0.0282 \angle -189.462^\circ + \cdots)$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\left[\begin{aligned} &0.2981\cos(t - 206.565^\circ) + 0.0647\cos(t - 190.036^\circ) \\ &\qquad + 0.0282\cos(t - 189.462^\circ) + \cdots \end{aligned}\right]$$

11. (a) The circuit given below shows a low-pass second-order active filter. Analyze the circuit and find the cut-off frequency. 8

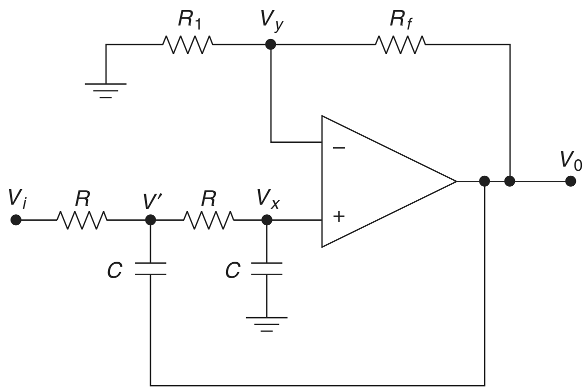


(b) For the second-order high-pass filter shown below, find the cut-off frequency and the high frequency gain. 7



Solution:

(a) The circuit is shown below.



Here, $V_y = \dfrac{V_0}{R_1 + R_f} R_1$ and $V_x = V_y$

Writing KCL at node V',

$$\frac{V' - V_i}{R} + \frac{V' - V_0}{1/sC} + \frac{V' - V_x}{R} = 0$$

or, $$(V' - V_i) + (V' - V_0)sRC + (V' - V_0) = 0$$

or, $$(-1)V_x + (2 + sRC)V' + (-sRC)V_0 = V_i \tag{1}$$

Writing KCL at node x,

$$\frac{V_x - V'}{R} + \frac{V_x}{1/sC} = 0$$

or, $$(1 + sRC)V_x + (-1)V' + (0)V_0 = 0 \tag{2}$$

Writing KCL at node y,

$$\frac{V_x}{R_1}+\frac{V_x-V_0}{R_f}=0$$

or $$(R_1+R_f)V_x+(0)V'+(-R_1)V_0=0 \tag{3}$$

Solving for V_0 from equations (1), (2), and (3), we get,

$$V_0=\frac{\begin{vmatrix}-1 & (2+sRC) & V_i\\ (1+sRC) & -1 & 0\\ (R_1+R_f) & 0 & 0\end{vmatrix}}{\begin{vmatrix}-1 & (2+sRC) & -sRC\\ (1+sRC) & -1 & 0\\ (R_1+R_f) & 0 & -R_1\end{vmatrix}}=V_i\frac{\dfrac{(R_1+R_f)}{R_1}\times\dfrac{1}{R^2C^2}}{s^2+3sRC-sRC\left(\dfrac{(R_1+R_f)}{R_1}\right)+\dfrac{1}{R^2C^2}}$$

or $$\frac{V_0(s)}{V_i(s)}=\frac{K/R^2C^2}{s^2+s\left(\dfrac{3-K}{RC}\right)+\left(\dfrac{1}{RC}\right)^2} \tag{4}$$

where $K=\dfrac{R_1+R_f}{R_1}$ = D. C. gain of the amplifier.

Substituting $s=j\omega$, the transfer function is,

$$H(j\omega)=\frac{V_0(j\omega)}{V_i(j\omega)}=\frac{K}{1+j(3-K)RC\omega-R^2C^2\omega^2}$$

The magnitude of the transfer function is,

$$|H(j\omega)|=\frac{K}{\sqrt{\left[1-\left(\dfrac{\omega}{\omega_c}\right)^2\right]^2+[3-K]^2\left(\dfrac{\omega}{\omega_c}\right)^2}}\text{ ; where, }\omega_c=\frac{1}{RC}$$

In the above equation, when $\omega\to 0$, $|H(j\omega)|=K$. Thus, the low-frequency gain of the filter is K and when $\omega\to\infty$, $|H(j\omega)|=0$, i.e., high-frequency gain is zero.
From the table of the Butterworth Filter, the transfer function for second-order (n = 2) filter is,

$$T(s)=\frac{K}{\left(\dfrac{s}{\omega_c}\right)^2+1.414\left(\dfrac{s}{\omega_c}\right)+1}=\frac{K\omega_c^2}{s^2+1.414\omega_c s+\omega_c^2} \tag{5}$$

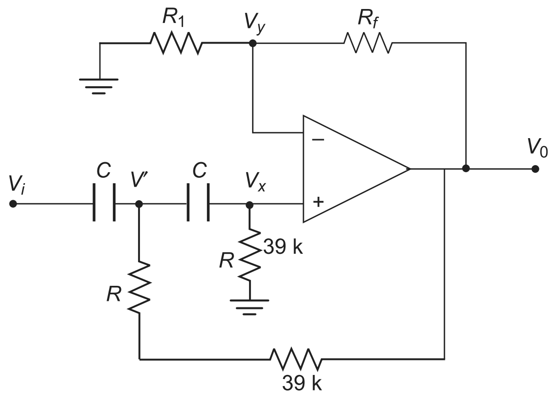
where, ω_c is the cut-off frequency. Comparing equations (4) and (5), we get,

$$\boxed{\omega_c=\frac{1}{RC}\quad\text{or,}\quad f_c=\frac{1}{2\pi RC}}$$

Putting the value R = 10 kΩ and C = 0.1 μF,

$$\omega_c = 10 \text{ rad/s} \quad \text{or,} \quad f_c = 1.59 \text{ Hz}$$

(b) The general circuit of a second-order active high-pass filter is shown in the figure below.



Here, $V_y = \dfrac{V_0}{R_1 + R_f} R_1$ and $V_x = V_y$

Writing KCL at node V',

$$\frac{V' - V_i}{1/sC} + \frac{V' - V_0}{R} + \frac{V' - V_x}{1/sC} = 0 \tag{1}$$

Writing KCL at node x,

$$\frac{V_x - V'}{1/sC} + \frac{V_x}{R} = 0 \tag{2}$$

Writing KCL at node y,

$$\frac{V_x}{R_1} + \frac{V_x - V_0}{R_f} = 0 \tag{3}$$

Solving for V_0 from equations (1), (2), and (3), we get,

$$\frac{V_0(s)}{V_i(s)} = \frac{Ks^2}{s^2 + s\left(\dfrac{3-K}{RC}\right) + \left(\dfrac{1}{RC}\right)^2}$$

or, (4)

where, $K = \dfrac{R_1 + R_f}{R_1}$ = dc gain of the amplifier.

Substituting $s = j\omega$, the transfer function is,

$$H(j\omega) = \frac{V_0(j\omega)}{V_i(j\omega)} = -\frac{KR^2C^2\omega^2}{1 + j(3-K)RC\omega - R^2C^2\omega^2}$$

The magnitude of the transfer function is,

$$|H(j\omega)| = \frac{K\left(\frac{\omega}{\omega_c}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_c}\right)^2\right]^2 + [3-K]^2\left(\frac{\omega}{\omega_c}\right)^2}}\text{; where, } \omega_c = \frac{1}{RC}$$

$$\boxed{\omega_c = \frac{1}{RC} \qquad \text{or,} \qquad f_c = \frac{1}{2\pi RC}}$$

Thus, the cut-off frequency of the filter given as,
Putting the values $R = 39\ \text{k}\Omega$ and $C = 1$ nF, we have,

$$f_c = \frac{1}{2\pi RC} = \frac{1}{2\pi \times 39 \times 10^3 \times 1 \times 10^{-9}} = 4.08\ \text{kHz}$$

In the above equation of transfer function, when $\omega \to 0$, $|H(j\omega)| = 0$. Thus, the low-frequency gain of the filter is zero. When $\omega \to \infty$, $|H(j\omega)| = K$, i.e., high frequency gain is K.

$$\boxed{K = (3 - 1.414) = 1.586}$$

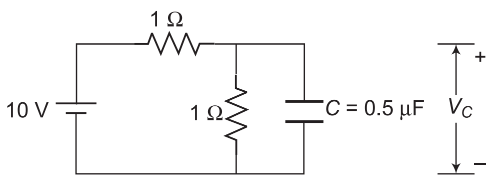
Circuit Theory & Networks (EE301) Solution of 2012 WBUT Paper

GROUP–A
(Multiple-Choice Questions)

1. Choose the correct alternatives for any *ten* of the following: **$10 \times 1 = 10$**

(i) The internal impedance of an ideal voltage source should be
(a) zero (b) infinite
(c) greater than zero but less than infinity (d) none of these

(ii) The steady-state voltage V_C in this given figure is



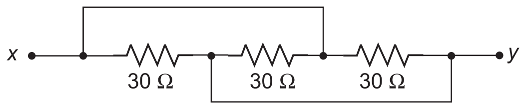
(a) 10 V (b) 15 V (c) 5 V (d) none of these.

(iii) What is the condition for reciprocity in terms of h parameters?
(a) $h_{11} = h_{22}$ (b) $h_{21}h_{12} = h_{11}h_{12}$ (c) h_{12} and $h_{21} = 0$ (d) $h_{12} = h_{21}$

(iv) An ideal filter should have
(a) zero attenuation in the pass band
(b) zero attenuation in the attenuation band
(c) infinite attenuation in the pass band
(d) none of these

(v) The number of links of a graph having n nodes and b branches are
(a) $b - n + 1$ (b) $n - b + 1$ (c) $b + n - 1$ (d) $b + n$

(vi) The equivalent resistance between x and y of the figure shown below is

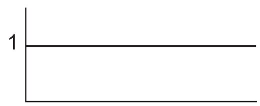


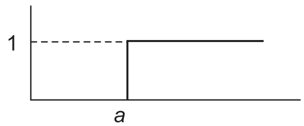
(a) 30 Ω (b) 50 Ω (c) 60 Ω (d) 10 Ω

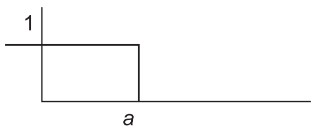
(vii) A periodic waveform having half-wave symmetry has no

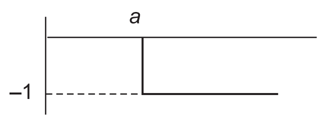
(a) odd harmonics (b) even harmonics (c) cosine terms (d) sine terms

(viii) Graphical representation of $u(a - t)$ is

(a) 

(b) 

(c) 

(d) 

(ix) A tie-set matrix has 3 rows and 7 branches. The number of twigs is

(a) 3 (b) 5 (c) 2 (d) 4

(x) Inverse Laplace of $F(s)=\dfrac{2}{s(s+1)}$ is

(a) $2 + e^{-2t}$ (b) $1 + 2e^{-t}$ (c) $2 + 2e^{-t}$ (d) $2 - 2e^{-t}$

(xi) Two networks can be dual when

(a) their nodal equations are same

(b) the loop equations of one network are the nodal equations of the other

(c) their loop equations are same

(d) none of these.

(xii) The dc gain of a system having the transfer function $H(s)=\dfrac{12}{(s+2)(s+3)}$ is

(a) 2 (b) 1 (c) 12 (d) 3

(e) 0

Solutions:

(i) (a) zero (ii) (c) 5 V (iii) correct answer is $h_{12} = -h_{21}$

(iv) (a) zero attenuation in the pass band (v) (a) $b - n + 1$ (vi) (d) 10 Ω

(vii) (b) even harmonics (viii) (c) (ix) (d) 4 (x) (d) $2 - 2e^{-t}$

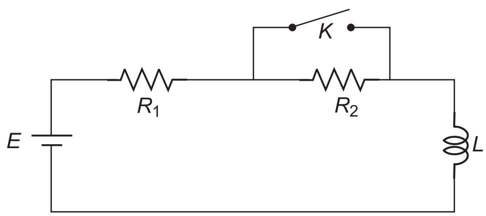
(xi) (b) the loop equations of one network are the nodal equations of the other

(xii) (a) 2

GROUP–B
(Short-Answer-Type Questions)

Answer any *three* of the following questions. **$3 \times 5 = 15$**

2. In the figure given below, the battery voltage is applied for a steady-state period. Obtain the q complete expression for the current for the current after closing the switch K. Assume $R_1 = 1\ \Omega$, $R_2 = 2\ \Omega$, $L = 1$ H, $E = 10$ V.

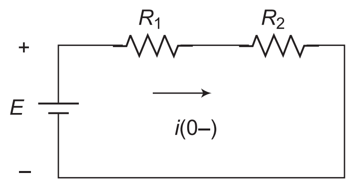


Solution:

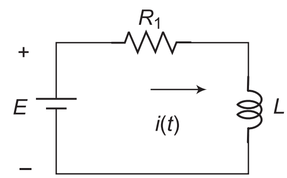
NB: The problem has some printing mistakes; however, we find the current as required.

For $t < 0$, as the circuit was in steady state with the switch in open position, the circuit becomes as shown below. The initial current is obtained as,

$$\therefore \qquad i(0-) = \frac{E}{R_1 + R_2} = \frac{10}{1+2} = \frac{10}{3} = 3.33\ \text{A}$$



Circuit for t < 0



Circuit for t > 0

For $t > 0$, the circuit becomes as shown.

By KVL,

$$R_1 I(s) + sLI(s) - Li(0-) = \frac{E}{s}$$

$$\Rightarrow \qquad [R_1 + sL]I(s) = \frac{E}{s} + \frac{EL}{R_1 + R_2}$$

$$\Rightarrow \qquad I(s) = \frac{E}{R_1}\left[\frac{1}{s\left(s + {R_1}/{L}\right)}\right] + \frac{E}{R_1 + R_2}\left(\frac{1}{\left(s + {R_1}/{L}\right)}\right)$$

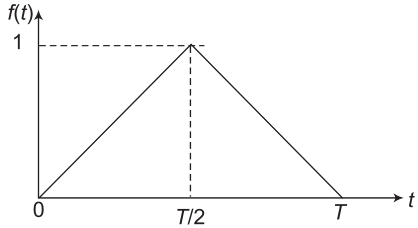
Taking inverse Laplace transform,

$$i(t) = \frac{E}{R_1}\left(1-e^{-\left(\frac{R_1}{L}\right)t}\right)+\frac{E}{R_1+R_2}e^{-\left(\frac{R_1}{L}\right)t} = \frac{10}{1}\left(1-e^{-\left(\frac{1}{1}\right)}\right)+\frac{10}{1+2}e^{-\left(\frac{1}{1}\right)}$$

$$= 10(1-e^{-t})+\frac{10}{3}e^{-t} = 10-\frac{20}{3}e^{-t}$$

$$= 10-6.67e^{-t} \quad (A), t>0$$ **Ans.**

3. Find the Laplace transform of the triangular waveform shown in the figure.



Solution: The equation of the given waveform can be written as,

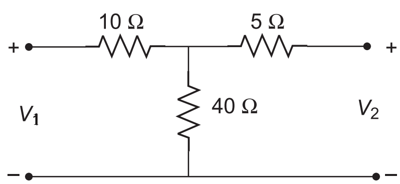
$$f(t) = \frac{2}{T}r(t)-\frac{4}{T}r\left(t-\frac{T}{2}\right)+\frac{2}{T}r(t-T)$$

$$F(s) = \frac{2}{T}\frac{1}{s^2}-\frac{4e^{-\frac{T}{2}s}}{Ts^2}+\frac{2e^{-Ts}}{Ts^2}$$

$$= \frac{2}{Ts^2}\left[1-2e^{-Ts/2}+e^{-Ts}\right] = \frac{2}{Ts^2}\left[1-e^{-Ts/2}\right]^2$$

$$\boxed{\therefore \quad F(s) = \frac{2}{Ts^2}\left[1-e^{-Ts/2}\right]^2}$$

4. Find the y-parameters for the following networks shown in the figure.



Solution: Since this is a T-network, the z-parameters are easily obtained as,

$z_{11} = (10 + 40) = 50\ \Omega$

$z_{12} = z_{21} = 40\ \Omega$

$$z_{13} = (5 + 40) = 45\ \Omega$$

Hence, the y-parameters are obtained as,

$$y_{11} = \frac{z_{22}}{\Delta z} = \frac{45}{50 \times 45 - 40^2} = \frac{9}{130}\ ℧$$

$$y_{12} = y_{21} = -\frac{z_{12}}{\Delta z} = -\frac{40}{650} = -\frac{4}{65}\ ℧$$

$$y_{22} = \frac{z_{11}}{\Delta z} = \frac{50}{50 \times 45 - 40^2} = \frac{1}{13}\ ℧$$

5. Define incidence matrix of a graph and draw the orientation graph from the reduced incidence matrix.

$$[A] = \begin{bmatrix} 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution:

- **Incidence Matrix:** The incidence matrix symbolically describes a network. It also facilitates the testing and identification of the independent variables. Incidence matrix is a matrix which represents a graph **uniquely**.

  For a given graph with n nodes and b branches, the complete incidence matrix A_a is a rectangular matrix of order $n \times b$, whose elements have the following values:

  Number of columns in [A] = Number of branches = b

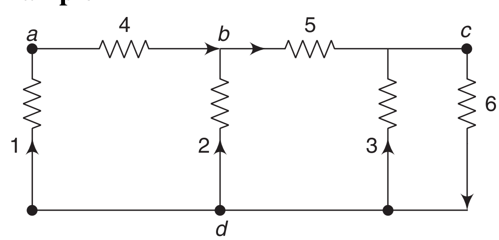
  Number of rows in [A] = Number of nodes = n

  A_{ij} = 1, if branch j is associated with node i and oriented away from node i.

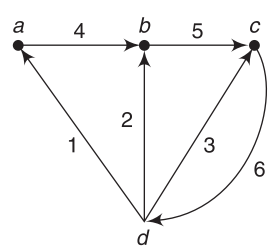
  = –1, if branch j is associated with node i and oriented towards node i.

  = 0, if branch j is not associated with node i.

  Example



(a) *Network*



(b) *Graph of Network*

		Branches						
		1	2	3	4	5	6	
	a	1	0	0	–1	0	0	Reduced incidence matrix A_I
Nodes	b	0	1	0	1	–1	0	
	c	0	0	1	0	1	–1	
Reference node	d	–1	–1	–1	0	0	1	

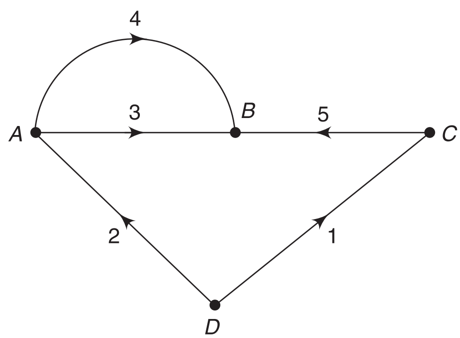
Reduced incidence matrix [A]: The matrix obtained from A_a by eliminating one of the rows is called reduced incidence matrix. In other words, suppression of the datum node (reference node) from the incidence matrix results in the reduced incidence matrix.

For the graph shown above, the reduced incidence matrix is given as,

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

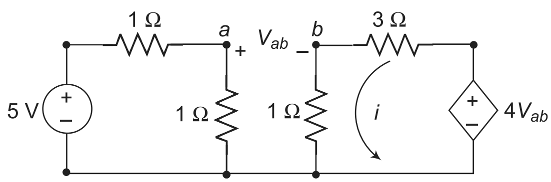
- **Solution to Numerical Problem:** From the property that for complete incidence matrix, the summation of all entries in any column must be zero, the complete incidence matrix is obtained as,

$$A_a = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

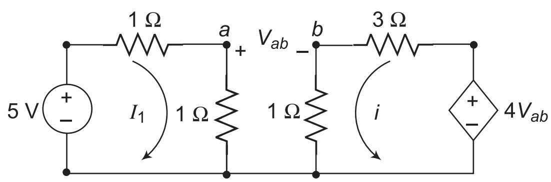


The graph is shown in the figure.

6. For the circuit shown in the figure, find the value of the current i.



Solution: By KVL for the meshes, we get,



$$1 \times I_1 + 1 \times I_1 = 5 \Rightarrow I_1 = 2.5 \text{ A}$$
$$3 \times i + 1 \times i = 4V_{ab} \Rightarrow \; V_{ab}$$

Now, from the middle loop, we have,

$$1 \times I_1 - 1 \times i - V_{ab} = 0 \Rightarrow i = \frac{I_1}{2} = 1.25 \text{ A}$$

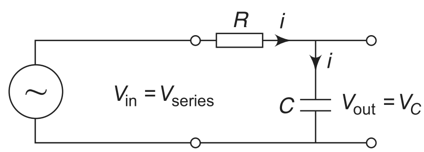
$\therefore$ $\qquad i = 1.25$ A **Ans.**

7. Explain under what condition, an *RC* series circuit behaves as
   (i) Low-pass filter
   (ii) Integrator

Solution:

(i) *RC* Series Circuit as Low-Pass Filter: If the *RC* series circuit is supplied with a frequency-varying source then it will act as a low-pass filter if the output is taken as the voltage across the capacitor.

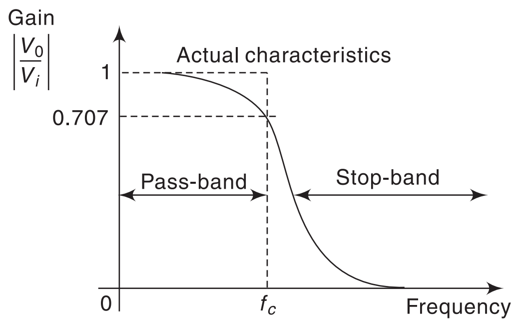
The voltage across the capacitor is $IX_C = 1/\omega C$. The voltage across the series combination is: $IZ = I\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$



So the gain is

$$g \equiv \frac{V_{out}}{V_{in}} = \frac{IX_C}{IZ} = \frac{\frac{1}{\omega C}}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}$$

$$\therefore \quad \boxed{g = \frac{1}{\sqrt{1+(\omega RC)^2}}}$$



Here, at low frequencies, capacitive reactance $\left(X_C = \frac{1}{j2\pi fC}\right)$ is very high and, therefore, the circuit can be considered an open circuit. Under these conditions, the input signal is equal to the output signal. At very high frequencies, the capacitive reactance $\left(X_C = \frac{1}{j2\pi fC}\right)$ is very low and, therefore, the output signal is very small as compared with the input signal. Thus, the circuit acts as a low-pass filter with the frequency characteristics as shown in the figure.

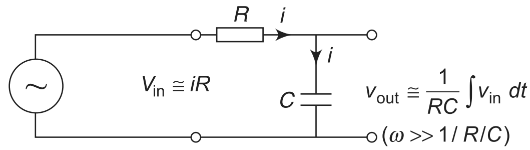
(ii) *RC* Series Circuit as Integrator: We have an ac source with voltage $v_{in}(t)$, input to an *RC* series circuit. The output is the voltage across the capacitor.

We consider only **high frequencies** $\omega >> 1/RC$, so that the capacitor has insufficient time to charge up, its voltage is small. So the input voltage approximately equals the voltage across the resistor.

$$V_{\text{in}} = IZ = I\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

But $\quad \omega C >> \frac{1}{R}, \quad$ so $\quad V_{\text{in}} \cong IR$

For frequencies, $\omega >> \frac{1}{RC}, \quad V_{\text{in}} \cong V_R$



$$\therefore \quad V_{\text{out}} = V_C = \frac{1}{C}\int i dt \cong \frac{1}{C}\int \frac{V_{\text{in}}}{R} dt$$

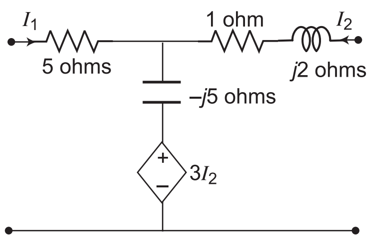
$$\therefore \quad \boxed{V_{\text{out}} \cong \frac{1}{RC}\int V_{\text{in}} dt}$$

Thus, the voltage v_C is the integration of the input voltage and, hence, the *RC series circuit acts as an integrator.*

GROUP–C
(Long-Answer-Type Questions)

Answer any *three* questions. **3 × 15 = 45**

8. (a) Find the *Z*-parameter and *ABCD* parameter of the circuit given in the figure.



(b) Express *h*-parameter in terms of *Y*-parameter of a two-port network.

(c) What is the cascade connection between two 2-port networks? Explain with a diagram.

7 + 4 + 4

Solution:

(a) We consider two cases:

Case (1): When $I_2 = 0$

By KVL, we get,

$$5I_1 - j5I_1 = V_1 \Rightarrow z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = (5 - j5)\,\Omega$$

$$\therefore \quad V_2 = I_1 \times (-j5) \Rightarrow z_{21} == \left.\frac{V_2}{I_1}\right|_{I_2=0} = -j5\,\Omega$$

Case (2): When $I_1 = 0$

By KVL, we get,

$$I_2\,(1 + j2 - j5) + 3I_2 = V_2 \Rightarrow z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = (4 - j3)\,\Omega$$

$$\therefore \qquad V_1 = I_2 \times (-j5) + 3I_2 \Rightarrow z_{12} == \left.\frac{V_1}{I_2}\right|_{I_1=0} = (3 - j5)\,\Omega$$

By interrelationship, the *ABCD* parameters are obtained as,

$$A = \frac{z_{11}}{z_{21}} = \frac{5 - j5}{-j5} = (1 + j1)$$

$$B = \frac{\Delta z}{z_{21}} = \frac{(5 - j5)\times(4 - j3) - (3 - j5)\times(-j5)}{-j5} = (4 + j6)\,\Omega$$

$$C = \frac{1}{z_{21}} = \frac{1}{-j5} = j0.2\,\mho$$

$$D = \frac{z_{22}}{z_{21}} = \frac{(4 - j3)}{-j5} = (0.6 + j0.8)$$

Hence, the required parameters are:

$$Z = \begin{bmatrix} (5 - j5) & (3 - j5) \\ -j5 & (4 - j3) \end{bmatrix}(\Omega) \text{ and } T = \begin{bmatrix} (1 + j1) & (4 + j6)\,\Omega \\ j0.2\,\mho & (0.6 + j0.8) \end{bmatrix}$$ **Ans.**

(b) ***h*-parameter in terms of *y*-parameters**

The *h*-parameter equations are

$$\begin{aligned} V_1 &= h_{11}I_1 + h_{12}V_2 \\ I_2 &= h_{21}I_1 + h_{22}V_2 \end{aligned} \tag{1}$$

The *y*-parameter equations are:

$$\begin{aligned} I_1 &= y_{11}V_1 + y_{12}V_2 \\ I_2 &= y_{21}V_1 + y_{22}V_2 \end{aligned} \tag{2}$$

We have to express Eq. (2) in the form of Eq. (1).

From the first equation of Eq. (2), we get,

$$V_1 = \left(\frac{1}{y_{11}}\right)I_1 + \left(-\frac{y_{12}}{y_{11}}\right)V_2 \tag{3}$$

Replacing this in the second equation of Eq. (2), we have,

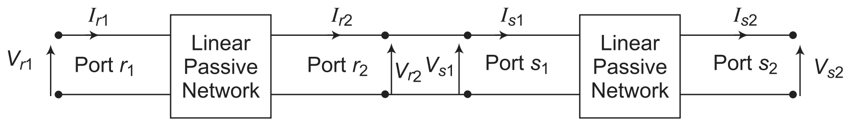
$$I_2 = y_{21}V_1 + y_{22}V_2 = y_{21}\left[\left(\frac{1}{y_{11}}\right)I_1 - \left(\frac{y_{12}}{y_{11}}\right)V_2\right] + y_{22}V_2 = \left(\frac{y_{21}}{y_{11}}\right)I_1 + \left(\frac{\Delta y}{y_{11}}\right)V_2 \quad (4)$$

(where, $\Delta y, = y_{11}\,y_{22} - y_{12}\,y_{21}$)

Comparing Eq. (1), (3) and (4), we get,

$$h_{11} = \frac{1}{y_{11}} \quad h_{12} = -\frac{y_{12}}{y_{11}} \qquad h_{21} = \frac{y_{21}}{y_{11}} \qquad h_{22} = \frac{\Delta y}{y_{11}}$$

(c) **Cascade Connection between Two 2-Port Networks:** A cascade connection is defined when the output of one network becomes the input to the next network.



It can be easily seen that $I_{r2} = I_{s1}$ and $V_{r2} = V_{s1}$

Therefore, it can easily be seen that the *ABCD* parameters are the most suitable to be used for this connection.

$$\begin{bmatrix} V_{r1} \\ I_{r1} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} V_{r2} \\ I_{r2} \end{bmatrix}, \begin{bmatrix} V_{s1} \\ I_{s1} \end{bmatrix} = \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix}\begin{bmatrix} V_{s2} \\ I_{s2} \end{bmatrix}$$

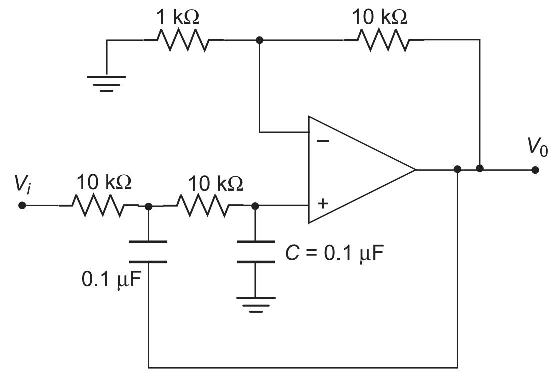
$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} V_{r1} \\ I_{r1} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} V_{r2} \\ I_{r2} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} V_{s1} \\ I_{s1} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix}\begin{bmatrix} V_{s2} \\ I_{s2} \end{bmatrix}$$

$$= \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix}\begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

Thus, it is seen that the ***overall ABCD matrix is the product of the two individual ABCD matrices***. This is a very useful property in practice, especially when analyzing transmission lines.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix}$$

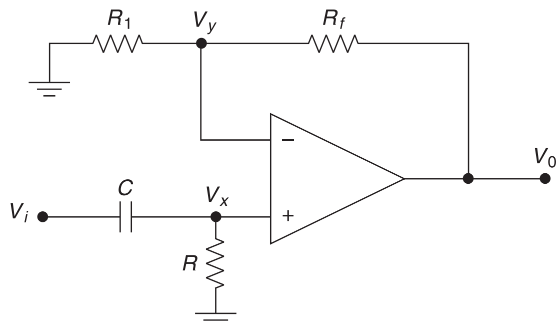
9. (a) Draw the circuit diagram of a first-order high-pass filter and find out the expression for the cut-off frequency.
   (b) Draw and explain the characteristics of an ideal band-pass and an ideal band-stop filter.
   (c) The circuit shown in the figure is a second-order low-pass filter. Analyze the circuit and find the cut-off frequency.



5 + 5 + 5

Solution:

(a) **First-Order High-Pass Active Filter:** The circuit is shown in the figure.



The filtering is done by the CR network and the op-amp is connected as a unity-gain follower. The feedback resistor, R_f is included to minimize dc offset.

Here, $$V_y = V_0 \frac{R_1}{R_1 + R_f}$$

Voltage across the resistor R,

$$V_x = \frac{R}{R + X_c} V_i = \frac{R}{R + \dfrac{1}{j\omega C}} V_i = \frac{j\omega RC}{1 + j\omega RC} V_i \tag{2}$$

Since op-amp gain is infinite,

$$V_x = V_y$$

$$\Rightarrow \quad \frac{V_0 R_1}{R_f + R_1} = \frac{j\omega RC}{1 + j\omega RC} V_i$$

$$\Rightarrow \quad \frac{V_0}{V_i} = \left(\frac{R_f + R_1}{R_1}\right)\left(\frac{j\omega RC}{1 + j\omega RC}\right) = A_F \times \frac{j2\pi fRC}{1 + j2\pi fRC}$$

where, $A_F = (1 + R_f/R_1)$ = pass-band gain of the filter,
f = frequency of the input signal (Hz),

$f_c = \dfrac{1}{2\pi RC}$ cut-off frequency of the filter (Hz).

The gain magnitude,

$$\left|\frac{V_0}{V_i}\right| = \frac{A_F(2\pi fRC)}{\sqrt{1+(2\pi fRC)^2}} = A_F \cdot \frac{\omega RC}{\sqrt{1+\omega^2 R^2 C^2}}$$

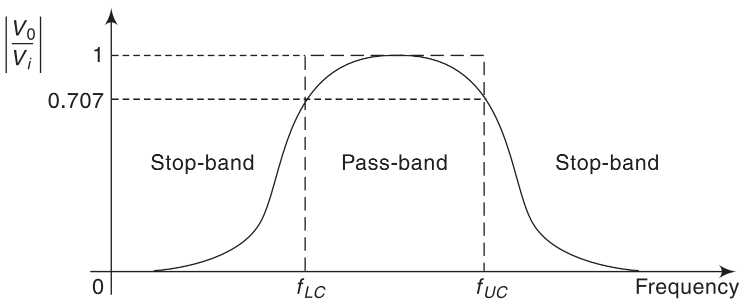
For this magnitude to be $\dfrac{A_F}{\sqrt{2}}$ at $f = f_c$, we have,

$$\frac{A_F}{\sqrt{2}} = \frac{A_F(2\pi_c fRC)}{\sqrt{1+(2\pi_c fRC)^2}}$$

or, $$\boxed{f_c = \frac{1}{2\pi RC}}$$

This is the cut-off frequency of the high-pass filter.

(b) **Band–Pass Filter:** It is a circuit that passes a band of frequencies and attenuates all frequencies outside the band.

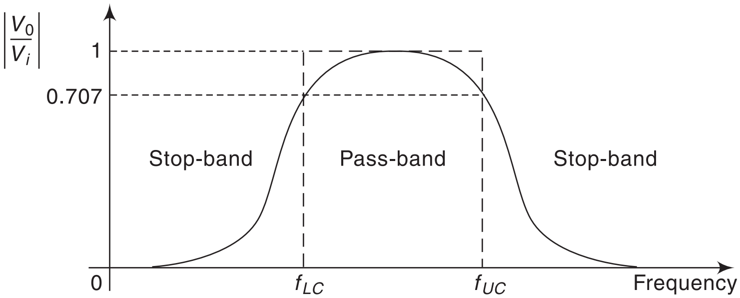


The bandwidth of a band-pass filter is the difference between the upper and lower cut-off frequencies. Depending on the value of bandwidth, band-pass filters are of two types:

1. **Wide Band-Pass Filter:** This is characterized by high bandwidth or low Q-factor, i.e., $Q \le 0.5$.
2. **Narrow Band-Pass Filters:** This is characterized by small bandwidth or high Q-factor, i.e., $Q > 0.5$.

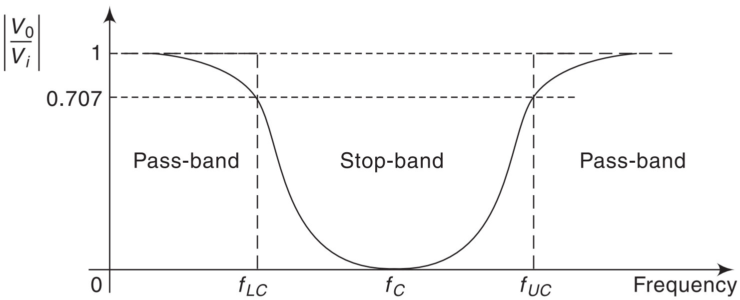
An ideal band-pass filter should have the following characteristics:

1. It should have a completely flat pass-band (e.g., with no gain attenuation throughout).
2. It should completely attenuate all frequencies outside the pass-band.
3. The transition out of the pass-band should be instantaneous in frequency.



In practice, no band-pass or band-stop filter is ideal. The filter does not attenuate all frequencies outside the desired frequency range completely; in particular, there is a region just outside the intended pass-band where frequencies are attenuated, but not rejected. This is known as the filter roll-off, and it is usually expressed in dB of attenuation per octave or decade of frequency.

Band Stop Filter: It rejects a specified band of frequencies while passing all other frequencies outside the band. If a band-stop filter has a narrow stop-band (i.e., high Q-factor) then the filter is known as a *notch filter*.



An ideal band-stop filter should have the following characteristics.

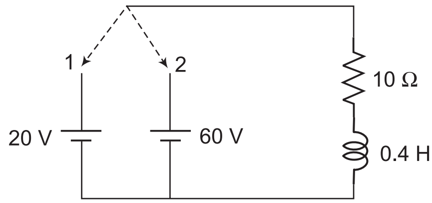
1. It should have a completely flat pass-band (e.g., with no gain attenuation throughout).
2. It should completely attenuate all frequencies within the stop-band.
3. The transition from pass-band to stop band should be instantaneous in frequency.

(c) [WBUT 2011 Q.11 (a)]

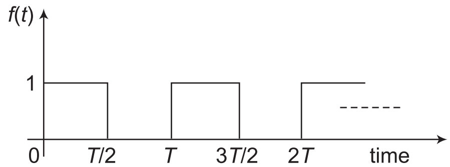
10. (a) Find the inverse Laplace transform of $F(s)$.

$$F(s) = \frac{s+1}{s(s^2+4s+4)}$$

(b) The circuit in the figure was in steady state with the switch in position 1. Find current $i(t)$ for $t > 0$ if the switch is moved from position 1 to 2 at $t = 0$.



(c) Determine the Laplace transform of the periodic square pulse train of amplitude as shown in the figure.



Solution:

(a)
$$F(s) = \frac{s+1}{s(s^2+4s+4)} = \frac{s+1}{s(s+2)^2}$$

Let,
$$F(s) = \frac{s+1}{s(s+2)^2} = \frac{A}{s} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)}$$

By residue method,

$$A = \left.\frac{s+1}{(s+2)^2}\right|_{s=0} = \frac{1}{4};\ B = \left.\frac{s+1}{s}\right|_{s=-2} = \frac{1}{2};$$

$$C = \left.\frac{d}{ds}\left[\frac{s+1}{s}\right]\right|_{s=-2} = -\left.\frac{1}{s^2}\right|_{s=-2} = -\frac{1}{4}$$

$\therefore$
$$F(s) = \frac{s+1}{s(s+2)^2} = \frac{1}{4s} + \frac{1}{2(s+2)^2} - \frac{1}{4(s+2)}$$

Taking inverse Laplace transform,

$$f(t) = \left[\frac{1}{4}u(t) - \frac{1}{4}e^{-2t} + \frac{1}{2}te^{-2t}\right]$$ **Ans.**

(b) When the switch is in position 1, steady state exists and the initial current through the inductor is,

$$i(0-) = \frac{20}{10} = 2 \text{ A}$$

After the switch is moved to position 2, the KVL gives, in Laplace transform,

$$10I(s) + 0.4sI(s) - 0.4 \times 2 = \frac{60}{s}$$

or, $$I(s) = \frac{150}{s(s+25)} + \frac{2}{s+25} = 6\left[\frac{1}{s} - \frac{1}{s+25}\right] + \frac{2}{s+25}$$

Taking inverse Laplace transform,

$$i(t) = 6 - 4e^{-25t} \quad (A);\ t > 0;$$ **Ans.**

(c) The given waveform can be written as,

$$f(t) = au(t) - Au\left(t - \frac{T}{2}\right) + Au(t-T) - Au\left(t - \frac{3T}{2}\right) + Au(t-2T) - ...$$

Taking Laplace transform,

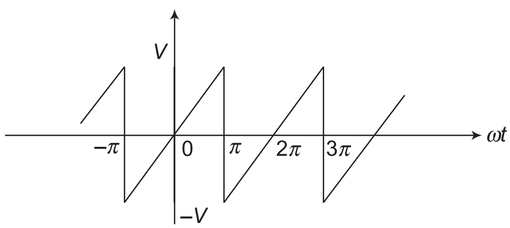
$$F(s) = \frac{A}{s} - \frac{A}{s}e^{-Ts/2} + \frac{A}{s}e^{-Ts} - \frac{A}{s}e^{-3Ts/2} + \frac{A}{s}e^{-2Ts} - ...$$

$$= \frac{A}{s}\left[1 - e^{-Ts/2} + e^{-Ts} - e^{-3Ts/2} + e^{-2Ts} - ...\right]$$

$$= \frac{A}{s}\left[\frac{1}{1 - e^{-Ts/2}}\right]$$ **Ans** $\left\{\because \text{since summation of an infinite GP series is} = \frac{1}{1-CR}\right\}$

$$\boxed{\therefore \quad F(s) = \frac{A}{s}\left[\frac{1}{1 - e^{-Ts/2}}\right]}$$

11. (a) Find the Fourier expansion of the following waveform shown in the figure.



(b) Determine the Fourier transform and sketch the amplitude and phase spectrums of the function

$$f(t) = Ve^{-t/a} \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t \leq 0$$

8 + 7

Solution:

(a) The waveform has both the odd and half-wave symmetry.

$\therefore$ $a_0 = 0 \ \ a_n = 0$

Also, the waveform will contain only the odd harmonics.

$$\therefore \quad b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t dt;\ n \text{ is odd only}$$

$$= \frac{4}{T}\int_0^{T/2}\frac{V}{\pi}t\sin n\omega t dt = \frac{4V}{\pi T}\left[\frac{-t\cos n\omega t}{n\omega}+\int\frac{\cos n\omega t}{n\omega}dt\right]_0^{T/2}$$

$$= \frac{4V}{\pi T}\left[-\frac{T}{2n\omega}\cos\left(\frac{n\omega T}{2}\right)+\frac{\sin n\omega t}{n^2\omega^2}\Bigg|_0^{T/2}\right]$$

$$= \frac{4V}{\pi 2\pi}\left[-\frac{2\pi}{2n}\cos n\pi+\frac{\sin n\pi}{n^2}\right] \qquad (\because\ T = 2\pi,\ \therefore\ \omega = 1)$$

$$= -\frac{2V}{n\pi}\cos n\pi \qquad (\because\ \sin n\pi = 0, \text{ for all } n)$$

$$= \frac{2V}{n\pi} \qquad (\because\ \cos n\pi = -1 \text{ for all } n)$$

Hence, the Fourier series of the given waveform is,

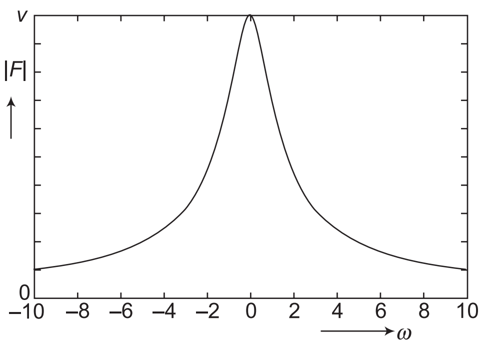
$$V(t) = \frac{2V}{\pi}\left(\sin\omega t+\frac{1}{3}\sin 3\omega t+\frac{1}{5}\sin 5\omega t+\frac{1}{7}\sin 7\omega t+\ldots\right)$$ **Ans.**

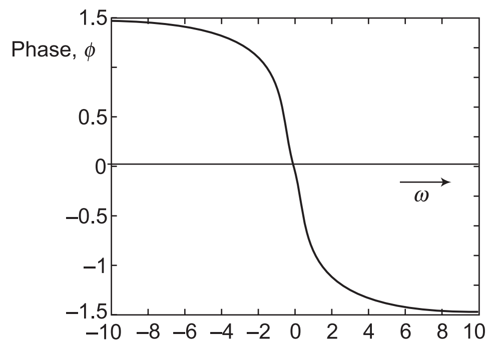
(b) Fourier transform of the function is,

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_0^{\infty} Ve^{-t/a}e^{-j\omega t}dt = V\int_{-\infty}^{\infty} e^{-\left(\frac{1}{a}+j\omega\right)t}dt$$

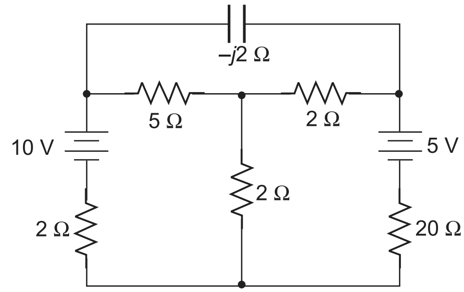
$$= V\left|\frac{e^{-\left(\frac{1}{a}+j\omega\right)t}}{-\left(\frac{1}{a}+j\omega\right)}\right|_0^{\infty} = \frac{Va}{1+j\omega a}$$ **Ans.**

The amplitude and phase are $|F(j\omega)| = \frac{Va}{\sqrt{1+\omega^2a^2}}$ and $\phi(j\omega) = -\tan^{-1}(\omega a)$



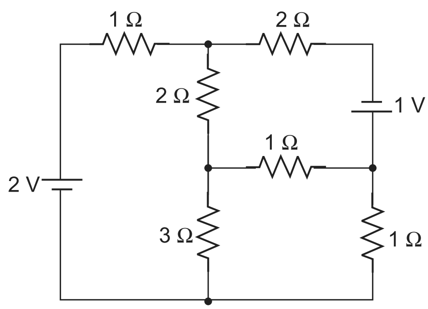


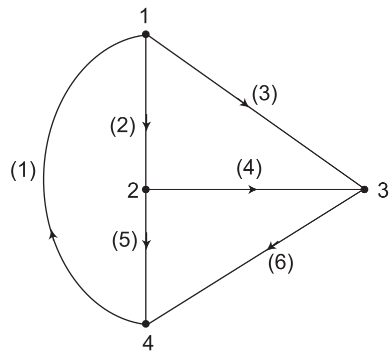
12. (a) What is the oriented graph of a network? Explain with a suitable example.
   (b) Develop at least three trees for your considered network. Mark the twigs and links.
   (c) For the network in the figure, draw the oriented graph, develop the incidence matrix, choose a tree and considering the tree, develop the tie-set matrix.



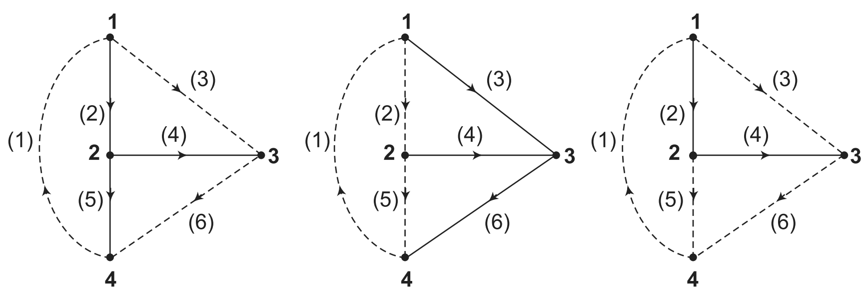
Solution:

(a) **Oriented Graph:** A graph whose branches are oriented, i.e. the branch current directions are shown by arrowheads, is called a *directed* or *oriented graph*.
For example, for the circuit shown in the figure, the oriented graph is shown in the figure.



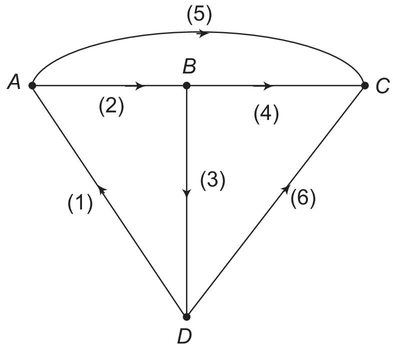


(b) The three trees of the considered network are shown in the figure below. The solid lines represent the twigs and the dashed lines represent the links.

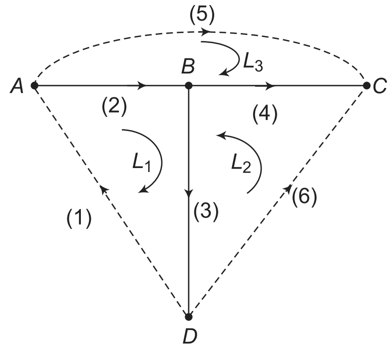


Tree 1 Tree 2 Tree 3

(c) The oriented graph of the network is shown below.



Graph



Tree

The incidence matrix is obtained as,

$$A_a = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

A tree has been chosen as shown in the figure. This tree creates three loops as shown in the figure.

The tie-set matrix is obtained as,

$$B_a = \begin{array}{c} \\ L_1 \\ L_2 \\ L_3 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

Circuit Theory & Networks (EE301) Solution of 2013 WBUT Paper

GROUP–A
(Multiple-Choice Questions)

1. Choose the correct alternatives for any *ten* of the following: **$10 \times 1 = 10$**

(i) Unit step function is the first derivative of
   (a) ramp function (b) impulse function
   (c) gate function (d) parabolic function

(ii) A practical current source is usually represented by
   (a) a resistance in series with an ideal current source
   (b) a resistance in parallel with an ideal current source
   (c) a resistance in parallel with an ideal voltage source
   (d) none of these

(iii) A two-port network is defined by the relations $I_1 = 2V_1 + V_2$ and $I_2 = 2V_1 + 3V_2$, then Z_{12} is
   (a) -2 ohm (b) -1 ohm (c) $-\frac{1}{2}$ ohm (d) $-\frac{1}{4}$ ohm

(iv) The Z-matrix of a 2-port network is given by $\begin{bmatrix} 0.9 & 0.2 \\ 0.2 & 0.6 \end{bmatrix}$. The element Y_{22} of the corresponding Y-matrix of the same network is given by
   (a) 1.2 (b) 0.4 (c) -0.4 (d) 1.8

(v) The Fourier series of the function $f(x) = \sin^2 x$ is
   (a) $\sin x + \sin 2x$ (b) $1 - \cos 2x$ (c) $\sin 2x + \cos 2x$ (d) $0.5 - 0.5 \cos 2x$

(vi) A rectangular pulse of duration t and magnitude I has the Laplace transform
   (a) $\frac{I}{s}$ (b) $\left(\frac{I}{s}\right)e^{-sT}$ (c) $\left(\frac{I}{s}\right)e^{sT}$ (d) $\left(\frac{I}{s}\right)(1-e^{-sT})$

(vii) The Laplace transform of a delayed unit impulse function $\delta(t-2)$ is
   (a) 1 (b) 0 (c) e^{-2s} (d) s

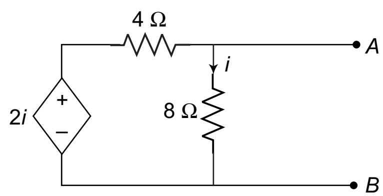
(viii) The convolution of $f(t)$ and $g(t)$ is

(a) $\int_0^t f(t)g(t-\tau)d\tau$ (b) $\int_0^t f(\tau)g(t-\tau)d\tau$ (c) $\int_0^t f(t-\tau)g(t)dt$ (d) $\int_0^t f(t)g(t-\tau)dt$

(ix) When applying the superposition theorem to any circuit,
(a) the voltage source is shorted, the current source is opened
(b) the voltage source is opened, the current source is shorted
(c) both are opened
(d) both are shorted

(x) A high-pass filter circuit is basically
(a) a differentiating circuit with low time constant
(b) a differentiating circuit with large time constant
(c) an integrating circuit with low time constant
(d) an integrating circuit with large time constant

(xi) The Thevenin's equivalent with respect to the terminals A and B would be only a resistance R_{th} equal to



(a) 2.66 Ω (b) 3.2 Ω (c) 8 Ω (d) 12 Ω

Solution:

(i) (a) ramp function (ii) (b) a resistance in parallel with an ideal current source

(iii) (c) $-\frac{1}{2}$ ohm (iv) (d) 1.8 (v) (d) 0.5–0.5cos2x

(vi) (d) $\left(\frac{I}{s}\right)(1-e^{-sT})$ (vii) (c) e^{-2s} (viii) (b) $\int_0^t f(\tau)g(t-\tau)d\tau$

(ix) (a) the voltage source is shorted, the current source is opened

(x) (a) a differentiating circuit with low time constant (xi) (b) 3.2 Ω

GROUP–B
(Short-Answer-Type Questions)

Answer any *three* of the following questions. **3 × 5 = 15**

2. State and prove maximum power transfer theorem for ac network.

Solution: **Maximum Power Transfer Theorem (for ac network):**

Statement: Maximum active power will be delivered from a source to a load when the load impedance is the complex conjugate of the source impedance.

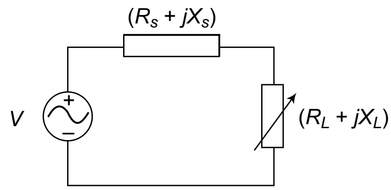
Proof: Let V be the voltage source, $(R_S + jX_S)$ be the internal impedance of the source and $(R_L + jX_L)$ be the load impedance.

$\therefore$ current, $I = \dfrac{V}{Z_S + Z_L} = \dfrac{V}{(R_S + R_L) + j(X_S + X_L)}$ (1)

Power delivered to the load is,

$$P = |I|^2 R_L = \frac{V^2 R_L}{(R_S + R_L)^2 + (X_S + X_L)^2} \quad (2)$$

where, $Z_S = R_S + jX_S$, $Z_L = R_L + jX_L$



Load impedance with variable resistance and variable reactance

For maximum power, $\dfrac{\partial P}{\partial X_L}$ must be zero.
Now,

$$\frac{\partial P}{\partial X_L} = \frac{-2(V)^2 R_L (X_L + X_S)}{\left[(R_L + R_S)^2 + (X_L + X_S)^2\right]^2} = 0$$

From which, $X_L + X_S = 0$ or $\boxed{X_L = -X_S}$

i.e. the reactance of the load impedance is of opposite sign to the reactance of the source impedance.

Putting $X_L = -X_S$ in Eq. (2) $P = \dfrac{V^2 R_L}{(R_L + R_S)^2}$

For maximum power, $\dfrac{\partial P}{\partial R_L} = \dfrac{V^2 (R_L + R_S)^2 - 2V^2 R_L (R_L + R_S)}{(R_L + R_S)^4} = 0$

or, $V^2(R_L + R_S) - 2V^2 R_L = 0$ or $R_L = R_S$

$$\boxed{\therefore \quad Z_L = Z_S\,^*}$$

3. What is time constant of an *R-L* series circuit and what does it signify? Explain it graphically. 2 + 3

Solution:

Time Constant of an *R-L* Series Circuit: The quantity $\tau = \dfrac{L}{R}$ in an *R-L* series circuit is known as the time-constant of the circuit and it is defined in three ways as follows.

Definitions of time-constant (τ)

1. It is the time taken for the current to reach 63% of its final value. Thus, it is a measure of the rapidity with which the steady state is reached.

   Also, at $t = 5\tau$, $i = 0.993 i_s$; the transient is, therefore, said to be practically disappeared in five time constants.

2. The tangent to the equation $i = \dfrac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$ at $t = 0$, intersects the straight line, $i = \dfrac{V}{R}$ at $t = \tau = \dfrac{L}{R}$. Thus, time-constant is the time in which steady state would be reached if the current increases at the initial rate.

Physically, time-constant represents the speed of the response of a circuit. A low value of time-constant represents a fast response and a high value of time-constant represents a sluggish response. Thus, unit of time constant is the unit of time, i.e. second.

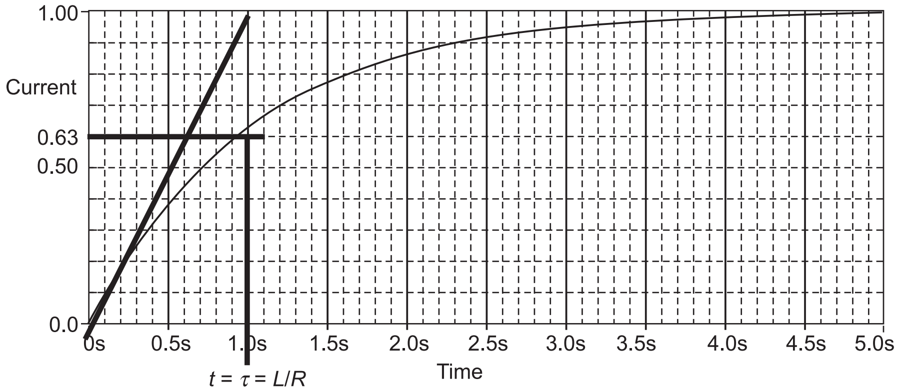
Graphical Representation of Time Constant

The current in an *R-L* series circuit with a dc voltage of *V* is given as,

$$i(t) = i(t) = \frac{V}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right)$$

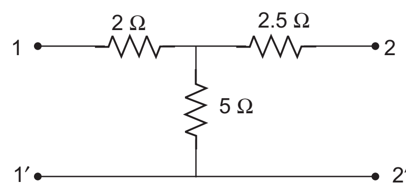
From the current equation at $t = \tau = \frac{L}{R}$, $i = \frac{V}{R}(1 - e^{-1}) = 0.63\frac{V}{R} = 0.63 i_s$

This is the time constant of the circuit as shown in the figure below.



Variation of current with time in R-L series circuit with step input

4. Find the equivalent π-network for the *T*-network as shown in the figure.



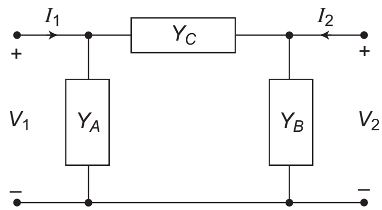
Solution: Let the equivalent π-network have Y_C as the series admittance and Y_A and Y_B as the shunt admittances at port-1 and port-2, respectively.

Now, the *z*-parameters are given as

$$z_{11} = (Z_A + Z_C) = 7\ \Omega,\ z_{12} = z_{21} = Z_C = 5\ \Omega,\ z_{22} = (Z_B + Z_C) = 7.5\ \Omega$$

$$\therefore \qquad \Delta z = (7 \times 7.5 - 5 \times 5) = 27.5\ \Omega^2$$

$$\therefore \qquad y_{11} = \frac{z_{22}}{\Delta z} = \frac{7.5}{27.5}\ \mho$$



$$y_{12} = y_{21} = -\frac{z_C}{\Delta z} = -\frac{5}{27.5}\ \mho$$

$$y_{22} = \frac{z_{11}}{\Delta z} = \frac{7}{27.5}\ \mho$$

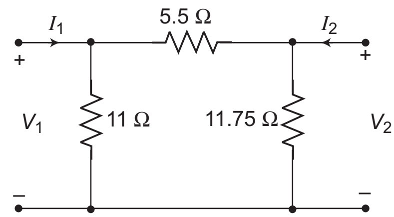
$\therefore$ $$Y_A = (y_{11} + y_{12}) = \frac{2.5}{27.5} = \frac{1}{11}\ \mho$$

$\therefore$ $$Y_B = (y_{22} + y_{12}) = \frac{2}{27.5}\ \mho$$

and $$Y_C = -y_{21} = \frac{5}{27.5} = \frac{2}{11}\ \mho$$

Thus, the impedances of the equivalent π-networks are

$$\left.\begin{aligned} Z_A &= \frac{1}{Y_A} = 11\ \Omega, \\ Z_B &= \frac{1}{Y_B} = 13.75\ \Omega \\ Z_C &= \frac{1}{Y_C} = 5.5\ \Omega \end{aligned}\right\}$$ **Ans.**



Equivalent π-network

NB: This problem can also be solved by using the result of star-delta conversion technique.

$$Z_A = 2 + 5 + \frac{2 \times 5}{2.5} = 11\ \Omega$$

$$Z_B = 2.5 + 5 + \frac{2.5 \times 5}{2} = 13.75\ \Omega$$

$$Z_C = 2 + 2.5 + \frac{2 \times 2.5}{5} = 5.5\ \Omega$$

5. Prove that the Laplace transform of a periodic function with period T_0 is equal to $\dfrac{1}{1 - e^{-T_0 s}}$ times the Laplace transform of the first cycle.

Solution: Let, $f(t)$ – be the periodic function,

T_0 – the time period,

$f_1(t), f_2(t), \ldots, f_n(t)$ – the functions representing the first, second, …, n^{th} cycle, respectively

$\therefore$ $$f(t) = f_1(t) + f_2(t) + \ldots + f_n(t) + \ldots = f_1(t) + f_1(t - T_0) + f_1(t - 2T_0) + \ldots$$

Taking Laplace transform,

$$\begin{aligned} L[f(t)] = F(s) &= L[f_1(t)] + L[f_1(t - T_0)] + L[f_1(t - 2T_0)] + \ldots \\ &= F_1(s) + e^{-T_0 s} F_1(s) + e^{-2T_0 s} F_1(s) + \ldots \\ &= F_1(s) + \left[1 + e^{-T_0 s} + e^{-2T_0 s} + e^{-3T_0 s} + \cdots\right] \end{aligned}$$

$$\boxed{F(s) = F_1(s)\left[\frac{1}{1-e^{-T_0 s}}\right]}$$

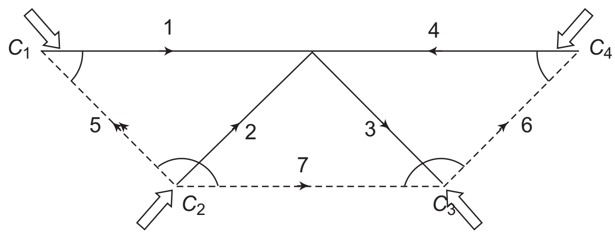
Therefore, it is proved that the Laplace transform of a periodic function with period T_0 is equal to $\dfrac{1}{1-e^{-T_0 s}}$ times the Laplace transform of the first cycle.

6. Draw the oriented graph of a network with fundamental cut-set matrix given below:

$$Q = \begin{array}{c} \begin{array}{cccc|ccc} \multicolumn{4}{c|}{\text{Twigs}} & \multicolumn{3}{c}{\text{Links}} \end{array} \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Solution: The graph has a total of seven branches out which four are tree branches (twigs) and the other three are links.

It is obvious that the graph must have 5 nodes. Hence, the graph is shown below:

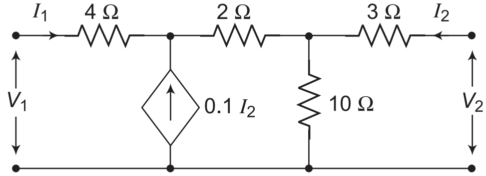


GROUP–C
(Long-Answer-Type Questions)

Answer any *three* questions. **3 × 15 = 45**

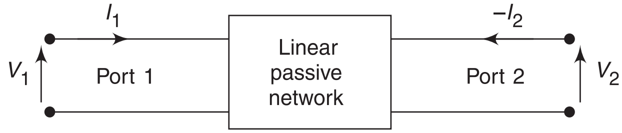
7. (a) What are *ABCD* parameters? Prove that $\Delta T = (AD - BC) = 1$. 7

(b) Find the z-parameter for the network shown in the figure below. Hence, find the h-parameter for the same network. 8



Solution:

(a) ***ABCD* Parameters:** The *ABCD* parameters represent the relation between the input quantities and the output quantities in a two-port network. They are thus voltage-current pairs.



The figure shows two-port current and voltage variables for calculation of transmission line parameters.

However, as the quantities are defined as an input-output relation, the output current is marked as going out rather than as coming into the port.

The transmission parameter matrix may be written as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \quad \text{or,} \quad \begin{aligned} V_1 &= AV_2 - BI_2 \\ I_1 &= CV_2 - DI_2 \end{aligned}$$

The parameters A, B, C, D can be defined in a similar manner with either Port 2 on short circuit or Port 2 on open circuit.

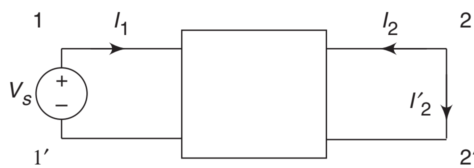
$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \text{Open circuit reverse voltage gain}$$

$$B = -\left.\frac{V_1}{I_2}\right|_{V_2=0} = \text{Short circuit transfer impedance}$$

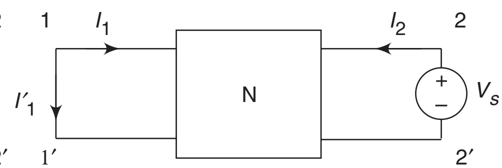
$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \text{Open circuit transfer admittance}$$

$$D = -\left.\frac{I_1}{I_2}\right|_{V_2=0} = \text{Short circuit reverse current gain}$$

- **To Prove *AD* – *BC* = 1**



(a) Reciprocal network



(b) Reciprocal network

From Fig. (a), writing the *ABCD*-parameter equations,

$$\begin{aligned} V_s &= A.0 - B(-I'_2) = BI'_2 \\ I_1 &= C.0 - D(-I'_2) = DI'_2 \end{aligned} \quad \Rightarrow \quad \frac{I'_2}{V_s} = \frac{1}{B} \tag{1}$$

From Fig. (b), writing the *ABCD*-parameter equations,

$$\begin{aligned} 0 &= AV_s - BI_2 \\ -I_1' &= CV_s - DI_2 \end{aligned} \quad \Rightarrow \quad \frac{I_1'}{V_s} = \frac{AD - BC}{B} \tag{2}$$

From the principle of reciprocity from Eq. (1) and (2), we get,

$$\frac{1}{B} = \frac{AD - BC}{B} \qquad \Rightarrow \quad AD - BC = 1$$

$$\boxed{\therefore \quad AD - BC = 1}$$

(b) We consider two cases:

Case (1): When $I_2 = 0$

The circuit is modified as shown in Fig. (a).

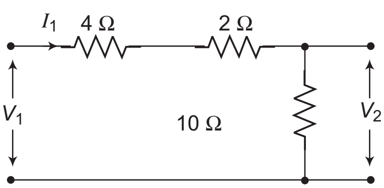


Fig. (a)

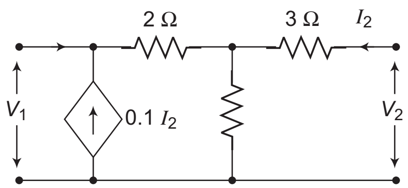


Fig. (b)

By KVL, we have,

$$V_1 = I_1(4 + 2 + 10) = 16 I_1 \Rightarrow z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = 16\ \Omega$$

$$\therefore \qquad V_2 = 10 \times I_1 \qquad \Rightarrow z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = 10\ \Omega$$

Case (2): When $I_1 = 0$

The circuit is modified as shown in Fig. (b).

By KVL, we have,

$$V_2 = 3I_2 + 10 \times (I_2 + 0.1 I_2) = 14 I_2 \Rightarrow z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = 14\ \Omega$$

$$\therefore \qquad V_1 = 2 \times 0.1 I_2 + 10 \times (I_2 + 0.1\, I_2) \Rightarrow z_{12} = \left.\frac{V_1}{I_2}\right|_{I_2=0} = 11.2\ \Omega$$

Hence, the *z*-parameters are given as,

$$[z] = \begin{bmatrix} 16 & 11.2 \\ 10 & 14 \end{bmatrix} (\Omega)$$

By interrelationships, the h-parameters are obtained as,

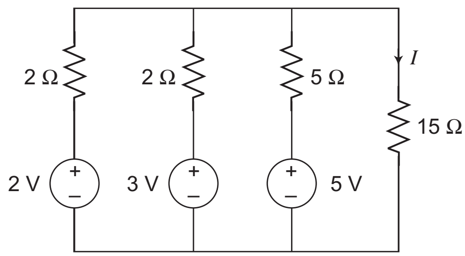
$$h_{11} = \frac{\Delta z}{z_{22}} = \frac{16\times 14 - 11.2\times 10}{14} = 8\ \Omega$$

$$h_{12} = \frac{z_{12}}{z_{22}} = \frac{11.2}{14} = 0.8$$

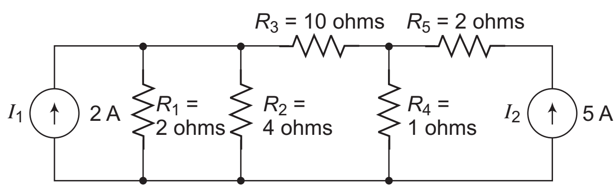
$$h_{13} = -\frac{z_{21}}{z_{22}} = -\frac{10}{14} = 0.714$$

$$h_{22} = \frac{1}{z_{22}} = \frac{1}{14} = 0.0714 \text{ mho}$$

8. (a) State and explain Millman's theorem. Calculate the load current I in the circuit in the figure by Millman's theorem. 2 + 6



(b) What is the power loss in the 10-ohm resistor? Use Thevenin's theorem in the figure below: 7



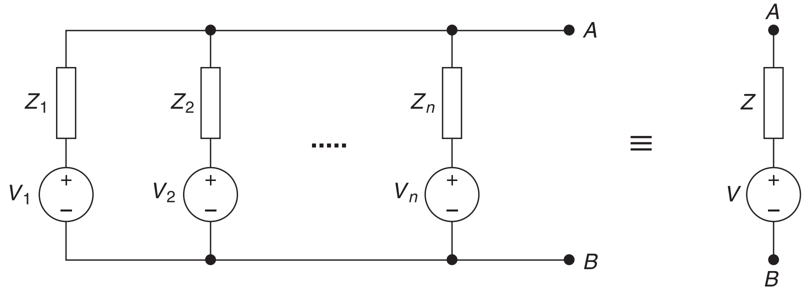
Solution:

(a)

- **Millman's Theorem**

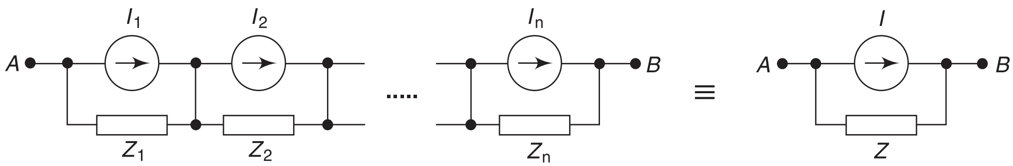
(I) This theorem states that if several ideal voltage sources (V_1, V_2, …) in series with impedances (Z_1, Z_2,…) are connected in parallel then the circuit may be replaced by a single ideal voltage source (V) in series with an impedance (Z), where,

$$V = \frac{\sum_{i=1}^{n} V_i Y_i}{\sum_{i=1}^{n} Y_i} \quad \text{and,} \quad Z = \frac{1}{\sum_{i=1}^{n} Y_i}$$



(II) If several ideal current sources (I_1, I_2,…) in parallel with impedances (Z_1, Z_2, …) are connected in series, then the circuit may be replaced by a single ideal current source (I) in parallel with an impedance (Z), where,

$$I = \frac{\sum_{i=1}^{n} I_i/Y_i}{\sum_{i=1}^{n} 1/Y_i} \quad \text{and,} \quad Y = \frac{1}{\sum_{i=1}^{n} 1/Y_i} \quad \text{or,} \quad Z = \sum_{i=1}^{n} Z_i$$



- **Solution to Numerical Problem**

By Millman's theorem,

$$V = \frac{\sum EY}{\sum Y} = \frac{\frac{2}{2}+\frac{3}{2}+\frac{5}{5}}{\frac{1}{2}+\frac{1}{2}+\frac{1}{5}} = \frac{35}{12} = 2.91667 \text{ V}$$

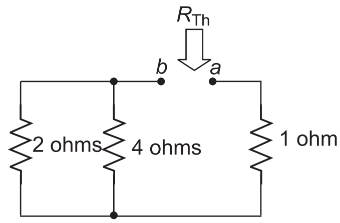
$$Z = \frac{1}{\sum Y} = \frac{1}{\frac{1}{2}+\frac{1}{2}+\frac{1}{5}} = \frac{10}{12} = 0.833\ \Omega$$

$$\therefore \qquad I = \frac{V}{Z+15} = \frac{2.91667}{0.833+15} = 0.184 \text{ A}$$ **Ans.**

(b) We find the Thevenin's equivalent circuit with respect to terminals a and b where the 10 Ω resistance is connected.

Thevenin equivalent resistance is obtained as,

$$R_{Th} = \frac{2\times 4}{2+4} + 1 = \frac{7}{3}\Omega$$

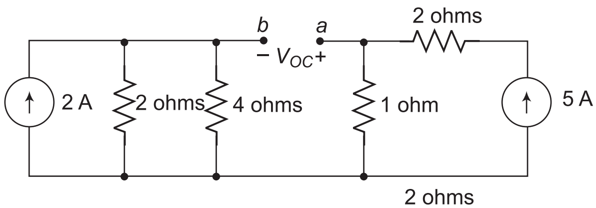


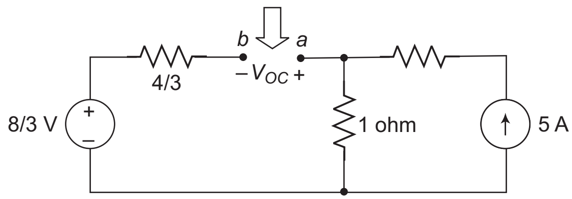
Thevenin voltage (open circuit voltage) is obtained as follows. Changing the current source into voltage source by source transformation, we simplify the circuit.

No, by KVL, we get,

$$-V_{OC} + 1\times 5 - 8/3 = 0$$

$$\Rightarrow \qquad V_{OC} = \frac{7}{3}\text{ V}$$





So, the current through the 10 Ω resistance is given as,

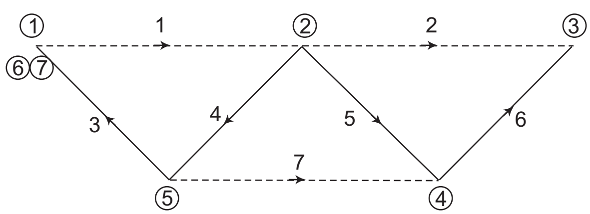
$$I = \frac{V_{OC}}{R_{Th} + R_L} = \frac{7/3}{7/3 + 10} = \frac{7}{37}\text{ A}$$

So, the power loss in the 10 Ω resistance is,

$$P = I^2 R_L = \left(\frac{7}{37}\right)^2 \times 10 = 0.358\text{ W}$$

9. (a) What is a tree? Discuss with a suitable example. 2

(b) A graph is shown in the figure below. Find the tie-set and cut-set matrices and obtain the KCL and KVL equation [bold lines indicate twigs and dotted lines the links]. 6



(c) Explain odd symmetry and even symmetry of periodic waveforms. 4

(d) Find the Fourier transform of $f(t) = e^{-a|t|}$ 3

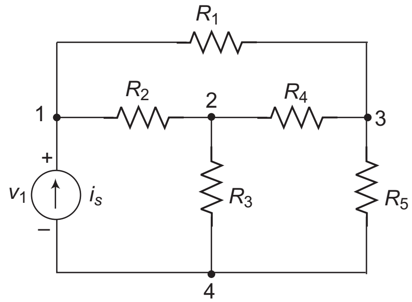
Solution:

(a) **Tree:** For a given connected graph of a network, a connected subgraph is known as a tree of the graph if the subgraph has all the nodes of the graph without containing any loop.

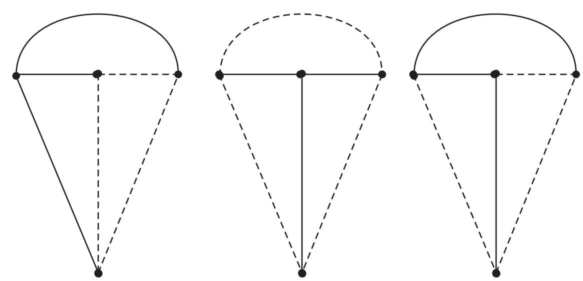
The branches of a tree are called twigs or tree branches. The number of branches or twigs, in any selected tree is always one less than the number of nodes, i.e.

Twigs = (n – 1), where n is the number of nodes of the graph.

For this case, twigs = (4 – 1) = 3 twigs. These are shown by solid lines in Fig. (b).



(a) Circuit

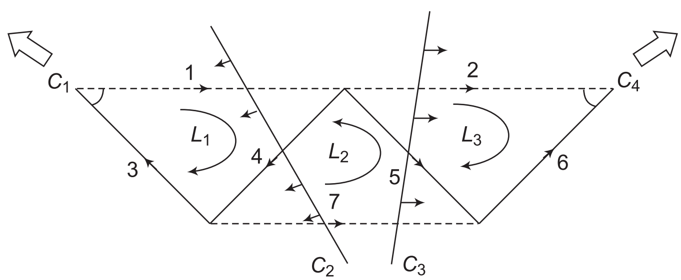


(b) Trees and links of circuit of Fig. (a)

If a graph for a network is known and a particular tree is specified, *the remaining branches* are referred to as the *links*. The *collection of links* is called a *co-tree*. So, co-tree is the complement of a tree. These are shown by dotted lines in Fig. ((b).

The branches of a co-tree may or may not be connected, whereas the branches of a tree are always connected.

(b) The graph has three loops.



The tie-set matrix is given as,

$$B_a = \begin{array}{c} \\ L_1 \\ L_2 \\ L_3 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 \end{bmatrix} \end{array}$$

The graph has 4 fundamental cut-sets. The cut-set matrix is obtained as,

$$Q = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

KCL is written in terms of cut-set matrix as,

$$QI_b = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix} = 0 \quad \Rightarrow \quad \left.\begin{array}{l} -i_1 + i_3 = 0 \\ -i_1 + i_4 - i_7 = 0 \\ i_2 + i_5 + i_7 = 0 \\ i_2 + i_6 = 0 \end{array}\right\}$$ **Ans**

KVL is written in terms of tie-set matrix as,

$$B_a V_b = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_{b1} \\ V_{b2} \\ V_{b3} \\ V_{b4} \\ V_{b5} \\ V_{b6} \\ V_{b7} \end{bmatrix} = 0 \quad \Rightarrow \quad \left.\begin{array}{l} V_{b1} + V_{b3} + V_{b4} = 0 \\ V_{b4} - V_{b5} + V_{b7} = 0 \\ V_{b2} - V_{b5} - V_{b6} = 0 \end{array}\right\}$$ **Ans**

(c) **Odd Symmetry:** A function $f(x)$ is said to be odd if,

$$f(x) = -f(-x)$$

We know that the Fourier coefficients are given as,

$$a_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\left[\int_{-T/2}^{0} f(t)dt + \int_0^{T/2} f(t)dt\right]$$

$$a_n = \frac{2}{T}\int_0^{T/2} [f(x) + f(-x)]\cos n\omega x dx \text{ and } b_n = \frac{2}{T}\int_0^{T/2} [f(x) - f(-x)]\sin n\omega x dx$$

Hence, for odd functions $a_0 = a_n = 0$ and $b_n = \frac{1}{T}\int_0^{T/2} f(x)\sin n\omega x$.

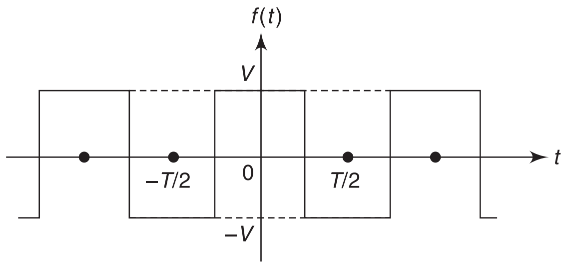
Thus, the *Fourier series expansion of an odd function contains only the sine terms, the constant and the cosine terms being zero.*

Even Symmetry A function $f(x)$ is said to be even, if

$$f(x) = f(-x)$$

$$\therefore \quad a_0 = \frac{2}{T}\int_0^{T/2} f(x)dx$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(x)\cos n\omega x dx$$



Even function

and $b_n = 0$

Thus, the *Fourier series expansion of an even periodic function contains only the cosine terms plus a constant, all sine terms being zero.*

(d) **Fourier Transform of $f(t) = e^{-a|t|}$, for all values of t**

$$F(j\omega) = \mathcal{F}\left[e^{-a|t|}\right] = \int_{-\infty}^{\infty} e^{-a|t|}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

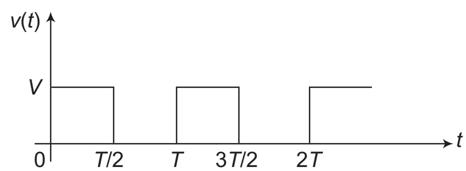
$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{2a}{a^2+\omega^2}$$ **Ans.**

10. (a) Define Fourier transform. How does Fourier transform differ from Laplace transform? 5

(b) What is impulse function? Find its Laplace transform. 3

(c) For the square wave shown in the figure, find the exponential Fourier series. 7



Solution:

(a) **Definition of Fourier Transform:** The Fourier Transform or the Fourier integral of a function $f(t)$ is denoted by $F(j\omega)$ and is defined by,

$$F(j\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad \text{(i)}$$

and the inverse Fourier transform is defined by,

$$f(t) = \mathcal{F}^{-1}[F(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega = \int_{-\infty}^{\infty} F(j2\pi f)e^{j2\pi f}\,df \qquad \text{(ii)}$$

Equations (i) and (ii) form the Fourier transform pair.

Difference between Laplace Transform and Fourier Transform: The defining equations are,

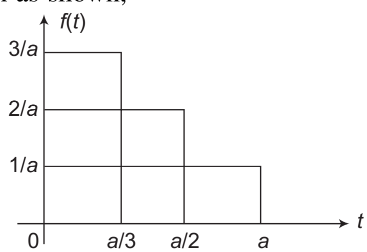
$$F(s) = \int_{0}^{\infty} f(t)e^{-st}\,dt \quad \text{and} \quad F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

The following are some differences and similarities:

1. Laplace transform is one-sided in the interval $0 < t < \infty$ and Fourier transform is double-sided in the interval $-\infty < t < \infty$. Thus, Laplace transform is applicable for positive time functions, $f(t)$, $t > 0$; while Fourier Transform is applicable for functions defined for all times.
2. Laplace transform includes the initial conditions and is applicable for transient analysis; while Fourier transform is only applicable for steady-state analysis.
3. For functions $f(t) = 0$ for $t < 0$ and $\int_{0}^{\infty} |f(t)|\,dt < \infty$, the two transforms are related as, $F(j\omega) = F(s)\big|_{s=j\omega}$. Thus, Laplace transform is associated with the entire s-plane, while, Fourier transform is restricted to the imaginary ($j\omega$) axis.
4. Laplace transform is applicable to a wider range of functions than the Fourier transform. On the other hand, Fourier transforms exist for signals that are not physically realizable and have no Laplace transform.

(b) **Impulse Function:** It is a function of a real variable t, such that the function is zero everywhere except at the instant $t = 0$. Physically, it is a very sharp pulse of infinitesimally small width and very large magnitude, the area under the curve being unity.

We consider a gate function as shown,



The function is compressed along the time-axis and stretched along the y-axis, keeping area under the pulse unity. As $a \to 0$, the value of $\frac{1}{a} \to \infty$ and the resulting function is known as impulse.

It is defined as, $\delta(t) = 0$ for $t \neq 0$

and $\int_{-\infty}^{\infty} \delta(t)dt = 1$

Also, $\delta(t) = \lim_{a \to 0} \frac{1}{a}[u(t) - u(t-a)]$

Laplace Transform of Impulse Function: The Laplace transform of the impulse function is obtained as,

$$L[\delta(t)] = \lim_{a\to 0} L\left\{\frac{1}{a}\left[u(t)-u(t-a)\right]\right\} = \lim_{a\to 0}\frac{1}{a}\left[\frac{1}{s}-\frac{e^{-as}}{s}\right] = \lim_{a\to 0}\frac{1-e^{-as}}{as}$$

$$= \lim_{a\to 0}\frac{se^{-as}}{s} \text{ [by L'Hospital's rule]}$$

$$= 1$$

(c) [WBUT 2012 Q.10 (c)]

11. (a) What are the advantages of active filter over passive filter? 4

(b) Design a high-pass active filter of cut-off frequency 1 kHz with a pass-band gain of 2. 5

(c) Draw the circuit diagram of a first order low-pass filter and find out the expression of the cut-off frequency. 6

Solution:

(a) **Advantages of Active Filter over Passive Filter**

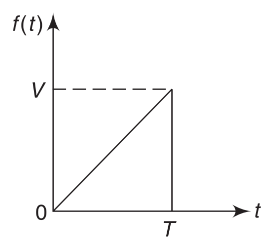
1. *Less Cost:* Active filters are inexpensive as compared to passive filters, due to the variety of cheaper op-amp and the absence of costly inductors.
2. *Gain and Frequency Adjustment Flexibility:* Since an op-amp is capable of providing a gain (which may also be variable), the input signal is not attenuated as it is in a passive filter. In addition, the active filter is easier to tune or adjust.
3. *No Loading Problem:* Active filters provide an excellent isolation between the individual stages due to the high input impedance (ranging from a few kΩ to a several thousand MΩ) and low output impedance (ranging from less than 1 Ω to a few hundred Ω). So, the active filter does not cause loading of the source or load.
4. *Size and Weight:* Active filters are small in size and less bulky (due to the absence of bulky 'L') and are rugged.
5. *Non-floating Input and Output:* Active filters generally have single-ended inputs and outputs which do not 'float' with respect to the system power supply or common. This property is different from that of the passive filters.

(b) Here, $f_c = 1$ kHz, $A_F = 2$

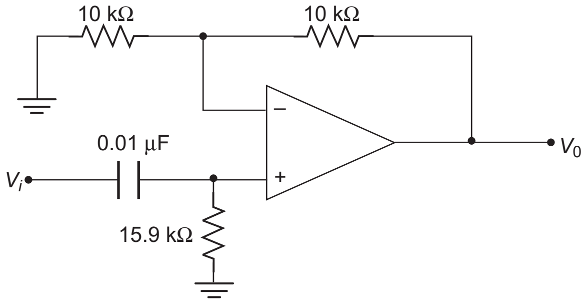
Let, $C = 0.01\ \mu$F

$\therefore$ $R = \dfrac{1}{2\pi f_c C} = \dfrac{1}{2\pi \times 10^3 \times 0.01 \times 10^{-6}} = 15.9$ kΩ

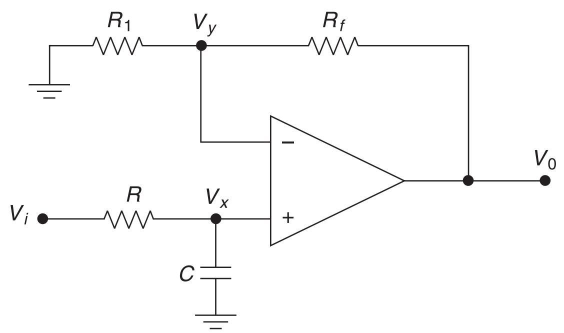
$\because$ $A_F = 2 = \left(1 + R_f / R_1\right) \Rightarrow R_f = R_1 = 10$ kΩ



So, the complete circuit is shown in the figure.



(c) *First-Order Low-Pass Active Filter:* The circuit of the figure is a commonly used low-pass active filter.



The filtering is done by the RC network, and the op-amp is used as a unity-gain amplifier. The resistor R_f (= R) is included for dc offset.
Here, all the voltages V_i, V_x, V_y, V_o are measured with respect to ground.
Since the input impedance of the op-amp is infinite, no current will flow into the input terminals.

$$V_y = \frac{V_0}{R_1 + R_f} \times R_1 \tag{1}$$

According to the voltage-divider rule, the voltage across the capacitor,

$$V_x = \frac{X_c}{R + X_c} V_i ; \qquad X_c = \frac{1}{j\omega C} = \frac{1}{j2\pi fC}$$

$$= \frac{1/j2\pi fC}{R + \dfrac{1}{j2\pi fC}} V_i$$

$$= \frac{Vi}{1 + j2\pi fRC} \tag{2}$$

Since the op-amp gain is infinite,

$\therefore \quad V_x = V_y$

or $$\frac{V_0 R_1}{R_1 + R_f} = \frac{V_i}{1 = j2\pi fRC}$$

$\Rightarrow$ $$\frac{V_0}{V_i} = \frac{\left(1 + \frac{R_f}{R_1}\right)}{1 + j2\pi fRC}$$

where, $A_F = \left(1 + \frac{R_f}{R_1}\right)$ = pass-band gain of the filter

f = frequency of the input signal

A_{cL} = closed-loop gain of the filter as a function of frequency

The gain magnitude,

$$|A_{cL}| = \left|\frac{V_0}{V_i}\right| = \frac{A_F}{\sqrt{1 + \omega_c{}^2 R^2 C^2}} = \frac{A_F}{\sqrt{1 + 4\pi^2 f_c{}^2 R^2 C^2}}$$

For this magnitude to be $\frac{A_F}{\sqrt{2}}$ at $f = f_c$, we have,

$$\frac{A_F}{\sqrt{2}} = \frac{A_F}{\sqrt{1 + (2\pi f_c RC)^2}}$$

or $$\boxed{f_c = \frac{1}{2\pi RC}}$$

This is the cut-off frequency of the low-pass filter.